# Engineering Mathematics I

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# Preface

I am deeply gratified with the enthusiastic response shown to all my books on Engineering Mathematics (for first year courses) by students and teachers throughout the country.

This book has been designed to meet the requirements of the students of first-year undergraduate course on Engineering Mathematics. The contents have been covered in adequate depth for Semester I of various universities/deemed universities across the country. It offers a balanced coverage of both theory and problems. Lucid writing style supported with step-by-step solutions to all problems enhances understanding of the concepts. The book has ample number of solved and unsolved problems of different types to help students and teachers learning and teaching the subject.

I hope that this book will be received by both the faculty and the students as enthusiastically as my other books on Engineering Mathematics. Critical evaluation and suggestions for the improvement of the book will be highly appreciated and acknowledged.

**T VEERARAJAN**

## Publisher's Note

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UNIT **1**

# Differential Calculus

## 1.1 FUNCTIONS

Students are familiar, to some extent, with the concept of *relation*. A relation can be thought of as a relationship of elements of a set to the elements of another set. In other words, when $A$ and $B$ are sets, a subset $R$ of the cartesian product $A \times B$ is called a relation from $A$ to $B$ viz., If $R$ is a relation from $A$ to $B$, $R$ is a set of ordered pairs $(a, b)$ where $a \in A$ and $b \in b$.

### 1.1.1 Definitions

A relation from set $X$ to another set $Y$ is called a *function,* if for every $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in f$.

In other words, a function from $X$ to $Y$ is an assignment of exactly one element of $Y$ to every element of $X$.

If $y$ is the unique element of $Y$ assigned by the functions $f$ to the element $x$ of $X$, we write $f(x) = y$ and say that $y$ is a function of $x$.

If $f$ is a function from $X$ to $Y$, we may also represent it as

$$f : X \to Y \quad \text{or} \quad X \xrightarrow{f} Y.$$

*Note* ☑ (i) Sometimes the terms transformations, mapping or correspondence are also used in the place of function.

(ii) If $y = f(x)$, then $x$ is called an argument or *preimage* and $y$ is called the *image* of $x$ under $f$ or the value of the function $f$ at $x$.

(iii) $X$ is the *domain* of $f$ denoted by $D_f$ and $Y$ is called the *codomain of f*

(iv) The set of the images of *all* elements of $X$ is called the *range* of $X$ denoted $R_f ncY R_f \leq y$.

(v) $x$ is called the independent variable and $y$ the dependent variable, if $y = f(x)$.

### 1.1.2 Representation of Functions

(1) A function can be represented or expressed by means of a mathematical rule of formula, such as $y = x^3 [\equiv f(x)]$ or

(2) it can be represented pictorially by means of two closed circles or two closed ellipses or any two closed curves. This representation is possible only if $D_f$ consists of finite number of elements. The elements of $D_f$ will be represented by points inside the first closed curve and those of range or co-domain of $f$ by

points inside the second closed curve. The points in $D_f$ and the corresponding $R_f$ will be connected by directed arrows as explained below:

Let $D_f = (1, 2, 3, 4)$ and $f(1) = b, f(2) = d, f(3) = a$ and $f(4) = b$

The pictorial representation is shown as follows:

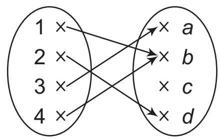


**Fig. 1.1**

Here $D_f = \{1, 2, 3, 4\}$, $R_f = \{a, b, d\}$ and co-domain of $f = \{a, b, c, d\}$

(3) A continuous function can be represented by means of a (curve) graph. For example, $y = x^2[= f(x)]$ is a continuous function of $x$;

The values of $x^2 = [f(x)]$ for different values of $x \in R$ lie on a parabola as in the figure given below:

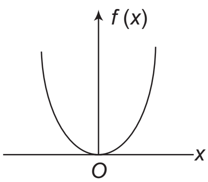


**Fig. 1.2**

*Note* ☑ (i) The curve $y = x^2$ drawn here is a one-piece without any break.

(ii) Discontinues functions can also be represented graphically but with a break in the neighbourhood of the point of discontinuity.

For example, $f(x) = \begin{cases} x+1, \text{ in}(-1, 0) \\ x-1, \text{ in}(0, 1) \end{cases}$ is a discontinues function, with the point $x = 0$ as a point of discontinuity. The graph of $y = f(x)$ consists of two line segments with a break near the origin as shown in the figure below:

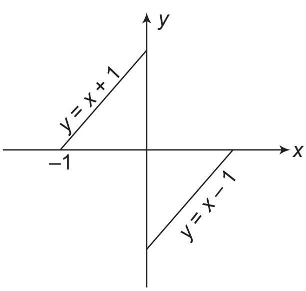


**Fig. 1.3**

*Note* ☑ Detailed discussion of continuous functions will be done later.

## 1.2 LIMIT OF A FUNCTION

Let us consider the function $y = f(x) = \dfrac{x^2 - 4}{x - 2}$. The value of $f(x)$ can not be found out at $x = 2$ by direct substitution though the values of $f(x)$ can be found out for all other values of $x$, however close they may be to 2.

As may be verified, $f(x)$ assume the values 3.9, 3.99, 3.999, 3.9999, etc. as $x$ take the values 1.9, 1.99, 1.999, 1.9999, etc.

Similarly $f(x)$ assume the value 4.1, 4.01, 4.001, 4.0001, etc. as $x$ takes the value 2.1, 2.01, 2.001, 2.0001, etc.

But if we put $x = 2$ in the definition of $f(x)$, we get $y = \frac{0}{0}$, which is meaningless and usually referred to as an *indeterminate form.*

The definition of limit of a function $f(x)$ does not require that the function $f(x)$ be defined at $x = 2$.

From the above example, we note that to make the difference between $f(x)$ and 4 as small as possible, we have to make the difference between $x$ and 2 correspondingly small. This fact is symbolically put as $\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = 4$

**Note** ☑ For all value of $x \neq 2$, $f(x) = \frac{(x-2)(x+2)}{x-2}$

$$= x + 2$$

$$\therefore \qquad \lim_{x\to 2}[f(x)] = 4$$

The formal definition of the limit of a function is given below:

$\lim_{x\to a}[f(x)] = l$, if and only if for any arbitrarily small positive number $\in$, there exists another small positive number $\delta$, such that

$$|f(x) - l| < \in, \text{ whenever } 0 < |x - a| < \delta.$$

**Note** ☑ When $x \to a$ through values greater than $a$, we say that $x$ approaches $a$ from the right (or from above) and write $\lim_{x\to a+}\{f(x)\} = l$. This is called *the right hand limit*. Similarly when $x \to a$ through values less than $a$ (or from below), we obtain *the left hand limit* and write $\lim_{x\to a-}\{f(x)\} = l$.

If $\lim_{x\to a}\{f(x)\}$ exists and equals $l$, it implies that both the left hand and right hand limits exist and are equal, each equal to $l$.

## 1.2.1 Some Fundamental Theorems on Limits (stated without proof)

If $\lim_{x\to a}[f(x)] = l$ and $\lim_{x\to a}[g(x)] = m$, then

(i) $\lim_{x\to a}[f(x) \pm g(x)] = l \pm m$

**Note** ☑ This theorem can be extended to the algebraic sum of a finite number of functions.

viz.,
$$\lim_{x\to a}[c_1 f_1(x) + c_2 f_2(x) + \cdots c_k f_k(x)] = c_1 \lim_{x\to a}[f_1(x)] + c_2 \lim_{x\to a}[f_2(x)] + \cdots c_k \lim_{x\to a}[f_x(x)]$$

(ii) $\lim\limits_{x \to a}[f(x) \cdot g(x)] = lm$

(iii) $\lim\limits_{x \to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{l}{m}$, provided $m \neq 0$.

## 1.2.2 Some Standard Limits

1. $\lim\limits_{x \to a}\left\{\dfrac{x^n - a^n}{x - a}\right\} = na^{n-1}$, for all rational values of $n$ and $a \neq 0$.

**Proof:** Let $x - a = h$. since $x \to a$, $h \to 0$

$$\lim_{x \to a}\left[\frac{x^n - a^n}{x - a}\right] = \lim_{h \to 0}\left\{\frac{(a+h)^n - a^n}{h}\right\}$$

$$= \lim_{h \to 0}\left[\frac{a^n\left(1 + \dfrac{h}{a}\right)^n - a^n}{h}\right] \qquad (1)$$

Since $x$ can approach a from the left or right,

$$|h| = |x - a| \to 0 \text{ and hence } \left|\frac{h}{a}\right| < 1$$

Using this fact and binomial theorem for a rational index in (1), we have

$$\lim_{x \to a}\left[\frac{x^n - a^n}{x - a}\right] = \lim_{h \to 0}\left[\frac{a^n\left\{1 + \binom{n}{1}\left(\dfrac{h}{a}\right) + \binom{n}{2}\left(\dfrac{h}{a}\right)^2 + \cdots \binom{n}{r}\left(\dfrac{h}{a}\right)^r + \cdots\right\} - a^n}{h}\right]$$

$$= \lim_{h \to 0}\left[\binom{n}{1}a^{n-1} + \binom{n}{2}a^{n-2}h + \cdots + \binom{n}{r}a^{n-r}h^{r-1} + \cdots\right]$$

$$= na^{n-1}$$

2. $\lim\limits_{\theta \to 0}\left(\dfrac{\sin\theta}{\theta}\right) = 1$, where $\theta$ is measured in radians and $< \dfrac{\pi}{2}$.

**Proof:**

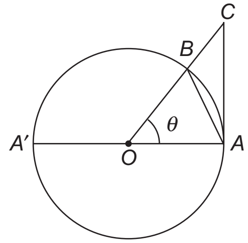


**Fig. 1.4**

Consider a circle with centre at 0 and radius $r = (= OA = OB)$

Let $\angle AOB = \theta$ radians and let the tangent at $A$ meet $OB$ produced at $C$,

Area of the $\Delta OAC = \frac{1}{2} OA \cdot AC$

$$= \frac{1}{2} r \cdot r \tan\theta \left( \because \frac{AC}{OA} = \tan\theta \text{ from the right angled } \Delta OAC \right)$$

$$= \frac{1}{2} r^2 \tan\theta$$

Area of the $\Delta OAB = \frac{1}{2} OA \cdot OB \cdot \sin\angle AOB$

$$= \frac{1}{2} r^2 \sin\theta$$

Area of the sector $OAB = \frac{1}{2} r^2\theta.$

Obviously, Area of the $\Delta OAB$ < Area of the sector $OAB$ < Area of $\Delta AOC$.

viz., $$\frac{1}{2} r^2 \sin\theta < \frac{1}{2} r^2\theta < \frac{1}{2} r^2 \tan\theta$$

viz., $$\sin\theta < \theta < \tan\theta$$

viz. $$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

or $$1 > \frac{\sin\theta}{\theta} > \cos\theta \quad \text{(considering the reciprocate)}$$

Taking limits as $\theta \to 0$, $\lim\limits_{\theta\to 0}(1) \geq \lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) \geq \lim\limits_{\theta\to 0}(\cos\theta)$

viz., $$1 \geq \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) \geq 1 \text{ or } \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1.$$

**Cor:** $\lim\limits_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right) = 1$, since $\lim\limits_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right) = \lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\left(\frac{1}{\cos\theta}\right) = 1 \times 1 = 1$

3. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$, which is the Naperian logarithmic base defined as $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty = 2.71828$

**Proof:**

Expanding $\left(1+\frac{1}{x}\right)^x$ by Binomial theorem, we have

$$\left(1+\frac{1}{x}\right)^x = 1 + x\cdot\frac{1}{x} + \frac{x(x-1)}{2!}\cdot\frac{1}{x^2} + \frac{x(x-1)(x-2)}{3!}\cdot\frac{1}{x^3} + \cdots$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{x}\right) + \frac{1}{3!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right) + \cdots$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

$$= e$$

***Note*** ☑ Obviously, $e > 2$.

Also $$e < 2 + \frac{1}{2} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 2\cdot 2} + \cdots$$

viz., $$e < 1 + \left\{1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right\}$$

viz., $$e < 1 + \frac{1}{1-\frac{1}{2}} \quad \left(\because\ a + ar + ar^2 + \cdots\infty = \frac{a}{1-r}\right)$$

viz., $e < 3$

Hence, $2 < e < 3$

**Cor (1)**: $$\lim_{x\to 0}(1+x)^{1/x} = e$$

Putting $\frac{1}{x} = y$ in result(3), we see that $y \to 0$ as $x \to \infty$

$\therefore$ $$\lim_{y\to 0}(1+y)^{1/y} = e$$

Replacing $y$ by $x$, the corollary follows:

**Cor (2):** $$\lim_{x\to 0}\left(\frac{e^x - 1}{x}\right) = 1$$

L. S. $$= \lim_{x\to 0}\frac{[(1+x)^{1/x}]^x - 1}{x} = \lim_{x\to 0}\left(\frac{x}{x}\right) = 1.$$

**Cor (3):** $$\lim_{x\to 0}\left(\frac{e^{mx} - 1}{x}\right) = m, \text{ as } \lim_{x\to 0}\left(\frac{e^{mx} - 1}{x}\right) = \lim_{y\to 0}\left(\frac{e^{y-1}}{y}\right) \times m = m$$

**Cor (4):** $$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{\frac{x}{a}}\right]^a = \lim_{\frac{x}{a}\to\infty}\left[\left(1+\frac{a}{x}\right)^{\frac{x}{a}}\right]^a = e^a$$

**Cor (5):** $$\lim_{x\to 0}\left(\frac{a^x - 1}{x}\right) = \log_{e^a}, \text{ since } \lim_{x\to 0}\left\{\frac{a^x - 1}{x}\right\} = \lim_{x\to 0}\left\{\frac{e^{x\log a} - 1}{x}\right\} = \log_{e^a},$$

by cor. (3)

## 1.3 CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be *continuous* at $x = a$, if

(i) $f(a)$ exists and is equal to $l$ and

(ii) $\lim_{x \to a} [f(x)]$ exists and is equal to $l$, viz., $\lim_{x \to a-} \{f(x)\} = \lim_{x \to a+} \{f(x)\} = l$

[*viz.*, the left hand and the right hand limits exist and are equal]

*Note* ☑ (1) If the graph of $y = f(x)$ is a continuous curve in an interval, the function is said to be continuous in that interval.

If the graph of $y = f(x)$ has a break at a point $x = a$, the function is said to be discontinuous at $x = a$. The point itself is called a point of discontinuity of $f(x)$.

(2) When a function $f(x)$ is continuous at every point in the interval $a < x < b$, $f(x)$ is said to be continuous in that interval.

(3) If a function $f(x)$ is continuous at a point $x = a$, a small change in the value of $x$ will produce only a small change in the value of $y = f(x)$.
For example, let $y = x^2$ and let $x$ be given a small change $\Delta x$. Let $\Delta y$ be the corresponding change in $y$.
Then $\Delta y = (x + \Delta x)^2 - y = 2x\,\Delta x + (\Delta x)^2$, which tends to 0 as $\Delta x \to 0$.

(4) When a continuous function changes from one values to another, it will pass at least once through every intermediate value

(5) A continuous function cannot change sign (either from +*ve* to –*ve* or –*ve* to +*ve*) without passing through the value 0.

## 1.4 DIFFERENTIABILITY OF A FUNCTION

Let $f'(x_0+) = \lim_{\Delta x \to 0} \left[ \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right]$ and

$f'(x_0-) = \lim_{\Delta x \to 0} \left[ \dfrac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} \right]$. If $f'(x_0+) = f'(x_0-)$, then

$f(x)$ is said to be *differentiable* at $x = x_0$ and the common value is denoted by $f'(x_0)$

*Note* ☑ (1) If the two limits exist but are unequal or if neither of them exists, then $f(x)$ is not differentiable at $x = x_0$.

(2) If a function is differentiable at a point, it follows from the definition that it is continuous at that point, but a function which is continuous at a point need not be differentiable at that point.

### WORKED EXAMPLES 1(a)

**Example 1.1** If $f(x) = x^4 - 3x^2 + 2$, show that $f(\sqrt{x+1}) - 2f(\sqrt{x}) + f(\sqrt{x-1})$ is a constant.

$$f(\sqrt{x}) = x^2 - 3x + 2;\ f(\sqrt{x+1}) = (x+1)^2 - 3(x+1) + 2 = x^2 - x;$$

$$f(\sqrt{x-1}) = (x-1)^2 - 3(x-1) + 2 = x^2 - 5x + 6$$

$$f(\sqrt{x+1}) + f(\sqrt{x-1}) = 2x^2 - 6x + 6$$

$$= 2(x^2 - 3x + 2) + 2$$

$$= 2\,(\sqrt{x}) + 2$$

$$\therefore \quad f(\sqrt{x+1}) - 2f(\sqrt{x}) + f\sqrt{(x-1)} = 2, \text{ a constant}.$$

**Example 1.2** Find the domain of the function $f(x) = \dfrac{x-2}{3x+1} - \dfrac{x-3}{3x-2}$ for which it is positive ($x$ is real).

$$f(x) = \frac{x-2}{3x+1} - \frac{x-3}{3x-2} = \frac{(x-2)(3x-2) - (x-3)(3x+1)}{(3x+1)(3x-2)}$$

$$= \frac{7}{(3x+1)(3x+2)}$$

Since $f(x) > 0$, $(3x + 1)(3x - 2) > 0$

$$\therefore \quad x > -\frac{1}{3} \text{ and } x > \frac{2}{3},\ \text{viz., } x > \frac{2}{3}$$

$$\text{and} \quad x < -\frac{1}{3} \text{ and } x < \frac{2}{3},\ \text{viz., } x < -\frac{1}{3}$$

$\therefore$ The required domain is $\left(-\infty, -\dfrac{1}{3}\right) \cup \left(\dfrac{2}{3}, \infty\right)$

**Example 1.3** If $x$ is real, find the range of the function.

$$f(x) = \frac{x^2 + 34x + 71}{x^2 + 2x - 7}$$

Let $$y = f(x) = \frac{x^2 + 34x - 71}{x^2 + 2x - 7}$$

$$\therefore \quad x^2(y - 1) + 2x(y - 17) + (71 - 7y) = 0 \qquad (1)$$

Since $x$ is real, discriminant of (1) $\geq 0$

viz., $4(y - 17)^2 - 4(y - 1)(71 - 7y) \geq 0$

viz., $8y^2 - 112y + 360 \geq 0$ or $y^2 - 14y + 45 \geq 0$

viz., $(y - 5)(y - 9) \geq 0$

$\therefore$ $y \geq 5$ and $y \geq 9$ or $y \leq 5$ and $y \leq 9$

viz., $y \geq 9$ or $y \leq 5$

$\therefore$ Rang of $f(x)$ is $(-\infty, 5)$ and $(9, \infty)$

i.e., $f(x)$ cannot have any values between 5 and 9.

**Example 1.4** Find $\lim_{x \to a}\left[\dfrac{\sqrt{2x+a}-\sqrt{3x}}{\sqrt{x+3a}-2\sqrt{x}}\right]$.

Put $x = a + h$. When $x \to a$, $h \to 0$.

$$\therefore \text{ Required limit } = \lim_{h \to 0}\left[\frac{\sqrt{(3a+2h)}-\sqrt{3(a+h)}}{\sqrt{4a+h}-\sqrt{4a+4h}}\right]$$

$$= \lim_{h \to 0}\left[\frac{\sqrt{3a}\left(1+\dfrac{2h}{3a}\right)^{\frac{1}{2}}-\sqrt{3a}\left(1+\dfrac{h}{a}\right)^{\frac{1}{2}}}{2\sqrt{a}\left(1+\dfrac{h}{4a}\right)^{\frac{1}{2}}-2\sqrt{a}\left(1+\dfrac{h}{a}\right)^{\frac{1}{2}}}\right]$$

$$= \lim_{h \to 0}\frac{\sqrt{3a}\left\{1+\dfrac{1}{2}\cdot\dfrac{2h}{3a}+0(h^2)\right\}-\sqrt{3a}\left\{1+\dfrac{h}{2a}+0(h^2)\right\}}{2\sqrt{a}\left\{1+\dfrac{1}{2}\cdot\dfrac{h}{4a}+0(h^2)\right\}-2\sqrt{a}\left\{1+\dfrac{1}{2}\cdot\dfrac{h}{a}+0(h^2)\right\}}$$

$$= \lim_{h \to 0}\left(\frac{h}{\sqrt{3a}}-\frac{\sqrt{3}}{2\sqrt{a}}h\right)\bigg/\left(\frac{h}{4\sqrt{a}}-\frac{h}{\sqrt{a}}\right)$$ ($\because$ terms involving $h^2$ and higher power of $h \to 0$)

$$= \left(\frac{1}{\sqrt{3}}-\frac{\sqrt{3}}{2}\right)\bigg/\left(\frac{1}{4}-1\right)$$

$$= -\frac{1}{2\sqrt{3}} \div \frac{-4}{3} = \frac{2}{3\sqrt{3}}$$

**Example 1.5** Evaluate $\lim_{x \to a}\left[\dfrac{\sqrt{x+5a}-\sqrt{7x-a}}{x^2-a^2}\right]$.

Multiplying the denominator and numerator by $\sqrt{x+5a}+\sqrt{7x-a}$, we get

$$\text{required limit } = \lim_{x \to a}\frac{(x+5a)-(7x-a)}{(x^2-a^2)\left\{\sqrt{x+5a}+\sqrt{7x-a}\right\}}$$

$$= \lim_{x \to a}\left[\frac{-6(x-a)}{(x^2-a^2)\left\{\sqrt{x+5a}+\sqrt{7x-a}\right\}}\right]$$

$$= \lim_{x \to a}\left[\frac{-6}{(x+a)\left\{\sqrt{x+5a}+\sqrt{7x-a}\right\}}\right]$$

$$= \lim_{x \to a}\left[\frac{-6}{2a\cdot 2\sqrt{ba}} = -\frac{3}{(2a)^{3/2}}\right]$$

**Example 1.6** Evaluate $\lim_{n\to\infty}\left(\dfrac{2^{n+1}+3^{n+1}}{2^n-3^n}\right)$.

$$\text{Required limit } = \lim_{n\to\infty}\left(\frac{2\times 2^n+3\times 3^n}{2^n-3^n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{2\times\left(\frac{2}{3}\right)^n+3}{\left(\frac{2}{3}\right)^n-1}\right]$$

$$= \frac{2\times 0+3}{0-1} \quad \left(\because \frac{2}{3}<1\right)$$

$$= -3$$

**Example 1.7** Evaluate $\lim_{x\to\frac{\pi}{4}}\left[\dfrac{\sin x-\cos x}{x-\frac{\pi}{4}}\right]$.

Put $x-\dfrac{\pi}{4}=y$. When $x\to\dfrac{\pi}{4}$, $y\to 0$

$$\therefore \text{ The required limit } = \lim_{y\to 0}\left[\frac{\sin\left(y+\frac{\pi}{4}\right)-\cos\left(y+\frac{\pi}{4}\right)}{y}\right]$$

$$= \lim_{y\to 0}\left[\frac{\left(\frac{1}{\sqrt{2}}\sin y+\frac{1}{\sqrt{2}}\cos y\right)-\left(\frac{1}{\sqrt{2}}\cos y-\frac{1}{\sqrt{2}}\sin y\right)}{y}\right]$$

$$= \lim_{y\to 0}\left[\sqrt{2}\left(\frac{\sin y}{y}\right)\right]=\sqrt{2}$$

**Example 1.8** Evaluate $\lim_{x\to 0}[x(\operatorname{cosec} x+2\operatorname{cosec} 2x)]$.

$$\text{Required limit } = \lim_{x\to 0}\left[x\left\{\frac{1}{\sin x}+\frac{2}{\sin 2x}\right\}\right]$$

$$= \lim_{x\to 0}\left[x\left(\frac{1+\cos x}{\sin x\cos x}\right)\right]$$

$$= \lim_{x\to 0}\left[\frac{x\cdot 2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}\cdot\cos x}\right]$$

$$= \lim_{x\to 0}\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)\cdot\left(\frac{2\cos\frac{x}{2}}{\cos x}\right)$$

$$= 1\times 2 = 2.$$

**Example 1.9** Evaluate $\lim_{x\to 0}\left(\frac{\tan x - \sin x}{x^3}\right)$.

$$L = \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1-\cos x}{x^2}\right)\cdot\frac{1}{\cos x}\right]$$

$$= \lim_{x\to 0}\left[1\times\frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2\times 4}\times\frac{1}{\cos x}\right]$$

$$= 1\times\frac{1}{2}\times 1^2\times 1 = \frac{1}{2}.$$

**Example 1.10** Evaluate $\lim_{h\to 0}\left[\frac{(a+h)^2\sin(a+h) - a^2\sin a}{h}\right]$.

$$L = \lim_{h\to 0}\left[\frac{(a^2+2ah+h^2)\sin(a+h) - a^2\sin a}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{a^2\{\sin(a+h)-\sin a\}}{h} + 2a\sin(a+h)\right], \text{ omitting } h^2,$$

$$= \lim_{h\to 0}\left[\frac{a^2\cdot 2\cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2}}{h} + 2a\sin(a+h)\right]$$

$$= a^2\cos a + 2a\sin a$$

**Example 1.11** Find $\lim_{x\to 0}(1+\sin x)^{\cot x}$.

$$L = \lim_{x\to 0}\{(1+\sin x)^{\frac{1}{\sin x}}\}^{\cos x}$$

$$= \lim_{x\to 0}(e^{\cos x}) = e, \text{ by cor (1) under standard limit (3)}$$

**Example 1.12** Find $\lim_{x\to y}\left(\frac{a^x - a^y}{x-y}\right)$.

$$L = \lim_{x\to y} a^y\left\{\frac{a^{x-y}-1}{x-y}\right\}$$

$$= a^y \lim_{(x-y)\to 0}\left(\frac{a^{x-y}-1}{x-y}\right)$$

$$= a^y \log a, \text{ by cor (5)}$$

**Example 1.13** Evaluate $\lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right]$.

$$L = \lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{x^2}\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right\}\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{x} + \frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} - \cdots\right]$$

**Example 1.14** Examine if the function $f(x)$ defined below is continuous at $x = 0$:

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Left hand limit of $f(x)$ at $(x = 0)$ $= \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\left[-h \cdot \sin\left(\frac{1}{-h}\right)\right]$$

$$= \lim_{h\to 0}\left[h \cdot \sin\left(\frac{1}{h}\right)\right]$$

$$= 0, \text{ since } \sin\left(\frac{1}{h}\right) \text{ is numerically less than 1}$$

Right hand limit of $f(x)$ at $(x = 0)$ $= \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} h \cdot \sin\left(\frac{1}{h}\right) = 0$$

Since $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0) = 0$, $f(x)$ is continuous at the origin.

**Example 1.15** Test the continuity of the function $f(x)$ at $x = 0$ defined as

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & \text{when } x \neq 0 \\ 0 & , \text{when } x = 0 \end{cases}$$

Left $\quad \lim_{\substack{x\to 0-h \\ h\to 0}}\left\{\frac{e^{-1/h} - 1}{e^{-1/h} + 1}\right\} = -1$

Right $\quad \lim_{\substack{x\to 0+h \\ h\to 0}}\left\{\frac{e^{1/h} - 1}{e^{1/h} + 1}\right\} = \lim_{h\to 0}\left\{\frac{1 - e^{-1/h}}{1 + e^{-1/h}}\right\} = 1$

Since $\lim\limits_{x \to 0-h} \{f(x)\} \neq \lim\limits_{x \to 0+h} \{f(x)\} \neq f(0)$, the function $f(x)$ is not continuous at the point $x = 0$.

**Example 1.16** A function $f(x)$ is defined as follows: Discuss the continuity of $f(x)$ at $x = 1$ and $x = 2$.

$$f(x) = \begin{cases} x, \text{ in } x < 1 \\ 2 - x, \text{ in } 1 \leq x \leq 2 \\ -2 + 3x - x^2, \text{ in } x > 2 \end{cases}$$

Left $\lim\limits_{x \to 1} \{f(x)\} = \lim\limits_{\substack{x \to 1-h \\ h \to 0}} (1 - h) = 1$

Right $\lim\limits_{x \to 1} \{f(x)\} = \lim\limits_{\substack{x \to 1+h \\ h \to 0}} \{2 - (1 + h)\} = 1$. Also $f(1) = 2 - 1 = 1$

$\therefore$ $f(x)$ is continuous at $x = 1$

Left $\lim\limits_{x \to 2} \{f(x)\} = \lim\limits_{\substack{x \to 2-h \\ h \to 0}} \{2 - (2 - h)\} = 0$

Right $\lim\limits_{x \to 2} \{f(x)\} = \lim\limits_{\substack{x \to 2+h \\ h \to 0}} \{-2 + 3\,(2 + h) - (2 + h)^2\} = 0$

Also, $f(2) = 2 - 2 = 0$

$\therefore$ $f(x)$ is continuous at $x = 2$ also.

**Example 1.17** A function $f(x)$ is defined as follows: Discuss the continuity of the function at $x = 0$ and at $x = 1$.

$$f(x) = \begin{cases} -x^2, \text{ if } x \leq 0 \\ 5x - 4, \text{ if } 0 < x \leq 1 \\ 4x^2 - 3x, \text{ if } 1 < x \leq 2 \end{cases}$$

Left $\lim\limits_{x \to 0} \{f(x)\} = \lim\limits_{\substack{x \to 0-h \\ h \to 0}} \{-(-h)^2\} \equiv 0$

Right $\lim\limits_{x \to 0} \{f(x)\} = \lim\limits_{\substack{x \to 0+h \\ h \to 0}} \{5h - 4\} = -4$

$\therefore$ $f(x)$ is not continuous at $x = 0$, since left lim $\neq$ right lim.

Left $\lim\limits_{x \to 1} \{f(x)\} = \lim\limits_{\substack{x \to 1-h \\ h \to 0}} \{5(1 - h) - 4\} = 1$

Right $\lim\limits_{x \to 1} \{f(x)\} = \lim\limits_{\substack{x \to 1+h \\ h \to 0}} \{4(1 + h)^2 - 3(1 + h)\} = 1$; Also $f(1) = 1$

Since left lim = right lim = $f(1)$, $f(x)$ is continuous at $x = 1$.

**Example 1.18** Find the values of $a$, $b$, $c$ for which the function defined below is continuous at the origin:

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x}, & \text{for } x<0\\ c, & \text{for } x=0\\ \dfrac{(x+bx^2)^{\frac{1}{2}}-x^{\frac{1}{2}}}{bx^{\frac{3}{2}}}, & \text{for } x>0\end{cases}$$

Left $\lim\limits_{\substack{x\to 0-h\\ h\to 0}}\left[\dfrac{\sin(a+1)(-h)+\sin(-h)}{-h}\right]=\lim\limits_{h\to 0}\left[\dfrac{\sin(a+1)h}{h}+\dfrac{\sinh}{h}\right]$

$$=a+1+1=a+2$$

Also, $f(0)=c$

Right $\lim\limits_{\substack{x\to 0+h\\ h\to 0}}\left[\dfrac{\sqrt{h+bh^2}-\sqrt{h}}{bh\sqrt{h}}\right]=\lim\limits_{h\to 0}\left[\dfrac{\{\sqrt{h+bh^2}-\sqrt{h}\}\{\sqrt{h+bh^2}+\sqrt{h}\}}{bh\sqrt{h}\{\sqrt{h+bh^2}+\sqrt{h}\}}\right]$

$$\lim_{h\to 0}\frac{bh^2}{bh^2\{\sqrt{1+bh+1}\}}=\frac{1}{2}$$

Since the function $f(x)$ is continuous at the origin,

$$a+2=c=\frac{1}{2}$$

$\therefore$ $a=-\dfrac{3}{2}$ and $c=\dfrac{1}{2}$; $b$ is arbitrary, but $\neq 0$

**Example 1.19** Examine the continuity and differentiability of $f(x)=x^2\sin\dfrac{1}{x}(x\neq 0)$ and $f(0)=0$ at the origin.

Left $\lim\limits_{\substack{x\to 0-h\\ h\to 0}}\left[(-h)^2\sin\left(-\dfrac{1}{h}\right)\right]=\lim\limits_{h\to 0}\left[h^2\sin\left(\dfrac{1}{h}\right)\right]$

$= 0$, since $\sin\left(\dfrac{1}{h}\right)$ is bounded

Right $\lim\limits_{\substack{x\to 0+h\\ h\to 0}}\left[h^2\sin\left(\dfrac{1}{h}\right)\right]=0$, Aslo $f(0)=0$

$\therefore f(x)$ is continuous at $x=0$

Now $\lim\limits_{h\to 0}\left[\dfrac{f(0+h)-f(0)}{h}\right]=\lim\limits_{h\to 0}\left[\dfrac{h^2\sin\left(\frac{1}{h}\right)-0}{h}\right]$

$$= \lim_{h \to 0}\left[h \sin\left(\frac{1}{h}\right)\right] = 0$$

$$\lim_{h \to 0}\left[\frac{f(0-h)-f(0)}{-h}\right] = \lim_{h \to 0}\left[\frac{h^2 \sin\left(\frac{1}{-h}\right) - 0}{-h}\right]$$

$$= \lim_{h \to 0}\left[h \sin\left(\frac{1}{h}\right)\right] = 0$$

$\therefore f(x)$ is also differentiable at the origin.

**Example 1.20** Examine the continuity and differentiability of $f(x)$ defined below at $x = 2$.

$$f(x) = \begin{cases} 1 + x, & \text{for } x \le 2 \\ 5 - x, & \text{for } x > 2 \end{cases}$$

Left $\quad \lim_{\substack{x \to 2-h \\ h \to 0}} \{1 + 2 - h\} = 3$

Right $\quad \lim_{\substack{x \to 2+h \\ h \to 0}} \{5 - (2 + h)\} = 3.$ Also $f(2) = 3$

$\therefore f(x)$ is continuous at the point $x = 2$.

Now $\quad \lim_{h \to 0}\left[\frac{f(2+h)-f(2)}{h}\right]$

$$= \lim_{h \to 0} \frac{5 - (2 + h) - 3}{h} = -1$$

and $\quad \lim_{h \to 0}\left[\frac{f(2-h)-f(2)}{-h}\right]$

$$= \lim_{h \to 0}\left[\frac{(1 + 2 - h - 3)}{-h}\right] = 1$$

Since the two limits are not equal, $f(x)$ is not differentiable at the point $x = 2$.

## EXERCISE 1(a)

***Part A***

(Short Answer Questions)

1. If $f(x) = a \cos^4 x + b \sin^2 x + c$, show that $f(\pi + x) - f(\pi - x) = f(x)$.
2. If $x$ is real, find the domain of the function of $f(x) = x^2 + x - 12$, for which it is negative.

3. If $x$ is real, find the domain of the function $f(x) = (x^2 - 3x - 18)$ for which it is positive.

Evaluate the following limits:

4. $\lim_{x\to 0}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\right\}$

5. $\lim_{x\to 1}\left\{\frac{\sqrt{5x-4}-\sqrt{x}}{x-1}\right\}$

6. $\lim_{x\to 0}\left\{\frac{x}{\sqrt{x+2}-\sqrt{2}}\right\}$

7. $\lim_{x\to a}\left[\frac{\sqrt{x}-\sqrt{a}}{3\sqrt{x}-3\sqrt{a}}\right]$

8. $\lim_{n\to\infty}\left(\frac{a^{n+1}+b^{n+1}}{a^n-b^n}\right), 0<a<b$

9. $\lim_{n\to\infty}\left(\frac{1+2+3+\cdots n}{n^2}\right)$

10. $\lim_{x\to 0}\left\{\frac{1-\cos mx}{1-\cos nx}\right\}$

11. $\lim_{x\to 0}\left\{\frac{1-\cos x}{x\sin x}\right\}$

12. $\lim_{x\to 0}\left\{\frac{\operatorname{cosec} x-\cot x}{x}\right\}$

13. $\lim_{\theta\to\alpha}\left\{\frac{\sin\theta-\sin\alpha}{\theta-\alpha}\right\}$

14. $\lim_{h\to 0}\left[\frac{\sin(x+h)-\sin(x-h)}{h}\right]$

15. $\lim_{x\to\frac{\pi}{4}}\left\{\frac{\operatorname{cosec} x-\sec x}{\cot x-\tan x}\right\}$

16. $\lim_{x\to\frac{\pi}{4}}(\sec 2x-\tan 2x)$

17. $\lim_{x\to 0}[(1+ax)^{\frac{b}{x}}]$

18. $\lim_{x\to 1}\{x^{\frac{1}{x-1}}\}$

19. $\lim_{x\to\frac{\pi}{2}}(1+\cos x)^{2\sec x}$

20. $\lim_{x\to\frac{\pi}{2}}\{(1+\cos x)^{\tan x}\}$

***Part B***

21. Find the domain of the function $f(x)=\frac{x-1}{4x+5}-\frac{x-3}{4x-3}$ for which $f(x)$ is negative, given that $x$ is real.

22. If $x$ is real, find the domain of the function $f(x)=\frac{3x-2}{x-1}-2(x\neq 1)$ for which $f(x)$ is negative.

23. If $x$ is real, find the range of the function $f(x)=\frac{x^2-2x+4}{x^2+2x+4}$.

24. If $x$ is real, prove that $f(x)=\frac{x}{x^2-5x+9}$ lies between $-\frac{1}{11}$ and 1.

25. If $x$ is real, prove that the range of the function $f(x)=\dfrac{x^2-3x+4}{x^2+3x+4}$ is $\left[\dfrac{1}{7}, 7\right]$.

26. If $y=f(x)=\dfrac{ax+b}{cx-a}$, prove that $x=f(y)$.

27. Evaluate $\lim\limits_{x\to\infty}\left[\dfrac{1^2+2^2+\cdots+n^2}{n(n^2+1)}\right]$.

28. Evaluate $\lim\limits_{x\to\infty}\left[\dfrac{(1^2+3^2+5^2+\cdots(2n-1)^2}{n^3}\right]$.

29. Evaluate $\lim\limits_{n\to\infty}\left[\dfrac{(1+2+3+\cdots n)(1^3+2^3+\cdots+n^3)}{n^3(1^2+2^2+\cdots+n^2}\right]$.

30. Evaluate $\lim\limits_{n\to\infty}\left[\dfrac{1.3+2.4+\cdots+n(n+2)}{1^2+2^2+\cdots+n^2}\right]$.

31. Evaluate $\lim\limits_{n\to\infty}\left[\dfrac{1^3+2^3+\cdots+n^3}{1\cdot2\cdot3+2\cdot3\cdot4+\cdots n(n+1)(n+2)}\right]$.

32. Evaluate $\lim\limits_{x\to 0}\left[\dfrac{2^x-1}{\sqrt{1+x}-1}\right]$. $\left\{\textbf{Hint:}\text{ Use }\lim\limits_{x\to 0}\left(\dfrac{a^x-1}{x}\right)\right\}$

33. Evaluate $\lim\limits_{x\to 0}\left(\dfrac{1-\cos 2x+\tan^2 x}{x\sin x}\right)$.

34. Evaluate $\lim\limits_{x\to 0}\left[\dfrac{e^x+\log(1+x)-(1-x)^{-2}}{x^2}\right]$. {**Hint:** Use expansions}

35. Evaluate $\lim\limits_{x\to 0}\left\{\dfrac{8^x-2^x}{x}\right\}$.

36. Evaluate $\lim\limits_{x\to 1}\left\{(1-x)\tan\dfrac{\pi x}{2}\right\}$.

37. Show that the function $f(x)$ defined below is continuous at $x = 0$ and at $x = 1$: Also draw the graph of $y = f(x)$.

$$f(x)=\begin{cases} -x, & \text{when } x\le 0 \\ x, & \text{when } 0<x<1 \\ 2-x, & \text{when } x\ge 1 \end{cases}$$

38. Test the continuity of the function $f(x)$ defined below at $x=\dfrac{3}{2}$.

$$f(x)=\begin{cases} \dfrac{|2x-3|}{2x-3}, & \text{for } x\ne\dfrac{3}{2} \\ 0, & \text{for } x=\dfrac{3}{2} \end{cases}$$

39. Examine the continuity of the function $f(x)$ defined below at $x = 0$.

$$f(x)=\begin{cases}\sin\dfrac{1}{x}, & \text{for } x\neq 0\\ 0\ , & \text{for } x=0\end{cases}$$

40. Find the values of the $a$ and $b$, if the function $f(x)$ defined below is continuous at $x = 3$ and at $x = 5$.

$$f(x)=\begin{cases}1, & \text{if } x\le 3\\ ax+b, & \text{if } 3<x<5\\ 7, & \text{if } x\ge 5\end{cases}$$

41. If $f(x)=\begin{cases}x-1, & \text{when } x>1\\ 0, & \text{when } x=1\\ 1-x, & \text{when } x<1\end{cases}$

shown that $f(x)$ is continuous at $x = 1$, but is not differentiable there.

42. Show that $f(x) = |x|$ is continuous at $x = 0$, but is not differentiable there.

43. Discuss the continuity and differentiability of the function $f(x)$ defined as

$$f(x)=\begin{cases}2x-3, & \text{for } 0\le x\le 2\\ x^2-3, & \text{for } x>2\end{cases}$$ at the point $x = 2$.

44. Discuss the continuity and differentiability of the function $f(x)$ defined below at $x = 1$ and at $x = 2$.

$f(x) = x$, for $x < 1$; $= 2 - x$, for $1 \le x \le 2$ and $= -2 + 3x - x^2$ for $x > 2$

45. Discuss the continuity and differentiability of the function $f(x) = |x-1| + |x-2|$ at the point $x = 1$ and $x = 2$.

## 1.5 DERIVATIVES

Let $y$ be a continuous function of $x$. Let $\Delta x$ be a small increment in the value of $x$ and let the corresponding increment in the value of $y$ be $\Delta y$. The limit of the ratio $\left(\dfrac{\Delta y}{\Delta x}\right)$ as $\Delta x \to 0$, if it exists is called the *differential coefficient* or *derivative* of $y$ with respect to $x$ and denoted by $\dfrac{dy}{dx}$. *viz*, $\lim\limits_{\Delta x\to 0}\left(\dfrac{\Delta y}{\Delta x}\right)=\dfrac{dy}{dx}$ or $Dy$

If $y$ is assumed as $f(x)$, then $\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}\left(\dfrac{\Delta y}{\Delta x}\right)=\lim\limits_{\Delta x\to 0}\left\{\dfrac{f(x+\Delta x)-f(x)}{\Delta x}\right\}$ and is denoted $f'(x)$ or $Df(x)$. The process of finding $\dfrac{dy}{dx}$ is called *differentiation.*

*Note* ☑ (1) $\frac{dy}{dx}$ is a composite symbol and should be understood neither as $\frac{\lim(\Delta y)}{\lim(\Delta x)}$ nor as $dy \div dx$. $\frac{d}{dx}$ must be interpreted as derivative w.r.t. $x$.

(2) The $\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$ is found out by the methods discussed in the previous section.

## 1.5.1 Derivatives of Elementary/Standard Functions

We shall follow the 4-step working procedure given below to find the derivatives of some elementary functions from first principles, viz., using the definition of $\frac{dy}{dx}$ or $f'(x)$.

(i) In the given function $y = f(x)$, replace $x$ by $x + \Delta x$. At the same time $y$ becomes $y + \Delta y$. viz., $y + \Delta y = f(x + \Delta x)$

(ii) Obtain $\Delta y = f(x + \Delta x) - f(x)$ and simplify as far as possible

(iii) Obatin $\frac{\Delta y}{\Delta x}$.

(iii) Find $\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$, which is the required derivative $\frac{dy}{dx}$.

**1. $\frac{d}{dx}(x^n) = nx^{n-1}$, where $n$ is rational number**

Let $y = x^n$. Let $\Delta x$ and $\Delta y$ be the small increments in $x$ and $y$ respectively.
Then $y + \Delta y = (x + \Delta x)^n$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\substack{\Delta x \to 0 \\ x+\Delta x \to x}}\left[\frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x}\right], \text{ by definition}$$

$$= nx^{n-1}, \text{ on using } \lim_{x \to a}\left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}$$

viz., $$\frac{d}{dx}(x^n) = nx^{n-1}$$

**2. $\frac{d}{dx}(\sin x) = \cos x$**

Let $y = \sin x$. Let $\Delta x$ and $\Delta y$ be the small increments in $x$ and $y$ respectively.
Then $y + \Delta y = \sin(x + \Delta x)$

$\therefore \qquad \Delta y = \sin(x + \Delta x) - \sin x$

$$= 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$$

$$\therefore \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left\{\cos\left(x + \frac{\Delta x}{2}\right) \cdot \frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\right\}$$

$$= \cos x \cdot 1 \left(\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right)$$

$$= \cos x$$

viz., $\frac{d}{dx}(\sin x) = \cos x$

**3. $\frac{d}{dx}(\cos x) = -\sin x$**

Let $\qquad y = \cos x$

Then $\qquad \Delta y = \cos(x + \Delta x) - \cos x$

$$= -2\sin\left(x + \frac{\Delta x}{2}\right) \cdot \sin\frac{\Delta x}{2}$$

$$\therefore \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left[-\sin\left(x + \frac{\Delta x}{2}\right) \cdot \left\{\frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\right\}\right]$$

$$= -\sin x \times 1 = -\sin x$$

viz., $\frac{d}{dx}(\cos x) = -\sin x$

**4. $\frac{d}{dx}(\tan x) = \sec^2 x$**

Let $\qquad y = \tan x$

Then $\qquad \Delta y = \tan(x + \Delta x) - \tan x$

$$= \frac{\sin(x + \Delta x)}{\cos(x + \Delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + \Delta x)\cos x - \cos(x + \Delta x)\sin x}{\cos(x + \Delta x)\cos x}$$

$$= \frac{\sin \Delta x}{\cos(x + \Delta x)\cos x}$$

$$\therefore \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\sin \Delta x}{\Delta x}\right) \cdot \frac{1}{\cos(x + \Delta x)\cos x}$$

$$= 1 \times \frac{1}{\cos^2 x} = \sec^2 x$$

viz., $\frac{d}{dx}(\tan x) = \sec^2 x.$

**5. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$**

Let $y = \cot x$

Then $\Delta y = \cot(x + \Delta x) - \cot x$

$$= \frac{\sin x \cos(x + \Delta x) - \cos x \sin(x + \Delta x)}{\sin x \sin(x + \Delta x)}$$

$$= \frac{\sin(-\Delta x)}{\sin x \sin(x + \Delta x)}$$

$\therefore$ $$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left\{-\left(\frac{\sin \Delta x}{\Delta x}\right) \cdot \frac{1}{\sin x \cdot \sin(x + \Delta x)}\right\}$$

$$= 1 \times \frac{1}{\sin^2 x} x - \operatorname{cosec}^2 x$$

viz., $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

**6. $\frac{d}{dx}(\sec x) = -\sec x \tan x$**

Let $y = \sec x$

Then $\Delta y = \sec(x + \Delta x) - \sec x$

$$= \frac{\cos x - \cos(x + \Delta x)}{\cos \cdot \cos(x + \Delta x)}$$

$$= \frac{2 \sin\left(x + \frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right)}{\cos x \cdot \cos(x + \Delta x)}$$

$\therefore$ $$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left[\frac{\sin\left(x + \frac{\Delta x}{2}\right)}{\cos x \cdot \cos(x + \Delta x)} \cdot \left(\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}\right)\right]$$

$$= \frac{\sin x}{\cos^2 x} \cdot 1 = \sec x \tan x$$

viz., $\frac{d}{dx}(\sec x) = \sec x \tan x$

**7.** $\dfrac{\mathbf{d}}{\mathbf{d}x}(\textbf{cosec}\, x) = -\textbf{cosec}\, x \,\textbf{cot}\, x$

Let $\quad y = \operatorname{cosec} x$

Then $\quad \Delta y = \operatorname{cosec}(x + \Delta x) - \operatorname{cosec} x$

$$= \frac{\sin x - \sin(x + \Delta x)}{\sin(x + \Delta x) \cdot \sin x}$$

$$= \frac{2\cos\left(x + \frac{\Delta x}{2}\right) \cdot \sin\left(-\frac{\Delta x}{2}\right)}{\sin(x + \Delta x)\sin x}$$

$\therefore$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left[\left\{\frac{\cos\left(x + \frac{\Delta x}{2}\right)}{\sin(x + \Delta x)\sin x}\right\}\left\{\frac{-\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right\}\right]$$

$$= \frac{\cos x}{\sin^2 x}(-1)$$

$$= -\operatorname{cosec} x \cot x$$

viz., $\quad \dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$

**8.** $\dfrac{\mathbf{d}}{\mathbf{d}x}(e^x) = e^x$

Let $\quad y = e^x$

Then $\quad \Delta y = e^{x + \Delta x} - e^x$

$\therefore$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{e^{x + \Delta x} - e^x}{\Delta x}\right)$$

$$= e^x \cdot \lim_{\Delta x \to 0}\left(\frac{e^{\Delta x} - 1}{\Delta x}\right)$$

$$= e^x \times 1, \left(\text{since } \lim_{h \to 1}\left(\frac{e^h - 1}{h}\right) = 1\right)$$

viz., $\quad \dfrac{d}{dx}(e^x) = e^x$

**9.** $\dfrac{\mathbf{d}}{\mathbf{d}x}(\mathbf{log}_e x) = \dfrac{\mathbf{1}}{x}$

Let $\quad y = \log_e x$

Then $\quad \Delta y = \log_e(x + \Delta x) - (\log_e x)$

$$= \log_e\left(\frac{x + \Delta x}{x}\right)$$

$$= \log_e\left(1 + \frac{\Delta x}{x}\right)$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left\{\frac{1}{\Delta x}\log_e\left(1+\frac{\Delta x}{x}\right)\right\}$$

$$= \lim_{\Delta x \to 0}\frac{1}{x}\log_e\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

$$= \frac{1}{x}\lim_{\left(\frac{\Delta x}{x}\right)\to 0}\left\{\log_e\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right\}$$

$$= \frac{1}{x}\log_e(e)\left[\because \lim_{h\to 0}(1+h)^{\frac{1}{h}} = e\right]$$

$$= \frac{1}{x}\cdot 1 = \frac{1}{x}$$

viz., $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}.$

## 1.6 RULES OF DIFFERENTIATION

**1. $\dfrac{d}{dx}(c) = 0$, where $c$ is a constant**

Let $y = c$

Then $\Delta y = c - c = 0$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{0}{\Delta x}\right) = 0$$

viz., $\dfrac{d}{dx}$ (any constant) = 0

**2. $\dfrac{d}{dx}(c\cdot u) = c\dfrac{du}{dx}$, where $c$ is a constant and $u$ is a function of $x$**

Let $y = c\cdot u$

Then $\Delta y = c(u + \Delta u) - cu = c\Delta u$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(c\frac{\Delta u}{\Delta x}\right) = c\cdot\frac{du}{dx}$$

**3. $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$, where $u$ and $v$ are functions of $x$**

Let $y = u \pm v$

Then $\Delta y = \{(u + \Delta u) \pm (v + \Delta v)\} - \{u \pm v\}$

$= \Delta u \pm \Delta v$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left\{\left(\frac{\Delta u}{\Delta x}\right) \pm \left(\frac{\Delta v}{\Delta x}\right)\right\}$$

$$= \frac{du}{dx} \pm \frac{dv}{dx}$$

This result can be extended as follows:

If $y = au \pm bv \pm cw \pm \cdots$, where $a, b, c, \cdots$ are constants and $u, v, w, \cdots$ are function of $x$, then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$

4. **Product rule, viz., $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, where $u$ are $v$ are different function of $x$**

Let $y = uv$

Then $\Delta y = u\Delta v + v\Delta u - \Delta u \cdot \Delta v$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0} u\left(\frac{\Delta v}{\Delta x}\right) + \lim_{\Delta x \to 0} v\frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\right)\Delta v$$

$$= u\frac{dv}{dx} + v\frac{du}{dx} + \frac{du}{dx} \times 0$$

$$= u\frac{dv}{dx} + v\frac{du}{dx}$$

viz., $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

**Extension**

$$\frac{d}{dx}(uvw) = \frac{d}{dx}\{u \cdot (vw)\}$$

$$= u\frac{d}{dx}(vw) + vw\frac{du}{dx}$$

$$= u\left\{v\frac{dw}{dx} + w\frac{dv}{dx}\right\} + vw\frac{du}{dx} \quad \text{(or)}$$

$$= uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}$$

**5. Quotient rule, viz., $\frac{d}{dx}\left(\frac{u}{v}\right) = \left\{\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right\}$**

Let $$y = \frac{u}{v}$$

Then $$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$= \frac{v\,\Delta u - u\,\Delta v}{v(v+\Delta v)}$$

$$\therefore \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0} \frac{v \lim\limits_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\right) - u \lim\limits_{\Delta x \to 0}\left(\frac{\Delta v}{\Delta x}\right)}{\lim\limits_{\Delta x \to 0} v(v+\Delta v)}$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v(v+0)} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

viz., $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

### 6. Derivative of a Function of Function

When $y$ is a function of $u$, where $u$ itself is a function of $x$, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$.

In this situation, $\Delta y = \frac{\Delta y}{\Delta u}\cdot \Delta u$

$$\therefore \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x}\right) = \frac{dy}{du}\cdot\frac{du}{dx}.$$

**Extension**

If $y$ is a function of $u$, $u$ a function of $u$, $v$ a function of $w$ and $w$ a function of $x$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}.$$

### 7. $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

If $y$ is a function of $x$, say $f(x)$, $x$ can also be considered as a function of $y$ and denoted as $f^{-1}(y)$

Then $$\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y} = 1$$

$$\therefore \qquad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}\left(\frac{\Delta y}{\Delta x}\right)\left(\frac{\Delta x}{\Delta y}\right) = 1$$

viz., $$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)\cdot \lim_{\Delta y \to 0}\left(\frac{\Delta x}{\Delta y}\right) = 1$$

viz., $$\frac{dy}{dx}\cdot\frac{dx}{dy} = 1 \text{ or } \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

## 1.7 DERIVATIVES OF HYPERBOLIC FUNCTION

1. $\dfrac{d}{dx}(\sinh x) = \cosh x$

Let
$$y = \sinh x$$
$$= \frac{e^x - e^{-x}}{2} \quad \text{[by definition]}$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left\{\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right\}$$
$$= \frac{1}{2}\left\{e^x - (e^{-x})\left(\frac{d}{dx}\right)(-1)\right\}$$
$$= \frac{e^x + e^{-x}}{2}$$
$$= \cosh x$$

2. Similarly we can prove that $\dfrac{d}{dx}(\cosh x) = \sinh x$

3. $\dfrac{d}{dx}(\tanh x) = \text{sech}^2\, x$

Let
$$y = \tanh x = \frac{\sinh x}{\cosh x}$$
$$\therefore \quad \frac{dy}{dx} = \frac{\cosh x \cdot \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{\cosh^2 x} \quad \text{[by quotient rule]}$$
$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$
$$= \frac{1}{\cosh^2 x} = \text{sech}^2\, x$$

4. Similarly, we can prove that $\dfrac{d}{dx}(\coth x) = -\,\text{cosech}^2\, x$

5. $\dfrac{d}{dx}(\text{sech}\, x) = -\,\text{sech}\, x \tanh x$

Let
$$y = \text{sech}\, x = \frac{1}{\cosh x}$$
$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\cosh^2 x} \cdot \frac{d}{dx}(\cosh x)$$

$$= -\frac{\sinh x}{\cosh^2 x}$$
$$= -\operatorname{sech} x \cdot \tanh x$$

6. Similarly, we can prove that $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$

## 1.7.1 Derivative of Inverse Circular and Inverse Hyperbolic Functions

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

   Let $y = \sin^{-1} x$; Then $x = \sin y$

   $\therefore \quad \frac{dx}{dy} = \cos y$

   $\therefore \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

2. Similarly, we can prove that $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

   Let $y = \tan^{-1} x$; Then $x = \tan y$

   $\therefore \quad \frac{dx}{dy} = \sec^2 y$

   $\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

4. Similarly, we can prove that $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

   Let $y = \sec^{-1} x$, Then $x = \sec y$

   $\therefore \quad \frac{dx}{dy} = \sec y \tan y$

   $\therefore \quad \frac{dy}{dx} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}}$

6. Similarly, we can prove that $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2 - 1}}$

7. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$

Let $y = \cosh^{-1} x$ Then $x = \cosh y$

$$\therefore \quad \frac{dx}{dy} = \sinh y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} \quad (\because \cosh^2 y - \sinh^2 y = 1)$$

$$= \frac{1}{\sqrt{x^2 - 1}}$$

8. Similarly, we can prove that $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}$

9. $$\frac{d}{dx}(\coth^{-1} x) = -\frac{1}{x^2 - 1}$$

Let $y = \coth^{-1} x$. Then $x = \coth y$

$$\therefore \quad \frac{dx}{dy} = -\text{cosech}^2 y$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\text{cosech}^2 y} = \frac{-1}{-1 + \coth^2 y} \quad (\because \coth^2 y - 1 = \text{cosech}^2 y)$$

$$= -\frac{1}{x^2 - 1}$$

10. Similarly, we can prove that $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}$

11. $\frac{d}{dx}(\text{cosech}^{-1} x) = \left| -\frac{1}{x\sqrt{1 + x^2}} \right|$

Let $y = \text{cosech}^{-1}$. Then $x = \text{cosech}\, y$

$$\therefore \quad \frac{dx}{dy} = -\text{cosech}\, y \cdot \coth y$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x\sqrt{1 + x^2}}$$

12. Similarly we can prove that $\frac{d}{dx}(\text{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$

## 1.8 METHODS OF DIFFERENTIATION

Though the rules of differentiation stated earlier can be applied to find the derivatives of many functions some functions can be differentiated using the methods indicated below:

### 1. Logarithmic Differentiation

When the given function $y$ is of the form $uv$, where $u$ and $v$ are functions of $x$, we use this method, in which we take logarithms on both sides of $y = uv$ and then differentiate w.r.t. $x$. This method will also simplify the work on differentiation of a function consisting of a number of products and quotients.

### 2. Differentiation of Implicit Functions

If $x$ and $y$ are implicitly related as $f(x, y) = 0$, viz., if $y$ cannot be expressed explicity as a function of $x$, the differentiation of $f(x, y) = 0$ w.r.t. $x$ is done, noting that $y$ is a function of $x$ whose derivation is $\frac{dy}{dx}$. On simplification, we will get $\frac{dy}{dx}$ as a mixed function of $x$ and $y$.

### 3. Differentiation by Trigonometric Substitution

Some apparently complicated functions of $x$, in particular, certain inverse trigonometric functions can be simplified by using appropriate trigonometric substitutions for $x$ and then differentiation is performed.

### 4. Differentiation from Parametric Equations

When $x$ and $y$ are both expressed as functions of a parameter, say '$t$', it is not recovery to get $y$ as a function of $x$, as $\frac{dy}{dx}$ can be given as a function of $t$.

In this case, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ or $\frac{dy}{dt} \Big/ \frac{dx}{dt}$

### 5. Differentiation of One Function w.r.t. another Function

If it is required to find the derivative of $u(x)$ w.r.t. $v(x)$, we treat that $u$ and $v$ are parametrically expressed in terms of the parameter $x$.

Then $\frac{du}{dv} = \frac{du}{dx} \Big/ \frac{dv}{dx}$

## WORKED EXAMPLES 1(b)

**Example 1.1** Find $\frac{dy}{dx}$ from first principles, when $y =$

(i) $\sqrt{1+3x}$, (ii) $\frac{x^2+8}{2x+3}$, (iii) $\sin^3 x$, (iv) $e^{\sqrt{x}}$, and (v) $\tan^{-1}(\sin x)$

(i) $y = \sqrt{1+3x}$. Let $\Delta x$ and $\Delta y$ be the increments in $x$ and $y$ respectively.

Then $\Delta y = \sqrt{1+3(x+\Delta y)} - \sqrt{1+3x}$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{\sqrt{1+3x+3\Delta x} - \sqrt{1+3x}}{\Delta x}\right]$$

$$= 3 \times \lim_{\Delta x \to 0}\left[\frac{(1+3x+3\Delta x)^{\frac{1}{2}} - (1+3x)^{\frac{1}{2}}}{(1+3x+3\Delta x) - (1+3x)}\right] = \frac{3}{2}(1+3x)^{\frac{1}{2}-1} = \frac{3}{2\sqrt{1+3x}}$$

(ii) $y = \frac{x^2+8}{2x+3}$

Let $\Delta x$ and $\Delta y$ be the increments in $x$ and $y$ respectively.

Then $$\Delta y = \frac{(x+\Delta x)^2+8}{2(x+\Delta x)+3} - \frac{x^2+8}{2x+3}$$

$$= \frac{(2x+3)(x^2+2x\Delta x+\Delta x^2+8) - (2x+2\Delta x+3)(x^2+8)}{(2x+3)(2x+2\Delta x+3)}$$

$$= \frac{(2x+3)(2x\Delta x+\Delta x^2) - 2\Delta x(x^2+8)}{(2x+3)(2x+2\Delta x+3)}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{(2x+3)(2x+\Delta x) - 2(x^2+8)}{(2x+3)(2x+2\Delta x+3)}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{(2x+3)\cdot 2x - 2(x^2+8)}{(2x+3)^2}$$

$$= \frac{4x^2+6x-2x^2-6}{(2x+3)^2}$$

$$= \frac{2x^2+6x-16}{(2x+3)^2}$$

(iii) $y = \sin^3 x$

Let $\Delta x$ and $\Delta y$ be the increments in $x$ and $y$ respectively.

Then $\Delta y = \sin^3(x+\Delta x) - \sin^3 x$

$$= \{\sin(x + \Delta x) - \sin x\}\ \{\sin^2(x + \Delta x) + \sin x \cdot \sin(x + \Delta x) + \sin^2 x\}$$

$$= \left\{2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}\right\}\{\sin^2(x + \Delta x) + \sin x \cdot \sin(x + \Delta x) + \sin^2 x\}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \left\{\cos\left(x + \frac{\Delta x}{2}\right) \cdot \frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\right\}\{\sin^2(x+\Delta x) + \sin x \sin(x+\Delta x) + \sin^2 x\}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \{\cos x \cdot 1\}\{\sin^2 x + \sin^2 x + \sin^2 x\}$$

$$= 3\sin^2 x \cos x.$$

(iv) $y = e^{\sqrt{x}}$

Let $\Delta x$ and $\Delta y$ be the increments in $x$ and $y$ respectively

Then $\quad \Delta y = e^{\sqrt{x + \Delta x}} - e^{\sqrt{x}}$

$$= e^{\sqrt{x}}\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{2}} - e^{\sqrt{x}}$$

$$= e^{\sqrt{x}}\left(1 + \frac{\Delta x}{2x}\right) - e^{\sqrt{x}},$$ expanding by Binomial theorem and omitting higher power of $\Delta x$

$$= e^{\sqrt{x}}\{e^{\frac{\Delta x}{2\sqrt{x}}} - 1\}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = e^{\sqrt{x}} \cdot \lim_{\Delta x \to 0}\left\{\frac{e^{\Delta x/2\sqrt{x}} - 1}{\frac{\Delta x}{2\sqrt{x}}}\right\} \times 2\sqrt{x}$$

$$= e^{\sqrt{x}} \times 1 \times 2\sqrt{x} \qquad \left\{\because \lim_{h \to 0}\left(\frac{e^{mh} - 1}{mh}\right) = 1\right\}$$

(v) $y = \tan^{-1}(x)$

Let $\Delta x$ and $\Delta y$ be the increments in $x$ and $y$ respectively.

Then $y + \Delta y = \tan{-1}\{(x + \Delta x)\}$

$\therefore \quad x = \tan y$ and $(x + \Delta x) = \tan(y + \Delta y)$

$\therefore \quad \Delta x = \tan(y + \Delta y) - \tan y$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 1}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta y \to 0}\left\{\frac{\Delta y}{\tan(y + \Delta y) - \tan y}\right\}$$

$\{\because$ when $\Delta x \to 0$, $\Delta y$ also $\to 0\}$

$$= \lim_{\Delta y \to 0}\left\{\frac{\cos y \cdot \cos(y + \Delta y) \cdot \Delta y}{\sin \Delta y}\right\}$$

$$= \cos^2 y \cdot \lim_{\Delta y \to 0}\left(\frac{\Delta y}{\sin y}\right) = \cos^2 y \cdot 1$$

$$= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

**Example 1.2** Find $\frac{dy}{dx}$, when $y = \sqrt{\sin^m x + \cos^n x}$.

$$y = \sqrt{\sin^m x + \cos^n x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{\sin^m x + \cos^n x}} \times \frac{d}{dx}(\sin^m x + \cos^n x)$$

$$= \frac{1}{2\sqrt{\sin^m x + \cos^n x}}\{m \sin^{m-1} x \cos x + n \cos^{n-1} x \cdot (-\sin x)\}$$

$$= \frac{\sin x \cos x (m \sin^{m-2} - n \cos^{n-2} x}{2\sqrt{\sin^m x + \cos^n x}}$$

**Example 1.3** Find $\frac{dy}{dx}$, where $y = \log_e\left(\frac{a + b\cos x}{a - b\cos x}\right)$.

$$y = \log(a + b\cos x) - \log(a - b\cos x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-b\sin x}{a + b\cos x} - \frac{b \sin x}{a - b\cos x}$$

$$= -b\sin x\left\{\frac{a - b\cos x + a + b\cos x}{a^2 - b^2\cos^2 x}\right\}$$

$$= \frac{-2ab\sin x}{a^2 - b^2\cos^2 x}$$

**Example 1.4** Find $\frac{dy}{dx}$, when $y = \log\left(\frac{x - \sqrt{1+x^2}}{x + \sqrt{1+x^2}}\right)$.

$$y = \log(x - \sqrt{1+x^2}) - \log(x + \sqrt{1+x^2})$$

$$\frac{dy}{dx} = \frac{1}{x - \sqrt{1+x^2}}\left\{1 - \frac{1}{2\sqrt{1+x^2}} \cdot (2x)\right\} - \frac{1}{x + \sqrt{1+x^2}}\left\{1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x\right\}$$

$$= \frac{1}{x - \sqrt{1-x^2}}\left\{\frac{\sqrt{1-x^2} - x}{\sqrt{1+x^2}}\right\} - \frac{1}{x + \sqrt{1+x^2}}\left\{\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}\right\}$$

$$= -\frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}} = -\frac{2}{\sqrt{1+x^2}}$$

**Example 1.5** Find $\dfrac{dy}{dx}$, when $y = \log\sqrt{\dfrac{\cos x + \sin x}{\cos x - \sin x}}$.

$$y = \frac{1}{2}[\log(\cos x + \sin x) - \log(\cos x - \sin x)]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{-\sin x + \cos x}{\cos x + \sin x} - \frac{(-\sin x - \cos x)}{\cos x - \sin x}\right]$$

$$= \frac{1}{2}\left[\frac{(\cos x - \sin x)^2 - (\cos x + \sin x)^2}{\cos^2 x - \sin^2 x}\right]$$

$$= \frac{1}{2} \times \frac{2(\cos^2 x + \sin^2 x)}{\cos^2 x - \sin^2 x} = \frac{1}{\cos^2 x - \sin^2 x}$$

**Example 1.6** Find $\dfrac{dy}{dx}$, when $y = (x + \sqrt{x^2 + a^2})^n + (-x + \sqrt{x^2 + a^2})^{-n}$.

$$\frac{dy}{dx} = n(x + \sqrt{x^2 + a^2})^{n-1}\left(1 + \frac{x}{\sqrt{x^2 + a^2}}\right)$$

$$- n(-x + \sqrt{x^2 + a^2})^{-n-1}\left(-1 + \frac{x}{\sqrt{x^2 + a^2}}\right)$$

$$= \frac{1}{\sqrt{x^2 + a^2}} \cdot n(x + \sqrt{x^2 + a^2})^n + \frac{1}{\sqrt{x^2 + a^2}} \cdot n(-x + \sqrt{x^2 + a^2})^{-n}$$

$$= \frac{ny}{\sqrt{x^2 + a^2}}$$

**Example 1.7** Find $\dfrac{dy}{dx}$, when $y = \left[\dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2}\log\{x + \sqrt{x^2 + a^2}\}\right]$.

$$\frac{dy}{dx} = \frac{x}{2} \cdot \frac{x}{\sqrt{x^2 + a^2}} + \frac{\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2}\left\{\frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left(1 + \frac{x}{x^2 + a^2}\right)\right\}$$

$$= \frac{1}{2}\left[\frac{2x^2 + a^2}{\sqrt{x^2 + a^2}} + \frac{a^2}{\sqrt{x^2 + a^2}}\right]$$

$$= \sqrt{x^2 + a^2}$$

**Example 1.8** Find $\dfrac{dy}{dx}$, when $y = \dfrac{1}{\sqrt{b^2 - a^2}}\log\left\{\dfrac{\sqrt{b + a} + \sqrt{b - a}\tan\dfrac{x}{2}}{\sqrt{b + a} - \sqrt{b - a}\tan\dfrac{x}{2}}\right\}$.

Putting $\sqrt{b+a}=c$ and $\sqrt{b-a}=d$, we get

$$y=\frac{1}{cd}\left[\log\left(c+d\tan\frac{x}{2}\right)-\log\left(c-d\tan\frac{x}{2}\right)\right],\ \text{where } c \text{ and } d \text{ are constants}$$

$$\frac{dy}{dx}=\frac{1}{cd}\left[\frac{\frac{d}{2}\sec^2\frac{x}{2}}{c+d\tan\frac{x}{2}}+\frac{\frac{d}{2}\sec^2\frac{x}{2}}{c-d\tan\frac{x}{2}}\right]$$

$$=\frac{1}{2c}\sec^2\frac{x}{2}\times\frac{c}{\left(c^2-d^2\tan^2\frac{x}{2}\right)}$$

$$=\frac{\sec^2\frac{x}{2}}{(b+a)-(b-a)\tan^2\frac{x}{2}}$$

$$=\frac{\sec^2\frac{x}{2}}{a\left(1+\tan^2\frac{x}{2}\right)+b\left(1-\tan^2\frac{x}{2}\right)}$$

$$=\frac{1}{a+b\frac{\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}=\frac{1}{a+b\cos x}$$

**Example 1.9** Find $\frac{dy}{dx}$, when $y=\cos^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$.

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-\left(\frac{a+b\cos}{b+a\cos x}\right)^2}}\cdot\left\{\frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)\cdot(-a\sin x)}{(b+a\cos x)^2}\right\}$$

$$=-\frac{(b+a\cos x)}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}}\cdot\frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$$

$$=-\frac{(a^2-b^2)\sin x}{\sqrt{(b^2-a^2)\sin^2 x}(b+a\cos x)}$$

$$=\frac{\sqrt{b^2-a^2}}{b+a\cos x}$$

**Example 1.10** Find $\frac{dy}{dx}$, when $y = \tan^{-1}\left(\frac{a\cos x + b\sin x}{b\cos x - a\sin x}\right)$.

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{a\cos x + b\sin x}{b\cos x - a\sin x}\right)^2}$$

$$\frac{\{b\cos x - a\sin x)(-a\sin x + b\cos x) - (a\cos x + b\sin x)(-b\sin x - a\cos x)\}}{(b\cos x - a\sin x)^2}$$

$$= \frac{1}{(b\cos x - a\sin x)^2 + (a\cos x + b\sin x)^2}(a^2 + b^2)$$

$$= \frac{a^2 + b^2}{a^2 + b^2} = 1$$

**Example 1.11** If $x = 2a\sin^{-1}\sqrt{\frac{y}{2a}} - \sqrt{2ay - y^2}$, show that $\frac{dy}{dx} = \sqrt{\frac{2a - y}{y}}$.

Differentiating w.r.t. $y$, we have

$$\frac{dx}{dy} = 2a \cdot \frac{1}{\sqrt{1 - \frac{y}{2a}}} \cdot \frac{d}{dy}\sqrt{\frac{y}{2a}} - \frac{1}{2\sqrt{2ay - y^2}}(2a - 2y)$$

$$= \frac{2a\sqrt{2a}}{\sqrt{2a - y}} \cdot \frac{1}{\sqrt{2a} \cdot 2\sqrt{y}} - \frac{a - y}{\sqrt{2ay - y^2}}$$

$$= \frac{a}{\sqrt{2ay - y^2}} - \frac{(a - y)}{\sqrt{2ay - y^2}}$$

$$= \frac{y}{\sqrt{2ay - y^2}}$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{2ay - y^2}{y}}$$

$$= \frac{\sqrt{y}\sqrt{2a - y}}{y}$$

$$= \sqrt{\frac{2a - y}{y}}$$

**Example 1.12** Find $\frac{dy}{dx}$, when $y = \sqrt{\sinh\sqrt{x}}$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\sinh\sqrt{x}}}\frac{d}{dx}(\sinh\sqrt{x})$$

$$= \frac{1}{2\sqrt{\sinh\sqrt{x}}}\cosh\sqrt{x}, \frac{d}{dx}\sqrt{x}$$

$$= \frac{\cosh\sqrt{x}}{4\sqrt{x\sinh\sqrt{x}}}$$

**Example 1.13** Find $\frac{dy}{dx}$, when $y = \sinh^{-1}\tan\left(\frac{1+x}{1-x}\right)$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\tan^2\left(\frac{1+x}{1-x}\right)}}\frac{d}{dx}\tan\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{\sec\left(\frac{1+x}{1-x}\right)}\sec^2\left(\frac{1+x}{1-x}\right)\cdot\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$= \sec\left(\frac{1+x}{1-x}\right)\left\{\frac{(1-x)-(1+x)(-1)}{(1-x)^2}\right\}$$

$$= \frac{2}{(1-x)^2}\sec\left(\frac{1+x}{1-x}\right)$$

**Example 1.14** Find $\frac{dy}{dx}$, when $y = \frac{x\sqrt[3]{x^2+4}}{\sqrt{x^2+3}}$.

Taking logarithms,

$$\log y = \log x + \frac{1}{3}\log(x^2+4) - \frac{1}{2}\log(x^2+3)$$

Differentiating w.r.t $x$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{2x}{3(x^2+4)} - \frac{x}{x^2+3}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{x\sqrt[3]{x^2+4}}{\sqrt{x^2+3}}\left\{\frac{1}{x} + \frac{2x}{3(x^2+4)} - \frac{x}{x+3}\right\}$$

**Example 1.15** Find $\frac{dy}{dx}$, when $y = x^{\sin x} + (\sin x)^x$.

Let $\quad y = u + v$, say.

Then $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \tag{1}$$

$$u = x^{\sin x}$$

Taking logarithms,

$$\log u = \sin x \log x$$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \frac{\sin x}{x} + \log x \cdot \cos x$$

$$\therefore \quad \frac{du}{dx} = x^{\sin x}\left\{\frac{1}{x}\sin x + \log x \cdot \cos x\right\} \tag{2}$$

$$v = (\sin x)^x$$

Taking logarithms,

$$\log v = x \log \sin x$$

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = x \cdot \frac{1}{\sin x}\cos x + \log \sin x$$

$$\therefore \quad \frac{dv}{dx} = (\sin x)^x\ (x \cot x + \log \sin x) \tag{3}$$

Using (2) and (3) in (1), we get $\frac{dy}{dx}$.

**Example 1.16** Find $\frac{dy}{dx}$, when $y = (1+x)^{\frac{1}{x}} + x^{1+\frac{1}{x}}$.

Let $$y = u + v, \text{ say}$$

Then $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \tag{1}$$

$$u = (1+x)^{\frac{1}{x}}$$

Taking logarithms, $\left|\log u = \frac{1}{x}\log(1+x)\right|$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \frac{1}{x}\cdot\frac{1}{1+x} - \frac{1}{x^2}\log(1+x)$$

$$\therefore \quad \frac{du}{dx} = (1+x)^{\frac{1}{x}}\left\{\frac{1}{x(1+x)} - \frac{1}{x^2}\log(1+x)\right\} \tag{2}$$

$$v = x^{1+\frac{1}{x}}$$

Taking logarithms, $\log v = \left(1 + \frac{1}{x}\right)\log x$

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = \left(1 + \frac{1}{x}\right) \cdot \frac{1}{x} - \frac{1}{x^2}\log x$$

$$\therefore \quad \frac{dv}{dx} = x^{1+\frac{1}{x}}\left\{\frac{x+1}{x^2} - \frac{1}{x^2}\log x\right\} \tag{3}$$

Using (2) and (3) in (1), we get $\frac{dy}{dx}$.

**Example 1.17** Find $\frac{dy}{dx}$, when $x$ and $y$ are connected by the relation $x^y + y^x = c$, where $c$ is a constant.

$$x^y + y^x = c, \text{ viz., } u + v = c$$

$$\therefore \quad \frac{du}{dx} + \frac{dv}{dx} = 0 \tag{1}$$

$$u = x^y$$

$$\therefore \quad \log u = y \log x$$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \log x \cdot \frac{dy}{dx}$$

$$\therefore \quad \frac{du}{dx} = x^y\left(\frac{y}{x} + \log x \frac{dy}{dx}\right) \tag{2}$$

$$v = y^x$$

$$\therefore \quad \log v = x \log y$$

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$$

$$\therefore \quad \frac{dv}{dx} = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) \tag{3}$$

Using (2) and (3) in (1), we get

$$x^y\left(\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0$$

$$\text{viz.,} \quad (x^y \log x + xy^{x-1})\frac{dy}{dx} = -(yx^{y-1} + y^x \log y)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{(yx^{y-1} + y^x \log y)}{(xy^{x-1} + x^y \log x)}$$

**Example 1.18** Find $\frac{dy}{dx}$, when $(\sin x)^{\cos y} = (\cos x)^{\sin y}$.

Taking logarithms on both sides of the given relation, we get

$$\cos y \cdot \log \sin x = \sin y \cdot \log \cos x \tag{1}$$

Differentiating both sides of (1) w.r.t. $x$,

$$\cos y \cdot \cot x - \log \sin x(-\sin y)\frac{dy}{dx} = -\sin y \tan x + \log \cos x(\cos y)\frac{dy}{dx}$$

viz., $$(\sin y \log \sin x - \cos y \log \cos x)\frac{dy}{dx} = -(\sin y \tan x + \cos y \cot x)$$

$\therefore$ $$\frac{dy}{dx} = \frac{\sin y \tan x + \cos y \cot x}{\cos y \log \cos x - \sin y \log \sin x}$$

**Example 1.19** If $x^m y^n = (x+y)^{m+n}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

Taking logarithms on both sides of the given relation, we get

$$m \log x + n \log y = (m+n)\log(x+y) \tag{1}$$

Differentiating both sides of (1) w.r.t. x,

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = (m+n)\cdot\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right)$$

viz., $$\left(\frac{m+n}{x+y} - \frac{n}{y}\right)\frac{dy}{dx} = \frac{m}{x} - \frac{m+n}{x+y}$$

viz., $$\frac{(my - nx)}{y(x+y)}\cdot\frac{dy}{dx} = \frac{my - nx}{x(x+y)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{y}{x}$$

**Example 1.20** If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.

Differentiating both sides of the give equation w.r.t. $x$, we get

$$\frac{-x}{\sqrt{1-x^2}} - \frac{y}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = a\left(1 - \frac{dy}{dx}\right) \tag{1}$$

Eliminating '$a$' between the given equation and (1), we get

$$\frac{-\dfrac{x}{\sqrt{1-x^2}} - \dfrac{y}{\sqrt{1-y^2}}\dfrac{dy}{dx}}{\sqrt{1-x^2} + \sqrt{1-y^2}} = \frac{1 - \dfrac{dy}{dx}}{x-y}$$

viz., $$\left[\sqrt{1-x^2} + \sqrt{1-y^2} - \frac{y(x-y)}{\sqrt{1-y^2}}\right]\frac{dy}{dx} = \sqrt{1-x^2} + \sqrt{1-y^2} + \frac{x(x-y)}{\sqrt{1-x^2}}$$

viz., $$\left[\frac{\sqrt{(1-x^2)(1-y^2)} + 1 - y^2 - xy + y^2}{\sqrt{1-y^2}}\right]\frac{dy}{dx} = \frac{1 - x^2 + \sqrt{(1-x^2)(1-y^2)} + x^2 - xy}{\sqrt{1-x^2}}$$

viz., $$\frac{\sqrt{(1-x^2)(1-y^2)}+(1-xy)}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}=\frac{\sqrt{(1-x^2)(1-y^2)}+(1-xy)}{\sqrt{1-x^2}}$$

$\therefore$ $$\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

**Example 1.21** Find $\frac{dy}{dx}$, when $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Put $$x=\tan\theta$$

Then $$y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$=\sin^{-1}(\sin 2\theta)+\cos^{-1}(\cos 2\theta)$$

$$=4\theta \text{ or } 4\tan^{-1}x$$

$\therefore$ $$\frac{dy}{dx}=\frac{4}{1+x^2}$$

**Example 1.22** Find $\frac{dy}{dx}$, when $y=\cos^2\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}$.

Put $$x=\cos\theta$$

Then $$y=\cos^2\tan^{-1}\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}$$

$$=\cos^2\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}$$

$$=\cos^2\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$=\cos^2\frac{\theta}{2}=\frac{1}{2}(1+\cos\theta)=\frac{1}{2}(1+x)$$

$\therefore$ $$\frac{dy}{dx}=\frac{1}{2}.$$

**Example 1.23** Find $\frac{dy}{dx}$, when $y=\tan^{-1}\left(\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}\right)$, where $a$ is a constant.

Put $$\sqrt{x}=\tan\theta \text{ and } \sqrt{a}=\tan\alpha$$

$$\therefore \quad y = \tan^{-1}\left(\frac{\tan\theta + \tan\alpha}{1 - \tan\theta \tan\alpha}\right) = \tan^{-1}\tan(\theta + \alpha)$$

$$= \theta + \alpha = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{a}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x}, \frac{1}{2\sqrt{x}} + 0 = \left|\frac{1}{2\sqrt{x}\,(1+x)}\right|$$

**Example 1.24** Find $\frac{dy}{dx}$, when $y = \tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$.

Put $\quad x^2 = \cos\theta$

Then $$y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2} - \sqrt{2}\sin\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left\{\frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4}\cdot\tan\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{-1}{\sqrt{1-x^4}}\cdot 2x = -\frac{x}{\sqrt{1-x^4}}$$

**Example 1.25** Find $\frac{dy}{dx}$, when $x = \frac{3at}{1+t}$ and $y = \frac{3at^2}{1+t^3}$.

$$x = \frac{3at}{1+t^3} \quad \therefore \quad \frac{dx}{dt} = \frac{3a\{1+t^3) - t\cdot 3t^2\}}{(1+t^3)^2}$$

$$= \frac{3a(1-2t^3)}{(1+t^3)^2}$$

$$y = \frac{3at^2}{1+t^3}$$

$$\therefore \quad \frac{dy}{dt} = \frac{3a\{(1+t^3)2t - t^2 \cdot 3t^2\}}{(1+t^3)^2}$$

$$= \frac{3a(2t - t^4)}{(1+t^3)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t - t^4}{1 - 2t^2}$$

**Example 1.26** Find $\frac{dy}{dx}$, when $x = a(\cos t + \log \tan \frac{t}{2})$ and $y = a \sin t$.

$$x = a\left(\cos t + \log \tan \frac{t}{2}\right)$$

$$\therefore \quad \frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2}\right\}$$

$$= a\left\{-\sin t + \frac{1}{2\sin \frac{t}{2} \cos \frac{t}{2}}\right\}$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a\cos^2 t}{\sin t}$$

$$y = a \sin t$$

$$\therefore \quad \frac{dy}{dt} = a \cos t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{a\cos t}{a\cos^2 t} \times \sin t = \tan t$$

**Example 1.27** Find $\frac{dy}{dx}$, when $y = \log(\sec\theta + \tan\theta)$ and $x = \sec\theta$.

$$y = \log(\sec\theta + \tan\theta)$$

$$\therefore \quad \frac{dy}{d\theta} = \frac{1}{\sec\theta + \tan\theta}(\sec\theta \tan\theta + \sec^2\theta)$$

$$= \sec\theta$$

$$x = \sec\theta$$

$$\therefore \quad \frac{dx}{d\theta} = \sec\theta \tan\theta$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{\sec\theta}{\sec\theta\tan\theta} = \cot\theta.$$

**Example 1.28** Differentiate $\tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$ w.r.t. $\sin^{-1}\left(\dfrac{2}{1 + x^2}\right)$.

Let $\qquad u = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$ and $v = \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right)$

Put $\qquad x = \tan\theta$

Then $\qquad u = \tan^{-1}(\tan 3\theta)$ and $v = \sin^{-1}(\sin 2\theta)$

viz., $\qquad u = 3\theta$ and $v = 2\theta$

$$\frac{du}{dv} = \frac{du}{d\theta} \div \frac{dv}{d\theta} = \frac{3}{2}$$

**Example 1.29** Find $\dfrac{du}{dv}$, where $u = e^{\sin^{-1} x}$ and $v = e^{\cos^{-1} x}$.

$$u = e^{\sin^{-1} x}$$

$$\therefore \qquad \frac{du}{dx} = e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1 - x^2}}$$

$$v = e^{\cos^{-1} x}$$

$$\therefore \qquad \frac{dv}{dx} = e^{\cos^{-1} x} \times -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{du}{dv} = \frac{du}{dx} \div \frac{dv}{dx} = -e^{(\sin^{-1} x - \cos^{-1} x)}$$

**Example 1.30** Differentiate $\tan^{-1}\dfrac{\sqrt{1 + x^2} - 1}{x}$ w.r.t. $\tan^{-1}x$.

Let $\qquad u = \tan^{-1}\left(\dfrac{\sqrt{1 + x^2} - 1}{x}\right)$ and $v = \tan^{-1} x$

Put $\qquad x = \tan\theta$

Then $\qquad u = \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\dfrac{1 - \cos\theta}{\sin\theta}\right)$

$$= \tan^{-1}\left(\frac{2\sin^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$v = \tan^{-1}(\tan\theta) = \theta$$

$$\frac{du}{dv} = \frac{du}{d\theta} \div \frac{dv}{d\theta} = \frac{1}{2}.$$

## EXERCISE 1(b)

***Part A***

(Short Answer Questions)

1. If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, show that $\frac{dy}{dx} = y$.

Using first principles, find $\frac{dy}{dx}$ when

2. $y = \frac{1}{\sqrt{x}}$
3. $y = \frac{x+2}{x-2}$
4. $y = \sqrt{\log x}$
5. $y = x^2 \sin x$
6. $y = \sin^{-1} x^2$

Find $\frac{dy}{dx}$ using the rules and methods of differentiation, when

7. $y = \log\left(\frac{1-\sin x}{1+\sin x}\right)$
8. $y = \tan\sqrt{1+x+x^2}$
9. $y = \cos^3 x \sin^2 x$
10. $y = \sin^{-1}(\cos x)$
11. $y = \tan^{-1}(\sinh x)$
12. $y = (\tan x)^{\cot x}$
13. $x^3 + y^3 = 3axy$
14. $y = a^{xy}$
15. $y = \sin^{-1}(2x\sqrt{1-x^2})$
16. $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$
17. $x = \cos^3 t$ and $y = \sin^3 t$
18. $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$
19. Differentiate $u = a \sec x$ w.r.t. $v = b \tan x$
20. Differentiate $u = \sqrt{\sin 2x}$ w.r.t. $v = \sqrt{\cos 2x}$

***Part B***

Differentiate the following function w.r.t. $x$:

21. $\sqrt{a^2 + b^2 - 2ab\cos x}$
22. $\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}$

23. $\dfrac{x\sin x+\cos x}{x\sin x-\cos x}$

24. $\sqrt{\dfrac{1-x+x^2}{1+x+x^2}}$

25. $e^{ax^2}\cos bx^3$

26. $\log_e\left(\dfrac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right)$

27. $\log\left\{\dfrac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x}-x}\right\}$

28. $\sec\log\sqrt{a^2-x^2}$

29. $\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$

30. $x\sqrt{x^2-a}-a^2\log(x+\sqrt{x^2-a^2}$

31. $\tan^{-1}(\cos\sqrt{x})$

32. $\dfrac{\cosh x+\cos x}{\sinh x+\sin x}$

33. $\sinh^{-1}\left(\dfrac{1-x}{1+x}\right)$

34. $\left\{\dfrac{(x+a)(x^2+b)}{x^3+c}\right\}^{\frac{1}{2}}$

35. $x^x+x^{\frac{1}{x}}$

36. $(\sin x)^{\tan x}+(\tan x)^{\sin x}$

37. Find $\dfrac{dy}{dx}$, when $x$ and $y$ are connected by $(x+y)^x=x^{x+y}$.

38. Find $\dfrac{dy}{dx}$, if $\sin y = x\sin(a+y)$.

39. Find $\dfrac{dy}{dx}$, if $(\sin x)^{\cos y}+(\cos y)^{\sin x}=1$.

40. Find $\dfrac{dy}{dx}$, if $(x-c)(y-c)=1+c^2$.

41. Find $\dfrac{dy}{dx}$, if $y=\sin^2\cot^{-1}\sqrt{\dfrac{1+x}{1-x}}$.

42. Find $\dfrac{dy}{dx}$, $y=\cos^{-1}(4x^3-3x)$.

43. Find $\dfrac{dy}{dx}$, if $y=\dfrac{\cos x-\sin x}{\cos x+\sin x}$.

44. Find $\dfrac{dy}{dx}$, if $y=\cos^{-1}[\sqrt{x^2-x^3}+\sqrt{x-x^3}]$.

45. Find $\frac{dy}{dx}$, if $x = \cos t + t \sin t$ and $y = \sin t - t \cos t$.

46. Find $\frac{dy}{dx}$, if $x = c \log \tan\theta$ and $y = \frac{c}{2}(\tan\theta + \cot\theta)$.

47. Find $\frac{dy}{dx}$, if $x = 3\sin t - \sin^3 t$ and $y = 3\cos t - \cos^3 t$.

48. Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

49. Differentiate $\tan^{-1}\left(\frac{3x-x^2}{1-3x^2}\right)$ w.r.t. $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

50. Differentiate $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ w.r.t. $\cos^{-1}(x^2)$.

## 1.9 MAXIMA AND MINIMA OF FUNCTIONS OF ONE VARIABLE

A function $f(x)$ is said to have a *maximum* at the point $x = a$, if $f(a) \geq f(a + h)$ for all positive and negative values of $h$ sufficiently near zero.

Similarly $f(x)$ is said to have a *minimum* at the point $x = b$, if $f(b) \leq f(b + h)$ for values of $h$ close to zero.

The figure given below represents the graph of $f(x)$, viz., $y = f(x)$ is the equation of the curve shown. The function has maximum at $A$ and $C$, while it has minimum at $B$ and $D$. In other words if the continuous function increases algebraically upto a certain value and then decreases, that value is called a *maximum value* of the function.

Similarly if the continuous function decreases algebraically upto a certain value and then increases, that value is called a *minimum value* of the function.

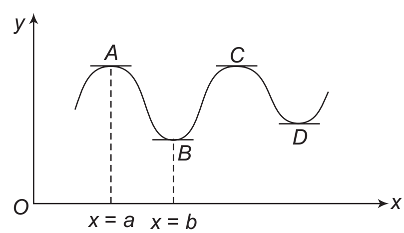

**Fig. 1.5**

*Note* ☑ (1) There are values of $f(x)$ which are greater then the maximum and values which are less than the minimum.

Hence $f(a)$ is a *relative maximum* value of $f(x)$. This means that $f(a)$ is algebraically greater than $f(a - h)$ and $f(a + h)$ where $h$ is a small positive quantity. Similarly, a *relative minimum* value of $f(x)$ occurs at $x = b$.

(2) The points $A, B, C, D$... at which the tangents to the curve $y = f(x)$ are parallel to the $x$-axis are called *stationary points* or *turning points* of the curve $y = f(x)$. The maximum or minimum values of $f(x)$ [at $x = a$ and $x = b$] are called *extreme values of* $f(x)$.

***Theorem***

If $f(x)$ is differentiable at $x = a$ and has a maximum or minimum there, then $f'(a) = 0$

By definition $f(a) \geq f(a + h)$, if $x = a$ is a maximum point.

By definition $f'(a) = \lim_{h \to 0} \left\{ \frac{f(a+h) - f(a)}{h} \right\}$, since $f'(x)$ exist at $x = a$.

This value is zero, for

$$\frac{f(a+h) - f(a)}{h} \leq 0, \text{ if } h > 0 \text{ and } \frac{f(a+h) - f(a)}{h} \geq 0, \text{ if } h \text{ is} < 0$$

viz., $$\lim_{h \to 0^+} \left[ \frac{f(a+h) - f(a)}{h} \right] \leq 0 \quad (1)$$

and $$\lim_{h \to 0^-} \left[ \frac{f(a+h) - f(a)}{h} \right] \geq 0 \quad (2)$$

(1) and (2) will be true, if and only if $f'(a) = 0$.

Similarly if $f(b)$ is a minimum value of $f(x)$, $f'(b) = 0$.

***Theorem*** $f$(a) is an extreme value of $f(x)$ if and only if $f'(x)$ change sign as $x$ passes through the point $x = a$.

Form the figure, it is clear that $f(x)$ is an increasing function before reaching $A$ and after passing through $A$ it is a decreasing function.

Thus $\frac{dy}{dx}$ [slope of the curve $y = f(x)$] changes sign from positive to negative as $x$ passes through $A$

viz., $\frac{d^2y}{dx^2}$ is negative at $A$.

Hence a function $y = f(x)$ is said to have a maximum (minimum) at $x = a$, if

(i) $\frac{dy}{dx} = 0$ at $x = a$

(ii) $\frac{d^2y}{dx^2} < 0$ at $x = a$ for a maximum

$\frac{d^2y}{dx^2} > 0$ at $x = a$ for a minimum

## Working Rule

(1) Find $f'(x)$ and solve the equation $f'(x) = 0$. Let the roots be $a, b, c, \ldots$

(2) Find $f''(x)$ and find $f''(a), f''(b)$, etc.

(3) If $f''(a) < 0$, there is a maximum at the point $x = a$ for $f(x)$.

If $f''(a) > 0$, there is a minimum point for $f(x)$ at $x = a$.

(4) If $f''(a) = 0$, this rule fails for testing whether $f(x)$ is maximum or minimum at $x = a$.

## WORKED EXAMPLES 1(c)

**Example 1.1** Find the maximum and minimum values of $f(x) = 2x^3 - 9x^2 - 24x - 20$.

$$f'(x) = 6x^2 - 18x - 24 = 6(x^2 - 3x - 4)$$
$$= 0, \text{ when } (x - 4)(x + 1) = 0$$

$\therefore$ The turning point of $f(x)$ are $x = 4$ and $x = -1$

$$f''(x) = 6(2x - 3)$$
$$f''(4) = 6 \times 5 > 0 \quad \therefore f(x) \text{ is minimum at } x = 4$$
$$f''(-1) = 6 \times 5 < 0 \quad \therefore f(x) \text{ is maximum at } x = -1$$

$\therefore$ Maximum of $f(x)$ (at $x = -1$) $= -2 - 9 + 24 - 20 = -7$.
and minimum value of $f(x)$ (at $x = 4$) $= 128 - 144 - 96 - 20 = -132$.

**Example 1.2** Find the maximum and minimum values of $y = 3\sin^2 x + 4\cos^2 x$.

$$\frac{dy}{dx} = 6\sin x\cos x - 8\cos x\sin x$$
$$= -2\sin x\cos x \text{ or } -\sin 2x$$
$$\frac{d^2y}{dx^2} = -2\cos 2x$$

The turning points of $y$ are given by $\sin x = 0$ and $\cos x = 0$

viz., $$x = 0 \text{ and } x = \frac{\pi}{2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = -2 < 0 \text{ and } \left(\frac{d^2y}{dx^2}\right)_{x=\frac{\pi}{2}} = 2 > 0$$

$\therefore$ $y$ is maximum at $x = 0$ and minimum at $x = \frac{\pi}{2}$

Maximum value of $y = 4$ and minimum value of $y = 3$

**Example 1.3** Find the maximum and minimum values of $y = kx^2 \log_e\left(\frac{1}{x}\right)$, where $x > 0$.

$$y = -kx^2 \log x$$
$$\frac{dy}{dx} = k\left[x^2 \cdot \frac{1}{x} + 2x\log x\right]$$
$$= kx(1 + 2\log x)$$
$$\frac{d^2y}{dx^2} = k\left[x \cdot \frac{2}{x} + 1 + 2\log x\right]$$
$$= k(3 + 2\log x)$$

$$\frac{dy}{dx} = 0, \text{ when } x = 0 \text{ and } x = \frac{1}{\sqrt{e}} \;(\because\; x > 0)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} \text{ is undefined and } \left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{\sqrt{2}}} = k\left\{3 - 2\log\sqrt{e}\right\} = -k < 0$$

$\therefore$ $y$ is maximum at $x = \dfrac{1}{\sqrt{e}}$

Maximum value of $y = k \cdot \dfrac{1}{e} \log \dfrac{1}{\sqrt{e}}$.

**Example 1.4** Find the maximum and minimum values of $y$, given by $x^2y = x^3 - 3x^2 + 4$.

$$y = \frac{x^3 - 3x^2 + 4}{x^2}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{x^2(3x^2 - 6x) - (x^3 - 3x^2 + 4)\cdot 2x}{x^4}$$

$$= \frac{x^4 - 8x}{x^4} \text{ or } 1 - \frac{8}{x^3}$$

$$\therefore \qquad \frac{dy}{dx} = 0, \text{ at } x^3 = 8 \text{ or } x = 2.$$

$$\frac{d^2y}{dx^2} = \frac{24}{x^6} > 0, \text{ at } x = 2.$$

$\therefore$ $y$ is minimum at $x = 2$

and the minimum value of $y = \dfrac{8 - 12 + 4}{4} = 0$

**Example 1.5** Find the minimum value of $(x^2 + y^2)$, given that $x$ and $y$ are connected by the relation $ax + by = 0$, where $a$, $b$, $c$ are constants.

From the given constant $y = \dfrac{c - ax}{b}$, (1)

using (1) in the given function, whose minimum value is required, we get

$$f(x) = x^2 + \frac{1}{b^2}(c - ax)^2 \qquad (2)$$

$$f'(x) = 2x + \frac{2}{b^2}(c - ax)(-a)$$

$$f'(x) = 0, \text{ where } x\left(2 + \frac{2a^2}{b^2}\right) = \frac{2ac}{b^2}$$

viz., when $$x = \frac{ac}{a^2 + b^2}$$

$$f''(x) = 2 + \frac{2a^2}{b^2} > 0, \text{ for values of } x.$$

$\therefore f(x)$ is minimum, when $x = \dfrac{ac}{a^2 + b^2}$ or $y = \dfrac{bc}{a^2 + b^2}$ get from (1), minimum value of $f(x) = \dfrac{c^2}{a^2 + b^2}$.

**Example 1.6** A rectangular sheet of metal has four equal square portions removed at the corners and the sides are then turned up so as to form an open rectangular box. Show that, when the volume of the box made is maximum, the depth will be $\dfrac{1}{\sin b}[(a+b) - \sqrt{a^2 - ab + b^2}]$, where $a$ and $b$ the sides of the original rectangle.

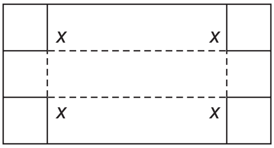

**Fig. 1.6**

Let $x$ be the side of each square metal removed.

Then the dimensions of the rectangular box made are $a - 2x$, $b - 2x$ and $x$

$\therefore$ Volume of the box made, $V = x(a - 2x)(b - 2x)$, where $x$ is the depth of the box

viz., $$V = 4x^3 - 2(a + b)x^2 + abx \qquad (1)$$

We have to find the value of $x$, for which $V$ is maximum.

$$\frac{dV}{dx} = 12x^2 - 4(a + b)x + ab \qquad (1)$$

$$\frac{d^2V}{dx^2} = 24x - 4(a + b) \qquad (2)$$

$$\frac{dV}{dx} = 0, \text{ when } 12x^2 - 4(a + b)x + ab = 0$$

viz., when $$x = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24}$$

$$= \frac{1}{6}\left[(a+b) \pm \sqrt{a^2 - ab + b^2}\right]$$

when $$x = \frac{1}{6}\left[(a+b) + \sqrt{a^2 - ab + b^2}\right] = \frac{d^2V}{dx^2} > 0$$

Hence $x = \dfrac{1}{6}\left[(a+b) - \sqrt{a^2 - ab + b^2}\right]$ is the admissible value of the depth for which $v$ is maximum.

**Example 1.7** The power output of a radio valve is proportional to $\frac{x}{(x+k)^2}$, where $k$, the valve resistance is a constant and $x$ is a variable impedance. Show that the output is a maximum when $x = k$.

$P$, the power output in given by

$$P = \frac{\lambda x}{(x+k)^2}, \text{ where } \lambda \text{ is the constant of proportion.}$$

$$\frac{dP}{dx} = \lambda\left[\frac{(x+k)^2 - x \cdot 2(x+k)}{(x+k)^4}\right]$$

$$= \frac{\lambda(k-x)}{(x+k)^3}$$

$$\frac{d^2P}{dx} = \lambda\left[\frac{(x+k)^3(-1)(k+x)^2(k-x)}{(x+k)^3}\right]$$

$$= \lambda\left[-1 - \frac{3(k-x)}{(k+x)^3}\right]$$

$$\frac{dP}{dx} = 0, \text{ when } x = k \text{ and } \frac{d^2P}{dx^2} = -\lambda < 0 \text{ at } x = k.$$

$\therefore$ $P$ is maximum, when $x = k$.

**Example 1.8** A man in a rowboat at $P$, 5 kms form the nearest point $A$ situated on a straight shore wishes to reach a point $B$, 6 kms from $A$ along the shore, in the shortest time. Where should be land on $AB$, if the can row 2 kms per hour and walk 4 kms per hours?

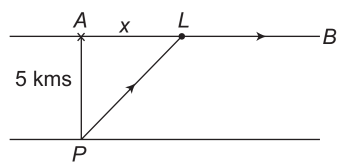

**Fig. 1.7**

Let $L$ be the point of landing such that $AL = x$

Rowing distance = $\sqrt{x^2 + 25}$

Walking distance = $6 - x$

The time taken by the man to go from $P$ to $B$ via $L$ is given by

$$T = \frac{1}{2}\sqrt{x^2+25} + \frac{6-x}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2+25}} = -\frac{1}{4}$$

$$\frac{d^2T}{dx^2} = \frac{1}{2}\left[\frac{\sqrt{x^2+25} - \frac{x^2}{\sqrt{x^2+25}}}{x^2+25}\right] = \frac{\frac{25}{2}}{(x^2+25)^{\frac{3}{2}}}$$

$$\frac{dT}{dx} = 0, \text{ where } \frac{x}{\sqrt{x^2 + 25}} = \frac{1}{2}$$

viz., $4x^2 = x^2 + 25$

viz., $x = \frac{5}{\sqrt{3}}$

$\frac{d^2T}{dx^2} > 0$, where $x = \frac{5}{\sqrt{3}}$ $\therefore$ $T$ is minimum, where $x = \frac{5}{\sqrt{3}}$.

$\therefore$ The man should land at $L$, where $AL = \frac{5}{\sqrt{3}}$ kms. to reach $B$ in the shortest times

**Example 1.9** The horse-power developed by an air-craft travelling horizontally with velocity $v$ metres per second is $H = \frac{AW^2}{v} + Bv^3$, where $A$, $B$, $W$are constants. Find for what value of $v$, the horse-power is minimum?

$$H = \frac{AW^2}{v} + Bv^3$$

$$\frac{dH}{dv} = -\frac{AW^2}{v^2} + 3Bv^2$$

$$\frac{d^2H}{dv^2} = \frac{2AW^2}{v^3} + 6Bv$$

$$\frac{dH}{dv} = 0, \text{ if } v^4 = \frac{AW^2}{3B} \text{ or } v = \left(\frac{AW^2}{3B}\right)^{\frac{1}{4}}$$

$$\left(\frac{d^2H}{dv^2}\right)_{v = \left(\frac{AW^2}{3B}\right)^{\frac{1}{4}}} > 0$$

$\therefore$ $H$ is minimum when $v = \left(\frac{AW^2}{3B}\right)^{\frac{1}{4}}$

**Example 1.10** What is the maximum volume of a right circular cone if the sum of the length of its height and base radius is a constant $k$?

Let '$x$' be the base radius and '$h$' the height of the right circular cone.

Then its volume $V = \frac{\pi}{3}x^3 y$ (1)

Given that $x + y = k$ (2)

Using (2) in (1), we have $V = \frac{\pi}{3}x^2(k - x)$ (3)

Differentiating (3) w.r.t. $x$;

$$\frac{dV}{dx} = \frac{\pi}{3}(2kx - 3x^2)$$

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}(2k - 6x)$$

$$\frac{dV}{dx} = 0, \text{ where } x(2k - 3x) = 0$$

viz., where $x = 0$ or $\dfrac{2k}{3}$

$x = 0$ is meaningless.

$$\left(\frac{d^2V}{dx^2}\right)_{x=\frac{2k}{3}} = \frac{2\pi}{3}(k - 2k) < 0$$

$\therefore$ $V$ is maximum, when $x = \dfrac{2k}{3}$.

**Example 1.11** An open cylindrical vessel is to be made with a given quantity of metal sheet. Find the ratio of the height to the diameter of the base, if the vessel is to have maximum capacity.

Let $r$ be the base radius and $h$ the height of the cylindrical vessel, assumed to be open at one end.

Given Quantity of metal used $= c$

viz., $2\pi rh + \pi r^2 = c$ (or) $2rh + r^2 = k$ (1)

The volume of the vessel $V$ is given by

$$V = \pi r^2 h \tag{2}$$

Using (1) in (2) we have $V = \pi r^2 \dfrac{(k - r^2)}{2r}$

viz., $$V = \frac{\pi}{2}(kr - r^3)$$

$\therefore$ $$\frac{dV}{dr} = \frac{\pi}{2}(k - 3r^2)$$

$$\frac{d^2V}{dr^2} = -3\pi r$$

$$\frac{dV}{dr} = 0, \text{ when } k - 3r^2 = 0$$

viz., when $$r = \sqrt{\frac{k}{3}}$$

For this value of $r$, $\dfrac{d^2V}{dr^2} < 0$.

$\therefore$ $V$ is maximum, when $r = \sqrt{\dfrac{k}{3}}$

When $r = \sqrt{\dfrac{k}{3}}$, from (2), $h = \dfrac{k - r^2}{2r} = \dfrac{k - \dfrac{k}{3}}{2\sqrt{\dfrac{k}{3}}}$ or $\sqrt{\dfrac{k}{3}}$

Thus when $V$ is maximum,

$$h : r = 1 : 1$$

$\therefore$ $h : d = 1 : 2$, where $d$ is the diameter of the base.

**Example 1.12** A normal window consists of a rectangle surmounted by a semi-circle. Given the perimeter of the window to be $k$, find its height and breadth, if the quantity of light admitted is to be maximum.

Let the length and breadth of the rectangular part of the window be $l$ and $2b$

Then the perimeter of the window is given by

$$2l + 2b + \pi b = k \qquad (1)$$

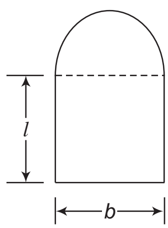

**Fig. 1.8**

If the light admitted is to be maximum, the area of the window $A = 2lb + \pi b^2$ is to be maximum.

Using (1) in the value of $A$,

viz., $$A = (k - 2b - \pi b)b + \frac{\pi}{2}b^2$$

viz., $$A = \left(k - 2b - \frac{\pi}{2}b\right)b$$

$$\frac{dA}{db} = \left(k - 2b - \frac{\pi}{2}b\right)\cdot 1 - \left(2 + \frac{\pi}{2}\right)b$$

$$= k - 4b - \pi b$$

$$\frac{d^2A}{db^2} = -4 - \pi < 0$$

$\therefore$ $A$ is maximum, when $b = \dfrac{k}{\pi + 4}$

When $b = \dfrac{k}{\pi + 4}$, from (1), $2l = k - (\pi + 2)b$

$$= k - (\pi + 2), \frac{k}{\pi + 4}$$

$$= \frac{2k}{\pi + 4} \text{ or } l = \frac{k}{\pi + 4}$$

$\therefore$ $A$ is maximum, when $l = b = \dfrac{k}{\pi + 4}$

viz., when the greatest height and breadth of the window are equal.

**Example 1.13** Find the dimensions of the rectangle of maximum area which can be inscribed in a circle of radius $R$.

When the rectangle inscribed in a circle is to have maximum area, it should be symmetrically situated inside the circle and the four corners should be on the circle. Choosing the centre of the circle as the origin, its equation become

$$x^2 + y^2 = R^2 \qquad (1)$$

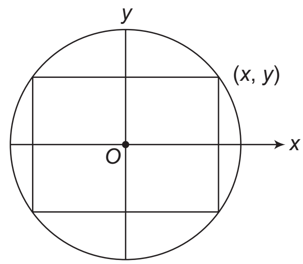


**Fig. 1.9**

If $2x$, $2y$ are the length and width of the rectangle, equation (1) holds good.

We have to find the maximum of

$$A = 4xy = 4x\sqrt{k^2 - x^2}$$

$$\frac{dA}{dx} = 4\left(\sqrt{k^2 - x^2} - \frac{x^2}{\sqrt{k^2 - x^2}}\right)$$

$$= \frac{4(k^2 - 2x^2)}{\sqrt{k^2 - x^2}}$$

$$\frac{d^2A}{dx^2} = 4\left[\frac{\sqrt{k^2 - x^2}(-4x) + (x^2 - 2x^2)}{k^2 - x^2} \cdot \frac{(x)}{\sqrt{k^2 - x^2}}\right]$$

$$= 4\left[\frac{-4x(k^2 - x^2) + x(k^2 - 2x^2)}{(k^2 - x^2)^{\frac{3}{2}}}\right]$$

$$= 4\left[\frac{-3k^2x + 2x^2}{(k^2 - x^2)^{\frac{3}{2}}}\right]$$

$$\frac{dA}{dx} = 0, \text{ when } x = \frac{k}{\sqrt{2}}.$$

When $\quad x = \dfrac{k}{\sqrt{2}}, \dfrac{d^2A}{dx^2} = 4 \cdot \dfrac{k}{\sqrt{2}}\left\{\dfrac{-3k^2 + 2k^2}{\left(\dfrac{k^2}{2}\right)^{\frac{3}{2}}}\right\} < 0$

$\therefore$ $A$ is maximum, when the length and breadth of the rectangle are equal, each equal to $k\sqrt{2}$.

**Example 1.14** Prove that the height of the cylinder of maximum value that can be inscribed in a sphere of radius $a$ is $\dfrac{2a}{\sqrt{3}}$.

Let $r$ and $2h$ be the base radius and height of the cylinder inscribed in the sphere.
By symmetry, $(r, h)$ lies on $x^2 + y^2 = a^2$

$\therefore \qquad r^2 + h^2 = a^2 \qquad (1)$

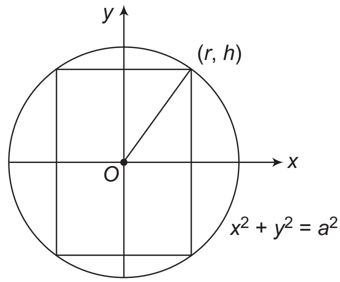


Fig. 1.10

The volume of the cylinder to be maximise is given by $V = \pi r^2 h$

$$= \pi(a^2 - h^2)h, \text{ from (1)}$$

$$\frac{dV}{dh} = \pi(a^2 - 3h^2)$$

$$\frac{d^2V}{dh^2} = 6\pi h < 0, \text{ for all } h$$

$$\frac{dV}{dh} = 0, \text{ when } h = \frac{a}{\sqrt{3}}$$

$$\frac{d^2V}{dh^2} < 0 \text{ at } h = \frac{a}{\sqrt{3}}$$

$\therefore$ $V$ is maximum, when the height of the cylinder $2h = \dfrac{2a}{\sqrt{3}}$.

**Example 1.15** Of all the right circular cones of given slant length $l$, find the dimensions and volume of the cone of maximum volume.

Let $r, h, l$ be the radius, height and slant length of the cone.
Given $r^2 + h^2 = \text{constant } (-l^2) \qquad (1)$

If $V$ is the volume of the cone, then $V = \dfrac{\pi}{3} r^2 h$

i.e., $$V = \frac{\pi}{3} h(l^2 - h^2)$$

$$\frac{dV}{dh} = \frac{\pi}{3}(l^2 - 3h^2)$$

$$\frac{d^2V}{dh^2} = -2\pi h < 0, \text{ for all } h$$

$$\frac{dV}{dh} = 0, \text{ when } h^2 = \frac{l^2}{3} \text{ or } h = \frac{l}{\sqrt{3}}; \; V \text{ is maximum, when } h = \frac{l}{\sqrt{3}}$$

When $h = \dfrac{l}{\sqrt{3}}$, from (1), $r^2 = \dfrac{2l^2}{3}$ or $r = \sqrt{\dfrac{2}{3}}\, l$

Maximum volume of the cone $= \frac{\pi}{3} r^2 h$

$$= \frac{\pi}{3} \cdot \frac{2}{3} l^2 \cdot \frac{l}{\sqrt{3}}$$

$$= \frac{2\pi}{9\sqrt{3}} l^3$$

## EXERCISE 1(c)

***Part A***

(Short Answer Questions)

1. Find the maximum and minimum values of $y = (x - 2)^2 (x - 3)$.
2. Find the maximum value of $\left(\frac{1}{x}\right)^x$.
3. Find the maximum and minimum values of $(\cos x + \cos 2x)$ in $(0, 2\pi)$.
4. Find the maximum and minimum values of $x + \frac{4}{x+2}$.
5. Find the minimum values of $x + y$, if $xy = 1$.

***Part B***

6. A battery whose internal resistance is $r$ ohms E.M.F is E sends a current through an internal resistance $R$. The power is given by $W = \frac{RE^2}{(R+r)^2}$. Given $E$ and $r$, find $R$ so that $W$ may be maximum.
7. The bending moment $M$ at a distance $x$ from one end of a beam of length $l$ uniformly loaded is given by $M = \frac{1}{2} wlx - \frac{1}{2} wx^2$, where $w$ = load per unit length . Show that the maximum bending moment is at the centre of the beam.
8. The cost of fuel per hour for running a train with uniform spend is proportional to be square of the speed in kms per hour and the cost is Rs. 48/= per hour when the speed is 16 kms per hour, what is the most economical speed if the other fixed charge are Rs. 300 per hour? Take the distance to be covered is a constant.
9. Of all rectangles of given area, show that the square has the least perimeter.
10. A rod $AB$ of given length $k$ slides between two perpendicular lines $0x$ and $0y$. Find when the area of the triangle is maximum.

11. An open tank is to be constructed with a square base and vertical sides so as to contain a given quantity of water. If the experience of lining it with lead is to be least find the ratio of the depth to the side of the base.

12. Given the sum of the area of curved surface and two circular ends of a right circular cylinder is a constant, find the height of the cylinder so that the volume may be a maximum.

13. A wire of length 'a' is cut into two parts which are bent respectively in the form of a square and a circle. Show that the least value of the sum of the areas so formed is $\dfrac{a^2}{4(\pi+4)}$.

14. Find the maximum volume of a right circular cylinder inscribed in a given sphere whose radius is $R$.

15. Find the attitude and radius of the cone of maximum volume that can be inscribed in a sphere of radius $R$.

16. a cylinder is inscribed in a cone of height '$h$' and semi vertical and '$\alpha$'. Prove that the volume of the greatest cylinder thus obtained is $\dfrac{4}{27}\pi h^3 \tan^2 \alpha$.

17. Show that a conical tent of given capacity will require the least amount of canvas when the height $\sqrt{2}$ time the radius of the base.

18. If the sum of the length of the hypotenuse and another side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between those sides is 60.

19. A sector is cut out of a circular piece of canvas and the bounding radii of the remaining part are drawn together to form a conical tent. What should be angle of the sector cut out, if the tent has maximum volume?

20. A cone is circumscribed to a sphere of radius $r$. Show that, when the volume of the cone is a minimum, its altitude is $4r$.

## ANSWERS

### Exercise 1(a)

| | | | |
|---|---|---|---|
| (2) $(-4, 3)$ | (3) $(-\infty, -3), (6, \infty)$ | (4) 2 | (5) 2 |
| (6) $2\sqrt{2}$ | (7) $\dfrac{3}{2}a^{\frac{1}{6}}$ | (8) $\dfrac{1}{2}$ | (9) $\dfrac{1}{2}$ |
| (10) $\dfrac{m^2}{n^2}$ | (11) $\dfrac{1}{2}$ | (12) $\cos\alpha$ | (15) $\dfrac{1}{\sqrt{2}}$ |
| (16) 0 | (17) $e^{ab}$ | (18) $e$ | (19) $e^2$ |

(20) $e$ (21) $\left(-\frac{5}{4}, \frac{3}{4}\right)$ (22) (0, 1) (23) $\left(\frac{1}{3}, 3\right)$

(27) $\frac{1}{3}$ (28) $\frac{4}{3}$ (29) $\frac{3}{8}$ (30) 1

(31) 1 (32) 2 log 2 (33) 3 (34) –3

(35) log 4 (36) $\frac{2}{\pi}$ (38) not continuous at $x = \frac{3}{2}$

(39) not continuous at $z = 0$ (40) $a = 3$, $b = -8$

(43) Continuous but not differentiable

(44) Continuous at both point; not differentiable at $x = 1$, but differentiable $x = 2$

(45) Continuous, but not differentiable at both the points.

## Exercise 1(b)

(2) $y' = \frac{1}{2x^{\frac{3}{2}}}$ (3) $y' = -\frac{4}{(x-2)^e}$ (4) $y' = \frac{1}{2x\sqrt{\log x}}$

(5) $y' = x^2 \cos x + 2x \sin x$ (6) $y' = \frac{2x}{\sqrt{1-x^4}}$

(7) $-2 \sec x$ (8) $\sec^2 \sqrt{1+x+x^2}\ \frac{1}{2\sqrt{1+x+x^2}} \cdot (2x+1)$

(9) $\sin x \cos^2 x(2 \cos^2 x - 3 \sin^2 x)$ (10) –1 (11) sech $x$

(12) $(\tan x)^{\cot x} \cdot \{\text{cosec}^2\, x(1 - \log \tan x)\}$ (13) $\frac{ay - x^2}{y^2 - ax}$

(14) $\frac{y^2 \log a}{1 - xy \log a}$ (15) $\frac{2}{\sqrt{1-x^2}}$ (16) $\frac{1}{\sqrt{1-x^2}}$

(17) $-\tan t$ (18) $\cot\frac{\theta}{2}$ (19) $\frac{a}{b}\sin x$ (20) $-\cot^{3/2}(2x)$

(21) $\frac{ab \sin x}{\sqrt{a^2 + b^2 - 2ab\cos x}}$ (22) $\sqrt{1-\sin 2x} + \sqrt{1+\sin 2x}$

(23) $-\frac{2(x + \sin x + \cos x)}{(x \sin x - \cos x)^2}$ (24) $\frac{x^2 - 1}{(1+x+x^2)^{3/2}\,(1-x+x^2)^{1/2}}$

(25) $xe^{ax^2}\{2a \cos bx^3 - 3bx^2 \sin bx^2\}$ (26) $\frac{1}{x\sqrt{1+x}}$ (27) $\frac{2}{\sqrt{a^2+x^2}}$

(28) $-\dfrac{x}{a^2-x^2}\sec\log\sqrt{a^2-x^2}\,\tan\log\sqrt{a^2x2y}$

(29) $\dfrac{x\sqrt{1-x^2}+\sin^{-1}x}{(1-x^2)^{3/2}}$ (30) $2\sqrt{x^2-a^2}$

(31) $\dfrac{-\sin\sqrt{x}}{2\sqrt{x}(1+\cos^2\sqrt{x})}$ (32) $-\dfrac{2(1+\cosh x\cos x)}{(\sinh x+\sin x)^2}$

(33) $\dfrac{-\sqrt{2}}{(1+x)\sqrt{(1+x^2)}}$

(34) $\left\{\dfrac{(x+a)(x^2+b)}{x^3+c}\right\}^{\frac{1}{2}}\left[\dfrac{1}{2(x+a)}+\dfrac{x}{x^2+b}-\dfrac{3x^2}{x^3+c}\right]$

(35) $x^x(1+x)+\dfrac{x^{\frac{1}{x}}(1-\log x)}{x^2}$

(36) $(\sin x)^{\tan x}\{1+\sec^2 x+\log\sin x)+(\tan x)^{\sin x}\{\sin x+\cos x\log\tan x\}$

(37) $\dfrac{x+y}{x}+\log x-\dfrac{x}{x+y}-\log(x+y)$ (38) $\dfrac{\sin^2(a+y)}{\sin a}$

(39) $\dfrac{(\sin x)^{\cos y}\cos y\cot x+(\cos y)^{\sin x}\cos x\log\cos y}{(\sin x)^{\cos y}\sin y\log\sin x+(\cos y)^{\sin x}\sin x\tan y}$ (40) $-\dfrac{(1+y^2)}{1+x^2}$

(41) $-\dfrac{1}{2}$ (42) $-\dfrac{3}{\sqrt{1-x^2}}$ (43) $-\dfrac{1}{1+x^2}$

(44) $-\dfrac{1}{\sqrt{1-x^2}}+\dfrac{1}{2\sqrt{x}\sqrt{1-x}}$ (45) $\tan t$ (46) $-\cot 2\theta$

(47) $-\tan^3 t$ (48) 1 (49) $\dfrac{3}{2}$ (50) $\dfrac{1}{2}$

## Exercise 1(c)

(1) Max $=\dfrac{2}{27}$ and min = 0 (2) $e^{\frac{1}{e}}$ Max = –6 and min = 2

(3) Max values 2, 0, 2 and min value $-\dfrac{2}{3}$ and $-2$ (6) $R = r$

(8) 40 km/hr. (10) When $OA = OB = \dfrac{k\sqrt{2}}{2}$ (11) 1 : 2

(13) $\sqrt{\dfrac{2k}{3\pi}}$, where $k$ is the total 5.A (14) $\dfrac{4}{3}\pi R^3\sqrt{3}$

(15) altitude $= \dfrac{4}{3}R$ and radius $= \dfrac{2\sqrt{2}}{3}R$

(19) 66° approximately

UNIT 2

# Functions of Several Variables

## 2.1 INTRODUCTION

The students have studied in the lower classes the concept of partial differentiation of a function of more than one variable. They were also exposed to homogeneous functions of several variables and Euler's theorem associated with such functions. In this chapter, we discuss some of the applications of the concept of partial differentiation, which are frequently required in engineering problems.

## 2.2 TOTAL DIFFERENTIATION

In partial differentiation of a function of two or more variables, it is assumed that only one of the independent variables varies at a time. In total differentiation, all the independent variables concerned are assumed to vary and so to take increments simultaneously.

Let $z = f(x, y)$, where $x$ and $y$ are continuous functions of another variable $t$.

Let $\Delta t$ be a small increment in $t$. Let the corresponding increments in $x, y, z$ be $\Delta x$, $\Delta y$ and $\Delta z$ respectively.

Then
$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$$
$$= \{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)\} + \{f(x, y+\Delta y) - f(x, y)\}$$

$$\therefore \quad \frac{\Delta z}{\Delta t} = \left\{\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right\} \cdot \frac{\Delta x}{\Delta t} + \left\{\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right\} \cdot \frac{\Delta y}{\Delta t} \qquad (1)$$

We note that $\Delta x$ and $\Delta y \to 0$ as $\Delta t \to 0$ and hence $\Delta z \to 0$ as $\Delta t \to 0$

Taking limits on both sides of (1) as $\Delta t \to 0$, we have $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dt}$

($\because$ $x, y$ and $z$ are functions of $t$ only and $f$ is a function of $x$ and $y$).

i.e., $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \text{ [since } f(x, y) \equiv z(x, y)\text{]}. \quad (2)$$

$\frac{dz}{dt}\left(\text{and also } \frac{dx}{dt} \text{ and } \frac{dy}{dt}\right)$ is called the *total differential coefficient* of $z$.

This name is given to distinguish it from the partial differential coefficients $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}\cdot$ Thus to differentiate $z$, which is directly a function of $x$ and $y$, (where $x$ and $y$ are functions of $t$) with respect to $t$, we need not express $z$ as a function of $t$ by substituting for $x$ and $y$. We can differentiate $z$ with respect to $t$ via $x$ and $y$ using the result (2).

**Corollary 1:** In the differential form, result (2) can be written as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad (3)$$

d$z$ is called the *total differential* of $z$.

**Corollary 2:** If $z$ is directly a function of two variables $u$ and $v$, which are in turn functions of two other variables $x$ and $y$, clearly $z$ is a function of $x$ and $y$ ultimately. Hence the total differentiation of $z$ is meaningless. We can find only $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by using the following results which can be derived as result (2) given above.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad (4)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \quad (5)$$

We note that the partial differentiation of $z$ is performed via the intermediate variables $u$ and $v$, which are functions of $x$ and $y$. Hence $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called *partial derivatives of a function of two functions.*

*Note* ☑ Results (2), (3), (4) and (5) can be extended to a function $z$ of several intermediate variables.

## 2.2.1 Small Errors and Approximations

Since $\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx}$, $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ approximately or $\Delta y \simeq \left(\frac{dy}{dx}\right)\Delta x$ (1)

If we assume that d$x$ and d$y$ are approximately equal to $\Delta x$ and $\Delta y$ respectively, result (1) can be derived from the differential relation.

$$\mathrm{d}y = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x \tag{2}$$

Though (2) is an exact relation, it can be made use of to get the approximate relation (1), by replacing d$x$ and d$y$ by $\Delta x$ and $\Delta y$ respectively.

Let $y = f(x)$. If we assume that the value of $x$ is obtained by measurement, it is likely that there is a small error $\Delta x$ in the measured value of $x$. This error in the value of $x$ will contribute a small error $\Delta y$ in the calculated value of $y$, as $x$ and $y$ are functionally related. The small increments $\Delta x$ and $\Delta y$ can be assumed to represent the small errors $\Delta x$ and $\Delta y$. Thus the relation between the errors $\Delta x$ and $\Delta y$ can be taken as

$$\Delta y \simeq f'(x)\,\Delta x$$

This concept can be extended to a function of several variables.

If $u = u(x, y, z)$ or $f(x, y, z)$ and if the value of $u$ is calculated on the measured values of $x, y, z$, the likely errors $\Delta x$, $\Delta y$, $\Delta z$ will result in an error $\Delta u$ in the calculated value of $u$, given by

$$\Delta u \simeq \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z,$$

which can be assumed as the approximate version of the total differential relation

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial z}\mathrm{d}z$$

***Note*** ☑ The error $\Delta x$ in $x$ is called *the absolute error* in $x$, while $\frac{\Delta x}{x}$ is called the *relative* or *proportional error* in $x$ and $\frac{100\,\Delta x}{x}$ is called *the percentage error* in $x$.

## 2.2.2 Differentiation of Implicit Functions

When $x$ and $y$ are connected by means of a relation of the form $f(x, y) = 0$, $x$ and $y$ are said to be implicitly related or $y$ is said to be an *implicit function* of $x$. When $x$ and $y$ are implicitly related, it may not be possible in many cases to express $y$ as a single valued function of $x$ explicitly. However $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be found out in such cases as a mixed function of $x$ and $y$ using partial derivatives as explained below:

Since $$f(x, y) = 0,\ \mathrm{d}f = 0$$

i.e., $\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = 0$, by definition of total differential. Dividing by d$x$, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} \tag{1}$$

If we denote $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ by the letters $p, q, r, s, t$ respectively, then

$$\frac{dy}{dx} = -\frac{p}{q} \tag{2}$$

We can express the second order derivative $\frac{d^2 y}{dx^2}$ in terms of $p, q, r, s, t$ as given below. Noting that $p$ and $q$ are functions of $x$ and $y$ and differentiating both sides of (2) with respect to $x$ totally, we have

$$\frac{d^2 y}{dx^2} = -\left(\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2}\right)$$

$$= \frac{p\left\{\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx}\right\} - q\left\{\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx}\right\}}{q^2}$$

$$= \frac{p\left\{s + t\left(\frac{-p}{q}\right)\right\} - q\left\{r + s\left(\frac{-p}{q}\right)\right\}}{q^2},$$

since $\quad \frac{\partial p}{\partial x} = \frac{\partial^2 f}{\partial x^2} = r;\ \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = s;\ \frac{\partial q}{\partial y} = \frac{\partial^2 f}{\partial y^2} = t.$

$$= \frac{p(qs - pt) - q(qr - ps)}{q^3}$$

$$= -\frac{(p^2 t - 2pqs + q^2 r)}{q^3}$$

## WORKED EXAMPLE 2(a)

**Example 2.1**

(i) If $u = xy + yz + zx$, where $x = e^t$, $y = e^{-t}$ and $z = \frac{1}{t}$, find $\frac{du}{dt}$

(ii) If $u = \sin^{-1}(x - y)$, where $x = 3t$ and $y = 4t^3$, show that $\frac{du}{dt} = \frac{3}{\sqrt{1 - t^2}}$

(i) $u = xy + yz + zx$

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= (y+z)e^t + (z+x)(-e^{-t}) + (x+y)\left(-\frac{1}{t^2}\right)$$

$$= \left(e^{-t} + \frac{1}{t}\right)e^t - \left(\frac{1}{t} + e^t\right)e^{-t} - \left(e^t + e^{-t}\right).\frac{1}{t^2}$$

$$= 1 + \frac{1}{t}e^t - \frac{1}{t}e^{-t} - 1 - \frac{1}{t^2}e^t - \frac{1}{t^2}e^{-t}$$

$$= \frac{2}{t}\sinh t - \frac{2}{t^2}\cosh t.$$

(ii) $u = \sin^{-1}(x - y)$

$$\therefore \quad \frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}\frac{dx}{dt} + \frac{1}{\sqrt{1-(x-y)^2}}\left(-\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}(3 - 12t^2) \tag{1}$$

Now $1-(x-y)^2 = 1-(3t-4t^3)^2$

$$= 1 - 9t^2 + 24t^4 - 16t^6$$

$$= (1-t^2)(1-8t^2+16t^4)$$

$$= (1-t^2)(1-4t^2)^2 \tag{2}$$

Using (2) in (1), we get

$$\frac{du}{dt} = \frac{1}{(1-4t^2)\sqrt{1-t^2}} \times 3(1-4t^2)$$

$$= \frac{3}{\sqrt{1-t^2}}.$$

**Example 2.2** If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

Let $$r = \frac{x}{y},\ s = \frac{y}{z} \text{ and } t = \frac{z}{x} \tag{1}$$

$\therefore$ $u = f(r, s, t)$, where $r, s, t$ are functions of $x, y, z$ as assumed in (1)

$\therefore$
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$
$$= \frac{1}{y}\cdot\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\cdot 0 - \frac{z}{x^2}\cdot\frac{\partial u}{\partial t} \qquad (2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$
$$= -\frac{x}{y^2}\cdot\frac{\partial u}{\partial r} + \frac{1}{z}\cdot\frac{\partial u}{\partial s} \qquad (3)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$
$$= -\frac{y}{z^2}\cdot\frac{\partial u}{\partial s} + \frac{1}{x}\cdot\frac{\partial u}{\partial t} \qquad (4)$$

From (2), (3) and (4), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \left(\frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t}\right)$$
$$+\left(-\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s}\right) + \left(-\frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}\right)$$
$$= 0.$$

**Example 2.3** If $z$ be a function of $x$ and $y$, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$= e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \qquad (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$
$$= -e^{-v}\frac{\partial z}{\partial x} - e^v\cdot\frac{\partial z}{\partial y} \qquad (2)$$

From (1) and (2), we have

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y}$$
$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

**Example 2.4** If $u = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\theta\cdot\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \qquad (1)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\cdot\frac{\partial u}{\partial x} + r\cos\theta\cdot\frac{\partial u}{\partial y}$$

i.e.,

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\cdot\frac{\partial u}{\partial y} \qquad (2)$$

Squaring both sides of (1) and (2) and adding, we get

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

**Example 2.5** Find the equivalent of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in polar co-ordinates.

$$u = u(x, y), \text{ where } x = r\cos\theta \text{ and } y = r\sin\theta$$

$\therefore$ $u$ can also be considered as $u(r, \theta)$, where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = tan^{-1}\left(\frac{y}{x}\right)$$

Now we proceed to find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ via $r$ and $\theta$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{x}{\sqrt{x^2 + y^2}}\frac{\partial u}{\partial r} + \frac{1}{1 + \frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)\cdot\frac{\partial u}{\partial \theta}$$

$$= \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \qquad (1)$$

From (1), we can infer that

$$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{2}$$

Now
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$
$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$
$$= \cos^2\theta\cdot\frac{\partial^2 u}{\partial r^2} - \sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)$$
$$-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r}\right) + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

$\left(\because r \text{ and } \theta \text{ are independent and } \frac{\partial u}{\partial r} \text{ and } \frac{\partial u}{\partial \theta} \text{ are functions of } r \text{ and } \theta\right)$.

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \sin\theta\,\cos\theta\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)$$
$$-\frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 u}{\partial\theta\,\partial r} - \sin\theta\frac{\partial u}{\partial r}\right) + \frac{\sin\theta}{r^2}\left(\sin\theta\frac{\partial^2 u}{\partial\theta^2} + \cos\theta\frac{\partial u}{\partial\theta}\right) \tag{3}$$

Now
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$
$$= \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(\frac{1}{x}\right)\frac{\partial u}{\partial\theta}$$
$$= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} \tag{4}$$

From (4) we infer that

$$\frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \tag{5}$$

$\therefore$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$
$$= \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$
$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \sin\theta\cos\theta\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)$$
$$+\frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 u}{\partial\theta\,\partial r} + \cos\theta\frac{\partial u}{\partial r}\right) + \frac{\cos\theta}{r^2}\left(\cos\theta\frac{\partial^2 u}{\partial\theta^2} - \sin\theta\frac{\partial u}{\partial\theta}\right) \tag{6}$$

Adding (3) and (6), we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

**Example 2.6** Given the transformations $u = e^x \cos y$ and $\upsilon = e^x \sin y$ and that $f$ is a function of $u$ and $\upsilon$ and also of $x$ and $y$, prove that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=(u^2+\upsilon^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial \upsilon^2}\right)$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial \upsilon}\cdot\frac{\partial \upsilon}{\partial x}$$

$$=e^x\cos y\cdot\frac{\partial f}{\partial u}+e^x\sin y\cdot\frac{\partial f}{\partial \upsilon}$$

$$=u\frac{\partial f}{\partial u}+\upsilon\frac{\partial f}{\partial \upsilon} \qquad (1)$$

$$\frac{\partial}{\partial x}\equiv u\frac{\partial}{\partial u}+\upsilon\frac{\partial}{\partial \upsilon} \qquad (2)$$

$$\frac{\partial^2 f}{\partial x^2}=\left(u\frac{\partial}{\partial u}+\upsilon\frac{\partial}{\partial \upsilon}\right)\left(u\frac{\partial f}{\partial u}+\upsilon\frac{\partial f}{\partial \upsilon}\right)$$

$$=u\left(u\cdot\frac{\partial^2 f}{\partial u^2}+\frac{\partial f}{\partial u}\right)+u\upsilon\frac{\partial^2 f}{\partial u\,\partial \upsilon}+u\upsilon\frac{\partial^2 f}{\partial \upsilon\,\partial u}$$

$$+\upsilon\left(\upsilon\frac{\partial^2 f}{\partial \upsilon^2}+\frac{\partial f}{\partial \upsilon}\right) \qquad (3)$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial \upsilon}\cdot\frac{\partial \upsilon}{\partial y}$$

$$=-e^x\sin y\cdot\frac{\partial f}{\partial u}+e^x\cos y\frac{\partial f}{\partial \upsilon}$$

$$=-\upsilon\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial \upsilon} \qquad (4)$$

$\therefore$

$$\frac{\partial}{\partial y}\equiv-\upsilon\frac{\partial}{\partial u}+u\frac{\partial}{\partial \upsilon} \qquad (5)$$

$$\frac{\partial^2 f}{\partial y^2}=\left(-\upsilon\frac{\partial}{\partial u}+u\frac{\partial}{\partial \upsilon}\right)\left(-\upsilon\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial \upsilon}\right)$$

$$=\upsilon^2\frac{\partial^2 f}{\partial u^2}-\upsilon\left(u\frac{\partial^2 f}{\partial u\,\partial \upsilon}+\frac{\partial f}{\partial \upsilon}\right)-u\left(\upsilon\frac{\partial^2 f}{\partial \upsilon\,\partial u}+\frac{\partial f}{\partial u}\right)$$

$$+u^2\frac{\partial^2 f}{\partial \upsilon^2} \qquad (6)$$

Adding (3) and (6), we get

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=(u^2+\upsilon^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial \upsilon^2}\right)$$

**Example 2.7** If $z = f(u,\upsilon)$, where $u = \cosh x \cos y$ and $\upsilon = \sinh x \sin y$, prove that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(\sinh^2 x+\sin^2 y\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial \upsilon^2}\right)$$

$$z_x = z_u \cdot u_x + z_\upsilon \cdot \upsilon_x, \text{ where } z_x \equiv \frac{\partial z}{\partial x} \text{ etc.}$$

$$= \sinh x \cdot \cos y \cdot z_u + \cosh x \sin y \cdot z_\upsilon$$

Since $z$ is a function of $u$ and $\upsilon$, $z_u$ and $z_\upsilon$ are also functions of $u$ and $\upsilon$. Hence to differentiate $z_u$ and $z_\upsilon$ with respect to $x$ or $y$, we have to do it via $u$ and $\upsilon$.

$$\therefore \quad z_{xx} = \cos y\left[\cosh x \cdot z_u + \sinh x\left\{z_{uu} \cdot \sinh x \cos y + z_{u\upsilon} \cosh x \sin y\right\}\right]$$

$$+\sin y\left[\sinh x \cdot z_\upsilon + \cosh x\left\{z_{\upsilon u} \cdot \sinh x \cos y + z_{\upsilon\upsilon} \cdot \cosh x \sin y\right\}\right]$$

i.e.,
$$z_{xx} = \cosh x \cos y \cdot z_u + \sinh x \cdot \sin y\, z_\upsilon + \sinh^2 x \cos^2 y \cdot z_{uu}$$

$$+2\sinh x \cosh x \sin y \cos y \cdot z_{u\upsilon} + \cosh^2 x \sin^2 y \cdot z_{\upsilon\upsilon} \qquad (1)$$

$$z_y = -z_u \cdot \cosh x \sin y + z_\upsilon \sinh x \cos y$$

$$z_{yy} = -\cosh x[\cos y \cdot z_u + \sin y\{z_{uu} \cdot (-\cosh x \sin y)$$

$$+z_{u\upsilon} \cdot \sinh x \cdot \cos y\} + \sinh x[-\sinh y \cdot z_\upsilon$$

$$+\cos y\{-z_{\upsilon u} \cdot \cosh x \sin y + z_{\upsilon\upsilon} \cdot \sinh x \cos y\}]$$

i.e.,
$$z_{yy} = -\cosh x \cos y \cdot z_u - \sinh x \cdot \sin y \cdot z_\upsilon$$

$$+\cosh^2 x \sin^2 y \cdot z_{uu} - 2\sinh x \cosh x \sin y \cos y \cdot z_{u\upsilon}$$

$$+\sinh^2 x \cos^2 y \cdot z_{\upsilon\upsilon} \qquad (2)$$

Adding (1) and (2), we get

$$z_{xx} + z_{yy} = (\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y)(z_{uu} + z_{\upsilon\upsilon})$$

$$= \{\sinh^2 x(1-\sin^2 y) + (1+\sinh^2 x)\sin^2 y\}(z_{uu} + z_{\upsilon\upsilon})$$

$$= (\sinh^2 x + \sin^2 y)(z_{uu} + z_{\upsilon\upsilon})$$

**Example 2.8** Find $\frac{dy}{dx}$, when (i) $x^3 + y^3 = 3ax^2y$ and (ii) $x^y + y^x = c$.

(i) $f(x, y) = x^3 + y^3 - 3ax^2y$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 6axy$$

$$q = \frac{\partial f}{\partial y} = 3y^2 - 3ax^2$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{3(x^2 - 2axy)}{3(y^2 - ax^2)} = \frac{x(2ay - x)}{y^2 - ax^2}$$

(ii) $f(x, y) = x^y + y^x - c$

$$p = \frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$q = \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{yx^{y-1} + y^x \log x}{xy^{x-1} + x^y \log x}.$$

**Example 2.9** If $ax^2 + 2hxy + by^2 = 1$, show that $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx+by)^3}$.

$f(x, y) = ax^2 + 2hxy + by^2 - 1$

$$p = \frac{\partial f}{\partial x} = 2(ax+hy);\ q = \frac{\partial f}{\partial y} = 2(hx+by)$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2a;\ s = \frac{\partial^2 f}{\partial x \partial y} = 2h;\ t = \frac{\partial^2 f}{\partial y^2} = 2b$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{(ax+hy)}{hx+by}$$

$$\frac{d^2y}{dx^2} = \frac{-(p^2t - 2pqs + q^2r)}{q^3}$$

(Refer to differentiation of implicit functions)

$$= \frac{-\{8b(ax+hy)^2 - 16h(ax+hy)(hx+by) + 8a(hx+by)^2\}}{8(hx+by)^3}$$

$$= \frac{1}{(hx+by)^3}[2h\{ahx^2 + (ab+h^2)xy + bhy^2\}$$

$$-\{a^2bx^2 + 2abhxy + h^2by^2\} - \{ah^2x^2 + 2abhxy + ab^2y^2\}]$$

$$= \frac{1}{(hx+by)^3}[a(h^2-ab)x^2+2h(h^2-ab)xy+b(h^2-ab)y^2]$$

$$= \frac{(h^2-ab)}{(hx+by)^3}(ax^2+2hxy+by^2) = \frac{(h^2-ab)\cdot 1}{(hx+by)^3} = \frac{h^2-ab}{(hx+by)^3}.$$

**Example 2.10** Find $\frac{\mathrm{d}u}{\mathrm{d}x}$ if (i) $u=\sin(x^2+y^2)$, where $a^2x^2+b^2y^2=c^2$ (i), $u=\tan^{-1}\left(\frac{y}{x}\right)$ where $x^2+y^2=a^2$, by treating $u$ as function of $x$ and $y$ only.

(i) $u = \sin(x^2+y^2)$

$$\therefore \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$= 2x\cos\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} \tag{1}$$

Now $a^2x^2+b^2y^2=c^2$

Differentiating with respect to $x$,

$$2a^2x+2b^2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{a^2x}{b^2y} \tag{2}$$

Using (2) in (1), we get

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x\cos\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\times\left(\frac{-a^2x}{b^2y}\right)$$

$$= 2x\cos\left(x^2+y^2\right)\left(b^2-a^2\right)/b^2$$

(ii)

$$u = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)+\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$= -\frac{y}{x^2+y^2}+\frac{x}{x^2+y^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} \tag{3}$$

$$x^2+y^2 = a^2$$

$$\therefore \quad 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

or $$\frac{dy}{dx} = -\frac{x}{y} \tag{4}$$

Using (4) in (3), we get

$$\frac{du}{dx} = -\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}\left(-\frac{x}{y}\right)$$
$$= -\frac{1}{y}.$$

**Example 2.11** If $u = x^2 - y^2$ and $\upsilon = xy$, find the values of $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial \upsilon}, \frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial \upsilon}$.

$x$ and $y$ cannot be easily expressed as single valued functions or $u$ and $\upsilon$.

Given $$x^2 - y^2 = u \tag{1}$$

and $$xy = \upsilon \tag{2}$$

Nothing that $x$ and $y$ are functions of $u$ and $\upsilon$ and differentiating both sides of (1) and (2) partially with respect to $u$, we have

$$2x\frac{\partial x}{\partial u} - 2y\frac{\partial y}{\partial u} = 1 \tag{3}$$

$$y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u} = 0 \tag{4}$$

Solving (3) and (4), we get

$$\frac{\partial x}{\partial u} = \frac{x}{2\left(x^2+y^2\right)} \text{ and } \frac{\partial y}{\partial u} = -\frac{y}{2\left(x^2+y^2\right)}$$

Differentiating both sides of (1) and (2) partially with respect to $\upsilon$, we have

$$2x\frac{\partial x}{\partial \upsilon} - 2y\frac{\partial y}{\partial \upsilon} = 0 \tag{5}$$

$$y\frac{\partial x}{\partial \upsilon} + x\frac{\partial y}{\partial \upsilon} = 1 \tag{6}$$

Solving (5) and (6), we get

$$\frac{\partial x}{\partial \upsilon} = \frac{y}{x^2+y^2} \text{ and } \frac{\partial y}{\partial \upsilon} = \frac{x}{x^2+y^2}.$$

**Example 2.12** If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$

Let $$\phi \equiv x^2 + y^2 + z^2 - 2xyz - 1 = 0 \quad (1)$$

$\therefore$ $$d\phi = 0$$

i.e., $$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \quad (2)$$

i.e., $$2(x - yz)dx + 2(y - zx)dy + 2(z - xy)dz = 0$$

Now $$(x - yz)^2 = x^2 - 2xyz + y^2z^2$$
$$= 1 - y^2 - z^2 + y^2z^2, \text{ from (1)}$$
$$= (1 - y^2)(1 - z^2)$$

$\therefore$ $$x - yz = \sqrt{(1-y^2)(1-z^2)}$$

Similarly, $$y - zx = \sqrt{(1-z^2)(1-x^2)}$$

and $$z - xy = \sqrt{(1-x^2)(1-y^2)}$$

Using these values in (2), we have

$$\sqrt{(1-y^2)(1-z^2)}\,dx + \sqrt{(1-z^2)(1-x^2)}\,dy + \sqrt{(1-x^2)(1-y^2)}\,dz = 0$$

Dividing by $\sqrt{(1-x^2)(1-y^2)(1-z^2)}$, we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

**Example 2.13** The specific gravity $s$ of a body is given by $s = \frac{W_1}{W_1 - W_2}$ where $W_1$ and $W_2$ are the weights of the body in air and in water respectively. Show that if there is an error of 1% in each weighing, $s$ is not affected. But if there is an error of 1% in $W_1$ and 2% in $W_2$, show that the percentage error in $s$ is $\frac{W_2}{W_1 - W_2}$.

$$s = \frac{W_1}{W_1 - W_2}$$

$\therefore$ $$\log s = \log W_1 - \log(W_1 - W_2)$$

Taking differentials on both sides,

$$\frac{1}{s}\mathrm{d}s = \frac{1}{W_1}\mathrm{d}W_1 - \frac{1}{W_1 - W_2}\left(dW_1 - dW_2\right)$$

∴ The relation among the errors is nearly

$$\frac{1}{s}\Delta s = \frac{1}{W_1}\Delta W_1 - \frac{1}{W_1 - W_2}\left(\Delta W_1 - \Delta W_2\right) \tag{1}$$

or
$$\frac{100\,\Delta s}{s} = \frac{100\,\Delta W_1}{W_1} - \frac{1}{W_1 - W_2}\left(100\,\Delta W_1 - 100\,\Delta W_2\right) \tag{2}$$

(i) Given that $\frac{100\,\Delta W_1}{W_1} = 1$ and $\frac{100\,\Delta W_2}{W_2} = 1$

Using these values in (2), we have

$$\frac{100\,\Delta s}{s} = 1 - \frac{1}{W_1 - W_2}\left(W_1 - W_2\right) = 0$$

∴ $s$ is not affected, viz., there is no error in $s$.

(ii) Given that $\frac{100\,\Delta W_1}{W_1} = 1$ and $\frac{100\,\Delta W_2}{W_2} = 2.$ Using these values in (2), we have

$$\frac{100\,\Delta s}{s} = 1 - \frac{1}{W_1 - W_2}\left(W_1 - 2W_2\right)$$

$$= \frac{W_2}{W_1 - W_2}$$

i.e., % error in $s = \frac{W_2}{W_1 - W_2}.$

**Example 2.14** The work that must be done to propel a ship of displacement $D$ for a distance $s$ in time $t$ is proportional to $s^2\,D^{3/2} \div t^2$. Find approximately the percentage increase of work necessary when the distance is increased by 1%, the time is diminished by 1% and the displacement of the ship is diminished by 3%.

Given that $W = ks^2\,D^{3/2}/t^2$, where $k$ is the constant of proportionality.

∴
$$\log W = \log k + 2\log s + \frac{3}{2}\log D - 2\log t.$$

Taking differentials on both sides,

$$\frac{\mathrm{d}W}{W} = 2\frac{\mathrm{d}s}{s} + \frac{3}{2}\frac{\mathrm{d}D}{D} - 2\frac{\mathrm{d}t}{t}$$

∴ The relation among the percentage errors is approximately,

$$\frac{100\,\Delta W}{W} = 2\times\frac{100\,\Delta s}{s}+\frac{3}{2}\cdot\frac{100\,\Delta D}{D}-2\times\frac{100\,\Delta t}{t} \qquad (1)$$

Given that $\dfrac{100\,\Delta s}{s} = 1,\ \dfrac{100\,\Delta t}{t} = -1$ and $\dfrac{100\,\Delta D}{D} = -3.$

Using these values in (1), we have

$$\begin{aligned}\frac{100\,\Delta W}{W} &= 2\times 1+\frac{3}{2}\times(-3)-2\times(-1)\\ &= -0.5\end{aligned}$$

i.e., percentage decrease of work = 0.5.

**Example 2.15** The period $T$ of a simple pendulum with small oscillations is given by $T = 2\pi\sqrt{\dfrac{l}{g}}$. If $T$ is computed using $l = 6$ cm and $g = 980$ cm/sec$^2$, find approximately the error in $T$, if the values are $l = 5.9$ cm and $g = 981$ cm/sec$^2$. Find also the percentage error.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

∴ $$\log T = \log 2+\log\pi+\frac{1}{2}\log l-\frac{1}{2}\log g$$

Taking differentials on both sides,

$$\frac{1}{T}\,\mathrm{d}T=\frac{1}{2l}\,\mathrm{d}l-\frac{1}{2g}\,\mathrm{d}g \qquad (1)$$

∴ $$\begin{aligned}\mathrm{d}T &= 2\pi\sqrt{\frac{l}{g}}\left\{\frac{1}{2l}\,\mathrm{d}l-\frac{1}{2g}\,\mathrm{d}g\right\}\\ &=\pi\left\{\frac{1}{\sqrt{l\mathrm{g}}}\,\mathrm{d}l-\frac{\sqrt{l}}{g\sqrt{g}}\,\mathrm{d}g\right\}\\ &= \pi\left\{\frac{0\cdot 1}{\sqrt{5.9\times 981}}-\frac{\sqrt{5\cdot 9}}{981\sqrt{981}}\times(-1)\right\}\end{aligned}$$

i.e., Error in $T = 0.0044$ sec.

$$\% \text{ error in } T = \frac{100\,\mathrm{d}T}{T}$$

$$= 50\left\{\frac{\mathrm{d}l}{l} - \frac{\mathrm{d}g}{g}\right\}, \text{by (1)}$$

$$= 50\left\{\frac{0\cdot 1}{5\cdot 9} - \frac{(-1)}{981}\right\}$$

$$= 0.8984$$

**Example 2.16** The base diameter and altitude of a right circular cone are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the value computed for the volume and lateral surface.

Volume of the right circular cone is given by $V = \frac{1}{3}\pi\cdot\left(\frac{D}{2}\right)^2 h$

$$\therefore \qquad \mathrm{d}V = \frac{\pi}{12}(D^2\cdot \mathrm{d}h + 2\,Dh\cdot \mathrm{d}D)$$

$$= \frac{\pi}{12}\{16\times 0.1 + 2\times 4\times 6\times 0.1\}$$

i.e., Error in $V = 1.6755$ cm$^3$.

Lateral surface area of the right circular cone is given by

$$S = \pi\frac{D}{2}l$$

$$= \frac{\pi}{4}D\sqrt{D^2 + 4h^2}$$

$$\therefore \qquad \mathrm{d}S = \frac{\pi}{4}\left[D\cdot\frac{1}{2\sqrt{D^2+4h^2}}(2D\,\mathrm{d}D + 8h\,\mathrm{d}h + \sqrt{D^2+4h^2}\;\mathrm{d}D\right]$$

$$= \frac{\pi}{4}\left[\frac{4}{\sqrt{16+144}}\{4\times 0\cdot 1 + 24\times 0\cdot 1\} + \sqrt{16+144}\times 0\cdot 1\right]$$

$$= 1.6889\,\mathrm{cm}^2.$$

**Example 2.17** The side $c$ of a triangle $ABC$ is calculated by using the measured values of its sides $a$, $b$ and the angle $C$. Show that the error in the side $c$ is given by

$$\Delta c = \cos B \cdot \Delta a + \cos A \cdot \Delta b + a \sin B \cdot \Delta C.$$

The side $c$ is given by the formula

$$c^2 = a^2 + b^2 - 2ab \cos C \tag{1}$$

Taking the differentials on both sides of (1),

$$2c\,\Delta c = 2a\,\Delta a + 2b\,\Delta b - 2\{b \cos C \cdot \Delta a + a \cos C \cdot \Delta b - ab \sin C \cdot \Delta C\}, \text{ nearly}$$

i.e.,
$$\Delta c = \frac{(a - b\cos C)\Delta a + (b - a\cos C)\Delta b + ab \sin C \cdot \Delta C}{c} \tag{2}$$

Now $b \cos C + c \cos B = a$

$$\therefore \quad \frac{a - b\cos C}{c} = \cos B \tag{3}$$

$$a \cos C + c \cos A = b$$

$$\therefore \quad \frac{b - a\cos C}{c} = \cos A \tag{4}$$

Also
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad b \sin C = c \sin B$$

$$\therefore \quad \frac{ab \sin C}{c} = a \sin B \tag{5}$$

Using (3), (4) and (5) in (2), we get

$$\Delta c = \cos B \cdot \Delta a + \cos A \cdot \Delta b + a \sin B \cdot \Delta C$$

**Example 2.18** The angles of a triangle $ABC$ are calculated from the sides $a$, $b$, $c$. If small changes $\delta a$, $\delta b$, $\delta c$ are made in the measurement of the sides, show that

$$\delta A = \frac{a}{2\Delta}(\delta a - \delta b \cos C - \delta c \cos B)$$

and $\delta B$ and $\delta C$ are given by similar expressions, where $\Delta$ is the area of the triangle. Verify that $\delta A + \delta B + \delta C = 0$.

In triangle $ABC$,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1}$$

Taking differentials on both sides of (1),

$$-2\sin A\cdot\delta A=\left[bc\left\{2b\,\delta b+2c\delta c-2a\delta a\right\}-\left(b^2+c^2-a^2\right)\left(b\delta c+c\delta b\right)\right]\div b^2\,c^2$$

$$=\frac{\left(b^2c-c^3+a^2c\right)\delta b+\left(bc^2-b^3+a^2\,b\right)\delta c-2abc\,\delta a}{b^2c^2}$$

$$=\frac{c\left(a^2+b^2-c^2\right)\delta b+b\left(c^2+a^2-b^2\right)\delta c-2abc\delta a}{b^2c^2}$$

$$=\frac{c\left(2\,ab\cos C\right)\delta b+b\left(2\,ca\cos B\right)\delta c-2abc\,\delta a}{b^2c^2},$$

by formulas similar to (1)

$$=\frac{2a}{bc}\left(\cos\,C\cdot\delta b+\cos B\cdot\delta c-\delta a\right)$$

$$\therefore\qquad \delta A=\frac{a}{bc\sin A}\left(\delta a-\cos C\cdot\delta b-\cos B\cdot\delta c\right)$$

$$=\frac{a}{2\Delta}\left(\delta a-\cos C\cdot\delta b-\cos B\cdot\delta c\right),\ \text{since } \Delta=\frac{1}{2}bc\sin A \qquad (2)$$

Similarly,

$$\delta B=\frac{b}{2\Delta}\left(\delta b-\cos A\cdot\delta c-\cos C\cdot\delta a\right) \qquad (3)$$

$$\delta C=\frac{c}{2\Delta}\left(\delta c-\cos B\cdot\delta a-\cos A\cdot\delta b\right) \qquad (4)$$

Adding (2), (3) and (4), we get

$$2\Delta\left(\delta A+\delta B+\delta C\right)=\left(a-b\cos C-c\cos B\right)\delta a$$

$$+\left(b-a\cos C-c\cos A\right)\delta b+\left(c-a\cos B-b\cos A\right)\delta c$$

$$=\left(a-a\right)\delta a+\left(b-b\right)\delta b+\left(c-c\right)\delta c$$

$$\left(\because\ b\cos C+c\cos B=a\ \text{etc.}\right)$$

$$=0$$

$$\therefore\qquad \delta A+\delta B+\delta C=0.$$

**Example 2.19** The area of a triangle $ABC$ is calculated from the lengths of the sides $a$, $b$, $c$. If $a$ is diminished and $b$ is increased by the same small amount $k$, prove that the consequent change in the area is given by

$$\frac{\delta\Delta}{\Delta} = \frac{2(a-b)k}{c^2-(a-b)^2}$$

The area of triangle *ABC* is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\text{ , where } 2s = a+b+c$$

$$\therefore \qquad \log\Delta = \frac{1}{2}\{\log s + \log(s-a) + \log(s-b) + \log(s-c)\}$$

Taking differentials on both sides, we get

$$\frac{\delta\Delta}{\Delta} = \frac{1}{2}\left\{\frac{\delta s}{s} + \frac{\delta s - \delta a}{s-a} + \frac{\delta s - \delta b}{s-b} + \frac{\delta s - \delta c}{s-c}\right\} \qquad (1)$$

Since $\quad 2s = a+b+c,\ 2\delta s = \delta a + \delta b + \delta c$

i.e.; $2\delta s = -k + k + 0 = 0$, by the given data.

$$\therefore \qquad \delta s = 0 \qquad (2)$$

Using (2) in (1), we have

$$\frac{\delta\Delta}{\Delta} = \frac{1}{2}\left(\frac{k}{s-a} - \frac{k}{s-b}\right)$$

$$= \frac{k}{2}\left\{\frac{2}{b+c-a} - \frac{2}{c+a-b}\right\} (\because 2s = a+b+c)$$

$$= \frac{k}{2} \times 2\left\{\frac{(c+a-b)-(b+c-a)}{[c-(a-b)][c+(a-b)]}\right\}$$

$$= \frac{2k(a-b)}{c^2-(a-b)^2}$$

## EXERCISE 2(a)

***Part A***

(Short Answer Questions)

1. What is meant by total differential? Why it is called so?
2. If $u = \sin(xy^2)$, express the total differential of $u$ in terms of those of $x$ and $y$.
3. If $u = x^y.y^x$, express d$u$ in terms of d$x$ and d$y$.
4. If $u = xy \log xy$, express d$u$ in terms of d$x$ and d$y$.
5. If $u = a^{xy}$, express d$u$ in terms of d$x$ and d$y$.

6. Find $\frac{du}{dt}$, if $u = x^3y^2 + x^2y^3$, where $x = at^2$, $y = 2at$.

7. Find $\frac{du}{dt}$, if $u = e^{xy}$, where $x = \sqrt{a^2 - t^2}$, $y = \sin^3 t$.

8. Find $\frac{du}{dt}$, if $u = \log(x + y + z)$, where $x = e^{-t}$, $y = \sin t$, $z = \cos t$.

9. Find $\frac{dy}{dx}$, using partial differentiation, if $x^3 + 3x^2y + 6xy^2 + y^3 = 1$.

10. If $x \sin(x - y) - (x + y) = 0$, use partial differentiation to prove that

$$\frac{dy}{dx} = \frac{y + x^2 \cos(x - y)}{x + x^2 \cos(x - y)}.$$

11. Find $\frac{dy}{dx}$, when $u = \sin(x^2 + y^2)$, where $x^2 + 4y^2 = 9$.

12. Find $\frac{dy}{dx}$, if $u = x^2y$, where $x^2 + xy + y^2 = 1$.
13. Define absolute, relative and percentage errors.
14. Using differentials, find the approximate value of $\sqrt{15}$.
15. Using differentials, find the approximate value of $2x^4 + 7x^3 - 8x^2 + 3x + 1$ when $x = 0.999$.
16. What error in the common logarithm of a number will be produced by an error of 1% in the number?
17. The radius of a sphere is found to be 10 cm with a possible error of 0.02 cm. Find the relative errors in computing the volume and surface area.
18. Find the percentage error in the area of an ellipse, when an error of 1% is made in measuring the lengths of its axes.
19. Find the approximate error in the surface of a rectangular parallelopiped of sides $a$, $b$, $c$ if an error of $k$ is made in measuring each side.
20. If the measured volume of a right circular cylinder is 2% too large and the measured length is 1% too small, find the percentage error in the calculated radius.

***Part B***

21. If $u = f(x - y, y - z, z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

22. If $f$ is a function of $u$, $v$, $w$, where $u = \sqrt{yz}$, $v = \sqrt{zx}$, and $w = \sqrt{xy}$ /show that

$$\sum u \frac{\partial f}{\partial u} = \sum x \frac{\partial f}{\partial x}.$$

23. If $f = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$, show that $x^2\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y} + z^2\frac{\partial f}{\partial z} = 0$.

24. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z} = 0.$$

25. If $f(cx - az, cy - bz) = 0$, where $z$ is a function of $x$ and $y$, prove that

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c.$$

26. If $z = f(u, v)$, where $u = x + y$ and $v = x - y$, show that $2\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$.

27. If $z = f(x, y)$, where $x = u^2 + v^2$, $y = 2uv$, prove that

$$u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = 2\sqrt{(x^2 - y^2)}\,\frac{\partial z}{\partial x}.$$

28. If $z = f(x, y)$, where $x = u + v$, $y = uv$, prove that

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}.$$

29. If $x = r\cos\theta$, $y = r\sin\theta$, prove that the equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ is equivalent to $\frac{\partial u}{\partial r} + \frac{1}{r}\tan\left(\frac{\pi}{4} - \theta\right)\frac{\partial u}{\partial \theta} = 0$.

30. If $z = f(u, v)$, where $u = x^2 - 2xy - y^2$ and $v = y$, show that the equation $(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$ is equivalent to $\frac{\partial z}{\partial v} = 0$.

31. If $z = f(u, v)$, where $u = x^2 - y^2$ and $v = 2xy$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x^2 + y^2)\left\{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right\}.$$

32. If $z = f(u, v)$, where $u = x^2 - y^2$ and $v = 2xy$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

33. If $z = f(x, y)$ where $x = X\cos a - Y\sin\alpha$ and $y = X\sin\alpha + Y\cos\alpha$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial X^2} + \frac{\partial^2 z}{\partial Y^2}$.

34. If $z = f(u, \upsilon)$, where $u = lx + my$ and $\upsilon = ly - mx$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial \upsilon^2}\right).$$

35. By changing the independent variables $x$ and $t$ to $u$ and $\upsilon$ by means of the transformations $u = x - at$ and $\upsilon = x + at$, show that $a^2\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 4a^2\frac{\partial^2 y}{\partial u\,\partial \upsilon}$.

36. By using the transformations $u = x + y$ and $\upsilon = x - y$, change the independent variables $x$ and $y$ in the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ to $u$ and $\upsilon$.

37. Transform the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ by changing the independent variables using $u = x - y$ and $\upsilon = x + y$.

38. Transform the equation $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$, by changing the independent variables using $u = x$ and $\upsilon = \frac{y^2}{x}$.

39. Transform the equation $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x\,\partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$, by changing the independent variables using $u = 2x + y$ and $\upsilon = 3x + y$.

40. Transform the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, by changing the independent variables using $z = x + iy$ and $z^* = x - iy$, where $i = \sqrt{-1}$.

41. Use partial differentiation to find $\frac{dy}{dx}$, when (i) $x^y = y^x$; (ii) $x^m y^n = (x + y)^{m+n}$; (iii) $(\cos x)^y = (\sin y)^x$; (iv) $(\sec x)^y = (\cot y)^x$; (v) $x^y = e^{x-y}$.

42. Use partial differentiation to find $\frac{d^2 y}{dx^2}$, when $x^3 + y^3 - 3axy = 0$.

43. Use partial differentiation to find $\frac{d^2y}{dx^2}$, when $x^4 + y^4 = 4a^2xy$.

44. Use partial differentiation to prove that $\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$, when $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

45. Use partial differentiation to prove that $\frac{d^2y}{dx^2} = \frac{b^2 - ac}{(ay + b)^3}$, when $ay^2 + 2by + c = x^2$.

46. If $x^2 - y^2 + u^2 + 2v^2 = 1$ and $x^2 + y^2 - u^2 - v^2 = 2$, prove that $\frac{\partial u}{\partial x} = \frac{3x}{u}$ and $\frac{\partial v}{\partial x} = -\frac{2x}{v}$.

47. The deflection at the centre of a rod of length $l$ and diameter $d$ supported at its ends and loaded at the centre with a weight $w$ is proportional to $wl^3/d^4$. What is the percentage increase in the deflection, if the percentage increases in $w$, $l$ and $d$ are 3, 2 and 1 respectively.

48 The torsional rigidity of a length of wire is obtained from the formula $N = \frac{8\pi Il}{t^2 r^4}$. If $l$ is decreased by 2%, $t$ is increased by 1.5% and $r$ is increased by 2%, show that the value of $N$ is decreased by 13% approximately.

49. The Current $C$ measured by a tangent galvanometer is given by the relation $C = k \tan \theta$, where $\theta$ is the angle of deflection. Show that the relative error in $C$ due to a given error in $\theta$ is minimum when $\theta = 45°$.

50. The range $R$ of a projectile projected with velocity $v$ at an elevation $\theta$ is given by $R = \frac{v^2}{g} \sin 2\theta$. Find the percentage error in $R$ due to errors of 1% in $v$ and $\frac{1}{2}$% in $\theta$, when $\theta = \frac{\pi}{6}$.

51. The velocity $v$ of a wave is given by $v^2 = \frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda}$, where $g$ and $\lambda$ are constants and $\rho$ and $T$ are variables. Prove that, if $\rho$ is increased by 1% and $T$ is decreased by 2%, then the percentage decrease in $v$ is approximately $\frac{3\pi T}{\lambda\rho v^2}$.

52. The focal length of a mirror is given by the formula $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$. If equal errors $k$ are made in the determination of $u$ and $v$, show that the percentage error in $f$ is $100k\left(\frac{1}{u} + \frac{1}{v}\right)$.

53. A closed rectangular box of dimensions $a$, $b$, $c$ has the edges slightly altered in length by amounts $\Delta a$, $\Delta b$ and $\Delta c$ respectively, so that both its volume and surface area remain unaltered. Show that $\dfrac{\Delta a}{a^2(b-c)} = \dfrac{\Delta b}{b^2(c-a)} = \dfrac{\Delta c}{c^2(a-b)}$.

    [**Hint:** Solve the equations d$V = 0$ and d$S = 0$ for $\Delta a$, $\Delta b$, $\Delta c$ using the method of cross-multiplication]

54. If a triangle $ABC$ is slightly disturbed so as to remain inscribed in the same circle, prove that

$$\frac{\Delta a}{\cos A}+\frac{\Delta b}{\cos B}+\frac{\Delta c}{\cos C} = 0.$$

55. The area of a triangle $ABC$ is calculated using the formula $\Delta = \dfrac{1}{2}\, bc \sin \text{A}$. Show that the relative error in $\Delta$ is given by

$$\frac{\delta\Delta}{\Delta} = \frac{\delta b}{b}+\frac{\delta c}{c}+\cot A\delta A.$$

    If an error of $5'$ is made in the measurement of $A$ which is taken as 60°, find the percentage error in $\Delta$.

56. Prove that the error in the area $\Delta$ of a triangle $ABC$ due to a small error in the measurement of $c$ is given by

$$\delta\Delta = \frac{\Delta}{4}\left(\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-c}\right)\delta c.$$

57. The area of a triangle $ABC$ is determined from the side $a$ and the two angles $B$ and $C$. If there are small errors in the values of $B$ and $C$, show that the resulting error in the calculated value of the area $\Delta$ will be $\dfrac{1}{2}(c^2\Delta B+b^2\Delta C)$.

$$\left[\textbf{Hint}:\Delta = \frac{1}{2}\frac{a^2 \sin B \sin C}{\sin(B+C)}\right]$$

### 2.2.3 Taylor's Series Expansion of a Function of Two Variables

Students are familiar with Taylor's series of a function of one variable viz. $f(x + h) = f(x)+\dfrac{h}{1!}f'(x)+\dfrac{h^2}{2!}f''(x)+\cdots\infty$, which is an infinite series of powers of $h$. This idea can be extended to expand $f(x + h, y + k)$ in an infinite series of powers of $h$ and $k$.

## Statement

If $f(x, y)$ and all its partial derivatives are finite and continuous at all points $(x, y)$, then

$$f(x+h, y+k) = f(x, y) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \cdots\infty$$

***Proof:***

If we assume $y$ to be a constant, $f(x + h, y + k)$ can be treated as a function of $x$ only.

Then $f(x+h, y+k) = f(x, y+k) + \dfrac{h}{1!}\dfrac{\partial f(x, y+k)}{\partial x} + \dfrac{h^2}{2!}\dfrac{\partial^2 f(x, y+k)}{\partial x^2} + \cdots$ (1)

Now treating $x$ as a constant,

$$f(x, y+k) = f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \cdots \quad (2)$$

Using (2) in (1), we have

$$f(x+h, y+k) = \left[f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \cdots\right]$$

$$+\frac{h}{1!}\frac{\partial}{\partial x}\left\{f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \cdots\right\}$$

$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + \frac{k}{1!}\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \cdots\right\} + \cdots\infty$$

$$= f(x, y) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2} + \cdots\infty\right)$$

$$= f(x, y) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \cdots\infty \quad (3)$$

Interchanging $x$ and $h$ and also $y$ and $k$ in (3) and then putting $h = k = 0$, we have

$$f(x, y) = f(0,0) + \frac{1}{1!}\left\{x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right\} + \frac{1}{2!}\left\{x^2\frac{\partial^2 f(0,0)}{\partial x^2}\right.$$

$$\left. + 2xy\frac{\partial^2 f(0,0)}{\partial x\,\partial y} + y^2\frac{\partial^2 f(0,0)}{\partial y^2}\right\} + \cdots \quad (4)$$

Series in (4) is the *Maclarin's series* of the function $f(x, y)$ in powers of $x$ and $y$. *Another form of Taylor's series of* $f(x, y)$

$$\begin{aligned}
f(x,y) &= f(a+\overline{x-a},\ b+\overline{y-b}) \\
&= f(a+h), (b+k), \text{ say} \\
&= f(a,b)+\frac{1}{1!}\left\{h\frac{\partial f(a,b)}{\partial x}+k\frac{\partial f(a,b)}{\partial y}\right\} \\
&\quad +\frac{1}{2!}\left\{h^2\frac{\partial^2 f(a,b)}{\partial x^2}+2kh\frac{\partial^2 f(a,b)}{\partial x\,\partial y}+k^2\frac{\partial^2 f(a,b)}{\partial y^2}\right\}+\cdots, \text{ by (3)} \\
&= f(a,b)+\frac{1}{1!}\left[(x-a)\frac{\partial f(a,b)}{\partial x}+(y-b)\frac{\partial f(a,b)}{\partial y}\right] \\
&\quad +\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2}+2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x\,\partial y}+(y-b)^2\frac{\partial^2 f(a,b)}{\partial y^2}\right]+\cdots \qquad (5)
\end{aligned}$$

(5) is called the Taylor's series of $f(x, y)$ at or near the point $(a, b)$.

Thus the Taylor's series of $f(x, y)$ at or near the point (0, 0) is Maclaurins series of $f(x, y)$.

## 2.3 JACOBIANS

If $u$ and $\upsilon$ are functions of two independent variables $x$ and $y$, then the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial \upsilon}{\partial x} & \dfrac{\partial \upsilon}{\partial y} \end{vmatrix}$$

is called *the Jacobian* or *functional determinant* of $u$, $\upsilon$ with respect to $x$ and $y$ and is written as

$$\frac{\partial(u,\upsilon)}{\partial(x,y)} \text{ or } J\left(\frac{u,\upsilon}{x,y}\right).$$

Similarly the Jacobian of $u$, $\upsilon$, $w$ with respect to $x$, $y$, $z$ is defined as

$$\frac{\partial(u,\upsilon,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial \upsilon}{\partial x} & \dfrac{\partial \upsilon}{\partial y} & \dfrac{\partial \upsilon}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

### Note ☑

1. To define the Jacobian of $n$ dependent variables, each of these must be a function of $n$ independent variables.
2. The concept of Jacobians is used when we change the variables in multiple integrals. (See property 4 given below)

## 2.3.1 Properties of Jacobians

1. If $u$ and $v$ are functions of $x$ and $y$, then $\dfrac{\partial(u, v)}{\partial(x, y)} \times \dfrac{\partial(x, y)}{\partial(u, v)} = 1.$

***Proof:***

Let $u = f(x, y)$ and $v = g(x, y)$. When we solve for $x$ and $y$, let $x = \phi(u, v)$ and $y = \psi(u, v)$.

Then

$$\left.\begin{aligned}
\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} &= \frac{\partial u}{\partial u} = 1\\
\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} &= \frac{\partial u}{\partial v} = 0\\
\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} &= \frac{\partial v}{\partial u} = 0\\
\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} &= \frac{\partial v}{\partial v} = 1
\end{aligned}\right\} \quad (1)$$

Now

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix},$$ by interchanging the rows and columns of the second determinant.

$$= \begin{vmatrix} \left(\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial u}\right) & \left(\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial v}\right) \\ \left(\dfrac{\partial v}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial v}{\partial y}\cdot\dfrac{\partial y}{\partial u}\right) & \left(\dfrac{\partial v}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial v}{\partial y}\cdot\dfrac{\partial y}{\partial v}\right) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{[by (1)]}$$

$$= 1$$

2. If $u$ and $v$ functions of $r$ and $s$, where $r$ and $s$ are functions of $x$ and $y$, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

***Proof:***

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix}$$, by rewriting the second determinant.

$$= \begin{vmatrix} \left(\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}\right) & \left(\frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}\right) \\ \left(\frac{\partial v}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial x}\right) & \left(\frac{\partial v}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial y}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}.$$

*Note* ☑ The two properties given above hold good for more than two variables too.

3. If $u$, $v$, $w$ are functionally dependent functions of three independent variables

$x, y, z$ then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

*Note* ☑ The functions $u$, $v$, $w$ are said to be functionally dependent, if each can be expressed in terms of the others or equivalently $f(u, v, w) = 0$. Linear dependence of functions is a particular case of functional dependence.

***Proof:***

Since $u$, $v$, $w$ are functionally dependent, $f(u, v, w) = 0$ (1)

Differentiating (1) partially with respect to $x$, $y$ and $z$, we have

$$f_u \,.\, u_x + f_v \,.\, v_x + f_w \,.\, w_x = 0 \quad (2)$$

$$f_u \,.\, u_y + f_v \,.\, v_y + f_w \,.\, w_y = 0 \quad (3)$$

$$f_u \,.\, u_z + f_v \,.\, v_z + f_w \,.\, w_z = 0 \quad (4)$$

Equations (2), (3) and (4) are homogeneous equations in the unknowns $f_u, f_v, f_w$. At least one of $f_u$, $f_v$ and $f_w$ is not zero, since if all of them are zero, then $f(u, v, w) \equiv$ constant, which is meaningless.

Thus the homogeneous equations (2), (3) and (4) possess a non-trivial solution. $\therefore$ Matrix of the coefficients of $f_u, f_v, f_w$ is singular.

i.e.,
$$\begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} = 0$$

i.e.,
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

***Note*** ☑ The converse of this property is also true. viz., if $u, v, w$ are functions of $x, y, z$ such that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$ then $u, v, w$ are functionally dependent. i.e., there exists a relationship among them.

4. If the transformations $x = x(u, v)$ and $y = y(u, v)$ are made in the double integral $\iint f(x, y)\,\mathrm{d}x\,\mathrm{d}y$, then $f(x, y) = F(u, v)$ and $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v$, where $J = \dfrac{\partial(x, y)}{\partial(u, v)}$.

***Proof:***
$\mathrm{d}x\,\mathrm{d}y$ = Elemental area of a rectangle with vertices $(x, y)$, $(x + \mathrm{d}x, y)$, $(x + \mathrm{d}x, y + \mathrm{d}y)$ and $(x, y + \mathrm{d}y)$

This elemental area can be regarded as equal to the area of the parallelogram with vertices $(x, y)$, $\left(x + \dfrac{\partial x}{\partial u}\mathrm{d}u, y + \dfrac{\partial y}{\partial u}\mathrm{d}u\right)$, $\left(x + \dfrac{\partial x}{\partial u}\mathrm{d}u + \dfrac{\partial x}{\partial v}\mathrm{d}v, y + \dfrac{\partial y}{\partial u}\mathrm{d}u + \dfrac{\partial y}{\partial v}\mathrm{d}v\right)$ and $\left(x + \dfrac{\partial x}{\partial v}\mathrm{d}v, y + \dfrac{\partial y}{\partial v}\mathrm{d}v\right)$, since $\mathrm{d}x$ and $\mathrm{d}y$ are infinitesimals.

Now the area of this parallelogram is equal to 2× area of the triangle with vertices $(x, y)$, $\left(x + \dfrac{\partial x}{\partial u}\mathrm{d}u, y + \dfrac{\partial y}{\partial u}\mathrm{d}u\right)$ and $\left(x + \dfrac{\partial x}{\partial v}\mathrm{d}v, y + \dfrac{\partial y}{\partial v}\mathrm{d}v\right)$

$$\therefore \quad \mathrm{d}x\,\mathrm{d}y = 2 \times \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x + \dfrac{\partial x}{\partial u}\mathrm{d}u & y + \dfrac{\partial y}{\partial u}\mathrm{d}u & 1 \\ x + \dfrac{\partial x}{\partial v}\mathrm{d}v & y + \dfrac{\partial y}{\partial v}\mathrm{d}v & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & y & 1 \\ \frac{\partial x}{\partial u}\mathrm{d}u & \frac{\partial y}{\partial u}\mathrm{d}u & 0 \\ \frac{\partial x}{\partial v}\mathrm{d}v & \frac{\partial y}{\partial v}\mathrm{d}v & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u}\mathrm{d}u & \frac{\partial y}{\partial u}\mathrm{d}u \\ \frac{\partial x}{\partial v}\mathrm{d}v & \frac{\partial y}{\partial v}\mathrm{d}v \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathrm{d}u\,\mathrm{d}v$$

i.e., $$\mathrm{d}x\,\mathrm{d}y = \frac{\partial(x, y)}{\partial(u, v)}\mathrm{d}u\,\mathrm{d}v.$$

Since both d$x$ d$y$ and d$u$ d$v$ are positive, $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v$, where $J = \frac{\partial(x, y)}{\partial(u, v)}$

Similarly, if we make the transformations

$$x = x(u, v, w),\ y = y(u, v, w) \text{ and } z = z(u, v, w)$$

in the triple integral $\iiint f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, then $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$, where $J = \frac{\partial(x, y, z)}{\partial(u, v, w)}$.

## 2.4 DIFFERENTIATION UNDER THE INTEGRAL SIGN

When a function $f(x, y)$ of two variables is integrated with respect to $y$ partially, viz., treating $x$ as a parameter, between the limits $a$ and $b$, then $\int_a^b f(x, y)\,\mathrm{d}y$ will be a function of $x$.

Let it be denoted by $F(x)$.

Now to find $F'(x)$, if it exists, we need not find $F(x)$ and then differentiate it with respect to $x$. $F'(x)$ can be found out without finding $F(x)$, by using *Leibnitz's* rules, given below:

### *1. Leibnitz's rule for constant limits of integration*

If $f(x, y)$ and $\frac{\partial f(x, y)}{\partial x}$ are continuous functions of $x$ and $y$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^b f(x, y)\mathrm{d}y\right] = \int_a^b \frac{\partial f(x, y)}{\partial x}\mathrm{d}y, \text{ where}$$

$a$ and $b$ are constants independent of $x$.

***Proof:***

Let $$\int_a^b f(x,y)\,\mathrm{d}y = F(x).$$

Then
$$F(x+\Delta x) - F(x) = \int_a^b f(x+\Delta x,\, y)\mathrm{d}y - \int_a^b f(x,y)\mathrm{d}y$$
$$= \int_a^b [f(x+\Delta x,\, y) - f(x,y)]\mathrm{d}y$$
$$= \Delta x \int_a^b \frac{\partial f(x+\theta\Delta\, x,\, y)}{\partial x}\mathrm{d}y,\ 0<\theta<1,$$

$$\left[\text{by Mean Value theorem, viz.,} f(x+h) - f(x) = h\frac{\mathrm{d}f(x+\theta h)}{\mathrm{d}x},\ 0<\theta<1\right]$$

$$\therefore \qquad \frac{F(x+\Delta x) - F(x)}{\Delta x} = \int_a^b \frac{\partial f(x+\theta\cdot\Delta x, y)}{\partial x}\mathrm{d}y \qquad (1)$$

Taking limits on both sides of (1) as $\Delta x \to 0$,

$$F'(x) = \int_a^b \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y$$

i.e., $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^b f(x,y)\mathrm{d}y\right] = \int_a^b \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y$$

***2. Leibnitz's rule for variable limits of integration***

If $f(x, y)$ and $\dfrac{\partial f(x,y)}{\partial x}$ are continuous functions of $x$ and $y$, then $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\displaystyle\int_{a(x)}^{b(x)} f(x,y)\mathrm{d}y\right]$

$= \displaystyle\int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\mathrm{d}y + f\{x,b(x)\}\frac{\mathrm{d}b}{\mathrm{d}x} - f\{x,a(x)\}\frac{\mathrm{d}a}{\mathrm{d}x}$, provided $a(x)$ and $b(x)$ possess continuous first order derivatives.

***Proof:***

Let $\int f(x,y)\mathrm{d}y = F(x,y)$, so that $\dfrac{\partial}{\partial y}F(x,y) = f(x,y)$ (1)

$$\therefore \qquad \int_{a(x)}^{b(x)} f(x,y)\mathrm{d}y = F\{x,b(x)\} - F\{x,a(x)\}$$

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,y)dy\right] = \frac{d}{dx}F\{x,b(x)\} - \frac{d}{dx}F\{x,a(x)\}$$

$$= \left[\frac{d}{dx}F(x,y)\right]_{y=b(x)} - \left[\frac{d}{dx}F(x,y)\right]_{y=a(x)}$$

$$= \left[\frac{\partial}{\partial x}F(x,y) + \frac{\partial}{\partial y}F(x,y)\cdot\frac{dy}{dx}\right]_{y=b(x)}$$

$$- \left[\frac{\partial}{\partial x}F(x,y) + \frac{\partial}{\partial y}F(x,y)\frac{dy}{dx}\right]_{y=a(x)}$$

by differentiation of implicit functions

$$= \left[\frac{\partial}{\partial x}F(x,y)\right]_{y=a(x)}^{y=b(x)} + \left[f(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{y=b(x)}$$

$$- \left[f(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{y=a(x)} \qquad \text{by (1)}$$

$$= \left[\frac{\partial}{\partial x}\int f(x,y)\,\mathrm{d}y\right]_{y=a(x)}^{y=b(x)} + f\{x,\, b(x)\}b'(x) - f\{x,\, a(x)\}a'(x)$$

$$= \left[\int \frac{\partial f(x,y)}{\partial x}\mathrm{d}y\right]_{y=a(x)}^{y=b(x)} + f\{x,\, b(x)\}b'(x) - f\{x,\, a(x)\}a'(x)$$

$$= \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\mathrm{d}y + f\{x,\, b(x)\}b'(x) - f\{x,\, a(x)\}a'(x)$$

## WORKED EXAMPLE 2(b)

**Example 2.1** Expand $e^x \cos y$ in powers of $x$ and $y$ as far as the terms of the third degree.

$$f(x,y) = e^x \cos y;\ \frac{\partial f(x,y)}{\partial x} = f_x(x,y) = e^x \cos y;$$

$$\frac{\partial f(x,y)}{\partial y} = f_y(x,y) = -e^x \sin y.$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = f_{xx}(x,y) = e^x \cos y; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = f_{yx}(x,y) = -e^x \sin y$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = f_{yy}(x,y) = -e^x \cos y.$$

Similarly $\quad f_{xxx}(x,y) = e^x \cos y; \ f_{xxy}(x,y) = -\,e^x \sin y;$

$$f_{xyy}(x,y) = -\,e^x \cos y; \ f_{yyy}(x,y) = e^x \sin y$$

$\therefore \quad f(0,0) = 1; \ f_x(0,0) = 1; \ f_y(0,0) = 0;$

$$f_{xx}(0,0) = 1; \ f_{xy}(0,0) = 0; \ f_{yy}(0,0) = -1;$$

$$f_{xxx}(0,0) = 1; \ f_{xxy}(0,0) = 0; \ f_{xyy}(0,0) = -1; \ f_{yyy}(0,0) = 0$$

Taylor's series of $f(x, y)$ in powers of $x$ and $y$ is

$$f(x,y) = f(0,0) + \frac{1}{1!}\{xf_x(0,0) + yf_y(0,0)\} +$$
$$\frac{1}{2!}\{x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\} +$$
$$\frac{1}{3!}\{x^3 f_{xxx}(0,0) + 3x^2 yf_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\} + \cdots$$

$$\therefore \ e^x \cos y = 1 + \frac{1}{1!}\{x \cdot 1 + y \cdot 0\} + \frac{1}{2!}\{x^2 \cdot 1 + 2xy \cdot 0 + y^2(-1)\}$$
$$+ \frac{1}{3!}\{x^3 \cdot 1 + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\} + \cdots$$
$$= 1 + \frac{x}{1!} + \frac{1}{2!} + (x^2 - y^2) + \frac{1}{3!}(x^3 - 3xy^2) + \cdots$$

## 2.4.1 Verification

$e^x \cos y =$ Real part of $e^{x+iy}$

$$= \text{R.P. of}\left[1 + \frac{x+iy}{1!} + \frac{(x+iy)^2}{2!} + \frac{(x+iy)^3}{3!} + \cdots\right],$$

by exponential theorem

$$= 1 + \frac{x}{1!} + \frac{1}{2!}(x^2 - y^2) + \frac{1}{3!}(x^3 - 3y^2) + \cdots$$

**Example 2.2** Expand $\dfrac{(x+h)(y+k)}{x+h+y+k}$ in a series of powers of $h$ and $k$ upto the second degree terms.

Let $$f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$$

$\therefore$ $$f(x, y) = \frac{xy}{x+y}.$$

Taylor's series of $f(x + h, y + k)$ in powers of $h$ and $k$ is

$$f(x+h, y+k) = f(x, y) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots \quad (1)$$

Now $$f_x = y\left\{\frac{(x+y)-x}{(x+y)^2}\right\} = \frac{y^2}{(x+y)^2}$$

$$f_y = x\left\{\frac{(x+y)-y}{(x+y)^2}\right\} = \frac{x^2}{(x+y)^2}$$

$$f_{xx} = -\frac{2y^2}{(x+y)^3}; \; f_{xy} = \frac{(x+y)^2 \cdot 2y - y^2 \cdot 2(x+y)}{(x+y)^4}$$

$$= \frac{2\{y(x+y) - y^2\}}{(x+y)^3} = \frac{2xy}{(x+y)^3}$$

$$f_{yy} = -\frac{2x^2}{(x+y)^3}.$$

Using these values in (1), we have

$$\frac{(x+h)(y+k)}{x+h+y+k} = \frac{xy}{x+y} + \frac{hy^2}{(x+y)^2} + \frac{kx^2}{(x+y)^2} - \frac{h^2y^2}{(x+y)^3} + \frac{2hkxy}{(x+y)^3} - \frac{k^2x^2}{(x+y)^3} + \cdots$$

**Example 2.3** Find the Taylor's series expansion of $x^y$ near the point (1, 1) upto the second degree terms.

Taylor's series of $f(x, y)$ near the point (1, 1) is $f(x,y) = f(1,1)+\frac{1}{1!}$

$$\{(x-1)f_x(1,1)+(y-1)f_y(1,1)\}+\frac{1}{2!}\{(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)f_{xy}(1,1)$$

$$+(y-1)^2 f_{yy}(1,1)\}+\cdots \tag{1}$$

$$f(x,y) = x^y;\ f_x(x,y) = yx^{y-1};\ f_y(x,y) = x^y \log x;$$

$$f_{xx}(x,y) = y(y-1)x^{y-2};\ f_{xy}(x,y) = x^{y-1}+yx^{y-1}\log x$$

$$f_{yy}(x,y) = x^y \cdot (\log x)^2 \cdot$$

$$f_{xx}(1,1) = 1;\ f_x(1,1) = 1;\ f_y(1,1) = 0;$$

$$f_{xx}(1,1) = 0;\ f_{xy}(1,1) = 1;\ f_{yy}(1,1) = 0$$

Using these values in (1), we get

$$x^y = 1+(x-1)+(x-1)(y-1)+\cdots$$

**Example 2.4** Find the Taylor's series expansion of $e^x \sin y$ near the point $\left(-1, \frac{\pi}{4}\right)$ upto the third degree terms.

Taylor's series of $f(x, y)$ near the point $\left(-1,\frac{\pi}{4}\right)$ is

$$f(x,y) = f\left(-1,\frac{\pi}{4}\right)+\frac{1}{1!}\left\{(x+1)f_x\left(-1,\frac{\pi}{4}\right)+\left(y-\frac{\pi}{4}\right)f_y\left(-1,\frac{\pi}{4}\right)\right\}$$

$$+\frac{1}{2!}\left\{(x+1)^2 f_{xx}\left(-1,\frac{\pi}{4}\right)+2(x+1)\left(y-\frac{\pi}{4}\right)f_{xy}\left(-1,\frac{\pi}{4}\right)\right.$$

$$\left.+\left(y-\frac{\pi}{4}\right)^2 f_{yy}\left(-1,\frac{\pi}{4}\right)\right\}+\cdots \tag{1}$$

$$f(x,y) = e^x \sin y;\ f_x = e^x \sin y;\ f_y = e^x \cos y;$$

$$f_{xx} = e^x \sin y; f_{xy} = e^x \cos y;\ f_{yy} = -e^x \sin y;$$

$$f_{xxx} = e^x \sin y;\ f_{xxy} = e^x \cos y;\ f_{xyy} = -e^x \sin y;$$

$$f_{yyy} = -e^x \cos y.$$

$$\therefore \quad f\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_x\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_y\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};$$

$$f_{xx}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_{xy}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_{yy}\left(-1,\frac{\pi}{4}\right) = -\frac{1}{e\sqrt{2}};$$

$$f_{xxx}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_{xxy}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_{xyy}\left(-1,\frac{\pi}{4}\right)$$
$$=\frac{1}{e\sqrt{2}};\ f_{yyy}\left(-1,\frac{\pi}{4}\right)=-\frac{1}{e\sqrt{2}}.$$

Using these values in (1), we get

$$e^x\sin y=\frac{1}{e\sqrt{2}}\left[1+\frac{1}{1!}\left\{(x+1)+\left(y-\frac{\pi}{4}\right)\right\}\right.$$
$$+\frac{1}{2!}\left\{(x+1)^2+2(x+1)\left(y-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)^2\right\}$$
$$\left.+\frac{1}{3!}\left\{(x+1)^3+3(x+1)^2\left(y-\frac{\pi}{4}\right)-3(x+1)\left(y-\frac{\pi}{4}\right)^2-\left(y-\frac{\pi}{4}\right)^3\right\}\right]+\cdots$$

**Example 2.5** Find the Taylor's series expansion of $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x + 2)$ and $(y - 1)$ upto the third powers.

Taylor's series of $f(x, y)$ in powers of $(x + 2)$ and $(y - 1)$ or near $(-2, 1)$ is

$$f(x,y)=f(-2,1)+\frac{1}{1!}\{(x+2)f_x(-2,1)+(y-1)f_y(-2,1)\}$$
$$+\frac{1}{2!}\{(x+2)^2f_{xx}(-2,1)+2(x+2)(y-1)f_{xy}(-2,1)$$
$$+(y-1)^2f_{yy}(-2,1)\}+\cdots \qquad (1)$$

$$f(x,y)=x^2y^2+2x^2y+3xy^2 \qquad f(-2,1)=6$$
$$f_x=2xy^2+4xy+3y^2 \qquad f_x(-2,1)=-9$$
$$f_y=2x^2y+2x^2+6xy \qquad f_y(-2,1)=4$$
$$f_{xx}=2y^2+4y \qquad f_{xx}(-2,1)=6$$
$$f_{xy}=4xy+4x+6y \qquad f_{xy}(-2,1)=-10$$
$$f_{yy}=2x^2+6x \qquad f_{yy}(-2,1)=-4$$
$$f_{xxx}=0 \qquad f_{xxx}(-2,1)=0$$
$$f_{xxy}=4y+4 \qquad f_{xxy}(-2,1)=8$$
$$f_{xyy}=4x+6 \qquad f_{xyy}(-2,1)=-2$$
$$f_{yyy}=0 \qquad f_{yyy}(-2,1)=0$$

Using these values in (1), we have

$$\begin{aligned} x^2y^2+2x^2y+3xy^2 &= 6+\frac{1}{1!}\{-9(x+2)+4(y-1)\} \\ &+\frac{1}{2!}\{6(x+2)^2-20(x+2)(y-1)-4(y-1)^2\} \\ &+\frac{1}{3!}\{24(x+2)^2(y-1)-6(x+2)(y-1)^2\}+\cdots \end{aligned}$$

**Example 2.6** Using Taylor's series, verify that

$$\log(1+x+y)=(x+y)-\frac{1}{2}(x+y)^2+\frac{1}{3}(x+y)^3-\cdots$$

The series given in the R.H.S. is a series of powers of $x$ and $y$.

So let us expand $f(x, y) = \log(1 + x + y)$ as a Taylor's series near (0, 0) or Maclaurin's series.

$$f_x=\frac{1}{1+x+y};\ f_y=\frac{1}{1+x+y}$$

$$f_{xx}=-\frac{1}{(1+x+y)^2}=f_{xy}=f_{yy}$$

$$f_{xxx}=\frac{2}{(1+x+y)^3}=f_{xxy}=f_{xyy}=f_{yyy}$$

$$f(0,0)=0;\ f_x(0,0)=f_y(0,0)=1;$$

$$f_{xx}(0,0)=f_{xy}(0,0)=f_{yy}(0,0)=-1;$$

$$f_{xxx}(0,0)=f_{xxy}(0,0)=f_{xyy}(0,0)=f_{yyy}(0,0)=2\,.$$

Maclaurin's series of $f(x, y)$ is given by

$$\begin{aligned} f(x,y) &= f(0,0)+\frac{1}{1!}\{xf_x(0,0)+yf_y(0,0)\} \\ &+\frac{1}{2!}\{x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\}+\cdots \quad (1) \end{aligned}$$

Using the relevant values in (1), we have

$$\begin{aligned} \log(1+x+y) &= (x+y)+\frac{1}{2}\{-x^2-2xy-y^2\}+ \\ &+\frac{1}{6}\{2x^3+6x^2y+6xy^2+2y^3\}+\cdots \end{aligned}$$

$$= (x+y) - \frac{1}{2}(x+y)^2 + \frac{1}{3}(x+y)^3 - \cdots$$

**Example 2.7** If $x = r\cos\theta$, $y = r\sin\theta$, verify that $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$.

$$x = r\cos\theta;\ y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Now $\qquad r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$

$\therefore \qquad 2r\frac{\partial r}{\partial x} = 2x \qquad \Big| \qquad \frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2}$

$\therefore \qquad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \Big| \qquad = -\frac{y}{x^2+y^2} = \frac{-y}{r^2}$

Similarly, $\qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \Big| \qquad$ Similarly $\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$\therefore \qquad \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \times \frac{1}{r} = 1.$$

**Example 2.8** If we transform from three dimensional cartesian co-ordinates $(x, y, z)$ to spherical polar co-ordinates $(r, \theta, \phi)$, show that the Jacobian of $x, y, z$ with respect to $r, \theta, \phi$ is $r^2 \sin\theta$.

The transformation equations are

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta.$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi,\ \frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\ \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi,\ \frac{\partial z}{\partial \phi} = 0.$$

Now $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= r^2[\sin\theta\cos\phi\left(0+\sin^2\theta\cos\phi\right)-\sin\theta\sin\phi$$

$$\times(0-\sin^2\theta\sin\phi)+\cos\theta(\sin\theta\cos\theta\cos^2\phi+\sin\theta\cos\theta\sin^2\phi)]$$

$$= r^2[\sin^3\theta\cos^2\phi+\sin^3\theta\sin^2\phi+\sin\theta\cos^2\theta]$$

$$= r^2(\sin^3\theta+\sin\theta\cos^2\theta)$$

$$= r^2\sin\theta.$$

**Example 2.9** If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$ and $y = r\sin\theta$, compute $\frac{\partial(u,v)}{\partial(r,\theta)}$.

By the property of Jacobians,

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \times \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= -4(y^2 + x^2) \times r(\cos^2\theta + \sin^2\theta)$$
$$= -4r^3.$$

**Example 2.10** Find the Jacobian of $y_1, y_2, y_3$ with respect to $x_1, x_2, x_3$, if

$$y_1 = \frac{x_2 x_3}{x_1},\ y_2 = \frac{x_3 x_1}{x_2},\ y_3 = \frac{x_1 x_2}{x_3}$$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_1 x_2 \\ x_2 x_3 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_3 x_1 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = 4$$

**Example 2.11** Express $\iiint \sqrt{xyz(1-x-y-z)}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in terms of $u$, $\upsilon$, $w$ given that $x + y + z = u$, $y + z = u\upsilon$ and $z = u\upsilon w$.

The given transformations are

$$x + y + z = u \tag{1}$$

$$y + z = uv \tag{2}$$

and

$$z = uvw \tag{3}$$

Using (3) in (2), we have $y = uv(1 - w)$
Using (2) in (1), we have $x = u(1 - v)$

$$dx\, dy\, dz = |J|\, du\, dv\, dw, \text{ where}$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & wu & uv \end{vmatrix}$$

$$= (1-v)\{u^2 v\,(1-w) + u^2 vw\} + u\{uv^2\ (1-w) + uv^2 w\}$$

$$= u^2 v\,(1-v) + u^2 v^2$$

$$= u^2 v$$

$$\therefore \qquad dx\,dy\,dz = u^2 v\, du\, dv\, dw \tag{4}$$

Using (1), (2), (3) and (4) in the given triple integral $I$, we have

$$I = \iiint \sqrt{u^3 v^2\, w(1-v)(1-w)(1-u)}\ u^2 v\, du\, dv\, dw$$

$$= \iiint u^{7/2}\, v^2\, w^{1/2}\, (1-u)^{\frac{1}{2}}\, (1-v)^{\frac{1}{2}}\, (1-w)^{\frac{1}{2}}\, du\, dv\, dw$$

**Example 2.12** Examine if the following functions are functionally dependent. If they are, find also the functional relationship.

(i) $u = \sin^{-1} x + \sin^{-1} y;\ v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

(ii) $u = y + z;\ v = x + 2z^2;\ w = x - 4yz - 2y^2$

(i) $u = \sin^{-1} x + \sin^{-1} y;\ v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$$\frac{\partial u}{\partial x} = \left(\frac{1}{\sqrt{1-x^2}}\right); \frac{\partial u}{\partial y} = \left(\frac{1}{\sqrt{1-y^2}}\right); \frac{\partial \upsilon}{\partial x} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}};$$

$$\frac{\partial \upsilon}{\partial y} = -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

Now $$\frac{\partial(u,\upsilon)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= \frac{-xy}{\sqrt{(1-x^2)(1-y^2)}} + 1 - 1 + \frac{xy}{\sqrt{(1-x^2)(1-y^2)}}$$

$$= 0.$$

$\therefore$ $u$ and $\upsilon$ are functionally dependent by property (3).

Now $$\sin u = \sin\,(\sin^{-1} x + \sin^{-1} y)$$

$$= \sin\,(\sin^{-1} x)\cos\,(\sin^{-1} y) + \cos\,(\sin^{-1} x)\sin\,(\sin^{-1} y)$$

$$= x \cdot \cos\left\{\cos^{-1}(\sqrt{1-y^2})\right\} + \cos\left\{\cos^{-1}(\sqrt{1-x^2})\right\} \cdot y$$

$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$= \upsilon.$$

$\therefore$ The functional relationship between $u$ and $\upsilon$ is $\upsilon = \sin u$.

(ii) $u = y + z;\ \upsilon = x + 2z^2;\ w = x - 4yz - 2y^2$

$$\frac{\partial u}{\partial x} = 0; \frac{\partial \upsilon}{\partial x} = 1; \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 1; \frac{\partial \upsilon}{\partial y} = 0; \frac{\partial w}{\partial y} = -4y - 4z$$

$$\frac{\partial u}{\partial z} = 1; \frac{\partial \upsilon}{\partial z} = 4z; \frac{\partial w}{\partial z} = -4y$$

Now $$\frac{\partial(u,\upsilon,w)}{\partial(x,y,z)} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4y-4z & -4y \end{vmatrix}$$

$$= -\{-4y + 4y + 4z\} + 4z = 0.$$

$\therefore$ $u$, $\upsilon$ and $w$ are functionally dependent

Now $$\upsilon - w = 2z^2 + 4yz + 2y^2$$
$$= 2(y+z)^2 = 2u^2$$

$\therefore$ The functional relationship among $u$, $\upsilon$ and $w$ is $2u^2 = \upsilon - w$.

**Example 2.13** Given that $\int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}} (a > b)$, find

$$\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} \text{ and } \int_0^{\pi} \frac{\cos x\, dx}{(a+b\cos x)^2}$$

$$\int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}} \tag{1}$$

Differentiating both sides of (1) with respect to $a$, we get

$$\int_0^{\pi} \frac{\partial}{\partial a}\left(\frac{1}{a + b\cos x}\right) dx = \frac{\partial}{\partial a} \cdot \frac{\pi}{\sqrt{a^2 - b^2}},$$ since the limits of integration are constants

i.e., $$\int_0^{\pi} \frac{-dx}{(a+b\cos x)^2} = \pi \times -1/2\left(a^2 - b^2\right)^{-3/2} 2a$$

i.e., $$\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{\left(a^2 - b^2\right)^{3/2}}$$

Differentiating both sides of (1) with respect to $b$, we get

$$\int_0^{\pi} \frac{\partial}{\partial b}\left(\frac{1}{a + b\cos x}\right) dx = \frac{\partial}{\partial b} \cdot \frac{\pi}{\sqrt{a^2 - b^2}}$$

i.e., $$\int_0^{\pi} -\frac{1}{(a+b\cos x)^2} \times \cos x\, dx = \pi \times -1/2\left(a^2 - b^2\right)^{-3/2} (-2b)$$

i.e., $$\int_0^{\pi} \frac{\cos x}{(a+b\cos x)^2} dx = -\frac{\pi b}{\left(a^2 - b^2\right)^{3/2}}.$$

**Example 2.14** By differentiating inside the integral, find the value of $\int_0^x \frac{\log(1+xy)}{1+y^2}\,dy$. Hence find the value of $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$.

Let

$$f(x) = \int_0^x \frac{\log(1+xy)}{1+y^2}\,dy \tag{1}$$

Differentiating both sides of (1) with respect to $x$, we have

$$f'(x) = \frac{d}{dx}\int_0^x \frac{\log(1+xy)}{1+y^2}\,dx$$

$$= \int_0^x \frac{\partial}{\partial x}\left\{\frac{\log(1+xy)}{1+y^2}\right\}dx + \frac{\log(1+x^2)}{1+x^2}\cdot\frac{d(x)}{dx}$$

(by Leibnitz's rule)

$$= \int_0^x \frac{y}{(1+xy)(1+y^2)}\,dy + \frac{\log(1+x^2)}{1+x^2}$$

$$= \int_0^x \left[\frac{-x}{(1+x^2)(1+xy)} + \frac{1}{1+x^2}\left(\frac{y+x}{1+y^2}\right)\right]dy + \frac{\log(1+x^2)}{1+x^2},$$

by resolving the integrand in the first term into partial fractions

$$= \left[-\frac{1}{1+x^2}\log(1+xy) + \frac{1}{2}\cdot\frac{1}{1+x^2}\log(1+y^2) + \frac{x}{1+x^2}\tan^{-1}y\right]_0^x + \frac{\log(1+x^2)}{1+x^2}$$

$$= \frac{1}{2}\cdot\frac{1}{1+x^2}\log(1+x^2) + \frac{x}{1+x^2}\tan^{-1}x \tag{2}$$

Integrating both sides of (2) with respect to $x$, we have

$$f(x) = \frac{1}{2}\int \log(1+x^2)\,d(\tan^{-1}x) + \int \frac{x}{1+x^2}\tan^{-1}x\,dx + c$$

$$= \frac{1}{2}\left[\tan^{-1}x\log(1+x^2) - \int \tan^{-1}x\cdot\frac{2x}{1+x^2}\,dx\right]$$

$$+ \int \frac{x\tan^{-1}x}{1+x^2}\,dx + c$$

$$= \frac{1}{2}\tan^{-1} x \cdot \log\left(1+x^2\right)+c \qquad (2)$$

Now putting $x = 0$ in (2), we get

$$c = f(0) = 0, \text{ by } (1)$$

$$\therefore \qquad f(x) = \int_0^x \frac{\log(1+xy)}{1+y^2}\,dy = \frac{1}{2}\tan^{-1} x \cdot \log\left(1+x^2\right) \qquad (3)$$

Putting $x = 1$ in (3), we get

$$\int_0^1 \frac{\log(1+y)}{1+y^2}\,dy = \frac{1}{2}\tan^{-1}(1) \cdot \log 2$$

$$= \frac{\pi}{8}\log 2$$

Since $y$ is only a dummy variable,

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$$

**Example 2.15** Show that $\dfrac{d}{da}\displaystyle\int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right)dx = 2a\tan^{-1}(a) - \frac{1}{2}\log\left(a^2+1\right)$.

$$\frac{d}{da}\int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right)dx = \int_0^{a^2} \frac{\partial}{\partial a}\tan^{-1}\left(\frac{x}{a}\right)dx + \tan^{-1}\left(\frac{a^2}{a}\right)\cdot\frac{d}{da}\left(a^2\right),$$

by Leibnitz's rule

$$= \int_0^{a^2} \frac{1}{1+\frac{x^2}{a^2}}\cdot\left(\frac{-x}{a^2}\right)dx + 2a\tan^{-1} a$$

$$= -\int_0^{a^2} \frac{x}{x^2+a^2}\,dx + 2a\tan^{-1} a$$

$$= -\frac{1}{2}\left[\log\left(x^2+a^2\right)\right]_0^{a^2} + 2a\tan^{-1} a$$

$$= -\frac{1}{2}\log\left(\frac{a^4 + a^2}{a^2}\right) + 2a\ \tan^{-1} a$$

$$= 2a \tan^{-1} a - \frac{1}{2}\log\left(a^2 + 1\right)$$

**Example 2.16** If $I = \int_0^\infty e^{-x^2 - \left(\frac{a}{x}\right)^2} \mathrm{d}x$, prove that $\frac{\mathrm{d}I}{\mathrm{d}a} = -2I$. Hence find the value of $I$.

$$I = \int_0^\infty e^{-x^2 - \left(\frac{a}{x}\right)^2}\ \mathrm{d}x \tag{1}$$

Differentiating both sides of (1) with respect to $a$, we have

$$\frac{\mathrm{d}I}{\mathrm{d}a} = \int_0^\infty \frac{\partial}{\partial a}\left\{e^{-x^2 - \frac{a^2}{x^2}}\right\}\mathrm{d}x$$

$$= \int_0^\infty e^{-x^2 - \frac{a^2}{x^2}} \cdot \left(-\frac{2a}{x^2}\right)\mathrm{d}x$$

$$= \int_\infty^0 e^{-\frac{a^2}{y^2} - y^2}\ 2\,\mathrm{d}y, \text{ on putting } x = \frac{a}{y} \text{ or } y = \frac{a}{x}$$

$$= -2\int_0^\infty e^{-y^2 - \left(\frac{a}{y}\right)^2}\ \mathrm{d}y$$

i.e.,
$$\frac{\mathrm{d}I}{\mathrm{d}a} = -2I \tag{2}$$

$\therefore$
$$\frac{\mathrm{d}I}{I} = -2\,\mathrm{d}a$$

Solving, we get $\log I = \log c - 2a$

$\therefore$
$$I = ce^{-2a} \tag{3}$$

When
$$a = 0,\ I = \int_0^\infty e^{-x^2}\ \mathrm{d}x = \frac{\sqrt{\pi}}{2} \tag{4}$$

Using (4) in (3), we get $c = \frac{\sqrt{\pi}}{2}$

Hence $$I = \frac{\sqrt{\pi}}{2} e^{-2a}.$$

## EXERCISE 2(b)

***Part A***

(Short Answer Question )

1. Write down the Taylor's series expansion of $f(x + h, y + k)$ in a series of (i) powers of $h$ and $k$ (ii) power of $x$ and $y$.
2. Write down the Maclaurin's series expansion of (i) $f(x, y)$, (ii) $f(x + h, y + k)$.
3. Write down the Taylor's series expansion of $f(x, y)$ near the point $(a, b)$.
4. Write down the Maclaurin's series for $e^{x+y}$.
5. Write down the Maclaurin's series for $\sin(x + y)$.
6. Define Jacobian.
7. State any three properties of Jacobians.
8. State the condition for the functional dependence of three functions $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$.
9. Prove that $\iint f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint f(r\cos\theta, r\sin\theta)\cdot r\,\mathrm{d}r\,\mathrm{d}\theta$.
10. Show that $\iint f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint f\{u(1-v), uv\}\cdot u\,\mathrm{d}u\,\mathrm{d}v$.
11. If $x = u(1 + v)$ and $y = v(1 + u)$, find the Jacobian of $x, y$ with respect to $u, v$.
12. State the Leibnitz's rule for differentiation under integral sign, when both the limits of integration are variables.
13. Write down the Leibnitz's formula for $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{b(x)} f(x, y)\,\mathrm{d}y$, where $a$ is a constant.
14. Write down the Leibnitz's formula for $\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b} f(x, y)\,\mathrm{d}y$, where $b$ is a constant.
15. Evaluate $\frac{\mathrm{d}}{\mathrm{d}y}\int_0^1 \log(x^2 + y^2)\,\mathrm{d}x$, without integrating the given function.

***Part B***

16. Expand $e^x \sin y$ in a series of powers of $x$ and $y$ as far as the terms of the third degree.
17. Find the Taylor's series expansion of $e^x \cos y$ in the neighbourhood of the point $\left(1, \frac{\pi}{4}\right)$ upto the second degree terms.

18. Find the Maclaurin's series expansion of $e^x \log(1+y)$ upto the terms of the third degree.
19. Find the Taylor's series expansion of $\tan^{-1}\left(\frac{y}{x}\right)$ in powers of $(x-1)$ and $(y-1)$ upto the second degree terms.
20. Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ upto the third degree terms.
21. Expand $xy^2 + 2x - 3y$ in powers of $(x+2)$ and $(y-1)$ upto the third degree terms.
22. Find the Taylor's series expansion of $y^x$ at (1, 1) upto the second degree terms.
23. Find the Taylor's series expansion of $e^{xy}$ at (1, 1) upto the third degree terms.
24. Using Taylor's series, verify that

$$\cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \cdots \infty$$

25. Using Taylor's series, verify that

$$\tan^{-1}(x+y) = (x+y) + \frac{1}{3}(x+y)^3 \cdots \infty$$

26. If $x = u(1-v)$, $y = uv$, verify that

$$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$$

27. (i) if $x = u^2 - v^2$ and $y = 2uv$, find the Jacobian of $x$ and $y$ with respect to $u$ and $v$.
    (ii) if $u = x^2$ and $v = y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$
28. If $x = a\cosh u \cos v$ and $y = a \sinh u \cdot \sin v$, show that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2}{2}(\cosh 2u - \cos 2v).$$

29. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$
30. If $F = xu + v - y$, $G = u^2 + vy + w$ and $H = zu - v + vw$, compute $\frac{\partial(F,G,H)}{\partial(u,v,w)}$.
31. If $u = xyz$, $v = xy + yz + zx$ and $w = x + y + z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.
32. Examine the functional dependence of the functions $u = \frac{x+y}{x-y}$ and $v = \frac{xy}{(x-y)^2}$. If they are dependent, find the relation between them.

33. Are the functions $u = \dfrac{x+y}{1-xy}$ and $\upsilon = \tan^{-1} x + \tan^{-1} y$ functionally dependent? If so, find the relation between them.
34. Are the functions $f_1 = x + y + z$, $f_2 = x^2 + y^2 + z^2$ and $f_3 = xy + yz + zx$ functionally dependent? If so, find the relation among $f_1$, $f_2$ and $f_3$.
35. If $\int_0^x \lambda e^{-\lambda(x-y)} f(y)\mathrm{d}y = \lambda^2 \cdot xe^{-\lambda x}$ prove that $f(x) = \lambda e^{-\lambda x}$. [**Hint:** Differentiate both sides with respect to $x$].

Use the concept of differentiation under integral sign to evaluate the following:

36. $\int_0^x \dfrac{\mathrm{d}x}{(x^2+a^2)^2}$ $\left[\textbf{Hint:} \text{ Use} \int_0^x \dfrac{\mathrm{d}x}{x^2+a^2}\right.$
37. $\int_0^1 x^m (\log x)^n \,\mathrm{d}x$ $\left[\textbf{Hint:} \text{ Use} \int_0^1 x^m \,\mathrm{d}x\right.$
38. $\int_0^\infty e^{-x^2} \cos 2ax \,\mathrm{d}x$
39. $\int_0^\infty \dfrac{e^{-ax}\sin x}{x}\mathrm{d}x$ and hence $\int_0^\infty \dfrac{\sin x}{x}\mathrm{d}x$
40. $\int_0^1 \dfrac{x^m - 1}{\log x}\mathrm{d}x,\ m \ge 0.$

## 2.5 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Students are familiar with the concept of maxima and minima of a function of one variable. Now we shall consider the maxima and minima of a function of two variables.

A function $f(x, y)$ is said to have a *relative maximum* (or simply maximum) at $x = a$ and $y = b$, if $f(a, b) > f(a + h, b + k)$ for all small values of $h$ and $k$.

A function $f(x, y)$ is said to have a *relative maximum* (or simply maximum) at $x = a$ and $y = b$, if $f(a, b) < f(\mathrm{a} + h, b + k)$ for all small values of $h$ and $k$.

A maximum or a minimum value of a function is called its *extreme value*. We give below *the working rule to find the extreme values of a function* $f(x, y)$:

(1) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(2) Solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously. Let the solutions be $(a, b); (c, d);\ldots$

**Note** ☑ The points like $(a, b)$ at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are called *stationary points* of the function $f(x, y)$. The values of $f(x, y)$ at the stationary points are called *stationary values* of $f(x, y)$.

(3) For each solution in step (2), find the values of $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial x \, dy}$, $C = \frac{\partial^2 f}{\partial y^2}$ and $\Delta = AC - B^2$.

(4) (i) If $\Delta > 0$ and $A$ (or $C$) $< 0$ for the solution $(a, b)$ then $f(x, y)$ has a maximum value at $(a, b)$.

(ii) If $\Delta > 0$ and $A$ (or $C$) $> 0$ for the solution $(a, b)$ then $f(x, y)$ has a minimum value at $(a, b)$.

(iii) If $\Delta < 0$ for the solution $(a, b)$, then $f(x, y)$ has neither a maximum nor a minimum value at $(a, b)$. In this case, the point $(a, b)$ is called *a saddle point* of the function $f(x, y)$.

(iv) If $\Delta = 0$ or $A = 0$, the case is doubtful and further investigations are required to decide the nature of the extreme values of the function $f(x, y)$.

## 2.5.1 Constrained Maxima and Minima

Sometimes we may require to find the extreme values of a function of three (or more) variables say $f(x, y, z)$ which are not independent but are connected by some given relation $\phi/(x, y, z) = 0$. The extreme values of $f(x, y, z)$ in such a situation are called *constrained extreme values.*

In such situations, we use $\phi/(x, y, z) = 0$ to eliminate one of the variables, say $z$ from the given function, thus converting the function as a function of only two variables and then find the unconstrained extreme values of the converted function. [Refer to examples (2.8), (2.9), (2.10)].

When this procedure is not practicable, we use Lagrange's method, which is comparatively simpler.

## 2.5.2 Lagrange's Method of Undetermined Multipliers

Let
$$u = f(x, y, z) \tag{1}$$
be the function whose extreme values are required to be found subject to the constraint

$$\phi(x, y, z) = 0 \tag{2}$$

The necessary conditions for the extreme values of $u$ are $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$

$$\therefore \qquad \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z = 0 \qquad (3)$$

From (2), we have

$$\frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z = 0 \qquad (4)$$

Now (3) + λ × (4), where $\lambda$ is an unknown multiplier, called *Langrange multiplier*, gives

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)\mathrm{d}y + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right) = 0 \qquad (5)$$

Equation (5) holds good, if

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \qquad (6)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \qquad (7)$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \qquad (8)$$

Solving the Equations (2), (6), (7) and (8), we get the values of $x$, $y$, $z$, $\lambda$, which give the extreme values of $u$.

*Note* ☑

(1) The Equations (2), (6), (7) and (8) are simply the necessary conditions for the extremum of the auxiliary function ($f + \lambda\phi$), where $\lambda$ is also treated as a variable.

(2) Lagrange's method does not specify whether the extreme value found out is a maximum value or a minimum value. It is decided from the physical consideration of the problem.

## WORKED EXAMPLE 2(c)

**Example 2.1** Examine $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ for extreme values.

$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$$f_x = 3x^2 + 3y^2 - 30x + 72$$

$$f_y = 6xy - 30y$$

$$f_{xx} = 6x - 30;\ f_{xy} = 6y;\ f_y = 6x - 30$$

The stationary points are given by $f_x = 0$ and $f_y = 0$

i.e., $$3(x^2 + y^2 - 10x + 24) = 0 \quad (1)$$

and $$6y(x-5) = 0 \quad (2)$$

From (2), $x = 5$ or $y = 0$
When $x = 5$, from (1), we get $y^2 - 1 = 0$; $\therefore\ y = \pm 1$
When $y = 0$, from (1), we get $x^2 - 10x + 24 = 0$

$$\therefore \quad x = 4, 6.$$

The stationary points are (5, 1), (5, −1), (4, 0) and (6, 0)
At the point (5, ±1), $A = f_{xx} = 0$; $B = f_{xy} = \pm 6$; $C = f_{yy} = 0$
Though $AC - B^2 < 0$, $A = 0$
$\therefore$ Nothing can be said about the maxima or minima of $f(x, y)$ at (5, ±1).
At the point (4, 0), $A = -6$, $B = 0$, $C = -6$

$$\therefore \quad AC - B^2 = 36 > 0 \quad \text{and} \quad A < 0$$

$\therefore f(x, y)$ is maximum at (4, 0) and maximum value of $f(x, y) = 112$.
At point (6, 0), $A = 6$, $B = 0$, $C = 6$

$$\therefore \quad AC - B^2 = 36 > 0 \text{ and } A > 0.$$

$\therefore f(x, y)$ is minimum at (6, 0) and the minimum value of $f(x, y) = 108$.

**Example 2.2** Examine the function $f(x, y) = x^3y^2(12 - x - y)$ for extreme values.

$$f(x,y) = 12x^3y^2 - x^4y^2 - x^3y^3$$
$$f_x = 36x^2y^2 - 4x^3y^2 - 3x^2y^3$$
$$f_y = 24x^3y - 2x^4y - 3x^3y^2$$
$$f_{xx} = 72xy^2 - 12x^2y^2 - 6xy^3$$
$$f_{xy} = 72x^2y - 8x^3y - 9x^2y^2$$
$$f_{yy} = 24x^3 - 2x^4 - 6x^3y$$

The stationary points are given by $f_x = 0$; $f_y = 0$

i.e., $$x^2y^2(36 - 4x - 3y) = 0 \quad (1)$$

and $$x^3y(24 - 2x - 3y) = 0 \quad (2)$$

Solving (1) and (2), the stationary points are (0, 0), (0, 8), (0, 12), (12, 0), (9, 0) and (6, 4).

At the first five points, $AC - B^2 = 0$.

$\therefore$ Further investigation is required to investigate the extremum at these points. At the point (6, 4), $A = -2304$, $B = -1728$, $C = -2592$ and $AC - B^2 > 0$.

Since $AC - B^2 > 0$ and $A < 0$, $f(x, y)$ has a maximum at the point (6, 4).

Maximum value of $f(x, y) = 6912$.

**Example 2.3** Discuss the maxima and minima of the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$$

$$f_x = 4(x^3 - x + y)$$

$$f_y = 4(y^3 + x - y)$$

$$f_{xx} = 4(3x^2 - 1);\ f_{xy} = 4;\ f_{yy} = 4(3y^2 - 1)$$

The possible extreme points are given by

$$f_x = 0 \text{ and } f_y = 0$$

i.e.,
$$x^3 - x + y = 0 \tag{1}$$

and
$$y^3 + x - y = 0 \tag{2}$$

Adding (1) and (2),
$$x^3 + y^3 = 0 \therefore y = -x \tag{3}$$

Using (3) in (1):
$$x^3 - 2x = 0$$

i.e.,
$$x(x^2 - 2) = 0 \therefore x = 0, +\sqrt{2}, -\sqrt{2}$$

and the corresponding values of $y$ are 0, $-\sqrt{2}, +\sqrt{2}$.

$\therefore$ The possible extreme points of $f(x, y)$ are (0, 0), $(+\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

At the point (0, 0), $A = -4$, $B = 4$ and $C = -4$

$$AC - B^2 = 0$$

$\therefore$ The nature of $f(x, y)$ is undecided at (0, 0). At the points $(\pm\sqrt{2}, \mp\sqrt{2})$, $A = 20$, $B = 4$, $C = 20$

$$AC - B^2 > 0$$

$\therefore f(x, y)$ is minimum at the points $(\pm\sqrt{2}, \mp\sqrt{2})$, and minimum value of $f(x, y) = 8$.

**Example 2.4** Examine the extrema of $f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$.

$$f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$$

$$f_x = 2x + y - \frac{1}{x^2}$$

$$f_y = x + 2y - \frac{1}{y^2}$$

$$f_{xx} = 2+\frac{2}{x^3};\ f_{xy} = 1;\ f_{yy} = 2+\frac{2}{y^3}$$

The possible extreme points are given by $f_x = 0$ and $f_y = 0$.

i.e., $$2x+y-\frac{1}{x^2} = 0 \tag{1}$$

and $$x+2y-\frac{1}{y^2} = 0 \tag{2}$$

(1) – (2) gives $$x-y+\frac{1}{y^2}-\frac{1}{x^2} = 0$$

i.e., $$x-y+\frac{x^2-y^2}{x^2y^2} = 0$$

i.e., $$(x-y)(x^2y^2+x+y) = 0$$

$\therefore$ $$x = y \tag{3}$$

Using (3) in (1), $3x^3 - 1 = 0$

$\therefore$ $$x = \left(\frac{1}{3}\right)^{\frac{1}{3}} = y$$

At the point $\left\{\left(\frac{1}{3}\right)^{\frac{1}{3}}, \left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}$, $A = 8$, $B = 1$ and $C = 8$

$\therefore$ $$AC - B^2 > 0$$

$\therefore f(x, y)$ is minimum at $\left\{\left(\frac{1}{3}\right)^{\frac{1}{3}}, \left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}$ and minimum value of $f(x, y) = 3^{\frac{4}{3}}$.

**Example 2.5** Discuss the extrema of the function $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^4$ at the origin

$$f(x,y) = x^2-2xy+y^2+x^3-y^3+x^4.$$

$$f_x = 2x-2y+3x^2+4x^3$$

$$f_y = -2x+2y-3y^2$$

$$f_{xx} = 2+6x+12x^2$$

$$f_{xy} = -2;\quad f_{yy} = 2-6y$$

The origin (0, 0) satisfies the equations $f_x = 0$ and $f_y = 0$.
$\therefore$ (0, 0) is a stationary point of $f(x, y)$.
At the origin, $A = 2$, $B = -2$ and $C = 2$

$$\therefore \qquad AC - B^2 = 0$$

Hence further investigation is required to find the nature of the extrema of $f(x, y)$ at the origin.

Let us consider the values of $f(x, y)$ at three points close to (0, 0), namely at $(h, 0)$, $(0, k)$ and $(h, h)$ which lie on the $x$-axis, the $y$-axis and the line $y = x$ respectively.

$$f(h,0) = h^2 + h^3 + h^4 > 0.$$
$$f(0,k) = k^2 - k^3 = k^2(1-k) > 0, \text{ when } 0 < k < 1$$
$$f(h,h) = h^4 > 0$$

Thus $f(x, y) > f(0, 0)$ in the neighbourhood of (0, 0).
$\therefore$ (0, 0) is a minimum point of $f(x, y)$ and minimum value of $f(x, y) = 0$.

**Example 2.6** Find the maximum and minimum values of

$$f(x,y) = \sin x \sin y \sin(x+y); 0 < x, y < \pi.$$

$$f(x,y) = \sin x \sin y \sin(x+y)$$
$$f_x = \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y)$$
$$f_y = \sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y)$$

i.e., $$f_x = \sin y \sin(2x+y)$$

and $$f_y = \sin x \cdot \sin(x+2y)$$
$$f_{xx} = 2\sin y \cos(2x+y)$$
$$f_{xy} = \sin y \cos(2x+y) + \cos y \cdot \sin(2x+y)$$
$$= \sin(2x+2y)$$
$$f_{yy} = 2\sin x \cos(x+2y)$$

For maximum or minimum values of $f(x, y)$, $f_x = 0$ and $f_y = 0$
i.e., $\sin y \sin(2x + y) = 0$ and $\sin x \cdot \sin(x + 2y) = 0$

i.e., $\frac{1}{2}[\cos 2x - \cos(2x+2y)] = 0$ and $\frac{1}{2}[\cos 2y - \cos(2x+2y)] = 0$

i.e., $$\cos 2x - \cos(2x+2y) = 0 \qquad (1)$$

and $$\cos 2y - \cos(2x+2y) = 0 \qquad (2)$$

From (1) and (2), $\cos 2x = \cos 2y$. Hence $x = y$ (3)
Using (3) in (1), $\cos 2x - \cos 4x = 0$
i.e., $2 \sin x \sin 3x = 0$
$\therefore \sin x = 0$ or $\sin 3x = 0$
$\therefore x = 0, \pi$ and $3x = 0, \pi, 2\pi$ i.e., $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore$ The admissible values of $x$ are 0, $\frac{\pi}{3}, \frac{2\pi}{3}$.

Thus the maxima and minima of $f(x, y)$ are given by (0, 0) $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$

At the point (0, 0), $A = B = C = 0$

$\therefore \quad AC - B^2 = 0$

Thus the extremum of $f(x, y)$ at (0, 0) is undecided.

At the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $A = -\sqrt{3}$, $B = -\frac{\sqrt{3}}{2}$ and $C = -\sqrt{3}$ and $AC - B^2 = 3 - \frac{3}{4} > 0\cdot$

As $AC - B^2 > 0$ and $A < 0$, $f(x, y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Maximum value of $f(x, y) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$.

At the point $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$, $A = \sqrt{3}, B = \frac{\sqrt{3}}{2}$ and $C = \sqrt{3}$ and $AC - B^2 = 3 - \frac{3}{4} > 0$.

As $AC - B^2 > 0$ and $A > 0$, $f(x, y)$ is maximum at $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$.

Minimum value of $f(x, y) = -\frac{3\sqrt{3}}{8}$.

**Example 2.7** Identify the saddle point and the extremum points of

$$f(x, y) = x^4 - y^4 - 2x^2 + 2y^2$$

$$f(x, y) = x^4 - y^4 - 2x^2 + 2y^2$$

$$f_x = 4x^3 - 4x; \; f_y = 4y - 4y^3$$

$$f_{xx} = 12x^2 - 4; \; f_{xy} = 0; \quad f_{yy} = 4 - 12y^2$$

The stationary points of $f(x, y)$ are given by $f_x = 0$ and $f_y = 0$

i.e., $\quad 4(x^3 - x) = 0$ and $4(y - y^3) = 0$

i.e., $\quad 4x(x^2 - 1) = 0$ and $4y(1 - y^2) = 0$

$\therefore x = 0$ or $\pm 1$ and $y = 0$ or $\pm 1$.

At the points (0, 0), ($\pm 1$, $\pm 1$), $AC - B^2 < 0$

$\therefore$ The points (0, 0), (1, 1), (1, –1), (–1, 1) and (–1, –1) are saddle points of the function $f(x, y)$.

At the point ($\pm 1$, 0), $AC - B^2 > 0$ and $A > 0$

$\therefore f(x, y)$ attains its minimum at ($\pm 1$, 0) and the minimum value is –1.

At the point (0, $\pm 1$), $AC - B^2 > 0$ and $A < 0$

$\therefore f(x, y)$ attains its maximum at $(0, \pm 1)$ and the maximum value is $+1$.

**Example 2.8** Find the minimum value of $x^2 + y^2 + z^2$, when $x + y + z = 3a$.

Here we try to find the conditional minimum of $x^2 + y^2 + z^2$, subject to the condition

$$x + y + z = 3a \quad (1)$$

Using (1), we first express the given function as a function of $x$ and $y$.

From (1), $z = 3a - x - y$.

Using this in the given function, we get

$$f(x, y) = x^2 + y^2 + (3a - x - y)^2$$

$$f_x = 2x - 2(3a - x - y)$$

$$f_y = 2y - 2(3a - x - y)$$

$$f_{xx} = 4;\ f_{xy} = 2;\ f_{yy} = 4$$

The possible extreme points are given by $f_x = 0$ and $f_y = 0$.

i.e., $$2x + y = 3a \quad (2)$$

and $$x + 2y = 3a \quad (3)$$

Solving (2) and (3), we get the only extreme point as $(a, a)$

At the point $(a, a)$, $AC - B^2 > 0$ and $A > 0$

$\therefore f(x, y)$ is minimum at $(a, a)$ and the minimum value of $f(x, y) = 3a^2$.

**Example 2.9** Show that, if the perimeter of a triangle is constant, its area is maximum when it is equilateral.

Let the sides of the triangle be $a$, $b$, $c$.

Given that $$a + b + c = \text{constant}$$

$$= 2k, \text{ say} \quad (1)$$

Area of the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad (2)$$

where $$s = \frac{a+b+c}{2}$$

Using (1) in (2),

$$A = \sqrt{k(k-a)(k-b)(k-c)} \quad (3)$$

$A$ is a function of three variables $a$, $b$, $c$

Again using (1) in (3), we get

$$A = \sqrt{k(k-a)(k-b)(a+b-k)}$$

$A$ is maximum or minimum, when $f(a, b) = \dfrac{A^2}{k} = (k - a)\ (k - b)(a + b - k)$ is maximum or minimum.

$$f_a = (k-b)\{(k-a)\cdot 1+(a+b-k)\cdot(-1)\}$$
$$= (k-b)(2k-2a-b)$$
$$f_b = (k-a)\{(k-b)\cdot 1+(a+b-k)\cdot(-1)\}$$
$$= (k-a)(2k-a-2b)$$
$$f_{aa} = -2(k-b);\ f_{ab} = -3k+2a+2b;$$
$$f_{ab} = -2(k-a)$$

The possible extreme points of $f(a, b)$ are given by

$f_a = 0$ and $f_b = 0$

i.e., $(k - b)\ (2k - 2a - b) = 0$ and $(k - a)(2k - a - 2b) = 0$

$\therefore\ b = k$ or $2a + b = 2k$ and $a = k$ or $a + 2b = 2k$

Thus the possible extreme points are given by

(i) $a = k$, $b = k$; (ii) $b = k$, $a + 2b = 2k$; (iii) $a = k$, $2a + b = 2k$ and (iv) $2a + b = 2k$, $a + 2b = 2k$.

(i) gives $a = k$, $b = k$ and hence $c = 0$.
(ii) gives $a = 0$, $b = k$ and hence $c = k$.
(iii) gives $a = k$, $b = 0$ and hence $c = k$.

All these lead to meaningless results.

Solving $2a + b = 2k$ and $a + 2b = 2k$, we get

$$a = \frac{2k}{3} \text{ and } b = \frac{2k}{3}$$

At the point $\left(\dfrac{2k}{3}, \dfrac{2k}{3}\right)$,

$$A = f_{aa} = -\frac{2k}{3};\ B = f_{ab} = -\frac{k}{3};\ C = f_{bb} = -\frac{2k}{3}$$

$AC - B^2 > 0$ and $A < 0$

$\therefore f(a, b)$ is maximum at $\left(\dfrac{2k}{3}, \dfrac{2k}{3}\right)$

Hence the area of the triangle is maximum when $a = \dfrac{2k}{3}$ and $b = \dfrac{2k}{3}$.

When $a = \dfrac{2k}{3}$, $b = \dfrac{2k}{3}$; $c = 2k-(a+b) = \dfrac{2k}{3}$

Thus the area of the triangle is maximum, when $a = b = c = \dfrac{2k}{3}$, i.e., when the triangle is equilateral.

**Example 2.10** In a triangle *ABC*, find the maximum value of cos *A* cos *B* cos *C*. In triangle *ABC*, $A + B + C = \pi$.

Using this condition, we express the given function as a function of *A* and *B*

Thus $\cos A \cos B \cos C = \cos A \cos B \cos\{\pi - (A+B)\}$

$$= -\cos A \cos B \cos(A+B)$$

Let
$$f(A,B) = -\cos A \cos B \cos(A+B)$$
$$f_A = -\cos B\{-\sin A \cos(A+B) - \cos A \sin(A+B)\}$$
$$= \cos B \sin(2A+B)$$
$$f_B = -\cos A\{-\sin B \cos(A+B) - \cos B \sin(A+B)\}$$
$$= \cos A \sin(A+2B)$$
$$f_{AA} = 2\cos B \cos(2A+B)$$
$$f_{AB} = \cos B \cos(2A+B) - \sin B \sin(2A+B)$$
$$= \cos(2A+2B)$$
$$f_{BB} = 2\cos A \cos(A+2B)$$

The possible extreme points are given by

$$f_A = 0 \text{ and } f_B = 0$$

i.e.,
$$\cos B \sin(2A+B) = 0 \quad (1)$$

and
$$\cos A \sin(A+2B) = 0$$

Thus the possible values of *A* and *B* are given by (i) $\cos B = 0$, $\cos A = 0$; (ii) $\cos B = 0$, $\sin(A + 2B) = 0$; (iii) $\sin(2A + B) = 0$, $\cos A = 0$ and (iv) $\sin(2A + B) = 0$, $\sin(A + 2B) = 0$

i.e., (i) $A = \frac{\pi}{2}$, $B = \frac{\pi}{2}$; (ii) $B = \frac{\pi}{2}$, $A = 0$ or $\pi$, (iii) $A = \frac{\pi}{2}$, $B = 0$ or $\pi$ and (iv) $2A + B = \pi$, $A + 2B = \pi$ or

$$A = \frac{\pi}{3},\ B = \frac{\pi}{3}$$

The first three sets of values of *A* and *B* lead to meaningless results.

Hence $A = \frac{\pi}{3}$, $B = \frac{\pi}{3}$ give the extreme point.

At this point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $A = f_{AA} = -1$; $B = f_{AB} = -\frac{1}{2}$; $f_{BB} = -1$ and $AC = B^2 > 0$.

Also $A < 0$

$\therefore f(A, B)$ is maximum at $A = B = \frac{\pi}{3}$ and the maximum value

$$= -\cos\frac{\pi}{3} \cdot \cos\cdot\frac{\pi}{3} \cos\frac{2\pi}{3} = \frac{1}{8}.$$

**Example 2.11** Find the maximum value of $x^m y^n z^p$, when $x + y + z = a$.

Let $f = x^m y^n z^p$ and $\phi = x + y + z - a$.

Using the Lagrange multiple $\lambda$, the auxiliary function is $g = (f + \lambda\phi)$.

This stationary points of $g = (f + \lambda\phi)$ are given by $g_x = 0$, $g_y = 0$, $g_z = 0$ and $g_\lambda = 0$

i.e.,
$$mx^{m-1} y^n z^p + \lambda = 0 \quad (1)$$
$$nx^m y^{n-1} z^p + \lambda = 0 \quad (2)$$
$$px^m y^n z^{p-1} + \lambda = 0 \quad (3)$$
$$x + y + z - a = 0 \quad (4)$$

From (1), (2) and (3), we have

$$-\lambda = mx^{m-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}.$$

i.e.,
$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z}$$
$$= \frac{m+n+p}{a}, \text{ by (4)}$$

$\therefore$ Maximum value of $f$ occurs,

when $x = \dfrac{am}{m+n+p}$, $y = \dfrac{an}{m+n+p}$, $z = \dfrac{ap}{m+n+p}$

Thus maximum value of $f = \dfrac{a^{m+n+p} \cdot m^m \cdot n^n \cdot p^p}{(m+n+p)^{m+n+p}}$

**Example 2.12** A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box, that requires the least material for its construction.

Let, $x$, $y$, $z$ be the length, breadth and height of the respectively.

The material for the construction of the box is least, when the area of surface of the box is least.

Hence we have to minimise

$$S = xy + 2yz + 2zx,$$

subject to the condition that the volume of the box, i.e., $xyz = 32$.

Here $f = xy + 2yz + 2zx$; $\phi = xyz - 32$.

The auxiliary function is $g = f + \lambda\phi$, where $\lambda$ is the Lagrange multiplier. The stationary points of g are given by $g_x = 0$, $g_y = 0$, $g_z = 0$ and $g_\lambda = 0$

i.e.,
$$y + 2z + \lambda yz = 0 \quad (1)$$
$$x + 2x + \lambda zx = 0 \quad (2)$$
$$2x + 2y + \lambda xy = 0 \quad (3)$$
$$xyz - 32 = 0 \quad (4)$$

From (1), (2) and (3), we have

$$\frac{1}{z}+\frac{2}{y} = -\lambda \tag{5}$$

$$\frac{1}{z}+\frac{2}{x} = -\lambda \tag{6}$$

$$\frac{2}{y}+\frac{2}{x} = -\lambda \tag{7}$$

Solving (5), (6) and (7), we get

$$x = -\frac{4}{\lambda},\ y = -\frac{4}{\lambda} \text{ and } z = -\frac{2}{\lambda}$$

Using these values in (4), we get

$$-\frac{32}{\lambda^3}-32 = 0$$

i.e., $\lambda = -1$

$\therefore$ $x = 4,\ y = 4,\ z = 2.$

Thus the dimensions of the box and 4 cm; 4 cm and 2 cm.

**Example 2.13** Find the volume of the greatest rectangular parallelopiped inscribed in the ellipsoid whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$

Let $2x$, $2y$, $2z$ be the dimensions of the required rectangular parallelopiped.

By symmetry, the centre of the parallelopiped coincides with that of the ellipsoid, namely, the origin and its faces are parallel to the co-ordinate planes.

Also one of the vertices of the parallelopiped has co-ordinates $(x, y, z)$, which satisfy the equation of the ellipsoid.

Thus, we have to maximise $V = 8xyz$, subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Here $f = 8xyz$ and $\phi = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

The auxiliary function is $g = f + \lambda\phi$, where $\lambda$ is the Lagrange multiplier. The stationary points of $g$ are given by

$$g_x = 0,\ g_y = 0,\ g_z = 0 \text{ and } g_\lambda = 0$$

i.e.,

$$8yz+\frac{2\lambda x}{a^2} = 0 \tag{1}$$

$$8zx+\frac{2\lambda y}{b^2} = 0 \tag{2}$$

$$8xy + \frac{2\lambda z}{c^2} = 0 \tag{3}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{4}$$

Multiplying (1) by $x$, $\dfrac{2\lambda x^2}{a^2} = -8xyz$

Similarly $\dfrac{2\lambda y^2}{b^2} = \dfrac{2\lambda z^2}{c^2} = -8xyz$ from (2) and (3)

Thus $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2} = k$ say

Using in (4), $3k = 1 \quad \therefore\ k = \dfrac{1}{3}$

$$\therefore\ x = \frac{a}{\sqrt{3}},\ y = \frac{b}{\sqrt{3}} \text{ and } z = \frac{c}{\sqrt{3}}$$

$$\therefore \text{ Maximum volume} = \frac{8abc}{3\sqrt{3}}.$$

**Example 2.14** Find the shortest and the longest distances from the point (1, 2, –1) to the sphere $x^2 + y^2 + z^2 = 24$.

Let $(x, y, z)$ be any point on the sphere. Distance of the point $(x, y, z)$ from (1, 2, –1) is given by $d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$.

We have to find the maximum and minimum values of $d$ or equivalently

$$d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2,$$

subject to the constant $x^2 + y^2 + z^2 - 24 = 0$

Here $\quad f = (x-1)^2 + (y-2)^2 + (z+1)^2$ and

$\quad \phi = x^2 + y^2 + z^2 - 24$

The auxiliary function is $g = f + \lambda\phi$, where $\lambda$ is the Lagrange multiplier. The stationary points of $g$ are given by $g_x = 0$, $g_y = 0$, $g_z = 0$ and $g_\lambda = 0$.

i.e.,

$$2(x-1) + 2\lambda x = 0 \tag{1}$$

$$2(y-2) + 2\lambda y = 0 \tag{2}$$

$$2(z+1) + 2\lambda z = 0 \tag{3}$$

$$x^2 + y^2 + z^2 = 24 \tag{4}$$

From (1), (2) and (3), we get

$$x = \frac{1}{1+\lambda},\ y = \frac{2}{1+\lambda},\ z = \frac{1}{1+\lambda}$$

Using these values in (4), we get

$$\frac{6}{(1+\lambda)^2} = 24 \text{ i.e., } (1+\lambda)^2 = \frac{1}{4}$$

$$\therefore \qquad \lambda = -\frac{1}{2} \text{ or } -\frac{3}{2}.$$

When $\lambda = -\frac{1}{2}$, the point on the sphere is (2, 4, –2)

When $\lambda = -\frac{3}{2}$, the point on the sphere is (–2, –4, 2)

When the point is (2, 4, –2), $d = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}$

When the point is (–2, –4, 2), $d = \sqrt{(-3)^2 + (-6)^2 + 3^2} = 3\sqrt{6}$

$\therefore$ Shortest and longest distances are $\sqrt{6}$ and $3\sqrt{6}$ respectively.

**Example 2.15** Find the point on the curve of intersection of the surfaces $z = xy + 5$ and $x + y + z = 1$ which is nearest to the origin.

Let $(x, y, z)$ be the required point.

It lies on both the given surfaces.

$$\therefore \qquad xy - z + 5 = 0 \quad \text{and} \quad x + y + z = 1$$

Distance of the point $(x, y, z)$ from the origin is given by $d = \sqrt{x^2 + y^2 + z^2}$.
We have to minimize $d$ or equivalently

$$d^2 = x^2 + y^2 + z^2,$$

subject to the constraints $xy - z + 5 = 0$ and $x + y + z - 1 = 0$.

***Note*** ☑ Here we have two constraint conditions. To find the extremum of $f(x, y, z)$ subject to the conditions $\phi_1(x, y, z) = 0$ and $\phi_2(x, y, z) = 0$, we form the auxiliary function
$g = f + \lambda_1\phi_1 + \lambda_2\phi_2$, where $\lambda_1$ and $\lambda_2$ are two Lagrange multipliers.

The stationary points of $g$ are given by $g_x = 0, g_y = 0, g_z = 0, g_{\lambda 1} = 0$ and $g_{\lambda 2} = 0$.

In this problem, $f = x^2 + y^2 + z^2$, $\phi_1 = xy - z + 5$ and $\phi_2 = x + y + z - 1$.

The auxiliary function is $g = f + \lambda_1\phi_1 + \lambda_2\phi_2$, where $\lambda_1, \lambda_2$ are Lagrange multipliers.
The stationary points of $g$ are given by

$$2x + \lambda_1 y + \lambda_2 = 0 \qquad (1)$$

$$2y + \lambda_1 x + \lambda_2 = 0 \qquad (2)$$

$$2z - \lambda_1 + \lambda_2 = 0 \qquad (3)$$

$$xy - z + 5 = 0 \tag{4}$$

$$x + y + z - 1 = 0 \tag{5}$$

Eliminating $\lambda_1$, $\lambda_2$ from (1), (2), (3), we have

$$\begin{vmatrix} 2x & y & 1 \\ 2y & x & 1 \\ 2z & -1 & 1 \end{vmatrix} = 0$$

i.e., $x\left(x+1\right) - y\left(y-2\right) - \left(y + zx\right) = 0$

i.e., $x^2 - y^2 + x - y - z\left(x - y\right) = 0$

i.e., $\left(x - y\right)\left(x + y - z + 1\right) = 0$

$\therefore$ $x = y$ or $x + y - z + 1 = 0$

Using $x = y$ in (4) and (5), we have

$$z = x^2 + 5 \tag{6}$$

and $$z = 1 - 2x \tag{7}$$

From (6) and (7), $x^2 + 2x + 4 = 0$, which gives only imaginary values for $x$.

Hence $$x + y - z + 1 = 0 \tag{8}$$

Solving (5) and (8), we get $x + y = 0$ (9)

and $$z = 1 \tag{10}$$

Using (10) in (4), we get $xy = -4$ (11)

Solving (9) and (11), we get $x = \pm 2$ and $y = \pm 2$.

$\therefore$ The required points are (2, –2, 1) and (–2, 2 ,1) and the shortest distance is 3.

## EXERCISE 2(c)

***Part A***

(Short Answer Questions)

1. Define relative maximum and relative minimum of a function of two variables.
2. State the conditions for the stationary point $(a, b)$ of $f(x, y)$ to be (i) a maximum point (ii) a minimum point and (iii) a saddle point.
3. Define saddle point of a function $f(x, y)$.
4. Write down the conditions to be satisfied by $f(x, y, z)$ and $\phi(x, y, z)$, when we extremise $f(x, y, z)$ subject to the condition $\phi(x, y, z) = 0$.
5. Find the minimum point of $f(x, y) = x^2 + y^2 + 6x + 12$.
6. Find the stationary point of $f(x, y) = x^2 - xy + y^2 - 2x + y$.
7. Find the stationary point of $f(x, y) = 4x^2 + 6xy + 9y^2 - 8x - 24y + 4$.

8. Find the possible extreme point of $f(x,y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$.
9. Find the nature of the stationary point (1, 1) of the function $f(x, y)$, if $f_{xx} = 6xy^3$, $f_{xy} = 9x^2 y^2$ and $f_{yy} = 6x^3 y$.
10. Given $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 6y$, find the nature of the stationary point (1, 2) of the function $f(x, y)$.

***Part B***

Examine the following functions for extreme values:

11. $x^3 + y^3 - 3axy$
12. $x^3 + y^3 - 12x - 3y + 20$
13. $x^4 + 2x^2 y - x^2 + 3y^2$
14. $x^3 y - 3x^2 - 2y^2 - 4y - 3$
15. $x^4 + x^2 y + y^2$ at the origin
16. $x^3 y^2 (a - x - y)$
17. $x^3 y^2 (12 - 3x - 4y)$
18. $xy + 27\left(\frac{1}{x} + \frac{1}{y}\right)$
19. $\sin x + \sin y + \sin(x + y), 0 \le x, y \le \frac{\pi}{2}$
20. Identify the saddle points and extreme points of the function $xy(3x + 2y + 1)$.
21. Find the minimum value of $x^2 + y^2 + z^2$, when (i) $xyz = a^3$ and (ii) $xy + yz + zx = 3a^2$.
22. Find the minimum value of $x^2 + y^2 + z^2$, when $ax + by + cz = p$.
23. Show that the minimum value of $(a^3 x^2 + b^3 y^2 + c^3 z^2)$, when $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{k}$, is $k^2 (a+b+c)^3$.
24. Split 24 into three parts such that the continued product of the first, square of the second and cube of the third may be minimum.
25. The temperature at any point $(x, y, z)$ in space is given by $T = k\,x\,y\,z^2$, where $k$ is a constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
26. Find the dimensions of a rectangular box, without top, of maximum capacity and surface area 432 square meters.
27. Show that, of all rectangular parallelopipeds of given volume, the cube has the least surface.
28. Show that, of all rectangular parallelopipeds with given surface area, the cube has the greatest volume.
29. Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.
30. Find the points on the surface $z^2 = xy + 1$ whose distance from the origin is minimum.
31. If the equation $5x^2 + 6xy + 5y^2 = 8$ represents an ellipse with centre at the origin, find the lengths of its major and minor axes.

(**Hint**: The longest distance of a point on the ellipse from its centre gives the length of the semi-major axis. The shortest distance of a point on the ellipse from its centre gives the length of the semi-minor axis).

32. Find the point on the surface $z = x^2 + y^2$, that is nearest to the point (3, –6, 4).
33. Find the minimum distance from the point (3, 4, 15) to the cone $x^2 + y^2 = 4z^2$.
34. Find the points on the ellipse obtained as the curve of intersection of the surfaces $x + y = 1$ and $x^2 + 2y^2 + z^2 = 1$, which are nearest to and farthest from the origin.
35. Find the greatest and least values of $z$, where $(x, y, z)$ lies on the ellipse formed by the intersection of the plane $x + y + z = 1$ and the ellipsoid $16x^2 + 4y^2 + z^2 = 16$.

**ANSWERS**

## Exercise 2(a)

(2) $du = \cos(xy^2)(y^2 dx + 2xy\, dy)$

(3) $du = x^{y-1} \cdot y^x (y + x \log y)\, dx + x^y y^{x-1}(x + y \log x)\, dy$

(4) $du = y(1 + \log xy)\, dx + x(1 + \log xy)\, dy$

(5) $du = (y \log a)\, a^{xy}\, dx + (x \log a)\, a^{xy}\, dx$

(6) $8\, a^5 t^6 (4t + 7)$;

(7) $e^{\sqrt{a^2-t^2}} \sin^3 t \left\{3\sqrt{a^2-t^2} \sin^2 t \cos t - t \sin^3 t / \sqrt{a^2-t^2}\right\}$

(8) $(\cos t - e^{-t} - \sin t)/(e^{-t} + \sin t + \cos t)$

(9) $-\dfrac{x^2 + 2xy + 2y^2}{x^2 + 4xy + y^2}$

(11) $\dfrac{3}{2} x \cos(x^2 + y^2)$

(12) $x(xy + 4y^2 - 2x^2)/(x + 2y)$

(14) 3.875

(15) 4.984

(16) 0.0043

(17) 0.006 cm$^3$; 0.004 cm$^2$

(18) 2

(19) $4(a + b + c)k$

(20) 1.5

(36) $\dfrac{\partial^2 z}{\partial u\, \partial v} = 0$

(37) $\dfrac{\partial^2 z}{\partial v^2} = 0$

(38) $\dfrac{\partial^2 z}{\partial u^2} = 0$

(39) $\dfrac{\partial^2 z}{\partial u \partial v} = 0$

(40) $\dfrac{\partial^2 u}{\partial z \cdot \partial z^*} = 0$

(41) (i) $\dfrac{y(y - x \log y)}{x(x - y \log x)}$

(ii) $\dfrac{y}{x}$

(iii) $\dfrac{y \tan x + \log \sin y}{\log \cos x - x \cot y}$

(iv) $\dfrac{\log \cot y - y \tan x}{\log \sec x + x \sec y \operatorname{cosec} y}$

(v) $\dfrac{x-y}{x(1+\log x)}$

(42) $2a^3xy/(ax-y^2)^3$

(43) $2a^2xy\,(3a^4+x^2y^2)/(a^2x-y^3)^3$

(47) 5%

(50) 2.3%

(55) $\dfrac{5\sqrt{3}\pi}{324}$

## Exercise 2(b)

(4) $1+(x+y)+\dfrac{(x+y)^2}{2}+\cdots$

(5) $(x+y)-\dfrac{1}{3!}(x+y)^3+\ldots$

(11) $u+\upsilon+1$

(15) $2\tan^{-1}\left(\dfrac{1}{y}\right)$

(16) $y+xy+\dfrac{x^2y}{2}-\dfrac{y^3}{6}+\cdots$

(17) $\dfrac{e}{\sqrt{2}}\left\{1+(x-1)-\left(y-\dfrac{\pi}{4}\right)+\dfrac{(x-1)^2}{2}-(x-1)\left(y-\dfrac{\pi}{4}\right)-\dfrac{1}{2}\left(y-\dfrac{\pi}{4}\right)^2+\cdots\right\}$

(18) $y+xy-\dfrac{y^2}{2}+\dfrac{1}{2}x^2y-\dfrac{1}{2}xy^2+\dfrac{1}{3}y^3+\cdots$

(19) $\dfrac{\pi}{4}-\dfrac{1}{2}(x-1)+\dfrac{1}{2}(y-1)+\dfrac{1}{4}(x-1)^2-\dfrac{1}{4}(y-1)^2+\cdots$

(20) $-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2)$

(21) $-9+3(x+2)-7(y-1)+2(x+2)(y-1)-2(y-1)^2+(x+2)(y-1)^2$

(22) $1+(y-1)+(x-1)(y-1)+\cdots$

(23) $e\Big[1+(x-1)+(y-1)+\dfrac{1}{2}(x-1)^2+2(x-1)(y-1)+(y-1)^2+\dfrac{1}{6}(x-1)^3$

$+\dfrac{3}{2}(x-1)^2(y-2)+\dfrac{3}{2}(x-1)(y-2)^2+\dfrac{1}{6}(y-2)^3\Big]$

(27) (i) $4(u^2+\upsilon^2)$

(ii) $4xy$

(28) $r$

(30) $x(y\upsilon+1-w)+z-2u\upsilon$

(31) $(x-y)(y-z)(z-x)$

(32) $u^2=\upsilon+1$

(33) $u\tan\upsilon$

(34) $f_1^2 = f_2 + 2f_3$

(36) $\frac{1}{2a^3}\{\tan^{-1}\frac{x}{a} + (ax)/(x^2 + a^2)\}$

(37) $\frac{(-1)^n\, n!}{(m+1)^{n+1}}$

(38) $\frac{1}{2}\sqrt{\pi}e^{-a^2}$

(39) $\tan^{-1}\left(\frac{1}{a}\right); \frac{\pi}{2}$

(40) log (1 + $m$)

## Exercise 2(c)

(5) (–3, 0) (6) (1, 0) (7) $\left(0, \frac{4}{3}\right)$

(8) (1, 1) (9) Saddle point (10) Minimum point

(11) Maximum at ($a$, $a$) if $a < 0$ and minimum at ($a$, $a$) if $a > 0$

(12) Minimum at (2, 1) and maximum at (–2, –1)

(13) Minimum at $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{4}\right)$

(14) Maximum at (0, –1) (15) Minimum at (0, 0)

(16) Maximum at $\left(\frac{a}{2}, \frac{a}{3}\right)$ (17) maximum at (2, 1) (18) Minimum at (3, 3)

(19) Maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and minimum at $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$

(20) Saddle point are (0, 0), $\left(-\frac{1}{3}, 0\right)$ and $\left(0, -\frac{1}{2}\right)$; maximum at $\left(-\frac{1}{9}, -\frac{1}{6}\right)$

(21) $3a^2$; $3a^2$ (22) $\frac{p^2}{a^2 + b^2 + c^2}$ (24) 4, 8, 12

(25) $\frac{ka^4}{8}$ (26) 12, 12 and 6 metres

(30) (0, 0, 1) and (0, 0, –1) (31) 4, 2 (32) (1, –2, 5)

(33) $5\sqrt{5}$ (34) $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$; (1, 0, 0) (35) $\frac{8}{3}; -\frac{8}{7}$

# UNIT 3
# Integral Calculus

## 3.1 INTRODUCTION

Integration can be considered as the reverse process of differentiation. viz, in integration, we are required to find the function $f(x)$ from its derivative which will be given as $g(x)$, say. In other words, the process of finding $f(x)$ from $g(x)$, given that $\frac{d}{dx}\{f(x)\} = g(x)$ is integration. In this situation, we say that $f(x)$ is *the integral* of $g(x)$ and write symbolically that $\int g(x)dx = f(x)$.

The symbol $\int$ is the symbol of integration, $g(x)$ is called the *integrand* and d$x$ indicates the variable ($x$) with respect to which integration is performed.

For example, $\int \cos x \, dx = \sin x$, since $\frac{d}{dx}(\sin x) = \cos x$

and $\int 3x^2 \, dx = x^3$, since $\frac{d}{dx}(x^3) = 3x^2$.

## 3.2 CONSTANT OF INTEGRATION

When $\int 3x^2 \, dx = x^3$, the result $\int 3x^2 \, dx = x^3 + c$ equally holds good, as $\frac{d}{dx}(x^3) = \frac{d}{dx}(x^3 + c)$, where $c$ is a constant. As $c$ can take any constant value, it is called *the arbitrary constant of integration*.

In general, when $\frac{d}{dx} f(x) = g(x)$, $\int g(x)dx = [f(x) + c] \cdot [f(x) + c]$ is called *the indefinite integral* of $g(x)$ due to indefinite nature, of $c$. For convenience, we normally omit $c$ when we evaluate an indefinite integral.

### 3.2.1 Definite Integrals

When $\int g(x)dx = f(x) + c$, then $[f(b) - f(a)]$ is called the *definite integral* of $g(x)$ between the limits (or end values) $a$ and $b$ and denoted by the symbol $\int_a^b g(x)dx$. $a$ is called the lower limit and $b$ called the upper limit and is denoted by $[f(x)]_a^b$.

Thus $\int_a^b g(x)\mathrm{d}x = [f(x)]_a^b = f(b) - f(a)$

*Note* ☑ The constant of integration '$c$' occurring in the indefinite integral does not find a place in the definite integral, for if $\int g(x)\mathrm{d}x = f(x) + c$,

Then
$$\int_a^b g(x)\mathrm{d}x = [f(x) + c]_a^b$$
$$= \{f(b) + c\} - f(a) + c$$
$$= f(b) - f(a)$$

Thus to evaluate $\int_a^b g(x)\mathrm{d}x$ first get $f(x)$ and omit the arbitrary constant $c$. Then we substitute $b$ and $a$ for the variable $x$ and obtain $f(b)$ and $f(a)$. Finally we get $[f(b) - f(a)] = \int_a^b g(x)\,\mathrm{d}x$

For example, $\int_0^{\pi/2} \cos x\ \mathrm{d}x = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$

and $\int_1^2 3x^2\,\mathrm{d}x = [x^3]_1^2 = 2^3 - 1^3 = 8 - 1 = 7.$

### 3.2.2 Standard Integrals

Using the knowledge of derivatives of elementary/standard functions, the following standard integrals are obtained. Students should not try to derive these results from differentiation results, but remember them as formulas of integral calculus. [Constants of integration are omitted in all the formulas]

1. $\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1}\ (n \neq -1)$; Extension: $\int (ax+b)^n\,\mathrm{d}x = \frac{(ax+b)^{n+1}}{(n+1)\cdot a}$

   $\int \frac{1}{x^2}\,dx \neq -\frac{1}{x}$ and $\int \frac{\mathrm{d}x}{x} = 2\sqrt{x}$

**An important note on the extension:**

In the place of the variable $x$ of the integrand of any standard integral, if we have a simple first degree expression $(ax + b)$, we have to replace $x$ by $(ax + b)$ in the corresponding result in the R.H.S. also and divide it by the coefficient of $x$ in $(ax + b)$, namely '$a$'. viz., if

$$\int g(x)\mathrm{d}x = f(x)\ \ldots \tag{1}$$

then

$$\int g(ax+b)\,\mathrm{d}x = \frac{1}{a} f(ax+b) \tag{2}$$

This result which can be considered as extended standard integral is obtained as follows:

If we put $ax + b = y$ in (1), then $a$ d$x$ = d$y$ or d$x$ = $\frac{1}{a}\, dy$

Thus (1) becomes $\int g(y)\frac{\mathrm{d}y}{a} = \frac{1}{a}\int g(y)\,\mathrm{d}y = \frac{1}{a}\,f(y)$ by (1).

*Particular cases of stand and formula(1):*

1. (a) $\int \frac{1}{x^2}\,\mathrm{d}x = -\frac{1}{x}\left[\textit{Extension:}\ \int \frac{\mathrm{d}x}{(ax+b)^2} = -\frac{1}{a(ax+b)}\right]$

   (b) $\int \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{x}\left[\textit{Extension:}\ \int \frac{\mathrm{d}x}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}\right]$

2. $\int \frac{\mathrm{d}x}{x} = \log_e x;\left[\textit{Extension:}\ \int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\log_e\,(ax+b)\right]$

3. $\int e^x\,\mathrm{d}x = e^x;\left[\textit{Extension:}\ \int \mathrm{a}^{ax+b}\,\mathrm{d}x = \frac{1}{a}\,e^{ax+b}\right]$

4. $\int \sin x\,\mathrm{d}x = -\cos x;\left[\textit{Extension:}\ \int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b)\right]$

**Note ☑** Extensions are omitted for the remaining standard integrals that follow, as they are obvious.

5. $\int \cos x\,\mathrm{d}x = \sin x$
6. $\int \tan x\,\mathrm{d}x = \log \sec x$
7. $\int \operatorname{cosec} x\,\mathrm{d}x = -\log(\operatorname{cosec} x + \cot x)$ or $\log \tan \frac{x}{2}$
8. $\int \sec x\,\mathrm{d}x = \log(\sec x + \tan x)$ or $\log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$
9. $\int \cot x\,\mathrm{d}x = \log \sin x$

**Note ☑** Formulas (6), (7), (8) and (9) are not derived from standard differentiation formulas.

10. $\int \sec^2 x\,\mathrm{d}x = \tan x$
11. $\int \operatorname{cosec}^2 x\,\mathrm{d}x = -\cot x$
12. $\int \sec x \tan x\,\mathrm{d}x = \sec x$

13. $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x$

14. $\int \sinh x \, dx = \cosh x$

15. $\int \cosh x \, dx = \sinh x$

16. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$ and $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$

17. $\int \frac{dx}{\sqrt{x^2+a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$ or $\log(x+\sqrt{x^2-a^2})$

18. $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$ or $\log(x+\sqrt{x^2-a^2})$

19. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x$ and $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}$

20. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ and hence $\int \frac{dx}{x^2+1} = \tan^{-1} x$

21. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$

22. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

23. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

24. $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$

25. $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

## 3.3 TECHNIQUES OF INTEGRATION

Before we proceed to discuss techniques of integration, we give below two basic properties of integration for which no proof is required as it is obvious.

(i) If $k$ is a constant, $\int k f(x)\, dx - k \int f(x)\, dx$.

(ii) If $k_1, k_2, \ldots, k_n$ are constants, then $\int [k_1 f_1(x) \pm k_2 f_2(x) \pm k_3 f_3(x)$
$\pm \cdots k_n f_n(x)]\, dx = k_1 \int f_1(x)\, dx \pm k_2 \int f_2(x)\, dx \pm \cdots k_m \int f_n(x)\, dx.$

### 3.3.1 Integration by Substitution

If the integral is of the form $\int F\{f(x)\} \cdot f'(x)\, dx$, where $f(x)$ is an elementary/standard function, the integral can be reduced to a simpler integrable form by putting

$y = f(x)$ so that $dy \simeq f'(x)dx$. The integral gets reduced to the form $\int F(y)\}\, dy$, which can be done by known methods or by using standard formulas.

As particular cases of this rule, we mention a few:

To evaluate

- $\int F(x^n)x^{n-1}dx$, we put $x^n = y$
- $\int F(x^2)\, x\, dx$, we put $x^2 = y$
- $\int F(\log x) \cdot \frac{dx}{x}$, we put $\log x = y$
- $\int F(e^x) \cdot e^x\, dx$, we put $e^x = y$
- $\int F(\sin x) \cos x\, dx$, we put $\sin x = y$
- $\int F(\tan x) \sec^2 x\, dx$, we put $\tan x = y$
- $\int F(\sin^{-1} x) \frac{dx}{\sqrt{1-x^2}}$, we put $\sin^{-1} x = y$
- $\int F(\tan^{-1} x) \frac{dx}{1+x^2}$, we put $\tan^{-1} x = y$

***Note*** ☑ The following two particular cases of $\int F\{f(x)\}f'(x)\, dx$ are of importance, as they will be used in integrating some rational and irrational functions.

(i) $\int \frac{f'(x)}{f(x)} dx \to \int \frac{dy}{y} = \log y \to \log f(x)$

(ii) $\int \frac{f'(x)}{\sqrt{f(x)}} dx \to \int \frac{dy}{\sqrt{y}} = 2\sqrt{y} \to 2\sqrt{f(x)}$

### 3.3.2 Integration by Trigonometric Substitution

If the integrand contains $\sqrt{a^2 - x^2}, \sqrt{x^2 + a^2}$ or $\sqrt{x^2 - a^2}$, it can be reduced to a rational or integrable form by making the trigonometic substitution $x = a \sin \theta$, $x = a \tan \theta$ or $x = a \sec \theta$ respectively

In the first case, we may even put $x = a \cos \theta$

In the second case, we may also put $x = a \cot \theta$ or $x = \sinh y$

In the third case, we may also put $x = a \operatorname{cosec} \theta$ or $x = \cosh y$.

### 3.3.3 Compound Trigonometric Substitution

If the integrand contains $\sqrt{(x-\alpha)(\beta - x)}$ or $\sqrt{\frac{(x-\alpha)}{(\beta - x)}}$, when $\beta > \alpha$, it can be retionalised or reduced to the integrable form by making the substitution $x = \alpha \cos^2 \theta + \beta \sin^2 \theta$.

## WORKED EXAMPLE 3(a)

**Example 3.1** Evaluate $\int \frac{x^3}{\sqrt{1+x^4}}\,dx.$

The integrand is of the form $f(1+x^4)\times x^3$

$\therefore$ Put $1+x^4 = y$ $\quad\therefore\quad 4x^3 dx = dy$

$$I = \int \frac{\frac{1}{4}dy}{\sqrt{y}} = \frac{1}{4}\times 2\sqrt{y} = \frac{1}{2}\sqrt{1+x^4}+c$$

**Example 3.2** Evaluate $\int_0^\infty \frac{dx}{a^2e^x+b^2e^{-x}}.$

$$I = \int_0^\infty \frac{dx}{a^2e^x+b^2e^{-x}} = \int_0^\infty \frac{e^x\,dx}{a^2e^{2x}+b^2}$$

Put $e^x = y$ $\therefore$ $e^x dx = dy$, since the integrand is of the form $f(e^x)\times e^x$.

*Note* ☑ Instead of evaluating the indefinite integral and using the limits in the end, we can express the limits for the new variable $y$ using the substitution used.

Thus, when $x = 0$, $y = 1$ and when $x = \infty$, $y = \infty$

$$\therefore \qquad I = \int_1^\infty \frac{dy}{a^2y^2+b^2} = \frac{1}{a^2}\int_1^\infty \frac{dy}{y^2+\left(\frac{b}{a}\right)^2}$$

$$= \frac{1}{a^2}\times\frac{a}{b}\left[\tan^{-1}\left(\frac{ay}{b}\right)\right]_1^\infty = \frac{1}{ab}\left(\frac{\pi}{2}-\tan^{-1}\frac{a}{b}\right)$$

**Example 3.3** Integrate $\frac{1}{x(1+\log x)^2}$ w.r.t $x$.

Since the integrand is the product of $\log x$ or $1+\log x$ and its derivative $\frac{1}{x}$, we put $1+\log x = y$ $\therefore$ $\frac{1}{x}dx = dy$

$$\therefore \qquad I = \int\frac{dy}{y^2} = -\frac{1}{y}+c \text{ or } -\frac{1}{1+\log x}+c.$$

**Example 3.4** Evaluate $\int_0^{\pi/4} \sin 3x\cos x\,dx.$

The integrand can be rewritten as the product of $f(\sin x)$ and $\cos x$. So we put $\sin x = y$ and $\therefore$ $\cos x\,dx = dy$

When $x = 0, y = 0$ and when $x = \frac{\pi}{4}, y = \frac{1}{\sqrt{2}}$

$$\therefore \quad I = \int_0^{\pi/4} (3 \sin x - 4 \sin^3 x) \cos x \, dx$$

$$= \int_0^{1/\sqrt{2}} (3y - 4y^3) \, dy$$

$$= \left(\frac{3}{2}y^2 - y^4\right)_0^{1/\sqrt{2}} = \frac{3}{2} \cdot \frac{1}{2} - \frac{1}{4} = \frac{1}{2}.$$

**Example 3.5** Evaluate $\int \left(\frac{1 - \sin x}{1 + \sin x}\right) dx.$

$$I = \int \left(\frac{(1 - \sin x)^2}{\cos^2 x}\right) dx = \int (\sec x - \tan x)^2 \, dx$$

$$= \int \left[\sec^2 x + (\sec^2 x - 1) - 2 \sec x \tan x\right] dx$$

$$= 2 \tan x - 2 \sec x - x + c$$

**Example 3.6** Evaluate $\int_0^{\pi/2} \sqrt{1 + \sin 2x} \, dx.$

$$I = \int_0^{\pi/2} \sqrt{\cos^2 x + \sin^2 x + 2 \cos x \sin x} \, dx$$

$$I = \int_0^{\pi/2} \sqrt{\cos x + \sin x)^2} \, dx$$

$$= \int_0^{\pi/2} (\cos x + \sin x) \, dx = (\sin x - \cos x)_0^{\pi/2} = 2$$

**Example 3.7** Evaluate $\int \operatorname{cosec}^8 x \, dx.$

$$I = \int \operatorname{cosec}^6 x \operatorname{cosec}^2 x \, dx$$

$$= \int (1 + \cot^2 x)^3 \operatorname{cosec}^2 x \, dx$$

As the integrand is of the form $f(\cot x)$. $\operatorname{cosec}^2 x$, we put $\cot x = y$ and $\operatorname{cosec}^2 x \, dx = -dy$

$$\therefore \quad I = \int (1 + y^2)^3 (-dy)$$

$$= -\int (1 + 3y^2 + 3y^4 + y^6) \, dy$$

$$= -\left[y + y^3 + 3y\frac{5}{5} + y\frac{7}{7}\right]$$

$$= -\left[\cot x + \cot^3 x + \frac{3}{5}\cot^5 x + \frac{1}{7}\cos^7 x\right] + c$$

**Example 3.8** Evaluate $I = \int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx.$

As the integrand is of the form $f(\sin^{-1} x)\cdot\frac{1}{\sqrt{1-x^2}}$, we put

$$\sin^{-1} x = y \text{ and } \therefore \frac{1}{\sqrt{1+x^2}} dx = dy$$

$$I = \int y^3 dy = \frac{y^4}{4} = \frac{1}{4}(\sin^{-1} x)^4 + c$$

**Example 3.9** Evaluate $\int \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx.$

As the integrand if of the form $f(\tan^{-1} x)\cdot\frac{1}{1+x^2}$, we put

$$\tan^{-1} x = y \text{ and } \therefore \frac{1}{1+x^2} dx = dy$$

$$I = \int \sqrt{y}\, dy = \frac{2}{3} y^{\frac{3}{2}} = \frac{2}{3}(\tan^{-1} x)^{\frac{3}{2}} + c$$

**Example 3.10** Evaluate $\int \frac{e^{\sec^{-1} x}}{x\sqrt{x^2-1}} dx.$

As the integrand is of the form $f(\sec^{-1} x)\cdot\frac{1}{x\sqrt{x^2-1}}$, we put

$$\sec^{-1} x = y \text{ and } \therefore \frac{1}{x\sqrt{x^2-1}} dx = dy$$

$$I = \int e^y dy = e^y = e^{\sec^{-1} x} + c$$

**Example 3.11** Evaluate $\int \sqrt{\frac{1-x}{1+x}} dx.$

Multiplying the Nr. and Dr. by $\sqrt{1-x}$, we get

$$I = \int \frac{1-x}{\sqrt{1-x^2}} dx$$

To reduce the integrand to the integrable form, we make the trigonometric substitution $x = \sin\theta$ and so $dx = \cos d\theta$

$$\therefore \qquad I = \int\left(\frac{1-\sin\theta}{\cos\theta}\right)\cos\theta\, d\theta$$

$$= \theta + \cos\theta$$

$$= \sin^{-1} x + \sqrt{1-x^2} + c$$

**Example 3.12** Evaluate $\int \frac{dx}{(1-x)\sqrt{1-x^2}}$.

Due to the occurrence of $\sqrt{1-x^2}$ in the integrand, we put

$$x = \sin\theta \text{ and so } dx = \cos\ d\theta$$

$$\therefore \qquad I = \int \frac{\cos\theta\, d\theta}{(1-\sin\theta)\cos\theta}$$

$$= \int \frac{1+\sin\theta}{\cos^2\theta} d\theta = \int (\sec^2\theta + \sec\theta\tan\theta) d\theta$$

$$= \tan\theta + \sec\theta$$

$$= \frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \text{ or } \frac{1+x}{\sqrt{1-x^2}} \text{ or } \sqrt{\frac{1+x}{1-x}} + c$$

**Example 3.13** Evaluate $\int \frac{x^2}{(4+x^2)^{5/2}} dx$.

Due to the presence of $(x^2+4)^{1/2}$ in the integrand, we put

$$x = 2\tan\theta \text{ and so } dx = 2\sec^2\theta\, d\theta$$

$$\therefore \qquad I = \int \frac{4\tan^2\theta \cdot 2\sec^2\theta\, d\theta}{(4\sec^2\theta)^{5/2}}$$

$$= \frac{8}{32}\int \frac{\tan^2\theta}{\sec^3\theta} d\theta$$

$$= \frac{1}{4}\int \sin^2\theta\cos\theta\, d\theta$$

$$= \frac{1}{4}\frac{\sin^3\theta}{3}, \text{ on putting } \sin\theta = t$$

$$= \frac{1}{12}\left(\frac{x}{\sqrt{x^2+4)}}\right)^3 \quad \text{or} \quad \frac{1}{12}\cdot\frac{x^3}{(x^2+4)^{3/2}} + c$$

**Example 3.14** Evaluate $\int \frac{\sqrt{x^2-a^2}}{x} dx$.

Due to the presence of $\sqrt{x^2-a^2}$ in the integrand, we put $x = a\sec\theta$ and so $dx = a\sec\theta\tan\theta\, d\theta$

Then $$I = \int \frac{\sqrt{a^2 \sec^2\theta - a^2}}{a\sec\theta} a\sec\theta \tan\theta\, d\theta$$

$$= a\int \tan^2\theta\, d\theta = a\int (\sec^2\theta - 1)d\theta$$

$$= a(\tan\theta - \theta)$$

$$= a\left\{\frac{\sqrt{x^2 - a^2}}{a} - \sec^{-1}\left(\frac{x}{a}\right)\right\} + c$$

**Example 3.15** Evaluate $\int \frac{\sqrt{a^2 - x^2}}{x^4} dx$.

Due to the presence of $\sqrt{a^2 - x^2}$ in the integrand, we put $x = a \sin\theta$ and so $dx = a \cos\theta \cos\theta\, d\theta$

Then $$I = \int \frac{a\cos\theta \cdot a\cos\theta\, d\theta}{a^4 \sin^4\theta}$$

$$= \frac{1}{a^2}\int \cot^2\theta \cdot \text{cosec}^2\theta\, d\theta$$

$$= \frac{1}{a^2}\int t^2(-dt), \text{ where } t = \cot\theta$$

$$= -\frac{1}{a^2}\frac{t^3}{3} = -\frac{1}{3a^2}\cot^3\theta$$

$$= -\frac{1}{3a^2}\left(\frac{\sqrt{a^2 - x^2}}{x}\right)^3$$

$$= -\frac{1}{3a^2}\frac{(a^2 - x^2)^{3/2}}{x^3} + c.$$

**Example 3.16** Evaluate $\int \sqrt{(x-3)(7-x)}\, dx$.

We make the compound trigonometric substitution

$$x = 3\cos^2\theta + 7\sin^2\theta$$

Then $x - 3 = 3\cos^2\theta + 7\sin^2\theta - 3 = 4\sin^2\theta$

and $7 - x = 7 - (3\cos^2\theta + 7\sin^2\theta) = 4\cos^2\theta$

$dx = (-6\cos\theta\sin\theta + 14\sin\theta\cos\theta)d\theta$

$= 8\sin\theta\cos\theta\, d\theta$

Then $$I = \int \sqrt{4\sin^2\theta \cdot 4\cos^2\theta} \times 8\sin\theta\cos\theta\, d\theta$$

$$= 32\int \sin^2\theta\cos^2\theta\, d\theta$$

$$= 8\int (2\sin\theta\cos\theta)^2 d\theta$$

$$= 8\int \sin^2 2\theta\, d\theta$$

$$= \frac{8}{2}\int (1-\cos 4\theta)\mathrm{d}\theta$$

$$= 4\left(\theta - \frac{\sin 4\theta}{4}\right)$$

$$= 4\sin^{-1}\left(\frac{\sqrt{x-3}}{2}\right) - 2\sin 2\theta \cos 2\theta$$

$$= 4\sin^{-1}\left(\frac{\sqrt{x-3}}{2}\right) - 4\sin\theta\cos\theta(2\cos^2\theta - 1)$$

$$= 4\sin^{-1}\left(\frac{\sqrt{x-3}}{2}\right) - 4\frac{\sqrt{x-3}}{2}\cdot\frac{\sqrt{7-x}}{2}\left(2\times\frac{(7-x)}{4} - 1\right)$$

$$= 4\sin^{-1}\left(\frac{\sqrt{x-3}}{2}\right) - \sqrt{(x-3)(7-x)}\left(\frac{5-x}{2}\right) + c$$

**Example 3.17** Evaluate $\int_1^2 \sqrt{\frac{x-1}{2-x}}\mathrm{d}x.$

Put $x = \cos^2\theta + 2\sin^2\theta$ $\quad\therefore\ \mathrm{d}x = (-2\cos\theta\sin\theta + 4\sin\theta\cos\theta)\,\mathrm{d}\theta$

$$= 2\sin\theta\cos\theta\,\mathrm{d}\theta$$

$$x - 1 = \cos^2\theta + 2\sin^2\theta - 1$$

$$= \sin^2\theta$$

$$2 - x = 2 - (\cos^2\theta + 2\sin^2\theta)$$

$$= \cos^2\theta$$

When $x = 1, \cos^2\theta + 2\sin^2\theta + 1$ viz., $\sin^2\theta = 0\ \therefore \theta = 0$

When $x = 2, \cos^2\theta + 2\sin^2\theta = 2$ viz., $\cos^2\theta = 0\ \therefore \theta = \frac{\pi}{2}$

Then $$I = \int_0^{\pi/2} \tan\theta\cdot 2\sin\theta\cos\theta\,\mathrm{d}\theta$$

$$= \int_0^{\pi/2} 2\sin^2\theta\,\mathrm{d}\theta = \int_0^{\pi/2}(1-\cos 2\theta)\mathrm{d}\theta$$

$$= \left(\theta - \frac{1}{2}\sin 2\theta\right)_0^{\frac{\pi}{2}} = \frac{\pi}{2}.$$

**Example 3.18** Evaluate $\int_\alpha^\beta \frac{\mathrm{d}x}{\sqrt{(x-\alpha)(\beta-x)}}\ (\beta > \alpha).$

Put $x = \alpha\cos^2\theta + \beta\sin^2\theta\ \therefore \mathrm{d}x = 2(\beta-\alpha)\sin\theta\cos\theta\,\mathrm{d}\theta$

$$x = \theta = \alpha\cos^2\theta + \beta\sin^2\theta - \theta = (\beta-\alpha)\sin^2\theta$$

$$\beta - x = \beta - (\alpha\cos^2\theta + \beta\sin^2\theta) = (\beta-\alpha)\cos^2\theta$$

When $\quad x=\alpha,\ \theta=0$ and when $x=\beta, \theta=\dfrac{\pi}{2}$

$$\therefore \quad I=\int_0^{\pi/2}\frac{2(\beta-\alpha)\sin\theta\cos\theta}{(\beta-\alpha)\sin\theta\cos\theta}\,d\theta=2\cdot\frac{\pi}{2}=\pi.$$

**Example 3.19** Evaluate $\displaystyle\int\sqrt{\frac{x-2}{5-x}}\,\mathrm{d}x.$

Put $x=2\cos^2\theta+5\sin^2\theta \quad \therefore = \mathrm{d}x=6\sin\theta\cos\theta\,\mathrm{d}\theta$

$$x-2=3\sin^2\theta \text{ and } 5-x=3\cos^2\theta$$

$$\therefore \quad I=\int\sqrt{\frac{3\sin^2\theta}{3\cos^2\theta}}\cdot 6\sin\theta\cos\theta\,\mathrm{d}\theta=\int 6\sin^2\theta\,\mathrm{d}\theta$$

$$=3\int(1-\cos 2\theta)\mathrm{d}\theta$$

$$=3\left[\theta-\frac{1}{2}\sin 2\theta\right]$$

$$=3\theta-3\sin\theta\cos\theta$$

$$=3\sin^{-1}\sqrt{\frac{x-2}{3}}-\sqrt[3]{\frac{x-2}{3}}\cdot\sqrt{\frac{5-x}{3}}$$

$$=3\sin^{-1}\sqrt{\frac{x-2}{3}}-\sqrt{(x-2)(5-x)}+c$$

**Example 3.20** Evaluate $\displaystyle\int_\alpha^\beta\frac{x}{\sqrt{(x-\alpha)(\beta-x)}}\,\mathrm{d}x.$

Putting $x=\alpha\cos^2\theta+\beta\sin^2\theta$ and proceeding as in Example (3.18),

we get
$$I=\int_0^{\pi/2}\frac{\alpha\cos^2\theta+\beta\sin^2\theta}{(\beta-\alpha)\sin\theta\cos\theta}\cdot 2(\beta-\alpha)\sin\theta\cos\theta\,\mathrm{d}\theta$$

$$=2\int_0^{\pi/2}\left[\alpha\left(\frac{1+\cos^2\theta}{2}\right)+\beta\left(\frac{1-\cos^2\theta}{2}\right)\right]\mathrm{d}\theta$$

$$=\alpha\left(\frac{\pi}{2}\right)+\beta\left(\frac{\pi}{2}\right) \text{ or } \frac{\pi}{2}(\alpha+\beta)$$

## EXERCISE 3(a)

***Part A***

(Short Answer Questions)

Integrate the following functions w.r.t. $x$:

(1) $\dfrac{x^2}{x+1}$ (2) $\dfrac{x^2-4x+3}{x-2}$ (3) $\cos mx \cos nx$

(4) $\dfrac{1}{1-\cos x}$ (5) $\dfrac{\sin^2 x}{1+\cos x}$ (6) $\dfrac{(1+\sqrt{x})^n}{\sqrt{x}}$

(7) $\dfrac{1}{2+e^x+e^{-x}}$ (8) $\dfrac{1}{x(\log x)^n}$ (9) $\sin^2 x \cdot \sin 2x$

(10) $\dfrac{\sin x+\cos x}{\sin x-\cos x}$ (11) $\dfrac{x+1}{x^2+2x+8}$ (12) $\dfrac{x^2+2x}{\sqrt{x^3+3x^2+2}}$

(13) $\sqrt{ax+b}+\dfrac{1}{\sqrt{cx+d}}$ (14) $\dfrac{x^{27}}{x^{14}-4}$

(15) $\dfrac{x^{n-1}}{\sqrt{a^{2n}-x^{2n}}}$ (16) $\dfrac{\sinh x}{\sqrt{\sinh^2 x+5}}$ (17) $\dfrac{x}{\sqrt{1+x^4}}$

(18) $\dfrac{1}{x\sqrt{1-(\log)^2}}$ (19) $\dfrac{\cos x}{\sqrt{9-\sin^2 x}}$ (20) $\dfrac{\sec^2 x}{\sqrt{16+\tan^2 x}}$

***Part B***

Evaluate the following integrals:

(21) $\dfrac{\cos^2\sqrt{x}}{\sqrt{x}}$ (22) $2\dfrac{\sin^3\sqrt{x}}{\sqrt{x}}$ (23) $\displaystyle\int_0^1 \frac{dx}{e^x+e^{-x}}$

(24) $\displaystyle\int_0^1 \frac{dx}{\sqrt{\sin^{-1}x(1-x^2)}}$ (25) $\displaystyle\int_0^\infty \frac{e^{\tan^{-1}x}}{1+x^2}$

(26) $\displaystyle\int_0^a x\sqrt{a^2-x^2}\,dx$ (27) $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$ (28) $\displaystyle\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$

(29) $\displaystyle\int_0^1 \frac{dx}{(2-x^2)\sqrt{4-x^2}}$

(30) $\displaystyle\int_0^a x\sqrt{\frac{a^2-x^2}{a^2+x^2}}dx$ [Hint: Put $x^2 = a^2\cos 2\theta$]

(31) $\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ (32) $\int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}$ [Hint: Put $x + 1 = y^2$]

(33) $\int_2^5 \sqrt{\frac{x-2}{5-x}}dx$ (34) $\int_2^3 \frac{dx}{\sqrt{(x-2)(3-x)}}$

(35) $\int_\alpha^\beta \frac{dx}{x\sqrt{(x-\alpha)(\beta-\alpha)}}$

## 3.4 INTEGRATION OF RATIONAL (ALGEBRAIC) FUNCTIONS

Integrals of the form $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{(lx+m)}{ax^2+bx+c}dx$ are two typical integrals of rational functions.

(1) To evaluate $\int \frac{dx}{ax^2+bx+c}$, we rewrite it as $\frac{1}{a}\int \frac{dx}{x^2+\frac{b}{a}x+\frac{c}{a}}$ (viz., the coefficient of $x^2$ is made unity)

Then $\int \frac{dx}{x^2+\frac{b}{a}x+\frac{c}{a}}$ is re-written in any one of the forms $\int \frac{dx}{(x+p)^2+q^2} \cdot \int \frac{dx}{(x+p)^2-q^2}$ and $\int \frac{dx}{q^2-(x+p^2)}$. There are extended standard integral formulas and hence easily evaluated.

(i) To evaluate $\int \frac{(lx+m)}{ax^2+bx+c}dx$, we express

$lx + m = A\cdot\frac{d}{dx}(ax^2+bx+c)+B$, where $A$ and $B$ are constants to be found out in individual problems.

Then $I = A\int \frac{\frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c}dx + B\int \frac{dx}{ax^2+bx+c}$

The first of these integrals is of the form $\int \frac{f'(x)}{f(x)}dx$ and hence the result is log $\{f(x)\}$ or log $(ax^2 + bx + c)$ and the second integral is evaluated as in case (i).

(ii) If the denominator of the integrand $f(x)$ in $\int f(x)dx$ can be factorised, $f(x)$ is split into partial fractions by algebraic method and the integration is performed term by term.

Integration of the trigonometric functions of the following form can be reduced to integration of rational functions, or integration by substitution.

(i) To evaluate integrals of the form $\int \frac{dx}{a\cos^2 x + b\sin^2 x + c}$ (where $a$, $b$, $c$ are constants, we multiply the numerator and denominator of the integrand by $\sec^2\theta$ and then the integral can be rewritten as $\int f(\tan\theta)\sec^2\theta\, d\theta$, which can be evaluated by earlier method by making the substitution $\tan\theta = u$.

(ii) To evaluate integrals of the form $\int \frac{dx}{a\cos x + b\sin x + c}$, we express $\cos x$ and $\sin x$ in terms of $\tan\frac{x}{2}$. On simplification, the integral takes the form $\int \frac{dt}{at^2 + bt + c}$ which can evaluated by the earlier method.

(iii) To evaluate integrals of the form $\int \left(\frac{l\cos x + m\sin x + n}{l'\cos x + m'\sin x + n'}\right)dx$, first we put integral in the form $Nr. = \left(A\,\text{Dr} + Bx\frac{d}{dx}\text{Dr}\right)$ where $A$, $B$, and $C$ constants to be found out in individual problems [Nr = numerator and Dr. = denomination]
Then the integral takes the form

$I = \int \left[A + B\frac{\frac{d}{dx}(\text{Dr})}{\text{Dr}} + c\right]dx$, the result of which is immediately obtained as

$Ax = B\log(\text{Dr.}) + c\int \frac{dx}{\text{Dr}}$. The third integral is a problem in the case (ii)

## WORKED EXAMPLE 3(b)

**Example 3.1** Evaluate $\int \frac{x\,dx}{x^4 + x^2 + 1}$.

The integrand is the product of a $f(x^2)$ and $x$. So we make substitution $x^2 = y$ and hence $dx = \frac{1}{2}dy$.

Then
$$I = \int \frac{\frac{1}{2}dy}{y^2 + y + 1} = \frac{1}{2}\int \frac{dy}{\left(y + \frac{1}{2}\right)^2 + \left(1 - \frac{1}{4}\right)}$$

$$= \frac{1}{2}\int \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$$

**Example 3.2** Evaluate $\int \frac{dx}{1+4x-4x^2}$.

$$I = \frac{1}{4}\int \frac{dx}{\frac{1}{4}+x-x^2} = \frac{1}{4}\int \frac{dx}{\frac{1}{4}-(x^2-x)}$$

$$= \frac{1}{4}\int \frac{dx}{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{4}\int \frac{dx}{\left(\frac{1}{\sqrt{2}}\right)^2-\left(x-\frac{1}{2}\right)^2} = \frac{1}{4}\cdot\frac{\sqrt{2}}{2}\log\left(\frac{\frac{1}{\sqrt{2}}+x-\frac{1}{2}}{\frac{1}{\sqrt{2}}-x+\frac{1}{2}}\right)$$

$$= \frac{1}{4\sqrt{2}} \log\left\{\frac{2-\sqrt{2}+2\sqrt{2}x}{2+\sqrt{2}-2\sqrt{2}x}\right\} + c$$

**Example 3.3** Evaluate $\int_0^1 \frac{dx}{x^2+2x\cos\theta+1}$; $0<\theta<1.$

$$I = \int_0^1 \frac{dx}{(x+\cos\theta)^2+(1-\cos^2\theta)}$$

$$= \int_0^1 \frac{dx}{(x+\cos\theta)^2+\sin^2\theta} = \left[\frac{1}{\sin\theta}\tan^{-1}\left(\frac{x+\cos\theta}{\sin\theta}\right)\right]_0^1$$

$$= \frac{1}{\sin\theta}\left\{\tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) - \tan^{-1}(\cot\theta)\right\}$$

$$= \frac{1}{\sin\theta}\left\{\tan^{-1}\left(\cot\frac{\theta}{2}\right) - \tan^{-1}(\cot\theta)\right\}$$

$$= \frac{1}{\sin\theta}\left\{\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - \tan^{-1}\tan\left(\frac{\pi}{2}-\theta\right)\right\}$$

$$= \frac{1}{\sin\theta}\cdot\left\{\frac{\pi}{2}-\frac{\theta}{2}-\frac{\pi}{2}+\theta\right\} = \frac{\theta}{2\sin\theta}.$$

**Example 3.4** Evaluate $\int \frac{x^8}{x^6+1}$.

Put $x^3 = y$ and so $x^2 \mathrm{d}x = \frac{1}{3}\mathrm{d}y$

$$I = \int \frac{\frac{1}{3}y^2 \mathrm{d}y}{y^2+1} = \frac{1}{3}\int\left(1 - \frac{1}{y^2+1}\right)\mathrm{d}y$$

$$= \frac{1}{3}[y - \tan^{-1} y] + c$$

$$= \frac{1}{3}[x^2 - \tan^{-1} x^3] + c$$

**Example 3.5** Evaluate $\int \frac{x-2}{2x^2+3x+1}\mathrm{d}x$.

Let $\quad x - 2 = A \cdot \frac{\mathrm{d}}{\mathrm{d}x}(2x^2+3x+1)$

$$= A(4x+3) + B$$

Comparing like terms; $A = \frac{1}{4}$ and $\frac{3}{4} + B = -2 \quad \therefore B = \frac{11}{4}$

Then
$$I = \int \frac{\frac{1}{4}(4x+3) - \frac{11}{4}}{2x^2+3x+1}\mathrm{d}x$$

$$= \frac{1}{4}\log(2x^2+3x+1) - \frac{11}{8}\int \frac{dx}{x^2 + \frac{3}{4}x + \frac{1}{2}}$$

$$= \frac{1}{4}\log(2x^2+3x+1) - \frac{11}{8}\int \frac{\mathrm{d}x}{\left(x+\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2}$$

$$= \frac{1}{4}\log(2x^2+3x+1) - \frac{11}{8} \times \frac{4}{2}\log\left(\frac{x+\frac{3}{4}-\frac{1}{4}}{x+\frac{3}{4}+\frac{1}{4}}\right)$$

$$= \frac{1}{4}\log(2x^2+3x+1) - \frac{11}{4}\log\left(\frac{2x+1}{2x+2}\right) + c$$

**Example 3.6** Evaluate $\int \frac{x+1}{6x+7-x^2}\mathrm{d}x$.

Let $\quad x + 1 = A(6-2x) + B$

Comparing like terms; $1 - 2A = 1 \;\therefore A = -\frac{1}{2}$ and $6A + B = 1 \;\therefore B = 4$

Then
$$I=\int\frac{-\frac{1}{2}(6-2x)+4}{6x+7-x^2}=-\frac{1}{2}\log(6x+7-x^2)+4\int\frac{dx}{7-(x^2-6x)}$$
$$=-\frac{1}{2}\log(6x+7-x^2)+4\int\frac{dx}{4^2-(x-3)^2}$$
$$=-\frac{1}{2}\log(6x+7-x^2)+4\times\frac{1}{2\times4}\log\left(\frac{4+x-3}{4-x+3}\right)+c$$
$$=-\frac{1}{2}\log(6x+7-x^2)+\frac{1}{2}\log\left(\frac{1+x}{7-x}\right)+c$$

**Example 3.7** Evaluate $\int\frac{(x^2+x+1)}{x^2-x+1}dx.$

Integrand is an improper function. So we express it as the sum of an integer and a proper function.

Then
$$I=\int\left(1+\frac{2x}{x^2-x+1}\right)dx$$
$$=x+\int\frac{1\cdot(2x-1)+1}{x^2-x+1}dx$$
$$=x+\log(x^2-x+1)+\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$
$$=x+\log(x^2-x+1)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c.$$

**Example 3.8** Evaluate $\int_0^2\frac{x^2+1}{4x^3+12x+7}dx.$

$$I=\int_0^2\frac{\frac{1}{12}\times(12x^2+12)}{4x^3+12x+7}dx=\frac{1}{12}\{\log(4x^3+12x+7)\}_0^2$$
$$=\frac{1}{12}[\log 63-\log 7]=\frac{1}{12}\log_e 9.$$

**Example 3.9** Evaluate $\int\frac{x^2+x+1}{(x-1)(x-2)^2}dx.$

As the denominator is the product of factors, we split the integrand into partial fractions.

Let $\frac{x^2+x+1}{(x-1)(x-2)^2}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$

$\therefore \qquad x^2 + x + 1 = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$

Putting $\qquad x = 1, \quad A = 3$

Putting $\qquad x = 2, \quad C = 7$

Equating coefficients of $x^2$ on both sides, we get $A + B = 1 \quad \therefore\ B = -2.$

Then
$$I = \int \left[\frac{3}{x-1} - \frac{2}{x-2} + \frac{7}{(x-2)^2}\right] \mathrm{d}x$$
$$= 3\log(x-1) - 2\log(x-2) - \frac{7}{x-2} + c$$

**Example 3.10** Evaluate $\displaystyle\int \frac{x^2-1}{x^4+x^2+1}\mathrm{d}x.$

Though the denominator of the integrand is not directly factorisable, it is made factorisable as explained below:

$$x^4 + x^2 + 1 = (x^4 + 2x^2 + 1) - x^2$$
$$= (x^2+1)^2 - x^2$$
$$= (x^2 + x + 1)(x^2 - x + 1)$$

Then
$$I = \frac{x^2-1}{(x^2-x+1)(x^2+x+1)}$$

Let
$$I = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+x+1}$$

$\therefore \qquad x^2 - 1 = (Ax+B)(x^2+x+1) + (cx+D)(x^2-x+1)$

Equating like coefficients, we get

$$A + C = 0$$
$$A + B - C + D = 1$$
$$A + B + C - D = 0$$
$$B + D = -1$$

Solving these equations, we get $A = 1$, $C = -1$, $B = -\frac{1}{2} = D$

$\therefore$
$$I = \int \left( \frac{x - \frac{1}{2}}{x^2 - x + 1} - \frac{x + \frac{1}{2}}{x^2 + x + 1} \right) \mathrm{d}x$$
$$= \frac{1}{2}\int \left( \frac{2x-1}{x^2-x+1} - \frac{2x+1}{x^2+x+1} \right) \mathrm{d}x$$
$$= \frac{1}{2}\log\left( \frac{x^2-x+1}{x^2+x+1} \right) + c$$

**Example 3.11** Evaluate $\int \frac{x\,\mathrm{d}x}{x^3+1}$.

The denominator of the integrand is factorisable as $(x + 1)(x^2 - x + 1)$

Let $$\frac{x}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$\therefore$ $$x = A(x^2 - x + 1) + (x+1)(Bx+0)$$

Putting $$x = -1,\ A = -\frac{1}{3}$$

Equating like terms, we get $A + B = 0 \ \therefore B = \frac{1}{3}$ and $A + C = 0 \quad \therefore C = \frac{1}{3}$

Then $$I = \int \left[ \frac{-\frac{1}{3}}{x+1} + \frac{1}{3}\frac{x+1}{x^2-x+1} \right] \mathrm{d}x$$

$$= -\frac{1}{3}\log(x+1) + \frac{1}{3}\int \frac{\frac{1}{2}(2x-1)+\frac{3}{2}}{x^2-x+1}\mathrm{d}x$$

$$= -\frac{1}{3}\log(x+1) + \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\int \frac{\mathrm{d}x}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log(x+1) + \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

**Example 3.12** Evaluate $\int_0^{\pi/2} \frac{\cos x\,\mathrm{d}x}{(1+\sin x)(2+\sin x)}$.

Put $\sin x = y$ and so $\cos x\,\mathrm{d}x = \mathrm{d}y$

When $x = 0$ and $\frac{\pi}{2}$, $y = 0$ and $1$ resepectively

$\therefore$ $$I = \int_0^1 \frac{\mathrm{d}y}{(1+y)(2+y)}$$

$$= \int_0^1 \left(\frac{1}{1+y} - \frac{1}{2+y}\right)\mathrm{d}y,$$

on splitting the integrand into partial fractions.

$\therefore$ $$I = [\log(1+y) - \log(2+y)]_0^1$$

$$= \log\frac{2}{3} - \log\frac{1}{2} = \log\left(\frac{4}{3}\right)$$

**Example 3.13** Evaluate $\int_0^\infty \frac{x}{(1+x)(1+x^2)}\,dx$.

Let $$\frac{x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$\therefore$ $$A(1+x^2) + (1+x)(Bx+c) = x$$

Putting $$x = -1,\ 2A = -1 \quad \therefore A = -\frac{1}{2}$$

Equating like terms, $$A + B = 0 \ \therefore B = \frac{1}{2}$$

$$\text{and}\ \ A + C = 0 \ \therefore C = \frac{1}{2}$$

Then $$I = \int_0^\infty \left( \frac{-\frac{1}{2}}{1+x} + \frac{1}{2}\cdot\frac{x+1}{1+x^2} \right) dx$$

$$= \left[ -\frac{1}{2}\log(1+x) + \frac{1}{4}\log(1+x^2) + \frac{1}{2}\tan^{-1} x \right]_0^\infty$$

$$= \log\left\{ \frac{(1+x^2)^{\frac{1}{4}}}{(1+x)^{\frac{1}{2}}} \right\}_0^\infty + \frac{1}{2}\tan^{-1}\infty - \frac{1}{2}\tan^{-1} 0$$

$$= \log\left\{ \frac{\left(\frac{1}{x^2}+1\right)^{\frac{1}{4}}}{\frac{1}{x}+1} \right\}_0^1 = \frac{\pi}{4}$$

**Example 3.14** Evaluate $\int \frac{dx}{\sin x + \sin 2x}$.

$$I = \int \frac{dx}{\sin x + 2\sin x\cos x}$$

$$= \int \frac{dx}{\sin x(1+2\cos x)} = \int \frac{\sin x\,dx}{(1-\cos^2 x)(1+2\cos x)}$$

Put $\cos x = y$ and so $\sin x\,dx = -dy$

$\therefore$ $$I = \int \frac{-dy}{(1-y)(1+y)(1+2y)}$$

Let $$\frac{1}{(1-y)(1+y)(1+2y)} = \frac{A}{1-y} + \frac{B}{1+y} + \frac{C}{1+2y}$$

$$=\frac{\frac{1}{6}}{1-y}-\frac{\frac{1}{2}}{1+y}+\frac{\frac{4}{3}}{1+2y},\text{ on splitting into partial fractions.}$$

$\therefore$
$$I=\int\left(\frac{-\frac{1}{6}}{1-y}+\frac{\frac{1}{2}}{1+y}-\frac{\frac{4}{3}}{1+2y}\right)\mathrm{d}y$$

$$=\frac{1}{6}\log(1-y+\frac{1}{2}\log(1+y)-\frac{1}{2}\cdot\frac{4}{3}\log(1+2y)$$

$$=\frac{1}{6}\log(1-\cos x)+\frac{1}{2}\log(1+\cos x)-\frac{2}{3}\log(1+2\cos x)+C.$$

**Example 3.15** Evaluate $\int\frac{\mathrm{d}x}{\sin^2 x+6\cos^2 x+3}$.

Multiplying the numerator and denominator by $\sec^2 x$,

we have
$$I=\int\frac{\sec^2 x\,\mathrm{d}x}{\tan^2 x+6+3(1+\tan^2 x)}$$

Since the integrand is the product of $f(\tan\theta)$ and $\sec^2\theta$, we make the substitution $\tan x=y$ and so $\sec^2 x\,\mathrm{d}x=\mathrm{d}y$

$\therefore$
$$I=\int\frac{\mathrm{d}y}{y^2+6+3(1+y^2)}$$

$$=\int\frac{\mathrm{d}y}{4y^2+9}=\frac{1}{4}\int\frac{\mathrm{d}y}{y^2+\left(\frac{3}{2}\right)^2}=\frac{1}{4}\cdot\frac{2}{3}\tan^{-1}\left(\frac{2y}{3}\right)$$

$$=\frac{1}{6}\tan^{-1}\left(\frac{2}{3}\tan x\right)+C$$

**Example 3.16** Evaluate $\int_0^{\pi/2}\frac{\mathrm{d}x}{a^2\cos^2 x+b^2\sin^2 x}$.

$$I=\int_0^{\pi/2}\frac{\sec^2 x\,\mathrm{d}x}{a^2+b^2\tan^2 x}$$

Put $\tan x=y$ and so $\sec^2 x\,\mathrm{d}x=\mathrm{d}y$

When $x=0,\ y=0$ and when $x=\frac{\pi}{2},\ y=\infty$

Then
$$I=\int_0^{\infty}\frac{\mathrm{d}y}{a^2+b^2y^2}=\frac{1}{b^2}\int_0^{\infty}\frac{\mathrm{d}y}{\left(\frac{a}{b}\right)^2+y^2}$$

$$= \frac{1}{b^2} \cdot \frac{b}{a} \left[ \tan^{-1} \left( \frac{by}{a} \right) \right]_0^{\infty}$$

$$= \frac{1}{ab} \left( \frac{\pi}{2} - 0 \right) = \frac{\pi}{2ab}$$

**Example 3.17** Evaluate $\int \frac{\mathrm{d}x}{4\cos x + 3\sin x + 5}$.

Expressing $\cos x$ and $\sin x$ in terms of $\tan\frac{x}{2}$ and putting $\tan\frac{x}{2} = t$ and so $\sec^2\frac{x}{2} \cdot \frac{1}{2}\mathrm{d}x = \mathrm{d}t$ or $\mathrm{d}x = \frac{2\mathrm{d}t}{1+t^2}$

we get

$$I = \int \frac{\frac{2\mathrm{d}t}{1+t^2}}{4\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right) + 5}$$

$$= 2\int \frac{\mathrm{d}t}{4(1-t^2) + 6t + 5(1+t^2)}$$

$$= 2\int \frac{\mathrm{d}t}{t^2 + 6t + 9} = 2\int \frac{\mathrm{d}t}{(t+3)^2}$$

$$= -\frac{2}{\tan\frac{x}{2} + 3} + c$$

**Example 3.18** Evaluate $\int_0^{\pi/2} \frac{\frac{\pi}{2}\mathrm{d}x}{12\cos x + 9\sin x}$.

Putting $\tan\frac{x}{2} = t$ and so $\mathrm{d}x = \frac{2\mathrm{d}t}{1+t^2}$ and changing the limits as 0 and 1, we get

$$I = \int_0^1 \frac{\frac{2\mathrm{d}t}{1+t^2}}{12\left(\frac{1-t^2}{1+t^2}\right) + 9 \times \frac{2t}{1+t^2}}$$

$$= 2\int_0^1 \frac{\mathrm{d}t}{12 - 12t^2 + 18t}$$

$$= \frac{1}{6}\int_0^1 \frac{\mathrm{d}t}{1 - \left(t^2 - \frac{3}{2}t\right)} = \frac{1}{6}\int_0^1 \frac{\mathrm{d}t}{\left(\frac{5}{4}\right)^2 - \left(t - \frac{3}{4}\right)^2}$$

$$= \frac{1}{6} \times \frac{2}{5} \log \left( \frac{\frac{5}{4} + t - \frac{3}{4}}{\frac{5}{4} - t + \frac{3}{4}} \right)_0^1$$

$$= \frac{1}{15} \left( \log \frac{3}{2} - \log \frac{1}{4} \right)$$

$$= \frac{1}{15} \log 6.$$

**Example 3.19** Evaluate $\int \frac{\sin x + 18 \cos x}{3 \sin x + 4 \cos x} dx.$

Let $\sin x + 18 \cos x = A(3 \sin x + 4 \cos x) + B(3 \cos x - 4 \sin x)$ Equating like terms, we get

$$3A - 4B = 1 \qquad (1)$$

and

$$4A + 3B = 18 \qquad (2)$$

Solving equation (1) and (2), we get $A = 3$ and $B = 2$

Then

$$I = \int \frac{3 \times (\text{Dr}) + 2 \frac{d}{dx} (\text{Dr.})}{\text{Dr.}} dx$$

$$= 3x + 2 \log(3 \sin x + 4 \cos x) + c$$

**Example 3.20** Evaluate $\int_0^{\pi/2} \frac{dx}{1 + \cot x}.$

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

Let $\sin x = A \cdot f(x) + B \cdot f'(x)$, where $f(x)$ is the denominator

$$= A(\sin x + \cos x) + B(\cos x - \sin x)$$

Equating like terms, we get $A - B = 1$ and $A + B = 0$

$\therefore$ $A = \frac{1}{2}$ and $B = -\frac{1}{2}$

Then

$$I = \int_0^{\pi/2} \left[ \frac{1}{2} - \frac{1}{2} \frac{f'(x)}{f(x)} \right] dx$$

$$= \left[ \frac{1}{2} x - \frac{1}{2} \log(\sin x + \cos x) \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left( \frac{\pi}{2} - 0 \right) - \frac{1}{2} (\log 1 - \log 1)$$

$$= \frac{\pi}{4}.$$

## EXERCISE 3(b)

***Part A***

(Short Answer Questions)

(1) $\dfrac{1}{x^2+a^2}$ (2) $\dfrac{1}{x^2-a^2}$ (3) $\dfrac{1}{a^2-x^2}$ (4) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x+1}{2}\right)$

(5) $\dfrac{x+1}{x^2+9}$ (6) $\dfrac{x}{x^4+a^4}$ (7) $\dfrac{1}{x(x+1)}$ (8) $\dfrac{\cos x}{\sin x(1+\sin x)}$

(9) $\dfrac{1}{\sin x}$ (10) $\dfrac{1}{\cos x}$

***Part B***

Integrate the following functions w.r.t. $x$:

(11) $\dfrac{x^2}{x^6+x^3+1}$ (12) $\dfrac{e^x}{e^{2x}+2e^x+10}$

(13) $\dfrac{2\log x+3}{x[(\log x)^2+2\log x+5}$ (14) $\dfrac{3x+7}{2x^2-3x+5}$

(15) $\dfrac{(5-4\sin x)\cos x}{1+2\sin x-\sin^2 x}$ (16) $\dfrac{x^2-x+1}{x^2+x+1}$

(17) Evaluate $\displaystyle\int_0^1 \frac{(x-3)}{x^2+2x-4}\,dx$ (18) Evaluate $\displaystyle\int_0^1 \frac{(x-3)}{x^2+2x-4}\,dx$

(19) Evaluate $\displaystyle\int \frac{dx}{x(x+1)(x+2)}$ (20) Evaluate $\displaystyle\int \frac{(7x-4)}{(x-1)^2(x+2)}\,dx$

(21) Evaluate $\displaystyle\int \frac{dx}{x^3+1}$ (22) Evaluate $\displaystyle\int \frac{2x^2+3}{x^3-1}\,dx$

Evaluate the following:

(23) $\displaystyle\int \frac{\cos x\,dx}{(1+\sin x)(1+\sin^2 x)}$ (24) $\displaystyle\int \frac{dx}{(x^2+1)(x^2+4)}$

(25) $\displaystyle\int \frac{dx}{\cos^2 x+2\sin^2 x+3}$ (26) $\displaystyle\int \frac{3\,dx}{2+7\cos^2 x}$

(27) $\displaystyle\int \frac{dx}{1+3\sin x+4\cos x}$ (28) $\displaystyle\int_0^{\pi} \frac{dx}{5+3\cos x}$

(29) $\displaystyle\int \frac{8\cos x + \sin x + 6}{3\cos x + 2\sin x + 4}\,dx$  (30) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\tan x}$

## 3.5 INTEGRATION OF IRRATIONAL FUNCTIONS

(i) To evaluate $\displaystyle\int \frac{dx}{\sqrt{ax^2+bx+c}}$, we make the coefficient of $x^2$ as unity, viz., I is rewritten as $\displaystyle\frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}$ which is put as $\displaystyle\frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{(x+p)^2 \pm q^2}}$ or $\displaystyle\frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{a^2-(x+p)^2}}$.

(ii) To evaluate $\displaystyle\int \frac{lx+m}{\sqrt{ax^2+bx+c}}\,dx$, we express $lx+m = A\times\frac{d}{dx}f(x)+B$, where $f(x) = ax^2+bx+c$ and $A$, $B$ are constants to be found out in individual problems.

Then $\displaystyle I = \int \frac{Af'(x)+B}{\sqrt{f(x)}}\,dx = A\times 2\sqrt{f(x)} + B\int \frac{dx}{\sqrt{f(x)}}$. The second integral is evaluated as in case (i).

(iii) $\displaystyle\int \sqrt{ax^2+bx+c}\,dx$:

Making coefficient of $x^2$ as unity, we get

$$I = \sqrt{a}\int \sqrt{(x+p)^2 \pm q^2}\,dx \text{ or } \sqrt{a}\int \sqrt{q^2-(x+p)^2}\,dx,$$

which are extended standard integrals.

(iv) $\displaystyle\int (lx+m)\sqrt{ax^2+bx+c}\,dx$

To evaluate this, we put $lx+m = Af'(x)+B$, when $A$ and $B$ are constants t to be found out in individual problems and $f(x) = ax^2+bx+c$.

Then
$$I = \int \{A\cdot f'(x)+B\}\sqrt{f(x)}\,dx$$
$$= A\cdot\frac{2}{3}[f(x)]^{3/2} + B\int \sqrt{f(x)}\,dx$$

Then second integral is evaluated as in case (iii).

(v) Integral of the form

$\int \frac{dx}{(x-k)\sqrt{ax^2+bx+c}}$ can be converted as $\int \frac{dx}{\sqrt{lx^2+mx+n}}$, by making the substitution $x-k=\frac{1}{y}$.

(vi) To evaluate $\int \frac{dx}{(px+q)\sqrt{lx+m}}$ also, the substitution $px+q=\frac{1}{y}$ can be made.

(vii) If a part of the integral contains $\sqrt{ax+b}$, we may put $ax+b=y^2$ and remove the irrational part from the integrand.

(viii) If the integral is of the form $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$, we put $x=\frac{1}{y}$. Then the integral takes the form which can be evaluated by substitution.

## WORKED EXAMPLE 3(c)

**Example 3.1** Evaluate $\int \frac{dx}{\sqrt{6x-x^2-5}}$.

$$I=\int \frac{dx}{\sqrt{-5-(x^2-6x)}}$$
$$=\int \frac{dx}{\sqrt{2^2-(x-3)^2}}$$
$$=\sin^{-1}\left(\frac{x-3}{2}\right)+c$$

**Example 3.2** Evaluate $\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}}$.

$$I=\int_0^1 \frac{dx}{\sqrt{(x+1)^2+1}}=\left[\log\{(x+1)+\sqrt{x^2+2x+2}\}\right]_0^1$$
$$=\log(2+\sqrt{5})-\log(1+\sqrt{2})$$
$$=\log\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$$

**Example 3.3** Evaluate $\int \frac{dx}{\sqrt{2x^2 - 7x + 5}}$.

$$I = \frac{1}{\sqrt{2}} \int \frac{dx}{x^2 - \frac{7}{2}x + \frac{5}{2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(x - \frac{7}{4}\right)^2 - \frac{9}{16}}}$$

$$= \frac{1}{\sqrt{2}} \cosh^{-1}\left(\frac{4x - 7}{3}\right), \text{ since } a^2 = \frac{9}{16}$$

**Example 3.4** Evaluate $\int \frac{x}{\sqrt{5x^2 - 4x}} dx$.

Let $$x = A \cdot \frac{d}{dx}(5x^2 - 4x) + B$$

$$A \cdot (10x - 4) + B$$

Equating like terms, we 10 $A = 1$ and so $A = \frac{1}{10}$.

$$-4A + B = 0 \text{ and so } B = \frac{2}{5}$$

Then $$I = \int \frac{\frac{1}{10}(10x - 4) + \frac{2}{5}}{\sqrt{5x^2 - 4x}}$$

$$= \frac{1}{10} \times 2\sqrt{5x^2 - 4x} + \frac{2}{5\sqrt{5}} \int \frac{dx}{\sqrt{x^2 - \frac{4}{5}x}}$$

$$= \frac{1}{5}\sqrt{5x^2 - 4x} + \frac{2}{5\sqrt{5}} \int \frac{dx}{\sqrt{\left(x - \frac{2}{5}\right)^2 - \left(\frac{2}{5}\right)^2}}$$

$$= \frac{1}{5}\sqrt{5x^2 - 4x} + \frac{2}{5\sqrt{5}} \cosh^{-1}\left(\frac{5x - 2}{2}\right) + c.$$

**Example 3.5** Evaluate $\int \frac{3x - 2}{\sqrt{4x^2 - 8x + 13}} dx$.

Let $$3x - 2 = A \times \frac{d}{dx}(4x^2 - 8x + 13) + B$$

$$= A \times (8x - 8) + B$$

Equating like terms, $A = \frac{3}{8}$ and $B - 3 = -2$ and so $B = 1$.

Then $$I = \int \frac{\frac{3}{8}(8x-8)+1}{\sqrt{4x^2-8x+13}}\,dx$$

$$= \frac{3}{4}\sqrt{4x^2-8x+13} + \frac{1}{2}\int \frac{dx}{\sqrt{x^2-2x+\frac{13}{4}}}$$

$$= \frac{3}{4}\sqrt{4x^2-8x+13} + \frac{1}{2}\int \frac{dx}{\sqrt{(x-1)^2+\left(\frac{3}{2}\right)^2}}$$

$$= \frac{3}{4}\sqrt{4x^2-8x+13} + \frac{1}{2}\sinh^{-1}\frac{2(x-1)}{3} + c$$

**Example 3.6** Evaluate $\int_2^5 \sqrt{\frac{x-2}{5-x}}\,dx.$

Multiplying the numerator and denominator of the integrand by $\sqrt{x-2}$, we get

$$I = \int_2^5 \frac{(x-2)}{\sqrt{(x-2)(5-x)}}\,dx \text{ or } \int_2^5 \frac{(x-2)}{\sqrt{-10+7x-x^2}}\,dx$$

Let $x-2 = A\times(7-2x)+B$

Equating like terms, $A = -\frac{1}{2}$ and $B = \frac{3}{2}$

$\therefore$ $$I = \int \frac{-\frac{1}{2}\times(7-2x)+\frac{3}{2}}{\sqrt{-10+7x-x^2}}$$

$$= -\sqrt{-10+7x-x^2} + \frac{3}{2}\int \frac{dx}{\sqrt{-10+7x-x^2}}$$

$$I = -\{\sqrt{(x-2)(5-x)}\}_2^5 + \frac{3}{2}\int_2^5 \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2}}$$

$$= 0 + \frac{3}{2}\left[\sin^{-1}\left(\frac{2x-7}{3}\right)\right]_2^5$$

$$= \frac{3}{2}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{3\pi}{2}$$

**Example 3.7** Evaluate $\int \sqrt{x(1-x)}\,\mathrm{d}x$.

$$\begin{aligned}
I &= \int \sqrt{x - x^2}\,\mathrm{d}x \\
&= \int \sqrt{-(x^2 - x)}\,\mathrm{d}x \\
&= \int \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}\,\mathrm{d}x \\
&= \frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{x(1-x)} + \frac{\left(\frac{1}{2}\right)^2}{2}\sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right) \\
&= \frac{(2x-1)}{4}\sqrt{x - x^2} + \frac{1}{8}\sin^{-1}(2x-1) + c
\end{aligned}$$

**Example 3.8** Evaluate $\int (x+1)\sqrt{x^2 - x + 1}\,\mathrm{d}x$.

Let
$$\begin{aligned}
x + 1 &= A\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - x + 1) + B \\
&= AX(2x-1) + B
\end{aligned}$$

Equating like terms, we get $2A = 1$ and so $A = \frac{1}{2}$ and $-A + B = 1$ and so $B = \frac{3}{2}$

Then $I = \int \left\{\frac{1}{2}f'(x) + \frac{3}{2}\right\}\sqrt{f(x)}\,\mathrm{d}x$, where $f(x) = x^2 - x + 1$

$$\begin{aligned}
&= \frac{1}{2}\frac{\{f(x)\}^{3/2}}{\frac{3}{2}} + \frac{3}{2}\int \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}x \\
&= \frac{1}{3}(x^2 - x + 1)^{3/2} + \frac{3}{2}\left[\frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{x^2 - x + 1} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\sinh^{-1}\left(\frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right] \\
&= \frac{1}{3}(x^2 - x + 1) + \frac{3}{8}(2x-1)\sqrt{x^2 - x + 1} + \frac{9}{16}\sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c
\end{aligned}$$

**Example 3.9** Evaluate $\int (x+1)\sqrt{\frac{x+2}{x-2}}\,\mathrm{d}x$.

$$I = \int (x+1)\frac{(x+2)}{\sqrt{x^2+4}}\,\mathrm{d}x,$$ on multiplying the numerator and denominator by $\sqrt{x+2}$.

Now $I = \int \frac{x^2+3x+2}{\sqrt{x^2-4}}\mathrm{d}x$. This integral is of form $\int \frac{lx^2+mx+n}{\sqrt{ax^2+bx+c}}\mathrm{d}x$

To evaluate this integral, we proceed as given below

Let $$x^2+3x+2 = A(x^2-4) + B\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x^2-4) + C$$

Equating like terms, we get $A=1$, $B=\frac{3}{2}$ and $C=6$

Then $$I = \int \sqrt{x^2-4}\,\mathrm{d}x + \frac{3}{2}\int \frac{\frac{\mathrm{d}}{\mathrm{d}x}(x^2-4)}{\sqrt{x^2-4}}\mathrm{d}x + 6\int \frac{dx}{x^2-4}$$

$$= \frac{x}{2}\sqrt{x^2-4} - \frac{4}{2}\cosh^{-1}\left(\frac{x}{2}\right) + \frac{3}{2}\times 2\sqrt{x^2-4} + 6\cosh^{-1}\frac{x}{2}$$

$$= \frac{x}{2}\sqrt{x^2-4} + 3\sqrt{x^2-4} + 4\cosh^{-1}\left(\frac{x}{2}\right) + C.$$

**Example 3.10** Evaluate $\int \frac{\mathrm{d}x}{(x+1)\sqrt{x^2+4x+2}}$.

Let $$x+1 = \frac{1}{y} \text{ and so } \mathrm{d}x = -\frac{1}{y^2}\mathrm{d}y \text{ and } x = \frac{1-y}{y}$$

Then $$= \int \frac{-\frac{1}{y^2}\mathrm{d}y}{\frac{1}{y}\sqrt{\left(\frac{1-y}{y}\right)^2 + 4\left(\frac{1-y}{y}\right) + 2}}$$

$$= -\int \frac{\mathrm{d}y}{\sqrt{(1-y)^2 + 4y(1-y) + 2y^2}}$$

$$= -\int \frac{\mathrm{d}y}{\sqrt{1+2y-y^2}} = -\int \frac{\mathrm{d}y}{\sqrt{(\sqrt{2})^2 - (y-1)^2}}$$

$$= \cos^{-1}\left(\frac{y-1}{\sqrt{2}}\right) \text{ or } \cos^{-1}\left[\frac{-x}{\sqrt{2}(x+1)}\right] + c$$

**Example 3.11** Evaluate $\int \frac{\mathrm{d}x}{(4x+1)\sqrt{1-x-x^2}}$.

Let $$4x+1 = \frac{1}{y} \text{ and so } \mathrm{d}x = -\frac{1}{4y^2}\mathrm{d}y \text{ and } x = \left(\frac{1}{y}-1\right) \div 4 \text{ or } \frac{1-y}{4y}$$

Then
$$I = \int \frac{-\frac{1}{4y^2}\,dy}{\frac{1}{y}\sqrt{1-\left(\frac{1-y}{4y}\right)-\left(\frac{1-y}{4y}\right)^2}}$$

$$= \int \frac{-dy}{\sqrt{16y^2 - 4y(1-y) - (1-y)^2}}$$

$$= -\int \frac{dy}{19y^2 - 2y - 1}$$

$$= -\frac{1}{\sqrt{19}}\int \frac{dy}{\sqrt{y^2 - \frac{2}{19}y - \frac{1}{19}}}$$

$$= -\frac{1}{\sqrt{19}}\int \frac{dy}{\sqrt{\left(y-\frac{1}{19}\right)^2 - \left(\frac{\sqrt{20}}{19}\right)^2}}$$

$$= -\frac{1}{\sqrt{19}}\cosh^{-1}\left\{\frac{\left(y-\frac{1}{19}\right)}{\left(\frac{\sqrt{20}}{19}\right)}\right\}$$

$$= -\frac{1}{\sqrt{19}}\cosh^{-1}\left(\frac{19y-1}{\sqrt{20}}\right)$$

$$= -\frac{1}{\sqrt{19}}\cosh^{-1}\left(\frac{19\times\frac{1}{4x+1}-1}{\sqrt{20}}\right)$$

$$= -\frac{1}{\sqrt{19}}\cosh^{-1}\left\{\frac{76x+18}{\sqrt{20}(4x+1)}\right\}$$

$$= -\frac{1}{\sqrt{19}}\cosh^{-1}\left\{\frac{38x+9}{\sqrt{5}(4x+1)}\right\} + c.$$

**Example 3.12** Evaluate $\int \frac{dx}{(x+2)\sqrt{x+3}}$.

Let $x + 2 = \frac{1}{y}$ and so $dx = -\frac{1}{y^2}dy$. Also $x + 3 = 1 + \frac{1}{y}$

Then
$$I = \int \frac{-\frac{1}{y^2}dy}{\frac{1}{y}\sqrt{\frac{1}{y}+1}} = -\int \frac{dy}{\sqrt{y}\cdot\sqrt{y+1}}$$

$$= -\int \frac{dy}{\sqrt{y^2+y}} = -\int \frac{dy}{\left(y+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= -\cosh^{-1}(2y+1) \text{ or } -\cosh\left(2\times\frac{1}{x+2}+1\right)$$

$$= -\cosh^{-1}\left(\frac{x+4}{x+2}\right) + c.$$

**Example 3.13** Evaluate $\int \frac{\sqrt{x}}{2+\sqrt{x}}\,dx$.

Let $x = y^2$ and so $dx = 2y\,dy$

Then
$$I = \int \frac{y\cdot 2y\,dy}{2+y}$$

$$= 2\int\left(y-2+\frac{4}{y+2}\right)dy,$$ as the improper function $\frac{y^2}{y+2}$ has been rewritten

$$= 2\left[\frac{y^2}{2} - 2y + 4\log(y+2)\right]$$
$$= x - 4\sqrt{x} + 8\log(\sqrt{x}+2) + c$$

**Example 3.14** Evaluate $\int \frac{dx}{(x^2+1)\sqrt{x^2-4}}$.

Let $x = \frac{1}{y}$ and so $dx = -\frac{1}{y^2}dy$

Then
$$I = \int \frac{-\frac{1}{y^2}dy}{\left(\frac{1}{y^2}+1\right)\sqrt{\frac{1}{y^2}-4}}$$

$$I = \int \frac{-y\,dy}{(y^2+1)\sqrt{1-4y^2}}$$

Let $1-4y^2 = u^2$ and so $-8y\,dy = 2u\,du$

Then
$$I = \int \frac{\frac{1}{4}u\,du}{\frac{1}{4}(5-u^2)\cdot u} = \int \frac{du}{(\sqrt{5})^2 - u^2}$$

$$= \frac{1}{2\sqrt{5}} \log\left(\frac{\sqrt{5}+u}{\sqrt{5}-u}\right)$$

$$= \frac{1}{2\sqrt{5}} \log\left\{\frac{\sqrt{5}+\sqrt{1-4y^2}}{\sqrt{5}-\sqrt{1-4y^2}}\right\}$$

$$= \frac{1}{2\sqrt{5}} \log\left\{\frac{\sqrt{5}+\sqrt{1-\frac{4}{x^2}}}{\sqrt{5}-\sqrt{1-\frac{4}{x^2}}}\right\}$$

$$= \frac{1}{2\sqrt{5}} \log\left\{\frac{x\sqrt{5}+\sqrt{x^2-4}}{x\sqrt{5}-\sqrt{x^2-4}}\right\} + c$$

**Example 3.15** Evaluate $\int_0^{1/\sqrt{3}} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

Let $x = \frac{1}{y}$ and so $dx = \frac{1}{y^2}dy$, when $x = 0$, $y = \infty$ and when

$x = \frac{1}{\sqrt{3}}$, $y = \sqrt{3}$

Then
$$I = \int_{\infty}^{\sqrt{3}} \frac{-\frac{1}{y^2}dy}{\left(1+\frac{1}{y^2}\right)\sqrt{1-\frac{1}{y^2}}}$$

$$= \int_{\sqrt{3}}^{\infty} \frac{y\,dy}{(y^2+1)\sqrt{y^2-1}}$$

Let $y^2 - 1 = u^2$ and so $y dy = u\,du$; when $y = \sqrt{3}$, $u = \sqrt{2}$ and when $y = \infty$, $u = \infty$

$$I = \int_{\sqrt{2}}^{\infty} \frac{u\,du}{(u^2+2)u}$$

Then
$$= \frac{1}{\sqrt{2}}\left(\tan^{-1}\frac{u}{\sqrt{2}}\right)_{\sqrt{2}}^{\infty}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$$

## EXERCISE 3(c)

***Part A***

(Short Answer Questions)

1. Evaluate $\int \frac{dx}{\sqrt{a^2 - x^2}}$
2. Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$
3. Evaluate $\int \frac{dx}{\sqrt{x^2 + a^2}}$
4. Evaluate $\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$
5. Evaluate $\int \frac{dx}{\sqrt{4 - 2x - x^2}}$
6. Evaluate $\int \frac{dx}{\sqrt{x^2 + 3x + 1}}$
7. Evaluate $\int \frac{(x+3)dx}{\sqrt{1 - x^2}}$
8. Evaluate $\int \frac{dx}{(x+4)\sqrt{x}}$
9. Evaluate $\int \frac{dx}{x\sqrt{1 - x}}$
10. Evaluate $\int \frac{dx}{\sqrt{2ax - x^2}}$

***Part B***

11. Evaluate $\int \frac{\frac{1}{\sqrt{3}}}{\sqrt{x(2 - 3x)}} dx$
12. Evaluate $\int_0^1 \frac{x dx}{\sqrt{1 + x^4}}$
13. Evaluate $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta - x)}} (\beta > \alpha)$

14. Evaluate $\int \frac{(2x+5)}{\sqrt{x^2-2x+10}}\,dx$

15. Evaluate $\int \sqrt{\frac{x+1}{2x-3}}\,dx$

16. Evaluate $\int_0^2 \frac{2x}{\sqrt{3x-x^2-2}}\,dx$

17. Evaluate $\int \frac{x^2+2x+3}{\sqrt{x^2+1}}\,dx$

18. Evaluate $\int \frac{dx}{(x+2)\sqrt{x^2+6x+7}}$

19. Evaluate $\int \frac{dx}{x\sqrt{7x^2-6x-1}}$

20. Evaluate $\int \frac{dx}{(2x+3)\sqrt{x+5}}$

21. Evaluate $\int \frac{dx}{(x+3)\sqrt{x-5}}$

22. Evaluate $\int_0^{1/\sqrt{3}} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

23. Evaluate $\int \frac{dx}{(x^2-1)\sqrt{x^2+1}}$

24. Evaluate $\int \sqrt{7x-10-x^2}\,dx$

25. Evaluate $\int (x+1)\sqrt{x^2+x+1}\,dx$

## 3.6 INTEGRATION BY PARTS

When $u$ and $v$ are function of $x$, then by the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

viz.,

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integration both side w.r.t. $x$, we have

$$\int u\,dv = uv - \int v\,du$$

Evaluation of $\int u\,\mathrm{d}v$ depends on $\int v\,\mathrm{d}u$. Thus when we want to evaluate an integral which is a product, it should be identified as the product of one factor ($u$) and the differential of another factor $v$. The choice of $u$ and $v$ should be carefully made so that $\int v\,du$ is easier then $\int u\,dv$.

## 3.6.1 Improper Integrals

The definite integral $I = \int_a^b f(x)\mathrm{d}x$ has meaning only when the limits $a$ and $b$ are finite and the integrand $f(x)$ is bounded in the interval $[a, b]$. Now we extend the definition when the range of integration is infinite or when the integrand is unbounded.

**Definition:** If $f(t)$ is bounded and integrable in $a \le t \le x$, where $a$ is a constant and $z$ is any number greater than a and if $\int f(t)\mathrm{d}t = F(t)$, then

$$\int_a^\infty f(t)\mathrm{d}t = \lim_{x\to\infty} \int_a^x f(t)\mathrm{d}t = \lim_{x\to\infty}\{F(x) - F(a)\}$$

is called *the improper* or infinite integral of $f(t)$, provided the limit exists. If the limits exists, the integral is said to *converge*. On the other hand, if $\int_a^x f(t)\mathrm{d}t \to \infty$ as $x \to \infty$, the integral is said to *diverge* to +∞ or said not to exist.

Similarly $\lim_{x\to-\infty} \int_x^b f(t)\mathrm{d}t = \lim_{x\to-\infty} \{F(b) - F(x)\}$, if the limit exists, is denoted by $\int_{-\infty}^b f(t)\mathrm{d}t$ and the infinite integral is said to converge

Finally $\int_{-\infty}^\infty f(t)dt = \lim_{x\to-\infty} \int_x^a f(t)\mathrm{d}t + \lim_{x\to\infty} \int_a^x f(t)\mathrm{d}t$ is defined the infinite integral in the in the L.S. is said to converge if both the integrals in the R.S. converge, '$a$' is arbitrary and the value of the integral does not depend on $a$.

## 3.6.2 Integral with Unbounded Integrands

If $f(t)$ is unbounded at $t \to a$ in $a \le t \le b$, then $\int_a^b f(t)dt = \lim_{\in\to 0} \int_{a+\in}^b f(t)dt(\in > 0)$, provided the limit exists.

Similarly if $f(t)$ becomes unbounded as to $\to b$ in $a \le t \le b$, then

$\int_a^b f(t)\mathrm{d}t = \lim_{\in\to 0} \int_a^{b-\in} f(t)\mathrm{d}t(\in > 0)$, provided the limit exists.

If both and $a$ and $b$ are points of discontinuity, then

$\int_a^b f(t)\mathrm{d}t = \lim_{\in \to 0} \int_{a+\in}^{c} f(t)\mathrm{d}t + \lim_{\in \to 0} \int_{c+\in_2}^{b-\in} f(t)\mathrm{d}t,\ a < c < b.$ If $f(t)$ is unbounded at an interior point $c$ such that $a < c < b$, then $\int_a^b f(t)\mathrm{d}t = \lim_{\in_1 \to 0} \int_a^{c-\in_1} f(t)at + \lim_{\in_2 \to 0} \int_{c+\in_2}^{b} f(t)\mathrm{d}t,$

where $\in_1$ and $\in_2$ are two arbitrary positive quantities tending to zero independently.

Although the two integrals in the R.S. may not exist independently, their sum may exist when $\in_1 = \in_2 = \in$.

The value of the sum is called the *Cauchy Principal value* and written as

$$P\int_a^b f(t)\mathrm{d}t = \lim_{\in \to 0}\left[\int_a^{c-\in} f(t)\mathrm{d}t + \int_{c+\in}^{b} f(t)\mathrm{d}t\right]$$

If $f(t)$ has a finite number of points of infinite discontinuity in $(a, b)$, say, $c_1, c_2, \ldots, c_n$ where $a \le c_1 \le c_2 \cdots \le c_n \le b$ then

$$\int_a^b f(t)\mathrm{d}t = \int_a^{c_1} f(t)\mathrm{d}t + \int_{c_1}^{c_2} f(t)\mathrm{d}t + \cdots + \int_{c_n}^{b} f(t)$$

### 3.6.3 Comparison Tests for Improper Integrals

Let $\int_a^b f(t)\mathrm{d}t$ be an improper integral. If there exists a $g(t)$ such that $|f(t)| \le g(t)$ in $a \le t \le b$ and $\int_a^b g(t)\mathrm{d}t$ converge, then $\int_a^b f(t)\mathrm{d}t$ also converges.

If there exists function $g(t)$ such that $f(t) \ge |g(t)|$ in $a \le t \le b$ and $\int_a^b |g(t)|\mathrm{d}t$ diverges, then $\int_a^b f(t)\mathrm{d}t$ also diverges.

### *Limit Form of Comparison Tests*

Let $f(x) > 0$ and $g(x) > 0$ for all $x \le a$.

If $\lim_{x \to +\infty}\left[\frac{f(x)}{g(x)}\right] = k$, where $k \ne 0$, then both the integrals $\int_a^\infty f(x)\mathrm{d}x$ and $\int_a^\infty g(x)\mathrm{d}x$ converge or diverge together.

It $k = 0$, we may conclude only that the convergence of $\int_a^\infty g(x)\mathrm{d}x$ implies that of $\int_a^\infty f(x)\mathrm{d}x$.

**WORKED EXAMPLE 3(d)**

**Example 3.1** Evaluate $\int x^3 e^{x^2}\,dx$.

Let $x^2 = t$ and so $x\,dx = \frac{1}{2}dt$

Then $I = \frac{1}{2}\int te^t dt$

$= \frac{1}{2}\int t d(e^t)$ $\quad$ [**Note :** $u = t$; $dv = e^t dt$. $\therefore v = e^t$]

$= \frac{1}{2}[te^t - \int e^t dt]$

$= \frac{1}{2}[x^2 e^{x^2} - e^{x^2}] + c$

**Example 3.2** Evaluate $\int x^3 e^{2x} dx$.

$$I = \int x^3 d\left(\frac{e^{2x}}{2}\right)$$

$$\left[\because u = x^3 \text{ and } dv = e^{2x}dx \text{ and so } v = \int e^{2x}dx = \frac{e^{2x}}{2}\right]$$

$$= \frac{1}{2}\left[x^3 e^{2x} - \int e^{2x}\cdot 3x^2 dx\right]$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\int x^2 d\left(\frac{e^{2x}}{2}\right)$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}\left[x^2 e^{2x} - \int e^{2x}\cdot 2x dx\right]$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\int x d\left(\frac{e^{2x}}{2}\right)$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}\left[xe^{2x} - \int e^{2x}dx\right]$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + c$$

**Example 3.3** Evaluate $\int x^2 \sin 2x\,dx$.

$$I = \int x^2 d\left(-\frac{\cos 2x}{2}\right)$$

$$\left[\because\ u = x^2 \text{ and } dv = \sin 2x\,dx \text{ and so } v = \int \sin 2dx = -\frac{\cos 2x}{2}\right]$$

$$= -\frac{x^2}{2}\cos 2x - \int -\frac{\cos 2x}{2}\cdot 2x\,dx$$

$$= -\frac{x^2}{2}\cos 2x + \int x d\left(\frac{\sin 2x}{2}\right)$$

$$= -\frac{x^2}{2}\cos 2x + \frac{1}{2}\left[x\sin 2x - \int \sin 2x\,dx\right]$$

$$= -\frac{x^2}{2}\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

**Example 3.4** Evaluate $\int x \tan^2 x\,dx$.

$$I = \int x(\sec^2 x - 1)dx = \int x d(\tan x) - \frac{x^2}{2}$$

$$= x\tan x - \int \tan x dx - \frac{x^2}{2}$$

$$= x\tan x - \log \sec x - \frac{x^2}{2} + c$$

**Example 3.5** Evaluate $\int_1^2 x^n \log x\,dx$.

Let
$$I = \int_1^2 x^n \log x\,dx = \int_1^2 \log x \cdot d\left(\frac{x^{n+1}}{n+1}\right)$$

$$= \left(\frac{x^{n+1}}{n+1}\log x\right)_1^2 - \int_1^2 \frac{x^{n+1}}{n+1}\cdot\frac{1}{x}dx$$

$$= \left(\frac{x^{n+1}}{n+1}\log x\right)_1^2 - \left\{\frac{x^{n+1}}{(n+1)^2}\right\}_1^2$$

$$= \frac{2^{n+1}}{n+1}\log 2 - \frac{2^{n+1}}{(n+1)^2} + \frac{1}{(n+1)^2}$$

**Example 3.6** Evaluate $\int \frac{\log x}{(x+1)^2}dx$.

Let
$$I = \int \log x \cdot d\left(-\frac{1}{1+x}\right)$$

$$= -\log x \cdot \frac{1}{x+1} - \int -\frac{1}{x+1}\cdot\frac{1}{x}dx$$

$$= -\frac{\log x}{x+1} + \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= -\frac{\log x}{x+1} + \log\left(\frac{x}{x+1}\right) + c$$

**Example 3.7** Evaluate $\int (\log x)^2 \, dx$.

***Note*** ☑ The integrand is not a product of two factors. So we assume that $u = (\log x)^2$ and $dx = dv$ so that $v = x$.

Then
$$I = \int (\log x)^2 \, d(x) = x(\log x)^2 - \int x \cdot 2\log x \cdot \frac{1}{x} dx$$
$$= x(\log x)^2 - 2\int (\log x) \, d(x)$$
$$= x(\log x)^2 - 2\left\{x\log x - \int x \cdot \frac{1}{x} dx\right\}$$
$$= x(\log x)^2 - 2x\log x + 2x + c$$

**Example 3.8** Evaluate $\int x \sin^{-1} x \, dx$.

Let
$$I = \int \sin^{-1} x \cdot d\left(\frac{x^2}{2}\right)$$
$$= \frac{x^2}{2}\sin^{-1} x - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} dx$$
$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sin^2\theta \, d\theta, \text{ on putting } x = \sin\theta$$
$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$
$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)$$
$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}(\theta - \sin\theta\cos\theta)$$
$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}(\sin^{-1} x - x\sqrt{1-x^2}) + c$$

**Example 3.9** Evaluate $\int \frac{\tan^{-1} x}{x^2} dx$.

$$I = \int \tan^{-1} x \cdot d\left(-\frac{1}{x}\right)$$
$$= -\frac{1}{x}\tan^{-1} x + \int \frac{1}{x} \cdot \frac{1}{1+x^2} dx$$

$$= -\frac{1}{x}\tan^{-1}x + \int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)\mathrm{d}x$$

$$= -\frac{1}{x}\tan^{-1}x + \log x - \frac{1}{2}\log(1+x^2) + c$$

**Example 3.10** Evaluate $\int \sin^{-1} x \cdot \mathrm{d}x$.

$$I = \int \sin^{-1} x \cdot \mathrm{d}(x)$$

$$= x\sin^{-1}x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$$

$$= x\sin^{-1}x - \int \frac{-\frac{1}{2}(-2x)}{\sqrt{1-x^2}}\mathrm{d}x$$

$$= x\sin^{-1}x + \frac{1}{2}\times 2\sqrt{1-x^2} + c$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + c$$

**Example 3.11** Evaluate $\int \frac{x+\sin x}{1+\cos x}\mathrm{d}x$.

Let
$$I = \int\left(\frac{x}{2\cos^2\frac{x}{2}} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)\mathrm{d}x$$

$$= \int\left(\frac{x}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)\mathrm{d}x$$

$$= \int\left[x\cdot \mathrm{d}\left(\tan\frac{x}{2}\right) + \int\tan\frac{x}{2}\right]\mathrm{d}x$$

$$= x\tan\frac{x}{2} - \int\tan\frac{x}{2}\mathrm{d}x + \int\tan\frac{x}{2}\mathrm{d}x$$

$$= x\tan\frac{x}{2} + c$$

**Example 3.12** Evaluate $\int \sqrt{x^2 - a^2}\mathrm{d}x$.

(*Note* ☑ This integral and $\int\sqrt{a^2-x^2}\mathrm{d}x$ and $\int\sqrt{x^2+a^2}\mathrm{d}x$ have been included in the list of standard integrals.)

Let
$$I = \int\sqrt{x^2-a^2}\,\mathrm{d}(x)$$

$$= x\sqrt{x^2 - a^2} - \int x \cdot \frac{2x}{2\sqrt{x^2 - a^2}}\,\mathrm{d}x$$

$$= x\sqrt{x^2 - a^2} - \frac{x^2 - a^2 + a^2}{\sqrt{x^2} - a}\,\mathrm{d}x$$

$$= x\sqrt{x^2 - a^2} - \int\left(\sqrt{x^2 - a^2} + \frac{a^2}{\sqrt{x^2 - a^2}}\right)\mathrm{d}x$$

$$= x\sqrt{x^2 - a^2}\; I - a^2 \cosh^{-1}\left(\frac{x}{a}\right)$$

$\therefore$
$$2I = x\sqrt{x^2 - a^2} - a^2 \cosh^{-1}\left(\frac{x}{a}\right)$$

$\therefore$
$$I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$$

**Example 3.13** Evaluate $I_1 = \int e^{ax}\cos bx\,\mathrm{d}x$ and $I_2 = \int e^{ax}\sin bx\,\mathrm{d}x$.

Let
$$I_1 = \int \cos bx \cdot \mathrm{d}\left(\frac{e^{ax}}{a}\right)$$

$$= \frac{1}{a}e^{ax}\cos bx - \int \frac{e^{ax}}{a}(-b\sin bx)\mathrm{d}x$$

$$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}I_2 \qquad (1)$$

Let
$$I_2 = \int \sin bx\,\mathrm{d}\left(\frac{e^{ax}}{a}\right)$$

$$= \frac{1}{a}e^{ax}\sin bx - \int \frac{e^{ax}}{a} \times b\cos bx\,\mathrm{d}x$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}I_1 \qquad (2)$$

Using (2) in (1), we get

$$I_1 = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left[\frac{1}{a}e^{ax}\sin bx - \frac{b}{a}I_1\right]$$

$\therefore$
$$\left(1 + \frac{b^2}{a^2}\right)I_1 = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx$$

$\therefore$
$$I_1 = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) \qquad (3)$$

Similarly, using (1) in (2) and simplifying, we can get

$$I_2 = \frac{e^{ax}}{a^2 + b^2}(a \sin bx + b \cos bx) \tag{4}$$

(*Note* ☑ Results (3) and (4) can be treated as formulas and remembered as such, as these will be used frequently in later situations)

**Example 3.14** Evaluate $\int_0^\infty e^{-3x} \sin 4x \, dx$.

Using the formula (4) derived in Example (3.13) above, we have

$$\int_0^\infty e^{-3x} \sin 4x \, dx = \left[\frac{e^{-3x}}{(-3)^2 + 4^2}(-3 \sin 4x - 4 \cos 4x\right]_0^\infty$$

$$= 0 - \frac{1}{25}(-4) = \frac{4}{25}$$

**Example 3.15** Evaluate $\int \cosh 3x \cos 4x \, dx$.

Let
$$I = \int \left(\frac{e^{3x} + e^{-3x}}{2}\right) \cos 4x \, dx$$

$$= \frac{1}{2}\int e^{3x} \cos 4x \, dx + \frac{1}{2}\int e^{-3x} \cos 4x \, dx$$

$$= \frac{1}{2}\frac{e^{3x}}{25}(3 \cos 4x + 4 \sin 4x)$$

$$+ \frac{1}{2} \cdot \frac{e^{-3x}}{25}(-3 \cos 4x + 4 \sin 4x) + c$$

$$= \frac{3}{25} \cos 4x \left(\frac{e^{3x} - e^{-3x}}{2}\right) + \frac{4}{25} \sin 4x \left(\frac{e^{3x} + e^{-3x}}{2}\right)$$

$$= \frac{3}{25} \cos 4x \sinh 3x + \frac{4}{25} \sin 4x \cosh 3x + c$$

**Example 3.16** Evaluate $\int e^x (\sin x + \cos x) \, dx$.

Let
$$I = \int e^x \sin x \, dx + \int e^x \cos x \, dx$$

$$= \int \sin x \cdot d(e^x) + \int e^x \cos x \, dx$$

$$= \int e^x \sin x - \int e^x \cos x \, dx + \int e^x \cos x \, dx$$

$$= e^x \sin x + c$$

**Example 3.17** Evaluate $\int e^x \left(\frac{1}{x} + \log x) \, dx\right.$

Let
$$I = \int e^x \cdot \frac{1}{x} dx + \int e^x \log x \, dx$$
$$= \int e^x d(\log x) + \int e^x \log x \, dx$$
$$= e^x \log x - \int \log x \, d(e^x) + \int e^x \log x \, dx$$
$$= e^x \log x - \int e^x \log x \, dx + \int e^x \log x \, dx$$
$$= e^x \log x + c$$

**Example 3.18** Evaluate $\int \frac{e^x (x^2 + 3x + 3)}{(x+2)^2} dx.$

$$I = \int e^x \left\{1 - \frac{x+1}{(x+2)^2}\right\} dx$$
$$= e^x - \int e^x \cdot \frac{x+1}{(x+2)^2} dx$$
$$= e^x + \int e^x (x+1) \, d\left(\frac{1}{x+2}\right)$$
$$= e^x + \left[\frac{e^x (x+1)}{x+2} - \int \frac{1}{x+2} d\{e^x (x+1)\}\right]$$
$$= e^x + \left[\frac{e^x (x+1)}{(x+2)} - \frac{1}{x+2} \{e^x + (x+1) e^x\}\right]$$
$$= e^x + \left[\frac{e^x (x+1)}{x+2} - e^x\right] = \left(\frac{x+1}{x+2}\right) e^x + c$$

**Example 3.19** Evaluate $\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx.$

Let
$$I = \int e^x \cdot \frac{1}{2\cos^2 \frac{x}{2}} dx + \int e^x \tan \frac{x}{2} dx$$
$$= \int e^x d\left(\tan \frac{x}{2}\right) + \int e^x \tan \frac{x}{2} dx$$
$$= e^x \tan \frac{x}{2} - \int \tan \frac{x}{2} \cdot e^x \, dx + \int e^x \tan \frac{x}{2} dx$$
$$= e^x \tan \frac{x}{2} + c$$

**Example 3.20** Examine the convergence of $\int_1^{\infty} \log x \, dx$.

The integrand ($\log x$) is unbounded as $x \to \infty$, viz. at $x = b$, where $b \to \infty$

$$\therefore \quad \int_1^{\infty} \log x \, dx = \lim_{b\to\infty} \int_1^{b} \log x \, dx$$

$$= \lim_{b\to\infty} (x \log x - x)_1^b$$

$$= \lim_{b\to\infty} [b \log b - b + 1] \to \infty \text{ as } b \to \infty$$

$$\therefore \quad \int_1^{\infty} \log x \, dx \text{ diverges to } +\infty$$

**Example 3.21** Test the convergence of the integral $\int_a^{\infty} \frac{1}{x^p} dx$, where $a > 0$ and $p \neq 1$.

$$\int_a^{\infty} \frac{1}{x^p} dx = \lim_{b\to\infty} \int_a^{b} \frac{1}{x^p} dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^b$$

$$= \lim_{b\to\infty} \left[\frac{-b^{-(p-1)}}{p-1}\right] + \frac{a^{1-p}}{p-1} \qquad (1)$$

Now $\lim_{b\to\infty} \left[\frac{-b^{-(p-1)}}{p-1}\right] \to 0$, if $p > 1$

$$\therefore \quad \int_a^{\infty} \frac{1}{x^p} dx = \frac{a^{1-p}}{p-1} \text{ by (1), if } p > 1$$

viz., I converges if $p > 1$ and diverges to $+\infty$, if $p < 1$

If $p = 1$, $\int_a^{\infty} \frac{1}{x} dx = \left[\lim_{b\to\infty} (\log b) - \log a\right] \to \infty$, as $b \to \infty$

$\therefore$ The integral diverges.

**Example 3.22** Examine the convergence of $\int_0^1 \frac{dx}{\sqrt{x}}$.

The integrand has an infinity at the lower limit

$$\therefore \quad I = \lim_{\epsilon\to 0} \left[\int_{0+\epsilon}^{1} \frac{dx}{\sqrt{x}}\right] = \lim_{\epsilon\to 0} (2\sqrt{x})_{\epsilon}^{1}$$

$$= \lim_{\epsilon\to 0} [2 - 2\sqrt{\epsilon}] = 2$$

$\therefore$ The given integral converges to 2.

**Example 3.23** Find the Cauchy's principal value of the integral $\int_2^4 \frac{dx}{(x-3)^3}$ $x = 3$ is the interior point of discontinuity for the integrand or has an infinity at $x = 3$

$$\int_2^4 \frac{dx}{(x-3)^3} = \lim_{\epsilon_1 \to 0} \int_2^{3-\epsilon_1} \frac{dx}{(x-3)^3} + \lim_{\epsilon_2 \to 0} \int_{3+\epsilon_2}^4 \frac{dx}{(x-3)^3}$$

$$= \lim_{\epsilon_1 \to 0} \left[ -\frac{1}{2(x-3)^2} \right]_2^{3-\epsilon_1} + \lim_{\epsilon_2 \to 0} \left[ -\frac{1}{2(x-3)^2} \right]_{3-\epsilon_2}^4$$

$$= \lim_{\epsilon_1 \to 0} \left[ +\frac{1}{2} - \frac{1}{2\epsilon_1^2} \right] + \lim_{\epsilon_2 \to 0} \left[ -\frac{1}{2} + \frac{1}{2\epsilon_2^2} \right]$$

Both the limits do not exist and hence the integral diverges. But if we put $\epsilon_1 = \epsilon_2 = \epsilon$, then

$$I = \lim_{\epsilon \to 0} \left[ \frac{1}{2} - \frac{1}{2\epsilon^2} - \frac{1}{2} + \frac{1}{2\epsilon^2} \right] = 0,$$ which is the Cauchy's principal value of the integral.

**Example 3.24** Test the convergence of $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$ and find its value.

Let $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$ the denoted by $f(x)$

Choose $g(x) = \int_0^3 \frac{dx}{x^{\frac{2}{3}}}$

$$\text{Now } \lim_{x \to \infty} \left[ \frac{f(x)}{g(x)} \right] = \lim_{x \to \infty} \left[ \frac{x^{\frac{2}{3}}}{(x-1)^{\frac{2}{3}}} \right]$$

$$= \lim_{x \to \infty} \left[ \left( 1 - \frac{1}{x} \right)^{-\frac{2}{3}} \right]$$

$$= 0$$

$\int_0^3 g(x)dx = \int_0^3 \frac{1}{x^{\frac{2}{3}}} dx$ converges. Hence by comparison test,

$\int_0^3 f(x)\,dx = \int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}$ also converges.

Required value $= \left[3(x-1)^{\frac{1}{3}}\right]_0^3 = 3 \times 2^{1/3} - 3(-1)^{1/3}$

$= 3\{1 + 2^{1/3}\}$

**Example 3.25** Test the convergence of $I = \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$.

$$I = \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{x}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{x}{1+x^2}\,dx + \lim_{l \to \infty} \int_{0}^{b} \frac{x}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \left\{\frac{1}{2}\log(1+x^2)\right\}_a^0 + \lim_{b \to \infty} \left\{\frac{1}{2}\log(1+x^2)\right\}_0^b$$

$$= \lim_{a \to \infty} \left[-\frac{1}{2}\log(1+a^2)\right] + \lim_{b \to \infty} \left[\frac{1}{2}\log(1+b^2)\right]$$

$$= 0$$

## EXERCISE 3(d)

***Part A***

(Short Answer Questions)

Evaluate the following integrals:

1. $\int xe^{ax}\,dx$
2. $\int x \sin mx\,dx$
3. $\int x \operatorname{cosec}^2 x\,dx$
4. $\int \frac{\log x}{x^2}\,dx$
5. $\int \log x\,dx$
6. $\int x \cos^2 x\,dx$
7. $\int \tan^{-1} x\,dx$
8. $\int \sinh^{-1} x\,dx$
9. $\int x \sec^{-1} x\,dx$
10. $\int x \log x\,dx$
11. Define improper integral.
12. Explain Cauchy's principal value of an improper integral.
13. When do you say that an improper integral converges or diverges?
14. State the comparison test used in testing convergence of an improper integral.
15. State the limit form of comparison test used for testing convergence of an improper integral.

***Part B***

Evaluate the following integrals:

16. $\int e^x (x-2)(2x+3)dx$

17. $\int e^{2x} (x+1)^2 dx$

18. $\int x^2 \sin^2 2x \, dx$

19. $\int x^2 \cos^3 x \, dx$

20. $\int_0^{\pi/2} x \sin x \cos x \, dx$

21. $\int x \operatorname{cosec}^2 x \, dx$

22. $\int_0^1 \sin^{-1} x \, dx$

23. $\int x^2 \tan^{-1} x \, dx$

24. $\int \sqrt{a^2 - x^2} dx$

25. $\int \sqrt{x^2 + a^2} \, dx$

26. $\int e^{2x+3} \sin(3x+1)dx$

27. $\int e^x (\sec x(1+\tan x)dx$

28. $\int e^x (x+1) \log x \, dx$

29. $\int_0^1 \frac{xe^x dx}{(1+x)^2}$

30. $\int_1^2 \frac{e^x (x^2+1)}{(x+1)^2} dx$

31. Test the convergence of $\int_1^2 \frac{dx}{(x-1)^2}$.

32. Test the convergence of $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}$.

33. Test the convergence of $\int_{-1}^1 \frac{1}{x} dx$.

34. Evaluate Cauchy's principal value of $\int_1^4 \frac{dx}{(x-2)^3}$.

35. Discuss the convergence of $\int_1^\infty e^{-x} x^2 \, dx$, using comparison test.

36. Test the convergence of $\int_0^\infty \frac{\sin x}{x} dx$.

37. Test the convergence $\int_0^1 \frac{dx}{x}$.

38. Test the convergence of $\int_0^1 \frac{1+x^2}{\sqrt{2x - x^2}} dx$.

39. Test the convergence of $\int_1^\infty \frac{e^{-x}}{x} dx$.

40. Test the convergence of $\int_1^{\infty} \frac{\sin x}{x^2}\,dx$.

**ANSWERS**

## Exercise 3(a)

(1) $\frac{x}{2} - x + \log(x+1)$

(2) $\frac{x^2}{2} - 2x - \log(x-2)$

(3) $\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]$

(4) $-\cot x - \operatorname{cosec} x$

(5) $x - \sin x$

(6) $\frac{2(1+\sqrt{x})^{n+1}}{n+1}$

(7) $-\frac{1}{1+e^x}$

(8) $-\frac{1}{(n-1)(\log x)n-1}$

(9) $\frac{\sin^3 x}{3}$

(10) $\log(\sin x - \cos x)$

(11) $\frac{1}{2}\log(x^2+2x+8)$

(12) $\frac{2}{3}\sqrt{x^3+3x^2+2}$

(13) $\frac{2}{3a}(ax+b)^{3/2} + \frac{2}{c}\sqrt{cx+d}$

(14) $\frac{1}{14}x^{14} + \frac{2}{7}\log(x^{14}-4)$

(15) $\frac{1}{a}\sin^{-1}\left(\frac{x^n}{a^n}\right)$

(16) $\log\{\cosh x + \sqrt{\cosh^2 x + 4}\}$

(17) $\frac{1}{2}\log\{x^2 + \sqrt{1+x^4}\}$

(18) $\sin^{-1}(\log x)$

(19) $\sin^{-1}\left(\frac{\sin x}{3}\right)$

(20) $\sinh^{-1}\left(\frac{\tan x}{4}\right)$

(21) $\sqrt{x} + \frac{1}{2}\sin 2\sqrt{x}$

(22) $-3\cos\sqrt{x} - \frac{1}{3}\cos 3\sqrt{x}$

(23) $\tan^{-1} 2 - \frac{\pi}{4}$

(24) $\sqrt{2\pi}$

(25) $e^{\pi/2} - 1$

(26) $\frac{1}{3}a^3$

(27) 1

(28) $\frac{1}{a^2\sqrt{2}}$

(29) $\frac{1}{4}\log\left(\frac{1}{\sqrt{3}}\right)$

(30) $a^2\left(\frac{\pi}{4} - \frac{1}{2}\right)$

(31) $\frac{\pi}{2\sqrt{2}}$

(32) $\frac{1}{2}\log\left(\frac{5}{3}\right)$

(33) $\frac{3}{2}\pi$

(34) $\pi$

(35) $\frac{\pi}{\sqrt{\alpha\beta}}$

## Exercise 3(b)

(1) $\frac{1}{a}\tan^{-1}\frac{x}{a}$ (2) $\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$ (3) $\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

(4) $\frac{1}{x^2+2x+5}$ (5) $\frac{1}{2}\log(x^2+9)+\frac{1}{3}\tan^{-1}\frac{x}{3}$

(6) $\frac{1}{2a^2}\tan^{-1}\left(\frac{x^2}{a^2}\right)$ (7) $\log\left(\frac{x}{x+1}\right)$ (8) $\log\left(\frac{\sin x}{1+\sin x}\right)$

(9) $\log\tan\frac{x}{2}$ (10) $\log\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$

(11) $\frac{2}{3\sqrt{3}}\tan^{1}\left(\frac{2x^3+1}{\sqrt{3}}\right)$ (12) $\frac{1}{3}\tan^{-1}\left(\frac{e^x+1}{3}\right)$

(13) $\log(y^2+2y+5)+\frac{1}{2}\tan^{-1}\left(\frac{y+1}{2}\right)$ where $y=\log x$

(14) $\frac{3}{4}\log(2x^2-3x+5)+\frac{37}{2\sqrt{31}}\times\tan^{-1}\left(\frac{4x-3}{\sqrt{31}}\right)$

(15) $2\log(1+2y-y^2)+\frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+y-1}{\sqrt{2}-y+1}\right)$, where $y=\sin x$

(16) $x-\log(x^2+x+1)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$

(17) $\frac{1}{2}\log\left(\frac{1}{4}\right)-\frac{2}{\sqrt{5}}\log\left(\frac{3-\sqrt{5}}{3+\sqrt{5}}\right)$

(19) $\frac{1}{2}\log x-\log(x+1)+\frac{1}{2}\log(x+2)$

(20) $2\log\left(\frac{x-1}{x+2}\right)-\frac{1}{x+1}$

(21) $\frac{1}{3}\log(x+1)-\frac{1}{6}\log(x^2-x+1)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$

(22) $\frac{5}{3}\log(x-1)+\frac{1}{6}\log(x^2+x+1)-\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$

(23) $\frac{1}{2}\log(1+\sin x)+\frac{1}{4}\log(1+\sin^2 x)+\frac{1}{2}\tan^{-1}(\sin x)$

(24) $\frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right)$ (25) $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}}{2}\tan x\right)$

(26) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}}{3}\tan x\right)$ (27) $\frac{1}{2\sqrt{6}}\log\left\{\frac{2\sqrt{2}+\sqrt{3}\left(\tan\frac{x}{2}-1\right)}{2\sqrt{2}-\sqrt{3}\left(\tan\frac{x}{2}-1\right)}\right\}$

(28) $\frac{\pi}{4}$

(29) $2x + \log(3\cos x + 2\sin x + 4) - 2\sqrt{3}\tan^{-1}\left(\frac{2\tan\frac{x}{2}}{\sqrt{3}}\right)$ (30) $\frac{\pi}{4}$

## Exercise 3(c)

(1) $\sin^{-1}\frac{x}{a}$ (2) $\cosh^{-1}\frac{x}{a}$ (3) $\sinh^{-1}\frac{x}{a}$

(4) $\sinh^{-1}\left(\frac{x+1}{2}\right)$ (5) $\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right)$ (6) $\cosh^{-1}\left(\frac{2x+3}{\sqrt{5}}\right)$

(7) $-\sqrt{1-x^2}+3\sin^{-1}x$ (8) $\tan^{-1}\frac{\sqrt{x}}{2}$ (9) $-\log\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right)$

(10) $\sin^{-1}\left(\frac{x-a}{a}\right)$ (11) $\frac{1}{\sqrt{3}}\sin^{-1}(3x-1)$ (12) $\frac{1}{2}\log(1+\sqrt{2})$

(13) $\pi$ (14) $2\sqrt{x^2-2x+10}+7\sinh^{-1}\left(\frac{x-1}{3}\right)$

(15) $\frac{1}{2}\sqrt{2x^2-x-3}+\frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{4x-1}{5}\right)$ (16) $3\pi$

(17) $\frac{x}{2}\sqrt{x^2+1}+2\sqrt{x^2+1}+\frac{5}{2}\sinh^{-1}x$

(18) $\cos^{-1}\left\{\frac{-(x+1)}{\sqrt{2}(x+2)}\right\}$ (19) $-\sin^{-1}\left(\frac{1+3x}{4x}\right)$

(20) $-\frac{1}{\sqrt{14}}\log\left[\frac{1}{2x+3}+\frac{1}{\sqrt{14}}+\sqrt{\frac{1}{(2x+3)^2}+\frac{1}{7(2x+3)}}\right]$

(21) $-\frac{1}{\sqrt{8}}\sin^{-1}\left(\frac{16}{x+3}-1\right)$ (22) $\frac{\pi}{4\sqrt{2}}$

(23) $\frac{1}{\sqrt{2}}\log\left\{\frac{x\sqrt{2}-\sqrt{x^2+1}}{x\sqrt{2}+\sqrt{x^2+1}}\right\}$

(24) $\left(\dfrac{2x-7}{4}\right)\sqrt{7x-10-x^2}+\dfrac{9}{8}\sin^{-1}\left(\dfrac{2x-7}{3}\right)$

(25) $\dfrac{1}{3}(x^2+x+1)^{3/2}+\dfrac{1}{3}(2x+1)\sqrt{x^2+x+1}+\dfrac{3}{16}\sinh^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$

## Exercise 3(d)

(1) $\dfrac{1}{a}xe^{ax}-\dfrac{1}{a^2}e^{ax}$ (2) $-\dfrac{x\cos mx}{m}+\dfrac{\sin mx}{m^2}$

(3) $-x\cot x+\log\sin x$ (4) $-\left(\dfrac{1}{x}\log x+\dfrac{1}{x}\right)$

(5) $x\log x-x$ (6) $\dfrac{x^2}{4}+\dfrac{x}{4}\sin 2x-\dfrac{1}{8}\cos 2x$

(7) $x\tan^{-1}x-\dfrac{1}{2}\log(1+x^2)$ (8) $x\sinh^{-1}x-\sqrt{x^2+1}$

(9) $\dfrac{x^2}{2}\sec^{-1}x-\dfrac{1}{2}\sqrt{x^2-1}$ (10) $\dfrac{x^2}{2}\log x-\dfrac{x^2}{4}$ (16) $e^x(2x^2-5x-1)$

(17) $\dfrac{e^{2x}}{4}(2x^2+2x+1)$ (18) $\dfrac{1}{6}x^3-\dfrac{1}{8}x^2\sin 4x-\dfrac{1}{16}x\cos 4x+\dfrac{1}{64}\sin 4x$

(19) $\dfrac{1}{4}\left[x^2(3\sin x+\dfrac{\sin 3x}{8}-2x\left(-3\cos x-\dfrac{\cos 3x}{9}\right)+2\left(-3\sin x-\dfrac{\sin 3x}{27}\right)\right]$

(20) $\dfrac{\pi}{8}$ (21) $-\dfrac{x}{3}\cot 3x+\dfrac{1}{9}\log\sin 3x$

(22) $\dfrac{\pi}{2}-1$ (23) $\dfrac{x^3}{3}\tan^{-1}x-\dfrac{x^2}{6}+\dfrac{1}{6}\log(1+x^2)$

(24) $\dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$ (25) $\dfrac{x}{2}\sqrt{x^2+a^2}+\dfrac{a^2}{2}\sinh^{-1}\left(\dfrac{x}{a}\right)$

(26) $\dfrac{1}{13}e^{2x+3}[2\sin(3x+1)-3\cos(3x+1)]$

(27) $e^x\sec x$ (28) $e^x(x\log x-1)$ (29) $\dfrac{e}{2}-1$

(30) $\dfrac{1}{3}e^2$ (31) divergent

(32) convergent to $\dfrac{\pi}{2}$ (33) convergent to 0

(34) $\dfrac{3}{8}$ (35) convergent

(36) convergent (37) divergent

(38) convergent (39) convergent

(40) convergent

UNIT **4**

# Multiple Integrals

## 4.1 INTRODUCTION

When a function $f(x)$ is integrated with respect to $x$ between the limits $a$ and $b$, we get the definite integral $\int_a^b f(x)\mathrm{d}x$.

If the integrand is a function $f(x,y)$ and if it is integrated with respect to $x$ and $y$ repeatedly between the limits $x_0$ and $x_1$ (for $x$) and between the limits $y_0$ and $y_1$ (for $y$),

we get a *double integral* that is denoted by the symbol $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y)\mathrm{d}x\,\mathrm{d}y$.

Extending the concept of double integral one step further, we get the *triple integral*

$$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

## 4.2 EVALUATION OF DOUBLE AND TRIPLE INTEGRALS

To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$, we first integrate $f(x, y)$ with respect to $x$ partially, i.e. treating $y$ as a constant temporarily, between $x_0$ and $x_1$. The resulting function got after the inner integration and substitution of limits will be a function of $y$. Then we integrate this function of $y$ with respect to $y$ between the limits $y_0$ and $y_1$ as usual.

The order in which the integrations are performed in the double integral is illustrated in Fig. 4.1.

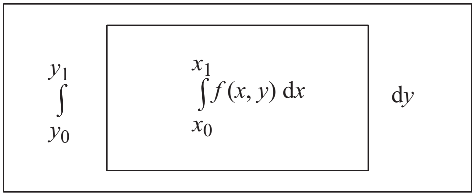


**Fig. 4.1**

*Note* ☑ Since the resulting function got after evaluating the inner integral is to be a function of $y$, the limits $x_0$ and $x_1$ may be either constants or functions of $y$.

The order in which the integrations are performed in a triple integral is illustrated in Fig. 4.2.

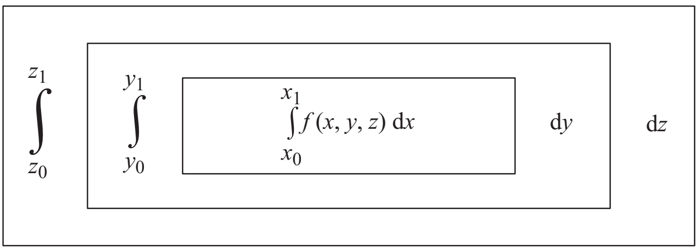


**Fig. 4.2**

When we first perform the innermost integration with respect to $x$, we treat $y$ and $z$ as constants temporarily. The limits $x_0$ and $x_1$ may be constants or functions of $y$ and $z$, so that the resulting function got after the innermost integration may be a function of $y$ and $z$. Then we perform the middle integration with respect to $y$, treating $z$ as a constant temporarily. The limits $y_0$ and $y_1$ may be constants or functions of $z$, so that the resulting function got after the middle integration may be a function of $z$ only. Finally we perform the outermost integration with respect to $z$ between the constant limits $z_0$ and $z_1$.

*Note* ☑ Sometimes $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y)dx\, dy$ is also denoted as $\int_{y_0}^{y_1} dy \int_{x_0}^{x_1} f(x, y)dx$ and $\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y, z)dx\, dy\, dz$ is also denoted as $\int_{z_0}^{z_1} dz \int_{y_0}^{y_1} dy \int_{x_0}^{x_1} f(x, y, z)\, dx$. If these notations are used to denote the double and triple integrals, the integrations are performed from right to left in order.

## 4.3 REGION OF INTEGRATION

Consider the double integral $\int_{c}^{d}\int_{\phi_1(y)}^{\phi_2(y)} f(x, y)\, dx\, dy$. As stated above $x$ varies from $\phi_1(y)$ to $\phi_2(y)$ and $y$ varies from $c$ to $d$.

i.e. $\phi_1(y) \le x \le \phi_2(y)$ and $c \le y \le d$.

These inequalities determine a region in the *xy*-plane, whose boundaries are the curves $x = \phi_1(y)$, $x = \phi_2(y)$ and the lines $y = c$, $y = d$ and which is shown in Fig. 4.3. This region *ABCD* is known as the region of integration of the above double integral.

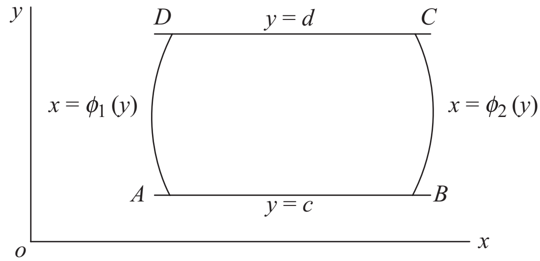


**Fig. 4.3**

Similarly, for the double integral $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\,dy\,dx$, the region of integration *ABCD*, whose boundaries are the curves $y = \phi_1(x)$, $y = \phi_2(x)$ and the lines $x = a$, $x = b$, is shown in Fig. 4.4.

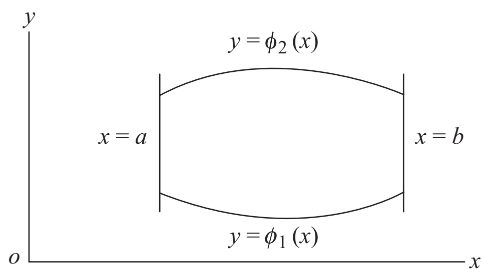


**Fig. 4.4**

For the triple integral $\int_{z_1}^{z_2} \int_{\psi_1(z)}^{\psi_2(z)} \int_{\phi_1(y,z)}^{\phi_2(y,z)} f(x, y, z)\,dx\,dy\,dz$, the inequalities $\phi_1(y, z) \le x \le \phi_2(y, z)$; $\psi_1(z) \le y \le \psi_2(z)$; $z_1 \le z \le z_2$ hold good. These inequalities determine a domain in space whose boundaries are the surfaces $x = \phi_1(y, z)$, $x = \phi_2(y, z)$, $y = \psi_1(z)$, $y = \psi_2(z)$, $z = z_1$ and $z = z_2$. This domain is called the domain of integration of the above triple integral.

## WORKED EXAMPLE 4(a)

**Example 4.1** Verify that $\int_1^2 \int_0^1 (x^2 + y^2)\,dx\,dy = \int_0^1 \int_1^2 (x^2 + y^2)\,dy\,dx$.

$$\text{L.S.} = \int_1^2 \left[ \int_0^1 (x^2 + y^2)\,dx \right] dy$$

$$= \int_1^2 \left[ \frac{x^3}{3} + y^2 x \right]_{x=0}^{x=1} \mathrm{d}y$$

*Note* ☑ $y$ is treated a constant during inner integration with respect to $x$.

$$= \int_1^2 \left( \frac{1}{3} + y^2 \right) \mathrm{d}y = \left( \frac{y}{3} + \frac{y^3}{3} \right)_1^2 = \frac{8}{3}$$

$$\text{R.S.} = \int_0^1 \left[ \int_1^2 (x^2 + y^2)\,\mathrm{d}y \right] \mathrm{d}x$$

$$= \int_0^1 \left[ x^2 y + \frac{y^3}{3} \right]_{y=1}^{y=2} \mathrm{d}x$$

*Note* ☑ $x$ is treated a constant during inner integration with respect to $y$.

$$= \int_0^1 \left( x^2 + \frac{7}{3} \right) \mathrm{d}x = \left( \frac{x^3}{3} + \frac{7}{3} x \right)_0^1 = \frac{8}{3}$$

Thus the two double integrals are equal.

*Note* ☑ From the above problem we note the following fact: If the limits of integration in a double integral are constants, then the order of integration is immaterial, provided the relevant limits are taken for the concerned variable and the integrand is continuous in the region of integration. This result holds good for a triple integral also.

**Example 4.2** Evaluate $\int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^4 \sin\phi \,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta$.

$$\text{The given integral} = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \mathrm{d}\phi \int_0^{a} r^4 \sin\phi \,\mathrm{d}r$$

$$= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \left( \frac{r^5}{5} \right)_0^a \sin\phi \,\mathrm{d}\phi$$

$$= \frac{a^5}{5} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \sin\phi \,\mathrm{d}\phi$$

$$= \frac{a^5}{5} \int_0^{2\pi} (-\cos\phi)_0^{\pi} \,\mathrm{d}\theta$$

$$= \frac{2}{5} a^5 \int_0^{2\pi} \mathrm{d}\theta$$

$$= \frac{4}{5} \pi a^5$$

**Example 4.3** Evaluate $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{1+x^2+y^2}$.

$$\text{The given integral} = \int_0^1 \left[ \int_0^{\sqrt{1+y^2}} \frac{1}{(1+y^2)+x^2}\,dx \right] dy$$

$$= \int_0^1 \left[ \frac{1}{\sqrt{1+y^2}} \tan^{-1} \frac{x}{\sqrt{1+y^2}} \right]_{x=0}^{x=\sqrt{1+y^2}} dy$$

$$= \frac{\pi}{4} \int_0^1 \frac{dy}{\sqrt{1+y^2}}$$

$$= \frac{\pi}{4} \left[ \log\left( y + \sqrt{1+y^2} \right) \right]_0^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

**Example 4.4** Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\,(x+y)\,dx\,dy$.

Since the limits for the inner integral are functions of $x$, the variable of inner integration should be $y$. Effecting this change, the given integral I becomes

$$I = \int_0^1 \left[ \int_x^{\sqrt{x}} xy\,(x+y)\,dy \right] dx$$

$$= \int_0^1 \left( x^2 \frac{y^2}{2} + x\frac{y^3}{3} \right)_{y=x}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left[ \left( \frac{x^3}{2} + \frac{1}{3} x^{5/2} \right) - \left( \frac{x^4}{2} + \frac{x^4}{3} \right) \right] dx$$

$$= \left( \frac{x^4}{8} + \frac{2}{21} x^{7/2} - \frac{x^5}{6} \right)_0^1$$

$$= \frac{1}{8} + \frac{2}{21} - \frac{1}{6} = \frac{3}{56}$$

**Example 4.5** Evaluate $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xyz \, dx \, dy \, dz$.

$$\text{The given integral } = \int_0^1 \int_0^{1-z} yz \left(\frac{x^2}{2}\right)_0^{1-y-z} dy \, dz$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-z} yz \,(1-y-z)^2 \, dy \, dz$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-z} yz \,\{(1-z)^2 - 2\,(1-z)y + y^2\} \, dy \, dz$$

$$= \frac{1}{2} \int_0^1 \left[ z(1-z)^2 \frac{y^2}{2} - 2z(1-z)\frac{y^3}{3} + z\frac{y^4}{4} \right]_{y=0}^{y=1-z} dz$$

$$= \frac{1}{2} \int_0^1 \left[ \frac{1}{2} z(1-z)^4 - \frac{2}{3} z(1-z)^4 + \frac{1}{4} z(1-z)^4 \right] dz$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) \int_0^1 z(1-z)^4 \, dz$$

$$= \frac{1}{24} \int_0^1 \{1-(1-z)\}\,(1-z)^4 \, dz$$

$$= \frac{1}{24} \left[ \frac{(1-z)^5}{-5} + \frac{(1-z)^6}{6} \right]_0^1$$

$$= \frac{1}{24}\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{720}$$

**Example 4.6** Evaluate $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dx \, dy \, dz$.

Since the upper limit for the innermost integration is a function of $x, y$, the corresponding variable of integration should be $z$. Since the upper limit for the middle integration is a function of $x$, the corresponding variable of integration should be $y$. The variable of integration for the outermost integration is then $x$. Effecting these changes, the given triple integral I becomes,

$$I = \int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx$$

$$= \int_0^{\log 2} dx \int_0^{x} dy \; e^{x+y} \,(e^z)_{z=0}^{z=x+y}$$

$$= \int_0^{\log 2} dx \int_0^{x} (e^{2x+2y} - e^{x+y}) \, dy$$

$$= \int_0^{\log 2} dx \left( e^{2x} \cdot \frac{e^{2y}}{2} - e^x \cdot e^y \right)_{y=0}^{y=x}$$

$$= \int_0^{\log 2} \left( \frac{1}{2} e^{4x} - \frac{3}{2} e^{2x} + e^x \right) dx$$

$$= \left( \frac{1}{8} e^{4x} - \frac{3}{4} e^{2x} + e^x \right)_0^{\log 2}$$

$$= \frac{5}{8}$$

**Example 4.7** Evaluate $\iint_R xy \, dx \, dy$ , where $R$ is the region bounded by the line $x + 2y = 2$, lying in the first quadrant.

We draw a rough sketch of the boundaries of $R$ and identify $R$.

The boundaries of $R$ are the lines $x = 0$, $y = 0$ and the segment of the line $\frac{x}{2} + \frac{y}{1} = 1$ lying in the first quadrant.

Now $R$ is the region as shown in Fig. 4.5.

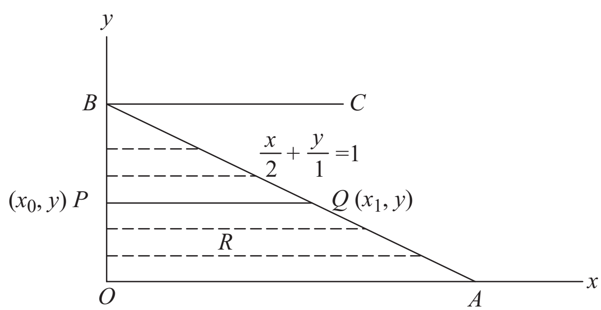


**Fig. 4.5**

Since the limits of the variables of integration are not given in the problem and to be fixed by us, we can choose the order of integration arbitrarily.

Let us integrate with respect to $x$ first and then with respect to $y$. Then the integral I becomes

$$I = \int \left[ \int_R xy \, dx \right] dy$$

When we perform the inner integration with respect to $x$, we have to treat $y$ as a constant temporarily and find the limits for $x$.

Geometrically, treating $y$ = constant is equivalent to drawing a line parallel to the $x$-axis arbitrarily lying within the region of integration $R$ as shown in the figure.

Finding the limits for $x$ (while $y$ is a constant) is equivalent to finding the variation of the $x$ co-ordinate of any point on the line $PQ$. We assume that the $y$ co-ordinates of all points on $PQ$ are $y$ each (since $y$ is constant on $PQ$) and $P \equiv (x_0, y)$ and $Q \equiv (x_1, y)$.

Thus $x$ varies from $x_0$ to $x_1$.

Wherever the line $PQ$ has been drawn, the left end $P$ lies on the $y$-axis and hence $x_0 = 0$ and the right end $Q$ lies on the line $x + 2y = 2$, and hence $x_1 + 2y = 2$ i.e. $x_1 = 2 - 2y$.

Thus the limits for the variable $x$ of inner integration are 0 and $2 - 2y$. When we go to the outer integration, we have to find the limits for $y$.

Geometrically we have to find the variation of the line $PQ$, so that the region $R$ is fully covered. To sweep the entire area of the region $R$, $PQ$ has to start from the position $OA$ where $y = 0$, move parallel to itself and go up to the position $BC$ where $y = 1$.

Thus the limits for $y$ are 0 and 1.

$$\therefore \qquad I = \int_0^1 \int_0^{2-2y} xy \, dx \, dy$$

$$= \int_0^1 y \left(\frac{x^2}{2}\right)_0^{2-2y} dy$$

$$= \int_0^1 \frac{y}{2} (2 - 2y)^2 \, dy$$

$$= 2 \int_0^1 y (1 - y)^2 \, dy$$

$$= 2 \left(\frac{y^2}{2} - 2\frac{y^3}{3} + \frac{y^4}{4}\right)_0^1$$

$$= \frac{1}{6}$$

## 4.3.1 Aliter

Let us integrate with respect to $y$ first and then with respect to $x$.

Then $I = \int_R \left[\int xy \, dy\right] dx$

As explained above, to find the limits for $y$, we draw a line parallel to the $y$-axis ($x$ = constant) in the region of integration and note the variation of $y$ on this line

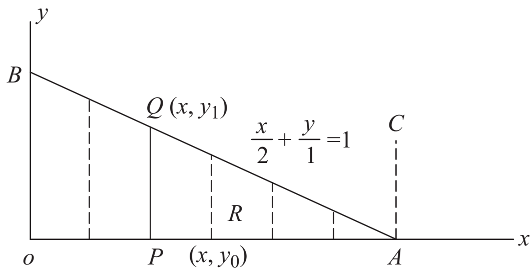


**Fig. 4.6**

$P(x, y_0)$ lies on the $x$-axis. $\therefore\ y_0 = 0$

$Q(x, y_1)$ lies on the line $x + 2y = 2$. $\therefore\ y_1 = \frac{1}{2}(2-x)$

i.e., the limits for $y$ are 0 and $\frac{1}{2}(2-x)$.

To cover the region of integration $OAB$, the line $PQ$ has to vary from $OB$ $(x = 0)$ to $AC$ $(x = 2)$

$\therefore$ The limits for $x$ are 0 and 2.

$$\therefore \qquad I = \int_0^2 \int_0^{\frac{1}{2}(2-x)} xy\,dy\,dx$$

$$= \int_0^2 x\left(\frac{y^2}{2}\right)_0^{\frac{1}{2}(2-x)} dx$$

$$= \frac{1}{8}\int_0^2 x(2-x)^2\,dx$$

$$= \frac{1}{8}\left(4\frac{x^2}{2} - 4\frac{x^3}{3} + \frac{x^4}{4}\right)_0^2$$

$$= \frac{1}{6}$$

**Example 4.8** Evaluate $\iint_R \frac{e^{-y}}{y}\,dx\,dy$, by choosing the order of integration suitably, given that $R$ is the region bounded by the lines $x = 0$, $x = y$ and $y = \infty$.

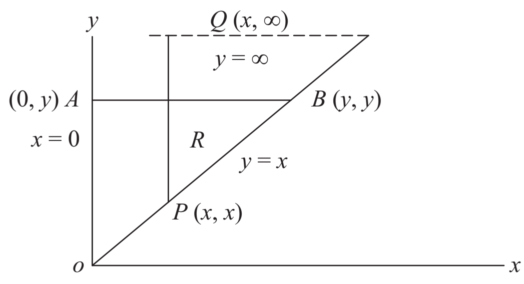


**Fig. 4.7**

Let $$I = \iint_R \frac{e^{-y}}{y}\,dx\,dy$$

Suppose we wish to integrate with respect to $y$ first.

Then $$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

We note that the choice of order of integration is wrong, as the inner integration cannot be performed. Hence we try to integrate with respect to $x$ first.

Then
$$I = \int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy$$
$$= \int_0^\infty \frac{e^{-y}}{y} (x)_0^y \, dy$$
$$= \int_0^\infty e^{-y} \, dy$$
$$= \left(e^{-y}\right)_\infty^0 = 1$$

*Note* ☑ From this example, we note that the choice of order of integration sometimes depends on the function to be integrated.

**Example 4.9** Evaluate $\iint_R xy \, dx \, dy$, where $R$ is the region bounded by the parabola $y^2 = x$ and the lines $y = 0$ and $x + y = 2$, lying in the first quadrant.

$R$ is the region $OABCDE$ shown in Fig. 4.8.

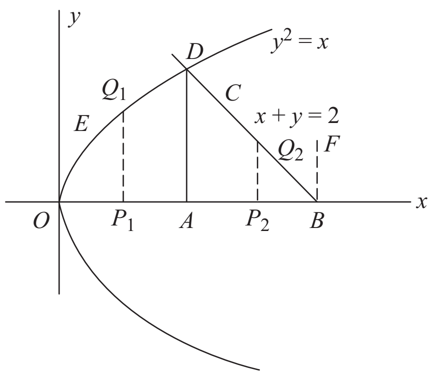


**Fig. 4.8**

Suppose we wish to integrate with respect to $y$ first. Then we will draw an arbitrary line parallel to $y$-axis ($x$ = constant). We note that such a line does not intersect the region of integration in the same fashion throughout.

If the line is drawn in the region $OADE$, the upper end of the line will lie on the parabola $y^2 = x$; on the other hand, if it is drawn in the region $ABCD$, the upper end of the line will lie on the line $x + y = 2$.

Hence in order to cover the entire region $R$, it should be divided into two, namely, $OADE$ and $ABCD$ and the line $P_1 Q_1$ should move from the $y$-axis to $AD$ and the line $P_2 Q_2$ should move from $AD$ to $BF$.

Accordingly, the given integral I is given by

$$I = \int_0^1 \int_0^{\sqrt{x}} xy \, dy \, dx + \int_1^2 \int_0^{2-x} xy \, dy \, dx$$

[$\because$ the co-ordinates of $D$ are (1, 1) and so the equation of $AD$ is $x = 1$]

$$I = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

***Note*** ☑ This approach results in splitting the double integral into two and evaluating two double integrals. On the other hand, had we integrated with respect to $x$ first, the problem would have been solved in a simpler way as indicated below. [Refer to Fig. 4.9]

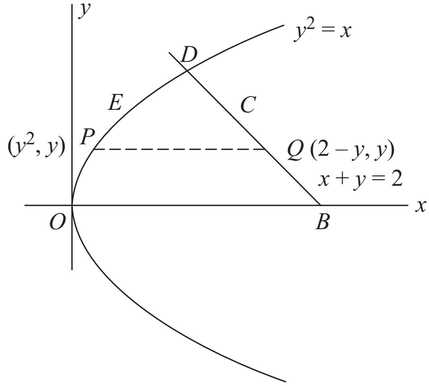


**Fig. 4.9**

$$I = \int_0^1 \int_{y^2}^{2-y} xy \, dx \, dy$$

$$= \frac{1}{2} \int_0^1 y\{(2-y)^2 - y^4\} \, dy$$

$$= \frac{1}{2} \int_0^1 (4y - 4y^2 + y^3 - y^5) \, dy$$

$$= \frac{3}{8}$$

***Note*** ☑ From this example, we note that the choice of order of integration is to be made by considering the region of integration so as to simplify the evaluation.

**Example 4.10** Evaluate $\iiint_V (x + y + z) \, dx \, dy \, dz$, where $V$ is the volume of the rectangular parallelopiped bounded by $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$ and $z = c$.

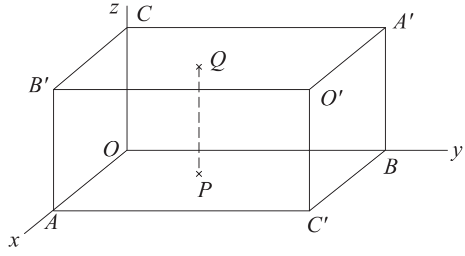


**Fig. 4.10**

The region of integration is the volume of the parallelopiped shown in Fig. 4.10, in which OA = $a$, OB = $b$, OC = $c$. Since the limits of the variables of integration are not given, we can choose the order of integration arbitrarily.

Let us take the given integral I as

$$I = \iiint_V (x+y+z)\, dz\, dy\, dx$$

The innermost integration is to be done with respect to $z$, treating $x$ and $y$ as constants.

Geometrically, $x$ = constant and $y$ = constant jointly represent a line parallel to the $z$-axis.

Hence we draw an arbitrary line $PQ$ in the region of integration and we note the variation of $z$ on this line so as to cover the entire volume. In this problem, $z$ varies from 0 to $c$. since $P \equiv (x, y, 0)$ and $Q \equiv (x, y, c)$

Having performed the innermost integration with respect to $z$ between the limits 0 and $c$, we get a double integral.

As $P$ take all positions inside the rectangle OAC′B in the $xy$-plane, the line $PQ$ covers the entire volume of the parallelopiped. Hence, the double integral got after the innermost integration is to be evaluated over the plane region OAC′B.

The limits for the double integral can be easily seen to be 0 and $b$ (for $y$) and 0 and $a$ (for $x$).

$$\therefore \qquad I = \int_0^a \int_0^b \int_0^c (x+y+z)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^b \left\{(x+y)z + \frac{z^2}{2}\right\}_{z=0}^{c} dy\, dx$$

$$= \int_0^a \int_0^b \left\{c\,(x+y) + \frac{c^2}{2}\right\} dy\, dx$$

$$= \int_0^a \left\{\left(cx + \frac{c^2}{2}\right) y + c\frac{y^2}{2}\right\}_0^b dx$$

$$= \int_0^a \left(bcx + \frac{bc^2}{2} + \frac{b^2 c}{2}\right) dx$$

$$= \left[bc\frac{x^2}{2} + \frac{bc}{2}(b+c)x\right]_0^a$$

$$= \frac{abc}{2}(a+b+c)$$

**Example 4.11** Evaluate $\iiint_V dx\, dy\, dz$, where $V$ is the finite region of space (terra-hedron) formed by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 3y + 4z = 12$.

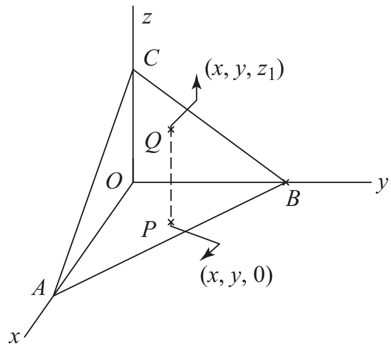


**Fig. 4.11**

Let I = the given integral.

Let I = $\iiint_V dz\, dy\, dx$

The limits for $z$, the variable of the innermost integral, are 0 and $z_1$, where $(x, y, z_1)$ lies on the plane $2x + 3y + 4z = 12$. [Refer to Fig. 4.11]

$$\therefore \qquad z_1 = \frac{1}{4}(12 - 2x - 3y)$$

After performing the innermost integration, the resulting double integral is evaluated over the orthogonal projection of the plane ABC on the $xy$-plane, i.e. over the triangular region OAB in the $xy$-plane as shown in Fig. 4.12.

In the double integral, the limits for $y$ are 0 and $\frac{1}{3}(12 - 2x)$ and those for $x$ are 0 and 6.

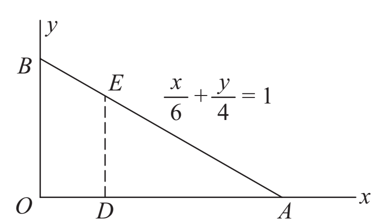


**Fig. 4.12**

$$\therefore \qquad I = \int_0^6 dx \int_0^{\frac{1}{3}(12-2x)} dy \int_0^{\frac{1}{4}(12-2x-3y)} dz$$

$$= \frac{1}{4}\int_0^6 dx \int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y)\, dy$$

$$= \frac{1}{4}\int_0^6 dx \left[(12 - 2x)y - \frac{3y^2}{2}\right]_{y=0}^{y=\frac{1}{3}(12-2x)}$$

$$= \frac{1}{24}\int_0^6 (12 - 2x)^2\, dx$$

$$= \frac{1}{6}\left\{\frac{(6 - x)^3}{-3}\right\}_0^6$$

$$= 12$$

**Example 4.12** Evaluate $\iiint\limits_V \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$, where $V$ is the region of space bounded by the co-ordinate planes and the sphere $x^2 + y^2 + z^2 = 1$ and contained in the positive octant.

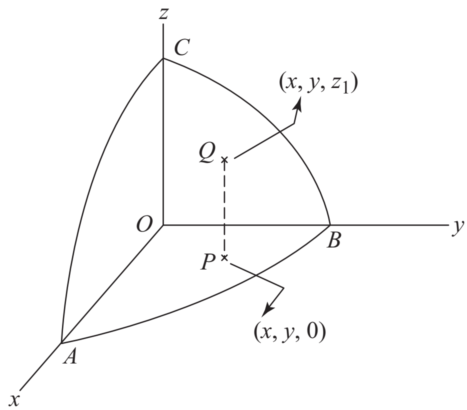


**Fig. 4.13**

*Note* ☑ In two dimensions, the $x$ and $y$-axes divide the entire $xy$-plane into 4 quadrants. The quadrant containing the positive $x$ and the positive $y$-axes is called the positive quadrant.

Similarly in three dimensions the $xy$, $yz$ and $zx$-planes divide the entire space into 8 parts, called octants. The octant containing the positive $x$, $y$ and $z$-axes is called the positive octant.

The region of space $V$ given in this problem is shown in Fig. 4.13.

Let
$$I = \iiint\limits_V \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$$

To find the limits for $z$, we draw a line $PQ$ parallel to the $z$-axis cutting the volume of integration.

The limits for $z$ and 0 and $z_1$, where $(x, y, z_1)$ lies on the sphere $x^2 + y^2 + z^2 = 1$

$\therefore$
$$z_1 = \sqrt{1-x^2-y^2} \quad (\because \text{ the point } Q \text{ lies in the positive octant})$$

After performing the innermost integration, the resulting double integral is evaluated over the orthogonal projection of the spherical surface on the $xy$-plane, i.e. over the circular region lying in the positive quadrant as shown in Fig. 4.14.

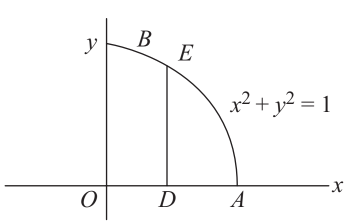


**Fig. 4.14**

In the double integral, the limits for $y$ are 0 and $\sqrt{1-x^2}$ and those for $x$ are 0 and 1.

$\therefore$

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{(1-x^2-y^2)-z^2}}$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \left( \sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}} \right)_{z=0}^{z=\sqrt{1-x^2-y^2}}$$

$$= \frac{\pi}{2} \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left( \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x \right)_0^1$$

$$= \frac{\pi}{2} \times \frac{\pi}{4} = \frac{\pi^2}{8}$$

## EXERCISE 4(a)

***Part A***

(Short Answer Questions)

1. Evaluate $\int_0^2 \int_0^1 4xy\, dx\, dy$.
2. Evaluate $\int_1^b \int_1^a \frac{dx\, dy}{xy}$.
3. Evaluate $\int_0^{\pi/2} \int_0^{\pi/2} \sin(\theta + \phi)\, d\theta\, d\phi$.
4. Evaluate $\int_0^1 \int_0^x dx\, dy$.
5. Evaluate $\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta$.
6. Evaluate $\int_0^1 \int_0^2 \int_0^3 xyz\, dx\, dy\, dz$.
7. Evaluate $\int_0^1 \int_0^z \int_0^{y+z} dz\, dy\, dx$.

Sketch roughly the region of integration for the following double integrals:

8. $\int_{-b}^{b}\int_{-a}^{a} f(x,y)\,dx\,dy.$

9. $\int_{0}^{1}\int_{0}^{x} f(x,y)\,dx\,dy.$

10. $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy.$

11. $\int_{0}^{b}\int_{0}^{\frac{a}{b}(b-y)} f(x,y)\,dx\,dy.$

Find the limits of integration in the double integral $\iint_R f(x,y)\,dx\,dy$, where $R$ is in the first quadrant and bounded by

12. $x = 0, y = 0, x + y = 1$

13. $x = 0,\ y = 0,\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

14. $x = 0, x = y, y = 1$

15. $x = 1, y = 0, y^2 = 4x$

***Part B***

16. Evaluate $\int_{0}^{4}\int_{y^2/4}^{y} \dfrac{y\,dx\,dy}{x^2+y^2}$ and also sketch the region of integration roughly.

17. Evaluate $\int_{0}^{a}\int_{a-x}^{\sqrt{a^2-x^2}} y\,dx\,dy$ and also sketch the region of integration roughly.

18. Evaluate $\int_{0}^{1}\int_{x}^{1} \dfrac{y\,dx\,dy}{x^2+y^2}$ and also sketch the region of integration roughly.

19. Evaluate $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy.$

20. Evaluate $\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-y} xyz\,dx\,dy\,dz.$

21. Evaluate $\int_{0}^{\log 2}\int_{0}^{x}\int_{0}^{x+\log y} e^{x+y+z}\,dz\,dy\,dx.$

22. Evaluate $\iint x e^{-\frac{x^2}{y}}\,\mathrm{d}x\,\mathrm{d}y$, over the region bounded by $x = 0$, $x = \infty$, $y = 0$ and $y = x$.
23. Evaluate $\iint xy\,\mathrm{d}x\,\mathrm{d}y$, over the region in the positive quadrant bounded by the line $2x + 3y = 6$.
24. Evaluate $\iint x\,\mathrm{d}x\,\mathrm{d}y$, over the region in the positive quadrant bounded by the circle $x^2 - 2ax + y^2 = 0$.
25. Evaluate $\iint (x + y)\,\mathrm{d}x\,\mathrm{d}y$, over the region in the positive quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
26. Evaluate $\iint (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$, over the area bounded by the parabola $y^2 = 4x$ and its latus rectum.
27. Evaluate $\iint_R x^2\,\mathrm{d}x\,\mathrm{d}y$, where $R$ is the region bounded by the hyperbola $xy = 4$, $y = 0$, $x = 1$ and $x = 2$.
28. Evaluate $\iiint_V (xy + yz + zx)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where $V$ is the region of space bounded by $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$ and $z = 3$.
29. Evaluate $\iiint_V \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(x + y + z + 1)^3}$, where $V$ is the region of space bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.
30. Evaluate $\iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where $V$ is the region of space bounded by the co-ordinate planes and the sphere $x^2 + y^2 + z^2 = 1$ and contained in the positive octant.

## 4.4 CHANGE OF ORDER OF INTEGRATION IN A DOUBLE INTEGRAL

In worked example (1) of the previous section, we have observed that if the limits of integration in a double integral are constants, then the order of integration can be changed, provided the relevant limits are taken for the concerned variables.

But when the limits for inner integration are functions of a variable, the change in the order of integration will result in changes in the limits of integration.

i.e. the double integral $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ will take the form $\int_a^b \int_{h_1(x)}^{h_2(x)} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$, when the order of integration is changed. This process of converting a given double integral into its equivalent double integral by changing the order of integration is often called *change of order of integration*. To effect the change of order of integration, the region of integration is identified first, a rough sketch of the region is drawn and then the new limits are fixed, as illustrated in the following worked examples.

## 4.5 PLANE AREA AS DOUBLE INTEGRAL

Plane area enclosed by one or more curves can be expressed as a double integral both in Cartesian coordinates and in polar coordinates. The formulas for plane areas in both the systems are derived below:

**(*i*) *Cartesian System***

Let $R$ be the plane region, the area of which is required. Let us divide the area into a large number of elemental areas like $PQRS$ (shaded) by drawing lines parallel to the $y$-axis at intervals of $\Delta x$ and lines parallel to the $x$-axis at intervals of $\Delta y$ (Fig. 4.15).

Area of the elemental rectangle $PQRS = \Delta x . \Delta y$. Required area $A$ of the region $R$ is the sum of elemental areas like $PQRS$.

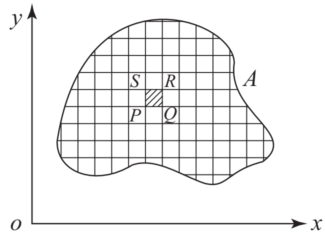


**Fig. 4.15**

viz.,
$$A = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Sigma\Sigma\ \Delta x\ \Delta y)$$
$$= \iint_R \mathrm{d}x\, \mathrm{d}y$$

**(*ii*) *Polar System***

We divide the area $A$ of the given region $R$ into a large number of elemental curvilinear rectangular areas like $PQRS$ (shaded) by drawing radial lines and concentric circular arcs, where $P$ and $R$ have polar coordinates $(r, \theta)$ and $(r + \Delta r, \theta + \Delta\theta)$ (Fig. 4.16)

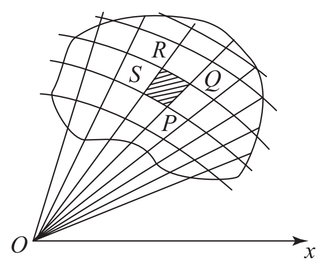


**Fig. 4.16**

Area of the element $PQRS = r\ \Delta r\ \Delta\theta$
$(\because\ PS = r\ \Delta\theta$ and $PQ = \Delta r)$

$\therefore$ Required area
$$A = \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} (\Sigma\Sigma\ r\ \Delta r\ \Delta\theta)$$
$$= \iint_R r\, \mathrm{d}r\, \mathrm{d}\theta.$$

### 4.5.1 Change of Variables

**(*i*) *From Cartesian Coordinates to Plane Polar Coordinates***

If the transformations $x = x(u, v)$ and $y = y(u, v)$ are made in the double integral $\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, then $f(x,y) \equiv g(u,v)$ and $\mathrm{d}x\ \mathrm{d}y = |J|\ \mathrm{d}u\ \mathrm{d}v$, where $J = \dfrac{\partial(x,y)}{\partial(u,v)}$.

[Refer to properties of Jacobians in the Unit 2, "Functions of Several Variables"].

When we transform from cartesian system to plane polar system,

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

In this case,

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Hence $\displaystyle\int\int_R f(x, y)\, dx\, dy = \int\int_R g(r, \theta)\, r\, dr\, d\theta$

In particular,

Area $A$ of the plane region $R$ is given by

$$A = \iint_R dx\, dy = \iint_R r\, dr\, d\theta$$

### (*ii*) *From Three Dimensional Cartesians to Cylindrical Coordinates*

Let us first define cylindrical coordinates of a point in space and derive the relations between cartesian and cylindrical coordinates (Fig. 4.17).

Let $P$ be the point $(x, y, z)$ in Cartesian coordinate system. Let $PM$ be drawn $\perp r$ to the $xoy$-plane and $MN$ parallel to $Oy$. Let $\angle NOM = \theta$ and $OM = r$. The triplet $(r, \theta, z)$ are called the cylindrical coordinates of $P$.

Clearly, $ON = x = r\cos\theta$ ; $NM = y = r\sin\theta$ and $MP = z$.

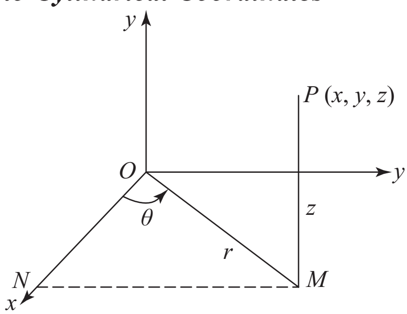


**Fig. 4.17**

Thus the transformations from three dimensional cartesians to cylindrical coordinates are $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

In this case,

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r$$

Hence $dx\, dy\, dz = r\, dr\, d\theta\, dz$

and $\displaystyle\iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_V g(r, \theta, z)\, r\, dr\, d\theta\, dz$

In particular, the volume of a region of space $V$ is given by

$$\iiint_V \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_V r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

*Note* ☑ Whenever $\iiint f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is to be evaluated throughout the volume of a right circular cylinder, it will be advantageous to evaluate the corresponding triple integral in cylindrical coordinates.

### (*iii*) *From Three Dimensional Cartesians to Spherical Polar Coordinates*

Let us first define spherical polar coordinates of a point in space and derive the relations between Cartesian and spherical polar coordinates (Fig. 4.18).

Let $P$ be the point whose Cartesian coordinates are $(x, y, z)$. Let $PM$ be drawn $\perp r$ to the $xOy$-plane. Let $MN$ be parallel to $y$-axis. Let $OP = r$, the angle made by $OP$ with the positive $z$-axis $= \theta$ and the angle made by $OM$ with $x$-axis $= \phi$.

The triplet $(r, \theta, \phi)$ are called the spherical polar coordinates of $P$.

Since $\lfloor OMP = 90°$, $MP = z = r\cos\theta$, $OM = r\sin\theta$, $ON = x = r\sin\theta\cos\phi$ and $NM = y = r\sin\theta\sin\phi$.

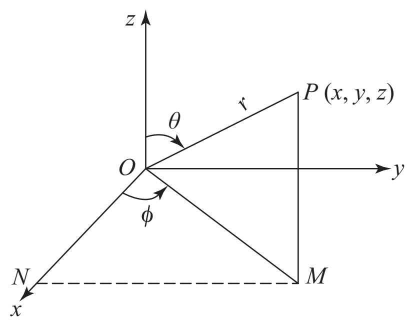


**Fig. 4.18**

Thus the transformations from three dimensional cartesians to spherical polar coordinates are

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

In this case,

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

[Refer to example (2.8) of Worked example set 2(b) in Unit 2 "Functions of Several Variables."]

Hence $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\theta\,\mathrm{d}r\,d\theta\,\mathrm{d}\phi$ and $\iiint_V f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_V g(r, \theta, \phi)\,r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$.

In particular, the volume of a region of space $V$ is given by

$$\iiint_V \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_V r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

*Note* ☑ Whenever $\iiint f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is to be evaluated throughout the volume of a sphere, hemisphere or octant of a sphere, it will be advantageous to use spherical polar coordinates.)

## WORKED EXAMPLE 4(b)

**Example 4.1** Change the order of integration in $\int_0^a \int_y^a \frac{x}{\sqrt{x^2 + y^2}}\, dx\, dy$ and then evaluate it.

The region of integration $R$ is defined by $y \le x \le a$ and $0 \le y \le a$. i.e. it is bounded by the lines $x = y$, $x = a$, $y = 0$ and $y = a$.

The rough sketch of the boundaries and the region $R$ is given in Fig. 4.19.

After changing the order of integration, the given integral I becomes

$$I = \iint_R \frac{x}{\sqrt{x^2 + y^2}}\, dy\, dx$$

The limits of inner integration are found by treating $x$ as a constant, i.e. by drawing a line parallel to the $y$-axis in the region of integration as explained in the previous section.

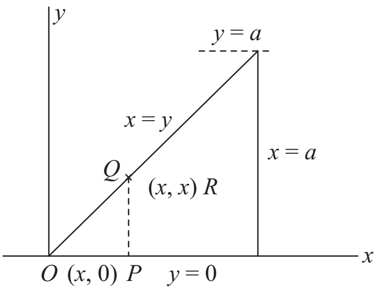


**Fig. 4.19**

Thus
$$I = \int_0^a \int_0^x \frac{x}{\sqrt{x^2 + y^2}}\, dy\, dx$$

$$= \int_0^a x \left\{ \log\left( y + \sqrt{y^2 + x^2} \right) \right\}_{y=0}^{y=x} dx$$

$$= \int_0^a x\, [\log (x + x\sqrt{2}) - \log x]\, dx$$

$$= \log (1 + \sqrt{2}) \cdot \left( \frac{x^2}{2} \right)_0^a = \frac{a^2}{2} \log\ (1 + \sqrt{2})$$

**Example 4.2** Change the order of integration in $\int_0^1 \int_x^1 \frac{x\, dx\, dy}{x^2 + y^2}$ and then evaluate it.

***Note*** ☑ Since the limits of inner integration are $x$ and 1, the corresponding variable of integration should be $y$. So we rewrite the given integral I in the corrected form first.

$$I = \int_0^1 \int_x^1 \frac{x\, dy\, dx}{x^2 + y^2}$$

The region of integration $R$ is bounded by the lines $x = 0$, $x = 1$, $y = x$ and $y = 1$ and is given in Fig. 4.20.

The limits for the inner integration (after changing the order of integration) with respect to $x$ are fixed as usual, by drawing a line parallel to $x$-axis ($y$ = constant)

$\therefore$
$$I = \int_0^1 \int_0^y \frac{x\, dx\, dy}{x^2 + y^2}$$

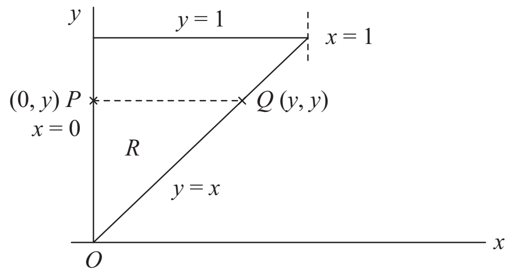


**Fig. 4.20**

$$= \int_0^1 \left[ \frac{1}{2} \log \ (x^2 + y^2) \right]_{x=0}^{x=y} dy$$

$$= \frac{1}{2} \int_0^1 \log \left( \frac{2y^2}{y^2} \right) dy$$

$$= \frac{1}{2} \log 2.$$

**Example 4.3** Change the order of integration in $\int_0^b \int_0^{\frac{a}{b}(b-y)} xy \, dx \, dy$ and then evaluate it.

The region of integration $R$ is bounded by the lines $x = 0$, $x = \frac{a}{b}(b - y)$ or $\frac{x}{a} + \frac{y}{b} = 1$, $y = 0$ and $y = b$ and is shown in Fig. 4.21.

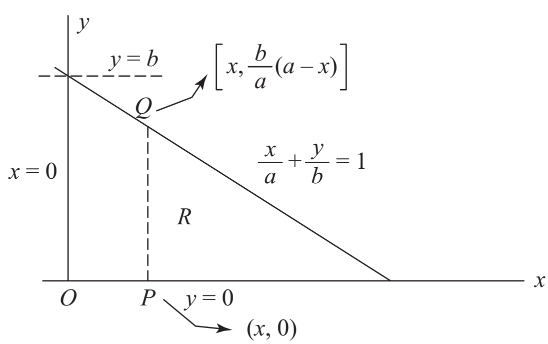


**Fig. 4.21**

After changing the order of integration, the integral becomes $I = \iint_R xy \, dy \, dx$. The limits are fixed as usual.

$$I = \int_0^a \int_0^{\frac{b}{a}(a-x)} xy \, dy \, dx$$

$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{\frac{b}{a}(a-x)} dx$$

$$= \frac{b^2}{2a^2}\int_0^a x\,(a-x)^2\, dx$$

$$= \frac{b^2}{2a^2}\left[a^2\frac{x^2}{2} - 2a\frac{x^3}{3} + \frac{x^4}{4}\right]_0^a$$

$$= \frac{a^2b^2}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{a^2b^2}{24}$$

**Example 4.4** Change the order of integration in $\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\, dy\, dx$ and then integrate it.

The region of integration $R$ is bounded by the lines $x = 0$, $x = a$, $y = 0$ and the curve $y = \frac{b}{a}\sqrt{a^2 - x^2}$ i.e. the curve $\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$, i.e. the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and is shown in Fig. 4.22.

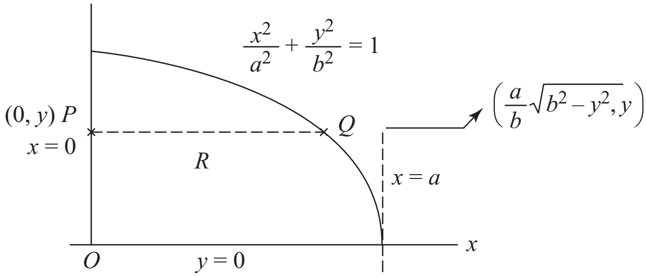


**Fig. 4.22**

After changing the order of integration, the integral becomes

$$\mathrm{I} = \iint_R x^2\, dx\, dy$$

The limits are fixed as usual.

$$\mathrm{I} = \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} x^2\, dx\, dy$$

$$= \int_0^b \left(\frac{x^3}{3}\right)_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \frac{a^3}{3b^3}\int_0^b (b^2 - y^2)^{\frac{3}{2}}\, dy$$

$$= \frac{a^3}{3b^3}\int_0^{\pi/2} b^4 \cos^4\theta\, d\theta \qquad \text{(on putting } y = b\sin\theta\text{)}$$

$$= \frac{a^3 b}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= \frac{\pi}{16}a^3 b$$

**Example 4.5** Change the order of integration in $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\, dx\, dy$ and then evaluate it.

The region of integration $R$ is bounded by the line $x = a - y$, the curve $x = \sqrt{a^2 - y^2}$, the lines $y = 0$ and $y = a$.
i.e. the line $x + y = a$, the circle $x^2 + y^2 = a^2$ and the lines $y = 0$, $y = a$. $R$ is shown in Fig. 4.23.

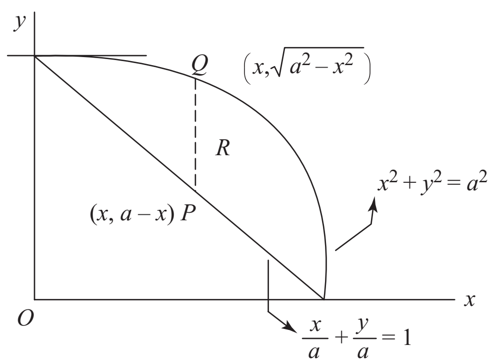


**Fig. 4.23**

After changing the order of integration, the integral I becomes,

$$I = \iint_R y\, dy\, dx$$

$$= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, dy\, dx$$

$$= \int_0^a \left(\frac{y^2}{2}\right)_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a (2ax - 2x^2)\, dx$$

$$= \left( a\frac{x^2}{2} - \frac{x^3}{3} \right)_0^a$$

$$= \frac{a^3}{6}.$$

**Example 4.6** Change the order of integration in $\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} \mathrm{d}y\, \mathrm{d}x$ and then evaluate it.

The region of integration $R$ is bounded by the curve $y = \frac{x^2}{4}$ i.e. the parabola $x^2 = 4y$, the curve $y = 2\sqrt{x}$ i.e. the parabola $y^2 = 4x$ and the lines $x = 0$, $x = 4$. $R$ is shown in Fig. 4.24.

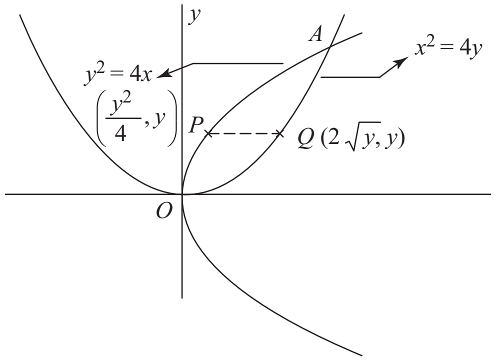


**Fig. 4.24**

The points of intersection of the two parabolas are obtained by solving the equations $x^2 = 4y$ and $y^2 = 4x$.

Solving them, we get $\left(\frac{x^2}{4}\right)^2 = 4x$

i.e $\quad x(x^3 - 64) = 0$

$\therefore \quad x = 0, \quad x = 4$

and $\quad y = 0, \quad y = 4$

i.e. the points of intersection are $O(0, 0)$ and $A(4, 4)$.

After changing the order of integration, the given integral

$$I = \iint_R \mathrm{d}x\, \mathrm{d}y$$

$$= \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} \mathrm{d}x\, \mathrm{d}y$$

$$= \int_0^4 \left( 2\sqrt{y} - \frac{y^2}{4} \right) \mathrm{d}y$$

$$=\left(\frac{4}{3}y^{\frac{3}{2}}-\frac{y^3}{12}\right)_0^4$$
$$=\frac{32}{3}-\frac{16}{3}$$
$$=\frac{16}{3}$$

**Example 4.7** Change the order of integration in $\int\limits_0^a \int\limits_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2+y^2}} xy\,dx\,dy$ and then evaluate it.

The region of integration $R$ is bounded by the curve $x=a\mp\sqrt{a^2-y^2}$, i.e. the circle $(x-a)^2+y^2=a^2$ and the lines $y=0$ and $y=a$. The region $R$ is shown in Fig. 4.25.

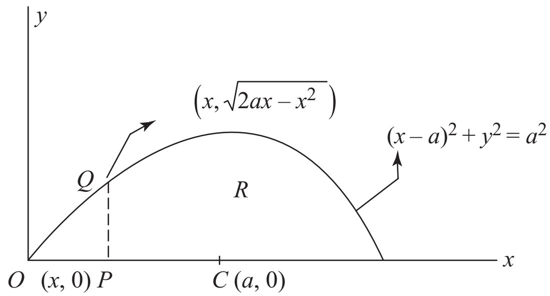


**Fig. 4.25**

After changing the order of integration, the integral I becomes

$$I=\iint\limits_R xy\,dy\,dx$$
$$=\int\limits_0^{2a}\int\limits_0^{\sqrt{2ax-x^2}} xy\,dy\,dx$$
$$=\int\limits_0^{2a} x\left(\frac{y^2}{2}\right)_0^{\sqrt{2ax-x^2}} dx$$
$$=\frac{1}{2}\int\limits_0^{2a}(2ax^2-x^3)dx$$
$$=\frac{1}{2}\left(2a\frac{x^3}{3}-\frac{x^4}{4}\right)_0^{2a}$$
$$=\frac{2}{3}a^4\cdot$$

**Example 4.8** Change the order of integration in $\int_0^1 \int_y^{2-y} xy \, dx \, dy$ and then evaluate it.

The region of integration $R$ is bounded by the lines $x = y$, $x + y = 2$, $y = 0$ and $y = 1$. It is shown in Fig. 4.26.

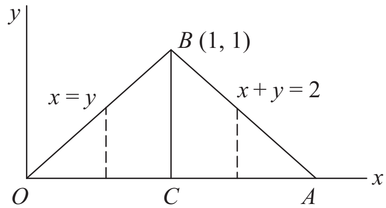


**Fig. 4.26**

After changing the order of integration, the integral I becomes

$$I = \iint_R xy \, dy \, dx$$

To fix the limits for $y$ in the inner integration, we have to draw a line parallel to $y$-axis (since $x$ = constant). The line drawn parallel to the $y$-axis does not intersect the region $R$ in the same fashion. If the line segment is drawn in the region $OCB$, its upper end lies on the line $y = x$; on the other hand, if it is drawn in the region $BCA$, its upper end lies on the line $x + y = 2$. In such situations, we divide the region into two sub-regions and fix the limits for each sub-region as illustrated below:

$$I = \iint_{\Delta OCB} xy \, dy \, dx + \iint_{\Delta BCA} xy \, dy \, dx$$

$$= \int_0^1 \int_0^x xy \, dy \, dx + \int_1^2 \int_0^{2-x} xy \, dy \, dx$$

$$= \int_0^1 x \left( \frac{y^2}{2} \right)_0^x dx + \int_1^2 x \left( \frac{y^2}{2} \right)_0^{2-x} dx$$

$$= \int_0^1 \frac{x^3}{2} dx + \int_1^2 \frac{x}{2} (2 - x)^2 \, dx$$

$$= \left( \frac{x^4}{8} \right)_0^1 + \frac{1}{2} \left( 2x^2 - \frac{4}{3} x^3 + \frac{x^4}{4} \right)_1^2$$

$$= \frac{1}{8} + \frac{5}{24}$$

$$= \frac{1}{3}$$

**Example 4.9** Change the order of integration in $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy \, dy \, dx$ and then evaluate it.

The region of integration $R$ is bounded by the curve $y = \frac{x^2}{a}$, i.e. the parabola $x^2 = ay$, the line $y = 2a - x$, i.e. $x + y = 2a$ and the lines $x = 0$ and $x = a$. It is shown in Fig. 4.27.

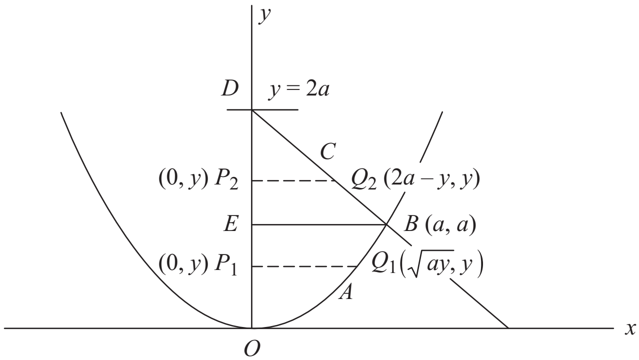


**Fig. 4.27**

After changing the order of integration, the integral I becomes

$$\mathrm{I} = \iint_R xy \, \mathrm{d}x \, \mathrm{d}y$$

When we draw a line parallel to $x$-axis for fixing the limits for the inner integration with respect to $x$, it does not intersect the region of integration in the same fashion. Hence the region $R$ is divided into two sub-regions $OABE$ and $EBCD$ and then the limits are fixed as given below:

$$\mathrm{I} = \iint_{OABE} xy \, \mathrm{d}x \, \mathrm{d}y + \iint_{EBCD} xy \, \mathrm{d}x \, \mathrm{d}y$$

$$= \int_0^a \int_0^{\sqrt{ay}} xy \, \mathrm{d}x \, \mathrm{d}y + \int_a^{2a} \int_0^{2a-y} xy \, \mathrm{d}x \, \mathrm{d}y$$

*Note* ☑ The co-ordinates of the point $B$ are obtained by solving the equations $x + y = 2a$ and $x^2 = ay$.

$B \equiv (a, a)$ and the equation of $EB$ is $y = a$.

$$\mathrm{I} = \int_0^a y \left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} \mathrm{d}y + \int_a^{2a} y \left(\frac{x^2}{2}\right)_0^{2a-y} \mathrm{d}y$$

$$= \frac{1}{2}\left[\int_0^a ay^2 \, \mathrm{d}y + \int_a^{2a} y(2a-y)^2 \, \mathrm{d}y\right]$$

$$= \frac{1}{2}\left[a\left(\frac{y^3}{3}\right)_0^a + \left(2a^2 y^2 - \frac{4a}{3} y^3 + \frac{y^4}{4}\right)_a^{2a}\right] = \frac{3}{8} a^4.$$

**Example 4.10** Change the order of integration in each of the double integrals $\int_0^1 \int_1^2 \frac{\mathrm{d}x \, \mathrm{d}y}{x^2 + y^2}$ and $\int_1^2 \int_y^2 \frac{\mathrm{d}x \, \mathrm{d}y}{x^2 + y^2}$ and hence express their sum as one double integral and evaluate it.

The region of integration $R_1$ for the first double integral $\mathrm{I}_1$ is bounded by the lines $x = 1$, $x = 2$, $y = 0$ and $y = 1$.

The region of integration $R_2$ for the second double integral $I_2$ is bounded by the lines $x = y$, $x = 2$, $y = 1$ and $y = 2$.

$R_1$ and $R_2$ are shown in Fig. 4.28.

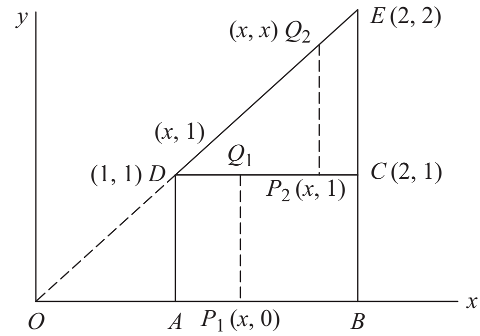


**Fig. 4.28**

After changing the order of integration,

$$I_1 = \int_1^2 \int_0^1 \frac{dy\,dx}{x^2 + y^2}$$

and $$I_2 = \int_1^2 \int_1^x \frac{dy\,dx}{x^2 + y^2}$$

Adding the integrals $I_1$ and $I_2$, we get

$$I = \int_1^2 dx \left( \int_0^1 \frac{dy}{x^2 + y^2} + \int_1^x \frac{dy}{x^2 + y^2} \right)$$

$$= \int_1^2 dx \int_0^x \frac{dy}{x^2 + y^2}$$

$$= \int_1^2 \left( \frac{1}{x} \tan^{-1} \frac{y}{x} \right)_{y=0}^{y=x} dx$$

$$= \int_1^2 \frac{\pi}{4} \frac{dx}{x} = \frac{\pi}{4} \log 2 .$$

**Example 4.11** Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = x$ by double integration.

The region, the area of which is required is bounded by the parabolas $(y - 0)^2 = -(x - 4)$ and $y^2 = x$ and is shown in Fig. 4.29.

Required area $= \int_{OC} \int_{AB} dx\,dy$

$$= 2 \int_{o} \int_{AB} dx\, dy, \text{ by symmetry}$$

$$= 2 \int_{0}^{\sqrt{2}} \int_{y^2}^{4-y^2} dx\, dy$$

$$= 2 \int_{0}^{\sqrt{2}} (4 - y^2 - y^2)\, dy$$

$$= 2 \left(4y - \frac{2}{3} y^3\right)_0^{\sqrt{2}}$$

$$= 2 \left(4\sqrt{2} - \frac{4}{3} \sqrt{2}\right)$$

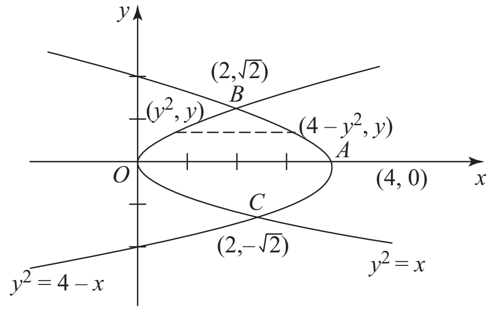


**Fig. 4.29**

$$= \frac{16}{3} \sqrt{2} \text{ square units}$$

**Example 4.12** Find the area between the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ lying in the first quadrant, by double integration.

The plane region, the area of which is required, is shown in Fig. 4.30.

$$\text{Required area } = \iint_{ABC} dx\, dy$$

$$= \int_{0}^{a} \int_{a-y}^{\sqrt{a^2 - y^2}} dx\, dy$$

$$= \int_{0}^{a} \left(\sqrt{a^2 - y^2} - a + y\right) dy$$

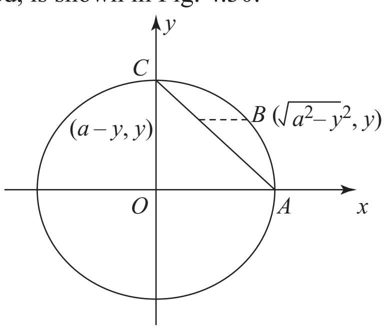


**Fig. 4.30**

$$= \left(\frac{y}{2} \sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} - ay + \frac{y^2}{2}\right)_0^a$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2} - a^2 + \frac{a^2}{2} = (\pi - 2) \frac{a^2}{4}$$

**Example 4.13** Find the area enclosed by the lemniscate $r^2 = a^2 \cos 2\theta$, by double integration.

As the equation $r^2 = a^2 \cos 2\theta$ remains unaltered on changing $\theta$ to $-\theta$, the curve is symmetrical about the initial line.

The points of intersection of the curve with the initial line $\theta = 0$ are given by $r^2 = a^2$ or $r = \pm a$.

Since $r^2 = a^2 \cos 2\alpha = a^2 \cos 2(\pi - \alpha)$, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

On putting $r = 0$, we get $\cos 2\theta = 0$. Hence $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$. Hence there is a loop of the curve between $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$ and another loop between $\theta = -\frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4}$.

Based on the observations given above the lemniscate is drawn in Fig. 4.31.

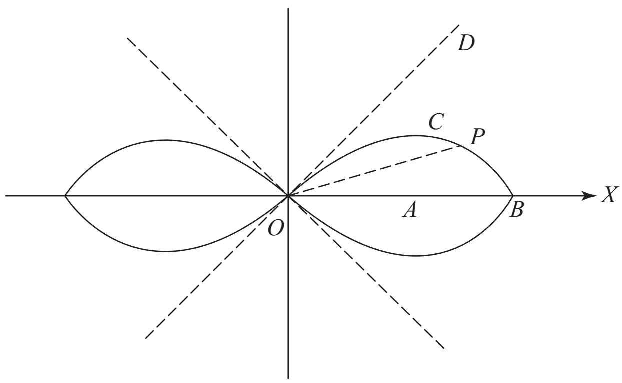


**Fig. 4.31**

Required area = 4 × area *OABC* (by symmetry)

$$= 4 \iint_{OAB} r \, \mathrm{d}r \, \mathrm{d}\theta$$

When we perform the inner integration with respect to $r$, we have to treat $\theta$ as a constant temporarily and find the limits for $r$.

Geometrically, treating $\theta$ = constant means drawing a line *OP* arbitrarily through the pole lying within the region of integration as shown in the figure.

Finding the limits for $r$ (while $\theta$ is a constant) is equivalent to finding the variation of the $r$ coordinate of any point on the line *OP*. Assuming that the $\theta$ coordinates of all points on *OP* are $\theta$ each (since $\theta$ is constant on *OP*), we take $O \equiv (0, \theta)$ and $P \equiv (r_1, \theta)$; viz., $r$ varies from 0 to $r_1$. Now wherever *OP* be drawn, the point $P(r_1, \theta)$ lies on the lemniscate.

Hence $r_1^2 = a^2 \cos 2\theta$ or $r_1 = a\sqrt{\cos 2\theta}$ (since $r$ coordinate of any point is + ve)

Thus the limits for inner integration are 0 and $a\sqrt{\cos 2\theta}$.

When we perform the outer integration, we have to find the limits for $\theta$. Geometrically, we have to find the variation of the line *OP* so that it sweeps the area of the region, namely *OABC*. To cover this area, the line *OP* has to start from the position $OA$ ($\theta = 0$) and move in the anticlockwise direction and go up to $OD\left(\theta = \dfrac{\pi}{4}\right)$. Thus the limits for $\theta$ are 0 and $\dfrac{\pi}{4}$.

$$\therefore \quad \text{Required area} = 4\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\int_0^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} \mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{4}} \cos 2\theta\,\mathrm{d}\theta$$

$$= a^2\,(\sin 2\theta)_0^{\frac{\pi}{4}} = a^2$$

**Example 4.14** Find the area that lies inside the cardioid $r = a\,(1 + \cos\theta)$ and outside the circle $r = a$, by double integration.

The cardioid $r = a\,(1 + \cos\theta)$ is symmetrical about the initial line. The point of intersection of the line $\theta = 0$ with the cardioid is given by $r = 2a$, viz., the point $(2a, 0)$.

Putting $r = 0$ in the equation, we get $\cos\theta = -1$ and $\theta = \pm\pi$. Hence the cardioid lies between the lines $\theta = -\pi$ and $\theta = \pi$.

The point of intersection of the line $\theta = \dfrac{\pi}{2}$ is $\left(a, \dfrac{\pi}{2}\right)$.

Noting the above properties, the cardioid is drawn as shown in Fig. 4.32. All the points on the curve $r = a$ have the same $r$ coordinate $a$, viz., they are at the same distance $a$ from the pole. Hence the equation $r = a$ represents a circle with centre at the pole and radius equal to $a$.

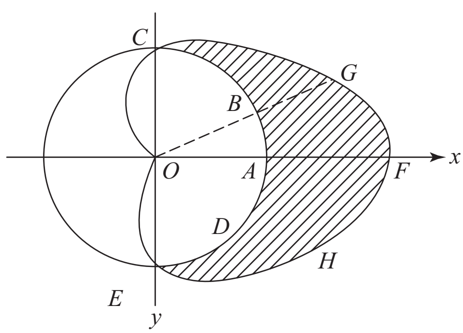

Fig. 4.32

Noting the above points, the circle $r = a$ is drawn as shown in Fig. 4.32. The area that lies outside the circle $r = a$ and inside the cardioid is shaded in the figure.

Both the curves are symmetric about the initial line. Hence the required area

$$= 2 \times AFGCB$$

$$= 2\int_0^{\frac{\pi}{2}}\int_{r_1}^{r_2} r\,dr\,d\theta$$, where $(r_1, \theta)$ lies on the circle $r = a$ and $(r_2, \theta)$ lies on the cardioid $r = a\,(1 + \cos\theta)$

$$= 2\int_0^{\frac{\pi}{2}}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}[(1+\cos\theta)^2 - 1]\,d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}\left(2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= a^2\left[2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= a^2\left(2 + \frac{\pi}{4}\right) = \frac{a^2}{4}(\pi + 8)$$

**Example 4.15** Express $\int_0^a\int_y^a \frac{x^2\,dx\,dy}{\left(x^2+y^2\right)^{3/2}}$ in polar coordinates and then evaluate it.

The region of integration is bounded by the lines $x = y$, $x = a$, $y = 0$ and $y = a$, whose equations in polar system are $\theta = \frac{\pi}{4}$, $r = a\sec\theta$, $\theta = 0$ and $r = a\operatorname{cosec}\theta$ respectively. The region is shown in Fig. 4.33.

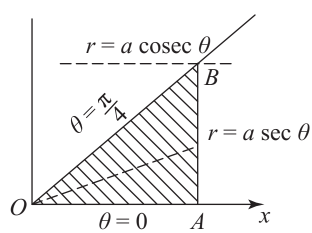


**Fig. 4.33**

Putting $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$ in the given double integral $I$, we get

$$I = \iint_{OAB} \frac{r^3\cos^2\theta}{r^3}\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} \cos^2\theta\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\pi/4} \cos^2\theta\cdot[r]_0^{a\sec\theta}\,\mathrm{d}\theta$$

$$= a\int_0^{\pi/4} \cos\theta\,\mathrm{d}\theta = a\,[\sin\theta]_0^{\pi/4} = \frac{a}{\sqrt{2}}$$

**Example 4.16** Transform the double integral $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{a^2-x^2-y^2}}$ in polar coordinates and then evaluate it.

The region of integration is bounded by the curves $y=\sqrt{ax-x^2}$, $y=\sqrt{a^2-x^2}$ and the lines $x = 0$ and $x = a$.

$$y=\sqrt{ax-x^2} \text{ is the curve } x^2 + y^2 - ax = 0$$

i.e.,
$$\left(x-\frac{a}{2}\right)^2 + (y-0)^2 = \left(\frac{a}{2}\right)^2$$

i.e. the circle with centre at $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$

$$y=\sqrt{a^2-x^2} \text{ is the curve } x^2 + y^2 = a^2$$

i.e. the circle with centre at the origin and radius $a$.

The polar equations of the boundaries of the region of integration are $r^2 - ar\cos\theta = 0$ or $r = a\cos\theta$, $r = a$, $r = a\sec\theta$ and $\theta = \frac{\pi}{2}$. The region of integration is shown in Fig. 4.34.

Putting $x = r\cos\theta$, $y = r\sin\theta$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ in the given double integral $I$, we get

$$I = \int_0^{\pi/2}\int_{a\cos\theta}^{a} \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{a^2-r^2}}$$

$$= \int_0^{\pi/2} \left\{-\frac{1}{2}\times 2\sqrt{a^2-r^2}\right\}_{a\cos\theta}^{a} \mathrm{d}\theta \text{, on putting } a^2 - r^2 = t$$

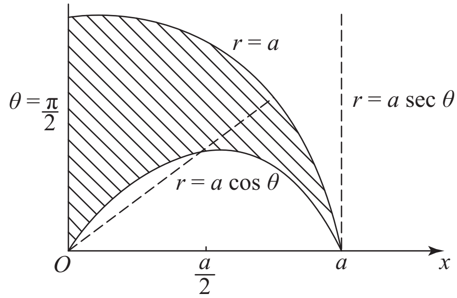


**Fig. 4.34**

$$= \int_0^{\pi/2} a \sin\theta \, d\theta = -a\left[\cos\theta\right]_0^{\frac{\pi}{2}} = a$$

**Example 4.17** By transforming into cylindrical coordinates, evaluate the integral $\iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz$ taken over the region of space defined by $x^2 + y^2 \le 1$ and $0 \le z \le 1$.

The region of space is the region enclosed by the cylinder $x^2 + y^2 = 1$ whose base radius is 1 and axis is the $z$-axis and the planes $z = 0$ and $z = 1$. The equation of the cylinder in cylindrical coordinates is $r = 1$. The region of space is shown in Fig. 4.35.

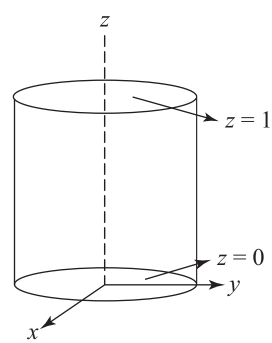


**Fig. 4.35**

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $dx\,dy\,dz = r\,dr\,d\theta\,dz$ in the given triple integral $I$, we get

$$I = \iiint_V (r^2 + z^2)\, r\, dr\, d\theta\, dz,$$

where $V$ is the volume of the region of space.

$$= \int_0^1 \int_0^{2\pi} \int_0^1 (r^2 + z^2)\, r\, dr\, d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{r^4}{4} + z^2 \frac{r^2}{2}\right)_0^1 d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{1}{4} + \frac{1}{2}z^2\right) d\theta\, dz$$

$$= 2\pi \int_0^1 \left(\frac{1}{4} + \frac{1}{2}z^2\right) dz$$

$$= 2\pi\left[\frac{z}{4} + \frac{z^3}{6}\right]_0^1$$

$$= \frac{5}{6}\pi$$

*Note* ☑ The intersection of $z$ = constant $c$ and the cylinder $x^2 + y^2 = 1$ is a circle with centre at $(0, 0, c)$ and radius 1. The limits for $r$ and $\theta$ have been fixed to cover the area of this circle and then the variation of $z$ has been used so as to cover the entire volume.]

**Example 4.18** Find the volume of the portion of the cylinder $x^2 + y^2 = 1$ intercepted between the plane $z = 0$ and the paraboloid $x^2 + y^2 = 4 - z$.

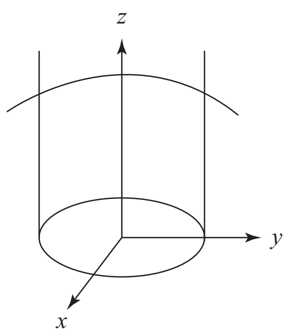


**Fig. 4.36**

Using cylindrical coordinates, the required volume $V$ is given by

$$V = \iiint r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \text{ taken throughout the region of space.}$$

Since the variation of $z$ is not between constant limits, we first integrate with respect to $z$ and then with respect to $r$ and $\theta$.

Changing to cylindrical coordinates, the boundaries of the region of space are $r = 1$, $z = 0$ and $z = 4 - r^2$.

$$\therefore \quad V = \int_0^{2\pi}\int_0^1\int_0^{4-r^2} \mathrm{d}z \, r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{2\pi}\int_0^1 r(4 - r^2) \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^1 \mathrm{d}\theta = \frac{7}{4}\int_0^{2\pi} \mathrm{d}\theta = \frac{7}{2}\pi$$

**Example 4.19** Evaluate $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} xyz \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$, by transforming to spherical polar coordinates.

The boundaries of the region of integration are $z = 0$, $z = \sqrt{a^2 - x^2 - y^2}$ or $x^2 + y^2 + z^2 = a^2$, $y = 0$, $y = \sqrt{a^2 - x^2}$ or $x^2 + y^2 = a^2$, $x = 0$ and $x = a$. From the boundaries, we note that the region of integration is the volume of the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

By putting $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$, the given triple integral $I$ becomes

$$I = \iiint_V r^3\sin^2\theta\cos\theta\sin\phi\cos\phi . r^2\sin\theta\,dr\,d\theta\,d\phi$$

where $V$ is the volume of the positive octant of the sphere $r = a$, which is shown in Fig. 4.37.

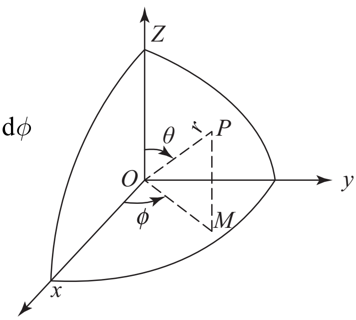


**Fig. 4.37**

To cover the volume $V$, $r$ has to vary from 0 to $a$, $\theta$ has to vary from 0 to $\frac{\pi}{2}$ and $\phi$ has to vary from 0 to $\frac{\pi}{2}$.

Thus $$I = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^a r^5\sin^3\theta\cos\theta\sin\phi\cos\phi\,dr\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}}\sin\phi\cos\phi\,d\phi . \int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta . \int_0^a r^5\,dr$$

[$\because$ the limits are constants]

$$= \left[\frac{\sin^2\phi}{2}\right]_0^{\frac{\pi}{2}} \cdot \left[\frac{\sin^4\theta}{4}\right]_0^{\frac{\pi}{2}} \cdot \left[\frac{r^6}{6}\right]_0^a$$

$$= \frac{1}{48}a^6 .$$

**Example 4.20** Evaluate $\iiint \sqrt{1 - x^2 - y^2 - z^2}\,dx\,dy\,dz$, taken throughout the volume of the sphere $x^2 + y^2 + z^2 = 1$, by transforming to spherical polar coordinates.

Changing to spherical polar coordinates, the given triple integral $I$ becomes

$$I = \iiint_V \sqrt{1 - r^2}\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

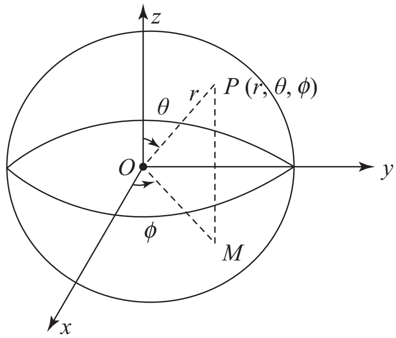


**Fig. 4.38**

To cover the entire volume $V$ of the sphere, $r$ has to vary from 0 to 1, $\theta$ has to vary from 0 to $\pi$ and $\phi$ has to vary from 0 to $2\pi$.

Thus
$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sqrt{1-r^2}\, r^2\, dr \cdot \sin\theta\, d\theta \cdot d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t\, dt \text{, by putting}$$

$r = \sin t$ in the innermost integral

$$= 2\pi \times (-\cos\theta)_0^{\pi} \times \left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= 4\pi \times \frac{\pi}{4} \times \frac{1}{4} = \frac{1}{4}\pi^2$$

## EXERCISE 4(b)

***Part A***

(Short Answer Questions)

1. Change the order of integration in $\int_0^a \int_0^x f(x, y)\, dy\, dx$.
2. Change the order of integration in $\int_0^1 \int_y^1 f(x, y)\, dx\, dy$.
3. Change the order of integration in $\int_0^a \int_x^a f(x, y)\, dy\, dx$.
4. Change the order of integration in $\int_0^1 \int_0^y f(x, y)\, dx\, dy$.
5. Change the order of integration in $\int_0^1 \int_0^{1-y} f(x, y)\, dx\, dy$.

6. Change the order of integration in $\int_0^a \int_0^{a-x} f(x, y)\,dy\,dx$.

7. Change the order of integration in $\int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y)\,dy\,dx$.

8. Change the order of integration in $\int_0^a \int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy$.

9. Change the order of integration in $\int_0^1 \int_0^{2\sqrt{x}} f(x, y)\,dy\,dx$.

10. Change the order of integration in $\int_0^\infty \int_0^{1/y} f(x, y)\,dx\,dy$.

***Part B***

Change the order of integration in the following integrals and then evaluate them:

11. $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$

12. $\int_0^2 \int_x^2 (x^2+y^2)\,dy\,dx$

13. $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}}\,dy\,dx$

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

15. $\int_0^1 \int_0^{1-x} e^{2x+y}\,dy\,dx$

16. $\int_0^2 \int_0^{\sqrt{4-y^2}} xy\,dx\,dy$

17. $\int_0^{2a} \int_{\frac{x^2}{4a}}^{a} (x+y)\,dy\,dx$

18. $\int_0^1 \int_{y^2}^{y} \frac{y\,dx\,dy}{x^2+y^2}$

19. $\int_0^3 \int_1^{\sqrt{4-x}} (x+y)\,dy\,dx$

20. $\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} y^2\,dx\,dy$

21. $\int_0^3 \int_{\frac{5}{9}x^2}^{\frac{5}{\sqrt{3}}x} dx\,dy$

22. $\int_1^2 \int_0^{4/x} xy\,dy\,dx$

23. $\int_0^1 \int_x^{\sqrt{2-x^2}} dy\,dx$

24. Change the order of integration in each of the double integrals $\int_0^1 \int_0^{\sqrt{x}} xy\,dy\,dx$ and $\int_1^2 \int_0^{2-x} xy\,dy\,dx$ and hence express their sum as one double integral and evaluate it.

25. Change the order of integration in each of the double integrals $\int_{-1}^{0}\int_{-x}^{1}(x^2+y^2)\,dy\,dx$ and $\int_{0}^{1}\int_{x}^{1}(x^2+y^2)\,dy\,dx$ and hence express their sum as one double integral and evaluate it.

Find the area specified in the following problems (26–35), using double integration:

26. The area bounded by the parabola $y = x^2$ and the straight line $2x - y + 3 = 0$.
27. The area included between the parabolas $y^2 = 4a(x + a)$ and $y^2 = 4a(a - x)$.
28. The area bounded by the two parabolas $y^2 = 4ax$ and $x^2 = 4by$.
29. The area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$.
30. The area bounded by the curve $y^2 = \dfrac{x^3}{2-x}$ and its asymptote.
31. The area of the cardioid $r = a(1 + \cos\theta)$.
32. The area common to the two circles $r = a$ and $r = 2a\cos\theta$.
33. The area common to the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.
34. The area that lies inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$.
35. The area that lies outside the circle $r = a\cos\theta$ and inside the circle $r = 2a\cos\theta$.

Change the following integrals (36 − 40), into polar coordinates and then evaluate them:

36. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} e^{-\left(x^2+y^2\right)}\,dx\,dy$

37. $\displaystyle\int_0^a\int_y^a \frac{x\,dx\,dy}{\left(x^2+y^2\right)}$

38. $\displaystyle\int_0^a\int_0^x \frac{x^3\,dx\,dy}{\sqrt{x^2+y^2}}$

39. $\displaystyle\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$

40. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{\left(a^2+x^2+y^2\right)^{3/2}}$

Evaluate the following integrals (41− 45) after transforming into cylindrical coordinates:

41. $\displaystyle\iiint_V (x+y+z)\,dx\,dy\,dz$, where $V$ is the region of space inside the cylinder $x^2 + y^2 = a^2$, that is bounded by the planes $z = 0$ and $z = h$.

42. $\displaystyle\iiint (x^2+y^2)\,dx\,dy\,dz$, taken throughout the volume of the cylinder $x^2 + y^2 = 1$ that is bounded by the planes $z = 0$ and $z = 4$.

43. $\displaystyle\iiint dx\,dy\,dz$, taken throughout the volume of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ and $y + z = 3$.

44. $\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, taken throughout the volume of the cylinder $x^2 + y^2 = 4$ bounded by the plane $z = 0$ and the surface $z = x^2 + y^2 + 2$.

45. $\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, taken throughout the volume bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylindrical surface $x^2 + y^2 - 2ay = 0$.

Evaluate the following integrals (46-50) after transforming into spherical polar coordinates:

46. $\iiint \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{x^2+y^2+z^2}$, taken throughout the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

47. $\iiint \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\sqrt{1-x^2-y^2-z^2}}$, taken throughout the volume contained in the positive octant of the sphere $x^2 + y^2 + z^2 = 1$.

48. $\iiint_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where $V$ is the region of space bounded by the sphere $x^2 + y^2 + z^2 = a^2$ above the $xOy$-plane.

49. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

50. $\int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\left(a^2+x^2+y^2+z^2\right)^{5/2}}$

## 4.6 LINE INTEGRAL

The concept of a line integral is a generalisation of the concept of a definite integral $\int_a^b f(x)\,\mathrm{d}x$.

In the definite integral, we integrate along the $x$-axis from $a$ to $b$ and the integrand $f(x)$ is defined at each point in $(a, b)$. In a line integral, we shall integrate along a curve $C$ in the plane (or space) and the integrand will be defined at each point of $C$. The formal definition of a line integral is as follows.

***Definition*** Let $C$ be the segment of a continuous curve joining $A(a, b)$ and $B(c, d)$ (Fig. 4.39).

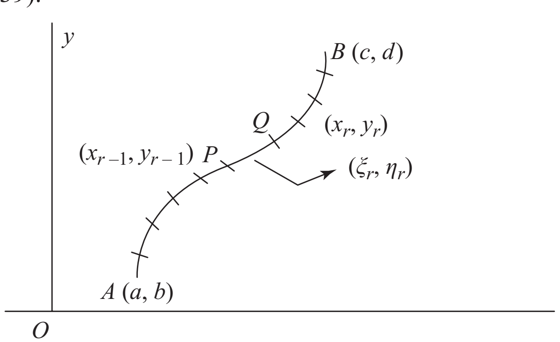


**Fig. 4.39**

Let $f(x, y)$, $f_1(x, y)$, $f_2(x, y)$ be single-valued and continuous functions of $x$ and $y$, defined at all points of $C$.
Divide $C$ into $n$ arcs at $(x_i, y_i)$ $[i = 1, 2, \ldots (n-1)]$
Let $x_0 = a$, $x_n = c$, $y_0 = b$, $y_n = d$.
Let $x_r - x_{r-1} = \Delta x_r$, $y_r - y_{r-1} = \Delta y_r$ and the arcual length of $PQ(\text{i.e.} \widehat{PQ}) = \Delta s_r$, where $P$ is $(x_{r-1}, y_{r-1})$ and $Q(x_r, y_r)$.
Let $(\xi_r, \eta_r)$ be any point on $C$ between $P$ and $Q$.

Then $$\lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r, \eta_r) \Delta s_r$$

or $$\lim_{n\to\infty} \sum_{r=1}^{n} \left[ f_1(\xi_r, \eta_r) \Delta x_r + f_2(\xi_r, \eta_r) \Delta y_r \right]$$

is defined as a line integral along the curve $C$ and denoted respectively as

$$\int_C f(x, y)\,\mathrm{d}s \quad \text{or} \quad \int_C [f_1(x, y)\,\mathrm{d}x + f_2(x, y)\,\mathrm{d}y]$$

### 4.6.1 Evaluation of a Line Integral

Using the equation $y = \phi(x)$ or $x = \psi(y)$ of the curve $C$, we express $\int_C [f_1(x, y)\,\mathrm{d}x + f_2(x, y)\,\mathrm{d}y]$ either in the form $\int_a^c g(x)\,\mathrm{d}x$ or in the form $\int_b^d h(y)\,\mathrm{d}y$ and evaluate it, which is only a definite integral.

If the line integral is in the form $\int_C f(x, y)\,\mathrm{d}s$, it is first rewritten as $\int_C f(x, y) \frac{\mathrm{d}s}{\mathrm{d}x}\,\mathrm{d}x = \int_C f(x, y) \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ or as $\int_C f(x, y) \frac{\mathrm{d}s}{\mathrm{d}y}\,\mathrm{d}y = \int_C f(x, y) \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$ and then evaluated after expressing it as a definite integral.

### 4.6.2 Evaluation when C is a Curve in Space

The definition of the line integral given above can be extended when $C$ is a curve in space. In this case, the line integral will take either the form $\int_C [f_1(x, y, z)\,\mathrm{d}x + f_2(x, y, z)\,\mathrm{d}y + f_3(x, y, z)\,\mathrm{d}z]$ or the form $\int_C f(x, y, z)\,\mathrm{d}s$. When $C$ is a curve in space, very often the parametric equations of $C$ will be known in the form $x = \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$. Using the parametric equations of $C$, the line integral can be expressed as a definite integral. In the case of $\int_C f(x, y, z)\,\mathrm{d}s$, it is rewritten as $\int_C f(x, y, z) \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t$, where

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}.$$

## 4.7 SURFACE INTEGRAL

The concept of a surface integral is a generalisation of the concept of a double integral. While a double integral is evaluated over the area of a plane surface, a surface integral is evaluated over the area of a curved surface in general. The formal definition of a surface integral is given below.

***Definition*** Let $S$ be a portion of a regular two-sided surface. Let $f(z, y, z)$ be a function defined and continuous at all points on $S$. Divide $S$ into $n$ sub-regions $\Delta s_1$, $\Delta S_2, \ldots, \Delta S_n$. Let $P(\xi_r, \eta_r, \zeta_r)$ be any point in $\Delta S_r$. Then $\lim\limits_{\substack{n\to\infty \\ \Delta S_r \to 0}} \sum_{r=1}^{n} f(\xi_r, \eta_r, \zeta_r)\,\Delta S_r$ is called the surface integral of $f(x, y, z)$ over the surface $S$ and denoted as $\int_S f(x, y, z)\,\mathrm{d}S$ or $\iint_S f(x, y, z)\,\mathrm{d}S$.

### 4.7.1 Evaluation of a Surface Integral

Let the surface integral be $\iint_S f(x, y, z)\,\mathrm{d}S$, where $S$ is the portion of the surface whose equation is $\phi(x, y, z) = c$ (Fig. 4.40).

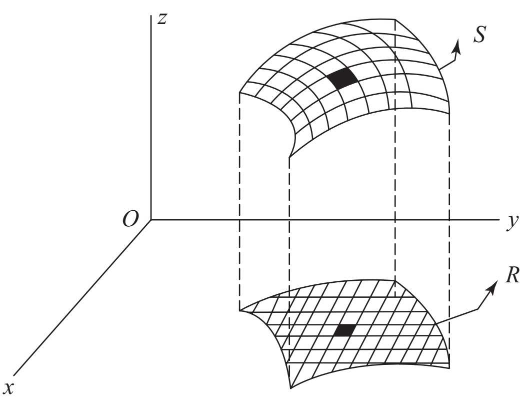


**Fig. 4.40**

Project the surface $S$ orthogonally on *xoy*-plane (or any one of the co-ordinate planes) so that the projection is a plane region $R$.

The projection of the typical elemental surface $\Delta S$ (shaded in the figure) is the typical elemental plane area $\Delta A$ (shaded in the figure).

We can divide the area of the region $R$ into elemental areas by drawing lines parallel to $x$ and $y$ axes at intervals of $\Delta y$ and $\Delta x$ respectively. Then $\Delta A = \Delta x \cdot \Delta y$.

Then $\Delta x \cdot \Delta y = \Delta S \cos\theta$, where $\theta$ is the angle between the surface $S$ and the plane $R$ (*xoy*-plane), i.e. $\theta$ is the angle between the normal to the surface $S$ at the typical point $(x, y, z)$ and the normal to the *xoy*-plane ($z$-axis). From Calculus, it is known that the direction ratios of the normal at the point $(x, y, z)$ to the surface $\phi(x, y, z) = c$ are $\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$. The direction cosines of the $z$-axis are (0, 0, 1)

$$\therefore \qquad \cos\theta = \frac{\dfrac{\partial\phi}{\partial z}}{\sqrt{\left(\dfrac{\partial\phi}{\partial x}\right)^2 + \left(\dfrac{\partial\phi}{\partial y}\right)^2 + \left(\dfrac{\partial\phi}{\partial z}\right)^2}}$$

Thus
$$\Delta S = \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z}\Delta x\,\Delta y\,.$$

$$\therefore \qquad \iint_S f(x, y, z)\,\mathrm{d}S = \iint_R f(x, y, z)\cdot\frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z}\,\mathrm{d}x\,\mathrm{d}y$$

Thus the surface integral is converted into a double integral by using the above relation, in which the limits for the double integration on the right side are fixed so as to cover the entire region $R$ and the integrand is converted into a function of $x$ and $y$, using the equation of $S$.

**Note** ☑ Had we projected the curved surface $S$ on the $yoz$-plane or $zox$-plane then the conversion formula would have been

$$\iint_S f(x, y, z)\mathrm{d}S = \iint_R f(x, y, z)\cdot\frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_x}\,\mathrm{d}y\,\mathrm{d}z$$

or
$$\iint_S f(x, y, z)\mathrm{d}S = \iint_R f(x, y, z)\cdot\frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_y}\,\mathrm{d}z\,\mathrm{d}x, \qquad \text{respectively.}$$

## 4.8 VOLUME INTEGRAL

***Definition*** Let $V$ be a region of space, bounded by a closed surface. Let $f(x, y, z)$ be a continuous function defined at all points of $V$. Divide $V$ into $n$ sub-regions $\Delta V_r$ by drawing planes parallel to the $yoz$, $zox$ and $xoy$-planes at intervals of $\Delta x$, $\Delta y$ and $\Delta z$ respectively. Then $\Delta V_r$ is a rectangular parallelopiped with dimensions $\Delta x$, $\Delta y$, $\Delta z$.

Let $P(\xi_r, \eta_r, \zeta_r)$ be any point in $\Delta V_r$.

Then $\lim\limits_{\substack{n\to\infty \\ \Delta V_r \to 0}} \sum\limits_{r=1}^{n} f(\xi_r, \eta_r, \zeta_r)\,\Delta V_r$ is called the volume integral of $f(x, y, z)$ over the region $V$ (or throughout the volume $V$) and denoted as

$$\int_V f(x, y, z)\,\mathrm{d}v \quad \text{or} \quad \iiint_V f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

## 4.8.1 Triple Integral versus Volume Integral

A triple integral discussed earlier is a three times repeated integral in which the limits of integration are given, whereas a volume integral is a triple integral in which the limits of integration will not be explicitly given, but the region of space in which it is to be evaluated will be specified. The limits of integration in a volume integral are fixed so as to cover the entire volume of the region of space $V$.

***Note*** ☑ Though the line integral and surface integral have been defined in the scalar form in this unit, they are also defined in the vector form.

**WORKED EXAMPLE 4(c)**

**Example 4.1** Evaluate $\int_C [(3xy^2 + y^3)\,\mathrm{d}x + (x^3 + 3xy^2)\,\mathrm{d}y]$ where $C$ is the parabola $y^2 = 4ax$ from the point (0, 0) to the point $(a, 2a)$.

The given integral

$$I = \int_{y^2 = 4ax} [(3xy^2 + y^3)\,\mathrm{d}x + (x^3 + 3\,xy^2)\,\mathrm{d}y]$$

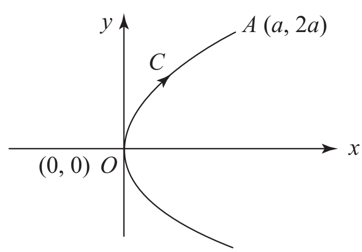


**Fig. 4.41**

In order to use the fact that the line integral is evaluated along the parabola $y^2 = 4ax$, we use this equation and the relation between d$x$ and d$y$ derived from it, namely, $2y\,\mathrm{d}y = 4a\,\mathrm{d}x$ and convert the body of the integral either to the form $f(x)\,\mathrm{d}x$ or to the form $\phi\,(y)\,\mathrm{d}y$. Then the resulting definite integral is evaluated between the concerned limits, got from the end points of $C$.

The choice of the form $f(x)\,\mathrm{d}x$ or $\phi\,(y)\,\mathrm{d}y$ for the body of the integral depends on convenience. In this problem, $x$ is expressed as $\frac{1}{4a}y^2$ more easily than expressing $y$ as $2\sqrt{ax}$.

***Note*** ☑ From $y^2 = 4ax$, we get $y = \pm 2\sqrt{ax}$. Since the arc $C$ lies in the first quadrant, $y$ is positive and hence $y = 2\sqrt{ax}$.

Thus $$I = \int_0^{2a} \left[\left(3 \cdot \frac{1}{4a} y^2 \cdot y^2 + y^3\right)\frac{y}{2a}\,\mathrm{d}y + \left(\frac{1}{64a^3} y^6 + 3 \cdot \frac{1}{4a} y^2 \cdot y^2\right)\mathrm{d}y\right]$$

(As the integration is done with respect to $y$, the limits for $y$ are the $y$ co-ordinates of the terminal points of the arc $C$).

$$I = \int_0^{2a} \left( \frac{5}{4a} y^4 + \frac{3}{8a^2} y^5 + \frac{1}{64a^3} y^6 \right) \mathrm{d}y$$

$$= \left( \frac{1}{4a} y^5 + \frac{1}{16a^2} y^6 + \frac{1}{448a^3} y^7 \right)_0^{2a}$$

$$= \frac{86}{7} a^4$$

**Example 4.2** Evaluate $\int_C [(2x - y)\,\mathrm{d}x + (x + y)\,\mathrm{d}y]$, where $C$ is the circle $x^2 + y^2 = 9$.

In this problem the line integral is evaluated around a closed curve. In such a situation the line integral is denoted as

$\oint_C [(2x - y)\,\mathrm{d}x + (x + y)\,\mathrm{d}y]$, where a small circle is put across the integral symbol.

When a line integral is evaluated around a closed curve, it is assumed to be described in the anticlockwise sense, unless specified otherwise. (Fig. 4.42)

In the case of a line integral around a closed curve $C$, any point on $C$ can be assumed to be the initial point, which will also be the terminal point.

Further if we take $x$ or $y$ as the variable of integration, the limits of integration will be the same, resulting in the value 'zero' of the line integral, which is meaningless. Hence whenever a line integral is evaluated around a closed curve, the parametric equations of the curve are used and hence the body of integral is converted to the form $f(t)\,\mathrm{d}t$ or $f(\theta)\,\mathrm{d}\theta$.

In this problem, the parametric equations of the circle $x^2 + y^2 = 9$ are $x = 3\cos\theta$ and $y = 3\sin\theta$.

$\therefore$ $\mathrm{d}x = -3\sin\theta\,\mathrm{d}\theta$ and $\mathrm{d}y = 3\cos\theta\,\mathrm{d}\theta$.

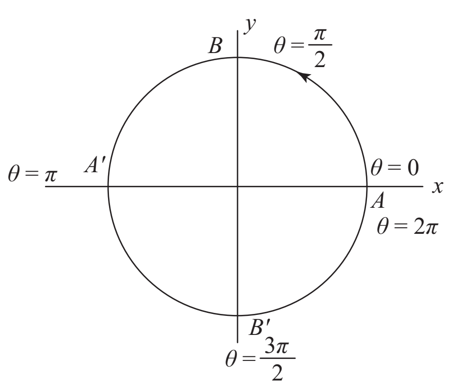


**Fig. 4.42**

The given integral = $\int_0^{2\pi} [(6\cos\theta - 3\sin\theta)\,(-3\sin\theta\,\mathrm{d}\theta)$

$$+ (3\cos\theta + 3\sin\theta)(3\cos\theta\, d\theta)]$$

$$= 9\int_0^{2\pi} (1 - \sin\theta\cos\theta)\, d\theta$$

$$= 9\left(\theta - \frac{\sin^2\theta}{2}\right)_0^{2\pi}$$

$$= 18\pi$$

**Example 4.3** Evaluate $\int_C xy\, ds$, where $C$ is the arc of the parabola $y^2 = 4x$ between the vertex and the positive end of the latus rectum.

Given integral $\quad I = \int_C xy \frac{ds}{dx} dx$

Equation of the parabola is $\quad y^2 = 4x$

Differentiating with respect to $x$, $\dfrac{dy}{dx} = \dfrac{2}{y}$

$\therefore \quad \dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + \dfrac{4}{y^2}}$

$\therefore \quad I = \int_C xy \dfrac{\sqrt{y^2+4}}{y} dx$

$$= \int_0^1 x\sqrt{4x+4}\, dx$$

$$= 2\int_1^{\sqrt{2}} (t^2 - 1)\cdot t \cdot 2t\, dt, \text{ on putting } x + 1 = t^2$$

$$= 4\int_1^{\sqrt{2}} (t^4 - t^2)\, dt$$

$$= 4\left(\frac{t^5}{5} - \frac{t^3}{3}\right)_1^{\sqrt{2}}$$

$$= \frac{8}{15}(1 + \sqrt{2})$$

**Example 4.4** Evaluate $\int_C (y^2 dx - x^2 dy)$, where $C$ is the boundary of the triangle whose vertices are (−1, 0), (1, 0) and (0, 1) (Fig. 4.43).
$C$ is made up of the lines $BC$, $CA$ and $AB$.

Equations of $BC$, $CA$ and $AB$ are respectively $y = 0$, $x + y = 1$ and $-x + y = 1$.

$$\text{Given integral} = \int\limits_{\substack{BC\\y=0\\dy=0}} + \int\limits_{\substack{CA\\x+y=1\\dy=-dx}} + \int\limits_{\substack{AB\\-x+y=1\\dy=dx}} (y^2 dx - x^2 dy)$$

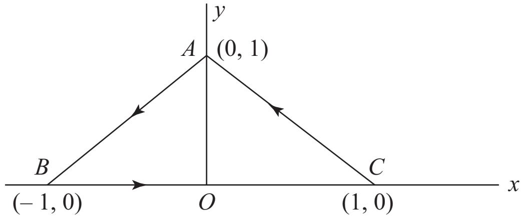


**Fig. 4.43**

$$= 0 + \int_1^0 [(1-x)^2 + x^2]\,dx + \int_0^{-1} [(1+x)^2 - x^2]\,dx$$

$$= \int_1^0 (1 - 2x + 2x^2)\,dx + \int_0^{-1} (1+2x)\,dx$$

$$= \left(x - x^2 + \frac{2x^3}{3}\right)_1^0 + (x + x^2)_0^{-1}$$

$$= -\frac{2}{3}$$

**Example 4.5** Evaluate $\int\limits_C [x^2 y\,dx + (x-z)\,dy + xyz\,dz]$, where $C$ is the arc of the parabola $y = x^2$ in the plane $z = 2$ from (0, 0, 2) to ( 1, 1, 2).

$$\text{Given integral} = \int\limits_{\left(\substack{y=x^2\\z=2}\right)} [x^2 y\,dx + (x-z)\,dy + xyz\,dz]$$

$$= \int\limits_{y=x^2} [x^2 y\,dx + (x-2)\,dy]$$

[$\because$ $dz = 0$, when $z = 2$]

$$= \int_0^1 [x^4 + (x-2)\,2x]\,dx = \left(\frac{x^5}{5} + \frac{2x^3}{3} - 2x^2\right)_0^1$$

$$= -\frac{17}{15}$$

**Example 4.6** Evaluate $\int\limits_C (x\,dx + xy\,dy + xyz\,dz)$, where $C$ is the arc of the twisted curve $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$.

The parametric equations of $C$ are $x = t$, $y = t^2$, $z = t^3$

$\therefore$ $dx = dt$, $dy = 2t\,dt$, $dz = 3t^2\,dt$ on $C$.

Using these values in the given integral I,

$$I=\int_0^1 (t+t^3\cdot 2t+t^6\cdot 3t^2)\,dt$$

$$=\left(\frac{t^2}{2}+2\frac{t^5}{5}+3\frac{t^9}{9}\right)_0^1$$

$$=\frac{17}{30}$$

**Example 4.7** Evaluate $\int_C (x^2+y^2+z^2)\,ds$, where $C$ is the arc of the circular helix $x = \cos t$, $y = \sin t$, $z = 3t$ from (1, 0, 0) to $(1, 0, 6\pi)$

The parametric of equations of $C$ are

$x = \cos t$, $y \sin t$, $z = 3t$.

$$\therefore \quad \frac{dx}{dt}=-\sin t, \qquad \frac{dy}{dx}=\cos t, \qquad \frac{dz}{dt}=3 \text{ on } C.$$

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$$

$$=\sqrt{\sin^2 t+\cos^2 t+9}=\sqrt{10}$$

Given integral $$I=\int_0^{2\pi} (\cos^2 t+\sin^2 t+9t^2)\frac{ds}{dt}\,dt$$

***Note*** ☑ The point (1, 0, 0) corresponds to $t = 0$ and $(1, 0, 6\pi)$ corresponds to $t = 2\pi$.

$$I=\left(t+3t^3\right)_0^{2\pi}\times\sqrt{10}$$

$$=2\sqrt{10}\pi(1+12\pi^2)$$

**Example 4.8** Evaluate $\iint_S xyz\,dS$, where $S$ is the surface of the rectangular parallelopiped formed by $x = 0$, $y = 0$, $z = 0$, $x = a$, $y = b$ and $z = c$ (Fig. 4.44).

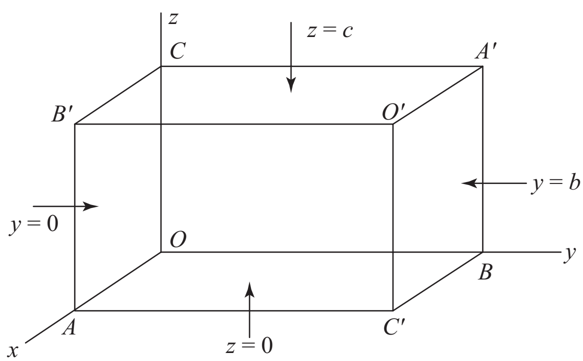


**Fig. 4.44**

Since $S$ is made up of 6 plane faces, the given surface integral I is expressed as

$$\text{I}=\iint\limits_{x=0}+\iint\limits_{x=a}+\iint\limits_{y=0}+\iint\limits_{y=b}+\iint\limits_{z=0}+\iint\limits_{z=c}(xyz\,\text{d}S)$$

Since all the faces are planes, the elemental curved surface area d$S$ becomes the elemental plane surface area d$A$.

On the planes $x = 0$ and $x = a$, d$A$ = d$y$ d$z$.
On the planes $y = 0$ and $y = b$, d$A$ = d$z$ d$x$.
On the planes $z = 0$ and $z = c$, d$A$ = d$x$ d$y$.

$\therefore$
$$\text{I}=\iint\limits_{x=0}+\iint\limits_{x=a}(xyz\,\text{d}y\,\text{d}z)+\iint\limits_{y=0}+\iint\limits_{y=b}(xyz\,\text{d}z\,\text{d}x)+\iint\limits_{z=0}+\iint\limits_{z=c}(xyz\,\text{d}x\,\text{d}y)$$

Simplifying the integrands using the equations of the planes over which the surface integrals are evaluated, we get

$$\text{I}=a\int_0^c\int_0^b yz\,\text{d}y\,\text{d}z+b\int_0^a\int_0^c zx\,\text{d}z\,\text{d}x+c\int_0^b\int_0^a xy\,\text{d}x\,\text{d}y$$

*Note* ☑ On the plane face $O'A'CB'$ ($z = c$), the limits for $x$ and $y$ are easily found to be 0, $a$ and 0, $b$. Similarly the limits are found on the faces $O'B'AC'$ ($x = a$) and $O'C'BA'$ ($y = b$).]

Now
$$\text{I}=a\frac{b^2}{2}\cdot\frac{c^2}{2}+b\cdot\frac{c^2}{2}\cdot\frac{a^2}{2}+c\cdot\frac{a^2}{2}\cdot\frac{b^2}{2}$$
$$=\frac{abc}{4}(ab+bc+ca)$$

**Example 4.9** Evaluate $\iint\limits_S (y+2z-2)\,\text{d}S$, where $S$ is the part of the plane $2x + 3y + 6z = 12$, that lies in the positive octant (Fig. 4.45).

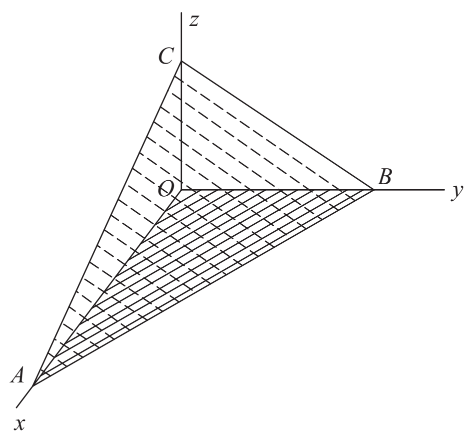


**Fig. 4.45**

Rewriting the equation of the (plane) surface $S$ in the intercept form, we get

$$\frac{x}{6}+\frac{y}{4}+\frac{z}{2}=1$$

∴ *S* is the plane that cuts off intercepts of lengths 6, 4 and 2 on the *x*, *y* and *z*-axes respectively and lies in the positive octant.

We note that the projection of the given plane surface *S* on the *xoy*-plane is the triangular region *OAB* shown in the two-dimensional Fig. 4.46.

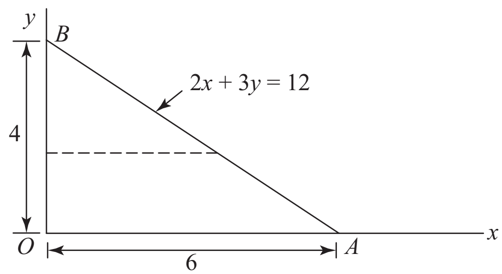


**Fig. 4.46**

Converting the given surface integral I as a double integral,

$$\mathrm{I}=\iint_{\Delta OAB}(y+2z-2)\frac{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}{\phi_z}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\phi = c$ is the equation of the given surface *S*.

Here $\phi = 2x + 3y + 6z$

$\therefore\ \phi_x = 2;\ \phi_y = 3;\ \phi_z = 6.$

$$\therefore \qquad \mathrm{I}=\iint_{\Delta OAB}(y+2z-2)\frac{\sqrt{4+9+36}}{6}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{7}{6}\iint_{\Delta OAB}(y+2z-2)\,\mathrm{d}x\,\mathrm{d}y \qquad (1)$$

Now the integrand is expressed as a function of *x* and *y*, by using the value of *z* (as a function of *x* and *y*) got from the equation of *S*, i.e. from the equation $2x + 3y + 6z = 12$

Thus
$$z=\frac{1}{6}(12-2x-3y) \qquad (2)$$

Using (2) in (1), we get

$$\mathrm{I}=\frac{7}{6}\iint_{\Delta OAB}\frac{1}{3}(6-2x)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{7}{18}\int_0^4 \int_0^{6-\frac{3}{2}y} (6-2x)\,dx\,dy$$

$$= \frac{7}{18}\int_0^4 \left(9y - \frac{9}{4}y^2\right) dy = \frac{28}{3}$$

**Example 4.10** Evaluate $\iint_S z^3\, dS$, where $S$ is the positive octant of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ (Fig. 4.47)

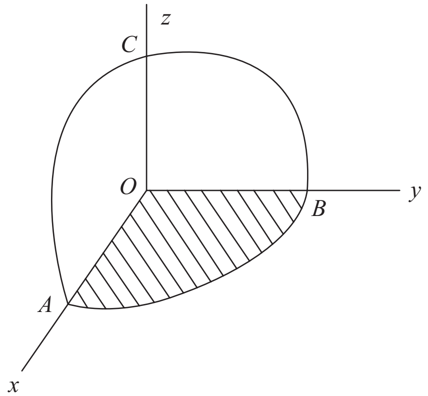


**Fig. 4.47**

The projection of the given surface of the sphere $x^2 + y^2 + z^2 = a^2$ (lying in the positive octant) in the $xoy$ - plane is the quadrant of the circular region $OAB$, shown in the two-dimensional Fig. 4.48.

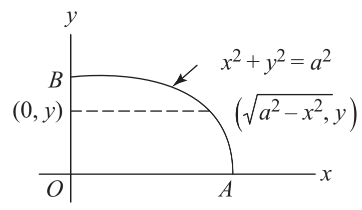


**Fig. 4.48**

Converting the given surface integral I as a double integral.

$$\mathrm{I} = \iint_{OAB} z^3 \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z}\,dx\,dy,$$

where $\phi \equiv x^2 + y^2 + z^2 = a^2$ is the equation of the given spherical surface.

$\phi_x = 2x;\ \phi_y = 2y;\ \phi_z = 2z.$

$\therefore$
$$\mathrm{I} = \iint_{OAB} z^3 \frac{\sqrt{4(x^2 + y^2 + z^2)}}{2z}\,dx\,dy$$

$$= \iint_{OAB} z^2 \sqrt{x^2 + y^2 + z^2}\,dx\,dy$$

$$= a \iint_{OAB} (a^2 - x^2 - y^2)\,dx\,dy \qquad [\because (x, y, z) \text{ lies on } x^2 + y^2 + z^2 = a^2]$$

$$= a \int_0^a \int_0^{\sqrt{a^2-y^2}} (a^2 - y^2 - x^2)\,dx\,dy$$

$$= a \int_0^a \left[(a^2 - y^2)x - \frac{x^3}{3}\right]_{x=0}^{x=\sqrt{a^2-y^2}} dy$$

$$= \frac{2}{3} a \int_0^a (a^2 - y^2)^{\frac{3}{2}}\,dy$$

$$= \frac{2}{3} a^5 \int_0^{\pi/2} \cos^4\theta\,d\theta, \text{ on putting } x = a\sin\theta.$$

$$= \frac{2}{3} a^5 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{8} a^5 \cdot$$

**Example 4.11** Evaluate $\iint_S y(z + x)\,dS$, where $S$ is the curved surface of the cylinder $x^2 + y^2 = 16$, that lies in the positive octant and that is included between the planes $z = 0$ and $z = 5$ (Fig. 4.49).

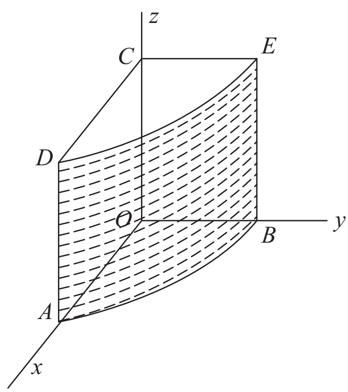


**Fig. 4.49**

We note that the projection of $S$ on the $xoy$-plane is not a plane (region) surface, but only the arc $AB$ of the circle whose centre is $O$ and radius equal to 4.

For converting the given surface integral into a double integral, the projection of $S$ must be a plane region. Hence we project $S$ on the $zox$-plane (or $yoz$-plane). The projection of $S$ in this case is the rectangular region $OCDA$, which is shown in Fig. 4.50.

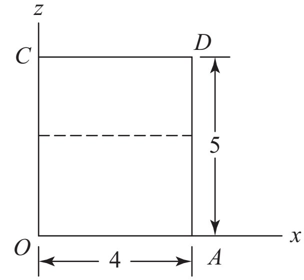


**Fig. 4.50**

Converting the given surface integral I as a double integral,

$$I = \iint_{OADC} y(z+x) \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi y} \, dz\, dx,$$

where $\phi \equiv x^2 + y^2 = 16$ is the equation of the given cylindrical surface. $\phi_x = 2x$; $\phi_y = 2y$; $\phi_z = 0$.

$$\therefore \quad I = \iint_{OADC} y(z+x) \frac{\sqrt{4(x^2+y^2)}}{2y} \, dz\, dx$$

$$= 4 \iint_{OADC} (z+x)\, dz\, dx \qquad [\because (x, y, z) \text{ lies on } x^2 + y^2 = 16]$$

$$= 4 \int_0^5 \int_0^4 (z+x)\, dx\, dz\, .$$

$$= 4 \int_0^5 (4z+8)\, dz$$

$$= 8(z^2 + 4z)_0^5$$

$$= 360.$$

**Example 4.12** Evaluate $\iiint_V xyz\, dx\, dy\, dz$, where $V$ is the region of space inside the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Vide worked Example 4.11 in the section on 'Double and triple integrals' for fixing the limits of the volume integral.

$$I = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} xyz\, dz\, dy\, dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} xy \left(\frac{z^2}{2}\right)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\, dx$$

$$= \frac{c^2}{2} \int_0^a \int_0^{bt} xy \left(t - \frac{y}{b}\right)^2 dy\, dx\, , \text{ where } t = 1 - \frac{x}{a}$$

$$= \frac{c^2}{2} \int_0^a x \left(t^2 \frac{y^2}{2} - \frac{2t}{b} \frac{y^3}{3} + \frac{1}{b^2} \frac{y^4}{4}\right)_0^{bt} dx$$

$$= \frac{c^2}{2} \int_0^a \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) b^2\, xt^4\, dx$$

$$= \frac{b^2 c^2}{24} \int_0^a x\left(1-\frac{x}{a}\right)^4 \mathrm{d}x$$

$$= \frac{b^2 c^2}{24} \int_0^a a\left[1-\left(1-\frac{x}{a}\right)\right]\cdot\left(1-\frac{x}{a}\right)^4 \mathrm{d}x$$

$$= \frac{ab^2 c^2}{24} \left[\frac{\left(1-\frac{x}{a}\right)^5}{-\frac{5}{a}} + \frac{\left(1-\frac{x}{a}\right)^6}{\frac{6}{a}}\right]_0^a$$

$$= \frac{a^2 b^2 c^2}{24}\left(\frac{1}{5}-\frac{1}{6}\right)$$

$$= \frac{1}{720} a^2 b^2 c^2 \cdot$$

**Example 4.13** Express the volume of the sphere $x^2 + y^2 + z^2 = a^2$ as a volume integral and hence evaluate it. [Refer to Fig. 4.51]

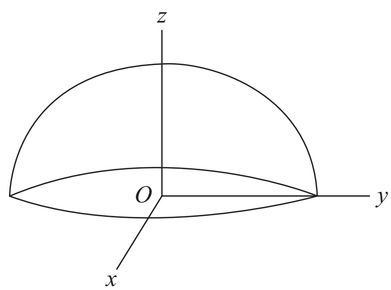


**Fig. 4.51**

Required volume = 2 × volume of the hemisphere above the *xoy*-plane. Vide worked Example 4.12 in the section on 'Double and Triple Integrals'.

$$\text{Required volume } = 2\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

$$= 2\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\, \mathrm{d}y\, \mathrm{d}x$$

Taking $a^2 - x^2 = b^2$, when integration with respect to $y$ is performed,

$$V = 2\int_{-a}^{a}\int_{-b}^{b} \sqrt{b^2-y^2}\, \mathrm{d}y\, \mathrm{d}x$$

$$= 4 \int_{-a}^{a} \int_{0}^{b} \sqrt{b^2 - y^2}\, dy\, dx \qquad [\because \sqrt{b^2 - y^2} \text{ is an even function of } y]$$

$$= 4 \int_{-a}^{a} \left( \frac{y}{2} \sqrt{b^2 - y^2} + \frac{b^2}{2} \sin^{-1} \frac{y}{b} \right)_0^b dx$$

$$= \pi \int_{-a}^{a} (a^2 - x^2)\, dx$$

$$= 2\pi \left( a^2 x - \frac{x^3}{3} \right)_0^a$$

$$= \frac{4}{3} \pi a^3$$

**Example 4.14** Evaluate $\iiint_V (x + y + z)\, dx\, dy\, dz$, where $V$ is the region of space inside the cylinder $x^2 + y^2 = a^2$ that is bounded by the planes $z = 0$ and $z = h$ [Refer to Fig. 4.52].

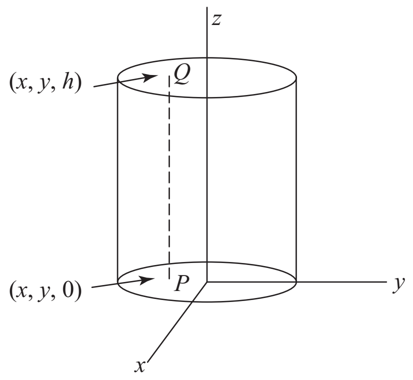


**Fig. 4.52**

***Note*** ☑ The equation $x^2 + y^2 = a^2$ (in three dimensions) represents the right circular cylinder whose axis is the $z$-axis and base circle is the one with centre at the origin and radius equal to $a$.

$$I = \int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \int_{0}^{h} (x + y + z)\, dz\, dy\, dx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \left[ (x + y)h + \frac{h^2}{2} \right] dy\, dx$$

$$= 2h \cdot \int_{-a}^{a} \int_{0}^{\sqrt{a^2 - x^2}} \left( x + \frac{h}{2} \right) dy\, dx$$

[by using properties of odd and even functions]

$$= 2h \int_{-a}^{a} \left( x + \frac{h}{2} \right) \sqrt{a^2 - x^2}\, dx$$

$$= 2h^2 \int_{0}^{a} \sqrt{a^2 - x^2}\, dx \qquad [\because x\sqrt{a^2 - x^2} \text{ is odd and } \sqrt{a^2 - x^2} \text{ is even}]$$

$$= 2h^2 \left( \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right)_0^a$$

$$= \frac{\pi}{2} a^2 h^2$$

## EXERCISE 4(c)

***Part A***

(Short Answer Questions)

1. Define a line integral.
2. What is the difference between a definite integral and a line integral?
3. Define a surface integral.
4. What is the difference between a double integral and a surface integral?
5. Define a volume integral.
6. What is the difference between a triple integral and a volume integral?
7. Write down the formula that converts a surface integral into a double integral.
8. Evaluate $\int_C (x^2\, dy + y^2\, dx)$ where $C$ is the path $y = x$ from (0, 0) to (1, 1).
9. Evaluate $\int_C \sqrt{(x^2 + y^2)}\, ds$, where $C$ is the path $y = -x$ from (0, 0) to (−1, 1).
10. Evaluate $\int_C (x\, dy - y\, dx)$, where $C$ is the circle $x^2 + y^2 = 1$ from (1, 0) to (0, 1) in the counterclockwise sense.
11. Evaluate $\iint_S dS$, where $S$ is the surface of the parallelopiped formed by $x = \pm 1, y = \pm 2, z = \pm 3$.

    [*Hint:* $\iint_S dS$ gives the area of the surface $S$]
12. Evaluate $\iint_S dS$, where $S$ is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
13. Evaluate $\iint_S dS$, where $S$ is the curved surface of the right circular cylinder $x^2 + y^2 = a^2$, included between $z = 0$ and $z = h$.
14. Evaluate $\iiint_V dV$, where $V$ is the region of space bounded by the planes $x = 0, x = a, y = 0, y = 2b, z = 0$ and $z = 3c$.

[*Hint:* $\iiint_V dV$ gives the volume of the region $V$]

15. Evaluate $\iiint_V dV$, where $V$ is the region of space bounded by $x^2 + y^2 + z^2 = 1$.
16. Evaluate $\iiint_V dV$, where $V$ is the region of space bounded by $x^2 + y^2 = a^2$, $z = -h, z = h$.

***Part B***

17. Evaluate $\int_{(0,0)}^{(1,3)} [x^2 y\, dx + (x^2 - y^2)\, dy]$ along the (i) curve $y = 3x^2$, (ii) line $y = 3x$.
18. Evaluate $\int_C [(x^2 - y^2 + x)\, dx - (2x\, y + y)\, dy]$ from (0, 0) to (1, 1), when $C$ is (i) $y^2 = x$, (ii) $y = x$.
19. Evaluate $\int_{(-a,0)}^{(a,0)} (y^2\, dx - x^2 dy)$ along the upper half of the circle $x^2 + y^2 = a^2$.
20. Evaluate $\int_C (x\, dy - y\, dx)$, where $C$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and described in the anticlockwise sense.
21. Evaluate $\int_C [(x^2 - y^2)\, dx + 2xy\, dy]$, where $C$ is the boundary of the rectangle formed by the lines $x = 0, x = 2, y = 0, y = 1$ and described in the anticlockwise sense.
22. Evaluate $\int_C [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$, where $C$ is the boundary of the region enclosed by $y^2 = x$ and $x^2 = y$ and described in the anticlockwise sense.
23. Evaluate $\int_C (x - y^2)\, ds$, where $C$ is the arc of the circle $x = a \cos \theta$, $y = a \sin \theta$ ; $0 \le \theta \le \frac{\pi}{2}$.
24. Evaluate $\int_C x\, ds$, where $C$ is the arc of the parabola $x^2 = 2y$ from (0, 0 ) to $\left(1, \frac{1}{2}\right)$.
25. Evaluate $\int_C [xy\, dx + (x^2 + z)\, dy + (y^2 + x)\, dz]$ from (0, 0, 0) to (1, 1, 1) along the curve $C$ given by $y = x^2$ and $z = x^3$.
26. Evaluate $\int_C [(3x^2 + 6y)\, dx - 14yz\, dy + 20xz^2\, dz]$, where $C$ is the segment of the straight line joining (0, 0, 0) and (1, 1, 1).

27. Evaluate $\int_C [3x^2\, dx + (2xy - y)\, dy - z\, dz]$ from $t = 0$ to $t = 1$ along the curve $C$ given by $x = 2t^2$, $y = t$, $z = 4t^3$.
28. Evaluate $\int xy\, ds$ along the arc of the curve given by the equations $x = a \tan \theta$, $y = a \cot \theta$, $z = \sqrt{2}\, a \log \tan \theta$ from the point $\theta = \frac{\pi}{4}$ to the point $\theta = \frac{\pi}{3}$.
29. Evaluate $\int_C (xy + z^2) ds$, where $C$ is the arc of the helix $x = \cos t$, $y = \sin t$, $z = t$ from $(1, 0, 0)$ to $(-1, 0, \pi)$.
30. Find the area of that part of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ that lies in the positive octant. $\left[\textit{Hint}: \text{Area of the surface} = \iint_S ds\right]$
31. Evaluate $\iint_S z\, dS$, where $S$ is the positive octant of the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
32. Evaluate $\iint xy\, dS$, where $S$ is the curved surface of the cylinder $x^2 + y^2 = a^2$, $0 \le z \le k$, included in the positive octant.
33. Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
34. Evaluate $\iiint_V z\, dx\, dy\, dz$, where $V$ is the region of space bounded by the sphere $x^2 + y^2 + z^2 = a^2$ above the $xoy$-plane.
35. Evaluate $\iiint_V (x^2 + y^2)\, dx\, dy\, dz$, where $V$ is the region of space inside the cylinder $x^2 + y^2 = a^2$ that is bounded by the planes $z = 0$ and $z = h$.

## 4.9 GAMMA AND BETA FUNCTIONS

***Definitions*** The definite integral $\int_0^\infty e^{-x}\, x^{n-1}\, dx$ exists only when $n > 0$ and when it exists, it is a function of $n$ and called *Gamma function* and denoted by $\Gamma(n)$ [read as "Gamma $n$"].

Thus
$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$$

The definite integral $\int_0^1 x^{m-1}\, (1 - x)^{n-1}\, dx$ exists only when $m > 0$ and $n > 0$ and when it exists, it is a function of $m$ and $n$ and called *Beta function* and denoted by $\beta$ $(m, n)$ [read as "Beta $m$, $n$"].

Thus
$$\beta(m, n) = \int_0^1 x^{m-1}\, (1 - x)^{n-1} dx:$$

*Note* ☑ $$\Gamma(1) = \int_0^\infty e^{-x}\, dx = (-e^{-x})_0^\infty = 1.$$

$$\beta(1,1) = \int_0^1 dx = 1.$$

## 4.9.1 Recurrence Formula for Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$$

$$= -(x^{n-1}\, e^{-x})_0^\infty + \int_0^\infty (n-1)\, e^{-x}\, x^{n-2}\, dx \quad \text{[integrating by parts]}$$

$$= (n-1)\,\Gamma(n-1), \text{ since } \lim_{n\to\infty}\left(\frac{x^{n-1}}{e^x}\right) = 0$$

This recurrence formula $\Gamma(n) = (n-1)\,\Gamma(n-1)$ is valid only when $n > 1$, as $\Gamma(n-1)$ exists only when $n > 1$.

**Cor.**

$\Gamma(n+1) = n\,!$, where $n$ is a positive integer.

$$\begin{aligned}\Gamma(n+1) &= n\,\Gamma(n)\\ &= n\,(n-1)\,\Gamma(n-1)\\ &= n\,(n-1)\,(n-2)\,\Gamma(n-2)\\ &= \ldots\ldots\ldots\ldots\ldots\ldots\\ &= n\,(n-1)\,(n-2)\ldots 3.2.1\,\Gamma(1)\\ &= n\,!\ (\because \Gamma(1) = 1)\end{aligned}$$

*Note* ☑
1. $\Gamma(n)$ does not exist (i.e. $= \infty$), when $n$ is 0 or a negative integer.
2. When $n$ is a negative fraction, $\Gamma(n)$ is defined by using the recurrence formula. i.e. when $n < 0$, but not an integer,

$$\Gamma(n) = \frac{1}{n}\,\Gamma(n+1)$$

For example, $$\begin{aligned}\Gamma(-3.5) &= \frac{1}{(-3.5)}\,\Gamma(-2.5)\\ &= \frac{1}{(-3.5)}\cdot\frac{1}{(-2.5)}\,\Gamma(-1.5)\\ &= \frac{1}{(3.5)\,(2.5)\,(-1.5)}\,\Gamma(-.5)\\ &= \frac{\Gamma\,(0.5)}{(3.5)\,(2.5)\,(1.5)\,(0.5)}\end{aligned}$$

The value of $\Gamma(0.5)$ can be obtained from the table of Gamma functions, though its value can be found out mathematically as given below.

**Value of** $\Gamma\left(\frac{1}{2}\right)$

By definition, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt$

$$= \int_0^\infty e^{-x^2} \cdot \frac{1}{x} \cdot 2x\, dx \qquad \text{(on putting } t = x^2\text{)}$$

$$= 2\int_0^\infty e^{-x^2}\, dx$$

Now $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 2\int_0^\infty e^{-x^2}\, dx \cdot 2\int_0^\infty e^{-y^2} dy$ [∵ the variable in a definite integral is only a dummy variable]

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy \qquad (1)$$

[∵ the product of two definite integrals can be expressed as a double integral, when the limits are constants].

Now the region of the double integral in (1) is given by $0 \le x < \infty$ and $0 \le y < \infty$, i.e. the entire first quadrant of the $xy$-plane.

Let us change over to polar co-ordinates through the transformations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Then $dx\, dy = |J|\, d\,r\, d\,\theta = r\, d\,r\, d\,\theta$

The region of the double integration is now given by $0 \le r < \infty$ and $0 \le \theta \le \frac{\pi}{2}$. Then, from (1), we have

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} d\theta \left(-\frac{1}{2}e^{-r^2}\right)_0^\infty$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= \pi$$

∴ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

## 4.9.2 Symmetry of Beta Function

$$\beta(m, n) = \beta(n, m)$$

By definition, $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \qquad (1)$

Using the property $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ in (1),

$$\beta(m, n) = \int_0^1 (1-x)^{m-1} \{1-(1-x)^{n-1} \, dx$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} \, dx$$

$$= \beta(n, m).$$

### 4.9.3 Trigonometric Form of Beta Function

By definition, $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$

Put $x = \sin^2 \theta$ $\therefore$ $dx = 2 \sin\theta \cos\theta \, d\theta$

The limits for $\theta$ are 0 and $\dfrac{\pi}{2}$.

$$\therefore \quad \beta(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

**Note** ☑ $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m, n)$

The first argument of the Beta function is obtained by adding 1 to the exponent of $\sin\theta$ and dividing the sum by 2. The second argument is obtained by adding 1 to the exponent of $\cos\theta$ and dividing the sum by 2.

Thus $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

### 4.9.4 Relation Between Gamma and Beta Functions

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Consider $$\Gamma(m)\,\Gamma(n) = \int_0^\infty e^{-t} t^{m-1} \, dt \cdot \int_0^\infty e^{-s} \cdot s^{n-1} \, ds$$

In the first integral, put $t = x^2$ and in the second, put $s = y^2$.

$$\therefore \quad \Gamma(m) \cdot \Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2m-1} \, dx \cdot 2\int_0^\infty e^{-y^2} y^{2n-1} \, dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\ x^{2m-1}\cdot y^{2n-1}\ \mathrm{d}x\ \mathrm{d}y$$

$$= 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\ (r\cos\theta)^{2m-1}\cdot(r\sin\theta)^{2n-1}\ r\ \mathrm{d}r\ \mathrm{d}\theta$$

[changing over to polar co-ordinates]

$$= 4\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\ \mathrm{d}\theta\cdot\int_0^{\infty} e^{-r^2}\ r^{2m+2n-2}\cdot r\ \mathrm{d}r$$

$$= \beta(m, n)\int_0^{\infty} e^{-r^2}\ r^{2(m+n-1)}\cdot 2r\ \mathrm{d}r$$

$$= \beta(m, n)\cdot\int_0^{\infty} e^{-u}\cdot u^{m+n-1}\ \mathrm{d}u \qquad \text{[putting } r^2 = u\text{]}$$

$$= \beta(m, n)\cdot\Gamma(m, n)$$

$$\therefore \qquad \beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

**Cor.**

Putting $m = n = \dfrac{1}{2}$ in the above relation, $\beta\left(\dfrac{1}{2}, \dfrac{1}{2}\right) = \dfrac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2}{\Gamma(1)}$

$$\therefore \qquad \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= 2\int_0^{\pi/2} \sin^0\theta\cdot\cos^0\theta\,\mathrm{d}\theta$$

$$= \pi$$

$$\therefore \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

**WORKED EXAMPLE 4(d)**

**Example 4.1** Prove that $\displaystyle\int_0^{\infty} e^{-ax}\ x^{n-1}\ \mathrm{d}x = \frac{\Gamma(n)}{a^n}$, where $a$ and $n$ are positive.

Hence find the value of $\displaystyle\int_0^1 x^{q-1}\left[\log\left(\frac{1}{x}\right)\right]^{p-1}\ \mathrm{d}x$.

In $\displaystyle\int_0^{\infty} e^{-ax}\ x^{n-1}\ \mathrm{d}x$, put $ax = t$, so that $\mathrm{d}x = \dfrac{\mathrm{d}t}{a}$

$$\therefore \qquad \int_0^\infty e^{-ax}\, x^{n-1}\, dx = \int_0^\infty e^{-t}\left(\frac{t}{a}\right)^{n-1}\cdot\frac{dt}{a}$$

$$= \frac{1}{a^n}\int_0^\infty e^{-t}\, t^{n-1}\, dt$$

$$= \frac{1}{a^n}\,\Gamma(n) \qquad (1)$$

In $I = \int_0^1 x^{q-1} \log\left(\frac{1}{x}\right)^{p-1} dx$,

put $\quad \frac{1}{x} = e^y$

i.e. $\quad x = e^{-y}$

Then $\quad dx = -\, e^{-y}\, dy$

Also the limits for $y$ are $\infty$ and 0.

$$\therefore \qquad I = \int_\infty^0 e^{-(q-1)y}\cdot y^{p-1}\cdot(-e^{-y})\, dy$$

$$= \int_0^\infty e^{-qy}\, y^{p-1}\, dy$$

$$= \frac{1}{q^p}\cdot\Gamma(p) \text{ [by (1)]}.$$

**Example 4.2** Prove that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$.

Hence deduce that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx$.

By definition, $\beta(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt \qquad (1)$

In (1), put $t = \frac{x}{1+x}$. Then $dt = \frac{1}{(1+x)^2}\, dx \qquad \left(\because x = \frac{t}{1-t}\right)$

When $t = 0$, $x = 0$; when $t = 1$, $x = \infty$

Then (1) becomes,

$$\beta(m,n) = \int_0^\infty \left(\frac{x}{1+x}\right)^{m-1}\cdot\left(\frac{1}{1+x}\right)^{n-1}\cdot\frac{1}{(1+x)^2}\, dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \qquad (2)$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \qquad (3)$$

In $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$, put $x = \frac{1}{y}$. Then $dx = -\frac{1}{y^2}\,dy$

When $x = 1$, $y = 1$; when $x = \infty$, $y = 0$

$$\therefore \qquad \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{\frac{1}{y^{(m-1)}}}{\left(1+\frac{1}{y}\right)^{m+n}} \cdot \left(-\frac{1}{y^2}\right) dy$$

$$= \int_0^1 \frac{y^{m+n}}{(1+y)^{m+n} \cdot y^{m+1}}\,dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

[changing the dummy variable] (4)

Using (4) in (3), we have

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$

**Example 4.3** Evaluate $\int_0^1 x^m (1-x^n)^p\,dx$ in terms of Gamma functions and hence find $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$.

In $$I = \int_0^1 x^m (1-x^n)^p\,dx,$$

put $\qquad x^n = t;$

then $\qquad nx^{n-1}\,dx = dt$

$$\therefore \qquad dx = \frac{1}{n} \cdot \frac{dt}{t^{1-\frac{1}{n}}}$$

When $x = 0$, $t = 0$; when $x = 1$, $t = 1$.

$$I = \int_0^1 t^{\frac{m}{n}} (1-t)^p \cdot \frac{1}{n} \cdot t^{\frac{1}{n}-1}\,dt$$

$$= \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1} \cdot (1-t)^p\,dt$$

$$= \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$$

$$= \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \cdot \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p + 1\right)} \qquad (1)$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \int_0^1 x^0 (1-x^n)^{-\frac{1}{2}}\, dx$$

Here $m = 0$, $n = n$, $p = -\frac{1}{2}$.

Using (1); we have

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$$

**Example 4.4** Prove that $\beta(n,n) = \dfrac{\sqrt{\pi}\, \Gamma(n)}{2^{2n-1} \Gamma\left(n + \frac{1}{2}\right)}$.

(or) $$\beta(n,n) = \frac{1}{2^{2n-1}} \cdot \beta\left(n, \frac{1}{2}\right)$$

$$\beta(n,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cdot \cos^{2n-1}\theta\, d\theta \qquad \text{[using trigonometric form]}$$

$$= 2\int_0^{\pi/2} (\sin\theta \cos\theta)^{2n-1} d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2n-1} d\theta$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} \sin^{2n-1} 2\theta\, d\theta$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi} \sin^{2n-1}\phi \frac{d\phi}{2}, \text{ putting } 2\theta = \phi$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} \sin^{2n-1}\phi \, d\phi \quad \left[ \because \int_0^{\pi} f(\sin\phi)\, d\phi = 2\int_0^{\pi/2} f(\sin\phi)\, d\phi \right]$$

$$= \frac{1}{2^{2n-2}} \cdot \frac{1}{2} \beta\left(n, \frac{1}{2}\right)$$

$$= \frac{1}{2^{2n-1}} \beta\left(n, \frac{1}{2}\right)$$

$$= \frac{1}{2^{2n-1}} \cdot \frac{\Gamma(n)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}\cdot\Gamma(n)}{2^{2n-1}\cdot\Gamma\left(n+\frac{1}{2}\right)}$$

**Example 4.5** Show that $\displaystyle\int_0^{\infty} x^n e^{-a^2x^2}\, dx = \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right)$.

Deduce that $\displaystyle\int_0^{\infty} e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{2a}$. Hence show that

$$\int_0^{\infty} \cos(x^2)\, dx = \int_0^{\infty} \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

In $\displaystyle I = \int_0^{\infty} x^n e^{-a^2x^2}\, dx$, put $ax = \sqrt{t}$; then $dx = \dfrac{dt}{2a\sqrt{t}}$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$.

$$\therefore \qquad I = \int_0^{\infty} \left(\frac{\sqrt{t}}{a}\right)^n e^{-t} \frac{dt}{2a\sqrt{t}}$$

$$= \frac{1}{2a^{n+1}} \int_0^{\infty} t^{\frac{n-1}{2}} \cdot e^{-t} dt$$

$$= \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right) \tag{1}$$

In (1), put $n = 0$.

Then
$$\int_0^{\infty} e^{-a^2x^2}\, dx = \frac{\Gamma\left(\frac{1}{2}\right)}{2a} = \frac{\sqrt{\pi}}{2a} \tag{2}$$

In (2), put $a = \dfrac{1-i}{\sqrt{2}}$; then $a^2 = -i$

$$\therefore \quad \int_0^\infty e^{ix^2}\,dx = \frac{\sqrt{\pi}}{\sqrt{2}(1-i)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i)$$

Equating the real parts on both sides,

$$\int_0^\infty \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Equating the imaginary parts on both sides,

$$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

**Example 4.6** Evaluate

(i) $\int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx$ and

(ii) $\int_{-a}^a (a+x)^{m-1}\cdot(a-x)^{n-1}\,dx$ in terms of Beta function.

(i) In $I_1 = \int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx$,

put $x - a = y$; then $dx = dy$

When $x = a$, $y = 0$; when $x = b$, $y = b - a$

$$\therefore \quad I_1 = \int_0^{b-a} y^{m-1}\{(b-a)-y\}^{n-1}\,dy$$

$$= (b-a)^{n-1}\int_0^{b-a} y^{m-1}\left(1-\frac{y}{b-a}\right)^{n-1} dy \qquad (1)$$

In (1), put $\frac{y}{b-a} = t$ ; then $dy = (b - a)\,dt$

When $y = 0$, $t = 0$; when $y = b - a$, $t = 1$.

$$\therefore \quad I_1 = (b-a)^{m+n-1}\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$= (b-a)^{m+n-1}\,\beta(m, n)$$

(ii) In $I_2 = \int_{-a}^a (a+x)^{m-1}\,(a-x)^{n-1}\,dx$,

put $a + x = y$; then $dx = dy$

When $x = -a$, $y = 0$; when $x = a$, $y = 2a$.

$$\therefore \quad I_2 = \int_0^{2a} y^{m-1}(2a-y)^{n-1}\,dy$$

$$= (2a)^{n-1} \int_0^{2a} y^{m-1} \left(1 - \frac{y}{2a}\right)^{n-1} \mathrm{d}y \tag{2}$$

In (2), put $\frac{y}{2a} = t$; then d$y$ = $2a$ d$t$.

When $y = 0$, $t = 0$; when $y = 2a$, $t = 1$.

$$\therefore \quad I_2 = (2a)^{m+n-1} \cdot \int_0^1 t^{m-1}(1-t)^{n-1} \, \mathrm{d}t \cdot$$

$$= (2a)^{m+n-1} \beta\,(m,n)$$

**Example 4.7** Prove that $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} \mathrm{d}x \times \int_0^{\infty} x^2 e^{-x^4} \mathrm{d}x = \frac{\pi}{4\sqrt{2}}$.

In $I_1 = \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} \, \mathrm{d}x$, put $x^2 = t$; then $\mathrm{d}x = \frac{\mathrm{d}t}{2x} = \frac{\mathrm{d}t}{2\sqrt{t}}$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$

$$\therefore \quad I_1 = \int_0^{\infty} \frac{e^{-t}}{t^{1/4}} \cdot \frac{\mathrm{d}t}{2\sqrt{t}} = \frac{1}{2}\int_0^{\infty} e^{-t} \cdot t^{-\frac{3}{4}} \, \mathrm{d}t$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

In $I_2 = \int_0^{\infty} x^2 \, e^{-x^4} \mathrm{d}x$, put $x^4 = s$; then $\mathrm{d}x = \frac{\mathrm{d}s}{4x^3} = \frac{\mathrm{d}s}{4s^{3/4}}$

When $x = 0$, $s = 0$; when $x = \infty$, $s = \infty$.

$$\therefore \quad I_2 = \int_0^{\infty} \sqrt{s} \, e^{-s} \cdot \frac{\mathrm{d}s}{4s^{3/4}} = \frac{1}{4}\int_0^{\infty} s^{-\frac{1}{4}} \, e^{-s} \mathrm{d}s$$

$$= \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\therefore \quad \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} \mathrm{d}x \times \int_0^{\infty} x^2 e^{-x^4} \, \mathrm{d}x = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \tag{1}$$

From Example 4.4;

$$\beta\,(n,\, n) = \frac{1}{2^{\,2n-1}} \cdot \beta\left(n,\, \frac{1}{2}\right)$$

i.e.

$$\frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \frac{1}{2^{\,2n-1}} \cdot \frac{\Gamma(n) \cdot \sqrt{\pi}}{\Gamma\left(n + \frac{1}{2}\right)}$$

$$\therefore \quad \Gamma(n) \cdot \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}\;\Gamma(2n)}{2^{2n-1}}$$

Putting $n=\frac{1}{4}$, we get

$$\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)=\frac{\sqrt{\pi}\cdot\Gamma\left(\frac{1}{2}\right)}{2^{-\frac{1}{2}}}$$

$$=\pi\sqrt{2} \tag{2}$$

Using (2) in (1);

$$\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx\times\int_0^\infty x^2e^{-x^4}\,dx=\frac{\pi\sqrt{2}}{8}=\frac{\pi}{4\sqrt{2}}.$$

**Example 4.8** Evaluate $\int_0^\infty \frac{x^{m-1}}{(1+x^n)^p}\,dx$ and deduce that $\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx$.

$=\frac{\pi}{n\sin\left(\frac{m\pi}{n}\right)}$. Hence show that $\int_0^\infty \frac{dx}{1+x^4}=\frac{\pi}{2\sqrt{2}}$.

In $I=\int_0^\infty \frac{x^{m-1}}{(1+x^n)^p}\,dx$, put $t=\frac{1}{1+x^n}$

Then $x^n=\frac{1-t}{t}$ $\quad\therefore\quad nx^{n-1}\,dx=-\frac{1}{t^2}\,dt$

When $x=0$, $t=1$; when $x=\infty$, $t=0$

$$\therefore\qquad I=\int_0^1\frac{t^{-\left(\frac{m-1}{n}\right)}\cdot(1-t)^{\frac{m-1}{n}}}{t^{-p}}\cdot\frac{dt}{nt^2\cdot t^{-\frac{(n-1)}{n}}(1-t)^{\frac{n-1}{n}}}$$

$$=\frac{1}{n}\int_0^1 t^{p-\frac{m}{n}-1}\cdot(1-t)^{\frac{m}{n}-1}\,dt$$

$$=\frac{1}{n}\beta\left(p-\frac{m}{n},\frac{m}{n}\right)$$

$$=\frac{1}{n}\frac{\Gamma\left(p-\frac{m}{n}\right)\cdot\Gamma\left(\frac{m}{n}\right)}{\Gamma(p)} \tag{1}$$

Putting $p=1$ in (1), we get

$$\int_0^\infty \frac{x^{m-1}\,dx}{1+x^n}=\frac{1}{n}\Gamma\left(1-\frac{m}{n}\right)\Gamma\left(\frac{m}{n}\right)$$

$$=\frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi m}{n}\right)\qquad\left[Hint: \text{Use}\,\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi}{\sin\alpha\pi}\right] \tag{2}$$

Taking $m = 1$ and $n = 4$ in (2), we get

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{4}\cdot \operatorname{cosec}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2\sqrt{2}}$$

**Example 4.9** Find the value of $\iint x^{m-1} y^{n-1}\, dx\, dy$, over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, in terms of Gamma functions.

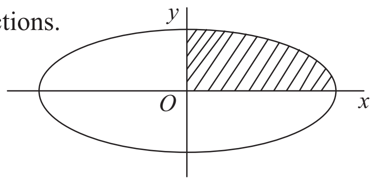


Fig. 4.53

Put $\frac{x}{a} = \sqrt{X}$ and $\frac{y}{b} = \sqrt{Y}$

Then $dx = \frac{a}{2\sqrt{X}}\, dX$ and $dy = \frac{b}{2\sqrt{Y}}\, dY$.

The region of double integration in the $xy$-plane is given by $x \geq 0$, $y \geq 0$ and

$\frac{x^2}{a^2}+\frac{y^2}{b^2} \leq 1$, shown in Fig. 4.53.

∴ The region of integration in the $XY$-plane is given by $X \geq 0$, $Y \geq 0$ and $X + Y \leq 1$, shown in Fig. 4.54.

The given integral

$$I = \iint_{\Delta OAB} (a\sqrt{X})^{m-1}\cdot(b\sqrt{Y})^{n-1}\, \frac{ab}{4\sqrt{X}\sqrt{Y}}\, dX\, dY$$

$$= \frac{a^m b^n}{4} \iint_{\Delta OAB} X^{\frac{m}{2}-1}\, Y^{\frac{n}{2}-1}\, dX\, dY$$

$$= \frac{a^m b^n}{4} \int_0^1 \int_0^{1-Y} X^{\frac{m}{2}-1}\, Y^{\frac{n}{2}-1}\, dX\, dY$$

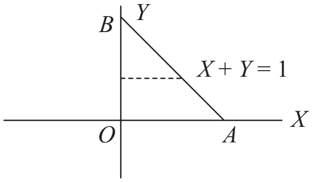


Fig. 4.54

$$I = \frac{a^m b^n}{4} \int_0^1 Y^{\frac{n}{2}-1} \cdot \frac{2}{m}\left(X^{m/2}\right)_0^{1-Y} dY$$

$$= \frac{a^m b^n}{2m} \int_0^1 Y^{\frac{n}{2}-1}\cdot(1-Y)^{m/2}\, dY$$

$$= \frac{a^m b^n}{2m}\, \beta\cdot\left(\frac{n}{2}, \frac{m}{2}+1\right)$$

$$= \frac{a^m b^n}{2m}\cdot \frac{\Gamma\left(\frac{n}{2}\right)\cdot\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)}$$

$$= \frac{a^m b^n}{2m} \cdot \frac{\frac{m}{2}\Gamma\left(\frac{m}{2}\right)\cdot\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)}$$

$$= \frac{a^m b^n}{4} \cdot \frac{\Gamma\left(\frac{m}{2}\right)\cdot\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m+n}{2}+1\right)}$$

**Example 4.10** Find the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, using Gamma functions.

By symmetry of the astroid, required area $A =$

$4 \times$ area of $OACB = 4\iint\limits_{OACB} \mathrm{d}x\,\mathrm{d}y$

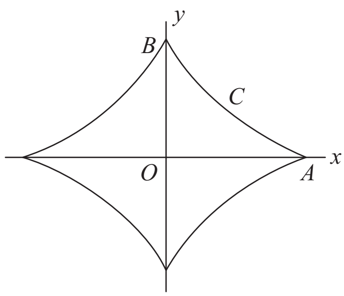

Fig. 4.55

Put $\left(\frac{x}{a}\right)^{2/3} = X$ and $\left(\frac{y}{a}\right)^{2/3} = Y$

i.e. $x = aX^{3/2}$ and $y = aY^{3/2}$

$\therefore$ $\mathrm{d}x = \frac{3}{2}aX^{1/2}$ and $\mathrm{d}y = \frac{3}{2}aY^{1/2}\,\mathrm{d}Y$

The region of integration in the $xy$-plane is given by $x \geq 0, y \geq 0$ and $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} \leq 1$ , as shown in Fig. 4.55.

$\therefore$ The region of integration in the $XY$-plane is given by $X \geq 0$, $Y \geq 0$ and $X + Y \leq 1$ as shown in Fig. 4.56.

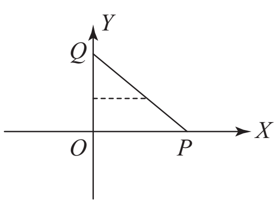

Fig. 4.56

$$\therefore \qquad A = 4\times\frac{9}{4}a^2 \iint\limits_{\Delta OPQ} X^{1/2}\,Y^{1/2}\,\mathrm{d}X\,\mathrm{d}Y$$

$$= 9a^2\cdot\int_0^1\int_0^{1-Y} X^{1/2}\,Y^{1/2}\,\mathrm{d}X\,\mathrm{d}Y$$

$$= 9a^2\int_0^1 Y^{1/2}\left(\frac{2}{3}X^{3/2}\right)_0^{1-Y}\mathrm{d}Y$$

$$= 6a^2\int_0^1 Y^{1/2}\,(1-Y)^{3/2}\,\mathrm{d}Y$$

$$= 6a^2\times\beta\left(\frac{3}{2},\frac{5}{2}\right)$$

$$= 6a^2\times\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(4)}$$

$$= a^2 \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \qquad [\because \Gamma(4) = 3!]$$

$$= \frac{3}{8}\pi a^2 \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

**Example 4.11** Evaluate $\iint [xy(1-x-y)]^{1/2}\,dx\,dy$, over the area enclosed by the lines $x = 0$, $y = 0$ and $x + y = 1$ in the positive quadrant.

Given Intergral $I = \int_0^1 \int_0^{1-x} x^{1/2}\, y^{1/2}\,(1-x-y)^{1/2}\,dy\,dx,$

$$= \int_0^1 x^{1/2}\,dx \int_0^a y^{1/2}\,(a-y)^{1/2}\,dy$$

where $a = 1 - x$. (1)

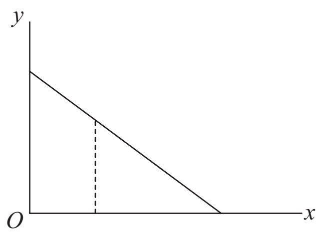


**Fig. 4.57**

Consider $\int_0^a y^{m-1}\,(a-y)^{n-1}\,dy$

$$= a^{n-1}\int_0^a y^{m-1}\left(1-\frac{y}{a}\right)^{n-1} dy$$

$$= a^{n-1}\int_0^1 a^{m-1}\, z^{m-1}\,(1-z)^{n-1}\,a\,dz \qquad \left(\text{putting } \frac{y}{a} = z\right)$$

$$= a^{m+n-1}\cdot\beta(m,n) \qquad (2)$$

***Note*** ☑ This result (2) will be of use in the following worked examples also.

Using (2) in (1) $\left[\text{note that } m = n = \frac{3}{2}\right]$,

$$I = \int_0^1 x^{1/2}\,(1-x)^2\,\beta\left(\frac{3}{2},\frac{3}{2}\right)dx$$

$$= \beta\left(\frac{3}{2},\frac{3}{2}\right) \times \beta\left(\frac{3}{2},3\right)$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} \times \frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma(3)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}\times\Gamma\left(\frac{3}{2}\right)}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{2\pi}{105}$$

**Example 4.12** Show that the volume of the region of space bounded by the coordinate planes and the surface $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$ is $\frac{abc}{90}$.

Required volume is given by

$$\text{Vol} = \iiint_V dz\, dy\, dx, \text{ where } V \text{ is the region of space given.}$$

Put $\sqrt{\frac{x}{a}} = X, \sqrt{\frac{y}{b}} = Y, \sqrt{\frac{z}{c}} = Z$

i.e. $x = aX^2, y = bY^2, z = cZ^2$

$\therefore$ $dx = 2aX\, dX, dy = 2bY\, dY, dz = 2cZ\, dZ$

$\therefore$ $\text{Vol} = \iiint_{V'} 8abc\, XYZ\, dZ\, dY\, dX$, where $V'$ is the region of space in $XYZ$-space defined by $X \geq 0,\ Y \geq 0,\ Z \geq 0,\ X + Y + Z \leq 1$ [Refer to Fig. 4.58]

$$\therefore \quad \text{Vol} = 8\, abc \int_0^1 dX \int_0^{1-X} dY \int_0^{1-X-Y} XYZ\, dZ$$

$$= 8\, abc \int_0^1 X\, dX \int_0^{1-X} Y\, dY \left(\frac{Z^2}{2}\right)_0^{1-X-Y}$$

$$= 4\, abc \int_0^1 X\, dX \int_0^{1-X} Y \cdot (1 - X - Y)^2\, dY$$

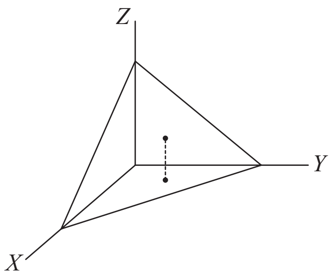


Fig. 4.58

$$= 4\, abc \int_0^1 X\, (1 - X)^4 \cdot \beta(2, 3)\, dX \quad \text{[by step (2) of Example (4.11)]}$$

$$= 4\, abc \cdot \frac{\Gamma(2)\, \Gamma(3)}{\Gamma(5)} \cdot \beta(2, 5)$$

$$= 4\, abc \times \frac{1 \times 2}{24} \times \frac{\Gamma(2)\, \Gamma(5)}{\Gamma(7)}$$

$$= \frac{abc}{3} \times \frac{1 \times 24}{720}$$

$$= \frac{abc}{90}$$

**Example 4.13** Evaluate $\iiint \frac{dx\, dy\, dz}{\sqrt{1 - x^2 - y^2 - z^2}}$, taken over the region of space in the positive octant bounded by the sphere $x^2 + y^2 + z^2 = 1$.

Put $x^2 = X, y^2 = Y, z^2 = Z$

$$\therefore \qquad dx = \frac{dX}{2\sqrt{X}}, \quad dy = \frac{dY}{2\sqrt{Y}}, \quad dz = \frac{dZ}{2\sqrt{Z}}$$

The region of integration in $xyz$-space is defined by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x^2 + y^2 + z^2 \leq 1$.

$\therefore$ The region of integration $V$ in the $XYZ$-space is defined by $X \geq 0$, $Y \geq 0$, $Z \geq 0$ and $X + Y + Z \leq 1$.

$\therefore$ Given integral
$$I = \iiint_V \frac{1}{8} X^{-\frac{1}{2}} Y^{-\frac{1}{2}} Z^{-\frac{1}{2}} (1 - X - Y - Z)^{-\frac{1}{2}} \, dX \cdot dY \cdot dZ$$

$$= \frac{1}{8} \int_0^1 X^{-\frac{1}{2}} \, dX \int_0^{1-X} Y^{-\frac{1}{2}} \, dY \int_0^{1-X-Y} Z^{-\frac{1}{2}} (1 - X - Y - Z)^{-\frac{1}{2}} \, dZ$$

$$= \frac{1}{8} \int_0^1 X^{-\frac{1}{2}} \, dX \int_0^{1-X} Y^{-\frac{1}{2}} \, dY \, (1 - X - Y)^{\frac{1}{2}+\frac{1}{2}-1} \cdot \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

[by step (2) of Example (4.11)]

$$= \frac{\pi}{8} \int_0^1 X^{-\frac{1}{2}} \, dX \left(2Y^{\frac{1}{2}}\right)_0^{1-X} \qquad \left\{\because \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi\right\}$$

$$= \frac{\pi}{4} \int_0^1 X^{-\frac{1}{2}} (1 - X)^{1/2} \, dX \cdot$$

$$= \frac{\pi}{4} \beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$= \frac{\pi}{4} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}$$

$$= \frac{\pi^2}{8}$$

**Example 4.14** Evaluate $\iiint_V \sqrt{a^2 b^2 c^2 - b^2 c^2 x^2 - c^2 a^2 y^2 - a^2 b^2 z^2} \, dx\, dy\, dz$,

where $V$ is the region defind by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$.

Put $\left(\frac{x}{a}\right)^2 = X, \left(\frac{y}{b}\right)^2 = Y, \left(\frac{z}{c}\right)^2 = Z$

i.e. $\quad x = a\sqrt{X}, y = b\sqrt{Y}, z = c\sqrt{Z}$

$$\therefore \qquad dx = \frac{a}{2\sqrt{X}} dX, dy = \frac{b}{2\sqrt{Y}} dY, dz = \frac{c}{2\sqrt{Z}} dZ$$

The region $V'$ of integration in the $XYZ$-space is defined by $X \geq 0$, $Y \geq 0$, $Z \geq 0$, $X + Y + Z \leq 1$.

$$\therefore \quad \text{Integral} = abc \iiint\limits_{V'} \sqrt{1 - X - Y - Z}\ \frac{abc\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{8\sqrt{X}\sqrt{Y}\sqrt{Z}}$$

$$= \frac{a^2b^2c^2}{8} \int_0^1 X^{-\frac{1}{2}}\,\mathrm{d}X \int_0^{1-X} Y^{-\frac{1}{2}}\,\mathrm{d}Y \int_0^{1-X-Y} Z^{-\frac{1}{2}}(1 - X - Y - Z)^{\frac{1}{2}}\,\mathrm{d}Z$$

$$= \frac{a^2b^2c^2}{8} \int_0^1 X^{-\frac{1}{2}}\,\mathrm{d}X \int_0^{1-X} Y^{-\frac{1}{2}}\,\mathrm{d}Y\,(1 - X - Y)\cdot\beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

[by step 2 of Example (4.11)]

$$= \frac{a^2b^2c^2}{8}\cdot\beta\left(\frac{1}{2}, \frac{3}{2}\right)\cdot\int_0^1 X^{-\frac{1}{2}}\,\mathrm{d}X\,(1 - X)^{\frac{3}{2}}\cdot\beta\left(\frac{1}{2}, 2\right)$$

[by step 2 of Example 4.11]

$$= \frac{a^2b^2c^2}{8}\cdot\beta\left(\frac{1}{2}, \frac{3}{2}\right)\cdot\beta\left(\frac{1}{2}, 2\right)\cdot\beta\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= \frac{a^2b^2c^2}{8}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)}$$

$$= \frac{\pi^2 a^2 b^2 c^2}{32}.$$

**Example 4.15** Find the value of $\iiint x^{l-1}\ y^{m-1}\ z^{n-1}\ (1-x-y-z)^{p-1}\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$, taken over the interior of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$, in terms of Gamma functions.

$$\text{Given integral} = \int_0^1 x^{l-1}\,\mathrm{d}x \int_0^{1-x} y^{m-1}\,\mathrm{d}y \int_0^{1-x-y} z^{n-1}\,(1-x-y-z)^{p-1}\,\mathrm{d}z,$$

$$= \int_0^1 x^{l-1}\,\mathrm{d}x \int_0^{1-x} y^{m-1}\,(1-x-y)^{n+p-1}\cdot\beta(n, p)\,\mathrm{d}y,$$

[by step (2) of Example (4.11)]

$$= \beta(n, p)\int_0^1 x^{l-1}\,(1-x)^{m+n+p-1}\,\beta(m, n+p)\,\mathrm{d}x,$$

[by step (2) of Example (4.11)]

$$= \beta(n, p)\cdot\beta(m, n+p)\cdot\beta(l, m+n+p)$$

$$= \frac{\Gamma(n)\,\Gamma(p)}{\Gamma(n+p)}\cdot\frac{\Gamma(m)\,\Gamma(n+p)}{\Gamma(m+n+p)}\cdot\frac{\Gamma(l)\cdot\Gamma(m+n+p)}{\Gamma(l+m+n+p)}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\,\Gamma(p)}{\Gamma(l+m+n+p)}$$

## EXERCISE 4(d)

***Part A***

(Short Answer Questions)

1. Prove that $\int_0^\infty x^4 e^{-x^2}\,dx = \frac{3}{8}\sqrt{\pi}$.

2. Evaluate $\int_0^\infty x\,e^{-x^3}\,dx$, given that $\Gamma\left(\frac{5}{3}\right) = 0.902$.

3. Find the value of $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\,dx$.

4. Find the value of $\int_0^{\pi/2} \sin^5\theta \cos^7\theta\,d\theta$.

5. Find the value of $\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$ in terms of Gamma functions.

6. Prove that $\int_0^{\pi/2} \sqrt{\cot\theta}\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$.

7. Find the value of $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$.

8. Prove that $\int_0^{\pi/2} \sqrt{\cos x}\,dx \times \int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}} = \pi$.

9. Prove that $\int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx = \Gamma(n)$.

10. Find the value of $\int_0^\infty \frac{x^n}{n^x}\,dx \ (n > 1)$.

11. Assuming that $\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin\alpha\pi}$, prove that $\Gamma(\alpha)\cdot\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}$, where α is neither zero nor an integer. [*Hint:* put $x = \tan^2\theta$]

12 Find the value of $\int_{-\infty}^\infty e^{-kx^2}\,dx$.

13. Prove that $\frac{\beta(m+1,n)}{\beta(m,n+1)} = \frac{m}{n}$.

14. Prove that $\frac{\beta(m+1,n)}{m}\,\frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n}$.

15. Find the value of $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$ in terms of a Beta function.

$\left[Hint\text{: put } x = \frac{a}{b}t\right]$

16. Prove that $\beta(m + 1, n) + \beta(m, n + 1) = \beta(m, n)$.
17. Find the value of $\int_0^2 (8-x^3)^{-\frac{1}{3}}\,dx$ in terms of Gamma functions.
18. Prove that $\int_0^a x^m (a-x)^n\,dx = a^{m+n+1}\cdot\beta(m+1, n+1)$.
19. Define Gamma and Beta functions.
20. Derive the recurrence formula for the Gamma function.
21. When $n$ is a positive integer, prove that $\Gamma(n + 1) = n!$
22. State the relation between Gamma and Beta functions and use it to find the value of $\Gamma\left(\frac{1}{2}\right)$.

***Part B***

23. Prove that $\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)}\, x^{2m-1}\, y^{2n-1}\,dx\,dy = \frac{1}{4a^m b^n}\Gamma(m)\Gamma(n)$.
24. When $n$ is a positive integer and $m > -1$, prove that $\int_0^1 x^m (\log x)^n\,dx = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$.
25. Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n (a+b)^m}$.

$\left[Hint\text{: Put } \frac{x}{a+bx} = \frac{z}{a+b}\right]$

26. Express $\beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right)$ in terms of Gamma functions in two different ways and hence prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}$.
27. Prove that $\int_0^\infty \sqrt{x}\, e^{-x^2}\,dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx = \frac{\pi}{2\sqrt{2}}$.
28. Prove that $\int_0^\infty x\, e^{-x^8}\,dx \times \int_0^\infty x^2 e^{-x^4}\,dx = \frac{\pi}{16\sqrt{2}}$.
29. Prove that $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4}$.

30. Evaluate (i) $\int_0^{\infty} \frac{dx}{1+x^4}$, (ii) $\int_0^{\infty} \frac{x^2\, dx}{(1+x^4)^2}$ and (iii) $\int_0^{\infty} \frac{x^2\, dx}{(1+x^4)^3}$

[*Hint:* put $x^2 = \tan^2 \theta$]

31. Find the value of $\iint x^m\, y^n\, dx\, dy$, taken over the area $x \geq 0, y \geq 0, x + y \leq 1$ in terms of Gamma functions, if $m, n > 0$.
32. Find the value, in terms of Gamma functions, of $\iiint x^m\, y^n\, z^p\, dx\, dy\, dz$ taken over the volume of the tetrahedron given by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x + y + z \leq 1$.
33. Find the area in the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ and the co-ordinate axes.
34. Evaluate $\iint x^{m-1}\, y^{n-1}\, (1-x-y)^{p-1}\, dx\, dy$, taken over the area in the first quadrant enclosed by the lines $x = 0$, $y = 0$, $x + y = 1$.
35. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in $A$, $B$ and $C$. Find the volume of the tatrahedron $OABC$.
36. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
37. Find the volume of the region of the space bounded by the co-ordinate planes and the surface $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n + \left(\frac{z}{c}\right)^n = 1$ and lying in the first octant.
38. Evaluate $\iiint \sqrt{xyz\,(1-x-y-z)}\, dx\, dy\, dz$, taken over the tetrahedral volume in the first octant enclosed by the plane $x=0, y=0, z=0$ and $x+y+z=1$.
39. Evaluate $\iiint x^2\, yz\, dx\, dy\, dz$, taken throughout the volume in the first octant bounded by $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
40. Evaluate $\iiint xyz\, dx\, dy\, dz$, taken over the space defined by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$.

## ANSWERS

## Exercise 4(a)

(1) 4 (2) $\log a \log b$ (3) 2

(4) $\frac{1}{2}$ (5) $\frac{\pi}{4}$ (6) $\frac{9}{2}$

(7) $\frac{1}{2}$

(8)

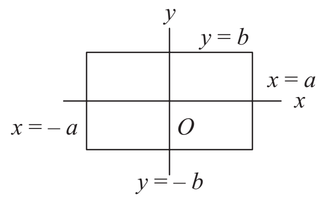


**Fig. 4.59**

(9)

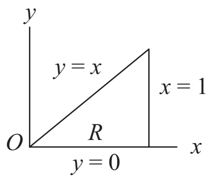


**Fig. 4.60**

(10)

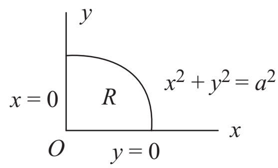


**Fig. 4.61**

(11)

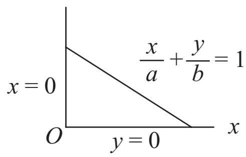


**Fig. 4.62**

(12) $\int_0^1 \int_0^{1-x} f(x, y)\,dy\,dx$ (or) $\int_0^1 \int_0^{1-y} f(x, y)\,dx\,dy$

(13) $\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} f(x, y)\,dy\,dx$ (or) $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} f(x, y)\,dx\,dy$

(14) $\int_0^1 \int_0^y f(x, y)\,dx\,dy$ (or) $\int_0^1 \int_x^1 f(x, y)\,dy\,dx$

(15) $\int_0^2 \int_{\frac{y^2}{4}}^1 f(x, y)\,dx\,dy$ (or) $\int_0^1 \int_0^{2\sqrt{x}} f(x, y)\,dy\,dx$

(16) 2 log 2 (17) $\frac{a^3}{6}$ (18) $\frac{\pi}{4}$

(19) $\frac{\pi a^3}{6}$ (20) $\frac{1}{720}$

(21) $\frac{8}{3}\log 2 - \frac{19}{9}$ (22) 1 (23) $\frac{3}{2}$

(24) $\frac{\pi}{2}a^3$ (25) $\frac{1}{3}ab(a+b)$ (26) $\frac{344}{105}$

(27) 6 (28) $\frac{33}{2}$

(29) $\frac{1}{16}(8\log 2 - 5)$ (30) $\frac{1}{48}$

## Exercise 4(b)

(1) $\int_0^a \int_y^a f(x,y)\,dx\,dy$

(2) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$

(3) $\int_0^a \int_0^y f(x,y)\,dx\,dy$

(4) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$

(5) $\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$

(6) $\int_0^a \int_0^{a-y} f(x,y)\,dx\,dy$

(7) $\int_0^1 \int_0^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$

(8) $\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

(9) $\int_0^2 \int_{\frac{y^2}{4}}^1 f(x,y)\,dx\,dy$

(10) $\int_0^\infty \int_0^{1/x} f(x,y)\,dy\,dx$

(11) $\frac{\pi a}{4}$

(12) $\frac{16}{3}$

(13) 1

(14) 1

(15) $\frac{1}{2}(e-1)^2$

(16) 2

(17) $\frac{9}{5}a^3$

(18) $\frac{1}{2}\log 2$

(19) $\frac{241}{60}$

(20) $\frac{\pi}{8}a^4$

(21) 3

(22) $8\log 2$

(23) $\frac{\pi}{4}$

(24) $\frac{3}{8}$

(25) $\frac{2}{3}$

(26) $\frac{32}{3}$

(27) $\frac{16}{3}a^2$

(28) $\frac{16}{3}ab$

(29) $\frac{\pi}{2}+\frac{1}{3}$

(30) $3\pi$

(31) $\frac{3}{2}\pi a^2$

(32) $a^2\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$

(33) $\frac{a^2}{2}(3\pi-8)$

(34) $\pi a^2$

(35) $\frac{3}{4}\pi a^2$

(36) $\frac{\pi}{4}\left(1-e^{-a^2}\right)$

(37) $\frac{\pi a}{4}$

(38) $\frac{a^4}{4}\log\left(1+\sqrt{2}\right)$

(39) $\frac{4}{3}a^3$

(40) $\frac{2\pi}{a}$

(41) $\frac{\pi}{2}a^2 h^2$

(42) $2\pi$

(43) $12\pi$

(44) $16\pi$

(45) $\frac{16}{9}a^3(3\pi-4)$

(46) $4\pi a$

(47) $\frac{\pi^2}{8}$ (48) $\frac{\pi}{4}a^4$ (49) $\frac{\pi}{16}a^4$

(50) $\frac{\pi}{6a^2}$

## Exercise 4(c)

(8) $\frac{2}{3}$ (9) 1 (10) $\frac{\pi}{2}$

(11) 88 (12) $4\pi a^2$ (13) $2\pi ah$

(14) 6 $abc$ (15) $\frac{4}{3}\pi$ (16) $2\pi a^2 h$

(17) $-\frac{69}{10};-\frac{29}{4}$ (18) $-\frac{2}{3};-\frac{2}{3}$ (19) $\frac{4}{3}a^3$

(20) $2\pi ab$ (21) 4 (22) $\frac{3}{2}$

(23) $a^2\left(1-\frac{\pi a}{4}\right)$ (24) $\frac{1}{3}\left(2\sqrt{2}-1\right)$. (25) $\frac{163}{70}$

(26) $\frac{13}{3}$ (27) $\frac{13}{6}$ (28) $\frac{2a^3}{\sqrt{3}}$

(29) $\frac{\sqrt{2}}{3}\pi^3$ (30) $\frac{1}{2}\sqrt{a^2 b^2+b^2 c^2+c^2 a^2}$

(31) $\frac{\pi a^3}{4}$ (32) $\frac{ka^3}{2}$ (33) $\frac{abc}{6}$

(34) $\frac{\pi}{4}a^4$ (35) $\frac{\pi}{2}a^4 h$

## Exercise 4(d)

(2) 0.456 (3) $\frac{8}{77}$ (4) $\frac{1}{120}$

(5) $\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (7) $\pi$ (10) $\frac{1}{(\log n)^{n+1}}\Gamma(n+1)$

(12) $\sqrt{\frac{\pi}{k}}$ (15) $\frac{1}{a^n b^m}\beta(m,n)$ (17) $\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)$

(22) $\sqrt{\pi}$ (30) $\frac{\pi}{2\sqrt{2}};\frac{\pi}{8\sqrt{2}};\frac{5\pi\sqrt{2}}{128}$

(31) $\Gamma(m+1)\,\Gamma(n+1)\,/\,\Gamma(m+n+3)$

(32) $\Gamma(m+1)\cdot\Gamma(n+1)\,\Gamma(p+1)\,/\,\Gamma(m+n+p+4)$

(33) $\dfrac{3\pi ab}{32}$ (34) $\Gamma(m)\,\Gamma(n)\,\Gamma(p)\,/\,\Gamma(m+n+p)$

(35) $\dfrac{abc}{6}$ (36) $\dfrac{4}{3}\pi\, abc$

(37) $abc\left\{\Gamma\left(\dfrac{1}{n}\right)\right\}^3 \Big/ 3n^2\ \Gamma\left(\dfrac{3}{n}\right)$ (38) $\pi^2/1920$

(39) $a^3b^2c^2/2520$ (40) $a^2b^2c^2/48$

UNIT **5**

# Differential Equations

## 5.1 EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

In the lower classes, the students have studied differential equations of the first order and first degree, such as variable separable equations and linear equations. Now we shall study differential equations of the first order and degree greater than or equal to two.

The general form of the differential equation of the first order and $n^{th}$ degree is

$$\left(\frac{dy}{dx}\right)^n + f_1(x,y)\left(\frac{dy}{dx}\right)^{n-1} + f_2(x,y)\left(\frac{dy}{dx}\right)^{n-2} + \cdots + f_{n-1}(x,y)\frac{dy}{dx} + f_n(x,y) = 0.$$

If we denote $\frac{dy}{dx}$ by $p$ for convenience, the general equation becomes

$$p^n + f_1(x,y)\,p^{n-1} + f_2(x,y)\,p^{n-2} + \cdots + f_{n-1}(x,y)\,p + f_n(x,y) = 0 \qquad (1)$$

Since (1) is an equation of the first order, its general solution will contain only one arbitrary constant. Solution of (1) will depend on solving one or more equations of the first order and first degree.

To solve (1), it is to be identified as an equation of any one of the following types and then solved as per the procedure indicated below:

(i) Equations solvable for $p$.
(ii) Equations solvable for $y$.
(iii) Equations solvable for $x$.
(iv) Clairaut's equations.

### 5.1.1 Type 1—Equations solvable for *p*

If equation (1) is of this type, then the L.H.S. of (1) can be resolved into $n$ linear factors. Then (1) becomes

$(p - F_1)(p - F_2) \ldots (p - F_n) = 0$, from which we get $p = F_1, p = F_2, \ldots, p = F_n$, where $F_1, F_2, \ldots, F_n$ are functions of $x$ and $y$.

Each of these $n$ equations is of the first order and first degree and can be solved by methods already known.

Let the solutions of the above $n$ component equations be $\phi_1(x, y, c) = 0$, $\phi_2(x, y, c) = 0, \ldots, \phi_n(x, y, c) = 0$. Then the general solution of (1) is got by combining the above solutions and given as $\phi_1(x, y, c)\, \phi_2(x, y, c) \ldots \phi_n(x, y, c) = 0$.

## 5.1.2 Type 2—Equations Solvable for *y*

If the given differential equation is of this type, then $y$ can be expressed explicitly as a single valued function of $x$ and $p$.
i.e. the equation of this type can be re-written as

$$y = f(x, p) \tag{1}$$

Differentiating (1) with respect to $x$, we get

$$p = \phi\left(x,\ p,\ \frac{dp}{dx}\right) \tag{2}$$

Equation (2) is a differential equation of the first order and first degree in the variables $x$ and $p$. It can be solved by methods already known. Let the solution of (2) be $\psi(x, p, c) = 0 \cdots (3)$, where $c$ is an arbitrary constant. If we eliminate $p$ between (1) and (3), the eliminant is the general solution of the given equation.

If $p$ cannot be easily eliminated between (1) and (3), they jointly provide the required solution in terms of the parameter $p$.

## 5.1.3 Type 3—Equations solvable for *x*

If the given differential equation is of this type, then $x$ can be expressed explicitly as a single valued function of $y$ and $p$. i.e the equation of this type can be re–written as

$$x = f(y, p) \tag{1}$$

Differentiating (1) with respect to $y$, we get

$$\frac{1}{p} = \phi\left(y,\ p,\ \frac{dp}{dy}\right) \tag{2}$$

Equation (2) is a differential equation of the first order and first degree in the variables $y$ and $p$. It can be solved by methods already known.

Let the solution of (2) be

$$\psi(y, p, c) = 0 \tag{3}$$

where $c$ is an arbitrary constant.

If we eliminate $p$ between (1) and (3), the eliminant is the general solution of the given equation.

If $p$ cannot be easily eliminated between (1) and (3), they jointly provide the required solution in terms of the parameter $p$.

**Note** ☑ Some differential equations can be put in both the forms $y = f(x, p)$ and $x = f(y, p)$. Both these forms may lead to the required solution. Sometimes one of the forms will lead to the required solution more easily than the other.

### 5.1.4 Type 4—Clairaut′s Equations

An equation of the form $y = px + f(p)$ is called Clairaut′s equation.

Clairaut′s equation is only a particular case of type–2 equation. Hence it can be solved in the same way in which a type–2 equation is solved, as explained below: Let the Clairaut′s equation be

$$y = px + f(p) \tag{1}$$

Differentiating (1) *w. r. t. x*,

$$p = p + \{x + f'(p)\}\frac{dp}{dx} \tag{2}$$

$$\therefore \quad \frac{dp}{dx} = 0 \ldots (2) \quad \text{or} \quad f'(p) + x = 0 \tag{3}$$

Solving (2), we get $p = c$ (4)

Eliminating $p$ between (1) and (4), we get the general solution of (1) as $y = cx + f(c)$.

Thus the general solution of a Clairaut′s equation is obtained by replacing $p$ by $c$ in the given equation.

Eliminating $p$ between (1) and (3), we also get a solution of (1).

This solution does not contain any arbitrary constant. Also it cannot be obtained as a particular case of the general solution. This solution is called *the singular solution* of the equation (1).

**Note** ☑ The singular solution of (1) is the eliminant of $p$ between $y = px + f(p)$ and $x + \frac{df}{dp} = 0$. Equivalently, the singular solution of (1) is the eliminant of $c$ between $y = cx + f(c)$ and $x + \frac{df}{dc} = 0$. Hence, we observe that if the general solution of (1), i.e. $y = cx + f(c)$ represents a family of straight lines, then the singular solution represents the envelope of the family of straight lines.

**WORKED EXAMPLE 5(a)**

**Example 5.1** Solve the equation $\left(\frac{dy}{dx}\right)^2 - 8\frac{dy}{dx} + 15 = 0$.

The given equation is $p^2 - 8p + 15 = 0$, which is solvable for $p$.

The equation is $(p - 3)(p - 5) = 0$

$$\therefore \quad \frac{dy}{dx} = 3 \quad \text{or} \quad \frac{dy}{dx} = 5$$

Solving these equations, we get $y = 3x + c$ and $y = 5x + c$.

To get the general solution of the given equation, we rewrite the solutions as $y - 3x - c = 0$ and $y - 5x - c = 0$ (in which the R.S. = 0) and combine them as $(y - 3x - c)(y - 5x - c) = 0$.

**Example 5.2** Solve the equation $p(p + y) = x(x + y)$.

The given equation is $p^2 + yp - (x^2 + xy) = 0$.

Solving for $p$, $p = \dfrac{-y \pm \sqrt{y^2 + 4\left(x^2 + xy\right)}}{2}$

$$= \frac{-y \pm \sqrt{(y + 2x)^2}}{2}$$

i.e. $$\frac{dy}{dx} = x \qquad (1)$$

or $$\frac{dy}{dx} = -x - y \qquad (2)$$

Solving (1), $y = \dfrac{x^2}{2} + \dfrac{c}{2}$.

i.e. $$2y - x^2 - c = 0 \qquad (3)$$

Rewriting (2), $\dfrac{dy}{dx} + y = -x$, which is a linear equation of the first order in $y$.

Integrating factor = $e^x$

Hence the solution of (2) is

$$y\, e^x = \int -x\, e^x\, dx + c$$

$$= -x\, e^x + e^x + c$$

i.e. $$y + x - 1 - c\, e^{-x} = 0 \qquad (4)$$

Combining (3) and (4), the required general solution is

$$(2y - x^2 - c)(y + x - 1 - ce^{-x}) = 0.$$

**Example 5.3** Solve the equation $p^2 - 2py \tan x - y^2$.

The given equation is $p^2 - 2py \tan x - y^2 = 0$.

Solving for $p$,

$$p = \frac{2y \tan x \pm \sqrt{4y^2 \tan^2 x + 4y^2}}{2}$$

$$= \frac{2y \tan x \pm \sqrt{4y^2\left(1 + \tan^2 x\right)}}{2} \qquad (1)$$

$$= y\left(\tan x \pm \sec x\right)$$

$\therefore$ $$\frac{dy}{dx} = y(\tan x + \sec x) \text{ or } \frac{dy}{dx} = y(\tan x - \sec x).$$

i.e. $$\frac{dy}{y} = (\tan x + \sec x)\, dx \ (1) \text{ and } \frac{dy}{y} = (\tan x - \sec x)\, dx \qquad (2)$$

Integrating (1), we get,

$$\log y = \log \sec x + \log(\sec x + \tan x) + \log c$$

i.e. $$y = c \sec x (\sec x + \tan x)$$

$$= \frac{c\,(1+\sin x)}{\cos^2 x}$$

$$= \frac{c}{1-\sin x}$$

i.e. $$y\,(1 - \sin x) - c = 0 \qquad (3)$$

Integrating (2), we get,

$$\log y = \log \sec x - \log (\sec x + \tan x) + \log c$$

i.e. $$y = \frac{c \sec x}{\sec x + \tan x}$$

$$= \frac{c}{1+\sin x}$$

i.e. $$y\,(1 + \sin x) - c = 0 \qquad (4)$$

Combining (3) and (4), the required general solution is $[y\,(1 - \sin x) - c]\,[y\,(1 + \sin x) - c] = 0$.

**Example 5.4** Solve the equation $xp^2 - 2py + x = 0$.

Solving the given equation for $p$, we get

$$p = \frac{2y \pm \sqrt{4y^2 - 4x^2}}{2x}$$

i.e. $$\frac{dy}{dx} = \frac{y}{x} \pm \sqrt{\left(\frac{y}{x}\right)^2 - 1} \qquad (1)$$

(1) is a homogeneous equation.

Putting $y = vx$, (1) becomes

$$v + x\frac{dv}{dx} = v \pm \sqrt{v^2 - 1}$$

i.e. $$\frac{dv}{\sqrt{v^2-1}} = \pm \frac{dx}{x} \qquad (2)$$

Integrating (2), we get,

$$\log\left(v + \sqrt{v^2 - 1}\right) = \pm \log x + \log c$$

∴ Solutions are

$$\frac{y + \sqrt{y^2 - x^2}}{x^2} = c \quad \text{and} \quad y + \sqrt{y^2 - x^2} = c$$

i.e. $$y + \sqrt{y^2 - x^2} - cx^2 = 0 \quad \text{and} \quad y + \sqrt{y^2 - x^2} - c = 0$$

∴ The general solution of the given equation is

$$\left\{y + \sqrt{y^2 - x^2} - cx^2\right\}\left\{y + \sqrt{y^2 - x^2} - c\right\} = 0\cdot$$

**Example 5.5** Solve the equation

$$p^3 - (x^2 + xy + y^2)\,p^2 + (x^3y + xy^3 + x^2y^2)\,p - x^3y^3 = 0.$$

The given equation is of the form

$$p^3 - (\alpha + \beta + \gamma)\, p^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)\, p - \alpha\beta\gamma = 0,$$

where $\alpha = x^2$, $\beta = xy$ and $\gamma = y^2$.

∴ The given equation can be rewritten as

$$(p - \alpha)(p - \beta)(p - \gamma) = 0$$

i.e. $$(p - x^2)(p - xy)(p - y^2) = 0.$$

∴ $$\frac{dy}{dx} = x^2, \frac{dy}{dx} = xy \text{ and } \frac{dy}{dx} = y^2$$

Solving these equations, we get

$$y = \frac{x^3}{3} + \frac{c}{3}; \log y = \frac{x^2}{2} + \log c; -\frac{1}{y} = x - c.$$

i.e. $$3y - x^3 - c = 0; y - ce^{x^2/2} = 0; x + \frac{1}{y} - c = 0.$$

∴ The required general solution is

$$\left(3y - x^3 - c\right)\left(y - ce^{\frac{x^2}{2}}\right)\left(x + \frac{1}{y} - c\right) = 0.$$

**Example 5.6** Solve the equation $p^2x - 2py - x - 0$.

Rewriting the given equation, we have

$$y = \frac{p^2 x - x}{2p} = \frac{1}{2}\left(px - \frac{x}{p}\right) \qquad (1)$$

We identify the equation as one solvable for $y$.

Differentiating (1) w.r.t. $x$, we get

$$p = \frac{1}{2}\left\{p + x\frac{dp}{dx} - \left(\frac{p - x\frac{dp}{dx}}{p^2}\right)\right\}$$

i.e. $$2p = p + x\frac{dp}{dx} - \frac{\left(p - x\frac{dp}{dx}\right)}{p^2}$$

i.e. $$p^3 = xp^2\frac{dp}{dx} - p + x\frac{dp}{dx}$$

i.e. $$x\frac{dp}{dx}\left(p^2 + 1\right) - p\left(p^2 + 1\right) = 0$$

i.e. $$x\frac{dp}{dx} = p \qquad (\because p^2 + 1 \neq 0)$$

Solving this equation, we get

$$\int \frac{\mathrm{d}p}{p} = \int \frac{\mathrm{d}x}{x} + \log c$$

i.e. $$p = cx \qquad (2)$$

Using (2) in the given equation, the required solution is $c^2x^2 = 2cy + 1$.

**Example 5.7** Solve the equation $16x^2 + 2p^2y - p^3x = 0$.

The given equation cannot be solved for $p$, nor for $x$. As it is solvable for $y$, the given equation is rewritten as

$$y = \frac{p^3x - 16x^2}{2p^2} \qquad (1)$$

Differentiating (1) w.r.t. $x$,

$$2p = \frac{p^2\left(p^3 + 3p^2x\frac{\mathrm{d}p}{\mathrm{d}x} - 32x\right) - \left(p^3x - 16x^2\right)2p\frac{\mathrm{d}p}{\mathrm{d}x}}{p^4}$$

i.e. $$2p^5 = p^5 + 3p^4x\frac{\mathrm{d}p}{\mathrm{d}x} - 32p^2x - 2p^4x\frac{\mathrm{d}p}{\mathrm{d}x} + 32px^2\frac{\mathrm{d}p}{\mathrm{d}x}$$

i.e. $$p^5 + 32p^2x = p^4x\frac{\mathrm{d}p}{\mathrm{d}x} + 32px^2\frac{\mathrm{d}p}{\mathrm{d}x}$$

i.e. $$p^2\left(p^3 + 32x\right) - px\left(p^3 + 32x\right)\frac{\mathrm{d}p}{\mathrm{d}x} = 0$$

i.e. $$p^2 - px\frac{\mathrm{d}p}{\mathrm{d}x} = 0 \qquad (2)$$

or $$p^3 + 32x = 0 \qquad (3)$$

(2) is differential equation in $p$ and (3) is an algebraic equation.

If we eliminate $p$ between the given equation and the solution of (2), we will get the general solution of (1).

If we eliminate $p$ between (1) and (3), we will get the singular solution of (1).

(2) can be rewritten as $\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}x}{x}$

Solving this, we get

$$p = cx \qquad (4)$$

Eliminating $p$ between the given equation and (4) we have

$$16x^2 + 2c^2x^2y - c^3x^4 = 0$$

i.e. $c^3x^2 - 2c^2y - 16 = 0$, which is the required general solution. Using (3) in the given equation,

$$16x^2 + 2p^2y + 32x^2 = 0$$

i.e. $p^2y = -24x^2$

i.e. $p^6y^3 = -(24)^3x^6$

i.e. $1024x^2y^3 + (24)^3x^6 = 0$

i.e. $16y^3 + 9x^4 = 0$, which is the singular solution.

**Example 5.8** Solve the equation $y = (1 + p)\,x + p^2$.

Treating the given equation as one solvable for $y$ and differentiating w.r.t. $x$,

$$p = 1 + p + x\frac{dp}{dx} + 2p\frac{dp}{dx} \tag{1}$$

i.e. $$(x + 2p)\frac{dp}{dx} + 1 = 0$$

i.e. $$\frac{dx}{dp} + x = -2p \tag{2}$$

This is a linear equation in $x$.

Solving (2), $$xe^p = \int -2pe^p\, dp + c$$

$$= -2\,(pe^p - e^p) + c$$

i.e. $$x = 2 - 2p + ce^{-p} \tag{3}$$

It is not easy to eliminate $p$ between (3) and the given equation.

∴ The parametric equations of the general solution are given as

$$x = 2 - 2p + ce^{-p} \text{ and}$$
$$y = (1 + p)\,(2 - 2p + ce^{-p}) + p^2$$

i.e. $$x = 2 - 2p + ce^{-p} \quad \text{and} \quad y = 2 - p^2 + c\,(1 + p)\,e^{-p}.$$

**Example 5.9** Solve the equation $y = x + p^2 - 2p$.

This equation can be treated as one solvable for $y$ and for $x$. We shall solve the equation in both ways.

***Method I:*** Let us assume $y = x + p^2 - 2p$ . . . (1) as solvable for $y$.

Differentiating (1) w.r.t. $x$,

$$p = 1 + (2p - 2)\frac{dp}{dx} \tag{1}$$

i.e. $$2(p - 1)\frac{dp}{dx} - (p - 1) = 0$$

∴ $$p - 1 = 0 \tag{2}$$

or $$2\frac{dp}{dx} = 1 \tag{3}$$

Eliminating $p$ between (1) and (2), we get the singular solution $y = x - 1$.

Solving (3), we have

$$2p = x + c \tag{4}$$

Eliminating $p$ between (1) and (4), $y - x = \dfrac{(x + c)^2}{4} - (x + c)$

i.e. $4\,(y + c) = (x + c)^2$, which is the general solution of (1).

***Method II:*** Treating (1) as one solvable for $x$ and rewriting, we have

$$x = y + 2p - p^2 \tag{1$'$}$$

Differentiating (1)′ w.r.t. $y$,

$$\frac{dx}{dy} = 1 + 2\frac{dp}{dy} - 2p\frac{dp}{dy}$$

i.e. $\frac{1}{p} - 1 = 2\,(1-p)\frac{dp}{dy}$

i.e. $(1-p)\left\{2p\frac{dp}{dy} - 1\right\} = 0.$

$$\therefore \quad p = 1 \qquad (5)$$

$$\text{or} \quad 2p\frac{dp}{dy} - 1 = 0 \qquad (6)$$

Eliminating $p$ between (1)′ and (5), we get the same singular solution as before. Solving (6), we have

$$p^2 = y + c \qquad (7)$$

Using (7) in (1)′, we get

$$x - y + y + c = 2p$$

Squaring and again using (7), we get the general solution as

$$(x + c)^2 = 4\,(y + c)$$

**Example 5.10** Solve the equation $p^2x + py - y^4 = 0.$

*Note* ☑ Though the equation appears to be solvable for $p$, the component equations are not easily solvable. Also the given equation is not solvable for $y$. Hence we treat it as one solvable for $x$.

Rewriting the given equation,

$$x = \frac{y^4 - py}{p^2} \qquad (1)$$

Differentiating (1) w.r.t. $y$,

$$\frac{dx}{dy} = \frac{p^2\left(4y^3 - p - y\frac{dp}{dy}\right) - \left(y^4 - py\right)2p\frac{dp}{dy}}{p^4}$$

i.e. $$\frac{1}{p} = \frac{p\left(4y^3 - p - y\frac{dp}{dy}\right) - \left(2y^4 - 2py\right)\frac{dp}{dy}}{p^3}$$

i.e. $$p^2 = 4py^3 - p^2 - \left(2y^4 - py\right)\frac{dp}{dy}$$

i.e. $$y\left(2y^3 - p\right)\frac{dp}{dy} - 2p\left(2y^3 - p\right) = 0$$

$$\therefore \quad y\frac{dp}{dy} - 2p = 0 \qquad (2)$$

$$\text{or} \quad 2y^3 - p = 0 \qquad (3)$$

(2) is $\frac{dp}{p} = \frac{2dy}{y}$

Integrating, $\log p = 2\log y + \log c$

i.e. $$p = cy^2 \qquad (4)$$

Eliminating $p$ between (1) and (4), we get

$$c^2xy^4 + cy^3 - y^4 = 0$$

i.e. $c^2xy + c = y$, which is the general solution of the given equation.

Eliminating $p$ between (1) and (3), we get

$$4xy^6 + y^4 = 0$$

i.e. $4xy^2 + 1 = 0$, which is the singular solution of the given equation.

**Example 5.11** Solve the equation $p^3 - 2\,x\,yp + 4y^2 = 0.$

The given equation is neither solvable for $p$ nor for $y$.

Rewriting the given equation,

$$2x = \frac{p^2}{y} + \frac{4y}{p} \tag{1}$$

Differentiating (1) w.r.t. $y$,

$$\frac{2}{p} = -\frac{p^2}{y^2} + \frac{2p}{y}\frac{dp}{dy} + \frac{4}{p} - \frac{4y}{p^2}\frac{dp}{dy}$$

i.e. $$\left(\frac{2p}{y} - \frac{4y}{p^2}\right)\frac{dp}{dy} + \left(\frac{2}{p} - \frac{p^2}{y^2}\right) = 0$$

i.e. $$\frac{2}{p^2y}\left(p^3 - 2y^2\right)\frac{dp}{dy} - \frac{1}{py^2}\left(p^3 - 2y^2\right) = 0$$

i.e. $$\left(p^3 - 2y^2\right)\left(2y\frac{dp}{dy} - p\right) = 0$$

∴ $$2y\frac{dp}{dy} - p = 0 \tag{2}$$

or $$p^3 - 2y^2 = 0 \tag{3}$$

Solving (2), $\displaystyle\int \frac{2dp}{p} = \int \frac{dy}{y} + \log c$

i.e. $2\log p = \log y + \log c$

i.e. $$p^2 = cy \tag{4}$$

Let us eliminate $p$ between (1) and (4).

Using (4) in (1), $cpy - 2xyp + 4y^2 = 0$

i.e. $$p(c - 2x) = -4y$$

Squaring and using (4),

$$cy(c - 2x)^2 = 16y^2$$

i.e. $c(c - 2x)^2 = 16y$, which is the general solution of the given equation.

Using (3) in (1), $2y^2 - 2\,x\,yp + 4y^2 = 0$

i.e. $$xp = 3y$$

Cubing and using (3),

$$2x^3y^2 = 27y^3$$

i.e. $2x^3 = 27y$, which is the singular solution of the given equation.

**Example 5.12** Solve the equaion $p^3x - p^2y - 1 = 0$.

The equation may be treated as one solvable for $x$. However if we treat it as one solvable for $y$, it becomes a Clairaut′s equation.

Rewriting the given equation, we get $y = px - \frac{1}{p^2}$, which is a Clairaut′s equation.

$\therefore$ Its general solution is

$$y = cx - \frac{1}{c^2} \tag{1}$$

Differentiating (1) partially w.r.t. $c$,

$$0 = x + \frac{2}{c^3} \tag{2}$$

Eliminating $c$ between (1) and (2), we get the singular solution of the given equation. From (2),

$$c^3 = -\frac{2}{x} \tag{3}$$

From (l), $c^2y = c^3x - 1$

$= -2 - 1$, using (3)

$\therefore$ $c^6y^3 = -27$

i.e. $4y^3 = -27x^2$, using (3)

This is the singular solution of the given equation.

**Example 5.13** Solve the equation $y = 2px + yp^2$.

The equation may be treated as one solvable for $x$. However we can convert it as a Clairaut′s equation by means of the transformation

$$y^2 = Y$$

$\therefore$ $$2y\frac{dy}{dx} = \frac{dY}{dx}$$

i.e. $2yp = P$, say.

Multiplying through out the given equation by $y$, it becomes

$$y^2 = 2ypx + y^2p^2 \tag{1}$$

On using the transformation, (1) becomes

$$Y + Px + \frac{P^2}{4}, \tag{2}$$

which is a Clairaut′s equation

$\therefore$ General solution of (2) is

$$Y = cx + \frac{c^2}{4} \tag{3}$$

$\therefore$ General solution of (1) is $y^2 = cx + \frac{c^2}{4}$.

Differentiating (3) partially w.r.t. $c$,

$$0 = x + \frac{c}{2} \tag{4}$$

Eliminating $c$ between (3) and (4), we get,

$$Y = -x^2$$

i.e. $x^2 + y^2 = 0$, which is the singular solution of (1).

**Example 5.14** Solve the equation $(px - y)(py + x) = 2p$.

Put $X = x^2$ and $Y = y^2$

$\therefore$ $dX = 2x\,dx$ and $dY = 2y\,dy$

$\therefore$ $\frac{dY}{dX} = \frac{y}{x}\frac{dy}{dx}$ i.e. $P = \frac{y}{x}p$ (say)

or $p = \frac{x}{y}P$

Then the given equation becomes

$$\left(x^2P - y^2\right)\frac{x}{y}(P+1) = 2\frac{x}{y}P$$

i.e. $PX - Y = \frac{2P}{P+1}$

i.e. $Y = PX - \frac{2P}{P+1}$, which is a Clairaut′s equation.

$\therefore$ General solution is $y^2 = cx^2 - \frac{2c}{c+1}$

**Example 5.15** Solve the equation $e^{4x}(p - 1) + e^{2y}p^2 = 0$

***Note*** ☑ In problems involving $e^{ax}$ and $e^{by}$ we make the transformations $X = e^{kx}$ and $Y = e^{ky}$ where $k$ is the H.C.F. of $a$ and $b$

In the given equation, put $X = e^{2x}$ and $Y = e^{2y}$

$\therefore$ $\frac{dY}{dX} = \frac{2e^{2y}}{2e^{2x}}\frac{dy}{dx}$

i.e. $p = \frac{X}{Y}P$, where $P = \frac{dY}{dX}$.

Then the given equation becomes

$$X^2\left(\frac{X}{Y}P - 1\right) + Y\cdot\frac{X^2}{Y^2}P^2 = 0$$

i.e. $XP - Y + P^2 = 0$

i.e. $Y = PX + P^2$, which is a Clairaut′s equation.

$\therefore$ The general solution of the given equation is $e^{2y} = ce^{2x} + c^2$.

## EXERCISE 5(a)

***Part A***

(Short Answer Questions)

1. Explain briefly how to find the general solution of the equation $[p - f_1(x, y)]$ $[p - f_2(x, y)] = 0$.
2. Explain briefly how to solve the equation, $y = f(x, p)$, if it is solvable for $y$.
3. Explain briefly how to solve the equation $x = f(y, p)$, if it is solvable for $x$.
4. Write down the standard form of a Clairaut′s equation and give its general solution.

5. How will you find the singular solution of the equation $y = px + f(p)$?
6. What does the singular solution represent geometrically?
7. Solve the equation $p^2 - 5p + 6 = 0$.
8. Solve the equation $p^2 - (x + y)p + xy = 0$.
9. Solve the equation $p^2 - (e^x + e^{-y})p + e^{x-y} = 0$.
10. Solve the equation $xyp^2 - (x + y)p + 1 = 0$.
11. Rewrite the equation $(y - px)(p - 1) = p$ as a Clairaut′s equation and write down its general solution.
12. Rewrite the equation $p = \log(px - y)$ as a Clairaut′s equation and give its general solution.
13. Rewrite the equation $p = \sin(y - px)$ as a Clairaut′s equation and give its general solution.
14. Find the singular solution of $y = px + \dfrac{1}{p}$.
15. Find the singular solution of $y = px - p^2$.

***Part B***

Solve the following equations:

16. $yp^2 + (x - y)p - x - 0$
17. $2p^2 - (x + 2y^2)p + xy^2 = 0$
18. $xyp^2 + (3x^2 - 2y^2)p - 6xy = 0$
19. $xyp^2 - (x^2 + y^2)p + xy = 0$
20. $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$
21. $p^2 + 2py \cot x = y^2$
22. $x^2p^2 - 2xyp + (2y^2 - x^2) = 0$
23. $y = -px + x^4p^2$
24. $4y = x^2 + p^2$
25. $y = 2px + p^n$
26. $y = 2px - p^2$
27. $y = x + 2\tan^{-1} p$
28. $y = (1 + p)x + e^p$
29. $y = 3x + \log p$, by considering the equation as one solvable for (i) $y$ and (ii) $x$.
30. $p^3 - 4xyp + 8y^2 = 0$
31. $x = y + p^2$
32. $y = 2px + 4yp^2$
33. $y = 2px + y^2p^3$
34. $y^2 \log y = xyp + p^2$
35. Find the singular solution of the equation $y = (x - 1)p + \tan^{-1} p$.
36. Find the singular solution of the equation $y + \log p = px$.

By suitable transformations, reduce the following equations into Clairaut′s equations and hence solve them:

37. $x^2(y - px) = yp^2$ [Put $X = x^2$, $Y = y^2$]
38. $y = 2px + p^2y$ [Put $X = 2x$, $Y = y^2$]
39. $(y + px)^2 = px^2$ [Put $Y = xy$]
40. $(p - 1)e^{3x} + p^3e^{2y} = 0$ [Put $X = e^x$, $Y = e^y$]

## 5.2 LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

### 5.2.1 Introduction

Students are already familiar with formation and solutions of some types of linear differential equations of second and higher order with constant coefficients. Before we take up the study of remaining types of such equations, we shall recall the various notions and working rules related to the solutions of such equations. This will be of use to pursue the study of not only the remaining types of linear differential equations with constant coeffcients, but also linear differential equations with variable coefficeints and simultaneous differential equations.

The general form of a linear differential equation of the $n^{th}$ order with constant coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X, \qquad (1)$$

where $a_0$ ($\neq 0$), $a_1$, $a_2$, . . . , $a_n$ are constants and $X$ is a function of $x$.

If we use the differential operator symbols $D \equiv \frac{d}{dx}, D^2 \equiv \frac{d^2}{dx^2}, \cdots D^n \equiv \frac{d^n}{dx^n}$,

equation (1) becomes

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)\, y = X \qquad (2)$$

or $f(D)\, y = X$, where $f(D)$ is a polynomial in $D$, which may be treated as an algebraic quantity.

When $X = 0$, (2) becomes

$$f(D)\, y = 0 \qquad (3)$$

(3) is called the homogeneous equation corresponding to equation (2).

*General Solution* of equation (2) is $y = u + v$, where $y = u$ is the general solution of (3), that contains $n$ arbitrary constants and $y = v$ is a particular solution of (2), that contains no arbitrary constants. $u$ is called the *complementary function* (C.F.) and $v$ is called the *particular integral* (P.I.) of the solution of Equation (2).

## 5.3 COMPLEMENTARY FUNCTION

To find the C.F. of the solution of Equation (2), we require the general solution of Equation (3). To get the solution of $f(D)\, y = 0$ or

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_n)\, y = 0 \qquad (3)'$$

we first write down the *auxiliary equation* (A.E.)

$$f(m) = 0 \quad \text{or} \quad a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0 \qquad (4)$$

which is obtained by simply replacing $D$ by $m$ in the operator polynomial, and then by equating it to zero.

The auxiliary equation is an $n^{th}$ degree algebraic equation in $m$.

The solution of Equation (3)′ depends on the nature of roots of the A.E. (4) as explained below.

***Case (i)*** The roots of the A.E. are real and distinct.

Let the roots of the A.E. be $m_1, m_2, \ldots, m_n$.

Then the solution of Equation (3)′ is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots c_n e^{m_n x}$, where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

∴ C.F. of the solution of Equation (2) is given by

$$u = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

***Case (ii)*** The A.E. has got real roots, some of which are equal.

Let the roots of the A.E. be $m_1, m_1, m_3, m_4, \ldots, m_n$ (the first two roots are equal).

Then the solution of Equation (3)′ is

$$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

If three roots of the A.E. are equal, i.e. if $m_l = m_2 = m_3$ (say), then the solution is

$$y = \left(c_1 x^2 + c_2 x + c_3\right) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}.$$

In general, if $r$ roots of the A.E. are equal, then the solution of (3)′ becomes.

$$y = (c_1 x^{r-1} + c_2 x^{r-2} + \cdots c_{r-1}\, x + c_r)\, e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}.$$

***Case (iii)*** Two roots of the A.E. are complex. As complex roots occur in conjugate pairs, let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

Then the solution of Equation (3)′ is

$$y = e^{ax}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

***Case (iv)*** Two pairs of complex roots of the A.E. are equal.

i.e. $m_l = m_3 = \alpha + i\beta$ and $m_2 = m_4 = \alpha - i\beta$.

Then the solution of Equation (3)′ is

$$y = e^{ax}\left[c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x\right] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}.$$

## 5.3.1 Particular Integral

The particular integral (P.I.) of the solution of the equation

$$f(D)\, y = X \tag{1}$$

is the function $v$, where $y = v$ is a particular solution of (1) containing no arbitrary constants. The particular integral depends on the value of the R.H.S. function $X$ and is defined as $\text{P.I.} = \frac{1}{f(D)} X$, where $\frac{1}{f(D)}$ is the inverse operator of $f(D)$.

i.e. $f(D)\left\{\frac{1}{f(D)} X\right\} = X.$

We give below the rules and working procedures to be adopted when $X$ is equal to some special function:

**Rule I** $X = e^{ax}$, where $\alpha$ is a constant.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{f(\alpha)} e^{ax}, \; f(\alpha) \neq 0$$

When $f(\alpha) = 0$, $(D - \alpha)^r$ is a factor of $f(D)$.
Let $f(D) = (D - \alpha)^r \phi(D)$, where $\phi(\alpha) \neq 0$.

Then
$$\text{P.I.} = \frac{1}{(D-\alpha)^r \phi(D)} e^{\alpha x}$$
$$= \frac{1}{\phi(\alpha)}\left[\frac{1}{(D-\alpha)^r} e^{\alpha x}\right]$$
$$= \frac{1}{\phi(\alpha)} \cdot \frac{x^r}{r} e^{\alpha x}.$$

In particular,
$$\frac{1}{D-\alpha} e^{\alpha x} = \frac{x}{1!} e^{\alpha x} \text{ and}$$
$$\frac{1}{(D-\alpha)^2} e^{\alpha x} = \frac{x^2}{2!} e^{\alpha x}.$$

**Rule 2** $X = \sin ax$ or $\cos ax$, where $\alpha$ is a constant. In this case, the following rules are used.

$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax$$
$$\frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax, \text{ if } \phi(-a^2) \neq 0.$$

When $\phi(-\alpha^2) = 0$, $(D^2 + \alpha^2)$ is a factor of $\phi(D^2)$

Let $\phi(D^2) = (D^2 + \alpha^2)\, \psi(D^2)$, where $\psi(-a^2) \neq 0$.

$$\therefore \quad \frac{1}{\phi(D^2)} \sin \alpha x = \frac{1}{\psi(D^2)(D^2 + \alpha^2)} \sin \alpha x.$$
$$= \frac{1}{\psi(-\alpha^2)}\left[\frac{1}{D^2 + \alpha^2} \sin \alpha x\right]$$

Similarly $\frac{1}{\phi(D^2)} \cos \alpha x = \frac{1}{\psi(-\alpha^2)}\left[\frac{1}{D^2 + \alpha^2} \cos \alpha x\right]$

Now
$$\frac{1}{D^2 + \alpha^2} \sin \alpha x = -\frac{x}{2\alpha} \cos \alpha x$$
$$= \frac{x}{2} \times \text{Intergral of } \sin \alpha x.$$

and
$$\frac{1}{D^2 + \alpha^2} \cos \alpha x = \frac{x}{2\alpha} \sin \alpha x$$
$$= \frac{x}{2} \times \text{Intergral } \cos \alpha x.$$

***Note*** ☑ When finding the P.I., the above rules are to be applied in parts, as $f(D)$ will not be, in general, of the form $\phi(D^2)$. This means that we have to first replace $D^2$ by $-\alpha^2$, $D^3$ by $-\alpha^2 D$, $D^4$ by $\alpha^4$, etc. After this has been done, $f(D)$ will take the form $(aD + b)$.

Then $\text{P.I.} = \frac{1}{f(D)}\sin\alpha x$

$= \frac{1}{(aD+b)}\sin\alpha x$

$= \frac{(aD-b)}{a^2D^2-b^2}\sin\alpha x,$

on multiplying the numerator and denominator by $(aD - b)$.

Then $$\text{P.I.} = \frac{1}{-\left(a^2\alpha^2+b^2\right)}\left(aD-b\right)\sin\alpha x,$$

using the rule in the denominator.

$$= -\frac{1}{\left(a^2\alpha^2+b^2\right)}\left(a\alpha\cos\alpha x - b\sin\alpha x\right),\text{ since } D \equiv \frac{d}{dx}\cdot$$

Similarly, $\frac{1}{f(D)}\cos\alpha x = \frac{1}{a^2\alpha^2+b^2}\left(a\alpha\sin\alpha x + b\cos\alpha x\right).$

**Rule 3** $X = x^m$, where $m$ is a positive integer.

$$\text{P.I.} = \frac{1}{f(D)}x^m$$

Rewrite $f(D)$ in terms of a standard binomial expression of the form $[1 \pm \phi(D)]$, by taking out the constant term or the lowest degree term from $f(D)$.

Thus $$\text{P.I.} = \frac{1}{aD^k\left[1\pm\phi(D)\right]}x^m$$

$$= \frac{1}{aD^k}\left[1\pm\phi(D)\right]^{-1}x^m$$

Now expand $[1 \pm \phi(D)]^{-1}$ in a series of ascending powers of $D$, by using binomial theorem, so that the simplified expansion of $\frac{1}{aD^k}\left[1\pm\phi(D)\right]^{-1}$ may contain terms up to $D^m$ and then operate by each term on $x^m$.

**Note** ☑ The ultimate expansion of $\frac{1}{f(D)}$ need not be considered beyond the $D^m$ term, since $D^{m+1}(x^m) = 0$, $D^{m+2}(x^m) = 0$ and so on.

**Rule 4** $X = e^{\alpha x} . V(x)$, where $V$ is of the form $\sin\beta x$, $\cos\beta x$ or $x^m$.

$$\text{P.I.} = \frac{1}{f(D)}e^{\alpha x}V(x) = e^{\alpha x}\cdot\frac{1}{f(D+\alpha)}V(x)$$

$\frac{1}{f(D+\alpha)}V(x)$ is evaluated by using the rule (3) or (4).

**Note** ☑ Thie rule is referred to as *Exponential shift rule*.

**Rule 5** $X = x. V(x)$, where $V(x)$ is of the form $\sin\alpha x$ or $\cos\alpha x$.

$$\text{P.I} = \frac{1}{f(D)} xV(x) = x \cdot \frac{1}{f(D)} V(x) + \frac{d}{dD}\left\{\frac{1}{f(D)}\right\} V(x)$$

or

$$= x \cdot \frac{1}{f(D)} V(x) - \frac{f'(D)}{\{f(D)\}^2} V(x)$$

By repeated applications of this rule, we can find the P.I. when $X = x^r V(x)$, where $r$ is a positive integer.

*Note* ☑ Instead of applying the Rule (5), we may adopt the following alternative procedure to find the P.I.,

when $X = x^r \cos ax$ or $x^r \sin ax$.

$$\frac{1}{f(D)} x^r \cos \alpha x = \frac{1}{f(D)} [\text{Real Part of } x^r e^{i\alpha x}]$$

$$= \text{R.P. of } \frac{1}{f(D)} x^r e^{i\alpha x}$$

$$= \text{R.P. of } e^{i\alpha x} \cdot \frac{1}{f(D+i\alpha)} x^r$$

Similarly,

$$\frac{1}{f(D)} x^r \sin \alpha x = \text{I.P. of } e^{i\alpha x} \cdot \frac{1}{f(D+i\alpha)} x^r \cdot$$

**Rule 6** $X$ is any other function of $x$.

$$\text{P.I} = \frac{1}{f(D)} X$$

$$= \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)} X,$$

resolving $f(D)$ into linear factors.

$$= \left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n}\right) X, \tag{1}$$

spliting into partial fractions.

Consider $\frac{1}{D-m} X = u$, say

$$\therefore \quad (D-m)\left\{\frac{1}{(D-m)} X\right\} = (D-m)u$$

i.e. $(D-m)u = X$ or $\frac{du}{dx} - mu = X.$

This is a linear equation of the first order.

$\therefore$ Its solution is $u\, e^{-mx} = \int e^{-mx} X \, dx$

The usual arbitrary constant is not added in the R.H.S, as $u$ is a part of the P.I. of the main problem.

$$\therefore \quad u = \frac{1}{D-m} X = e^{mx} \cdot \int e^{-mx} X \, \mathrm{d}x \qquad (2)$$

Inserting (2) in (1), the required

$$\text{P.I.} = A_1 e^{m_1 x} \int e^{-m_1 x} X \cdot \mathrm{d}x + A_2 e^{m_2 x} \int e^{-m_2 x} X \, \mathrm{d}x + \cdots + A_n e^{m} n^{x} \cdot \int e^{-m} n^{x} X \, \mathrm{d}x$$

## WORKED EXAMPLE 5(b)

**Example 5.1** Solve the equation $(D^2 - 4D + 3)\,y = \sin 3x + x^2$.

A.E. is $\quad m^2 - 4m + 3 = 0.$

i.e. $\quad (m-1)(m-3) = 0; \quad \therefore m = 1, 3$

$\therefore \quad \text{C.F.} = c_1 e^x + c_2 e^{3x}.$

$$\text{P. I.} = \frac{1}{D^2 - 4D + 3} (\sin 3x + x^2)$$

$$= \frac{1}{D^2 - 4D + 3} (\sin 3x) + \frac{1}{D^2 - 4D + 3} x^2$$

$$= \text{P.I.}_1 + \text{P.I.}_2 \text{ (say)}$$

$$\text{P. I.}_1 = \frac{1}{D^2 - 4D + 3} \sin 3x$$

$$= \frac{1}{-9 - 4D + 3} \sin 3x$$

$$= -\frac{1}{2(2D+3)} \sin 3x$$

$$= -\frac{1}{2} \cdot \frac{(2D-3)}{4D^2 - 9} \sin 3x$$

$$= \frac{1}{90} (2D - 3) \sin 3x$$

$$= \frac{1}{90} (6 \cos 3x - 3 \sin 3x)$$

$$= \frac{1}{30} (2 \cos 3x - \sin 3x)$$

$$\text{P.I.}_2 = \frac{1}{D^2 - 4D + 3} x^2$$

$$= \frac{1}{3\left[1 - \frac{D(4-D)}{3}\right]} x^2$$

$$= \frac{1}{3}\left[1 - \frac{D(4-D)}{3}\right]^{-1} x^2$$

$$= \frac{1}{3}\left[1 + \frac{D(4-D)}{3} + \frac{D^2(4-D)^2}{9}\right]x^2$$

$$= \frac{1}{3}\left(1 + \frac{4}{3}D + \frac{13}{9}D^2\right)x^2$$

$$= \frac{1}{3}\left(x^2 + \frac{8}{3}x + \frac{26}{9}\right)$$

$\therefore$ The general solution of the given equation is $y$ = C.F + P.I.$_1$ + P.I.$_2$

i.e. $$y = c_1 e^x + c_2 e^{3x} + \frac{1}{30}(2\cos 3x - \sin 3x) + \frac{1}{3}\left(x^2 + \frac{8}{3}x + \frac{26}{9}\right)$$

**Example 5.2** Solve $(D^2 + 4)\,y = x^4 + \cos^2 x$.

A.E. is $m^2 + 4 = 0$.
The roots are $m = \pm\, i\, 2$.
$\therefore$ C.F. $= A\cos 2x + B\sin 2x$.

$$\text{P.I.}_1 = \frac{1}{D^2+4}x^4$$

$$= \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1} x^4$$

$$= \frac{1}{4}\left(1 - \frac{D^2}{4} + \frac{D^4}{16}\right)x^4$$

$$= \frac{1}{4}\left(x^4 - 3x^2 + \frac{3}{2}\right)$$

$$\text{P.I.}_2 = \frac{1}{D^2+4}\cos^2 x$$

$$= \frac{1}{D^2+4}\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)$$

$$= \frac{1}{2}\cdot\left[\frac{1}{D^2+4}e^{0.x} + \frac{1}{D^2+4}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{x}{2}\cdot\frac{\sin 2x}{2}\right]$$

$$= \frac{1}{8}(1 + x\sin 2x)$$

∴ General solution of the given equation is

$$y = A\cos 2x + B\sin 2x + \frac{1}{8}\left(4 - 6x^2 + 2x^4 + x\sin 2x\right)$$

**Example 5.3** Solve $(D^3 + 8)y = x^4 + 2x + 1 + \cosh 2x$.

A.E. is $m^3 + 8 = 0$.

i.e. $(m + 2)(m^2 - 2m + 4) = 0$.

$$\therefore \quad m = -2,\ m = \frac{2 \pm \sqrt{4-16}}{2} \quad \text{or} \quad 1 \pm i\sqrt{3}$$

$$\therefore \quad \text{C.F.} = A\,e^{-2x} + e^{x}\,(B\cos\sqrt{3}x + C\sin\sqrt{3}x)$$

$$\text{P.I.}_1 = \frac{1}{D^3+8}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 + \frac{D^3}{8}\right)^{-1}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 - \frac{D^3}{8}\right)(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left[(x^4 + 2x + 1) - 3x\right]$$

$$= \frac{1}{8}(x^4 - x + 1)$$

$$\text{P.I.}_2 = \frac{1}{D^3+8}\left(\frac{e^{2x} + e^{-2x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^3+8}e^{2x} + \frac{1}{(D+2)(D^2 - 2D + 4)}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{16}e^{2x} + \frac{1}{12}\cdot\frac{1}{D+2}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{16}e^{2x} + \frac{1}{12}\cdot\frac{x}{1!}e^{-2x}\right] \qquad \left[\because \frac{1}{D-\alpha}e^{\alpha x} = \frac{x}{1!}e^{\alpha x}\right]$$

$$= \frac{1}{96}(3\,e^{2x} + 4x\,e^{-2x})$$

∴ General solution is $y = \text{C.F.} + \text{P.I.}_1 + \text{P.I.}_2$

**Example 5.4** Solve $(D^4 - 2D^3 + D^2)y = x^2 + e^x$.

A.E. is $m^4 - 2m^3 + m^2 = 0$.

i.e. $m^2(m^2 - 2m + 1) = 0$

∴ The roots are $m = 0, 0, 1, 1$.

$\therefore$ $$\text{C.F.} = (c_1x + c_2) + (c_3x + c_2)e^x.$$

$$\text{P.I.}_1 = \frac{1}{D^2(D-1)^2}x^2 = \frac{1}{D^2}(1-D)^{-2}x^2$$

$$= \frac{1}{D^2}(1 + 2D + 3D^2 + 4D^3 + 5D^4)x^2$$

$$= \left(\frac{1}{D^2} + \frac{2}{D} + 3 + 4D + 5D^2\right)x^2$$

$$= \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 + 8x + 10 \qquad \left(\because \frac{1}{D}x^2 = \int x^2\,\mathrm{d}x\right)$$

$$\text{P.I.}_2 = \frac{1}{D^2(D-1)^2}e^x$$

$$= \frac{1}{1^2}\cdot\frac{1}{(D-1)^2}e^x$$

$$= \frac{x^2}{2!}e^x$$

$\therefore$ General solution is

$$y = (c_1x + c_2) + (c_3x + c_4)e^x + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 + 8x + 10 + \frac{x^2}{2}e^x.$$

***Note*** ☑ The two terms $(8x + 10)$ in $\text{P.I.}_1$ can be merged with the two terms $(c_1x + c_2)$ of the C.F.

$\therefore$ The general solution may be given as

$$y = (c_1x + c_2) + (c_3x + c_4)e^x + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 + \frac{x^2}{2}e^x.$$

**Example 5.5** Solve $(D^2 + 1)^2 y = x^4 + 2\sin x\cos 3x$.

A.E. is $(m^2 + 1)^2 = 0$

The roots are $m = i, i, -i, -i$ (i.e. two pair's of equal complex conjugate roots)

$\therefore$ $$\text{C.F.} = (c_1x + c_2)\cos x + (c_3x + c_4)\sin x.$$

$$\text{P.I.}_1 = \frac{1}{(1+D^2)^2}x^4$$

$$= (1+D^2)^{-2}x^4$$

$$= (1 - 2D^2 + 3D^4)x^4$$

$$= x^4 - 24x^2 + 72.$$

$$\text{P.I.}_2 = \frac{1}{(D^2+1)^2}\,2\sin x\cos 3x.$$

$$= \frac{1}{(D^2+1)^2}(\sin 4x - \sin 2x)$$

$$= \frac{1}{(-16+1)^2}\sin 4x - \frac{1}{(-4+1)^2}\sin 2x$$

$$= \frac{1}{225}\sin 4x - \frac{1}{9}\sin 2x.$$

$\therefore$ General solution of the given equation is $y$ = C.F. + P.I.$_1$ + P.I.$_2$.

**Example 5.6** Solve $(D^2 + 6D + 9)y = e^{-2x} x^3$

A.E. is $\quad m^2 + 6m + 9 = 0$

i.e. $\quad (m + 3)^2 = 0$

The roots are $m = -3, -3$

$\therefore \quad$ C.F. $= (c_1 x + c_2)\, e^{-3x}$

$$\text{P.I.} = \frac{1}{(D+3)^2} e^{-2x} x^3$$

$$= e^{-2x} \cdot \frac{1}{(D-2+3)^2} x^3, \text{ by Exponential shift rule}$$

$$= e^{-2x}(1+D)^{-2} x^3$$

$$= e^{-2x}(1-2D+3D^2-4D^3)x^3$$

$$= e^{-2x}(x^3-6x^2+18x-24)$$

$\therefore$ General solution is $y$ = C.F. + P.I.

**Example 5.7** Solve $(D^3 - 3D^2 + 3D - 1)\, y = e^{-x}x^3$.

A.E. is $\quad m^3 - 3m^2 + 3m - 1 = 0$

i.e. $\quad (m - 1)^3 = 0$

The roots are $m = 1, 1, 1$

$\therefore \quad$ C.F. $= (c_1 x^2 + c_2 x + c_3)\, e^x$

$$\text{P.I.} = \frac{1}{(D-1)^3} e^{-x} x^3 \cdot$$

$$= e^{-x} . \frac{1}{(D-2)^3} x^3$$

$$= -\frac{1}{8} e^{-x} \cdot \left(1 - \frac{D}{2}\right)^{-3} x^3$$

$$= -\frac{1}{8} e^{-x} \cdot \frac{1}{1\cdot 2}\left(1\cdot 2 + 2\cdot 3 + \frac{D}{2} + 3\cdot 4\cdot\frac{D^2}{4} + 4\cdot 5\cdot\frac{D^3}{8}\right)x^3$$

$$= -\frac{1}{16} e^{-x}\left(2 + 3D + 3D^2 + \frac{5}{2}D^3\right)x^3$$

$$= -\frac{1}{16} e^{-x}(2x^3 + 9x^2 + 18x + 15)$$

$\therefore$ the general solution is $y$ = C.F. + P.I.

**Example 5.8** Solve $(D^2 - 4)\, y = x^2 \cosh 2x$.

A.E. is $m^2 - 4 = 0$

i.e. $(m + 2)(m - 2) = 0$

$\therefore$ The roots are $m = -2, 2$

$\therefore$ C.F. $= A\, e^{-2x} + B\, e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4} x^2 \cosh 2x$$

$$= \frac{1}{D^2 - 4} \frac{x^2}{2}\left(e^{2x} + e^{-2x}\right)$$

$$= \frac{1}{2} e^{2x} \cdot \frac{1}{(D+2)^2 - 4} x^2 + \frac{1}{2} e^{-2x} \cdot \frac{1}{(D-2)^2 - 4} x^2$$

$$= \frac{1}{2} e^{2x} \cdot \frac{1}{D^2 + 4D} x^2 + \frac{1}{2} e^{-2x} \cdot \frac{1}{D^2 - 4D} x^2$$

$$= \frac{1}{2} e^{2x} \cdot \frac{1}{4D}\left(1 + \frac{D}{4}\right)^{-1} x^2 + \frac{1}{2} e^{-2x} \cdot \frac{1}{(-4D)}\left(1 - \frac{D}{4}\right)^{-1} x^2$$

$$= \frac{1}{8} e^{2x} \cdot \frac{1}{D}\left(1 - \frac{D}{4} + \frac{D^2}{16} - \frac{D^3}{64}\right) x^2 - \frac{1}{8} e^{-2x}$$

$$\times \frac{1}{D}\left(1 + \frac{D}{4} + \frac{D^2}{16} + \frac{D^3}{64}\right) x^2$$

$$= \frac{1}{8} e^{2x}\left(\frac{x^3}{3} - \frac{x^2}{4} + \frac{x}{8} - \frac{1}{32}\right)$$

$$- \frac{1}{8} e^{-2x}\left(\frac{x^3}{3} + \frac{x^2}{4} + \frac{x}{8} + \frac{1}{32}\right)$$

$$= \frac{x^3}{12} \sinh 2x - \frac{x^2}{16} \cosh 2x + \frac{x}{32} \sinh 2x$$

(the terms $-\frac{1}{256} e^{2x}$ and $-\frac{1}{256} e^{-2x}$ are omitted, as they may be considered to have been included in the C.F.)

Then the G.S. is

$$y = A\, e^{-2x} + B\, e^{2x} + \frac{x}{96}\left(8x^2 \sinh 2x - 6x \cosh 2x + 3 \sinh 2x\right)$$

**Example 5.9** Solve $(D^4 - 2D^2 + 1)y = (x + 1)e^{2x}$.

A.E. is $m^4 - 2m^2 + 1 = 0$

i.e. $(m^2 - 1)^2 = 0$

$\therefore$ The roots are $m = \pm 1, \pm 1$

$\therefore$ C.F. $= (c_1 x + c_2)\, e^{x} + (c_3 x + c_4)\, e^{-x}$.

$$\text{P.I.} = \frac{1}{\left(D^2-1\right)^2} e^{2x}(x+1) = e^{2x} \cdot \frac{1}{\left\{(D+2)^2-1\right\}^2}(x+1)$$

$$= e^{2x} \cdot \frac{1}{\left\{D^2+4D+3\right\}^2}(x+1)$$

$$= e^{2x} \cdot \frac{1}{9}\left\{1+\frac{D(D+4)}{3}\right\}^{-2}(x+1)$$

$$= \frac{1}{9} e^{2x} \cdot \left\{1-\frac{2D}{3}(D+4)\right\}(x+1)$$

$$= \frac{1}{9} e^{2x}\left(1-\frac{8}{3}D\right)(x+1)$$

$$= \frac{1}{9} e^{2x}\left(x-\frac{5}{3}\right)$$

Then the general solution is $y$ = C.F. + P.I.

**Example 5.10** Solve $(D^2 + 2D - 1)y = (x + e^x)^2$.

A.E. is $m^2 + 2m - 1 = 0$.

i.e. $(m + 1)^2 = 2$

$\therefore\ m = -1 \pm \sqrt{2}$ (real roots)

$$\therefore \quad \text{C.F.} = A\, e^{\left(-1+\sqrt{2}\right)x} + B\, e^{\left(-1-\sqrt{2}\right)x} = e^{-x}\left(A\, e^{\sqrt{2}x} + B\, e^{-\sqrt{2}x}\right)$$

$$\text{P.I.} = \frac{1}{D^2+2D-1}\left(x+e^x\right)^2$$

$$= \frac{1}{D^2+2D-1}\left(x^2+e^{2x}+2xe^x\right)$$

$$= \text{P.I.}_1 + \text{P.I.}_2 + \text{P.I.}_3 \text{ (say)}$$

$$\text{P.I.}_1 = \frac{1}{-\left\{1-D(D+2)\right\}} x^2 = -\left\{1+D(D+2)+D^2(D+2)^2\right\}x^2$$

$$= -\left\{1+2D+5D^2\right\}x^2$$

$$= -\left(x^2+4x+10\right).$$

$$\text{P.I.}_2 = \frac{1}{D^2+2D-1} e^{2x} = \frac{1}{7} e^{2x}$$

$$\text{P.I}_3 = \frac{1}{D^2 + 2D - 1} 2xe^x$$
$$= 2e^x \cdot \frac{1}{(D+1)^2 + 2(D+1) - 1} x$$
$$= 2e^x \cdot \frac{1}{D^2 + 4D + 2} x$$
$$= 2e^x \cdot \frac{1}{2}\left\{1 + \frac{D(D+4)}{2}\right\}^{-1} x$$
$$= e^x \cdot \{1 + 2D\} x$$
$$= e^x \cdot (x+2)$$

Then the general solution is $y$ = C.F. + $\text{P.I.}_1$ + $\text{P.I.}_2$ + $\text{P.I.}_3$.

**Example 5.11** Solve $(D^2 + 5D + 4)\, y = e^{-x} \sin 2x$.

A.E. is $m^2 + 5m + 4 = 0$.

i.e. $(m + 1)(m + 4) = 0$.

∴ The roots are $m = -1, -4$

∴ $\text{C.F.} = Ae^{-x} + B\, e^{-4x}$.

$$\text{P.I.} = \frac{1}{D^2 + 5D + 4} e^{-x} \sin 2x$$
$$= e^{-x} \cdot \frac{1}{(D-1)^2 + 5(D-1) + 4} \sin 2x$$
$$= e^{-x} \cdot \frac{1}{D^2 + 3D} \sin 2x$$
$$= e^{-x} \cdot \frac{1}{3D - 4} \sin 2x$$
$$= e^{-x} \cdot \frac{(3D+4)}{9D^2 - 16} \sin 2x$$
$$= -\frac{1}{52} e^{-x} (6 \cos 2x + 4 \sin 2x)$$
$$= -\frac{1}{26} e^{-x} (3 \cos 2x + 2 \sin 2x)$$

Then the general solution is $y$ = C.F. + P.I.

**Example 5.12** Solve $(D^4 - 1)y = \cos 2x \cosh x$.

A.E. is $m^4 - 1 = 0$

i.e. $(m - 1)(m + 1)(m^2 + 1) = 0$

∴ The roots are $m = 1, -1, \pm i$

∴ $\text{C.F.} = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.

$$\text{P.I.} = \frac{1}{D^4-1}\cos 2x\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{1}{2}e^x \cdot \frac{1}{(D+1)^4-1}\cos 2x + \frac{1}{2}e^{-x}\cdot\frac{1}{(D-1)^4-1}\cos 2x$$

$$= \frac{1}{2}e^x \cdot \frac{1}{D^4+4D^3+6D^2+4D}\cos 2x$$

$$+ \frac{1}{2}e^{-x} \cdot \frac{1}{D^4-4D^3+6D^2-4D}\cos 2x$$

$$= \frac{1}{2}e^x \cdot \frac{1}{16-16D-24+4D}\cos 2x + \frac{1}{2}e^{-x}\cdot\frac{1}{16+16D-24-4D}\cos 2x$$

$$= -\frac{1}{8}e^x \cdot \frac{1}{3D+2}\cos 2x + \frac{1}{8}e^{-x}\cdot\frac{1}{3D-2}\cos 2x$$

$$= -\frac{1}{8}e^x \cdot \frac{(3D-2)}{9D^2-4}\cos 2x + \frac{1}{8}\cdot e^{-x}\cdot\frac{(3D+2)}{9D^2-4}\cos 2x$$

$$= \frac{1}{320}e^x(-6\sin 2x - 2\cos 2x) - \frac{1}{320}e^{-x}(-6\sin 2x + 2\cos 2x)$$

$$= -\frac{3}{80}\sin 2x\left(\frac{e^x-e^{-x}}{2}\right) - \frac{1}{80}\cos 2x\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= -\frac{1}{80}(3\sin 2x\sinh x + \cos 2x\cosh x)$$

∴ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

**Example 5.13** Solve $(D^2 - 4D + 13)\ y = e^{2x}\cos 3x$.

A.E. is $m^2 - 4m + 13 = 0$

i.e. $(m-2)^2 = -9$

∴ The roots are $m = 2 \pm i3$

∴ $\text{C.F.} = e^{2x}(A\cos 3x + B\sin 3x)$

$$\text{P.I.} = \frac{1}{D^2-4D+13}e^{2x}\cdot\cos 3x$$

$$= e^{2x}\cdot\frac{1}{(D+2)^2-4(D+2)+13}\cos 3x$$

$$= e^{2x}\cdot\frac{1}{D^2+9}\cos 3x = e^{2x}\cdot\frac{x}{2}\cdot\frac{\sin 3x}{3}$$

$$= \frac{1}{6}x\,e^{2x}\sin 3x$$

$\therefore$ The general solution is $y$ = C.F. + P.I.

**Example 5.14** Solve $(D^2 + D + 1)\,y = e^{-x}\sin^2\dfrac{x}{2}$.

A.E. is $m^2 + m + 1 = 0$

The roots are $m = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

$$\text{C.F.} = e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^2+D+1}e^{-x}\left(\frac{1-\cos x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+D+1}e^{-x} - \frac{1}{D^2+D+1}e^{-x}\cos x\right]$$

$$= \frac{1}{2}e^{-x} - \frac{1}{2}e^{-x}\cdot\frac{1}{(D-1)^2+(D-1)+1}\cos x$$

$$= \frac{1}{2}e^{-x} - \frac{1}{2}e^{-x}\cdot\frac{1}{D^2-D+1}\cos x$$

$$= \frac{1}{2}e^{-x} - \frac{1}{2}e^{-x}\cdot\left(\frac{1}{-D}\cos x\right)$$

$$= \frac{1}{2}e^{-x}(1+\sin x)$$

$\therefore$ General solution is $y$ = C.F. + P.I.

**Example 5.15** Solve $(D^2 + 2D + 5)\,y = e^x\cos^3 x$.

A.E. is $m^2 + 2m + 5 = 0$

The roots are $m = -1 \pm i\,2$

$$\text{C.F.} = e^{-x}(A\cos 2x + B\sin 2x).$$

$$\text{P.I.} = \frac{1}{D^2+2D+5}e^x\cos^3 x$$

$$= \frac{1}{D^2+2D+5}e^x\left(\frac{3}{4}\cos x + \frac{1}{4}\cos 3x\right)$$

$$= \frac{3}{4}e^x\cdot\frac{1}{(D+1)^2+2(D+1)+5}\cos x$$

$$+ \frac{1}{4}e^x\cdot\frac{1}{(D+1)^2+2(D+1)+5}\cos 3x$$

$$= \frac{3}{4}e^x\cdot\frac{1}{D^2+4D+8}\cos x + \frac{1}{4}e^x\cdot\frac{1}{D^2+4D+8}\cos 3x$$

$$= \frac{3}{4} e^x \frac{1}{4D+7} \cos x + \frac{1}{4} e^x \cdot \frac{1}{4D-1} \cos 3x$$

$$= \frac{3}{4} e^x \cdot \frac{(4D-7)}{16D^2-49} \cos x + \frac{1}{4} e^x \cdot \frac{(4D+1)}{16D^2-1} \cos 3x$$

$$= \frac{3}{260} e^x (4 \sin x + 7 \cos x) + \frac{1}{580} e^x (12 \sin 3x - \cos 3x)$$

∴ The general solution is $y$ = C.F. + P.I.

**Example 5.16** Solve $(D^2 + 4D + 8)\, y = 12e^{-2x} \sin x \sin 2x$.

A.E. is $m^2 + 4m + 8 = 0$.

The roots are $m = -2 \pm i\,2$

∴ $$\text{C.F.} = e^{-2x} (A \cos 2x + B \sin 2x)$$

$$\text{P.I.} = \frac{1}{D^2+4D+8} 6e^{-2x} (\cos x - \cos 3x)$$

$$= 6e^{-2x} \cdot \frac{1}{(D-2)^2 + 4(D-2) + 8} (\cos x - \cos 3x)$$

$$= 6e^{-2x} \cdot \frac{1}{D^2+4} (\cos x - \cos 3x)$$

$$= 6e^{-2x} \left\{ \frac{1}{3} \cos x + \frac{1}{5} \cos 3x \right\}$$

$$= \frac{2}{5} e^{-2x} (5 \cos x + 3 \cos 3x)$$

∴ The general solution is $y$ = C.F. + P.I.

**Example 5.17** Solve $(D^3 - 1)\, y = x \sin x$.

A.E. is $m^3 - 1 = 0$

i.e. $(m - 1)(m^2 + m + 1) = 0$

∴ The roots are $m = 1, -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$

∴ $$\text{C.F.} = c_1 e^x + e^{-x/2} \left( c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right)$$

$$\text{P.I.} = \frac{1}{D^3-1} x \sin x$$

$$= x \cdot \frac{1}{D^3-1} \sin x - \frac{3D^2}{(D^3-1)^2} \sin x \quad \left[ \because \frac{1}{f(D)} xV = x \cdot \frac{1}{f(D)} V - \frac{f'(D)}{\{f(D)\}^2} V \right]$$

$$= -x \cdot \frac{1}{D+1} \sin x + \frac{3}{(D+1)^2} \sin x$$

$$= -x \cdot \frac{(D-1)}{D^2-1} \sin x + \frac{3}{D^2+2D+1} \sin x$$

$$= \frac{x}{2} (\cos x - \sin x) - \frac{3}{2} \cos x.$$

$\therefore$ The general solution is $y$ = C.F. + P.I.

***Alternative method for finding P.I.***

$$\text{P.I.} = \frac{1}{D^3-1} x \sin x$$

$$= \frac{1}{D^3-1} (\text{Imaginary part of } x\, e^{ix})$$

$$= \text{I.P. of } \frac{1}{D^3-1} x\, e^{ix}$$

$$= \text{I.P. of } e^{ix} \cdot \frac{1}{(D+i)^3-1} x$$

$$= \text{I.P. of } e^{ix} \cdot \frac{1}{D^3+3iD^2-3D-i-1} x$$

$$= \text{I.P. of } -\frac{e^{ix}}{1+i} \left\{1 - \frac{D}{1+i}(-3+3iD+D^2)\right\}^{-1} x$$

$$= \text{I.P. of } -\frac{e^{ix}}{1+i} \left(1 - \frac{3D}{1+i}\right) x$$

$$= \text{I.P. of } -\frac{e^{ix}}{1+i} \left(x - \frac{3}{1+i}\right)$$

$$= \text{I.P. of } -\frac{(1-i)}{2} (\cos x + i \sin x) \left\{x - \frac{3}{2}(1-i)\right\}$$

$$= \text{I.P. of } -\frac{1}{2} \{(\cos x + \sin x) + i(\sin x - \cos x)\} \left\{x - \frac{3}{2}(1-i)\right\}$$

$$= -\frac{1}{2}\left[\frac{3}{2}(\cos x + \sin x) + \left(x - \frac{3}{2}\right)(\sin x - \cos x)\right]$$

$$= -\frac{3}{2} \cos x + \frac{x}{2} (\cos x - \sin x)$$

**Example 5.18** Solve the equation $(D^2 + 4)\, y = x^2 \cos 2x$.

A.E. is $m^2 + 4 = 0$

The roots are $m = \pm i\, 2$

$\therefore$ C.F. $= A \cos 2x + B \sin 2x$.

$$\text{P.I.} = \frac{1}{D^2+4} \text{R.P. of } x^2\, e^{i2x}$$

$$= \text{R.P. of } e^{i2x} \cdot \frac{1}{(D+i2)^2+4} x^2$$

$$= \text{R.P. of } e^{i2x} \cdot \frac{1}{D^2+4iD} x^2$$

$$= \text{R.P. of } \frac{e^{i2x}}{4iD}\left(1-\frac{iD}{4}\right)^{-1} x^2$$

$$= \text{R.P. of } \frac{e^{i2x}}{4iD}\left(1+\frac{iD}{4}-\frac{D^2}{16}-\frac{iD^3}{64}\right) x^2$$

$$= \text{R.P. of } -\frac{i}{4} e^{i2x}\left(\frac{x^3}{3}+\frac{i}{4}x^2-\frac{x}{8}-\frac{i}{32}\right)$$

$$= \text{R.P. of } \frac{1}{4}(\sin 2x - i\cos 2x)\left\{\left(\frac{x^3}{3}-\frac{x}{8}\right)+\frac{i}{4}\left(x^2-\frac{1}{8}\right)\right\}$$

$$= \frac{1}{4}\left\{\left(\frac{x^3}{3}-\frac{x}{8}\right)\sin 2x+\frac{1}{4}\left(x^2-\frac{1}{8}\right)\cos 2x\right\}$$

∴ General solution i s $y$ = C.F. + P.I.

**Example 5.19** Solve $(D^2 - 4D + 4)\, y = 8x^2\, e^{2x} \sin 2x$.

A.E. is $m^2 - 4m + 4 = 0$

i.e. $(m-2)^2 = 0$

∴ Roots are $m = 2, 2$.

∴ C.F. $= (c_1 x + c_2)\, e^{2x}$.

$$\text{P.I.} = \frac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x \;\; = 8\, e^{2x} \cdot \frac{1}{D^2} x^2 \sin 2x$$

$$= 8\, e^{2x} \cdot \frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - 2x\left(\frac{-\sin 2x}{4}\right) + 2\,\frac{\cos 2x}{8}\right\}$$

(by applying Bernouilli's formula)

$$= e^{2x} \cdot \left[\frac{1}{D}(-4x^2\cos 2x) + \frac{1}{D}(4x\sin 2x) + \frac{1}{D}(2\cos 2x)\right]$$

$$= e^{2x}\left[-4\left\{x^2\left(\frac{\sin 2x}{2}\right) - 2x\left(\frac{-\cos 2x}{4}\right) + 2\left(\frac{-\sin 2x}{8}\right)\right\}\right.$$

$$\left. + 4\left\{x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right\} + \sin 2x\right]$$

$$= e^{2x}\left[(3-2x^2)\sin 2x - 4x\cos 2x\right]$$

∴ The general solution is $y$ = C.F. + P.I.

**Example 5.20** Solve $(D^2 + a^2)\, y = \sec a\, x$.

A.E. is $m^2 + a^2 = 0$

The roots are $m = \pm\, ia$

$\therefore$ C.F. = $A \cos a\, x + B \sin a\, x$.

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec a\, x$$

$$= \frac{1}{(D - ia)(D + ia)} \sec a\, x$$

$$= \left( \frac{\frac{1}{2ia}}{D - ia} - \frac{\frac{1}{2ia}}{D + ia} \right) \sec a\, x$$

$$= \frac{1}{2ia} \cdot e^{ia\,x} \int e^{-ia\,x} \sec a x\, \mathrm{d}x - \frac{1}{2ia} e^{-ia\,x} \int e^{ia\,x} \sec a x\, \mathrm{d}x$$

$$\left[ \because \frac{1}{D - m} X = e^{mx} . \int X e^{-mx}\, \mathrm{d}x \right]$$

$$= \frac{1}{2ia} e^{ia\,x} \int (1 - i \tan a\, x)\, \mathrm{d}\, x - \frac{1}{2ia} e^{-ia\,x} \int (1 + i \tan a\, x)\, \mathrm{d}\, x$$

$$= \frac{1}{2ia} e^{ia\,x} \left( x - \frac{i}{a} \log \sec a\, x \right) - \frac{1}{2ia} e^{-ia\,x} \left( x + \frac{i}{a} \log \sec a\, x \right)$$

$$= \frac{x}{a} \left( \frac{e^{iax} - e^{-ia\,x}}{2i} \right) - \frac{1}{a^2} \log \sec a\, x \left( \frac{e^{ia\,x} + e^{-ia\,x}}{2} \right)$$

$$= \frac{x}{a} \sin a\, x - \frac{1}{a^2} \cos a\, x \log \sec a\, x$$

$\therefore$ General solution is $y$ = C.F. + P.I.

## EXERCISE 5(b)

***Part A***

(Short Answer Questions)

1. Solve the equation $(D^2 - D + 1)^2 y = 0$.
2. Find the particular integral of $(D - 1)^3\, y = 2 \cosh x$.
3. Find the particular integral of $(D^2 + a^2)\, y = b \cos ax + c \sin ax$.
4. Find the particular integral of $(D^2 + 4D + 4)\, y = x\, e^{-2x}$.
5. Find the particular integral of $(D - 3)^2 y = x\, e^{-2x}$.
6. Find the particular integral of $(D + 1)^2 y = e^{-x} \cos x$.
7. Find the particular integral of $(D^2 - 2D + 5)\, y = e^{\,x} \sin 2x$.
8. Find the particular integral of $(D^2 + 4D + 5)\, y = e^{-2x} \cos x$.

9. Find the particular integral of $(D^2 - 2D + 6)y = e^{x}(4 \sin x + \cos x)$.
10. Find a formula for $\dfrac{1}{D-a} f(x)$.

***Part B***

Solve the following differential equations.

11. $(D^3 + D^2 + D + 1)y = x^2 + 2e^{-x}$
12. $(D^2 + 9)y = x^2 + \cosh x$
13. $(D^2 + 2D + 1)y = x^3 + \cos 2x$
14. $(D^2 - 8D + 9)y = 8 \sin 5x + x^2$
15. $(D^2 + 3D + 2)y = 2 \sin^2 x + 2x^2$
16. $(D^4 + D^3 + D^2)y = 12x^2 + 2 \cos 2x \cos x$
17. $(D^2 - 1)y = 12e^x (x + 1)^2$
18. $(D^3 - 6D^2 + 12D - 8)y = 16x^3 e^{4x}$
19. $(D^3 + 2D^2 + D)y = x^2 e^{2x}$
20. $(D^2 - 4)\, y = x \sinh x$
21. $(\mathrm{D}^2 + 1)^2 y = 2x^2 e^{-x}$
22. $(D^2 - 5D + 4)y = (2x + e^{-x})^2$
23. $(D^2 - 4D + 3)y = 8e^x \cos 2x$
24. $(D^4 - 1)y = \cos x \cosh x$
25. $(D^2 - 2D + 5)y = e^x (\sin x + \cos x)^2$
26. $(D^3 + 1)y = e^{-x} \cos^2 \dfrac{x}{2}$
27. $(D^2 + 4)y = 4\, e^{2x} \sin^3 x$
28. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$
29. $(D^2 - 2D + 1)y = x\, e^x \sin x$
30. $(D^2 + D)y = x \cos x$
31. $(D^2 - 4D + 4)y = x \sin x$
32. $(D^2 - 1)y = x^2 \cos x$
33. $(D^2 + 4)^2\, y = \cos 2x$
34. $(D^2 + 1)y = x^2 \sin 2x$
35. $(D^2 + 4)y = 4 \tan 2x$

## 5.4 EULER′S HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

The equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \qquad (1)$$

where $a_0, a_1, \ldots, a_n$ are constants and $X$ is a function of $x$ is called Euler′s homogeneous linear differential equation.

*Note* ☑ In each term in the L.H.S. of Equation (1), the power of $x$ and the order of the derivative are equal. It is because of this property that the equation is called a homogeneous equation.

Equation (1) can be reduced to a linear differential equation with constant coefficients by changing the independent variable from $x$ to $t$ by means of the transformation.

$$x = e^t \quad \text{or} \quad t = \log x$$

as explained below:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\therefore \qquad x\frac{dy}{dx} = \frac{dy}{dt} \qquad (2)$$

Differentiating both sides of (2) w.r.t. $x$,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dt^2}\frac{1}{x}$$

i.e.
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dt^2}$$

i.e.
$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}\left[\text{by}(2)\right] \qquad (3)$$

Denoting $\frac{d}{dx}$ by $D$ and $\frac{d}{dt}$ by $\theta$,

(2) gives $xD = \theta$ and

(3) gives $x^2 D^2 = \theta^2 - \theta = \theta(\theta - 1)$

Similarly we can show that

$$x^3 D^3 = \theta(\theta - 1)(\theta - 2)$$

$$x^4 D^4 = \theta(\theta - 1)(\theta - 2)(\theta - 3) \text{ and so on.}$$

If this transformation is made, then Eqn. (1) becomes $\left[a_0\, \theta(\theta-1)\cdots\left(\theta-\overline{n-1}\right)+ a_1\, \theta(\theta-1)\cdots\left(\theta-\overline{n-2}\right)+\cdots a_n\right] y = 0$, which is a linear differential equation with constant coefficients and can be solved by methods discussed in the previous section.

The more general form of Euler′s homogeneous equation is

$$a_0(ax+b)^n\frac{d^ny}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots$$

$$+ a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = X \qquad (2)$$

Equation (2) can be reduced to a linear differential equation with constant coefficients by the substitution $ax + b = e^t$.

Equation (2) is called *Legendre's linear differential equation*.

## 5.5 SIMULTANEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

If $x$ and $y$ are two dependent variables and $t$ is the independent variable, then the pair of equations of the form

$$f_1(D)x + f_2(D)y = T_1 \quad (1)$$

$$\phi_1(D)x + \phi_2(D)y = T_2 \quad (2)$$

where $f_1, f_2, \phi_1, \phi_2$, are polynomials in the operator $D \equiv \frac{d}{dt}$ and $T_1$ and $T_2$ are functions of $t$ is called a pair of simultaneous differential equations.

It is not possible to solve for the two dependent variables (unknowns), if only one of the above equations is given.

If there are more than 2 dependent variables, we should have as many equations as the number of dependent variables.

To solve Equations (1) and (2) simultaneously, we proceed as in solving simultaneous algebraic equations.

We operate both sides of (1 ) by $\phi_2(D)$ and both sides of (2) by $f_2(D)$ and subtract to eliminate $y$.

Thus we get

$$\left[f_1(D)\phi_2(D) - f_2(D)\phi_1(D)\right]x = \phi_2(D)T_1 - f_2(D)T_2 \quad \text{or} \quad f(D)x = T \quad (3)$$

which is a linear equation in $x$ and $t$ with constant coefficients and can be solved by the methods discussed already.

The value of $x$ obtained by the solution of (3) is substituted either in (1) or (2) to get the value of $y$.

The number of arbitrary constants that appear in the values of $x$ and $y$ should be equal to the order of the resultant equation (3).

If more arbitrary constants are introduced in the process of solving the equations, the extra ones should be expressed in terms of the other constants.

***Note*** ☑ We can also eliminate $x$, get a linear equation in $y$ and $t$ and solve it first.

**WORKED EXAMPLE 5(c)**

**Example 5.1** Solve the equation $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^2 + \frac{1}{x^2}$.

The given equation is $\left(x^2D^2 + 4xD + 2\right)y = x^2 + \frac{1}{x^2}$, where $D \equiv \frac{d}{dx}$

Put $x = e^t$ or $t = \log x$ and denote $\frac{d}{dt}$ by $\theta$.

Then the given equation becomes

$$\left[\theta(\theta - 1) + 4\theta + 2\right]y = e^{2t} + e^{-2t}$$

i.e.

$$\left(\theta^2 + 3\theta + 2\right)y = e^{2t} + e^{-2t}$$

A.E. is $m^2 + 3m + 2 = 0$

i.e. $(m + 1)(m + 2) = 0$

$\therefore$ The roots are $m = -1, -2$.

$$\therefore \quad \text{C.F.} = A\,e^{-t} + B\,e^{-2t} = \frac{A}{x} + \frac{B}{x^2}$$

$$\text{P.I.} = \frac{1}{(\theta+1)(\theta+2)}(e^{2t} + e^{-2t})$$

$$= \frac{1}{12}e^{2t} - \frac{1}{\theta+2}e^{-2t}$$

$$= \frac{1}{12}e^{2t} - t\,e^{-2t} = \frac{1}{12}x^2 - \frac{1}{x^2}\log x$$

$\therefore$ The general solution is $y$ = C.F. + P.I.

**Example 5.2** Solve $(x^2 D^2 + xD + 1)\,y = \sin(2\log x)\cdot\sin(\log x)$.

Putting $x = e^t$ or $t = \log x$ and denoting $\frac{d}{dt}$ by $\theta$, the given equation becomes

$$[\theta(\theta-1) + \theta + 1]\,y = \sin 2t \sin t$$

i.e. $$(\theta^2 + 1)\,y = \frac{1}{2}(\sin 3t + \sin t)$$

A.E. is $m^2 + 1 = 0$.

The roots are $m = \pm i$

$\therefore$ C.F. $= A\cos t + B\sin t = A\cos(\log x) + B\sin(\log x)$.

$$\text{P.I.} = \frac{1}{\theta^2+1}\,\frac{1}{2}(\sin 3t + \sin t)$$

$$= \frac{1}{2}\left\{-\frac{1}{8}\sin 3t + \frac{t}{2}(-\cos t)\right\}$$

$$= -\frac{1}{16}\sin(3\log x) - \frac{1}{4}\log x\cos(\log x)$$

$\therefore$ General solution is $y$ = C.F.+ P.I.

**Example 5.3** Solve $(x^2 D^2 - 2\,xD - 4)\,y = 32\,(\log x)^2$

Putting $x = e^t$ or $t = \log x$ and denoting $\frac{d}{dt}$ by $\theta$, the given equation becomes

$$[\theta(\theta-1) - 2\theta - 4]\,y = 32\,t^2$$

i.e. $$(\theta^2 - 3\theta - 4)\,y = 32t^2$$

A.E. is $$m^2 - 3m - 4 = 0$$

i.e. $$(m-4)(m+1) = 0$$

$\therefore$ The roots are $m = 4, -1$.

$$\therefore \quad \text{C.F.} = A\,e^{4t} + B\,e^{-t}$$

$$= Ax^4 + \frac{B}{x}$$

$$\text{P.I.} = \frac{1}{\theta^2 - 3\theta - 4} 32t^2$$

$$= -8 \cdot \left\{1 - \frac{\theta}{4}(\theta - 3)\right\}^{-1} t^2$$

$$= -8\left[1 + \frac{\theta}{4}(\theta - 3) + \frac{\theta^2}{16}(\theta - 3)^2\right] t^2$$

$$= -8\left[1 - \frac{3\theta}{4} + \frac{13}{16}\theta^2\right] t^2$$

$$= -8\left(t^2 - \frac{3}{2}t + \frac{13}{8}\right)$$

$$= -[8(\log x)^2 - 12(\log x) + 13]$$

∴ General solution is $y$ = C.F. + P.I.

**Example 5.4** Solve$(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x.}\right)^2$.

Putting $x = e^t$ or $t = \log x$ and denoting $\frac{d}{dt}$ by $\theta$, the given equation becomes

$$[\theta(\theta - 1) - \theta + 1)]\, y = t^2 e^{-2t}$$

i.e. $(\theta^2 - 2\theta + 1)\, y = t^2 e^{-2t}$ .

A.E. is $m^2 - 2m + 1 = 0$.

i.e. $(m - 1)^2 = 0$

∴ The roots are $m = 1, 1$.

∴ $\text{C.F.} = (At + B)e^t = (A\log x + B)\cdot\frac{1}{x}$

$$\text{P.I.} = \frac{1}{(\theta - 1)^2} e^{-2t} \cdot t^2$$

$$= e^{-2t} \cdot \frac{1}{(\theta - 3)^2} t^2$$

$$= e^{-2t} \cdot \frac{1}{9}\left(1 - \frac{\theta}{3}\right)^{-2} t^2$$

$$= \frac{1}{9} e^{-2t}\left(1 + \frac{2\theta}{3} + 3\cdot\frac{\theta^2}{9}\right) t^2$$

$$= \frac{1}{9} e^{-2t}\left(t^2 + \frac{4}{3}t + \frac{2}{3}\right)$$

$$= \frac{1}{27x^2}\left\{3(\log x)^2 + 4\log x + 2\right\}$$

$\therefore$ General solution is $y$ = C.F. + P.I.

**Example 5.5** Solve $(x^2D^2 - xD + 4)\, y = x^2 \sin(\log x)$.

Putting $x = e^t$ or $t = \log x$ and denoting $\frac{d}{dt}$ by $\theta$, the given equation becomes

$$\left[\theta(\theta-1)-\theta+4\right]y = e^{2t}\sin t$$

i.e. $$\left(\theta^2 - 2\theta + 4\right)y = e^{2t}\sin t$$

A.E. is $$m^2 - 2m + 4 = 0$$

The roots are $m = 1 \pm i\sqrt{3}$

$$\therefore \quad \text{C.F.} = e^t\left(A\cos\sqrt{3}\,t + B\sin\sqrt{3}\,t\right)$$

$$= x\left\{A\cos\left(\sqrt{3}\log x\right) + B\sin\left(\sqrt{3}\log x\right)\right\}$$

$$\text{P.I.} = \frac{1}{\theta^2 - 2\theta + 4}e^{2t}\sin t$$

$$= e^{2t}\cdot\frac{1}{(\theta+2)^2 - 2(\theta+2) + 4}\sin t$$

$$= e^{2t}\cdot\frac{1}{\theta^2 + 2\theta + 4}\sin t$$

$$= e^{2t}\cdot\frac{1}{2\theta + 3}\sin t$$

$$= e^{2t}\cdot\frac{2\theta - 3}{4\theta^2 - 9}\sin t$$

$$= -\frac{1}{13}e^{2t}\left(2\cos t - 3\sin t\right)$$

$$= -\frac{1}{13}x^2\left\{2\cos(\log x) - 3\sin(\log x)\right\}$$

$\therefore$ General solution is $y$ = C.F. + P.I.

**Example 5.6** Solve $(2x+3)^2\frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$.

The given equation is a Legendre's linear equation.
Put $2x + 3 = e^t$ or $t = \log(2x + 3)$

Then $$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{2}{2x+3}$$

i.e. $$(2x+3)\frac{dy}{dx} = 2\theta y$$

Similarly $$(2x+3)^2 \frac{d^2 y}{dx^2} = 4\theta(\theta-1)y$$

The given equation then becomes

$$\left[4\theta(\theta-1)-2\times 2\theta-12\right]y=3\left(e^t-3\right)$$

i.e. $$\left(4\theta^2-8\theta-12\right)y=3\left(e^t-3\right)$$

i.e. $$\left(\theta^2-2\theta-3\right)y=\frac{3}{4}\left(e^t-3\right)$$

A.E. is $$m^2-2m-3=0$$

The roots are $m = 3, -1$

$\therefore$ $$\text{C.F.} = Ae^{3t}+Be^{-t}$$

$$= A(2x+3)^3+\frac{B}{2x+3}$$

$$\text{P.I.} = \frac{1}{\theta^2-2\theta-3}\frac{3}{4}\left(e^t-3\right)$$

$$= \frac{3}{4}\left[-\frac{1}{4}e^t+1\right]$$

$$= -\frac{3}{16}(2x+3)+\frac{3}{4}$$

$\therefore$ General solution is $y$ = C.F. + P.I.

**Example 5.7** Solve $(x^2D^2 + xD + 1)\,y = \log x \sin(\log x)$.

Putting $x = e^t$ or $t = \log x$ and denoting $\frac{d}{dt}$ by $\theta$, the given equation becomes

$$\left[\theta(\theta-1)+\theta+1\right]y = t\sin t$$

i.e. $$\left(\theta^2+1\right)y = t\sin t$$

A.E.is $$m^2+1=0$$

The roots are $m = \pm i$

$\therefore$ $$\text{C.F.} = A\cos t + B\sin t$$

$$\text{P.I.} = \frac{1}{\theta^2+1}\text{ Imaginary part of } e^{it}\cdot t$$

$$= \text{I.P. of } e^{it}\cdot\frac{1}{(\theta+i)^2+1}t = \text{I.P. of } e^{it}\cdot\frac{1}{2i\theta}\left(1-\frac{i\theta}{2}\right)^{-1}t$$

$$= \text{I.P. of } -\frac{i}{2}e^{it}\cdot\frac{1}{\theta}\left(1+\frac{i\theta}{2}\right)t = \text{I.P. of } -\frac{i}{2}e^{it}\left(\frac{t^2}{2}+\frac{it}{2}\right)$$

$$= \text{I.P. of } \frac{1}{2}(\sin t - i\cos t)\left(\frac{t^2}{2}+\frac{it}{2}\right)$$

$$= \frac{t}{4}\sin t - \frac{1}{4}t^2\cos t$$

$$= \frac{1}{4} \log x \cdot \sin(\log x) - \frac{1}{4}(\log x)^2 \cos(\log x)$$

$\therefore$ General solution is $y$ = C.F. + P.I.

**Example 5.8** Solve $(x^2 D^2 + 4xD + 2)\, y = \sin x$.

Putting $x = e^t$ or $\log x = t$ and denoting $\frac{d}{dt}$ by $\theta$, the given equation becomes

$$[\theta(\theta - 1) + 4\theta + 2]y = \sin(e^{-t})$$

i.e. $(\theta^2 + 3\theta + 2)\, y = \sin(e^{-t})$

A.E. is $m^2 + 3m + 2 = 0$

The roots are $m = -1, -2$.

$\therefore$ C.F. $= A\, e^{-t} + B\, e^{-2t}$.

$$= \frac{A}{x} + \frac{B}{x^2}$$

$$\text{P.I.} = \frac{1}{(\theta+1)(\theta+2)} \sin(e^t)$$

$$= \left(\frac{1}{\theta+1} - \frac{1}{\theta+2}\right) \sin(e^t)$$

$$= e^{-t} \int \sin(e^t) e^t \, dt - e^{-2t} \int \sin(e^t) \cdot e^{2t} \, dt$$

$$\left[\because \frac{1}{D-m} X = e^{mx} \int X e^{-mx} \, dx\right]$$

$$= e^{-t} \int \sin u \, du - e^{-2t} \cdot \int u \sin u \, du, \text{ putting } e^t = u$$

$$= -e^{-t} \cos u - e^{-2t}(-u \cos u + \sin u)$$

$$= -e^{-t} \cos(e^t) + e^{-t} \cos(e^t) - e^{-2t} \sin(e^t)$$

$$= -\frac{1}{x^2} \sin x$$

$\therefore$ General solution is $y$ = C.F. + P.I.

**Example 5.9** Solve the simultaneous equations

$$\frac{dx}{dt} + 2x - 3y = 5t$$

$$\frac{dy}{dt} - 3x + 2y = 2e^{2t}$$

Denoting $\frac{d}{dt}$ by $D$, the given equations become

$$(D + 2)x - 3y = 5t \qquad (1)$$

$$-3x + (D + 2)\, y = 2e^{2t} \qquad (2)$$

Operating (1) by

$$(D + 2);\ (D + 2)^2 x - 3(D + 2)y = 5 + 10t \qquad (1)'$$

Multiplying (2) by 3;

$$-9x + 3(D + 2)y = 6e^{2t} \qquad (2)'$$

Adding (1)′ and (2)′, we get

$$(D^2 + 4D - 5)\,x = 5(1 + 2t) + 6e^{2t} \tag{3}$$

A.E. is $m^2 + 4m - 5 = 0$

The roots are $m = 1, -5$

$\therefore$ C.F. $= A\,e^{t} + B\,e^{-5t}$.

$$\text{P.I.} = \frac{1}{D^2+4D-5}5(1+2t) + 6\cdot\frac{1}{D^2+4D-5}e^{2t}$$

$$= -1\times\left\{1-\frac{D(D+4)}{5}\right\}^{-1}(1+2t) + \frac{6}{7}e^{2t}$$

$$= -1\times\left\{1+\frac{4D}{5}\right\}(1+2t) + \frac{6}{7}e^{2t}$$

$$= -\left(1+2t+\frac{8}{5}\right) + \frac{6}{7}e^{2t}$$

$$= -2t - \frac{13}{5} + \frac{6}{7}e^{2t}$$

$\therefore$ $$x = A\,e^{t} + B\,e^{-5t} - 2t - \frac{13}{5} + \frac{6}{7}e^{2t}.$$

To find $y$, we may follow any one of the following two methods:

***Method 1***

If we eliminate $x$ between Equations (1) and (2), we will get

$$(D^2 + 4D - 5)\,y = 15t + 8e^{2t} \tag{4}$$

Solving (4) in the usual manner, we will get

$$y = C\,e^{t} + D\,e^{-5t} - 3t - \frac{12}{5} + \frac{8}{7}e^{2t}$$

*Note* ☑ In the solution of $y$, we should not use the same arbitrary constants $A$ and $B$ used in the solution of $x$. Though the values of $x$ and $y$ have been found out, they are expressed in terms of four arbitrary constants. The solutions for $x$ and $y$ should contain as many constants (in this problem, it is 2) as the order of the Equation (3) or (4). Hence the values of $C$ and $D$ should be expressed in terms of $A$ and $B$ as explained below:

Inserting the values of $x$ and $y$ in Equation (1),

$$\left(Ae^{t} - 5Be^{-5t} + \frac{12}{7}e^{2t} - 2\right) + \left(2Ae^{t} + 2Be^{-5t} + \frac{12}{7}e^{2t} - 4t - \frac{26}{5}\right)$$
$$-\left(3Ce^{t} + 3De^{-5t} + \frac{24}{7}e^{2t} - 9t - \frac{36}{5}\right) = 5t$$

i.e. $$3(A - C)\,e^{t} - 3(B + D)\,e^{-5t} = 0$$

$\therefore$ $$C = A \quad \text{and} \quad D = -B.$$

$\therefore$ The required solutions of the given equations are

$$x = A e^{t} + Be^{-5t} + \frac{6}{7} e^{2t} - 2t - \frac{13}{5}$$

and $$y = A e^{t} - B e^{-5t} + \frac{8}{7} e^{2t} - 3t - \frac{12}{5}$$

***Method 2***

We eliminate $Dy$ from Equation (1) and (2).

Operating (1) by $D$; $\quad D^2x + 2Dx - 3Dy = 5$

Multiplying (2) by 3;

$$-9x + 6y + 3Dy = 6e^{2t}$$

Adding; $$6y + x'' + x' - 9x = 5 + 6e^{2t}$$

$\therefore$ $$y = -\frac{1}{6}\left[x'' + x' - 9x - 6e^{2t} - 5\right] \qquad (5)$$

$x$ is given by $$x = A e^{t} + B e^{-5t} + \frac{6}{7} e^{2t} - 2t - \frac{13}{5} \qquad (6)$$

Differentiating $x$ w.r.t. $t$;

$$x' = Ae^{t} - 5e^{-5t} + \frac{12}{7} e^{2t} - 2 \qquad (7)$$

Further differentiating w.r.t. $t$;

$$x'' = Ae^{t} + 25Be^{-5t} + \frac{24}{7} e^{2t} \qquad (8)$$

Using (6), (7) and (8) in (5) and simplifying,

$$y = -\frac{1}{6}\left[-6\,A\,e^{t} + 6\,Be^{-5t} - \frac{48}{7} e^{2t} + 18t + \frac{72}{5}\right]$$

i.e. $$y = Ae^{t} - Be^{-5t} + \frac{8}{7} e^{2t} - 3t - \frac{12}{5}$$

**Example 5.10** Solve $Dx - (D - 2)\,y = \cos 2t$.

$$(D - 2)\,x + Dy = \sin 2t$$

$$Dx - (D - 2)\,y = \cos 2t \qquad (1)$$

$$(D - 2)\,x + Dy = \sin 2t \qquad (2)$$

Operating (1) by $D$; $\quad D^2x - D(D - 2)\,y = -2\sin 2t$

Operating (2) by $(D - 2)$; $\quad (D - 2)^2x + D(D - 2)y = 2\cos 2t - 2\sin 2t.$

Adding; $\quad (2D^2 - 4D + 4)\,x = 2\cos 2t - 4\sin 2t$

i.e. $$(D^2 - 2D + 2)\,x = \cos 2t - 2\sin 2t \qquad (3)$$

A.E. is $m^2 - 2m + 2 = 0$

The roots are $m = 1 \pm i$

$\therefore$ $$\text{C.F.} = e^t (A \cos t + B \sin t)$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}\left(\cos 2t - 2\sin 2t\right)$$

$$= -\frac{1}{2}\cdot\frac{1}{D+1}\left(\cos 2t - 2\sin 2t\right) = -\frac{1}{2}\cdot\frac{(D-1)}{-5}\left(\cos 2t - 2\sin 2t\right)$$

$$= \frac{1}{10}\left(-2\sin 2t - 4\cos 2t - \cos 2t + 2\sin 2t\right) = -\frac{1}{2}\cos 2t$$

$\therefore$ $$x = e^t\left(A\cos t + B\sin t\right) - \frac{1}{2}\cos 2t \tag{4}$$

Adding (1) and (2);

$$2Dx - 2x + 2y = \sin 2t + \cos 2t$$

$\therefore$ $$2y = 2x - 2x' + \sin 2t + \cos 2t \tag{5}$$

Differentiating both sides of (4) w.r.t. $t$;

$$x' = e^t (A \cos t + B \sin t) + e^t (-A \sin t + B \cos t) + \sin 2t \tag{6}$$

Using (4) and (6) in (5), we get,

$$2y = 2A\, e^t \sin t - 2B\, e^t \cos t - \sin 2t$$

$\therefore$ $$y = e^t(A \sin t - B \cos t) - \frac{1}{2}\sin 2t \tag{7}$$

Now (4) and (7) constitute the solutions of the given equations.

**Example 5.11** Solve $D^2x - Dy - 2x = 2t$.

$$Dx + 4Dy - 3y = 0$$

Rewriting the given equation,

$$(D^2 - 2)x - Dy = 2t \tag{1}$$

$$Dx + (4D - 3)y = 0 \tag{2}$$

Operating (1) by $(AD - 3)$;

$$(D^2 - 2)(4D - 3)\, x - D(4D - 3)\, y = 8 - 6t \tag{1$'$}$$

Operating (2) by $D$:

$$D^2x + D(4D - 3)\, y = 0 \tag{2$'$}$$

Adding (1)′ and (2)′, we get

$$(4D^3 - 2D^2 - 8D + 6)x = 8 - 6t$$

i.e. $$(2D^3 - D^2 - 4D + 3)x = 4 - 3t \tag{3}$$

A.E. is $$2m^3 - m^2 - 4m + 3 = 0$$

i.e. $$(m-1)(2m^2 + m - 3) = 0$$

i.e. $$(m-1)(m-1)(2m+3) = 0$$

$\therefore$ The roots are $m = 1,\ 1,\ -\dfrac{3}{2}$

$\therefore$ $$\text{C.F.} = (At+B)e^t + Ce^{-\frac{3}{2}t}$$

$$\text{P.I.} = \frac{1}{(D-1)^2(2D+3)}(4-3t) = \frac{1}{3}\left(1+\frac{2D}{3}\right)^{-1}(1-D)^{-2}(4-3t)$$

$$= \frac{1}{3}\left(1-\frac{2D}{3}\right)(1+2D)(4-3t) = \frac{1}{3}\left(1+\frac{4}{3}D\right)(4-3t)$$

$$= \frac{1}{3}(4-3t-4)$$

$$= -t.$$

$\therefore$ $$x = (At+B)e^t + Ce^{-\frac{3}{2}t} - t \qquad (4)$$

Eliminating $Dy$ from (1) and (2), we get

$$4D^2x + Dx - 8x - 3y = 8t$$

$\therefore$ $$y = \frac{1}{3}(4x'' + x' - 8x - 8t) \qquad (5)$$

Form (4) $$x' = (At+B)e^t + Ae^t - \frac{3}{2}Ce^{-\frac{3}{2}t} - 1$$

and $$x'' = (At+B)e^t + 2Ae^t + \frac{9}{4}Ce^{-\frac{3}{2}t}$$

Using the values of $x$, $x'$ and $x''$ in (5),

$$y = \{(3-t)A - B\}e^t - \frac{1}{6}Ce^{-\frac{3}{2}t} - \frac{1}{3}$$

**Example 5.12** Solve $\dfrac{d^2x}{dt^2} - 3x - 4y = 0.$

$$\frac{d^2y}{dt^2} + x + y = 0 \quad \text{if } x = y = 1$$

and $$\frac{dx}{dt}=\frac{dy}{dt}=0, \quad \text{when } t=0$$

The given equations are

$$(D^2-3)x-4y=0 \tag{1}$$

$$x+(D^2+1)y=0 \tag{2}$$

Operating (1) by $(D^2+1)$;

$$(D^2+1)(D^2-3)x-4(D^2+1)y=0$$

Multiplying (2) by 4;

$$4x+4(D^2+1)y=0$$

Adding, $$(D^4-2D^2+1)x=0$$

A.E. is $(m^2-1)^2=0$

Roots are $m=\pm 1, \pm 1$

$\therefore$ Solution is $$x=(At+B)e^t+(Ct+D)e^{-t} \tag{3}$$

Differentiating $x$ w.r.t. $t$;

$$x'=(At+B)e^t+Ae^t-(Ct+D)e^{-t}+Ce^{-t} \tag{4}$$

Differentiating further

$$x''=(At+B)e^t+2Ae^t+(Ct+D)e^{-t}-2Ce^{-t} \tag{5}$$

From (1),

$$y=\frac{1}{4}(x''-3x)$$

$$=\frac{1}{4}\left[-2(At+B)e^t+2Ae^t-2(Ct+D)e^{-t}-2Ce^{-t}\right]$$

$$=-\frac{1}{2}(At+B)e^t+\frac{A}{2}e^t-\frac{1}{2}(Ct+D)e^{-t}-\frac{C}{2}e^{-t} \tag{6}$$

$\therefore$ $$y'=-\frac{1}{2}(At+B)e^t+\frac{1}{2}(Ct+D)e^{-t} \tag{7}$$

Using the condition $x=1$ when $t=0$ in (3),

$$B+D=1 \tag{8}$$

Using the condition $y=1$ when $t=0$ in (6),

$$\frac{A-B}{2}-\frac{C+D}{2}=1$$

i.e $$A - B - C - D = 1 \qquad (9)$$

Using the condition $x' = 0$ when $t = 0$ in (4),

$$A + B + C - D = 0 \qquad (10)$$

Using the condition $y' = 0$ when $t = 0$ in (7),

$$-B + D = 0 \qquad (11)$$

Solving equations (8), (9), (10), and (11), we get

$$A = \frac{3}{2}, \quad B = \frac{1}{2}, \quad C = -\frac{3}{2}, \quad D = \frac{1}{2}$$

Using these values in (3) and (6), the required particular solutions are

$$x = \left(\frac{1+3t}{2}\right)e^{t} + \left(\frac{1-3t}{2}\right)e^{-t}$$

and $$y = \left(\frac{1}{2} - \frac{3}{4}t\right)e^{t} + \left(\frac{1}{2} + \frac{3}{4}t\right)e^{-t}$$

**Example 5.13** Solve $(D^2 - 5)x + 3y = \sin t$.

$$-3x + (D^2 + 5)\,y = t$$

Eliminating $y$ from the given equations, we get $(D^4 - 16)x = 4 \sin t - 3t$
A.E. is $m^4 - 16 = 0$

The roots are $m = \pm 2, \pm i\,2$.

$\therefore$ $$\text{C.F.} = A\,e^{2t} + B\,e^{-2t} + C \cos 2t + D \sin 2t$$

$$\text{P.I.} = 4.\frac{1}{D^4 - 16}\sin t - \frac{3}{D^4 - 16}t$$

$$= -\frac{4}{15}\sin t + \frac{3}{16}\left(1 - \frac{D^4}{16}\right)^{-1} t$$

$$= -\frac{4}{15}\sin t + \frac{3}{16}\left(1 + \frac{D^4}{16}\right)^{-1} t$$

$$= -\frac{4}{15}\sin t + \frac{3}{16}t$$

$$x = A e^{2t} + B e^{-2t} + C \cos 2t + D \sin 2t - \frac{4}{15}\sin t + \frac{3}{16}t \qquad (1)$$

$\therefore$ $$x' = 2A e^{2t} - 2B e^{-2t} - 2C \sin 2t + 2D \cos 2t - \frac{4}{15}\cos t$$

and $$x'' = 4A e^{2t} + 4B e^{-2t} - 4C \cos 2t - 4D \sin 2t + \frac{4}{15}\sin t$$

From the given equation,

$$y = \frac{1}{3}\left[5x - x'' + \sin t\right]$$
$$= \frac{A}{3}e^{2t} + \frac{B}{3}e^{-2t} + 3C\cos 2t + 3D\sin 2t - \frac{1}{5}\sin t + \frac{5}{16}t$$

## EXERCISE 5(c)

***Part A***

(Short Answer Questions)

1. Transform the equation $xy'' + y' + 1 = 0$ into a linear equation with constant coefficients and hence solve it.
2. Solve the equation $x^2y'' - xy' + y = 0$.
3. Convert the equation $xy'' - 3y' + x^{-1}y = x^2$ as a linear equation with constant coefficients.
4. Convert the equation $x^4y''' - x^3y'' + x^2y' = 1$ as a linear equation with constant coefficients.
5. Solve the equation $x^2y'' - 2nxy' + n(n + 1)y = 0$.
6. Solve the equation $x^3y''' + 3x^2y'' + xy' + y = 0$.
7. Solve for $x$ from the equations $x' - y = t$ and $x + y' = 1$.

***Part B***

Solve the following equations $\left(D \equiv \frac{d}{dx}\right)$

8. $(x^2D^2 + 2xD - 20)y = (x^2 + 1)^2$
9. $(x^4D^3 - x^3D^2 + x^2D)y = 1$
10. $(x^3D^3 - x^2D^2 + 2xD - 2)y = \cos(2\log x)$
11. $(x^2D^2 + xD - 9)y = \sin^3(\log x)$
12. $(x^2D^2 + 9xD + 25)y = (\log x)^2$
13. $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = (1 + \log x)^2$
14. $(x^2D^2 - 3xD + 4)y = x(\log x)^2$
15. $(x^4D^4 + 2x^3D^3 + x^2D^2 - xD + 1)y = x^2\log x$
16. $\left(x^2D^2 - xD - 3\right)y = \frac{1}{x}\cos\left(2\log x\right)$
17. $(x^2D^2 + 3xD + 5)y = x\cos(\log x)$
18. $[(3x + 2)^2D^2 + 3(3x + 2)D - 36]\,y = 3x^2 + 4x + 1$
19. $[(x + 1)^2D^2 + (x + 1)D + 1]\,y = 4\cos\log(x + 1)$

Solve the following simultaneous equations: $\left(D \equiv \frac{d}{dt}\right)$

20. $(D + 4)x + 3y = t$
    $2x + (D + 5)y = e^{2t}$

21. $(2D + 1)\,x + (3D + 1)\,y = e^t$
    $(D + 5)x + (D + 7)\,y = 2e^t$
22. $Dx + y = \sin t \quad x + Dy = \cos t$ given that $x = 2$ and $y = 0$ at $t = 0$
23. $2D^2x - Dy - 4x = 2t$
    $2Dx + 4Dy - 3y = 0$
24. $D^2x + y = 3e^{2t}$
    $Dx - Dy = 3e^{2t}$
25. $(D^2 + 4)\,x + y = 0$
    $(D^2 + 1)\,y - 2x = 1 + \cos^2 t$
26. $D^2x - 2Dy - x = e^t \cos t$
    $D^2y + 2Dx - y = e^t \sin t$
27. $(D^2 + 4)\,x + 5y = t^2$
    $(D^2 + 4)\,y + 5x = t + 1$

## 5.6 LINEAR EQUATIONS OF SECOND ORDER WITH VARIABLE COEFFICIENTS

In the previous section we have discussed the solution of Euler′s homogeneous linear differential equations of the second (and higher) order, which are a particular case of linear equations of second order with variable coefficients, that are functions of $x$. The general form of such a differential equation will be taken as

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = r(x),$$

in which the coefficient of $\frac{d^2y}{dx^2}$ is unity and $p(x)$, $q(x)$ and $r(x)$ are functions of $x$. In this section, we shall discuss a few methods of solving such equations.

### 5.6.1 Method of Reduction of Order-Transformation of the Equation by Changing the Dependent Variable

Let the given equation be

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = r(x) \tag{1}$$

Let us assume that one solution of the corresponding homogeneous equation, namely,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0 \tag{2}$$

is known. Let it be $y = u(x)$ (3)

We then assume that $$y = u(x) \cdot v(x) \tag{4}$$
is a solution of equation (1).

From (4), we get $$y' = uv' + u'v \tag{5}$$

and $$y'' = uv'' + 2u'v' + u''v \tag{6}$$

Using (4), (5) and (6) in (1), we get

$$uv'' + (2u' + pu)\, v' + (u'' + pu' + qu)\, v = r$$

i.e. $uv'' + (2u' + pu)\, v' = r$, by (2) and (3)

i.e. $$v'' + \left(2\frac{u'}{u} + p\right)v' = \frac{r}{u}$$

i.e. $$v'' = p_1 v' = r_1,\ \text{where } p_1 = \frac{2u'}{u} + p \text{ and } r_1 = \frac{r}{u}. \tag{7}$$

Putting $v' = w$ in $(7)$, it becomes

$$\frac{dw}{dx} + p_1 w = r_1 \tag{8}$$

Now equation (8) is a linear equation of the first order in $w$, which can be solved. Thus by changing the dependent variable $y$ to $w$, we have reduced the order of the equation by one.

**Note** ☑ Had the given equation (1) been homogeneous, viz., $r(x) = 0$, then equation (7) would have become

$$v'' + \left(\frac{2u'}{u} + p\right)v' = 0$$

i.e. $$\frac{d(v')}{v'} = -\left(\frac{2u'}{u} + p\right)$$

Integrating both sides with respect to $x$, we get $\log v' = -2\log u - \int pdx + \log A$

$$= \log\left[Au^{-2}\, e^{-\int pdx}\right]$$

$\therefore$ $$v' = Au^{-2}\, e^{-\int pd.} \tag{9}$$

Solving equation (9), we get $v$ and hence the solution of equation (1).

## 5.6.2 Definition

A second order differential equation in $y$ not containing the term in the first derivative $y'$ is said to be in the *canonical* or *normal form*.

## 5.6.3 Reduction to Canonical or Normal Form

The second order linear differential equation $y'' + p(x)\, y' + q(x)y = r(x)$ can be transformed to the canonical or normal form $v'' + f(x)v = g(x)$, where $f(x) =$ $q(x) - \frac{1}{4}\{p(x)\}^2 - \frac{1}{2}\, p'(x)$ and $g(x) = r(x)\,.\, e^{1/2 \int p(x)\, dx}$, by using the substitution $y = ve^{-1/2 \int p(x)\, dx}$.

***Proof:***

Let us assume that $y = uv$ is a solution of the equation $y'' + p(x)\, y' + q(x)\, y = r(x)$(1)

Differentiating $\quad y = uv \quad (2)$

with respect to $x$,

we get $\quad y' = u'v + uv' \quad (3)$

and $\quad y'' = u''v + 2u'v' + uv'' \quad (4)$

The values of $y$, $y'$ and $y''$ given in steps (2), (3), (4) should satisfy equation (1).

i.e. $\quad u''v + 2u'v' + uv'' + p(u'v + uv') + quv = r$

i.e. $\quad uv'' + (2u' + pu)\, v' + (u'' + pu' + qu)\, v = r \quad (5)$

Equation (5) should be in the canonical form, viz., it should not contain the $v'$ term.

$$\therefore \quad 2u' + pu = 0$$

$$\text{i.e.} \quad \frac{u'}{u} = -\frac{p}{2}$$

$$\text{i.e.} \quad \log u = -\int \frac{pdx}{2} + c$$

Assuming $c = 0$, we get $u = e^{-1/2 \int pdx}$

Thus the substitution $y = uv = ve^{-1/2 \int pdx}$

transforms equation $(1)$ into the canonical form

$$\text{When} \quad u = e^{-1/2 \int pdx}, u' = -\frac{p}{2} \cdot e^{-1/2 \int pdx}$$

$$\text{and} \quad u'' = \frac{p^2}{4} \cdot e^{-1/2 \int pdx} - \frac{p'}{2} \cdot e^{-1/2 \int pdx}$$

Using the values of $u$, $u'$ and $u''$ in the canonical form, viz., in $v'' + \frac{1}{u}(u'' + pu' + qu)$

$v = \frac{1}{u} r$ it becomes

$$v'' + \left(\frac{p^2}{4} - \frac{p'}{2} - \frac{p^2}{2} + q\right)v = re^{1/2\int p dx}$$

i.e. $$v'' + \left(q - \frac{1}{4}p^2 - \frac{1}{2}p'\right)v = re^{1/2\int p dx}$$

i.e. $$v'' + f(x)\cdot v = g(x), \text{where}$$

$$f(x) = q(x) - \frac{1}{4}\{p(x)\}^2 - \frac{1}{2}p'(x) \text{and}$$

$$g(x) = r(x)\cdot e^{1/2\int p(x)dx}$$

## 5.6.4 Method of Reduction of Order—Special Types of Equations

**Type 1. Equations of the form $\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right)$, in which $y$ is explicitly absent.**

Putting $\frac{dy}{dx} = p$, we get $\frac{d^2y}{dx^2} = \frac{dp}{dx}$

The equation gets transformed as $\frac{dp}{dx} = f(x, p)$, which is only a first order equation.

Solving this transformed equation, we get

$$p = \frac{dy}{dx} = \phi(x, c_1)$$

Again, solving this equation, we get

$$y = \psi\,(x, c_1, c_2).$$

***Extension:***

Equations of the form $\frac{d^ny}{dx^n} = f\left(x, \frac{d^{n-1}y}{dx^{n-1}}\right)$ can be solved by putting $\frac{d^{n-1}y}{dx^{n-1}} = p$ and reducing the order successively.

**Type 2. Equation of the form $\frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right)$ in which $x$ is explicitly absent.**

We put $\frac{dy}{dx} = p$ and treat $p$ as a function of $y$. Then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = p\,\frac{dp}{dy}$

The equation becomes $\frac{dp}{dy} = f(y, p)$, which is a first order equation.

Solving this transformed equation, we get

$$p = \frac{dy}{dx} = \phi(y, c_1)$$

Solving this equation further, we get

$$x = \psi(y, c_1, c_2)$$

***Extension:***

Equations of the form $f\{y, y', y'', \dots y^{(n)}\} = 0$ can be solved by the above technique, viz., by putting $\frac{dy}{dx} = p$ and treating $p$ as a function of $y$.

**Type 3. Equations $f(x, y, y', y'') = 0$, which are homogeneous in $y$, $y'$ and $y''$ (but not in $x$)**

By putting $y = e^{\int z dx}$, the order of the equation can be reduced by one and hence solved. When $y = e^{\int z dx}$, $y' = z e^{\int z dx}$ and $y'' = (z^2 + z')e^{\int z dx}$. Thus, the order of the transformed equation in the dependent variable $z$ will be one less than that of the given equation.

**Type 4. Equations f(x, y, y′, y″) = 0 which are exact, viz. which can be expressed as $\frac{d}{dx}\{\phi(x, y, y')\} = 0$.**

The first integral of the equation $\frac{d}{dx}\{\phi(x, y, y')\} = 0$ is $\phi(x, y, y') = c_1$, which is a first order equation, solving which we get the solution of the given second order equation.

*Note* ☑ The equation $[p_0(x)D^2 + p_1(x)D + p_2(x)]y = r(x)$ is exact if and only if $p''_0 - p'_1 + p_2 = 0$.

Let $(p_0 D^2 + p_1 D + p_2)y = D(q_0 D + q_1)y = [q_0 D^2 + (q'_0 + q_1)D + q'_1]y$

Comparing like terms, we get

$$p_0 = q_0;\ p_1 = q'_0 + q_1 \text{ and } p_2 = q'_1$$

Differentiating both sides of $p_1 = q'_0 + q_1$,

we get $$p'_1 = q''_0 + q'_1$$
$$= p''_0 + p_2$$

∴ $$p''_0 - p'_1 + p_2 = 0$$

Conversely, when $p''_0 - p'_1 + p_2 = 0$, we have

$$(p_0 D^2 + p_1 D + p_2)y = (p_0 D^2 + p_1 D + p'_1 - p''_0)y$$
$$= p_0 y'' + p_1 y' + p'_1 y - p''_0 y$$

$$= (p_0 y'' + p'_0 y') + (p_1 y' + p'_1 y) - (p''_0 y + p'_0 y')$$

$$= D(p_0 y') + D(p_l y) - D(p'_0 y)$$

$$= D(p_0 y' + p_l y - p'_0 y)$$

Thus $(p_0 D^2 + p_1 D + p_2)y = 0$ is an exact equation, when $p''_0 - p'_1 + p_2 = 0$.

## WORKED EXAMPLE 5(d)

**Example 5.1** Solve the equation $xy'' - 2(x + 1)y' + (x + 2)y = (x - 2)e^{2x}$, by finding one solution of the corresponding homogeneous equation by inspection and reducing the order of the equation.

### *Important Notes* ☑

To find one solution of the equation $p_0(x)\, y'' + p_1(x)\, y' + p_2(x).\, y = 0$, the following hints may be useful:

(i) If $p_0(x) + p_1(x) + p_2(x) = 0$, $y = e^x$ is a solution of the equation.
(ii) If $p_0(x) - p_1(x) + p_2(x) = 0$, $y = e^{-x}$ is a solution of the equation.
(iii) If $p_1(x) + xp_2(x) = 0$, $y = x$ is a solution of the equation.

In the given problem, $p_0 = x$, $p_1 = -2(x + 1)$ and $p_2 = x + 2$.

The condition $p_0 + p_1 + p_2 = 0$ is satisfied.

$\therefore$ $y = e^x$ is a solution of the homogeneous equation corresponding to the given equation.

Let $y = ve^x$ be a solution of the given equation.

Then
$$y' = ve^x + v'e^x$$
$$y'' = v''e^x + 2v'e^x + ve^x$$

Using these values of $y, y', y''$ in the given equation, it becomes

$$x(e^x v'' + 2e^x v' + e^x v) - 2(x + 1)(e^x v' + e^x v) + (x + 2)e^x v = (x - 2)e^{2x}$$

i.e.,
$$xv'' - 2v' = (x - 2)e^x$$

i.e.,
$$\frac{dp}{dx} - \frac{2}{x}p = \left(1 - \frac{2}{x}\right)e^x, \text{ where } p = \frac{dv}{dx}$$

This is a linear equation of the first order.

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$$

Solution of this equation is

$$\frac{p}{x^2} = \int \left(\frac{1}{x^2} - \frac{2}{x^3}\right)e^x\, dx + 3c_1$$

$$= \int d\left(\frac{1}{x^2} e^x\right) + 3c_1$$

$$= \frac{1}{x^2} e^x + 3c_1$$

i.e.
$$\frac{dv}{dx} = 3c_1 x^2 + e^x$$

Solving this equation, we get

$$v = c_1 x^3 + c_2 + e^x$$

$\therefore$ The solution of the given equation is

$$y = e^x (c_1 x^3 + c_2 + c^x)$$

**Example 5.2** Solve the equation $\frac{d^2 y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y \cot x = \sin^2 x$, by the method of reduction of order.

The given equation is $y'' + (1 - \cot x)\, y' - y \cot x = \sin^2 x$ (1)

Here $p_0 = 1$; $p_1 = 1 - \cot x$; $p_2 = -\cot x$

We observe that $p_0 - p_1 + p_2 = 0$

$\therefore$ $y = e^{-x}$ is a solution of the equation

$$y'' + (1 - \cot x)y' - y \cot x = 0 \quad (2)$$

Let $y = ve^{-x}$ be a solution of equation (1)

Then
$$y' = v' e^{-x} - ve^{-x}$$

and
$$y'' = v'' e^{-x} - 2v' e^{-x} + ve^{-x}$$

Using these values of $y$, $y'$ and $y''$ in (1), we have

$$(v'' e^{-x} - 2v' e^{-x} + ve^{-x}) + (1 - \cot x)(v' e^{-x} - ve^{-x}) - ve^{-x} \cot x = \sin^2 x$$

i.e.
$$v'' - (1 + \cot x)\, v' = e^x \sin^2 x \quad (3)$$

Putting $v' = p$ in (3), it becomes

$$\frac{dp}{dx} - (1 + \cot x)\, p = e^x \sin^2 x \quad (4)$$

(4) in a linear equation of the first order

$$\text{I.F.} = e^{-\int (1+\cot x)dx}$$

$$= e^{-x - \log \sin x} = \frac{e^{-x}}{\sin x}$$

$\therefore$ Solution of equation (4) is

$$\frac{pe^{-x}}{\sin x} = \int e^x \sin^2 x \cdot \frac{e^{-x}}{\sin x}\, dx + c_1$$

$$= -\cos x + c_1$$

i.e.
$$p = \frac{dv}{dx} = e^x \left(c_1 \sin x - \sin x \cos x\right) \tag{5}$$

Integrating (5) with respect to $x$,

$$v = c_1 \int e^x \sin x\, dx - \frac{1}{2}\int e^x \sin 2x\, dx + c_2$$

$$= \frac{c_1}{2} e^x \left(\sin x - \cos x\right) - \frac{1}{2} \cdot \frac{e^x}{5}\left(\sin 2x - 2\cos 2x\right) + c_2$$

$\therefore$ The solution of the given equation (1) is

$$y = A\left(\sin x - \cos x\right) - \frac{1}{10}\left(\sin 2x - 2\cos 2x\right) + Be^{-x}.$$

**Example 5.3** Solve the equation $\left(1-x^2\right)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2y = 2,$ by the method of reduction of order.

The given equation is

$$(1 - x^2)y'' - 2xy' + 2y = 2 \tag{1}$$

Here $p_0 = 1 - x^2$; $p_1 = -2x$; $p_2 = 2$

We observe that $p_1 + p_2 x = 0$.

$\therefore$ $y = x$ is a solution of

$$(1 - x^2)y'' - 2xy' + 2y = 0 \tag{2}$$

Let $y = vx$ be a solution of equation (1).

Then
$$y' = v'x + v;\ y'' = v''x + 2v'$$

$y, y', y''$ satisfy equation (1)

i.e.
$$(1 - x^2)(v''x + 2v') - 2x(v'x + v) + 2vx = 2$$

i.e.
$$x(1 - x^2)v'' + [2(1 - x^2) - 2x^2]v' = 2$$

i.e.
$$\frac{dp}{dx} + \left(\frac{2}{x} - \frac{2x}{1-x^2}\right)p = \frac{2}{x\left(1-x^2\right)} \tag{3}$$

where
$$p = v'$$

Equation (3) is a linear equation of the first order.

$$\text{I.F.} = e^{\int\left(\frac{2}{x}-\frac{2x}{1-x^2}\right)dx}$$

$$= e^{2\log x+\log\left(1-x^2\right)}$$

$$= x^2\left(1-x^2\right)$$

∴ Solution of equation (3) is

$$px^2\left(1-x^2\right) = \int \frac{2}{x\left(1-x^2\right)}\cdot x^2\left(1-x^2\right)dx + c_1$$

$$= x^2 + c_1$$

i.e.

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{1-x^2} + \frac{c_1}{x^2\left(1-x^2\right)}$$

$$= \frac{1}{1-x^2} + c_1\left(\frac{1}{x^2}+\frac{1}{1-x^2}\right) \tag{4}$$

Integrating (4) with respect to $x$

$$v = (c_1+1)\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) - \frac{c_1}{x} + c_2$$

∴ Solution of equation (1) is

$$y = \left(\frac{c_1+1}{2}\right)x\log\left(\frac{1+x}{1-x}\right) - c_1 + c_2x$$

**Example 5.4** Solve the equation $x\frac{d^2y}{dx^2}+(2-x)\frac{dy}{dx}-y=0$, given that $y=\frac{1}{x}$ is a solution.

[Refer to the note under the discussion of the method of reduction of order]

If $y = u$ is a solution of the equation

$y'' + p(x)y' + q(x)y = 0$, then $y = uv$ will also be a solution of $y'' + p(x)\,y' + q(x)y = 0$, where $v' = c_1u^{-2}e^{-\int p(x)\,dx}$

The given equation can be rewritten as

$$y'' + \left(\frac{2}{x}-1\right)y' - \frac{1}{x}y = 0 \tag{1}$$

Here $p(x) = \frac{2}{x} - 1$ and $q(x) = -\frac{1}{x}$

Since $y = \frac{1}{x}$ is a solution of equation (1),

$y = \frac{1}{x} v$ is also a solution of (1), where

$$v' = c_1 x^2 \, e^{-\int\left(\frac{2}{x}-1\right)dx}$$

$$= c_1 x^2 \, e^{-2\log x + x}$$

$$= c_1 x^2 \cdot x^{-2} e^x = c_1 e^x$$

$\therefore$ $$v = c_1 e^x + c_2$$

$\therefore$ The general solution of equation (1) is

$$y = \frac{1}{x}\left(c_1 e^x + c_2\right) \quad \text{or} \quad xy = c_1 e^x + c_2$$

**Example 5.5** Solve the equation $\frac{d^2 y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 0$ given that $y = \frac{1}{x}\sin x$ is a solution.

$y = \frac{1}{x}\sin x \cdot v$ is also a solution of the given equation, where

$$v' = c_1 \frac{x^2}{\sin^2 x} \cdot e^{-\int \frac{2}{x}dx}, \text{ since } u = \frac{1}{x}\sin x \text{ and } p(x) = \frac{2}{x}$$

$$= c_1 \frac{x^2}{\sin^2 x} \cdot e^{-2\log x} = \frac{c_1}{\sin^2 x}$$

Integrating, we get

$$v = -c_1 \cot x + c_2 \quad \text{or} \quad A \cot x + B$$

$\therefore$ The general solution of the given equation is $y = \frac{1}{x}\sin x\left(A\cot x + B\right)$

i.e. $$xy = A\cos x + B\sin x$$

**Example 5.6** Solve the equation $\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} + x^2 y = 0$, by reducing it to the canonical form.

The given equation is $y'' + 2xy' + x^2y = 0$ (1)

Comparing equation (1) with the standard equation

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

We have $p(x) = 2x$; $q(x) = x^2$; $r(x) = 0$.

Putting $y = uv$, where $u = e^{-\int \frac{p(x)dx}{2}} = e^{-x^2/2}$, in (1), it becomes

$$v'' + \left(q - \frac{p^2}{4} - \frac{p'}{2}\right)v = re^{\int p/2\,\mathrm{d}x}$$

[Refer to the discussion of reduction to normal form]

i.e. $$v'' + (x^2 - x^2 - 1)\,v = 0$$

i.e. $$v'' = v \tag{3}$$

which is of the form $v'' = f(v, v')$

Putting $v' = p$ and treating $p$ as a function of $v$, we have $v'' = p\dfrac{\mathrm{d}p}{\mathrm{d}v}$

∴ The equation becomes

$$p\frac{\mathrm{d}p}{\mathrm{d}v} = v \tag{4}$$

Solving this equation, we get $p^2 = v^2 + c_1^2$

or $$p = \frac{dv}{dx} = \sqrt{v^2 + c_1^2} \tag{5}$$

Solving equation (5), we have $\sinh^{-1}\left(\dfrac{v}{c_1}\right) = x + c_2$

or $$v = c_1 \sinh\left(x + c_2\right)$$

$$= c_1\left\{\frac{e^{x+c_2} - e^{-(x+c_2)}}{2}\right\}$$

$$= Ae^x + Be^{-x}$$

∴ The required solution of equation (1) is

$$y = \left(Ae^x + Be^{-x}\right)e^{-x^2/2}$$

**Example 5.7** Solve the equation $4x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - 1\right)y = 0$, by reducing it to the normal form.

The given equation can be rewritten as

$$y'' + \frac{1}{x}y' + \frac{1}{4}\left(1 - \frac{1}{x^2}\right)y = 0 \qquad (1)$$

Here $p = \frac{1}{x}, q = \frac{1}{4}\left(1 - \frac{1}{x^2}\right)$ and $r = 0$

Let $y = uv$ be a solution of (1), where

$$u = e^{-1/2\int p\,dx} = e^{-1/2\int \frac{dx}{x}} = \frac{1}{\sqrt{x}}$$

Putting $y = \frac{1}{\sqrt{x}}v$ in (1), it becomes

$$v'' + \left(q - \frac{p^2}{4} - \frac{p'}{2}\right)v = re^{\int \frac{p}{2}dx}$$

i.e. $$v'' + \left\{\frac{1}{4}\left(1 - \frac{1}{x^2}\right) - \frac{1}{4x^2} + \frac{1}{2x^2}\right\}v = 0$$

i.e. $$v'' + \frac{1}{4}v = 0 \qquad (3)$$

Solving equation (3), we have $v = A\cos\frac{x}{2} + B\sin\frac{x}{2}$

∴ Solution of equation (1) is $y = \frac{1}{\sqrt{x}}\left(A\cos\frac{x}{2} + B\sin\frac{x}{2}\right)$

**Example 5.8** Solve the equation $x\frac{d^2y}{dx^2} - \frac{dy}{dx} = x^2e^x$, given that $y(0) = -1$ and $y'(0) = 0$.

The given equation does not contain $y$ explicitly.

Putting $\frac{dy}{dx} = p$ and treating $p$ as a function of $x$, we have $\frac{d^2y}{dx^2} = \frac{dp}{dx}$; the equation becomes

$$x\frac{dp}{dx} - p = x^2e^x$$

i.e. $$\frac{dp}{dx} - \frac{1}{x}p = xe^x \qquad (1)$$

Equation (1) in a linear equation of first order in $p$.

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

Solution of (1) is
$$p \cdot \frac{1}{x} = \int \frac{1}{x} \cdot xe^x dx + 2c_1$$

i.e.
$$p = x(e^x + 2c_1)$$

or
$$\frac{dy}{dx} = xe^x + 2c_1 x \tag{2}$$

Solving equation (2), we have

$$y = \int xe^x \, dx + \int 2c_1 x \, dx + c_2$$

i.e., the solution of the given equation is

$$y = (x - 1)\, e^x + c_1x^2 + c_2.$$

Using the condition $y(0) = -1$, $c_2 = 0$. Using the condition $y'(0) = 0$ in (2), $c_1$ is arbitrary. Taking $c_1 = 0$, the required solution is $y = (x - 1)e^x$.

**Example 5.9** Solve the equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$, given that $y(0) = 0$, $y'(0) = 0$

***Method 1***

The given equation

$$y'' + y'^2 = 1 \tag{1}$$

can be considered as one not containing $y$ explicitly.

Putting $\frac{dy}{dx} = p$ and treating $p$ as a function of $x$, we have $\frac{d^2y}{dx^2} = \frac{dp}{dx}$

Then equation (1) becomes

$$\frac{dp}{dx} = 1 - p^2 \tag{2}$$

Solution of equation (2) is

$$\int \frac{dp}{1 - p^2} = x + c_1$$

i.e.
$$\frac{1}{2} \log\left(\frac{1+p}{1-p}\right) = x + c_1$$

Given that $p = 0$, when $x = 0$

$\therefore$
$$c_1 = 0$$

Thus we have $\dfrac{1+p}{1-p}=e^{2x}$

$$\therefore \qquad p=\frac{dy}{dx}=\frac{e^{2x}-1}{e^{2x}+1}$$

$$=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\tanh x \tag{3}$$

Solving (3), we have $y = \log\cosh x + c_2$

Using the condition $y(0) = 0$, we get $c_2 = 0$

$\therefore$ Solution of equation (1) is $y = \log\cosh x$.

***Method 2***

The equation

$$y'' + y'^2 = 1 \tag{1}$$

can be considered as one not containing $x$ explicitly.

Putting $\dfrac{dy}{dx}=p$ and treating $p$ as a function of $y$, we have $\dfrac{d^2y}{dx^2}=p\dfrac{dp}{dy}$

Then equation (1) becomes

$$p\frac{dp}{dy}=1-p^2 \tag{2}$$

Solution of equation (2) is

$$\int\frac{p\,dp}{1-p^2}=y+c_1$$

i.e.

$$-\frac{1}{2}\log\left(1-p^2\right)=y+c_1$$

Given that $y = 0$ and $p = 0$, when $x = 0$

or when $y = 0$, $p = 0$

$$\therefore \qquad c_1 = 0$$

Thus

$$1 - p^2 = e^{-2y}$$

i.e.

$$p=\frac{dy}{dx}=\sqrt{1-e^{-2y}} \tag{3}$$

Solving equation (3),

$$\int\frac{dy}{\sqrt{1-e^{-2y}}}=x+c_2$$

i.e. $$\int \frac{e^y\,\mathrm{d}y}{\sqrt{e^{2y}-1}} = x + c_2$$

i.e. $$\cosh^{-1}(e^y) = x + c_2$$

When $x = 0, y = 0. \therefore\ c_2 = 0$

$\therefore$ The required solution of equation (1) is

$$e^y = \cosh x \text{ or } y = \log\cosh x.$$

**Example 5.10** Solve the equation $x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}y}{\mathrm{d}x}\cdot\log\left(\dfrac{1}{x}\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$

The given equation is $y'' = \dfrac{y'}{x}\log\left(\dfrac{y'}{x}\right)$ (1)

It is of the form $y'' = f(x, y')$, which does not contain $y$ explicitly.

Putting $y' = p$ and treating $p$ as a function of $x$, we have $y'' = \dfrac{\mathrm{d}p}{\mathrm{d}x}$ .

Then equation (1) becomes

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{p}{x}\log\left(\frac{p}{x}\right) \tag{2}$$

Putting $p = xe^v$, we have $\dfrac{\mathrm{d}p}{\mathrm{d}x} = e^v + xe^v\dfrac{\mathrm{d}v}{\mathrm{d}x}$

Then equation (2) becomes

$$e^v\left(1 + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = e^v\log\left(e^v\right)$$

i.e. $$x\frac{\mathrm{d}v}{\mathrm{d}x} = v - 1 \tag{3}$$

$\therefore$ Solution of (3) is

$$\int\frac{\mathrm{d}v}{v-1} = \int\frac{\mathrm{d}x}{x} + \log c$$

i.e. $$v = 1 + cx$$

i.e. $$\log\left(\frac{p}{x}\right) = 1 + cx$$

$\therefore$ $$p = \frac{\mathrm{d}y}{\mathrm{d}x} = xe^{1+cx} \tag{4}$$

Solving equation (4), we have

$$y=\frac{x\cdot e^{1+cx}}{c}-\frac{e^{1+cx}}{c^2}+c'$$

$\therefore$ The required solution of equation (1) is

$$c^2y = (cx - 1)e^{1+cx}+c'$$

**Example 5.11** Solve the equation $\frac{d^2y}{dx^2}+\frac{2}{1-y}\left(\frac{dy}{dx}\right)^2=0$ The given equation is

$$y''+\frac{2}{1-y}\cdot y'^2=0 \tag{1}$$

It is of the form $y'' = f(y, y')$, which does not contain $x$ explicitly.

Putting $y' = p$ and treating $p$ as a function of $y$, we have $y''=p\frac{dp}{dy}$.

Then equation (1) becomes

$$p\frac{dp}{dy}+\frac{2}{1-y}p^2=0 \tag{2}$$

i.e.

$$\frac{dp}{p}+\frac{2}{1-y}dy=0$$

$\therefore$ solution of (2) is

$$\log p - 2\log(1-y) = \log c_1$$

i.e.

$$\frac{dy}{dx}=p=c_1(1-y)^2 \tag{3}$$

Solving (3), we have

$$\int\frac{dy}{(1-y)^2}=c_1x+c_2$$

i.e.

$$\frac{1}{1-y}=c_1x+c_2$$

$\therefore$ The required solution of equation (1) is

$$y=1-\frac{1}{c_1x+c_2}.$$

**Example 5.12** Solve the equation $xy\dfrac{d^2y}{dx^2}-x\left(\dfrac{dy}{dx}\right)^2-y\dfrac{dy}{dx}=0\cdot$

The given equation $xyy''-xy'^2-yy'=0$ (1)
is homogeneous of degree 2 in $y, y', y''$.

Putting $y=e^{\int z\,dx}$ and hence $y'=ze^{\int z\,dx}$ and $y''=\left(z^2+z'\right)e^{\int z\,dx}$ in equation (1), we have

$$\left[e^{\int z\,dx}\right]^2\left\{x\left(z^2+z'\right)-xz^2-z\right\}=0$$

i.e. $$xz'-z=0 \qquad (2)$$

Solving (2), we get $$z=c_1x$$

$\therefore$ $$y=e^{\int c_1 x\,dx}=e^{c_1\frac{x^2}{2}+c_2}$$

or $$y=Ae^{Bx^2}$$

**Example 5.13** Solve the equation $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=\dfrac{y\dfrac{dy}{dx}}{\sqrt{1+x^2}}\cdot$

The given equation $yy''+y'^2=yy'\big/\sqrt{1+x^2}$ (1)

is homogeneous of degree 2 in $y$, $y'$ and $y''$.

Putting $y=e^{\int z\,dx}$ and hence $y'=ze^{\int z\,dx}$ and $y''=\left(z^2+z^1\right)e^{\int z\,dx}$ in equation (1), we have

$$\left[e^{\int z\,dx}\right]^2\left\{\left(z^2+z^1\right)+z^2-\frac{z}{\sqrt{1+x^2}}\right\}=0$$

i.e. $$\frac{dz}{dx}+2z^2-\frac{z}{\sqrt{1+x^2}}=0 \qquad (2)$$

i.e. $$-\frac{1}{z^2}\frac{dz}{dx}+\frac{1}{z}\cdot\frac{1}{\sqrt{1+x^2}}=2$$

i.e. $$\frac{du}{dx}+\frac{1}{\sqrt{1+x^2}}u=2 \qquad (3)$$

where $$u=\frac{1}{z}$$

Equation (3) is a linear equation of the first order.

$$\text{I.F.}= e^{\int \frac{1}{\sqrt{1+x^2}}dx}$$

$$= e^{\log\left(x+\sqrt{1+x^2}\right)} = x+\sqrt{1+x^2}$$

Solution of equation (3) is

$$\left(x+\sqrt{1+x^2}\right)u=\int 2\left(x+\sqrt{1+x^2}\right)dx+c_1$$

$$=x^2+x\sqrt{1+x^2}+\log\left(x+\sqrt{1+x^2}\right)+c_1$$

i.e. $$z=\frac{x+\sqrt{1+x^2}}{x^2+x\sqrt{1+x^2}+\log\left(x+\sqrt{1+x^2}\right)+c_1}$$

$$=\frac{1}{2}\frac{d}{dx}\log\left\{x^2+x\sqrt{1+x^2}+\log\left(x+\sqrt{1+x^2}\right)+c_1\right\}$$

$\therefore$ $$y=e^{\int z\,dx}$$

$$=e^{1/2\log\left\{x^2+x\sqrt{1+x^2}+\log\left(x+\sqrt{1+x^2}\right)+c_1\right\}+c_2}$$

$$=\left\{x^2+x\sqrt{1+x^2}+\log\left(x+\sqrt{1+x^2}\right)+c_1\right\}^{\frac{1}{2}}\cdot c_3$$

or $$y^2=A\left\{x^2+x\sqrt{1+x^2}+\log\left(x+\sqrt{1+x^2}\right)+B\right\}$$

**Example 5.14** Show that the equation $x\frac{d^2y}{dx^2}+(x+2)\frac{dy}{dx}+y=0$ is exact and hence solve it.

The given equation is $xy''+(x+2)y'+y=0$ (1)

Comparing equation (1) with

$$p_0y''+p_1y'+p_2y=0.$$

We have $$p_0 = x, p_1 = x + 2, p_2 = 1$$

Now $$p_0'' - p_1' + p_2 = 0 - 1 + 1 = 0$$

Thus the condition for exactness is satisfied.

Now equation (1) can be rewritten as

$$(xy'' + y') + (xy' + y') + y' = 0$$

i.e. $$\frac{d}{dx}(xy') + \frac{d}{dx}(xy) + \frac{dy}{dx} = 0$$

or $$\frac{d}{dx}(xy' + xy + y) = 0 \tag{2}$$

$\therefore$ Solution of (2) is $x\frac{dy}{dx} + (x+1)y = c_1$

i.e. $$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = \frac{c_1}{x} \tag{3}$$

Equation (3) is a linear equation of the first order.

$$\text{I.F.} = e^{\int\left(1+\frac{1}{x}\right)dx}$$
$$= e^{x+\log x} = xe^x$$

$\therefore$ Solution of equation (3) and hence equation (1) is

$$y \cdot x \cdot e^x = \int c_1 e^x dx + c_2$$

i.e. $$xye^x = c_1 e^x + c_2$$

**Example 5.15** Solve the equation $(\sin x)\frac{d^2y}{dx^2} - (\cos x)\frac{dy}{dx} + 2(\sin x)y = \cos x$

Comparing the given equation with

$$p_0 y'' + p_1 y' + p_2 y = 0, \text{ we have}$$

$$p_0 = \sin x, p_1 = -\cos x \text{ and } p_2 = 2 \sin x.$$

Now $p''_0 - p'_1 + p_2 = -\sin x - \sin x + 2 \sin x = 0$

Hence the given equation is exact. It can be rewritten as

$$\frac{d}{dx}(p_0 y' + p_1 y - p_0' y) = \cos x$$

i.e. $$\frac{d}{dx}\left[(\sin x)\,y' - (\cos x)\,y - (\cos x)\,y\right] = \cos x$$

$\therefore$ The first solution of the given equation is $(\sin x)\,y' - 2(\cos x)\,y = \sin x + c_1$

i.e. $$y' - 2(\cot x)y = 1 + c_1 \operatorname{cosec} x \tag{1}$$

Equation (1) is a linear equation of the first order

$$\text{I.F.} = e^{-\int 2\cot x\,dx}$$
$$= e^{-2\log\sin x} = \frac{1}{\sin^2 x}$$

$\therefore$ Solution of (1) is $$\frac{y}{\sin^2 x} = -\cot x + c_1 \int \operatorname{cosec}^3 x\,dx + c_2$$

$$= -\cot x + \frac{c_1}{2}\left(-\operatorname{cosec} x\cot x + \log\tan\frac{x}{2}\right) + c_2$$

i.e. $$y = -\sin x\cos x + \frac{c_1}{2}\left(\sin^2 x\log\tan\frac{x}{2} - \cos x\right) + c_2\sin^2 x$$

## EXERCISE 5(d)

***Part A***

(Short Answer Questions)

1. If $y = u(x)$ and $y = u(x)\,v(x)$ are solutions of the equation $y'' + p(x)\,y' + q(x)\,y = 0$, write down the first order equation satisfied by $v(x)$.
2. When is a second order linear differential equation said to be in the canonical form?
3. Write down the transformation which will convert the equation $y'' + p(x)\,y' + q(x)\,y = r(x)$ into the normal form.
4. When the equation $y'' + p(x)\,y' + q(x)\,y = r(x)$ is transformed as $v'' + f(x)\,v = g(x)$ by the substitution $y = v\ \exp\left[-1/2\int p(x)\,dx\right]$, what are the values of $f(x)$ and $g(x)$?
5. What is the substitution to be made to convert the equation $y'' = f(x, y')$ and $y'' = f(y, y')$ into first order equation? Indicate the difference in the subsequent procedures.
6. Write down the substitution to be made to convert the equation $f(x, y, y', y'') = 0$ that is homogeneous in $y, y', y''$, into a first order equation.
7. Write down the condition for the equation $p_0(x)\,y'' + p_1(x)\,y' + p_2(x)\,y = r(x)$ to be exact.
8. If the equation $p_0(x)\,y'' + p_1(x)\,y' + p_2(x)\,y = r(x)$ is exact, what is its first integral?

9. How will you identify a second order linear differential equation in $y$ with variable coefficients that has (i) $y = e^x$ as a solution (ii) $y = e^{-x}$ as a solution?
10. How will you identify a second order linear differential equation in $y$ with variable coefficients that has $y = x$ as a solution?

***Part B***

Solve the following equations by the method of reduction of order, after finding one solution of the corresponding homogeneous equation by inspection:

11. $(x+1)\dfrac{d^2y}{dx^2} - 2(x+3)\dfrac{dy}{dx} + (x+5)y = e^x$

12. $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = e^x$

13. $\dfrac{d^2y}{dx^2} - \cot x \cdot \dfrac{dy}{dx} - (1-\cot x)\, y = e^x \sin x$

14. $x^2\dfrac{d^2y}{dx^2} + 2x(x-1)\dfrac{dy}{dx} + x(x-2)y = 0$

15. $(x-1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 1$

16. $x\dfrac{d^2y}{dx^2} + (2x+1)\dfrac{dy}{dx} + (x+1)\, y = 0$

17. $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 6(1-x^2)$

18. $\dfrac{d^2y}{dx^2} - \dfrac{4x}{2x-1}\dfrac{dy}{dx} + \dfrac{4}{2x-1}y = 0$

19. $x^2\dfrac{d^2y}{dx^2} - (x^2+2x)\dfrac{dy}{dx} + (x+2)y = x^3e^x$

20. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 3y = x^2(2x-1)$

Solve the following equations by the method of reduction:

21. $\dfrac{d^2y}{dx^2} + \left(\dfrac{x^2-2x-2}{x^2+x}\right)\dfrac{dy}{dx} - \left(\dfrac{2x^2-2x-2}{x^3+x^2}\right)y = 0$, given that $y = x^2$ is a solution.

22. $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = 0$, given that $y = x^3$ is a solution.

23. $\dfrac{d^2y}{dx^2} + y = \sec x$, given that $y = \cos x$ is a solution of the corresponding homogeneous equation.

24. $\dfrac{d^2y}{dx^2} + 4y = 4\tan 2x$, given that $y = \sin 2x$ is a solution of the corresponding homogeneous equation.

25. $x^2\dfrac{d^2y}{dx^2} + \left(x^2 - x\tan x - \dfrac{3}{4}\right)y = 0$, given that $y = \dfrac{1}{\sqrt{x}}\cos x$ is a solution.

26. $x(1+3x^2)\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 6xy = 0$, given that $y = \dfrac{1}{x}$ is a solution.

27. $(x^2-1)\dfrac{d^2y}{dx^2} - 6y = 1$, given that $y = x - x^3$ is a solution of the corresponding homogeneous equation.

28. $\cos^2 x \cdot \dfrac{d^2y}{dx^2} = 2y$, given that $y = \tan x$ is a solution.

29. $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is a solution.

30. Find the values of $a$ and $b$ if $y = x$ is a solution of the homogeneous equation corresponding to the equation $x^2\dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + (ax+b)y = x^3$. For these values of $a$ and $b$, solve the equation completely.

31. Solve the equation $(x\sin x + \cos x)\dfrac{d^2y}{dx^2} - x\cos x\dfrac{dy}{dx} + y\cos x = 0$, given that $y = x^m$ is a solution of the equation.

Reduce the following equations to the canonical form and hence solve them:

32. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - \left(x^2 + \dfrac{1}{4}\right)y = 0$

33. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + \left(x^2 - \dfrac{1}{4}\right)y = 0$

34. $x^2 \frac{d^2 y}{dx^2} - 2\left(x^2 + x\right) \frac{dy}{dx} + (x^2 + 2x + 2)y = 0$

35. $4x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + (16x^2 - 1)y = 4x^{\frac{3}{2}} \sin x$

Reduce the order of the following equations by suitable transformations and hence solve them:

36. $\frac{d^2 y}{dx^2} + \tan x \cdot \frac{dy}{dx} = \sin 2x$, given that $y(0) = -1$ and $y'(0) = 0$.

37. $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 1$ , given that $y(0) = 0$ and $y(1) = \frac{\pi^2}{4}$.

38. $(1 + x^2) \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$

39. $\frac{d^2 y}{dx^2} = x\left(\frac{dy}{dx}\right)^3$

40. $y \frac{d^2 y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^3 = 0$

41. $y \cdot \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} = \left(\frac{dy}{dx}\right)^3 + \left(\frac{d^2 y}{dx^2}\right)^2$

42. $y \frac{d^2 y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$

43. $\left(\frac{d^2 y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = a^2$, given that $y(0) = -1$ and $y'(0) = 0$.

44. $y \frac{d^2 y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 6xy^2$

45. $y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 2y^2$

46. $y \frac{d^2 y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$

Show that the following equations are exact and hence solve them

47. $x^2.\dfrac{d^2 y}{dx^2}+x\dfrac{dy}{dx}-y=0$

48. $(1-x^2)\dfrac{d^2 y}{dx^2}-3x\dfrac{dy}{dx}-y=1$

49. $(x^2+1)\dfrac{d^2 y}{dx^2}+3x\dfrac{dy}{dx}+y=1$

50. $x\dfrac{d^2 y}{dx^2}+(x+2)\dfrac{dy}{dx}+y=e^{-x}$

## 5.7 METHOD OF VARIATION OF PARAMETERS

The method of Variation of Parameters is another method for solving a linear differential equation, either of the first order or of the second order. If the given equation is of the form $f(D)y = x$, this method can be applied to get the general solution, provided the corresponding homogeneous equation, viz. $f(D)y = 0$ can be solved by earlier methods. The procedures to solve linear equations of the first and second orders are the following.

*Solution of the equation* $\dfrac{dy}{dx}+Py=Q$ (1)

*where P and Q are functions of x.*

The homogeneous equation corresponding to equation (1) is

$$\frac{dy}{dx}+Py=0 \tag{2}$$

i.e.
$$\frac{dy}{y}=-P\text{d}x$$

∴
$$\log y=-\int P\text{d}x+\log c=\log ce^{-\int P\text{d}x}$$

∴ solution of Eq. (2) is

$$y=ce^{-\int P\text{d}x} \tag{3}$$

Now we treat the arbitrary constant $c$ as a function of $x$ and assume that (3) is the required solution of (1).

Differentiating (3) with respect to $x$, we have

$$\frac{dy}{dx} = c \cdot e^{-\int P dx} \cdot (-P) + \frac{dc}{dx} \cdot e^{-\int Pd} \tag{4}$$

Since (3) is assumed as the solution of (1), (3) and (4) satisfy (1)

$$\therefore \; -cPe^{-\int P dx} + \frac{dc}{dx} e^{-\int P dx} + cPe^{-\int P dx} = Q$$

i.e.

$$\frac{dc}{dx} = Qe^{-\int P dx}$$

$\therefore$

$$c = \int Q \cdot e^{\int P dx} .dx + A \tag{5}$$

Using (5) in (3), the required general solution of (1) is

$$y = e^{-\int P dx} \left[ \int Qe^{\int P dx} \cdot dx + A \right]$$

$$ye^{\int P dx} = \int Q \cdot e^{\int P dx} .dx + A \tag{6}$$

*Note* ☑ Solution (6) should not be treated as a formula and hence should not be directly used in problems. The procedure used in obtaining (6) alone should be used in solving problems.

*Solution of the equation* $\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R$ (1)

*where P, Q and R are functions of x or constants.*
The homogeneous equation corresponding to equation (1) is

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \tag{2}$$

Let the general solution of equation (2) be

$$y = Au + Bv \tag{3}$$

where $A$ and $B$ are arbitrary constants (parameters) and $y = u(x)$ and $y = v(x)$ are independent particular solutions of Eq. (2).

Now we treat $A$ and $B$ as functions of $x$ and assume (3) to be the general solution of (1). Differentiating (3) with respect to $x$, we have

$$\frac{dy}{dx} = (Au' + Bv') + (A'u + B'v) \tag{4}$$

We choose $A$ and $B$ such that

$$A'u + B'v = 0 \tag{5}$$

Then (4) becomes

$$\frac{dy}{dx} = Au' + Bv' \tag{6}$$

Differentiating (6) with respect to $x$, we have

$$\frac{d^2y}{dx^2} = Au'' + Bv'' + A'u' + B'v' \tag{7}$$

Since (3) is a solution of Eq. (1), (3), (6) and (7) satisfy (1).

$$\therefore \quad (Au'' + Bv'' + A'u' + B'v') + P(Au' + Bv') + Q(Au + Bv) = R$$

i.e. $$A(u'' + Pu' + Qu) + B(v'' + Pv' + Qv) + A'u' + B'v' = R \tag{8}$$

Since $y = u$ is a solution of Equation (2)

$$\therefore \quad u'' + Pu' + Qu = 0$$

Similarly $$v'' + Pv' + Qv = 0$$

Inserting these values in (8), it reduces to

$$A'u' + B'v' = R \tag{9}$$

Solving (5) and (9), we get the values of $A'$ and $B'$ integrating which, we get the values of $A$ and $B$ as functions of $x$. Using these values in (3), we get the required general solution of Eq. (1).

## Notes ☑

1. To solve Eq. (1) by the method of variation of parameters, one should know the complementary function of (1) and remember (5) and (9), solving which the values of $A$ and $B$ are obtained.
2. Equations (5) and (9) hold good, only if the coefficient of $\frac{d^2y}{dx^2}$ in the given differential equation is unity.
3. The method is known as variation of parameters, as we treat the parameters (arbitrary constants) $A$ and $B$ as varying functions of $x$.

## WORKED EXAMPLE 5(e)

**Example 5.1** Solve the equation $(x^2 + 1)\frac{dy}{dx} + 4xy = \frac{1}{x^2 + 1}$, by using the method of variation of parameters.

The homogeneous equation corresponding to the given equation is

$$\left(x^2+1\right)\frac{dy}{xy}+4xy=0 \tag{1}$$

i.e. $$\frac{dy}{y}+\frac{4x}{x^2+1}\,dx=0$$

Integrating, we get $\log y + 2\log(x^2+1) = \log c$

i.e. $$y=\frac{c}{\left(x^2+1\right)^2} \text{ is the solution of (1)} \tag{2}$$

Treating $c$ as a function of $x$ and differentiating (2) with respect to $x$, we have

$$\frac{dy}{dx}=\frac{\left(x^2+1\right)^2 c' - c\cdot 2\left(x^2+1\right)\cdot 2x}{\left(x^2+1\right)^4} \tag{3}$$

Using (2) and (3) in the given equation, we have

$$\frac{\left(x^2+1\right)^2 c' - 4cx\left(x^2+1\right)}{\left(x^2+1\right)^3}+\frac{4cx}{\left(x^2+1\right)^2}=\frac{1}{x^2+1}$$

i.e. $$(x^2+1)^2 c' - 4cx\,(x^2+1) + 4cx\,(x^2+1) = (x^2+1)^2$$

i.e. $$c' = 1 \qquad \therefore\ c = x + k \tag{4}$$

Using (4) in (2), the required general solution of the given equation is $(x^2+1)^2\,y = x + k$, where $k$ is an arbitrary constant.

**Example 5.2** Solve the equation $\dfrac{dy}{dx} + x\,\sin 2y = x^3\cos^2 y$, by the method of variation of parameters.

The method of variation of parameters can be applied to solve only a linear differential equation. The given equation is not linear. We shall convert the given equation into a linear equation and then apply the method of variation of parameters. Dividing the given equation by $\cos^2 y$, we get

$$\sec^2 y\,\frac{dy}{dx}+2x\tan y = x^3 \tag{1}$$

Putting $\tan y = z$, Eq. (1) becomes

$$\frac{dz}{dx}+2x\cdot z = x^3\text{, which is linear.} \tag{2}$$

The homogeneous equation corresponding to Eq. (2) is

$$\frac{dz}{dx} + 2xz = 0 \quad \text{or} \quad \frac{dz}{z} + 2x dx = 0 \tag{3}$$

Integrating, we get $\log z = \log c - x^2$

i.e. the solution of Eq. (3) is

$$z = ce^{-x^2} \tag{4}$$

Treating $c$ as a function of $x$ and differentiating (4) with respect to $x$,

$$\frac{dz}{dx} = -2cxe^{-x^2} + c'e^{-x^2} \tag{5}$$

Using (4) and (5) in (2),

$$-2cxe^{-x2} + c'e^{-x2} + 2cxe^{-x2} = x^3$$

i.e. $$c' = x^3 e^{x2}$$

$\therefore$ $$c = \int x^3 e^{x2} dx + k$$

$$= \frac{1}{2}\int t\, e^t \, dt + k \text{, on putting } x^2 = t$$

$$= \frac{1}{2}\left(te^t - e^t\right) + k$$

$$= \frac{1}{2}\left(x^2 - 1\right) e^{x^2} + k \tag{6}$$

Using (6) in (4), the required general solution of Eq. (2) is

$$z = \frac{1}{2}\left(x^2 - 1\right) + ke^{-x^2}$$

Therefore the general solution of the given equation is

$$\tan y = \frac{1}{2}\left(x^2 - 1\right) + ke^{-x^2}$$

where $k$ is an arbitrary constant

**Example 5.3** Solve the equation $\frac{d^2y}{dx^2} + y = x \cos x$, by the method of variation of parameters.

$$\frac{d^2y}{dx^2} + y = x \cos x \tag{1}$$

The homogeneous equation corresponding to Eq. (1) is

$$\frac{d^2y}{dx^2} + y = 0 \tag{2}$$

The solution of Eq. (2) is

$$y = A \cos x + B \sin x, \tag{3}$$

where $A$ and $B$ are parameters.

Treating $A$ and $B$ as functions of $x$, $A'$ and $B'$ are given by

$$-A' \sin x + B' \cos x = x \cos x \tag{4}$$

and $$A' \cos x + B' \sin x = 0 \tag{5}$$

by Eq. (5) and (9) of the discussion of the method of variation of parameters. Solving (4) and (5), we get

$$A' = -x \sin x \cos x \quad \text{or} \quad -\frac{1}{2} x \sin 2x \text{ and} \tag{6}$$

$$B' = x \cos^2 x \quad \text{or} \quad \frac{1}{2} x(1 + \cos 2x) \tag{7}$$

Integrating (6) and (7) with respect to $x$, we get

$$A = -\frac{1}{2}\left[\frac{-x \cos 2x}{2} + \frac{\sin 2x}{4}\right] + c_1 \tag{8}$$

and $$B = \frac{x^2}{4} + \frac{1}{2}\left[\frac{x \sin 2x}{2} + \frac{\cos 2x}{4}\right] + c_2 \tag{9}$$

Using (8) and (9) in (3), the general solution of Eq. (1) is

$$y = \left(c_1 + \frac{x}{4}\cos 2x - \frac{1}{8}\sin 2x\right)\cos x + \left(c_2 + \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x\right)\sin x$$

i.e. $$y = c_1 \cos x + c_2 \sin x - \frac{1}{8}\sin x + \frac{x^2}{4}\sin x + \frac{x}{4}\cos x$$

or $$y = c_1 \cos x + c_3 \sin x + \frac{x^2}{4}\sin x + \frac{x}{4}\cos x$$

where $\left(c_2 - \frac{1}{8}\right)$ has been assumed as $c_3$.

**Example 5.4** Solve the equation $\frac{d^2y}{dx^2} + a^2y = \tan ax$, by the method of variation of parameters.

$$\frac{d^2y}{dx^2} + a^2y = \tan ax \tag{1}$$

The homogeneous equation corresponding to Eq. (1) is

$$\frac{d^2y}{dx^2} + a^2y = 0 \tag{2}$$

The solution of Eq. (2) is

$$y = A\cos ax + B\sin ax \tag{3}$$

If we treat $A$ and $B$ as functions of $x$, $A'$ and $B'$are given by

$$-aA'\sin ax + aB'\cos ax = \tan ax \tag{4}$$

and
$$A'\cos ax + B'\sin ax = 0 \tag{5}$$

Solving (4) and (5), we get

$$A' = -\frac{1}{a}\frac{\sin^2 ax}{\cos ax} \tag{6}$$

and
$$B' = \frac{1}{a}\sin ax \tag{7}$$

Integrating (6) and (7) with respect to $x$, we get

$$A = \frac{1}{a^2}\left[\sin ax - \log\left(\sec ax + \tan ax\right)\right] + c_1 \tag{8}$$

and
$$B = -\frac{1}{a^2}\cos ax + c_2 \tag{9}$$

Using (8) and (9) in (3), the general solution of Eq. (1) is

i.e.
$$y = \left[c_1 + \frac{1}{a^2}\left\{\sin ax - \log\left(\sec ax + \tan ax\right)\right\}\right]\cos ax$$

$$+\left(c_2 - \frac{1}{a^2}\cos ax\right)\sin ax$$

i.e.
$$y = c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos ax \cdot \log\left(\sec ax + \tan ax\right)$$

**Example 5.5** Solve the equation $(2D^2 - D - 3)\,y = 25e^{-x}$, by the method of variation of parameters.

As the formulae (5) and (9) given in the discussion of the procedure can be applied, only if the coefficient of $D^2y$ is unity, we rewrite the given equation as

$$\left(D^2 - \frac{1}{2}D - \frac{3}{2}\right)y = \frac{25}{2}e^{-x} \tag{1}$$

The homogeneous equation corresponding to (1) is

$$\left(D^2 - \frac{1}{2}D - \frac{3}{2}\right)y = 0 \tag{2}$$

The auxiliary equation corresponding to Eq. (2) is

$$m^2 - \frac{1}{2}m - \frac{3}{2} = 0$$

or

$$\left(m - \frac{3}{2}\right)(m+1) = 0$$

$\therefore m = \dfrac{3}{2}$ and $-1$.

Therefore the solution of Equation (2) is

$$y = Ae^{\frac{3}{2}x} + Be^{-x} \tag{3}$$

Treating $A$ and $B$ as functions of $x$, $A'$ and $B'$ are given by

$$\frac{3}{2}A'\cdot e^{\frac{3}{2}x} + B'e^{-x} = \frac{25}{2}e^{-x} \tag{4}$$

and

$$A'e^{\frac{3}{2}x} + B'e^{-x} = 0 \tag{5}$$

Solving (4) and (5), we get

$$A' = 5e^{-\frac{5}{2}x} \text{ and } B' = -5$$

Integrating, we get,

$$A = -2e^{-\frac{5}{2}x} + c_1 \text{ and } B = -5x + c_2$$

Using these values in (3), the general solution of Eq. (1) is

$$y = \left(c_1 - 2e^{-\frac{5}{2}x}\right)e^{\frac{3}{2}x} + (c_2 - 5x)\,e^{-x}$$

i.e. $$y = c_1 e^{\frac{3}{2}x} + c_2 e^{-x} - 2e^{-x} - 5xe^{-x}$$

**Example 5.6** Solve the equation $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = \sin(\log x)$, by the method of variation of parameters.

To make the coefficient of $\frac{d^2y}{dx^2}$ as unity, we rewrite the given equation as

$$\frac{d^2y}{dx^2} - \frac{4}{x}\frac{dy}{dx} + \frac{6}{x^2}y = \frac{1}{x^2}\sin(\log x) \quad (1)$$

The homogeneous equation corresponding to (1) is

$$\frac{d^2y}{dx^2} - \frac{4}{x}\frac{dy}{dx} + \frac{6}{x^2}y = 0 \text{ or}$$

$$x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0 \quad (2)$$

Putting $x = e^t$ or $\log x = t$ and denoting $\frac{d}{dt}$ by $\theta$, Eq. (2) becomes

$$[\theta(\theta - 1) - 4\theta + 6]\, y = 0$$

i.e. $$(\theta^2 - 5\theta + 6)\, y = 0 \quad (3)$$

The auxiliary equation corresponding to (3) is

$$m^2 - 5m + 6 = 0$$

$\therefore$ $$m = 2, 3$$

Therefore the solution of Eq. (2) is

$$y = Ae^{2t} + Be^{3t}$$

or $$y = Ax^2 + Bx^3 \quad (4)$$

Treating $A$ and $B$ as functions of $x$, $A'$ and $B'$ are given by

$$2A'x + 3B'x^2 = \frac{1}{x^2}\sin(\log x) \quad (5)$$

and $$A'x^2 + B'x^3 = 0 \quad (6)$$

Solving (5) and (6), we get

$$A' = \frac{1}{x^3}\sin\log x \quad \text{and} \quad B' = \frac{1}{x^4}\sin\log x.$$

$$\therefore \qquad A = -\int \frac{1}{x^3} \sin \log x \, dx + c_1 \text{ and } B = \int \frac{1}{x^4} \sin \log x \, dx + c_2$$

i.e. $$A = -\int e^{-2t} \sin t \, dt + c_1 \text{ and } B = \int e^{-3t} \sin t \, dt + c_2,$$

on putting $\log x = t$ or $x = e^t$

i.e. $$A = c_1 - \frac{e^{-2t}}{5}(-2 \sin t - \cos t) \text{ and}$$

$$B = c_2 + \frac{e^{-3t}}{10}(-3 \sin t - \cos t)$$

i.e. $$A = c_1 + \frac{1}{5x^2}(2 \sin \log x + \cos \log x) \text{ and}$$

$$B = c_2 - \frac{1}{10x^3}(3 \sin \log x + \cos \log x)$$

Using these values of $A$ and $B$ in (4), the required solution of Eq. (1) is

$$y = \left[c_1 + \frac{1}{5x^2}(2 \sin \log x + \cos \log x)\right] x^2$$

$$+ \left[c_2 - \frac{1}{10x^3}(3 \sin \log x + \cos \log x)\right] x^3$$

i.e. $$y = c_1 x^2 + c_2 x^3 + \left(\frac{2}{5} - \frac{3}{10}\right) \sin \log x + \left(\frac{1}{5} - \frac{1}{10}\right) \cos \log x$$

i.e. $$y = c_1 x^2 + c_2 x^3 + \frac{1}{10}\{\sin \log x + \cos \log x\}.$$

**Example 5.7** Solve the equation $x^2 \frac{d^2 y}{d x^2} - x \frac{d y}{d x} + y = x \log x$, by the method of variation of parameters.

To make the coefficient of $\frac{d^2 y}{d x^2}$ as unity, we rewrite the given equation as

$$\frac{d^2 y}{d x^2} - \frac{1}{x}\frac{d y}{d x} + \frac{1}{x^2} y = \frac{1}{x} \log x \qquad (1)$$

The homogeneous equation corresponding to Eq. (1) is

$$\frac{d^2 y}{d x^2} - \frac{1}{x}\frac{d y}{d x} + \frac{1}{x^2} y = 0 \text{ or}$$

$$x^2 \frac{d^2 y}{d x^2} - x\frac{d y}{d x} + y = 0 \qquad (2)$$

Putting $x = e^t$ or $t = \log x$ and denoting $\frac{d}{dt}$ by $\theta$ , Eq. (2) becomes

$$\left[\theta\left(\theta-1\right)-\theta+1\right]y=0$$

i.e. $$\left(\theta-1\right)^2 y = 0 \qquad (3)$$

Therefore the solution of Eq. (3) is

$$y = \left(At + B\right)e^t \text{ or } y = Ax \log x + Bx \qquad (4)$$

Treating $A$ and $B$ as functions of $x$, $A'$ and $B'$ are given by

$$A'\left(1 + \log x\right) + B' = \frac{1}{x}\log x \qquad (5)$$

and $$A' x \log x + B' x = 0 \qquad (6)$$

Solving (5) and (6), we get

$$A' = \frac{1}{x}\log x \text{ and } B' = -\frac{1}{x}\left(\log x\right)^2$$

$\therefore$ $$A = \int \frac{1}{x}\log x \, dx \text{ and } B = -\int \frac{1}{x}\left(\log x\right)^2 dx$$

i.e. $$A = \int t \, dt \text{ and } B = -\int t^2 dt \text{ on putting } \log x = t$$

$\therefore$ $$A = \frac{1}{2}\left(\log x\right)^2 + c_1 \text{ and } B = -\frac{1}{3}\left(\log x\right)^3 + c_2$$

Using these values of $A$ and $B$ in (4), the required solution of Eq. (1) is

$$y = \left[\frac{1}{2}\left(\log x\right)^2 + c_1\right] x \log x + \left[-\frac{1}{3}\left(\log x\right)^3 + c_2\right] x$$

i.e. $$y = c_1 \, x \log x + c_2 x + \frac{1}{6} x \left(\log x\right)^3$$

## EXERCISE 5(e)

Solve the following equations by the method of variation of parameters.

1. $\frac{dy}{dx} - y\tan x = e^x \sec x$
2. $x\frac{dy}{dx} + (1+x)y = e^{-x}$
3. $\frac{d^2y}{d^2x} + y = x\sin x$
4. $\frac{d^2y}{d^2x} + a^2y = \sec ax$
4. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x$
6. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{1}{x^2}e^{3x}$
7. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^2 + \frac{1}{x^2}$
8. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 32(\log x)^2$

## ANSWERS

**Exercise 5(a)**

(7) $(y-2x-C)(y-3x-C)=0$

(8) $\left(y-\frac{x^2}{2}-C\right)(\log y-x-C)=0$

(9) $(y-e^x-C)(e^y-x-C)=0$

(10) $(y-\log x-C)\left(\frac{y^2}{2}-x-C\right)=0$

(11) $y=px+\frac{p}{p-1}$; $y=Cx+\frac{C}{C-1}$

(12) $y=px-e^p$; $y=Cx-e^C$

(13) $y=px+\sin^{-1}p$; $y=Cx+\sin^{-1}C$

(14) $y^2=4x$

(15) $x^2=4y$

(16) $(y-x-C)(x^2+y^2-C)=0$

(17) $\left(y-\frac{x^2}{4}-C\right)\left(x+\frac{1}{y}+C\right)=0$

(18) $(y-Cx^2)(3x^2-y^2-C)=0$

(19) $(y - Cx)(x^2 - y^2 - C) = 0$ (20) $(xy - C)(x^2 - y^2 - C) = 0$

(21) $[(y(1 + \cos x) - C][y(1 - \cos x) - C] = 0$

(22) $\left(\sin^{-1}\frac{y}{x} + \log x - C\right)\left(\sin^{-1}\frac{y}{x} - \log x - C\right) = 0$

(23) $y + \frac{C}{x} = C^2;\ y + \frac{1}{4x^2} = 0$

(24) Eliminant of $p$ between $4y = x^2 + p^2$ and $\log(p - x) = \frac{x}{p - x} + C.$

(25) Eliminant of $p$ between $x = -\frac{n}{n+1}p^{n-1} + \frac{C}{p^2}$ and the given equation.

(26) Eliminant of $p$ between $x = \frac{2p}{3} + \frac{C}{p^2}$ and $y = \frac{p^2}{3} + \frac{2C}{p}$.

(27) Eliminant of $p$ between $x + C = \log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1}p$ and the given equation.

(28) Eliminant of $p$ between $x = Ce^{-p} - \frac{e^p}{2}$ and $y = C(1+p)e^{-p} + \frac{1}{2}(1-p)e^p$

(29) $y = 3x + \log\left(\frac{3}{1 - Ce^{3x}}\right)$

(30) $64y = C(C - 4x)^2;\ 4x^3 = 27y$

(31) Eliminant of $p$ between $y = C - [2\log(p - 1) + 2p + p^2]$ and $x = C - [2\log(p - 1) + 2p]$

(32) $y^2 = 2Cx + 4C^2$ (33) $y^2 = 2Cx + C^3$ (34) $\log y = Cx + C^2$

(35) $y = -\sqrt{x(1-x)} + \tan^{-1}\sqrt{\frac{x}{1-x}}$ (36) $y = 1 + \log x$

(37) $y^2 = Cx^2 + C^2$ (38) $y^2 = 2Cx + C^2$

(39) $xy = Cx - C^2$ (40) $e^y = Ce^x + C^3$

**Exercise 5(b)**

(1) $y = e^{\frac{x}{2}}\left[(C_1x + C_2)\cos\frac{\sqrt{3}}{2}x + (C_3x + C_4)\sin\frac{\sqrt{3}}{2}x\right].$

(2) $\frac{x^3}{6}e^x - \frac{1}{8}e^{-x}$ (3) $\frac{x}{2a}(b\sin ax - c\cos ax).$

(4) $\frac{x^3}{6}e^{-2x}$ (5) $(x - 2)e^{2x}$

(6) $-e^{-x}\cos x$ (7) $-\frac{1}{4}xe^x\cos 2x$

(8) $\frac{1}{2} x e^{-2x} \sin x$

(9) $e^x (4 \sin x + \cos x)$

(11) $y = C_1 e^{-x} + C_2 \cos x + C_3 \sin x + x^2 - 2x + xe^{-x}$

(12) $y = C_1 \cos 3x + C_2 \sin 3x + \frac{x^2}{9} - \frac{2}{81} + \frac{1}{10} \cosh x$

(13) $y = (Ax + B)e^{-x} + x^3 - 6x^2 + 18x - 24 + \frac{1}{25}(4 \sin 2x - 3 \cos 2x)$

(14) $y = e^{4x}\left(Ae^{\sqrt{7}x} + Be^{-\sqrt{7}x}\right) + \frac{1}{29}(5 \cos 5x - 2 \sin 5x) + \frac{1}{9}\left(x^2 + \frac{16}{9}x + \frac{110}{81}\right)$

(15) $y = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{20}(\cos 2x - 3 \sin 2x) + x^2 - 3x + 4$

(16) $y = (C_1 x + C_2) + e^{-\frac{x}{2}}\left(C_3 \cos\frac{\sqrt{3}}{2}x + C_4 \sin\frac{\sqrt{3}}{2}x\right) + x^4 - 4x^3 +$

$\frac{1}{657}(8 \cos 3x - 3 \sin 3x) - \sin x$

(17) $y = C_1 e^x + C_2 e^{-x} + xe^x + (3 + 3x + 2x^2)$

(18) $y = (C_1 x^2 + C_2 x + C_3)\, e^{2x} + (2x^3 - 9x^2 + 18x - 15)\, e^{4x}$

(19) $y = C_1 + (C_2 x + C_3)e^{-x} + \frac{e^{2x}}{18}\left(x^2 - \frac{7}{3}x + \frac{11}{6}\right)$

(20) $y = C_1 e^{2x} + C_2 e^{-2x} - \frac{x}{3} \sinh x - \frac{2}{9} \cosh x$

(21) $y = (C_1 x + C_2) \cos x + (C_3 x + C_4) \sin x + (x^2 + 4x + 4)\, e^{-x}$

(22) $y = C_1 e^x + C_2 e^{4x} + \frac{1}{18}e^{-2x} + \frac{2}{5}e^{-x}\left(x - \frac{7}{10}\right) + \left(x^2 - \frac{5}{2}x + \frac{21}{8}\right)$

(23) $y = C_1 e^x + C_2 e^{3x} - e^x(\text{cox } 2x + \sin 2x)$

(24) $y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x - \frac{1}{5} \sin x \cosh x$

(25) $y = e^x(C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{4}e^x - \frac{1}{4}xe^x \cos 2x$

(26) $y = C_1 e^{-x} + e^{\frac{x}{2}}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{6}xe^{-x} + \frac{1}{26}e^{-x}(2 \sin x$

$+ 3 \cos x)$

(27) $y = C_1 \cos 2x + C_2 \sin 2x + e^{2x}\Big[\frac{3}{65}(7 \sin x - 4 \cos x)$

$+ \frac{1}{145}(12 \cos 3x + \sin 3x)\Big]$

(28) $y = C_1e^x + C_2e^{3x} + \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x).$

(29) $y = (Ax + B)e^x - e^x(x\sin x + 2\cos x)$

(30) $y = A + Be^{-x} + \frac{x}{2}(\sin x - \cos x) + \cos x + \frac{\sin x}{2}$

(31) $y = (Ax + B)e^{2x} + \frac{x}{25}(3\sin x + 4\cos x) + \frac{2}{125}(11\cos x + 2\sin x)$

(32) $y = Ae^x + Be^{-x} + \frac{1}{2}(1 - x^2)\cos x + x\sin x.$

(33) $y = (C_1x + C_2)\cos 2x + (C_3x + C_4)\sin 2x + \frac{x}{64}(\sin 2x - 2x\cos 2x)$

(34) $y = A\cos x + B\sin x - \frac{1}{27}\left[24x\cos 2x + (9x^2 - 26)\sin 2x\right].$

(35) $y = A\cos 2x + B\sin 2x - \cos 2x\log(\sec 2x + \tan 2x).$

**Exercise 5(c)**

(1) $y = A\log x + B - x$

(2) $y = x(A\log x + B)$

(3) $(\theta^2 - 4\theta + 1)y = e^{3t}$

(4) $(\theta^3 - 4\theta^2 + 4\theta)y = e^{-t}$

(5) $y = Ax^n + Bx^{n+1}.$

(6) $y = \frac{A}{x} + \sqrt{x}\left\{B\cos\left(\frac{\sqrt{3}}{2}\log x\right) + C\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right\}$

(7) $x = A\cos t + B\sin t + 2$

(8) $y = Ax^4 + \frac{B}{x^5} + \frac{1}{9}x^4\log x - \frac{1}{7}x^2 - \frac{1}{20}.$

(9) $y = A + x^2(B + C\log x) - \frac{1}{9x}$

(10) $y = (A\log x + B)x + Cx^2 + \frac{1}{100}\{\sin(2\log x) + 7\cos(2\log x)\}.$

(11) $y = Ax^3 + \frac{B}{x^3} - \frac{3}{40}\sin(\log x) + \frac{1}{72}\sin(3\log x).$

(12) $y = \frac{1}{x^4}\left[A\cos(3\log x) + B\sin(3\log x)\right] + \frac{1}{25}\left[(\log x)^2 - \frac{16}{25}\log x + \frac{78}{625}\right]$

(13) $y = (A\log x + B)\cos(\log x) + (C\log x + D)\sin(\log x) + (\log x)^2 + 2\log x - 3$

(14) $y = x^2(A\log x + B) + x\{(\log x)^2 - 4\log x + 6\}$

(15) $y = \left[A(\log x)^3 + B(\log x)^2 + C\log x + D\right]x + x^2(\log x - 4)$

(16) $y = Ax^3 + \frac{B}{x} - \frac{1}{20x}\{2\sin(2\log x) + \cos(2\log x)\}$

(17) $y = \frac{1}{x}\left\{A\cos(2\log x) + B\sin(2\log x) + \frac{x}{65}\{4\sin(\log x) + 7\cos(\log x)\}\right\}$

(18) $y = A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2 \log(3x+2) + 1\right]$

(19) $y = A \cos \log (x + 1) + B \sin \log (x + 1) + 2 \log (x + 1) \sin \log (x + 1)$

(20) $x = Ae^{-2t} + Be^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{12}e^{2t}$

$y = -\frac{2}{3}Ae^{-2t} + Be^{-7t} - \frac{1}{7}t + \frac{9}{98} + \frac{1}{6}e^{2t}$

(21) $x = -\frac{5}{3}Ae^{-2t} - \frac{4B}{3}e^{t} + \frac{1}{3}e^{t}$

$y = Ae^{-2t} + Be^{t}$

(22) $x = 2 \cosh t; y = \sin t - 2 \sinh t$

(23) $x = (A + Bt)e^{t} + Ce^{-\frac{3t}{2}} - \frac{t}{2}$

$y = (-2A + 6B - 2Bt)e^{t} - \frac{C}{3}e^{-\frac{3t}{2}} - \frac{1}{3}$

(24) $x = A + B\cos t + C\sin t + \frac{9}{10}e^{2t}$

$y = B\cos t + C\sin t - \frac{6}{10}e^{2t}$

(25) $x = A\cos\sqrt{2}\,t + B\sin\sqrt{2}\,t + C\cos\sqrt{3}\,t + D\sin\sqrt{3}\,t - \frac{1}{4}(1 + \cos 2t)$

$y = 1 - 2A\cos\sqrt{2}\,t - 2B\sin\sqrt{2}\,t - C\cos\sqrt{3}\,t - D\sin\sqrt{3}\,t$

(26) $x = (At + B)\cos t + (Ct + D)\sin t + \frac{1}{25}e^{t}(4 \sin t - 3 \cos t)$

$y = -(At + B)\sin t + (Ct + D)\cos t - \frac{1}{25}e^{t}(3 \sin t + 4 \cos t)$

(27) $x = Ae^{t} + Be^{-t} + C\cos 3t + D\sin 3t - \frac{4}{9}t^2 + \frac{5}{9}t - \frac{37}{81}.$

$y = -Ae^{t} - Be^{-t} + C\cos 3t + D\sin 3t + \frac{5}{9}t^2 - \frac{4}{9}t + \frac{44}{81}.$

**Exercise 5(d)**

(11) $y = A(x+1)^5 e^{x} - \frac{x}{4}e^{x} + Be^{x}.$

(12) $y = e^{x}(c_1 \log x + c_2 + x).$

(13) $y = Ae^{-x}\left(2\sin x + \cos x\right) + Be^{x} - \frac{1}{2}e^{x}\cos x$

(14) $y = (Ax^3 + B)e^{-x}$

(15) $y = Axe^{-2x} + Be^{-x}$

(16) $y = e^{-x}(c_1 \log x + c_2)$

(17) $y = \left(\frac{c_1}{2} + \frac{3}{4}\right)x \log\left(\frac{1+x}{1-x}\right) - c_1 + c_2 x + \frac{3}{2}x^2$

(18) $y = c_1 e^{2x} + c_2 x$

(19) $y = c_1 xe^{x} + c_2 x + x(x-1)e^{x}$

(20) $y = c_1 x^3 + c_2 x + x^3 \log x + x^2$

(21) $y = c_1 xe^{x} + c_2 x^2$

(22) $y = c_1 x^3 + c_2 x^{-3}$

(23) $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \log \cos x$

(24) $y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$

(25) $y = \frac{1}{\sqrt{x}}\cos x\left[A + B\left(x\tan x + \log\cos x\right)\right]$

(26) $y = c_1\left(1 + x^2\right) + c_2 \cdot \frac{1}{x}$

(27) $y = c_1\left(x - x^3\right) + c_2\left(4 - 3x - 6x^2 + 3x^3\right)\log\left(\frac{1+x}{1-x}\right) - \frac{1}{6}$ ·

(28) $y = c_1 \tan x - c_2(1 + x\tan x)$

(29) $y = c_1(1 - x\cot x) + c_2 \cot x$

(30) $y = \left(Ae^{2x} + B\right)x - \frac{x^2}{2}$

(31) $y = c_1 x - c_2 \cos x$

(32) $y = \frac{1}{\sqrt{x}}\left(A\cosh x + B\sinh x\right)$

(33) $y = \frac{1}{\sqrt{x}}\left(A\cos x + B\sin x\right)$

(34) $y = xe^{x}(Ax + B)$

(35) $y = \frac{1}{\sqrt{x}}\left(A\cos 2x + B\sin 2x + \frac{1}{3}\sin x\right)$

(36) $y = 2\sin x - \sin x\cos x - x - 1$

(37) $y = \frac{1}{2}\left(\sin^{-1} x\right)^2 + \frac{\pi}{4}\sin^{-1} x$

(38) $y = -\frac{1}{c_1}x + \left(1 + \frac{1}{c_1^2}\right)\log\left(1 + c_1 x\right) + c_2$

(39) $x = c_1 \sin (y - c_2)$

(40) $y + c_1 \log y = x + c_2$

(41) $y = c_2\ e^{c_1 x} + c_1$

(42) $y = c_2\ e^{c_1 x} + \dfrac{1}{c_1}$

(43) $y = a \cos x - (a + 1)$

(44) $y = c_1\ e^{(x^3 + c_2 x)}$

(45) $y^2 = c_1 \cosh (2x + c_2)$

(46) $y^3 = Ax + B$

(47) $y = Ax + \dfrac{B}{x}$

(48) $y = c_1 \sin^{-1} x / \sqrt{1 - x^2} + c_2 / \sqrt{1 - x^2} + 1$

(49) $y \sqrt{1 + x^2} = \sqrt{1 + x^2} + c_1 \sinh^{-1} x + c_2$

(50) $xye^x = c_1\ e^x + c_2 - x.$

**Exercise 5(e)**

(1) $y \cos x = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + c$

(2) $xy\ e^x = x + c$

(3) $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{4} x \sin x - \dfrac{1}{4} x^2 \cos x$

(4) $y = c_1 \cos ax + c_2 \sin ax + \dfrac{1}{a^2} \cos ax \log \cos ax + \dfrac{1}{a} x \sin ax$

(5) $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$

(6) $y = (c_1 x + c_2)\ e^{3x} - e^{3x} \log x$

(7) $y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{1}{12} x^2 - \dfrac{1}{x^2} \log x$

(8) $y = c_1 x^4 + c_2 \cdot \dfrac{1}{x} - 8 \left(\log x\right)^2 + 12 \log x + 13$