Dirac Operators in Riemannian Geometry
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ABSTRACT. This text examines the Dirac operator on Riemannian manifolds, especially its connection with the underlying geometry and topology of the manifold. The presentation includes a review of preliminary material, including spin and spin^c structures.

An important link between the geometry and the analysis is provided by estimates for the eigenvalues of the Dirac operator in terms of the scalar curvature and the sectional curvature. Considerations of Killing spinors and solutions of the twistor equation on M lead to results about whether M is an Einstein manifold or conformally equivalent to one. An appendix contains a concise introduction to the Seiberg-Witten invariants, which are a powerful tool for the study of four-manifolds.

This book is suitable as a text for courses in advanced differential geometry and global analysis, and can serve as an introduction for further study in these areas.
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Introduction

It is well-known that a smooth complex-valued function $f : \mathcal{O} \to \mathbb{C}$ defined on an open subset $\mathcal{O} \subset \mathbb{R}^2$ is holomorphic if and only if it satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{with} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Geometrically, we consider $\mathbb{R}^2$ here as flat Euclidean space with fixed orientation. Changing this orientation results in replacing the operator $\frac{\partial}{\partial \bar{z}}$ by the differential operator $\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$. Taking both operators together we obtain a differential operator $P : C^\infty(\mathbb{R}^2; \mathbb{C}^2) \to C^\infty(\mathbb{R}^2; \mathbb{C}^2)$ acting via

$$P \left( \begin{array}{c} f \\ g \end{array} \right) = 2i \left( \begin{array}{c} \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial \bar{z}} \end{array} \right),$$

on pairs of complex-valued functions. An easy calculation leads to the following alternative formula for $P$:

$$P = \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) \frac{\partial}{\partial x} + \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \frac{\partial}{\partial y}.$$

Denoting the matrices occurring in this formula by $\gamma_x$ and $\gamma_y$,

$$\gamma_x = \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right), \quad \gamma_y = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right),$$

yields

$$P = \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y}.$$
as well as
\[ \gamma_x^2 = -E = \gamma_y^2, \quad \gamma_x \gamma_y + \gamma_y \gamma_x = 0. \]
The square of the operator \( P \) coincides with the Laplacian \( \Delta \) on \( \mathbb{R}^2 \):
\[ P^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \Delta. \]
Thus we have found a square root \( P = \sqrt{\Delta} \) of the Laplacian within the class of first order differential operators, and its kernel is, moreover, the space of holomorphic (anti-holomorphic) functions.

In higher-dimensional Euclidean spaces the question whether there exists a square root \( \sqrt{\Delta} \) of the Laplacian was raised in the following discussion by P.A.M. Dirac (1928). Let \( T \) be a free classical particle in \( \mathbb{R}^3 \) with spin \( \frac{1}{2} \) whose motion is to be studied in special relativity. Denoting its mass by \( m \), its energy by \( E \) and its momentum by \( p = \frac{\hbar v}{\sqrt{1-v^2/c^2}} \), we have
\[ E = \sqrt{c^2 p^2 + m^2 c^4}. \]
In quantum mechanics \( T \) is described by a state function \( \psi(t, x) \) defined on \( \mathbb{R}^1 \times \mathbb{R}^3 \), and energy as well as momentum are to be replaced by the differential operators
\[ E \mapsto i\hbar \frac{\partial}{\partial t} \quad \text{and} \quad p \mapsto -i\hbar \text{grad}, \]
respectively. The state function \( \psi \) then becomes a solution of the equation
\[ i\hbar \frac{\partial \psi}{\partial t} = \sqrt{c^2 \hbar^2 \Delta + m^2 c^4} \psi \]
involving the 3-dimensional Laplacian \( \Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \). Mathematically speaking we now move to an \( n \)-dimensional Euclidean space and look for a square root \( P = \sqrt{\Delta} \) of the Laplacian \( \Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \). The obvious assumption that \( P \) should be a first order differential operator with constant coefficients leads to the ansatz
\[ P = \sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i}. \]
Now the equation \( P^2 = \Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) holds if and only if the coefficients \( \gamma_i \) of \( P \) satisfy the conditions
\[ \gamma_i^2 = -E, \quad i=1,\ldots,n; \quad \gamma_i \gamma_j + \gamma_j \gamma_i = 0, \quad i \neq j. \]
For \( n = 3 \), there is an obvious solution to these equations. The vector space \( \mathbb{C}^2 \) can be identified with the set of quaternions via \( \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 + jz_2 \), and \( \gamma_1, \gamma_2, \gamma_3 : \mathbb{C}^2 \rightarrow \mathbb{H} = \mathbb{C}^2 \) then correspond to multiplication by the quaternions \( i, j, k \in \mathbb{H} \), respectively. Writing these as complex \((2 \times 2)\)-matrices, we obtain
\[
\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]
The algebra multiplicatively generated by \( n \) elements \( \gamma_1, \ldots, \gamma_n \) satisfying the relations
\[
\gamma_i^2 = -1, \quad \gamma_i \gamma_j + \gamma_j \gamma_i = 0 \quad (i \neq j),
\]
is called the Clifford algebra \( \mathcal{C}_n \) (W.K. Clifford, 1845-1879) of the negative definite quadratic form \((\mathbb{R}^n, -x_1^2 - \ldots - x_n^2)\). Thus, the question whether there is a square root \( \sqrt{\Delta} \) of the Laplacian leads to the study of complex representations \( \kappa : \mathcal{C}_n \rightarrow \text{End}(V) \) of the Clifford algebra. It turns out that \( \mathcal{C}_n \) has a smallest representation of dimension \( \dim_\mathbb{C} V = 2^{[\frac{n}{2}]} \). The corresponding vector space is denoted by \( \Delta_n \) and its elements are the Dirac spinors. Moreover, \( \sqrt{\Delta} \) is a constant coefficient first order differential operator acting on the space \( \mathcal{C}^\infty(\mathbb{R}^n; \Delta_n) \) of smooth \( \Delta_n \)-valued functions on \( \mathbb{R}^n \).

Spinors can be multiplied by vectors from Euclidean space. In order to define this product we represent a vector \( x \in \mathbb{R}^n \) as a linear combination with respect to an orthonormal basis \( e_1, \ldots, e_n \),
\[
x = \sum_{i=1}^{n} x^i e_i,
\]
and then define its product \( x \cdot \psi \) by a spinor \( \psi \in \Delta_n \) as
\[
x \cdot \psi = \sum_{i=1}^{n} x^i \kappa(\gamma_i)(\psi).
\]
From the defining relations of the Clifford algebra one immediately deduces the formula
\[
x \cdot (x \cdot \psi) = -||x||^2 \psi.
\]
In particular, the product \( x \cdot \psi \) vanishes if and only if either the vector \( x \in \mathbb{R}^n \) or the spinor \( \psi \in \Delta_n \) is equal to zero. There is no non-trivial representation \( \varepsilon \) of the linear or the orthogonal group in the space \( \Delta_n \) of spinors that is compatible with Clifford multiplication, i.e. which satisfies the relation
\[
A(x) \cdot \varepsilon(A)(\psi) = \varepsilon(A)(x \cdot \psi).
\]
for every $A \in SO(n; \mathbb{R})$, $x \in \mathbb{R}^n$ and $\psi \in \Delta_n$. Hence spinors on Riemannian manifolds cannot be defined as sections of a vector bundle that is associated with the frame bundle of the manifold. It is for this reason that in differential geometry the question to what extent the concept of spinors could be transferred from flat space to general Riemannian manifolds remained open for decades. In 1938 Elie Cartan expressed this difficulty in his book "Leçons sur la théorie des spineurs" with the following words:

"With the geometric sense we have given to the word 'spinor' it is impossible to introduce fields of spinors into the classical Riemannian technique."

Only the development of the framework of principal fibre bundles and their associated bundles as well as the general theory of connections within differential geometry at the end of the forties made it possible to overcome this difficulty. The group $SO(n; \mathbb{R})$ is not simply connected. For $n \geq 3$ its universal covering, the group denoted by $Spin(n)$, is compact and covers $SO(n; \mathbb{R})$ twice. On the other hand, there exists a representation $\bar{\sigma} : Spin(n) \rightarrow GL(\Delta_n)$ of the spin group which is compatible with Clifford multiplication. Considering now those special Riemannian manifolds $M^n$, today called spin manifolds, the frame bundle of which allows a reduction to the double cover $Spin(n)$ of the structure group $SO(n; \mathbb{R})$, we can define the vector bundle $S$ associated with this reduction via the representation $\bar{\sigma} : Spin(n) \rightarrow GL(\Delta_n)$, the so-called spinor bundle of $M^n$. Then spinor fields over $M^n$ are sections of the bundle $S$ and, as in the Euclidean case, the Dirac operator $D$ can be introduced by the formula

$$D\psi = \sum_{i=1}^{n} e_i \cdot \nabla e_i \psi.$$ 

Here $\nabla$ denotes the covariant derivative corresponding to the Levi-Civita connection of the Riemannian manifold.

Therefore, spinor fields and Dirac operators cannot be introduced on every Riemannian space; but, nevertheless, they can be introduced for a large class. The existence of a $Spin(n)$-reduction of the frame bundle of $M^n$ translates into a topological condition on the manifold, i.e. the first two Stiefel-Whitney classes have to vanish:

$$w_1(M^n) = 0 = w_2(M^n).$$

In dimension $n = 4$, for a compact simply connected manifold $M^4$, this topological condition is equivalent to the condition that the intersection form on $H^2(M^4; \mathbb{Z})$, considered as a quadratic form over the ring $\mathbb{Z}$, is even and unimodular. The algebraic theory of quadratic $\mathbb{Z}$-forms then implies that the signature $\sigma$ is divisible by 8. Surprisingly, in 1952 Rokhlin proved
a further divisibility by 2: the signature $\sigma(M^4)$ of a smooth compact 4-dimensional spin manifold $M^4$ is divisible by 16:

$$\sigma(M^4)/16 \in \mathbb{Z}.$$ 

This additional divisibility of the signature of a 4-dimensional spin manifold, which does not result from purely algebraic considerations, was an essential aspect for the introduction of spinor fields and Dirac operators into mathematics. The consideration behind that may be outlined as follows. Could it be possible that there exists an elliptic operator $P$ on every compact smooth 4-dimensional manifold with even intersection form on $H^2(M^4; \mathbb{Z})$, the index of which coincides with $\sigma/16$? Today we know the answer to that question: it is essentially given by the Dirac operator on a spin manifold, eventually introduced for Riemannian manifolds by M.F. Atiyah in 1962 in connection with his elaboration of the index theory for elliptic operators. Since then it has occurred in many branches of mathematics and has become one of the basic elliptic differential operators in analysis and geometry.

This book was written after a one-semester course held at Humboldt-University in Berlin during 1996/97. It contains an introduction into the theory of spinors and Dirac operators on Riemannian manifolds. The reader is assumed to have only basic knowledge of algebra and geometry, such as a two or three year study in mathematics or physics should provide. The presentation starts with an algebraic part comprising Clifford algebras, spin groups and the spin representation. The topological aspects concerning the existence and classification of spin reductions of principal $SO(n)$-bundles are discussed in Chapter 2. Here the approach essentially requires only elementary covering theory of topological spaces. At the same time, each result will also be translated into the cohomological language of characteristic classes. The subsequent Chapter 3 deals with analysis in the spinor bundle, the twistor operator and the Dirac operator in detail. Here the general techniques of principal bundles and the theory of connections are applied systematically. To make the book more self-contained, these results of modern differential geometry are presented without proof in Appendix B. Chapter 4 contains special proofs for the analytic properties of Dirac operators (essential self-adjointness, Fredholm property) avoiding the general theory for elliptic pseudo-differential operators. Eigenvalue estimates and solution spaces of special spinorial field equations (Killing spinors, twistor spinors) are the topic of Chapter 5. We mainly discuss the general approach, referring to the literature for detailed investigations of these problems. The book is concluded in Appendix A with an extended version of a talk on

Since the eighties a group of younger mathematicians at Humboldt-University in Berlin has been working on spectral properties of Dirac operators and solution spaces of spinorial field equations. Many of the results from this period are collected in the references. On the other hand, the present book may serve as an introduction for a closer study. I would like to thank all those students and colleagues whose remarks and hints had an impact on the contents of this text in various ways.

I am particularly grateful to Dr. Ines Kath for her careful and detailed corrections of the text, and to Heike Pahlisch, whose typing of the manuscript took into account every single wish.

Thomas Friedrich
Berlin, March 1997

The English translation of this book has been prepared in the beginning of the year 2000. It does not differ essentially from the original text, although I made many changes in details which are not worth listing. During the last three years many new results have been published in this still dynamic area of mathematics. I included the corresponding references in the bibliography of the translation. During the academic year 1996/97 Dr. Andreas Nestke provided the exercises for students of my lectures at Humboldt University which furnished the starting point for this book. Two years later he had to leave the University. I thank him, as well as Dr. Ilka Agricola and Heike Pahlisch, for all the work and help related with the preparation of the English edition of this book.

Thomas Friedrich
Berlin, March 2000
Chapter 1

Clifford Algebras and Spin Representation

1.1. Linear algebra of quadratic forms

We will start by recalling some facts from linear algebra. Let $\mathbb{K}$ be a field of characteristic $\neq 2$. A bilinear form is a pair $(V, B)$ consisting of a $\mathbb{K}$-vector space $V$ and a symmetric bilinear map

$$B : V \times V \rightarrow \mathbb{K}.$$ 

The function $Q : V \rightarrow \mathbb{K}$ defined by $Q(v) = B(v, v)$ is the corresponding quadratic form. Thus,

$$Q(\lambda v) = \lambda^2 Q(v) \quad v \in V, \quad \lambda \in \mathbb{K}.$$ 

On the other hand, $B$ can also be expressed by $Q$:

$$B(v_1, v_2) = \frac{1}{2} [Q(v_1 + v_2) - Q(v_1) - Q(v_2)],$$

hence we will often identify $B$ and $Q$. A bilinear form $(V, B)$ determines a linear map from the vector space $V$ to its dual $V^*$,

$$V \ni v \mapsto B(v, \cdot) \in V^*.$$ 

$(V, B)$ is called a non-degenerate bilinear form, if this map is injective. So, $(V, B)$ is a non-degenerate bilinear form if and only if

$$\forall \ 0 \neq v \in V \ \exists \ w \in V : B(v, w) \neq 0.$$ 

Let $V$ be a finite-dimensional $\mathbb{K}$-vector space and $v_1, \ldots, v_n$ ($n = \dim_{\mathbb{K}} V$) a basis. Then the matrix

$$M(V, B) = (B(v_i, v_j))_{i,j=1}^n$$
is symmetric. It is called the matrix of the form.

**Definition.** The rank of the bilinear form \((V, B)\) is defined to be the rank of the matrix \(M(V, B)\), \(\text{rank}(V, B) := \text{rank}(M(V, B))\).

**Theorem (Lagrange).** Let \((V, B)\) be a finite-dimensional bilinear form. Then there exists a basis \(v_1, \ldots, v_n\) for the vector space \(V\) with the property that the matrix \(M(V, B)\) is diagonal (bases with this property are called canonical):

\[
M(V, B) = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & & \\
& \ddots & \ddots & 0 \\
0 & & \cdots & \lambda_r
\end{pmatrix}, \quad r = \text{rank}(V, B).
\]

If the field is \(K = \mathbb{R}\) or \(\mathbb{C}\), then even more holds:

**Theorem (Sylvester).** For every quadratic form over \(\mathbb{R}\) there exists a canonical basis with respect to which the form has the following matrix:

\[
M(V, B) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
n & -1 & \cdots & 0 \\
& & \ddots & \ddots \\
0 & & & 1
\end{pmatrix}.
\]

The number \(p\) of \((+1)\)-entries and the number \(q\) of \((-1)\)-entries in this diagonal matrix do not depend on the choice of this basis. The pair \((p, q)\) is called the signature of the form, the number \(q\) its index.
Theorem. For every quadratic \( \mathbb{C} \)-form there exists a canonical basis with respect to which the form has the following matrix:

\[
M(V, B) = \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}.
\]

We now consider a finite-dimensional non-degenerate bilinear form \((V, B)\). If \( W \subset V \) is a subspace, we define the \( B \)-orthogonal complement \( W^\perp \) as

\[
W^\perp = \{ v \in V : B(v, w) = 0 \ \forall w \in W \}.
\]

\( W \) is called an isotropic subspace if \( W \cap W^\perp \neq \{0\} \). \( W \) is called a null-subspace, if \( W \subset W^\perp \) holds. Obviously, \( W \) is a null-subspace if and only if \( Q|_W \equiv 0 \) (or \( B|_{W \times W} \equiv 0 \)), i.e. \( Q(B) \) vanishes identically on \( W \). Every null-subspace is isotropic but not vice versa.

Theorem (Witt Decomposition). Let \((V, B)\) be a finite-dimensional non-degenerate quadratic \( \mathbb{K} \)-form and \( W \subset V \) a maximal null-subspace. Then there exists a maximal null-subspace \( U \subset V \) such that

a) \( \dim U = \dim W \), \( U \cap W = \{0\} \).

b) \( V = W \oplus U \oplus (W \oplus U)^\perp \).

c) For every vector \( 0 \neq v \in (W \oplus U)^\perp \) \( Q(v) \neq 0 \).

Moreover, for every basis \( w_1 \ldots w_k \) in \( W \) there exists a basis \( u_1 \ldots u_k \) in \( U \) satisfying

\[
B(u_i, w_j) = \delta_{ij}, \quad 1 \leq i, j \leq k \quad \text{(Witt basis)}.
\]

Corollary. Let \( \mathbb{K} \) be an algebraically closed field and \((V, B)\) a finite-dimensional non-degenerate form. Then for every maximal null-subspace \( W \)

\[
\dim W \leq \left\lfloor \frac{\dim V}{2} \right\rfloor.
\]

Proof. Consider the Witt decomposition

\[
V = W \oplus U \oplus (W \oplus U)^\perp.
\]

It is sufficient to prove \( \dim(W \oplus U)^\perp \leq 1 \). Let \( v, v' \in (W \oplus U)^\perp \) be two non-trivial vectors. According to the preceding proposition, Part c), we have for \( \lambda, \mu \in \mathbb{K} \)

\[
\lambda v + \mu v' = 0 \iff Q(\lambda v + \mu v') = 0.
\]
For fixed $0 \neq \mu \in \mathbb{K}$ the quadratic equation in $\lambda$

$$\frac{Q(\lambda v + \mu v')}{Q(v)} = \lambda^2 + \frac{2\mu B(v, v')}{Q(v)} \lambda + \frac{\mu^2 Q(v')}{Q(v)} = 0$$

has a solution. Hence, $Q(\lambda v + \mu v') = 0$, i.e. $\lambda v + \mu v' = 0$, so $v$ and $v'$ are proportional.$\blacksquare$

### 1.2. The Clifford algebra of a quadratic form

Let $(V, Q)$ be a quadratic form over the field $\mathbb{K}$.

**Definition.** A pair $(C(Q), j)$ is called a *Clifford algebra* for $(V, Q)$ if

1) $C(Q)$ is an associative $\mathbb{K}$-algebra with $1$;

2) $j : V \to C(Q)$ is a linear map and

$$j(v)^2 = Q(v) \cdot 1$$

for all $v \in V$;

3) if $A$ is another $\mathbb{K}$-algebra with $1$ and $u : V \to A$ a linear map satisfying $u(v)^2 = Q(v) \cdot 1$, then there exists one and only one algebra homomorphism $\tilde{U} : C(Q) \to A$ such that $u = \tilde{U} \circ j$.

\[
\begin{array}{c}
V \\
\downarrow j \\
C(Q) \\
\downarrow \tilde{U}
\end{array}
\quad \quad
\begin{array}{c}
A \\
\downarrow u
\end{array}
\]

**Proposition.**

a) For every quadratic form $(V, Q)$ there exists a Clifford algebra $(C(Q), j)$.

b) If $(C(Q), j)$ and $(C'(Q), j')$ are both Clifford algebras for the same quadratic form $(V, Q)$, then there exists an isomorphism $f : C(Q) \to C'(Q)$ of the algebras satisfying $f \circ j = j'$.

\[
\begin{array}{c}
C(Q) \\
\downarrow j \\
V
\end{array}
\quad \quad
\begin{array}{c}
\downarrow j' \\
C'(Q)
\end{array}
\quad \quad
\begin{array}{c}
f
\end{array}
\]
1.2. The Clifford algebra of a quadratic form

Proof. To show existence, consider the tensor algebra $T(V) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus \ldots$ of the vector space $V$, and denote by $I(Q)$ the two-sided ideal generated by all elements of the form

$$\{v \otimes v - Q(v) : v \in V\}.$$ 

Set $C(Q) = T(V)/I(Q)$. If $\pi : T(V) \to C(Q)$ denotes the projection and $i : V \to T(V)$ the natural embedding of the vector space into its tensor algebra, then

$$j = \pi \circ i$$

determines a linear map $j : V \to C(Q)$ for which, by construction, $j(v)^2 = Q(v) \cdot 1$. Moreover, each linear map $u : V \to A$ into any algebra extends via

$$U(v_1 \otimes \ldots \otimes v_k) = u(v_1) \ldots u(v_k)$$

to an algebra homomorphism $U : T(V) \to A$. If, in addition, $u(v)^2 = Q(v) \cdot 1$, then we have $I(Q) \subset \ker(U)$, and thus $U$ induces a homomorphism $\tilde{U} : C(Q) \to A$ satisfying the relation we were looking for. Given another homomorphism $\tilde{U}_1 : C(Q) \to A$ satisfying

$$\tilde{U}_1 \circ j = U \circ j.$$

then $u = \tilde{U}_1 \circ j = \tilde{U} \circ j$. Hence $\tilde{U}$ and $\tilde{U}_1$ coincide on $V \subset C(Q)$. On the other hand, the vectors from $V$ generate the tensor algebra $T(V)$ multiplicatively and hence the algebra $C(Q)$ as well. Thus we immediately conclude $\tilde{U} = \tilde{U}_1$. Uniqueness is a straightforward consequence of the third defining property of a Clifford algebra.

Corollary. The linear map $j : V \to C(Q)$ from the vector space $V$ of the quadratic form into its Clifford algebra is injective. The set $j(V) \subset C(Q)$ generates the algebra multiplicatively.

For this reason we will often view the vector space $V$ as a linear subspace of $C(Q)$.

Proposition. The Clifford algebra $C(Q)$ of a quadratic form is equipped with an involution $\beta : C(Q) \to C(Q)$ such that

a) $\beta$ is an algebra homomorphism and an involution, i.e. $\beta^2 = \text{Id}$.

b) Setting $C^0(Q) = \{x \in C(Q) : \beta(x) = x\}$, $C^1(Q) = \{x \in C(Q) : \beta(x) = -x\}$ we have the splitting

$$C(Q) = C^0(Q) \oplus C^1(Q).$$
and the relations $C^0(Q) \cdot C^0(Q) \subset C^0(Q)$, $C^0(Q) \cdot C^1(Q) \subset C^1(Q)$ as well as $C^1(Q) \cdot C^1(Q) \subset C^0(Q)$.

In particular, $C^0(Q) \subset C(Q)$ is a subalgebra.

**Proof.** Consider the linear map $u : V \to C(Q)$, $u(v) = -j(v)$. Since $u^2(v) = (-j(v))^2 = j(v)^2 = Q(v) \cdot 1$, there exists an algebra homomorphism $\beta : C(Q) \to C(Q)$ satisfying

$$\beta \circ j(v) = -j(v)$$

for all $v \in V$. Since, by construction,

$$(\beta \circ \beta \circ j)(v) = \beta(-j(v)) = -\beta j(v) = j(v),$$

$\beta^2$ is the identity on the set $j(V) \subset C(Q)$. Thus $\beta^2 = \text{Id}$ holds in general. $\square$

In addition to the involution $\beta : C(Q) \to C(Q)$ each Clifford algebra also carries an anti-involution $\gamma : C(Q) \to C(Q)$. To construct it, we begin with a preliminary remark: If $A$ is a $K$-algebra, then one can define a new algebra $\hat{A}$ on the set $A$ by introducing the product

$$x \ast y := y \cdot x.$$ 

Now let $(V, Q)$ be a quadratic form and $C(Q)$ its Clifford algebra. Consider the algebra $A = \overline{C(Q)}$. For the linear map

$$V \xrightarrow{j} A = \overline{C(Q)}$$

the relation $j(v) \ast j(v) = j(v) \cdot j(v) = Q(v) \cdot 1$ holds in the algebra $A$. Thus there exists a unique algebra homomorphism

$$\gamma : C(Q) \to \overline{C(Q)}$$

such that

$$j(v) = \gamma j(v), \quad v \in V.$$ 

The properties of $\gamma$ as a map from $C(Q)$ to itself are stated in the following

**Proposition.** For every Clifford algebra there exists a linear map $\gamma : C(Q) \to C(Q)$ with the following properties:

1) $\gamma$ is linear.
2) $\gamma \circ \gamma = \text{Id}$ ($\gamma$ is an involution).
3) $\gamma(v) = v$ for all $v \in V \subset C(Q)$.
4) $\gamma(x \cdot y) = \gamma(y) \gamma(x)$, $x, y \in C(Q)$. 

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The properties of $\gamma$ as a map from $C(Q)$ to itself are stated in the following
Given two bilinear forms \((V_1, B_1)\) and \((V_2, B_2)\), their direct sum is defined as the bilinear form \((V, B)\) with
\[
V = V_1 \oplus V_2, \quad B(V_1, V_2) = 0, \quad B|_{V_1 \times V_1} = B_1, \quad B|_{V_2 \times V_2} = B_2.
\]
Correspondingly, we will consider the direct sum \(Q_1 \oplus Q_2\) of two quadratic forms.

**Proposition.** The Clifford algebra \(C(Q_1 \oplus Q_2)\) is isomorphic to the tensor product \(C(Q_1) \hat{\otimes} C(Q_2)\),
\[
C(Q_1 \oplus Q_2) = C(Q_1) \hat{\otimes} C(Q_2).
\]

**Remark.** The tensor product \(C(Q_1) \hat{\otimes} C(Q_2)\) is to be understood as a tensor product of \(\mathbb{Z}_2\)-graded algebras. For two \(\mathbb{Z}_2\)-graded algebras \(A = A^0 \oplus A^1, B = B^0 \oplus B^1\) the tensor product \(A \hat{\otimes} B\) is the following \(\mathbb{Z}_2\)-graded algebra:
\[
(A \otimes B)^0 = (A^0 \otimes B^0) \oplus (A^1 \otimes B^1), \quad (A \otimes B)^1 = (A^1 \otimes B^0) \oplus (A^0 \otimes B^1)
\]
with product
\[
(a \otimes b^j) \cdot (a^i \otimes b) = (-1)^{ij}(aa^i) \otimes (b^j b)
\]
for any \(a \in A, b \in B, a^i \in A^i\) and \(b^j \in B^j\).

**Proof of the proposition.** Consider the linear map
\[
u : V_1 \oplus V_2 \longrightarrow C(Q_1) \otimes C(Q_2)
\]
defined by the formula
\[
u(v_1 + v_2) = j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2).
\]
The multiplication rule in the algebra \(C(Q_1) \otimes C(Q_2)\) implies
\[
u^2(v_1 + v_2) = (v_1 \otimes 1 + 1 \otimes v_2)^2 = v_1^2 \otimes 1 + v_1 \otimes v_2 - v_1 \otimes v_2 + 1 \otimes v_2^2,
\]
since \(1 \in C^0(Q_1), v_1 \in C^1(Q_1),\) and \(v_2 \in C^1(Q_2)\) belong to the corresponding sets of the \(\mathbb{Z}_2\)-grading. Thus,
\[
u^2(v_1 + v_2) = (Q_1(v_1) + Q_1(v_2))1 \otimes 1 = Q(v_1 + v_2) \cdot 1.
\]
Therefore, \(\nu\) induces an algebra homomorphism
\[
\tilde{\nu} : C(Q_1 \oplus Q_2) \longrightarrow C(Q_1) \otimes C(Q_2)
\]
furnishing the isomorphism to be constructed.

**Proposition.** Let \(V\) be an \(n\)-dimensional vector space. Then the vector space \(C(Q)\) has dimension \(2^n\),
\[
\dim_{\mathbb{K}} C(Q) = 2^n.
\]
Proof. By the Lagrange theorem the quadratic form is the sum of \( n \) one-dimensional quadratic forms:

\[ Q = Q_1 \oplus \ldots \oplus Q_n. \]

But the Clifford algebra of a one-dimensional quadratic form \( B : \mathbb{K} \times \mathbb{K} \to \mathbb{K} \) is easily computed,

\[ C^0(Q) = \mathbb{K}, \quad C^1(Q) = \mathbb{K} \cdot e, \quad \text{and} \quad e^2 = Q(1). \]

Hence, \( \dim_{\mathbb{K}} C(Q) = 2 \) for a one-dimensional quadratic form. From the last proposition we conclude

\[ C(Q) = C(Q_1) \otimes \ldots \otimes C(Q_n) \]

and hence \( \dim_{\mathbb{K}} C(Q) = 2^n. \)

Proposition. Let \((V, B)\) be a quadratic form and \(v_1, \ldots, v_n\) a basis of \(V\) such that

\[ B(v_i, v_j) = 0, \quad i \neq j. \]

Then the Clifford algebra \( C(Q) \) is multiplicatively generated by the elements \(v_1, \ldots, v_n \in V \subset C(Q)\) which satisfy the relations

\[ v_i^2 = Q(v_i), \quad v_i v_j + v_j v_i = 0, \quad i \neq j. \]

A particular basis of the vector space \( C(Q) \) is formed by the elements \(1\) and \(v_{i_1} \cdot \ldots \cdot v_{i_s}\), where \(1 \leq i_1 < i_2 < \cdots < i_s \leq n\) with \(1 \leq s \leq n\).

Proof. \(V \subset C(Q)\) generates \(C(Q)\) multiplicatively, and \(v_1, \ldots, v_n\) are a basis of the vector space \(V\). Hence the vectors \(v_1, \ldots, v_n\) generate the algebra \(C(Q)\), too. Moreover, \(v_i^2 = Q(v_i)\) for trivial reasons, and

\[ (v_i + v_j)^2 = Q(v_i + v_j) = Q(v_i) + Q(v_j) = v_i^2 + v_j^2 \]

immediately implies

\[ v_i v_j + v_j v_i = 0, \quad i \neq j. \]

These \(2^n\) elements generate \(C(Q)\) linearly. Finally, since \(\dim C(Q) = 2^n\), this system has to be a basis.

Example. Let us consider the trivial quadratic form \(Q \equiv 0\) on the vector space \(V\). Then \(C(Q) = \Lambda^*(V)\) is nothing but the exterior algebra of \(V\).

Example. Let \(\mathbb{K} = \mathbb{R}\) and take \(V = \mathbb{R}^2\) with the quadratic form \(Q = -x^2 - y^2\). Then \(C(Q) = \mathbb{H}\) is the algebra of quaternions.
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Proof. $C(\mathbb{R}^2, Q)$ is generated by $e_1, e_2 \in \mathbb{R}^2$. Thus $1, e_1, e_2, e_1 \cdot e_2$ are a basis of the vector space $C(Q)$. With the notation

$$i := e_1, \quad j := e_2, \quad k := e_1 \cdot e_2,$$

the fundamental relations take the form

$$i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j, \quad j^2 = k^2 = -1,$$

since $e_1^2 = -1 = e_2^2$, $e_1 e_2 = -e_2 e_1$. □

Example. Consider $\mathbb{K} = \mathbb{R}, V = \mathbb{R}^1$ and $Q = -x^2$. In this case, $C(Q)$ coincides with the algebra of complex numbers.

Proof. $1$ and $e_1 \in \mathbb{R}^1$ form a special linear basis of $C(Q)$ subject to the single relation $e_1^2 = -1$. □

Next we want to determine the centers of $C(Q)$ and $C^0(Q)$, respectively, in the case of a non-degenerate form. In order to achieve this, consider a basis $v_1, \ldots, v_n$ of the vector space $V$ such that

$$B(v_i, v_i) = \lambda_i \neq 0 \quad \text{and} \quad B(v_i, v_j) = 0 \quad \text{for} \quad i \neq j, \quad i = 1, \ldots, n.$$

Let $\mathcal{P}$ be the set of all strictly ordered subsets of $\{1, \ldots, n\}$: an element of $\mathcal{P}$ is a subset $P = \{p_1, \ldots, p_k\}$ where $1 \leq p_1 < p_2 < \ldots < p_k \leq n$. As an abbreviation we set

$$v_P = \left\{ v_{p_1} \cdot \ldots \cdot v_{p_k} : P = \{p_1 < p_2 < \ldots < p_k\} \right\}.$$

Lemma. For $P, R \in \mathcal{P}$ the equality

$$v_P v_R v_P^{-1} = (-1)^{|P||R|-|P \cap R|} v_R$$

holds in $C(Q)$, where $| \cdot |$ denotes the cardinality of the corresponding set.

Proof. Note that the element $v_P$ is invertible in $C(Q)$. Its inverse is given by

$$v_P^{-1} = \frac{1}{\lambda_{p_1}} \cdot \ldots \cdot \frac{1}{\lambda_{p_k}} v_{p_k} \cdot \ldots \cdot v_{p_1}.$$

The formula to be proved then immediately follows from the relation $v_i \cdot v_j = -v_j \cdot v_i$ for $i \neq j$. □

Proposition.

1) The center of the algebra $C(Q)$ is equal to

$$\mathcal{Z}(C(Q)) = \begin{cases} \mathbb{K} & \text{for dim} \, V \text{ even,} \\ \mathbb{K} \oplus \mathbb{K}[v_1 \cdot \ldots \cdot v_n] & \text{for dim} \, V \text{ odd.} \end{cases}$$
2) The center of the algebra $C^0(Q)$ is equal to

$$Z(C^0(Q)) = \begin{cases} \mathbb{K} \oplus \mathbb{K}[v_1 \ldots v_n] & \text{for dim } V \text{ even,} \\ \mathbb{K} & \text{for dim } V \text{ odd.} \end{cases}$$

**Proof.** Every element of the algebra $C(Q)$ is of the form

$$a = \sum_{P \in \mathcal{P}} a_P v_P.$$ 

If $a$ belongs to $Z(C(Q))$ or to $Z(C^0(Q))$, then, in particular, $v_R a = a v_R$ necessarily holds for each set $R \in \mathcal{P}$ with $|R| = 2$. This implies

$$\sum_{P \in \mathcal{P}} a_P v_R v_P v_R^{-1} = \sum_{P \in \mathcal{P}} a_P v_P.$$

Since $|R| = 2$, we conclude

$$\sum_{P \in \mathcal{P}} (-1)^{|P \cap R|} a_P v_P = \sum_{P \in \mathcal{P}} a_P v_P,$$

so that $(-1)^{|P \cap R|} a_P = a_P$ for each set $R \subset \mathcal{P}$ with $|R| = 2$. This relation is a necessary condition to be satisfied by any central element. From this it follows straightaway that

$$a_P = 0, \quad \text{if } P \neq \emptyset, \{1, \ldots, n\},$$

i.e.

$$Z(C(Q)), Z(C^0(Q)) \subset \mathbb{K} \oplus \mathbb{K}[v_1 \ldots v_n].$$

If $n$ is odd, then $v_1 \ldots v_n$ does not belong to $C^0(Q)$, hence $Z(C^0(Q)) = \mathbb{K}$. On the other hand, for $n$ odd and $P = \{1, \ldots, n\}$ the lemma implies

$$v_P v_R v_P^{-1} = (-1)^{|R|-|R|} v_R,$$

yielding $v_P v_R = v_R v_P$. In this case, the element $v_1 \ldots v_n$ commutes with all elements of the algebra $C(Q)$, and we arrive at

$$Z(C(Q)) = \mathbb{K} \oplus \mathbb{K}[v_1 \ldots v_n].$$

The case $\dim V \equiv 0 \mod 2$ is treated analogously. \qed

1.3. Clifford algebras of real negative definite quadratic forms

Let us first fix notations for some special Clifford algebras which will play an eminent role through the rest of the book:

$$C_n = C(\mathbb{R}^n, -x_1^2 - \ldots - x_n^2)$$ - Clifford algebra of the $n$-dimensional real negative definite form,

$$C'_n = C(\mathbb{R}^n, x_1^2 + \ldots + x_n^2)$$ - Clifford algebra of the $n$-dimensional real positive definite quadratic form,
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\[ C_n^c = C(\mathbb{C}^n, z_1^2 + \ldots + z_n^2) \] - Clifford algebra of the n-dimensional complex quadratic form.

**Remark.** If \( Q \) is a non-degenerate complex quadratic form of dimension \( n \) (e.g. \( Q = -z_1^2 - \ldots - z_n^2 \)), then \( Q \) is equivalent to the form \((\mathbb{C}^n, z_1 + \ldots + z_n)\) over the complex numbers and hence both their Clifford algebras coincide,

\[ C(Q) = C_n^c. \]

Let us begin with the following general consideration concerning complexification. For a real quadratic form \((V, B)\) the complexification \((V \otimes_{\mathbb{R}} \mathbb{C}, B_{\mathbb{C}})\) is defined as

\[ B_{\mathbb{C}}(v_1 \otimes z, v_2 \otimes w) = B(v_1, v_2)zw. \]

On the other hand, if \( A \) is any real algebra, then its complexification \( A \otimes_{\mathbb{R}} \mathbb{C} \) carries the product structure

\[ (a_1 \otimes z) \cdot (a_2 \otimes w) = (a_1a_2) \otimes (zw) \]

turning \( A \otimes \mathbb{C} \) into a complex algebra.

**Proposition.** Let \((V, B)\) be a real quadratic form and \((V_{\mathbb{C}}, B_{\mathbb{C}})\) its complexification. Then, in the sense of isomorphic \( \mathbb{C} \)-algebras,

\[ C(V_{\mathbb{C}}, B_{\mathbb{C}}) = C(V, B) \otimes_{\mathbb{R}} \mathbb{C}. \]

**Proof.** Define \( u : V \otimes_{\mathbb{R}} \mathbb{C} \rightarrow C(V, B) \otimes \mathbb{C} \) by \( u(v \otimes z) = j(v) \otimes z \), where \( j : V \rightarrow C(V, B) \) is the embedding of the real vector space into the Clifford algebra. Then,

\[ u(v \otimes z)^2 = (j(v) \otimes z)^2 = j(v)^2 \otimes z^2 \]
\[ = B(v, v)z^2 1 \otimes 1 = B_{\mathbb{C}}(v \otimes z, v \otimes z) \cdot 1. \]

Hence \( u \) extends to a homomorphism \( \tilde{u} : C(V_{\mathbb{C}}, B_{\mathbb{C}}) \rightarrow C(V, B) \otimes_{\mathbb{R}} \mathbb{C} \) of the complex algebras. It is easy to check that \( \tilde{u} \) is an isomorphism. \( \square \)

**Corollary.** One has the following identity of complexifications:

\[ C_n^c = C_n \otimes_{\mathbb{R}} \mathbb{C} = C_n^' \otimes_{\mathbb{R}} \mathbb{C}. \]

**Proof.** The complexifications of the real forms \( x_1^2 + \ldots + x_n^2 \) and \( -x_1^2 - \ldots - x_n^2 \) are equivalent. \( \square \)

**Proposition.** The following graded real algebras are isomorphic:

\[ C_{n+2} = C_n^' \otimes_{\mathbb{R}} C_2, \quad C_{n+2}^' = C_n \otimes C_2'. \]

The tensor product is to be understood as the usual one for algebras.
Proof. Choose an orthonormal basis of the vector space $\mathbb{R}^{n+2}$ consisting of the vectors $e_1, \ldots, e_{n+2}$. The first $n$ vectors then generate the algebras $C_n$ and $C_n'$, where $e_1', \ldots, e_n'$ denote these same vectors considered this time as generators of $C_n'$. Now define

$$u : \mathbb{R}^{n+2} \to C_n' \otimes \mathbb{R} C_2$$

by $u(e_1) = 1 \otimes e_1$, $u(e_2) = 1 \otimes e_2$, $u(e_i) = e_{i-2}' \otimes e_1 e_2$, $3 \leq i \leq n+2$. Then, in $C_n' \otimes \mathbb{R} C_2$ the following equations hold:

$$u(e_1)^2 = (1 \otimes e_1)(1 \otimes e_1) = 1 \otimes e_1^2 = -1, \quad u(e_2)^2 = (1 \otimes e_2)(1 \otimes e_2) = -1,$$

$$u(e_i)^2 = (e_{i-2}' \otimes e_1 e_2)(e_{i-2}' \otimes e_1 e_2) = e_{i-2}'^2 \otimes e_1 e_2 e_1 e_2 = 1 \otimes e_1 e_2 e_1 e_2 = -1,$$

and of course for mixed terms

$$u(e_i)u(e_j) + u(e_j)u(e_i) = 0.$$ 

Consequently, there exists an algebra homomorphism

$$\tilde{u} : C_{n+2} \to C_n' \otimes \mathbb{R} C_2$$

fitting into the commutative diagram

Now $\tilde{u}$ preserves the $\mathbb{Z}_2$-grading and is the desired isomorphism. □

Example. The algebra $C_2^c = C_2 \otimes \mathbb{C}$ is the complexification of $C_2$. The latter algebra is generated by $e_1, e_2$ satisfying the relations

$$e_1^2 = -1 = e_2^2, \quad e_1 e_2 + e_2 e_1 = 0.$$ 

On the other hand, the underlying vector space of the complex algebra $M_2(\mathbb{C})$ has the basis

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

with the relations

$$g_1^2 = -1 = g_2^2, \quad g_1 g_2 + g_2 g_1 = 0.$$ 

This implies $C_2^c = C_2 \otimes \mathbb{C} = M_2(\mathbb{C})$.

Making repeated use of this example, we arrive at the following isomorphism.

Proposition. There is an isomorphism $C_{n+2}^c = C_n^c \otimes \mathbb{C} M_2(\mathbb{C})$.

Proof. $C_{n+2}^c = C_{n+2} \otimes \mathbb{R} \mathbb{C} = (C_n' \otimes \mathbb{R} C_2) \otimes \mathbb{R} \mathbb{C} = (C_n' \otimes \mathbb{R} C_2) \otimes \mathbb{C} (C_2 \otimes \mathbb{C}) = C_n^c \otimes \mathbb{C} C_2^c = C_n^c \otimes \mathbb{C} M_2(\mathbb{C})$. □
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This isomorphism explicitly involves the one described in the proof of the previous proposition, i.e. $C_{n+2} = C_n' \otimes_R C_2$. Thus we obtain an explicit isomorphism $C_{n+2}^c = C^c \otimes M_2(\mathbb{C})$ which we spell out once again.

Corollary. Let $e_1, \ldots, e_{n+2}$ be the generating elements of the algebra $C_{n+2}^c$ and, correspondingly, $e_1^*, \ldots, e_n^*$ those for the algebra $C_n^c$. Moreover, denote by

$$g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

the generating elements of the algebra $M_2(\mathbb{C})$. The isomorphism $C_{n+2}^c \approx C_n^c \otimes_C M_2(\mathbb{C})$ is then given by

$$e_1 \mapsto 1 \otimes g_1, \quad e_2 \mapsto 1 \otimes g_2, \quad e_j \mapsto (i e_{j-2}^*) \otimes g_1 g_2, \quad 3 \leq j \leq n + 2.$$

We will now apply these isomorphisms repeatedly and take into account the descriptions of the Clifford algebras involved, i.e.

$$C_1^c = C_1 \otimes_R C = C \otimes_R C = C \oplus C, \quad C_2^c = M_2(\mathbb{C}).$$

In summary, we immediately obtain the following proposition:

Proposition.

a) If $n = 2k$ is even, then

$$C_n^c = \underbrace{M_2(\mathbb{C}) \otimes \ldots \otimes M_2(\mathbb{C})}_{k \text{ times}} = \text{End}(\underbrace{C^2 \otimes \ldots \otimes C^2}_{k \text{ times}}) = \text{End}(C^{2^k}).$$

b) If $n = 2k + 1$ is odd, then

$$C_n^c = \{M_2(\mathbb{C}) \otimes \ldots \otimes M_2(\mathbb{C})\} \oplus \{M_2(\mathbb{C}) \otimes \ldots \otimes M_2(\mathbb{C})\} = \text{End}(C^{2^k}) \oplus \text{End}(C^{2^k}).$$

These isomorphisms are explicitly described by the following formulas:

The case $n = 2k$ even:

$$e_j \mapsto E \otimes \ldots \otimes E \otimes g_{\alpha(j)} \otimes T \otimes \ldots \otimes T \quad \text{["\(i-1\)/2" times]}$$

where

$$\alpha(j) = \begin{cases} 1 & \text{if } j \text{ is odd,} \\ 2 & \text{if } j \text{ is even.} \end{cases}$$

The case $n = 2k + 1$ odd: If $1 \leq j \leq 2k$, $e_j$ is mapped to

$$e_j \mapsto (E \otimes \ldots \otimes E \otimes g_{\alpha(j)} \otimes T \otimes \ldots \otimes T, \quad E \otimes \ldots \otimes E \otimes g_{\alpha(j)} \otimes T \otimes \ldots \otimes T) \quad \text{["\(i-1\)/2" times]}$$

The endomorphism $e_{2k+1}$ is realized by $e_{2k+1} \mapsto (iT \otimes \ldots \otimes T, -iT \otimes \ldots \otimes T)$. 
Definition (complex n-spinors, Dirac spinors). The vector space of complex n-spinors is
\[ \Delta_n := \mathbb{C}^{2^k} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \] for \( n = 2k, 2k + 1 \).

The elements of \( \Delta_n \) are called complex spinors.

Using this notation we obtain
\[ C_n^c = \text{End}(\Delta_n), \] if \( n = 2k \) is even,
\[ C_n^c = \text{End}(\Delta_n) \oplus \text{End}(\Delta_n), \] if \( n = 2k + 1 \) is odd.

Moreover, the following diagram is commutative:
\[
\begin{array}{ccc}
C_{2k} & \xrightarrow{\varphi} & C_{2k+1} \\
\downarrow & & \downarrow \\
\text{End}(\Delta_{2k}) & \xrightarrow{\varphi} & \text{End}(\Delta_{2k+1}) \oplus \text{End}(\Delta_{2k+1})
\end{array}
\]

Here \( \varphi \) is the diagonal map \( \varphi(A) = (A, A) \) and the vector spaces \( \Delta_{2k} = \Delta_{2k+1} \) coincide.

Denote by \( \kappa_n \) the so-called spin representation of the Clifford algebra \( C_n^c \). In the case of an even dimension, \( n = 2k \), this is nothing but the isomorphism explained before,
\[ \kappa_n : C_n^c \xrightarrow{\sim} \text{End}(\Delta_n). \]

If \( n = 2k + 1 \) is odd, then \( \kappa_n \) consists of the isomorphism \( C_n^c = \text{End}(\Delta_n) \oplus \text{End}(\Delta_n) \) followed by the projection onto the first component,
\[ \kappa_n : C_n^c \xrightarrow{\sim} \text{End}(\Delta_n) \oplus \text{End}(\Delta_n) \xrightarrow{\text{proj}_1} \text{End}(\Delta_n). \]

The vector space of complex n-spinors is thus turned into a module over the Clifford algebra \( C_n^c \).

1.4. The pin and the spin group

Consider the real vector space \( \mathbb{R}^n \) and the Clifford algebra \( C_n \) of the quadratic form \(-x_1^2 - \ldots - x_n^2\). \( \mathbb{R}^n \) itself is a linear subspace of \( C_n \), \( \mathbb{R}^n \subset C_n \). For every vector \( x \in \mathbb{R}^n \), the equality
\[ x \cdot x = -||x||^2 \]
holds in \( C_n \) and hence the inverse element \( x^{-1} \) is given by
\[ x^{-1} = \frac{-x}{||x||^2}. \]
Definition. \( \text{Pin}(n) \subset C_n \) is the group which is multiplicatively generated by all vectors \( x \in S^{n-1} \). Therefore, the elements of \( \text{Pin}(n) \) are the products \( x_1 \ldots x_m \) with \( x_i \in \mathbb{R}^n, ||x_i|| = 1 \). The spin group, \( \text{Spin}(n) \), is defined as \( \text{Spin}(n) = \text{Pin}(n) \cap C_n^0 \).

Now we will define a homomorphism

\[ \lambda : \text{Pin}(n) \rightarrow O(n) \]

from the group \( \text{Pin}(n) \) onto the group \( O(n) \) of orthogonal transformations of the Euclidean space \( \mathbb{R}^n \). To this end, recall the anti-involution

\[ \gamma : C_n \rightarrow C_n \]

existing in every Clifford algebra. It has the particular property that \( \gamma(x) = x \) holds for all vectors \( x \in \mathbb{R}^n \subset C_n \).

Lemma. If \( y \in \mathbb{R}^n \subset C_n \) and \( x \in \text{Pin}(n) \subset C_n \), then the element \( x \cdot y \cdot \gamma(x) \) belongs to the subspace \( \mathbb{R}^n \subset C_n \).

Proof. \( x \) is, by definition, the product \( x = x_1 \ldots x_m \) of vectors from the sphere \( S^{n-1} \). Since

\[ x \cdot y \cdot \gamma(x) = x_1 \ldots x_m \cdot y \cdot \gamma(x_m) \ldots \gamma(x_1), \]

we can suppose without loss of generality that \( m = 1 \), i.e. \( x \in S^{n-1} \). Next we choose an orthonormal basis in \( \mathbb{R}^n \) with \( x = e_1 \) as the first vector. Then, writing

\[ y = \sum_{i=1}^{n} y_i e_i, \]

we get

\[ x \cdot y \cdot \gamma(x) = e_1 \cdot \left( \sum_{i=1}^{n} y_i e_i \right) \cdot e_1 = -y_1 e_1 + \sum_{i=2}^{n} y_i e_i, \]

and hence \( x \cdot y \cdot \gamma(x) \) is the vector in \( \mathbb{R}^n \) which is the image of \( y \) under the reflection in the plane perpendicular to \( x \).

For \( x \in \text{Pin}(n) \) we now define

\[ \lambda(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \lambda(x)y = x \cdot y \cdot \gamma(x). \]

Then, obviously,

\[ \lambda(x_1 x_2)y = x_1 x_2 y \gamma(x_1 x_2) = x_1 x_2 y \gamma(x_2) \gamma(x_1) = \lambda(x_1)(\lambda(x_2)y) \]

and hence \( \lambda : \text{Pin}(n) \rightarrow GL(n) \) is a group homomorphism. Every element \( x \) of \( \text{Pin}(n) \) is the product \( x = x_1 \ldots x_m \) of vectors \( x_i \in S^{n-1} \), and the calculation above shows that \( \lambda(x_i) \) is a reflection. Therefore, \( \lambda(x) \) itself is the superposition of reflections, so, in particular, an orthogonal transformation.
Proposition.

a) \( \lambda : \text{Pin}(n) \to O(n) \) is a continuous surjective group homomorphism.

b) \( \lambda^{-1}(SO(n)) = \text{Spin}(n) \).

c) \( \ker(\lambda) = \{1, -1\} \cong \mathbb{Z}_2 \).

d) For \( n \geq 2 \), \( \text{Spin}(n) \) is a connected group.

e) For \( n \geq 3 \), \( \text{Spin}(n) \) is simply connected and \( \lambda : \text{Spin}(n) \to SO(n) \) is the universal covering of the group \( SO(n) \).

Proof. a) directly follows from the well-known fact that every orthogonal linear map \( A : \mathbb{R}^n \to \mathbb{R}^n \) is the superposition of reflections. Let \( \beta : C_n \to C_n \) be the involution of the Clifford algebra which defines the decomposition \( C_n = C_n^0 \oplus C_n^1 \). For every vector \( x \in \mathbb{R}^n \) we have \( \beta(x) = -x \). Now apply \( \beta \) to any element \( x = x_1 \ldots x_m \): 

\[
\beta(x) = \beta(x_1) \ldots \beta(x_m) = (-1)^m x_1 \ldots x_m.
\]

From this we see that \( x = x_1 \ldots x_m \) belongs to \( \text{Spin}(n) \) if and only if \( m \) is even. On the other hand, \( A(x) = \lambda(x_1) \ldots \lambda(x_m) \) is the superposition of \( m \) reflections and, hence, \( \lambda(x) \) lies in \( SO(n) \) if and only if \( m \) is even. This proves b). Now suppose that \( \lambda(x) = 1 \) holds in \( O(n) \). Then \( xy\gamma(x) = y \) for all \( y \in \mathbb{R}^n \). Moreover, since \( \lambda(x) \in SO(n) \), \( x \) is the product of an even number of vectors, \( x = x_1 \ldots x_m \) and \( m \equiv 0 \mod 2 \). Thus,

\[
x_1 \ldots x_my_1 \ldots x_m = y.
\]

Multiplying this equation from the right by \( x_1 \ldots x_m \) and taking into account that \( x_1^2 = \ldots = x_m^2 = -1 \) as well as \( m \equiv 0 \mod 2 \), we obtain

\[
x_1 \ldots x_my = yx_1 \ldots x_m.
\]

The element \( x = x_1 \ldots x_m \) \((m \equiv 0 \mod 2)\) of the Clifford algebra \( C_n \) thus commutes with every vector from \( \mathbb{R}^n \). Therefore, \( x \) belongs to the center of \( C_n \) and, at the same time, to the center of \( C_n^0 \). But, for every Clifford algebra,

\[
\mathbb{Z}(\mathbb{C}(Q)) \cap \mathbb{Z}(\mathbb{C}^0(Q)) = \mathbb{K}.
\]

Hence, \( x \in \mathbb{R}^1 \), and from \( |x| = 1 \) we see that \( x = \pm 1 \). It remains to be shown that \( \text{Spin}(n) \) is a connected group. As

\[
\lambda : \text{Spin}(n) \to SO(n)
\]

is surjective with \( \ker \lambda = \{1, -1\} \), it suffices to find a path in \( \text{Spin}(n) \) connecting the element \((-1) \in \text{Spin}(n) \) with the neutral element \( 1 \in \text{Spin}(n) \). In the case \( n \geq 2 \), one such path is given by \((0 \leq t \leq 1)\)

\[
\gamma(t) = -\cos(\pi t) - \sin(\pi t)e_1e_2.
\]
\[ \gamma(t) \text{ indeed lies in } \text{Spin}(n), \text{ since} \]
\[ \gamma(t) = \left( \cos \left( \frac{\pi}{2} t \right) e_1 + \sin \left( \frac{\pi}{2} t \right) e_2 \right) \cdot \left( \cos \left( \frac{\pi}{2} t \right) e_1 - \sin \left( \frac{\pi}{2} t \right) e_2 \right) \]
in \( C_n \).

Now we will describe the Lie algebra of the group \( \text{Spin}(n) \). For this we need the following general remark: Let \( A \) be a finite-dimensional associative real algebra and \( A^* \subset A \) the group of its invertible elements. \( A^* \) is an open subset of \( A \) and, furthermore, a Lie group. Its Lie algebra \( a^* \), which can be identified with \( T_1(A^*) \cong A \), thus coincides with \( A \),

\[ a^* = A. \]

The commutator of two elements \( a_1, a_2 \in A = a^* \) is given by \([a_1, a_2] = a_1 a_2 - a_2 a_1\), and the exponential map

\[ \exp : a^* = A \longrightarrow A^* \]
is defined by the power series of the exponential function:

\[ \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}. \]

We are going to apply this observation in the case of the real Clifford algebra \( C_n \). The spin group is a subgroup of \( C_n^* \),

\[ \text{Spin}(n) \subset C_n^*, \]
and hence, to describe the Lie algebra \( \text{spin}(n) \), it suffices to determine the tangent space \( T_1(\text{Spin}(n)) \subset C_n \). We will proceed as follows: Let \( \gamma(t) = x_1(t) \cdots x_{2m}(t) \) be a path in \( \text{Spin}(n) \) with \( x_i(t) \in S^{n-1} \) and \( \gamma(0) = 1 \). The tangent to \( \gamma \) at \( t = 0 \) is

\[ \frac{d\gamma}{dt}(0) = \frac{dx_1}{dt}(0) \cdot x_2(0) \cdots x_{2m}(0) + x_1(0)x_2(0) \cdots \frac{dx_{2m}}{dt}(0). \]

Let \( m_2 \subset C_n \) be the subspace of \( C_n \) spanned by the Clifford products \( e_i \cdot e_j, \quad 1 \leq i < j \leq n \). We will prove that each summand of \( \frac{d\gamma}{dt}(0) \) belongs to \( m_2 \). Since \( \gamma(0) = 1 \), the first summand is equal to

\[ \frac{dx_1}{dt}(0) \cdot x_1^{-1}(0) = -\frac{dx_1}{dt}(0) \cdot x_1(0). \]

But \( x_1(t) \cdot x_1(t) \equiv 1 \) implies

\[ \frac{dx_1}{dt}(0)x_1(0) + x_1(0)\frac{dx_1}{dt}(0) = 0 \]
and, because of the relations in the Clifford algebra, \( \frac{dx_1}{dt}(0) \) and \( x_1(0) \) are perpendicular as vectors in \( \mathbb{R}^n \). Therefore, the first summand belongs to \( m_2 \).
Analogously, the second summand coincides with
\[
x_1(0)\frac{dx_2}{dt}(0)x_2^{-1}(0)x_1^{-1}(0) = -x_1(0)\frac{dx_2}{dt}(0)x_2(0)x_1^{-1}(0)
\]
\[
= -\left\{x_1(0)\frac{dx_2}{dt}(0)x_2^{-1}(0)\right\} \cdot \left\{x_1(0)x_2(0)x_1^{-1}(0)\right\}.
\]
As \(\frac{dx_2}{dt}(0)\) and \(x_2(0)\) are perpendicular, the vectors \(x_1(0)\frac{dx_2}{dt}(0)x_2^{-1}(0)\) and \(x_1(0)x_2(0)x_1^{-1}(0)\) are perpendicular, too, and the second summand lies in \(m_2\). Altogether we arrive at the

**Proposition.**

a) The linear subspace \(m_2 \subset C_n\)
\[
m_2 = \text{Lin}(e_i e_j : 1 \leq i < j \leq n)
\]
equipped with the commutator
\[
[x, y] = xy - yx
\]
is a Lie algebra which coincides with the Lie algebra of the group \(\text{Spin}(n) \subset C_n\).

b) The exponential map \(\exp : m_2 \rightarrow \text{Spin}(n)\) is given by
\[
\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}.
\]

b) If \(\sigma : C_n \rightarrow \text{End}(W)\) is a (real or complex) representation of the Clifford algebra and
\[
\sigma|_{\text{Spin}(n)} : \text{Spin}(n) \rightarrow \text{Aut}(W),
\]
the group homomorphism defined by restriction, then its differential
\[
(\sigma|_{\text{Spin}(n)})_* : \text{spin}(n) = m_2 \rightarrow \text{End}(W)
\]
is given by the formula \((\sigma|_{\text{Spin}(n)})_* = \sigma|_{m_2}\).

**Proof.** The above calculation, first of all, shows that the Lie algebra \(\text{spin}(n)\) is contained in the subspace \(m_2\). Since
\[
\dim_{\mathbb{R}}(m_2) = \frac{n(n - 1)}{2}
\]
and \(\dim(\text{Spin}(n)) = \dim(\text{SO}(n)) = n(n - 1)/2\), the two spaces have to coincide. This proves a) and b). c) follows from general facts concerning the differential of a homomorphism \(f : H \rightarrow G\) between two Lie groups.
Namely, $f_* : \mathfrak{h} \to \mathfrak{g}$ is uniquely determined by the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{h} & \xrightarrow{f_*} & \mathfrak{g} \\
\exp & \downarrow & \exp \\
H & \xrightarrow{f} & G
\end{array}
\]

as a homomorphism of Lie algebras. But in our situation the diagram

\[
\begin{array}{ccc}
m_2 & \xrightarrow{\sigma} & \text{End}(W) \\
\exp & \downarrow & \exp \\
\text{Spin}(n) & \xrightarrow{\sigma} & \text{Aut}(W)
\end{array}
\]

commutes, because $\sigma : C_n \to \text{End}(W)$ is a homomorphism of algebras. This completes the proof. □

Consider now the universal covering

$\lambda : \text{Spin}(n) \to \text{SO}(n)$

and let us compute its differential

$\lambda_* : \text{spin}(n) \to \text{so}(n)$. 

As before, we identify $\text{spin}(n)$ with the vector space

$m_2 = \text{Lin}(e_i e_j : 1 \leq i < j \leq n)$

and $\text{so}(n)$ with the set of all skew-symmetric matrices. A particular basis of the vector space $\text{so}(n)$ is formed by the matrices $E_{ij} (i < j)$ defined by

\[
E_{ij} = \begin{pmatrix}
i & j \\
0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\
0 & -1 & 0 & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ldots & 1 & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & \ldots & 0 & 0 \\
0 & \ldots & \ldots & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

Now we want to prove the formula

$\lambda_*(e_i e_j) = 2E_{ij}$

for $\lambda_* : \text{spin}(n) = m_2 \to \text{so}(n)$. The path

$\gamma(t) = \cos(t) + \sin(t)e_i e_j = -(\cos(t/2)e_i + \sin(t/2)e_j)(\cos(t/2)e_i - \sin(t/2)e_j)$

is a subgroup $\gamma \subset \text{Spin}(n)$ with $\frac{d\gamma}{dt}(0) = e_i e_j$. Then, for $k \neq i, j$,

$\lambda(\gamma(t))e_k = e_k$. 

and, e.g. in the case $k = i$, we have

$$\lambda(\gamma(t))e_i = (\cos(t) + \sin(t)e_i e_j)e_i (\cos(t) + \sin(t)e_j e_i)$$

$$= \cos^2(t)e_i + 2\sin(t)\cos(t)e_j - \sin^2(t)e_i = \cos(2t)e_i + \sin(2t)e_j.$$  

Thus $\frac{d}{dt}(\lambda(\gamma(t))e_i) = 2e_j$. This computation proves the formula $\lambda_*(e_i e_j) = 2E_{ij}$. The result can also be written invariantly.

**Proposition.** If $z \in \text{spin}(n)$ is an element of the Lie algebra, then for the differential $\lambda_* : \text{spin}(n) \to \text{so}(n)$ the relation

$$\lambda_*(z)x = zx - xz$$

holds for every $x \in \mathbb{R}^n$. In particular, the Clifford product $zx - xz$ belongs to $\mathbb{R}^n (z \in m_2, x \in \mathbb{R}^n)$.

### 1.5. The spin representation

Consider the $C_\mathbb{C}$-module of $n$-spinors $\Delta_n$. Via

$$\text{Spin}(n) \subset C_\mathbb{C} \subset C_\mathbb{C}^{\kappa_n} \xrightarrow{\kappa_n} \text{End}(\Delta_n),$$

we obtain a representation $\kappa$ of the group $\text{Spin}(n)$ by restriction:

$$\kappa = \kappa_n|_{\text{Spin}(n)} : \text{Spin}(n) \to \text{Aut}(\Delta_n),$$

the spin representation of the group $\text{Spin}(n)$.

**Proposition.** The spin representation is a faithful representation of the group $\text{Spin}(n)$.

**Proof.** If $n = 2k$ is an even number, then $C_\mathbb{C}^\kappa = \text{End}(\Delta_n)$ and the statement is trivial. This leaves the case $n = 2k + 1$. As vector spaces, $\Delta_{2k+1} = \Delta_{2k}$, and the diagram

$$
\begin{array}{ccc}
\text{Spin}(2k) & \xrightarrow{\kappa_{2k}} & GL(\Delta_{2k}) \\
\downarrow & & \downarrow \\
\text{Spin}(2k + 1) & \xrightarrow{\kappa_{2k+1}} & GL(\Delta_{2k+1})
\end{array}
$$

commutes. This implies that the normal subgroup $H := \ker(\kappa_{2k+1})$ has only the neutral element in common with $\text{Spin}(2k)$,

$$H \cap \text{Spin}(2k) = \{1\}.$$  

The subgroup $\lambda(H) \subset SO(2k + 1)$ is normal, since $\lambda : \text{Spin}(2k + 1) \to SO(2k + 1)$ is surjective. Moreover,

$$\lambda(H) \cap SO(2k) = \{E\}.$$  

Let $A \in \lambda(H) \subset SO(2k + 1)$ be an element from this normal subgroup. The characteristic polynomial of $A$ has odd degree, hence there exists a
1.5. The spin representation

unit vector $v_0$ such that $A(v_0) = v_0$. This means there is an element $B \in SO(2k + 1)$ for which

$$BAB^{-1} \in SO(2k).$$

As $\lambda(H)$ is normal, this implies $BAB^{-1} \in \lambda(H) \cap SO(2k)$, i.e. $BAB^{-1} = E$. Thus we proved that $\lambda(H)$ is the trivial subgroup of $SO(2k+1)$. For $H$ itself this leaves two possibilities, either $H = \{1\}$ or $H = \{1, -1\}$. But $(-1) \in Spin(2k + 1)$ does not belong to the kernel of the spin representation. □

A vector $x \in \mathbb{R}^n \subset C_n \subset C_n^C \rightarrow End(\Delta_n)$ can be considered as an endomorphism of $\Delta_n$. This leads to the so-called Clifford multiplication of vectors and spinors, which is a linear map

$$\mu : \mathbb{R}^n \otimes \Delta_n \rightarrow \Delta_n.$$

Here $\mu(x \otimes \psi)$ is defined for $x \in \mathbb{R}^n$ and $\psi \in \Delta_n$ by

$$\mu(x \otimes \psi) = \kappa_n(x)(\psi).$$

Instead of $\mu(x \otimes \psi)$, we will often simply write $x \cdot \psi$. This Clifford multiplication extends to a homomorphism

$$\mu : \Lambda(\mathbb{R}^n) \otimes \Delta_n \rightarrow \Delta_n$$

as follows: Using the orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, each element of the exterior algebra $\Lambda(\mathbb{R}^n)$ can be written as

$$w^k = \sum_{i_1 < \ldots < i_k} w_{i_1 i_2 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}.$$

Define

$$\mu(w^k \otimes \psi) = w^k \cdot \psi = \sum_{i_1 < \ldots < i_k} w_{i_1 \ldots i_k} e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \psi,$$

where, as before, $e_{i_\alpha} \cdot \varphi$ denotes Clifford multiplication of the vector $e_{i_\alpha}$ by the spinor $\varphi$. A straightforward calculation leads to the formula

$$(x \wedge w^k) \cdot \psi = x \cdot (w^k \cdot \psi) + (x \cdot w^k) \cdot \psi$$

for a vector $x \in \mathbb{R}^n$ and a multi-vector $w^k \in \Lambda(\mathbb{R}^n)$. Now we will prove the

**Proposition.** Clifford multiplication $\mu : \Lambda(\mathbb{R}^n) \otimes \Delta_n \rightarrow \Delta_n$ is a homomorphism of $Spin(n)$-representations.

**Proof.** For every element $g \in Spin(n)$ we have to check the formula

$$\kappa(g)(w^k \cdot \psi) = (\lambda(g)w^k)(\kappa(g)\psi).$$

We will do this by induction on the degree $k$ of the multi-vector $w^k$. For $k = 1$, $w^k$ is an ordinary vector $x \in \mathbb{R}^n$ and the equation results from the
following computation:
\[
\kappa(g)(x \cdot \psi) = \kappa(g)\kappa_n(x)(\psi) = \kappa(g)\kappa_n(x)\kappa(g^{-1})\kappa(g)(\psi) \\
= \kappa_n(gxg^{-1})\kappa(g)\psi = \kappa_n(\lambda(g)x)(\kappa(g)\psi) \\
= (\lambda(g)x) \cdot (\kappa(g)\psi).
\]

Now we assume that the claim holds for multi-vectors of degree \( \leq k \) and consider \( w^{k+1} = x \wedge w^k \). Then,
\[
\kappa(g)((x \wedge w^k) \cdot \psi) \\
= \kappa(g)(x \cdot (w^k \cdot \psi)) + \kappa(g)((x \wedge w^k) \cdot \psi) \\
= \lambda(g)x \cdot (\kappa(g)(w^k \cdot \psi)) + \lambda(g)(x \wedge w^k) \cdot \kappa(g)\psi \\
= (\lambda(g)x) \cdot ((\lambda(g)w^k) \cdot \kappa(g)\psi) + (\lambda(g)x \wedge \lambda(g)w^k) \cdot \kappa(g)\psi \\
= ((\lambda(g)x) \wedge (\lambda(g)w^k)) \cdot \kappa(g)\psi = (\lambda(g)w^{k+1})\kappa(g)\psi.
\]

This proves the proposition.

Summarizing, we know now that vectors and multi-vectors can be multiplied by spinors. The product is always a spinor, and this multiplication is equivariant for the actions of the group \( \text{Spin}(n) \) on corresponding spaces. Next we will consider the case \( n = 2k \) in greater detail. In this case, the element \( e_1 \cdots e_{2k} \) belongs to the center of the algebra \( C_0^0 \). As \( \text{Spin}(n) \subset C_0^0 \), this element commutes with all elements from \( \text{Spin}(n) \). Hence the endomorphism
\[
f = i^k\kappa(e_1 \cdots e_{2k}) : \Delta_{2k} \longrightarrow \Delta_{2k}
\]
is an automorphism of the \( \text{Spin}(n) \)-representation, i.e. \( f(\kappa(g)\psi) = \kappa(g)f(\psi) \) for all \( g \in \text{Spin}(n) \) and \( \psi \in \Delta_{2k} \). Since \( (e_1 \cdots e_{2k})^2 = (-1)^k \), \( f \) is an involution, \( f^2 = \text{Id}_{\Delta_{2k}} \). Thus the spin representation \( \Delta_{2k} \) decomposes into the eigensubspaces of \( f \):
\[
\Delta_{2k} = \Delta_{2k}^+ \oplus \Delta_{2k}^-,
\]
\[
\Delta_{2k}^\pm = \{ \psi \in \Delta_{2k} : f(\psi) = \pm \psi \}.
\]

**Definition** (Weyl spinors). The spinors belonging to the subspaces \( \Delta_{2k}^\pm \) are called (positive or negative, respectively) Weyl spinors.

**Proposition.**

a) \( \dim C \Delta_{2k}^+ = \dim C \Delta_{2k}^- = 2^{k-1} \).

b) If \( x \in \mathbb{R}^{2k} \) is a vector and \( \psi^\pm \in \Delta_{2k}^\pm \), then the spinor \( x \cdot \psi^\pm \) belongs to \( \Delta_{2k}^\mp \). Thus Clifford multiplication induces homomorphisms
\[
\mu : \mathbb{R}^{2k} \otimes \mathbb{R} \Delta_{2k}^\pm \longrightarrow \Delta_{2k}^\mp.
\]

**Proof.** If \( x \in \mathbb{R}^n \) is a vector (e.g. \( x = e_1 \)), then, in the algebra \( C_n \), we have the relation
\[
x \cdot (e_1 \cdots e_{2k}) = -(e_1 \cdots e_{2k}) \cdot x.
\]
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Hence Clifford multiplication by the vector $x$ anti-commutes with the involution $f$. Consequently, Clifford multiplication by a vector $0 \neq x \in \mathbb{R}^n$ maps the space $\Delta_{2k}^\pm$ bijectively onto the space $\Delta_{2k}^\mp$. □

We will now prove that the spin representations $\Delta_{2k}^+, \Delta_{2k}^-$ and $\Delta_{2k+1}$ all are irreducible representations of the spin group. To do this, we need the following preparation from linear algebra: Let $V$ be a complex vector space and $A = \text{End}(V)$ the algebra of all endomorphisms. Then $A$ is a simple algebra, i.e. $A$ does not contain any proper ideal. From this we immediately conclude the

**Lemma.** Let $V$ and $W$ be complex vector spaces with $\dim W < \dim V$. Then each homomorphism of algebras,

$$f : \text{End}(V) \to \text{End}(W),$$

is trivial, $f \equiv 0$. □

**Proposition.** The Spin$(2k)$-representation $\Delta_{2k}^\pm$ is irreducible.

**Proof.** Consider the inclusions

$$\text{Spin}(2k) \subset (C_{2k}^c)^0 \subset C_{2k}^c = \text{End}(\Delta_{2k}^\pm \oplus \Delta_{2k}^-)$$

and assume that $\{0\} \neq W \not\subset \Delta_{2k}^\pm$ is a $\text{Spin}(2k)$-invariant subspace. The products $e_i e_j$ ($i < j$) belong to the group $\text{Spin}(2k)$, and hence $W$ is left invariant by them. On the other hand, the products $e_i e_j$ ($i < j$) multiplicatively generate the algebra $(C_{2k}^c)^0$. Thus we obtain a representation of this algebra in the vector space $W$:

$$(C_{2k}^c)^0 \rightarrow \text{End}(W).$$

As $(C_{2k}^c)^0 \approx C_{2k-1}^c = \text{End}(\Delta_{2k-1}) \oplus \text{End}(\Delta_{2k-1})$, $\dim \Delta_{2k-1} = 2^{k-1}$ and $\dim W < \dim \Delta_{2k}^\pm = 2^{k-1}$, this representation $f$ has to be trivial, an obvious contradiction. □

The same argument shows

**Proposition.** The Spin$(2k+1)$-representation $\Delta_{2k+1}$ is irreducible.

**Proof.** In this case,

$$\text{Spin}(2k+1) \subset (C_{2k+1}^c)^0 \subset C_{2k+1}^c = \text{End}(\Delta_{2k+1}) \oplus \text{End}(\Delta_{2k+1}).$$

Again, let $\{0\} \neq W \neq \Delta_{2k+1}$ be a $\text{Spin}(2k+1)$-invariant subspace. Then, as above, $W$ is left invariant by the action of the algebra $(C_{2k+1}^c)^0$ and gives rise to a representation

$$f : (C_{2k+1}^c)^0 \rightarrow \text{End}(W).$$

Since $(C_{2k+1}^c)^0 = C_{2k}^c = \text{End}(\Delta_{2k})$ and $\dim W < \dim \Delta_{2k+1} = \dim \Delta_{2k}$, the representation $f$ is trivial, a contradiction. □
The spin representation $\kappa : Spin(n) \to GL(\Delta_n)$ is the representation of a compact group in a complex vector space. Thus there exists a $Spin(n)$-invariant Hermitian scalar product in $\Delta_n$. We will construct one such product satisfying an even stronger invariance property.

**Proposition.** In the vector space of $n$-spinors, $\Delta_n$, there exists a positive definite Hermitian scalar product $(\cdot , \cdot )$ with the invariance property

$$(x \cdot \psi, \varphi) + (\psi, x \cdot \varphi) = 0,$$

$x \in \mathbb{R}^n, \varphi, \psi \in \Delta_n$. The spin representation $\kappa : Spin(n) \to GL(\Delta_n)$ is a unitary representation with respect to this scalar product.

**Proof.** Let $m_2 = \text{Lin}(e_i e_j : i < j) \subset C_n$ be the Lie algebra of the group $Spin(n)$. Consider $g = \mathbb{R}^n \oplus m_2 \subset C_n$. A simple calculation shows that $g$ is a Lie algebra with the commutator

$$[z, w] = z \cdot w - w \cdot z, \quad z, w \in g.$$

The map $\varphi : g \to C_{n+1}$ given by

$$\varphi|_{m_2} = \text{Id}, \quad \varphi(e_i) = e_i e_{n+1} \quad \text{for } 1 \leq i \leq n$$

is the restriction of an algebra homomorphism $\Phi : C_n \to C_{n+1}$. $\Phi$ is induced by the map $\Phi : \mathbb{R}^n \to C_{n+1}$, $\Phi(e_i) = e_i e_{n+1}$, as can be seen from the equations

$$\Phi(e_i)^2 = -1, \quad 1 \leq i \leq n,$$

$$\Phi(e_i)\Phi(e_j) + \Phi(e_j)\Phi(e_i) = 0, \quad 1 \leq i < j \leq n.$$

Thus $\varphi : g \to C_{n+1}$ maps the Lie algebra $g$ bijectively onto the Lie algebra of the group $Spin(n+1)$ and is, moreover, an isomorphism of these Lie algebras. Hence $g$ is a compact Lie algebra. In view of the following remark this proves the assertion. $\square$

**Remark.** In the previous proof we made use of the following general result: Let $g$ be a compact real Lie algebra and $\kappa : g \to \text{End}(W)$ a representation in a complex vector space. Then in $W$ there is a positive definite Hermitian scalar product $(\cdot , \cdot )$ with the invariance property

$$(\kappa(x)w_1, w_2) + (w_1, \kappa(x)w_2) = 0$$

for $x \in g, w_1, w_2 \in W$. To prove this we consider the compact group $G$ corresponding to the Lie algebra $g$, an arbitrary positive definite product $(\cdot , \cdot )^*$ in $W$, and we set

$$(w_1, w_2) = \int_G (gw_1, gw_2)^* dg,$$

where $dg$ is the Haar measure of the group $G$.  


1.6. The group Spin

**Proposition.** If $\kappa : Spin(n) \to U(\Delta_n)$ is the spin representation, then

$$\det(\kappa(g)) = 1$$

for every group element $g \in Spin(n)$. In other words, the spin representation is a representation into the special unitary group $SU(\Delta_n)$ of the space of $n$-spinors.

**Proof.** Basically, this is not a property special to the spin representation, but a consequence of the following observation: Consider the group homomorphism

$$f : Spin(n) \to S^1, \quad f(g) = \det(\kappa(g)).$$

As $Spin(n)$ is simply connected, there exists a lift $F : Spin(n) \to \mathbb{R}$ to the universal covering of $S^1$, $f(g) = e^{2\pi i F(g)}$,

which is a group homomorphism as well. $Spin(n)$ is compact. Hence $F(Spin(n)) \subset \mathbb{R}$ is a subgroup which is contained in a bounded interval. This implies that $F \equiv 0$ and thus $f(g) = \det(\kappa(g)) \equiv 1$. \qed

1.6. The group Spin$^C$

The complex Clifford algebra $C_n^c$ comprises the group $Spin(n)$ as well as the group $S^1$ of all complex numbers of modulus 1. Together they generate a group which we want to denote by $Spin^C(n)$. Since $Spin(n) \cap S^1 = \{1, -1\}$, the group $Spin^C(n)$ is apparently given by

$$Spin^C(n) = (Spin(n) \times S^1)/\{\pm 1\} = Spin(n) \times_{\mathbb{Z}_2} S^1.$$ 

The elements of $Spin^C(n)$ are thus classes $[g, z]$ of pairs $(g, z) \in Spin(n) \times S^1$ under the equivalence relation $(g, z) \sim (-g, -z)$.

We define several homomorphisms:

a) Let $\lambda : Spin^C(n) \to SO(n)$ be given by $\lambda[g, z] = \lambda(g)$.

b) $i : Spin(n) \to Spin^C(n)$ is the natural inclusion, $i(g) = [g, 1]$.

c) $j : S^1 \to Spin^C(n)$ is the natural inclusion, $j(z) = [1, z]$.

d) Let $l : Spin^C(n) \to S^1$ be given by $l[g, z] = z^2$.

e) $p : Spin^C(n) \to SO(n) \times S^1$ is given by $p([g, z]) = (\lambda(g), z^2)$. Hence, $p = \lambda \times l$. 

Then the following diagram commutes:

$$
\begin{array}{cccccc}
\vdots & 1 & \downarrow & 1 & \downarrow & 1 \\
& S^1 & \downarrow j & S^1 & \downarrow l & S^1 \\
1 & \rightarrow & Spin(n) & \rightarrow & Spin^c(n) & \rightarrow & 1 \\
\downarrow \lambda & \downarrow & \downarrow \lambda & \downarrow & \downarrow & \downarrow & \downarrow \\
& SO(n) & \rightarrow & & & & 1 \\
& 1 & \rightarrow & & & & 1
\end{array}
$$

and all its rows and columns are exact. Moreover, \( p \) is a 2-fold covering of the group \( Spin^c(n) \) over \( SO(n) \times S^1 \). We will use this diagram to compute the fundamental group.

**Proposition.** Let \( n \geq 3 \). Then,

a) The fundamental group \( \pi_1(Spin^c(n)) \) is isomorphic to \( \mathbb{Z} \), and \( l^\ast : \pi_1(Spin^c(n)) \rightarrow \pi_1(S^1) = \mathbb{Z} \) is an isomorphism.

b) Choose generators for the following groups:

\( a \in \pi_1(Spin^c(n)) \), \( \beta \in \pi_1(SO(n)) \), \( \gamma \in \pi_1(S^1) \)

with \( l_\beta(a) = \gamma \). Then for the homomorphism induced by the 2-fold covering \( p_\beta : \pi_1(Spin^c(n)) \rightarrow \pi_1(SO(n)) \times \pi_1(S^1) \) the following formula holds:

\[ p_\beta(a) = \beta + \gamma. \]

**Proof.** a) follows directly from the exact sequence

\[ 1 \rightarrow Spin(n) \rightarrow Spin^c(n) \rightarrow S^1 \rightarrow 1 \]

combined with \( \pi_1(Spin(n)) = 1 \) for \( n \geq 3 \). As \( p = \lambda \times l \), we still have to prove \( \lambda_\beta(\alpha) = \beta \). \( \pi_1(SO(n)) \) is isomorphic to \( \mathbb{Z}_2 \), and hence this is equivalent to \( \lambda_\beta(\alpha) \neq 0 \). Suppose that \( \lambda_\beta(\alpha) = 0 \). From the corresponding exact column sequence we see that there exists an element \( \delta \in \pi_1(S^1) \) with \( j_\beta(\delta) = \alpha \). This implies

\[ \gamma = l_\beta(a) = l_\beta j_\beta(\delta) = 2\delta \]

and \( \gamma \) (as well as \( \alpha \)) is not the generating element of the group \( \pi_1(S^1) \). \( \square \)
Let \( n = 2k \) be an even number. The unitary group \( U(k) \) is a subgroup of \( SO(2k) \). Consider the homomorphism

\[
f : U(k) \rightarrow SO(2k) \times S^1, \quad f(A) = (A, \det A).
\]

**Proposition.** There exists a homomorphism \( F : U(k) \rightarrow Spin^C(2k) \) such that the diagram

\[
\begin{array}{ccc}
Spin^C(2k) & \xrightarrow{F} & U(k) \xrightarrow{f} SO(2k) \times S^1 \\
\downarrow p & & \downarrow f \\
Spin^C(2k) & \xrightarrow{p} & SO(2k) \times S^1
\end{array}
\]

commutes.

**Proof.** We have to prove that the group \( f_\#(\pi_1(U(k))) \) is contained in the set \( p_\#(Spin^C(2k)) \). If we choose a generating element \( \delta \in \pi_1(U(k)) = \mathbb{Z} \) in addition to the generating elements \( \alpha, \beta, \gamma \), then we have \( f_\#(\delta) = \beta + \gamma \), and the assertion follows by covering theory. \( \square \)

The same argument yields a lift of the homomorphism

\[
f_1 : U(k) \rightarrow SO(2k) \times S^1, \quad f_1(A) = \left( A, \frac{1}{\det A} \right).
\]

**Proposition.** There exists a homomorphism \( F_1 : U(k) \rightarrow Spin^C(2k) \) such that the diagram

\[
\begin{array}{ccc}
Spin^C(2k) & \xrightarrow{F_1} & U(k) \xrightarrow{f_1} SO(2k) \times S^1 \\
\downarrow p & & \downarrow f_1 \\
Spin^C(2k) & \xrightarrow{p} & SO(2k) \times S^1
\end{array}
\]

commutes. \( \square \)

**Remark.** The homomorphism \( F : U(k) \rightarrow Spin^C(2k) \) can be explicitly described. Let \( A \in U(k) \). Then there is a unitary basis \( f_1, \ldots, f_k \) in \( \mathbb{C}^k \) with respect to which \( A \) has diagonal form

\[
A = \begin{pmatrix}
e^{i\theta_1} & & 0 \\
& \ddots & \ & \ & \\
0 & \ & e^{i\theta_k}
\end{pmatrix}.
\]

If \( J : \mathbb{C}^k \rightarrow \mathbb{C}^k \) is the complex structure of \( \mathbb{C}^k \); then \( f_j \) and \( J(f_j), 1 \leq j \leq k \), are elements of the complex Clifford algebra \( C^c_{2n} \). We then define a
homomorphism $F : U(k) \to Spin^C(2k) = Spin(2k) \times \mathbb{Z}_2 S^1$ by the formula

$$F(A) = \prod_{j=1}^k \left( \cos \left( \frac{\theta_j}{2} \right) + \sin \left( \frac{\theta_j}{2} \right) f_j J(f_j) \right) \times e^{\frac{i}{2} \sum_{j=1}^k \theta_j}.$$ 

Since $Spin(n)$ is contained in the Clifford algebra $C_n$, the spin representation of the group $Spin(n)$ extends to a $Spin^C(n)$-representation. For an element $[g, z]$ from $Spin^C(n)$ and any spinor $\psi \in \Delta_n$ we then have

$$\kappa[g, z] \psi = z \cdot \kappa(g)(\psi).$$

Hence the space $\Delta_n$ of $n$-spinors becomes a $Spin^C(n)$-representation. The determinant of the endomorphism $\kappa[g, z] : \Delta_n \to \Delta_n$ is given by the formula

$$\det \kappa[g, z] = z^{\dim(\Delta_n)}.$$ 

In the case of an even dimension, $n = 2k$, the splitting $\Delta_{2k} = \Delta_{2k}^+ \oplus \Delta_{2k}^-$ is $Spin^C(2k)$-invariant and for the corresponding representations we have

$$\det \kappa^\pm[g, z] = z^{\dim(\Delta_{2k}^\pm)}.$$ 

This implies that the $Spin^C(n)$-representations $\det(\Delta_n^\pm) = \Lambda^{\dim(\Delta_{n}^\pm)}(\Delta_{n}^\pm)$ and

$$(l)^{\dim \Delta_n^\pm/2} = l \otimes \ldots \otimes l \quad \left( \text{dim} \Delta_n^\pm \text{ times} \right)$$

are equivalent. For the particular case $n = 4$ we obtain the equation

$$\Lambda^2(\Delta_4^+) = \Lambda^2(\Delta_4^-) = l$$

in the sense of $Spin^C(4)$-representations.

**Proposition.** There exists an injective homomorphism $f : Spin^C(n) \to Spin(n + 2)$ such that the diagram

$$\begin{array}{ccc}
Spin^C(n) & \xrightarrow{f} & Spin(n + 2) \\
\downarrow p & & \downarrow \lambda \\
SO(n) \times S^1 & \rightarrow & SO(n) \times SO(2) \rightarrow SO(n + 2)
\end{array}$$

commutes.

**Proof.** $Spin(n)$ is a subgroup of $Spin(n + 2)$, and $S^1$ can be realized as a subgroup of $Spin(n + 2)$ by the elements

$$\cos t + \sin t e_{n+1} e_{n+2}$$

$$= (\cos (t/2) e_{n+1} + \sin (t/2) e_{n+2}) (\cos (t/2) e_{n+1} - \sin (t/2) e_{n+2}).$$

The intersection of these subsets of $Spin(n + 2)$ is $\{\pm 1\}$, and so we obtain a subgroup of $Spin(n + 2)$ isomorphic to $Spin^C(n)$. □
The Lie algebra of the group $\text{Spin}^C(n) \subset C_n$ is the direct sum

$$\text{spin}^C(n) = m_2 \oplus i\mathbb{R}$$

and the differential $p_* : \text{spin}^C(n) \to so(n) \times i\mathbb{R}$ of the covering $p$ is described by

$$p_*(e_\alpha e_\beta, it) = (2E_{\alpha\beta}, 2it), \quad 1 \leq \alpha < \beta \leq n.$$
Proof. We realize \( \Delta_3 \) as the vector space \( \Delta_3 = \mathbb{C}^2 \) and make use of the realization of the spin representation of the Clifford algebra \( C_3 \) described in Section 1.3:

\[
e_1 = g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 = iT = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Define \( \alpha, \beta : \Delta_3 \to \Delta_3 \) by the formulas

\[
\alpha \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \left( \begin{array}{c} -\bar{z}_2 \\ \bar{z}_1 \end{array} \right), \quad \beta \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \left( \begin{array}{c} \bar{z}_1 \\ \bar{z}_2 \end{array} \right).
\]

A straightforward calculation then leads to the result, e.g.

\[
\beta(e_2 \cdot \psi) = \beta \left( \begin{array}{c} iz_2 \\ iz_1 \end{array} \right) = \left( \begin{array}{c} -i\bar{z}_2 \\ -i\bar{z}_1 \end{array} \right), \quad e_2 \cdot \beta(\psi) = e_2 \left( \begin{array}{c} \bar{z}_1 \\ \bar{z}_2 \end{array} \right) = \left( \begin{array}{c} i\bar{z}_2 \\ i\bar{z}_1 \end{array} \right),
\]

\[
\beta(e_3 \cdot \psi) = \beta \left( \begin{array}{c} z_2 \\ -z_1 \end{array} \right) = \left( \begin{array}{c} \bar{z}_2 \\ -\bar{z}_1 \end{array} \right), \quad e_3 \cdot \beta(\psi) = e_3 \left( \begin{array}{c} \bar{z}_1 \\ \bar{z}_2 \end{array} \right) = \left( \begin{array}{c} \bar{z}_2 \\ -\bar{z}_1 \end{array} \right).
\]

Hence, \( \beta(e_2 \cdot \psi) = -e_2\beta(\psi) \) as well as \( \beta(e_3 \cdot \psi) = e_3\beta(\psi) \).

In order to continue the construction, we will need the following algebraic preparation. If \( \alpha : V \to V \) and \( \beta : W \to W \) are real or quaternionic structures, then their tensor product

\[
\alpha \otimes \beta : V \otimes \mathbb{C} W \to V \otimes \mathbb{C} W
\]

can be defined by

\[
(\alpha \otimes \beta)(v \otimes w) = \alpha(v) \otimes \beta(w).
\]

\( \alpha \otimes \beta \) is \( \mathbb{R} \)-linear and anti-commutes with multiplication by \( i \), since

\[
(\alpha \otimes \beta)(i(v \otimes w)) = (\alpha \otimes \beta)(iv \otimes w) = \alpha(iv) \otimes \beta(w) = -i\alpha(v) \otimes \beta(w) = -i(\alpha \otimes \beta)(v \otimes w).
\]

Note that \( \alpha \otimes \beta \) is correctly defined. In \( V \otimes \mathbb{C} W \), the identity \( v \otimes w = -(iv) \otimes (iw) \) holds, and we have

\[
(\alpha \otimes \beta)(-(iv) \otimes (iw)) = -\alpha(iv) \otimes \beta(iw) = -(-i\alpha(v)) \otimes (-i\beta(w)) = \alpha(v) \otimes \beta(w).
\]

Thus \( \alpha \otimes \beta \) is well-defined. \( \alpha \otimes \beta \) again is a real (quaternionic) structure, since \( (\alpha \otimes \beta)^2 = \alpha^2 \otimes \beta^2 = \pm \text{Id} \).

Recall the real and quaternionic structures \( \alpha, \beta : \mathbb{C}^2 \to \mathbb{C}^2 \) defined above and their commutation relations (note that \( e_3 = iT \) is replaced by \( T \))

\[
\alpha g_1 = g_1\alpha, \quad \alpha g_2 = g_2\alpha, \quad \alpha T = -T\alpha,
\]

\[
\beta g_1 = -g_1\beta, \quad \beta g_2 = -g_2\beta, \quad \beta T = -T\beta.
\]

Now we will construct one of these structures in each vector space \( \Delta_n \) of \( n \)-spinors \( \alpha_n : \Delta_n \to \Delta_n \).
1.7 Real and quaternionic structures in the space of \( n \)-spinors

Proposition.

1) Let \( n = 8k, 8k + 1 \). In \( \Delta_n \) there exists a real \( \text{Spin}(n) \)-equivariant structure \( \alpha_n : \Delta_n \to \Delta_n \) which anti-commutes with Clifford multiplication:

\[
\alpha_n(x \cdot \psi) = -x \cdot \alpha_n(\psi), \quad x \in \mathbb{R}^n \quad \text{and} \quad \psi \in \Delta_n.
\]

2) Let \( n = 8k + 2, 8k + 3 \). In \( \Delta_n \) there exists a quaternionic \( \text{Spin}(n) \)-equivariant structure \( \alpha_n : \Delta_n \to \Delta_n \) which commutes with Clifford multiplication:

\[
\alpha_n(x \cdot \psi) = x \cdot \alpha_n(\psi), \quad x \in \mathbb{R}^n \quad \text{and} \quad \psi \in \Delta_n.
\]

3) Let \( n = 8k + 4, 8k + 5 \). In \( \Delta_n \) there exists a quaternionic \( \text{Spin}(n) \)-equivariant structure \( \alpha_n : \Delta_n \to \Delta_n \) which anti-commutes with Clifford multiplication:

\[
\alpha_n(x \cdot \psi) = -x \cdot \alpha_n(\psi), \quad x \in \mathbb{R}^n \quad \text{and} \quad \psi \in \Delta_n.
\]

4) Let \( n = 8k + 6, 8k + 7 \). In \( \Delta_n \) there exists a real \( \text{Spin}(n) \)-equivariant structure \( \alpha_n : \Delta_n \to \Delta_n \) which commutes with Clifford multiplication:

\[
\alpha_n(x \cdot \psi) = x \cdot \alpha_n(\psi), \quad x \in \mathbb{R}^n \quad \text{and} \quad \psi \in \Delta_n.
\]

Proof. We define \( \alpha_n \) case by case.

First case. \( n = 8k, 8k + 1 \).

We have \( \Delta_n = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \) \((4k \text{ times})\), and we set

\[
\alpha_n = (\alpha \otimes \beta) \otimes \ldots \otimes (\alpha \otimes \beta) \quad (2k \text{ times}).
\]

Second case. \( n = 8k + 2, 8k + 3 \).

We have \( \Delta_n = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \) \((4k+1 \text{ times})\), and we set

\[
\alpha_n = \alpha \otimes (\beta \otimes \alpha) \otimes \ldots \otimes (\beta \otimes \alpha) \quad (2k \text{ times}).
\]

Third case. \( n = 8k + 4, 8k + 5 \).

We have \( \Delta_n = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \) \((4k+2 \text{ times})\), and we set

\[
\alpha_n = (\alpha \otimes \beta) \otimes \ldots \otimes (\alpha \otimes \beta) \quad (2k + 1 \text{ times}).
\]

Fourth case \( n = 8k + 6, 8k + 7 \).

We have \( \Delta_n = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \) \((4k+3 \text{ times})\), and we set

\[
\alpha_n = \alpha \otimes (\beta \otimes \alpha) \otimes \ldots \otimes (\beta \otimes \alpha) \quad (2k + 1 \text{ times}).
\]

Using the presentation of Clifford multiplication and the commutation relations between \( \alpha, \beta \) and \( g_1, g_2, T \), respectively, described in Section 1.3, the properties listed above are easily checked.

\( \Box \)
Summarizing, we arrive at the following table for the real or quaternionic structures in $\Delta_n$:

<table>
<thead>
<tr>
<th>$\alpha_n$</th>
<th>real structures</th>
<th>quaternionic structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>commutes with Clifford multiplication</td>
<td>$n \equiv 6, 7 \mod 8$</td>
<td>$n \equiv 2, 3 \mod 8$</td>
</tr>
<tr>
<td>anti-commutes with Clifford multiplication</td>
<td>$n \equiv 0, 1 \mod 8$</td>
<td>$n \equiv 4, 5 \mod 8$</td>
</tr>
</tbody>
</table>

We now ask whether, in the case of an even dimension $n$, the structures just constructed are compatible with the decomposition of Dirac spinors $\Delta_n$ into the sum of Weyl spinors $\Delta^+_n \oplus \Delta^-_n$. Let $n = 8k + 2\varepsilon$ ($\varepsilon = 0, 1, 2, \text{ or } 3$). The decomposition is defined by the operator

$$f = i^{4k+\varepsilon}e_1 \cdots e_{8k+2\varepsilon} = i^\varepsilon e_1 \cdots e_{8k+2\varepsilon}.$$

For $\varepsilon = 1, 3$, Clifford multiplication commutes with $\alpha_n$. Since $\alpha_n$ is complex anti-linear, in these cases, $\alpha_n$ anti-commutes with $f = \pm ie_1 \cdots e_{8k+2\varepsilon}$. For dimensions $n = 8k + 2, 8k + 6$ this implies that the real (or quaternionic) structure $\alpha_n : \Delta_n \to \Delta_n$ interchanges the summands in the decomposition $\Delta_n = \Delta^+_n \oplus \Delta^-_n$:

$$\alpha_n(\Delta^\pm_n) \subset \Delta^\mp_n.$$

For $\varepsilon = 0, 2$, Clifford multiplication anti-commutes with $\alpha_n$. Thus $\alpha_n$ commutes with $f = \pm e_1 \cdots e_{8k+2\varepsilon}$, $\alpha_n \cdot f = f \cdot \alpha_n$, and $\alpha_n$ preserves the decomposition $\Delta_n = \Delta^+_n \oplus \Delta^-_n$. In summary, we obtain the

**Proposition.**

1) The representation $\Delta^\pm_{8k}$ admits a real $\text{Spin}(8k)$-equivariant structure.

2) The representation $\Delta^\pm_{8k+4}$ admits a quaternionic $\text{Spin}(8k + 4)$-equivariant structure.

1.8. References and exercises


1.8. References and exercises


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**Exercise 1.** Let \((V, Q)\) be a non-degenerate quadratic form. Determine all elements in the Clifford algebra \(C(Q)\) which anti-commute with every element of \(V\).

**Exercise 2.** Let \((V_1, Q_1)\) and \((V_2, Q_2)\) be two quadratic forms and \(f : V_1 \to V_2\) a linear map such that

\[Q_1(v_1) = Q_2(f(v_1))\]

for all vectors \(v_1 \in V_1\). Prove that there is a homomorphism \(C(f) : C(Q_1) \to C(Q_2)\) of Clifford algebras such that the diagram

\[
\begin{array}{ccc}
C(Q_1) & \xrightarrow{C(f)} & C(Q_2) \\
\uparrow & & \uparrow \\
V_1 & \xrightarrow{f} & V_2
\end{array}
\]

commutes.

**Exercise 3.** In the text, the equations \(C_1 = \mathbb{C}, \ C_2 = \mathbb{H}\) were proved. Prove the following additional isomorphisms:

\[C_3 = \mathbb{H} \oplus \mathbb{H}, \quad C_4 = M_2(\mathbb{H}), \quad C_5 = M_4(\mathbb{C}),\]

\[C_6 = M_8(\mathbb{R}), \quad C_7 = M_8(\mathbb{R}) \oplus M_8(\mathbb{R}), \quad C_8 = M_{16}(\mathbb{R}).\]

**Exercise 4.** Prove the isomorphisms

\[C'_1 = \mathbb{R} \oplus \mathbb{R}, \quad C'_2 = M_2(\mathbb{R}), \quad C'_3 = M_2(\mathbb{C}), \quad C'_4 = M_2(\mathbb{H}),\]

\[C'_5 = M_2(\mathbb{H}) \oplus M_2(\mathbb{H}), \quad C'_6 = M_4(\mathbb{H}), \quad C'_8 = M_8(\mathbb{C}).\]

**Exercise 5.** Prove the equation \(C_{k-1} = C^0_k\).

Hint: If \(C_{k-1} = C^0_{k-1} \oplus C_{k-1}^1\) is the decomposition of the algebra \(C_{k-1}\) and \(e_k \in \mathbb{R}^k\) the last vector, then setting

\[f(x_0 + x_1) = x_0 + e_k x_1\]

defines a homomorphism \(f : C_{k-1} \to C_k\).
Exercise 6. Prove that

\[ \text{Spin}(3) \cong SU(2) = \{ q \in \mathbb{H} : \|q\| = 1 \}, \]
\[ \text{Spin}(4) \cong SU(2) \times SU(2), \]
\[ \text{Spin}(5) \cong Sp(2), \]
\[ \text{Spin}(6) \cong SU(4). \]

Exercise 7. Prove that the group \( \text{Spin}^C(4) \) is isomorphic to the following subgroup \( H \) of \( U(2) \times U(2) \):

\[ H = \{ (A, B) \in U(2) \times U(2) : \det(A) = \det(B) \}. \]

Exercise 8. Let \( e_1, \ldots, e_n \) be an orthonormal basis of \( \mathbb{R}^n \) and \( \lambda_* : \text{spin}(n) \rightarrow \text{so}(n) \) the differential of the 2-fold covering \( \lambda : \text{Spin}(n) \rightarrow \text{SO}(n) \). Prove that for every element \( z \in \text{spin}(n) \) the following equation holds:

\[ z = \frac{1}{2} \sum_{i<j} (\lambda_*(z) e_i, e_j) e_i e_j = \frac{1}{4} \sum_{i,j} (\lambda_*(z) e_i, e_j) e_i, e_j. \]
2.1. Spin structures on $SO(n)$-principal bundles

Let $X$ be a connected CW-complex and let $(Q, \pi, X; SO(n))$ denote an $SO(n)$-principal bundle over $X$.

**Definition.** A *spin structure* on the principal bundle $Q$ is a pair $(P, \Lambda)$ where

a) $P$ is a $Spin(n)$-principal bundle over $X$,

b) $\Lambda : P \to Q$ is a 2-fold covering for which the diagram

\[
P \times Spin(n) \longrightarrow P \quad \xrightarrow{\Lambda \times \lambda} \quad Q \times SO(n) \longrightarrow Q
\]

\[
\Lambda \quad \xrightarrow{\pi} \quad X
\]

commutes. Here the rows contain the action of the respective group on the corresponding principal bundle.

**Definition.** Two spin structures $(P_1, \Lambda_1)$ and $(P_2, \Lambda_2)$ are called *equivalent* if there exists a $Spin(n)$-equivariant map $f : P_1 \to P_2$ compatible with the coverings $\Lambda_1$ and $\Lambda_2$:

\[
P_1 \xrightarrow{f} P_2 \quad \xrightarrow{\Lambda_1} \quad Q \quad \xrightarrow{\Lambda_2} \quad P_2
\]
Denote by $F$ a fibre of the $SO(n)$-principal bundle $Q$. $F$ is diffeomorphic to the group $SO(n)$ and, since $n \geq 3$, the fundamental group $\pi_1(F)$ consists of two elements:

$$\pi_1(F) = \mathbb{Z}_2.$$ 

Let $\alpha \in \pi_1(F)$ be the non-trivial element. Denote by $i : F \to Q$ the embedding of $F$ into the space $Q$. Then

$$\alpha_F := i_\#(\alpha)$$

is an element of the fundamental group $\pi_1(Q)$. Consider a spin structure $\Lambda : P \to Q$ on $Q$. To this covering there corresponds the subgroup

$$H(P, \Lambda) := \Lambda_\#(\pi_1(P)) \subset \pi_1(Q).$$

$H(P, \Lambda) \subset \pi_1(Q)$ is a subgroup of index 2.

**Proposition.** The element $\alpha_F$ does not belong to $H(P, \Lambda)$, $\alpha_F \notin H(P, \Lambda)$.

**Proof.** Suppose that $\alpha_F \in H(P, \Lambda)$. Then the inclusion $i : F \to Q$ lifts to a continuous map $I : F \to P$ such that the diagram

$$\begin{array}{ccc}
  P & \xrightarrow{I} & F \\
  \downarrow{\Lambda} & & \downarrow{i} \\
  F & \xrightarrow{i} & Q
\end{array}$$

commutes. $I(F) \subset P$ is contained in one fibre $F'$ of the $Spin(n)$-principal bundle $P$, and hence we obtain a map

$$I : F = SO(n) \to F' = Spin(n) \quad \text{with} \quad \lambda \circ I = \text{Id}_{SO(n)}.$$ 

For the induced homomorphisms of fundamental groups this implies $\lambda_\#I_\# = \text{Id}_{\pi_1(SO(n))}$. Since $\pi_1(SO(n)) = \mathbb{Z}_2$ and $\pi_1(Spin(n)) = 1$, this is a contradiction. \hfill $\square$

**Proposition.** The equivalence classes of spin structures on an $SO(n)$-principal bundle $Q$ over a connected CW-complex $X$ are in bijective correspondence with those subgroups $H \subset \pi_1(Q)$ of index 2 which do not contain $\alpha_F$, $\alpha_F \not\in H$.

**Proof.** Let a subgroup $H \subset \pi_1(Q)$ of index 2 with $\alpha_F \not\in H$ be given. This subgroup defines a 2-fold covering

$$\Lambda : P \to Q$$
with connected total space $P$. Fix a point $p_0 \in P$ and denote by $\mu : Q \times SO(n) \to Q$ the action of the group $SO(n)$ on $Q$. The map

$$
P \times Spin(n) \xrightarrow{\lambda \times \lambda} Q \times SO(n) \xrightarrow{\mu} Q \times SO(n) \xrightarrow{\mu} Q
$$

induces the homomorphism $\mu_\# \circ (\Lambda \times \lambda)_\#$,

$$
\pi_1(P) = \pi_1(P \times Spin(n)) \xrightarrow{(\Lambda \times \lambda)_\#} \pi_1(Q) \oplus \pi_1(SO(n)) \xrightarrow{\mu_\#} \pi_1(Q),
$$

the image in $\pi_1(Q)$ of which coincides with $H$. Hence there exists a unique continuous map $\tilde{\mu} : P \times Spin(n) \to P$ with $\tilde{\mu}(p_0, 1) = p_0$ fitting into the commutative diagram

$$
P \times Spin(n) \xrightarrow{\tilde{\mu}} P
$$

It is easy to show that $\tilde{\mu}$ is an action of $Spin(n)$ on $P$. For example, the map $f(p) = \tilde{\mu}(p, 1) : P \to P$ is the lift of the map $\Lambda : P \to Q$,

$$
P \xrightarrow{f} P
$$

with initial condition $f(p_0) = p_0$. Uniqueness of the lift then implies $f = \text{Id}_P$, i.e.

$$
\tilde{\mu}(p, 1) = p \quad \text{for all } p \in P.
$$

It remains to be proved that $Spin(n)$ acts simply transitively on each fibre of the map $\pi \circ \Lambda : P \to X$ over $X$. To show this, it suffices to check that $(-1) \in Spin(n)$ does not have any fixed point. Suppose that $\mu(p_0, (-1)) = p_0$.

Choose a path $\gamma(t)$ ($0 \leq t \leq 1$) from 1 to $(-1)$ in $Spin(n)$. Then $\gamma^*(t) = \tilde{\mu}(p_0, \gamma(t))$ is a loop in $P$; hence it defines an element of the fundamental group $\pi_1(P)$. The equivalence class of the loop $\Lambda \gamma^*(t)$ thus belongs to the subgroup $H \subset \pi_1(Q)$ of the covering $\Lambda : P \to Q$. On the other hand,

$$
\Lambda \gamma^*(t) = \Lambda \tilde{\mu}(p_0, \gamma(t)) = \mu(\Lambda(p_0), \lambda \circ \gamma(t)).
$$
Now $\lambda \circ \gamma(t)$ is a loop in $SO(n)$ representing the non-trivial element $\alpha \in \pi_1(SO(n))$. Finally, for the fibre $F = \lambda(p_0) \cdot SO(n) \subset Q$ we obtain the inclusion

$$\alpha_F = \Lambda \gamma^*(t) \in H \subset \pi_1(Q),$$

in contradiction with the assumption on the subgroup $H$.

Let us look at the exact homotopy sequence of the $SO(n)$-fibration $\pi : Q \to X$:

\[
\cdots \to \pi_2(X) \xrightarrow{\partial} \pi_1(F) \xrightarrow{i^\#} \pi_1(Q) \xrightarrow{\pi^\#} \pi_1(X) \to 1.
\]

A subgroup $H \subset \pi_1(Q)$ of index 2 defines a non-trivial homomorphism

$$f_H : \pi_1(Q) \to \pi_1(Q)/H = \mathbb{Z}_2$$

and vice versa. The condition $\alpha_F \not\in H$ is equivalent to requiring that the homomorphism

$$f_H \circ i^\# : \pi_2 = \pi_1(F) \to \pi_1(Q) \to \pi_1(Q)/H = \mathbb{Z}_2$$

is the identity. Hence we obtain the

**Corollary.** The spin structures on an $SO(n)$-principal bundle $Q$ over a connected CW-complex $X$ are in one-to-one correspondence with the homomorphisms splitting the sequence $(\ast)$:

$$f : \pi_1(Q) \to \pi_1(F) \quad \text{and} \quad f \circ i^\# = \text{Id}_{\pi_1(F)}.$$  

**Corollary.** If the $SO(n)$-principal bundle has a spin structure, then the following groups are isomorphic:

a) $\pi_1(Q) = \pi_1(F) \oplus \pi_1(X)$,  

b) $\pi_2(Q) = \pi_2(X)$.

**Corollary.** Let $X$ be a simply connected CW-complex. An $SO(n)$-principal bundle $Q$ has a spin structure if and only if

$$\pi_1(Q) = \mathbb{Z}_2.$$  

In this case, the spin structure is uniquely determined.

The last corollary can be generalized using the extension theory of finite groups. The preceding considerations show that an $SO(n)$-principal bundle $Q$ over a CW-complex $X$ has a spin structure if

a) $i^\# : \pi_1(F) = \mathbb{Z}_2 \to \pi_1(Q)$ is injective;  

b) the exact sequence $1 \to \pi_1(F) \xrightarrow{i^\#} \pi_1(Q) \to \pi_1(X) \to 1$ splits.
We now suppose that $\pi_1(X)$ is a finite group and, moreover, that condition a) is satisfied. Then $\pi_1(Q)$ is an extension of the group $\mathbb{Z}_2 = \pi_1(F)$ by $\pi_1(X)$. Recall the notion of a 2-Sylow subgroup of a finite group $G$. If $|G|$ denotes the order of the group $G$ and $2^k$ is the largest power of 2 dividing $|G|$, then each subgroup of $G$ of order $2^k$ is called a 2-Sylow subgroup. It is well-known that there exists at least one such subgroup in every group $G$. If

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

is an extension of $\mathbb{Z}_2$ by $\Gamma$, then we have the following criterion for the splitting of this extension.

**Proposition (Schur-Zassenhaus, Gaschütz).** The extension $G$ of the group $\mathbb{Z}_2$ by the finite group $\Gamma$ splits if and only if for every 2-Sylow subgroup $G_2 \subset G$ the extension

$$1 \longrightarrow (G_2 \cap \mathbb{Z}_2) \longrightarrow G_2 \longrightarrow G_2/(G_2 \cap \mathbb{Z}_2) \longrightarrow 1$$

splits.


A first conclusion is that for $\pi_1(X)$ finite, the principal bundle $Q$ admits a spin structure if and only if the following conditions are satisfied:

a') $i_\#: \pi_1(F) \rightarrow \pi_1(Q)$ is injective;

b') for each 2-Sylow subgroup $G_2 \subset \pi_1(Q)$, the corresponding extension

$$1 \longrightarrow (G_2 \cap \mathbb{Z}_2) \longrightarrow G_2 \longrightarrow G_2/(G_2 \cap \mathbb{Z}_2) \longrightarrow 1$$

splits.

If $G_2 \subset \pi_1(Q)$ is a 2-Sylow subgroup of $\pi_1(Q)$, then $G_2^* = p_\#(G_2) \subset \pi_1(X)$ is a 2-Sylow subgroup of $\pi_1(X)$ and vice versa. Thus, if every extension of $\mathbb{Z}_2$ by $G_2^*$ splits (or, equivalently, $H^2(G_2^*; \mathbb{Z}_2) = 0$), then condition b') is automatically satisfied. We obtain the

**Corollary.** Let $X$ be a CW-complex with finite fundamental group satisfying the condition

$$H^2(G_2^*; \mathbb{Z}_2) = 0$$

for every 2-Sylow subgroup $G_2^* \subset \pi_1(X)$. Then an $SO(n)$-principal bundle $Q$ over $X$ has a spin structure if

$$i_\#: \pi_1(F) \longrightarrow \pi_1(Q)$$

is injective.
In particular, this corollary applies in the case where the fundamental group \( \pi_1(X) \) is of odd order. The only condition for the existence of a spin structure then consists in requiring the injectivity of the homomorphism

\[
i_\#: \pi_1(F) \longrightarrow \pi_1(Q).
\]

We will now reformulate the description of spin structures thus obtained in the language of cohomology. For every CW-complex \( Y \) we have

\[
H^1(Y; \mathbb{Z}_2) = \text{Hom}(H_1(Y); \mathbb{Z}_2) = \text{Hom}(\pi_1(Y)/[\pi_1(Y), \pi_1(Y)]; \mathbb{Z}_2) = \text{Hom}(\pi_1(Y); \mathbb{Z}_2).
\]

Hence the homomorphism \( f : \pi_1(Q) \to \pi_1(F) = \mathbb{Z}_2 \) defines an element in the cohomology group, \( f \in H^1(Q; \mathbb{Z}_2) \). On the other hand, \( H^1(F; \mathbb{Z}_2) = \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \), and the condition \( f \circ i_\# = \text{Id}_{\pi_1(F)} \) is equivalent to the requirement that the element \( f \in H^1(Q; \mathbb{Z}_2) \) remains non-trivial after restriction to the fibre \( i^* : H^1(Q; \mathbb{Z}_2) \to H^1(F; \mathbb{Z}_2) = \mathbb{Z}_2 \). From this we obtain the

**Proposition.** The spin structures on an \( SO(n) \)-principal bundle over a connected CW-complex \( X \) are in one-to-one correspondence with those elements \( f \in H^1(Q; \mathbb{Z}_2) \) for which \( i^*(f) \neq 0 \) holds in \( H^1(F; \mathbb{Z}_2) = \mathbb{Z}_2 \).

The \( SO(n) \)-fibration \( \pi : Q \to X \) induces the following exact sequence of cohomology groups:

\[
1 \longrightarrow H^1(X; \mathbb{Z}_2) \longrightarrow H^1(Q; \mathbb{Z}_2) \xrightarrow{i^*} H^1(F; \mathbb{Z}_2) = \mathbb{Z}_2 \xrightarrow{\partial} H^2(X; \mathbb{Z}_2) \longrightarrow \ldots
\]

If \( 1 \in \mathbb{Z}_2 = H^1(F; \mathbb{Z}_2) \) denotes the non-trivial element, then

\[
w_2(Q) := \partial(1) \in H^2(X; \mathbb{Z}_2)
\]

is called the second Stiefel-Whitney class of the \( SO(n) \)-principal bundle. The above sequence, moreover, immediately implies the

**Proposition.** An \( SO(n) \)-principal bundle over a connected CW-complex \( X \) has a spin structure if and only if \( w_2(Q) \) vanishes,

\[
w_2(Q) = 0.
\]

*In this case, the spin structures are classified by \( H^1(X; \mathbb{Z}_2) \).*

**Example.** Consider the complex projective space \( \mathbb{C}P^n \). The group \( SU(n + 1) \) acts transitively on \( \mathbb{C}P^n \):

\[
\mathbb{C}P^n = SU(n + 1)/S(U(n) \times U(1)).
\]

The isotropy representation

\[
\sigma : S(U(n) \times U(1)) \to U(n) \subset SO(2n)
\]
2.1. Spin structures on $SO(n)$-principal bundles

is given by the formula

$$\sigma\left(\begin{array}{cc} B & 0 \\ 0 & \frac{1}{\det B} \end{array}\right) = \det B \cdot B.$$ 

Let $R = SU(n + 1) \times_{\sigma} SO(2n)$ be the frame bundle. As $\mathbb{CP}^n$ is simply connected, the fundamental group $\pi_1(R)$ has at most two elements; it is the surjective image of $\pi_1(SO(n)) = \mathbb{Z}_2$. In order to decide whether $R$ admits a spin structure, we first compute the homomorphism $\sigma_\#$ between fundamental groups induced from the isotropy representation $\sigma$. The generating element of the fundamental group $\pi_1(S(U(n) \times U(1)))$ is represented by the loop

$$\gamma(t) = \left(\begin{array}{cccc} e^{it} & 0 & & \\
 & 1 & & \\
 & & \ddots & \\
 & & & e^{-it} \end{array}\right), \quad 0 \leq t \leq 2\pi.$$ 

Thus $\sigma(\gamma(t))$, as a path in $U(n)$, is given by

$$\sigma(\gamma(t)) = \left(\begin{array}{cccc} e^{2it} & 0 & & \\
 & e^{it} & & \\
 & & \ddots & \\
 & & & e^{it} \end{array}\right)$$ 

and hence is equal to the $(n + 1)$-fold power of the generating element of $\pi_1(U(n))$. This implies that the homomorphism

$$\sigma_\# : \pi_1(S(U(n) \times U(1))) = \mathbb{Z} \longrightarrow \pi_1(SO(2n)) = \mathbb{Z}_2$$

is described by $\sigma_\#(1) = (n + 1) \mod 2$. Consider first the case that $n$ is odd. Then $\sigma_\#$ is the trivial homomorphism, and thus there exists a lift $\tilde{\sigma} : S(U(n) \times U(1)) \rightarrow Spin(2n)$ of the isotropy representation $\sigma$ to the spin group. Hence

$$P := SU(n + 1) \times_{\tilde{\sigma}} Spin(2n)$$

is a spin structure on the frame bundle $R$. Finally, we also discuss the case that $n$ is even. Consider now the fibration of $R = SU(n + 1) \times_{\sigma} SO(2n)$ over the space $SO(2n)/\sigma(S(U(n) \times U(1))) = SO(2n)/U(n)$ defined by the formula

$$[A, B] \longrightarrow B \mod \sigma(S(U(n) \times U(1))),$$

where $A \in SU(n + 1)$ and $B \in SO(2n)$. The fibre of this fibration is $SU(n + 1)/\mathbb{Z}_{n+1}$. Hence the resulting exact sequence has the form

$$\ldots \longrightarrow \mathbb{Z}_{n+1} \longrightarrow \pi_1(R) \longrightarrow \pi_1(SO(2n)/U(n)) = 1.$$
Since $\pi_1(R)$ contains at most two elements, and $n$ is even, in this case, $\pi_1(R) = 1$. Summarizing, we obtain

$$\pi_1(R) = \begin{cases} 
\mathbb{Z}_2 & \text{if } n \text{ is odd}, \\
1 & \text{if } n \text{ is even}.
\end{cases}$$

Since $\mathbb{C}P^n$ is simply connected, we can apply the corresponding criterion for the existence of a spin structure on $R$. The bundle $R$ has a spin structure if and only if $\pi_1(R) = \mathbb{Z}_2$. This leads to the

**Proposition.** The frame bundle $R$ of the complex projective space $\mathbb{C}P^n$ admits a spin structure if and only if $n$ is odd, $n \equiv 1 \mod 2$.

**Remark.** It is important to note that two spin structures on one and the same $SO(n)$-principal bundle may not be equivalent even if the corresponding $Spin(n)$-principal bundles over $X$ are equivalent. To see this, consider e.g. $X = \mathbb{R}P^2$ and the trivial bundle $Q = \mathbb{R}P^2 \times SO(n)$. Because $H^1(\mathbb{R}P^2; \mathbb{Z}_2) = \mathbb{Z}_2$, there are two spin structures on $Q$. On the other hand, let $P \to X = \mathbb{R}P^2$ be any $Spin(n)$-principal bundle. Since

$$\dim(\mathbb{R}P^2) = 2, \quad \pi_1(Spin(n)) = 0,$$

this bundle has a section and is thus trivial. Hence the spin bundles of both these spin structures are isomorphic as principal bundles over $X = \mathbb{R}P^2$. The same effect occurs if we replace $\mathbb{R}P^2$ by an arbitrary 2-dimensional manifold.

### 2.2. Spin structures in covering spaces

Consider a simply connected CW-complex $X$ ($\pi_1(X) = 1$) and an $SO(n)$-principal bundle $Q$ which is assumed to have (exactly) one spin structure $(P, \Lambda)$. Let the discrete group $\Gamma$ act from the left on the space $X$ and on $Q$ as a group of $SO(n)$-bundle morphisms. Set

$$X^* = \Gamma \backslash X, \quad Q^* = \Gamma \backslash Q$$

and let $X \to X^* = \Gamma \backslash X$ be the covering map. Then $Q^*$ is an $SO(n)$-principal bundle over the space $X^*$. We want to study the question of whether $Q^*$ admits a spin structure.

Since $\pi_1(P) = \pi_1(X) = 1$, for every group element $\gamma \in \Gamma$ there exist two lifts $\gamma^\pm$ of the transformation $\gamma : Q \to Q$ such that the following diagram commutes:

$$\begin{array}{ccc}
P & \xrightarrow{\gamma^\pm} & P \\
\Lambda \downarrow & & \Lambda \downarrow \\
Q & \xrightarrow{\gamma} & Q
\end{array}$$

If $\varepsilon$ is a left action of $\Gamma$ on $P$ with $\varepsilon(\gamma) = \gamma^\pm$, then $P^* = \Gamma \backslash P$ obviously becomes a spin structure on $Q^*$. Conversely, if $(\tilde{P}^*, \tilde{\Lambda})$ is a spin structure
on $Q^*$ and $q : X \to X^*$ denotes the projection, then $Q$ is the bundle $q^*(Q^*)$ induced from $Q^*$ by means of this projection. Therefore, $q^*(P^*)$ is a spin structure on $Q$ on which $\Gamma$ acts as a group of automorphisms. But, since $\pi_1(X) = 1$, the spin structure $q^*(P^*)$ is equivalent to $(P, \Lambda)$. Thus, in summary, we conclude:

**Proposition.** The spin structures in the principal bundle $Q^*$ over $X^*$ are in one-to-one correspondence with all left actions $\varepsilon$ of $\Gamma$ on $P$ satisfying

$$\varepsilon(\gamma) = \gamma^\pm.$$  

We can also study the existence question for a spin structure on $Q^*$ by looking at the fundamental group. From

$$
\begin{array}{ccc}
Q & \longrightarrow & Q^* \\
\downarrow & & \downarrow \\
X & \longrightarrow & X^*
\end{array}
$$

we obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \\
\uparrow & & \uparrow & & \\
\mathbb{Z}_2 = \pi_1(F) & \sim & \pi_1(F^*) & \\
\downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(Q) & \longrightarrow & \pi_1(Q^*) & \longrightarrow & \Gamma & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
1 = \pi_1(X) & \longrightarrow & \pi_1(X^*) = \Gamma
\end{array}
$$

By assumption, $Q$ has a spin structure and hence $\pi_1(F) \to \pi_1(Q)$ is injective. Thus $\pi_1(F^*) \to \pi_1(Q^*)$ is also injective, and we arrive at the

**Proposition.** The $SO(n)$-principal bundle $Q^*$ over the space $X^*$ has a spin structure if and only if the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 = \pi_1(F^*) \longrightarrow \pi_1(Q^*) \longrightarrow \Gamma \longrightarrow 1$$

splits.

**Example 1.** Let $X = S^n = SO(n+1)/SO(n)$ be the $n$-dimensional sphere and $Q = SO(n+1)$ its frame bundle. Moreover, let the group $\Gamma \subset SO(n+1)$ act freely on the sphere from the left. Then

$$Q^* = \Gamma \backslash SO(n+1)$$
is the frame bundle of the space $X^* = \Gamma \backslash S^n$. $Q$ itself, as a bundle over $S^n$, has the spin structure $P = \text{Spin}(n+1)$. Thus $Q^*$ admits a spin structure if and only if the sequence

$$1 \to \mathbb{Z}_2 = \pi_1(SO(n)) \to \pi_1(\Gamma \backslash SO(n+1)) \to \Gamma \to 1$$

splits. Considering the covering $\lambda : \text{Spin}(n+1) \to SO(n+1)$, we obtain

$$\pi_1(\Gamma \backslash SO(n+1)) = \lambda^{-1}(\Gamma) \backslash \text{Spin}(n+1)$$

as well as $\pi_1(\lambda^{-1}(\Gamma) \backslash \text{Spin}(n+1)) = \lambda^{-1}(\Gamma)$. Hence the sequence in question is

$$1 \to \mathbb{Z}_2 \to \lambda^{-1}(\Gamma) \to \Gamma \to 1,$$

and the manifold $X^* = \Gamma \backslash S^n$ is a spin manifold if and only if this sequence splits. For a group $\Gamma$ of odd order $|\Gamma|$ this sequence always splits. Otherwise, the splitting of the above exact sequence is equivalent to the splitting of the sequence

$$1 \to (\lambda^{-1}(\Gamma_2) \cap \mathbb{Z}_2) \to \lambda^{-1}(\Gamma_2) \to \Gamma_2 \to 1,$$

where $\Gamma_2 \subset \Gamma$ is a 2-Sylow subgroup. Summarizing, we obtain the

**Proposition.** Let $\Gamma \subset SO(n+1)$ be a finite subgroup acting freely on the sphere $S^n$. The manifold $\Gamma \backslash S^n$ has a spin structure if and only if for every 2-Sylow subgroup $\Gamma_2 \subset \Gamma$ the sequence

$$1 \to (\lambda^{-1}(\Gamma_2) \cap \mathbb{Z}_2) \to \lambda^{-1}(\Gamma_2) \to \Gamma_2 \to 1$$

splits.

Let us consider the case $n = 4k + 1 \equiv 1 \mod 4$ separately.

**Proposition.** Let $n = 4k + 1$, and let $\Gamma \subset SO(n+1)$ be a finite subgroup acting without fixed points on the sphere $S^n$. Then $\Gamma \backslash S^n$ has a spin structure if and only if $\Gamma$ contains no elements of order 2. In this case,

$$H^1(\Gamma \backslash S^n ; \mathbb{Z}_2) = 0,$$

and hence the spin structure of $\Gamma \backslash S^n$ is unique.

**Proof.** Suppose that $\Gamma$ does not contain any elements of order 2. Then $\Gamma$ has no proper 2-Sylow subgroups, too, i.e. on $\Gamma \backslash S^n$ there exists a spin structure. Let $A \in \Gamma$ be an element of order 2. Since $\langle Ax, Ay \rangle = \langle x, y \rangle$ for $x, y \in \mathbb{R}^{n+1}$ and $A^2 = 1$, the matrix $A$ is symmetric: $\langle Ax, y \rangle = \langle x, Ay \rangle$. Hence $A$ is diagonalizable with eigenvalues $\lambda_1 = \ldots = \lambda_n = \pm 1$. If 1 is an eigenvalue, then $A$ has a fixed point on the sphere. This implies $A = -I$. 
Set $\Gamma_0 = \{E, -E\}$. This leads to the following commutative diagram of coverings:

$$
\begin{array}{ccc}
S^{4k+1} & \rightarrow & \Gamma_0 \backslash S^{4k+1} = \mathbb{R}P^{4k+1} \\
\downarrow & & \\
\Gamma \backslash S^{4k+1}
\end{array}
$$

If $\Gamma \backslash S^{4k+1}$ has a spin structure, then pulling it back we construct a spin structure on the real projective space $\mathbb{R}P^{4k+1}$. Thus we arrive at a contradiction, since $\mathbb{R}P^n$ admits a spin structure only for $n \equiv 3 \mod 4$. Finally, we will prove that

$$H^1(\Gamma \backslash S^n; \mathbb{Z}_2) = \text{Hom}(\Gamma; \mathbb{Z}_2) = 1,$$

if $\Gamma$ contains no elements of order 2. Actually, in this case, the order $|\Gamma|$ of the group $\Gamma$ is odd, i.e. $\Gamma$ has an odd number of elements. A non-trivial homomorphism $f: \Gamma \rightarrow \mathbb{Z}_2$ defines by means of $\Gamma_0 = \ker(f)$ and any $\gamma_0 \notin \ker(f)$ a partition of the set $\Gamma$ into $\Gamma = \Gamma_0 \cup \gamma_0 \cdot \Gamma_0$, and hence $|\Gamma| = 2|\Gamma_0| \equiv 0 \mod 2$, in contradiction with the assumption. \qed

### 2.3. Spin structures on $G$-principal bundles

Let $G \subset SO(n)$ be a connected compact subgroup for which, moreover, in this section, it will generally be assumed that the inclusion induces an epimorphism

$$i_\# : \pi_1(G) \longrightarrow \pi_1(SO(n)).$$

Then the homogeneous space $SO(n)/G$ is simply connected, $\pi_1(SO(n)/G) = \{1\}$. Consider a $G$-principal bundle $(Q, \pi, X; G)$ over a CW-complex $X$. Then the associated bundle $Q^* = Q \times_G SO(n)$ is an $SO(n)$-principal bundle over $X$.

**Definition.** The $G$-principal bundle $Q$ admits a spin structure if the $SO(n)$-principal bundle $Q^*$ admits one.

Now we will derive a condition for the existence of a spin structure in this sense. To do so, denote by $F = G$ and $F^* = SO(n)$ the fibres in the bundles $Q$ and $Q^*$, respectively. Consider the commutative diagram

$$
\begin{array}{ccc}
F = G & \xrightarrow{i} & SO(n) = F^* \\
\downarrow & & \downarrow \\
Q & \xrightarrow{j} & Q^* \\
\downarrow & & \downarrow \\
X & \xrightarrow{l} & SO(n)/G
\end{array}
$$
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where the maps \( j, l, m \) are defined as follows for \( q \in Q, A \in SO(n) \):

\[
\begin{align*}
 j(q) &= [q, 1], \\
l[q, A] &= A \mod G, \\
m(A) &= A \mod G.
\end{align*}
\]

The row \( Q \to Q^* \to SO(n)/G \) as well as the columns are fibrations. Hence we obtain the following commutative diagram of homotopy groups:

\[
\begin{array}{cccc}
\pi_2(X) & \downarrow & \pi_2(X) & \downarrow \\
\pi_1(F) & \downarrow & \pi_1(F^*) & \downarrow \\
\pi_1(Q) & \downarrow & \pi_1(Q^*) & \downarrow \\
\pi_1(X) & \downarrow & \pi_1(SO(n)/G) & = 1 \\
\end{array}
\]

Suppose that \( Q^* \) admits a spin structure, and let \( f^* : \pi_1(Q^*) \to \pi_1(F^*) \) be a homomorphism for which \( \pi_1(F^*) \xrightarrow{\varepsilon^*} \pi_1(Q^*) \to \pi_1(F^*) \) is the identity. Then consider the homomorphism

\[
f = f^* \circ j_\#: \pi_1(Q) \to \pi_1(F^*) = \pi_1(SO(n))
\]

and conclude that

\[
f \circ \varepsilon = f^* \circ j_\# \circ \varepsilon = f^* \circ \varepsilon^* \circ i_\# = i_#.
\]

Hence the following diagram commutes:

\[
\begin{array}{ccc}
\pi_1(F) & \xrightarrow{i_\#} & \pi_1(F^*) \\
\downarrow \varepsilon & & \downarrow f \\
\pi_1(Q) & & \\
\end{array}
\]

Conversely, let \( f \) with this commutative diagram be given. Define \( f^* : \pi_1(Q^*) \to \pi_1(F^*) \) as follows: For each element \( x \in \pi_1(Q^*) \) there exists an element \( y \in \pi_1(Q) \) such that \( j_\#(y) = x \). Set \( f^*(x) = f(y) \). If \( y_1 \) is another element from \( \pi_1(Q) \) with \( j_\#(y_1) = x \), then there exists an element \( z \in \pi_2(SO(n)/G) \) such that \( y_1 = y \cdot \partial(z) \). But this implies

\[
f(y_1) = f(y)f(\partial(z)) = f(y)f(\varepsilon(\partial(z))) = f(y)i_\#\partial(z) = f(y) \cdot 1
\]

in \( \pi_1(F^*) \). Thus \( f^* : \pi_1(Q^*) \to \pi_1(F^*) \) is uniquely defined. We still have to check the condition \( f^* \circ \varepsilon^* = \text{Id} \) on \( \pi_1(F^*) \). Choose \( \alpha \in \pi_1(F) \) with \( i_\#(\alpha) = \alpha^* \neq 1 \) in \( \pi_1(F^*) \). Then,

\[
\alpha^* = i_\#(\alpha) = f\varepsilon(\alpha) = f^*j_\#\varepsilon(\alpha) = f^*\varepsilon^*i_\#(\alpha) = f^*\varepsilon^*(\alpha^*).
\]
2.4. Existence of spin$^C$ structures

Summarizing, we obtain the

**Proposition.** Let $G \subset SO(n)$ be a connected compact subgroup with
\[ \pi_1(SO(n)/G) = 1. \]

A $G$-principal bundle $Q$ over a connected CW-complex $X$ has a spin structure if and only if there exists a homomorphism $f : \pi_1(Q) \to \pi_1(SO(n))$ for which the diagram
\[
\begin{array}{ccc}
\pi_1(F) & \xrightarrow{i^\#} & \pi_1(SO(n)) \\
\downarrow & & \downarrow \quad f \\
\pi_1(Q) & & \\
\end{array}
\]
commutes.

This condition can again be reformulated cohomologically. The homomorphism $f$ defines an element
\[ f \in H^1(Q; \mathbb{Z}_2) = Hom(\pi_1(Q), \mathbb{Z}_2) \]
whose restriction to the fibre $F$, $i^*(f) \in H^1(F; \mathbb{Z}_2) = Hom(\pi_1(G), \mathbb{Z}_2)$, has to coincide with $i_\# : \pi_1(G) \to \pi_1(SO(n)) = \mathbb{Z}_2$. Hence we have the

**Proposition.** Let $G \subset SO(n)$ be a connected compact group with
\[ \pi_1(SO(n)/G) = 1. \]

A $G$-principal bundle $Q$ over a connected CW-complex has a spin structure if and only if there is an element $f \in H^1(Q; \mathbb{Z}_2)$ for which the restriction to the fibre $F$ coincides with $i_\#$, $i^*(f) = i_\#$.

2.4. Existence of spin$^C$ structures

Consider an $SO(n)$-principal bundle $Q$ with base space $X$. In analogy with a spin structure, we define

**Definition.** A spin$^C$ structure on $Q$ is a pair $(P, \Lambda)$ consisting of a Spin$^C$-principal bundle $P$ over the space $X$ and a map $\Lambda : P \to Q$ such that the diagram
\[
\begin{array}{ccc}
P \times Spin^C(n) & \to & P \\
\downarrow_{\Lambda \times \lambda} & & \downarrow \Lambda \\
Q \times SO(n) & \to & Q \\
\end{array}
\]
commutes.

**Example.** Because of the inclusion $i : Spin(n) \to Spin^C(n)$, each spin structure on $Q$ induces a spin$^C$ structure.
Example. Suppose that the $SO(n)$-bundle $Q$ ($n = 2k$) has a $U(k)$-reduction, i.e. there exists a $U(k)$-bundle $R$ with

$$Q = R \times_{U(k)} SO(2k).$$

In Section 1.6 we constructed a homomorphism $F : U(k) \to Spin^C(2k)$ for which the diagram

$$\begin{array}{ccc}
Spin^C(2k) & \xrightarrow{F} & U(k) \\
\downarrow & & \downarrow \\
SO(2k) & \to & SO(2k)
\end{array}$$

commutes. Hence

$$P := R \times_{U(k)} Spin^C(2k)$$

is a $spin^C$ structure on $Q$. In other words: Every $U(k)$-reduction of the $SO(n)$-bundle $Q$ induces a $spin^C$ structure in $Q$.

The groups $Spin(n)$ and $S^1$ are subgroups of $Spin^C(n)$ whose elements commute with each other. Hence, if $(P, \Lambda)$ is a $spin^C$ structure, then

1) $P/S^1$ is a $Spin(n)/\{\pm 1\} = SO(n)$-bundle isomorphic to $Q$,

$$P/S^1 = Q,$$

2) $P_1 := P/Spin(n)$ is an $S^1/\{\pm 1\} = S^1$-bundle over $X$,

and the combination of bundle morphisms over the base space $P \to Q \times P_1$ is a 2-fold covering. Here $Q \times P_1$ denotes the fibre-product of the $SO(n)$-principal bundle $Q$ with the $S^1 = SO(2)$-principal bundle $P_1$ over $X$. $Q \times P_1$ is an $(SO(n) \times SO(2))$-principal bundle over $X$. Because of the diagram (compare Section 1.6)

$$\begin{array}{ccc}
Spin^C(n) & \to & Spin(n + 2) \\
\downarrow & & \downarrow \\
SO(n) \times SO(2) & \to & SO(n + 2)
\end{array}$$

the $(SO(n) \times SO(2))$-principal bundle $Q \times P_1$ admits a spin structure in the sense of Section 2.3. All in all, we obtain the

**Proposition.** If the $SO(n)$-principal bundle $Q$ admits a $spin^C$ structure, then there exists an $S^1$-principal bundle $P_1$ over $X$ such that the fibre product $Q \times P_1$ has a spin structure. Conversely, if such a bundle $P_1$ with the given property exists, then $Q$ has a $spin^C$ structure.

**Remark.** Making use of characteristic classes, we can formulate this equivalently as follows:

a) $Q$ has a $spin^C$ structure;
b) there exists an $S^1$-bundle $P_1$ such that $w_2(Q \times P_1) = 0$;

c) there exists an $S^1$-bundle $P_1$ such that $w_2(Q) \equiv c_1(P_1) \mod 2$;

d) there exists a cohomology class $z \in H^2(X; \mathbb{Z})$ such that $w_2(Q) \equiv z \mod 2$.

Hence, $Q$ has a spin$^C$ structure if and only if the Stiefel-Whitney class $w_2(Q) \in H^2(X; \mathbb{Z})$ is the $\mathbb{Z}_2$-reduction of an integral class $z \in H^2(X; \mathbb{Z})$.

**Corollary.** If $H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}_2)$ is surjective, then every $SO(n)$-bundle $Q$ over $X$ admits a spin$^C$ structure.

We will now discuss the existence question for a spin$^C$ structure on an $SO(n)$-principal bundle over a special class of base spaces $X$, and show that, sometimes, one can obtain a complete answer. Concerning the base space $X$ we assume that $\pi_1(X) = 1$ and $\pi_2(X)$ is a finite group.

Let $Q$ be an $SO(n)$-principal bundle and $P_1$ an $S^1$-bundle over $X$. On the one hand, the fibre-product $Q \times P_1$ is an $(SO(n) \times SO(2))$-principal bundle over $X$, and, on the other, a fibration over $Q$ with fibre $S^1$. Together, this yields the diagram

$$
\begin{array}{cccccccc}
\vdots \\
p_2(X) & \downarrow \partial & \\
\mathbb{Z}_2 \oplus \mathbb{Z} & \alpha \rightarrow (0, \alpha) & \\
Z = \pi_1(S^1) & \rightarrow & \pi_1(Q \times P_1) & \rightarrow & \pi_1(Q) & \rightarrow & 1 \\
\pi_1(X) = 1 & \\
\end{array}
$$

of homotopy groups in which the column and the row are exact. As $\pi_2(X)$ is finite, the image of $\partial$ is contained in the subgroup $\mathbb{Z}_2$ and, therefore, there are two cases.

**First case:** $\partial \equiv 0$.

Then $\mathbb{Z}_2 \oplus \mathbb{Z} = \pi_1(SO(n) \times SO(2)) \hookrightarrow \pi_1(Q \times P_1)$ is an isomorphism. Hence the bundle $Q \times P_1$ has a spin structure, i.e. $Q$ itself admits a spin$^C$ structure. Moreover, the exact row immediately implies $\pi_1(Q) = \mathbb{Z}_2$. 
Second case: $\text{im}(\partial) = \mathbb{Z}_2$.

Then the generating element of the group $\mathbb{Z}_2 = \pi_1(SO(n))$ belongs to the kernel of the homomorphism

$$\pi_1(SO(n) \times SO(2)) \to \pi_1(Q \times P_1).$$

Hence there cannot exist any homomorphism $f$ of the groups

$$\pi_1(SO(n) \times SO(2)) \to \pi_1(SO(n+2))$$

such that the given diagram commutes ($i$ is the embedding of $SO(n) \times SO(2)$ into $SO(n + 2)$). Consequently, $Q \times P_1$ admits no spin structure, i.e. $Q$ itself has no spin$^C$ structure. Finally, $\pi_1(Q \times P_1) = \mathbb{Z}$, and, therefore, the homomorphism $\pi_1(S^1) \to \pi_1(Q \times P_1)$ in the row of the diagram is surjective. This implies $\pi_1(Q) = 1$.

Summarizing this argument and applying, in addition, the criterion for the existence of a spin structure on an $SO(n)$-principal bundle over a simply connected base space, we conclude:

**Proposition.** Let $X$ be a CW-complex with $\pi_1(X) = 1$ and $\pi_2(X)$ a finite group. If $Q$ is an $SO(n)$-principal bundle over $X$, then the following conditions are equivalent:

1) $Q$ has a spin structure.

2) $Q$ has a spin$^C$ structure.

3) $\pi_1(Q) = \mathbb{Z}_2$.

If $Q$ has no spin (spin$^C$) structure, then $\pi_1(Q) = 1$. \hfill \Box

**Example.** Consider the homogeneous space $X^5 = SU(3)/SO(3)$. The isotropy representation $\varphi : SO(3) \to SO(5)$ of this space induces an isomorphism $\varphi_\# : \pi_1(SO(3)) \to \pi_1(SO(5))$ of fundamental groups. Let $Q = SU(3) \times_{SO(3)} SO(5)$ be the frame bundle of $X^5$. $Q$ is an $SO(5)$-principal bundle over $X^5$. We will show that $Q$ has no spin$^C$ structure. To do so, start by computing the first and the second homotopy groups of $X^5 = SU(3)/SO(3)$ from the exact sequence

$$\ldots \to \pi_2(SU(3)) = 1 \to \pi_2(X^5) \to \pi_1(SO(3)) = \mathbb{Z}_2 \to \pi_1(SU(3)) = 1 \to \pi_1(X^5) \to 1.$$ 

We obtain

$$\pi_1(X^5) = 1, \quad \pi_2(X^5) = \mathbb{Z}_2.$$
2.4. Existence of spin$^\mathbb{C}$ structures

The fundamental group of $Q$ is analogously determined from the sequence

$$\cdots \to \pi_2(SU(3) \times_{SO(3)} SO(5)) \xrightarrow{\partial} \pi_1(SO(3)) \xrightarrow{\varphi^\#} \pi_1(SU(3) \times SO(5)) \to \pi_1(Q) \to 1.$$ 

Since $\varphi^\#$ is an isomorphism, we obtain $\pi_1(Q) = 1$, i.e. $Q$ admits no spin$^\mathbb{C}$ structure.

Next we are going to discuss when two spin$^\mathbb{C}$ structures $(P, \Lambda)$, $(P^*, \Lambda^*)$ on an $SO(n)$-principal $Q$ bundle will be considered as equivalent.

**Definition.** Two spin$^\mathbb{C}$ structures $(P, \Lambda)$, $(P^*, \Lambda^*)$ on an $SO(n)$-principal bundle $Q$ are called *equivalent* if

1) There exists a bundle isomorphism

$$\psi : P_1 \to P_1^*$$

of the $S^1$-principal bundles $P_1 = P/\text{Spin}(n)$ and $P_1^* = P^*/\text{Spin}(n)$.

2) There exists a spin$^\mathbb{C}$-bundle isomorphism $\Phi : P \to P^*$ fitting into the commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\Phi} & P^* \\
& \downarrow & \\
Q \times P_1 & \xrightarrow{\text{Id} \times \psi} & Q \times P_1^*
\end{array}
$$

**Remark.** Given a bundle isomorphism $\Phi : P \to P^*$ with the commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\Phi} & P^* \\
& \downarrow & \\
\Lambda & & \Lambda^*
\end{array}
$$

$\Phi$ induces an isomorphism $\psi : P_1 = P/\text{Spin}(n) \to P_1^* = P^*/\text{Spin}(n)$ of $S^1$-bundles, and the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\Phi} & P^* \\
& \downarrow & \\
Q \times P_1 & \xrightarrow{\text{Id} \times \psi} & Q \times P_1^*
\end{array}
$$

commutes. The equality of two spin$^\mathbb{C}$ structures can thus be formulated equivalently by requiring the existence of a bundle isomorphism $\Phi : P \to P^*$ with the commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\Phi} & P^* \\
& \downarrow & \\
\Lambda & & \Lambda^*
\end{array}
$$
Definition. If \((P, \Lambda)\) is a spin\(^C\) structure on \(Q\), then the line bundle 

\[ L = P_1 \times_{U(1)} \mathbb{C} = P \times_{\text{Spin}^c(n)} \mathbb{C} \]

is called the determinant bundle of the spin\(^C\) structure.

\(L\) is a complex line bundle over \(X\), and the above consideration of characteristic classes yields the condition 

\[ w_2(Q) \equiv c_1(L) \mod 2 \]

in \(H^2(X; \mathbb{Z}_2)\). Thus we obtain the map 

\[ \text{Spin}^C(Q) \longrightarrow \{ \alpha \in H^2(X; \mathbb{Z}_2) : \alpha \equiv w_2(Q) \mod 2 \} \]

from the set of all spin\(^C\) structures on \(Q\) to \(H^2(X; \mathbb{Z})\). Recalling that the existence of a spin\(^C\) structure on \(Q\) is equivalent to the existence of an element \(\alpha \in H^2(X; \mathbb{Z})\) with \(w_2(Q) \equiv \alpha \mod 2\), we conclude that the map from \(\text{Spin}^C(X; \mathbb{Z})\) to \(\{ \alpha \in H^2(X; \mathbb{Z}) : \alpha \equiv w_2(Q) \mod 2 \}\) is surjective. For fixed \(\alpha \in H^2(X; \mathbb{Z})\) we have, on the other hand, the \(SO(n) \times S^1\)-bundle \(Q \times P_1\), and a spin\(^C\) structure on \(Q\) is then only a reduction of this \((SO(n) \times S^1)\)-bundle onto the group \(\text{Spin}^C(n) \longrightarrow SO(n) \times S^1\). These reductions are labeled by the elements of \(H^1(X; \mathbb{Z}_2)\). However, in this last step we still allow for a gauge transformation of the \(S^1\)-bundle \(P_1\). But the ambiguity resulting thereby may be controlled. Let \(P_1\) be an \(S^1\)-bundle and \(P_1^*\) a reduction onto the 2-fold covering \(z \rightarrow z^2\) of \(S^1\). A gauge transformation \(F\) of \(P_1\) is determined by a function \(f : X \rightarrow S^1\) with 

\[ F(p_1) = p_1 \cdot f(\pi(p_1)) \]

The reductions \(P_1^*\) and \(F^*(P_1^*)\) are equivalent as reductions of \(P_1\) if and only if the map 

\[ f_\# : \pi_1(X) \longrightarrow \pi_1(S^1) = \mathbb{Z} \]

takes values in \(2\mathbb{Z} \subset \mathbb{Z}\) (\(f\) has a square root). The gauge transformation itself corresponds to an element of \(H^1(X; \mathbb{Z})\), and the exact sequence 

\[ H^1(X; \mathbb{Z}) \xrightarrow{2} H^1(X; \mathbb{Z}) \xrightarrow{\varphi} H^1(X; \mathbb{Z}_2) \xrightarrow{\beta} H^2(X; \mathbb{Z}) \xrightarrow{\psi} H^2(X; \mathbb{Z}_2) \longrightarrow \ldots \]

then implies that, for a fixed isomorphism class of the \(S^1\)-bundle \(P_1\), the spin\(^C\) structures are labeled by 

\[ H^1(X; \mathbb{Z}_2)/\text{Im}(\varphi) = H^1(X; \mathbb{Z}_2)/\ker(\beta) = \text{im}(\beta). \]

On the other hand, 

\[ \{ \alpha \in H^2(X; \mathbb{Z}) : \alpha \equiv w_2(Q) \mod 2 \} = \text{im}(\psi) = H^2(X; \mathbb{Z})/\ker(\psi) = H^2(X; \mathbb{Z})/\text{im}(\beta), \]

and thus we obtain the
Proposition. Let $Q$ be an $SO(n)$-principal bundle with spin$^C$ structure. Then all the spin$^C$ structures on $Q$ are classified by $H^2(X; \mathbb{Z})$.

Example 2. Let $Q$ be an $SO(2k)$-bundle admitting a reduction $R$ onto the subgroup $U(k) \subset SO(2k)$. Let $(P, \Lambda)$ be the canonical spin$^C$ structure. Then for the determinant bundle we have

$$L = P \times_{Spin^c} \mathbb{C} = (R \times_{U(k)} Spin^C(2k)) \times_{Spin^c(2k)} \mathbb{C}.$$  

The homomorphism $U(k) \to Spin^C(2k) \to S^1$ is given by $A \to \det(A)$. This implies

$$L = R \times_{\det} \mathbb{C}.$$  

On the other hand, if $E = R \times_{U(k)} \mathbb{C}^k$ is the associated complex vector bundle, then

$$\Lambda^k(E) = R \times_{\det} \mathbb{C}.$$  

This implies for the determinant bundle $L$ of the spin$^C$ structure the formula

$$L = \Lambda^k(E).$$

2.5. Associated spinor bundles

Consider an $SO(n)$-principal bundle $(Q, \pi, X; SO(n))$ and denote by

$$T = Q \times_{SO(n)} \mathbb{R}^n$$

the associated real $n$-dimensional vector bundle. Let $(P, \Lambda)$ be a spin or spin$^C$ structure. The spin representation

$$\kappa : Spin(n), Spin^C(n) \to U(\Delta_n)$$

now allows us to consider the associated complex vector bundle

$$S = P \times_{\kappa} \Delta_n.$$  

$S$ is called the spinor bundle for a given spin or spin$^C$ structure, respectively. $S$ is a complex vector bundle over $X$ with an Hermitian metric. In the case $n = 2k$, this vector bundle splits into the sum of two subbundles $S^+, S^-:

$$S = S^+ \oplus S^-, \quad S^\pm = P \times_{\kappa} \Delta_n^\pm.$$  

Clifford multiplication,

$$\mu : \mathbb{R}^n \otimes_{\mathbb{R}} \Delta_n \to \Delta_n \quad \text{or} \quad \mu : \Lambda(\mathbb{R}^n) \otimes_{\mathbb{R}} \Delta_n \to \Delta_n,$$

is a homomorphism of the spin or spin$^C$ representation, respectively, and, since

$$T = Q \times_{SO(n)} \mathbb{R}^n = P \times_{\chi} \mathbb{R}^n,$$

$\mu$ induces a bundle morphism of the associated bundles

$$\mu : T \otimes S \to S \quad \text{or} \quad \mu : \Lambda(T) \otimes S \to S.$$
The real or quaternionic structures in $\Delta_n$, respectively, pass on, in the case of a spin structure (not for a spin$^C$ structure!), to the corresponding bundles.

The spin$^C$ representations $\det(\Delta_n) = \Lambda^{\dim(\Delta_n)}(\Delta_n)$ and $\ell^{\dim(\Delta_n)/2}$ are equivalent where $\ell : Spin^C(n) \to S^1$ is the homomorphism constructed above. This implies for the determinant bundle $L$ of the spin$^C$ structure the formula

$$L^{\dim(S)/2} = \Lambda^{\dim(S)}(S),$$

and, in the case of even dimension $n = 2k$, we analogously obtain

$$L^{\dim(S^\pm)/2} = \Lambda^{\dim(S^\pm)}(S^\pm).$$

In forming the associated spin bundle $S$ starting from a spin or spin$^C$ structure, it may well happen that different spin structures lead to isomorphic spin bundles. We will now study this question in the case of a spin structure in greater detail.

Let $(Q, \pi, X; SO(n))$ be a principal bundle. We think of $Q$ as being given by

- a covering $X = \bigcup_{i \in I} U_i$ of $X$ by open sets $U_i \subset X$,
- a system of transition functions $g_{ij} : U_i \cap U_j \to SO(n)$ such that $g_{ij}g_{jk} = g_{ik}$, $g_{ii} = 1$.

Then, a spin structure $(P, \Lambda)$ on $Q$ is completely determined by a system of transition functions $\tilde{g}_{ij} : U_i \cap U_j \to Spin(n)$ satisfying the conditions

$$\lambda \circ \tilde{g}_{ij} = g_{ij}, \quad \tilde{g}_{ij}g_{jk} = \tilde{g}_{jk}, \quad \tilde{g}_{ii} = 1.$$ 

Hence two spin structures $(P, \Lambda)$ and $(P^*, \Lambda^*)$ are described by maps $\tilde{g}_{ij}, \tilde{g}_{ij}^* : U_i \cap U_j \to Spin(n)$ with $\lambda \circ \tilde{g}_{ij} = g_{ij} = \lambda \circ \tilde{g}_{ij}^*$.

Set $\varepsilon_{ij} = \tilde{g}_{ij} \cdot [\tilde{g}_{ij}^*]^{-1}$. Then $\varepsilon_{ij}$ is a map from $U_i \cap U_j$ to $\ker(\lambda) = \mathbb{Z}_2 \subset Spin(n)$,

$$\varepsilon_{ij} : U_i \cap U_j \to \mathbb{Z}_2,$$

and $\varepsilon_{ij} \varepsilon_{jk} = \varepsilon_{ik}$. Thus the system of transition functions $\varepsilon_{ij}$ defines a real 1-dimensional bundle $E$. Since

$$\tilde{g}_{ij} = \tilde{g}_{ij}^* \varepsilon_{ij},$$

for the spinor bundle $S$ of the spin structure $(P, \Lambda)$ ($S^*$ for the spin structure $(P^*, \Lambda^*)$) there are isomorphisms

$$S = S^* \otimes_{\mathbb{R}} E = S^* \otimes_{\mathbb{C}} E^c,$$
where $E^c = E \otimes_R \mathbb{C}$ is the complexification of $E$. As $\pm 1$ belongs to the centre of the Clifford algebra, this isomorphism is compatible with Clifford multiplication, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
T \otimes S & \xrightarrow{\mu} & S \\
\downarrow & & \downarrow \\
T \otimes S^* \otimes E & \xrightarrow{\mu \otimes \text{Id}_E} & S^* \otimes E
\end{array}
$$

From these considerations we obtain, e.g., the following

**Proposition.** Let $X$ be a CW-complex whose second integral cohomology has no 2-torsion. Then the spinor bundles corresponding to possibly different spin structures on an $SO(n)$-principal bundle are isomorphic.

**Proof.** The complex line bundle $E^c$ is the complexification of a real line bundle, and hence, for the first Chern class, $2c_1(E^c) = 0$ in $H^2(X; \mathbb{Z})$. Since $H^2(X; \mathbb{Z})$ has no 2-torsion, this implies $c_1(E^c) = 0$, i.e. $E^c$ is a trivial bundle. \qed

**Remark.** The spin structures $(P, \Lambda)$ and $(P^*, \Lambda^*)$ are described by elements $f, f^* \in H^1(Q; \mathbb{Z}_2)$ whose restrictions to $H^1(\text{fibre}; \mathbb{Z}_2) = \mathbb{Z}_2$ are non-trivial. Thus, $f - f^*$ vanishes after restriction to the fibre. From the exact sequence of the $SO(n)$-principal bundle,

$$
0 \rightarrow H^1(X; \mathbb{Z}_2) \rightarrow H^1(Q; \mathbb{Z}_2) \rightarrow H^1(SO(n); \mathbb{Z}_2) \xrightarrow{\delta} \cdots ,
$$

we thus conclude that $f - f^*$ is an element of $H^1(X; \mathbb{Z}_2)$. This implies

$$
w_1(E) = f - f^*,
$$

where $w_1$ is the first Stiefel-Whitney class of the real bundle $E$.

**Example.** Let $X = \mathbb{R}P^5$ be the real projective space of dimension five and $Q = \mathbb{R}P^5 \times SO(3)$ the trivial $SO(3)$-principal bundle. Since $H^1(\mathbb{R}P^5; \mathbb{Z}_2) = \mathbb{Z}_2$ on $Q$, there are two spin structures $(P, \Lambda)$ and $(P^*, \Lambda^*)$. The bundle $E$ is uniquely determined by $w_1(E) \neq 0$ in $H^1(\mathbb{R}P^5; \mathbb{Z}_2)$. The first spin structure is trivial, and hence so is the corresponding spin bundle $S$. This implies for $S^*$

$$
S^* = 2E^c = E^c \oplus E^c.
$$

But, in general,

$$
c_2(2E^c) = c_1(E^c)^2.
$$

However, the bundle $E^c$ is the associated bundle

$$
E^c = S^5 \times_{\mathbb{Z}_2} \mathbb{C}.
$$
If $E^c$ is trivial, then there exists a mapping $f : S^5 \to S^1$ with $f(-x) = -f(x)$, in contradiction with the Borsuk-Ulam theorem. Thus $E^c$ is non-trivial, i.e.

$$c_1(E^c) \neq 0 \text{ in } H^2(\mathbb{R}P^5; \mathbb{Z}).$$

Since the map

$$Z = H^2(\mathbb{R}P^5; \mathbb{Z}) \ni \alpha \longmapsto \alpha^2 \in H^4(\mathbb{R}P^5; \mathbb{Z})$$

is injective, this implies $c_2(2E^c) \neq 0$. Hence $S^*$ is not the trivial bundle.

### 2.6. References and exercises


**Exercise 1.** Let $\mathbb{R}P^n$ be the $n$-dimensional real projective space. Prove that

- a) $\mathbb{R}P^n$ is orientable $\iff n \equiv 1 \text{ mod } 2$.
- b) $\mathbb{R}P^n$ has a spin structure $\iff n \equiv 3 \text{ mod } 4$.

**Exercise 2.** Which of the Grassmannian manifolds

$$G_{n+k,k} = SO(n+k)/[SO(n) \times SO(k)]$$

have a spin structure?

**Exercise 3.** Let $M^3$ be a compact, closed, and orientable 3-manifold. Then every $SO(n)$-principal bundle $Q$ over $M^3$ has a spin$^C$ structure.

**Hint:** If $M^3$ is orientable, then by Poincaré duality we have

$$H^2(M^3; \mathbb{Z}) = H_1(M^3; \mathbb{Z}) = \frac{\pi_1(M^3)}{[\pi_1(M^3), \pi_1(M^3)]}$$

and

$$H^2(M^3; \mathbb{Z}_2) = H_1(M^3; \mathbb{Z}_2) = \frac{\pi_1}{[\pi_1, \pi_1]} \otimes \mathbb{Z}_2.$$
3.1. Connections in spinor bundles

Let \((M^n, g)\) be an oriented connected Riemannian manifold and \(Q \rightarrow M^n\) the \(SO(n)\)-principal bundle of positively oriented orthonormal frames. The Riemannian manifold has a uniquely determined torsion-free metric connection. Considering it as a covariant derivative on vector fields, we will denote this Levi-Civita connection by \(\nabla\). Viewed as a connection in the \(SO(n)\)-principal bundle, however, it is the \(so(n)\)-valued 1-form

\[ Z : TQ \rightarrow so(n). \]

Let, in addition, a spin\(^C\) structure \((P, \Lambda)\) together with the corresponding \(U(1)\)-bundle \(P_1\) and 2-fold covering \(\pi\) be given,

\[ \pi : P \rightarrow Q\tilde{\times}P_1. \]

Finally, we also fix a connection \(A\) in the principal bundle \(P_1\),

\[ A : TP_1 \rightarrow i\mathbb{R}, \]

where we identify the Lie algebra of the group \(S^1 = U(1)\) with the purely imaginary numbers. The connections \(Z\) and \(A\) together define a connection

\[ Z \times A : T(Q\tilde{\times}P_1) \rightarrow so(n) \oplus i\mathbb{R} \]

in the fibre-product \(Q\tilde{\times}P_1\). It is not difficult to see that this connection lifts to the 2-fold covering \(\pi : P \rightarrow Q\tilde{\times}P_1\) as a connection \(\widetilde{Z \times A}\) in the spin\(^C\)
principal bundle. The following diagram commutes:

\[
\begin{array}{ccc}
T(P) & \xrightarrow{Z \times A} & \text{spin}^C(n) = m_2 \oplus i\mathbb{R} \\
\downarrow \text{d} \pi & & \downarrow p_* \\
T(Q \times P_1) & \xrightarrow{Z \times A} & \text{so}(n) \oplus i\mathbb{R}
\end{array}
\]

where \( p_* : \text{spin}^C(n) \to \text{so}(n) \oplus i\mathbb{R} \) is the differential of the 2-fold covering \( p : Spin^C(n) \to SO(n) \times S^1 \). The spin representation \( \kappa : Spin^C(n) \to GL(\Delta_n) \) induces the spinor bundle

\[ S = P \times_{Spin^C(n)} \Delta_n. \]

The sections \( \psi \in \Gamma(S) \) of the spinor bundle can be identified with the mappings \( \psi : P \to \Delta_n \) obeying the transformation rule \( \psi(p \cdot g) = \kappa(g^{-1})\psi(p), \quad g \in Spin^C(n) \). On the one hand, the absolute differential of a section \( \psi \) with respect to the connection \( Z \times A \) is computed by

\[ D^A \psi = d\psi + \kappa_*(Z \times A)\psi \]

and determines, on the other hand, a covariant derivative

\[ \nabla^A : \Gamma(S) \to \Gamma(T^*M \otimes S) \]

in the spinor bundle. A vector field \( X \) on the manifold \( M^n \) can be considered as a function \( X : P \to \mathbb{R}^n \) with \( X(p \cdot g) = \lambda(g^{-1})X(p) \), since the tangent bundle is an associated vector bundle to the principal bundle \( P \). Then the Clifford product \( X \cdot \psi \) is given by the function \( X \cdot \psi : P \to \Delta_n \)

\[ (X \cdot \psi)(p) = X(p) \cdot \psi(p). \]

Using this, we obtain

\[ D^A(X \cdot \psi) = d(X \cdot \psi) + \kappa_*(Z \times A)(X \cdot \psi) \]

\[ = dX \cdot \psi + X \cdot d\psi + \kappa_*(Z \times A)(X \cdot \psi). \]

Now let us transform \( \kappa_*(Z \times A)(X \cdot \psi) \) algebraically. To do this, we insert a vector \( t \in T(P) \). Then \( Z \times A(t) \) is an element of \( m_2 \oplus i\mathbb{R} = \text{spin}^C(n) \) and

\[ \kappa_*(Z \times A(t)) = (y + is) \cdot X \cdot \psi = y \cdot X \cdot \psi + X \cdot (is\psi). \]

However, since \( y \in m_2 \) and \( X \in \mathbb{R}^n \), the following formula holds in the Clifford algebra \( C_n \):

\[ y \cdot X = X \cdot y + \lambda_*(y)(X). \]

Here \( \lambda_* : \text{spin}(n) \to \text{so}(n) \) denotes the differential. In our situation, \( \lambda_*(y) = Z(d\pi(t)) \) and thus

\[ \kappa_*(Z \times A)(X \cdot \psi) = X \cdot \{ \kappa_*(Z \times A)\psi \} + (\lambda_*(Z)X) \cdot \psi. \]
Inserting this implies
\[ D^A(X \cdot \psi) = X \cdot D^A \psi + (dX + \lambda_*(Z)X) \cdot \psi = X \cdot D^A \psi + (\nabla X) \cdot \psi. \]

All in all, we have thus proved:

**Proposition.** Let \( X, Y \) be vector fields on \( M^n \) and \( \psi \in \Gamma(S) \) a spinor field. Then, for the spinor derivative with respect to any connection \( A \) in the \( U(1) \)-bundle \( P_1 \),
\[ \nabla^A_Y (X \cdot \psi) = X \cdot (\nabla^A_Y \psi) + (\nabla Y X) \cdot \psi. \]

The spinor derivative \( \nabla^A \) is metric with respect to the Hermitian product in \( S \), i.e.
\[ X(\psi, \psi_1) = (\nabla^A_X \psi, \psi_1) + (\psi, \nabla^A_X \psi_1). \]

**Proof.** The last formula is a consequence of the fact that the spin\(^C\) representation \( \kappa : \text{Spin}^C \to GL(\Delta) \) is unitary.

We also want to specify local formulas for the connections. Let \( e : U \subset M^n \to Q \) be a local section of the frame bundle \( Q \). \( e \) consists of an orthonormal frame \( e = (e_1, \ldots, e_n) \) of vector fields defined on the open set \( U \subset M^n \). The local connection form \( Z^e = e^*(Z) : TU \to \mathfrak{so}(n) \) is given by the formula
\[ Z^e = \sum_{i<j} w_{ij} E_{ij} \]
where the 1-forms \( w_{ij} \) denote the forms defining the Levi-Civita connection, \( w_{ij} = g(V e_i, e_j) \), and \( E_{ij} \in \mathfrak{so}(n) \) are the standard basis matrices of the Lie algebra \( \mathfrak{so}(n) \). Analogously, we fix a section \( s : U \to P_1 \) of the \( U(1) \)-principal bundle and obtain the local connection form
\[ A^s = s^*(A) : TU \to i\mathbb{R}. \]

\( A^s \) is an imaginary-valued 1-form defined on the set \( U \), and \( e \times s : U \to Q \times P_1 \) is a local section of the principal bundle \( Q \times P_1 \). Let \( e \times s \) be a lift of this section to the 2-fold covering \( \pi : P \to Q \times P_1 \). Since
\[ (e \times s)^*(p_*(Z \times A)) = (e \times s)^* \pi^*(Z \times A) = (Z^e, A^s) = \left( \sum_{i<j} w_{ij} E_{ij}, A^s \right) \]
and
\[
\begin{array}{ccc}
T(P) & \xrightarrow{Z \times A} & \text{spin}^C(n) = m_2 \oplus i\mathbb{R} \\
\downarrow d\pi & & \downarrow p_* \\
T(U) & \xrightarrow{d(e \times s)} & T(Q \times P_1) \xrightarrow{Z \times A} \mathfrak{so}(n) \oplus i\mathbb{R}
\end{array}
\]
the local connection form $\widetilde{Z \times A}^{(e \times s)}$ is given by the formula

$$\widetilde{Z \times A}^{(e \times s)} = \left( \frac{1}{2} \sum_{i<j} w_{ij} e_i e_j, \frac{1}{2} A^s \right).$$

With respect to the section $e \times s$, the section $\psi \in \Gamma(U, S)$ of the spinor bundle over $U$ is described by a function $\psi : U \to \Delta_n$, and its covariant derivative is computed according to the formula

$$\nabla^A \psi = d\psi + \frac{1}{2} \sum_{i<j} w_{ij} e_i e_j \psi + \frac{1}{2} A^s \psi.$$

**Remark (Special case of a spin structure).** If $(P, \Lambda)$ is a spin structure on $Q$, then $P \times_{\text{Spin}(n)} \text{Spin}^C(n)$ is an induced spin$^C$ structure. The $U(1)$-bundle $P_1$, in this case, is trivial with a canonical global section $s : M^n \to P_1$. If we choose the connection $A_0$ in $P_1$ for which $A_0 = 0$ holds, all the formulas simplify correspondingly. Given a spin structure, we will simply denote the covariant derivative in the spinor bundle $S$ with respect to this canonical connection $A_0$ by $\nabla$.

**Remark (Special case of a $U(k)$-reduction).** Let $n = 2k$ be even, and let a topological $U(k)$-reduction $R$ of the $SO(2k)$-principal bundle $Q$ be given, $R \subset Q$. This is equivalent to saying that there is an almost-complex structure $J : T(M^{2n}) \to T(M^{2n})$ which is compatible with the metric $g$:

$$g(J(\tilde{t}_1), J(\tilde{t}_2)) = g(\tilde{t}_1, \tilde{t}_2).$$

The lift $F : U(k) \to \text{Spin}^C(2k)$ fitting into the commutative diagram

$$\begin{array}{ccc}
\text{Spin}^C(2k) & \xrightarrow{F} & U(k) \\
\downarrow{p} & & \downarrow{f} \\
SO(2k) \times S^1 & \xrightarrow{f} & SO(2k) \times S^1
\end{array}$$

together with $f(A) = (A, \det(A))$, $A \in U(k)$, induces a spin$^C$ structure on $Q$ via

$$P = R \times_{U(k)} \text{Spin}^C(2k).$$

The corresponding $U(1)$-bundle $P_1$ is

$$P_1 = R \times_{\det} S^1$$

with the 1-dimensional complex vector bundle $E = R \times_{\det} \mathbb{C}$. This vector bundle can also be described differently. $T(M^{2k})$ becomes a $k$-dimensional complex vector bundle by means of the almost-complex structure $J$ and, by construction,

$$E = \Lambda^k_C(T).$$
Summarizing, we conclude that each connection $A$ in the $U(1)$-bundle $P_1 = R \times_{\det} S^1$ (or in the vector bundle $\Lambda^2(S(T))$) induces a covariant derivative $\nabla^A$ in the spinor bundle. Particularly important is the case that a connection can be distinguished in the principal bundle $P_1$ in an "obvious" way. This happens, e.g., if the Levi-Civita connection $Z$ reduces to the $U(k)$-reduction $R$ to a connection $Z^*$. Then $(M^{2k}, g, J)$ is a Kähler manifold. In this case, $Z^*$ in turn induces a special connection $A_0$ in the associated bundle $P_1 = R \times_{\det} S^1$, and we again obtain a distinguished covariant derivative which only depends upon the geometry of the base space.

Now we return to the general situation and consider two connections $A$ and $A'$ in $P_1$. The difference $A - A'$ is an $i\mathbb{R}$-valued 1-form on the manifold $M^n$ which will be denoted by $\eta$:

$$A - A' = \eta.$$  

From the local formula for the covariant derivative $\nabla^A$ we immediately conclude that

$$\nabla^A_X\psi - \nabla^{A'}_X\psi = \frac{1}{2}\eta(X) \cdot \psi$$

for all spinor fields $\psi \in \Gamma(S)$ and all vectors $X \in T(M^n)$. A gauge transformation $f : P_1 \to P_1$ of the $U(1)$-bundle $P_1$ is described by a mapping $\mu_f : M^n \to S^1$ satisfying

$$f(p_1) = p_1 \cdot \mu_f(\pi(p_1)),$$

where $\pi : P_1 \to M^n$ is the projection in the bundle. The connection $f^*(A)$ is given by

$$f^*(A) = A + \pi^* \mu_f^*(\Theta)$$

with the Maurer-Cartan form $\Theta = \frac{dz}{z}$ of the group $U(1) = S^1$. This implies the formula

$$\nabla^{f^*(A)}_X\psi - \nabla^A_X\psi = \frac{1}{2} \frac{d\mu_f(X)}{\mu_f} \cdot \psi$$

for the corresponding covariant derivatives.

We now turn to the description of the curvature form of the connection $Z \times A$. Let $\Omega^Z : TQ \times TQ \to so(n)$ be the curvature form of the Levi-Civita connection with the components

$$\Omega_{ij} : TQ \times TQ \to \mathbb{R}.$$  

The curvature $\Omega^A$ as a 2-form on $P_1$ is simply $\Omega^A = dA$. The commutative diagram defining the connection $Z \times A$ immediately implies the formula

$$\Omega^{Z \times A} = \frac{1}{2} \sum_{i<j} \pi^*(\Omega_{ij}) e_i e_j + \frac{1}{2} \pi^*(dA).$$
The 2-form $dA$ is a form on the base space $M^n$. From the general equation $DZDZ\phi = \rho_*(\Omega^Z)\phi$ for the 2-fold absolute differential with respect to the connection $Z$ we obtain the formula

$$\nabla^A(\nabla^A\psi) = \frac{1}{2} \sum_{i<j} \Omega_{ij} e_i \cdot e_j + \frac{1}{2} dA \cdot \psi.$$  

Here $e = (e_1, \ldots, e_n)$ is a local orthonormal frame and $(\Omega^Z)^e = e^*(\Omega^Z) = \sum_{i<j} \Omega_{ij} e_{ij}$ defines the corresponding components of the curvature form of the Levi-Civita connection. Using the structure equations of the Riemannian space (or, more generally, the formula for the curvature form, $\Omega^Z = dZ + \frac{1}{2}[Z, Z]$) we can express the 2-forms $\Omega_{ij}$ in terms of the forms $\omega_{ij} = g(\nabla e_i, e_j)$ of the Levi-Civita connection as well as the components

$$R_{ijkl} = g(\nabla_{e_i} \nabla_{e_j} e_k - \nabla_{e_j} \nabla_{e_i} e_k - \nabla_{[e_i, e_j]} e_k, e_l).$$

To this end, we begin with the following general remarks. Let $(P, \pi, M; G)$ be a $G$-principal bundle, $Z : T(P) \to g$ a connection and $\rho : G \to GL(V_0)$ a representation. The curvature form $\Omega^Z = dZ + \frac{1}{2}[Z, Z]$ defines a 2-form $\rho_*(\Omega^Z)$ on the manifold with values in the endomorphisms of the associated vector bundle $V = P \times_{\rho} V_0$. On the other hand, a section $\phi$ of this bundle can be identified with a function $\phi : P \to V_0$ obeying the transformation rule $\phi(p g) = \rho(g^{-1})\phi(p)$. Then $D^Z\phi$ is a tensorial 1-form of type $\rho$ and hence a 1-form on the manifold $M$ with values in $V$. The induced covariant derivative in $V$ is thus given by

$$\nabla^Z_X \phi = D^Z(\phi)(X^*),$$

where $X^*$ is a (horizontal) lift of $X$. The equation

$$R^Z(X, Y) = \nabla^Z_X \nabla^Z_Y \phi - \nabla^Z_Y \nabla^Z_X \phi - \nabla^Z_{[X, Y]} \phi$$

determines the curvature tensor $R^Z$, which is a 2-form with values in $\text{End}(V)$, too. The curvature form and the curvature tensor are related by the following well-known formula.

**Lemma.** One has the identity $R^Z = \rho_*(\Omega^Z)$.  

**Proof.** To prove this, consider vector fields $X$, $Y$ on the manifold $M$ and denote by $X^*, Y^*$ the corresponding $Z$-horizontal lifts. The section $\nabla^Z_Y \phi$ is given by $D^Z(\phi)(X^*) = d\phi(Y^*) + Z(Y^*)\phi = d\phi(Y^*)$. An analogous calculation shows that $\nabla^Z_X \nabla^Z_Y \phi$ coincides with $X^*Y^*\phi$, and thus

$$\nabla^Z_X \nabla^Z_Y \phi - \nabla^Z_Y \nabla^Z_X \phi - \nabla^Z_{[X, Y]} \phi = X^*Y^*(\phi) - Y^*X^*(\phi) - [X, Y]^{\text{hor}}(\phi)$$

$$= [X^*, Y^*](\phi) - [X^*, Y^*]^{\text{hor}}(\phi)$$

$$= [X^*, Y^*]^{\text{vert}}(\phi).$$
3.1. Connections in spinor bundles

On the other hand, the structure equation of the connection immediately implies \([X^*, Y^*]_{\text{vert}} = Z[X^*, Y^*] = -\Omega(X^*, Y^*)\). Hence,

\[ R^Z(X, Y)\phi = -\Omega\tilde{Z}(X^*, Y^*)(\phi). \]

Finally, if \(W \in \mathfrak{g}\) is an element of the Lie algebra and \(\tilde{W}\) the corresponding fundamental vector field, then

\[
\tilde{W}(\phi)(p) = \lim_{t \to 0} \frac{\phi(p \cdot e^{tW}) - \phi(p)}{t} = \lim_{t \to 0} \left( \frac{\rho(e^{-tW}) - 1}{t} \right) \phi(p) = -\rho_*(W)(\phi(p)).
\]

This implies the formula we wanted to prove,

\[ R^Z(X, Y)\phi = \rho_*(\Omega(X, Y))\phi. \]

Applying this to the components of the Levi-Civita connection, we obtain

\[
\Omega_{ij}(X, Y) = \langle \Omega\tilde{Z}(X, Y)e_i, e_j \rangle = \langle R(X, Y)e_i, e_j \rangle = \sum_{k,l} R_{kij} \sigma^k(X)\sigma^l(Y)
\]

\[ = \frac{1}{2} \sum_{k,l} R_{ijkl}(\sigma^k \wedge \sigma^l)(X, Y), \]

where \(\sigma^1, \ldots, \sigma^n\) is the frame dual to \(e_1, \ldots, e_n\). Thus we arrive at the local formula for the curvature form \(\Omega\tilde{Z} \times A\) of the connection \(\tilde{Z} \times A\),

\[
\Omega\tilde{Z} \times A = \frac{1}{4} \sum_{i<j} \left( \sum_{k,l} R_{ijkl}(\sigma^k \wedge \sigma^l) \right) e_i e_j + \frac{1}{2} dA,
\]

and the 2-form \(\nabla^A \nabla^A\) with values in the spinor bundle is calculated as follows:

\[
\nabla^A \nabla^A \psi = \frac{1}{4} \sum_{i<j} \left( \sum_{k,l} R_{ijkl}(\sigma^k \wedge \sigma^l) \right) e_i e_j \cdot \psi + \frac{1}{2} dA \cdot \psi.
\]

Here \(\nabla^A \nabla^A \psi\) is the 2-fold absolute differential of the spinor field \(\psi\) and hence a section of \(\Gamma(A^2 \otimes S)\). Starting from this 2-form with values in the spinor bundle, we construct a spinor-valued 1-form \(H^A\) by a suitable contraction:

**Definition.** For a vector \(X \in T(M^n)\) the 1-form \(H^A\) is defined by

\[ H^A(X) = \sum_{\alpha=1}^n e_\alpha \cdot (\nabla^A \nabla^A \psi)(X, e_\alpha). \]

Then the following holds.
Proposition. Let Ric : $T(M^n) \to T(M^n)$ be the Ricci tensor of the Riemannian space considered as a symmetric endomorphism of the tangent bundle. Then one has the relation

$$H^A_\psi(X) = -\frac{1}{2} \text{Ric}(X) \cdot \psi + \frac{1}{2} (X \cdot dA) \cdot \psi.$$ 

Proof. We are going to use the formula for $\nabla^A \nabla^A \psi$ stated above, and we have to prove the following two relations:

1) $\sum_{\alpha=1}^{n} e_{\alpha} \cdot (dA(X, e_{\alpha})) \cdot \psi = (X \cdot dA) \cdot \psi,$

2) $\sum_{i<j} \sum_{k,l} R_{ijkl} \sigma^k \wedge \sigma^l(X, e_{\alpha}) e_{\alpha} e_i e_j \cdot \psi = -2 \text{Ric}(X) \cdot \psi.$

The first one is trivial since the sum $\sum_{\alpha=1}^{n} dA(X, e_{\alpha}) \cdot e_{\alpha}$ represents the decomposition of the form $X \cdot dA$ with respect to the basis $e_1, \ldots, e_n$. The following calculation takes place in the Clifford algebra $C_n$:

$$\sum_{a, k, l \atop i<j} R_{ijkl} \sigma^k \wedge \sigma^l(X, e_{\alpha}) e_{\alpha} e_i e_j$$

$$= \sum_{a, k, i<j} R_{ijk\alpha} \sigma^k(X) e_{\alpha} e_i e_j - \sum_{a, l, i<j} R_{ijl\alpha} \sigma^l(X) e_{\alpha} e_i e_j$$

$$= 2 \sum_{a, k, i<j} R_{ijk\alpha} \sigma^k(X) e_{\alpha} e_i e_j$$

$$= -2 \sum_{k, i<j} R_{ijk\alpha} \sigma^k(X) e_j + 2 \sum_{k, i<j} R_{ijk\alpha} \sigma^k(X) e_i + 2 \sum_{k, i<j \atop \alpha \neq i, j} R_{ijk\alpha} \sigma^k(X) e_{\alpha} e_i e_j$$

$$= -2 \sum_{k, i<j} R_{ijk\alpha} \sigma^k(X) e_j + 2 \sum_{k, i<j \atop \alpha \neq i, j} R_{ijk\alpha} \sigma^k(X) e_{\alpha} e_i e_j$$

$$= -2 \text{Ric}(X) + 2 \sum_{k, i<j \atop \alpha \neq i, j} R_{ijk\alpha} \sigma^k(X) e_{\alpha} e_i e_j.$$ 

However, the second summand vanishes. Namely, for a fixed index $k$ this sum contains the Clifford product $e_{p} e_{q} e_{r}$, $p < q < r$, precisely three times, for the triples $(\alpha, i, j) = (p, q, r)$, $(q, p, r)$ and $(r, p, q)$. Hence the coefficient at $e_{p} e_{q} e_{r}$ is proportional to the sum

$$R_{qrkp} - R_{prkq} + R_{pqkr} = R_{qrkp} + R_{rpkq} + R_{pqkr} = 0$$

(Bianchi identity for the curvature tensor).

Remark. Taking into account that

$$(\nabla^A \nabla^A \psi)(X, e_{\alpha}) = R^S(X, e_{\alpha}) \psi = \nabla^A_X \nabla^A_{e_{\alpha}} \psi - \nabla^A_{e_{\alpha}} \nabla^A_X \psi - \nabla^A_{[X, e_{\alpha}]} \psi,$$
the formula
\[\sum_{\alpha=1}^{n} e_{\alpha} \cdot (\nabla^{A}_X \nabla^{A}_Y \psi)(X, e_{\alpha}) = -\frac{1}{2} \text{Ric}(X) \psi + \frac{1}{2} (X \lrcorner dA) \cdot \psi\]
can also be written as
\[\sum_{\alpha=1}^{n} e_{\alpha} \cdot R^{S}(X, e_{\alpha}) \psi = -\frac{1}{2} \text{Ric}(X) \psi + \frac{1}{2} (X \lrcorner dA) \cdot \psi,\]
where \(R^{S}(X,Y)\psi = \nabla^{A}_X \nabla^{A}_Y \psi - \nabla^{A}_Y \nabla^{A}_X \psi - \nabla^{A}_{[X,Y]} \psi\) is the curvature tensor in the spinor bundle.

A spinor field \(\psi \in \Gamma(S)\) is called \(\nabla^{A}\)-parallel (or simply parallel) if
\[\nabla^{A}_X \psi = 0.\]

Since
\[X \| \psi \|^2 = (\nabla^{A}_X \psi, \psi) + (\psi, \nabla^{A}_X \psi) = 0,\]
the length of a parallel spinor field is constant. In the following we suppose that \(\psi\) does not vanish identically. \(\nabla^{A}_X \psi = 0\) implies \(\nabla^{A}_X \nabla^{A}_Y \psi = 0,\) and hence the 1-form \(H^{A}_\psi\) vanishes identically. Thus, for every vector \(X \in T(M^n),\)
\[\text{Ric}(X) \cdot \psi = (X \lrcorner dA) \cdot \psi.\]

\(\text{Ric}(X)\) is a real vector while \(X \lrcorner dA\) is purely imaginary. For the evaluation of the last equation we need the following

**Lemma.** Let \(\psi \in \Delta_n\) be a non-trivial spinor and \(Z_1, Z_2 \in \mathbb{R}^n\) two real vectors. If
\[(Z_1 + iZ_2) \cdot \psi = 0,\]
then for the vectors \(Z_1, Z_2\)
1) \(|Z_1| = |Z_2|,\)
2) \(\langle Z_1, Z_2 \rangle = 0.\)

**Proof.** Multiply the equation \((Z_1 + iZ_2) \psi = 0\) once again by \((Z_1 + iZ_2)\). In the complexified Clifford algebra \(C^n\)
\[(Z_1 + iZ_2)(Z_1 + iZ_2) = Z_1^2 - Z_2^2 + i(Z_1 Z_2 + Z_2 Z_1)\]
\[= \{-|Z_1|^2 + |Z_2|^2\} + i\{-2(Z_1, Z_2)\}.\]
From \((Z_1 + iZ_2)(Z_1 + iZ_2) \psi = 0\) and \(\psi \neq 0\) we then obtain \(|Z_1|^2 = |Z_2|^2\) as well as \(\langle Z_1, Z_2 \rangle = 0.\)

Consequently, the condition \(\text{Ric}(X) \cdot \psi = (X \lrcorner dA) \cdot \psi\) implies
1) \(|\text{Ric}(X)| = \frac{1}{i} (X \lrcorner dA)|\) for all \(X \in T(M^n)\),
2) \(\langle \text{Ric}(X), \frac{1}{i} (X \lrcorner dA) \rangle = 0\) for all \(X \in T(M^n)\).
For the sake of brevity we denote by $S$ the symmetric endomorphism $S(X) = \text{Ric}(X)$, and by $A$ the anti-symmetric endomorphism $A(X) = \frac{1}{i}(X \wedge dA)$ of the tangent bundle. The second equation implies

$$\langle S(X), A(X) \rangle = 0$$

for every vector $X$. Inserting $X + Y$ now leads to

$$\langle S(X), A(Y) \rangle + \langle S(Y), A(X) \rangle = 0$$

and hence

$$\langle X, SA(Y) \rangle - \langle X, AS(Y) \rangle = 0.$$ 

Thus, $SA = AS$, i.e. $A$ and $S$ commute. Hence, at a fixed point $m \in M^n$, $A$ and $S$ can be diagonalized simultaneously: $A$ has the form

$$A = \begin{pmatrix}
0 & -\omega_1 & & \\
\omega_1 & 0 & & \\
& & \ddots & \\
& & & 0
\end{pmatrix}$$

and $S$ is similar to

$$S = \begin{pmatrix}
\lambda_1 & 0 & & \\
& \ddots & & \\
& & \lambda_n & \\
0 & & & 0
\end{pmatrix}.$$ 

The condition $|S(X)| = |A(X)|$, $X \in T$, now leads to the equations

$$\lambda_1 = \lambda_2 = \pm \omega_1, \ldots, \lambda_{2k-1} = \lambda_{2k} = \pm \omega_k, \lambda_{2k+1} = \ldots = \lambda_n = 0.$$ 

For the lengths of the endomorphisms,

$$\|S\|^2 = \text{tr}(S \circ S^T) = \text{tr}(S^2) = \sum_{i=1}^{n} \lambda_i^2,$$

$$\|A\|^2 = \text{tr}(AA^T) = -\text{tr}(A^2) = 2 \sum_{i=1}^{k} \omega_i^2,$$
we then obtain
\[ \| \text{Ric} \| = \| A \|. \]
However, the square of the length of \( A \) as an anti-symmetric mapping is twice the square of the length of the 2-form \( \frac{1}{2} dA \),
\[ \| A \|^2 = 2 \| dA \|^2. \]

To summarize:

**Proposition.** Let \((M^n, g)\) be a Riemannian manifold with a spin\(^C\) structure and \( A \) a connection in the \( U(1) \)-principal bundle \( P_1 \) induced from this spin\(^C\) structure. \( \nabla^A \) denotes the induced covariant derivative in the spinor bundle. If there exists a \( \nabla^A \)-parallel spinor \( \psi \),
\[ \nabla^A \psi = 0, \]
then the following necessary conditions are satisfied:

1) \( \| \text{Ric} \| = 2 \| \Omega^A \| \), where \( \Omega^A = \frac{1}{2} dA \) is the curvature form of the connection,
2) \( \text{rank}(\Omega^A) = \text{rank}(\text{Ric}) \), and
3) the endomorphisms \( \text{Ric} \) and \( \Omega^A \) of the tangent bundle commute.

**Corollary.** Let \((M^n, g)\) be a connected Riemannian manifold with a fixed spin structure. \( \nabla \) denotes the canonical covariant derivative in the spinor bundle \( S \) \((A = 0)\). If there exists a non-trivial parallel spinor \( \psi \), then the Ricci tensor of \( M^n \) vanishes identically, \( \text{Ric} \equiv 0 \).

**Remark.** The conditions for the existence of parallel spinor fields listed above are only necessary. These conditions are not even locally sufficient — compare the exercises.

**Corollary.** Let \((M^n, g)\) be a connected Riemannian manifold which admits a spin\(^C\) structure. If the Ricci tensor has odd rank at least at one point, then \( M^n \) has no \( \nabla^A \)-parallel spinor fields for any spin\(^C\) structure and for any connection \( A \) in the \( U(1) \)-bundle of the spin\(^C\) structure.

**Remark.** In 1997 A. Moroianu studied the classification of Riemannian manifolds with parallel spin\(^C\) spinors in greater detail.

### 3.2. The Dirac and the Laplace operator in the spinor bundle

We start from a Riemannian manifold \((M^n, g)\) with a fixed spin\(^C\) structure and a connection \( A \) in the \( U(1) \)-principal bundle \( P_1 \). These induce a covariant derivative
\[ \nabla^A : \Gamma(S) \rightarrow \Gamma(T^* \otimes S) = \Gamma(T \otimes S) \]
3. Dirac Operators

in the associated spinor bundle $S$. Here and in what follows the cotangent bundle $T^*$ of $M^n$ and the tangent bundle $T$ will be identified by means of the metric $g$. Clifford multiplication and the Hermitian metric $(,)$ in $S$ behave as follows with respect to the covariant derivative $\nabla^A (X, Y \in \Gamma(T), \psi_1, \psi_2 \in \Gamma(S))$:

$$\nabla^A_Y (X \cdot \psi) = (\nabla_Y X) \cdot \psi + X \cdot \nabla^A_Y \psi,$$

$$(\nabla^A_X \psi_1, \psi_2) + (\psi_1, \nabla^A_X \psi_2) = X(\psi_1, \psi_2).$$

As in every vector bundle with connection, we can define the Laplace operator $\Delta$ in the bundle $S$.

**Definition** (Laplace operator on spinors). If $\psi \in \Gamma(S)$ is a spinor field, then $\Delta(\psi)$ is defined by

$$\Delta(\psi) = -\sum_{i=1}^{n} \nabla_{e_i}^A \nabla_{e_i}^A \psi - \sum_{i=1}^{n} \text{div}(e_i) \nabla_{e_i}^A \psi.$$ 

Using Stokes' theorem one then derives — as for every Laplace operator in a Hermitian vector bundle — the formula

$$\int_{M^n} (\Delta_A(\psi_1), \psi_2) = \int_{M^n} (\nabla^A \psi_1, \nabla^A \psi_2) = \int_{M^n} (\psi_1, \Delta_A(\psi_2))$$

for two spinor fields $\psi_1, \psi_2$ with compact support contained in the interior of the manifold $M^n$. Here, $(\nabla^A \psi_1, \nabla^A \psi_2)$ is the scalar product on 1-forms, i.e.

$$(\nabla^A \psi_1, \nabla^A \psi_2) = \sum_{i=1}^{n} (\nabla_{e_i}^A \psi_1, \nabla_{e_i}^A \psi_2).$$

The Dirac operator in turn results from the composition of the canonical derivative with Clifford multiplication.

**Definition** (Dirac operator). The composition

$$D_A = \mu \circ \nabla^A : \Gamma(S) \longrightarrow \Gamma(T^* \otimes S) = \Gamma(T \otimes S) \longrightarrow \Gamma(S)$$

is called the **Dirac operator**. With respect to a (local) orthonormal frame $e = (e_1, \ldots, e_n)$ on the manifold $M^n$, 

$$D_A \psi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}^A \psi.$$ 

Obviously, $D_A$ is a first order differential operator. Moreover, $D_A$ is an elliptic operator. Its symbol $\sigma(D_A)(X) : S \to S$, for a vector $X \in T$, is given by Clifford multiplication:

$$\sigma(D_A)(X)(\psi) = X \cdot \psi.$$
This immediately follows from the formula
\[
D_A(f \cdot \psi) = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}^A (f \cdot \psi) = \sum_{i=1}^{n} e_i \cdot \{df(e_i)\psi + f \nabla_{e_i}^A\psi\}
\]
\[
= \text{grad}(f) \cdot \psi + fD_A(\psi).
\]

We compute \((D_A\psi, \psi_1)\):
\[
(D_A\psi, \psi_1) = \sum_{i=1}^{n} (e_i \cdot \nabla_{e_i}^A \psi, \psi_1) = -\sum_{i=1}^{n} (\nabla_{e_i}^A \psi, e_i \cdot \psi_1)
\]
\[
= -\sum_{i=1}^{n} \{e_i(\psi, e_i \cdot \psi_1) - (\psi, (\nabla_{e_i}e_i) \cdot \psi_1) - (\psi, e_i \cdot \nabla_{e_i}^A \psi_1)\}
\]
\[
= -\sum_{i=1}^{n} e_i(\psi, e_i \cdot \psi_1) - \sum_{i=1}^{n} \text{div}(e_i)(\psi, e_i \cdot \psi_1) + (\psi, D_A\psi_1).
\]

Considering the 1-form \(M^{\psi, \psi_1}(X) = (\psi, X \cdot \psi_1)\), we see that the first two summands are just its divergence, \(\delta M^{\psi, \psi_1}\). Thus,
\[
(D_A\psi, \psi_1) = (\psi, D_A\psi_1) + \delta M^{\psi, \psi_1}.
\]

This implies that the Dirac operator is a symmetric operator with respect to the \(L^2\)-product.

**Proposition.** Let \(\psi\) and \(\psi_1\) be spinor fields with compact support (contained in the interior of the manifold). Then,
\[
\int_{M^n} (D_A\psi, \psi_1) = \int_{M^n} (\psi, D_A\psi_1).
\]

**Remark.** If the dimension \(n = 2k\) is even, then the spinor bundle splits into the sum \(S = S^+ \oplus S^-\) of Dirac spinors. Since Clifford multiplication by vectors interchanges these summands, the Dirac operator decomposes into the sum of two operators, \(D_A^\pm : \Gamma(S^\pm) \to \Gamma(S^\mp)\).

Now we want to discuss a third operator acting on spinor fields, the so-called twistor operator \(T_A\). To this end, we need several preparations: First, Clifford multiplication \(\mu : T \otimes S \to S\) is a surjective homomorphism. Let \(\ker(\mu) \subset T \otimes S\) denote its kernel.

**Lemma.** The formula
\[
P(X \otimes \psi) = X \otimes \psi + \frac{1}{n} \sum_{i=1}^{n} e_i \otimes e_i X \cdot \psi
\]
defines a projection from the bundle \(T \otimes S\) onto the bundle \(\ker(\mu) \subset T \otimes S\).
Proof. A straightforward calculation shows that the image of $P$ is contained in $\ker(\mu)$:

$$\mu \circ P(X \otimes \psi) = X \cdot \psi + \frac{1}{n} \sum_{i=1}^{n} e_i e_i X \cdot \psi = X \cdot \psi - X \cdot \psi = 0.$$ 

Analogously, one shows that $P$ acts as the identity on $\ker(\mu)$.

Definition. The twistor operator $T_A = P \circ \nabla^A$ is defined as the superposition of the covariant derivative with the projection onto the kernel of Clifford multiplication,

$$T_A : \Gamma(S) \to \Gamma(\ker(\mu)).$$

From $\nabla^A \psi = \sum_{i=1}^{n} e_i \otimes \nabla^A_{e_i} \psi$ we obtain the following formula for $T_A$:

$$T_A(\psi) = \sum_{i=1}^{n} e_i \otimes \nabla^A_{e_i} \psi + \frac{1}{n} \sum_{i=1}^{n} e_i \otimes e_i \cdot D_A(\psi)$$

$$= \sum_{i=1}^{n} \nabla^A_{e_i} \psi + \frac{1}{n} e_i \cdot D_A(\psi).$$

Corollary. A spinor field $\psi \in \Gamma(S)$ belongs to the kernel of the twistor operator $T_A$ if and only if, for every vector $X \in T$,

$$\nabla^A_{X} \psi + \frac{1}{n} X \cdot D_A(\psi) = 0.$$ 

Example. Consider $\mathbb{R}^2$ with its Euclidean metric and coordinates $x, y$. Then $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}$ is an orthonormal frame. For the forms $w_{ij}$ of the Levi-Civita connection we have $w_{ij} = 0$. A spinor field is simply a mapping $\psi : \mathbb{R}^2 \to \Delta_2 = \mathbb{C}^2$. The covariant derivative $\nabla \psi$ coincides with the differential $d\psi$, since $w_{ij} = 0$. Contrary to the convention valid up to now, this time we will employ an equivalent though different realization of the Clifford algebra described by the matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$ 

If $\psi = \begin{pmatrix} f \\ g \end{pmatrix} : \mathbb{R}^2 \to \mathbb{C}^2$ is a spinor field, then

$$D(\psi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial x} \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial y} \end{pmatrix} = 2 \begin{pmatrix} \frac{\partial g}{\partial z} \\ -\frac{\partial f}{\partial z} \end{pmatrix},$$
where
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

The kernel of the Dirac operator \((D\psi = 0)\) thus consists of the pairs of complex-valued functions \(f, g : \mathbb{R}^2 \to \mathbb{C}\) which satisfy the Cauchy-Riemann equations
\[
\frac{\partial f}{\partial z} = \frac{\partial g}{\partial \bar{z}} = 0.
\]

Fix a vector \((\begin{array}{c} A \\ B \end{array}) \in \mathbb{C}^2\) and consider the spinor field \(\psi : \mathbb{R}^2 \to \mathbb{C}^2\) defined by
\[
\psi(x, y) = x \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + y \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.
\]

Then
\[
\frac{\partial \psi}{\partial x} = \begin{pmatrix} B \\ -A \end{pmatrix}, \quad \frac{\partial \psi}{\partial y} = \begin{pmatrix} iB \\ iA \end{pmatrix},
\]
and hence
\[
D(\psi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial \psi}{\partial x} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial \psi}{\partial y} = -2 \begin{pmatrix} A \\ B \end{pmatrix}.
\]

We want to show that \(\psi\) is a solution to the twistor equation. To this end, we have to check that
\[
\frac{\partial \psi}{\partial x} + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D(\psi) = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial y} + \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} D(\psi) = 0.
\]

But this immediately follows from
\[
\frac{\partial \psi}{\partial x} + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D(\psi) = \begin{pmatrix} B \\ -A \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0,
\]
\[
\frac{\partial \psi}{\partial y} + \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} D(\psi) = \begin{pmatrix} iB \\ iA \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.
\]

3.3. The Schrödinger-Lichnerowicz formula

The square \(D_A^2\) of the Dirac operator as well as that of the Laplace operator \(\Delta_A\) are second order differential operators. We compare these operators computing their difference \(D_A^2 - \Delta_A\):
\[ D^2_A \psi - \Delta_A \psi = \sum_{i,j} e_i \cdot \nabla e_i^A(e_j \cdot \nabla e_j^A \psi) + \sum_i \nabla e_i^A \nabla e_i^A \psi + \sum i \ \text{div}(e_i) \nabla e_i^A \psi \]
\[ = \sum_{i,j} e_i \cdot \{(\nabla e_i e_j) \cdot \nabla e_j^A \psi + e_j \cdot \nabla e_i^A \nabla e_j \psi\} + \sum_i \nabla e_i^A \nabla e_i^A \psi + \sum \text{div}(e_i) \nabla e_i^A \psi \]
\[ = \sum_{i,j,k} g(\nabla e_i e_j, e_k) e_i e_k \cdot \nabla e_j^A \psi + \sum_{i,j} e_i e_j \cdot \nabla e_i^A \nabla e_j \psi \]
\[ + \sum \nabla e_i^A \nabla e_j \psi + \sum \text{div}(e_i) \nabla e_i^A \psi \]
\[ = \sum_{j \neq k} \sum_{i} g(\nabla e_i e_j, e_k) e_i e_k \cdot \nabla e_j^A \psi + \sum_{i \neq j} e_i e_j \cdot \nabla e_i^A \nabla e_j \psi. \]

The latter of these equations is a consequence of the definition of the divergence, i.e.
\[ \sum_j \sum_{i=k} g(\nabla e_i e_j, e_k) e_i e_k \nabla e_j \psi = - \sum_j \text{div}(e_j) \nabla e_j \psi. \]

Now rewrite the following endomorphism:
\[ \sum_{i \neq k} g(\nabla e_i e_j, e_k) e_i e_k = - \sum_{i \neq k} g(e_j, \nabla e_i e_k) e_i e_k \]
\[ = - \sum_{i < k} g(e_j, \nabla e_i e_k - \nabla e_k e_i) e_i e_k \]
\[ = \sum_{i < k} g(e_j, [e_k, e_i]) e_i e_k. \]

This implies
\[ D^2_A - \Delta_A \psi \]
\[ = \sum_j \sum_{i < k} g(e_j, [e_k, e_i]) e_i e_k \nabla e_j \psi + \sum_{i < j} e_i e_j (\nabla e_i^A \nabla e_j^A - \nabla e_j^A \nabla e_i^A) \psi \]
\[ = \sum_{i < j} e_i e_j (\nabla e_i^A \nabla e_j - \nabla e_i^A \nabla e_j) \psi \]
\[ = \frac{1}{2} \sum_{i,j} e_i e_j R^S(e_i, e_j) \psi. \]

In Section 3.2, the identity
\[ \sum_j e_j \cdot R^S(e_i, e_j) \psi = - \frac{1}{2} \text{Ric}(e_i) \cdot \psi + \frac{1}{2} (e_i \omega dA) \cdot \psi \]
was proved. Multiplying by \( e_i \) and summing over the index \( i \) yields
\[ \sum_{i,j} e_i e_j R^S(e_i, e_j) \psi = - \frac{1}{2} \sum_i e_i \cdot \text{Ric}(e_i) \cdot \psi + \frac{1}{2} \sum_i e_i \cdot (e_i \omega dA) \cdot \psi. \]
But, in the Clifford algebra,
\[ \sum_i e_i \text{Ric}(e_i) = \sum_{i,j} R_{ij} e_i e_j = - \sum_i R_{ii} = -R. \]

Moreover, it is easily checked that for every 2-form \( \eta^2 \) the equality
\[ \sum_i e_i \cdot (e_i \cdot \eta^2) = 2\eta^2 \]
holds in the Clifford algebra \( \mathcal{C}_n \). Thus, altogether we obtain
\[ D^2_A \psi - \Delta_A \psi = \frac{1}{2} \left( \frac{1}{2} R + dA \right) \psi = \frac{R}{4} \psi + \frac{1}{2} dA \cdot \psi. \]

Summarizing this yields the following formula, first proved by E. Schrödinger in 1932.

**Proposition** (Schrödinger-Lichnerowicz formula). Denote by \( R \) the scalar curvature of the Riemannian manifold and let \( dA = \Omega^A \) be the imaginary-valued curvature 2-form of the connection \( A \) in the \( U(1) \)-bundle associated with the spin\( ^C \) structure. Then one has
\[ D^2_A \psi = \Delta_A \psi + \frac{R}{4} \psi + \frac{1}{2} dA \cdot \psi. \]

\[ \square \]

### 3.4. Hermitian manifolds and spinors

Consider an almost-complex manifold \((M^{2k}, J)\) with almost-complex structure \( J, J^2 = -\text{Id} \). \( J \) acts on a 1-form \( w^1 \in T^*(M^{2k}) \) via
\[ (Jw^1)(X) = w^1(JX), \quad X \in T(M^{2k}), \]
and the complexification \( T^*(M^{2k}) \otimes_{R} \mathbb{C} \) splits into the \((\pm i)\)-eigensubspaces of \( J \):
\[ \Lambda^1 = T^*(M^{2k}) \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \]
where
\[ \Lambda^{1,0} = \{ w^1 \in T^*(M^{2k}) \otimes \mathbb{C} : J(w^1) = i w^1 \}, \]
\[ \Lambda^{0,1} = \{ w^1 \in T^*(M^{2k}) \otimes \mathbb{C} : J(w^1) = -i w^1 \}. \]

Let \( \Lambda^{p,q} \) be the linear span of all elements \( u \wedge w \) with \( u \in \Lambda^p(\Lambda^{1,0}) \) and \( w \in \Lambda^q(\Lambda^{0,1}) \). Then,
\[ \Lambda^r = \sum_{p+q=r} \Lambda^{p,q}. \]

Denote by \( \Omega^r \) and \( \Omega^{p,q} \) the space of sections of the bundle \( \Lambda^r \) and \( \Lambda^{p,q} \), respectively. The exterior differential acting on \( r \)-forms,
\[ d : \Omega^r \rightarrow \Omega^{r+1}, \]
decomposes with respect to this splitting. In particular, define the operators
\[ \partial : \Omega^{p,q} \rightarrow \Omega^{p+1,q}, \quad \partial = \pi_{\Lambda^{p+1,q}} \circ d, \]
\[ \bar{\partial} : \Omega^{p,q} \rightarrow \Omega^{p,q+1}, \quad \bar{\partial} = \pi_{\Lambda^{p,q+1}} \circ d. \]

In general, \( \partial + \bar{\partial} \) does not coincide with \( d \). However, by induction one can show the following.

**Lemma.** The exterior differential \( d \) maps \( \Omega^{p,q} \) into the sum
\[ \Omega^{p-1,q+2} \oplus \Omega^{p,q+1} \oplus \Omega^{p+1,q} \oplus \Omega^{p+2,q-1}. \]
Thus,
\[ d|_{\Omega^{p,q}} = \partial + \bar{\partial} \mod \Omega^{p-1,q+2} \oplus \Omega^{p+2,q-1}. \]
The forms from \( \Omega^{1,1} \) are exterior products \( \alpha \wedge \beta \). On the other hand, \( J \) acts on a 2-form \( w^2 \) by
\[ (Jw^2)(X,Y) = w^2(JX, JY). \]
Hence, \( J(\alpha \wedge \beta) = \alpha \wedge \beta \), and this implies the

**Lemma.**
\[ \Omega^{1,1} = \{ w^2 \in \Omega^2 : J(w^2) = w^2 \}, \]
\[ \Omega^{2,0} \oplus \Omega^{0,2} = \{ w^2 \in \Omega^2 : J(w^2) = -w^2 \}. \]
Moreover, fix an Hermitian metric \( g \) compatible with the almost-complex manifold \((M^{2k}, J)\), i.e. a Riemannian metric \( g \) with the property
\[ g(JX, JY) = g(X, Y). \]
Then \( \Omega(X,Y) = g(JX, Y) \) is a 2-form and, since
\[ \Omega(JX, JY) = g(J^2X, JY) = -g(X, JY) = -\Omega(Y, X) = \Omega(X, Y), \]
the 2-form \( \Omega \) belongs to \( \Omega^{1,1} \). Choose a local orthonormal frame of vector fields
\[ e_1, e_2 = J(e_1), \ldots, e_{2k-1}, e_{2k} = J(e_{2k-1}). \]
By means of this, the 2-form \( \Omega \) can be expressed as
\[ \Omega = e_1 \wedge e_2 + \ldots + e_{2k-1} \wedge e_{2k}. \]
Since \( J(e_1 + i e_2) = e_2 - i e_1 = -i(e_1 + i e_2) \), the forms
\[ (e_1 + i e_2), \ldots, (e_{2k-1} + i e_{2k}) \]
are a basis of the fibre \( \Lambda^{0,1} \) at every point, and any \( \Lambda^{0,r} \)-form is a linear combination of exterior products of \( r \) forms of this type. Next we will compute some algebraic identities. First,
\[ e_{2\alpha-1}(e_{2\alpha} \wedge (e_{2\beta-1} + ie_{2\beta})) = e_{2\alpha-1} \wedge (e_{2\alpha-1}(e_{2\beta-1} + ie_{2\beta})) = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ i(e_{2\beta-1} + ie_{2\beta}) & \text{if } \alpha = \beta. \end{cases} \]
3.4. Hermitian manifolds and spinors

and

\[ e_{2\alpha-1}(e_{2\alpha-1} \wedge (e_{2\beta-1} + ie_{2\beta})) + e_{2\alpha} \wedge (e_{2\alpha-1}(e_{2\beta-1} + ie_{2\beta})) = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ -i(e_{2\beta-1} + ie_{2\beta}) & \text{if } \alpha = \beta. \end{cases} \]

This implies the

**Lemma.** For every \((0, r)\)-form \(\eta^{0,r} \in \Lambda^{0,r}\)

\[
\sum_{\alpha=1}^{k} e_{2\alpha-1}(e_{2\alpha} \wedge \eta^{0,r}) + \sum_{\alpha=1}^{k} e_{2\alpha-1} \wedge (e_{2\alpha-1}) = ir\eta^{0,r},
\]

\[
\sum_{\alpha=1}^{k} e_{2\alpha-1}(e_{2\alpha-1} \wedge \eta^{0,r}) + \sum_{\alpha=1}^{k} e_{2\alpha} \wedge (e_{2\alpha-1} \eta^{0,r}) = -ir\eta^{0,r}.
\]

Now let \((P, \Lambda)\) be a spinC structure on the bundle of \(SO(n)\)-frames of the Hermitian manifold \((M^{2k}, J, g)\), and denote by \(S\) the associated spinor bundle. The 2-form \(\Omega\) acts as an endomorphism in the bundle \(S\),

\[ \Omega : S \longrightarrow S. \]

We compute the eigenvalues of this endomorphism.

**Proposition.** \(\Omega : S \rightarrow S\) has the eigenvalues \(i(k-2r), 0 \leq r \leq k\), and the corresponding eigensubspaces have dimension \(\binom{k}{r}\), respectively. The spinor bundle splits into \(S = S_0 \oplus S_1 \oplus \ldots \oplus S_k\), where

\[ S_r = \{ \psi \in S : \Omega \psi = i(k-2r)\psi \}. \]

**Proof.** We will use the spin representation explicitly described in Section 1.3. In the space of Dirac spinors \(\Delta_{2k} = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2\) \((k\text{ times})\) the operator \(e_{2\alpha-1}e_{2\alpha}\), \(1 \leq \alpha \leq k\), is given by the matrix

\[ e_{2\alpha-1}e_{2\alpha} = E \otimes \ldots \otimes E \otimes g_{1g_2} \otimes E \otimes \ldots \otimes E \]

with \(g_{1g_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). Thus \(\Omega = e_1 \wedge e_2 + \ldots + e_{2k-1} \wedge e_{2k}\) is represented, as an endomorphism of \(\Delta_{2k}\), by

\[ \Omega = (g_{1g_2}) \otimes E \otimes \ldots \otimes E + \ldots + E \otimes \ldots \otimes (g_{1g_2}). \]

The matrix \(g_{1g_2}\) has eigenvalues \(\pm i\). Let \(v(+1)\) and \(v(-1)\) be a basis of \(\mathbb{C}^2\) consisting of corresponding eigenvectors. Then \(v(\varepsilon_1) \otimes \ldots \otimes v(\varepsilon_k) (\varepsilon_\alpha = \pm 1)\) is a basis of \(\Delta_{2k} = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2\), and \(\Omega\) acts on these basis elements by

\[ \Omega(v(\varepsilon_1) \otimes \ldots \otimes v(\varepsilon_k)) = i \left( \sum_{\alpha=1}^{k} \varepsilon_\alpha \right) v(\varepsilon_1) \otimes \ldots \otimes v(\varepsilon_k). \]

From this we conclude the assertion.
Now we will present an explicit isomorphism from the \((0, r)\)-forms with values in \(S_0\) onto the bundle \(S_r\),

\[ \Lambda^{0,r} \otimes S_0 \cong S_r, \quad 0 \leq r \leq k. \]

**Proposition.** The mapping \(\alpha_r : \Lambda^{0,r} \otimes S_0 \rightarrow S_r\) defined by

\[ \eta^{0,r} \otimes \psi_0 \mapsto \frac{1}{2\pi} \eta^{0,r} \cdot \psi_0 \]

induces an isomorphism between the bundles \(\Lambda^{0,r} \otimes S_0\) and \(S_r\) preserving inner products.

**Proof.** We first have to prove that the product \(\eta^{0,r} \cdot \psi_0\) belongs to \(S_r\). To do so, we will use the formula

\[ (x \wedge w^k)\psi = x \cdot (w^k \cdot \psi) + (x \wedge w^k) \cdot \psi \]

for \(x \in \mathbb{R}^n, w^k \in \Lambda^k\) and \(\psi \in \Delta_n\). Next we obtain

\[
e_{2\alpha-1}e_{2\alpha}(\eta^{0,r} \cdot \psi_0) = e_{2\alpha-1} \cdot \{(e_{2\alpha} \wedge \eta^{0,r}) \cdot \psi_0 - (e_{2\alpha-1} \cdot \eta^{0,r}) \cdot \psi_0\} = (\eta^{0,r} \wedge e_{2\alpha-1} \cdot e_{2\alpha}) \cdot \psi_0 - (e_{2\alpha-1} \wedge (e_{2\alpha} \wedge \eta^{0,r})) \cdot \psi_0 - (e_{2\alpha-1} \wedge (e_{2\alpha} \wedge \eta^{0,r})) \cdot \psi_0 + 0,
\]

since \(e_{2\alpha-1} \wedge e_{2\alpha} \cdot \eta^{0,r} = 0\) for every form \(\eta^{0,r} \in \Lambda^{0,r}\). Now we similarly rewrite the following expression:

\[ (\eta^{0,r} \wedge e_{2\alpha-1} \cdot e_{2\alpha}) \cdot \psi_0 = (\eta^{0,r} \wedge e_{2\alpha-1}) \cdot e_{2\alpha} \cdot \psi_0 - (-1)^{r+1}(e_{2\alpha-1} \wedge (e_{2\alpha} \wedge \eta^{0,r})) \cdot \psi_0 + (e_{2\alpha-1} \wedge (e_{2\alpha} \wedge \eta^{0,r})) \cdot \psi_0 = \eta^{0,r} \cdot e_{2\alpha-1}e_{2\alpha} \psi_0 + e_{2\alpha} \wedge (e_{2\alpha-1} \cdot \eta^{0,r}) \cdot \psi_0 + e_{2\alpha-1} \wedge (e_{2\alpha-1} \cdot \eta^{0,r}) \cdot \psi_0. \]

This computation implies

\[ \Omega(\eta^{0,r} \cdot \psi_0) = \eta^{0,r} \cdot (\Omega \psi_0) \sum_{\alpha=1}^{k} e_{2\alpha-1} \wedge (e_{2\alpha} \wedge \eta^{0,r}) \cdot \psi_0 - \sum_{\alpha=1}^{k} e_{2\alpha-1} \wedge (e_{2\alpha} \wedge \eta^{0,r}) \cdot \psi_0 + e_{2\alpha-1} \wedge (e_{2\alpha} \wedge \eta^{0,r}) \cdot \psi_0 + (e_{2\alpha-1} \wedge (e_{2\alpha} \wedge \eta^{0,r})) \cdot \psi_0 \]

\[ = \eta^{0,r} (\Omega \psi_0) - 2ir \eta^{0,r} \cdot \psi_0 = i\eta^{0,r} (k - 2r) \psi_0 = i(k - 2r) \eta^{0,r} \cdot \psi_0. \]

Hence the spinor \(\eta^{0,r} \cdot \psi_0\) belongs to \(S_r\). The remaining assertions follow directly from algebraic calculations. \(\square\)

**Remark.** The factor \(\frac{1}{2r/2} \) is necessary for the following reason: If \(\eta^{0,1} = e_1 + ie_2\) is a \((0,1)\)-form, then \(|\eta^{0,1}|^2 = 2\). On the other hand, from \(e_1 e_2 \psi_0 = i \psi_0\) we obtain \(|(e_1 + ie_2) \psi_0|^2 = 4|\psi_0|^2\).
Remark. In the isomorphism specified above we have to use the complex-conjugate bundle $\Lambda^{0,r}$, since to apply Clifford multiplication we first have to change the $r$-form into the corresponding $r$-vector. However, an Hermitian metric induces a complex anti-linear identification of vectors with covectors. An analogous calculation leads to the following

**Proposition.** The mapping $\beta_r : \Lambda^{0,r} \otimes S_k \to S_{k-r}$ defined by $\eta^{r,0} \otimes \psi_k \to \frac{1}{2^{r/2}} \eta^{r,0} \cdot \psi_k$ is an isometry.

**Corollary.** The spinor bundle $S$ of an Hermitian manifold (with respect to an arbitrary $\text{spin}^C$ structure) is isomorphic to

$$S = (\Lambda^{0,0} + \ldots + \Lambda^{0,k}) \otimes S_0 = (\Lambda^{0,0} + \ldots + \Lambda^{k,0}) \otimes S_k,$$

where

$$S_0 = \{ \psi \in S : \Omega \psi = i k \psi \}, \quad S_k = \{ \psi \in S : \Omega \psi = -i k \psi \}.$$  

In particular, $S_0 = \overline{\Lambda^{k,0}} \otimes S_k$ and $S_k = \overline{\Lambda^{0,k}} \otimes S_0$.

We will now consider the case of the canonical $\text{spin}^C$ structure of the Hermitian manifold $(M^{2k}, j, g)$, i.e. the one defined by the lift of the group homomorphism constructed before:

$$\begin{array}{ccc}
\text{Spin}^C(2k) & \rightarrow & \text{SO}(2k) \\
\beta & & \\
U(k) & \rightarrow & \text{SO}(2k)
\end{array}$$

The subbundles $S_0, S_1, \ldots$ precisely correspond to the irreducible $U(k)$-components of the representation $\Delta_{2k}$. If $A \in U(k)$ has diagonal form,

$$A = \begin{pmatrix}
e^{i\Theta_1} & 0 \\
0 & \ddots \\
0 & e^{i\Theta_k}
\end{pmatrix},$$

then

$$F(A) = e^{\frac{i}{2} \sum_{j=1}^{k} \Theta_j \prod_{j=1}^{k} \left( \cos \left( \frac{\Theta_j}{2} \right) + \sin \left( \frac{\Theta_j}{2} \right) e_{2j-1}e_{2j} \right)}.$$  

$S_0$ corresponds to the basis vector $v(1) \otimes \ldots \otimes v(1)$ and $S_k$ to the basis vector $v(-1) \otimes \ldots \otimes v(-1)$. Hence the endomorphism $e_{2j-1}e_{2j}$ acts on $S_0$ as multiplication by $(+i)$ and on $S_k$ as multiplication by $(-i)$. Thus the bundle $S_0$ coincides with the highest power of the complex tangent bundle $(TM^{2k}, J)$, while $S_k$ is trivial. On the other hand, the determinant bundle $\mathcal{L}$ of the spin$^C$ structure is again $\Lambda^k(TM^{2k}, J)$; hence

$$S_0 = \mathcal{L} = \Lambda^k(T), \quad S_k = \Theta^1.$$
in the case of the canonical spin\(_C\) structure. Now we turn to the following calculation involving first Chern classes. In general, for every spin\(_C\) structure we have
\[ L^{\operatorname{dim}(S)/2} = \Lambda^{\operatorname{dim}(S)}(S). \]
This implies
\[ 2^{k-1}c_1(L) = c_1(S) = c_1((\Lambda^{0,0} + \ldots + \Lambda^{0,k}) \otimes S_0) = 2^k c_1(S_0) + c_1(\Lambda^{0,0} + \ldots + \Lambda^{0,k}). \]
For the canonical spin\(_C\) structure, \( S_0 = L = \Lambda^k(T) \) implies
\[ -2^{k-1}c_1(M^{2k}) = c_1(\Lambda^{0,0} + \ldots + \Lambda^{0,k}). \]
Inserting this, we obtain the

**Proposition.** Let \((P, \Lambda)\) be an arbitrary spin\(_C\) structure on an Hermitian manifold, \(L\) the determinant bundle of this spin\(_C\) structure and \(S_0\) the corresponding subbundle of the spinor bundle. Then for the Chern classes the following relations hold:
\[ c_1(L) + c_1(M^{2k}) = 2c_1(S_0), \quad c_1(L) - c_1(M^{2k}) = 2c_1(S_k). \]

**Remark.** For an Hermitian manifold \((M^{2k}, J, g)\) we defined the canonical spin\(_C\) structure by the lift \(F\) of the homomorphism \(f(A) = (A, \det A)\): \[ \begin{array}{ccc} \operatorname{Spin}(2k) & \xrightarrow{F} & U(k) \\ \downarrow & & \downarrow f \\ SO(2k) \times U(1) \end{array} \]
In this case, \(L = S_0 = \Lambda^k(T)\). However, there is a second lift from \(U(k)\) to \(\operatorname{Spin}(2k)\) related to the homomorphism \(f_1(A) = (A, \frac{1}{\det A})\). We will call the spin\(_C\) structure corresponding to this the anti-canonical spin\(_C\) structure. In this case, \(L = \Lambda^k(T^*)\) and \(S_0 = \Theta^1\). Furthermore, we will discuss the case that the given spin\(_C\) structure on \((M^{2k}, J, g)\) originates from a spin structure \((P, \Lambda)\). Given this, one obtains a spin\(_C\) structure by \(P^c = P \times_{\operatorname{Spin}(2k)} \operatorname{Spin}(2k)\). The determinant bundle \(L\) is then trivial, \(L = \Theta^1\), and the spinor bundle again splits into
\[ S = (\Lambda^{0,0} + \ldots + \Lambda^{0,k}) \otimes S_0. \]
However,
\[ (S_0)^2 = \Lambda^k(T). \]
Indeed, the spinor bundle in this case is a vector bundle associated with the group \(\operatorname{Spin}(2k)\). On the other hand, \(\Delta_{2k}\) has a real (quaternionic) structure \(j : \Delta_{2k} \to \Delta_{2k}\) which is spin equivariant and commutes (or anti-commutes,
respectively) with Clifford multiplication (compare Section 1.7). \( j \) induces a morphism in the spinor bundle which is complex anti-linear and, since

\[
\Omega j(\psi) = j\Omega(\psi) = j(i(k - 2r)\psi) = -(k - 2r)ij(\psi) = (k - 2(k - r))ij(\psi),
\]

it maps the subbundle \( S_r \) into \( S_{k-r} \). Going to the conjugate bundle \( \bar{S}_{k-r} \) \( j \) can be considered as a complex linear morphism,

\[
j : S_r \rightarrow \bar{S}_{k-r}.
\]

Composing \( j : S_0 \rightarrow \bar{S}_k \) with \( \beta_k : \Lambda_{k,0} \otimes S_k \rightarrow S_0 \) now defines an isomorphism

\[
S_0 \otimes S_0 \xrightarrow{1 \otimes j} S_0 \otimes \bar{S}_k \xrightarrow{\beta_k^{-1} \otimes 1} \Lambda_{k,0} \otimes S_k \otimes \bar{S}_k = \Lambda_{k,0}.
\]

Hence,

\[
S_0^2 = \Lambda_{k,0} = \Lambda^k(\Lambda^0,1) = \Lambda^{0,k}.
\]

Altogether, we obtain the following table.

<table>
<thead>
<tr>
<th>( \mathcal{L} )</th>
<th>( S_0 )</th>
<th>( S_k )</th>
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<tbody>
<tr>
<td>canonical spin( C ) structure</td>
<td>( \Lambda^k(T) )</td>
<td>( \Lambda^k(T) )</td>
</tr>
<tr>
<td>anti-canonical spin( C ) structure</td>
<td>( \Lambda^k(T^*) )</td>
<td>( \Theta^1 )</td>
</tr>
<tr>
<td>spin structure</td>
<td>( \Theta^1 )</td>
<td>( S_0^2 = \Lambda^{0,k} )</td>
</tr>
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</table>

Fix a connection \( A \) in the \( U(1) \)-principal bundle \( P_1 \) of the spin\( C \) structure (or, equivalently, in \( \mathcal{L} \)) as well as a connection \( A_0 \) in the Hermitian vector bundle \( S_0 \). Then, on the one hand, there is the Dirac operator,

\[
D_A : \Gamma(S) \rightarrow \Gamma(S),
\]

and, on the other, \( \bar{\partial}_{A_0} \) as well as \( \bar{\partial}_{A_0}^* \) are operators in the bundle \( (\sum_{i=0}^k \Lambda^{0,i}) \otimes S_0 \). Here \( \bar{\partial}_{A_0} \) and \( \bar{\partial}_{A_0}^* \) are defined by

\[
\bar{\partial}_{A_0}(\eta^{0,r} \otimes \psi_0) = (\bar{\partial}_0^{0,r}) \otimes \psi_0 + \left( \sum_{i=1}^{2k} e_i \wedge \eta^{0,r} \right)^{0,r+1} \otimes \nabla_{e_i} A_0 \psi_0,
\]

\[
\bar{\partial}_{A_0}^*(\eta^{0,r} \otimes \psi_0) = (\bar{\partial}^{*0,r}) \otimes \psi_0 - \left( \sum_{i=1}^{2k} e_i \eta^{0,r} \right) \otimes \nabla_{e_i} A_0 \psi_0.
\]
Note that for every vector $X \in T(M^{2k})$ and each $(0, r)$-form $\eta^{0, r}$, the form $X \eta^{0, r}$ is a $(0, r - 1)$-form. Thus in the formula for $\bar{\partial}_A^*\bar{\partial}_{A_0}$ the projection onto the $(0, r - 1)$-component is no longer necessary. The isomorphisms already defined,

$$\alpha_r : \Lambda^{0,r} \otimes S_0 \rightarrow S_r,$$

combine to an isomorphism $\alpha = \sum_{r=0}^{k} \alpha_r$ between the bundle $\sum_{r=0}^{k} \Lambda^{0,r} \otimes S_0$ and $S$. Hence we can compare $DA$ with $\bar{\partial}_A + \bar{\partial}_{A_0}^*$. Note that $\alpha^{-1}DA\alpha$ becomes a complex linear operator in the bundle $\sum_{r=0}^{k} \Lambda^{0,r} \otimes S$.

**Proposition.** There exists an endomorphism

$$E : \left( \sum_{r=0}^{k} \Lambda^{0,r} \right) \otimes S_0 \rightarrow \left( \sum_{r=0}^{k} \Lambda^{0,r} \right) \otimes S_0$$

such that

$$\alpha^{-1}DA\alpha = \sqrt{2} (\bar{\partial}_A + \bar{\partial}_{A_0}^*) + E.$$

$E$ depends on the almost complex structure $J$, on the Hermitian metric $g$ and on the connections $A, A_0$.

**Proof.** $\alpha^{-1}DA\alpha$ and $\sqrt{2} (\bar{\partial}_A + \bar{\partial}_{A_0}^*)$ are first order differential operators. Hence it suffices to show that their symbols coincide. Fix a vector (covector) $X$. Then the symbol $\sigma(\sigma_A)(X) : S \rightarrow S$ is Clifford multiplication by the vector $X$. Without loss of generality we choose the Hermitian basis $e_1, J(e_1), \ldots e_{2k-1}, J(e_{2k-1})$ with $X = e_1$. Now compute

$$\sigma(\alpha^{-1}DA\alpha)(X) : \left( \sum_{r=0}^{k} \Lambda^{0,r} \right) \otimes S_0 \rightarrow \left( \sum_{r=0}^{k} \Lambda^{0,r} \right) \otimes S_0.$$

To this end, first decompose a given $(0, r)$-form $\eta^{0,r}$ into

$$\eta^{0,r} = (e_1 + ie_2) \wedge \eta^{0,r-1} + \eta^{*0,r} \quad \text{with} \quad e_1 \eta^{*0,r} = e_2 \eta^{*0,r} = 0.$$

Then,

$$\sigma(\sigma_A)(X)\alpha(\eta^{0,r} \otimes \psi_0) = \frac{1}{2^{r/2}} e_1 \cdot \eta^{0,r} \cdot \psi_0$$

$$= \frac{1}{2^{r/2}} e_1 \cdot (e_1 + ie_2) \cdot \eta^{0,r-1} \cdot \psi_0 + \frac{1}{2^{r/2}} e_1 \cdot \eta^{*0,r} \cdot \psi_0$$

$$= \frac{1}{2^{r/2}} (-1 + i e_1 e_2) \cdot \eta^{0,r-1} \cdot \psi_0 + \frac{1}{2^{r/2}} (e_1 \wedge \eta^{*0,r}) \cdot \psi_0.$$

Clifford multiplication by $e_1 e_2$ commutes with $\eta^{0,r-1}$, and $e_1 e_2 \psi_0 = i \psi_0$. Moreover, $e_1 \eta^{*0,r} = 0$. Hence,

$$\sigma(\sigma_A)(X)\alpha(\eta^{0,r} \otimes \psi_0) = -\frac{2}{2^{r/2}} \eta^{0,r-1} \cdot \psi_0 + \frac{1}{2^{r/2}} (e_1 + i e_2) \wedge \eta^{*0,r} \cdot \psi_0.$$
Applying $\alpha^{-1}$, we conclude that
\[
\alpha^{-1}\sigma(D_A)(X)\alpha(\eta^{0,r} \otimes \psi_0)
= \left( -\sqrt{2}\eta^{0,r-1} + \sqrt{2} \left( \frac{e_1 + ie_2}{2} \right) \wedge \eta^{*,0,r} \right) \otimes \psi_0
= \left( -\sqrt{2}X \eta^{0,r} + \sqrt{2} \left( X + iJX \right) \wedge \eta^{0,r} \right) \otimes \psi_0.
\]
Thus the symbol of $\alpha^{-1}D_A\alpha$ is computed. For the symbols of $\bar{\partial}_{A_0}$ and $\bar{\partial}_{A_0}^*$ we have
\[
\sigma(\bar{\partial}_{A_0})(X) = \sigma(\bar{\partial})(X) \otimes \text{Id}_{S_0}, \quad \sigma(\bar{\partial}_{A_0}^*)(X) = \sigma(\bar{\partial}^*)(X) \otimes \text{Id}_{S_0},
\]
and the mappings $\sigma(\bar{\partial})(X) : \Lambda^{0,r} \to \Lambda^{0,r+1}$ and $\sigma(\bar{\partial})^*(X) : \Lambda^{0,r} \to \Lambda^{0,r-1}$, respectively, are given by
\[
\sigma(\bar{\partial})(X)(\eta^{0,r}) = \frac{X + iJX}{2} \wedge \eta^{0,r}, \quad \sigma(\bar{\partial})^*(X)(\eta^{0,r}) = -X \eta^{0,r}.
\]

Consider the special case of a Kähler manifold $(M^{2k}, J, g)$. If $Q$ is the bundle of $SO(2k)$-frames and $R$ its $U(k)$-reduction, then the Levi-Civita connection reduces to the $U(k)$-principal bundle $R$. Choosing, furthermore, the anti-canonical spin$^C$ structure, we know that $S_0 = \Theta^1$, and $\mathcal{L} = \Lambda^k(T^*) = R \times (\det)^{-1} \mathbb{C}$. Thus the Levi-Civita connection induces a connection $A$ in $\mathcal{L}$. As the connection $A_0$ in $S_0 = \Theta^1$ we take the trivial one. In this case, the corresponding Dirac operator $D$ just agrees with $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$:

**Proposition.** Let $(M^{2k}, J, g)$ be a Kähler manifold with the anti-canonical spin$^C$ structure. Then,

1) $S \approx \Lambda^{0,0} + \ldots + \Lambda^{0,k}$, and

2) the Dirac operator defined by the Levi-Civita connection coincides with $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$.

**Corollary.** The space of harmonic spinors, $\{\psi \in \Gamma(S) : D\psi = 0\}$, of a compact Kähler manifold (with respect to the anti-canonical spin$^C$ structure) is isomorphic to
\[
\sum_{r=0}^{k} H^r(M^{2k}; \mathcal{O}).
\]

Finally, we discuss the case of a Kähler manifold $(M^{2k}, J, g)$ with fixed spin structure. Then,
\[
\mathcal{L} = \Theta^1, \quad S_0^2 = \Lambda^{0,k}, \quad S_k^2 = \Lambda^{k,0}.
\]
Thus $S_k$ is a square root of the canonical bundle $K = \Lambda^k(T^*) = \Lambda^{k,0}$ of the Kähler manifold, and the spinor bundle is isomorphic to

$$S = (\Lambda^{0,0} + \Lambda^{0,1} + \ldots + \Lambda^{0,k}) \otimes S_k.$$ 

Choose the trivial connection $A$ in $L = \Theta^1$. As $S_k^2 = \Lambda^{k,0}$, the connection in $S_k$ is induced from the Levi-Civita connection. Summarizing, we have for the corresponding Dirac operator $D$ the

**Proposition.** Let $(M^{2k}, J, g)$ be a Kähler manifold with fixed spin structure. Then the following assertions hold:

1) $S_k$ is a square root of the canonical bundle, $S_k^2 = K = \Lambda^{k,0}$.

2) The spinor bundle is isomorphic to $S = \left(\sum_{r=0}^{k} \Lambda^{0,r}\right) \otimes S_k$.

3) The Dirac operator coincides with $\sqrt{2}(\bar{\partial} + \partial^*)$.

**Proposition.** The space of harmonic spinors, $\{\psi \in \Gamma(S) : D\psi = 0\}$, on a compact Kähler manifold with spin structure is isomorphic to

$$\sum_{r=0}^{k} H^r(M^{2k}, \mathcal{O}(S_k)),$$

where $S_k$ is a line bundle for which $S_k^2 = K = \Lambda^{k,0}$ (describing the spin structure).

3.5. The Dirac operator of a Riemannian symmetric space

Consider a Riemannian symmetric space $M^m$ with isotropy group $G$. If $K$ is the isotropy group of a fixed point $m_0 \in M^m$, then $M^m$ can be identified with the homogeneous space $G/K$. The Lie algebra $g$ of the group $G$ splits into

$$g = \mathfrak{k} + \mathfrak{m}$$

and the following commutation relations hold:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

Moreover, $\mathfrak{m}$ is an $\text{Ad}(K)$-invariant subspace of the Lie algebra $g$,

$$\text{Ad}(k)(\mathfrak{m}) \subset \mathfrak{m} \quad \text{for} \quad k \in K.$$

Let $\langle , \rangle$ be a scalar product in the vector space $g$ which is positive definite on $\mathfrak{m}$ and has the invariance property

$$\langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle = 0$$
for all $X, Y, Z \in \mathfrak{g}$. This scalar product defines a Riemannian metric on $M^n$ which we also want to denote by $\langle \cdot, \cdot \rangle$. Right and left translations in the group $G$ will be denoted by $R_g$ and $L_g$, respectively,

$$R_g(g_1) = g_1 g, \quad L_g(g_1) = g g_1.$$ 

The projection $\pi : G \to G/K = M^m$ is a $K$-principal bundle. This principal bundle has a canonical connection $Z$. The group $K$ acts on $G$ from the right. For $X \in \mathfrak{k}$, the fundamental vector field of the $K$-action at the point $g \in G$ is given by

$$\tilde{X}(g) = \frac{d}{dt} (g \cdot e^{tX})|_{t=0}.$$ 

But this is precisely the left invariant vector field $X$ determined by the vector $X \in \mathfrak{k}$. Thus the vertical tangent space of the $K$-principal bundle $\pi : G \to G/K$ at the point $g \in G$ coincides with the space $dL_g(\mathfrak{k})$,

$$T^v_g(G) = dL_g(\mathfrak{k}).$$

Define the connection in the $K$-principal bundle as the splitting

$$T_g(G) = T^v_g(G) + T^h_g(G)$$

with $T^h_g(G) = dL_g(\mathfrak{m})$. We have to check that this distribution

$$\{T^h_g(G) = dL_g(\mathfrak{m})\}$$

is right invariant under the $K$-action. However,

$$T^h_{gk}(G) = dL_g dL_k(\mathfrak{m}) = dR_k dR_k^{-1} dL_g dL_k(\mathfrak{m}).$$

Each right translation commutes with every left translation, and thus

$$T^h_{gk}(G) = dR_k dL_g dR_k^{-1} dL_k(\mathfrak{m}) = dR_k dL_g \text{Ad}(k)(\mathfrak{m}) = dR_k(T^h_g(G)).$$

The canonical connection in the principal bundle $(G, \pi, G/K; K)$ as a 1-form $Z : TG \to \mathfrak{k}$ is easily described. Let $\Theta$ be the Maurer-Cartan form of the Lie group $G$,

$$\Theta : T(G) \longrightarrow \mathfrak{g}, \quad \Theta(\tilde{\mathfrak{v}}) = dL_{g^{-1}}(\tilde{\mathfrak{v}}_g).$$

Then $Z = \text{pr}_\mathfrak{k} \circ \Theta$. Indeed, for $X \in \mathfrak{k}$, the fundamental vector field $\tilde{X}$ of the $K$-action is described by the left invariant vector field $X$, and this implies $\Theta(\tilde{X}) = X$, i.e. $Z(\tilde{X}) = X$. On the other hand, the kernel of $Z$ is exactly $T^h(G)$. We also compute the curvature form of this canonical connection:

$$\Omega^Z = dZ + \frac{1}{2}[Z, Z] = \text{pr}_\mathfrak{k}(d\Theta) + \frac{1}{2}[\text{pr}_\mathfrak{k}\Theta, \text{pr}_\mathfrak{k}\Theta].$$

If, however, $\Theta = \Theta_\mathfrak{k} + \Theta_\mathfrak{m}$ is the decomposition of the Maurer-Cartan form, then

$$[\Theta, \Theta] = [\Theta_\mathfrak{k}, \Theta_\mathfrak{k}] + [\Theta_\mathfrak{k}, \Theta_\mathfrak{m}] + [\Theta_\mathfrak{m}, \Theta_\mathfrak{k}] + [\Theta_\mathfrak{m}, \Theta_\mathfrak{m}],$$

and the commutator relations imply

$$\text{pr}_\mathfrak{k}[\Theta, \Theta] = [\Theta_\mathfrak{k}, \Theta_\mathfrak{k}] + [\Theta_\mathfrak{m}, \Theta_\mathfrak{m}].$$
Thus,

\[ \Omega^Z = \text{pr}_\mathfrak{k} \left( d\Theta + \frac{1}{2}[[\Theta, \Theta]] - \frac{1}{2}[\Theta_m, \Theta_m] \right) \]

and the structure equation \( d\Theta + \frac{1}{2}[[\Theta, \Theta]] = 0 \) of the Lie group \( G \) leads to

\[ \Omega^Z = -\frac{1}{2}[\text{pr}_m \Theta, \text{pr}_m \Theta]. \]

Let \( Q \) be the bundle of orthonormal frames on \( M^m \). Then there exists an inclusion \( i : G \rightarrow Q \) such that the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{i} & Q \\
\downarrow{\pi} & & \downarrow{\pi} \\
G/K & \leftrightarrow & M^m
\end{array}
\]

commutes. To this end, fix an orthonormal basis \( e_1, \ldots, e_m \) at the point \( m_0 \) and define \( i(g) = (dl_g(e_1), \ldots, dl_g(e_m)) \), where \( l_g : M^m \rightarrow M^m \) is the action of \( g \in G \) on \( M^m \). Now we want to see that the Levi-Civita connection reduces to the \( K \)-principal bundle \((G, \pi, G/K; K)\) and coincides there with the constructed connection \( Z \). Note that the tangent bundle \( T(M^m) \) of \( M^m = G/K \) is

\[ T(M^m) = G \times_{\text{Ad}(K)} \mathfrak{m}, \]

the bundle associated with the representation \( \text{Ad} : K \rightarrow \mathcal{O}(\mathfrak{m}) \). Hence a vector field \( T \) on the manifold \( M^m \) is a mapping \( T : G \rightarrow \mathfrak{m} \) with the invariance property \( T(gk) = \text{Ad}(k^{-1})T(g) \). \( Z \) induces a covariant derivative \( \nabla^Z \) in \( T(M^m) = G \times_{\text{Ad}(K)} \mathfrak{m} \), and

\[ \nabla^Z T = dT + [\text{pr}_\mathfrak{k} \Theta, T]. \]

By the invariance property of \( \langle \cdot, \cdot \rangle \) this implies

\[
\langle \nabla^Z T, T_1 \rangle + \langle T, \nabla^Z T_1 \rangle = \langle dT, T_1 \rangle + \langle T, dT_1 \rangle + \langle [\text{pr}_\mathfrak{k} \Theta, T], T_1 \rangle + \langle T, [\text{pr}_\mathfrak{k} \Theta, T_1] \rangle
= \langle dT, T_1 \rangle + \langle T, dT_1 \rangle = d\langle T, T_1 \rangle,
\]

i.e. \( \nabla^Z \) preserves the Riemannian metric. Analogously, one shows that \( \nabla^Z \) is torsion-free. However, these two conditions uniquely determine the Levi-Civita connection of \( M^m \).
Fix a homogeneous spin structure of the symmetric space $G/K$, i.e. a homomorphism $\widetilde{Ad} : K \rightarrow Spin(m)$ such that the diagram

$$
\begin{array}{ccc}
Spin(m) & \xrightarrow{\lambda} & SO(m) \\
\widetilde{Ad} & \downarrow & \text{Ad} \\
K & \rightarrow & K
\end{array}
$$

commutes. Let $\kappa : Spin(m) \rightarrow GL(\Delta)$ be the spin representation. Then a spinor field $\psi$ is identified with a function $\psi : G \rightarrow \Delta$ satisfying the invariance condition

$$
\psi(gk) = \kappa \widetilde{Ad}(k^{-1})\psi(g).
$$

Let $X$ be a left invariant vector field on the group $G$ with $X \in \mathfrak{k}$. Then,

$$
X\psi(g) = \frac{d}{dt} \psi(ge^{tx}) = \frac{d}{dt} \kappa \widetilde{Ad}(e^{-tx})\psi(g) = -\kappa^* \widetilde{Ad}_*(X)\psi(g),
$$

and hence

$$
X\psi = -\kappa^* \widetilde{Ad}(X)\psi = -\widetilde{Ad}_*(X) \cdot \psi,
$$

where $\widetilde{Ad}(X) \cdot \psi$ is Clifford multiplication of the spinor $\psi \in \Delta = \Delta(m)$ by the element $\widetilde{Ad}_*(X) \in \text{spin}(m) \subset \mathfrak{cliff}(m)$ of the Clifford algebra. Consider the 1-form $DZ\psi$ with values in $\Delta$. Then,

$$
(D^2\psi)(X) = D\psi(X) + \kappa^* \widetilde{Ad}_*(Z(X))\psi.
$$

For $X \in \mathfrak{k}$, the last computation ($Z(X) = X!$) immediately shows that $D^2\psi(X) = 0$. For $X \in \mathfrak{m}$ we obtain $Z(X) = 0$, and thus $D^2\psi(X) = X(\psi)$. Choosing an orthonormal basis $X_1, \ldots, X_m$ in $\mathfrak{m}$, we obtain

$$
D^2\psi = \sum_{i=1}^{m} X_i \otimes X_i(\psi),
$$

and, for the Dirac operator, this implies the simple formula

$$
D\psi = \sum_{i=1}^{m} X_i \cdot X_i(\psi).
$$

We will now compute $D^2$:

$$
D^2\psi = \sum_{i,j=1}^{m} X_i \cdot X_j \cdot (X_iX_j(\psi)) = -\sum_{i=1}^{m} X_i^2(\psi) + \frac{1}{2} \sum_{i,j=1}^{m} X_i \cdot X_j \cdot ([X_i, X_j]_\psi).
$$

However, the vector field $[X_i, X_j]$ belongs to $\mathfrak{k}$, and hence the above calculation shows that

$$
[X_i, X_j](\psi) = -\widetilde{Ad}_*([X_i, X_j]) \cdot \psi.
$$
Inserting this yields

\[ D^2 \psi = - \sum_{i=1}^{m} X_i^2(\psi) - \frac{1}{2} \sum_{i,j=1}^{m} X_i \cdot X_j \cdot \widetilde{\text{Ad}}_*([X_i, X_j]) \cdot \psi. \]

Choose an orthonormal basis in the Lie algebra \( \mathfrak{t}, Y_1, \ldots, Y_k \). Then, from

\[ Y_\alpha(\psi) = -\widetilde{\text{Ad}}_*(Y_\alpha) \cdot \psi \]

we immediately conclude that

\[ Y_\alpha^2(\psi) = \widetilde{\text{Ad}}_*(Y_\alpha) \cdot \widetilde{\text{Ad}}_*(Y_\alpha) \cdot \psi, \]

and, finally, using the Casimir operator \( \Omega_G = - \sum_{i=1}^{m} X_i^2 - \sum_{\alpha=1}^{k} Y_\alpha^2 \) of the group \( G \), we arrive at the formula

\[ D^2 \psi = \Omega_G(\psi) + \sum_{\alpha=1}^{k} \widetilde{\text{Ad}}_*(Y_\alpha) \cdot \widetilde{\text{Ad}}_*(Y_\alpha) \cdot \psi - \frac{1}{2} \sum_{i,j=1}^{m} X_i \cdot X_j \cdot \widetilde{\text{Ad}}_*( [X_i, X_j] ) \cdot \psi. \]

To treat the remaining term, we compute in the Clifford algebra \( \text{Cliff}(m) \) as follows. Let \( \widetilde{\text{Ad}}_*(Y_\alpha) \in \text{spin}(m) \), and let \( X_1, \ldots, X_m \) be an orthonormal basis in \( m \). Then,

\[ \widetilde{\text{Ad}}_*(Y_\alpha) = \frac{1}{4} \sum_{i,j=1}^{m} \langle \lambda_\alpha \widetilde{\text{Ad}}_*(Y_\alpha)(X_i), X_j \rangle X_i X_j \]

\[ = \frac{1}{4} \sum_{i,j=1}^{m} \langle \text{Ad}_*(Y_\alpha)(X_i), X_j \rangle X_i \cdot X_j \]

\[ = \frac{1}{4} \sum_{i,j=1}^{m} \langle [Y_\alpha, X_i], X_j \rangle X_i \cdot X_j \]

and, analogously,

\[ \widetilde{\text{Ad}}_*([X_i, X_j]) = \frac{1}{4} \sum_{p,q=1}^{m} \langle [[X_i, X_j], X_p], X_q \rangle X_p \cdot X_q. \]

The remainder term thus coincides with

\[ \frac{1}{16} \sum_{\alpha=1}^{k} \sum_{i,j=1}^{m} \sum_{p,q=1}^{m} \langle [Y_\alpha, X_i], X_j \rangle \langle [Y_\alpha, X_p], X_q \rangle X_i X_j X_p X_q \]

\[ - \frac{1}{8} \sum_{i,j=1}^{m} \sum_{p,q=1}^{m} \langle [[X_i, X_j], X_p], X_q \rangle X_i X_j X_p X_q. \]
3.5. The Dirac operator of a Riemannian symmetric space

Because of the invariance property of the scalar product $\langle \cdot , \cdot \rangle$ in $\mathfrak{g}$ we also have

$$\sum_{\alpha=1}^{k} \langle [Y_\alpha, X_i], X_j \rangle \langle [Y_\alpha, X_p], X_q \rangle$$

$$= \sum_{\alpha=1}^{k} \langle Y_\alpha, [X_i, X_j] \rangle \langle Y_\alpha, [X_p, X_q] \rangle = \langle [X_i, X_j], [X_p, X_q] \rangle.$$

Hence the term under consideration simplifies to the expression

$$-\frac{1}{16} \sum_{i,j,p,q} \langle [X_i, X_j], [X_p, X_q] \rangle X_i X_j X_p X_q.$$

The Levi-Civita connection is induced from the connection $Z$ in the $K$-principal bundle. Thus for the curvature tensor $R$ of the Riemannian space the following formula holds:

$$R(X_i, X_j) = [QZ(X_i, X_j), \cdot],$$

i.e.

$$\langle R(X_i, X_j) X_p, X_q \rangle = \langle [QZ(X_i, X_j), X_p], X_q \rangle = -\langle [X_i, X_j], [X_p, X_q] \rangle.$$

In the Clifford algebra $\text{cliff}(m) = C_m$, the remaining term has $C_m^0$ and $C_m^2$-as well as $C_m^4$-parts. The $C_m^2$- and $C_m^4$-parts vanish, and for the $C_m^0$-part we have

$$-\frac{1}{4} \sum_{i<j} \langle [X_i, X_j], [X_i, X_j] \rangle X_i X_j X_i X_j$$

$$= \frac{1}{4} \sum_{i<j} \| [X_i, X_j] \|^2 = \frac{1}{8} \sum_{i,j} \| [X_i, X_j] \|^2 = \frac{1}{8} R,$$

where $R$ is the scalar curvature of the space $G/K$. Summarizing, we arrive at the

**Proposition.** Let $M^m = G/K$ be a compact Riemannian symmetric space with a homogeneous spin structure. Let $\Omega_G$ denote the Casimir operator of the Lie group $G$. Then,

$$D^2 = \Omega_G + \frac{1}{8} R.$$

**Remark.** The importance of this formula lies in the fact that it allows us to compute the eigenvalues of $D^2$ purely by means of representation theory. Let $\lambda : G \to GL(V_\lambda)$ be an irreducible complex representation and
\( \lambda_* : g \to \mathfrak{gl}(V_{\lambda}) \) its differential. Then,

\[
\lambda_*(\Omega_G) = - \sum_{i=1}^{m} \lambda_*(X_i)^2 - \sum_{\alpha=1}^{k} \lambda_*(Y_{\alpha})^2
\]

is an operator in \( V_{\lambda} \). It is easy to show that this operator commutes with all automorphisms \( \lambda(g) : V_{\lambda} \to V_{\lambda} \) (\( g \in G \)). This follows immediately from the formula

\[
\lambda(g) \lambda_*(X) \lambda(g^{-1}) = \lambda_*(\text{Ad}(g)X),
\]

\( X \in g, g \in G \), which itself is the result of a straightforward calculation using the fact that \( \text{Ad}(g) : g \to g \) maps the orthonormal basis consisting of \( \{X_1, \ldots, X_m, Y_1, \ldots, Y_k\} \) again onto an orthonormal basis of \( g \). Let \( \mu \) be an eigenvalue of \( \lambda_*(\Omega_G) \) and \( W_\mu \subset V_{\lambda} \) the corresponding eigensubspace. Then \( W_\mu \) is invariant under all \( \lambda(g), g \in G \). The irreducibility of \( V_{\lambda} \) implies \( W_\mu = 0 \) or \( V_{\lambda} \). But \( W_\mu = 0 \) is excluded, as \( \lambda_*(\Omega_G) \) indeed has eigenvalues over \( \mathbb{C} \). Hence we have \( W_\mu = V_{\lambda} \), i.e. \( \lambda_*(\Omega_G) \) is a multiple of the identity.

Finally, consider the Hilbert space \( L^2(S) = L^2(G/K; S) \) of all square-integrable sections of the spinor bundle over the compact Riemannian symmetric space. Then \( G \) acts as a group of unitary transformations on \( L^2 \). Decompose \( L^2 \) into the direct sum

\[
L^2(S) = \bigoplus_{\lambda \in \Lambda} V_{\lambda}
\]

of finite-dimensional irreducible \( G \)-representations \( V_{\lambda} \), and compute each time the number \( c(\lambda) \) with \( \lambda_*(\Omega_G) = c(\lambda) I_{V_{\lambda}} \). The spectrum of \( D^2 \) is then given by

\[
\text{Spec } (D^2) = \left\{ c(\lambda) + \frac{1}{8} R : \lambda \in \Lambda \right\}.
\]

These computations were, e.g. carried out in the case of spheres, certain Grassmannian manifolds, and for the complex projective spaces \( \mathbb{C}P^{2k+1} \) (compare the references).

3.6. References and Exercises


S. Seifarth, U. Semmelmann. The spectrum of the Dirac operator on the odd-dimensional complex projective space \(\mathbb{CP}^{2m-1}\), Preprint des SFB 288 ”Differentialgeometrie und Quantenphysik” No. 95 (1993).


Exercise 1. Let \((M^4, g)\) be a 4-dimensional Riemannian spin manifold with non-trivial parallel spinors \(\psi^+, \psi^-\) in the bundles \(S^+\) and \(S^-\), respectively. Prove that \((M^4, g)\) is flat.

Exercise 2. Prove that every parallel spinor \(\psi^+\) in the bundle \(S^+\) over a 4-dimensional Riemannian manifold \((M^4, g)\) induces a complex structure \(J : TM^4 \to TM^4\) such that \((M^4, g, J)\) is a Kähler manifold.

Exercise 3. Prove that, in 2-dimensional Euclidian space \(\mathbb{R}^2\), the general solution to the twistor equation \(T(\psi) = 0\) is given by

\[
\psi(x, y) = \begin{pmatrix} C \\ D \end{pmatrix} - \begin{pmatrix} 0 & x + iy \\ -x + iy & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
\]

with arbitrary vectors \(\begin{pmatrix} A \\ B \end{pmatrix}, \begin{pmatrix} C \\ D \end{pmatrix} \in \mathbb{C}^2 = \Delta_2\).

Exercise 4. The metric on \(M^4 \subset \mathbb{R}^4\),

\[
M^4 = \{(x_1, \ldots, x_4) \in \mathbb{R}^4 : x_1 > 0, 0 < x_2 < \pi\},
\]
3. Dirac Operators

defined by

\[ ds^2 = \frac{x_1}{x_1 + c} (dx_1)^2 + x_1^2 (dx_2)^2 + x_1 \sin^2(x_2) (dx_3)^2 + \frac{x_1 + c}{x_1} (dx_4)^2 \]

\((c > 0)\) is Ricci-flat but does not admit any parallel spinors.

**Exercise 5.** Let \((M^4, g)\) be a Riemannian spin manifold and \(\psi \in \Gamma(S)\) a spinor field. If

\[ \nabla_X \psi = w(X) \cdot \psi \]

with a real-valued 1-form \(w\), then

a) \(\text{Ric} \equiv 0\),

b) \(dw = 0\).

Prove by examples that this does not hold for general complex-valued forms.
Analytical Properties of Dirac Operators

4.1. The essential self-adjointness of the Dirac operator in $L^2$

Let us recall some notions from the spectral theory of linear operators in complex Hilbert spaces. Let $A$ be an (in general unbounded) operator with dense domain of definition, $\mathcal{D}(A)$, in the complex Hilbert space $H$. Denote the range of $A$ by $R(A)$. The graph, $\Gamma(A) \subset H \times H$, consists of all pairs $(x, Ax)$, $x \in \mathcal{D}(A)$. In the sequel we will assume that its closed hull $\overline{\Gamma(A)} \subset H \times H$ again is the graph of an operator $\tilde{A}$ which then is called the closure of $A$. Then $\tilde{A}$ acts via the formula

$$\tilde{A}(x) = \lim_{n \to \infty} A(x_n)$$

and its domain of definition, $\mathcal{D}(\tilde{A})$, consists of all vectors $x \in H$ for which there exists a sequence $x_n \in \mathcal{D}(A)$ with $\lim_{n \to \infty} x_n = x$ such that, moreover, the sequence $A(x_n)$ converges in $H$. Differential operators always have a closure in this sense. The spectrum of an operator consists of three parts. First, there are the eigenvalues of $A$ which form the so-called point spectrum $\sigma_p(A)$:

$$\sigma_p(A) = \{ \lambda \in \mathbb{C} : \ker(A - \lambda) \neq \{0\} \}.$$

Furthermore, there are the residual spectrum, $\sigma_r(A)$, as well as the continuous spectrum, $\sigma_c(A)$:

$$\sigma_r(A) = \{ \lambda \in \mathbb{C} : \ker(A - \lambda) = 0, \overline{R(A - \lambda)} \neq H \},$$

$$\sigma_c(A) = \{ \lambda \in \mathbb{C} : \ker(A - \lambda) \neq \{0\}, \overline{R(A - \lambda)} = H \}.$$
σ_c(A) = \{λ ∈ ℂ : ker(A - λ) = 0, \overline{R(A - λ)} = H, (A - λ)^{-1} \text{ is unbounded}\}.

The remaining complex numbers form the resolvent set ρ(A):

ρ(A) = \{λ ∈ ℂ : (A - λ)^{-1} \text{ is a bounded operator defined on } \overline{R(A - λ)} = H\}.

To each operator A there corresponds an adjoint operator A* with the domain of definition

D(A*) = \{x ∈ H : ∃ y ∈ H ∀ z ∈ D(A) : ⟨Az, x⟩ = ⟨z, y⟩\}

and A*(x) = y. This implies the relation

⟨Az, x⟩ = ⟨z, A*(x)⟩

for all vectors z ∈ D(A), x ∈ D(A*). Let A be a symmetric operator, i.e.

⟨Ax, y⟩ = ⟨x, Ay⟩, \quad x, y ∈ D(A).

Then D(A) is contained in D(A*) and, in addition, A^*|_{D(A)} = A. (We will abbreviate this as A ⊂ A*.) The double adjoint operator A** coincides with the closure A (von Neumann theorem):

A = A** ⊂ A*.

Definition. The operator A is called self-adjoint if A = A*.

In particular, self-adjoint operators are closed, A = A.

Definition. The operator A is called essentially self-adjoint if its closure A is self-adjoint, i.e. A = A*.

The spectrum of essentially self-adjoint operators is real,

σ(A) = σ(A) = σ(A*) ⊂ ℝ^1.

Finally, recall the notion of spectral measure. If S ⊂ ℂ is a set of complex numbers and B(S) the σ-algebra of all its Borel subsets, then a spectral measure F is a mapping

F : B(S) → Proj(H)

from the σ-algebra B(S) into the set Proj(H) of all projectors in the Hilbert space with the following properties:

a) F(S) = Id_H,

b) for every x ∈ H the equality μ_x(B) = ⟨F(B)x, x⟩, B ∈ B(S), defines a measure on the σ-algebra B(S).
Given two vectors $x, y \in H$, then
\[ \mu_{x,y}(B) = (F(B)x, y) \]
is a complex-valued measure. Any measurable function $f : S \to \mathbb{C}$ can be integrated against a spectral measure:
\[ \int_S f(s) dF(s). \]
The value of this integral is an operator in the Hilbert space $H$ which is bounded for a bounded function. Moreover, we have the formula
\[ \left\langle \left( \int_S f(s) dF(s) \right)x, y \right\rangle = \int_S f(s) d\mu_{x,y}(s). \]
The spectral theorem for self-adjoint operators can now be formulated as follows:

**Theorem.** Let $A$ be a self-adjoint operator in $H$ with spectrum $\sigma(A) \subseteq \mathbb{R}$. Then there exists exactly one spectral measure $F$ on the $\sigma$-algebra $\mathcal{B}(\sigma(A))$ with
\[ A = \int_{\sigma(A)} \lambda dF(\lambda). \]

We now turn to the situation for the Dirac operator $D_A$ on a Riemannian manifold $(M^n, g)$ with fixed spin$^C$ structure and fixed connection $A$ in the determinant bundle of the spin$^C$ structure. The space $\Gamma_c(S)$ of all sections of the spinor bundle with compact support carries the scalar product
\[ (\psi_1, \psi_2) = \int_{M^n} (\psi_1(x), \psi_2(x)) dM^n. \]
Let $L^2(S)$ be the completion of this space. $D_A$ is a symmetric operator in $L^2(S)$ with domain $\mathcal{D}(D_A) = \Gamma_c(S)$ (compare Section 3.2). In this section, we will prove that $D_A$ is essentially self-adjoint whenever $(M^n, g)$ is a complete Riemannian manifold. To start with, we state the following formula for $D_A$:

**Proposition.** If $f$ is a smooth function defined on the manifold $M^n$, $\text{grad}(f)$ its gradient field and $\psi$ a spinor field, then
\[ D_A(f \cdot \psi) = f D_A(\psi) + \text{grad}(f) \cdot D_A(\psi). \]
4. Analytical Properties of Dirac Operators

Proof. This formula is obtained by a straightforward computation:

\[
D_A(f \cdot \psi) = \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i} (f \cdot \psi) = \sum_{i=1}^{n} e_i \cdot (e_i(f) \psi + f \nabla^A_{e_i} \psi) = \text{grad}(f) \cdot \psi + f D_A(\psi).
\]

Now we start to prove the essential self-adjointness of the Dirac operator \(D_A\). The argument will proceed along the lines of the the article by J. Wolf (see the references at the end of this chapter). The proof is subdivided into several steps. Let \(D_A^*\) be the adjoint to \(D_A\). In the domain \(D(D_A^*)\) we introduce the norm

\[
N(\psi) = \sqrt{||\psi||^2 + ||D_A^* \psi||^2},
\]

where \(|| \cdot ||\) denotes the norm in the Hilbert space \(L^2\).

Lemma 1. Let \(\Gamma_c(S) \subset D(D_A^*)\) be dense with respect to the \(N\)-norm. Then \(D_A\) is essentially self-adjoint.

Proof. Under the assumptions above we have to prove that \(D(D_A^*) \subset D(D_A)\). Let \(\psi \in D(D_A^*)\). By assumption, there exists a sequence \(\psi_n \in \Gamma_c(S)\) with \(\lim_{n \to \infty} N(\psi - \psi_n) = 0\). This implies that \(\lim_{n \to \infty} \psi_n = \psi\) in \(L^2(S)\) and, moreover, that \(D_A^*(\psi_n)\) converges to \(D_A^*(\psi)\) in \(L^2\). However, \(\psi_n\) is a smooth spinor field with compact support, and hence \(D_A^*(\psi_n) = D_A(\psi_n)\). Thus the sequence \(D_A(\psi_n)\) converges in \(L^2\). But the latter just means that \(\psi \in D(D_A)\).

Next we introduce the linear subspace

\[
D_c(D_A^*) = \{ \psi \in D(D_A^*) : \psi \text{ has compact support} \}.
\]

Lemma 2. \(\Gamma_c(S)\) is dense in \(D_c(D_A^*)\) with respect to the \(N\)-norm.

Proof. Choose a locally finite covering of the manifold \(M^n\) by charts \((U_i, h_i)\) indexed by the set \(i \in I\) such that each \(U_i \subset M^n\) is a compact subset. Let \(\{f_i\}_{i \in I}\) be a partition of unity subordinate to this covering with \(\text{supp}(f_i) \subset U_i\).

For a given spinor \(\psi \in D_c(D_A^*)\), there are only finitely many indices \(i \in I\) such that \(\text{supp}(f_i) \cap \text{supp}(\psi) \neq \emptyset\).

Denote those by \(i_1, \ldots, i_l \in I\). Then, \(\psi = \psi_1 + \cdots + \psi_e\) with \(\psi_j = f_j \cdot \psi\), where \(1 \leq j \leq l\). The spinor field \(\psi_j\) can be considered as a \(2^{[n/2]}\)-tuple of functions with compact support defined on the space \(\mathbb{R}^n\). Approximate \(\psi_j\)
by the convolutions with an approximation of the delta-distribution. Let $h : \mathbb{R}^n \to \mathbb{R}^1$ be given by

$$h(x) = \begin{cases} 
0 & \text{for } |x| \geq 1, \\
\frac{1}{e^{-\frac{1}{1-|x|^2}}} & \text{for } |x| < 1,
\end{cases}$$

and let $h_\varepsilon : \mathbb{R}^n \to \mathbb{R}^1$ be defined by $h_\varepsilon = \varepsilon h(\frac{x}{\varepsilon})$. Then $\psi_j * h_\varepsilon$ is a smooth function with compact support approximating $\psi_j$ in $L^2$. Since $D^*_A(\psi_j)$ belongs to $L^2$, the sequence $D^*_A(\psi_j * h_\varepsilon)$ thus converges to $D^*_A(\psi_j)$. Applying the chart mapping again, we obtain smooth spinor fields with compact support $\psi_{j,1}, \psi_{j,2}, ...$ defined in the Riemannian space with

$$\lim_{k \to \infty} N(\psi_j - \psi_{j,k}) = 0.$$ 

Forming $\psi_k = \psi_{1,k} + ... + \psi_{l,k}$, we now conclude that $\psi_k$ belongs to $\Gamma_c(S)$ and $\lim_{k \to \infty} N(\psi - \psi_k) = 0$. $\square$

**Lemma 3.** If $(M^n, g)$ is a complete Riemannian manifold, then $D_c(D^*_A)$ is dense in $D(D^*_A)$ with respect to the $N$-norm.

**Proof.** Denote the interior distance between two points $m_1$ and $m_2$ in the Riemannian manifold by $d(m_1, m_2)$. Let $m_0 \in M^n$ be a fixed point and $\rho(m) = d(m_0, m)$ the distance from $m$ to $m_0$. The triangle inequality implies

$$|\rho(m_1) - \rho(m_2)| \leq d(m_1, m_2),$$

i.e. $\rho$ is a Lipschitz continuous function. Hence $\rho(m)$ is differentiable almost everywhere and the gradient $\text{grad}(\rho)$ exists a.e., too. Furthermore, at each of these points

$$||\text{grad}(\rho)|| \leq 1.$$ 

Consider the geodesic ball

$$K(r) = \{ m \in M^n : \rho(m) < r \}.$$ 

Since $(M^n, g)$ is a complete Riemannian manifold, the closure $\bar{K}_r$ is a compact subset of $M^n$. Choose a $C^\infty$-function, $\alpha : \mathbb{R}^1 \to [0, 1]$, with the following properties:

(i) $\alpha(t) \equiv 1$ for $-\infty < t \leq 1$,

(ii) $\alpha(t) \equiv 0$ for $2 \leq t < \infty$.

Let the constant $M$ be the maximum of the derivative $|\alpha'(t)|:

$$M = \sup_{1 \leq t \leq 2} |\alpha'(t)|.$$ 

The function $b_r : M^n \to [0, 1]$ is defined by

$$b_r(m) = \alpha\left(\frac{\rho(m)}{r}\right).$$
Then, $b_r \equiv 1$ on $\mathcal{K}(r)$ and $\text{supp} \,(b_r) \subset \mathcal{K}(2r)$. Moreover, $b_r$ is Lipschitz continuous and the following inequality holds a.e.:

$$||\text{grad}(b_r)||^2 = \frac{1}{r^2} \left| \alpha' \left( \frac{\rho}{r} \right) \right|^2 \cdot ||\text{grad}(\rho)||^2 \leq \frac{M^2}{r^2}.$$ 

Let $\psi \in \mathcal{D}_c(D_A)$ be given. Then $\psi_r = b_r \cdot \psi$ belongs to $\mathcal{D}(D_A)$. Furthermore,

$$D_A^*(\psi_r) = \text{grad}(b_r) \cdot \psi + b_r D_A^*(\psi),$$

and this implies for the corresponding $L^2$-norms

$$||D_A^*(\psi - \psi_r)||^2 = ||(1 - b_r)D_A^* \psi - \text{grad}(b_r)\psi||^2 \leq \int_{\mathbb{M}^n \setminus \mathcal{K}(r)} 2||D_A^*(\psi)||^2 + \frac{2M^2}{r^2} \int_{\mathbb{M}^n} ||\psi||^2.$$

Thus we obtain the following estimate for the $N$-norm:

$$N^2(\psi - \psi_r) = ||\psi - \psi_r||^2 + ||D_A^*(\psi - \psi_r)||^2 \leq \int_{\mathbb{M}^n \setminus \mathcal{K}(r)} ||\psi||^2 + \int_{\mathbb{M}^n \setminus \mathcal{K}(r)} 2||D_A^*(\psi)||^2 + \frac{2M^2}{r^2} \int_{\mathbb{M}^n} ||\psi||^2.$$

From $\int_{\mathbb{M}^n} ||\psi||^2 < \infty$ and $\int_{\mathbb{M}^n} ||D_A^*(\psi)||^2 < \infty$ we then immediately conclude that

$$\lim_{r \to \infty} N^2(\psi - \psi_r) = 0.$$ 

In particular, we have proved the following:

**Proposition.** Let $(\mathbb{M}^n, g)$ be a complete Riemannian manifold with spin$^C$ structure. Then the Dirac operator $D_A$ is essentially self-adjoint in $L^2(S)$.

We will show, in addition, that the kernels of $D_A$ and $D_A^2$ in $L^2$ coincide. This will result from a more general inequality we are going to prove first.

**Proposition.** Let $(\mathbb{M}^n, g)$ be a complete Riemannian manifold and $\psi$ a spinor field of class $C^2$. Then, for the $L^2$-norm $|| \cdot ||$ and any number $t > 0$,

$$||D_A(\psi)||^2 \leq t||D_A^2(\psi)||^2 + \frac{1}{t} ||\psi||^2.$$ 

**Proof.** We will use the same balls $\mathcal{K}(r)$ and functions $b_r$ as in the proof of the previous proposition. The equality

$$D_A(b_r^2 \psi) = 2b_r \text{grad}(b_r) \cdot \psi + b_r^2 D_A(\psi)$$
implies that for every positive number $\varepsilon > 0$
\[
\int_{\mathcal{K}(2r+\varepsilon)} ||b_r D_A(\psi)||^2 = \int_{\mathcal{K}(2r+\varepsilon)} (b_r^2 D_A(\psi), D_A(\psi)) = \int_{\mathcal{K}(2r+\varepsilon)} (D_A(b_r^2 D_A(\psi)), \psi) \\
= 2 \int_{\mathcal{K}(2r+\varepsilon)} (b_r \text{grad}(b_r) \cdot D_A(\psi), \psi) + \int_{\mathcal{K}(2r+\varepsilon)} (b_r^2 D_A^2(\psi), \psi).
\]
Here we used the fact that the support of the spinor $b_r^2 D_A(\psi)$ is contained in $\mathcal{K}(2r)$. For $\varepsilon \to 0$ this implies
\[
\int_{\mathcal{K}(2r)} ||b_r D_A(\psi)||^2 = \int_{\mathcal{K}(2r)} (D_A^2(\psi), b_r^2 \psi) - \int_{\mathcal{K}(2r)} (b_r D_A(\psi), 2\text{grad}(b_r) \cdot \psi).
\]
Now we will apply Schwarz’ inequality,
\[
|\langle x, y \rangle| \leq ||x|| ||y|| \leq \frac{t}{2} ||x||^2 + \frac{1}{2t} ||y||^2
\]
for all $t > 0$. This allows us to estimate the last term in the equation (with $t = 1$):
\[
\left| \int_{\mathcal{K}(2r)} (b_r D_A(\psi), 2\text{grad}(b_r) \cdot \psi) \right| \leq \frac{1}{2} \int_{\mathcal{K}(2r)} ||b_r D_A(\psi)||^2 + 2 \int_{\mathcal{K}(2r)} ||\text{grad}(b_r) \cdot \psi||^2 \\
\leq \frac{1}{2} \int_{\mathcal{K}(2r)} ||b_r D_A(\psi)||^2 + \frac{2M^2}{r^2} \int_{\mathcal{K}(2r)} ||\psi||^2.
\]
From $0 \leq b_r \leq 1$ and another application of Schwarz’ inequality, this time to the first term, we obtain
\[
\left| \int_{\mathcal{K}(2r)} (D_A^2(\psi), b_r^2 \psi) \right| \leq \frac{t}{2} \int_{\mathcal{K}(2r)} ||D_A^2(\psi)||^2 + \frac{1}{2t} \int_{\mathcal{K}(2r)} ||\psi||^2.
\]
Combining both relations yields
\[
\int_{\mathcal{K}(r)} ||D_A(\psi)||^2 \leq \int_{\mathcal{K}(2r)} ||b_r D_A(\psi)||^2 = 2 \int_{\mathcal{K}(2r)} ||b_r D_A(\psi)||^2 - \int_{\mathcal{K}(2r)} ||b_r D_A(\psi)||^2 \\
\leq \left\{ \frac{1}{2} ||b_r D_A(\psi)||^2 + \frac{2M^2}{r^2} ||\psi||^2 + \frac{t}{2} ||D_A^2(\psi)||^2 + \frac{1}{2t} ||\psi||^2 \right\} \\
- \int_{\mathcal{K}(2r)} ||b_r D_A(\psi)||^2 \\
= \int_{\mathcal{K}(2r)} \left\{ t ||D_A^2(\psi)||^2 + \left( \frac{1}{t} + \frac{4M^2}{r^2} \right) ||\psi||^2 \right\}.
\]
In the case \( \int_{M^n} ||\psi||^2 = \infty \), the inequality to be proved is trivial. If, however, the integral is finite, then we obtain the inequality to be proved for \( r \to \infty \),

\[
||D_A(\psi)||^2 \leq t||D_A^2(\psi)||^2 + \frac{1}{t}||\psi||^2.
\]

\[\square\]

**Corollary.** For a complete Riemannian manifold \((M^n, g)\), the kernels of the operators \(D_A\) and \(D_A^2\) in \(L^2(S)\) coincide,

\[
\ker(D_A) = \ker(D_A^2).
\]

**Proof.** Let \( \psi \in L^2(S) \) satisfy \( D_A^2 \psi = 0 \). By the regularity theorem for solutions of elliptic differential equations we first conclude that \( \psi \) is smooth. Hence we can apply the inequality from the previous proposition and thus obtain

\[
||D_A(\psi)||^2 \leq t||D_A^2(\psi)||^2 + \frac{1}{t}||\psi||^2 = \frac{1}{t}||\psi||^2.
\]

If \( ||\psi||^2 < \infty \) then implies for \( t \to \infty \) that \( D_A(\psi) \to 0 \).

\[\square\]

### 4.2. The spectrum of Dirac operators over compact manifolds

Going from an operator \( A \) to its closure \( A \) does not change the spectrum,

\[
\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A):
\]

\[
\sigma(A) = \sigma(\bar{A}).
\]

For a Dirac operator this means, in particular, that \( \sigma(D_A) = \sigma(\bar{D}_A) \). In the case of a complete Riemannian manifold \( \bar{D}_A \) is a self-adjoint operator; hence it has no residual spectrum:

\[
\sigma_r(\bar{D}_A) = \emptyset.
\]

Thus \( \sigma(\bar{D}_A) \) only consists of the point spectrum, \( \sigma_p(\bar{D}_A) \), and the continuous spectrum, \( \sigma_c(\bar{D}_A) \):

\[
\sigma(D_A) = \sigma(\bar{D}_A) = \sigma_p(\bar{D}_A) \cup \sigma_c(\bar{D}_A):
\]

For a compact Riemannian manifold the domain of definition of the operator \( D_A \) coincides with \( \Gamma(S) \). If \( \lambda \in \sigma_p(\bar{D}_A) \) is an eigenvalue of the closure, then there exists a spinor field \( \psi \in L^2(S) \) with

\[
D_A(\psi) = \lambda \psi, \quad \psi \in L^2.
\]

The regularity theorem for elliptic differential operators then implies that \( \psi \) is smooth. Hence we have \( \psi \in \Gamma(S) = D(D_A) \), i.e. the point spectrum of \( D_A \) coincides with the point spectrum of the closure:

\[
\sigma_p(D_A) = \sigma_p(\bar{D}_A).
\]
4.2. The spectrum of Dirac operators over compact manifolds

Continuous spectrum does not occur in the case of a compact base space. However, we have the

**Proposition.** Let \((M^n, g)\) be a compact Riemannian manifold with spin\(^C\) structure. For the spectrum of the Dirac operator the following relations hold:

i) \[ \sigma_p(D_A) = \sigma(\bar{D}_A), \]

ii) \[ \sigma_c(\bar{D}_A) = \emptyset = \sigma_r(\bar{D}_A). \]

Together, these imply

iii) \[ \sigma(D_A) = \sigma(\bar{D}_A) = \sigma_p(D_A) = \sigma_p(\bar{D}_A). \]

**Proof.** We give a sketch of the proof. First, it is easy to see that the residual spectrum of \(D_A\) is empty. Indeed, for \(\lambda \in \sigma_r(D_A)\), there exists a spinor field \(\varphi \in L^2(S)\) such that

\[ (D_A(\psi) - \lambda \psi, \varphi)_{L^2} = 0 \]

for all \(\psi \in \Gamma(S)\). Choosing \(\psi\) with support in a chart and transferring this equation to Euclidean space, we obtain an elliptic differential operator \(P = D_A - \lambda\) as well as a function \(\varphi \in L^2(\mathbb{R}^n)\) such that

\[ (P(\psi), \varphi)_{L^2} = 0 \]

for all \(\psi \in C_0^\infty(\mathbb{R}^n)\). By the regularity theorem for elliptic operators, \(\varphi\) is smooth. In this case, we can write the equation \((D_A(\psi) - \lambda \psi, \varphi)_{L^2} = 0\) as \((\psi, (D_A - \lambda)\varphi)_{L^2} = 0\). This in turn implies \(D_A\varphi = \lambda\varphi\) and \(\varphi \in \Gamma(S) = \mathcal{D}(D_A)\), i.e. \(\lambda\) is an eigenvalue of \(D_A\). If now \(\sigma_r(D_A)\) is empty, we obtain

\[ \sigma_p(D_A) \cup \sigma_c(D_A) = \sigma(D_A) = \sigma(\bar{D}_A) = \sigma_p(\bar{D}_A) \cup \sigma_c(\bar{D}_A). \]

The approximation spectrum \(\sigma_\alpha(A)\) of an arbitrary operator \(A : \mathcal{D}(A) \to R(A)\) in a Hilbert space is defined by

\[ \{ \lambda \in \mathbb{C} : \exists \text{ a sequence } x_n \in \mathcal{D}(A) \text{ s. t. } \|x_n\| = 1, \|A(x_n) - \lambda x_n\| \to 0 \}. \]

In general,

\[ \sigma_p(A) \cup \sigma_c(A) \subset \sigma_\alpha(A) \subset \sigma(A). \]

Applying this inclusion to the case of a Dirac operator over a compact manifold, we at once conclude from the facts proved so far that

\[ \sigma_\alpha(D_A) = \sigma(\bar{D}_A) = \sigma(D_A) = \sigma(\bar{D}_A). \]

Finally, it remains to be shown that \(\sigma_\alpha(D_A) = \sigma_p(D_A)\). Let \(\lambda \in \sigma_\alpha(D_A)\). Then there is a sequence of spinor fields \(\psi_n \in \Gamma(S)\) with

\[ \|\psi_n\|_{L^2} = 1, \quad \|D_A(\psi_n) - \lambda \psi_n\|_{L^2} \to 0. \]
ψₙ is smooth and we can thus apply the Schrödinger-Lichnerowicz formula:

\[
\frac{1}{2} \cdot ||D_A(ψₙ) - λψₙ||_{L^2}^2 \leq ||D_A(ψₙ)||^2 + λ^2||ψₙ||^2
\]

\[
= ||∇^Aψₙ||_{L^2}^2 + \int_{M^n} \frac{R}{4} ||ψₙ||^2 + \frac{1}{2} \int_{M^n} (dA \cdot ψₙ, ψₙ) + λ^2||ψₙ||_{L^2}^2.
\]

Since \(M^n\) is compact and \(||ψₙ||_{L^2}^2 \equiv 1\) as well as \(||D_A(ψₙ) - λψₙ||_{L^2}^2 \to 0\), the \(L^2\)-lengths \(||∇^Aψₙ||_{L^2}^2\) of the 1-forms \(∇^Aψₙ\),

\[
||∇^Aψₙ||_{L^2}^2 = \int_{M^n} \sum_{i=1}^{n} (∇_{e_i}ψₙ, ∇_{e_i}ψₙ) dM^n,
\]

are bounded. Hence \(ψₙ\) is a bounded sequence in the Sobolev space \(H^1(S)\). By the Rellich lemma, the embedding \(H^1(S) \to L^2(S)\) is compact. Thus we can assume that \(ψₙ\) converges to a spinor field \(ψ_0\) in \(L^2\), and this immediately implies \(D_A(ψ₀) = λψ₀\). So we have proved that \(λ\) is an eigenvalue of \(D_A\), and hence \(σ_α(D_A) = σ_p(D_A)\).

We still suppose that \((M^n, g)\) is a compact Riemannian manifold with fixed spin\(^{C}\) structure. The first Sobolev norm of a smooth spinor field \(ψ \in Γ(S)\) is given by

\[
||ψ||_{H^1}^2 = ||ψ||_{L^2}^2 + ||∇^Aψ||_{L^2}^2
\]

and the corresponding Sobolev space \(H^1(S)\) is the completion of \(Γ(S)\) with respect to this norm. Since \(M^n\) is compact, different connections \(A\) induce equivalent norms in the determinant bundle of the spin\(^{C}\) structure. Moreover, the embedding

\[
H^1(S) \to L^2(S)
\]

is a compact operator (Rellich lemma). The Dirac operator \(D_A\) is a continuous operator \(D_A : H^1(S) \to L^2(S)\). This follows, e.g., from the estimate below for the norm of \(D_A(ψ)\):

\[
||D_A(ψ)||_{L^2}^2 = \sum_{i,j=1}^{n} \int_{M^n} (e_i∇^Aψ, e_j∇^Aψ) \leq \sum_{i,j=1}^{n} \int_{M^n} |∇^Aψ||∇^Aψ|
\]

\[
\leq \frac{1}{2} \sum_{i,j} \int_{M^n} (|∇_{e_i}ψ|^2 + |∇_{e_j}ψ|^2) = n||∇^Aψ||_{L^2}^2 \leq n||ψ||_{H^1}^2.
\]

Applying the Schrödinger-Lichnerowicz formula for \(D^2_A(ψ)\), we can, on the other hand, rewrite \(||D_A(ψ)||_{L^2}^2\) as follows:

\[
||D_A(ψ)||_{L^2}^2 = (D^2_A(ψ), ψ)_{L^2} = ||∇^Aψ||_{L^2}^2 + \int_{M^n} \frac{R}{4} ||ψ||^2 + \int_{M^n} \frac{1}{2} (dA \cdot ψ, ψ).
\]
4.2. The spectrum of Dirac operators over compact manifolds

The endomorphism $\frac{1}{2}dA : S \to S$ is bounded, i.e. there exists a constant $c > 0$ such that for every point of the manifold $M^n$ and any spinor $\psi$ we have the pointwise inequality

$$-c\|\psi\|^2 \leq \left| \left( \frac{1}{2}dA \cdot \psi, \psi \right) \right| \leq c\|\psi\|^2.$$

Inserting this, we obtain the inequality

(*) \quad \|\psi\|^2_{H^1} + \left( \frac{R_{\min}}{4} - c - 1 \right) \|\psi\|^2_{L^2} \leq \|D_A\psi\|^2_{L^2}

\leq \|\psi\|^2_{H^1} + \left( \frac{R_{\max}}{4} + c - 1 \right) \|\psi\|^2_{L^2},

where $R_{\max} = \max \{R(m) : m \in M^n\}$ is the maximum of the scalar curvature and $R_{\min}$ its minimum. We will use this inequality to prove the following:

**Proposition.** Let $D_A$ be a Dirac operator over a compact manifold. The closure $\overline{D}_A = D_A^*$ of the Dirac operator $D_A$ is defined on the subspace $H^1(S) \subset L^2(S)$.

**Proof.** If $\psi \in D(\overline{D}_A)$ belongs to the domain of $\overline{D}_A$, then there is a sequence $\psi_n \in \Gamma(S)$ such that $\psi_n \to \psi$ in $L^2(S)$ and $D_A(\psi_n)$ converges in $L^2$. From inequality (*) one immediately sees that $\psi_n$ is a Cauchy sequence in $H^1(S)$. This implies that $\psi_n$ converges to $\psi^*$ in $H^1(S)$. The embedding $H^1(S) \to L^2(S)$ is continuous, hence $\psi^*$ coincides with $\psi$. Thus $\psi$ belongs to $H^1(S)$. The converse is trivial.

**Proposition.** Let $\lambda \notin \sigma(\overline{D}_A)$ be a number which is not in the spectrum of the operator $\overline{D}_A$. Then

$$\overline{(D_A - \lambda)^{-1}} : L^2(S) \to L^2(S)$$

is a compact operator.

**Proof.** The inequality (*) can be written as

$$\|(D_A - \lambda)^{-1}(D_A - \lambda)\psi\|^2_{H^1} \leq \|(D_A - \lambda)\psi\|^2_{L^2} + \left( c + 1 + \lambda^2 - \frac{R_{\min}}{4} \right) \|\psi\|^2_{L^2}.$$

Setting $\varphi = (D_A - \lambda)\psi \in \text{im}(D_A - \lambda)$, we have

$$\|(D_A - \lambda)^{-1}\varphi\|^2_{H^1} \leq \|\varphi\|^2_{L^2} + C\|(D_A - \lambda)^{-1}\varphi\|^2_{L^2}.$$

However, the operator $(D_A - \lambda)^{-1}$ is continuous in $L^2$. Hence, for a suitable constant $C^*$, we obtain the estimate

$$\|(D_A - \lambda)^{-1}\varphi\|^2_{H^1} \leq C^*\|\varphi\|^2_{L^2}.$$
Thus the image of the operator \((D_A - \lambda)^{-1} (\lambda \not\in \sigma(\overline{D}_A))\) is contained in the Sobolev space \(H^1(S)\). The assertion then follows from the compactness of the embedding \(H^1(S) \to L^2(S)\).

**Proposition.** There exists a complete orthonormal basis \(\psi_1, \psi_2, \ldots\) of the Hilbert space \(L^2(S)\) consisting of eigenspinors of the Dirac operator \(D_A\),

\[
D_A(\psi_n) = \lambda_n \psi_n.
\]

Moreover, \(\lim_{n \to \infty} |\lambda_n| = \infty\).

**Proof.** Choose the real number \(\lambda \not\in \sigma(\overline{D}_A)\) in such a way that \(\lambda\) is not in the spectrum of \(\overline{D}_A\). The operator \((D_A - \lambda)^{-1} : L^2(S) \to L^2(S)\) is compact and self-adjoint. The spectral theory of these operators leads to a complete orthonormal basis \(\psi_1, \psi_2, \ldots\) in \(L^2(S)\) with

\[
(D_A - \lambda)^{-1} \psi_n = \mu_n \psi_n
\]

and \(\lim_{n \to \infty} \mu_n = 0\). Thus \((\mu_n \neq 0)\),

\[
D_A(\psi_n) = \left(\frac{1}{\mu_n} + \lambda\right) \psi_n,
\]

i.e. the spinor field \(\psi_n\) is an eigenspinor of the Dirac operator for the eigenvalue \(\lambda_n = \frac{1}{\mu_n} + \lambda\).

**Corollary.** There exists a constant \(C > 0\) such that for all spinor fields \(\varphi \in H^1(S)\) orthogonal to the kernel \(\ker(D_A)\) the inequality

\[
|\langle D_A(\varphi), \varphi \rangle|_{L^2} \geq C ||\varphi||_{L^2}^2
\]

holds.

Inequality (*) admits still another interpretation. Since \(\|\nabla^A \psi\|_{L^2}^2 \geq \frac{1}{n} ||D_A \psi||_{L^2}^2\), we have

\[
\frac{1}{n} \left\{ ||\psi||_{L^2}^2 + ||D_A \psi||_{L^2}^2 \right\} \leq ||\psi||_{H^1}^2 \leq ||D_A \psi||_{L^2}^2 + \left(c + 1 - \frac{R_{\min}}{4}\right) ||\psi||_{L^2}^2.
\]

Thus \(||\psi||_{H^1}^2\) and \(||\psi||_{L^2}^2 + ||D_A \psi||_{L^2}^2\) are equivalent norms. In other words, the Sobolev space \(H^1(S)\) can be defined as the completion of the space \(\Gamma(S)\) with respect to the norm

\[
||\psi||_{\star}^2 = ||\psi||_{L^2}^2 + ||D_A \psi||_{L^2}^2.
\]

The \(k\)-th Sobolev space \(H^k(S)\) is then the completion of \(\Gamma(S)\) with respect to the norm

\[
||\psi||_{H^k}^2 = \sum_{i=0}^k ||D_A^i(\psi)||_{L^2}^2.
\]
4.2. The spectrum of Dirac operators over compact manifolds

If \( \psi \in \Gamma(S) \) is a smooth spinor field and \( \psi = \sum_{n=1}^{\infty} A_n \psi_n \) its decomposition in \( L^2 \) with respect to the complete orthonormal system \( \psi_1, \psi_2, \ldots \) consisting of eigenspinors of \( D_A \), one easily computes

\[
\|D^k_A(\psi)\|_{L^2}^2 = \sum_{n=1}^{\infty} |A_n|^2 \lambda_n^{2k}.
\]

This implies the

**Proposition.** Let \( \psi_1, \psi_2, \ldots \) be the complete basis of the Hilbert space \( L^2(S) \) consisting of eigenspinors. If

\[
\psi = \sum_{n=1}^{\infty} A_n \psi_n
\]

is the \( L^2 \)-representation of the spinor field \( \psi \in L^2(S) \), then \( \psi \) belongs to the Sobolev space \( H^k(S) \) if and only if the sum

\[
\sum_{n=1}^{\infty} |A_n|^2 \lambda_n^{2k} < \infty
\]

is finite.

We will use the previous proposition to define the \( \zeta \)- and the \( \eta \)-function for the Dirac operator. First, we prove that the embedding \( H^k \to L^2 \) is a Hilbert-Schmidt operator for \( k > \frac{1}{2} \dim(M^n) \).

**Lemma.** Let \( E \) be a complex vector bundle with Hermitian metric and metric connection over a compact Riemannian manifold \( (M^n, g) \), and let \( H^k(E) \) denote the \( k \)-th Sobolev space. If \( k > \frac{1}{2} \dim(M^n) \), then the embedding

\[
H^k(E) \to L^2(E)
\]

is a Hilbert-Schmidt operator.

**Proof.** The Sobolev embedding theorem states that \( H^k(E) \) is contained in \( C^0(E) \) and that this embedding is continuous for \( k > \frac{1}{2} n \). Fix a point \( m_0 \in M^n \) and an orthonormal basis \( e_1(m_0), \ldots, e_l(m_0) \) in the fibre \( E_{m_0} \). Let \( \varphi : H^k(E) \to E_{m_0} \) be the linear mapping \( \varphi(s) = s(m_0) \). \( \varphi \) is continuous, and the norm of \( \varphi \) can be estimated as follows:

\[
\|\varphi\| = \sup_{s \in H^k} \frac{\|s(m_0)\|}{\|s\|_{H^k}} = \sup_{s \in H^k} \frac{\|s\|_{C^0}}{\|s\|_{H^k}} =: C.
\]

By Riesz' lemma there exist elements \( s_1, \ldots, s_l \in H^k \) with

\[
s(m_0) = \varphi(s) = (s, s_1)_{H^k} e_1(m_0) + \ldots + (s, s_l)_{H^k} e_l(m_0).
\]
The norm $\|\varphi\|^2$ is then computed to be

$$\|\varphi\|^2 = \sup_{s \in H^k} \frac{||s(m_0)||^2}{\|s\|^2_{H^k}} = \sup_{s \in H^k} \sum_{i=1}^{l} \frac{|(s, s_i)_{H^k}|^2}{\|s\|^2_{H^k}},$$

and hence, for each index $1 \leq i \leq l$,

$$\|\varphi\|^2 \geq \sup_{s \in H^k} \frac{|(s, s_i)_{H^k}|^2}{\|s\|^2_{H^k}} = \|s_i\|^2_{H^k}.$$

Now let $f_1, f_2, \ldots$ be an orthonormal basis in $H^k(E)$. Then,

$$\sum_{j=1}^{\infty} |f_j(m_0)|^2 = \sum_{j=1}^{\infty} |\varphi(f_j)|^2 = 0$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{l} |(f_j, s_i)_{H^k}|^2 = \sum_{i=1}^{l} \|s_i\|^2_{H^k} \leq l \|\varphi\|^2 \leq lC^2.$$

Integrating this inequality over the compact manifold $M^n$ yields

$$\sum_{j=1}^{\infty} ||f_j||_{L_2}^2 \leq l \cdot C^2 \cdot \text{vol}(M^n).$$

The spectrum $\sigma(D_A)$ of a Dirac operator only consists of eigenvalues, and the closure $\tilde{D}_A$ defined on $H^1(S)$ is self-adjoint. Let $F : B(\sigma) \to \text{Proj}(L^2(S))$ be the spectral measure. Then,

$$\tilde{D}_A = \int dF(\lambda).$$

For fixed $t > 0$, $e^{-t\lambda^2}$ is a bounded function. Hence

$$S_t = \int e^{-t\lambda^2} dF(\lambda)$$

is a bounded operator in $L^2(S)$.

**Proposition.**

i) The operators $S_t, t \geq 0$, form a semigroup, $S_{t_1+t_2} = S_{t_1}S_{t_2}$, of bounded operators with norm $\|S_t\| \leq 1$.

ii) The generator of this semigroup is $D_A^2$.

iii) For $t > 0$, $S_t : L^2(S) \to L^2(S)$ is a Hilbert-Schmidt operator.
Proof. Only the last point needs to be proved. First, note that for all \( \varphi \in L^2(S) \) the \( H^k \)-norm \( \| S_t(\varphi) \|_{H^k} \) is finite. Indeed, since

\[
\| \psi \|_{H^k}^2 = \sum_{i=0}^{k} \| D_A^i(\psi) \|_{H^k}^2,
\]

it is sufficient to prove that \( \| D_A^k(S_t(\varphi)) \|_{L^2}^2 \) is finite. The operator \( D_A^k \circ S_t \) is given by the integral

\[
\left( \int_{\sigma} \lambda^k dF(\lambda) \right) \cdot \left( \int_{\sigma} e^{-t\lambda^2} dF(\lambda) \right) = \int_{\sigma} \lambda^k e^{-t\lambda^2} dF(\lambda)
\]

and \( \lambda^k e^{-t\lambda^2} \) is a bounded function of \( \lambda \) \((t > 0)\). Thus \( D_A^k \circ S_t \) is continuous in \( L^2 \). The argument above now proves that the image of each operator \( S_t \) is contained in \( \Gamma(S) \),

\[
S_t(L^2(S)) \subset \bigcap_{k=0}^{\infty} H^k(S) = \Gamma(S).
\]

Moreover, as an operator from \( L^2 \) to \( H^k(S) \), \( S_t \) is continuous. Choosing \( k > \frac{1}{2} \dim M^n \), we obtain

\[
L^2(S) \overset{S_t}{\to} H^k(S) \to L^2(S),
\]

where \( H^k(S) \to L^2(S) \) is a Hilbert-Schmidt operator. But then \( S_t : L^2(S) \to L^2(S) \) is a Hilbert-Schmidt operator, too.

Corollary. The function \( \sum_{\lambda \in \sigma(D_A)} e^{-t\lambda^2} := \zeta_{D_A}^2(t) \) is finite for \( t > 0 \).

Proof. The Hilbert-Schmidt norm of the operator \( S_t : L^2(S) \to L^2(S) \) is

\[
\| S_t \|_{H-S}^2 = \sum_{\lambda \in \sigma(D_A)} \| S_t(\psi_\lambda) \|_{L^2}^2 = \sum_{\lambda \in \sigma(D_A)} e^{-2t\lambda^2} = \zeta_{D_A}^2(2t),
\]

where \( \psi_\lambda \) is a complete orthonormal basis in \( L^2(S) \) consisting of eigenspinors of \( D_A \).

By the same method we can define other spectral functions of the operator \( D_A \). Particularly important here is the so-called \( \eta \)-function. To define it, suppose that the kernel of \( D_A \) is trivial. Then zero has positive distance to the spectrum \( \sigma(D_A) \). Let \( z \in \mathbb{C} \) be a complex number with positive real part, \( \text{Re}(z) > 0 \). The function

\[
\text{sgn}(\lambda) \frac{1}{|\lambda|^z}
\]
is bounded on the set $\sigma(D_A)$. The corresponding operator

$$T(z) = \int_{\sigma} \text{sgn} \left( \frac{1}{|\lambda|^2} \right) dF(\lambda)$$

is bounded in $L^2(S)$. The superposition $D_A^k \circ T(z)$ is given by the function $\text{sgn}(\lambda) \frac{\lambda^k}{|\lambda|^z}$, which is bounded if $k \leq \text{Re}(z)$. Now assume $\text{Re}(z) > \frac{\dim(M)}{2}$.

Choose $k$ with $\text{Re}(z) > k > \frac{\dim(M)}{2}$. Then the operators $D_A \circ T(z), \ldots, D_A^k \circ T(z)$ are bounded in $L^2$. Hence $T(z)$ maps the space $L^2(S)$ into the space $H^k(S)$. The embedding $H^k(S) \to L^2(S)$ is a Hilbert-Schmidt operator. Eventually, we obtain the

**Proposition.** If $\text{Re}(z) > \frac{1}{2} \dim(M^n)$ and $\ker(D_A) = 0$, then the operator

$$T(z) = \int_{\sigma} \text{sgn}(\lambda) \frac{1}{|\lambda|^z} dF(\lambda)$$

is a Hilbert-Schmidt operator in $L^2(S)$.

The Hilbert-Schmidt norm is now computed by

$$||T(z)||^2_{HS} = \sum_{\lambda \in \sigma} |\lambda|^{-2 \text{Re}(z)}.$$

Define the so-called $\eta$-function of the Dirac operator $D_A$ as

$$\eta_{D_A}(z) = \sum_{0 \neq \lambda \in \sigma} \text{sgn}(\lambda)|\lambda|^{-z}.$$

Then we obtain the following result.

**Proposition.** Suppose that $\ker(D_A) = 0$. Then the function $\eta_{D_A}(z)$ is analytic in the half-plane $\text{Re}(z) > \dim(M^n)$.

**Remark.** $\eta_{D_A}(z)$ has a meromorphic continuation onto the complex plane and is, in particular, analytic at the point $z = 0$. The $\eta$-invariant of $D_A$ is then defined to be $\eta_{D_A}(0)$.

We also want to discuss the boundary case $z = \dim(M^n)$. Since

$$D_A^{n/2} \circ T^* = \int_{\sigma} \frac{\lambda^{n/2}}{|\lambda|^{n/2}} dF(\lambda) = \int (\text{sgn}(\lambda))^{n/2} dF(\lambda),$$

the operator

$$T^* = \int_{\sigma} \frac{1}{|\lambda|^{n/2}} dF(\lambda)$$

maps the space $L^2(S)$ into $H^{n/2}(S)$ and $D_A^{n/2} \circ T^* : L^2 \to H^{n/2} \to L^2$ is invertible, $(D_A^{n/2} \circ T^*)^2 = \text{Id}_{L^2}$. The kernel of $D_A$ is trivial and hence
4.3. Dirac operators are Fredholm operators

Proposition. The Dirac operator $D_A : H^1(S) \to L^2(S)$ over a compact Riemannian manifold is a Fredholm operator of index zero.

Proof. We have to prove that $\ker(D_A)$ and $L^2/\text{im}(D_A)$ are finite-dimensional vector spaces of equal dimension. In the vector space $\ker(D_A)$ consider the balls

$$K^1 = \{ \psi \in H^1(S) : D_A(\psi) = 0, \quad ||\psi||_{H^1}^2 = ||\psi||_{L^2}^2 + ||D_A\psi||_{L^2}^2 \leq 1 \},$$

$$K^0 = \{ \psi \in H^1(S) : D_A(\psi) = 0, \quad ||\psi||_{L^2} \leq 1 \}.$$

Obviously, $K^0 = K^1$. On the other hand, $H^1(S) \to L^2(S)$ is a compact operator and hence $K^0 = K^1 \subset L^2(S)$ is compact. Considering $\ker(D_A)$ now as a subspace of $L^2$, the balls are compact in the $L^2$-norm. Hence $\ker(D_A)$ is a finite-dimensional vector space. We determine the orthogonal complement of $D_A(H^1)$ in $L^2$. A spinor $\varphi$ from this space satisfies the condition

$$(D_A(\psi), \varphi)_{L^2} = 0$$

for all $\psi \in H^1(S)$. Similarly to the proof of the fact that the residual spectrum $\sigma_r(D_A)$ of $D_A$ in $L^2$ is empty, we conclude first the smoothness of $\varphi$ and then $D_A(\varphi) = 0$. Thus,

$$\left(D_A(H^1)\right)^\perp = \ker(D_A)$$

and, last, to complete the proof it remains to be shown that $D_A(H^1) \subset L^2$ is a closed subspace. Assume that the sequence $D_A(\psi_n)$ converges to $\psi \in L^2(S)$ in $L^2$. Without loss of generality we can suppose that $\psi_n$ is orthogonal to the kernel $\ker(D_A)$. But then,

$$||D_A(\psi_n)||_{L^2} \geq C||\psi_n||_{L^2}$$

(compare the corollary in Section 4.2) and $\psi_n$ is a Cauchy sequence in $L^2$. Inequality (*) implies that $\psi_n$ is a Cauchy sequence in $H^1(S)$ as well. Thus the limit $\lim_{n \to \infty} \psi_n = \psi^*$ exists in $H^1(S)$. The operator $D_A : H^1(S) \to L^2(S)$ is continuous, and

$$\psi = \lim_{n \to \infty} D_A(\psi_n) = D_A(\lim_{n \to \infty} \psi_n) = D_A(\psi^*)$$
for $\psi^* \in H^1(S)$. This means that $\psi$ belongs to the image $D_A(H^1)$. □

In the case of a manifold $M^{2k}$ of even dimension, $n = 2k$, consider the Dirac operators

$$D_A^\pm : \Gamma(S^\pm) \to \Gamma(S^\mp).$$

They are Fredholm operators

$$D_A^\pm : H^1(S^\pm) \to L^2(S^\mp).$$

The index of $D_A^+$ will be denoted by $\text{Index}(D_A^+)$. Since $D_A$ is a self-adjoint operator,

$$\text{Index}(D_A^+) = \dim \ker(D_A^+) - \dim \ker(D_A^-).$$

The index of $D_A^+$ depends on the characteristic classes of the manifold $M^{2k}$ as well as on the first Chern class of the determinant bundle $L$ of the spin$^C$ structure. We quote the corresponding formula without proof.

The power series of the even function

$$\frac{t}{\sinh(t/2)} = \frac{t}{e^{t/2} - e^{-t/2}}$$

can be represented in the form

$$\frac{t}{e^{t/2} - e^{-t/2}} = 1 + A_2 t^2 + A_4 t^4 + \cdots.$$

An easy calculation shows, e.g., that

$$A_2 = -\frac{1}{24}, \quad A_4 = \frac{7}{10 \cdot 24 \cdot 24} = \frac{7}{5760}.$$

Denote the Pontrjagin classes of a $4k$-dimensional compact manifold $M^{4k}$ by $p_1, p_2, \ldots, p_k$. The class $p_j$, $1 \leq j \leq k$, is an element of the $4j$-th cohomology group $H^{4j}(M^{4k})$. Introduce $k$ formal variables $x_1, \ldots, x_k$ and represent $p_1, \ldots, p_k$ as the elementary symmetric functions in the squares of these variables:

$$x_1^2 + \cdots + x_k^2 = p_1, \quad \ldots, \quad x_1^2 \cdots x_k^2 = p_k.$$

Then

$$\prod_{i=1}^k \frac{x_i/2}{\sinh(x_i/2)}$$

is a symmetric power series in the variables $x_1^2, \ldots, x_k^2$ and hence defines a polynomial in the Pontrjagin classes. Denote this cohomology class by $\hat{A}(M^{4k})$:

$$\hat{A}(M^{4k}) = \prod_{i=1}^k \frac{x_i/2}{\sinh(x_i/2)}.$$

For example, for $k = 1, 2$ we obtain the formulas

$$\hat{A}(M^4) = 1 - \frac{1}{24} p_1, \quad k = 1,$$
4.3. Dirac operators are Fredholm operators

\[ \hat{A}(M^8) = 1 - \frac{1}{24} p_1 + \frac{7}{5760} p_1^2 - \frac{1}{1740} p_2, \quad k = 2. \]

A manifold of dimension $4k + 2$ also has $k$ Pontrjagin classes, and we define $\hat{A}(M^{4k+2})$ by the same formulas. The index theorem for Dirac operators now reads as follows:

**Proposition.** Let $(M^{2k}, g)$ be a compact oriented Riemannian manifold with spin$^C$ structure, and let $c = c_1(L)$ denote the first Chern class of the determinant bundle of the spin$^C$ structure. The index of the Dirac operator $D^+_A$ associated with a connection $A$ in the $U(1)$-bundle of the spin$^C$ structure is equal to

\[ \text{Index} (D^+_A) = \int_{M^{2k}} e^{\frac{1}{2} c} \hat{A}(M^{2k}). \]

As a consequence of this one can prove that certain characteristic numbers are integers.

**Corollary.** Let $M^{2k}$ be an oriented compact smooth manifold and take $c \in H^2(M^{2k}; \mathbb{Z})$ to be a cohomology class whose $\mathbb{Z}_2$-reduction coincides with the second Stiefel-Whitney class of $M^{2k}$, $c \equiv w_2(M^{2k}) \mod 2$. Then

\[ \int_{M^{2k}} e^{\frac{1}{2} c} \hat{A}(M^{2k}) \]

is an integer.

As an example, we discuss the case of a 4-dimensional manifold $M^4$ in greater detail. The intersection form of $M^4$ is defined by the cup-product in $H^2$:

\[ H^2(M^4; \mathbb{Z}) \times H^2(M^4; \mathbb{Z}) \rightarrow \mathbb{Z}, \quad (\alpha, \beta) \rightarrow (\alpha \cup \beta)[M^4]. \]

Considering this quadratic form over the ring $\mathbb{Z}$ of integers as a real quadratic form, the resulting form has a signature $(p, q)$. By Poincaré duality we have $p + q = \dim H^2(M^4; \mathbb{R})$. The number $\sigma(M^4) = p - q$ is called the signature of the 4-dimensional manifold $M^4$. This signature is closely related to the Pontrjagin numbers and, by the Hirzebruch signature theorem,

\[ \sigma(M^4) = \frac{1}{3} \int_{M^4} p_1. \]

The $\hat{A}(M^4)$-class can thus be written as $\hat{A}(M^4) = 1 - \frac{1}{8} \sigma$, and the polynomial $e^{\frac{1}{2} c} \hat{A} = (1 + \frac{1}{2} c + \frac{1}{8} c^2)(1 - \frac{1}{8} \sigma)$ yields the formula

\[ \text{Index} (D^+_A) = \frac{1}{8} (c^2 - \sigma). \]

As an application of the formula just stated we prove the following result, originally due to Rokhlin.
Proposition. Let $M^4$ be a smooth compact orientable 4-dimensional manifold with spin structure, $w_2(M^4) = 0$. Then the signature $\sigma(M^4)$ is divisible by 16.

Proof. Since $w_2 = 0$, the manifold $M^4$ has a $Spin(4)$-structure. Consider the corresponding Dirac operator in the spinor bundle $S$. Then,

$$\text{Index}(D^+) = -\frac{1}{8}\sigma(M^4).$$

But $S$ is a vector bundle associated with the group $Spin(4)$. From the considerations in Section 1.7 we know that the spin representations $\Delta^\pm_4$ have equivariant quaternionic structures. These in turn induce parallel quaternionic structures in the spinor bundles $S^\pm$, and hence the complex vector spaces $\ker(D^+), \ker(D^-)$ also carry quaternionic structures. Thus,

$$\dim_{\mathbb{C}} \ker(D^\pm) \equiv 0 \mod 2,$$

which immediately implies $\text{Index}(D^+) \equiv 0 \mod 2$. This means that $\sigma(M^4) \equiv 0 \mod 16$. □

Remark. The condition $w_2(M^4) = 0$ is equivalent to the integral intersection form in $H^2(M^4; \mathbb{Z})$ being an even $\mathbb{Z}$-form,

$$\alpha^2 \equiv 0 \mod 2 \quad \forall \alpha \in H^2(M^4; \mathbb{Z}).$$

However, it is a well-known algebraic fact that the signature $\sigma$ of even forms over the ring $\mathbb{Z}$ is divisible by 8. The Rokhlin theorem thus states an additional divisibility of the signature by 2 in the case of smooth manifolds with even intersection form! The smoothness of $M^4$ is indeed necessary. There exist simply connected topological manifolds $M^4_{\text{top}}$ with $w_2(M^4_{\text{top}}) = 0$ and $\sigma(M^4_{\text{top}}) = 8$.

Remark. The parallel quaternionic structures in the spinor bundle $S^\pm$ used in the proof of the Rokhlin theorem exist in all dimensions $n = 8k + 4 \equiv 4 \mod 8$, if $S^\pm$ is associated with the group $Spin$ (trivial spinC structure). Thus we have e.g.: Let $M^{8k+4}$ be an orientable compact smooth spin manifold. Then,

$$\frac{1}{2} \int_{M^{8k+4}} \hat{A}$$

is an integer.

Apart from results concerning the integrality of special characteristic numbers, a further application of the index formula for Dirac operators consists in deriving topological obstructions for the existence of Riemannian metrics with positive scalar curvature. The Schrödinger-Lichnerowicz formula,

$$D^2_{\hat{A}} = \Delta_A + \frac{1}{4}R + \frac{1}{2}dA,$$
immediately implies \( \ker(D_A) = 0 \) and hence \( \text{Index}(D^+_A) = 0 \), if all eigenvalues of the self-adjoint endomorphism \( \frac{1}{4} R + \frac{1}{2} dA : S \to S \) are positive. Thus we have the

**Proposition.** Let \((M^{2k}, g)\) be a compact oriented Riemannian manifold with spin\(^C\) structure, and denote by \(c = c_1(L)\) the first Chern class of the determinant bundle \(L\) of the spin\(^C\) structure. If \(L\) has an Hermitian connection \(A\) such that all eigenvalues of the endomorphism

\[
\frac{1}{4} R + \frac{1}{2} dA = \frac{1}{4} R + \frac{1}{2} \Omega^A : S \to S
\]

are positive, then

\[
\int_{M^{2k}} e^{\frac{1}{2} c} \hat{A}(M^{2k}) = 0.
\]

**Corollary.** Let \(M^{4k}\) be a compact oriented spin manifold \((w_2(M^{4k}) = 0)\), and assume that

\[
\int_{M^{4k}} \hat{A} \neq 0.
\]

Then \(M^{4k}\) admits no Riemannian metric of positive scalar curvature.

**Example.** In the last corollary the assumption \(w_2(M^{4k}) = 0\) is necessary. The complex-projective plane \(M^4 = \mathbb{CP}^2\) with the Fubini-Study metric has a Riemannian metric of positive scalar curvature, and

\[
\hat{A}(\mathbb{CP}^2) = -\frac{1}{8} \sigma(\mathbb{CP}^2) = -\frac{1}{8} \neq 0.
\]

However, \(\mathbb{CP}^2\) is not a spin manifold.

4.4. References and Exercises


P.B. Gilkey. Invariance theory, the heat equation and the Atiyah-Singer index theorem, Publish or Perish 1984.


**Exercise 1.** Let \((M^4, g)\) be a compact Riemannian manifold with spin\(^C\) structure and \(D_A\) the Dirac operator. Consider the heat equation,
\[
\frac{\partial}{\partial t} \psi(t, m) = -D_A^2 \psi(t, m), \quad t \geq 0,
\]
with the Cauchy initial condition \(\psi(0, m) = \psi_0(m)\). Prove:

1) This Cauchy problem has at most one solution.

2) If \(S_t : L^2(S) \to L^2(S)\) is defined by
\[
S_t = \int_{\sigma} e^{-t\lambda^2} dF(\lambda),
\]
then \(\psi(t, m) = S_t(\psi_0(m))\) is the unique solution to the Cauchy problem.

**Exercise 2.** Let \(\psi_\lambda, \lambda \in \sigma(D_A)\), be a complete basis of \(L^2(S)\) consisting of eigenspinors of the Dirac operator. Prove that the section \(E(t, m_1, m_2)\) in the bundle \(S \times S\) over \(M^n \times M^n\) defined by
\[
E(t, m_1, m_2) = \sum_{\lambda \in \sigma} e^{-t\lambda^2} \psi_\lambda(m_1) \otimes \psi_\lambda(m_2)
\]
for \(t > 0\) is smooth. Show, in addition, the following properties:

1) \(\frac{\partial}{\partial t} E(t, m_1, m_2) = -D_A^2(E(t, m_1, m_2))\),

2) \(\psi(m) = \lim_{t \to 0} \int_{M^n} E(t, m_1, m_2)\psi(m_2)dm_2\) for \(\psi \in L^2(S)\).

**Exercise 3.** The operator \(S_t : L^2(S) \to L^2(S)\) is an integral operator with kernel \(E(t, m_1, m_2)\), i.e.
\[
(S_t \psi)(m) = \int_{M^n} E(t, m, m_2)\psi(m_2)dm_2.
\]

**Exercise 4.** Let \(0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \ldots\) be the eigenvalues of \(D_A^2\). Prove the following asymptotic formula for \(k \to \infty\):
\[
\lambda_k^2 \sim Ck^{2/n}.
\]

Hint: \(\sum_{i=1}^{\infty} \frac{1}{(\lambda_i^2)^z}\) converges for \(\text{Re}(z) > n/2\) and has a singularity at the point \(z = n/2\).

**Exercise 5.** Let \(D\) be the Dirac operator of a compact Riemannian spin manifold \((M^n, g)\). Prove that in the cases \(n \not\equiv 3, 7\) mod 8 the \(\eta\)-function of \(D\) vanishes identically.
Eigenvalue Estimates for the Dirac Operator and Twistor Spinors

5.1. Lower estimates for the eigenvalues of the Dirac operator

In this chapter we will consider a compact Riemannian manifold $(M^n, g)$ with fixed spin structure and its Dirac operator $D$ which, in this case, is exclusively determined by the Levi-Civita connection. By integration, from the Schrödinger-Lichnerowicz formula,

$$D^2 = \Delta + \frac{1}{4} R,$$

we immediately obtain the inequality $\lambda^2 \geq \frac{1}{4} R_0$ for every eigenvalue $\lambda$ of the Dirac operator, where $R_0 = \min\{R(m) : m \in M^n\}$ is the minimum of the scalar curvature. However, this estimate is not optimal. We have (Th. Friedrich, 1980)

**Proposition.** Let $(M^n, g)$ be a compact Riemannian manifold with spin structure, and $\lambda$ an eigenvalue of the Dirac operator $D$. Then,

$$\lambda^2 \geq \frac{1}{4 \frac{n}{n-1}} R_0.$$
Moreover, if \( \lambda = \pm \frac{1}{2} \sqrt{\frac{n}{n-1} R_0} \) is an eigenvalue of the Dirac operator and \( \psi \) a corresponding eigenspinor, then \( \psi \) is a solution of the field equation
\[
\nabla_X \psi = \pm \frac{1}{2} \sqrt{\frac{R_0}{n(n-1)}} X \cdot \psi
\]
and the scalar curvature \( R \) is constant.

**Proof.** The idea of the proof is based on not using the Levi-Civita connection but, instead, considering a suitably modified covariant derivative in the spinor bundle. To this end, fix a real-valued function \( f : M^n \to \mathbb{R}^1 \) and introduce the covariant derivative \( \nabla^f \) in the spinor bundle \( S \) by the formula
\[
\nabla^f_X \psi = \nabla_X \psi + f X \cdot \psi.
\]
The algebraic properties of Clifford multiplication imply that \( \nabla^f \) is a metric covariant derivative in the spinor bundle \( S \):
\[
X(\psi, \psi_1) = (\nabla^f_X \psi, \psi_1) + (\psi, \nabla^f_X \psi_1).
\]
Let \( \Delta^f = - \sum_{i=1}^n \nabla_{e_i}^f \nabla_{e_i}^f - \sum_{i=1}^n \text{div}(e_i) \nabla_{e_i}^f \) be the corresponding Laplace operator, and denote by
\[
|\nabla^f \psi|^2 = \sum_{i=1}^n |\nabla_{e_i}^f \psi|^2 = \sum_{i=1}^n |\nabla_{e_i} \psi + f e_i \cdot \psi|^2
\]
the length of the 1-form \( \nabla^f \psi \). We will compute the operator \( (D - f)^2 \). First,
\[
(D - f)^2 = (D - f)(D - f) = D^2 - 2fD - \text{grad}(f) + f^2,
\]
and the Schrödinger-Lichnerowicz formula implies
\[
(D - f)^2 = \Delta + \frac{1}{4} R - 2fD - \text{grad}(f) + f^2.
\]
On the other hand,
\[
\Delta^f = - \sum_{i=1}^n (\nabla_{e_i} + f e_i)(\nabla_{e_i} + f e_i) - \sum_{i=1}^n \text{div}(e_i)(\nabla_{e_i} + f e_i)
\]
\[
= \Delta - 2fD - \text{grad}(f) + nf^2.
\]
Summing up, this yields
\[
(D - f)^2 = \Delta^f + \frac{1}{4} R + (1 - n)f^2,
\]
and, by integration over \( M^n \), we obtain the formula
\[
\int_{M^n} ((D - f)^2 \psi, \psi) = \int_{M^n} \left\{ |\nabla^f \psi|^2 + \frac{1}{4} R|\psi|^2 + (1 - n)f^2|\psi|^2 \right\}.
\]
Suppose now that $D\psi = \lambda \psi$. Then we can insert the function $f = \frac{\lambda}{n}$ into the last formula and obtain
\[
\lambda^2 \left( \frac{n-1}{n} \right)^2 \|\psi\|_{L^2}^2 = \|\nabla_n^0 \psi\|_{L^2}^2 + \lambda^2 \frac{1-n}{n^2} \|\psi\|_{L^2}^2 + \frac{1}{4} \int_{M^n} R|\psi|^2.
\]
An algebraic transformation yields
\[
\lambda^2 \frac{n-1}{n} \|\psi\|_{L^2}^2 = \|\nabla_n^0 \psi\|_{L^2}^2 + \frac{1}{4} \int_{M^n} R|\psi|^2 \geq \frac{1}{4} R_0 \|\psi\|_{L^2}^2,
\]
i.e. $\lambda^2 \geq \frac{1}{4} \frac{n}{n-1} R_0$. Discussing the boundary case in this estimate, we immediately obtain the remaining assertions of the proposition. \(\square\)

The method of proof applied here may be refined in various ways. Consider, for example, for a fixed smooth real-valued function $f : M^n \to \mathbb{R}^1$ the (non-metric) covariant derivative
\[
\nabla_X \psi = \nabla_X \psi + \frac{\lambda}{n} X \cdot \psi + \mu X \cdot \text{grad}(f) \cdot \psi + \nu d f(X) \psi
\]
with the "optimal" parameters $\mu = -\frac{1}{n-1}$, $\nu = -\frac{n}{n-1}$, and perform a calculation with the length $\|e^{\mu f} \nabla \psi\|_{L^2}$ similar to the one in the proof above. Then one obtains the inequality (O. Hijazi, 1986)

**Proposition.**
\[
\lambda^2 \geq \frac{n}{n-1} \min \left\{ \frac{1}{4} R + \Delta(f) - \frac{n-2}{n-1} |\text{grad } f|^2 \right\}.
\]
In particular, in dimension 2 the summand $|\text{grad } f|^2$ drops out. Then the formula simplifies to
\[
\lambda^2 \geq \min \left\{ \frac{1}{2} R + 2 \Delta(f) \right\}.
\]
The Gauß curvature $K$ of the Riemann surface $(M^2, g)$ is equal to $K = \frac{1}{2} R$, and we can choose $f$ as a solution to the differential equation
\[
2\Delta(f) = -K + \frac{1}{\text{vol}(M^2, g)} \int_{M^2} K = -K + \frac{2\pi \chi(M^2)}{\text{vol}(M^2)}.
\]
Thus $\frac{1}{2} R + 2\Delta(f) = \frac{2\pi \chi(M^2)}{\text{vol}(M^2)}$ is constant, and we obtain
\[
\lambda^2 \geq \frac{2\pi \chi(M^2)}{\text{vol}(M^2)}.
\]
Of course, the last inequality is interesting only for 2-dimensional Riemannian manifolds which, topologically, are spheres. Summarizing, we obtain the following proposition, originally due to Lott, Hijazi, and Bär.

**Proposition.** If \((S^2, g)\) is a Riemannian metric on \(S^2\), then, for the first eigenvalue of the Dirac operator, we have

\[
\lambda^2 \geq \frac{4\pi}{\text{vol}(S^2, g)}.
\]

The method we have outlined for estimating the eigenvalues of the Dirac operator may be refined even further when the Riemannian manifold carries additional geometric structures. Let us consider e.g. the case of a Kähler manifold \((M^{2k}, J, g)\) with complex structure \(J: T(M^{2k}) \to T(M^{2k})\). In this situation, consider the covariant derivative

\[
\tilde{\nabla}_X \psi = \nabla_X \psi + f X \cdot \psi + hJ(X) \cdot \psi
\]

depending on two parameters \(f\) and \(h\) which can be chosen freely. Elaborating on the Weitzenböck formulas for Riemannian manifolds with additional geometric structures, one will in general obtain better estimates than in the general case of a Riemannian manifold. For example, the following inequality, first proved by K.-D. Kirchberg, holds for Kähler manifolds:

**Proposition.** Let \((M^{2k}, J, g)\) be a compact Kähler spin manifold and \(\lambda\) an eigenvalue of the Dirac operator. Then,

\[
\frac{1}{4} \frac{k+1}{k} R_0 \text{ if } k = \dim_{\mathbb{C}} M \text{ is odd,}
\]

\[
\frac{1}{4} \frac{k}{k-1} R_0 \text{ if } k = \dim_{\mathbb{C}} M \text{ is even.}
\]

**Remark.** The quaternionic Kähler case has been investigated by Kramer, Semmelmann, and Weingart in 1997/98.

### 5.2. Riemannian manifolds with Killing spinors

By the proposition proved in Section 5.1, a spinor field \(\psi\) which is an eigen-spinor for the eigenvalue \(\pm \frac{1}{2} \sqrt{\frac{n}{n-1} R_0}\) solves the stronger field equation

\[
\nabla_X \psi = \pm \frac{1}{2} \sqrt{\frac{R_0}{n(n-1)}} X \cdot \psi.
\]

This leads to the general notion of Killing spinors.

**Definition.** A spinor field \(\psi\) defined on a Riemannian spin manifold \((M^n, g)\) is called a Killing spinor, if there exists a complex number \(\mu\) such that

\[
\nabla_X \psi = \mu X \cdot \psi
\]

for all vectors \(X \in T\). \(\mu\) itself is called the Killing number of \(\psi\).
We begin by listing a few elementary properties of Killing spinors.

**Proposition.** Let \((M^n, g)\) be a connected Riemannian manifold.

1) A not identically vanishing Killing spinor has no zeroes.

2) Every Killing spinor \(\psi\) belongs to the kernel of the twistor operator \(T\). Moreover, \(\psi\) is an eigenspinor of the Dirac operator, \(D(\psi) = -n\mu \psi\).

3) If \(\psi\) is a Killing spinor corresponding to a real Killing number \(\mu \in \mathbb{R}\), then the vector field

\[
V^\psi = \sum_{i=1}^{n} (e_i \cdot \psi, \psi)e_i
\]

is a Killing vector field of the Riemannian manifold \((M^n, g)\).

**Proof.** A Killing spinor restricted to the curve \(\gamma(t), \psi(t) = \psi(\gamma(t))\), satisfies the following first order ordinary differential equation along this curve:

\[
\frac{d}{dt}\psi(t) = \mu \gamma'(t) \cdot \psi(t).
\]

Now \(\psi(0) = 0\) immediately implies \(\psi(\gamma(t)) \equiv 0\), and this in turn yields property (1). Starting from \(\nabla_X \psi = \mu X \cdot \psi\), we compute

\[
D\psi = \sum_{i=1}^{n} e_i \nabla_{e_i} \psi = \mu \sum_{i=1}^{n} e_i \cdot e_i \cdot \psi = -n\mu \psi
\]

and thus obtain

\[
T(\psi) = \sum_{i=1}^{n} e_i \otimes \left( \nabla_{e_i} \psi + \frac{1}{n} e_i \cdot D\psi \right) = \sum_{i=1}^{n} e_i \otimes (\mu e_i \psi - \mu e_i \cdot \psi) = 0.
\]

For a fixed point \(m_0 \in M^n\) and a local orthonormal frame \(e_1, \ldots, e_n\) with \(\nabla e_i(m_0) = 0\) we compute the covariant derivative \(\nabla_X V^\psi:\)

\[
\nabla_X V^\psi = \sum_{i=1}^{n} (e_i \cdot \nabla_X \psi, \psi)e_i + \sum_{i=1}^{n} (e_i \cdot \psi, \nabla_X \psi)e_i
\]

\[
= \mu \sum_{i=1}^{n} (e_i \cdot X \cdot \psi, \psi)e_i + \sum_{i=1}^{n} \mu (e_i \psi, X \cdot \psi)e_i
\]

\[
= \mu \sum_{i=1}^{n} ((e_i \cdot X - X e_i) \cdot \psi, \psi)e_i.
\]

This implies \(g(\nabla_X V^\psi, Y) = \mu((YX - XY) \cdot \psi, \psi)\); hence \(g(\nabla_X V^\psi, Y)\) is antisymmetric in \(X, Y\). But this property characterizes Killing vector fields on a Riemannian manifold. \(\square\)
Not every Riemannian manifold allows Killing spinors $\psi \neq 0$, and not every number $\mu \in \mathbb{C}$ occurs as a Killing number. We now derive a series of necessary conditions. To this end, recall the Weyl tensor of a Riemannian manifold. Let

$$R_{ijkl} = g(\nabla_{e_i} \nabla_{e_j} e_k - \nabla_{e_j} \nabla_{e_i} e_k - \nabla_{[e_i,e_j]} e_k, e_l)$$

be the components of the curvature tensor and

$$R_{ij} = \sum_{\alpha=1}^{n} R_{ij}\alpha$$

those of the Ricci tensor. Then define two new tensors $K$ and $W$ by

$$K_{ij} = \frac{1}{n-2} \left\{ \frac{R}{2(n-1)} g_{ij} - R_{ij} \right\},$$

$$W_{\alpha \beta \gamma \delta} = R_{\alpha \beta \gamma \delta} - g_{\beta \delta} K_{\alpha \gamma} - g_{\alpha \gamma} K_{\beta \delta} + g_{\beta \gamma} K_{\alpha \delta} + g_{\alpha \delta} K_{\beta \gamma}.$$ 

$W$ is called the Weyl tensor of the Riemannian manifold. Because of its symmetry properties the Weyl tensor can be considered as a bundle morphism defined on the 2-forms of $(M^n, g)$:

$$W : \Lambda^2(M^n) \to \Lambda^2(M^n), \quad W(e_i \wedge e_j) = \sum_{k<l} W_{ijkl} e_k \wedge e_l.$$ 

With these notations we have the following

**Proposition.** Let $(M^n, g)$ be a connected Riemannian spin manifold with a non-trivial Killing spinor $\psi$ for the Killing number $\mu$. Then:

1) $\mu^2 = \frac{1}{4n(n-1)} R$ at each point. In particular, the scalar curvature of $(M^n, g)$ is constant and $\mu$ is either real or purely imaginary.

2) $(M^n, g)$ is an Einstein space.

3) $W(w^2) \cdot \psi = 0$ for every 2-form $w^2 \in \Lambda^2(M^n)$.

**Proof.** $\nabla_X \psi = \mu X \cdot \psi$ implies $\nabla_X \nabla_Y \psi = \mu(\nabla_X Y) \cdot \psi + \mu^2 Y \cdot X \cdot \psi$ and hence

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} )\psi = \mu^2 (Y \cdot X - X \cdot Y) \psi.$$ 

Computing $\sum_{\alpha=1}^{n} e_{\alpha} \cdot R(X, e_{\alpha}) \cdot \psi$ now yields

$$\sum_{\alpha=1}^{n} e_{\alpha} \cdot R(X, e_{\alpha})\psi = \mu^2 \sum_{\alpha=1}^{n} e_{\alpha}(e_{\alpha} X - X e_{\alpha}) \cdot \psi = 2(1-n)\mu^2 \psi.$$ 

On the other hand, in Section 3.1 we proved the formula

$$\sum_{\alpha=1}^{n} e_{\alpha} \cdot R(X, e_{\alpha})\psi = -\frac{1}{2} \text{Ric}(X) \cdot \psi.$$ 

Hence, \( \text{Ric}(X) \cdot \psi = 4(n - 1)\mu^2 X \cdot \psi \), and, since \( \psi \) does not vanish at any point, this implies
\[
\text{Ric}(X) = 4(n - 1)\mu^2 X.
\]
Thus \( (M^n, g) \) is an Einstein space of scalar curvature \( R = 4n(n - 1)\mu^2 \). The curvature tensor \( R(X, Y) \) in the spinor bundle \( S \) is related to the curvature tensor \( R(X, Y)Z \) of the Riemannian manifold \( (M^n, g) \) via the formula
\[
R(X, Y)\psi = \frac{1}{4} \sum_{\alpha=1}^{n} e_\alpha \cdot R(X, Y)e_\alpha \cdot \psi.
\]
Hence, because \( 4\mu^2 = \frac{R}{n(n-1)} \), the equation
\[
\nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X,Y]} \psi = \mu^2 (YX - XY) \cdot \psi
\]
can also be written as
\[
\left\{ \sum_{\alpha=1}^{n} e_\alpha \cdot R(X, Y)e_\alpha + \frac{R}{n(n-1)} (X \cdot Y - Y \cdot X) \right\} \psi = 0
\]
and, for an Einstein space,
\[
\sum_{\alpha=1}^{n} e_\alpha \wedge R(X, Y)e_\alpha + \frac{R}{n(n-1)} (X \wedge Y - Y \wedge X)
\]
coincides with \( W(X \wedge Y) \). This, eventually, implies
\[
W(w^2) \cdot \psi = 0.
\]
From the proof of the preceding proposition we can also deduce the following geometrical property of manifolds with Killing spinors:

**Proposition.** A Riemannian spin manifold admitting a Killing spinor \( \psi \neq 0 \) with Killing number \( \mu \neq 0 \) is locally irreducible.

**Proof.** If \( M^n \) is locally the Riemannian product \( M^n = M^k_1 \times M^{n-k}_2 \), then we may consider vectors \( X, Y \) tangent to \( M^k_1 \) and \( M^{n-k}_2 \), respectively. This implies \( R(X, Y)Z = 0 \), and from
\[
\left\{ \sum_{\alpha=1}^{n} e_\alpha R(X, Y)e_\alpha + \frac{R}{n(n-1)} (XY - YX) \right\} \psi = 0
\]
we obtain
\[
R \cdot X \cdot Y \cdot \psi = 0.
\]
Since \( \mu \neq 0 \), the scalar curvature is different from zero. Moreover, \( X \) and \( Y \) are orthogonal vectors. But this implies \( \psi = 0 \), hence a contradiction. \( \square \)
The next-to-last proposition shows that Killing spinors are divided into two types, depending on whether the Killing number $\mu$ is real or imaginary ($\mu \neq 0$):

<table>
<thead>
<tr>
<th>Type</th>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>real Killing spinors</td>
<td>$\mu \in \mathbb{R}$</td>
<td>$M^n$ is an Einstein space of pos. scalar curvature $R &gt; 0$</td>
</tr>
<tr>
<td>imaginary Killing spinors</td>
<td>$\mu \in i \cdot \mathbb{R}$</td>
<td>$M^n$ is an Einstein space of neg. scalar curvature $R &lt; 0$</td>
</tr>
</tbody>
</table>

Since $R = 4n(n - 1)\mu^2$, real Killing spinors precisely correspond to eigen-spinors of the Dirac operator for the eigenvalue $\pm \frac{1}{2} \sqrt{\frac{n}{n-1}} R$. The field equation $\nabla \psi = \mu X \cdot \psi$ could be generalized by allowing $\mu : M^n \to \mathbb{C}$ to be a complex-valued function. However, due to a result by A. Lichnerowicz, this does not lead to an actual generalization:

**Proposition.** Let $(M^n, g)$ be a connected spin manifold, $\mu : M^n \to \mathbb{C}$ a smooth function and $\psi$ a non-trivial solution of the equation

$$\nabla_X \psi = \mu X \cdot \psi.$$  

If the real part $\text{Re}(\mu) \neq 0$ is not identically zero, then $\mu$ is constant and real. Hence $\psi$ is a real Killing spinor.

In low dimensions $n = \dim (M^n)$, the geometrical conditions for the existence of real or imaginary Killing spinors, respectively, are rather restrictive, and for $n \leq 4$ only Riemannian spaces of constant sectional curvature admit this kind of spinor fields. Consider e.g. the case $n = 3$. Then $(M^3, g)$ is necessarily a 3-dimensional Einstein space, hence a space form. We meet the same situation in dimension $n = 4$.

**Proposition.** Let $(M^4, g)$ be a connected Riemannian spin manifold with a non-trivial Killing spinor $\psi$ for the Killing number $\mu \neq 0$. Then $(M^4, g)$ is a space of constant sectional curvature.

**Proof.** Decompose the Killing spinor $\psi = \psi^+ + \psi^-$ according to the splitting of the spinor bundle, $S = S^+ \oplus S^-$. The equation for the Killing spinor then takes the form

$$\nabla_X \psi^+ = \mu X \psi^-, \quad \nabla_X \psi^- = \mu X \psi^+.$$  

Define the set

$$N = \{ m \in M^4 : \psi^+(m) = 0 \text{ or } \psi^-(m) = 0 \}.$$
$N \subset M^4$ is a closed subset without inner points. Indeed, if $N$ has inner points, then there is an open subset $U \subset N \subset M^n$ on which, e.g. $\psi^+$ vanishes, $\psi^+_U = 0$. This implies $\nabla \psi^+_U \equiv 0$ and, since $\mu \neq 0$, from the Killing equation we obtain $\psi^-_U \equiv 0$. Hence $\psi = \psi^+ + \psi^-$ vanishes identically on the subset $U$, a contradiction to the fact proved above that non-trivial Killing spinors have no zeroes. Thus $U := M^4 \setminus N$ is a dense open subset of $M^4$.

The condition on the Weyl tensor $W$ of $M^4$ now takes the form

$$W(w^2)\psi^+ = 0 \text{ and } W(w^2)\psi^- = 0.$$ 

However, a simple algebraic computation using the realization of the $C_4$-module $\Delta_4 = \Delta_4^+ \oplus \Delta_4^-$ explicitly given in Sections 1.3 and 1.5 proves the following fact.

If $\eta^2 \in \Lambda^2(\mathbb{R}^4)$ is a 2-form and $\psi^+ \in \Delta_4^+, \psi^- \in \Delta_4^-$ are two non-trivial spinors, then $\eta^2 \cdot \psi^+ = 0 = \eta^2 \cdot \psi^-$ implies that the 2-form $\eta^2$ is trivial, $\eta^2 = 0$.

Applying this, we immediately conclude that the Weyl tensor $W$ vanishes on the set $M^4 \setminus N$. However, this set is dense. Hence $(M^4, g)$ is a 4-dimensional Einstein space with vanishing Weyl tensor, i.e. a space of constant sectional curvature.

In view of this proposition, the search for necessary and sufficient conditions on a Riemannian space to have (real or imaginary) Killing spinors is interesting only in dimensions $n \geq 5$. There are extensive series of examples and studies, for which we refer to the book [BFGK] and the supplementary article [Bär] using the holonomy theory. In the case of real Killing spinors (e.g. in the important dimension $n = 7$) this classification problem has not been finally settled yet (compare the paper [FKMS]).

5.3. The twistor equation

A spinor field $\psi$ belongs to the kernel of the twistor operator $T$ if and only if

$$\nabla_X \psi + \frac{1}{n} X \cdot D(\psi) = 0$$

for all vectors $X \in T(M^n)$. Here $D(\psi)$ denotes the application of the Dirac operator to $\psi$ (compare Section 3.2). Solutions $\psi$ of this field equation are called twistor spinors. Killing spinors are special solutions of this twistor equation. The essential difference between the field equation for Killing spinors and the twistor equation consists in the fact that the latter is conformally invariant. By a straightforward and elementary calculation one easily proves the following
Proposition. Let \( g \) and \( g^* \) be two conformally equivalent Riemannian metrics on a manifold \( M^n \). Then there exists an (explicit) isomorphism \( \ker(T) \approx \ker(T^*) \) between the kernels of the twistor operators \( T \) and \( T^* \).

Of course, the field equation for Killing spinors does not show this conformal invariance, because the (not conformally invariant) Einstein condition is necessary for the existence of Killing spinors. On the other hand, if \( (M^n, g) \) is a Riemannian spin manifold with Killing spinors and \( g^* \) a Riemannian metric conformally equivalent to \( g \), then, in general, \( (M^n, g^*) \) carries no Killing spinors although it admits twistor spinors. In the compact case, this is the only difference between these two equations.

Proposition. Let \( (M^n, g) \) be a compact connected Riemannian spin manifold with \( \ker(T) \neq 0 \). Then there exists an Einstein metric \( g^* \) conformally equivalent to \( g \) such that the space

\[
\ker(T) \approx \ker(T^*)
\]

coincides with the space of real Killing spinors on \( (M^n, g^*) \).

Proof. To begin with, the solution to the Yamabe problem for compact Riemannian manifolds implies that the metric \( g \) can be replaced by a conformally equivalent metric \( g^* \) of constant scalar curvature. Hence, without loss of generality, we can assume that \( g \) itself has constant scalar curvature and that \( \ker(T) \neq 0 \). We have to prove that, in this case, every solution of the twistor equation is representable as the sum of real Killing spinors. The equation

\[
\nabla_X \psi + \frac{1}{n} X D(\psi) = 0
\]

implies

\[
0 = \sum_{\alpha=1}^{n} \nabla_{e_{\alpha}} \nabla_{e_{\alpha}} \psi + \frac{1}{n} \sum_{\alpha=1}^{n} \nabla_{e_{\alpha}} (e_{\alpha} \cdot D(\psi)) = -\Delta(\psi) + \frac{1}{n} D^2(\psi).
\]

Applying the Schrödinger-Lichnerowicz formula, \( D^2 = \Delta + \frac{1}{4} R \), we obtain

\[
D^2(\psi) = \frac{1}{4} \frac{n}{n-1} R \psi.
\]

If the (constant) scalar curvature vanishes, \( R = 0 \), then \( D^2(\psi) = 0 \) and, since

\[
0 = \int_{M^n} (D^2(\psi), \psi) = \int_{M^n} |\nabla \psi|^2,
\]

\( \psi \) is a parallel section. In case \( R \neq 0 \), decompose \( \psi \) into

\[
\psi = \sqrt{\frac{n-1}{nR}} (\varphi_+ + \varphi_-)
\]
with \( \varphi_{\pm} = \frac{1}{2} \sqrt{\frac{nR}{n-1}} \psi \pm D\psi \). The scalar curvature is positive for \( R \neq 0 \), since all eigenvalues of \( D^2 \) are positive. Moreover,

\[
D(\varphi_{\pm}) = \frac{1}{2} \sqrt{\frac{nR}{n-1}} D(\psi) \pm D^2(\psi) = \pm \frac{1}{2} \sqrt{\frac{nR}{n-1}} \varphi_{\pm}.
\]

The spinor fields \( \varphi_{\pm} \) are thus eigenspinors for the smallest possible eigenvalue \( \pm \frac{1}{2} \sqrt{\frac{nR}{n-1}} \). From the proposition proved in Section 5.1 we conclude that \( \varphi_{\pm} \) are Killing spinors.

Now we want to prove a local version of the preceding proposition. To this end, we need the

**Lemma.** Let \( \psi \) be a twistor spinor. Then,

\[
\nabla_X(D(\psi)) = \frac{n}{2(n-2)} \left( \frac{R}{2(n-1)} X - \text{Ric}(X) \right) \cdot \psi.
\]

**Proof.** Differentiating once more, we derive from \( \nabla_X\psi + \frac{1}{n} X \cdot D(\psi) = 0 \) the equations

\[
\nabla_{e_\alpha} \nabla_X\psi + \frac{1}{n} (\nabla_{e_\alpha} X) \cdot D(\psi) + \frac{1}{n} X \cdot \nabla_{e_\alpha}(D(\psi)) = 0.
\]

\[
\nabla_X \nabla_{e_\alpha} + \frac{1}{n} (\nabla_X e_\alpha) \cdot D(\psi) + \frac{1}{n} e_\alpha \cdot \nabla_X(D(\psi)) = 0.
\]

These imply

\[
R(X, e_\alpha)\psi + \frac{1}{n} e_\alpha \cdot \nabla_X(D(\psi)) - \frac{1}{n} X \cdot \nabla_{e_\alpha}(D(\psi)) = 0.
\]

Multiplying by \( e_\alpha \) and adding up yields

\[
\sum_{\alpha=1}^{n} e_\alpha \cdot R(X, e_\alpha)\psi = \nabla_X(D(\psi)) - \frac{1}{n} X \cdot D^2(\psi) - \frac{2}{n} \nabla_X(D(\psi)).
\]

Inserting \( D^2(\psi) = \frac{1}{4} \frac{Rn}{n-1} \psi \) and \( \sum_{\alpha=1}^{n} e_\alpha \cdot R(X, e_\alpha) \cdot \psi = -\frac{1}{2} \text{Ric}(X) \cdot \psi \), we immediately arrive at the asserted formula.

A consequence of the formula is that the equality

\[
\text{Re}(\nabla_X(D\psi), \psi) = 0
\]

holds for every twistor spinor \( \psi \). Hence we can define the following first integrals:
Proposition. Let \( \psi \) be a twistor spinor defined on a connected Riemannian manifold. Then the functions

\[
C(\psi) = \text{Re}(\psi, D\psi),
\]

\[
Q(\psi) = |\psi|^2|D\psi|^2 - C^2(\psi) - \sum_{\alpha=1}^{n} [\text{Re}(D\psi, e_\alpha \cdot \psi)]^2
\]

are constant.

Proof. Differentiating \( C(\psi) \) yields

\[
X(C(\psi)) = \text{Re}(\nabla_X \psi, D\psi) + \text{Re}(\psi, \nabla_X (D\psi))
\]

\[
= \text{Re} \left( -\frac{1}{n} X \cdot D\psi, D\psi \right) + 0 = 0.
\]

An analogous, if a little longer, computation using the formula of the previous lemma also shows that

\[
X(Q(\psi)) = 0.
\]

Remark. The first integral \( C(\psi) \) was introduced by A. Lichnerowicz (1987), the invariant \( Q(\psi) \) by Th. Friedrich (1989).

These first integrals of a solution \( \psi \) to the twistor equation occur in the following local result which shows that, locally outside its zero set, a given twistor spinor \( \psi \) can always be transformed conformally into the sum of two real Killing spinors. The proof relies on using the conformal weight of the twistor operator and can be found in [F2] (compare also [BFGK]).

Proposition. Let \((M^n, g)\) be a Riemannian spin manifold with non-trivial twistor spinor \( \psi \), and denote by \( N = \{ m \in M^n : \psi(m) = 0 \} \) its zero set. Then \( N \) only consists of isolated points, and with the Riemannian metric \( g^* = \frac{1}{|\psi|^4} g \) defined over \( M^n \setminus N \) the latter becomes an Einstein space with scalar curvature

\[
R^* = \frac{4(n-1)}{n} (C^2(\psi) + Q(\psi)).
\]

With respect to the metric \( g^* \), the spinor field \( \frac{1}{|\psi(m)|} \psi(m) \) is the sum of two real Killing spinors. \( \square \)

Finally, we note that many examples of Riemannian manifolds with solutions \( \psi \) of the twistor equation admitting zeroes are known (see Kühnel and Rademacher). The classification of Riemannian manifolds with these solutions of the twistor equation is not yet fully understood.
5.4. Upper estimates for the eigenvalues of the Dirac operator

Estimates from above for the eigenvalues of the Dirac operator depending upon the geometry of the base space can be attained in various ways. By constructing suitable test spinors and considering the corresponding Rayleigh quotients one obtains bounds depending on the injectivity radius and curvature data of the Riemannian manifold (compare [Bal]). A further method, based on a comparison with the sphere, was suggested by Vafa and Witten in 1984 and geometrically elaborated by H. Baum (compare [Ba]). In this section, we will describe the results obtained along these lines, but refer to the original papers for the details. The starting point of this method is the observation that the spinor bundle $S_0$ of the sphere $S^{2m} \subset \mathbb{R}^{2m+1}$ is trivial. Therefore, one can choose a global trivialization of this bundle over $S^{2m}$ by spinors $\psi_1, \ldots, \psi_{2m}$ whose covariant derivatives $\nabla \psi_i$ $(1 \leq i \leq 2m)$ are known (take, e.g. real Killing spinors on $S^{2m}$). Then the space $\Gamma(S_0)$ of all spinors can be identified with the space $C^\infty(S^{2m}; \Lambda_{2m}) = C^\infty(S^{2m}; \mathbb{C}^{2m})$ of $\mathbb{C}^{2m}$-valued functions. For the covariant derivative in the spinor bundle the formula

$$\nabla_X(u) = X(u) + \frac{1}{2}(-1)^miX \cdot (u^+ - u^-)$$

holds, where $u = u^+ + u^-$ is the decomposition of $u \in C^\infty(S^{2m}; \Lambda_{2m})$ according to the splitting $\Lambda_{2m} = \Lambda^+_{2m} \oplus \Lambda^-_{2m}$.

Now let $f : M^{2m} \to S^{2m}$ be a smooth mapping from the compact connected Riemannian spin manifold $M^{2m}$ into the sphere $S^{2m}$. If $S$ is the spinor bundle on $M^{2m}$ and $f^*(S_0)$ its pull-back by the mapping $f$, then there are two operators of Dirac-type in the bundle $S \otimes f^*(S_0)$. The Levi-Civita connection of the sphere $S^{2m}$ defines a connection $\nabla^S_0$ in the induced bundle $f^*(S_0)$, and hence

$$D_0 = \sum_{i=1}^{2m} e_i \otimes (\nabla_{e_i} \otimes 1 + 1 \otimes \nabla^S_{e_i})$$

becomes a Dirac operator in $S \otimes f^*(S_0)$. On the other hand, $S_0$, and thus $f^*(S_0)$ as well, has a flat connection $\nabla^0$ defined by requiring the constant functions $u \in C^\infty(S^{2m}; \Lambda_{2m})$ to be $\nabla^0$-parallel. Then $f^*(S_0)$ is a flat bundle, and we can consider

$$D_0 = \sum_{i=1}^{2m} e_i \otimes (\nabla_{e_i} \otimes 1 + 1 \otimes \nabla^0_{e_i}).$$

Of course, $D_0$ is unitarily equivalent to $2^m$ copies of the Dirac operator $D$ of the Riemannian manifold $M^{2m}$. The difference of the operators, $L_f = D_f - D_0$, is a self-adjoint bundle morphism in the vector bundle $S \otimes f^*(S_0)$,
and its $L^2$-norm as an operator in the Hilbert space $L^2(S \otimes f^*(S_0))$ can be controlled:

$$||L_f|| \leq 2^{m-1}\sqrt{m}||df||_\infty.$$ 

Here $||df||_\infty = \max\{||df_x|| : x \in M^{2m}\}$. Since $D_0 = D_f - L_f$, a perturbation argument shows that there are at least as many eigenvalues of $D_0$ in the interval $[-||L_f||, ||L_f||]$ as the dimension $\dim \ker(D_f)$ of the kernel of the operator $D_f$ prescribes. But this dimension in turn can be computed using index theory. We have

$$\text{Index}(D^\pm_f) = 2^{m-1}\hat{\Delta}(M^{2m}) \pm \deg(f),$$

where $\deg(f)$ is the degree of the mapping $f : M^{2m} \to S^{2m}$. This immediately implies the inequality

$$\dim \ker(D_f) \geq |2^{m-1}\hat{\Delta}(M^{2m}) + \deg(f)| + |2^{m-1}\hat{\Delta}(M^{2m}) - \deg(f)| \geq 2|\deg(f)|.$$

Assuming now that the mapping degree $\deg(f)$ is greater than

$$2^{m-1}(m_0 + \ldots + m_{k-1}) + 1,$$

where $0 \leq \lambda^2_0 < \lambda^2_1 < \ldots$ denote the eigenvalues of $D^2$ over $M^{2m}$ and $m_0, m_1, \ldots$ the dimension of the corresponding eigensubspaces, we see that the operator $D_0 = 2^m \cdot D$ has at least $2^m(m_0 + \ldots + m_{k-1}) + 2$ eigenvalues in the interval

$$[-2^{m-1}\sqrt{m}||df||_\infty, 2^{m-1}\sqrt{m}||df||_\infty].$$

Thus the Dirac operator $D$ of the Riemannian manifold $M^{2m}$ has its $k$-th eigenvalue still in this interval. Summarizing this argument leads to the following result.

**Proposition** (compare [Bau]). Let $f : M^{2m} \to S^{2m}$ be a smooth mapping from the compact connected Riemannian spin manifold $M^{2m}$ into the sphere. Denote by $m_0, m_1, \ldots$ the dimension of the spaces of eigenspinors of $D^2$ for the eigenvalues $\lambda^2_j (0 \leq \lambda^2_0 < \lambda^2_1 < \ldots)$. If

$$\deg(f) \geq 2^{m-1}(m_0 + \ldots + m_{k-1}) + 1,$$

then the $k$-th eigenvalue $\lambda^2_k$ is bounded by

$$|\lambda_k| \leq 2^{m-1}\sqrt{m}||df||_\infty.$$

The construction of special mappings $f : M^{2m} \to S^{2m}$ now allows us to derive geometrical bounds on the eigenvalues of Dirac operators from this proposition. For example, one obtains the
Corollary. Let \((M^{2m}, g)\) be a compact connected Riemannian spin manifold of even dimension with positive sectional curvature \(K\). Denote by \(K_{\text{max}}\) the maximum of the sectional curvature and by \(\lambda\) the first eigenvalue of the Dirac operator. Then
\[
|\lambda| \leq 2^{m-1}\sqrt{m}\sqrt{K_{\text{max}}}.
\]

If \(M^{2m}\) is a submanifold of the Euclidean space \(\mathbb{R}^{2m+1}\), then, in particular, we can choose the Gauß mapping as the mapping \(f : M^{2m} \rightarrow S^{2m}\). For surfaces \(M^2 \subset \mathbb{R}^3\) in 3-dimensional Euclidean space this leads to the estimate
\[
|\lambda| \leq C(M^2) \max\{|\mu(m)| : m \in M^2\}
\]
where
\[
C(M^2) = \begin{cases} 
1 & \text{if the genus of } M^2 \text{ is } g = 0, \\
3 & \text{if the genus of } M^2 \text{ is } g = 2 \text{ or } 3, \\
2 & \text{if the genus of } M^2 \text{ is } g \geq 4,
\end{cases}
\]
and \(\mu(m)\) is the greater of the two principal curvatures at the point \(m \in M^2\).

To conclude this chapter we want to quote an inequality relating the first eigenvalue of the Dirac operator to the first eigenvalue of a Schrödinger operator in the case of a surface \(M^2 \subset \mathbb{R}^3\). The detailed discussion can be found in the cited paper.

Proposition (compare [Agr/Fri]). Let \(M^2 \subset \mathbb{R}^3\) be a closed oriented surface with mean curvature \(H\), and denote by \(D\) the Dirac operator on \(M^2\) corresponding to the induced spin structure. Then,
\[
\lambda_1^2(D) \leq \mu(\Delta + H^2),
\]
where \(\lambda_1\) is the first eigenvalue of the Dirac operator \(D\) and \(\mu_1\) the first eigenvalue of the Schrödinger operator \(\Delta + H^2\) on the surface \(M^2\).

5.5. References and Exercises


Exercise 1. Let $(M^n, g)$ be a compact connected Riemannian spin manifold and denote by $\mathcal{K}_\pm$ the space of Killing spinors:

$$\mathcal{K}_\pm = \left\{ \psi \in \Gamma(S) : \nabla_X \psi = \pm \frac{1}{2} \sqrt{\frac{R}{n(n-1)}} X \cdot \psi \right\}.$$

If $\dim(\mathcal{K}_+) + \dim(\mathcal{K}_-) \geq 2^{[n/2]}$, then $(M^n, g)$ is isometric to the sphere $S^n$.

Exercise 2. Compute the space $\mathcal{K}_\pm$ of Killing spinors for the sphere $S^n$ and for Euclidean space $\mathbb{R}^n$. Determine all solutions of the twistor equation for hyperbolic space $\mathbb{H}^n$.

Exercise 3. $\mathbb{R}P^3$ has two spin structures. Each of the numbers $\pm \frac{1}{2} \sqrt{\frac{3}{2}}$ is an eigenvalue of the Dirac operator associated with exactly one of these spin structures.

Exercise 4. Let $(M^n, g)$ be a Riemannian spin manifold and $N^{n-1} \subset M^n$ an umbilic submanifold. Prove that the restriction of a Killing spinor on $M^n$ to the submanifold $N^{n-1}$ is a solution of the twistor equation on $N^{n-1}$. 
Appendix A

Seiberg-Witten Invariants

A.1. On the topology of 4-dimensional manifolds

The topology of manifolds looks back at a long history which began in the last century (Riemann). In the works of many mathematicians (Poincaré, Brouwer, Hopf, Morse etc.) during a first period lasting until the middle of the thirties of this century homological properties of manifolds were studied, the calculus of variations was developed and, in particular, complete proofs were given for the classification of compact 2-dimensional manifolds. The characteristic features in the topology of manifolds between 1935 and 1960 were the theory of characteristic classes (Whitney, Pontryagin), the calculation of the bordism ring (Thom) and the discovery of exotic differential structures (Milnor). In particular, it became obvious that in dimensions \( n > 4 \) the category \( \text{Top}(n) \) of \( n \)-dimensional topological manifolds does not coincide with the category \( \text{Diff}(n) \) of smooth manifolds, i.e. the natural mapping

\[
\text{Diff}(n) \to \text{Top}(n)
\]

forgetting the differential structure is neither injective nor surjective. In connection with the solution of the Poincaré conjecture in dimensions \( n \geq 5 \) and the proof of the \( h \)-cobordism theorem (Smale), the so-called surgery techniques were developed in the sixties (Wall, Browder, Novikov). They led to a far-reaching classification theory for certain classes of smooth compact manifolds in dimensions \( n \geq 5 \).
The situation in low dimensions, \( n = 3, 4 \), is rather special. On the one hand, the possible variety of forms is considerably larger already in dimension \( n = 3 \) than for surfaces, and the classification question is much more difficult. On the other hand, even in a 4-dimensional manifold there is not enough room to apply the surgery techniques which were so successful in higher dimensions. Dimension \( n = 3 \) turned out to be the last one in which \( \text{Top} \) and \( \text{Diff} \) coincide: every compact 3-dimensional manifold is triangulizable, every two triangulizations are combinatorially equivalent and, moreover, every such manifold admits exactly one differential structure. The Poincaré conjecture in this dimension is still unsettled. Apart from the fundamental group \( \pi_1(M^4) \), the integral intersection form \( H^2(M^4; \mathbb{Z}) \) is the most important invariant of an orientable compact 4-dimensional manifold. In 1982, M. Freedman proved that for simply connected compact topological 4-manifolds this intersection form almost determines the manifold itself in \( \text{Top}(4) \). In particular, each unimodular quadratic form over the ring \( \mathbb{Z} \) of integers can be realized as the intersection form of a compact simply connected topological manifold \( M^4 \). On the other hand, it was already known for a long time (Rokhlin 1952) that if a unimodular form of even type can be realized by a closed smooth 4-manifold, then its signature is divisible by 16. Take, e.g., the positive definite unimodular \( \mathbb{Z} \)-form \( E_8 \) of dimension 8,

\[
E_8 = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}.
\]

\( E_8 \) is of even type and has signature \( 8, \sigma(E_8) = \dim E_8 = 8 \). Hence there exists a simply connected topological manifold \( M_0^4 \) with intersection form \( E_8 \), and \( M_0^4 \) is by no means smooth, i.e. the mapping

\[
\text{Diff}(4) \to \text{Top}(4)
\]

is not surjective (and not injective as well). Thus the smooth topology in dimension 4 is already a completely different topic than the continuous.

Likewise, at the beginning of the eighties, S.K. Donaldson introduced a method for the study of \( \text{Diff}(4) \) based on associating with every smooth 4-dimensional manifold the moduli space of solutions of the self-dual Yang-Mills equation in a non-abelian gauge field theory as an invariant, and on deriving new invariants from it. In this way, he was able to exclude further unimodular quadratic forms over the ring \( \mathbb{Z} \) as intersection forms of smooth
simply connected and closed 4-manifolds $M^4$. For example, if $H^2(M^4; \mathbb{Z})$ is positive definite, then this intersection form has to be trivial. In particular, $E_8 \oplus E_8$ cannot occur as the intersection form of a smooth manifold even though the Rokhlin condition $\sigma/16 \in \mathbb{Z}$ is satisfied. This obstruction eventually led to the proof that there exist exotic differential structures in $\mathbb{R}^4$. On the other hand, using known algebraic surfaces, many unimodular $\mathbb{Z}$-forms can be realized as intersection forms. Computations for connected sums of $K3$-surfaces with $(S^2 \times S^2)$ resulted in the conjecture that the form

$$2k(-E_8) \oplus m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

does occur as the intersection form of a smooth 4-manifold if and only if $m \geq 3k$ (the so-called 11/8 conjecture; this inequality is equivalent to $b_2(M^4)/|\sigma(M^4)| \geq \frac{11}{8}$). Within the framework of Donaldson theory, this inequality could only be proved for small $k$. Another application of the invariants constructed by means of non-abelian gauge field theory concerns splitting questions. A compact simply connected complex surface $S$ with $b_2(S) > 3$ cannot be smoothly represented as a connected sum $S = X_1 \# X_2$ with $b_2^+(X_i) > 0$. This led to the construction of different differential structures on compact simply connected 4-manifolds.

In autumn 1994, E. Witten suggested that all these results, and others reaching even further, could be obtained by considering the moduli space of a system of equations for a pair consisting of a spinor and an abelian connection. The spinor has to be harmonic with respect to the abelian gauge field and, on the other hand, algebraically related to the curvature form of the abelian connection (Seiberg-Witten equation). This system of equations is the 4-dimensional analogue to the 2-dimensional Ginzburg-Landau model (1950) of superconductivity. Witten’s claim was elaborated by many mathematicians in the following months and turned out to be right. In contrast to non-abelian gauge field theories with their non-linear equations, one could now return to an abelian theory, and the analytical theory of smooth 4-dimensional topology gets drastically simplified!

The first observation forming the base of Seiberg-Witten theory is that each orientable compact 4-dimensional manifold $M^4$ has a spin$^C$ structure (though, possibly, no spin structure!). Hence spinors may be defined on it. We briefly sketch a proof: The universal coefficient theorem implies the formula

$$H^3(M^4; \mathbb{Z}) = \{H_3(M^4; \mathbb{Z})/\text{Tor}(H_3(M^4; \mathbb{Z}))\} \oplus \text{Tor}(H_2(M^4; \mathbb{Z})), $$
and, from Poincaré duality, \( H_2(M^4; \mathbb{Z}) = H^2(M^4; \mathbb{Z}) \), we conclude the relations
\[
\text{Tor}(H^3(M^4; \mathbb{Z})) = \text{Tor}(H_2(M^4; \mathbb{Z})) = \text{Tor}(H^2(M^4; \mathbb{Z})).
\]
Let \( T \subset H^2(M^4; \mathbb{Z}) \) be the torsion subgroup. Consider the exact sequence
\[
H^2(M^4; \mathbb{Z}) \xrightarrow{\beta_*} H^3(M^4; \mathbb{Z}) \xrightarrow{\alpha} H^2(M^4; \mathbb{Z}),
\]
Then,
\[
\text{im}(\beta_*) = \{ \alpha^3 \in \text{Tor}(H^3(M^4; \mathbb{Z})) : 2\alpha^3 = 0 \} \approx \{ \gamma^2 \in T : 2\gamma^2 = 0 \}.
\]
The sequence \( \{ \gamma^2 \in T : 2\gamma^2 = 0 \} \to T \xrightarrow{\beta} T \to T/2T \) is an exact sequence of \( \mathbb{Z}_2 \)-vector spaces. Thus, \( \dim_{\mathbb{Z}_2}(T/2T) = \dim_{\mathbb{Z}_2} \{ \gamma^2 \in T : 2\gamma^2 = 0 \} \) and, since \( r(T) = T/2T \), we obtain
\[
\dim_{\mathbb{Z}_2}(H^2(M^4; \mathbb{Z}_2)) = \dim_{\mathbb{Z}_2}(\text{im}(r)) + \dim_{\mathbb{Z}_2}(\text{im}(\beta_*))
\]
\[
= \dim_{\mathbb{Z}_2}(\text{im}(r)) + \dim_{\mathbb{Z}_2}(r(T)).
\]
The following inclusion is obvious:
\[
r(T) \subset \text{im}(r) \subset H^2(M^4; \mathbb{Z}_2).
\]
Consider \( x \in \text{im}(r) \) with \( x = r(\alpha) \) and \( \alpha \in H^2(M^4; \mathbb{Z}) \). If \( y \in r(T) \), then there exists an element \( \beta \in T \subset H^2(M^4; \mathbb{Z}) \) with \( r(\beta) = y \). \( \beta \) is torsion and hence \( \alpha \cup \beta \) in \( H^4(M^4; \mathbb{Z}) \approx \mathbb{Z} \) vanishes. This in turn implies
\[
x \cup y = 0 \quad \text{for} \quad x \in \text{im}(r), y \in r(T).
\]
The set \( \Gamma = \{ \gamma \in H^2(M^4; \mathbb{Z}) : \forall y \in r(T) \gamma \cup y = 0 \} \) thus contains \( \text{im}(r) \). On the other hand,
\[
\dim_{\mathbb{Z}_2}(\Gamma) = \dim_{\mathbb{Z}_2}(H^2(M^4; \mathbb{Z}_2)) - \dim_{\mathbb{Z}_2}(r(T)) = \dim_{\mathbb{Z}_2}(\text{im}(r))
\]
by \( \mathbb{Z}_2 \)-Poincaré duality. Hence, \( \text{im}(r) = \Gamma \), i.e.
\[
\text{im}(r) = \{ \gamma \in H^2(M^4; \mathbb{Z}) : \forall y \in r(T) \gamma \cup y = 0 \}.
\]
So we arrive at a more precise description of the image of the \( \mathbb{Z}_2 \)-reduction \( r : H^2(M^4; \mathbb{Z}) \to H^2(M^4; \mathbb{Z}_2) \). We are going to use this to prove that \( M^4 \) has a spin\(^C \) structure. The necessary and sufficient condition to be considered is that the second Stiefel-Whitney class \( w_2 \) belongs to the image \( \text{Im}(r) \). We thus have to check whether \( w_2 \cup y = 0 \) for all \( y \in r(T) \). By the Wu formulas for an orientable 4-dimensional manifold, \( w_2 \) is the only cohomology class \( w_2 \in H^2(M^4; \mathbb{Z}_2) \) satisfying the condition \( x^2 = w_2 \cup x \) for all \( x \in H^2(M^4; \mathbb{Z}_2) \). Now for \( y \in r(T) \) we have \( y^2 = 0 \), since \( y \) is the \( \mathbb{Z}_2 \)-reduction of a torsion element from \( H^2(M^4; \mathbb{Z}) \). This implies
\[
w_2 \cup y = y^2 = 0
\]
for all \( y \in r(T) \), i.e. the second Stiefel-Whitney class \( w_2(M^4) \) is the \( \mathbb{Z}_2 \)-reduction of an integral cohomology class. Altogether we obtain the

**Proposition** (Wu 1950; Hirzebruch and Hopf 1958). Every compact orientable 4-dimensional manifold \( M^4 \) has a spin\(^C(4) \) structure.

Next we will collect a few special formulas from 4-dimensional spin algebra. The Hodge operator \( * : \Lambda^2(\mathbb{R}^4) \to \Lambda^2(\mathbb{R}^4) \) acting on 2-forms splits \( \Lambda^2 \) into the self-dual and the anti-self-dual 2-forms:

\[
\Lambda^2(\mathbb{R}^4) = \Lambda^2_+(\mathbb{R}^4) \oplus \Lambda^2_-(\mathbb{R}^4).
\]

The 4-dimensional spin representation \( \Delta_4 \) also decomposes into \( \Delta_4 = \Delta^+_4 \oplus \Delta^-_4 \) with \( \Delta^+_4 \cong \mathbb{C}^2 \). The endomorphisms \( e_i \cdot e_j : \Delta^+_4 \to \Delta^+_4 \) induced by Clifford multiplication have the following matrices:

\[
e_1e_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_1e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_1e_4 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},
\]

\[
e_2e_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_2e_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3e_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

In particular, as endomorphisms in \( \Delta^+_4 \)

\[
e_1e_2 - e_3e_4 = 0, \quad e_1e_3 + e_2e_4 = 0, \quad e_1e_4 - e_2e_3 = 0.
\]

For a spinor \( \Psi \in \Delta^+_4 \) we define a 2-form \( w^\Psi \) by the formula

\[
w^\Psi(X,Y) = \langle X \cdot Y \cdot \Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2,
\]

where \( X, Y \in \mathbb{R}^4 \). \( w^\Psi \) is a 2-form with imaginary values. This results from the following calculations:

\[
w^\Psi(X,Y) = \langle X \cdot Y \cdot \Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2
= \langle (-YX - 2\langle X, Y \rangle \Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2
= -\langle YX\Phi, \Phi \rangle - \langle X, Y \rangle |\Phi|^2 = -w^\Psi(Y,X),
\]

\[
\overline{w^\Psi(X,Y)} = \langle X \cdot Y \cdot \Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2 = \langle \overline{\Psi}, X \cdot Y \cdot \Phi \rangle + \langle X, Y \rangle |\Phi|^2
= \langle Y \cdot X\Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2 = w^\Psi(Y,X) = -w^\Psi(X,Y).
\]

Using the explicitly described spin representation, we easily obtain the proof of the next

**Proposition.**

1) Let \( \Phi \in \Delta^+_4 \) and \( w^2 \in \Lambda^2_- \). Then, \( w^2 \cdot \Phi = 0 \).

2) \( \langle w^\Phi \cdot \Phi, \Phi \rangle = -2|\Phi|^4 \) and \( |w^\Phi|^2 = 2|\Phi|^4 \).
Proof. Setting $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in \mathbb{C}^2 = \Delta_+^4$ yields

$$w^\Phi(e_1, e_2) = i\{|\Phi_1|^2 - |\Phi_2|^2\} = w^\Phi(e_3, e_4),$$

$$w^\Phi(e_1, e_3) = -\Phi_2 \bar{\Phi}_1 + \Phi_1 \bar{\Phi}_2 = -w^\Phi(e_2, e_4),$$

$$w^\Phi(e_1, e_4) = -i\Phi_2 \bar{\Phi}_1 - i\Phi_1 \bar{\Phi}_2 = w^\Phi(e_2, e_3),$$

and the formulas simply result from inserting these entities. □

A.2. The Seiberg-Witten equation

Let $M^4$ be an oriented (compact) 4-dimensional manifold. Fix a $\text{Spin}^{c}(4)$-structure and a connection $A \in \mathcal{C}(P)$ in the $U(1)$-principal bundle associated with the spin$^c$ structure. For $\Phi \in \Gamma(S^+)$, consider the equations (Seiberg-Witten equation):

$$D_A \Phi = 0, \quad \Omega_A^+ = -\frac{1}{4} \omega^\Phi.$$  

To understand the Seiberg-Witten equation we compute, for an arbitrary section $\Phi \in \Gamma(S^+)$ and any connection $A \in \mathcal{C}(P)$, the integral

$$\int_{M^4} \left| \Omega_A^+ + \frac{1}{4} \omega^\Phi \right|^2 + |D_A \Phi|^2.$$  

Since $\Omega_A^+$ and $\omega^\Phi$ are forms with purely imaginary values, calculating their length amounts to

$$\left| \Omega_A^+ + \frac{1}{4} \omega^\Phi \right|^2 = -\sum_{i<j} \left[ \Omega_A^+(e_i, e_j) + \frac{1}{4} \omega^\Phi(e_i, e_j) \right]^2$$

$$= |\Omega_A^+|^2 + \frac{1}{16} |\omega^\Phi|^2 - \frac{1}{2} \sum_{i<j} \Omega_A^+ \langle e_i, e_j \rangle \langle e_i e_j \Phi, \Phi \rangle$$

$$= |\Omega_A^+|^2 + \frac{1}{16} |\omega^\Phi|^2 - \frac{1}{2} \langle \Omega_A^+ \Phi, \Phi \rangle$$

$$= |\Omega_A^+|^2 + \frac{1}{8} |\Phi|^4 - \frac{1}{2} \langle \Omega_A^+ \Phi, \Phi \rangle.$$  

On the other hand, by the Schrödinger-Lichnerowicz formula we have

$$\int_{M^4} |D_A \Phi|^2 = \int_{M^4} \left[ |\nabla^A \Phi|^2 + \frac{R}{4} |\Phi|^2 + \frac{1}{2} \langle \Omega_A^+ \Phi, \Phi \rangle \right].$$  

This implies

$$\int_{M^4} \left| \Omega_A^+ + \frac{1}{4} \omega^\Phi \right|^2 + |D_A \Phi|^2 = \int_{M^4} \left[ |\Omega_A^+|^2 + |D^A \Phi|^2 + \frac{R}{4} |\Phi|^2 + \frac{1}{8} |\Phi|^4 \right].$$
Hence the functional
\[ \varepsilon(\Phi, A) = \int_{M^4} \left[ |\Omega_A^+|^2 + |\nabla A \Phi|^2 + \frac{R}{4} |\Phi|^2 + \frac{1}{8} |\Phi|^4 \right] \]
is non-negative and its zeroes are precisely the solutions of the Seiberg-Witten equation.

**Proposition.** Let \((\Phi, A)\) be a solution of \(D_A \Phi = 0, \Omega_A^+ = -\frac{1}{4} \omega \Phi\) over a compact Riemannian manifold \((M^4, g)\) with scalar curvature \(R\). Then, at each point,
\[ |\Phi(x)|^2 \leq -R_{\text{min}}, \quad \text{where} \quad R_{\text{min}} = \min \{ R(m) : m \in M^4 \}. \]

**Proof.** At a point \(x\) where \(|\Phi(x)|^2\) attains its maximum we have \(0 \leq \nabla |\Phi|^2\). However,
\[
0 \leq \Delta |\Phi|^2 = 2(\langle \nabla A \phi \phi, \phi \rangle - 2 \langle \nabla A \phi, \nabla A \phi \rangle) \leq 2(\langle \nabla A \phi \phi, \phi \rangle - \langle \Omega_A^+ \phi, \phi \rangle)
\]
\[
= 2 \left\{ \frac{R}{4} |\Phi|^2 - \frac{1}{2} \langle \Omega_A^+ \phi, \phi \rangle \right\} = -\frac{R}{2} |\Phi|^2 - \frac{1}{2} |\Phi|^4.
\]

If now \(|\Phi|_{\text{max}}^2 > 0\), then \(0 \leq -\frac{R}{2} - 1/2 |\Phi|_{\text{max}}^2\) and \(|\Phi(x_{\text{max}})|^2 \leq -R_{\text{min}}\).

\[ \square \]

Let \(P \to M\) be a \(U(1)\)-principal bundle and \(A : TP \to \mathcal{E}^1 = i\mathbb{R}^1\) a connection. A gauge transformation on \(P\) is given by a function \(f : M^n \to S^1\) acting via \(f(p) = p \cdot f(\pi(p))\). Let \(\Theta = \frac{dz}{z} = \bar{z}dz\) be the 1-form \(\Theta : TS^1 \to \mathcal{G}^1 \approx i\mathbb{R}^1\). Then the action of the gauge group \(G(P) = \text{Map}(M^n, S^1)\) on \(C(P)\) is described by
\[ f^*(A) = A + \pi^* f^*(\Theta) \]
and the curvature forms of \(A\) and \(f^*(A)\), respectively, are
\[ \Omega_A = dA, \quad \Omega_{f^*(A)} = \Omega_A. \]

We will describe the action of the gauge group \(G(P) = G(L) = \text{Map}(M^4, S^1)\) in greater detail. Consider \(f : M^4 \to S^1\). Then \(f\) acts on \(C(P)\) by
\[ f^*(A) = A + \pi^* \frac{df}{f}. \]

For connections in the fibre product \(R \times P\) of the frame bundle with the \(U(1)\)-bundle of the spin\(^C\) structure we thus obtain
\[ LC \oplus A \to LC \oplus A + \pi^* \frac{df}{f}. \]

Here \(LC\) denotes the Levi-Civita connection of \(R\). Hence, \(LC \oplus (A + \pi^* \frac{df}{f}) - LC \oplus A\) is a 1-form on \(R \times P\) with values in \(\mathcal{G}^1\) vanishing on \(TR\). Lifting
both connections to the $Spin^C(4)$-structure, we obtain, for the covariant
derivatives in $S$,
\[
\nabla^f_{t}^*(A) - \nabla^A_t \Phi = \frac{df(t)}{f} \cdot \Phi.
\]
This implies that for the Dirac operators $D_A, D_{f^*(A)} : \Gamma(S) \to \Gamma(S)$ the
following formula holds:
\[
D_{f^*(A)} \Phi - D_A \Phi = \frac{1}{f} \text{grad}(f) \cdot \Phi.
\]
As the group $G(P) = \{ f : M^4 \to S^1 \}$ acts on $\Gamma(S) \times C(P)$ by $f \cdot (\Phi, A) = (\frac{1}{f} \cdot \Phi, f^*(A))$, we have
\[
D_{f^*(A)} (\Phi/f) = D_A (\Phi/f) + \frac{1}{f} \text{grad}(f) \cdot (\Phi/f)
\]
\[
= \frac{1}{f} D_A (\Phi) - \frac{1}{f^2} \text{grad}(f) \cdot \Phi + \frac{1}{f^2} \text{grad}(f) \cdot \Phi = \frac{1}{f} D_A (\Phi).
\]
This implies that if $(\Phi, A)$ is a solution of the Seiberg-Witten equation, so
is $f \cdot (\Phi, A) = (\Phi/f, f^*(A))$.

Now we turn to the definition of the moduli space for Seiberg-Witten theory. If $M^4$ is a compact oriented Riemannian spin$^C$ manifold, $P$ the $U(1)$-bundle
of the spin$^C$ structure and $L$ the line bundle, then define
\[
\mathcal{M}_L = \left\{ (\Phi, A) \in \Gamma(S^+) \times C(P) : D_A \Phi = 0, \quad \Omega_A^+ = -\frac{1}{4} \omega^\Phi \right\} / G.
\]

**Proposition.** $\mathcal{M}_L$ is compact.

**Proof.** Let
\[
F(L) = F = \{ \omega^2 \in \Lambda^2(M^4) : \quad d\omega^2 = 0, \quad [\omega^2]_{DR} = c_1(L) \},
\]
and let $\omega^2_{\text{harm}}$ be the only harmonic form in $F(L)$. Since the curvature form $\Omega_A$ of $A \in C(P)$ is gauge invariant, we obtain a mapping
\[
P : \mathcal{M}_L \to F(L), \quad P[A, \Phi] = \Omega_A.
\]

**First Step.** $P(\mathcal{M}_L) \subset F(L)$ is a compact subset (relative to the $L^2$-topology
on $F(L)$).

**Proof.** Suppose that $D_A \Phi = 0$ and $\Omega_A^+ = -\frac{1}{4} \omega^\Phi$. Write the curvature form $\Omega_A$ as
\[
\Omega_A = (\Omega_A^+) + (\Omega_A^-) = \omega^2_{\text{harm}} + d\eta^1.
\]
Denote by $\mathcal{H}^1$ the harmonic 1-forms and $\text{im}(d^0) = \text{im}(d : \Lambda^0 \to \Lambda^1)$. Then we can choose $\eta^1$ to be orthogonal to $\text{im}(d^0) \oplus \mathcal{H}^1$ in $\Gamma(\Lambda^1)$. From

$$\Delta |\Phi|^2 = 2\langle (\nabla^A)^* \nabla^A \Phi, \Phi \rangle - 2\langle \nabla^A \Phi, \nabla^A \Phi \rangle = -\frac{R}{2} |\Phi|^2 - \frac{1}{2} |\Phi|^4 - 2\langle \nabla^A \Phi, \nabla^A \Phi \rangle$$

and $\int_{M^4} \Delta |\Phi|^2 = 0$ we obtain

$$2\|\nabla^A \Phi\|_{L^2}^2 = \int_{M^4} \left( -\frac{R}{2} |\Phi|^2 - \frac{1}{2} |\Phi|^4 \right) \leq \int_{\{R < 0\}} (-\frac{R}{2}) |\Phi|^2 \leq \int_{R < 0} (-\frac{R}{2} \cdot (-R_{\text{min}})) =: C_1(R).$$

Hence there is an $L^2$-bound $\|\nabla^A \Phi\|_{L^2}^2 \leq C_1$. Moreover,

$$\delta\Omega^+_A = *d \ast (\Omega_A + *\Omega_A) = *d \ast \Omega_A,$$

$$\delta\Omega^-_A = *d \ast (\Omega_A - *\Omega_A) = *d \ast \Omega_A,$$

i.e. $\delta\Omega_A = 2\delta\Omega^+_A = 1/2 \delta(\omega^\Phi)$. Now we compute $\delta\omega^\Phi$:

$$\frac{1}{2} \delta(\omega^\Phi) = \delta\Omega_A(\iota) = \frac{1}{2} \langle (\Phi, \nabla^A \Phi) - \langle \nabla^A \Phi, \Phi \rangle \rangle.$$

This implies

$$\|\delta\Omega_A\|_{L^2}^2 \leq C_2^* \|\Phi\|_{L^2}^2 \|\nabla^A \Phi\|_{L^2}^2$$

and, by the $C^0$-bound for $|\Phi|$,

$$\|\delta\Omega_A\|_{L^2}^2 \leq C_2.$$

Since $\Omega_2 = \omega_{\text{harm}} + d\eta^1$, this implies $\|\delta d\eta^1\|_{L^2}^2 \leq C_2$. Now $\eta^1$ was chosen orthogonal to $\text{im}(d^0)$, i.e. $\delta\eta^1 = 0$. As $\Delta\eta^1 = \delta d\eta^1$, we obtain $\|\Delta \eta^1\|_{L^2}^2 \leq C_2$. Since $\eta^1 \perp \mathcal{H}^1 = \text{ker}(\Delta)$ and, because the spectrum of $\Delta$ is discrete, we have

$$\|\eta^1\|_{L^2}^2 \leq C_3^* \|\Delta \eta^1\|_{L^2}^2.$$

Thus,

$$\|\eta^1\|_{L^2}^2 \leq C_3, \quad \|\Delta \eta^1\|_{L^2}^2 \leq C_2.$$

So far we have shown that the set of 1-forms $\eta^1$ in question is bounded in the Sobolev space $H^2(\Lambda^1(M^4))$. Since the map

$$d : H^2(\Lambda^1(M^4)) \to H^1(\Lambda^2(M^4))$$

is continuous, the set of all $d\eta^1$, i.e. the set of all $\Omega_A$ in the image of $P : \mathcal{M}_L \to F(L)$, is bounded in $H^1(\Lambda^2(M^4))$. Lastly, by the Rellich lemma,

$$H^1(\Lambda^2(M^4)) \to L^2(\Lambda^2(M^4))$$

is a compact embedding. Hence $P(\mathcal{M}_L) \subset F(L)$ is a compact subset.
A. Seiberg-Witten Invariants

Second Step. Consider \( P_1 : \mathcal{M}_L \to \mathcal{C}(P)/\mathcal{G}(P) \), \( P_1[\Phi, A] = [A] \). Then \( P_1(\mathcal{M}_L) \subset \mathcal{C}(P)/\mathcal{G}(P) \) is a compact subset.

Proof. This is Weyl’s theorem. Namely, the mapping \( \mathcal{C}(P) \to F(P) \), \( A \to \Omega_A \) is a fibration with compact fibre \( \text{Pic}(M^4) = H^1(M^4; \mathbb{R})/H^1(M^4; \mathbb{Z}) \approx T^b_1(M^4) \). Since the diagram

\[
\begin{array}{ccc}
\mathcal{M}_L & \xrightarrow{P_1} & \mathcal{C}(P)/\mathcal{G}(P) \\
\downarrow{P} & & \downarrow{F(L)} \\
F(L) & & \\
\end{array}
\]

commutes and \( P(\mathcal{M}_L) \subset F(L) \) is compact, the same also holds for \( P_1(\mathcal{M}_L) \subset \mathcal{C}(P)/\mathcal{G}(P) \).

Third Step. \( \mathcal{M}_L \) is compact.

Proof. For a given \( A \in \mathcal{C}(P)/\mathcal{G}(P) \) the pre-image \( P_1^{-1}([A]) \subset \mathcal{M}_L \) consists of the solutions of

\[
DA\Phi = 0, \quad \max |\Phi(x)| \leq -R_{\text{min}}.
\]

This is a bounded ball in a finite-dimensional vector space.

A.3. The Seiberg-Witten invariant

In this section, we will study the linearization of the space \( \mathcal{M}_L \). The Seiberg-Witten equations are

\[
DA\Phi = 0, \quad \Omega_A^+ = -1/4 \omega^\Phi.
\]

The linearization of these equations at the point \((\Phi, A)\) is an operator

\[
P_{(\Phi,A)} : \Gamma(S^+) \oplus \Gamma(A^1) \to \Gamma(S^-) \oplus \Gamma(A^2_+) .
\]

For given \( \Psi \in \Gamma(S^+) \) and \( \eta^1 \in \Gamma(A^1) \), consider the variation

\[
A_t = A + t\eta^1, \quad \Phi_t = \Phi + t\Psi.
\]

Then \( \Omega_{A_t} = dA + td\eta^1 \), and thus, \( \frac{d}{dt}(\Omega_{A_t}^+)|_{t=0} = (d\eta^1)^+ \). Moreover, from

\[
\omega^\Phi(X,Y) = \langle XY\Phi, \Phi \rangle + \langle X,Y \rangle |\Phi|^2 \quad \text{we obtain}
\]

\[
\left( \frac{d}{dt}\omega^\Phi_{|t=0} \right)(X,Y)
\]

\[
= \langle XY(\frac{d\Phi}{dt})|_{t=0}, \Phi \rangle + \langle XY\Phi, \frac{d\Phi}{dt}|_{t=0} \rangle + \langle X,Y \rangle \langle \Psi, \Phi \rangle + \langle X,Y \rangle \langle \Phi, \Psi \rangle
\]

\[
= \langle XY\Psi, \Phi \rangle + \langle XY\Phi, \Psi \rangle + \langle X,Y \rangle \langle \Psi, \Phi \rangle + \langle X,Y \rangle \langle \Phi, \Psi \rangle
\]
\[ = (XY \Psi, \Phi) + \langle \Phi, YX \Psi \rangle + \langle X, Y \rangle \langle \Psi, \Phi \rangle + \langle X, Y \rangle \langle \Phi, \Psi \rangle \]

\[ = (XY \Psi, \Phi) + \langle YX \Psi, \Phi \rangle + \langle X, Y \rangle \langle \Psi, \Phi \rangle + \langle X, Y \rangle \langle \Phi, \Psi \rangle \]

\[ = (XY \Psi, \Phi) + \mathcal{L}(XY - 2\langle X, Y \rangle) \Psi, \Phi \rangle + \langle X, Y \rangle \langle \Psi, \Phi \rangle + \langle X, Y \rangle \langle \Phi, \Psi \rangle \]

\[ = (XY \Psi, \Phi) - (XY \Psi, \Phi) - 2\langle X, Y \rangle \langle \Psi, \Phi \rangle + \langle X, Y \rangle \langle \Psi, \Phi \rangle + \langle X, Y \rangle \langle \Phi, \Psi \rangle \]

\[ = (XY \Psi, \Phi) - (XY \Psi, \Phi) + \langle X, Y \rangle \{ \langle \Psi, \Phi \rangle - \langle \Psi, \Phi \rangle \} \quad \text{def} \quad \omega^{\Phi, \Psi}(X, Y). \]

Hence \( \Omega^+_A = -\frac{1}{4} \omega^\Phi \) is linearized by \((d\eta^1)^+ = \frac{1}{4} \omega^{\Phi, \Psi} \). Analogously, \( D_A \Phi = 0 \) linearizes to \( \eta^1 \Phi + D_A \Psi = 0 \).

Altogether, \( P_{(\Phi, A)} : \Gamma(S^+) \oplus \Gamma(\Lambda^1) \to \Gamma(S^-) \oplus \Gamma(\Lambda^2) \) is given by

\[ P_{(\Phi, A)}(\Psi, \eta^1) = (\eta^1 \Phi + D_A \Psi, d\eta^1_+ + \frac{1}{4} \omega^{\Phi, \Psi}). \]

We compute the tangent space to the orbit \( G(P) \cdot (\Phi, A) \) at the point \( (\Phi, A) \) as follows: If \( f_t : M \to S^1 \) is a smooth family of gauge transformations, \( f_0 = 1 \), then

\[ \frac{d}{dt} \left( \frac{1}{f_t} \Phi \right) = \frac{d}{dt} (f_t \Phi) = \frac{d}{dt} \Phi = -h \Phi \]

where \( h := \frac{d}{dt} f_t |_{t=0} \) and \( f_t^* A - A = \frac{df_t}{f_t} \). Thus,

\[ \frac{d}{dt} (f_t^* A - A) |_{t=0} = dh. \]

Hence \( P_{(\Phi, A)}^0 : \Gamma(\Lambda^0) \to \Gamma(S^+) \oplus \Gamma(\Lambda^1) \) is given by

\[ P_{(\Phi, A)}^0(h) = (-h \Phi, dh). \]

Indeed, we have \( P_{(\Phi, A)}^1 \circ P_{(\Phi, A)}^0 = 0 \). Namely, if \( \Psi = -h \Phi \) and \( \eta^1 = dh \), then

\[ \eta^1 \Phi + D_A(\Psi) = dh \cdot \Phi - D_A(h \Phi) = dh \Phi - dh \cdot \Phi + hD_A(\Phi) = hD_A(\Phi) = 0 \]

and \( d\eta^1 = 0 \) as well as (since \( \Psi = h \Phi \), \( h \) is purely imaginary)

\[ \omega^{\Phi, \Psi}(X, Y) = \langle XY h \Phi, \Phi \rangle - h(XY \Phi, \Phi) + \langle XY \rangle \{ h(\Phi, \Phi) + h(\Phi, \Phi) \} \]

\[ = h\{ (XY \Phi, \Phi) + \langle \Phi, XY \Phi \rangle + 2\langle X, Y \rangle \langle \Phi, \Phi \rangle \} = 0. \]

Next we want to compute the index of the elliptic complex (over the real numbers)

\[ \Gamma(\Lambda^0) \xrightarrow{P_{(\Phi, A)}^0} \Gamma(S^+) \oplus \Gamma(\Lambda^1) \xrightarrow{P_{(\Phi, A)}^1} \Gamma(S^-) \oplus \Gamma(\Lambda^2). \]
Since operators of lower order than that determining the principal symbol do not contribute, the index we are looking for is equal to that of the complex
\[ \Gamma(\Lambda^0) \xrightarrow{(0,d)} \Gamma(S^+) \oplus \Gamma(\Lambda^1) \xrightarrow{(D_A, pr+d)} \Gamma(S^-) \oplus \Gamma(\Lambda_+^2). \]
However, the latter is the sum of two complexes and, moreover, the index is additive:

\[
\begin{align*}
\text{Index}_R &= \text{Index}_R(\Gamma(\Lambda^0) \xrightarrow{d} \Gamma(\Lambda^1) \xrightarrow{pr+d} \Gamma(\Lambda_+^2)) \\
&\quad + \text{Index}_R(\Gamma(S^+) \xrightarrow{D_A} \Gamma(S^-)) \\
&= \text{Index}_R(\Gamma(\Lambda^0) \xrightarrow{d} \Gamma(\Lambda^1) \xrightarrow{pr+d} \Gamma(\Lambda_+^2)) - 2 \text{Index}_C(D_A^+) .
\end{align*}
\]
By the index formula we have
\[ \text{Index}_C(D_A^+) = \frac{1}{8} c^2 - \frac{1}{8} \sigma. \]
This implies
\[ \text{Index}_R = \text{Index}_R(\Gamma(\Lambda^0) \xrightarrow{d} \Gamma(\Lambda^1) \xrightarrow{pr+d} \Gamma(\Lambda_+^2)) - \frac{1}{4} c^2 + \frac{1}{4} \sigma. \]
The index of the complex \( \Gamma(\Lambda^0) \xrightarrow{d} \Gamma(\Lambda^1) \rightarrow \Gamma(\Lambda_+^2) \) equals \( \frac{1}{2} \chi + \frac{1}{2} \sigma \). Thus,
\[ \text{Index}_R(*) = \frac{1}{2} \chi + \frac{1}{2} \sigma - \frac{1}{4} c^2 + \frac{1}{4} \sigma = \frac{1}{4} (2 \chi + 3 \sigma) - \frac{1}{4} c^2. \]
This calculation yields
\[ \dim_R \ker(P_{(\Phi,A)}^0) - \dim_R \mathcal{H}^1 + \dim_R \mathcal{H}^2 = \frac{1}{4} (2 \chi + 3 \sigma) - \frac{1}{4} c^2, \]
where \( \mathcal{H}^1, \mathcal{H}^2 \) are the first and the second cohomology of the complex (*), respectively. In conclusion, we obtain the formula
\[ \dim_R \mathcal{H}^1 + \dim_R \mathcal{H}^2 - \dim_R \ker(P_{(\Phi,A)}^0) = \frac{1}{4} c^2 - \frac{1}{4} (2 \chi + 3 \sigma). \]

**Definition.** A solution of the Seiberg-Witten equation is called reducible if \( \Phi \equiv 0 \). Then \( \Omega_{A}^+ \equiv 0 \). If \( (\Phi, A) \) is irreducible, then \( \ker(P_{(\Phi,A)}^0) = 0 \). This implies
\[ \dim_R \mathcal{H}^1 - \dim_R \mathcal{H}^2 = \frac{1}{4} c^2 - \frac{1}{4} (2 \chi + 3 \sigma). \]
Hence we have computed the virtual dimension \( v = \dim_R \mathcal{M}_L \) of the moduli space.

**Proposition.** One has \( v - \dim_R \mathcal{M}_L = \frac{1}{4} c^2 - \frac{1}{4} (2 \chi + 3 \sigma) \).

Now we want to prove the following generic vanishing theorem for the second cohomology \( \mathcal{H}^2 \) of the complex (*).
Proposition. Let $M^4$ be a compact oriented manifold and $c \in H^2(M^4; \mathbb{Z})$ a $\text{Spin}^C(4)$ structure. For a generic set of metrics, $g \in \text{Met}(M^4)$, the second cohomology $H^2$ of the complex $(\ast)$ is trivial for every irreducible solution $(\Phi \neq 0, A)$ of the Seiberg-Witten equation.

Proof. Fix a metric $g \in \text{Met}(M^4)$ and consider only local perturbations of the metric in a neighbourhood $U$, $g \in U \subset \text{Met}(M^4)$, so that we may identify the spinor bundles corresponding to different metrics. Then regard the Seiberg-Witten equation $D_A^g \Phi = 0$, $(\Omega_A)^{+(g)} = -\frac{1}{4} \omega^\Phi$ as depending also upon the metric $g \in U$. If $g_t = g + t \cdot h$ is a variation of the metric, then the variation of the operator is given by

$$P_{(\Phi, A, g)} : \Gamma(S^+) \oplus \Gamma(\Lambda^1) \oplus \Gamma(S^2(T)) \longrightarrow \Gamma(S^-) \oplus \Gamma(\Lambda^2), \quad P = (P^{-}, P^{\Lambda^2})$$

with components

$$P_{(\Phi, A, g)}^{-}(\varphi, \varepsilon, h) = \varepsilon \Phi + D_A^g \Phi + \frac{d}{dt} \left( D_A^{g+th} \Phi \right)_{t=0},$$

$$P_{(\Phi, A, g)}^{\Lambda^2}(\varphi, \varepsilon, h) = \left( d(\text{tr}(h)) \cdot \Phi, \Psi \right) - \frac{1}{2} \omega^\Phi, \Psi + \frac{d}{dt} (\Omega_A)^{+(g+th)}.$$

Of course, we only have to take into account variations of the metric transversal to the action of the diffeomorphism group of $M^4$, i.e. we can assume $\delta^g(h) = 0$. Hence the variation of the Dirac operator involves, among others, the Clifford product $d(\text{tr}(h)) \cdot \Phi$ and, on the other hand, the variation of the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ only depends on the traceless part of the symmetric tensor $h$, since the Hodge operator $*$ of a 4-dimensional manifold acting on $\Lambda^2$ is conformally invariant. If now $(\Psi, \eta) \in \Gamma(S^-) \oplus \Gamma(\Lambda^2)$ belongs to the cokernel of the operator $P_{(\Phi, A, g)}$, then we deduce the conditions

$$(d(\text{tr}(h)) \cdot \Phi, \Psi) \equiv 0, \quad (\rho(h), \eta) \equiv 0,$$

where $\rho(h)$ describes the change of the $*$-operator. Varying $h \in \Gamma(S^2(T))$, we immediately obtain $\eta \equiv 0$ and $\Psi \equiv 0$, since, as a solution of the elliptic differential equation $D_A^g \Phi = 0 (\Phi \neq 0)$, $\Phi$ does not vanish on a dense set in $M^4$. \hfill $\square$

Corollary. Let $M^4$ be a spin manifold with its canonical $\text{Spin}^C(4)$ structure ($L \equiv \theta^1$). Then,

$$v - \dim M_{g_1} = -1 + b_1 - b_2^+ - \frac{\sigma}{4}.$$

$\square$

Let $L \to M^4$ be a line bundle over $M^4$ and $c_1(L) \in H^2_{DR}(M^4; \mathbb{R})$ its Chern class in de Rham cohomology. A Riemannian metric $g$ on $M^4$ is called "$L$-nice" if there is no 2-form $\omega^2$ such that

1) $\Delta \omega^2 = 0;$
A. Seiberg-Witten Invariants

2) \(*\omega^2 = -\omega^2;\)
3) \([\omega^2]_{DR} = c_1(L).\)

We first discuss under which conditions there exists a “nice” metric. The ∧-product defines a bilinear form

\[ \wedge : H^2_{DR}(M^4; \mathbb{R}) \times H^2_{DR}(M^4; \mathbb{R}) \to \mathbb{R}. \]

For a metric \(g\), denote by \(H^2(g)\) the space of all harmonic forms. Then there is an isomorphism

\[ H^2(g) \approx H^2_{DR}(M^4; \mathbb{R}). \]

The Hodge operator \(* : H^2(g) \to H^2(g)\) is an involution. Hence each metric \(g\) defines an involution on de Rham cohomology,

\[ *_g : H^2_{DR}(M^4; \mathbb{R}) \to H^2_{DR}(M^4; \mathbb{R}), \]

and \(*_g\) preserves the ∧-product. Let \(E^+(g)\) and \(E^-(g)\) be the eigensubspaces for \(\pm 1\) of \(*_g\). Then, \(\dim E^+(g) = b_2^+\).

First case. If \(b_2^+ = 0\) and \(c_1(L) \neq 0\), then there exists a \(g\)-harmonic form \(\omega_2\) in \(c_1(L)\) such that \(\Delta \omega_2 = 0\). As \(b_2^+ = 0\), we have \(*\omega_2 = -\omega_2\). Hence, in this case, there is no nice metric for \(c_1(L)\) at all.

Second case. \(b_2^+ = 1\). In this case, \(H^2_{DR}\) is a pseudo-Euclidean space with index \((1, b_2^-)\). If \(b_2^- = 0\), then every metric is nice. Now suppose that \(b_2^- > 0\). Then we have \((H^2_{DR}, \Lambda) \simeq (\mathbb{R}^{b_2^+}, z^2 - x_1^2 - \cdots - x_{b_2^-}^2)\) and, for transversality reasons, it is obvious that an \(L\)-nice metric \(g\) can be chosen. However, two such metrics cannot necessarily be joined, since \(E^-(g_\varepsilon)\) has to stay in the range \(z^2 - x_1^2 - \cdots - x_{b_2^-}^2 < 0\)! Altogether, this implies:

If \(b_2^+ = 1, b_2^- > 0\) and \(c_1^2(L) < 0\), then the space of \(L\)-nice metrics is not connected. (If \(c_1^2(L) > 0\), then every metric is \(L\)-nice.)

Third case. \(b_2^+ \geq 2\). It is clear that any two metrics in \(E^-(g)\) can be deformed within \(x_1^2 + x_2^2 + \cdots + x_{b_2^+}^2 - y_1^2 - \cdots - y_{b_2^+}^2 < 0\) in such a way that \(c_1^2(L)\) is not met. This implies

a) if \(b_2^+ \geq 2\) and \(c_1^2(L) > 0\), then every metric is \(L\)-nice;

b) if \(b_2^+ \geq 2\) and \(c_1^2(L) < 0\), then the space of \(L\)-nice metrics is connected.

Proposition. Let \((M^4, g)\) be an oriented Riemannian manifold and \(c\) a \(\text{Spin}^C(4)\) structure with line bundle \(L\). Let \(g\) be an \(L\)-nice metric. Then there are no reducible solutions of the Seiberg-Witten equation.
Proof. $D_A \Phi = 0$, $\Omega_A^+ = -\frac{1}{4} \omega^\Phi$ and $\Phi \equiv 0$ together immediately imply $\Omega_A^+ = 0$. Hence $\Omega_A$ is an anti-self-dual 2-form, $*\Omega_A = -\Omega_A$. Thus $\Omega_A$ is also harmonic, $\Delta \Omega_A = 0$, since $d\Omega_A = 0$ and $[\frac{i}{2\pi} \Omega_A]_{DR} = c_1(L)$. However, by the assumption on the metric, there is no such form $\Omega_A$. □

For the moduli space $\mathcal{M}_L = \mathcal{M}_L(g)$ the following picture results.

First case. $b_2^+(M^4) \geq 2$. Then, for a generic metric $g$ on $M^4$, we have that $g$ is an $L$-nice metric, $\mathcal{H}^2 \approx \ker((P_{(\Phi,A)}^1)^*) = 0$, and the space of these metrics is pathwise connected. Consequently, for such a metric, $\mathcal{M}_L(g)$ is empty or a compact smooth manifold of dimension

$$\dim \mathcal{M}_L(g) = \frac{1}{4} c^2 - \frac{1}{4} (2\chi + 3\sigma)$$

and the bordism class in the unoriented bordism ring $\mathcal{N}_*$ is independent of the metric. Define the Seiberg-Witten invariant as

$$S - W(M^4, c) = [\mathcal{M}_L(g)] \in \mathcal{N}_*.$$

Second case. $b_2^+(M^4) = 1$.

Case 2.1. $c_1^2(L) > 0$, $b_2^+(M^4) = 1$. In this case, each metric $g$ is $L$-nice and, generically, $\mathcal{H}^2 \approx \ker((P_{(\Phi,A)}^1)^*) = 0$. As above, the bordism class $\mathcal{M}_L(g)$ is uniquely determined.

Case 2.2. $c_1^2(L) < 0$, $b_2^+(M^4) = 1$. A generic metric $g$ is $L$-nice and we have $\mathcal{H}^2 \approx \ker((P_{(\Phi,A)}^1)^*) = 0$. Hence $\mathcal{M}_L(g)$ is empty or a compact smooth manifold, but the space of the metrics in question is not necessarily connected. A definition of

$$S - W(M^4, c) = [\mathcal{M}_L(g)] \in \mathcal{N}_*$$

is only possible for a chosen connected component in the corresponding space of metrics (or following a different line of argument).

Third case. $b_2^+(M^4) = 0$. In this case, we have $\mathcal{H}^2 \approx \ker((P_{(\Phi,A)}^1)^*) = 0$ for a generic metric. However, now there exist reducible solutions. If $\Phi \equiv 0$, then $\Omega_A^+ \equiv 0$ implies that the curvature $\Omega_A$ is the only $g$-harmonic form in $c_1(L)$. By the Weyl theorem, the set

$$\left\{ (\Phi \equiv 0, A) : \left[ \frac{i}{2\pi} \Omega_A \right] = c_1(L) \right\} / \mathcal{G}(P)$$

is diffeomorphic to the Picard manifold $Pic(M^4) = H^2(M^4; \mathbb{R})/H^2(M^4; \mathbb{Z})$ of $M^4$. The moduli space $\mathcal{M}_L(g)$ is empty or a compact manifold with singular set $Pic(M^4) = T^{b_1(M^4)}$. Its "bordism class" is again unique.
Remark. To define the Seiberg-Witten invariant within the unoriented bordism ring by

\[ S - W(M^4, c) = [\mathcal{M}_L(g)] \in \mathcal{N}_* \]

is the simplest possibility. More refined invariants are obtained by the observations to follow.

Observation 1. On \( \mathcal{M}_L(g) \) there exists a universal \( S^1 \)-principal bundle. If \( m_0 \in M^4 \) is a fixed point, then we consider the subgroup \( G_0 \subset G(P) \) of the gauge group given by

\[ G_0 = \{ f : M^4 \to S^1 : f(m_0) = 1 \}. \]

Then \( \mathcal{M}_L^0(g) = \{ (\Phi, A) : D_A \Phi = 0, \Omega_A^+ = -\frac{1}{2} \omega^\Phi \}/G_0 \) is a principal bundle with structure group \( G/G_0 = S^1 \) over \( \mathcal{M}_L(g) \). Hence \( \mathcal{M}_L(g) \) has a distinguished cohomology class \( e \in H^2(\mathcal{M}_L(g), \mathbb{Z}) \). Thus the Seiberg-Witten invariant can be understood as a pair \( (\mathcal{M}_L(g), e) \) consisting of a bordism class and a cohomology element from \( H^2 \).

Observation 2. The tangent space \( T_{(\Phi, A)} \mathcal{M}_L(g) \) can, for a generic metric, roughly speaking be identified with the sum

\[ \{ \Psi \in \Gamma(S^+) : D_A \Psi = 0 \} \oplus \{ \eta^1 \in \Gamma(\Lambda^1) : \delta \eta^1 = 0, (d\eta^1)^+ = 0 \}. \]

The former of these spaces is a complex vector space and hence has a canonical orientation. The second space results from the complex \( \Gamma(\Lambda^0) \xrightarrow{\delta} \Gamma(\Lambda^1) \xrightarrow{d^+} \Gamma(\Lambda^2_+) \) or from the operator \( \delta \oplus d^+ : \Gamma(\Lambda^1) \to \Gamma(\Lambda^0) \oplus \Lambda(\Lambda^2_+) \). Thus the determinant of the second space is given by

\[ \det(\delta \oplus d^+) = \det \ker(\delta \oplus d^+) \otimes \det \coker(\delta \oplus d^+) = \det H^1 \otimes \det H^2. \]

Now if an orientation in \( H^1(M^4; \mathbb{R}) \oplus H^2_+(M^4; \mathbb{R}) \) is fixed, then \( \mathcal{M}_L(g) \) can be oriented. Considering the triple \( (M^4, c, D) \) consisting of

1) a 4-dimensional compact and oriented manifold \( M^4 \),
2) a \( \text{Spin}^C(4) \) structure \( c \in H^2(M^4, \mathbb{Z}) \),
3) an orientation \( D \) in \( H^1(M^4; \mathbb{R}) \oplus H^2_+(M^4; \mathbb{R}) \),

then, in the cases \( b_2^+(M^4) \geq 2 \) or \( b_2^+(M^4) = 1 \) and \( c^2 < 0 \), the Seiberg-Witten invariant is defined in the oriented bordism ring \( \Omega_*^{SO} \).

A.4. Vanishing theorems

Consider a 4-dimensional manifold \( M^4 \) with \( b_2^+ \geq 1 \) and assume that \( M^4 \) has a metric with positive scalar curvature, \( R > 0 \). Perturbing the metric, we can arrange it to be an \( L \)-nice metric as well. Then, \( \mathcal{M}_L(g) \) is the space of all flat connections,

\[ \mathcal{M}_L(g) = \{ (\Phi \equiv 0, A) : \Omega_A^+ = 0 \}/G = \{ A \in C(P) : \Omega_A = 0 \}/G. \]
If $c_1(L) \in H^2(M^4)$ is not a torsion element, then $\mathcal{M}_L(g) = \emptyset$. Thus we have the

**Proposition.** Let $M^4$ admit a metric of positive scalar curvature and suppose that $c_1(L) \in H^2(M^4, \mathbb{Z})$ is not a torsion element. Then,

$$S - W(M^4, c) = 0.$$  

This holds for $b_2^+(M^4) \geq 2$ in general. In the case $b_2^+(M^4) = 1$, consider as $S - W(M^4, c)$ the Seiberg-Witten invariant computed with respect to the component containing the metric of positive scalar curvature. \hfill \Box

Let $g$ be a fixed Riemannian metric and $(\Phi, A)$ a solution of

$$D_A \Phi = 0, \quad \Omega_A^+ = -\frac{1}{4} \omega \Phi.$$

We already know that at each point $|\Phi(x)|^2 \leq -R_{\min}$. By the Schrödinger-Lichnerowicz formula,

$$\Delta |\Phi|^2 = -\frac{R}{2} |\Phi|^2 - \frac{1}{2} |\Phi|^4 - 2|\nabla A \Phi|^2,$$

and hence

$$\int_{M^4} |\Phi|^4 \leq \int_{M^4} (-R)|\Phi|^2 \leq R_{\min}^2 \text{vol}(M^4).$$

This implies

$$\int_{M^4} |\Omega_A^+|^2 = \frac{1}{16} \int_{M^4} |\omega|^2 = \frac{1}{8} \int_{M^4} |\Phi|^4 \leq \frac{1}{8} \cdot R_{\min}^2 \text{vol}(M^4).$$

As $\Omega_A$ is a curvature in $L$,

$$c_1^2(L) = \frac{1}{4\pi^2} \int_{M^4} (|\Omega_A^+|^2 - |\Omega_A^-|^2).$$

If $\mathcal{M}_L(g) \neq \emptyset$, this implies $c_1^2(L) - (2\chi + 3\sigma) \geq 0$, and hence

$$2\chi + 3\sigma \leq c_1^2(L) = \frac{1}{4\pi^2} \int_{M^4} (|\Omega_A^+|^2 - |\Omega_A^-|^2),$$

i.e.

$$\int_{M^4} |\Omega_A^-|^2 \leq \int_{M^4} |\Omega_A^+|^2 - 4\pi^2(2\chi + 3\sigma).$$

Thus $\int |\Omega_A^+|^2$ as well as $\int |\Omega_A^-|^2$ is bounded, if $\mathcal{M}_L$ is not empty. This implies the
Proposition. Let \((M^4, g)\) be a compact oriented manifold with fixed Riemannian metric \(g\). Then there exist at most finitely many \(\text{Spin}^C(4)\) structures \(c \in H^2(M^4, \mathbb{Z})\) for which \(\mathcal{M}_L(g)\) is not empty.

A.5. The case \(\dim \mathcal{M}_L(g) = 0\)

The Seiberg-Witten invariant as an element in the bordism ring becomes a numerical invariant for \(\dim \mathcal{M}_L(g) = 0\). If \(M^4\) is a compact oriented 4-dimensional manifold and \(c \in H^2(M^4; \mathbb{Z})\) a \(\text{Spin}^C(4)\) structure with \[c^2 - (2\chi + 3\sigma) = 0,\]

then define the \(\mathbb{Z}_2\)-invariant \[n_L(g) = \text{number of points in the moduli space } \mathcal{M}_L(g) \text{ mod } 2.\]

In the case \(b_2^+ \geq 2\), the invariant \(n_L(g) \in \mathbb{Z}_2\) is uniquely determined for a generic metric; for \(b_2^+ = 1\) a "component" in the space of \(L\)-nice metrics has to be fixed additionally. If, moreover, an orientation \(O\) in \(H^1(M^4; \mathbb{R}) \oplus H^2_+(M^4; \mathbb{R})\) is given, then \(n^O_L(g) \in \mathbb{Z}\) is defined as an integral invariant to be the alternating sum of the points in \(\mathcal{M}_L(g)\) which now carry a sign. However, this numerical Seiberg-Witten invariant does not exist for every manifold \(M^4\). Namely, the existence of a cohomology element \(c \in H^2(M^4; \mathbb{R})\) with \(c^2 = 2\chi + 3\sigma\) and \(c \equiv \omega_2(M^4) \mod 2\) is a necessary condition. The number \(2\chi + 3\sigma = (2e + p_1)[M^4]\) has to be representable by an element \(c \in H^2(M^4; \mathbb{R})\). Here \(e \in H^4(M^4; \mathbb{R})\) denotes the Euler class and \(p_1\) the first Pontryagin class. In other words, the resulting necessary condition on \(M^4\) requires that the cohomology class \(2e + p_1 \in H^4(M^4; \mathbb{R})\) has to belong to the image of the quadratic intersection form \(H^2(M^4; \mathbb{R}) \in c \rightarrow c^2 \in H^4(M^4; \mathbb{R})\). Suppose that \(M^4\) has an almost-complex structure \(J : TM^4 \rightarrow TM^4\) inducing the orientation. Then \(TM^4\) is a complex 2-dimensional vector bundle over \(M^4\) having Chern classes \(c_1 \in H^2(M^4; \mathbb{Z})\) and \(c_2 \in H^2(M^4; \mathbb{Z})\). Since the complex structure \(J\) is compatible with the orientation, the real characteristic classes \(\omega_2, e\) and \(p_1\) of \(M^4\) are related to \(c_1\) and \(c_2\) via \[e \equiv c_2, \quad \omega_2 \equiv c_1 \mod 2, \quad p_1 = c_1^2 - 2c_2.\]

This immediately implies \(2e + p_1 = c_1^2\). The argument can also be reversed. The following proposition holds.

Proposition (Hirzebruch and Hopf, 1958). Let \(M^4\) be a compact oriented 4-dimensional manifold. Then there exists an almost-complex structure on \(M^4\) inducing the orientation if and only if there is an element \(c \in H^2(M^4; \mathbb{Z})\) with \(c^2 = 2e + p_1\).
A.6. The Kähler case

Let \((M^4, g, J)\) be a Kähler manifold (or, for the moment, just an Hermitian manifold) and denote by \(P_J \subset P(M^4, g)\) the corresponding \(U(2)\)-principal bundle. Then \(M^4\) has the canonical \(Spin^C(4)\) structure

\[ Q = P_J \times_i Spin^C(4) \]

with the associated line bundle \(L = \Lambda^2(TM^4)\). Consequently, for the canonical \(Spin^C(4)\) structure, \(c_1(L) = c_1(M^4)\). As is well-known, the signature and the Euler characteristic of the Kähler manifold \(M^4\) are given by

\[ \sigma = \frac{1}{3}(c_1^2 - 2c_2), \quad \chi = c_2. \]

This implies

\[ v - \dim \mathfrak{M}_L(g) = \frac{1}{4}c_1^2(L) - \frac{1}{4}(2\chi + 3\sigma) = \frac{1}{4}\{c_1^2 - (2c_2 + c_1^2 - 2c_2)\} = 0. \]

Hence the virtual dimension of the moduli space \(\mathfrak{M}_L(g)\) vanishes for a Kähler manifold with respect to the canonical \(Spin^C(4)\) structure.

**Remark.** This computation of the virtual dimension can also be performed in a more general situation. Consider an oriented manifold \(M^4\) and an almost-complex structure \(J : TM^4 \to TM^4, \ J^2 = -\text{Id}\). Then, \(\text{det}(J) = 1\). The set of almost-complex structures has two components. One component consists of those \(J\) for which \(\{X, JX, Y, JY\}\) defines the given orientation, and the other contains those \(J\) defining the opposite one. Let \(\mathcal{J}^+(M^4, \mathcal{O})\) and \(\mathcal{J}^-(M^4, \mathcal{O})\) be the corresponding bundles. If \(J\) is a section in \(\mathcal{J}^+(M^4, \mathcal{O})\), then \((TM^4, J)\) is a complex vector bundle and the second Chern class, \(c_2\), coincides with the Euler class of \(M^4\). For \(c = c_1(TM^4, J) \in H^2(M^4; \mathbb{Z})\),

\[ v - \dim \mathfrak{M}_L(g) = \frac{1}{4}c_1^2 - \frac{1}{4}(2\chi + 3\sigma) = \frac{1}{4}c_1^2 - \frac{1}{4}(2c_2 + p_1) = \frac{1}{4}c_1^2 - \frac{1}{4}(2c_2 + c_1^2 - 2c_2) = 0. \]

For the Pontryagin class \(p_1\) of the oriented real bundle \(TM^4\) we made use of the formula \(p_1(TM^4) = c_1^2(TM^4, J) - 2c_2(TM^4, J)\).

Let \(\Omega(X, Y) = g(X, JY)\) be the Kähler form of \((M^4, g, J)\). \(\Omega\) acts as an endomorphism in the spinor bundle \(\Omega : S^+ \to S^+\) and has the eigenvalues \(\pm 2i\). Indeed, with respect to a basis from the principal bundle \(P_J\),

\[ \Omega = e_1 \wedge e_2 + e_3 \wedge e_4, \]

the endomorphisms \(e_1 \wedge e_2 = e_3 \wedge e_4 : S^+ \to S^+\) coincide and are both given by the matrix

\[
\begin{pmatrix}
i & 0 \\0 & -i\end{pmatrix}.
\]
Let $S^+(\pm 2i) \subset S^+$ be the corresponding subbundles. Furthermore, let $\Phi \in S^+$ be a spinor from $S^+(2i)$, i.e. $\Omega \Phi = 2i \Phi$. Identifying by means of the basis $S^+ \cong \mathbb{C}^2$, the spinor $\Phi$ has the components $\Phi = \begin{pmatrix} \Phi_1 \\ 0 \end{pmatrix}$. Computing the 2-form $\omega^\Phi$ yields $\omega^\Phi = i |\Phi|^2 \Omega$ for $\Phi \in S^+(2i)$. Analogously, one computes $\omega^\Phi = -i |\Phi|^2 \Omega$ for spinors $\Phi \in S^+(-2i)$. The bundles $S^+(2i)$ and $S^+(-2i)$ are isomorphic to $\Lambda^{0,2}$ and $\Lambda^{0,0}$, respectively (compare Section 3.4).

Denote by $\Phi_0$ the spinor in $S^+(-2i) \cong \Lambda^{0,0}$ corresponding to the function 1. Then, in the chosen coordinates,

$$\Phi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega^{\Phi_0} = -i \Omega.$$ 

The bundle $L = \Lambda^2 T$ of the canonical spin$^C$ structure has a distinguished connection $A_0$, the Levi-Civita connection. Considering the corresponding Dirac operator

$$D_{A_0} : \Gamma(S^+) \to \Gamma(S^-)$$

and identifying as above

$$S^+ \cong \Lambda^{0,0} \oplus \Lambda^{0,2}, \quad S^- = \Lambda^{0,1},$$

we see that $D_{A_0}$ coincides with the $\bar{\partial}$-operator of the Dolbeault complex:

$$D_{A_0} = \sqrt{2}(\bar{\partial}_0 \oplus \bar{\partial}_2^*).$$

Now suppose that the scalar curvature $R$ of the Kähler manifold $(M^4, g, J)$ is negative and constant, $R \equiv \text{const} < 0$. Then $\Phi = \sqrt{-R} \Phi_0$ is a section in $S^+$, and

$$D_{A_0} \Phi = 0, \quad \omega^\Phi = -|\Phi|^2 i \Omega = -(-R) i \Omega = R i \Omega.$$ 

On the other hand, the curvature $\Omega_{A_0}$ in the line bundle $L = \Lambda^2 T$ is given by the Ricci form $\rho$,

$$\Omega_{A_0} = i \rho$$

with $\rho(X, Y) = g(X, J \text{Ric} Y)$. Here $\text{Ric} : T \to T$ denotes the Ricci tensor. In a basis, $J : T \to T$ and the Ricci tensor are respectively given by the matrices

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{Ric} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{12} & R_{22} & R_{23} & R_{24} \\ R_{13} & R_{13} & R_{33} & R_{34} \\ R_{14} & R_{24} & R_{34} & R_{44} \end{pmatrix}.$$
Since $J$ and Ric commute, we obtain the following special shape for the skew-symmetric endomorphism:

$$J \circ \text{Ric} = \begin{pmatrix} 0 & -A & D & -C \\ A & 0 & C & D \\ -D & -C & 0 & -B \\ C & -D & B & 0 \end{pmatrix}, \quad A = R_{11} = R_{22}, \quad B = R_{33} = R_{44},$$

and hence $\rho$ takes the form

$$\rho = Ae_1 \wedge e_2 + Be_3 \wedge e_4 + C(e_1 \wedge e_4 - e_2 \wedge e_3) - D(e_1 \wedge e_3 + e_2 \wedge e_4).$$

Now $e_1 \wedge e_4 - e_2 \wedge e_3$ and $e_1 \wedge e_3 + e_2 \wedge e_4$ are 2-forms in $\Lambda^2_+$. For the projection $\rho^+$ of the form $\rho$ onto the subbundle $\Lambda^2_+$ this implies the formula

$$\rho^+ = \frac{A + B}{2}(e_1 \wedge e_2 + e_3 \wedge e_4).$$

Since $A + B = R_{11} + R_{33} = \frac{1}{2}(R_{11} + R_{22} + R_{33} + R_{44})$, we thus obtain the following relation, holding in every 4-dimensional Kähler manifold:

$$\rho^+ = \frac{R}{4} \Omega.$$  

As

$$\Omega_{A_0} = i \rho = \frac{i}{4} R \Omega = \frac{1}{4} \omega^\Phi,$$

the pair $(\Phi, A_0) = (\sqrt{-R} \Phi_0, A_0)$ is a solution of the Seiberg-Witten equation.

**Theorem** (LeBrun 1994). Let $(M^4, J, g)$ be a compact Kähler manifold of constant negative scalar curvature, $R < 0$, and choose the canonical Spin$^C(4)$ structure. Then, for the Seiberg-Witten invariant,

$$n_c(g) = \begin{cases} 1 & \text{in } \mathbb{Z}_2 \\ b_2^+ \geq 2 & \text{for the Seiberg-Witten invariant to be defined independently of the metric}. \end{cases}$$

**Proof.** Let $(\Phi, A)$ be an arbitrary solution of the equation

$$DA \Phi = 0, \quad \Omega^+_A = \frac{1}{4} \omega^\Phi.$$  

Then,

$$|\Phi(m)|^2 \leq -R_{\text{min}} = - \frac{R}{4} \quad \text{and} \quad |\Omega^+_A|^2 = \frac{1}{16} |\omega^\Phi|^2 = \frac{1}{8} |\Phi|^4 \leq \frac{R^2}{8}.$$  

Since $\rho^+ = \frac{R}{4} \Omega$ and $|\Omega|^2 = 2$, this implies

$$\int_{M^4} |\Omega^+_A|^2 \leq \int_{M^4} \frac{R^2}{8} = \int_{M^4} \frac{R^2 |\Omega|^2}{16} = \int_{M^4} \left( \frac{R |\Omega|}{4} \right)^2 = \int_{M^4} |\rho^+|^2.$$
On the other hand, since $\rho$ is the curvature form of the Levi-Civita connection in the bundle $\Lambda^2 T$, it follows that

$$c_1^2(\Lambda^2 T) = \frac{1}{4\pi^2} \int_{M^4} (|\rho^+|^2 - |\rho^-|^2),$$

and hence

$$\int_{M^4} |\rho^+|^2 = 2\pi^2 c_1^2(\Lambda^2 T) + \frac{1}{2} \int_{M^4} |\rho|^2.$$

The scalar curvature of the Kähler manifold is constant. Therefore, the Ricci form $\rho$ is a harmonic 2-form, a consequence of the Bianchi identity. The harmonic form realizes the minimum of the $L^2$-norm in each cohomology class. $A$ is a connection in $\Lambda^2 T$, and hence

$$\int_{M^4} |\rho|^2 \leq \int_{M^4} |\Omega_A|^2.$$

Thus,

$$\int_{M^4} |\rho^+|^2 \leq 2\pi^2 c_1^2(L) + \frac{1}{2} \int_{M^4} |\Omega_A|^2 = \int_{M^4} |\Omega_A^+|^2. \quad (**)$$

Combining (*) and (**) yields $\Omega_A = i\rho$ and $|\Phi|^2 \equiv -R$. The proof of the estimate $|\Phi|^2 \leq -R_{\min}$ then immediately implies $\nabla^A \Phi \equiv 0$. Decompose $\Phi$ according to the splitting of the spinor bundle $S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ into

$$\Phi = \Phi^{0,0} \oplus \Phi^{0,2}.$$ 

$\Phi^{0,2}$ is a $\nabla^A$-parallel section in $\Lambda^{0,2}$. If it is non-trivial, then we conclude that $c_1(\Lambda^{0,2}) = c_1(\Lambda^2 T) = 0$. However, the curvature form of the Levi-Civita connection $A_0$ in $\Lambda^2 T$ is

$$\Omega^+_{A_0} = i\rho^+ = i \frac{R}{4} \Omega \neq 0,$$

since $R < 0$. Hence $\Phi^{0,2}$ vanishes and $\Phi$ is proportional to the standard spinor, $\Phi = f \Phi_0$. The length $|f|^2 = -R$ is constant, since $|\Phi|^2$ is constant. The connection $A$ differs from the Levi-Civita connection $A_0$ by an imaginary-valued 1-form $\eta^1$, $A - A_0 = \eta^1$. As $\Omega_A = i\rho$, we have $d\eta^1 = 0$. But

$$0 = D_A \Phi = D_{A_0} \Phi + \eta^1 \cdot \Phi = \text{grad}(f) \cdot \Phi_0 + \eta^1 f \Phi_0.$$ 

This implies $\text{grad} f + \eta^1 f = 0$. Now consider the gauge transformation

$$g = \frac{f}{|f|} : M^4 \to S^1.$$
Locally we can write \( f \) as \( f = \sqrt{-R} e^{iF} \) and \( dg = \frac{1}{R} df \). Then, \( g = e^{iF} = \frac{1}{R} df \). Since \( \text{grad}(f) + f \eta^1 = 0 \), this implies \( \eta^1 = -\frac{dg}{g} \). In addition, we have found a gauge transformation \( g : M^4 \to \mathbb{R}^1 \) with
\[
A = A_0 - \frac{dg}{g}, \quad \Phi = g \cdot (\sqrt{-R} \Phi_0),
\]
i.e. \((\Phi, A)\) is equivalent to \((\sqrt{-R} \Phi_0, A_0)\).

**Corollary.** Let \( M^4 \) be a compact oriented manifold with \( b_2^+ \geq 2 \). If \( M^4 \) has a Kahler structure with constant negative scalar curvature, then \( M^4 \) admits no Riemannian metric of positive scalar curvature.

**Remark.** The preceding corollary is the special case of a more general fact. Consider a 4-dimensional compact and oriented manifold \( M^4 \) with a symplectic structure \( \omega \) and assume that \( \omega \wedge \omega > 0 \) defines the fixed orientation. Choose an almost-complex structure \( J : TM^4 \to TM^4, \ J^2 = -\text{Id}, \) for which \( g(X, Y) = \omega(X, JY) \) is a Riemannian metric. From \( \omega \wedge \omega > 0 \) one easily concludes that \( J \) is a section in the bundle \( J^+(M^4, \mathcal{O}) \). Let \( c = c(\omega) \in H^2(M^4; \mathbb{Z}) \) be the first Chern class of the complex vector bundle \((TM^4, J)\). Then, by the observations above,
\[
v - \dim \mathcal{M}_L = 0,
\]
i.e. for a generic metric the moduli space is discrete.

**Theorem** (Taubes, 1994). Let \((M^4, \mathcal{O})\) be an oriented compact 4-dimensional manifold and \( \omega \) a symplectic form with \( \omega \wedge \omega > 0 \). Moreover, assume that \( b_2^+(M^4) \geq 2, \) i.e. the Seiberg-Witten invariant is defined. If \( c = c(\omega) \in H^2(M^4; \mathbb{Z}) \) is the Chern class of an almost-complex structure associated with \( \omega \), then
\[
n_c(M^4) = 1 \mod 2.
\]

**Corollary.** Let \((M^4, \mathcal{O})\) be an oriented compact 4-dimensional manifold satisfying the following two conditions:

1) \( b_2^+(M^4) \geq 2, \)

2) \( M^4 \) has a symplectic structure \( \omega \) with \( \omega \wedge \omega > 0 \).

Then \( M^4 \) admits no Riemannian metric of positive scalar curvature.

We will now turn to the case of a Kahler manifold \((M^4, g, J)\) where the \( \text{Spin}^C(4) \) structure \( c \in H^2(M^4; \mathbb{Z}) \) is not necessarily the canonical one of \( M^4 \). As before, denote the line bundle by \( L \). The Kahler form \( \Omega \) acts as an endomorphism on the spinor bundle and has the eigenvalues \( \pm 2i \) there. Hence the spinor bundle splits,
\[
S^+ = S^+(2i) \oplus S^+(-2i).
\]
$L$ is isomorphic to $\Lambda^2 S^+$ for each $\text{Spin}^c(4)$ structure, and hence we obtain the isomorphism
\[ S^+(2i) \otimes S^+(-2i) = L. \]
For every connection $A \in C(L)$ the Kähler form $\Omega$ is parallel, and hence the decomposition $S^+ = S^+(i) \oplus S^+(-i)$ is $\nabla^A$-parallel, too. For a spinor $\Phi = \Phi_+ + \Phi_-$ the integral formula now takes the following form:
\[
\int_{M^4} |\Omega_A^+ + \frac{1}{4} \omega^\Phi|^2 + |D_A \Phi|^2 = \\
\int_{M^4} \left\{ |\Omega_A^+|^2 + |\nabla^A \Phi_+|^2 + |\nabla^A \Phi_-|^2 + \frac{R}{4} (|\Phi_+|^2 + |\Phi_-|^2) + \frac{1}{8} (|\Phi_+|^2 + |\Phi_-|^2)^2 \right\}.
\]
This implies that for a given solution $(\Phi = \Phi_+ + \Phi_-, A)$ the pair $(\hat{\Phi} = \Phi_+ - \Phi_-, A)$ also solves the Seiberg-Witten equation. This remark, in turn, allows us to reduce the Seiberg-Witten equation considerably. Start from a solution of the equation,
\[
D_A \Phi = 0, \quad \Omega_A^+ = -\frac{1}{4} \omega^\Phi
\]
with $\Phi = \Phi_+ + \Phi_-$, and note that, moreover,
\[
D_A \hat{\Phi} = 0, \quad \Omega_A^+ = -\frac{1}{4} \omega^{\hat{\Phi}}
\]
with $\hat{\Phi} = \Phi_+ - \Phi_-$. In a local orthonormal frame $e_1, \ldots, e_4$ on $M^4$ the Kähler form $\Omega$ has the form
\[
\Omega = e_1 \wedge e_2 + e_3 \wedge e_4
\]
and the spinor $\Phi = (\Phi_+, \Phi_-)$ is given by its components. By the definition of $\omega^\Phi$,
\[
\omega^\Phi = i(|\Phi_+|^2 - |\Phi_-|^2) (e_1 \wedge e_2 + e_3 \wedge e_4) \\
+ (\Phi_+ \Phi_- - \Phi_- \Phi_+) (e_1 \wedge e_3 - e_2 \wedge e_4) \\
- i(\Phi_+ \Phi_- + \Phi_- \Phi_+) (e_1 \wedge e_4 + e_2 \wedge e_3).
\]
Therefore,
\[
\omega^\Phi + \omega^{\hat{\Phi}} = 2i(|\Phi_+|^2 - |\Phi_-|^2) \Omega.
\]
Since $\Omega_A^+ = -\frac{1}{4} \omega^\Phi = -\frac{1}{4} \omega^{\hat{\Phi}}$, we obtain
\[
-4\Omega_A^+ = i(|\Phi_+|^2 - |\Phi_-|^2) \Omega \quad \text{and} \quad \Phi_+ \cdot \Phi_- \equiv 0.
\]
Thus, either $\Phi_+ \equiv 0$ or $\Phi_- \equiv 0$. Moreover, $\Omega$ is a $\Lambda^{1,1}$-form, and hence, since $\Lambda^{0,2} \cap \Lambda_2 = \{0\} = \Lambda^{2,0} \cap \Lambda_2$, we at once arrive at
\[
\Omega_A^{0,2} = 0 = \Omega_A^{2,0}.
\]
The curvature form of the connection $A$ in $L$ is thus a $(1,1)$-form. Hence $A$ defines a holomorphic structure in $L$, and, therefore, in $S^+(\pm 2i)$ as well. In this case, the Dirac equation, $D_A \Phi_+ = 0$ (or $D_A \Phi_- = 0$, respectively), means that $\Phi_\pm$ is holomorphic. The Chern class of $L$ is given by

$$c_1(L) = \frac{i}{2\pi} \Omega_A, \quad c_1^+(L) = \frac{i}{2\pi} \Omega_A^+ = \frac{1}{8\pi} (|\Phi_+|^2 - |\Phi_-|^2) \Omega,$$

and hence, from $\Omega \wedge c_1 = \Omega \wedge c_1^+$, we conclude that

$$J := \int_{M^4} \Omega \wedge c_1(L) = \frac{1}{8\pi} \int_{M^4} (|\Phi_+|^2 - |\Phi_-|^2) \Omega \wedge \Omega.$$

$J$ is the cup product of $\Omega$ and $c_1(L)$, hence a topological invariant. If $J < 0$, then $\Phi_+ \equiv 0$; in case $J > 0$ we thus have $\Phi_- \equiv 0$. Taking into account, in addition, the action of the gauge group leads to the

**Theorem** (Witten, 1994). Every solution of the Seiberg-Witten equation over a Kähler manifold $M^4$ for the spin$^C$ structure $c \in H^2(M^4; \mathbb{Z})$ corresponds to a pair consisting of a holomorphic structure in the bundle $S^+(\pm 2i)$ and an element of

$$\mathbb{P}H^0(M^4; S^+(\pm 2i)) \quad \text{if} \quad \Omega \wedge c < 0,$$

$$\mathbb{P}H^0(M^4; S^+(\pm 2i)) \quad \text{if} \quad \Omega \wedge c > 0.$$

For $b_1(M^4) = 0$, the holomorphic structure in $S^+(\pm 2i)$ is unique and

$$\mathcal{M}_c(L) \approx \mathbb{P}H^0(M^4; S^+(\pm 2i)).$$

**Remark.** In general, $\mathcal{M}_c(L)$ cannot be used to determine the Seiberg-Witten invariant $S - W(M^4, c) \in \mathcal{N}$, since the Kähler metric $g$ is not generic.

**A.7. References**


Principal Bundles and Connections

B.1. Principal fibre bundles

Definition. Let $E, X$ and $F$ be three topological spaces. A mapping $\pi : E \to X$ is called a locally trivial fibration with fibre $F$ if for each point $x_0 \in X$ there exist a neighborhood $U(x_0)$ and a homeomorphism $\Phi_{U(x_0)} : p^{-1}(U(x_0)) \to U(x_0) \times F$ such that the diagram

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\Phi_u} & U \times F \\
\downarrow \pi & & \downarrow \text{pr}_1 \\
U & & \\
\end{array}
\]

commutes. Then $E$ is called the total space of the fibration, $X$ its base and $F = E_x = \pi^{-1}(x)$ the fibre over $x \in X$.

Example 1. Consider $E = X \times F$ and the projection $\pi : E \to X$. Then $(E, \pi, X; F)$ is a locally trivial fibration.

Example 2. Let $\pi : E \to X$ be an unramified covering, and let $F$ be a discrete space whose points can be mapped bijectively onto $\pi^{-1}(x)$. Then $\pi : E \to X$ is a locally trivial fibration with fibre $F$.

Example 3. Let $X = M^n$ be a smooth manifold, $E = T(M^n)$ and $\pi : E \to X$ the projection of the tangent bundle. Then $(T(M^n), \pi, M^n; \mathbb{R}^n)$ is a locally trivial fibration with fibre $F = \mathbb{R}^n$.

Example 4. Let $G$ be a Lie group, $H \subset G$ a closed subgroup and $E = G$, $X = G/H$. Then the projection $\pi : G \to G/H$ is a smooth locally trivial fibration with fibre $H$. 

Definition. Let $E$ and $E^*$ be two fibrations over $X$ with projections $\pi$ and $\pi^*$, respectively. These fibrations are called equivalent if there is a homeomorphism $f : E \to E^*$ fitting into the commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E^* \\
\downarrow{\pi} & & \downarrow{\pi^*} \\
X & \xrightarrow{f} & X
\end{array}
\]

Definition. A locally trivial fibration $(E, \pi, X; F)$ is called trivial if it is equivalent to $(X \times F, \pi, X; F)$.

Example 5. Not every fibration is trivial. Let $E = T(S^2)$ be the tangent bundle of the sphere $X = S^2$. Then $T(S^2) \to S^2$ cannot be the trivial fibration, since its Euler characteristic is two, $\chi(S^2) = 2 \neq 0$, and hence, by the Hopf theorem, there are no vector fields without zeroes.

Definition. Let $p : E \to X$ be a fibration. A mapping $s : X \to E$ is called a section if $\pi \circ s = \text{Id}_X$. In case $s$ is defined on an open set $U \subset X$ only, it is called a local section.

Let $(E, \pi, X; F)$ be a fibration over $X$ and $f : Y \to X$ a (e.g. continuous or smooth) mapping. Then define a bundle $f^*E$ over $Y$ by

\[f^*E = \{(y, e) \in Y \times E : f(y) = \pi(e)\}\]

with projection $\pi^*(y, e) = y$. The fibration $\pi^* : f^*E \to Y$ is called the fibration induced by $f$.

Proposition. $p^* : f^*E \to Y$ is a locally trivial fibration with fibre $F$.

Definition. A 4-tuple $(P, \pi, X; G)$ is called a $G$-principal bundle if

1) $P$ is a topological space and $G$ a topological group acting freely from the right on $P$;

2) $\pi : P \to X$ is continuous and surjective, and $\pi(p_1) = \pi(p_2)$ if and only if there exists an element $g \in G$ such that $p_1g = p_2$ holds; and

3) $\pi : P \to X$ is a locally trivial fibration in the sense of principal bundles, i.e. for every $x \in X$ there exist $U, x \in U \subset X$ and $\Phi_U : \pi^{-1}(U) \to U \times G$ such that

\[\Phi_U(p) = (\pi(p), \varphi_U(p))\]

and $\varphi_U : \pi^{-1}(U) \to G$ has the property $\varphi_U(p \cdot g) = \varphi_U(p) \cdot g$. 

Remark.

a) Every $G$-principal bundle is a locally trivial fibration with fibre $F = G$.

b) If $(P, \pi, X; G)$ is a $G$-principal bundle and $f : Y \to X$ is continuous, then $(f^*P, \pi^*, Y; G)$ is again a $G$-principal bundle.

c) If $P, G, X$ and all mappings are smooth, then the principal bundle is also smooth.

Definition. Let $(P, \pi, X; G)$ and $(P_1, \pi_1, X; G)$ be two principal bundles over the same base $X$ with the same structure group $G$. These two principal bundles are called isomorphic if there exists a homeomorphism $f : P \to P_1$, satisfying the following conditions:

1) The diagram

\[
\begin{array}{ccc}
\pi & \to & \pi_1 \\
P \bigg\downarrow f \bigg\uparrow & & \bigg\downarrow \pi_1 \\
X
\end{array}
\]

commutes.

2) $f(p \cdot g) = f(p) \cdot g$, i.e. $f$ is compatible with the action of $G$.

Proposition. If the principal bundle $(P, \pi, X; G)$ has a section, then this principal bundle is isomorphic to the trivial $G$-principal bundle

$(X \times G, \pi, X; G)$.

Example 1. Let $G$ be a Lie group and $H$ a closed subgroup. Then, setting $P = G$, $X = G/H$ and $G = H$, we see that $(G, \pi, G/H; H)$ defines an $H$-principal bundle over $G/H$.

Remark. Non-isomorphic principal bundles may well be equivalent as fibrations.

Example 2. Take $X = S^2 = \mathbb{CP}^1$ and $P = S^3 = \{(\omega_1, \omega_2) \in \mathbb{C}^2 : |\omega_1|^2 + |\omega_2|^2 = 1\}$. Consider the fibration

$\pi : S^3 \to \mathbb{CP}^1, \quad \pi(\omega_1, \omega_2) = [\omega_1 : \omega_2],$

and two principal bundles $\xi_1 = (S^3, \pi, \mathbb{CP}^1; S^1)$ and $\xi_2 = (S^3, \pi, \mathbb{CP}^1; S^1)$ differing only in the action of the group $S^1$ on $S^3$. In $\xi_1$ let the group $S^1$ act on $S^3$ by

$$(\omega_1, \omega_2) \cdot z = (\omega_1 z, \omega_2 z)$$

and in $\xi_2$ by

$$(\omega_1, \omega_2) \cdot z = (\omega_1 z^{-1}, \omega_2 z^{-1}).$$

Then $\xi_1$ as well as $\xi_2$ are principal bundles. $\xi_1$ and $\xi_2$ are not isomorphic as $S^1$-principal bundles. The principal bundle $\xi_1$ is called the Hopf fibration.
Example 3. Let $M^n$ be a smooth manifold and $L_x(M^n) = \{(v_1, \ldots, v_n) \in T_xM^n \mid \det(v_1, \ldots, v_n) \neq 0\}$ the set of all frames at the point $x \in M^n$. The union $L(M^n) = \bigcup_{x \in M^n} L_x(M^n)$ is called the frame bundle of $M$. Let $GL(n; \mathbb{R})$ act on $L(M^n)$ by
\[
(v_1, \ldots, v_n) \cdot \begin{pmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{pmatrix} = \begin{pmatrix}
\sum_{i=1}^n v_i A_{i1} & \cdots & \sum_{i=1}^n v_i A_{in}
\end{pmatrix}.
\]
Then $(L(M^n), \pi, M^n; GL(n, \mathbb{R}))$ is a $GL(n, \mathbb{R})$-principal bundle over $M^n$.

Example 4. Let $(M^n, g)$ be a Riemannian manifold and $O(M^n; g) = \{(v_1, \ldots, v_n) \in L(M^n) : g(v_i, v_j) = \delta_{ij}\}$. Then $O(M^n; g)$ is an $O(n; \mathbb{R})$-principal bundle over $M^n$.

Example 5. Let $(M^{2n}, \omega)$ be a symplectic manifold and $Sp(M^{2n}; \omega)$
\[
= \left\{(v_1 \ldots v_n, w_1 \ldots w_n) \in L(M^{2n}) : \begin{aligned}
\omega(v_i, v_j) &= \omega(w_i, w_j) = 0 \\
\omega(v_i, w_j) &= \delta_{ij}
\end{aligned}\right\}.
\]
Then $Sp(M^{2n}, \omega)$ is a $Sp(2n; \mathbb{R})$-principal bundle over $M^{2n}$.

Example 6. Let $M^n$ be a manifold with a fixed orientation $O$. Set
\[
P = \{(v_1, \ldots, v_n) \in L(M^n) : \{v_1, \ldots, v_n\} = O\}.
\]
Then $P$ is a $GL^+(n; \mathbb{R})$-principal bundle.

Definition. Let $\rho = (P, \pi, X; G)$ be a $G$-principal bundle and $\lambda : G_1 \to G$ a continuous (smooth) group homomorphism. A $\lambda$-reduction is a pair $(\mu, f)$ consisting of a $G_1$-principal bundle $\mu = (Q, \pi, X; G_1)$ over $X$ and a mapping $f : Q \to P$ such that
1) the diagram
\[
\begin{array}{ccc}
Q & \xrightarrow{f} & P \\
\downarrow{\pi} & & \downarrow{\pi} \\
X & \xrightarrow{} & 
\end{array}
\]
commutes, and
2) $f(g \cdot g_1) = f(g) \cdot \lambda(g_1)$.

Examples 4, 5, 6 describe different reductions of the frame bundle to the groups $O(n; \mathbb{R})$, $Sp(2n; \mathbb{R})$ and $GL^+(n; \mathbb{R})$, respectively.
Definition. Two $\lambda$-reductions $(\mu, f), (\bar{\mu}, \bar{f})$ of the principal bundle $\rho$ are called equivalent if there exists an isomorphism $\Phi : Q \rightarrow \bar{Q}$ of the $G_1$-principal bundles such that the diagram

$$
\begin{array}{ccc}
Q & \xrightarrow{\Phi} & \bar{Q} \\
\downarrow f & & \downarrow \bar{f} \\
P & \xrightarrow{\mu} & \bar{P}
\end{array}
$$

commutes.

Proposition. Let $\lambda : O(n; \mathbb{R}) \rightarrow GL(n; \mathbb{R})$ be the canonical embedding of groups and $M^n$ a smooth manifold. The set of all $\lambda$-reductions of the frame bundle $L(M^n)$ is in bijective correspondence with the set of all Riemannian metrics on $M^n$.

Proposition. Let $\lambda : GL^+(n; \mathbb{R}) \rightarrow GL(n; \mathbb{R})$ be the canonical embedding of groups and $M^n$ a smooth manifold. The set of all $\lambda$-reductions of the frame bundle $L(M^n)$ is in bijective correspondence with the set of all orientations on $M^n$.

Remark. For a given $M^{2n}$, each symplectic structure $\omega$ defines an $Sp(2n; \mathbb{R})$-reduction of $L(M^{2n})$; compare Example 5. Conversely, given an $Sp(2n)$-reduction, one can define at each point $x \in M^{2n}$ a 2-form $\omega_x : T_xM^{2n} \times T_xM^{2n} \rightarrow \mathbb{R}$. The resulting 2-form $\omega$ will be non-degenerate. However, in general, $d\omega = 0$ will not hold. Thus not every reduction of the frame bundle $L(M^{2n})$ to the subgroup $Sp(2n; \mathbb{R})$ can be identified with a symplectic structure on $M^{2n}$.

Consider now a $G$-principal bundle $(P, \pi, X; G)$ and a topological space $F$ on which $G$ acts from the left, $G \times F \rightarrow F$. Let $G$ act from the right on $P \times F$ by $(p, f) \cdot g = (p \cdot g, g^{-1} f)$. Set

$$
E = P \times F/G := P \times_G F
$$

and take the projection $\pi : E \rightarrow X$ defined by

$$
\pi(e) = \pi[p, f] = \pi(p).
$$

The 4-tuple $(E, p, X; F)$ is a locally trivial fibration, i.e. we have the

Proposition. If $(P, \pi, X; G)$ is a principal bundle and $F$ a space on which $G$ acts from the left, then $(E, p, X; F)$ with $E = P \times_G F$ is a locally trivial fibration. The fibration $(E, \pi, X; F)$ is called the bundle with fibre $F$ associated to the principal bundle.

Proposition. Let $(P, \pi, X; G)$ be a $G$-principal bundle and $F$ a $G$-space defining the associated bundle $E = P \times_G F$. Then there exists a bijection between the sections $s$ in the bundle $(E, \pi, X; F)$ and the maps $s^* : P \rightarrow F$ with $s^*(p \cdot g) = g^{-1}s^*(p)$. 
Now let $G$ be a group and $H \subseteq G$ a closed subgroup. Consider the $G$-space $G/H = F$ and the bundle $E = P \times_G (G/H)$ associated to the principal bundle $(P, \pi, X; G)$.

**Proposition.** The bundle $(E, \pi, X; G/H)$ has a section if and only if the $G$-principal bundle $(P, \pi, X; G)$ has a reduction to the subgroup $H \hookrightarrow G$.

**Example 7.** Let $G$ be a group and $H$ a closed subgroup. Consider the trivial $G$-principal bundle $P = X \times G$. A section in $P \times_G (G/H) \cong X \times G/H$ is then simply a map $\bar{s} : X \rightarrow G/H$, and the $H$-principal bundle corresponding to this section is

$$Q = \{ (x, g) \in X \times G : g^{-1}\bar{s}(x) = e \cdot H \}$$

with the projection $(x, g) \rightarrow x$. Now fix

$$G = SO(3), \quad H = SO(2) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \right\} \hookrightarrow SO(3)$$

and $X = S^2 = SO(3)/SO(2)$, where the identification $SO(3)/SO(2) \cong S^2$ is defined by $A \mapsto A(e_3)$; $e_3$ is the third basis vector of the Euclidean space $\mathbb{R}^3$. Since $e \cdot H \leftrightarrow e_3$ under this identification, we conclude that, for every mapping $f : X = S^2 \rightarrow SO(3)/SO(2) \cong S^2$,

$$Q = \{ (x, g) \in S^2 \times SO(3) : g^{-1}f(x) = e_3 \}$$

is an $S^1$-principal bundle which is a reduction of the trivial bundle $P = X \times G = S^2 \times SO(3)$. Take, e.g., $f : S^2 \rightarrow S^2$ to be the identity. Then the resulting $S^1 = SO(2)$-principal bundle is

$$Q = \{ (x, g) \in S^2 \times SO(3) : g^{-1}x = e_3 \}.$$

Though $Q$ is the reduction of the trivial $SO(3)$-principal bundle over $S^2$, $Q$ itself is not a trivial $S^1$-principal bundle over $S^2$. This example shows that reductions of trivial bundles may well be non-trivial principal bundles.

As the last topic in this section we want to discuss vector bundles. To this end, consider again a $G$-principal bundle $(P, \pi, X; G)$ and, in addition, a vector space $F = V$ (complex or real). Let $G$ act on $V$ via a representation $\rho : G \rightarrow GL(V)$. Then we obtain the associated bundle $E = P \times_G V = P \times_\rho V$ with projection $\pi : E \rightarrow X$. Now define $\mathbb{R} \times E \rightarrow E$ or $\mathbb{C} \times E \rightarrow E$, respectively, by

$$E \ni e = [p, v] \mapsto \lambda \cdot e = [p, \lambda v] \in E$$

and an addition of two elements $e_1, e_2$ satisfying $\pi(e_1) = \pi(e_2)$ by

$$e_1 = [p, v_1], \quad e_2 = [p, v_2] \rightarrow e_1 + e_2 = [p, v_1 + v_2].$$

As $G$ acts linearly on $V$, these operations are uniquely defined. This results in the following structure: Each fibre of $P \times_\rho V = E$ is a vector space over
B.1. Principal fibre bundles

\( \mathbb{R} \) or \( \mathbb{C} \), respectively, and for every \( x \in X \), there exist a neighborhood \( U, x \in U \subset X \), and a mapping \( \Phi_U : p^{-1}(U) \to U \times V \) which is linear in each fibre.

**Definition.** A (real, complex) vector bundle over the space \( X \) is a fibration \( (E, \pi, X; V^n) \) such that each fibre is a vector space and for every \( x_0 \in X \) there exist an open set \( U, x_0 \in U \subset X \), and a homeomorphism \( \Phi_U : p^{-1}(U) \to U \times \mathbb{R}^n (U \times \mathbb{C}^n) \) which is linear in each fibre.

**Example 8.** Let \( M^n \) be a manifold and \( (L(M^n), \pi, M^n; GL(n; \mathbb{R})) \) its frame bundle. Moreover, let \( \rho : GL(n) \to GL(\mathbb{R}^n) \) be the usual representation. Then the associated vector bundle is isomorphic to the tangent bundle:

\[
L(M^n) \times_{GL(n)} \mathbb{R}^n \simeq T(M^n).
\]

To show this, define a mapping \( f : L(M^n) \times_{GL(n)} \mathbb{R}^n \to T(M^n) \) by

\[
f(\vec{v}_1, \ldots, \vec{v}_n; c_1, \ldots, c_n) = \sum_i \vec{v}_i c_i = \vec{v} \cdot c.
\]

For \( A \in GL(n; \mathbb{R}) \) we have \((\vec{v}, c) = (\vec{v}A, A^{-1}c)\) and, at the same time, \( \vec{v}AA^{-1}c = \vec{v} \cdot c \). Hence, \( f : L(M^n) \times_{GL(n)} \mathbb{R}^n \to T(M^n) \) is uniquely defined and, in addition, an isomorphism of vector bundles.

**Example 9.** \( T^*M^n \simeq L(M^n) \times_{\rho^*} (\mathbb{R}^n)^* \), where the mapping \( \rho^* : GL(n; \mathbb{R}) \to GL((\mathbb{R}^n)^*) \) is the dual representation in the dual space \((\mathbb{R}^n)^*\).

**Example 10.** Let \( \rho_k : GL(n; \mathbb{R}) \to GL(\Lambda^k((\mathbb{R}^n)^*)) \) be the representation in the space of \( k \)-forms of \( \mathbb{R}^n \). Then,

\[
\Lambda^k(M^n) = L(M^n) \times_{\rho_k} \Lambda^k((\mathbb{R}^n)^*).
\]

**Example 11.** Let \( \xi = (S^3, \pi, \mathbb{C}P^1; S^1) \) be the Hopf fibration and \( H = S^3 \times_{\rho} \mathbb{C} \) the associated bundle, where \( \rho : S^1 \to GL(\mathbb{C}) \) is given by \( \rho(z)w = zw \). \( H \) consists of the equivalence classes \([(w_1, w_2), w] \) of pairs of complex numbers under the identification

\[
\{(w_1, w_2), w \} \sim \{(w_1z, w_2z), z^{-1}w \}.
\]

The mapping \( f : H \to \hat{H} = \{(l, \xi) \in \mathbb{C}P^1 \times \mathbb{C}^2 : \xi \in l \} \) defined by

\[
f(w_1, w_2; w) = ([w_1 : w_2], (w_1w_2, w_2w))
\]

\( \in \mathbb{C}P^1 \subset \mathbb{C}^2 \)

obviously defines an isomorphism between \( H \) and \( \hat{H} \). Hence \( H \to \mathbb{C}P^1 \), as a vector bundle, is equal to

\[
H = \{(l, \xi) \in \mathbb{C}P^1 \times \mathbb{C}^2 : \xi \in l \}
\]

with projection \( p : H \to \mathbb{C}P^1, p(l, \xi) = l \). \( H (\hat{H}) \) is the so-called Hopf bundle (tautological bundle over \( \mathbb{C}P^1 \)).
B. Principal Bundles and Connections

B.2. The classification of principal bundles

In this section, we will study the following question:

Let a space $X$ be given. How many isomorphism classes of $G$-principal bundles $(P, \pi, X; G)$ with structure group $G$ exist over $X$?

**Theorem** (First homotopy classification theorem). Let $\xi = (P, \pi, Y; G)$ be a $G$-principal bundle over the topological space $Y$, $X$ a paracompact space and $f_1, f_2 : X \to Y$ two homotopic mappings, $f_1 \sim f_2$. Then the induced principal bundles $f_1^*\xi$, $f_2^*\xi$ over $X$ are isomorphic.

**Definition.** If $X$ is a paracompact space, then denote by $HFBG(X)$ the set of all isomorphism classes of principal bundles over $X$ with structure group $G$.

**Definition.** Let $G$ be a topological group. A universal $G$-bundle is a $G$-principal bundle $\xi_G = (E_G, \pi, B_G; G)$ such that for every CW-complex $X$ the assignment

$$[X; B_G] \in [f] \mapsto f^*\xi_G \in HFBG(X)$$

is a bijection. In other words, the following two conditions have to be satisfied:

1) For every $G$-principal bundle $\xi$ over $X$ there exists a mapping $f : X \to B_G$ with $\xi \approx f^*(\xi_G)$.

2) For $f_0, f_1 : X \to B_G$ such that $f_0^*\xi_G \approx f_1^*\xi_G$ we have $f_0 \sim f_1$.

$B_G$ is called the classifying space of the topological group $G$.

Obviously, the classifying space of a topological group is not uniquely determined. Namely, we have the

**Proposition.** If $\xi_G = (E_G, \pi, B_G; G)$ is a universal bundle and $B$ a topological space which is homotopy equivalent to $B_G$, then over $B$ there is also a $G$-principal bundle which is universal.

**Proposition.** Let $\xi_G = (E_G, \pi, B_G; G)$ and $\tilde{\xi}_G = (\tilde{E}_G, \pi, \tilde{B}_G, G)$ be two universal $G$-principal bundles with the CW-complexes $B_G, \tilde{B}_G$. Then there exist homotopy equivalences $\psi : B_G \to \tilde{B}_G$ and $\phi : \tilde{B}_G \to B_G$ with

$$\psi \circ \phi \sim \text{Id}_{\tilde{B}_G}, \quad \phi \circ \psi \sim \text{Id} \quad \text{and} \quad \xi_G = \psi^*\tilde{\xi}_G, \quad \tilde{\xi}_G = \phi^*\xi_G.$$ 

In the class of CW-complexes, the homotopy type of the classifying space of a group, if one exists, is uniquely determined. Now we can formulate the second homotopy classification theorem for principal bundles:
**Theorem** (Second homotopy classification theorem).

1) For every topological group \( G \) there exists a universal \( G \)-principal bundle \( \xi_G = (E_G, \pi, B_G; G) \). This bundle, moreover, has the property that for every paracompact space \( X \) the assignment

\[
[X; B_G] \ni [f] \to f^*\xi_G \in HFB_G(X)
\]

is bijective.

2) For every topological group \( G \) there exists a universal \( G \)-principal bundle \( (E_G^*, \pi, B_G^*; G) \) such that \( B_G^* \) is a CW-complex.

**Example** \((G = \mathbb{Z}_2)\). The set \( PB_{\mathbb{Z}_2}(X) = [X; B(\mathbb{Z}_2)] = [X; \mathbb{RP}^{\infty}] \) (\( X \) a CW-complex) is in bijective correspondence with the elements of \( H^1(X; \mathbb{Z}_2) \). If \( \xi \) is a \( \mathbb{Z}_2 \)-principal bundle, then the element in \( H^1(X; \mathbb{Z}_2) \) corresponding to this principal bundle is called the first Stiefel-Whitney class \( w_1(\xi) \in H^1(X; \mathbb{Z}_2) \).

**Example** \((G = S^1)\). If \( X \) is a CW-complex, then \( PB_{S^1}(X) = [X; B(S^1)] = [X; \mathbb{CP}^{\infty}] \) is in bijective correspondence with the elements of \( H^2(X; \mathbb{Z}) \). For an \( S^1 \)-principal bundle \( \xi \), the element in \( H^2(X; \mathbb{Z}) \) corresponding to this fibration is called the first Chern class of \( \xi \) and will be denoted by \( c_1(\xi) \in H^2(X; \mathbb{Z}) \).

### B.3. Connections in principal bundles

Consider a smooth principal bundle \((P, \pi, M^n; G)\). The vertical space \( T^v_p(P) \) of the projection \( \pi \) is the space \( T^v_p(P) = \{ X \in T_p(P) : d\pi(X) = 0 \} \).

**Lemma 1.** For \( X \in \mathfrak{g} \) denote by \( \tilde{X}(p) = \frac{d}{dt}(p \cdot \exp(tX))|_{t=0} \) the fundamental vector field of the \( G \)-action on \( P \). Then

\[
g \in X \to \tilde{X}(p) \in T^v_p(P)
\]

is a linear isomorphism.

**Definition.** The assignment \( T^h : p \in P \to T^h_p(P) \subset T_pP \) (geometric distribution, Pfaff system) is called a connection on \((P, \pi, M; G)\) if

1. \( T_pP = T^h_p(P) \oplus T^v_p(P) \),
2. \( dR_g(T^h_p(P)) = T^h_{pg}(P) \); and
3. \( T^h \) is smooth.

The projection onto the vertical space

\[
\tilde{X}(p) \oplus Y : T^v_p(P) \oplus T^h_p(P) \to X \in \mathfrak{g}
\]
defines a \( \mathfrak{g} \)-valued 1-form \( Z \) on \( P \), \( Z : TP \to \mathfrak{g} \). Then we have the
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Proposition.

a) \( Z(X) = X \) for every \( X \in g \),

b) \( (R_g)^*Z = \text{Ad}(g^{-1})Z \).

Conversely, given a \( g \)-valued 1-form \( Z : TP \to g \) satisfying a) and b), then \( T^h_P(P) = \{ X \in T_P P : Z(X) = 0 \} \) is a connection.

Now we will describe the local characterization of a connection: Let \( s : U \subset M \to P \) be a section. \( Z^s = Z \circ ds = s^*(Z) : TU \to g \) is the local connection form. For two sections \( s_i : U_i \to P, s_j : U_j \to P \) and \( U_i \cap U_j \neq \emptyset \), define \( g_{ij} : U_i \cap U_j \to G \) by
\[
s_i(x) = s_j(x) \cdot g_{ij}(x).
\]

Denote the Maurer-Cartan form of the group \( G \) by \( \Theta : TG \to g, \quad \Theta(T_g) = dL_{g^{-1}}(T_g) \). The corresponding 1-forms on \( U_i \cap U_j \) are \( \Theta_{ij} = g_{ij}^*(\Theta) \):
\[
\Theta_{ij} = g_{ij}^* \Theta = dL_{g_{ij}^{-1}}dg_{ij}.
\]

Proposition.

1) \( Z_{s_i} = \text{Ad}(g_{ij}^{-1}(x))Z_{s_j} + \Theta_{ij} \).

2) Let a family \( \{(U_i, s_i)\} \) with \( \bigcup U_i = M \) be given and let \( Z_i : T_{U_i} \to g \) be a family of 1-forms such that
\[
Z_i = \text{Ad}(g_{ij}^{-1})Z_j + \Theta_{ij}.
\]

Then there exists precisely one connection \( Z : TP \to g \) with \( Z_{s_i} = Z_i \).

Example. Let \( (M^n, g) \) be a Riemannian manifold, \( \nabla : TM \to T^*M \otimes TM \) the Levi-Civita connection, and \( O(M, g) \) the bundle of orthonormal frames. In the Lie algebra \( \mathfrak{so}(n) \) we choose the basis \( \{ X_{ij} \}_{i<j} \), where \( X_{ij} = E_{ij} - E_{ji} \) and \( E_{ij} \) denotes the matrix with 1 in the \( i \)-th row and \( j \)-th column (and zero otherwise). Let \( s : U \to O(M, g), s = (s_1, \cdots, s_n) \) be a local section and set \( Z^s = \sum_{i<j} g(\nabla s_i, s_j)X_{ij} \). Then \( \{Z^s, s\} \) defines a connection \( Z \) in \( O(M, g) \) and the 1-forms \( w_{ij} = g(\nabla e_i, e_j) \) are the connection forms of the Levi-Civita connection.

Example. Let \( M^n \) be a differentiable manifold. The set of connections on the frame bundle \( L(M^n) \) is in bijective correspondence with the set of affine connections \( \nabla : TM \to T^*M \otimes TM \).

Example. \( X = G/H \) is called reductive if there exists a decomposition \( g = \mathfrak{h} \oplus \mathfrak{m} \) with \( \text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m} \). Consider the principal bundle \( \xi = (G; \pi, G/H; H) \) and the distribution \( G \ni g \to T^h_{g}G = dL_g(\mathfrak{m}) \). This is a connection in \( \xi \) with \( dL_a(T^h_{g}G) = T^h_{ag}G \).

Let a principal bundle \( (P, \pi, M; G) \) and a representation \( \rho : G \to GL(V) \) be given.
Definition. A $q$-form $w \in \Lambda^q(P,V)$ with values in $V$ is called tensorial of type $\rho$ if

1) $w_p(\tilde{t}_1, \ldots, \tilde{t}_q) = 0$ if one of the vectors $\tilde{t}_i \in T^v_p$ is vertical, and

2) $R^*_g w = \rho(g^{-1})w$, \quad $g \in G$.

Example. If $Z, \tilde{Z} : TP \to g$ are two connections, then $Z - \tilde{Z}$ is a tensorial 1-form of type Ad on $P$ with values in $g$. Conversely, if $Z$ is a connection and $w : TP \to g$ a tensorial 1-form of type Ad, then $Z + w$ is again a connection.

Proposition. The vector space of tensorial $q$-forms of type $\rho$ on $P$ with values in $V$ is isomorphic to the vector space $\Lambda^q(M; E)$ of $q$-forms on $M$ with values in the associated vector bundle $E = P \times \rho V$.

Corollary. Let $C(P)$ be the set of all connections on $P$. $C(P)$ is an affine space with vector space $\Lambda^1(M; g)$. Here $g$ denotes the bundle $g = P \times \text{Ad} g$ associated by means of the representation Ad.

Definition. Let $(P, \pi, M; G)$ be a principal bundle with connection. If $X \in \mathcal{X}(M)$ is a vector field on $M$ and $X^* \in \mathcal{X}(P)$ a vector field on $P$, then $X^*$ is called a horizontal lift of $X$ if

a) $X^*(p)$ is horizontal at every point $p \in P$, and

b) $d\pi(X^*(p)) = X(\pi(p))$.

Proposition.

1) Let $X \in \mathcal{X}(M)$ be given. Then there exists a uniquely determined horizontal lift $X^* \in \mathcal{X}(P)$ of $X$. $X^*$ is right invariant. If, on the other hand, $Y \in \mathcal{X}(P)$ is a right invariant horizontal vector field, then there exists a vector field $X \in \mathcal{X}(M)$ with $X^* = Y$.

2) $X, Y \in \mathcal{X}(M) \implies X^* + Y^* = (X + Y)^*$,

   $(fX)^* = (f \circ \pi)X^*$,

   $[X, Y]^* = \text{proj}_{\text{hor}}[X^*, Y^*]$.

3) If $Y$ is a horizontal vector field on $P$, then $[\tilde{X}, Y]$ is horizontal for all $X \in g$.

4) In particular, $[\tilde{X}, Z^*] = 0$ for $X \in g$, $Z \in \mathcal{X}(M^n)$.

Let $\gamma : [a, b] \to M$ be a curve (continuous, piecewise $C^2$).

Definition. A curve $\gamma^* : [a, b] \to P$ is called a horizontal lift of $\gamma$ if

1) $\pi \gamma^* = \gamma$, and

2) $\dot{\gamma}^*$ is horizontal.

Proposition. Let $\gamma : I \to M$ be a curve in $M$ and $u \in P_{\gamma(0)}$ a fixed point. Then there exists precisely one horizontal lift $\gamma^*_u$ of $\gamma$ with $\gamma^*_u(0) = u$. 
Definition. Let \( \tau_\gamma : P_{\gamma(a)} \to P_{\gamma(b)} \) be defined by \( \tau_\gamma(u) = \gamma_u^r(b) \). \( \tau_\gamma \) is called the parallel transport along \( \gamma \).

Proposition.
1) \( \tau_\gamma \) does not depend on the parametrization of \( \gamma \).
2) \( \tau_\gamma R_g = R_g \tau_\gamma \quad \forall g \in G \).
3) \( \tau_\gamma \) is bijective.

Proposition. Let \((P, \pi, M; G)\) be a principal bundle with connection \( Z \), and suppose that \( M \) is connected. If the parallel transport \( \tau \) does not depend on the curve, then there exists a horizontal section in \( P \) and, moreover, the principal bundle is isomorphic to the trivial bundle \((M \times G, \text{pr}_1, M; G)\) with flat connection.

For \( E = P \times_G F \) (\( F \) an arbitrary space), the parallel transport in \( E \), \( \tau_\gamma^E : E_{\gamma(a)} \to E_{\gamma(b)} \), is defined by \( \tau_\gamma^E[p, v] = [\tau_\gamma(p), v] \). A connection in the principal bundle induces a parallel transport in every associated bundle.

B.4. Absolute differential and curvature

Definition. Let \((P, \pi, M; G)\) be a principal bundle with connection \( Z \), \( V \) an arbitrary vector space, and let \( w \in \Lambda^q(P, V) \) denote a \( q \)-form on \( P \) with values in \( V \). Define \( Dw \in \Lambda^{q+1}(P, V) \) by

\[
(Dw)_p(t_0, \ldots, t_q) = dw(pr_{t_0}, \ldots, pr_{t_q}).
\]

\( Dw \) is called the absolute differential of \( w \). It defines a linear mapping

\[
D : \Lambda^q(P, V) \to \Lambda^{q+1}(P, V).
\]

Proposition.
1) If \( w \) is a \( q \)-form of type \( \rho \), then \( Dw \) is a tensorial \((q+1)\)-form of type \( \rho \).
2) Suppose that \( w \in \Lambda^q(P; V) \) is tensorial of type \( \rho \). Then, \( Dw = dw + \rho_*(Z) \wedge w \) with \( \rho_* : g \to gl(V) \) and

\[
(\rho_*(Z) \wedge w)(t_0, \ldots, t_q) = \sum_{\alpha=0}^{q} (-1)^{\alpha} \rho_*(Z(t_\alpha))w(t_0, \ldots, \hat{t_\alpha}, \ldots, t_q).
\]

Now let \( \rho : G \to GL(V) \) be a representation and set \( E = P \times_\rho V \). The tensorial forms of type \( \rho \) coincide with the forms \( \Lambda^\rho(M, E) \) on \( M \) with values in the vector bundle \( E \). Hence the absolute differential can be viewed as an operator \( D : \Lambda^q(M; E) \to \Lambda^{q+1}(M; E) \).

Definition. Let \((E, \pi, M)\) be a vector bundle over \( M \) and \( \Gamma(E) \) the space of smooth sections. A mapping \( \nabla : \Gamma(E) \to \Gamma(TM \otimes E) \) is called a covariant derivative in \( E \) if the following conditions are satisfied:
1) $\nabla$ is linear, $\nabla(e_1 + e_2) = \nabla e_1 + \nabla e_2$.
2) $\nabla(fe) = df \otimes e + f\nabla e$ for $f \in C^\infty(M), e \in \Gamma(E)$.

The 1-form $\nabla e$ is also written as $(\nabla e)(X) = \nabla_X e$.

**Proposition.** Let $(P, \pi, M; G)$ be a principal bundle with connection, $\rho : G \to GL(V)$ a representation and $E = P \times_\rho V$ the associated vector bundle. Then the absolute differential $D : \Gamma(E) \to \Lambda^1(M; E) = \Gamma(T^*M \otimes E)$ is a covariant derivative.

$D$ is called the covariant derivative in $E$ associated with $Z$. It is occasionally also denoted by $\nabla^Z, \nabla^E, \ldots$.

**Proposition.** Let $s \in \Gamma(E)$ be a smooth section. Then,

$$(Ds)(X) = \frac{d}{dt} (\tau^E_{t,0}(s(\gamma(t))))|_{t=0}, \quad X \in T_pM,$$

where $\gamma(t) \subset M$ is a curve with $\gamma(0) = p, \dot{\gamma}(0) = X$, and $\tau^E_{t,0} : E_{\gamma(t)} \to E_{\gamma(0)}$ is the parallel transport.

**Corollary.** Let $\gamma(t) \subset M$ be a curve and $s(t) \subset E$ a curve over $\gamma(t)$. Then $s(t)$ is the parallel transport of $s(0)$ along $\gamma(t)$ if and only if $\frac{d}{dt} (Ds(\gamma(t))) = 0$.

**Definition.** Let $(P, \pi, M; G)$ be a principal bundle and $Z : TP \to g$ a connection. Then $Z$ is a 1-form on $P$ of type $\text{Ad}$ (i.e. $R^*_g Z = \text{Ad}(g^{-1})Z$). By the preceding theorem,

$$\Omega := DZ$$

is a 2-form on $P$ with values in $g$ which is tensorial and of type $\text{Ad}$. $\Omega$ is called the curvature form of the connection.

By the general identification

$$\{\text{tensorial} q\text{-form of type } \rho\} \iff \Lambda^q(M; P \times_\rho V),$$

$\Omega$ can also be considered as a 2-form on $M$ with values in $g = P \times \text{Ad } g$, $\Omega \in \Lambda^2(M; g)$. We introduce a few notations: Let $w \in \Lambda^i(P; g), \tau \in \Lambda^j(P; g)$ be two forms on $P$ with values in $g$ (or else, $w \in \Lambda^i(M; g), \tau \in \Lambda^j(M; g)$ two forms on $M$ with values in the bundle $g$). Then define a form $[w, \tau] \in \Lambda^{i+j}(P; g)$ (or $[w, \tau] \in \Lambda^{i+j}(M; g)$), respectively) by

$$[w, \tau](X_1, \ldots, X_{i+j}) = \frac{1}{i!j!} \sum_{g \in S_{i+j}} (-1)^g [w(X_{g(1)}, \ldots, X_{g(i)}), \tau(X_{g(i+1)}, \ldots, X_{g(i+j)})].$$

If $A_1, \ldots, A_e$ is a basis in $g$ and $w = w^i A_i, \tau = \tau^j A_j$, then, obviously,

$$[w, \tau] = w^i \wedge \tau^j [A_i, A_j].$$
This bracket has the following properties:

a) \([w, r] = (-1)^{ij+1}[r, w]\).

b) For \(w \in \Lambda^i(P, g), r \in \Lambda^j(P, g), \varphi \in \Lambda^k(P, g)\),
\((-1)^{ik}[w, r, \varphi] + (-1)^{kj}[[\varphi, w], r] + (-1)^{ji}[[r, \varphi], w] = 0.\)

c) \(d[w, r] = [dw, r] + (-1)^i[w, dr].\)

d) If \(w\) is a 1-form and \(w \in \Lambda^1(P, g)\) (or \(w \in \Lambda^1(M, \mathfrak{g})\)), then
\(\frac{1}{2}[w, w](X, Y) = [w(X), w(Y)].\)

**Proposition.** Let \((P, \pi, M; G)\) be a principal bundle, \(Z : TP \to g\) a connection and \(\Omega = DZ\) its curvature form. Then we have:

1) The structure equation: \(\Omega = dZ + \frac{1}{2}[Z, Z].\)

2) The Bianchi identity: \(D\Omega = 0.\)

3) If \(w \in \Lambda^q(P; V)\) is tensorial of type \(\rho : G \to GL(V)\), then
\(DDw = \rho_*(\Omega) \wedge w.\)

4) If \(w \in \Lambda^q(P, g)\) is tensorial of type \(\text{Ad} : G \to GL(g)\), then
\(Dw = dw + [Z, w].\)

**Corollary.** Let \(X, Y\) be horizontal vector fields. Then,
\(Z([X, Y]) = -\Omega(X, Y).\)

**Proof.** \(\Omega = dZ + \frac{1}{2}[Z, Z].\) Inserting horizontal fields we obtain \(Z(X) = 0 = Z(Y)\), hence
\(\Omega(X, Y) = -Z[X, Y].\)

**Proposition.** Let \((P, \pi, M; G)\) be a principal bundle and \(Z, \bar{Z}\) two connections with curvatures \(\Omega = DZ, \bar{\Omega} = \bar{DZ}\). Then, \(\eta = \bar{Z} - Z\) is a tensorial 1-form of type \(\text{Ad}\) and
\(\bar{\Omega} = \Omega + D\eta + \frac{1}{2}[\eta, \eta].\)

Considering \(\bar{\Omega}, \Omega\) as 2-forms on \(M\) and \(\eta\) as a 1-form on \(M\) with values in \(\mathfrak{g} = P \times _{Ad} g\), \(\bar{\Omega}, \Omega \in \Lambda^2(M, \mathfrak{g}), \eta \in \Lambda^1(M, \mathfrak{g})\), we have the same formula, this time with the operator
\(D : \Lambda^1(M; \mathfrak{g}) \to \Lambda^2(M; \mathfrak{g}).\)

**Definition.** A connection \(Z\) on \((P, \pi, M; G)\) is called (locally) flat if there exists an open covering \(U_i\) of \(M\) such that \((P|_{U_i}, Z)\) is isomorphic to \((U_i \times G, pr_1, \pi_i; G)\) with the canonical connection.
Proposition. $Z$ is a locally flat connection $\iff \Omega \equiv 0 \iff$ the bundle $T^h(P) \subset TP$ of horizontal vectors is involutive.

Proposition. Let $\pi_1(M) = 0$ and let $(P, \pi, M; G)$ be a principal bundle with a locally flat connection $Z$. Then $(P; Z)$ is isomorphic to $(M \times G)$ with the canonical connection.

Definition. Let $(P, \pi, M; G)$ be a principal bundle. A gauge transformation is a diffeomorphism $f : P \to P$ with

1) $\pi \circ f = \pi$, and
2) $f(p \cdot g) = f(p) \cdot g$.

Denote by $\mathcal{G}(P)$ the group of all gauge transformations.

Proposition.

1) If $Z \in \mathcal{C}(P)$ is a connection and $f \in \mathcal{G}(P)$ a gauge transformation, then $f^*Z \in \mathcal{C}(P)$ is again a connection. In other words, the group of gauge transformations acts on the set of all connections.

2) Each gauge transformation $f$ is given by a mapping $\mu_f : P \to G$, $f(p) = p \cdot \mu_f(p)$. Then,

$$(f^*Z)_p = \text{Ad}(\mu_f(p)^{-1})Z_p + d\mu_f(p)^{-1}d\mu_f|_p = \text{Ad}(\mu_f^{-1}(p))Z_p + \mu_p^*\Theta.$$ 

Proposition. Let the gauge transformation $f : P \to P$ be given by $\mu_f : P \to G$, and let $Z$ be a connection. Then, for the curvature forms $\Omega^Z$ and $\Omega^{f^*Z}$,

$$\Omega^{f^*Z} = \text{Ad}(\mu_f^{-1})\Omega^Z.$$ 

B.5. Connections in $U(1)$-principal bundles and the Weyl theorem

In this section, we deal with the group $G = S^1 = U(1) = \{z \in \mathbb{C} : |z| = 1\}$. If $\gamma(t)$ is a curve in $G$ with $\gamma(0) = 1$, then $\dot{\gamma}(0) \in T_1S^1 = \mathfrak{g}^1$. On the other hand, $|\gamma(t)|^2 \equiv 1$ implies $\dot{\gamma}(0) \in i\mathbb{R}$. Hence we obtain an identification $\mathfrak{g}^1 \ni \dot{\gamma}(0) \to \dot{\gamma}(0) \in i\mathbb{R}$. The Lie algebra $\mathfrak{g}^1$ can be identified with $i\mathbb{R}$ in such a way that the diagram

$$\begin{array}{ccc}
\mathfrak{g}^1 & \longrightarrow & i\mathbb{R} \\
\exp \downarrow & & \searrow e \\
S^1 & \longrightarrow &
\end{array}$$

with $e : i\mathbb{R} \to S^1$, $e(ix) = e^{ix}$, commutes. Consider now the canonical form (Maurer-Cartan form) $\Theta : TS^1 \to \mathfrak{g}^1 \cong i\mathbb{R}$ of the group. We will show that

$$\Theta = \frac{dz}{z} = \bar{z}dz.$$
In fact, if \( z \in S^1, \vec{t} \in T_z S^1 \) and \( \gamma \) is a curve with \( \gamma(0) = z, \dot{\gamma}(0) = \vec{t} \), then

\[
\Theta(\vec{t}) = (dL_{z^{-1}}(\dot{t})) = \frac{d}{dt} \left( \frac{1}{z} \gamma(t) \right)_{t=0} = \frac{1}{z} \frac{d\gamma}{dt}_{t=0} = \frac{1}{z} \vec{t} = \frac{dz}{z} (t),
\]

i.e. \( \Theta = \frac{dz}{z} \). Moreover, \( \int_{S^1} \Theta = \int dz/z = 2\pi i \). Hence, \( \varphi := \frac{1}{2\pi i} \Theta : TS^1 \rightarrow \mathbb{R} \) is a real-valued 1-form on \( S^1 \) with \( \int_{S^1} \varphi = 1 \).

Now let \((P, \pi, M^n; S^1)\) be an \( S^1 \)-principal bundle over \( M \). If \( f : P \rightarrow P \) is a gauge transformation, then set \( f(p) = p \cdot \mu_f(p), \mu_f : P \rightarrow S^1 \). Since \( f(p \cdot z) = f(p) \cdot z \),

\[
p \cdot z \cdot \mu_f(p \cdot z) = p \cdot \mu_f(p) \cdot z.
\]

Hence, \( z \mu_f(p \cdot z) = \mu_f(p) \cdot z \) and, since \( S^1 \) is abelian, we have \( \mu_f(p \cdot z) = \mu_f(p) \). Thus \( \mu_f : P \rightarrow S^1 \) is constant on the fibres and induces a mapping \( \bar{\mu}_f : M^n \rightarrow S^1 \). Conversely, if a mapping \( \bar{\mu} : M^n \rightarrow S^1 \) is given, then

\[
f(p) = p \cdot \bar{\mu}(\pi(p))
\]

defines a gauge transformation. The group of gauge transformations thus coincides with the group of all mappings \( \mu : M^n \rightarrow S^1 \):

\[
\mathcal{G}(P) \simeq \{ \mu : M^n \rightarrow S^1 \}.
\]

Fix a connection \( Z : TP \rightarrow i\mathbb{R} \) on \( P \). If \( f : P \rightarrow P \) is the gauge transformation corresponding to \( \bar{\mu}_f : M^n \rightarrow S^1 \), then

\[
f^* Z = \text{Ad}(\mu_f^{-1})Z + \mu_f^* \Theta = Z + \mu_f^* \Theta = Z + \pi^* \bar{\mu}_f^* \Theta = Z + 2\pi \bar{\mu}_f \varphi.
\]

Consider \( \Omega = \Omega^Z : TP \times TP \rightarrow i\mathbb{R} \), the connection form of \( Z \). Since

\[
R^*_{\pi} \Omega = \text{Ad}(z^{-1}) \Omega = \Omega,
\]

\( \Omega \) is a tensorial 2-form on \( TP \) invariant under the action of right translations. Thus \( \Omega \) is simply a 2-form on \( M^n \) with values in \( i\mathbb{R} \),

\[
\Omega^Z : TM^n \times TM^n \rightarrow i\mathbb{R}.
\]

As \( 0 = D\Omega^Z = d\Omega^Z + \text{Ad}_\pi(Z) \wedge \Omega^Z = d\Omega^Z \), \( \Omega^Z \) is a closed 2-form, i.e.

\[
d\Omega^Z = 0.
\]

If \( \tilde{Z} \) is another connection, then

\[
\Omega^{\tilde{Z}} = \Omega^Z + D\eta + \frac{1}{2} [\eta, \eta] = \Omega^Z + D\eta
\]

with \( \eta = \tilde{Z} - Z \). Now, \( \eta \) again is a tensorial 1-form with \( R^*_{\pi} \eta = \text{Ad}(Z^{-1}) \eta = \eta \), hence a 1-form on \( M^n \) with values in \( i\mathbb{R} \). This implies:

1) The curvature form \( \Omega^Z \) of an arbitrary connection in \( P \) is a closed 2-form on \( M^n \) with values in \( i\mathbb{R} \).
2) If $\Omega^\tilde{Z}, \Omega^Z$ are the curvature forms of two connections, then there exists a 1-form on $M^n$ with values in $i\mathbb{R}$ such that

$$\Omega^\tilde{Z} - \Omega^Z = d\eta.$$

The de Rham cohomology of a compact manifold is defined by

$$H^2_{DR}(M^n; \mathbb{R}) = Z^2(M^n)/B^2(M^n),$$

where

$$Z^2(M^n) = \{w^2 : w^2 \text{ is a 2-form and } dw^2 = 0\},$$

$$B^2(M^n) = \{w^2 : \text{there exists a 1-form } \mu^1 \text{ with } w^2 = d\mu^1\}.$$  

1) and 2) imply that the class $[-\frac{1}{2\pi i} \Omega^Z] \in H^2_{DR}(M^n; \mathbb{R})$ is a uniquely determined element of the de Rham cohomology of $M^n$ not depending on the choice of $Z$, but only on the principal bundle. This class will be denoted by $c_1(P) \in H^2_{DR}(M; \mathbb{R})$.

It is called the real Chern class of the $S^1$ principal bundle $P$. Set

$$\mathcal{F}(P) = \{w^2 : w^2 \text{ is a 2-form with } dw^2 = 0, \ [w^2] = c_1(P)\}.$$  

Then,

$$\mathcal{C}(P) \ni Z \rightarrow -\frac{\Omega^Z}{2\pi i} \in \mathcal{F}(P)$$

obviously defines a mapping $\psi : \mathcal{C}(P) \rightarrow \mathcal{F}(P)$. We state some of its properties:

1) If $Z$ and $\tilde{Z}$ are gauge equivalent connections, then $\psi(Z) = \psi(\tilde{Z})$.

2) $\psi$ is surjective.

From 1) and 2) we obtain a surjective mapping

$$\psi : \mathcal{C}(P)/G(P) \rightarrow \mathcal{F}(P).$$

The first de Rham cohomology is defined by $H^1_{DR}(M^n; \mathbb{R}) = Z^1(M)/B^1(M)$, where $Z^1$ and $B^1$ are the following spaces:

$$Z^1(M) = \{w^1 : w^1 \text{ is a closed 1-form, } dw^1 = 0\},$$

$$B^1(M) = \{w^1 : \text{there exists a function } f \text{ on } M \text{ with } w^1 = df\}.$$  

Let, moreover,

$$Z^1(M; \mathbb{Z}) = \left\{ w^1 \in Z^1(M) : \int_{\gamma} w^1 \in \mathbb{Z} \text{ for all closed curves } \gamma \right\}.$$

Since $\int_{\gamma} df = \int_{\gamma} f = 0$, we obviously have $Z^1(M; \mathbb{Z}) \supset B^1(M)$. Let

$$H^1_{DR}(M^n; \mathbb{Z}) = Z^1(M^n; \mathbb{Z})/B^1(M^n).$$
be the so-called integral de Rham cohomology. Again consider \( w^2 \in \mathcal{F}(P) \) and denote by \( C_{w^2}(P) \) the set
\[
C_{w^2}(P) = \psi^{-1}(w^2) = \left\{ Z \in C(P) : -\frac{1}{2\pi i} \Omega^2 = w^2 \right\}.
\]
3) \( C_{w^2}(P) \) is a \( \mathcal{G}(P) \)-invariant affine space with vector space \( Z^1(M^n) \).
4) The set \( C_{w^2}(P)/\mathcal{G}(P) \) is in bijective correspondence with
\[
\text{Pic}(M^n) = H^1_{DR}(M^n; \mathbb{R})/H^1_{DR}(M^n; \mathbb{Z}).
\]
Summarizing, we state the so-called Weyl theorem.

**Theorem** (Weyl theorem). Let \((P, \pi, M; S^1)\) be an \( S^1 \)-principal bundle over the compact manifold \( M^n \) with first Chern class \( c_1(P) \in H^2_{DR}(M; \mathbb{R}) \), and set
\[
\mathcal{F}(P) = \{ w^2 : dw^2 = 0, [w^2] = c_1(P) \}.
\]
Define a surjective mapping \( \psi : C(P)/\mathcal{G}(P) \rightarrow \mathcal{F}(P) \) by the assignment
\[
Z \mapsto -\bar{\Omega}Z = -\frac{1}{2\pi i} \Omega Z.
\]
Then each fibre \( \psi^{-1}(w^2) \) of \( \psi \) is diffeomorphic to the Picard manifold
\[
\text{Pic}(M^n) = H^1_{DR}(M^n; \mathbb{R})/H^1_{DR}(M^n; \mathbb{Z})
\]
of \( M^n \). In particular, \( H^1_{DR}(M^n; \mathbb{R}) = 0 \) (e.g. for simply connected \( M^n \)) implies that \( \psi \) is bijective.

**Example.** Consider the Hopf fibration \( \pi : S^3 \rightarrow \mathbb{C}P^1 = S^2 \) with
\[
S^3 = \{(w_1, w_2) \in \mathbb{C}^2 : |w_1|^2 + |w_2|^2 = 1\}
\]
and the \( S^1 \)-action \( S^3 \times S^1 \rightarrow S^3 \),
\[
((w_1, w_2); z) = (w_1 z, w_2 z).
\]
We will construct a connection in this \( S^1 \)-principal bundle. Set
\[
Z = \frac{1}{2} \{ \bar{w}_1 dw_1 - w_1 d\bar{w}_1 + \bar{w}_2 dw_2 - w_2 d\bar{w}_2 \}.
\]
Since \( z - \bar{z} \in i\mathbb{R} \) for every \( z \in \mathbb{C} \), it follows that \( Z \) is a 1-form on \( S^3 \) with values in \( i\mathbb{R} \). It has the following properties:
1) \( Z \) is invariant under the \( S^1 \)-action, i.e. \( (R_z)^*Z = Z, z \in S^1 \).
2) For \( ix \in i\mathbb{R} \) and the corresponding fundamental vector field \( (ix) \) on \( S^3 \) we have \( Z(ix) = ix \).
Thus \( Z \) is a connection in the bundle \( \pi : S^3 \rightarrow S^2 \). We are going to compute its curvature. As \( S^1 \) is abelian, \( \Omega = dZ \), and thus,
\[
\Omega = -dw_1 \wedge d\bar{w}_1 - dw_2 \wedge d\bar{w}_2.
\]
as a 2-form on $S^3$ with values in $i\mathbb{R}$. Since $\Omega$ is a curvature form, $\Omega = \pi^*\Omega$ for a 2-form $\Omega$ on $\mathbb{CP}^1 = S^2$. Let $\gamma : S^2 \setminus \{\text{north pole}\} \to \mathbb{C}$ denote stereographic projection, and let $\bar{\Omega}$ be a 2-form on $\mathbb{C}$ with $\gamma^*\bar{\Omega} = \bar{\Omega}$.

Then $\gamma \circ \pi : S^3 \setminus \{(w_1, w_2) : w_2 \neq 0\} \to \mathbb{C}$ is given by $\gamma \circ \pi(w_1, w_2) = \frac{w_1}{w_2}$, and $\bar{\Omega}$ has the form

$$\bar{\Omega} = -\frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$ 

Hence, for the curvature form $\bar{\Omega}$ of the connection $\mathcal{Z}$, we have $\int_{S^2} \bar{\Omega}/2\pi i = 1$.

This equation means that $c_1(\text{Hopf fibration}) = -1$.

**Remark.** For the $S^1$-principal bundle $(S^3, \pi, \mathbb{CP}^1; S^1)$ with $S^1$-action

$$((w_1, w_2); z) = (w_1z^{-1}, w_2z^{-1}),$$

a similar argument shows that $Z^* = -Z : T\mathbb{S}^3 \to i\mathbb{R}$ is a connection in that bundle. This implies

$$\begin{align*}
\Omega^* &= dZ^* = -\Omega = dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2, \\
\bar{\Omega}^* &= -\bar{\Omega} = \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}, \quad \frac{1}{2\pi i} \int_{S^2} \bar{\Omega}^* = -1.
\end{align*}$$

Hence, $c_1(\xi_2) = +1$, and we have once again proved that this $S^1$-principal bundle is not isomorphic to the Hopf fibration.

### B.6. Reductions of connections

Let $(P, \pi, M; G)$ be a $G$-principal bundle over $M^n$, and let $(P', \pi, M'; G')$ together with $f : P' \to P$ be a $\lambda$-reduction of this bundle, i.e.:

1) $\lambda : G' \to G$ is a group homomorphism,

2) $f : P' \to P$ is smooth and the diagram

$$\begin{array}{ccc}
P' & \xrightarrow{f} & P \\
\downarrow \pi & & \downarrow \pi \\
M & \xleftarrow{\pi} & M
\end{array}$$

commutes,

3) $f(p'g') = f(p')\lambda(g')$. 

Proposition. Let $Z' : TP' \to g'$ be a connection in the $G'$-principal bundle $(P', \pi, M; G')$.

1) There exists one and only one connection $Z : TP \to g$ such that $df : TP' \to TP$ maps the horizontal spaces with respect to $Z'$ onto the horizontal spaces with respect to $Z$.

2) $\lambda_* Z' = f^* Z$ and, for the curvature forms, $\lambda_* \Omega' = f^* \Omega$.

Remark.

1) The connection $Z$ constructed starting from the connection $Z'$ is called the induced connection or the $\lambda$-extension of $Z'$.

2) If $G'$ is a subgroup of $G$ and $\lambda : G' \to G$ the embedding, then $Z'$ is called a reduction of the connection $Z$ onto the subbundle $(P', \pi, M; G)$.

Consider now a principal bundle $(P, \pi, M; G)$ with connection, as well as a subgroup $H \subset G$ and a subbundle $(Q; \pi, M; H)$. We ask the following question:

When does a connection $Z'$ exist in $(Q, \pi, M; H)$ such that $Z'$ is a reduction of $Z$?

A provisional answer to this question is contained in the following:

Proposition. In the above notation, with the additional assumption that there exists a decomposition of the Lie algebra

$$g = \mathfrak{h} \oplus \mathfrak{m} \text{ with } \text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m},$$

we have: If $Z : TP \to g$ is a connection in $(P, \pi, M; G)$, then $Z' = pr_{\mathfrak{h}} \circ Z |_{TQ} : TQ \to \mathfrak{h}$ is a connection in $Q$.

If, in particular, $Z : TP \to g$ takes only values in the subalgebra $\mathfrak{h}$, then $Z$ reduces to a connection $Z' : TQ \to \mathfrak{h}$.

B.7. Frobenius' theorem

The local Frobenius theorem in Euclidean space can be formulated as follows:

Theorem (Local Frobenius theorem in $\mathbb{R}^n$). Let $U \subset \mathbb{R}^n$ be an open subset and $w^1, \ldots, w^r$ 1-forms on $U$, $n = r + s$. Moreover, suppose that

a) $w^1, \ldots, w^r$ are linearly independent at every point and

b) there exist 1-forms $\Theta^i_j$ on $U$ with

$$dw^i = \sum_{j=1}^r \Theta^i_j \wedge w^j.$$
For $x \in U$ define
\[
\sum^s(x) = \{ \vec{t} \in T_xU : w^1(\vec{t}) = \cdots = w^r(\vec{t}) = 0 \}.
\]
Then for every point $x_0 \in U$ there exists a regular $s$-dimensional surface piece $F^s$ with
1) $x_0 \in F^s$,
2) $y \in F^s \Rightarrow T_yF^s = \sum^s(y)$.

Replacing $\mathbb{R}^n$ by an $n$-dimensional manifold, this leads to the notion of a distribution:

**Definition.** Let $M^n$ be a manifold. A *distribution* on $M^n$ is a selection of $k$-dimensional subspaces $E^k_x \subset T_xM^n$ in every tangent space such that $E^k_x$ depends smoothly on the point $x$ in the following sense:

For every $x \in M^n$ there exist a neighborhood $U(x)$ and vector fields $\vec{t}_1, \cdots, \vec{t}_k$ on $U(x)$ with $E^k_y = \text{Lin}(\vec{t}_1(y), \cdots, \vec{t}_k(y))$ for all $y \in U(x)$.

Then $E^k = \bigcup_x E^k_x$ is a smooth subvector bundle of the tangent bundle.

**Definition.** Let $E^k \subset TM^n$ be a $k$-dimensional distribution. $E^k$ is called integrable if the following condition is satisfied: For two vector fields $\vec{t}_1, \vec{t}_2$ on $M^n$ with values in $E^k$, the commutator $[\vec{t}_1, \vec{t}_2]$ also has values in $E^k$.

**Theorem** (Local Frobenius theorem on manifolds). Let $E^k \subset TM^n$ be an integrable $k$-dimensional distribution. For every point $x \in M$ there exist a neighborhood $U_x$ and a submanifold $x \in F^k \subset U_x$ with $T_yF^k = E^k_y$ for all $y \in F^k$.

Before turning to the global version of the Frobenius theorem we have to explain or extend a few general notions. To this end, recall the following definitions:

**Definition.** A smooth manifold *without boundary* is a pair $(M, D)$, where
1) $M$ is a topological $\mathcal{T}_2$-space with countable basis, and
2) $D$ is a differentiable structure on $M$, i.e. a family $D = \{(U_i, h_i)\}_{i \in \tau}$ where $U_i \subset M$ is open, $h_i : U_i \rightarrow V_i \subset \mathbb{R}^n$ are homeomorphisms and the $h_i h_j^{-1}$ are smooth.

**Definition.** Let $(M, D)$ be a smooth manifold without boundary. A subset $A \subset M$ is called a *$k$-dimensional submanifold* if
\[
\forall a \in A \exists (U, \varphi) \in D, a \in U, \varphi : U \rightarrow V \subset \mathbb{R}^n : \\
\varphi(A \cap U) \text{ is an open subset of } \mathbb{R}^k \times \{0\}.
\]
One then shows that \((M, D(M))\) is a manifold in the induced topology and with the atlas \(D(A) = \{(A \cap U, \varphi|_{A \cap U})\}\). Moreover, the embedding \(i: A \to M\) is smooth and \(di: TA \to TM\) is injective.

**Definition.** Let \((M, D(M))\) be a smooth manifold. A subset \(A \subset M\) is called a weak submanifold if there exist a smooth manifold \((N, D(N))\) and a differentiable mapping \(f: N \to M\) with the following properties:

1) \(f\) is injective.
2) \(f(N) = A\).
3) \(df: T_nN \to T_{f(n)}M\) is injective.

If \(a \in A\) is a point in the weak submanifold, then there is one and only one \(n \in N\) with \(f(n) = a\). The space \(df_n(T_nN) =: T_aA\) is called the tangent space of \(A\) at the point \(a \in A\).

**Example.** Take \(M = T^2\) and let \(\varphi(t) = (e^{iat}, e^{ibt}), \alpha/\beta\) irrational, \(A = \varphi(\mathbb{R}^1)\). Then \(A \subset T^2\) is a (dense) weak submanifold which is not a submanifold.

**Proposition.** Let \(A \subset M\) be a weak submanifold and \(f: N \to A\) a model. For every point \(n \in N\) there exists a neighborhood \(u \in U(n) \subset N\) with the following properties:

1) \(f(U(n))\) is a submanifold of \(M\).
2) \(f: U(n) \to f(U(n))\) is a diffeomorphism.

**Corollary.** Let \(A \subset M\) be a weak submanifold, and let \(f: N \to A, f_1: N_1 \to A\) be two models. Then \(f_1^{-1} \circ f: N \to N_1\) is a diffeomorphism.

**Definition.** Let \(E^k \subset TM\) be a distribution. An integral manifold of \(E^k\) is a weak submanifold \(A \subset M\) with \(TA = E^y_y\) for all \(y \in A\).

**Theorem** (Global Frobenius theorem on manifolds). Let \(E^k \subset TM\) be an integrable distribution on a manifold \(M\). Then for every point \(x \in M\) there exists a weak submanifold \(A(x)\) with the following properties:

1) \(A(x)\) is an integral manifold of \(E^k\), i.e. \(T_yA(x) = E^k_y \forall y \in A(x)\).
2) \(A(x)\) is connected.
3) \(A(x)\) is maximal, i.e., if \(B\) is a connected integral manifold of \(E^k\) with \(A(x) \subset B\), then \(A(x) = B\).
B.8. The Freudenthal-Yamabe theorem

Definition. Let $G$ be a Lie group. A subset $H \subset G$ is called a \textit{(weak) Lie subgroup} if the following conditions are satisfied.

1) $H$ is a subgroup.

2) $H$ is a weak submanifold respecting the group structure, i.e. there exist a Lie group $\hat{H}$ and a smooth mapping $f : \hat{H} \to G$ such that
   a) $f$ is injective,
   b) $f(\hat{H}) = H$,
   c) $df$ is injective,
   d) $f$ is a group homomorphism.

Theorem (Freudenthal-Yamabe). Let $G$ be a Lie group and $H \subset G$ a subgroup with the following property: Each element of $H$ can be connected with the neutral element $e \in G$ by a piecewise smooth curve, and this curve lies in $H$. Then $H$ is a weak Lie subgroup.

B.9. Holonomy theory

Let $(P, \pi, M; G)$ be a principal bundle and $Z : TP \to g$ a connection. (In this section, we suppose that $M$ is connected.) Take, moreover, $p \in P$ and $x = \pi(p)$. If $\gamma$ is a curve in $M$ (piecewise smooth) starting and ending at $x$, then we can consider the parallel displacement

$$\tau_\gamma : P_x \to P_x.$$ 

Let $\tau_\gamma(p) = p \cdot g_\gamma$. Since $\tau_\gamma(p \cdot h) = \tau_\gamma(p) \cdot h = p \cdot g_\gamma \cdot h$ for all $h$, it obviously follows that $\tau_\gamma : P_x \to P_x$ is completely described by $g_\gamma \in G$. The set $\phi(p) := \{g \in G : \text{there exists a loop } \gamma \text{ at } x \text{ with } \tau_\gamma(p) = p \cdot g\}$ is an (algebraic) subgroup of $G$,

$$\phi(p) \subset G, \quad p \in P.$$ 

In fact, if $\gamma_1, \gamma_2$ are two loops at $x$, then $\tau_{\gamma_1 \cdot \gamma_2} = \tau_{\gamma_1} \circ \tau_{\gamma_2}$, and hence

$$g_{\gamma_1 \cdot \gamma_2} = g_{\gamma_1} \cdot g_{\gamma_2}.$$ 

Definition. The group $\phi(p)$ is called the \textit{holonomy group} of the connection $Z$ with respect to the base point $p \in P$.

Furthermore, define

$$\phi^0(p) = \{g \in G : \exists \text{ a loop } \gamma \text{ at } x \text{ which is null-homotopic to } \tau_\gamma(p) = p \cdot g\}.$$ 

For trivial reasons, $\phi^0(p) \subset \phi(p) \subset G$ is a subgroup.
B. Principal Bundles and Connections

Proposition.

1) $\phi^0(p)$ is a weak Lie subgroup of $G$.

2) $\phi^0(p)$ is normal in $\phi(p)$, and $\phi(p)/\phi^0(p)$ is countable.

Theorem (Reduction theorem of holonomy theory). Let $(P, \pi, M; G)$ be a principal bundle with connected base $M^n$ and $Z : TP \to g$ a connection. For a fixed point $p_0 \in P$, denote by $\phi(p_0)$ the holonomy group and by $P(p_0)$ the set

$$P(p_0) = \{ p \in P : \text{there exists a horizontal path from } p_0 \text{ to } p \}.$$ 

Then $(P(p_0), \pi, M; \phi(p_0))$ is a reduction of the principal bundle $(P, \pi, M; G)$, and the connection $Z$ reduces to this bundle.

Theorem (Ambrose-Singer, 1953). Let $(P, \pi, M; G)$ be a principal bundle, $M$ connected, and let $Z : TP \to g$ be a connection with curvature form $\Omega = DZ$. Let $p_0 \in P$ be a fixed point and $(P(p_0), \pi, M, \phi(p_0))$ the reduction. Then the Lie algebra of the holonomy group $\phi(p_0)$ is generated by the elements $\Omega(X, Y)$ with $X, Y \in (TP)_p$ and $p \in P(p_0)$.

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