

# Mathematics for IIT-JEE

Two Dimensional Coordinate Geometry  
Vector And Three Dimensional Geometry  
Integral Calculus  
Algebra

## Volume-2

Second Edition 2015

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This book contains theory and a large collection of about 7500 questions and is useful for students and learners of IIT-JEE, Higher and Technical Mathematics; and also for the students who are preparing for Standardized Tests, Achievement Tests, Aptitude Tests and other competitive examinations all over the world

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**First Edition 2014**

**Second Edition 2015**

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**ISBN-13: 978-1511497176**

**ISBN-10: 1511497173**

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## ABOUT THE AUTHOR



Indian Institute of Technology Joint Entrance Examination (IIT-JEE) is considered to be one of the toughest competitive entrance examinations in the world with success ratio well below 1%. The author of this book Sanjiva Dayal was selected in IIT-JEE in his first attempt. After completing B.Tech. from IIT Kanpur, since the last more than 25 years, he has been teaching IIT-JEE Mathematics to the students aspiring for success in IIT-JEE and other engineering entrance examinations with an objective to provide training of the highest order by way of specialized and personalized oral coaching, teaching, training, guidance, educational facilities including tutorials, tests, assignments, reading materials etc. and all such facilities which are helpful to the students seeking admission in the IIT and other Engineering Institutes of India.

His excellent teaching and result-oriented methods have combined to produce outstanding performance by his students in the past years. A large number of his students were selected in their First Attempt with top ranks. Most of his students have secured admission in the IIT and other Engineering Institutes of India and they are doing well in their engineering education and professional career also.

Conceptual understanding of Mathematics provided by him not only helps his students to do well in the IIT-JEE and other engineering entrance examinations but also helps them to do well in their engineering studies where they are pitted against the best students of the country and competition is much more intense.

## ACKNOWLEDGEMENTS

I thank the Almighty God for my very existence and for providing me the opportunity and resources to write this book.

Mother is the first teacher and father is the second teacher. My respects and deepest gratitude to my mother Late Pushpa Dayal, my father Late Chandra Mauleshwar Dayal, my grandfather Late Lakshmeshwar Dayal, my grandmother Late Sarju Dayal, my maternal grandfather Late Ranbir Jang Bahadur and my maternal grandmother Late Chandra Kishori Bahadur who are no longer in this world but their blessings are always with me.

I am extremely thankful to all my school teachers and all my IIT Kanpur Professors who have taught and trained me.

I am also extremely thankful to all my batch mates in IIT Kanpur because studying and training with such brilliant persons was an amazing experience for me which will be cherished by me throughout my life.

I am also extremely thankful to all my students because teaching and interacting with such brilliant minds has been an amazing learning experience for me which will be cherished by me throughout my life.

I am also extremely thankful to my wife Dr. Vidya Dayal, my sister Ms. Smita Prakash and my niece Ms. Somya Prakash for their love, cooperation and moral support.

I am also extremely thankful to my childhood friends Mr. Navneet Butan, Mr. Pankaj Agarwal, Mr. Akhil Srivastava and Mr. Subir Nigam for their love, cooperation and moral support.

I am also thankful to all other persons who have ever helped me directly or indirectly.

I am also thankful to my student Mr. Ashutosh Verma for his help and efforts in publishing this book.

Place: Kanpur, India.

Date: 28<sup>th</sup> March, 2015.

Sanjiva Dayal

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## PREFACE TO THE SECOND EDITION 2015

**"If there is a God, he's a great mathematician."~Paul Dirac**

**"How is an error possible in mathematics?"~Henri Poincare**

**"Go down deep enough into anything and you will find mathematics."~Dean Schlicter**

- **About Indian Institute of Technology Joint Entrance Examination (IIT-JEE)**

Indian Institute of Technology (IIT) are the premier Engineering Institutes of India. Career in engineering calls for high level of confidence, motivation and capacity to do hard work besides intelligence and analytical & deductive thinking. Therefore, in order to select the brightest students for admission in the Indian Institute of Technology (IIT), National Institute of Technology (NIT) and other Engineering Institutes of India, the competitive entrance examination Indian Institute of Technology Joint Entrance Examination (IIT-JEE) is conducted on all India basis every year, along with other state level engineering entrance examinations. IIT-JEE is considered to be one of the toughest competitive entrance examinations in the world with success ratio well below 1%. The number of candidates appearing in IIT-JEE as well as the level of competition has grown tremendously in the past years thus creating a need for highly specialized course content and result-oriented coaching and training.

- **About the student**

An engineering aspirant should possess sharp mental reflexes, acumen and skill to understand and master the fundamental concepts of Science and Mathematics; should have the basic qualities of an ideal student; should be capable of going through the rigorous and tough training schedule to achieve the standards set by the IIT-JEE and other engineering entrance examinations; and must be hungry for achievement. With specialized coaching and training, such talented students can achieve success in IIT-JEE and other engineering entrance examinations.

- **Purpose of the book**

This book is useful for students and learners of IIT-JEE, Higher and Technical Mathematics; and also for the students who are preparing for Standardized Tests, Achievement Tests, Aptitude Tests and other competitive examinations all over the world.

- **Organization of the book**

This book is a part of book series which is divided into two volumes, seven parts and twenty eight chapters as under:-

Volume-1

Part-I: Differential Calculus

- Chapter-1: Real Functions, Domain, Range
- Chapter-2: Limit
- Chapter-3: Continuity
- Chapter-4: Derivatives
- Chapter-5: Applications Of Derivatives
- Chapter-6: Investigation Of Functions And Their Graphs

Part-II: Algebra

- Chapter-7: Equations And Inequalities
- Chapter-8: Quadratic Expressions
- Chapter-9: Progressions
- Chapter-10: Determinants And Matrices

Part-III: Trigonometry

- Chapter-11: Trigonometric Identities And Expressions
- Chapter-12: Trigonometric And Inverse Trigonometric Functions, Equations And Inequalities
- Chapter-13: Properties Of Triangles

Volume-2

Part-IV: Two Dimensional Coordinate Geometry

- Chapter-14: Point And Straight Line
- Chapter-15: Circle
- Chapter-16: Parabola
- Chapter-17: Ellipse
- Chapter-18: Hyperbola

Part-V: Vector And Three Dimensional Geometry

- Chapter-19: Vector
- Chapter-20: Three Dimensional Coordinate Geometry

Part-VI: Integral Calculus

- Chapter-21: Indefinite Integrals
- Chapter-22: Definite Integral
- Chapter-23: Differential Equations
- Chapter-24: Applications Of Calculus

Part-VII: Algebra

- Chapter-25: Complex Numbers
- Chapter-26: Binomial Theorem

Chapter-27: Permutations and Combinations  
Chapter-28: Probability

This book contains theory and a large collection of about 7500 questions. In each chapter, theory is divided into Sections and questions are divided into Question Categories. For each Section there are one or more corresponding Question Category/ Categories in order to make this book more readable and more useful for the readers and students.

- **Using the book**

Each Section and its corresponding Question Category/ Categories is the prerequisite of its following Sections and their corresponding Question Category/ Categories. It is recommended that readers and students should read a Section and then solve the questions of its corresponding Question Category/ Categories; and thereafter go to the next Section and so on. This approach is necessary for proper conceptual development and problem solving skills.

- **Reader's feedback**

All readers and students are requested and welcome to send their feedback, comments, corrections, suggestions, improvements, mathematical theory and questions on my email ID [sanjivadaya1@yahoo.com](mailto:sanjivadaya1@yahoo.com). This will help in continuous improvement and updation of this book. All readers and students are invited to join me as friends on Facebook at URL [www.facebook.com/sanjiva.dayal](http://www.facebook.com/sanjiva.dayal) and they are also invited to join my Facebook Group "Mathematics By Sanjiva Dayal".

Place: Kanpur, India.

Date: 28<sup>th</sup> March, 2015.

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# Mathematics for IIT-JEE

By **Er. Sanjiva Dayal,** *B.Tech. (I.I.T. Kanpur)*

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## *PART-IV*

### *TWO DIMENSIONAL COORDINATE GEOMETRY*

#### *CHAPTER-14*

#### *POINT AND STRAIGHT LINE*

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## **CHAPTER-14**

### **POINT AND STRAIGHT LINE**

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#### **LIST OF THEORY SECTIONS**

- 14.1. Locus, Coordinates, Equation
- 14.2. Points
- 14.3. General Equation Of A Straight Line
- 14.4. Equation Of A Straight Line In Different Standard Forms
- 14.5. Transformation Of General Equation Of A Line In Different Standard Forms
- 14.6. A Straight Line And Point(s)
- 14.7. Two Or More Lines
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- 14.1. Locus Plotting
- 14.2. Distance Formula
- 14.3. Section Formula
- 14.4. Area Of A Triangle, Collinearity Of Points
- 14.5. Coordinates Of Centroid, Incenter, Excenters Of A Triangle
- 14.6. Finding Equation Of A Straight Line From Its Properties
- 14.7. Finding Properties Of A Straight Line By Its Equation
- 14.8. Position Of Point(s) W.R.T. A Straight Line, Length Of Perpendicular From A Point On A Straight Line
- 14.9. Angle Between Two Lines, Parallel And Perpendicular Lines
- 14.10. Distance Between Two Parallel Lines
- 14.11. Point Of Intersection Of Two Non-Parallel Lines
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- 14.14. Concurrency Of Three Lines
- 14.15. A Variable Line Passing Through A Fixed Point
- 14.16. Equation Representing More Than One Straight Lines
- 14.17. A General Second Degree Equation Representing A Pair Of Straight Lines
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- 14.20. Locus Problems In Straight Lines
- 14.21. Transformation Of Axes
- 14.22. Additional Questions

## CHAPTER-14

### POINT AND STRAIGHT LINE

#### SECTION-14.1. LOCUS, COORDINATES, EQUATION

##### 1. Locus

When a point moves so as to satisfy a given condition, or conditions, the path it traces out is called its Locus under these conditions. Locus of a point is a point, curve or region.

##### 2. Representation of a point in two-dimensional space by coordinates

- i. Coordinates of a point  $P$  in two-dimensional space w.r.t.  $OXY$  axes is ordered pair of real numbers written as  $(x, y)$  such that the coordinates are the distances from the origin of the feet of the perpendiculars from the point  $P$  on the respective coordinate axes.
- ii. Coordinates of origin is  $(0, 0)$ .

##### 3. Equation of a curve/ region

- i. The equation of a curve/ region is the relation which exists between the coordinates of every point on the curve/ region, and which holds for no other points except those lying on the curve/ region.
- ii. Equation of  $x$ -axis is  $y = 0$ ; equation of  $y$ -axis is  $x = 0$ .

#### SECTION-14.2. POINTS

##### 1. Distance formula

- i. Distance between two points  $(x_1, y_1)$  and  $(x_2, y_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ .
- ii. Distance of a point  $(x_1, y_1)$  from origin  $= \sqrt{x_1^2 + y_1^2}$ .

##### 2. Section formula

- i. Coordinates of the point which divides the line segment joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  in a given ratio  $m_1 : m_2$  internally is  $\left( \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right)$ .
- ii. Coordinates of the point which divides the line segment joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  in a given ratio  $m_1 : m_2$  ( $m_1 \neq m_2$ ) externally is  $\left( \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2} \right)$ .
- iii. Coordinates of the mid-point of the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$ .

##### 3. Area of a triangle

- i. Area of a triangle whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  is  $\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ .
- ii. If the coordinates of the vertices are written in anti-clockwise order, then the value of the determinant  $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$  is positive, but if written in clockwise order, then the value of this determinant is negative.

**4. Condition of collinearity of three points**

Three points  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  are collinear iff  $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$  and are non-collinear if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \neq 0.$$

**5. Coordinates of centroid of a triangle**

Coordinates of the centroid of a triangle whose vertices are  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  is

$$G \equiv \left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

**6. Coordinates of incenter and ex-centers of a triangle**

i. Coordinates of the incentre of a triangle whose vertices are  $A \equiv (x_1, y_1)$ ,  $B \equiv (x_2, y_2)$ ,  $C \equiv (x_3, y_3)$  is

$$I \equiv \left( \frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right)$$

ii. Coordinates of the ex-centers are

$$I_1 \equiv \left( \frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c} \right)$$

$$I_2 \equiv \left( \frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c} \right)$$

$$I_3 \equiv \left( \frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c} \right)$$

**SECTION-14.3. GENERAL EQUATION OF A STRAIGHT LINE****1. Definition of a straight line**

- A straight line is a curve such that every point on the line segment joining any two points on it lies on it.
- Every first degree equation in  $x, y$ , i.e.  $ax + by + c = 0$ , represents a straight line.

**2. General equation of a straight line**

- General equation of a straight line is  $ax + by + c = 0$ , where  $a$  and  $b$  both cannot be zero.
- If  $a = 0$ , then the line is parallel to  $x$ -axis. If  $b = 0$ , then the line is parallel to  $y$ -axis.

**3. Two (or more) equations representing the same straight line**

- $ax + by + c = 0 \equiv kax + kby + kc = 0$ , ( $k \neq 0$ ), therefore equations  $ax + by + c = 0$  and  $kax + kby + kc = 0$  represent the same straight line.
- Conversely, equations  $a_1x + b_1y + c_1 = 0$  &  $a_2x + b_2y + c_2 = 0$  represents the same line if

$$a_1 = ka_2, b_1 = kb_2, c_1 = kc_2 \text{ or } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

- The number of arbitrary constant in the equation of a straight line is two, hence to completely determine the equation of a straight line, two conditions are required to determine the two arbitrary constants.

**4. Inclination and slope (gradient) of a straight line**



- i. Inclination of a line, denoted by  $\theta$ , is the angle which the line makes with positive  $x$ -axis and  $0 \leq \theta < \pi$ .
- ii. Slope (gradient) of a line, denoted by  $m$ , is  $\tan \theta$ .
- iii. Slope of lines parallel to  $y$ -axis is not defined.

### SECTION-14.4. EQUATION OF A STRAIGHT LINE IN DIFFERENT STANDARD FORMS

#### 1. Slope-intercept form

- i.  $y = mx + c$   
 where  
 $m = \text{slope}$ .  
 $c = y\text{-intercept}$ .
- ii. Therefore equation of a line passing through origin is  $y = mx$ .

#### 2. Point-slope form

$$y - y_1 = m(x - x_1)$$

where  $(x_1, y_1)$  is a given point on the line and  $m = \text{slope}$ .

#### 3. Two point form

$$y - y_1 = \left( \frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1)$$

where  $(x_1, y_1)$  and  $(x_2, y_2)$  are two given points on the line and  $m = \frac{y_2 - y_1}{x_2 - x_1}$ .

#### 4. Intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

where  
 $a = x\text{-intercept}$   
 $b = y\text{-intercept}$ .

#### 5. Normal form

$$x \cos \alpha + y \sin \alpha = p$$

where  
 $p = \text{length of the perpendicular from origin}$   
 $\alpha = \text{angle that this perpendicular makes with positive } x\text{-axis. } (0 \leq \alpha < 2\pi)$ .

#### 6. Parametric form

- i.  $\frac{x-h}{\cos \theta} = \frac{y-k}{\sin \theta} = r$  or  $x = h + r \cos \theta$ ,  $y = k + r \sin \theta$  ( $r \in R$ )  
 where  $(h, k)$  is a given point on the line  
 $\theta = \text{inclination of the line}$   
 $|r| = \text{distance of the point } (x, y) \text{ on the line from the point } (h, k)$ .
- ii. For all real  $r$ , the point  $(h + r \cos \theta, k + r \sin \theta)$  always lies on the line passing through the point  $(h, k)$  and having inclination  $\theta$ .  $|r|$  is the distance of the point  $(h + r \cos \theta, k + r \sin \theta)$  from  $(h, k)$ .

### SECTION-14.5. TRANSFORMATION OF GENERAL EQUATION OF A LINE IN DIFFERENT STANDARD FORMS

**1. Transformation of  $Ax + By + C = 0$  in the slope-intercept form ( $y = mx + c$ )**

$$Ax + By + C = 0$$

$$\equiv By = -Ax - C$$

$$\equiv y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

Therefore,

$$\text{Slope } (m) = -\frac{A}{B}$$

$$y - \text{intercept} = -\frac{C}{B}.$$

**2. Transformation of  $Ax + By + C = 0$  in the intercept form  $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$** 

$$Ax + By + C = 0$$

$$\equiv Ax + By = -C$$

$$\equiv \frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$$

Therefore,

$$x - \text{intercept} = -\frac{C}{A}$$

$$y - \text{intercept} = -\frac{C}{B}.$$

**3. Transformation of  $Ax + By + C = 0$  in the normal form ( $x \cos \alpha + y \sin \alpha = p$ )**

$$Ax + By + C = 0$$

$$\equiv Ax + By = -C$$

$$\equiv \frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y = -\frac{C}{\sqrt{A^2 + B^2}}$$

$$\equiv \left(\pm \frac{A}{\sqrt{A^2 + B^2}}\right)x + \left(\pm \frac{B}{\sqrt{A^2 + B^2}}\right)y = \mp \frac{C}{\sqrt{A^2 + B^2}}$$

Therefore,

$$p = \frac{|C|}{\sqrt{A^2 + B^2}}$$

$$\text{If } C > 0, \text{ then } \cos \alpha = -\frac{A}{\sqrt{A^2 + B^2}}, \sin \alpha = -\frac{B}{\sqrt{A^2 + B^2}}.$$

$$\text{If } C < 0, \text{ then } \cos \alpha = \frac{A}{\sqrt{A^2 + B^2}}, \sin \alpha = \frac{B}{\sqrt{A^2 + B^2}}.$$

**SECTION-14.6. A STRAIGHT LINE AND POINT(S)****1. Position of a point w.r.t. a line**

A point  $(x_1, y_1)$  lies above, on or below the line  $ax + by + c = 0$  when  $\frac{ax_1 + by_1 + c}{b}$  is  $>$ ,  $=$  or  $< 0$ .

## 2. Position of two points w.r.t. a line

Two points  $(x_1, y_1)$  and  $(x_2, y_2)$  lie on the same side or different side of the line when  $(ax_1 + by_1 + c)(ax_2 + by_2 + c)$  is  $>$  or  $< 0$ .

## 3. Length of perpendicular from a point on a line

i. Length of perpendicular from a point  $(x_1, y_1)$  on a line  $ax + by + c = 0$  is  $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ .

ii. Therefore, length of perpendicular from origin on the line  $ax + by + c = 0$  is  $\frac{|c|}{\sqrt{a^2 + b^2}}$ .

iii. Algebraic length of perpendicular from a point  $(x_1, y_1)$  on a line  $ax + by + c = 0$  is  $\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$ .

## SECTION-14.7. TWO OR MORE LINES

### 1. Angle between two lines

i. Acute angle between two lines having gradients  $m_1$  and  $m_2$  is  $\tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$ , therefore obtuse angle is

$$\pi - \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|.$$

ii. Lines are parallel when  $m_1 = m_2$ .

iii. Lines are perpendicular when  $m_1 m_2 = -1$ .

iv. Acute angle of a line with  $x$ -axis =  $\tan^{-1} |m|$ .

v. Acute angle of a line with  $y$ -axis =  $\frac{\pi}{2} - \tan^{-1} |m| = \cot^{-1} |m| = \tan^{-1} \frac{1}{|m|}$ .

### 2. Equation of a line parallel to a given line

Equation of a line parallel to a given line  $ax + by + c = 0$  is  $ax + by + \lambda = 0$ , where  $\lambda$  is any constant.

### 3. Equation of a line perpendicular to a given line

Equation of a line perpendicular to a given line  $ax + by + c = 0$  is  $bx - ay + \lambda = 0$ , where  $\lambda$  is any constant.

### 4. Distance between two parallel lines

Distance between two parallel lines  $y = mx + c_1$  and  $y = mx + c_2$  is  $\frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$ .

### 5. Point of intersection of two non-parallel lines

Point of intersection of two non-parallel lines  $a_1 x + b_1 y + c_1 = 0$ ,  $a_2 x + b_2 y + c_2 = 0$  is

$$\left( \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1} \right) \text{ (By Cramer's rule).}$$

### 6. Equation of angular bisectors of two lines

i. Equation of angular bisectors of two lines  $a_1 x + b_1 y + c_1 = 0$ ,  $a_2 x + b_2 y + c_2 = 0$  are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

- ii. Let the equation of the two lines be written so that  $c_1$  and  $c_2$  are both positive, then

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ is the bisector of the angle containing the origin.}$$

- iii. Let the equation of the two lines be written so that  $c_1$  and  $c_2$  are both positive. If  $a_1a_2 + b_1b_2 > 0$  then

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = - \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ is the acute angle bisector. If } a_1a_2 + b_1b_2 < 0 \text{ then}$$

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ is the acute angle bisector.}$$

- iv. Angle between the acute angle bisector and each of the lines is less than  $45^\circ$ . Angle between the obtuse angle bisector and each of the lines is more than  $45^\circ$ .

### 7. Equation of family of lines passing through the points of intersection of two given lines

Equation of family of lines passing through the points of intersection of two given lines

$a_1x + b_1y + c_1 = 0$ ,  $a_2x + b_2y + c_2 = 0$  is  $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$ , where  $\lambda$  is any constant.

### 8. Test of concurrency of three lines

Three lines  $a_1x + b_1y + c_1 = 0$ ,  $a_2x + b_2y + c_2 = 0$ ,  $a_3x + b_3y + c_3 = 0$  are concurrent if  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$  and

are non-concurrent if  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ .

## SECTION-14.8. GENERAL SECOND DEGREE EQUATION REPRESENTING A PAIR OF LINES

### 1. Equation representing $n$ straight lines

- i. The equation representing  $n$  straight lines,  $a_ix + b_iy + c_i = 0$ ,  $i = 1, 2, \dots, n$ , is

$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) \dots (a_nx + b_ny + c_n) = 0$ , which is a  $n$ th degree equation in  $x, y$ .

- ii. A  $n$ th degree equation in  $x, y$  may represent  $n$  straight lines iff it can be factorized into  $n$  linear factors.

### 2. Second degree equation representing two straight lines

- i. A second degree equation representing two straight lines  $a_1x + b_1y + c_1 = 0$  and  $a_2x + b_2y + c_2 = 0$  is

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0.$$

- ii. Conditions that a general second degree equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  may represent two straight lines are:-

$$\begin{aligned} \text{a. } (gh - af)^2 - (h^2 - ab)(g^2 - ac) &= 0 \\ \Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 &= 0 \end{aligned}$$

$$\Rightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

$$abc + 2fgh - af^2 - bg^2 - ch^2 \text{ or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \text{ is denoted by } \Delta.$$

b.  $h^2 - ab \geq 0$ .

Case 1: If  $\Delta = 0$ ,  $h^2 - ab > 0 \Rightarrow$  Two different lines;

Case 2: If  $\Delta = 0$ ,  $h^2 - ab = 0 \Rightarrow$  Two parallel lines or coincident lines (one line) or no locus;

Case 3: If  $\Delta = 0$ ,  $h^2 - ab < 0 \Rightarrow$  A point.

iii. If the lines are  $y = m_1x + c_1$  and  $y = m_2x + c_2$  then  $m_1 + m_2 = -\frac{2h}{b}$ ,  $m_1m_2 = \frac{a}{b}$ ,  $c_1 + c_2 = -\frac{2f}{b}$ ,

$$c_1c_2 = \frac{c}{b}, \quad m_1c_2 + m_2c_1 = \frac{2g}{b}.$$

iv.  $\theta = \text{acute angle between the lines} = \tan^{-1} \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$

a. if  $a + b = 0$  then lines are perpendicular

b. if  $h^2 = ab$  then lines are parallel.

v. Point of intersection of the lines is  $\left( \frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab} \right)$ .

vi. Equation of bisectors is

$$\frac{(x - x_0)^2 - (y - y_0)^2}{a - b} = \frac{(x - x_0)(y - y_0)}{h}$$

where  $(x_0, y_0)$  is the point of intersection of lines.

a. If  $a = b$ , the equation of bisectors is  $(x - x_0)^2 - (y - y_0)^2 = 0 \equiv y = x + (y_0 - x_0); y = -x + (y_0 + x_0)$ .

b. If  $h = 0$ , the equation of bisectors is  $(x - x_0)(y - y_0) = 0 \equiv x = x_0; y = y_0$ .

### 3. Homogeneous second degree equation representing a pair of straight lines

- The equation representing  $n$  straight lines passing through origin,  $a_ix + b_iy = 0, i = 1, 2, \dots, n$ , is  $(a_1x + b_1y)(a_2x + b_2y) \dots (a_nx + b_ny) = 0$ , which is a homogeneous  $n$ th degree equation in  $x, y$ .
- A homogeneous  $n$ th degree equation in  $x, y$  may represent  $n$  straight lines passing through origin iff it can be factorized into  $n$  linear factors.
- A homogeneous second degree equation representing two straight lines  $a_1x + b_1y = 0$  and  $a_2x + b_2y = 0$  is  $(a_1x + b_1y)(a_2x + b_2y) = 0$ .
- Condition that a homogeneous second degree equation  $ax^2 + 2hxy + by^2 = 0$  may represent two straight lines passing through origin is  $h^2 \geq ab$ ,  $\Delta$  being zero.

Case 1: If  $h^2 - ab > 0 \Rightarrow$  Two different lines passing through origin.

Case 2: If  $h^2 - ab = 0 \Rightarrow$  Coincident lines (one line) passing through origin.

Case 3: If  $h^2 - ab < 0 \Rightarrow$  A point (origin).

v. If the lines are  $y = m_1x$  and  $y = m_2x$  then  $m_1 + m_2 = -\frac{2h}{b}$  and  $m_1m_2 = \frac{a}{b}$ .

vi.  $\theta = \text{angle between the lines} = \tan^{-1} \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$

a. if  $a + b = 0$  then lines are perpendicular

b. if  $h^2 = ab$  then lines are coincident (one line).

vii. Point of intersection of lines is origin.

viii. Equation of the bisectors is  $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

a. If  $a = b$ , the equation of the bisectors is  $x^2 - y^2 = 0 \equiv y = \pm x$

b. If  $h = 0$ , the equation of the bisectors is  $xy = 0 \equiv x = 0; y = 0$ .

#### 4. Homogeneous equation of lines joining the origin to the points of intersection of a given line and a given second degree curve

Equation of lines joining the origin to the points in which the straight line  $lx + my + n = 0$  meets the second degree curve  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  is

$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$ , which is a homogeneous second degree equation.

### SECTION-14.9. TRANSFORMATION OF AXES

#### 1. Parallel transformation of axes

Let  $\overline{O}\overline{x}\overline{y}$  be the axes parallel to the axes  $Oxy$  such that  $\overline{O} \equiv (h, k)$ , then

$$\begin{cases} x = \overline{x} + h \\ y = \overline{y} + k \end{cases} \text{ and } \begin{cases} \overline{x} = x - h \\ \overline{y} = y - k \end{cases}$$

#### 2. Angular transformation of axes

Let  $\overline{O}\overline{x}\overline{y}$  be the axes rotated through an angle  $\theta$  w.r.t. the axes  $Oxy$ , then

$$\begin{cases} x = \overline{x} \cos \theta - \overline{y} \sin \theta \\ y = \overline{x} \sin \theta + \overline{y} \cos \theta \end{cases} \text{ and } \begin{cases} \overline{x} = x \cos \theta + y \sin \theta \\ \overline{y} = -x \sin \theta + y \cos \theta \end{cases}$$

## EXERCISE-14

### CATEGORY-14.1. LOCUS PLOTTING

Plot locus on  $x - y$  plane:-

1.  $x + y \leq 1, -x - y \leq 1.$
2.  $3x + 2y + 1 \geq 0, 3x + 2y - 3 \leq 0.$
3.  $x - y \leq 1, x + y \leq 2, y \leq 2.$
4.  $x + y - 2 \leq 0, 2y + 5x \geq 10, 5x - 2y - 10 \leq 0.$
5.  $2y - x \leq 6, 9x + 4y \leq 56, 3x + 5y \geq 4.$
6.  $x - 3y + 13 \leq 0, y + 5 \leq 5x, 4y + 28 \geq 7x.$
7.  $x^2 - 4x - y + 3 \leq 0, 2x - y - 2 \geq 0.$
8.  $y \geq x^2 - 2x - 3, y < x + 1.$
9.  $y \geq x^2, y \leq 4 - x^2.$
10.  $y \geq x^2 - 4x + 3, y < x^2 + 4x + 3.$
11.  $2y \geq x^2, y \leq -2x^2 + 3x.$
12.  $y + x^2 \leq 0, y - 2x + 3 \geq 0, y + 1 \leq 0.$
13.  $y - |\log_2 x| > 0, y - 2 \geq 0.$
14.  $y \leq \log_2 x, x - y - 1 \leq 0.$
15.  $x^2 - x < y - xy.$
16.  $2y - x^2 < 0, x^2 - y^2 \geq 0.$
17.  $2x + y^2 \leq 0, x + 2 \geq 0, y^2 - 1 \leq 0.$
18.  $xy \leq 1, x + y \geq 0, y - x \leq 0.$
19.  $\sqrt{x + y} \geq x, y \leq 2.$
20.  $|xy| < 1.$
21.  $|y| + \frac{1}{2} \leq e^{-|x|}.$
22.  $|y| + 2|x| \leq x^2 + 1.$
23.  $\frac{y - |x|}{xy^2} \geq 0.$
24.  $\frac{y - 1 + |x - 1|}{y - x^2 + 2x} \geq 0.$
25.  $\log_{\frac{1}{3}}(2x + y - 2) > \log_{\frac{1}{3}}(y + 1), \sqrt{y - 2x - 3} < \sqrt{3 - 2x}$
26.  $\log_{\frac{1}{2}}(x + y - 1) > \log_{\frac{1}{2}} y, \sqrt{y - x - 1} < \sqrt{2 - x}.$
27.  $\log_2(2y - x^2 + 1) > \log_2 y, \sqrt{y - 2x + 4} < \sqrt{8 - x}.$
28.  $|x| + |y| < 4, \log_2(2y - x^2 + 4) > \log_2(y + 1).$

$$29. |x+y|+|x-y| \leq 4, |x| \leq 1, y \geq \sqrt{x^2-2x+1}.$$

$$30. [x] + \{y\} = 1$$

$$31. \{x\} + [y] = \frac{3}{2}$$

$$32. \max([x], [y]) = 1$$

$$33. \max([x], [y]) = \frac{3}{2}$$

$$34. \max([x], \{y\}) = 1$$

$$35. \max([x], \{y\}) = \frac{1}{2}$$

$$36. \max([x], \{y\}) = \frac{3}{2}$$

$$37. \max([x], \{y\}) = -1$$

### CATEGORY-14.2. DISTANCE FORMULA

38. Find the distance between the following pairs of points:-

i. (2, 3) and (5, 7). {Ans. 5}

ii. (4, -7) and (-1, 5). {Ans. 13}

iii. (a, 0) and (0, b). {Ans.  $\sqrt{a^2+b^2}$ }

iv. (b+c, c+a) and (c+a, a+b). {Ans.  $\sqrt{a^2+2b^2+c^2-2ab-2bc}$ }

v. (a cos  $\alpha$ , a sin  $\alpha$ ) and (a cos  $\beta$ , a sin  $\beta$ ). {Ans.  $2|a \sin \frac{\alpha-\beta}{2}|$ }

vi. ( $am_1^2, 2am_1$ ) and ( $am_2^2, 2am_2$ ). {Ans.  $|a(m_1-m_2)|\sqrt{(m_1+m_2)^2+4}$ }

39. Find the value of h if the distance between the points (h, 2) and (3, 4) be 8. {Ans.  $3 \pm 2\sqrt{15}$ }

40. A line is of length 10 and one end is at the point (2, -3); if the abscissa of the other end be 10, prove that its ordinate must be 3 or -9.

41. Find the area of the circle centered at (1,2) and passing through (4,6). {Ans.  $25\pi$ }

42. Prove that the points (2a, 4a), (2a, 6a) and ( $2a + \sqrt{3}a, 5a$ ) are the vertices of an equilateral triangle whose side is 2a.

43. Prove that the points (2, -2), (8, 4), (5, 7) and (-1, 1) are at the vertices of a rectangle.

44. Prove that the points (-2, -1), (1, 0), (4, 3) and (1, 2) are at the vertices of a parallelogram.

45. Vertices of a quadrilateral ABCD are A(0,0), B(3,4), C(7,7) and D(4,3). Check whether the quadrilateral ABCD is a square or rectangle or parallelogram or rhombus. {Ans. Rhombus}

46. Prove that the point  $\left(-\frac{1}{14}, \frac{39}{14}\right)$  is the center of the circle circumscribing the triangle whose vertices are (1, 1), (2, 3) and (-2, 2).

47. If O be the origin, and if the coordinates of any two points  $P_1$  and  $P_2$  be respectively  $(x_1, y_1)$  and  $(x_2, y_2)$ , prove that  $OP_1 \cdot OP_2 \cdot \cos P_1OP_2 = x_1x_2 + y_1y_2$ .

48. Prove that a point can be found which is at the same distance from each of the four points



$$\left(am_1, \frac{a}{m_1}\right), \left(am_2, \frac{a}{m_2}\right), \left(am_3, \frac{a}{m_3}\right) \text{ and } \left(\frac{a}{m_1 m_2 m_3}, am_1 m_2 m_3\right).$$

**CATEGORY-14.3. SECTION FORMULA**

49. Find the coordinates of the points which
- divides, internally and externally, the line joining the points (1, 3) and (2, 7) in the ratio 3 : 4. {Ans.  $\left(\frac{10}{7}, \frac{33}{7}\right), (-2, -9)$ }
  - divides, internally and externally, the line joining (-1, 2) to (4, -5) in the ratio 2 : 3. {Ans.  $\left(1, -\frac{4}{5}\right), (-11, 16)$ }
  - divides, internally and externally, the line joining (-3, -4) to (-8, 7) in the ratio 7 : 5. {Ans.  $\left(-\frac{71}{12}, \frac{29}{12}\right), \left(-\frac{41}{2}, \frac{69}{2}\right)$ }
50. The line joining the points (1, -2) and (-3, 4) is trisected; find the coordinates of the points of trisection. {Ans.  $\left(-\frac{1}{3}, 0\right), \left(-\frac{5}{3}, 2\right)$ }
51. The line joining the points (-6, 8) and (8, -6) is divided into four equal parts; find the coordinates of the points of section. {Ans.  $\left(-\frac{5}{2}, \frac{9}{2}\right), (1, 1), \left(\frac{9}{2}, -\frac{5}{2}\right)$ }
52. Find the ratio in which the line segment joining the points (-3, -4) and (1, -2) is divided by y - axis. {Ans. 3:1}
53. In a triangle ABC, find the length of the median from B on AC where A(-1, 3), B(1, -1), C(5, 1). {Ans.  $\sqrt{10}$ }
54. Find the coordinates of the points which divide, internally and externally, the line joining the points (a+b, a-b) to the point (a-b, a+b) in the ratio a : b. {Ans.  $\left(\frac{a^2+b^2}{a+b}, \frac{a^2+2ab-b^2}{a+b}\right), \left(\frac{a^2-2ab-b^2}{a-b}, \frac{a^2+b^2}{a-b}\right)$ }
55. The coordinates of the vertices of a triangle are (x<sub>1</sub>, y<sub>1</sub>), (x<sub>2</sub>, y<sub>2</sub>) and (x<sub>3</sub>, y<sub>3</sub>). The line joining the first two is divided in the ratio l : k, and the line joining this point of division to the opposite angular point is then divided in the ratio m : k + l. Find the coordinates of the latter point of section. {Ans.  $\left(\frac{kx_1+lx_2+mx_3}{k+l+m}, \frac{ky_1+ly_2+my_3}{k+l+m}\right)$ }
56. Prove that the coordinates, x and y, of the middle point of the line joining the point (2, 3) to the point (3, 4) satisfy the equation x - y + 1 = 0.
57. Prove that the lines joining the middle points of opposite sides of a quadrilateral and the line joining the middle points of its diagonals meet in a point and bisect one another.
58. A, B, C, D, ..... are n points in a plane whose coordinates are (x<sub>1</sub>, y<sub>1</sub>), (x<sub>2</sub>, y<sub>2</sub>), (x<sub>3</sub>, y<sub>3</sub>)..... AB is bisected in the point G<sub>1</sub>; G<sub>1</sub>C is divided at G<sub>2</sub> in the ratio 1 : 2; G<sub>2</sub>D is divided at G<sub>3</sub> in the ratio 1 : 3; G<sub>3</sub>E at G<sub>4</sub> in the ratio 1 : 4, and so on until all the points are exhausted. Show that the coordinates of the final point so obtained is  $\left(\frac{x_1 + x_2 + x_3 + \dots + x_n}{n}, \frac{y_1 + y_2 + y_3 + \dots + y_n}{n}\right)$ .

**CATEGORY-14.4. AREA OF A TRIANGLE, COLLINEARITY OF POINTS**

59. Find the area of the triangle the coordinates of whose vertices are:-
- (1, 3), (-7, 6) and (5, -1). {Ans. 10}
  - (0, 4), (3, 6) and (-8, -2). {Ans. 1}
  - (5, 2), (-9, -3) and (-3, -5). {Ans. 29}
  - (a, b+c), (a, b-c) and (-a, c). {Ans. 2/ac}
  - (a, c+a), (a, c) and (-a, c-a) {Ans. a<sup>2</sup>}

- vi.  $(a \cos \phi_1, b \sin \phi_1), (a \cos \phi_2, b \sin \phi_2)$  and  $(a \cos \phi_3, b \sin \phi_3)$ . {Ans.  $2 \left| ab \sin \frac{\phi_2 - \phi_3}{2} \sin \frac{\phi_3 - \phi_1}{2} \sin \frac{\phi_1 - \phi_2}{2} \right|$ }
- vii.  $(am_1^2, 2am_1), (am_2^2, 2am_2)$  and  $(am_3^2, 2am_3)$ . {Ans.  $a^2 |(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)|$ }
- viii.  $\{am_1m_2, a(m_1 + m_2)\}, \{am_2m_3, a(m_2 + m_3)\}$  and  $\{am_3m_1, a(m_3 + m_1)\}$  {Ans.  $\frac{a^2}{2} |(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)|$ }
- ix.  $\left(am_1, \frac{a}{m_1}\right), \left(am_2, \frac{a}{m_2}\right)$  and  $\left(am_3, \frac{a}{m_3}\right)$  {Ans.  $\frac{a^2}{2} \left| \frac{(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)}{m_1m_2m_3} \right|$ }
60. An equilateral triangle has each side equal to  $a$ . If the coordinates of its vertices are  $(x_1, y_1), (x_2, y_2)$  and  $(x_3, y_3)$ , then find the value of the determinant  $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ . {Ans.  $\frac{\sqrt{3}a^2}{2}$ }
61. Check whether that the following sets of points are collinear or not:-
- $(1, 4), (3, -2)$  and  $(-3, 14)$ . {Ans. non-collinear}
  - $\left(-\frac{1}{2}, 3\right), (-5, 6)$  and  $(-8, 8)$ . {Ans. collinear}
  - $(a, b+c), (b, c+a)$  and  $(c, a+b)$ . {Ans. collinear}
  - $(-a, -b), (0, 0), (a, b)$  and  $(a^2, ab)$ . {Ans. collinear}
62. Find the area of the quadrilateral the coordinates of whose vertices, taken in order, are
- $(1, 1), (3, 4), (5, -2)$  and  $(4, -7)$ . {Ans.  $\frac{41}{2}$ }
  - $(-1, 6), (-3, -9), (5, -8)$  and  $(3, 9)$ . {Ans. 96}

#### CATEGORY-14.5. COORDINATES OF CENTROID, INCENTER, EX-CENTERS OF A TRIANGLE

63. Find the coordinates of centroid, incenter, ex-centers of the triangle the coordinates of whose vertices are  $(1, 2), (5, 2)$  and  $(1, 5)$ . {Ans.  $G \equiv \left(\frac{7}{3}, 3\right), I \equiv (2, 3), (7, 8), (-1, 4), (4, -1)$ }
64. If  $G$  be the centroid of a triangle  $ABC$  and  $P$  be any point prove that  $3(GA^2 + GB^2 + GC^2) = BC^2 + CA^2 + AB^2$  and  $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3GP^2$ .

#### CATEGORY-14.6. FINDING EQUATION OF A STRAIGHT LINE FROM ITS PROPERTIES

65. Find the equations to the straight lines passing through the following pairs of points:-
- $(0, 0)$  and  $(2, -2)$ . {Ans.  $x + y = 0$ }
  - $(3, 4)$  and  $(5, 6)$ . {Ans.  $y - x = 1$ }
  - $(-1, 3)$  and  $(6, -7)$ . {Ans.  $7y + 10x = 11$ }
  - $(0, -a)$  and  $(b, 0)$ . {Ans.  $ax - by = ab$ }
  - $(a, b)$  and  $(a + b, a - b)$ . {Ans.  $(a - 2b)x - by + b^2 + 2ab - a^2 = 0$ }
  - $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$ . {Ans.  $y(t_1 + t_2) - 2x = 2at_1t_2$ }
  - $\left(at_1, \frac{a}{t_1}\right)$  and  $\left(at_2, \frac{a}{t_2}\right)$ . {Ans.  $t_1t_2y + x = a(t_1 + t_2)$ }

- viii.  $(a \cos \phi_1, a \sin \phi_1)$  and  $(a \cos \phi_2, a \sin \phi_2)$ . {Ans.  $x \cos \frac{1}{2}(\phi_1 + \phi_2) + y \sin \frac{1}{2}(\phi_1 + \phi_2) = a \cos \frac{1}{2}(\phi_1 - \phi_2)$ }
- ix.  $(a \cos \phi_1, b \sin \phi_1)$  and  $(a \cos \phi_2, b \sin \phi_2)$ . {Ans.  $\frac{x}{a} \cos \frac{1}{2}(\phi_1 + \phi_2) + \frac{y}{b} \sin \frac{1}{2}(\phi_1 + \phi_2) = \cos \frac{1}{2}(\phi_1 - \phi_2)$ }
- x.  $(a \sec \phi_1, b \tan \phi_1)$  and  $(a \sec \phi_2, b \tan \phi_2)$ . {Ans.  $bx \cos \frac{1}{2}(\phi_1 - \phi_2) - ay \sin \frac{1}{2}(\phi_1 + \phi_2) = ab \cos \frac{1}{2}(\phi_1 + \phi_2)$ }
66. Find the equation of the straight line
- cutting off an intercept unity from the positive direction of the axis of  $y$  and inclined at  $45^\circ$  to the axis of  $x$ . {Ans.  $y = x + 1$ }
  - cutting off an intercept  $-5$  from the axis of  $y$  and being equally inclined to the axes. {Ans.  $x - y - 5 = 0$ }
  - cutting off an intercept  $2$  from the negative direction of the axis of  $y$  and inclined at  $30^\circ$  to  $OX$ . {Ans.  $x - \sqrt{3}y - 2\sqrt{3} = 0$ }
  - cutting off an intercept  $-3$  from the axis of  $y$  and inclined at an angle  $\tan^{-1} \frac{3}{5}$  to the axis of  $x$ . {Ans.  $5y - 3x + 15 = 0$ }
  - cutting off an intercepts  $3$  and  $2$  from the axes. {Ans.  $2x + 3y = 6$ }
  - cutting off an intercept  $-5$  and  $6$  from the axes. {Ans.  $6x - 5y + 30 = 0$ }
67. Find the equation to the straight line which passes through the point  $(5, 6)$  and has intercepts on the axes
- equal in magnitude and both positive, {Ans.  $x + y = 11$ }
  - equal in magnitude but opposite in sign. {Ans.  $y - x = 1$ }
68. Find the equations to the straight lines which pass through the point  $(1, -2)$  and cut off equal distances from the two axes. {Ans.  $x + y + 1 = 0, x - y = 3$ }
69. If the point  $(5, 2)$  bisects the intercept of a line between the axes, then find the equation of the line. {Ans.  $2x + 5y = 20$ }
70. Find the equation of the line through  $P(1, 2)$  such that its intercept between the axes is bisected at  $P$ . {Ans.  $2x + y - 4 = 0$ }
71. Points  $A(1, 3)$  and  $C(5, 1)$  are opposite vertices of a rectangle  $ABCD$ . If the slope of  $BD$  is  $2$ , then find its equation. {Ans.  $2x - y = 4$ }
72. Find the equation to the straight line which passes through the given point  $(x', y')$  and is such that the given point bisect the part intercepted between the axes. {Ans.  $xy' + x'y = 2x'y'$ }
73. Find the equation to the straight line which passes through the point  $(-4, 3)$  and is such that the portion of it between the axes is divided by the point in the ratio  $5 : 3$ . {Ans.  $20y - 9x = 96; 5x - 4y + 32 = 0$ }
74. Find the equations to the sides of the triangles the coordinates of whose vertices are respectively
- $(1, 4), (2, -3)$  and  $(-1, -2)$ . {Ans.  $x + 3y + 7 = 0, y - 3x = 1, y + 7x = 11$ }
  - $(0, 1), (2, 0)$  and  $(-1, -2)$ . {Ans.  $2x - 3y = 4, y - 3x = 1, x + 2y = 2$ }
75. Find the equations to the diagonals of the rectangle the equations of whose sides are  $x = a, x = a', y = b$ , and  $y = b'$ . {Ans.  $y(a' - a) - x(b' - b) = a'b - ab', y(a' - a) + x(b' - b) = a'b' - ab$ }
76. Find the equation to the straight line which bisects the distance between the points  $(a, b)$  and  $(a', b')$  and also bisects the distance between the points  $(-a, b)$  and  $(a', -b')$ . {Ans.  $2ay - 2b'x = ab - a'b'$ }
77. Find the equation to the straight lines which go through the origin and trisect the portion of the straight line  $3x + y = 12$  which is intercepted between the axes of coordinates. {Ans.  $y = 6x, 2y = 3x$ }

**CATEGORY-14.7. FINDING PROPERTIES OF A STRAIGHT LINE BY ITS EQUATION**

78. Find slope, x-intercept, y-intercept,  $p$  and  $\alpha$  of the following straight lines:-

- i.  $x + 2y + 3 = 0$ .
- ii.  $5x - 7y - 9 = 0$ .
- iii.  $3x + 7y = 0$ .
- iv.  $2x - 3y + 4 = 0$ .

79. Prove that the point whose coordinates are respectively (5, 1), (1, -1) and (11, 4) lie on a straight line, and find its intercepts on the axes. {Ans.  $3, -\frac{3}{2}$ }

**CATEGORY-14.8. POSITION OF POINT(S) W.R.T. A STRAIGHT LINE, LENGTH OF PERPENDICULAR FROM A POINT ON A STRAIGHT LINE**

80. Given points (3,4) and (5,-2) and a line  $x - 2y - 3 = 0$ . Determine the position of the points w.r.t. the line. {Ans. (3,4) is above; (5,-2) is below; points are in different sides}

81. Find the position of the points (1,2) and (-2,1) w.r.t. the line  $4x + 2y = 1$ . {Ans. points are in different sides}

82. Find the length of the perpendicular drawn from

i. the point (4, 5) upon the straight line  $3x + 4y = 10$ . {Ans.  $\frac{22}{5}$ }

ii. the origin upon the straight line  $\frac{x}{3} - \frac{y}{4} = 1$ . {Ans.  $\frac{12}{5}$ }

iii. the point (-3, -4) upon the straight line  $12(x + 6) = 5(y - 2)$ . {Ans.  $\frac{66}{13}$ }

iv. origin on the line  $\frac{x}{a} + \frac{y}{b} = 1$ . {Ans.  $\frac{|ab|}{\sqrt{a^2 + b^2}}$ }

v. the point (b, a) upon the straight line  $\frac{x}{a} - \frac{y}{b} = 1$ . {Ans.  $\frac{|a^2 + ab - b^2|}{\sqrt{a^2 + b^2}}$ }

83. Find the length of the perpendicular from the origin upon the straight line joining the two points whose coordinates are  $(a \cos \alpha, a \sin \alpha)$  and  $(a \cos \beta, a \sin \beta)$ . {Ans.  $a \cos \frac{1}{2}(\alpha - \beta)$ }

84. Show that the product of the perpendiculars drawn from the two points  $(\pm \sqrt{a^2 - b^2}, 0)$  upon the straight line  $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$  is  $b^2$ .

85. If  $p$  and  $p'$  be the perpendiculars from the origin upon the straight lines whose equations are  $x \sec \theta + y \operatorname{cosec} \theta = a$  and  $x \cos \theta - y \sin \theta = a \cos 2\theta$ , prove that  $4p^2 + p'^2 = a^2$ .

86. What is the points on the axis of  $x$  whose perpendicular distance from the straight line  $\frac{x}{a} + \frac{y}{b} = 1$  is  $a$ ?

{Ans.  $\left\{ \frac{a}{b} \left( b \pm \sqrt{a^2 + b^2} \right), 0 \right\}}$ }

87. Show that the perpendiculars let fall from any point of the straight line  $2x + 11y = 5$  upon the two straight lines  $24x + 7y = 20$  and  $4x - 3y = 2$  are equal to each other.

88. Find the coordinates of the centers and the radii of the four circles which touch the sides of the triangle the coordinates of whose vertices are the points (6, 0), (0, 6) and (7, 7). {Ans.  $\left(\frac{9}{2}, \frac{9}{2}\right), \frac{3}{\sqrt{2}}$ ;  $(2, 12), 4\sqrt{2}$ ;

$$(12, 2), 4\sqrt{2}; (-3, -3), 6\sqrt{2} \}$$

**CATEGORY-14.9. ANGLE BETWEEN TWO LINES, PARALLEL AND PERPENDICULAR LINES**

89. Find the angles between the pairs of straight lines
- $x - y\sqrt{3} = 5$  and  $\sqrt{3}x + y = 7$ . {Ans.  $90^\circ$ }
  - $x - 4y = 3$  and  $6x - y = 11$ . {Ans.  $\tan^{-1} \frac{23}{10}$ }
  - $y = 3x + 7$  and  $3y - x = 8$ . {Ans.  $\tan^{-1} \frac{4}{3}$ }
  - $y = (2 - \sqrt{3})x + 5$  and  $y = (2 + \sqrt{3})x - 7$ . {Ans.  $60^\circ$ }
  - $2x + y + 1 = 0$  and  $x = 3$ . {Ans.  $\cot^{-1} 2$ }
  - $(m^2 - mn)y = (mn + n^2)x + n^3$  and  $(mn + m^2)y = (mn - n^2)x + m^3$ . {Ans.  $\tan^{-1} \left| \frac{4m^2n^2}{m^4 - n^4} \right|$ }
90. Find the tangent of the angle between the lines whose intercepts on the axes are respectively  $a$ ,  $-b$  and  $b$ ,  $-a$ . {Ans.  $\tan^{-1} \left| \frac{a^2 - b^2}{2ab} \right|$ }
91. Prove that the points  $(2, -1)$ ,  $(0, 2)$ ,  $(2, 3)$ , and  $(4, 0)$  are the coordinates of the vertices of a parallelogram and find the angle between its diagonals. {Ans.  $\tan^{-1} 2$ }
92. Find the equation to the straight line
- passing through the point  $(2, 3)$  and perpendicular to the straight line  $4x - 3y = 10$ . {Ans.  $4y + 3x = 18$ }
  - passing through the point  $(-6, 10)$  and perpendicular to the straight line  $7x + 8y = 5$ . {Ans.  $7y - 8x = 118$ }
  - passing through the point  $(2, -3)$  and perpendicular to the straight line joining the points  $(5, 7)$  and  $(-6, 3)$ . {Ans.  $4y + 11x = 10$ }
  - passing through the point  $(-4, -3)$  and perpendicular to the straight line joining  $(1, 3)$ ,  $(2, 7)$ . {Ans.  $x + 4y + 16 = 0$ }
93. A rectangle has two opposite vertices at the points  $(1, 2)$  and  $(5, 5)$ . If the other vertices lie on the line  $x = 3$ , then find their coordinates. {Ans.  $(3, 1)$ ,  $(3, 6)$ }
94. Find the equation of the line which is perpendicular to the line  $5x - y = 0$  and forms a triangle with coordinate axes of area 5 sq. units. {Ans.  $x + 5y \pm 5\sqrt{2} = 0$ }
95. If the foot of the perpendicular from the origin to a straight line is at the point  $(3, -4)$ , find the equation of the line. {Ans.  $3x - 4y = 25$ }
96. Find the perpendicular distance of the origin from the perpendicular from the point  $(1, 2)$  upon the straight line  $x - \sqrt{3}y + 4 = 0$ . {Ans.  $\frac{2+\sqrt{3}}{2}$ }
97. If  $A(1, 1)$ ,  $B(\sqrt{3} + 1, 2)$  and  $C(\sqrt{3}, \sqrt{3} + 2)$  be three vertices of a square, then find the diagonal through  $B$ .
98. Find the equation to the straight line drawn at right angles to the straight line  $\frac{x}{a} - \frac{y}{b} = 1$  through the point where it meets the axis of  $x$ . {Ans.  $ax + by = a^2$ }
99. Find the equation to the straight line which bisect, and is perpendicular to, the straight line joining the points  $(a, b)$  and  $(a', b')$ . {Ans.  $2x(a - a') + 2y(b - b') = a^2 - a'^2 + b^2 - b'^2$ }
100. Prove that the equation to the straight line which passes through the point  $(a \cos^3 \theta, a \sin^3 \theta)$  and is

perpendicular to the straight line  $x \sec \theta + y \cos \theta = a$  is  $x \cos \theta - y \sin \theta = a \cos 2\theta$ .

101. Find the equations to the straight line which passing through  $(x', y')$  and respectively perpendicular to the straight lines  $xx' + yy' = a^2$ ,  $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$  and  $x'y + xy' = a^2$ . {Ans.  $yx' - xy' = 0, a^2xy' + b^2x'y = (a^2 + b^2)x'y', xx' - yy' = x'^2 - y'^2$ }
102. Find the equations of the straight lines which divide, internally and externally, the line joining  $(-3, 7)$  to  $(5, -4)$  in the ratio of  $4 : 7$  and which are perpendicular to this line. {Ans.  $121y - 88x = 371, 33y - 24x = 1043$ }
103. Show that the equations to the straight lines passing through the point  $(3, -2)$  and inclined at  $60^\circ$  to the line  $\sqrt{3}x + y = 1$  are  $y + 2 = 0$  and  $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$ .
104. Find the equations to the straight lines which pass through the origin and are inclined at  $75^\circ$  to the straight line  $x + y + \sqrt{3}(y - x) = a$ . {Ans.  $x = 0, y + \sqrt{3}x = 0$ }
105. Find the equations to the straight lines which pass through the point  $(h, k)$  and are inclined at an angle  $\tan^{-1} m$  to the straight line  $y = mx + c$ . {Ans.  $y = k, (1 - m^2)(y - k) = 2m(x - h)$ }
106. Find the angle between the two straight lines  $3x = 4y + 7$  and  $5y = 12x + 6$  and also the equations to the two straight lines which pass through the point  $(4, 5)$  and make equal angles with the two given lines. {Ans.  $\tan^{-1} \frac{33}{56}, 9x - 7y = 1, 7x + 9y = 73$ }
107. One side of a square is inclined to the axis of  $x$  at an angle  $\alpha$  and one of its extremities is at the origin; prove that the equations to its diagonals are  $y(\cos \alpha - \sin \alpha) = x(\sin \alpha + \cos \alpha)$  and  $y(\sin \alpha + \cos \alpha) + x(\cos \alpha - \sin \alpha) = a$ , where  $a$  is the length of the side of the square.

#### CATEGORY-14.10. DISTANCE BETWEEN TWO PARALLEL LINES

108. Find the distance between the lines  $x + y - 2 = 0$  and  $2x + 2y + 1 = 0$ . {Ans.  $\frac{5}{\sqrt{8}}$ }
109. Find the distance between the lines  $4x + 3y = 11$  and  $8x + 6y = 15$ . {Ans.  $\frac{7}{10}$ }
110. Find the distance between the parallel lines  $y = 2x + 4$  and  $6x = 3y + 5$ . {Ans.  $\frac{17}{3\sqrt{5}}$ }
111. Find the length of the sides of the rectangle whose sides are  $x + 2y + 1 = 0$ ,  $2x - y + 1 = 0$ ,  $x + 2y + 3 = 0$  and  $4x - 2y + 1 = 0$ .

#### CATEGORY-14.11. POINT OF INTERSECTION OF TWO NON-PARALLEL LINES

112. Find the coordinates of the points of intersection of the straight lines whose equations are
- $2x - 3y + 5 = 0$  and  $7x + 4y = 3$ . {Ans.  $(-\frac{11}{29}, \frac{41}{29})$ }
  - $\frac{x}{a} + \frac{y}{b} = 1$  and  $\frac{x}{b} + \frac{y}{a} = 1$ . {Ans.  $(\frac{ab}{a+b}, \frac{ab}{a+b})$ }
  - $y = m_1x + \frac{a}{m_1}$  and  $y = m_2x + \frac{a}{m_2}$ . {Ans.  $(\frac{a}{m_1m_2}, a(\frac{1}{m_1} + \frac{1}{m_2}))$ }
  - $x \cos \phi_1 + y \sin \phi_1 = a$  and  $x \cos \phi_2 + y \sin \phi_2 = a$ . {Ans.  $(a \cos \frac{\phi_1 + \phi_2}{2} \sec \frac{\phi_1 - \phi_2}{2}, a \sin \frac{\phi_1 + \phi_2}{2} \sec \frac{\phi_1 - \phi_2}{2})$ }
113. Two straight lines cut the axis of  $x$  at distances  $a$  and  $-a$  and the axis of  $y$  at distances  $b$  and  $b'$  respectively;

- find the coordinates of their point of intersection. {Ans.  $\left(\frac{a(b-b')}{b+b'}, \frac{2bb'}{b+b'}\right)$ }
114. Find the distance of the point of intersection of the two straight lines  $2x - 3y + 5 = 0$  and  $3x + 4y = 0$  from the straight line  $5x - 2y = 0$ . {Ans.  $\frac{130}{17\sqrt{29}}$ }
115. Find the foot of the perpendicular from the point  $(0,5)$  on the line  $3x - 4y - 5 = 0$ . {Ans.  $(3,1)$ }
116. Find the nearest point on the line  $3x - 4y = 25$  from the origin. {Ans.  $(3,-4)$ }
117. Find the ratio in which the line  $3x - 2y + 5 = 0$  divides the join of  $(6,-7)$  and  $(-2,3)$ . {Ans.  $37:7$ }
118. Show that the perpendicular from the origin upon the straight line joining the points  $(a \cos \alpha, a \sin \alpha)$  and  $(a \cos \beta, a \sin \beta)$  bisects the distance between them.
119. Find the equations of the two straight lines drawn through the point  $(0, a)$  on which the perpendiculars let fall from the point  $(2a, 2a)$  are each of length  $a$ . Prove also that the equation of the straight line joining the feet of these perpendiculars is  $y + 2x = 5a$ . {Ans.  $y = a, 3y = 4x + 3a$ }
120. Find the point of intersection and the inclination of the two lines  $Ax + By = A + B$  and  $A(x - y) + B(x + y) = 2B$ . {Ans.  $(1,1), 45^\circ$ }
121. Find the coordinates of the point in which the line  $2y - 3x + 7 = 0$  meets the line joining the two points  $(6, -2)$  and  $(-8, 7)$ . Find also the angle between them. {Ans.  $\left(\frac{5}{2}, \frac{1}{4}\right), \tan^{-1} 60$ }
122. Find the coordinates of the feet of the perpendiculars let fall from the point  $(5, 0)$  upon the sides of the triangle formed by joining the three points  $(4, 3)$ ,  $(-4, 3)$  and  $(0, -5)$ ; prove also that the points so determined lie on a straight line. {Ans.  $(-1,-3), (3,1), (5,3)$ }
123. Find the coordinates of the point of intersection of the straight lines  $2x - 3y = 1$  and  $5y - x = 3$  and determine also the angle at which they cut one another. {Ans.  $(2,1), \tan^{-1} \frac{7}{17}$ }
124. Find the angle between the two lines  $3x + y + 12 = 0$  and  $x + 2y - 1 = 0$ . Find also the coordinates of their point of intersection and the equations of lines drawn perpendicular to them from the point  $(3, -2)$ . {Ans.  $45^\circ, (-5,3), x - 3y = 9, 2x - y = 8$ }
125. Through the point  $(3, 4)$  are drawn two straight lines each inclined at  $45^\circ$  to the straight line  $x - y = 2$ . Find their equations and find also the area included by the three lines. {Ans.  $x = 3, y = 4, \frac{9}{2}$ }
126.  $A(-5,0)$  and  $B(3,0)$  are two of the vertices of a triangle  $ABC$ . Its area is 20 square units. The vertex  $C$  lies on the line  $x - y = 2$ . Find the coordinates of  $C$ . {Ans.  $(-3,-5)$  or  $(7,5)$ }
127. If line  $x + 2y - 8 = 0$  is the perpendicular bisector of  $AB$ . If  $B \equiv (3,5)$ , then find the coordinates of  $A$ . {Ans.  $(1,1)$ }
128. Find the coordinates of the orthocenter of the triangles whose angular points are  
i.  $(0, 0)$ ,  $(2, -1)$  and  $(-1, 3)$ . {Ans.  $(-4,-3)$ }
- ii.  $(1, 0)$ ,  $(2, -4)$  and  $(-5, -2)$ . {Ans.  $\left(\frac{11}{13}, -\frac{7}{13}\right)$ }
129. Find the center and radius of the in-circle of the triangle whose sides are  
i.  $x = 0$ ,  $y = 0$  and  $3x + 4y = 12$ . {Ans.  $(1,1), 1$ }
- ii.  $3x + 4y + 2 = 0, 3x - 4y + 12 = 0$  and  $4x - 3y = 0$ . {Ans.  $\left(-\frac{13}{28}, \frac{5}{4}\right), \frac{157}{140}$ }
- iii.  $2x + 4y + 3 = 0, 4x + 3y + 3 = 0$  and  $x + 1 = 0$ . {Ans.  $\left(\frac{-85-7\sqrt{5}}{120}, \frac{21\sqrt{5}-65}{120}\right), \frac{35-7\sqrt{5}}{120}$ }
- iv.  $y = 0, 12x - 5y = 0$  and  $3x + 4y - 7 = 0$ . {Ans.  $\left(\frac{7}{9}, \frac{14}{27}\right), \frac{14}{27}$ }
130. Find the position of the center of the circle circumscribing the triangle whose vertices are the points  $(2, 3)$ ,

- (3, 4) and (6, 8). {Ans.  $(-\frac{27}{2}, \frac{39}{2})$ }
131. Given the lines  $3x - 4y + 4a = 0$ ,  $2x - 3y + 4a = 0$  and  $5x - y + a = 0$ , prove that the feet of the perpendiculars from the origin upon them are collinear.
132. Find the area of the triangle formed by the straight lines whose equations are
- $y = x$ ,  $y = 2x$  and  $y = 3x + 4$ . {Ans. 4}
  - $y + x = 0$ ,  $y = x + 6$  and  $y = 7x + 5$ . {Ans.  $\frac{361}{48}$ }
  - $2y + x - 5 = 0$ ,  $y + 2x - 7 = 0$  and  $x - y + 1 = 0$ . {Ans.  $\frac{3}{2}$ }
  - $y = ax - bc$ ,  $y = bx - ca$  and  $y = cx - ab$ . {Ans.  $\frac{1}{2} |(b-c)(c-a)(a-b)|$ }
  - $y = m_1x + \frac{a}{m_1}$ ,  $y = m_2x + \frac{a}{m_2}$  and  $y = m_3x + \frac{a}{m_3}$ . {Ans.  $\frac{a^2 |(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)|}{2m_1^2 m_2^2 m_3^2}$ }
  - $y = m_1x + c_1$ ,  $y = m_2x + c_2$  and the axis of  $y$ . {Ans.  $\frac{(c_1 - c_2)^2}{2|(m_1 - m_2)|}$ }
  - $y = m_1x + c_1$ ,  $y = m_2x + c_2$  and  $y = m_3x + c_3$ . {Ans.  $\frac{1}{2} \left| \frac{(c_2 - c_3)^2}{m_2 - m_3} + \frac{(c_3 - c_1)^2}{m_3 - m_1} + \frac{(c_1 - c_2)^2}{m_1 - m_2} \right|$ }
133. Prove that the area of the triangle formed by the three straight lines  $x \cos \alpha + y \sin \alpha - p_1 = 0$ ,  $x \cos \beta + y \sin \beta - p_2 = 0$  and  $x \cos \gamma + y \sin \gamma - p_3 = 0$  is
- $$\frac{1}{2} \frac{\{p_1 \sin(\gamma - \beta) + p_2 \sin(\alpha - \gamma) + p_3 \sin(\beta - \alpha)\}^2}{\sin(\gamma - \beta) \sin(\alpha - \gamma) \sin(\beta - \alpha)}.$$
134.  $P(2,1)$ ,  $Q(4,-1)$ ,  $R(3,2)$  are the vertices of a triangle. If through  $P$  and  $R$  lines parallel to opposite sides are drawn to intersect in  $S$ , then find the area of  $PQRS$ . {Ans. 4}
135. Prove that the area of the parallelogram contained by the lines  $4y - 3x - a = 0$ ,  $3y - 4x + a = 0$ ,  $4y - 3x - 3a = 0$  and  $3y - 4x + 2a = 0$  is  $\frac{2}{7} a^2$ .
136. Prove that the area of the parallelogram whose sides are the straight lines  $a_1x + b_1y + c_1 = 0$ ,  $a_1x + b_1y + d_1 = 0$ ,  $a_2x + b_2y + c_2 = 0$  and  $a_2x + b_2y + d_2 = 0$  is  $\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}$ .
137. The vertices of a quadrilateral, taken in order; are the points (0, 0), (4, 0), (6, 7) and (0, 3); find the coordinates of the point of intersection of the two lines joining the middle points of opposite sides. {Ans.  $(\frac{5}{2}, \frac{5}{2})$ }
138.  $A$  and  $B$  are two fixed points whose coordinates are (3, 2) and (5, 1) respectively.  $ABP$  is an equilateral triangle on the side of  $AB$  remote from the origin. Find the coordinates of  $P$  and the orthocentre of the triangle  $ABP$ . {Ans.  $(\frac{8+\sqrt{3}}{2}, \frac{3+2\sqrt{3}}{2}), (\frac{24+\sqrt{3}}{6}, \frac{9+2\sqrt{3}}{6})$ }
139. If three vertices of a rhombus taken in order are (2, -1), (3, 4) and (-2, 3), then find the fourth vertex. {Ans. (-3, -2)}
140. The lines  $x + y + 1 = 0$ ,  $x - y + 2 = 0$ ,  $4x + 2y + 3 = 0$  and  $x + 2y - 4 = 0$  are the equations to the sides of a quadrilateral taken in order; find the equation to its diagonals and the equation to the line on which their middle point lie. {Ans.  $10y + 32x + 43 = 0$ ,  $29y + 25x + 5 = 0$ ,  $80y + 52x = 47$ }
141. Find the direction in which a straight line must be drawn through the point (1, 2) so that its point of intersection with the line  $x + y = 4$  may be at a distance  $\frac{1}{3}\sqrt{6}$  from this point. {Ans.  $15^\circ$  or  $75^\circ$  from  $x$ -



axis}

142. Find the image of the point  $(-1, 3)$  in the line  $x - y = 0$ . {Ans.  $(3, -1)$ }
143. Find the image of the point  $(1, 3)$  in the line  $x + y - 6 = 0$ . {Ans.  $(3, 5)$ }
144. Find the distance of the point  $(3, 5)$  from the line  $2x + 3y - 14 = 0$  measured parallel to the line  $x - 2y = 1$ .  
{Ans.  $\sqrt{5}$ }

**CATEGORY-14.12. ANGULAR BISECTOR OF TWO LINES**

145. Find the equations of the angular bisectors of the following pairs of straight lines, specify the bisector of the angle in which the origin lies and also specify the acute angle bisector:-
- $x + y\sqrt{3} = 6 + 2\sqrt{3}$  and  $x - y\sqrt{3} = 6 - 2\sqrt{3}$ . {Ans.  $y = 2, x = 6$ }
  - $12x + 5y - 4 = 0$  and  $3x + 4y + 7 = 0$ . {Ans.  $99x + 77y + 71 = 0, 7x - 9y - 37 = 0$ }
  - $4x + 3y - 7 = 0$  and  $24x + 7y - 31 = 0$ . {Ans.  $x - 2y + 1 = 0, 2x + y = 3$ }
  - $2x + y = 4$  and  $y + 3x = 5$ . {Ans.  $(2\sqrt{2} - 3)x + (\sqrt{2} - 1)y = 4\sqrt{2} - 5, (2\sqrt{2} + 3)x + (\sqrt{2} + 1)y = 4\sqrt{2} + 5$ }
146. Find the bisectors of the angles between the straight lines  $y - b = \frac{2m}{1 - m^2}(x - a)$  and  $y - b = \frac{2m'}{1 - m'^2}(x - a)$ . {Ans.  $(y - b)(m + m') + (x - a)(1 - mm') = 0$ ,  
 $(y - b)(1 - mm') - (x - a)(m + m') = 0$ }
147. Show that the straight lines  $x + y = 0$ ,  $3x + y = 4$ ,  $x + 3y - 4 = 0$  form an isosceles triangle.
148. Two sides of an isosceles triangle are given by the equation  $7x - y + 3 = 0$  and  $x + y - 3 = 0$ . If its third side passes through the point  $(1, -10)$ , then find its equation. {Ans.  $x - 3y - 31 = 0$  or  $3x + y + 7 = 0$ }
149. Find the equations to the bisectors of the internal angles of the triangles the equations of whose sides are respectively
- $3x + 4y = 6, 12x - 5y = 3$  and  $4x - 3y + 12 = 0$ . {Ans.  $33x + 9y = 31, 112x - 64y + 141 = 0, 7y - x = 18$ }
  - $3x + 5y = 15, x + y = 4$  and  $2x + y = 6$ . {Ans.  $(3 + \sqrt{17})x + (5 + \sqrt{17})y = 15 + 4\sqrt{17}$ ,  
 $(4 + \sqrt{10})x + (2 + \sqrt{10})y = 12 + 4\sqrt{10}$ }
150. Find the equations to the straight lines passing through the foot of the perpendicular from the point  $(h, k)$  upon the straight line  $Ax + By + C = 0$  and bisecting the angles between the perpendicular and the given straight line. {Ans.  $A(y - k) - B(x - h) = \pm(Ax + By + C)$ }

**CATEGORY-14.13. FAMILY OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO GIVEN LINES**

151. Find the equation to the straight line passing through
- the point  $(3, 2)$  and the point of intersection of the lines  $2x + 3y = 1$  and  $3x - 4y = 6$ . {Ans.  $43x - 29y = 71$ }
  - the point  $(2, -9)$  and the intersection of the lines  $2x + 5y - 8 = 0$  and  $3x - 4y = 35$ . {Ans.  $x - y = 11$ }
  - the origin and the point of intersection of  $x - y - 4 = 0$  and  $7x + y + 20 = 0$  proving that it bisects the angle between them. {Ans.  $y = 3x$ }
  - the origin and the point of intersection of the lines  $\frac{x}{a} + \frac{y}{b} = 1$  and  $\frac{x}{b} + \frac{y}{a} = 1$ . {Ans.  $y = x$ }

- v. the point  $(a, b)$  and the intersection of the lines  $\frac{x}{a} + \frac{y}{b} = 1$  and  $\frac{x}{b} + \frac{y}{a} = 1$ . {Ans.  $a^2y - b^2x = ab(a - b)$ }
- vi. the intersection of the lines  $x - 2y - a = 0$  and  $x + 3y - 2a = 0$  and parallel to the straight line  $3x + 4y = 0$ . {Ans.  $3x + 4y = 5a$ }
- vii. the intersection of the lines  $x + 2y + 3 = 0$  and  $3x + 4y + 7 = 0$  and perpendicular to the straight line  $y - x = 8$ . {Ans.  $x + y + 2 = 0$ }
- viii. the intersection of the lines  $3x - 4y + 1 = 0$  and  $5x + y - 1 = 0$  and cutting off equal intercepts from the axes. {Ans.  $23x + 23y = 11$ ;  $8x - 3y = 0$ }
- ix. the intersection of the lines  $2x - 3y = 10$  and  $x + 2y = 6$  and the intersection of the lines  $16x - 10y = 33$  and  $12x + 14y + 29 = 0$ . {Ans.  $13x - 23y = 64$ }
152. Find the equations to the straight lines passing through the point of intersection of the straight lines  $Ax + By + C = 0$  and  $A'x + B'y + C' = 0$  and
- passing through the origin, {Ans.  $Ax + By + C - \frac{C}{C'}(A'x + B'y + C') = 0$ }
  - parallel to the axis of  $y$ , {Ans.  $Ax + By + C - \frac{B}{B'}(A'x + B'y + C') = 0$ }
  - cutting of a given distance  $a$  from the axis of  $y$  {Ans.  $Ax + By + C - \left(\frac{Ba+C}{B'a+C'}\right)(A'x + B'y + C') = 0$ }
  - passing through a given point  $(x', y')$ . {Ans.  $Ax + By + C - \left(\frac{Ax'+By'+C}{A'x'+B'y'+C'}\right)(A'x + B'y + C') = 0$ }
153. Prove that the diagonals of the parallelogram formed by the four straight lines  $\sqrt{3}x + y = 0$ ,  $\sqrt{3}y + x = 0$ ,  $\sqrt{3}x + y = 1$  and  $\sqrt{3}y + x = 1$  are at right angles to one another. Prove the same property for the parallelogram whose sides are  $\frac{x}{a} + \frac{y}{b} = 1$ ,  $\frac{x}{b} + \frac{y}{a} = 1$ ,  $\frac{x}{a} + \frac{y}{b} = 2$  and  $\frac{x}{b} + \frac{y}{a} = 2$ .
154. Find the orthocentre of the triangle whose sides are given by  $4x - 7y + 10 = 0$ ,  $x + y - 5 = 0$  and  $7x + 4y - 15 = 0$ . {Ans.  $(1, 2)$ }
155. Show that the orthocenter of the triangle formed by the three straight lines  $y = m_1x + \frac{a}{m_1}$ ,  $y = m_2x + \frac{a}{m_2}$  and  $y = m_3x + \frac{a}{m_3}$  is the point  $\left\{-a, a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1m_2m_3}\right)\right\}$ .

#### CATEGORY-14.14. CONCURRENCY OF THREE LINES

156. Prove that the following sets of three lines meet in a point:-
- $2x - 3y = 7$ ,  $3x - 4y = 13$  and  $8x - 11y = 33$ .
  - $3x + 4y + 6 = 0$ ,  $6x + 5y + 9 = 0$  and  $3x + 3y + 5 = 0$ .
  - $\frac{x}{a} + \frac{y}{b} = 1$  and  $\frac{x}{b} + \frac{y}{a} = 1$  and  $y = x$ .
157. Check whether the lines  $3x + 4y + 6 = 0$ ,  $\sqrt{2}x + \sqrt{3}y + 2\sqrt{2} = 0$  and  $4x + 7y + 5 = 0$  are concurrent or not. {Ans. not concurrent}
158. For what values of  $a$ , the lines  $2x + y - 1 = 0$ ,  $ax + 3y - 3 = 0$  and  $3x + 2y - 2 = 0$  are concurrent? {Ans.  $a \in R$ }
159. If the line  $y = mx$  meets the lines  $x + 2y - 1 = 0$  and  $2x - y + 3 = 0$  at the same point, then find the value of  $m$ . {Ans.  $-1$ }

160. If the lines  $ax + 2y + 1 = 0$ ,  $bx + 3y + 1 = 0$ ,  $cx + 4y + 1 = 0$  are concurrent, then show that  $a, b, c$  are in A.P.
161. Prove that the three straight lines whose equations are  $15x - 18y + 1 = 0$ ,  $12x + 10y - 3 = 0$  and  $6x + 66y - 11 = 0$  all meet in a point. Show also that the third line bisects the angle between the other two.
162. Find the conditions that the straight lines  $y = m_1x + a_1$  and  $y = m_2x + a_2$  and  $y = m_3x + a_3$  may meet in a point. {Ans.  $m_1(a_2 - a_3) + m_2(a_3 - a_1) + m_3(a_1 - a_2) = 0$ }
163. In any triangle  $ABC$ , prove that
- the medians meet at a point.
  - the perpendicular bisectors of the sides meet at a point.
  - the altitudes meet at a point.
  - the bisectors of the angles  $A$ ,  $B$  and  $C$  meet at a point.
164. If through the angular points of a triangle straight lines be drawn parallel to the sides, and if the intersections of these lines be joined to the opposite angular points of the triangle, show that the joining lines so obtained will meet in a point.

**CATEGORY-14.15. A VARIABLE LINE PASSING THROUGH A FIXED POINT**

165. If  $a, b, c$  are in AP, then show that  $ax + by + c = 0$  will always pass through a fixed point and find the coordinates of this fixed point. {Ans.  $(1, -2)$ }
166. If a straight line move so that the sum of the algebraic length of perpendiculars let fall on it from the two fixed points  $(3, 4)$  and  $(7, 2)$  is equal to three times the algebraic length of perpendicular on it from a third fixed point  $(1, 3)$ , prove that there is another fixed point through which this line always passes and find its coordinates. {Ans.  $(-7, 3)$ }
167. A straight line is such that the sum of the algebraic length of perpendiculars let fall upon it from any number of fixed points is zero. Show that it always passes through a fixed point.
168. Two fixed straight lines  $OX$  and  $OY$  are cut by a variable line in the points  $A$  and  $B$  respectively and  $P$  and  $Q$  are the feet of the perpendiculars drawn from  $A$  and  $B$  upon the lines  $OBY$  and  $OAX$ . Show that, if  $AB$  pass through a fixed point, then  $PQ$  will also pass through a fixed point.
169. If the equal sides  $AB$  and  $AC$  of an isosceles triangle be produced to  $E$  and  $F$  so that  $BE \cdot CF = AB^2$ , show that the line  $EF$  will always pass through a fixed point.
170. A variable straight line cuts off from  $n$  given concurrent straight lines intercepts the sum of the reciprocals of which is constant. Show that it always passes through a fixed point.

**CATEGORY-14.16. EQUATION REPRESENTING MORE THAN ONE STRAIGHT LINES**

171. Find the joint equation of the straight lines  $x + y = 1$  and  $x - y = 4$ . {Ans.  $x^2 - y^2 - 5x - 3y + 4 = 0$ }
172. What loci are represented by the following equations:-
- $x^2 - y^2 = 0$ .
  - $x^2 - xy = 0$ .
  - $xy - ay = 0$ .
  - $x^3 - x^2 - x + 1 = 0$ .
  - $x^3 - xy^2 = 0$ .
  - $x^3 + y^3 = 0$ .
  - $x^2 + y^2 = 0$ .

- viii.  $x^2y = 0$ .  
 ix.  $(x^2 - 1)(y^2 - 4) = 0$ .  
 x.  $(x^2 - 1)^2 + (y^2 - 4)^2 = 0$ .  
 xi.  $(y - mx - c)^2 + (y - m'x - c')^2 = 0$ .  
 xii.  $(x^2 - a^2)^2(x^2 - b^2)^2 + c^4(y^2 - a^2)^2 = 0$ .  
 xiii.  $(x - a)^2 - y^2 = 0$ .  
 xiv.  $(x + y)^2 - c^2 = 0$ .

**CATEGORY-14.17. A GENERAL SECOND DEGREE EQUATION REPRESENTING A PAIR OF STRAIGHT LINES**

173. What loci are represented by the following equations:-

- i.  $6x^2 + 11xy - 10y^2 + x + 31y - 15 = 0$ . {Ans.  $3x - 2y + 5 = 0$ ;  $2x + 5y - 3 = 0$ }  
 ii.  $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ . {Ans.  $x + 3y - 1 = 0$ ;  $x + 3y + 5 = 0$ }  
 iii.  $x^2 + 2xy + y^2 - 2x - 2y + 1 = 0$ . {Ans.  $x + y - 1 = 0$ }  
 iv.  $x^2 + 2xy + y^2 + 2x + 2y + 2 = 0$ . {Ans. no locus}  
 v.  $2x^2 + 6xy + 5y^2 + 4x + 6y + 2 = 0$ . {Ans.  $(-1, 0)$ }

174. Find the value of  $k$  so that the following equations may represent pairs of straight lines and find their equations also:-

- i.  $6x^2 + xy + ky^2 - 11x + 43y - 35 = 0$ . {Ans.  $k = -12$ ,  $3x - 4y + 5 = 0$ ,  $2x + 3y - 7 = 0$ }  
 ii.  $2x^2 + xy - y^2 + kx + 6y - 9 = 0$ . {Ans.  $k = -3$ ,  $2x - y + 3 = 0$ ,  $x + y - 3 = 0$ }  
 iii.  $12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0$ . {Ans.  $k = -10$ ,  $4x - 2y + 1 = 0$ ,  $3x - y + 2 = 0$ ;  
 $k = -\frac{35}{2}$ ,  $3x - 4y + 2 = 0$ ,  $8x - y + 2 = 0$ }

iv.  $kxy - 8x + 9y - 12 = 0$ . {Ans.  $k = 6$ ,  $x = -\frac{3}{2}$ ,  $y = \frac{4}{3}$ }

v.  $x^2 + kxy + y^2 - 5x - 7y + 6 = 0$ . {Ans.  $k = \frac{5}{2}$ ,  $2x + y - 6 = 0$ ,  $x + 2y - 2 = 0$ ;

$k = \frac{10}{3}$ ,  $3x + y - 6 = 0$ ,  $x + 3y - 3 = 0$ }

vi.  $kx^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ . {Ans. 2}

vii.  $3x^2 - 11xy + 10y^2 - 7x + 13y + k = 0$ . {Ans. 4}

viii.  $x^2 - kxy + y^2 + 2y + 2 = 0$ . {Ans.  $k \in \phi$ }

ix.  $4x^2 + 8xy + ky^2 = 9$ . {Ans. 4}

175. What relations must hold between the coefficients of the equations so that each of them may represent a pair of straight lines?

i.  $ax^2 + by^2 + cx + cy = 0$ ; {Ans.  $\left\{ \begin{array}{l} ab < 0 \\ c = 0; a + b = 0 \end{array} \right\}$ }

ii.  $ay^2 + bxy + cy + dx = 0$ ; {Ans.  $d = 0$ ;  $ad = bc$ }

- iii.  $x^2 + y^2 + 2gx + 2fy + 1 = 0$ . {Ans.  $f^2 + g^2 = 1$ }
176. Prove that the following equations represent two straight lines; find the angle between them; their point of intersection; and the equation of angular bisectors:-
- i.  $6y^2 - xy - x^2 + 30y + 36 = 0$ . {Ans.  $45^\circ$ ,  $(\frac{6}{5}, -\frac{12}{5})$ ,  $x^2 - 14xy - y^2 - 36x + 12y + 36 = 0$ }
- ii.  $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$ . {Ans.  $\tan^{-1} \frac{3}{5}$ ,  $(2, 1)$ ,  $5x^2 - 6xy - 5y^2 - 14x + 22y + 3 = 0$ }
- iii.  $3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0$ . {Ans.  $90^\circ$ ,  $(-\frac{3}{2}, -\frac{5}{2})$ ,  $8x^2 - 12xy - 8y^2 - 6x - 58y - 77 = 0$ }
- iv.  $y^2 + xy - 2x^2 - 5x - y - 2 = 0$ . {Ans.  $\tan^{-1} 3$ ,  $(-1, 1)$ ,  $x^2 + 6xy - y^2 - 4x + 8y - 6 = 0$ }
- v.  $x^2 - y^2 + 4x + 4y = 0$ . {Ans.  $90^\circ$ ,  $(-2, 2)$ ,  $xy - 2x + 2y - 4 = 0$ }
- vi.  $2x^2 - 5xy + 2y^2 - 3x + 3y + 1 = 0$ . {Ans.  $\tan^{-1} \frac{3}{2}$ ,  $(\frac{1}{3}, -\frac{1}{3})$ ,  $3x^2 - 3y^2 - 2x - 2y = 0$ }
- vii.  $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ . {Ans.  $\tan^{-1} \frac{1}{5}$ ,  $(1, -2)$ ,  $5x^2 + 2xy - 5y^2 - 6x - 22y - 19 = 0$ }
- viii.  $2x^2 + 5xy + 2y^2 + 3x + 3y + 1 = 0$ . {Ans.  $\cos^{-1}(\frac{4}{5})$ }
177. The equation  $12x^2 + 7xy + by^2 + gx + 7y - 1 = 0$  represents a pair of perpendicular lines, then find  $b$  and  $g$ .  
{Ans.  $b = -12$ ,  $g = -1$ }
178. Find the combined equation of the lines through  $(-1, -1)$  and making angle  $45^\circ$  with the line  $x + y = 0$ .  
{Ans.  $xy + x + y + 1 = 0$ }
179. The equations to a pair of opposite sides of a parallelogram are  $x^2 - 7x + 6 = 0$  and  $y^2 - 14y + 40 = 0$ ; find the equations to its diagonals. {Ans.  $5y + 6x = 56$ ,  $5y - 6x = 14$ }
180. Find the distance between the parallel lines given by  $x^2 + 2\sqrt{2}xy + 2y^2 + 4x - 8 = 0$ . {Ans. 4}
181. Find the distance between the parallel lines given by  $4x^2 + 12xy + 9y^2 - 6x - 9y + 2 = 0$ . {Ans.  $\frac{1}{\sqrt{13}}$ }
182. Prove that the general equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents two parallel straight lines if  $h^2 = ab$  and  $bg^2 = af^2$ . Prove also that the distance between them is  $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$ .
183. If the equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represent a pair of straight lines, prove that the equation to the third pair of straight lines passing through the points where these meet the axes is  $ax^2 + 2\left(\frac{2fg}{c} - h\right)xy + by^2 + 2gx + 2fy + c = 0$ .
184. If the equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents two straight lines, prove that the square of the distance of their point of intersection from the origin is  $\frac{c(a+b) - f^2 - g^2}{ab - h^2}$ .

**CATEGORY-14.18. HOMOGENEOUS SECOND DEGREE EQUATION REPRESENTING A PAIR OF STRAIGHT LINES**

185. Find what straight lines are represented by the following equations, determine the angles between them and find the equation of the bisectors:-

- i.  $x^2 - 7xy + 12y^2 = 0$ . {Ans.  $x - 3y = 0$ ;  $x - 4y = 0$ ;  $\tan^{-1} \frac{1}{13}$ ,  $7x^2 - 22xy - 7y^2 = 0$ }
- ii.  $4x^2 - 24xy + 11y^2 = 0$ . {Ans.  $2x - 11y = 0$ ;  $2x - y = 0$ ,  $\tan^{-1} \frac{4}{3}$ ,  $12x^2 - 7xy - 12y^2 = 0$ }
- iii.  $33x^2 - 71xy - 14y^2 = 0$ . {Ans.  $11x + 2y = 0$ ;  $3x - 7y = 0$ ,  $\tan^{-1} \frac{83}{19}$ ,  $71x^2 + 94xy - 71y^2 = 0$ }
- iv.  $2x^2 - 7xy + 3y^2 = 0$ . {Ans.  $y = 2x$ ;  $x - 3y = 0$ ,  $45^\circ$ ,  $7x^2 - 2xy - 7y^2 = 0$ }
186. If the slopes of the lines given by  $ax^2 + 2hxy + by^2 = 0$  are in the ratio 3:1, then find  $h^2$ . {Ans.  $\frac{4ab}{3}$ }
187. If the slope of one line in the pair  $ax^2 + 4xy + y^2 = 0$  is three times the other, then find  $a$ . {Ans. 3}
188. The lines represented by  $xy - y^2 = 0$  are rotated through an angle  $\frac{\pi}{2}$ . Find the equation of the lines in the new position. {Ans.  $x^2 + xy = 0$ }
189. Find the combined equation to a pair of straight lines passing through the origin and inclined at  $30^\circ$  and  $60^\circ$  respectively with  $x$ -axis. {Ans.  $\sqrt{3}(x^2 + y^2) = 4xy$ }
190. Find the angle between the pair of straight lines  $y^2 \sin^2 \theta - xy \sin^2 \theta + x^2 (\cos^2 \theta - 1) = 0$ .
191. If the sum of the slopes of the lines given by  $4x^2 + 2kxy - 7y^2 = 0$  is equal to the product of the slopes, then find  $k$ . {Ans. -2}
192. Show that the lines  $x^2 - 2xy - y^2 = 0$  and  $x + y = 3$  form a right-angled triangle.
193. Find the equation to the pair of lines perpendicular to the pair of lines  $3x^2 - 4xy + y^2 = 0$ . {Ans.  $x^2 + 4xy + 3y^2 = 0$ }
194. Prove that the two straight lines  $x^2 \sin^2 \alpha \cos^2 \theta + 4xy \sin \alpha \sin \theta + y^2 [4 \cos \alpha - (1 + \cos \alpha)^2 \cos^2 \theta] = 0$  meet at angle  $\alpha$ .
195. Prove that the two straight lines  $(x^2 + y^2)(\cos^2 \theta \sin^2 \alpha + \sin^2 \theta) = (x \tan \alpha - y \sin \theta)^2$  include an angle  $2\alpha$ .
196. If the lines represented by  $x^2 - 2pxy - y^2 = 0$  are rotated about the origin through an angle  $\theta$ , one in clockwise direction and other in anti-clockwise direction, then find the equation of the bisectors of the angle between the lines in the new position. {Ans.  $px^2 + 2xy - py^2 = 0$ }
197. Show that the two straight lines  $x^2(\tan^2 \theta + \cos^2 \theta) - 2xy \tan \theta + y^2 \sin^2 \theta = 0$  make with the axis of  $x$  angles such that the difference of their tangents is 2.
198. Prove that the equations to the straight lines passing through the origin which make an angle  $\alpha$  with the straight line  $y + x = 0$  are given by the equation  $x^2 + 2xy \sec 2\alpha + y^2 = 0$ .
199. Show that the straight lines  $(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$  form with the line  $Ax + By + C = 0$  an equilateral triangle whose area is  $\frac{C^2}{\sqrt{3}(A^2 + B^2)}$ .
200. Prove that the equation  $y^2(\cos \alpha + \sqrt{3} \sin \alpha) \cos \alpha - xy(\sin 2\alpha - \sqrt{3} \cos 2\alpha) + x^2(\sin \alpha - \sqrt{3} \cos \alpha) \sin \alpha = 0$  represents two straight lines inclined at  $60^\circ$  to each other. Prove also that the area of the triangle formed with them by the straight line  $(\cos \alpha - \sqrt{3} \sin \alpha)y - (\sin \alpha + \sqrt{3} \cos \alpha)x + a = 0$  is  $\frac{a^2}{4\sqrt{3}}$ , and that this triangle is equilateral.

201. Show that the equation  $bx^2 - 2hxy + ay^2 = 0$  represents a pair of straight lines which are at right angles to the pair given by the equation  $ax^2 + 2hxy + by^2 = 0$ .
202. If pairs of straight lines  $x^2 - 2pxy - y^2 = 0$  and  $x^2 - 2qxy - y^2 = 0$  be such that each pair bisects the angles between the other pair, prove that  $pq = -1$ .
203. Prove that the pair of lines  $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$  is equally inclined to the pair  $ax^2 + 2hxy + by^2 = 0$ . Show also that the pair  $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$  is equally inclined to the same pair.
204. If one of the straight lines given by the equation  $ax^2 + 2hxy + by^2 = 0$  coincide with one of those given by  $a'x^2 + 2h'xy + b'y^2 = 0$ , and the other lines represented by them be perpendicular, prove that  $\frac{ha'b'}{b' - a'} = \frac{h'ab}{b - a} = \frac{1}{2} \sqrt{-aa'bb'}$ .
205. Show that the distance between the points of intersection of the straight line  $x \cos \alpha + y \sin \alpha - p = 0$  with the straight lines  $ax^2 + 2hxy + by^2 = 0$  is  $\frac{2p\sqrt{h^2 - ab}}{b \cos^2 \alpha - 2h \cos \alpha \sin \alpha + a \sin^2 \alpha}$ . Deduce the area of the triangle formed by them.
206. Prove that the product of the perpendiculars let fall from the point  $(x', y')$  upon the pair of straight lines  $ax^2 + 2hxy + by^2 = 0$  is  $\frac{|ax'^2 + 2hx'y' + by'^2|}{\sqrt{(a-b)^2 + 4h^2}}$ .
207. Prove that the equation  $y^3 - x^3 + 3xy(y - x) = 0$  represents three straight lines equally inclined to one another.
208. Prove that one of the lines represented by the equation  $ax^3 + bx^2y + cxy^2 + dy^3 = 0$  will bisect the angle between the other two if  $(3a + c)(bc + 2cd - 3ad) = (b + 3d)^2(bc + 2ab - 3ad)$ .
209. Show that two of the straight lines represented by the equation  $ay^4 + bxy^3 + cx^2y^2 + dx^3y + ex^4 = 0$  will be at right angles if  $(b + d)(ad + be) + (e - a)^2(a + c + e) = 0$ .
210. Prove that two of the lines represented by the equation  $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$  will bisect the angles between the other two if  $c + 6a = 0$  and  $b + d = 0$ .

**CATEGORY-14.19. HOMOGENEOUS EQUATION OF LINES JOINING THE ORIGIN TO THE POINTS OF INTERSECTION OF A GIVEN LINE AND A GIVEN SECOND DEGREE CURVE**

211. Find the lines joining the origin to the points of intersection of  $2x^2 + 3xy - 4x + 1 = 0$  and  $3x + y = 1$ .  
{ Ans.  $x^2 - y^2 - 5xy = 0$  }
212. Prove that the straight lines joining the origin to the points of intersection of the straight line  $x - y = 2$  and the curve  $5x^2 + 12xy - 8y^2 + 8x - 4y + 12 = 0$  make equal angles with the axes.
213. Prove that the angle between the straight lines joining the origin to the intersection of the straight line  $y = 3x + 2$  with the curve  $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$  is  $\tan^{-1} \frac{2\sqrt{2}}{3}$ .
214. Find the equation to the pair of straight lines joining the origin to the intersections of the straight line

- $y = mx + c$  and the curve  $x^2 + y^2 = a^2$ . Prove that they are at right angles if  $2c^2 = a^2(1 + m^2)$ .
215. Prove that the straight lines joining the origin to the points of intersection of the straight line  $kx + hy = 2hk$  with the curve  $(x - h)^2 + (y - k)^2 = c^2$  are at right angles if  $h^2 + k^2 = c^2$ .
216. Show that the straight lines joining the origin to the other two point of intersection of the curves whose equations are  $ax^2 + 2hxy + by^2 + 2gx = 0$  and  $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$  will be at right angles if  $g(a' + b') - g'(a + b) = 0$ .

### CATEGORY-14.20. LOCUS PROBLEMS IN STRAIGHT LINES

217. The base  $BC (= 2a)$  of a triangle  $ABC$  is fixed; the axes being  $BC$  and a perpendicular to it through its middle point, find the locus of the vertex  $A$ , when
- the difference of the base angles is given  $(= \alpha)$ . {Ans.  $x^2 + 2xy \cot \alpha - y^2 = a^2$ }
  - the product of the tangents of the base angles is given  $(= \lambda)$ . {Ans.  $y^2 + \lambda x^2 = \lambda a^2$ }
  - the tangent of one base angle is  $m$  times the tangent of the other. {Ans.  $(m + 1)x = (m - 1)a$ }
  - $m$  times the square of one side added to  $n$  times the square of the other side is equal to a constant quantity  $c^2$ . {Ans.  $(m + n)(x^2 + y^2 + a^2) - 2ax(m - n) = c^2$ }
218. From a point  $P$  perpendiculars  $PM$  and  $PN$  are drawn upon two fixed lines which are inclined at an angle  $\omega$  and meet in  $O$ . Find the locus of  $P$ .
- if  $OM + ON$  be equal to  $2c$ .
  - if  $OM - ON$  be equal to  $2d$ .
  - if  $PM + PN$  be equal to  $2c$ .
  - if  $PM - PN$  be equal to  $2c$ .
  - if  $MN$  be equal to  $2c$ .
  - if  $MN$  pass through the fixed point  $(a, b)$ .
  - if  $MN$  be parallel to the given line  $y = mx$ .
219. Two fixed points  $A$  and  $B$  are taken on the axes such that  $OA = a$  and  $BO = b$ . Two variable points  $A'$  and  $B'$  are taken on the same axes. Find the locus of the intersection of  $AB'$  and  $A'B$  when
- $OA' + OB' = OA + OB$
  - $\frac{1}{OA'} - \frac{1}{OB'} = \frac{1}{OA} - \frac{1}{OB}$ .
220.  $OX$  and  $OY$  are two straight lines at right angles to one another. On  $OY$  is taken a fixed point  $A$  and on  $OX$  any point  $B$ . On  $AB$  an equilateral triangle is described, its vertex  $C$  being on the side of  $AB$  away from  $O$ . Show that the locus of  $C$  is a straight line.
221. If a straight line pass through a fixed point, find the locus of the middle point of the portion of it which is intercepted between two given straight lines.
222.  $A$  and  $B$  are two fixed points. If  $PA$  and  $PB$  intersect a constant distance  $2c$  from a given straight line, find the locus of  $P$ .
223. Through a fixed point  $O$  are drawn two straight lines at right angles to meet two fixed straight lines, which are also at right angles, in the points  $P$  and  $Q$ . Show that the locus of the foot of the perpendicular from  $O$  on  $PQ$  is a straight line.
224. Find the locus of a point at which two given portions of the same straight line subtend equal angles.
225. Find the locus of a point which moves so that the difference of its distances from two fixed straight lines at right angles is equal to its distance from a fixed straight line.
226. Having given the bases and the sum of the areas of a number of triangles which have a common vertex,



show that the locus of this vertex is a straight line.

227. Through a given point  $O$  a straight line is drawn to cut two given straight lines in  $R$  and  $S$ . Find the locus of a point  $P$  on this variable straight line, which is such that
- $2OP = OR + OS$
  - $OP^2 = OR \cdot OS$ .
228. Given  $n$  straight lines and a fixed point  $O$ . Through  $O$  is drawn a straight line meeting these lines in the points  $R_1, R_2, R_3, \dots, R_n$ , and on it is taken a point  $R$  such that  $\frac{n}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} + \frac{1}{OR_3} + \dots + \frac{1}{OR_n}$ . Show that the locus of  $R$  is a straight line.
229. A right angled triangle  $ABC$ , having  $C$  a right angle, is of given magnitude and the angular points  $A$  and  $B$  slides along two given perpendicular axes. Show that the locus of  $C$  is the pair of straight lines whose equations are  $y = \pm \frac{b}{a}x$ .
230. Two given straight lines meet in  $O$  and through a given point  $P$  is drawn a straight line to meet them in  $Q$  and  $R$ . If the parallelogram  $OQSR$  be completed find the equation to the locus of  $S$ .
231. Through a given point  $O$  is drawn a straight line to meet two given parallel straight lines in  $P$  and  $Q$ . Through  $P$  and  $Q$  are drawn straight lines in given directions to meet in  $R$ . Prove that the locus of  $R$  is a straight line.
232. Show that the orthocenter of the triangle formed by the straight lines  $ax^2 + 2hxy + by^2 = 0$  and  $lx + my = 1$  is a point  $(x', y')$  such that  $\frac{x'}{l} = \frac{y'}{m} = \frac{a+b}{am^2 - 2hlm + bl^2}$ . Hence find the locus of the orthocenter of a triangle of which two sides are given in position and whose third side goes through a fixed point.

#### CATEGORY-14.21. TRANSFORMATION OF AXES

233. The origin is shifted to the point  $(-2, 3)$  then what are the co-ordinates of the point  $(3, -5)$  in the new position? {Ans.  $(5, -8)$ }
234. If the origin is shifted to the point  $(1, -2)$  then the coordinates of point  $A$  becomes  $(2, 3)$ . What are the original coordinates of  $A$ ? {Ans.  $(3, 1)$ }
235. Transform to parallel axes through the point  $(1, -2)$  the equations  $y^2 - 4x + 4y + 8 = 0$  and  $2x^2 + y^2 - 4x + 4y = 0$ . {Ans.  $\bar{y}^2 = 4\bar{x}$ ,  $2\bar{x}^2 + \bar{y}^2 = 6$ }
236. What does the equation  $(a-b)(x^2 + y^2) - 2abx = 0$  become if the origin be moved to the point  $\left(\frac{ab}{a-b}, 0\right)$ ? {Ans.  $(a-b)^2(\bar{x}^2 + \bar{y}^2) = a^2b^2$ }
237. By transforming to parallel axes through a properly chosen point  $(h, k)$ , prove that the equation  $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$  can be reduced to one containing only terms of the second degree. Also find  $(h, k)$ . {Ans.  $\left(-\frac{3}{2}, -\frac{5}{2}\right)$ }
238. What will be the coordinates of the point  $(4, 2\sqrt{3})$  when the axes are rotated through an angle of  $30^\circ$  in clockwise sense? {Ans.  $(\sqrt{3}, 5)$ }
239. What will be the coordinates of the point in original position if its coordinates after rotation of axes

through an angle  $60^\circ$  be  $(2, -\sqrt{3})$ ? {Ans.  $\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$ }

240. Transform to axes inclined at  $45^\circ$  to the original axes the equations  $x^2 - y^2 = 1$ ,  $17x^2 - 16xy + 17y^2 = 225$  and  $x^4 + y^4 + 6x^2y^2 = 2$ . {Ans.  $2\bar{x}\bar{y} + 1 = 0$ ,  $9\bar{x}^2 + 25\bar{y}^2 = 225$ ,  $\bar{x}^4 + \bar{y}^4 = 1$ }
241. If the axes are turned through an angle  $\tan^{-1} 2$ , what does the equation  $4xy - 3x^2 = 1$  become? {Ans.  $\bar{x}^2 - 4\bar{y}^2 = 1$ }
242. Find the angle through which the axes may be turned so that the equation  $ax + by + c = 0$  can be reduced to the form  $x = \text{constant}$  and determine the value of this constant. {Ans.  $\tan^{-1} \frac{b}{a}$ ,  $-\frac{c}{\sqrt{a^2 + b^2}}$ }
243. If mixed term  $xy$  is to be removed from the general equation of second degree  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ , then prove that one should rotate the axes through an angle  $\theta$  given by  $\tan 2\theta = \frac{2h}{a-b}$ .

#### CATEGORY-14.22. ADDITIONAL QUESTIONS

244. If a line joining two points  $A(2,0)$  and  $B(3,1)$  is rotated about  $A$  in anticlockwise direction through an angle  $15^\circ$  such that the point  $B$  goes to  $C$  in the new position, then find the coordinates of  $C$ . {Ans.  $\left(2 + \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$ }
245. If  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , then find the points of intersection of the two curves  $y = \cos x$  and  $y = \sin 3x$ . {Ans.  $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(\frac{\pi}{8}, \cos \frac{\pi}{8}\right), \left(-\frac{3\pi}{8}, \cos \frac{3\pi}{8}\right)$ }

# Mathematics for IIT-JEE

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*PART-IV*  
*TWO DIMENSIONAL COORDINATE GEOMETRY*

*CHAPTER-15*  
*CIRCLE*

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## ***CHAPTER-15***

### ***CIRCLE***

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#### **LIST OF THEORY SECTIONS**

- 15.1. Equation Of A Circle
- 15.2. Position Of A Point/Line W.R.T. A Circle
- 15.3. Tangent, Normal, Chord And Polar Of A Circle
- 15.4. System Of Circles

#### **LIST OF QUESTION CATEGORIES**

- 15.1. Finding Equation Of A Circle From Its Properties
- 15.2. Finding Properties Of A Circle From Its Equation
- 15.3. Position Of A Point/Line W.R.T. A Circle
- 15.4. Tangent And Normal Of A Circle
- 15.5. Chord Of A Circle
- 15.6. Pole And Polar Of A Circle
- 15.7. Position Of Two Circles W.R.T. Each Other
- 15.8. Circles Passing Through The Points Of Intersection Of A Given Circle And A Given Line/Circle
- 15.9. Common Chord Of Two Circles
- 15.10. Common Tangent Of Two Circles
- 15.11. Angle Of Intersection Of Two Circles, Orthogonal Circles
- 15.12. Radical Axis, Radical Center
- 15.13. Coaxial System Of Circles
- 15.14. Locus Problems In Circles
- 15.15. Additional Questions

## CHAPTER-15

### CIRCLE

#### SECTION-15.1. EQUATION OF A CIRCLE

##### 1. Definition of a circle

A circle is the locus of a point which moves in a plane such that its distance from a fixed point in that plane is always constant. The fixed point is called the center of the circle and the constant distance is called the radius of the circle.

##### 2. Standard equation of a circle

- i. Equation of a circle with centre at  $(h, k)$  and radius  $a$  is  $(x-h)^2 + (y-k)^2 = a^2$
- ii. Therefore, equation of a circle with centre at origin and radius  $a$  is  $x^2 + y^2 = a^2$ .

##### 3. General equation of a circle

- i. Therefore, the equation of a circle is a second degree equation in which coefficient of  $x^2$  and  $y^2$  are equal and there is no term containing  $xy$ , i.e. the coefficient of  $xy$  is zero.
- ii. Therefore, a general second degree equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  may represent a circle when  $a = b$  &  $h = 0$  (necessary condition).
- iii. General equation of a circle is  $x^2 + y^2 + 2gx + 2fy + c = 0$ , which can be written as  $(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$ .
- iv. If  $g^2 + f^2 - c > 0$ , this equation represents a circle with radius  $\sqrt{g^2 + f^2 - c}$  and center  $(-g, -f)$ .
- v. If  $g^2 + f^2 - c = 0$ , this equation represents the point  $(-g, -f)$  (point circle).
- vi. If  $g^2 + f^2 - c < 0$ , this equation represents no locus (imaginary circle).
- vii. The general equation of a circle contains three arbitrary constants, i.e.  $g, f$  and  $c$ .

##### 4. Parametric equation of a circle

- i. Point  $(a\cos\theta, a\sin\theta)$  always lies on the circle  $x^2 + y^2 = a^2$ .
- ii. Point  $(h + a\cos\theta, k + a\sin\theta)$  always lies on the circle  $(x-h)^2 + (y-k)^2 = a^2$ .

##### 5. Equation of circle with given diametrically opposite points

Equation of a circle whose diametrically opposite points are  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$ .

#### SECTION-15.2. POSITION OF A POINT/LINE W.R.T. A CIRCLE

##### 1. Position of a point w.r.t. a circle

- i. A point  $(x_1, y_1)$  is inside, on or outside a circle  $x^2 + y^2 = a^2$  if  $x_1^2 + y_1^2 - a^2$  is  $<$ ,  $=$  or  $> 0$ .
- ii. A point  $(x_1, y_1)$  is inside, on or outside a circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  if  $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$  is  $<$ ,  $=$  or  $> 0$ .

##### 2. Position of a line w.r.t. a circle

- i. A line cuts, touches or does not cut a circle if length of perpendicular from the center on it is  $<$ ,  $=$  or  $>$  the radius.
- ii. Point(s) of intersection of a circle and a line can be obtained by solving their equations simultaneously.

#### SECTION-15.3. TANGENT, NORMAL, CHORD AND POLAR OF A CIRCLE

**1. Tangent to a circle**

- Definition of Tangent:** The tangent at any point on the curve is the straight line which is the limiting position of chord. The point is called the point of contact of the tangent.
- Equation of the tangent at the point  $(x_1, y_1)$  on the circle  $x^2 + y^2 = a^2$  is  $xx_1 + yy_1 = a^2$ .
- Equation of the tangent at the point  $(x_1, y_1)$  on the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  is  $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ .
- The line  $y = mx + c$  is tangent to the circle  $x^2 + y^2 = a^2$  if  $c = \pm a\sqrt{1+m^2}$ . Therefore, the line  $y = mx \pm a\sqrt{1+m^2}$  is always a tangent to the circle  $x^2 + y^2 = a^2$  at the point  $\left( \pm \frac{am}{\sqrt{1+m^2}}, \mp \frac{a}{\sqrt{1+m^2}} \right)$ .
- Equation to the pair of tangents drawn from  $(x_1, y_1)$  to the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  is  $(x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) = [xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c]^2$

**2. Length of tangent**

Length of tangent from an external point  $(x_1, y_1)$  to the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  is

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.$$

**3. Normal to a circle**

- Definition of Normal:** The normal at any point on the curve is the straight line which is perpendicular to the tangent at that point.
- All normals to a circle pass through its center.

**4. Length of chord**

- Definition of Chord:** A chord of the curve is the straight line which joins two points on the curve.
- Length of chord of a circle  $= 2\sqrt{r^2 - l^2}$ , where  $r$  is the radius of the circle and  $l$  is the distance of the chord from the center of the circle.

**5. Chord of contact of tangents**

- Definition of Chord of contact of tangents:** The chord joining the points of contact of two tangents to the curve drawn from a given point, is called the chord of contact of tangents.
- Equation of chord of contact of tangents drawn from an external point  $(x_1, y_1)$  to the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  is  $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ .

**6. Chord with a given mid-point**

- The equation of the chord with mid-point  $(h, k)$  of the circle  $x^2 + y^2 = a^2$  is  $hx + ky = h^2 + k^2$ .
- The equation of the chord with mid-point  $(h, k)$  of the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  is  $hx + ky + g(x + h) + f(y + k) + c = h^2 + k^2 + 2gh + 2fk + c$ .

**7. Polar and Pole**

- Definition of Polar of a point w.r.t. a conic section and Pole of the Polar:** If through a point  $P$  (inside, on or outside the conic section), drawn any straight line to meet the given curve at  $Q$  and  $R$ , the locus of the point of intersection of the tangents at  $Q$  and  $R$  is called the Polar of point  $P$ ; and the point  $P$  is called the pole of its polar.
- Equation of the polar of the point  $P \equiv (x_1, y_1)$  w.r.t. the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  is the line  $L \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ . The point  $P$  is called the pole of its polar line  $L$ .
- If the point  $P$  lies outside the circle, its polar is chord of contact of tangents from  $P$ ; if the point  $P$  lies on the circle, then its polar is tangent at point  $P$ ; if the point  $P$  lies inside the circle then its polar is a line

outside the circle.

- iv. The center of the circle has no polar. A line passing through the center of the circle has no pole.
- v. Therefore, chord with a given mid-point  $(h, k)$  is parallel to the polar of the point  $(h, k)$ .

### SECTION-15.4. SYSTEM OF CIRCLES

#### 1. Position of two circles w.r.t. each other

Two circles having radius  $r_1$  and  $r_2$  and distance between their centers is  $d$ , then if

- i.  $d > r_1 + r_2 \Rightarrow$  circles are outside of each other
- ii.  $d = r_1 + r_2 \Rightarrow$  circles touch externally
- iii.  $r_1 \sim r_2 < d < r_1 + r_2 \Rightarrow$  circles cut each other at two points
- iv.  $d = r_1 \sim r_2 \Rightarrow$  circles touch internally
- v.  $d < r_1 \sim r_2 \Rightarrow$  one circle is completely inside the other

#### 2. General equation of circles passing through the points of intersection of a given circle and a given line

General equation of circles passing through the points of intersection of a given circle

$x^2 + y^2 + 2gx + 2fy + c = 0$  and a given line  $lx + my + n = 0$  is  $x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0$ , where  $\lambda$  is any constant.

#### 3. General equations of circles passing through the points of intersection of two given circles

General equations of circles passing through the points of intersection of two given circles

$x^2 + y^2 + 2gx + 2fy + c = 0$  &  $x^2 + y^2 + 2g'x + 2f'y + c' = 0$  is

$(x^2 + y^2 + 2gx + 2fy + c) + \lambda(x^2 + y^2 + 2g'x + 2f'y + c') = 0$ , where  $\lambda$  is any constant  $\neq -1$ .

#### 4. Equation of common chord/common tangent

Equation of common chord/common tangent of two intersecting/touching circles is

$(x^2 + y^2 + 2gx + 2fy + c) - (x^2 + y^2 + 2g'x + 2f'y + c') = 0$ .

#### 5. Common tangents to two circles

- i. If circles are outside of each other  $\Rightarrow$  Four common tangents (Two direct and two indirect)
- ii. If circles touch externally  $\Rightarrow$  Three common tangents
- iii. If circles cut each other at two points  $\Rightarrow$  Two common tangents
- iv. If circles touch internally  $\Rightarrow$  One common tangent
- v. If one circle is completely inside the other  $\Rightarrow$  No common tangent
- vi. Direct tangents cut the line segment joining the centers in the ratio  $r_1 : r_2$  externally.
- vii. Indirect tangents cut the line segment joining the center in the ratio  $r_1 : r_2$  internally.
- viii. If radii of the two circles are equal, then direct tangents are parallel to the line joining their centers and they do not intersect.

#### 6. Angle of intersection of two intersecting circles

- i. **Definition of Angle of intersection of two intersecting curves:** If two curves intersect at a point, then the angle between their tangents at the point of intersection is called the angle of intersection of the two curves at the point of intersection.
- ii. **Orthogonal curves:** Two curves are said to intersect orthogonally when their angle of intersection is  $90^\circ$ , i.e. their tangents at the point of intersection are perpendicular, and such curves are called orthogonal curves.
- iii. If the angle of intersection is  $0^\circ$ , then the two curves have same tangent at the point of intersection.

- iv. Angle of intersection of two intersecting circles is  $\theta = \cos^{-1} \frac{|r_1^2 + r_2^2 - d^2|}{2r_1 r_2}$ , where  $r_1, r_2$  are the radii of the two circles and  $d$  is the distance between their centers.
- v. If angle of intersection is  $90^\circ$ , the two intersecting circles are said to be orthogonal circles. Orthogonality condition is  $r_1^2 + r_2^2 - d^2 = 0 \Rightarrow 2gg' + 2ff' = c + c'$ .

### 7. Radical axis and Radical center

- Definition of Radical axis:** The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.
- Equation of Radical axis of two circles is  $(x^2 + y^2 + 2gx + 2fy + c) - (x^2 + y^2 + 2g'x + 2f'y + c') = 0$ .
- Radical axis is always perpendicular to the line joining the centers.
- If circles cut each other then Radical axis is common chord; if circles touch each other then Radical axis is common tangent.
- Concentric circles cannot have equal tangents from a point, therefore concentric circles cannot have radical axis.
- Definition of Radical center:** The radical axes of three circles whose centers are non-collinear, taken in pairs, meet in a point. This point is called the Radical Center of the three circles.

### 8. Coaxial Circles

- Definition of Coaxial circles:** A system of three or more circles is said to be coaxial when each pair of circles of the system has the same radical axis.
- Centers of the coaxial circles are collinear and the line passing through their centers is perpendicular to their common radical axis.
- There are three kinds of coaxial system of circles:-
 

**Case I:** All circles of the coaxial system cut each other at two points on the common radical axis.

**Case II:** All circles of the coaxial system touch each other at one point on the common radical axis.

**Case III:** Circles of the coaxial system does not intersect each other nor cut the common radical axis.
- If the common radical axis is taken as  $y$ -axis and the line passing through the centers of the circles is taken as  $x$ -axis, then the equation of a coaxial system of circles is in simplest form and is  $x^2 + y^2 + 2g_i x + c = 0$ , where  $g_i$  is a variable and  $c$  is a constant.
- If  $x^2 + y^2 + 2gx + 2fy + c = 0$  is one of the circles of a coaxial system of circles and  $lx + my + n = 0$  is the common radical axis, then the general equation of the coaxial system of circles is  $x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0$ , where  $\lambda$  is any constant.
- If  $x^2 + y^2 + 2gx + 2fy + c = 0$  and  $x^2 + y^2 + 2g'x + 2f'y + c' = 0$  are the two circles of a coaxial system of circles then the general equation of the coaxial system of circles is  $(x^2 + y^2 + 2gx + 2fy + c) + \lambda(x^2 + y^2 + 2g'x + 2f'y + c') = 0$ , where  $\lambda$  is any constant  $\neq -1$ .



## EXERCISE-15

### CATEGORY-15.1. FINDING EQUATION OF A CIRCLE FROM ITS PROPERTIES

- Find the equation to the circle
  - whose radius is 3 and whose centre is  $(-1, 2)$ . {Ans.  $x^2 + y^2 + 2x - 4y = 4$ }
  - whose radius is 10 and whose centre is  $(-5, -6)$ . {Ans.  $x^2 + y^2 + 10x + 12y = 39$ }
  - whose radius is  $a + b$  and whose centre is  $(a, -b)$ . {Ans.  $x^2 + y^2 - 2ax + 2by = 2ab$ }
  - whose radius is  $\sqrt{a^2 - b^2}$  and whose centre is  $(-a, -b)$ . {Ans.  $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$ }
- Find the equation of the circle with centre on the  $y$ -axis and passing through the origin and  $(2, 3)$ . {Ans.  $3x^2 + 3y^2 - 13y = 0$ }
- Find the equation to the circle which passes through the points  $(1, -2)$  and  $(4, -3)$  and which has its centre on the straight line  $3x + 4y = 7$ . {Ans.  $15x^2 + 15y^2 - 94x + 18y + 55 = 0$ }
- Find the equation to the circle passing through the points  $(0, a)$  and  $(b, h)$  and having its centre on the axis of  $x$ . {Ans.  $b(x^2 + y^2 - a^2) = x(b^2 + h^2 - a^2)$ }
- Find the equations of the circle which pass through the points
  - $(0, 0)$ ,  $(a, 0)$  and  $(0, b)$ . {Ans.  $x^2 + y^2 - ax - by = 0$ }
  - $(1, 2)$ ,  $(3, -4)$  and  $(5, -6)$ . {Ans.  $x^2 + y^2 - 22x - 4y + 25 = 0$ }
  - $(1, 1)$ ,  $(2, -1)$  and  $(3, 2)$ . {Ans.  $x^2 + y^2 - 5x - y + 4 = 0$ }
  - $(5, 7)$ ,  $(8, 1)$  and  $(1, 3)$ . {Ans.  $3x^2 + 3y^2 - 29x - 19y + 56 = 0$ }
  - $(a, b)$ ,  $(a, -b)$  and  $(a + b, a - b)$ . {Ans.  $b(x^2 + y^2) - (a^2 + b^2)x + (a - b)(a^2 + b^2) = 0$ }
- Find the equation of the unit circle concentric with  $x^2 + y^2 + 8x + 4y - 8 = 0$ . {Ans.  $x^2 + y^2 + 8x + 4y + 19 = 0$ }
- Find the equation of the circle concentric with  $x^2 + y^2 - 3x + 4y - c = 0$  and passing through  $(-1, -2)$ . {Ans.  $x^2 + y^2 - 3x + 4y = 0$ }
- Find the equation of the circle which passes through  $(1, 0)$  and  $(0, 1)$  and has radius as small as possible. {Ans.  $x^2 + y^2 - x - y = 0$ }
- $ABCD$  is a square whose side is  $a$ . Find the equation of the circle circumscribing the square, taking  $AB$  and  $AD$  as axes of reference. {Ans.  $x^2 + y^2 - ax - ay = 0$ }
- $ABCD$  is a square whose side is  $a$ . Taking  $AB$  and  $AD$  as axes, prove that the equation to the circle circumscribing the square is  $x^2 + y^2 = a(x + y)$ .
- Find the equation to the circle which passes through the origin and cuts off intercepts equal to 3 and 4 from the axes. {Ans.  $x^2 + y^2 - 3x - 4y = 0$ }
- Find the equation to the circle passing through the origin and the points  $(a, b)$ , and  $(b, a)$ . Find the lengths of the chords that it cuts off from the axes. {Ans.  $x^2 + y^2 - \left(\frac{a^2 + b^2}{a + b}\right)(x + y) = 0, \frac{a^2 + b^2}{a + b}$ }
- Find the equation to the circle which goes through the origin and cuts off intercepts equal to  $h$  and  $k$  from the positive parts of the axes. {Ans.  $x^2 + y^2 - hx - ky = 0$ }
- A circle is described on the line joining the points  $(0, 1)$ ,  $(a, b)$  as diameter. Show that it cuts the  $x$ -axis in

points whose abscissae are roots of the equation  $x^2 - ax + b = 0$ .

15. Find the equation to the circle, of radius  $a$ , which passes through the two points on the axis of  $x$  which are at a distance  $b$  from the origin. {Ans.  $x^2 + y^2 \pm 2y\sqrt{a^2 - b^2} = b^2$ }
16. Find the equation to the circle which
  - i. touches each axis at a distance 5 from the origin. {Ans.  $x^2 + y^2 \pm 10x \pm 10y + 25 = 0$ }
  - ii. touches both axes with radius 5. {Ans.  $x^2 + y^2 \pm 10x \pm 10y + 25 = 0$ }
  - iii. touches each axis and is of radius  $a$ . {Ans.  $x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$ }
  - iv. touches both axes and passes through the point  $(3, 6)$ . {Ans.  $x^2 + y^2 - 6x - 6y + 9 = 0$ }
  - v. touches both axes and passes through the point  $(-2, -3)$ . {Ans.  $x^2 + y^2 + 2(5 \pm \sqrt{12})(x + y) + 37 \pm 10\sqrt{12} = 0$ }
  - vi. touches the axis of  $x$  and passes through the two points  $(1, -2)$  and  $(3, -4)$ . {Ans.  $x^2 + y^2 - 6x + 4y + 9 = 0, x^2 + y^2 + 10x + 20y + 25 = 0$ }
  - vii. touches the axis of  $y$  at the origin and passes through the point  $(b, c)$ . {Ans.  $b(x^2 + y^2) = x(b^2 + c^2)$ }
  - viii. touches the axis of  $x$  at a distance 3 from the origin and intercepts a distance 6 on the axis of  $y$ . {Ans.  $x^2 + y^2 \pm 6x \pm 6\sqrt{2}y + 9 = 0$ }
17. Points  $(1, 0)$  and  $(2, 0)$  are taken on the axis of  $x$ . On the line joining these points an equilateral triangle is described, its vertex being in the positive quadrant. Find the equations to the circles described on its sides as diameters. {Ans.  $x^2 + y^2 - 3x + 2 = 0, 2x^2 + 2y^2 - 5x - \sqrt{3}y + 3 = 0, 2x^2 + 2y^2 - 7x - \sqrt{3}y + 6 = 0$ }
18. Find the equation to the circle passing through the points  $(12, 43)$ ,  $(18, 39)$  and  $(42, 3)$  and prove that it also passes through the points  $(-54, -69)$  and  $(-81, -38)$ . {Ans.  $(x + 21)^2 + (y + 13)^2 = 65^2$ }
19. Find the equation to the circle circumscribing the quadrilateral formed by the straight lines  $2x + 3y = 2$ ,  $3x - 2y = 4$ ,  $x + 2y = 3$  and  $2x - y = 3$ . {Ans.  $8x^2 + 8y^2 - 25x - 3y + 18 = 0$ }
20. Prove that the equation to the circle of which the points  $(x_1, y_1)$  and  $(x_2, y_2)$  are the ends of a chord of a segment containing an angle  $\theta$  is
 
$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) \pm \cot \theta [(x - x_1)(y - y_2) - (x - x_2)(y - y_1)] = 0.$$
21. Find the equations to the circles in which the line joining the points  $(a, b)$  and  $(b, -a)$  is a chord subtending an angle of  $45^\circ$  at any point on its circumference. {Ans.  $x^2 + y^2 = a^2 + b^2, x^2 + y^2 - 2(a + b)x + 2(a - b)y + a^2 + b^2 = 0$ }
22. Find the general equation of a circle referred to two perpendicular tangents as axes. {Ans.  $x^2 + y^2 - 2cx - 2cy + c^2 = 0$ }
23. The angular points of a triangle are the points  $(a \cos \alpha, a \sin \alpha)$ ,  $(a \cos \beta, a \sin \beta)$  and  $(a \cos \gamma, a \sin \gamma)$ . Prove that the coordinates of the orthocentre of the triangle are  $(a(\cos \alpha + \cos \beta + \cos \gamma), a(\sin \alpha + \sin \beta + \sin \gamma))$ . Hence prove that if  $A, B, C$  and  $D$  be four points on a circle the orthocentres of the four triangles  $ABC, BCD, CDA$  and  $DAB$  lie on a circle.

### CATEGORY-15.2. FINDING PROPERTIES OF A CIRCLE FROM ITS EQUATION

24. Find the coordinates of the centers and the radii of the circles whose equations are
  - i.  $x^2 + y^2 - 4x - 8y = 41$ . {Ans.  $(2, 4), \sqrt{61}$ }

- ii.  $3x^2 + 3y^2 - 5x - 6y + 4 = 0$ . {Ans.  $(\frac{5}{6}, 1), \frac{\sqrt{13}}{6}$  }
- iii.  $x^2 + y^2 = k(x + k)$ . {Ans.  $(\frac{k}{2}, 0), \frac{\sqrt{5}}{2}k$  }
- iv.  $x^2 + y^2 = 2gx - 2fy$ . {Ans.  $(g, -f), \sqrt{f^2 + g^2}$  }
- v.  $\sqrt{1 + m^2}(x^2 + y^2) - 2cx - 2mcy = 0$ . {Ans.  $(\frac{c}{\sqrt{1+m^2}}, \frac{mc}{\sqrt{1+m^2}}), c$  }
25. Find the coordinates of the point on the circle  $x^2 + y^2 - 12x - 4y + 30 = 0$  which is farthest from the origin. {Ans.  $(9, 3)$  }
26. Find the area of an equilateral triangle inscribed in the circle  $x^2 + y^2 - 6x - 8y - 25 = 0$ . {Ans.  $\frac{225\sqrt{3}}{6}$  }

### CATEGORY-15.3. POSITION OF A POINT/LINE W.R.T. A CIRCLE

27. Determine the position of the following points w.r.t. the circle  $x^2 + y^2 - 4x - 6y + 5 = 0$  :-
- i.  $(1, 2)$ . {Ans. inside}
- ii.  $(-2, 3)$ . {Ans. outside}
28. Determine the position of the following lines w.r.t. the circle  $x^2 + y^2 - 22x - 4y + 25 = 0$  and also find points of intersection, if any:-
- i.  $3x + y - 5 = 0$ . {Ans. cuts at  $(1, 2)$  and  $(3, -4)$ }
- ii.  $x + y + 2 = 0$ . {Ans. does not cut}
29. How many tangents can be drawn from  $(1, 2)$  to  $x^2 + y^2 = 5$ . {Ans. 1}
30. How many tangents can be drawn from  $(2, 2)$  to the circle  $x^2 + y^2 - 6x - 4y + 3 = 0$ . {Ans. 0}
31. If the line  $y = mx$  passes outside the circle  $x^2 + y^2 - 10x + 16 = 0$ , then find the value of  $m$ . {Ans.  $|m| > \frac{3}{4}$  }
32. Show that the straight line  $y = mx + c$  cuts the circle  $x^2 + y^2 = a^2$  in real points if  $\sqrt{a^2(1 + m^2)} > c$ .
33. Show that  $x + 3y = 0$  is one of the diameters of the circle  $x^2 + y^2 - 12x + 4y + 6 = 0$ .
34. Prove that the straight line  $y = x + c\sqrt{2}$  touches the circle  $x^2 + y^2 = c^2$  and find its point of contact. {Ans.  $(-\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}})$  }
35. Find the condition that the straight line  $cx - by + b^2 = 0$  may touch the circle  $x^2 + y^2 = ax + by$  and find the point of contact. {Ans.  $c = a$  or  $b = 0, (0, b)$  }
36. Find whether the straight line  $x + y = 2 + \sqrt{2}$  touches the circle  $x^2 + y^2 - 2x - 2y + 1 = 0$ . {Ans. yes}
37. Find the condition that the straight line  $3x + 4y = k$  may touch the circle  $x^2 + y^2 = 10x$ . {Ans.  $k = 40, -10$ }
38. Find the value of  $p$  so that the straight line  $x \cos \alpha + y \sin \alpha - p = 0$  may touch the circle  $x^2 + y^2 - 2ax \cos \alpha - 2by \sin \alpha - a^2 \sin^2 \alpha = 0$ . {Ans.  $a \cos^2 \alpha + b \sin^2 \alpha \pm \sqrt{a^2 + b^2 \sin^2 \alpha}$  }
39. Find the condition that the straight line  $Ax + By + C = 0$  may touch the circle  $(x - a)^2 + (y - b)^2 = c^2$ . {Ans.  $Aa + Bb + C = \pm c\sqrt{A^2 + B^2}$  }
40. A square is inscribed in the circle  $x^2 + y^2 - 2x + 4y + 3 = 0$  and its sides are parallel to the coordinate axes

then find the vertices of the square.

41. Find the equation to the straight lines joining the origin to the points in which the straight line  $y = mx + c$  cuts the circle  $x^2 + y^2 = 2ax + 2by$ . Hence find the condition that these points may subtend a right angle at the origin. Hence find also the condition that the straight line may touch the circle. {Ans.  $(c + 2am)x^2 - 2(a - bm)xy + (c - 2b)y^2 = 0$ ,  $c = b - am$ ,  $c = b - am \pm \sqrt{(1 + m^2)(a^2 + b^2)}$ }
42. Find the equation to the circle which
- has its centre at the point (3, 4) and touches the straight line  $5x + 12y = 1$ . {Ans.  $x^2 + y^2 - 6x - 8y + \frac{381}{169} = 0$ }
  - has its centre at the point (1, -3) and touches the straight line  $2x - y - 4 = 0$ . {Ans.  $5x^2 + 5y^2 - 10x + 30y + 49 = 0$ }
  - touches the axes of coordinates and also the line  $\frac{x}{a} + \frac{y}{b} = 1$ , the centre being in the positive quadrant.  
{Ans.  $x^2 + y^2 - \left(a + b \pm \sqrt{a^2 + b^2}\right)x - \left(a + b \pm \sqrt{a^2 + b^2}\right)y + \frac{(a + b \pm \sqrt{a^2 + b^2})^2}{2} = 0$ }

#### CATEGORY-15.4. TANGENT AND NORMAL OF A CIRCLE

43. Write down the equation of the tangent to the circle
- $x^2 + y^2 - 3x + 10y = 15$  at the point (4, -11). {Ans.  $5x - 12y = 152$ }
  - $4x^2 + 4y^2 - 16x + 24y = 117$  at the point  $(-4, -\frac{11}{2})$ . {Ans.  $24x + 10y + 151 = 0$ }
  - $x^2 + y^2 = 4$  which are parallel to the line  $x + 2y + 3 = 0$ . {Ans.  $x + 2y = \pm 2\sqrt{5}$ }
  - $x^2 + y^2 + 2gx + 2fy + c = 0$  which are parallel to the line  $x + 2y - 6 = 0$ . {Ans.  $x + 2y + g + 2f = \pm \sqrt{5}\sqrt{g^2 + f^2 - c}$ }
44. Find the equation to the tangent to the circle  $x^2 + y^2 = a^2$  which
- is parallel to the straight line  $y = mx + c$ , {Ans.  $y = mx \pm a\sqrt{1 + m^2}$ }
  - is perpendicular to the straight line  $y = mx + c$ , {Ans.  $my + x = \pm a\sqrt{1 + m^2}$ }
  - passes through the point (b, 0) {Ans.  $ax \pm y\sqrt{b^2 - a^2} = ab$ }
  - makes with the axes a triangle whose area is  $a^2$ . {Ans.  $\pm x \pm y = a\sqrt{2}$ }
45. Find the slope of the tangent at the point (h, h) of the circle  $x^2 + y^2 = a^2$ . {Ans. -1}
46. Find the equation to a circle of radius  $r$  which touches the axis of  $y$  at a point distant  $h$  from the origin, the centre of the circle being in the positive quadrant. Prove also that the equation to the other tangent which passes through the origin is  $(r^2 - h^2)x + 2rhy = 0$ . {Ans.  $(x - r)^2 + (y - h)^2 = r^2$ }
47. Find the equation to the circle whose centre is at the point ( $\alpha$ ,  $\beta$ ) and which passes through the origin and prove that the equation of the tangent at the origin is  $\alpha x + \beta y = 0$ . {Ans.  $x^2 + y^2 - 2\alpha x - 2\beta y = 0$ }
48. A circle passes through the points (-1, 1), (0, 6) and (5, 5). Find the points on this circle the tangents at which are parallel to the straight line joining the origin to its centre. {Ans. (5, 1), (-1, 5)}
49. Find the lengths of the tangents drawn
- to the circle  $2x^2 + 2y^2 = 3$  from the point (-2, 3). {Ans.  $\frac{\sqrt{46}}{2}$ }
  - to the circle  $3x^2 + 3y^2 - 7x - 6y = 12$  from the point (6, -7). {Ans. 9}

- iii. to the circle  $x^2 + y^2 + 2bx - 3b^2 = 0$  from the point  $(a + b, a - b)$ . {Ans.  $\sqrt{2a^2 + 2ab + b^2}$ }
- iv. to the circle  $2(x^2 + y^2) + x - y + 5 = 0$  from  $(0,0)$ . {Ans.  $\sqrt{\frac{5}{2}}$ }
- v. to the circle  $x^2 + y^2 - 2x - y - 7 = 0$  from  $(-1, -3)$ . {Ans.  $2\sqrt{2}$ }
50. The distances from the origin of the centres of three circles  $x^2 + y^2 - 2\lambda x = c^2$  (where  $c$  is a constant and  $\lambda$  a variable) are in geometrical progression. Prove that the lengths of the tangents drawn to them from any point on the circle  $x^2 + y^2 = c^2$  are also in geometrical progression.
51. Find the equation to the pair of tangents drawn
- from the point  $(11, 3)$  to the circle  $x^2 + y^2 = 65$ , {Ans.  $7x - 4y = 65, 4x + 7y = 65$ }
  - from the point  $(4, 5)$  to the circle  $2x^2 + 2y^2 - 8x + 12y + 21 = 0$ . {Ans.  $(10\sqrt{3} - 6)x + 13y = \frac{22+50\sqrt{3}}{13}, (6+10\sqrt{3})x - 13y + 22 - 50\sqrt{3} = 0$ }
52. From any point on the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  tangents are drawn to the circle  $x^2 + y^2 + 2gx + 2fy + c \sin^2 \alpha + (g^2 + f^2) \cos^2 \alpha = 0$ . Prove that the angle between them is  $2\alpha$ .

#### CATEGORY-15.5. CHORD OF A CIRCLE

53. Find the length of the chord cut off by  $y = 2x + 1$  from the circle  $x^2 + y^2 = 2$ . {Ans.  $\frac{6}{\sqrt{5}}$ }
54. Find the length of the intercept which the circle  $x^2 + y^2 + 4x - 7y + 12 = 0$  cuts on  $y$ -axis. {Ans. 1}
55. Find the area of the circle in which a chord of length  $\sqrt{2}$  makes an angle  $\frac{\pi}{2}$  at the centre. {Ans.  $\pi$ }
56. If  $(x, 3)$  and  $(3, 5)$  are the extremities of a diameter of a circle with centre at  $(2, y)$ , then find the values of  $x$  and  $y$ . {Ans.  $x = 1, y = 4$ }
57. Find the equation of the diameter of the circle  $x^2 + y^2 - 2x + 4y = 0$  which passes through the origin. {Ans.  $2x + y = 0$ }
58. Find the equation of the chord of the circle  $x^2 + y^2 - 4x = 0$  whose mid point is  $(1, 0)$ . {Ans.  $x = 1$ }
59. Find the equation to the chord of the circle  $x^2 + y^2 = 9$  whose middle point is  $(1, -2)$ . {Ans.  $x - 2y - 5 = 0$ }
60. Find the coordinates of the middle point of the chord cut off by  $2x - 5y + 18 = 0$  by the circle  $x^2 + y^2 - 6x + 2y - 54 = 0$ . {Ans.  $(1, 4)$ }
61. A square is inscribed in the circle  $x^2 + y^2 - 2x + 4y - 93 = 0$  with its sides parallel to the coordinate axes. Find the coordinates of its vertices. {Ans.  $(-6, -9), (-6, 5), (8, -9), (8, 5)$ }
62. The lines  $2x - 3y - 5 = 0$  and  $3x - 4y = 7$  are diameters of a circle of the area 154 sq. units, then find the equation of the circle. {Ans.  $x^2 + y^2 - 2x + 2y - 47 = 0$ }
63. Find the length of the chord joining the points in which the straight line  $\frac{x}{a} + \frac{y}{b} = 1$  meets the circle  $x^2 + y^2 = r^2$ . {Ans.  $2\sqrt{r^2 - \frac{a^2b^2}{a^2+b^2}}$ }
64. Find the equation to the circles which pass through the origin and cut off equal chords  $a$  from the straight

lines  $y = x$  and  $y = -x$ . {Ans.  $x^2 + y^2 \pm \sqrt{2}ax = 0, x^2 + y^2 \pm \sqrt{2}ay = 0$ }

65. If the chord of contact of tangents drawn from a point on the circle  $x^2 + y^2 = a^2$  to the circle  $x^2 + y^2 = b^2$  touches the circle  $x^2 + y^2 = c^2$ , then show that  $a, b, c$  are in G.P.
66. Find the condition that the chord of contact of tangents from the point  $(x', y')$  to the circle  $x^2 + y^2 = a^2$  should subtend a right angle at the centre. {Ans.  $x'^2 + y'^2 = 2a^2$ }
67. A line is drawn through a fixed point  $P(\alpha, \beta)$  to cut the circle  $x^2 + y^2 = r^2$  at  $A$  and  $B$ . Show that  $PA \cdot PB$  is equal to  $\alpha^2 + \beta^2 - r^2$ .
68. Tangents are drawn from the point  $(h, k)$  to the circle  $x^2 + y^2 = a^2$ . Prove that the area of the triangle formed by them and the straight line joining their points of contact is  $\frac{a(h^2 + k^2 - a^2)^{\frac{3}{2}}}{h^2 + k^2}$ .
69. In any circle prove that the perpendicular from any point of it on the line joining the points of contact of two tangents is geometric mean of the perpendiculars from the point upon the two tangents.

#### CATEGORY-15.6. POLE AND POLAR OF A CIRCLE

70. Find the polar of the points
- $(1, 2)$  with respect to the circle  $x^2 + y^2 = 7$ . {Ans.  $x + 2y = 7$ }
  - $(4, -1)$  with respect to the circle  $2x^2 + 2y^2 = 11$ . {Ans.  $8x - 2y = 11$ }
  - $(-2, 3)$  with respect to the circle  $x^2 + y^2 - 4x - 6y + 5 = 0$ . {Ans.  $x = 0$ }
  - $(5, -\frac{1}{2})$  with respect to the circle  $3x^2 + 3y^2 - 7x + 8y - 9 = 0$ . {Ans.  $23x + 5y = 57$ }
  - $(a, -b)$  with respect to the circle  $x^2 + y^2 + 2ax - 2by + a^2 - b^2 = 0$ . {Ans.  $by - ax = a^2$ }
71. Find the pole of the straight line
- $x + 2y = 1$  with respect to the circle  $x^2 + y^2 = 5$ . {Ans.  $(5, 10)$ }
  - $2x - y = 6$  with respect to the circle  $5x^2 + 5y^2 = 9$ . {Ans.  $(\frac{3}{5}, -\frac{3}{10})$ }
  - $2x + y + 12 = 0$  with respect to the circle  $x^2 + y^2 - 4x + 3y - 1 = 0$ . {Ans.  $(1, -2)$ }
  - $48x - 54y + 53 = 0$  with respect to the circle  $3x^2 + 3y^2 + 5x - 7y + 2 = 0$ . {Ans.  $(\frac{1}{2}, -\frac{1}{3})$ }
  - $ax + by + 3a^2 + 3b^2 = 0$  with respect to the circle  $x^2 + y^2 + 2ax + 2by = a^2 + b^2$ . {Ans.  $(-2a, -2b)$ }
72. Find the equation to that chord of the circle  $x^2 + y^2 = 81$  which is bisected at the point  $(-2, 3)$ , and its pole with respect to the circle. {Ans.  $3y - 2x = 13, (-\frac{162}{13}, \frac{243}{13})$ }
73. Prove that the polars of the point  $(1, -2)$  with respect to the circles whose equations are  $x^2 + y^2 + 6y + 5 = 0$  and  $x^2 + y^2 + 2x + 8y + 5 = 0$  coincide; prove also that there is another point the polars of which with respect to these circles are the same and find its coordinates. {Ans.  $(2, -1)$ }
74. Prove that the distances of two points,  $P$  and  $Q$ , each from the polar of the other with respect to a circle, are to one another as the distances of the points from the centre of the circle.
75. Prove that the polars of a given point with respect to any one of the circles  $x^2 + y^2 - 2kx + c^2 = 0$ , where  $k$  is variable, always passes through a fixed point, whatever be the value of  $k$ .

#### CATEGORY-15.7. POSITION OF TWO CIRCLES W.R.T. EACH OTHER

76. Show that the circles  $(x - a)^2 + (y - b)^2 = c^2$  and  $(x - b)^2 + (y - a)^2 = c^2$  touch each other if  $a = b \pm \sqrt{2}c$ .

77. Show that the two circles  $x^2 + y^2 - 5 = 0$  and  $x^2 + y^2 - 2x - 4y - 15 = 0$  touch each other internally.
78. If the circles  $x^2 + y^2 = 9$  and  $x^2 + y^2 + 8y + c = 0$  touch each other, then find  $c$ . {Ans. 15}
79. If the circles  $x^2 + y^2 = a$  and  $x^2 + y^2 - 6x - 8y + 9 = 0$ , touch externally, then find  $a$ . {Ans. 1}
80. Show that the circle  $(x-2)^2 + (y-5)^2 = a^2$  will be inside the circle  $(x-3)^2 + (y-6)^2 = b^2$  if  $b > a + \sqrt{2}$ .
81. Show that the circles  $x^2 + (y-1)^2 = 9$ ,  $(x-1)^2 + y^2 = 25$  are such that one of these circles lies entirely inside the other.
82. Show that the circles  $x^2 + y^2 - 10x + 16 = 0$  and  $x^2 + y^2 = r^2$  intersect each other in two distinct points if  $2 < r < 8$
83. If two circles  $(x-1)^2 + (y-3)^2 = r^2$  and  $x^2 + y^2 - 8x + 2y + 8 = 0$  intersect in two distinct points, then find the value of  $r$ . {Ans.  $2 < r < 8$ }

**CATEGORY-15.8. CIRCLES PASSING THROUGH THE POINTS OF INTERSECTION OF A GIVEN CIRCLE AND A GIVEN LINE/CIRCLE**

84. Find the equation of the circle passing through  $(1, -3)$  and the points common to the two circles  $x^2 + y^2 - 6x + 8y - 16 = 0$  and  $x^2 + y^2 + 4x - 2y - 8 = 0$ . {Ans.  $x^2 + y^2 + \frac{3}{2}x + \frac{1}{2}y - 10 = 0$ }
85. Find the equation of a circle passing through  $(1, 1)$  and points of intersection of  $x^2 + y^2 + 13x - 3y = 0$  and  $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ . {Ans.  $4x^2 + 4y^2 + 30x - 13y - 25 = 0$ }
86. Two circles  $x^2 + y^2 = 6$  and  $x^2 + y^2 - 6x + 8 = 0$  are given. Find the equation of the circle through their point of intersection and the point  $(1, 1)$ . {Ans.  $x^2 + y^2 - 3x + 1 = 0$ }
87. If  $y = mx$  be the equation of a chord of a circle whose radius is  $a$ , the origin of coordinates being one extremity of the chord and the axis of  $x$  being a diameter of the circle, prove that the equation of a circle of which this chord is the diameter is  $(1 + m^2)(x^2 + y^2) - 2a(x + my) = 0$ .
88. A variable circle passes through the point of intersection  $O$  of any two straight lines and cuts off from them portions  $OP$  and  $OQ$  such that  $m OP + n OQ$  is equal to unity. Prove that this circle always passes through a fixed point.

**CATEGORY-15.9. COMMON CHORD OF TWO CIRCLES**

89. Find the length of the common chord of the circles  $x^2 + y^2 - 2ax - 4ay - 4a^2 = 0$  and  $x^2 + y^2 - 3ax + 4ay = 0$ . {Ans.  $8a\sqrt{\frac{14}{65}}$ }
90. Prove that the length of the common chord of the two circles whose equations are  $(x-a)^2 + (y-b)^2 = c^2$  and  $(x-b)^2 + (y-a)^2 = c^2$  is  $\sqrt{4c^2 - 2(a-b)^2}$ . Hence find the condition that the two circles may touch.
91. Find the length of the common chord of the circles, whose equations are  $(x-a)^2 + y^2 = a^2$  and  $x^2 + (y-b)^2 = b^2$  and prove that the equation to the circle whose diameter is this common chord is  $(a^2 + b^2)(x^2 + y^2) = 2ab(bx + ay)$ . {Ans.  $\frac{2ab}{\sqrt{a^2 + b^2}}$ }
92. Find the equations to the straight lines joining the origin to the points of intersection of  $x^2 + y^2 - 4x - 2y = 4$  and  $x^2 + y^2 - 2x - 4y - 4 = 0$ . {Ans.  $y = x$ }
93. Find the angle subtended by the common chord of  $x^2 + y^2 - 4x - 4y = 0$  and  $x^2 + y^2 = 16$  at the origin.

$$\left\{ \text{Ans. } \frac{\pi}{2} \right\}$$

94. Tangents are drawn to the circle  $x^2 + y^2 = 12$  at the points where it is met by the circle  $x^2 + y^2 - 5x + 3y - 2 = 0$ . Find the point of intersection of these tangents. {Ans.  $(6, -\frac{18}{5})$ }

#### CATEGORY-15.10. COMMON TANGENT OF TWO CIRCLES

95. Find the number of common tangents to the circles  $x^2 + y^2 - x = 0$ ,  $x^2 + y^2 + x = 0$ . {Ans. 3}
96. Find the equations to the common tangents of the circles  $x^2 + y^2 - 2x - 6y + 9 = 0$  and  $x^2 + y^2 + 6x - 2y + 1 = 0$ . {Ans.  $x = 0$ ,  $3x + 4y = 10$ ,  $y = 4$ ,  $3y = 4x$ }
97. Given the circles  $x^2 + y^2 - 2ax - 4ay - 4a^2 = 0$  and  $x^2 + y^2 - 3ax + 4ay = 0$ , find the equations of the common tangents and show that the length of each is  $4a$ . {Ans.  $x = 4a$ ,  $63x + 16y + 100a = 0$ }

#### CATEGORY-15.11. ANGLE OF INTERSECTION OF TWO CIRCLES, ORTHOGONAL CIRCLES

98. Find the angle of intersection of the circles  $x^2 + y^2 - 6x + 8y - 16 = 0$  and  $x^2 + y^2 + 4x - 2y - 8 = 0$ .  
{Ans.  $\cos^{-1} \frac{2}{\sqrt{533}}$ }
99. The two circles are drawn through the points  $(a, 5a)$  and  $(4a, a)$  to touch the axis of  $y$ . Prove that they intersect at an angle  $\tan^{-1} \frac{40}{9}$ .
100. Prove that the following pairs of circles intersect orthogonally:-  
i.  $x^2 + y^2 + x + y = 0$  and  $x^2 + y^2 + x - y = 0$ .  
ii.  $x^2 + y^2 - 2ax + c = 0$  and  $x^2 + y^2 + 2by - c = 0$ .  
iii.  $x^2 + y^2 - 2ax + 2by + c = 0$  and  $x^2 + y^2 + 2bx + 2ay - c = 0$ .
101. Find the equation to the circle which passes through the origin and cuts orthogonally each of the circles  $x^2 + y^2 - 6x + 8 = 0$  and  $x^2 + y^2 - 2x - 2y = 7$ . {Ans.  $3x^2 + 3y^2 - 8x + 29y = 0$ }
102. Prove that the two circles, which pass through the two points  $(0, a)$  and  $(0, -a)$  and touch the straight line  $y = mx + c$ , will cut orthogonally if  $c^2 = a^2(2 + m^2)$ .
103. If two circles cut orthogonally, prove that the polar of any point  $P$  on the first circle with respect to the second passes through the other end of the diameter of the first circle which goes through  $P$ .

#### CATEGORY-15.12. RADICAL AXIS, RADICAL CENTER

104. Find the radical axis of the pair of circles:-  
i.  $x^2 + y^2 = 144$  and  $x^2 + y^2 - 15x + 11y = 0$  {Ans.  $15x - 11y = 144$ }  
ii.  $x^2 + y^2 - 3x - 4y + 5 = 0$  and  $3x^2 + 3y^2 - 7x + 8y + 11 = 0$  {Ans.  $x + 10y = 2$ }
105. Find the radical centre of the sets of circles  
i.  $x^2 + y^2 + x + 2y + 3 = 0$ ,  $x^2 + y^2 + 2x + 4y + 5 = 0$  and  $x^2 + y^2 - 7x - 8y - 9 = 0$ . {Ans.  $(-\frac{2}{3}, -\frac{2}{3})$ }  
ii.  $(x-2)^2 + (y-3)^2 = 36$ ,  $(x+3)^2 + (y+2)^2 = 49$  and  $(x-4)^2 + (y+5)^2 = 64$ . {Ans.  $(\frac{26}{25}, \frac{13}{50})$ }
106. Given the three circles  $x^2 + y^2 - 16x + 60 = 0$ ,  $3x^2 + 3y^2 - 36x + 81 = 0$  and  $x^2 + y^2 - 16x - 12y + 84 = 0$ , find the point from



which the tangents to them are equal in length. Also find the length of these tangents. {Ans.  $(\frac{33}{4}, 2), \frac{1}{4}$ }

107. Show that the radical centre of three circles described on the three sides of a triangle as diameter is the orthocenter of the triangle.
108. Prove that the square of the tangent that can be drawn from any point on one circle to another circle is equal to twice the product of the perpendicular distance of the point from the radical axis of the two circles, and the distance between their centres.
109. Prove that a common tangent to two circles is bisected by the radical axis. Hence, by joining the middle points of any two of the common tangents, we have a construction for the radical axis.

### CATEGORY-15.13. COAXIAL SYSTEM OF CIRCLES

110. Find the equation of a circle which is coaxial with the circles  $2x^2 + 2y^2 - 2x + 6y - 3 = 0$  and  $x^2 + y^2 + 4x + 2y + 1 = 0$ . It is given that the center of the circle to be determined lies on the radical axis of these circles. {Ans.  $4x^2 + 4y^2 + 6x + 10y - 1 = 0$ }
111. Find the equation of the circle of radius  $\sqrt{5}$ , which is coaxial with the circles  $x^2 + y^2 = 4$  and  $x^2 + y^2 + 2x + 4y - 6 = 0$ . {Ans.  $x^2 + y^2 + 2x + 4y - 6 \pm \sqrt{6}(x^2 + y^2 - 4) = 0$ }
112. Show that as  $\lambda$  varies, the circles  $x^2 + y^2 + 2ax + 2by + 2\lambda(ax - by) = 0$  form a coaxial system. Find the equation of the radical axis and also the equation of the circle which are orthogonal to the circles of the above system.
113. Prove that the circles with respect to which the given line  $lx + my + n = 0$  is polar of the origin, form a coaxial system with radical axis  $2lx + 2my + n = 0$ .

### CATEGORY-15.14. LOCUS PROBLEMS IN CIRCLES

114. Plot locus
- $x^2 + y^2 \geq 1, x^2 + y^2 \leq 16$
  - $x^2 + y^2 \leq 9, x + y \geq 0$
  - $y^2 \leq 4 - x^2, 3x + 2y - 3 \leq 0$
  - $y \geq \sqrt{1 - x^2}, y + |x| \leq 4$
  - $x^2 + y^2 - 4 \leq 0, y - x^2 + 3 < 0, x \geq 0$
  - $x^2 + y^2 \leq 9, y \geq x^2 - 3, |x| \leq 1$
115. A point moves so that the sum of the squares of its distances from the four sides of a square is constant. Prove that it always lies on a circle.
116. A point moves so that the sum of the squares of the perpendiculars let fall from it on the sides of an equilateral triangle is constant. Prove that its locus is a circle.
117. A point moves so that the sum of the squares of its distances from the angular points of a triangle is constant. Prove that its locus is a circle.
118. Find the locus of a point which moves so that the square of the tangent drawn from it to the circle  $x^2 + y^2 = a^2$  is equal to  $c$  times its distance from the straight line  $lx + my + n = 0$ . {Ans. A circle}
119. Find the locus of a point whose distance from a fixed point is in a constant ratio to the length of the tangent drawn from it to a given circle. {Ans. A circle}
120. A straight line  $AB$  whose length is  $c$ , slides between two given lines which meet at  $O$ ; find the locus of the

- orthocentre of the triangle  $OAB$ . {Ans. A circle with centre  $O$ }
121. Find the locus of the vertex of a triangle, given (1) its base and the sum of the squares of its sides, (2) its base and the sum of  $m$  times the square of one side and  $n$  times the square of the other. {Ans. A circle}
122. A point moves so that the sum of the squares of its distances from  $n$  fixed points is given. Prove that its locus is a circle.
123. Whatever be the value of  $\alpha$ , prove that the locus of the intersection of the straight lines  $x\cos\alpha + y\sin\alpha = a$  and  $x\sin\alpha - y\cos\alpha = b$ , is a circle.
124. From a point  $P$  on a circle, perpendiculars  $PM$  and  $PN$  are drawn to two radii of the circle which are not at right angles. Find the locus of the middle point of  $MN$ .
125. Tangents are drawn to a circle from a point which always lies on a given line. Prove that the locus of the middle point of the chord of contact is another circle.
126. Find the locus of the middle points of chords of the circle  $x^2 + y^2 = a^2$  which pass through the fixed point  $(h, k)$ . {Ans. A circle}
127. Find the locus of the middle points of chords of the circle  $x^2 + y^2 = a^2$  which subtend a right angle at the point  $(c, 0)$ . {Ans. A circle}
128.  $O$  is a fixed point and  $P$  any point on a fixed circle. On  $OP$  is taken a point  $Q$  such that  $OQ$  is in a constant ratio to  $OP$ . Prove that the locus of  $Q$  is a circle.
129.  $O$  is a fixed point and  $P$  any point on a given straight line.  $OP$  is joined and on it is taken a point  $Q$  such that  $OP \cdot OQ = k^2$ . Prove that the locus of  $Q$  is a circle which passes through  $O$ .
130. One vertex of a triangle of given angles is fixed, and another moves along the circumference of a fixed circle. Prove that the locus of the remaining vertex is a circle and find its radius.
131.  $O$  is any point in the plane of a circle, and  $OP_1P_2$  any chord of the circle which passes through  $O$  and meets the circle in  $P_1$  and  $P_2$ . On this chord is taken a point  $Q$  such that  $OQ$  is equal to (1) the arithmetic, (2) the geometric and (3) the harmonic mean between  $OP_1$  and  $OP_2$ . In each case find the equation to the locus of  $Q$ . {Ans. A circle, circle, straight line}
132. Find the locus of the point of intersection of the tangent to a given circle and the perpendicular let fall on this tangent from a fixed point on the circle.
133. A circle touches the axis of  $x$  and cuts off a constant length  $2l$  from the axis of  $y$ . Prove that the equation of the locus of its centre is  $y^2 - x^2 = l^2$ .
134. A straight line moves so that the product of the perpendiculars on it from two fixed points is constant. Prove that the locus of the feet of the perpendiculars from each of these points upon the straight line is a circle, the same for each.
135.  $O$  is a fixed point and  $AP$  and  $BQ$  are two fixed parallel straight lines.  $BOA$  is perpendicular to both and  $POQ$  is a right angle. Prove that the locus of the foot of the perpendicular drawn from  $O$  upon  $PQ$  is the circle on  $AB$  as diameter.
136. Two rods of lengths  $a$  and  $b$ , slide along the axes in such a manner that their ends are always concyclic. Prove that the locus of the centre of the circle passing through these ends is the curve  $4(x^2 - y^2) = a^2 - b^2$ .
137. Show that the locus of a point, which is such that the tangents from it to two given concentric circles are inversely proportional to their radii, is a concentric circle, the square of whose radius is equal to the sum of the squares of the radii of the given circles.
138. Show that if the length of the tangent from a point  $P$  to the circle  $x^2 + y^2 = a^2$  be four times the length of the tangent from it to the circle  $(x - a)^2 + y^2 = a^2$ , then  $P$  lies on the circle  $15x^2 + 15y^2 - 32ax + a^2 = 0$ .

Prove also that these three circles pass through two points and that the distance between the centres of the first and third circles is sixteen times the distance between the centres of the second and third circles.

139. Find the locus of the foot of the perpendicular let fall from the origin upon any chord of the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  which subtends a right angle at the origin. Find also the locus of the middle points of these chords. {Ans. A circle}
140. Through a fixed point  $O$  are drawn two straight lines  $OPQ$  and  $ORS$  to meet a circle in  $P$  and  $Q$ , and  $R$  and  $S$ , respectively. Prove that the locus of the point of intersection of  $PS$  and  $QR$ , as also that of the point of intersection of  $PR$  and  $QS$ , is the polar of  $O$  with respect to the circle.
141.  $A, B, C$  and  $D$  are four points in a straight line. Prove that the locus of a point  $P$ , such that the angles  $APB$  and  $CPD$  are equal, is a circle.
142. The polar of  $P$  with respect to the circle  $x^2 + y^2 = a^2$  touches the circle  $(x - \alpha)^2 + (y - \beta)^2 = b^2$ ; prove that the locus is the curve given by the equation  $(\alpha x + \beta y - a^2)^2 = b^2(x^2 + y^2)$
143. Find the locus of the centre of the circle which cuts two given circles orthogonally.

### CATEGORY-15.15. ADDITIONAL QUESTIONS

144. If the points  $(2,0)$ ,  $(0,1)$ ,  $(4,5)$  and  $(0,c)$  are concyclic, then find the value of  $c$ . {Ans.  $\frac{14}{3}$ }
145. Show that four distinct points  $(2k,3k)$ ,  $(1,0)$ ,  $(0,1)$  and  $(0,0)$  lie on a circle for two values of  $k$ .
146. A circle  $C$  of radius 1 is inscribed in an equilateral triangle  $PQR$ . The point of contact of  $C$  with the sides  $PQ, QR, RP$  are  $D, E, F$  respectively. The line  $PQ$  is given by the equation  $\sqrt{3}x + y - 6 = 0$  and the point  $D$  is  $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ . Further, it is given that the origin and the center of  $C$  are on the same side of the line  $PQ$ . Find
- the equation of circle  $C$ . {Ans.  $x^2 + y^2 - 2\sqrt{3}x - 2y + 3 = 0$ }
  - the point  $E$  and  $F$ . {Ans.  $(\sqrt{3},0), \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$ }
  - the equations of  $QR$  and  $RP$ . {Ans.  $y = 0, y = \sqrt{3}x$ }
147. Find the no. of points of intersection of the curve  $\sin x = \cos y$  and circle  $x^2 + y^2 = 1$ . {Ans. 0}

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# Mathematics for IIT-JEE

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*PART-IV*  
*TWO DIMENSIONAL COORDINATE GEOMETRY*

*CHAPTER-16*  
*PARABOLA*

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## ***CHAPTER-16***

### ***PARABOLA***

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#### **LIST OF THEORY SECTIONS**

- 16.1. Conic Sections
- 16.2. Equation Of A Parabola
- 16.3. Position Of A Point/Line W.R.T. A Parabola
- 16.4. Tangent, Normal, Chord And Polar Of A Parabola
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#### **LIST OF QUESTION CATEGORIES**

- 16.1. Finding Equation Of A Parabola From Its Properties
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- 16.3. Position Of A Point/Line W.R.T. A Parabola
- 16.4. Tangent And Normal Of A Parabola
- 16.5. Chord Of A Parabola
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## CHAPTER-16

### PARABOLA

#### SECTION-16.1. CONIC SECTIONS

##### 1. Definition of Conic sections

A conic section is the locus of a point such that its distance from a fixed point ( $S$ ) always bears a constant ratio ( $e$ ) to its distance from a fixed line ( $D$ ).

##### 2. Definition of various terms of Conic sections

- i. **Focus ( $S$ ):** The fixed point is called the focus of the conic section.
- ii. **Directrix ( $D$ ):** The fixed line is called the directrix of the conic section.
- iii. **Eccentricity ( $e$ ):** The constant ratio is called the eccentricity of the conic section.
- iv. **Axis:** The line passing through the focus and perpendicular to the directrix is called the axis of the conic section.
- v. **Vertex ( $A$ ):** The point of intersection of the conic section and the axis is called the vertex of the conic section.
- vi. **Foot of Directrix ( $Z$ ):** The point of intersection of the directrix and the axis is called the foot of directrix.
- vii. **Latus rectum ( $LL'$ ):** The chord passing through the focus and perpendicular to the axis is called latus rectum.
- viii. **Focal chord:** A chord passing through the focus is called a focal chord.
- ix. **Ordinate ( $PN$ ):** Let  $P$  be a point on the conic section then the line segment  $PN$  perpendicular to the axis is called the ordinate of point  $P$ .
- x. **Double ordinate ( $PNP'$ ):** Let  $P$  be a point on the conic section then the chord  $PNP'$  perpendicular to the axis is called the double ordinate of point  $P$ .
- xi. **Focal distance:** Let  $P$  be a point on the conic section then the distance of  $P$  from the focus is called the focal distance of point  $P$ .
- xii. **Ellipse:** If  $0 < e < 1$ , then the conic section is called an ellipse.
- xiii. **Parabola:** If  $e = 1$ , then the conic section is called a parabola.
- xiv. **Hyperbola:** If  $e > 1$ , then the conic section is called a hyperbola.

##### 3. General equation of a conic section is a second degree equation

- i. Let  $S \equiv (h, k)$  be the focus,  $lx + my + n = 0$  be the directrix and  $e$  be the eccentricity of a conic section,

then by definition its equation is  $(x - h)^2 + (y - k)^2 = e^2 \frac{(lx + my + n)^2}{(l^2 + m^2)}$ , which is a second degree equation.

- ii. A general second degree equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  may represent
  - a. a pair of straight lines if  $\Delta = 0$  &  $h^2 \geq ab$ .
  - b. a circle if  $\Delta \neq 0$ ,  $a = b$  &  $h = 0$ .
  - c. a parabola if  $\Delta \neq 0$  &  $h^2 = ab$ .
  - d. an ellipse if  $\Delta \neq 0$  &  $h^2 < ab$ .
  - e. a hyperbola if  $\Delta \neq 0$  &  $h^2 > ab$ .

#### SECTION-16.2. EQUATION OF A PARABOLA

**1. Equation of a parabola with given Focus and Directrix**

- i. Equation of the parabola with Focus at  $(h, k)$  and Directrix  $lx + my + n = 0$  is

$$\begin{aligned}\sqrt{(x-h)^2 + (y-k)^2} &= \frac{|lx + my + n|}{\sqrt{l^2 + m^2}} \\ \equiv (x-h)^2 + (y-k)^2 &= \frac{(lx + my + n)^2}{l^2 + m^2} \\ \equiv (mx - ly)^2 + 2gx + 2fy + c &= 0.\end{aligned}$$

- ii. Second degree terms form a perfect square.

- iii.  $mx - ly = 0$  is perpendicular to the Directrix  $lx + my + n = 0$  and parallel to the axis.

**2. Standard equation of a parabola**

Equation of a standard parabola having vertex at origin, axis as  $x$ -axis and focus at positive  $x$ -axis is  $y^2 = 4ax$ , where  $a$  is the distance between the vertex and the focus.

**3. Properties of standard parabolas**

		$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
1.	Coordinates of Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
2.	Equation of Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
3.	Equation of Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
4.	Coordinates of Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
5.	Coordinates of Foot of Directrix	$(-a, 0)$	$(a, 0)$	$(0, -a)$	$(0, a)$
6.	Coordinates of the end points of the Latus Rectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
7.	Length of Latus Rectum	$4a$	$4a$	$4a$	$4a$
8.	Equations of Latus rectums	$x = a$	$x = -a$	$y = a$	$y = -a$

**4. Tracing a general parabola**

- i. A general parabola is traced by parallel and angular transformation of axes.  
 ii. Equation of a parabola having vertex at  $(h, k)$  and axis parallel to  $x$ -axis is  $(y - k)^2 = \pm 4a(x - h)$ .  
 Equation of a parabola having vertex at  $(h, k)$  and axis parallel to  $y$ -axis is  $(x - h)^2 = \pm 4a(y - k)$ .

**5. Parametric equation of standard parabola**

Point  $(at^2, 2at)$  always lies on the standard parabola  $y^2 = 4ax$ . This point is called point 't'.

**SECTION-16.3. POSITION OF A POINT/LINE W.R.T. A PARABOLA****1. Position of a point w.r.t. a parabola**

The point  $(x_1, y_1)$  lies inside, on or outside the parabola  $y^2 = 4ax$  according as  $y_1^2 - 4ax_1 <, =$  or  $> 0$ .

**2. Position of a line w.r.t. a parabola**

- i. The line  $y = c$ , which is parallel to the axis of the parabola  $y^2 = 4ax$ , cuts the parabola at only one



point  $\left(\frac{c^2}{4a}, c\right)$ .

- ii. The line  $y = mx + c$  ( $m \neq 0$ ) cuts, touches or does not cut the parabola  $y^2 = 4ax$  according as  $mc <, =$  or  $> a$ .
- iii. The line  $x = c$  cuts, touches or does not cut the parabola  $y^2 = 4ax$  according as  $c >, =$  or  $< 0$ .
- iv. Points of intersection of a parabola and a line can be obtained by solving their equations simultaneously.

## SECTION-16.4. TANGENT, NORMAL, CHORD AND POLAR OF A PARABOLA

### 1. Tangent to a parabola

- i. Slope of tangent at the point  $(x_1, y_1)$  on the parabola  $y^2 = 4ax$  is  $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{2a}{y_1}$ .
- ii. Equation of tangent at the point  $(x_1, y_1)$  on the parabola  $y^2 = 4ax$  is  $yy_1 = 2a(x + x_1)$ .
- iii. Equation of tangent at  $(x_1, y_1)$  on a second degree curve  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  is  $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$ .
- iv. Slope of tangent at the point  $(at^2, 2at)$  on the parabola  $y^2 = 4ax$  is  $\left(\frac{dy}{dx}\right)_{(at^2, 2at)} = \frac{1}{t}$ .
- v. Equation of tangent at the point  $(at^2, 2at)$  on the parabola  $y^2 = 4ax$  is  $ty = x + at^2$ .
- vi. Straight line  $y = mx + \frac{a}{m}$  is always tangent to the parabola  $y^2 = 4ax$  at  $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ .
- vii. The combined equation of the pair of tangents drawn from an external point  $(x_1, y_1)$  to the parabola  $y^2 = 4ax$  is  $(y^2 - 4ax)(y_1^2 - 4ax_1) = [yy_1 - 2a(x + x_1)]^2$ .
- viii. The tangent at two points  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$  on the parabola  $y^2 = 4ax$  intersect at the point  $(at_1t_2, a(t_1 + t_2))$ .
- ix. Length of tangent at point  $P$  on the conic section =  $PT$ , where  $T$  is the point where the tangent cuts the axis.

### 2. Normal to a parabola

- i. Slope of normal at the point  $(x_1, y_1)$  on the parabola  $y^2 = 4ax$  is  $-\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = -\frac{y_1}{2a}$ .
- ii. Equation of normal at the point  $(x_1, y_1)$  on the parabola  $y^2 = 4ax$  is  $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ .
- iii. Slope of normal at the point  $(at^2, 2at)$  on the parabola  $y^2 = 4ax$  is  $-\left(\frac{dx}{dy}\right)_{(at^2, 2at)} = -t$ .
- iv. Equation of normal at the point  $(at^2, 2at)$  on the parabola  $y^2 = 4ax$  is  $y + tx = 2at + at^3$ .
- v. Straight line  $y = mx - 2am - am^3$  is always normal to the parabola  $y^2 = 4ax$  at  $(am^2, -2am)$ .
- vi. Maximum three normals and minimum one normal can be drawn from a point to a parabola. The three points on the parabola at which the normals pass through a common point are called co-normal points.
- vii. The sum of the slopes of the normals at co-normal points is zero.

viii. The sum of the ordinates of the co-normal points is zero.

ix. Length of normal at point  $P$  on the conic section =  $PG$ , where  $G$  is the point where the normal cuts the axis.

### 3. Chord of a parabola

i. The equation of the chord joining two points  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$  on the parabola  $y^2 = 4ax$  is  $y(t_1 + t_2) = 2x + 2at_1t_2$ .

ii. If the end points of a focal chord are  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$  then  $t_1t_2 = -1$ . Therefore, if one end point of a focal chord is  $(at^2, 2at)$  then the other end point is  $(\frac{a}{t^2}, -\frac{2a}{t})$  and its length is  $a\left(t + \frac{1}{t}\right)^2$ .

Because  $\left(t + \frac{1}{t}\right)^2 \geq 4$ , therefore length of smallest focal chord is  $4a$ , i.e. latus rectum is the smallest focal chord.

### 4. Chord of contact of tangents

Equation of chord of contact of tangents drawn from an external point  $(x_1, y_1)$  to the parabola  $y^2 = 4ax$  is  $yy_1 = 2a(x + x_1)$ .

### 5. Chord with a given mid-point

The equation of the chord with mid-point  $(h, k)$  of the parabola  $y^2 = 4ax$  is  $ky - 2a(x + h) = k^2 - 4ah$ .

### 6. Polar and Pole

i. Equation of the polar of the point  $P \equiv (x_1, y_1)$  w.r.t. the parabola  $y^2 = 4ax$  is the line

$L \equiv yy_1 = 2a(x + x_1)$ . The point  $P$  is called the pole of its polar line  $L$ .

ii. Therefore, chord with a given mid-point  $(h, k)$  is parallel to the polar of the point  $(h, k)$ .

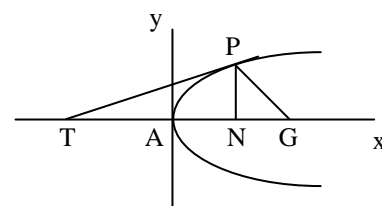
## SECTION-16.5. SUBTANGENT, SUBNORMAL AND DIAMETER OF A PARABOLA

### 1. Subtangent of a point on the conic section

i. If the tangent at any point  $P$  of a conic section meets the axis in  $T$  and  $PN$  be the ordinate at  $P$ , then  $NT$  is called the Subtangent of  $P$ .

ii. Length of subtangent of a point  $P(x_1, y_1)$  on the parabola  $y^2 = 4ax$  is  $2x_1$ .

iii. Vertex is the mid-point of the subtangent.



### 2. Subnormal of a point on the conic section

i. If the normal at any point  $P$  of a conic section meets the axis in  $G$  and  $PN$  be the ordinate at  $P$ , then  $NG$  is called the Subnormal of  $P$ .

ii. Length of subnormal of a point  $P(x_1, y_1)$  on the parabola  $y^2 = 4ax$  is  $2a$ , which is a constant.

### 3. Diameter of a conic section

i. Diameter of a conic section is the locus of mid-point of a chord of given slope. Diameter of a conic section is a straight line.

ii. Diameter of the parabola  $y^2 = 4ax$ , bisecting chords of gradient  $m$ , is the straight line,  $y = \frac{2a}{m}$ . Hence, diameter of a parabola is a straight line parallel to its axis.

## EXERCISE-16

### CATEGORY-16.1. FINDING EQUATION OF A PARABOLA FROM ITS PROPERTIES

- Find the equation to the parabola with
  - focus  $(0, -3)$  and directrix  $y = 3$ . {Ans.  $x^2 = -12y$ }
  - focus  $(-3, 0)$  and directrix  $x = 3$ . {Ans.  $y^2 = -12x$ }
  - focus  $(3, -4)$  and directrix  $6x - 7y + 5 = 0$ . {Ans.  $(7x + 6y)^2 - 570x + 750y + 2100 = 0$ }
  - focus  $(a, b)$  and directrix  $\frac{x}{a} + \frac{y}{b} = 1$ . {Ans.  $(ax - by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0$ }
- If the focus and vertex of a parabola are the points  $(0, 2)$  and  $(0, 4)$  respectively, then find its equation. {Ans.  $x^2 + 8y = 32$ }
- If the vertex of a parabola is the point  $(-3, 0)$  and the directrix is the line  $x + 5 = 0$ , then find its equation. {Ans.  $y^2 = 8(x + 3)$ }
- If  $(2, 0)$  is the vertex and  $y$ -axis the directrix of a parabola, then find its focus and its equation. {Ans.  $(4, 0)$ }
- Find the value of  $p$  when the parabola  $y^2 = 4px$  goes through the point (i)  $(3, -2)$  and (ii)  $(9, -12)$ . {Ans.  $\frac{1}{3}, 4$ }
- Prove that the equation to the parabola, whose vertex and focus are on the axis of  $x$  at distances  $a$  and  $a'$  from the origin respectively, is  $y^2 = 4(a' - a)(x - a)$ .
- The vertex of a parabola is the point  $(a, b)$  and latus rectum is of length  $l$ . If the axis of the parabola is along the positive direction of  $y$ -axis, then find its equation. {Ans.  $(x - a)^2 = \frac{l}{2}(2y - 2b)$ }

### CATEGORY-16.2. FINDING PROPERTIES OF A PARABOLA FROM ITS EQUATION

- Find the vertex, axis, latus rectum and focus of the parabolas
  - $y^2 = 4x + 4y$ . {Ans.  $(-1, 2)$ ,  $y = 2$ ,  $4$ ,  $(0, 2)$ }
  - $x^2 + 2y = 8x - 7$ . {Ans.  $(4, \frac{9}{2})$ ,  $x = 4$ ,  $2$ ,  $(4, 4)$ }
  - $x^2 - 4x + 4y = 0$ . {Ans.  $(2, 1)$ ,  $x = 2$ ,  $4$ ,  $(2, 0)$ }
  - $y^2 = 4y - 4x$ . {Ans.  $(1, 2)$ ,  $y = 2$ ,  $4$ ,  $(0, 2)$ }
  - $x^2 + 8x + 12y + 4 = 0$ . {Ans.  $(-4, 1)$ ,  $x = -4$ ,  $12$ ,  $(-4, -2)$ }
  - $y^2 - 4y - 8x + 4 = 0$ . {Ans.  $(0, 2)$ ,  $y = 2$ ,  $8$ ,  $(2, 2)$ }
  - $y^2 - x - 2y + 2 = 0$ . {Ans.  $(1, 1)$ ,  $y = 1$ ,  $1$ ,  $(\frac{5}{4}, 1)$ }
  - $x^2 - 4x - 8y - 4 = 0$ . {Ans.  $(2, -1)$ ,  $x = 2$ ,  $8$ ,  $(2, 1)$ }
- Find the equation of the directrix of the parabola  $x^2 - 4x - 3y + 10 = 0$ . {Ans.  $y = \frac{5}{4}$ }
- Find the vertex and latus rectum of the parabola  $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$ . {Ans.

$$\left(-\frac{29}{25}, \frac{22}{25}\right), 3\}$$

11. Find the coordinates of a point on the parabola  $y^2 = 8x$  whose focal distance is 4. {Ans.  $(2, \pm 4)$ }
12. For what point of the parabola  $y^2 = 18x$  is the ordinate equal to three times the abscissa? {Ans.  $(2, 6), (0, 0)$ }
13. The two ends of latus rectum of a parabola are the points  $(3, 6)$  and  $(-5, 6)$ , then find its focus. {Ans.  $(-1, 6)$ }
14. If the parabola  $y^2 = 4ax$  passes through  $(3, 2)$ , then find the length of its latusrectum. {Ans.  $\frac{4}{3}$ }
15. If the vertex of the parabola  $y = x^2 - 8x + c$  lies on  $x$ -axis, then find the value of  $c$ . {Ans. 16}
16. Prove that the equation  $y^2 + 2Ax + 2By + C = 0$  represents a parabola, whose axis is parallel to the axis of  $x$  and find its vertex and the equation to its latus rectum. {Ans.  $\left(\frac{B^2 - C}{2A}, -B\right), x = \frac{B^2 - A^2 - C}{2A}$ }
17. A point on a parabola, the foot of the perpendicular from it upon the directrix, and the focus are the vertices of an equilateral triangle. Prove that the focal distance of the point is equal to the latus rectum.

#### CATEGORY-16.3. POSITION OF A POINT/LINE W.R.T. A PARABOLA

18. Determine the position of the following points w.r.t. the parabola  $y^2 = 7x$ :-
  - i.  $(1, 2)$ . {Ans. inside}
  - ii.  $(3, -4)$ . {Ans. inside}
  - iii.  $(9, 8)$ . {Ans. outside}
19. Determine the position of the line  $4x - y + 6 = 0$  w.r.t. the parabola  $y^2 = 4x + 4y$ . {Ans. cuts at  $\left(-\frac{3}{4}, 3\right)$  and  $(-1, 2)$ }
20. Determine the position of the line  $x + y + 5 = 0$  and  $8x + 6y + 7 = 0$  w.r.t. the parabola  $16x^2 + 24xy + 9y^2 - 5x - 10y + 1 = 0$ . {Ans. does not cut, cuts at one point}
21. Find the point of contact of the line  $x - 2y - 1 = 0$  with the parabola  $y^2 = 2(x - 3)$ . {Ans.  $(5, 2)$ }

#### CATEGORY-16.4. TANGENT AND NORMAL OF A PARABOLA

22. Write down the equations to the tangent and normal
  - i. at the point  $(4, 6)$  of the parabola  $y^2 = 9x$ , {Ans.  $4y = 3x + 12, 4x + 3y = 34$ }
  - ii. at the point  $(3, 6)$  of the parabola  $y^2 = 12x$ , {Ans.  $x + y = 9$ }
  - iii. at the point of the parabola  $y^2 = 6x$  whose ordinate is 12, {Ans.  $4y - x = 24, 4x + y = 108$ }
  - iv. at the ends of the latus rectum of the parabola  $y^2 = 12x$  {Ans.  $y - x = 3, y + x = 9, x + y + 3 = 0, x - y = 9$ }
  - v. at the ends of the latus rectum of the parabola  $y^2 = 4a(x - a)$  {Ans.  $y = x, x + y = 4a, y + x = 0, x - y = 4a$ }
23. Find the equation of the normal to the parabola  $y^2 = 8x$  having slope 1. {Ans.  $x - y - 6 = 0$ }
24. Find the equation of the tangent at the vertex of the parabola  $x^2 + 4x + 2y = 0$ . {Ans.  $y = 2$ }

25. Show that the tangents at the points  $(at_1^2, 2at_1)$ ,  $(at_2^2, 2at_2)$  on the parabola  $y^2 = 4ax$  are at right angles if  $t_1 t_2 = -1$ .
26. The tangents to the parabola  $y^2 = 4ax$  at  $(at_1^2, 2at_1)$ ,  $(at_2^2, 2at_2)$  intersect on its axis, then show that  $t_1 = -t_2$ .
27. Find the equation to that tangent to the parabola  $y^2 = 7x$  which is parallel to the straight line  $4y - x + 3 = 0$ . Find also its point of contact. {Ans.  $4y = x + 28$ ,  $(28, 14)$ }
28. A tangent to the parabola  $y^2 = 4ax$  makes an angle of  $60^\circ$  with the axis. Find its point of contact. {Ans.  $(\frac{a}{3}, \frac{2a}{\sqrt{3}})$ }
29. A tangent of the parabola  $y^2 = 8x$  makes an angle of  $45^\circ$  with the straight line  $y = 3x + 5$ . Find its equation and its point of contact. {Ans.  $y + 2x + 1 = 0$ ,  $(\frac{1}{2}, -2)$ ,  $2y = x + 8$ ,  $(8, 8)$ }
30. Find the points of the parabola  $y^2 = 4x$  at which (i) the tangent, and (ii) the normal is inclined at  $30^\circ$  to the axis. {Ans.  $(3, 2\sqrt{3})$ ,  $(\frac{1}{3}, -\frac{2\sqrt{3}}{3})$ }
31. Find the equation to the tangents to the parabola  $y^2 = 9x$  which goes through the point  $(4, 10)$ . {Ans.  $4y = 9x + 4$ ,  $4y = x + 36$ }
32. If the line  $x + y = 1$  touches the parabola  $y^2 - y + x = 0$ , then find the coordinates of the point of contact. {Ans.  $(0, 1)$ }
33. Prove that the straight line  $x + y = 1$  touches the parabola  $y = x - x^2$ .
34. If the line  $x + y - 1 = 0$  touches the parabola  $y^2 = kx$ , then find the value of  $k$ . {Ans.  $-4$ }
35. For what value of  $m$ , the line  $y = mx + 1$  is a tangent to the parabola  $y^2 = 4x$ . {Ans.  $1$ }
36. Prove that the straight line  $y = mx + c$  touches the parabola  $y^2 = 4a(x + a)$  if  $c = ma + \frac{a}{m}$ .
37. Prove that the straight line  $lx + my + n = 0$  touches the parabola  $y^2 = 4ax$  if  $ln = am^2$ .
38. Find the value of  $\lambda$  for which the line  $2x + y + \lambda = 0$  is a normal to the parabola  $y^2 = -8x$ . {Ans.  $24$ }
39. If  $x = my + c$  is a normal to the parabola  $x^2 = 4ay$ , then find the value of  $c$ . {Ans.  $-2am - am^3$ }
40. Find the equation to the common tangents of
- the parabolas  $y^2 = 4ax$  and  $x^2 = 4by$ , {Ans.  $b^{\frac{1}{3}}y + a^{\frac{1}{3}}x + a^{\frac{2}{3}}b^{\frac{2}{3}} = 0$ }
  - the circle  $x^2 + y^2 = 4ax$  and the parabola  $y^2 = 4ax$ . {Ans.  $x = 0$ }
41. Two equal parabolas have the same vertex and their axes are at right angles. Prove that the common tangent touches each at the end of a latus rectum.
42. If perpendiculars be drawn on any tangent to a parabola from two fixed points on the axis, which are equidistant from the focus, prove that the difference of their squares is constant.
43. If  $P$ ,  $Q$  and  $R$  be three points on a parabola whose abscissa are in geometrical progression, prove that tangents at  $P$  and  $R$  meet on the ordinate of  $Q$ .
44. If the tangents at the points  $(x', y')$  and  $(x'', y'')$  meet at the point  $(x_1, y_1)$ , and the normals at the same points meet at the point  $(x_2, y_2)$ , prove that
- $x_1 = \frac{y'y''}{4a}$  and  $y_1 = \frac{y' + y''}{2}$ ,

- ii.  $x_2 = 2a + \frac{y'^2 + y'y'' + y''^2}{4a}$  and  $y_2 = -y'y'' \frac{y' + y''}{8a^2}$ ,
- iii.  $x_2 = 2a + \frac{y_1^2}{a} - x_1$  and  $y_2 = -\frac{x_1 y_1}{a}$ ,
- iv. if tangents be drawn to the parabola  $y^2 = 4ax$  from any point on the parabola  $y^2 = a(x + b)$ , then the normals at the points of contact meet on a fixed straight line.
45. Find the lengths of the normals drawn from the point on the axis of the parabola  $y^2 = 8ax$  whose distance from the focus is  $8a$ . {Ans.  $6a, 10a$ }
46. Prove that the distance between a tangent to the parabola and the parallel normal is  $a \operatorname{cosec} \theta \sec^2 \theta$ , where  $\theta$  is the angle that either makes with the axis.
47. If a normal to a parabola make an angle  $\phi$  with the axis, show that it will cut the curve again at an angle  $\tan^{-1} \left( \frac{1}{2} \tan \phi \right)$ .
48. Prove that the two parabolas  $y^2 = 4ax$  and  $y^2 = 4c(x - b)$  cannot have a common normal, other than the axis, unless  $\frac{b}{a - c} > 2$ .
49. If  $a^2 > 8b^2$ , prove that a point can be found such that the two tangents from it to the parabola  $y^2 = 4ax$  are normals to the parabola  $x^2 = 4by$ .
50. Prove that three tangents to a parabola, which are such that the tangents of their inclinations to the axis are in a given harmonical progression, form a triangle whose area is constant.
51. Prove that the parabolas  $y^2 = 4ax$  and  $x^2 = 4by$  intersect at an angle  $\tan^{-1} \frac{3a^{\frac{1}{3}} b^{\frac{1}{3}}}{2 \left( a^{\frac{2}{3}} + b^{\frac{2}{3}} \right)}$ .
52. Prove that two equal parabolas, having the same focus and their axes in opposite directions, cut at right angles.
53. Show that the two parabolas  $x^2 + 4a(y - 2b - a) = 0$  and  $y^2 = 4b(x - 2a + b)$  intersect at right angles at the common end of the latus rectum of each.
54. A parabola is drawn touching the axis of  $x$  at the origin and having its vertex at a given distance  $k$  from this axis. Prove that the axis of the parabola is a tangent to the parabola  $x^2 = -8k(y - 2k)$ .
55. Two equal parabolas with axes in opposite directions touch at a point  $O$ . From a point  $P$  on one of them are drawn tangents  $PQ$  and  $PQ'$  to the other. Prove that  $QQ'$  will touch the first parabola in  $P'$  where  $PP'$  is parallel to the common tangent at  $O$ .
56. If  $T$  be any point on the tangent at any point  $P$  of a parabola, and if  $TL$  be perpendicular to the focal radius  $SP$  and  $TN$  be perpendicular to the directrix, prove that  $SL = TN$ . Hence obtain a geometrical construction for the pair of tangents drawn to the parabola from any point  $T$ .
57. The normal at the point  $(at_1^2, 2at_1)$  meets the parabola again in the point  $(at_2^2, 2at_2)$ ; prove that  $t_2 = -t_1 - \frac{2}{t_1}$ .
58. Prove that the length of the intercept on the normal at the point  $(at^2, 2at)$  made by the circle, which is described on the focal distance of the given point as diameter, is  $a\sqrt{1 + t^2}$ .

59. Prove that the area of the triangle formed by the normals to the parabola at the points  $(at_1^2, 2at_1)$ ,  $(at_2^2, 2at_2)$  and  $(at_3^2, 2at_3)$  is  $\frac{a^2}{2}(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)(t_1 + t_2 + t_3)^2$ .
60. Prove that the orthocentres of the triangles formed by three tangents and the corresponding three normals to a parabola are equidistant from the axis.
61. Prove that the equation to the circle passing through the points  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$  and the intersection of the tangents to the parabola at these points is  $x^2 + y^2 - ax[(t_1 + t_2)^2 + 2] - ay(t_1 + t_2)(1 - t_1 t_2) + a^2 t_1 t_2 (2 - t_1 t_2) = 0$ .
62. Through the vertex  $A$  of the parabola  $y^2 = 4ax$  two chords  $AP$  and  $AQ$  are drawn, and the circles on  $AP$  and  $AQ$  as diameters intersect in  $R$ . Prove that, if  $\theta_1$ ,  $\theta_2$  and  $\phi$  be the angles made with the axis by the tangents at  $P$  and  $Q$  and by  $AR$ , then  $\cot \theta_1 + \cot \theta_2 + 2 \tan \phi = 0$ .

### CATEGORY-16.5. CHORD OF A PARABOLA

63. In the parabola  $y^2 = 6x$ , find  
 i. the equation to the chord through the vertex and the negative end of the latus rectum, and  
 ii. the equation to any chord through the point on the curve whose abscissa is 24. {Ans.  $y = -2x$ ,  $y \pm 12 = m(x - 24)$ }
64. The vertex  $A$  of a parabola is joined to any point  $P$  on the curve and  $PQ$  is drawn at right angles to  $AP$  to meet the axis in  $Q$ . Prove that the projection of  $PQ$  on the axis is always equal to the latus rectum.
65. A double ordinate of the curve  $y^2 = 4px$  is of length  $8p$ . Prove that the lines from the vertex to its two ends are at right angles.
66. Prove that the length of the chord joining the points of contact of tangents drawn from the point  $(x_1, y_1)$  is  $\frac{\sqrt{y_1^2 + 4a^2} \sqrt{y_1^2 - 4ax_1}}{a}$ .
67. Prove that the area of the triangle formed by the tangents from the point  $(x_1, y_1)$  and the chord of contact is  $\frac{(y_1^2 - 4ax_1)^{\frac{3}{2}}}{2a}$ .
68. What is the equation to the chord of the parabola  $y^2 = 8x$  which is bisected at the point  $(2, -3)$ ? {Ans.  $4x + 3y + 1 = 0$ }
69. Find the mid-point of the chord  $2x + y - 4 = 0$  of the parabola  $y^2 = 4x$ . {Ans.  $\left(\frac{5}{2}, -1\right)$ }
70. If  $\omega$  be the angle which a focal chord of a parabola makes with the axis, prove that the length of the chord is  $4a \operatorname{cosec}^2 \omega$  and that the perpendicular on it from the vertex is  $a \sin \omega$ .
71. Prove that the semi-latus-rectum is a harmonic mean between the segments of any focal chord.
72. Prove that the chord of the parabola  $y^2 = 4ax$ , whose equation is  $y - x\sqrt{2} + 4a\sqrt{2} = 0$ , is a normal to the curve and that its length is  $6\sqrt{3a}$ .
73. A chord is a normal to a parabola and is inclined at an angle  $\theta$  to the axis; prove that the area of the triangle formed by it and the tangents at its extremities is  $4a^2 \sec^3 \theta \operatorname{cosec}^3 \theta$ .
74. Prove that the normal chord at the point whose ordinate is equal to its abscissa subtends a right angle at the focus.

75. A chord of a parabola passes through a point on the axis (outside the parabola) whose distance from the vertex is half the latus rectum; prove that the normals at its extremities meet on the curve.
76. Prove that on the axis of any parabola there is a certain point  $K$  which has the property that, if a chord  $PQ$  of the parabola be drawn through it, then  $\frac{1}{PK^2} + \frac{1}{QK^2}$  is the same for all positions of the chord.
77. If  $r_1$  and  $r_2$  be the lengths of radii vectors of the parabola which are drawn at right angles to one another from the vertex, prove that  $r_1^{\frac{4}{3}} r_2^{\frac{4}{3}} = 16a^2 (r_1^{\frac{2}{3}} + r_2^{\frac{2}{3}})$ .
78. Prove that all circles described on focal chords as diameters touch the directrix of the curve, and that all circles on focal radii as diameters touch the tangent at the vertex.
79. A circle is described on a focal chord as diameter; if  $m$  be the tangent of the inclination of the chord to the axis, prove that the equation to the circle is  $x^2 + y^2 - 2ax \left( 1 + \frac{2}{m^2} \right) - \frac{4ay}{m} - 3a^2 = 0$ .
80.  $LOL'$  and  $MOM'$  are two chords of a parabola passing through a point  $O$  on its axis. Prove that the radical axis of the circles described on  $LL'$  and  $MM'$  as diameters passes through the vertex of the parabola.
81. A circle and a parabola intersect in four points; show that the algebraic sum of the ordinates of the four points is zero. Show also that the line joining one pair of these four points and the line joining the other pair are equally inclined to the axis.
82. Through each point of the straight line  $x = my + h$  is drawn the chord of the parabola  $y^2 = 4ax$  which is bisected at the point; prove that it always touches the parabola  $(y + 2am)^2 = 8a(x - h)$ .

#### CATEGORY-16.6. POLE AND POLAR OF A PARABOLA

83. Given the parabola  $y^2 = 10x$ , find polar of the point  $(1, 2)$  and pole of the line  $x + y + 1 = 0$ . {Ans.  $5x - 2y + 5 = 0, (1, -5)$ }
84. In a parabola, show that the polar of the focus is the directrix.
85. If a perpendicular be let fall from any point  $P$  upon its polar prove that the distance of the foot of this perpendicular from the focus is equal to the distance of the point  $P$  from the directrix.
86.  $T$  is the pole of the chord  $PQ$ ; prove that the perpendiculars from  $P, T$  and  $Q$  upon any tangent to the parabola are in geometrical progression.

#### CATEGORY-16.7. SUBTANGENT, SUBNORMAL AND DIAMETER OF A PARABOLA

87. Find the length of the subtangent to the parabola  $y^2 = 16x$  at the point whose abscissa is 4. {Ans. 8}
88. For what point of the parabola  $y^2 = 4ax$  is
- the normal equal to twice the subtangent?
  - the normal equal to the difference between the subtangent and the subnormal?
89.  $PN$  is an ordinate of the parabola. A straight line is drawn parallel to the axis to bisect  $NP$  and meet the curve in  $Q$ . Prove that  $NQ$  meets the tangent at the vertex in a point  $T$  such that  $AT = \frac{2}{3} NP$ .
90. The general equation to a system of parallel chords in the parabola  $y^2 = \frac{25}{7}x$  is  $4x - y + k = 0$ . What is the equation to the corresponding diameter? {Ans.  $56y = 25$ }
91.  $P, Q$  and  $R$  are three points on a parabola and the chord  $PQ$  cuts the diameter through  $R$  in  $V$ . Ordinates  $PM$  and  $QN$  are drawn to this diameter. Prove that  $RM \cdot RN = RV^2$ .
92. The normal at a point  $P$  of a parabola meets the curve again in  $Q$ , and  $T$  is the pole of  $PQ$ ; show that  $T$  lies



on the diameter passing through the other end of the focal chord passing through  $P$ , and that  $PT$  is bisected by the directrix.

93. A parabola touches the sides of a triangle  $ABC$  in the points  $D, E$  and  $F$  respectively; if  $DE$  and  $DF$  cut the diameter through the point  $A$  in  $b$  and  $c$  respectively, prove that  $Bb$  and  $Cc$  are parallel.

### CATEGORY-16.8. LOCUS PROBLEMS IN PARABOLA

94. Prove that the locus of the middle points of all chords of the parabola  $y^2 = 4ax$ , which are drawn through the vertex, is the parabola  $y^2 = 2ax$ .
95. Prove that the locus of the centre of a circle, which intercepts a chord of given length  $2a$  on the axis of  $x$  and passes through a given point on the axis of  $y$  distant  $b$  from the origin, is the curve  $x^2 - 2yb + b^2 = a^2$ . Trace this parabola.
96.  $PQ$  is a double ordinate of a parabola. Find the locus of its points of trisection. {Ans.  $9y^2 = 4ax$ }
97. Prove that the locus of a point, which moves so that its distance from a fixed line is equal to the length of the tangent drawn from it to a given circle, is a parabola. Find the position of the focus and directrix.
98. If a circle be drawn so as always to touch a given straight line and also a given circle, prove that the locus of its centre is a parabola.
99. If on a given base triangles be described such that the sum of the tangents of the base angles is constant, prove that the locus of the vertices is a parabola.
100. Two parabolas have a common axis and concavities in opposite directions. If any line parallel to the common axis meet the parabolas in  $P$  and  $P'$ , prove that the locus of the middle point of  $PP'$  is another parabola, provided that the latus recta of the given parabolas are unequal.
101. A parabola is drawn to pass through  $A$  and  $B$ , the ends of a diameter of a given circle of radius  $a$ , and to have as directrix a tangent to a concentric circle of radius  $b$ ; the axes being  $AB$  and a perpendicular diameter, prove that the locus of the focus of the parabola is  $\frac{x^2}{b^2} + \frac{y^2}{b^2 - a^2} = 1$ .
102. Prove that two straight lines, one a tangent to the parabola  $y^2 = 4a(x + a)$  and the other to the parabola  $y^2 = 4a'(x + a')$ , which are at right angles to one another, meet on the straight line  $x + a + a' = 0$ . Show also that this straight line is the common chord of the two parabolas.
103. Tangents are drawn to a parabola at points whose abscissa are in the ratio  $\mu : 1$ . Prove that they intersect on the curve  $y^2 = \left(\mu^{\frac{1}{4}} + \mu^{-\frac{1}{4}}\right)^2 ax$ .
104. Prove that the locus of the middle point of the portion of a normal intersected between the curve and the axis is a parabola whose vertex is the focus and whose latus rectum is one quarter of that of the original parabola.
105.  $PNP'$  is a double ordinate of the parabola. Prove that the locus of the point of intersection of the normal at  $P$  and the straight line through  $P'$  parallel to the axis is the equal parabola  $y^2 = 4a(x - 4a)$ .
106. The normal at any point  $P$  meets the axis in  $G$  and the tangent at the vertex in  $G'$ . If  $A$  be the vertex and the rectangle  $AGQG'$  be completed, prove that the equation to the locus of  $Q$  is  $x^3 = 2ax^2 + ay^2$ .
107. Two equal parabolas have the same focus and their axes are at right angles. A normal to one is perpendicular to a normal to the other. Prove that the locus of the point of intersection of these normals is another parabola.
108. If  $PQ$  be a normal chord of the parabola and if  $S$  be the focus, prove that the locus of the centroid of the triangle  $SPQ$  is the curve  $36ay^2(3x - 5a) - 81y^4 = 128a^4$ .

109. If from the vertex of a parabola a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, prove that the locus of the further angle of the rectangle is the parabola  $y^2 = 4a(x - 8a)$ .
110. A series of chords is drawn so that their projections on a straight line, which is inclined at an angle  $\alpha$  to the axis, are all of constant length  $c$ ; prove that the locus of their middle point is the curve  $(y^2 - 4ax)(y \cos \alpha + 2a \sin \alpha)^2 + a^2 c^2 = 0$ .
111. Prove that the locus of the poles of chords, which subtend a right angle at a fixed point  $(h, k)$ , is  $ax^2 - hy^2 + (4a^2 + 2ah)x - 2aky + a(h^2 + k^2) = 0$ .
112. Prove that the locus of the middle points of all tangents drawn from points on the directrix to the parabola is  $y^2(2x + a) = a(3x + a)^2$ .
113. Circles are drawn through the vertex of the parabola to cut the parabola orthogonally at the other point of intersection. Prove that the locus of the centres of the circles is the curve  $2y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2$ .
114. From the external point  $P$  tangents are drawn to the parabola; find the equation to the locus of  $P$  when these tangents make angles  $\theta_1$  and  $\theta_2$  with the axis, such that
- $\tan \theta_1 + \tan \theta_2$  is constant  $(= b)$ . {Ans.  $y = bx$ }
  - $\tan \theta_1 \tan \theta_2$  is constant  $(= c)$ . {Ans.  $cx = a$ }
  - $\cot \theta_1 + \cot \theta_2$  is constant  $(= d)$ . {Ans.  $y = ad$ }
  - $\theta_1 + \theta_2$  is constant  $(= 2\alpha)$ . {Ans.  $y = (x - a) \tan 2\alpha$ }
  - $\tan^2 \theta_1 + \tan^2 \theta_2$  is constant  $(= \lambda)$ . {Ans.  $y^2 - \lambda x^2 = 2ax$ }
  - $\cos \theta_1 \cos \theta_2$  is constant  $(= \mu)$ . {Ans.  $x^2 = \mu^2[(x - a)^2 + y^2]$ }
115. Two tangents to a parabola meet at an angle of  $45^\circ$ ; prove that the locus of their point of intersection is the curve  $y^2 - 4ax = (x + a)^2$ . If they meet at an angle of  $60^\circ$ , prove that the locus is  $y^2 - 3x^2 - 10ax - 3a^2 = 0$ .
116. A pair of tangents are drawn which are equally inclined to a straight line whose inclination to the axis is  $\alpha$ ; prove that the locus of their point of intersection is the straight line  $y = (x - a) \tan 2\alpha$ .
117. Prove that the locus of the point of intersection of two tangents which intercept a given distance  $4c$  on the tangent at the vertex is an equal parabola.
118. Show that the locus of the point of intersection of two tangents, which with the tangent at the vertex form a triangle of constant area  $c^2$ , is the curve  $x^2(y^2 - 4ax) = 4c^2$ .
119. If the normals at  $P$  and  $Q$  meet on the parabola, prove that the point of intersection of the tangents at  $P$  and  $Q$  lies either on a certain straight line, which is parallel to the tangent at the vertex, or on the curve whose equation is  $y^2(x + 2a) + 4a^3 = 0$ .
120. Two tangents to a parabola intercept on a fixed tangent segments whose product is constant; prove that the locus of their point of intersection is a straight line.
121. Show that the locus of the poles of chords, which subtend a constant angle  $\alpha$  at the vertex, is the curve  $(x + 4a)^2 = 4 \cot^2 \alpha (y^2 - 4ax)$ . Also show that if the constant angle be a right angle the locus is a straight line perpendicular to the axis.
122. A point  $P$  is such that the straight line drawn through it perpendicular to its polar with respect to the parabola  $y^2 = 4ax$  touches the parabola  $x^2 = 4by$ . Prove that its locus is the straight line  $2ax + by + 4a^2 = 0$ .

123. Two equal parabolas,  $A$  and  $B$ , have the same vertex and axis but have their concavities turned in opposite directions; prove that the locus of poles with respect to  $B$  of tangents to  $A$  is the parabola  $A$ .
124. Prove that the locus of the poles of tangents to the parabola  $y^2 = 4ax$  with respect to the circle  $x^2 + y^2 = 2ax$  is the circle  $x^2 + y^2 = ax$ .
125. Show the locus of the poles of tangents to the parabola  $y^2 = 4ax$  with respect to the parabola  $y^2 = 4bx$  is the parabola  $y^2 = \frac{4b^2}{a}x$ .
126. Find the locus of the middle points of chords of the parabola which
- pass through the focus. { Ans.  $y^2 = 2a(x - a)$  }
  - pass through the fixed point  $(h, k)$ . { Ans.  $y^2 - ky = 2a(x - h)$  }
  - are normal to the curve. { Ans.  $y^2(y^2 - 2ax + 4a^2) + 8a^4 = 0$  }
  - subtend a constant angle  $\alpha$  at the vertex. { Ans.  $(8a^2 + y^2 - 2ax)^2 \tan^2 \alpha = 16a^2(4ax - y^2)$  }
  - are of given length  $l$ . { Ans.  $y^4 + 4ay^2(a - x) - 16a^3x + a^2l^2 = 0$  }
  - are such that the normals at their extremities meet on the parabola. { Ans.  $y^2 = 2a(x + 2a)$  }
127. Two parabolas have the same axis and tangents are drawn to the second from points on the first; prove that the locus of the middle points of the chords of contact with the second parabola all lies on a fixed parabola.

### CATEGORY-16.9. ADDITIONAL QUESTIONS

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# Mathematics for IIT-JEE

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*PART-IV*  
*TWO DIMENSIONAL COORDINATE GEOMETRY*

*CHAPTER-17*  
*ELLIPSE*

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## ***CHAPTER-17***

### ***ELLIPSE***

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#### **LIST OF THEORY SECTIONS**

- 17.1. Equation Of An Ellipse
- 17.2. Position Of A Point/Line W.R.T. An Ellipse
- 17.3. Tangent, Normal, Chord And Polar Of An Ellipse
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- 17.1. Finding Equation Of An Ellipse From Its Properties
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## CHAPTER-17

### ELLIPSE

#### SECTION-17.1. EQUATION OF AN ELLIPSE

##### 1. Equation of an ellipse with given Focus, Directrix and Eccentricity

Equation of the ellipse with Focus at  $(h, k)$ , Directrix  $lx + my + n = 0$  and Eccentricity  $e$  is

$$\sqrt{(x-h)^2 + (y-k)^2} = e \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$

$$\equiv (x-h)^2 + (y-k)^2 = e^2 \frac{(lx + my + n)^2}{l^2 + m^2}.$$

##### 2. Standard equation of an ellipse

- i. An ellipse has two vertices, denoted by  $A$  and  $A'$ . The mid-point of the two vertices is called the center of the ellipse, denoted by  $C$ . The line segment  $AA'$  is called the major axis of the ellipse.
- ii. Equation of an standard ellipse having center at origin and axis as  $x$ -axis is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , where  $2a$  is the distance between the vertices and  $b^2 = a^2(1 - e^2) \Rightarrow b = a\sqrt{1 - e^2} \Rightarrow b < a$ .
- iii. The line segment  $BB'$ , which is bisected at  $C$ , is perpendicular to  $AA'$  and having length  $2b$ , is called the minor axis of the ellipse.
- iv. An ellipse has two pairs of Focus and Directrix, denoted by  $S, S'$  and  $D, D'$  respectively.
- v. An ellipse has two Latus rectums, denoted by  $L_1L'_1$  and  $L_2L'_2$ .

##### 3. Properties of standard ellipses

		$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
1.	Coordinates of Foci	$(\pm ae, 0)$	$(0, \pm ae)$
2.	Equation of Directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{a}{e}$
3.	Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \frac{b^2}{a^2}}$
4.	Coordinates of the vertices	$(\pm a, 0)$	$(0, \pm a)$
5.	Coordinates of the center	$(0, 0)$	$(0, 0)$
6.	Equation of major axis	$y = 0$	$x = 0$
7.	Equation of minor axis	$x = 0$	$y = 0$
8.	Length of major axis	$2a$	$2a$
9.	Length of minor axis	$2b$	$2b$
10.	Coordinates of the end points of the minor axis	$(0, \pm b)$	$(\pm b, 0)$

11.	Coordinates of Foot of Directrices	$\left(\pm \frac{a}{e}, 0\right)$	$\left(0, \pm \frac{a}{e}\right)$
12.	Coordinates of the end points of the Latus Rectums	$\left(ae, \pm \frac{b^2}{a}\right) \& \left(-ae, \pm \frac{b^2}{a}\right)$	$\left(\pm \frac{b^2}{a}, ae\right) \& \left(\pm \frac{b^2}{a}, -ae\right)$
13.	Lengths of Latus Rectums	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
14.	Equations of Latus Rectums	$x = \pm ae$	$y = \pm ae$

#### 4. Tracing a general ellipse

i. A general ellipse is traced by parallel and angular transformation of axes.

ii. Equation of an ellipse having center at  $(h, k)$  and major axis parallel to  $x$ -axis is  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ .

Equation of an ellipse having center at  $(h, k)$  and major axis parallel to  $y$ -axis is  $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$ .

#### 5. Auxiliary circle of the ellipse

i. The circle described on the major axis of an ellipse as diameter, is called the Auxiliary circle of the ellipse.

ii. Equation of Auxiliary circle of standard ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $x^2 + y^2 = a^2$ .

#### 6. Eccentric angle of a point on the ellipse

If  $P$  is any point on the ellipse and the ordinate  $NP$  is produced to meet the Auxiliary circle at  $Q$ , then angle  $\phi = \angle ACQ$  ( $0 \leq \phi < 2\pi$ ) is called the Eccentric angle of the point  $P$ .

#### 7. Parametric equation of standard ellipse

Point  $(a \cos \phi, b \sin \phi)$  always lies on the standard ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , where  $\phi$  is the eccentric angle of the point. This point is called point ' $\phi$ '.

#### 8. Alternate definition of an ellipse

i. The sum of the focal distances of any point on an ellipse is a constant and equal to the length of the major axis ( $2a$ ).

ii. Therefore, an ellipse is the locus of a point such that the sum of its distances from two fixed points (Foci) is always a constant ( $2a$ ).

#### 9. Area of an ellipse

i. Area of ellipse =  $\pi ab$ .

### SECTION-17.2. POSITION OF A POINT/LINE W.R.T. AN ELLIPSE

#### 1. Position of a point w.r.t. an ellipse

The point  $(x_1, y_1)$  lies inside, on or outside the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  according as  $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$  is  $<$ ,  $=$  or  $>$  0.

#### 2. Position of a line w.r.t. an ellipse



- i. The line  $y = mx + c$  cuts, touches or does not cut the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  according as  $c^2 <, =$  or  $> a^2 m^2 + b^2$ .
- ii. The line  $x = c$  cuts, touches or does not cut the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  according as  $c^2 <, =$  or  $> a^2$ .
- iii. Points of intersection of an ellipse and a line can be obtained by solving their equations simultaneously.

### SECTION-17.3. TANGENT, NORMAL, CHORD AND POLAR OF AN ELLIPSE

#### 1. Tangent to an ellipse

- i. Slope of tangent at the point  $(x_1, y_1)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{b^2 x_1}{a^2 y_1}$ .
- ii. Equation of tangent at the point  $(x_1, y_1)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ .
- iii. Equation of tangent at  $(x_1, y_1)$  on a second degree curve  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  is  $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$ .
- iv. Slope of tangent at the point  $(a \cos \phi, b \sin \phi)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $\left(\frac{dy}{dx}\right)_{(a \cos \phi, b \sin \phi)} = -\frac{b}{a} \cot \phi$ .
- v. Equation of tangent at the point  $(a \cos \phi, b \sin \phi)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1$ .
- vi. Straight line  $y = mx \pm \sqrt{a^2 m^2 + b^2}$  is always tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at  $\left(\mp \frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 m^2 + b^2}}\right)$ .
- vii. The combined equation of the pair of tangents drawn from an external point  $(x_1, y_1)$  to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$ .
- viii. **Definition of Director circle:** The locus of the point of intersection of two perpendicular tangents to an ellipse is a circle and is called Director circle. Equation of Director circle of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $x^2 + y^2 = a^2 + b^2$ .

#### 2. Normal to an ellipse

- i. Slope of normal at the point  $(x_1, y_1)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $-\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = \frac{a^2 y_1}{b^2 x_1}$ .
- ii. Equation of normal at the point  $(x_1, y_1)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is

$$\frac{x-x_1}{\left(\frac{x_1}{a^2}\right)} = \frac{y-y_1}{\left(\frac{y_1}{b^2}\right)} \equiv \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2.$$

iii. Slope of normal at the point  $(a \cos \phi, b \sin \phi)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $-\left(\frac{dx}{dy}\right)_{(a \cos \phi, b \sin \phi)} = \frac{a}{b} \tan \phi$ .

iv. Equation of normal at the point  $(a \cos \phi, b \sin \phi)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2.$$

v. Straight line  $y = mx - \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2 m^2}}$  is always normal to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

vi. Maximum four normals can be drawn from a point to an ellipse. The four points on the ellipse at which the normals pass through a common point are called co-normal points.

vii. The sum of the eccentric angles of the co-normal points on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is equal to odd multiple of  $\pi$ .

### 3. Chord of an ellipse

The equation of the chord joining two points  $(a \cos \theta, b \sin \theta)$  and  $(a \cos \phi, b \sin \phi)$  on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{x}{a} \cos\left(\frac{\theta + \phi}{2}\right) + \frac{y}{b} \sin\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta - \phi}{2}\right).$$

### 4. Chord of contact of tangents

Equation of chord of contact of tangents drawn from an external point  $(x_1, y_1)$  to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

### 5. Chord with a given mid-point

The equation of the chord with mid-point  $(h, k)$  of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$ .

### 6. Polar and Pole

i. Equation of the polar of the point  $P \equiv (x_1, y_1)$  w.r.t. the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is the line

$$L \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \text{ The point } P \text{ is called the pole of its polar line } L.$$

ii. Therefore, chord with a given mid-point  $(h, k)$  is parallel to the polar of the point  $(h, k)$ .

## SECTION-17.4. SUBTANGENT, SUBNORMAL AND DIAMETER OF AN ELLIPSE

### 1. Subtangent of a point on the ellipse

i. Length of subtangent of a point  $P(x_1, y_1)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $\frac{a^2}{x_1} - x_1$ .

### 2. Subnormal of a point on the ellipse

- i. Length of subnormal of a point  $P(x_1, y_1)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $(1 - e^2)x_1$ .

### 3. Diameter of an ellipse

- i. Diameter of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , bisecting chords of gradient  $m$ , is the straight line  $y = -\frac{b^2}{a^2 m} x$ .  
Hence, diameter of an ellipse is a straight line passing through its center.
- ii. Two diameters of an ellipse are said to be conjugate diameters if each bisects chords parallel to the other.
- iii. The two diameters  $y = m_1 x$  &  $y = m_2 x$  are conjugate if  $m_1 m_2 = -\frac{b^2}{a^2}$ .
- iv. Major axis and minor axis are conjugate diameters and are the only perpendicular conjugate diameters.
- v. The eccentric angles of the end points of conjugate diameters of an ellipse differ by a right angle.

**EXERCISE-17****CATEGORY-17.1. FINDING EQUATION OF AN ELLIPSE FROM ITS PROPERTIES**

- Find the equation to the ellipse, whose focus is the point  $(-1, 1)$ , whose directrix is the straight line  $x - y + 3 = 0$  and whose eccentricity is  $\frac{1}{2}$ . {Ans.  $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$ }
- Find the equation to the ellipses, whose centres are the origin, whose axes are the axes of coordinates, and which pass through
  - the points  $(2, 2)$  and  $(3, 1)$ . {Ans.  $3x^2 + 5y^2 = 32$ }
  - the points  $(1, 4)$  and  $(-6, 1)$ . {Ans.  $3x^2 + 7y^2 = 115$ }
- Find the equation of the ellipse referred to its centre
  - whose latus rectum is 5 and whose eccentricity is  $\frac{2}{3}$ , {Ans.  $20x^2 + 36y^2 = 405$ }
  - whose minor axis is equal to the distance between the foci and whose latus rectum is 10, {Ans.  $x^2 + 2y^2 = 100$ }
  - whose foci are the points  $(4, 0)$  and  $(-4, 0)$  and whose eccentricity is  $\frac{1}{3}$ . {Ans.  $8x^2 + 9y^2 = 1152$ }
  - which passes through the point  $(-3, 1)$  and has eccentricity  $\sqrt{\frac{2}{5}}$ . {Ans.  $3x^2 + 5y^2 = 32$ }
  - whose one focus is at  $(4, 0)$  and whose eccentricity is  $\frac{4}{5}$ . {Ans.  $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ }
  - which passes through  $(2, 1)$  having  $e = \frac{1}{2}$ . {Ans.  $3x^2 + 4y^2 = 16$ }

**CATEGORY-17.2. FINDING PROPERTIES OF AN ELLIPSE FROM ITS EQUATION**

- Find the latus rectum, the eccentricity, and the coordinates of the foci, of the following ellipses:-
  - $x^2 + 3y^2 = 9$ . {Ans.  $2, \frac{\sqrt{6}}{3}, (\pm\sqrt{6}, 0)$ }
  - $5x^2 + 4y^2 = 1$  {Ans.  $\frac{4}{5}, \frac{1}{\sqrt{5}}, (0, \pm\frac{1}{2\sqrt{5}})$ }
  - $3x^2 + y^2 = 12$ . {Ans.  $\frac{4}{\sqrt{3}}$ }
  - $5x^2 + 9y^2 = 45$ . {Ans.  $\frac{10}{3}$ }
- Find the center, length of the axes, eccentricity, the coordinates of the foci, equation of directrices and latus rectum of the following ellipses:-
  - $9x^2 + 5y^2 - 30y = 0$ .
  - $2x^2 + y^2 - 4x - 4y + 2 = 0$ .
  - $4x^2 + 9y^2 + 16x - 18y - 11 = 0$ .
- Find the eccentricity of the ellipse  $x^2 - 4x + 4y^2 = 12$ . {Ans.  $\frac{\sqrt{3}}{2}$ }
- Whether the equation  $y^2 - x^2 + 2x - 1 = 0$  represents an ellipse or not? {Ans. No}
- Find the center, eccentricity and latus rectum of the ellipse  $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ . {Ans.

$$(2,3), \frac{1}{\sqrt{3}}, \frac{8}{\sqrt{6}} \}$$

9. Find the eccentricity of an ellipse, if its latus rectum be equal to one half its minor axis. {Ans.  $\frac{\sqrt{3}}{2}$ }
10. Find the eccentricity of an ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  whose latus-rectum is half of its major axis. {Ans.  $\frac{1}{\sqrt{2}}$ }
11. If the length of the latus rectum of an ellipse is one third of its major axis, then find its eccentricity. {Ans.  $\sqrt{\frac{2}{3}}$ }
12. If the length of the major axis of an ellipse is three times the length of its minor axis, then find its eccentricity. {Ans.  $\frac{2\sqrt{2}}{3}$ }
13. In an ellipse the distance between its foci is 6 and its minor axis is 8. Find its eccentricity. {Ans.  $\frac{3}{5}$ }
14.  $S$  and  $T$  are the foci of an ellipse and  $B$  is an end of the minor axis. If  $STB$  is an equilateral triangle, find the eccentricity of the ellipse. {Ans.  $\frac{1}{2}$ }
15. Find the radius of the circle passing through the foci of the ellipse  $\frac{x^2}{16} + \frac{y^2}{9} = 1$  and having its centre  $(0,3)$ . {Ans. 4}
16. Find the lengths of, and the equations to, the focal radii drawn to the point  $(4\sqrt{3}, 5)$  of the ellipse  $25x^2 + 16y^2 = 1600$ . {Ans.  $x + 4\sqrt{3}y = 24\sqrt{3}$ ,  $11x - 4\sqrt{3}y = 24\sqrt{3}$ , 7 & 13 }

### CATEGORY-17.3. AUXILIARY CIRCLE AND ECCENTRIC ANGLE IN AN ELLIPSE

17. Find the equation of the auxiliary circle of the following ellipses:-
  - i.  $x^2 + 3y^2 = 9$ .
  - ii.  $2x^2 + y^2 + 4x - 4y + 2 = 0$ .
18. Find the eccentric angle of the point  $(\sqrt{2}, -1)$  on the ellipse  $x^2 + 2y^2 = 4$ . {Ans.  $\frac{7\pi}{4}$ }
19.  $C$  is the center of an ellipse and  $Q$  is the point on the auxiliary circle corresponding to  $P$  on the ellipse;  $PLM$  is drawn parallel to  $CQ$  to meet the axes in  $L$  and  $M$ ; prove that  $PL = b$  and  $PM = a$ .
20. If  $\alpha, \beta, \gamma$  and  $\delta$  be the eccentric angles of the four points of intersection of the ellipse and any circle, prove that  $\alpha + \beta + \gamma + \delta$  is an even multiple of  $\pi$  radians.
21. Prove that the circle on any focal distance as diameter touches the auxiliary circle of the ellipse.
22. Prove that the directrices of the two parabolas that can be drawn to have their foci at any given point  $P$  of the ellipse and to pass through its foci meet at an angle which is equal to twice the eccentric angle of  $P$ .
23. Prove that the area of the triangle formed by three points on an ellipse, whose eccentric angles are  $\theta, \phi$  and  $\psi$ , is  $2ab \sin \frac{\phi - \psi}{2} \sin \frac{\psi - \theta}{2} \sin \frac{\theta - \phi}{2}$ . Prove also that its area is to the area of the triangle formed by the corresponding points on the auxiliary circle as  $b : a$ , and hence that its area is a maximum when the latter triangle is equilateral, i.e. when  $\phi - \theta = \psi - \phi = \frac{2\pi}{3}$ . Prove also that the maximum area of the triangle

inscribed in the ellipse is  $\frac{3\sqrt{3}}{4}ab$ . Find the maximum area of the triangle inscribed in the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1. \text{ {Ans. } } 9\sqrt{3} \text{ sq. units}$$

#### CATEGORY-17.4. POSITION OF A POINT/LINE W.R.T. AN ELLIPSE

24. Is the point  $(4, -3)$  inside or outside the ellipse  $5x^2 + 7y^2 = 11$ ? {Ans. outside}  
 25. Determine the position of the line  $x + y + 1 = 0$  w.r.t. the ellipse  $5x^2 + 4y^2 = 1$ . {Ans. does not cut}

#### CATEGORY-17.5. TANGENT AND NORMAL OF AN ELLIPSE

26. Find the equation to the tangent and normal  
 i. at the point  $(1, \frac{4}{3})$  of the ellipse  $4x^2 + 9y^2 = 20$ , {Ans.  $x + 3y = 5$ ,  $9x - 3y - 5 = 0$ }  
 ii. at the point of the ellipse  $5x^2 + 3y^2 = 137$  whose ordinate is 2, {Ans.  $25x + 6y = 137$ ,  $6x - 25y + 20 = 0$ }  
 iii. at the ends of the latus recta of the ellipse  $9x^2 + 16y^2 = 144$ . {Ans.  $\pm\sqrt{7}x \pm 4y = 16$ ,  $\pm 4x \mp \sqrt{7}y = \frac{7\sqrt{7}}{4}$ }  
 27. Prove that the straight line  $y = x + \sqrt{\frac{7}{12}}$  touches the ellipse  $3x^2 + 4y^2 = 1$ .  
 28. If  $y = mx + c$  is a tangent to the ellipse  $x^2 + 2y^2 = 6$ , then show that  $c^2 = 6m^2 + 3$ .  
 29. Find the equations to the tangents to the ellipse  $4x^2 + 3y^2 = 5$  which are parallel to the straight line  $y = 3x + 7$ . Find also the equation of the tangents which are inclined at  $60^\circ$  to the axis of  $x$  and the coordinates of its points of contact. {Ans.  $y = 3x \pm \frac{1}{2}\sqrt{\frac{155}{3}}$ ,  $y = \sqrt{3}x \pm \sqrt{\frac{65}{12}}$ ,  $(\pm \frac{3\sqrt{65}}{26}, \mp \frac{2\sqrt{195}}{39})$ }  
 30. Find the equations to the tangents at the ends of the latus rectum of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , and show that they pass through the intersections of the axis and the directrices.  
 31. If  $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$  touches the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at  $P$ , then find the eccentric angle of  $P$ . {Ans.  $45^\circ$ }  
 32. Find the points on the ellipse such that the tangent at each of them makes equal angles with the axes. Prove also that the length of the perpendicular from the center on either of these tangents is  $\sqrt{\frac{a^2 + b^2}{2}}$ . {Ans.,  $(\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}}), (\mp \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}})$ }  
 33. In an ellipse, prove that the sum of the squares of the perpendiculars on any tangent from two points on the minor axis, each distant  $\sqrt{a^2 - b^2}$  from the centre, is  $2a^2$ .  
 34. In an ellipse, find the equations to the normals at the ends of the latus rectum, and prove that each passes through an end of the minor axis if  $e^4 + e^2 = 1$ .  
 35. In an ellipse, if any ordinate  $MP$  meet the tangent at  $L$  in  $Q$ , prove that  $MQ$  and  $SP$  are equal.  
 36. Two tangents to the ellipse intersect at right angles; prove that the sum of the squares of the chords, which the auxiliary circle intercepts on them, is constant, and equal to the square on the line joining the foci.  
 37. If  $P$  be a point on the ellipse, whose ordinate is  $y'$ , prove that the angle between the tangent at  $P$  and the

focal distance of  $P$  is  $\tan^{-1} \frac{b^2}{aey'}$ .

38. Show that the angle between the tangents to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  and the circle  $x^2 + y^2 = ab$  at their points of intersection is  $\tan^{-1} \frac{a-b}{\sqrt{ab}}$ .
39. A circle of radius  $r$ , is concentric with the ellipse. Prove that the common tangent is inclined to the major axis at an angle  $\tan^{-1} \sqrt{\frac{r^2 - b^2}{a^2 - r^2}}$  and find its length.
40. Prove that the common tangent of the ellipses  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2x}{c}$  and  $\frac{x^2}{b^2} + \frac{y^2}{a^2} + \frac{2x}{c} = 0$  subtends a right angle at the origin.
41. The tangent at  $P$  meets the axes in  $T$  and  $t$ , and  $CY$  is the perpendicular on it from the centre of the ellipse. Prove that
- $Tt \cdot PY = a^2 - b^2$
  - the least value of  $Tt$  is  $a + b$ .
42. Prove that the perpendicular from the focus of the ellipse upon any tangent and the line joining the centre to the point of contact meet on the corresponding directrix.
43. Prove that the straight lines, joining each focus of the ellipse to the foot of the perpendicular from the other focus upon the tangent at any point  $P$ , meet on the normal  $PG$  and bisect it.
44. In an ellipse, find the tangent of the angle between  $CP$  and the normal at  $P$ , and prove that its greatest value is  $\frac{a^2 - b^2}{2ab}$ .
45. Prove that the straight line  $lx + my = n$  is a normal to the ellipse, if  $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$ .
46. If a number of ellipses be described, having the same major axis, but a variable minor axis, prove that the tangents at the ends of their latus rectum pass through one or other of two fixed points.
47.  $PM$  and  $PN$  are perpendiculars upon the axes from any point  $P$  on the ellipse. Prove that  $MN$  is always normal to a fixed concentric ellipse.
48. A tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  meets the ellipse  $\frac{x^2}{a} + \frac{y^2}{b} = a + b$  in the points  $P$  and  $Q$ . Prove that the tangents at  $P$  and  $Q$  are at right angles.
49. Write down the equation of the pair of tangents drawn to the ellipse  $3x^2 + 2y^2 = 5$  from the point  $(1, 2)$  and prove that the angle between them is  $\tan^{-1} \frac{12\sqrt{5}}{5}$ . {Ans.  $9x^2 - 24xy - 4y^2 + 30x + 40y - 55 = 0$ }
50. If tangents  $TP$  and  $TQ$  be drawn to the ellipse from a point  $T$ , whose coordinates are  $h$  and  $k$ , prove that the area of the triangle  $TPQ$  is  $ab \left( \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 \right)^{\frac{3}{2}} \div \left( \frac{h^2}{a^2} + \frac{k^2}{b^2} \right)$ , and that the area of the quadrilateral  $CPTQ$  is

$$ab \left( \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 \right)^{\frac{1}{2}}.$$

51. Tangents are drawn to the ellipse from the point  $\left( \frac{a^2}{\sqrt{a^2 - b^2}}, \sqrt{a^2 + b^2} \right)$ ; prove that they intercept on the ordinate through the nearer focus a distance equal to the major axis.
52. Prove that the angle between the tangents that can be drawn from any point  $(x_1, y_1)$  to the ellipse is  $\tan^{-1} \frac{2ab \sqrt{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1}}{x_1^2 + y_1^2 - a^2 - b^2}$ .
53. If  $T$  be the point  $(x_1, y_1)$ , show that the equation to the straight lines joining it to the foci,  $S$  and  $S'$ , is  $(x_1 y - x y_1)^2 - a^2 e^2 (y - y_1)^2 = 0$ . Prove that the bisector of the angle between these lines also bisects the angle between the tangents  $TP$  and  $TQ$  that can be drawn from  $T$ , and hence that  $\angle STP = \angle S'TQ$ .
54. Prove that the sum of the angles that the four normals drawn from any point to an ellipse make with the axis is equal to the sum of the angles that the two tangents from the same point make with the axis.

#### CATEGORY-17.6. CHORD OF AN ELLIPSE

55. In the ellipse  $\frac{x^2}{36} + \frac{y^2}{9} = 1$ , find the equation to the chord which passes through the point  $(2, 1)$  and is bisected at that point. {Ans.  $x + 2y = 4$ }
56. Find the equation of the chord of the ellipse  $2x^2 + 5y^2 = 20$  which is bisected at the point  $(2, 1)$ . {Ans.  $4x + 5y = 13$ }
57. Tangents are drawn from the point  $(3, 2)$  to the ellipse  $x^2 + 4y^2 = 9$ . Find the equation to their chord of contact and the equation of the straight line joining  $(3, 2)$  to the middle point of this chord of contact. {Ans.  $3x + 8y = 9$ ,  $2x = 3y$ }
58. Any point  $P$  of an ellipse is joined to the extremities of the major axis; prove that the portion of a directrix intercepted by them subtends a right angle at the corresponding focus.
59. If the straight line  $y = mx + c$  meets the ellipse, prove that the equation to the circle, described on the line joining the points of intersection as diameter, is  $(a^2 m^2 + b^2)(x^2 + y^2) + 2ma^2 cx - 2b^2 cy + c^2(a^2 + b^2) - a^2 b^2(1 + m^2) = 0$ .

#### CATEGORY-17.7. POLE AND POLAR OF AN ELLIPSE

60. Find with respect to the ellipse  $4x^2 + 7y^2 = 8$ ,
- the polar of the point  $(-\frac{1}{2}, 1)$  and
  - the pole of the straight line  $12x + 7y + 16 = 0$ . {Ans.  $2x - 7y + 8 = 0, (-\frac{3}{2}, -\frac{1}{2})$ }
61. In an ellipse, show that the polar of the focus is the corresponding directrix.
62. Show that the four lines which join the foci to two points  $P$  and  $Q$  on an ellipse all touch a circle whose centre is the pole of  $PQ$ .
63. In an ellipse, if the pole of the normal at  $P$  lie on the normal at  $Q$ , then show that the pole of the normal at  $Q$  lies on the normal at  $P$ .



64.  $CK$  is the perpendicular from the centre on the polar of any point  $P$ , and  $PM$  is the perpendicular from  $P$  on the same polar and is produced to meet the major axis in  $L$ . Show that (i)  $CK.PL = b^2$ , and (ii) the product of the perpendiculars from the foci on the polar  $= CK.LM$ . What do these theorems become when  $P$  is on the ellipse? If  $PN$  be the ordinate of  $P$  and the polar meet the axis in  $T$ , show that  $CL = e^2$ .  $CN$  and  $CT.CN = a^2$ .

**CATEGORY-17.8. SUBTANGENT, SUBNORMAL AND DIAMETER OF AN ELLIPSE**

65. Prove that the sum of the squares of the reciprocals of two perpendicular diameters of an ellipse is constant.
66. Find the inclination to the major axis of the diameter of the ellipse the square of whose length is (1) the arithmetical mean, (2) the geometrical mean, and (3) the harmonical mean, between the squares on the major and minor axes. {Ans.  $\tan^{-1} \frac{b}{a}$ ,  $\tan^{-1} \sqrt{\frac{b}{a}}$ ,  $45^\circ$ }
67. Show that the perpendiculars from the centre upon all chords, which join the ends of perpendicular diameters, are of constant length.
68. In an ellipse, referred to its centre, the length of the sub-tangent corresponding to the point  $(3, \frac{12}{5})$  is  $\frac{16}{3}$ ; prove that the eccentricity is  $\frac{4}{5}$ .
69. In the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , write down the equations to the diameters which are conjugate to the diameters whose equations are  $x - y = 0$ ,  $x + y = 0$ ,  $y = \frac{a}{b}x$  and  $y = \frac{b}{a}x$ . {Ans.  $a^2y + b^2x = 0$ ,  $a^2y - b^2x = 0$ ,  $a^3y + b^3x = 0$ ,  $ay + bx = 0$ }
70. Show that the diameters whose equations are  $y + 3x = 0$  and  $4y - x = 0$ , are conjugate diameters of the ellipse  $3x^2 + 4y^2 = 5$ .
71. Prove that the straight lines joining the centre to the inter-sections of the straight line  $y = mx + \sqrt{\frac{a^2m^2 + b^2}{2}}$  with the ellipse are conjugate diameters.
72. Any tangent to an ellipse meets the director circle in  $P$  and  $Q$ ; prove that  $CP$  and  $CQ$  are the conjugate diameters of the ellipse.
73. In an ellipse. if  $CP$  be conjugate to the diameter parallel to the normal at  $Q$ , prove that  $CQ$  is conjugate to the diameter parallel to the normal at  $P$ .
74. If a fixed straight line parallel to either axis meet a pair of conjugate diameters in the points  $K$  and  $L$ , show that the circle described on  $KL$  as diameter passes through two fixed points on the other axis.
75. Prove that the chord, which joins the ends of a pair of conjugate diameters of an ellipse, always touches a similar ellipse.
76. The eccentric angles of two points  $P$  and  $Q$  on the ellipse are  $\phi_1$  and  $\phi_2$ ; prove that the area of the parallelogram formed by the tangents at the ends of the diameters through  $P$  and  $Q$  is  $4ab \operatorname{cosec}(\phi_1 - \phi_2)$ , and hence that it is least when  $P$  and  $Q$  are at the end of conjugate diameters.
77. A pair of conjugate diameters is produced to meet the directrix; show that the orthocentre of the triangle so formed is at the focus.
78. If the tangent at any point  $P$  meets in the points  $L$  and  $L'$  (i) two parallel tangents, or (ii) two conjugate diameters, prove that in each case  $LP.PL'$  is equal to the square of the semi-diameter which is parallel to the tangent at  $P$ .

79. Tangents are drawn from any point on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  to the circle  $x^2 + y^2 = r^2$ ; prove that the chords of contact are tangents to the ellipse  $a^2x^2 + b^2y^2 = r^4$ . If  $\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2}$ , prove that the lines joining the centre to the points of contact with the circle are conjugate diameters of the second ellipse.

**CATEGORY-17.9. LOCUS PROBLEMS IN ELLIPSE**

80. Find the locus of the middle points of chords of an ellipse which are drawn through the positive end of the minor axis. {Ans.  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{y}{b}$ }
81. Find the locus of mid-points of a focal chord of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . {Ans.  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{ex}{a}$ }
82. In an ellipse, prove that the locus of the intersection of  $AP$  with the straight line through  $A'$  perpendicular to  $A'P$  is a straight line which is perpendicular to the major axis.
83. The tangent at any point  $P$  of a circle meets the tangent at a fixed point  $A$  in  $T$ , and  $T$  is joined to  $B$ , the other end of the diameter through  $A$ . Prove that the locus of the intersection of  $AP$  and  $BT$  is an ellipse whose eccentricity is  $\frac{1}{\sqrt{2}}$ .
84. From any point  $P$  on the ellipse,  $PN$  is drawn perpendicular to the axis and produced to  $Q$ , so that  $NQ$  equals  $PS$ , where  $S$  is a focus. Prove that the locus of  $Q$  is the two straight lines  $y \pm (ex - a) = 0$ .
85. Given the base of a triangle and the sum of its sides, prove that the locus of the centre of its incircle is an ellipse.
86. With a given point and line as focus and directrix, a series of ellipses are described. Prove that the locus of the extremities of their minor axes is a parabola.
87. A line of fixed length  $a + b$  moves so that its ends are always on two fixed perpendicular straight lines. Prove that the locus of a point, which divides this line into portions of length  $a$  and  $b$ , is an ellipse.
88. Prove that the extremities of the latus recta of all ellipses, having a given major axis  $2a$ , lie on the parabola  $x^2 = -a(y - a)$  or on the parabola  $x^2 = a(y + a)$ .
89. If ' $t$ ' is a variable, find the locus of the point of intersection of the two straight lines  $\frac{tx}{a} - \frac{y}{b} + t = 0$  and  $\frac{x}{a} + \frac{ty}{b} - 1 = 0$ .
90. Prove that the locus of the mid-point of the portion of tangent of an ellipse included between the axes is the curve  $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$ .
91. Any ordinate  $NP$  of an ellipse meets the auxiliary circle in  $Q$ ; prove that the locus of the intersection of the normals at  $P$  and  $Q$  is the circle  $x^2 + y^2 = (a + b)^2$ .
92. The normal at  $P$  meets the axes in  $G$  and  $g$ . Show that the loci of the middle points of  $PG$  and  $Gg$  are respectively the ellipses  $\frac{4x^2}{a^2(1+e^2)^2} + \frac{4y^2}{b^2} = 1$  and  $a^2x^2 + b^2y^2 = \frac{1}{4}(a^2 - b^2)^2$ .
93. Find the locus of the feet of the perpendicular drawn from the centre upon any tangent to the ellipse.
94. The normal  $GP$  is produced to  $Q$ , so that  $GQ = n \cdot GP$ . Prove that the locus of  $Q$  is the ellipse

$$\frac{x^2}{a^2(n+e^2-ne^2)^2} + \frac{y^2}{n^2b^2} = 1.$$

95. If the product of the perpendiculars from the foci upon the polar of  $P$  be constant and equal to  $c^2$ , prove that the locus of  $P$  is the ellipse  $b^4x^2(c^2+a^2e^2)+c^2a^4y^2=a^4b^4$ .
96. If two tangents to an ellipse and one of its foci be given, prove that the locus of its centre is a straight line.
97. A point is such that the perpendicular from the centre on its polar with respect to the ellipse is constant and equal to  $c$ ; show that its locus is the ellipse  $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$ .
98. If the circles be described on two semi-conjugate diameters of the ellipse as diameters, prove that the locus of their second points of intersection is the curve  $2(x^2+y^2)^2 = a^2x^2 + b^2y^2$ .
99. The tangents drawn from a point  $P$  to the ellipse make angles  $\theta_1$  and  $\theta_2$  with the major axis; find the locus of  $P$  when
- $\theta_1 + \theta_2$  is constant ( $= 2\alpha$ ). {Ans.  $x^2 - 2xy \cot 2\alpha - y^2 = a^2 - b^2$ }
  - $\tan \theta_1 + \tan \theta_2$  is constant ( $= c$ ). {Ans.  $cx^2 - 2xy = ca^2$ }
  - $\tan \theta_1 - \tan \theta_2$  is constant ( $= d$ ). {Ans.  $d^2(x^2 - a^2)^2 = 4(b^2x^2 + a^2y^2 - a^2b^2)$ }
  - $\tan^2 \theta_1 + \tan^2 \theta_2$  is constant ( $= \lambda$ ). {Ans.  $\lambda(x^2 - a^2)^2 = 2(x^2y^2 + b^2x^2 + a^2y^2 - a^2b^2)$ }
100. Find the locus of the intersection of tangents
- which meet at a given angle  $\alpha$ . {Ans.  $(x^2 + y^2 - a^2 - b^2)^2 = 4\cot^2 \alpha (b^2x^2 + a^2y^2 - a^2b^2)$ }
  - if the sum of the eccentric angles of their points of contact be equal to a constant angle  $2\alpha$ . {Ans.  $ay = bx \tan \alpha$ }
  - if the difference of these eccentric angles be  $120^\circ$ . {Ans.  $b^2x^2 + a^2y^2 = 4a^2b^2$ }
  - if the lines joining the points of contact to the centre be perpendicular. {Ans.  $b^4x^2 + a^4y^2 = a^2b^2(a^2 + b^2)$ }
  - if the sum of the ordinates of the points of contact be equal to  $b$ . {Ans.  $b^2x^2 + a^2y^2 = 2a^2by$ }
101. Find the locus of the middle points of chords of an ellipse
- whose distance from the centre is the constant length  $c$ . {Ans.  $(b^2x^2 + a^2y^2)^2 = c^2(b^4x^2 + a^4y^2)$ }
  - which subtend a right angle at the centre. {Ans.  $(a^2 + b^2)(b^2x^2 + a^2y^2)^2 = a^2b^2(b^4x^2 + a^4y^2)$ }
  - which pass through the given point  $(h, k)$ . {Ans.  $b^2x(x-h) + a^2y(y-k) = 0$ }
  - whose length is constant  $= (2c)$ . {Ans.  $c^2a^2b^2(b^2x^2 + a^2y^2) + (b^2x^2 + a^2y^2 - a^2b^2)(b^4x^2 + a^4y^2) = 0$ }
  - whose poles are on the auxiliary circle. {Ans.  $(b^2x^2 + a^2y^2)^2 = a^2b^4(x^2 + y^2)$ }
  - the tangents at the ends of which intersect at right angles. {Ans.  $a^4b^4(x^2 + y^2) = (a^2 + b^2)(b^2x^2 + a^2y^2)^2$ }
102. Prove that the locus of the intersection of normals at the ends of conjugate diameters is the curve  $2(a^2x^2 + b^2y^2)^3 = (a^2 - b^2)^2(a^2x^2 - b^2y^2)^2$ .
103. Prove that the locus of the intersection of normals at the ends of chords, parallel to the tangent at the point whose eccentric angle is  $\alpha$ , is the conic  $2(ax \sin \alpha + by \cos \alpha)(ax \cos \alpha + by \sin \alpha) = (a^2 - b^2)^2 \sin 2\alpha \cos^2 2\alpha$ .
104. A parallelogram circumscribes the ellipse and two of its opposite angular points lie on the straight lines

$x^2 = h^2$ ; prove that the locus of the other two is the conic  $\frac{x^2}{a^2} + \frac{y^2}{b^2} \left(1 - \frac{a^2}{h^2}\right) = 1$ .

105. Circles of constant radius  $c$  are drawn to pass through the ends of a variable diameter of the ellipse. Prove that the locus of their centres is the curve  $(x^2 + y^2)(a^2x^2 + b^2y^2 + a^2b^2) = c^2(a^2x^2 + b^2y^2)$ .
106. The polar of a point  $P$  with respect to an ellipse touches a fixed circle, whose centre is on the major axis and which passes through the centre of the ellipse. Show that the locus of  $P$  is a parabola.
107. Prove that the locus of the pole, with respect to the ellipse, of any tangent to the auxiliary circle is the curve  $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$ .
108. Show that the locus of the pole, with respect to the auxiliary circle, of a tangent to the ellipse is a similar concentric ellipse, whose major axis is at right angles to that of the original ellipse.
109. Chords of the ellipse touch the parabola  $ay^2 = -2b^2x$ ; prove that the locus of their poles is the parabola  $ay^2 = 2b^2x$ .
110. If the polar with respect to  $y^2 = 4ax$  touches the ellipse  $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ , find the locus of its pole. {Ans.
- $$\frac{x^2}{\alpha^2} - \frac{y^2}{\left(\frac{4a^2\alpha^2}{\beta^2}\right)} = 1 \}$$
111. An ellipse is rotated through a right angle in its own plane about its centre, which is fixed; prove that the locus of the point of intersection of a tangent to the ellipse in its original position with the tangent at the same point of the curve in its new position is  $(x^2 + y^2)(x^2 + y^2 - a^2 - b^2) = 2(a^2 - b^2)xy$ .

### CATEGORY-17.10. ADDITIONAL QUESTIONS

# Mathematics for IIT-JEE

By **Er. Sanjiva Dayal,** *B.Tech. (I.I.T. Kanpur)*

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*PART-IV*  
*TWO DIMENSIONAL COORDINATE GEOMETRY*

*CHAPTER-18*  
*HYPERBOLA*

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## ***CHAPTER-18***

## ***HYPERBOLA***

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### **LIST OF THEORY SECTIONS**

- 18.1. Equation Of A Hyperbola
- 18.2. Position Of A Point/Line W.R.T. A Hyperbola
- 18.3. Tangent, Normal, Chord And Polar Of A Hyperbola
- 18.4. Subtangent, Subnormal And Diameter Of A Hyperbola
- 18.5. Asymptotes Of A Hyperbola
- 18.6. Conjugate Hyperbola
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### **LIST OF QUESTION CATEGORIES**

- 18.1. Finding Equation Of A Hyperbola Fro Its Properties
- 18.2. Finding Properties Of A Hyperbola From Its Equation
- 18.3. Position Of A Point/Line W.R.T. A Hyperbola
- 18.4. Tangent And Normal Of A Hyperbola
- 18.5. Chord Of A Hyperbola
- 18.6. Pole And Polar Of A Hyperbola
- 18.7. Subtangent, Subnormal And Diameter Of A Hyperbola
- 18.8. Asymptotes Of A Hyperbola
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- 18.10. Rectangular Hyperbola
- 18.11. Locus Problems In Hyperbola
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## CHAPTER-18

### HYPERBOLA

#### SECTION-18.1. EQUATION OF A HYPERBOLA

##### 1. Equation of a hyperbola with given Focus, Directrix and Eccentricity

Equation of the hyperbola with Focus at  $(h,k)$ , Directrix  $lx + my + n = 0$  and Eccentricity  $e$  is

$$\sqrt{(x-h)^2 + (y-k)^2} = e \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$

$$\equiv (x-h)^2 + (y-k)^2 = e^2 \frac{(lx + my + n)^2}{l^2 + m^2}.$$

##### 2. Standard equation of a hyperbola

i. A hyperbola has two vertices, denoted by  $A$  and  $A'$ . The mid-point of the two vertices is called the center of the hyperbola, denoted by  $C$ . The line segment  $AA'$  is called the transverse axis of the hyperbola.

ii. Equation of a standard hyperbola having center at origin and axis as  $x$ -axis is  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , where  $2a$  is

the distance between the vertices and  $b^2 = a^2(e^2 - 1) \Rightarrow b = a\sqrt{e^2 - 1}$ .

(a). If  $1 < e < \sqrt{2}$  then  $b < a$ .

(b). If  $e = \sqrt{2}$  then  $b = a$  (Rectangular hyperbola).

(c). If  $e > \sqrt{2}$  then  $b > a$ .

iii. The line segment  $BB'$ , which is bisected at  $C$ , is perpendicular to  $AA'$  and having length  $2b$ , is called the conjugate axis of the hyperbola.

iv. A hyperbola has two pairs of Focus and Directrix, denoted by  $S, S'$  and  $D, D'$  respectively.

v. A hyperbola has two Latus rectums, denoted by  $L_1L_1'$  and  $L_2L_2'$ .

##### 3. Properties of standard hyperbola

		$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
1.	Coordinates of Foci	$(\pm ae, 0)$	$(0, \pm ae)$
2.	Equation of Directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{a}{e}$
3.	Eccentricity	$e = \sqrt{1 + \frac{b^2}{a^2}}$	$e = \sqrt{1 + \frac{b^2}{a^2}}$
4.	Coordinates of the vertices	$(\pm a, 0)$	$(0, \pm a)$
5.	Coordinates of the center	$(0, 0)$	$(0, 0)$
6.	Equation of transverse axis	$y = 0$	$x = 0$
7.	Equation of conjugate axis	$x = 0$	$y = 0$
8.	Length of transverse axis	$2a$	$2a$

9.	Length of conjugate axis	$2b$	$2b$
10.	Coordinates of the end points of the conjugate axis	$(0, \pm b)$	$(\pm b, 0)$
11.	Coordinates of Foot of Directrices	$\left(\pm \frac{a}{e}, 0\right)$	$\left(0, \pm \frac{a}{e}\right)$
12.	Coordinates of the end points of the Latus Rectums	$\left(ae, \pm \frac{b^2}{a}\right) \& \left(-ae, \pm \frac{b^2}{a}\right)$	$\left(\pm \frac{b^2}{a}, ae\right) \& \left(\pm \frac{b^2}{a}, -ae\right)$
13.	Lengths of Latus Rectums	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
14.	Equations of Latus Rectums	$x = \pm ae$	$y = \pm ae$

#### 4. Tracing a general hyperbola

- A general hyperbola is traced by parallel and angular transformation of axes.
- Equation of a hyperbola having center at  $(h, k)$  and transverse axis parallel to  $x$ -axis is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1. \text{ Equation of a hyperbola having center at } (h, k) \text{ and transverse axis parallel to } y\text{-axis is } \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

#### 5. Auxiliary circle of the hyperbola

- The circle described on the transverse axis of a hyperbola as diameter, is called the Auxiliary circle of the hyperbola.
- Equation of Auxiliary circle of standard hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $x^2 + y^2 = a^2$ .

#### 6. Parametric equation of standard hyperbola

Point  $(a \sec \phi, b \tan \phi)$  always lies on the standard hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .

#### 7. Alternate definition of a hyperbola

- The difference of the focal distances of any point on a hyperbola is a constant and equal to the length of the transverse axis  $(2a)$ .
- Therefore, a hyperbola is the locus of a point such that the difference of its distances from two fixed points (Foci) is always a constant  $(2a)$ .

### SECTION-18.2. POSITION OF A POINT/LINE W.R.T. A HYPERBOLA

#### 1. Position of a point w.r.t. a hyperbola

The point  $(x_1, y_1)$  lies inside, on or outside the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  according as  $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$  is  $>$ ,  $=$  or  $< 0$ .

#### 2. Position of a line w.r.t. a hyperbola

- The line  $y = mx + c$   $\left(m \neq \pm \frac{b}{a}\right)$  cuts, touches or does not cut the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  according as  $c^2$



$>, =$  or  $< a^2m^2 - b^2$ .

- ii. The line  $y = mx + c$   $\left(m = \pm \frac{b}{a}\right)$  cuts or does not cut the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  according as  $c \neq$  or  $= 0$ .
- iii. The line  $x = c$  cuts, touches or does not cut the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  according as  $c^2 >, =$  or  $< a^2$ .
- iv. Points of intersection of a hyperbola and a line can be obtained by solving their equations simultaneously.

### SECTION-18.3. TANGENT, NORMAL, CHORD AND POLAR OF A HYPERBOLA

#### 1. Tangent to a hyperbola

- i. Slope of tangent at the point  $(x_1, y_1)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{b^2 x_1}{a^2 y_1}$ .
- ii. Equation of tangent at the point  $(x_1, y_1)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ .
- iii. Equation of tangent at  $(x_1, y_1)$  on a second degree curve  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  is  $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$ .
- iv. Slope of tangent at the point  $(a \sec \phi, b \tan \phi)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $\left(\frac{dy}{dx}\right)_{(a \sec \phi, b \tan \phi)} = \frac{b}{a} \operatorname{cosec} \phi$ .
- v. Equation of tangent at the point  $(a \sec \phi, b \tan \phi)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1$ .
- vi. Straight line  $y = mx \pm \sqrt{a^2m^2 - b^2}$  is always tangent to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  at  $\left(\mp \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \mp \frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$ .
- vii. The combined equation of the pair of tangents drawn from an external point  $(x_1, y_1)$  to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right) \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$ .
- viii. **Definition of Director circle:** The locus of the point of intersection of two perpendicular tangents to a hyperbola is a circle and is called Director circle. Equation of Director circle of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $x^2 + y^2 = a^2 - b^2$ , if  $a > b$  ( $1 < e < \sqrt{2}$ ). If  $a \leq b$  ( $e \geq \sqrt{2}$ ), then there cannot be perpendicular tangents and hence there is no Director circle.

#### 2. Normal to a hyperbola

- i. Slope of normal at the point  $(x_1, y_1)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $-\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = -\frac{a^2 y_1}{b^2 x_1}$ .

ii. Equation of normal at the point  $(x_1, y_1)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is

$$\frac{x - x_1}{\left(\frac{x_1}{a^2}\right)} = \frac{y - y_1}{-\left(\frac{y_1}{b^2}\right)} \equiv \frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2.$$

iii. Slope of normal at the point  $(a \sec \phi, b \tan \phi)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is

$$-\left(\frac{dx}{dy}\right)_{(a \sec \phi, b \tan \phi)} = -\frac{a}{b} \sin \phi.$$

iv. Equation of normal at the point  $(a \sec \phi, b \tan \phi)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is

$$ax \cos \phi + by \cot \phi = a^2 + b^2.$$

v. Straight line  $y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2 m^2}}$  is always normal to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .

vi. Maximum four normals can be drawn from a point to a hyperbola. The four points on the hyperbola at which the normals pass through a common point are called co-normal points.

### 3. Chord of a hyperbola

The equation of the chord joining two points  $(a \sec \theta, b \tan \theta)$  and  $(a \sec \phi, b \tan \phi)$  on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{x}{a} \cos\left(\frac{\theta - \phi}{2}\right) - \frac{y}{b} \sin\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta + \phi}{2}\right).$$

### 4. Chord of contact of tangents

Equation of chord of contact of tangents drawn from an external point  $(x_1, y_1)$  to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\text{is } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

### 5. Chord with a given mid-point

The equation of the chord with mid-point  $(h, k)$  of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$ .

### 6. Polar and Pole

i. Equation of the polar of the point  $P \equiv (x_1, y_1)$  w.r.t. the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is the line

$$L \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1. \text{ The point } P \text{ is called the pole of its polar line } L.$$

ii. Therefore, chord with a given mid-point  $(h, k)$  is parallel to the polar of the point  $(h, k)$ .

## SECTION-18.4. SUBTANGENT, SUBNORMAL AND DIAMETER OF A HYPERBOLA

### 1. Subtangent of a point on the hyperbola

i. Length of subtangent of a point  $P(x_1, y_1)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $x_1 - \frac{a^2}{x_1}$ .

**2. Subnormal of a point on the hyperbola**

- i. Length of subnormal of a point  $P(x_1, y_1)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $(e^2 - 1)x_1$ .

**3. Diameter of a hyperbola**

- i. Diameter of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , bisecting chords of gradient  $m$ , is the straight line  $y = \frac{b^2}{a^2 m} x$ .  
Hence, diameter of a hyperbola is a straight line passing through its center.
- ii. Two diameters of a ellipse are said to be conjugate diameters if each bisects chords parallel to the other.
- iii. The two diameters  $y = m_1 x$  &  $y = m_2 x$  are conjugate if  $m_1 m_2 = \frac{b^2}{a^2}$ .
- iv. Transverse axis and conjugate axis are conjugate diameters and are the only perpendicular conjugate diameters.

**SECTION-18.5. ASYMPTOTES OF A HYPERBOLA****1. Definition of Asymptotes**

Asymptote to a curve is a straight line which touches the curve at infinity.

**2. Asymptotes of a hyperbola**

- i. The two asymptotes of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  are  $y = \pm \frac{b}{a} x \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ .
- ii. The asymptotes pass through the center of the hyperbola.
- iii. The bisectors of the angles between the asymptotes are the transverse and conjugate axes.
- iv. The combined equation of the pair of asymptotes differ the equation of the hyperbola by constant only, i.e.

$$\text{Equation of hyperbola} \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0;$$

$$\text{Combined equation of asymptotes} \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0.$$

**SECTION-18.6. CONJUGATE HYPERBOLA****1. Definition of Conjugate hyperbola**

Two hyperbolas are said to be conjugate hyperbolas if transverse axis of one is the conjugate axis of the other and conjugate axis of one is the transverse axis of the other.

**2. Equation of Conjugate hyperbola**

- i. Equation of conjugate hyperbola of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ .
- ii. A hyperbola and its conjugate hyperbola have the same asymptotes.
- iii. The combined equation of the pair of asymptotes differ the equation of the hyperbola and its conjugate hyperbola by the same constant only, i.e.

$$\text{Equation of hyperbola} \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0;$$

$$\text{Combined equation of asymptotes} \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0;$$

$$\text{Equation of conjugate hyperbola} \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c + 2\lambda = 0.$$

**SECTION-18.7. RECTANGULAR (EQUILATERAL) HYPERBOLA****1. Definition of Rectangular hyperbola**

- i. A hyperbola whose transverse and conjugate axes are equal in length, is called a rectangular hyperbola.
  - ii. In a rectangular hyperbola,  $a = b$  and  $e = \sqrt{2}$ .
  - iii. A general second degree equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents a rectangular hyperbola when  $\Delta \neq 0$ ,  $h^2 > ab$  and  $a + b = 0$ .
- 2. Equation of a standard rectangular hyperbola**  
Equation of standard rectangular hyperbola is  $x^2 - y^2 = a^2$ , where  $a$  is the length of its transverse and conjugate axes.
- 3. Asymptotes of a rectangular hyperbola**  
Equation of asymptotes of the rectangular hyperbola  $x^2 - y^2 = a^2$  are  $y = \pm x \equiv x^2 - y^2 = 0$ , which are perpendicular lines.
- 4. Equation of a rectangular hyperbola referred to its asymptotes as the coordinate axes**
- i. Equation of a rectangular hyperbola with its asymptotes taken as coordinate axes is  $xy = \frac{a^2}{2}$ .
  - ii. Therefore, the equation  $xy = c^2$  represents a rectangular hyperbola, where  $c = \frac{a}{\sqrt{2}} \Rightarrow a = \sqrt{2}c$ .
  - iii. The point  $\left(ct, \frac{c}{t}\right)$  always lies on the rectangular hyperbola  $xy = c^2$ .
- 5. Tangent to the rectangular hyperbola  $xy = c^2$**
- i. The equation of tangent at  $(x_1, y_1)$  to the hyperbola  $xy = c^2$  is  $xy_1 + yx_1 = 2c^2 \equiv \frac{x}{x_1} + \frac{y}{y_1} = 2$ .
  - ii. The equation of tangent at  $\left(ct, \frac{c}{t}\right)$  to the hyperbola  $xy = c^2$  is  $\frac{x}{t} + yt = 2c$ .
  - iii. Point of intersection of tangents at the points  $\left(ct_1, \frac{c}{t_1}\right)$  and  $\left(ct_2, \frac{c}{t_2}\right)$  is  $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$ .
- 6. Normal to the rectangular hyperbola  $xy = c^2$**
- i. The equation of normal at  $(x_1, y_1)$  to the hyperbola  $xy = c^2$  is  $xx_1 - yy_1 = x_1^2 - y_1^2$ .
  - ii. The equation of normal at  $\left(ct, \frac{c}{t}\right)$  to the hyperbola  $xy = c^2$  is  $xt^3 - yt - ct^4 + c = 0$ .
- 7. Chord of the rectangular hyperbola  $xy = c^2$**   
Equation of chord joining points  $\left(ct_1, \frac{c}{t_1}\right)$  and  $\left(ct_2, \frac{c}{t_2}\right)$  on the hyperbola  $xy = c^2$  is  $x + yt_1t_2 - c(t_1 + t_2) = 0$ .
- 8. Chord with a given mid-point of the rectangular hyperbola  $xy = c^2$**   
Equation of a chord of the hyperbola  $xy = c^2$ , whose mid-point is  $(h, k)$  is  $xh + yk = 2hk$ .
- 9. Asymptotes of the rectangular hyperbola  $xy = c^2$**   
Asymptotes of the rectangular hyperbola  $xy = c^2$  are the coordinate axes, i.e.  $xy = 0$ .
- 10. Conjugate hyperbola of the rectangular hyperbola  $xy = c^2$**   
Equation of a conjugate hyperbola of rectangular hyperbola  $xy = c^2$  is  $xy = -c^2$ .

## EXERCISE-18

### CATEGORY-18.1. FINDING EQUATION OF A HYPERBOLA FROM ITS PROPERTIES

- Find the equation of the hyperbola whose directrix is  $x + 2y = 1$ , focus  $(2,1)$  and eccentricity 2. {Ans.  $x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$ }
- Find the equation of the conic with the focus at  $(1,-1)$ , directrix along  $x - y + 1 = 0$  and with eccentricity  $\sqrt{2}$ . {Ans.  $2xy - 4x + 4y + 1 = 0$ }
- Find the equation to the hyperbola, referred to its axes as axes of coordinates
  - whose transverse and conjugate axes are respectively 3 and 4. {Ans.  $16x^2 - 9y^2 = 36$ }
  - whose conjugate axis is 5 and the distances between whose foci is 13. {Ans.  $25x^2 - 144y^2 = 900$ }
  - whose conjugate axis is 7 and which passes through the point  $(3, -2)$ . {Ans.  $65x^2 - 36y^2 = 441$ }
  - the distance between whose foci is 16 and whose eccentricity is  $\sqrt{2}$ . {Ans.  $x^2 - y^2 = 32$ }
- Find the equation to the hyperbola whose transverse axis is 2 and whose vertex bisects the distance between the centre and the focus. {Ans.  $3x^2 - y^2 = 3$ }
- Find the equation of the hyperbola with vertices  $(3,0)$  and  $(-3,0)$  and semi-latusrectum 4. {Ans.  $4x^2 - 3y^2 - 36 = 0$ }
- Find the equation to the hyperbola, whose eccentricity is  $\frac{5}{4}$ , whose focus is  $(a, 0)$ , and whose directrix is  $4x - 3y = a$ . Find also the coordinates of the centre and the equation to the other directrix. {Ans.  $7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0, (-\frac{a}{3}, a), 12x - 9y + 29a = 0$ }
- The equation of one of the directrices of hyperbola is  $2x + y = 1$ , the corresponding focus is  $(1,2)$  and  $e = \sqrt{3}$ . Find the equation of the hyperbola and coordinates of the center and second focus. {Ans.  $7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0, (-\frac{4}{5}, \frac{11}{10}), (-\frac{13}{5}, \frac{1}{5})$ }
- Find the equation of the hyperbola the coordinates of whose foci are  $(\pm 5,0)$  and the length of transverse axis is 8. {Ans.  $9x^2 - 16y^2 = 144$ }
- Find the equation of the hyperbola whose foci are  $(6,4)$  and  $(-4,4)$  and eccentricity is 2. {Ans.  $12x^2 - 4y^2 - 24x + 32y - 127 = 0$ }
- Find the locus of a point which moves so that the difference of its distances from the points  $(5,0)$  and  $(-5,0)$  is 2. {Ans.  $24x^2 - y^2 = 24$ }
- The coordinates of the foci of a hyperbola are  $(\pm 6,0)$  and its latus-rectum is of 10 units. Find the equation of the hyperbola taking the coordinate axes as the axis of the hyperbola. {Ans.  $5x^2 - 4y^2 = 80$ }

### CATEGORY-18.2. FINDING PROPERTIES OF A HYPERBOLA FROM ITS EQUATION

- Find the eccentricity of the hyperbola  $3x^2 - 4y^2 = -12$ . {Ans.  $\sqrt{\frac{7}{3}}$ }
- In the hyperbola  $4x^2 - 9y^2 = 36$ , find the length of the axes, the coordinates of the foci, the eccentricity and the latus rectum. {Ans.  $6, 4, (\pm\sqrt{13}, 0), \frac{\sqrt{13}}{3}, \frac{8}{3}$ }

14. Find the center, eccentricity, foci, directrices and the lengths of the transverse and conjugate axes of the hyperbola  $x^2 - 2y^2 - 2x + 8y - 1 = 0$ . {Ans.  $C \equiv (1, 2)$ ,  $e = \sqrt{3}$ ,  $S \equiv (1, 5) \& (1, -1)$ ,  $D \equiv y = 3 \& y = 1$ ,  $a = \sqrt{3}$ ,  $b = \sqrt{6}$ }
15. Find the centre of the hyperbola  $9x^2 - 36x - 16y^2 + 96y - 252 = 0$ . {Ans.  $(2, 3)$ }
16. Find the coordinates of foci, the eccentricity and latus-rectum, equations of directrices of the hyperbola  $9x^2 - 16y^2 - 72x + 96y - 144 = 0$ . {Ans.  $S \equiv (9, 3) \& (-1, 3)$ ,  $e = \frac{5}{4}$ ,  $l = \frac{9}{2}$ ,  $D \equiv x = \frac{4}{5} \& x = \frac{36}{5}$ }
17. Find the coordinates of center, eccentricity and latus-rectum of the hyperbola  $x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$ . {Ans.  $(-2, 2)$ }
18. Find the eccentricity of the hyperbola with latusrectum 12 and semi-conjugate axis  $2\sqrt{3}$ . {Ans. 2}
19. For what values of  $k$  the equation  $\frac{x^2}{12-k} + \frac{y^2}{8-k} = 1$  represents a hyperbola? {Ans.  $8 < k < 12$ }
20. The foci of the ellipse  $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$  and the hyperbola  $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$  coincide, then find the value of  $b^2$ . {Ans. 7}
21. Find the eccentricity of the hyperbola  $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . {Ans.  $e = \sqrt{\frac{a^2 + b^2}{b^2}}$ }
22. The hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  passes through the point of intersection of the lines  $7x + 13y - 87 = 0$  and  $5x - 8y + 7 = 0$  and its latus rectum is  $\frac{32\sqrt{2}}{5}$ . Find  $a$  and  $b$ . {Ans.  $a = \frac{5}{\sqrt{2}}$ ,  $b = 4$ }
23. Show that the curve represented by  $x = a(\cosh\theta + \sinh\theta)$ ,  $y = b(\cosh\theta - \sinh\theta)$  is a hyperbola.

### CATEGORY-18.3. POSITION OF A POINT/LINE W.R.T. A HYPERBOLA

24. Is the point  $(4, -5)$  inside or outside the hyperbola  $16x^2 - 9y^2 = 36$ ? {Ans. outside}
25. Find the points common to the hyperbola  $25x^2 - 9y^2 = 225$  and the straight line  $25x + 12y - 45 = 0$ . {Ans.  $(5, -\frac{20}{3})$ }

### CATEGORY-18.4. TANGENT AND NORMAL OF A HYPERBOLA

26. Find the equation of the tangent to the hyperbola  $4y^2 = x^2 - 1$  at the point  $(1, 0)$ . {Ans.  $x = 1$ }
27. Find the equation of the tangent to the conic  $x^2 - y^2 - 8x + 2y + 11 = 0$  at  $(2, 1)$ . {Ans.  $x - 2 = 0$ }
28. Find the equation of a tangent parallel to  $y = x$  drawn to  $\frac{x^2}{3} - \frac{y^2}{2} = 1$ . {Ans.  $x - y + 1 = 0$ }
29. Find the equation of the tangent to the hyperbola  $4x^2 - 9y^2 = 1$  which is parallel to the line  $4y = 5x + 7$ . {Ans.  $24y - 30x = \pm\sqrt{161}$ }
30. Find the equation of the tangents to the hyperbola  $x^2 - 2y^2 = 18$  which are perpendicular to the line  $x - y = 0$ . {Ans.  $x + y = \pm 3$ }
31. Find the equation of the tangent to the hyperbola  $x^2 - 4y^2 = 36$  which is perpendicular to the line

$$x - y + 4 = 0. \{ \text{Ans. } x + y \pm 3\sqrt{3} = 0 \}$$

32. If the line  $y = 3x + \lambda$  touches the hyperbola  $9x^2 - 5y^2 = 45$ , then find the value of  $\lambda$ . {Ans. 6}
33. Find the value of  $m$  for which  $y = mx + 6$  is a tangent to the hyperbola  $\frac{x^2}{100} - \frac{y^2}{49} = 1$ . {Ans.  $\sqrt{\frac{17}{20}}$ }
34. Tangents are drawn to the hyperbola  $3x^2 - 2y^2 = 25$  from the point  $\left(0, \frac{5}{2}\right)$ . Find their equations. {Ans.  $3x - 2y + 5 = 0, 3x + 2y - 5 = 0$ }
35. Find the equation of the common tangents to the hyperbola  $3x^2 - y^2 = 3$  and the parabola  $y^2 = 8x$ . {Ans.  $y = 2x + 1, y = -2x - 1$ }
36. Prove that the straight line  $21x + 5y = 116$  touches the hyperbola  $7x^2 - 5y^2 = 232$ . Also find its point of contact. {Ans.  $(6, -2)$ }
37. Prove that a circle can be drawn through the foci of a hyperbola and the points in which any tangent meets the tangents at the vertices.
38. In both an ellipse and a hyperbola, prove that the focal distance of any point and the perpendicular from the centre upon the tangent at it meet on a circle whose centre is the focus and whose radius is the semi-transverse axis.
39. Find the equation to, and the length of, the common tangent to the two hyperbolas  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  and  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ . {Ans.  $y = \pm x \pm \sqrt{a^2 - b^2}, (a^2 + b^2)\sqrt{\frac{2}{a^2 - b^2}}$ }
40. Find the condition that  $x \cos \alpha + y \sin \alpha = p$ , should touch the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . {Ans.  $a^2 \cos^2 \alpha - b^2 \sin^2 \alpha = p^2$ }
41. Show that the product of the lengths of the perpendiculars drawn from foci on any tangent to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $b^2$ . Prove further that the feet of the perpendiculars lie on the auxiliary circle  $x^2 + y^2 = a^2$ .
42. Prove that the part of the tangent at any point of an hyperbola intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendiculars drawn from the foci on the normal at the same point.
43. The tangent at a point  $P$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  meet one of its directrices in  $Q$ . Show that  $PQ$  subtends a right angle at the corresponding focus.
44. If  $m_1, m_2$  are slopes of the tangents to the hyperbola  $\frac{x^2}{25} - \frac{y^2}{16} = 1$  which pass through  $(6, 2)$ , find the value of  $m_1 + m_2$  and  $m_1 m_2$ . {Ans.  $\frac{24}{11}, \frac{20}{11}$ }
45. If the normal at  $P$  to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  meets the transverse axis in  $Q$  and conjugate axis in  $R$  and  $CF$  be perpendicular to the normal from the center then prove that  $PF \cdot PQ = b^2, PF \cdot PR = a^2$

46. If the normal at a point  $P$  to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  meets at the  $x$ -axis at  $G$ , show that  $SG = e \cdot SP$ .

**CATEGORY-18.5. CHORD OF A HYPERBOLA**

47. Find the equation to the chord of the hyperbola  $25x^2 - 16y^2 = 400$  which is bisected at the point  $(5, 3)$ .  
 { Ans.  $125x - 48y = 481$  }
48. Find the equation to the chord of the hyperbola  $x^2 - y^2 = 9$  which is bisected at  $(5, -3)$ . { Ans.  $5x + 3y = 16$  }
49. Let  $PQ$  be a double ordinate of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . If  $O$  be the center of the hyperbola and  $OPQ$  is an equilateral triangle, then prove that the eccentricity  $e > \frac{2}{\sqrt{3}}$ .

**CATEGORY-18.6. POLE AND POLAR OF A HYPERBOLA**

50. Given the hyperbola  $25x^2 - 16y^2 = 400$ , find polar of the point  $(1, 2)$  and pole of the line  $x + y + 1 = 0$ .  
 { Ans.  $25x - 32y = 400, (-16, 25)$  }
51. In a hyperbola, show that the polar of the focus is the corresponding directrix.
52. An ellipse and a hyperbola have the same principal axes. Show that the polar of any point on either curve with respect to the other touches the first curve.

**CATEGORY-18.7. SUBTANGENT, SUBNORMAL AND DIAMETER OF A HYPERBOLA**

53. Find the equation of that diameter which bisects the chord  $7x + y - 20 = 0$  of the hyperbola  $\frac{x^2}{3} - \frac{y^2}{7} = 1$ .  
 { Ans.  $x + 3y = 0$  }
54. In the hyperbola  $16x^2 - 9y^2 = 144$ , find the equation to the diameter which is conjugate to the diameter whose equation is  $x = 2y$ . { Ans.  $9y = 32x$  }

**CATEGORY-18.8. ASYMPTOTES OF A HYPERBOLA**

55. Find the equations of the asymptotes of the hyperbola  $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$ . { Ans.  $3x + 4y + 5 = 0, x + 2y + 3 = 0$  }
56. Find the equations of the asymptotes of the hyperbola  $xy - 4x + 3y = 0$ . { Ans.  $x = -3, y = 4$  }
57. Find the equation of the hyperbola which has  $3x - 4y + 7 = 0$  and  $4x + 3y + 1 = 0$  for its asymptotes and which passes through the origin. { Ans.  $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$  }
58. Show that the angle between the asymptotes of  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is equal to  $2 \tan^{-1} \left( \frac{b}{a} \right)$ .
59. Chords of contact of a point  $P$  w.r.t. the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  and its auxiliary circle  $x^2 + y^2 = a^2$  are at right angle. Prove that  $P$  lies on one of the asymptotes.
60. Prove that the tangent to a hyperbola makes with the asymptotes a triangle of constant area and the portion of the tangent is bisected at the point of contact.



**CATEGORY-18.9. CONJUGATE HYPERBOLA**

61. Find the asymptotes of the hyperbola  $x^2 + 3xy + 2y^2 + 2x + 3y = 0$  and the equation to its conjugate hyperbola. {Ans.  $x^2 + 3xy + 2y^2 + 2x + 3y + 1 = 0$ ,  $x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$ }
62. Through the positive vertex of the hyperbola a tangent is drawn; where does it meet the conjugate hyperbola? {Ans.  $(a, \pm b\sqrt{2})$ }
63. If  $e$  and  $e'$  be the eccentricities of a hyperbola and its conjugate, prove that  $\frac{1}{e^2} + \frac{1}{e'^2} = 1$
64. Prove that chords of a hyperbola, which touch the conjugate hyperbola, are bisected at the point of contact.
65. Tangents are drawn to a hyperbola from any point on one of the branches of the conjugate hyperbola; show that their chord of contact will touch the other branch of the conjugate hyperbola.
66. Show that the chord, which joins the points in which a pair of conjugate diameters meets the hyperbola or its conjugate hyperbola, is parallel to one asymptote and is bisected by the other.
67. A straight line is drawn parallel to the conjugate axis of a hyperbola to meet it and the conjugate hyperbola in the points  $P$  and  $Q$ ; show that the tangents at  $P$  and  $Q$  meet on the curve  $\frac{y^4}{b^4} \left( \frac{y^2}{b^2} - \frac{x^2}{a^2} \right) = \frac{4x^2}{a^2}$ , and that the normals meet on the axis of  $x$ .

**CATEGORY-18.10. RECTANGULAR HYPERBOLA**

68. If  $5x^2 + \lambda y^2 = 20$  represents a rectangular hyperbola, then find  $\lambda$ . {Ans. -5}
69. Show that the equation  $3x^2 - 3y^2 - 18x + 12y + 2 = 0$  represents a rectangular hyperbola. Find its center and foci. {Ans.  $C \equiv (3, 2)$ ,  $S \equiv \left( 3 \pm \sqrt{\frac{26}{3}}, 2 \right)$ }
70. If  $e$  and  $e_1$ , are the eccentricities of the hyperbolas  $xy = c^2$  and  $x^2 - y^2 = c^2$ , then find the value of  $e^2 + e_1^2$ . {Ans. 4}
71. Find the equations of the two tangents to the hyperbola  $xy = 27$  which are perpendicular to the straight line  $4x - 3y = 7$ . Also find the points of contacts of these tangents. {Ans.  $4y + 3x = \pm 36$ }
72. In a rectangular hyperbola prove that
- $SP \cdot S'P = CP^2$
  - the distance of any point from the centre varies inversely as the perpendicular from the centre upon its polar.
  - if the normal at  $P$  meet the axes in  $G$  and  $g$ , then  $PG = Pg = PC$
  - the angle subtended by any chord at the centre is the supplement of the angle between the tangents at the ends of the chord
  - the angles subtended at its vertices by any chord which is parallel to its conjugate axis are supplementary.
73. If tangent and normal to a rectangular hyperbola cut off intercepts  $a_1$  and  $a_2$  on one axis and  $b_1$  and  $b_2$  on the other, show that  $a_1 a_2 + b_1 b_2 = 0$ .
74. Show that the locus represented by  $x = \frac{1}{2}a \left( t + \frac{1}{t} \right)$ ;  $y = \frac{1}{2}a \left( t - \frac{1}{t} \right)$  is a rectangular hyperbola. Show also

that the equation to the normal at the point  $t$  is  $\frac{x}{(t^2+1)} + \frac{y}{(t^2-1)} = \frac{a}{t}$ .

75. If a triangle is inscribed in a rectangular hyperbola, prove that its orthocentre lies on the curve.
76. Determine the constant  $c$  such that the straight line joining the points  $(0,3)$  and  $(5,-2)$  is tangent to the curve  $y = \frac{c}{(x+1)}$ . {Ans.  $c = 4$ }
77. If the normal to the rectangular hyperbola  $xy = c^2$  at the point " $t$ " on it intersects the hyperbola at " $t_1$ ", prove that  $t^3 t_1 = -1$ .
78. A circle cuts a rectangular hyperbola  $xy = c^2$  in  $A, B, C, D$  and the parameters of these four points are  $t_1, t_2, t_3, t_4$  respectively. Prove that  $t_1 t_2 t_3 t_4 = 1$ .
79. If the normals at four points  $P_i(x_i, y_i); i = 1, 2, 3, 4$  on the rectangular hyperbola  $xy = c^2$  meet at the point  $Q(h, k)$ , prove that
- $x_1 + x_2 + x_3 + x_4 = h$ ,
  - $y_1 + y_2 + y_3 + y_4 = k$ ,
  - $x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = -c^4$ .
80. Prove that if the normals at  $P, Q, R, S$  on a rectangular hyperbola intersect in a point, then the circle through  $P, Q, R$  passes through the other extremity of the diameter through  $S$ .

### CATEGORY-18.11. LOCUS PROBLEMS IN HYPERBOLA

81. If  $m$  is a variable, prove that the straight lines  $\frac{x}{a} - \frac{y}{b} = m$  and  $\frac{x}{a} + \frac{y}{b} = \frac{1}{m}$  always meet on the hyperbola.
82. The normal to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  meets the axes in  $M$  and  $N$ , and lines  $MP$  and  $NP$  are drawn at right angles to the axes. Prove that the locus of  $P$  is the hyperbola  $a^2 x^2 - b^2 y^2 = (a^2 + b^2)^2$ .
83. If one axis of a varying central conic be fixed in magnitude and position, prove that the locus of the point of contact of a tangent drawn to it from a fixed point on the other axis is a parabola.
84. If the ordinate  $MP$  of a hyperbola be produced to  $Q$ , so that  $MQ$  is equal to either of the focal distances of  $P$ , prove that the locus of  $Q$  is one or other of a pair of parallel straight lines.
85. Show that the locus of the centre of a circle, which touches externally two given circles, is a hyperbola.
86. On a level plain the crack of the rifle and the thud of the ball striking the target are heard at the same instant. Prove that the locus of the hearer is a hyperbola.
87. Given the base of a triangle and the ratio of the tangents of half the base angles, prove that the vertex moves on a hyperbola whose foci are the extremities of the base.
88. Prove that the locus of the poles of normal chords with respect to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is the curve  $y^2 a^6 - x^2 b^6 = (a^2 + b^2)^2 x^2 y^2$ .
89. Find the locus of the pole of a chord of the hyperbola which subtends a right angle at (1) the centre, (2) the vertex; and (3) the focus of the curve. {Ans.  $b^4 x^2 + a^4 y^2 = a^2 b^2 (b^2 - a^2), x = a \cdot \frac{a^2 - b^2}{a^2 + b^2}, x^2 (a^2 + 2b^2) - a^2 y^2 - 2a^3 ex + a^2 (a^2 - b^2) = 0$ }

90. Show that the locus of poles with respect to the parabola  $y^2 = 4ax$  of tangents to the hyperbola  $x^2 - y^2 = a^2$  is the ellipse  $4x^2 + y^2 = 4a^2$ .
91. Prove that the locus of the pole with respect to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  of any tangent to the circle, whose diameter is the line joining the foci, is the ellipse  $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}$ .
92. Prove that the locus of the intersection of tangents to a hyperbola, which meet at a constant angle  $\beta$ , is the curve  $(x^2 + y^2 + b^2 - a^2)^2 = 4 \cot^2 \beta (a^2 y^2 - b^2 x^2 + a^2 b^2)$ .
93. From points on the circle  $x^2 + y^2 = a^2$  tangents are drawn to the hyperbola  $x^2 - y^2 = a^2$ . Prove that the locus of the middle points of the chords of contact is the curve  $(x^2 - y^2)^2 = a^2(x^2 + y^2)$ .
94. Chords of the hyperbola are drawn, all passing through the fixed point  $(h, k)$ . Prove that the locus of their middle points is a hyperbola whose centre is the point  $(\frac{h}{2}, \frac{k}{2})$ .
95. Two straight lines pass through the fixed points  $(\pm a, 0)$  and have slopes whose products is  $p > 0$ . Show that the locus of the points of intersection of the lines is a hyperbola.
96.  $AOB, COD$  are two straight lines which bisect one another at right angles, show that the locus of a point  $P$  which moves so that  $PA \cdot PB = PC \cdot PD$  is a rectangular hyperbola.
97. Show that the locus of the foot of the perpendicular drawn from the center  $C$  of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  on any normal is  $(a^2 y^2 - b^2 x^2)(x^2 + y^2)^2 = (a^2 + b^2)^2 x^2 y^2$ .
98. Find the locus of the point of intersection of tangents to the hyperbola  $4x^2 - 9y^2 = 36$  which meet at a constant angle  $\frac{\pi}{4}$ . {Ans.  $(x^2 + y^2 - 5)^2 = 4(9y^2 - 4x^2 + 36)$ }
99. The chord of contact of the tangents through  $P$  to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  subtends a right angle at the centre. Prove that the locus of  $P$  is the ellipse  $b^4 x^2 + a^4 y^2 = a^2 b^2 (b^2 - a^2)$ .
100. Show that the locus of the middle points of normal chords of the rectangular hyperbola  $x^2 - y^2 = a^2$ , is the curve  $(y^2 - x^2)^3 = 4a^2 x^2 y^2$ .
101. Chords of the hyperbola  $x^2 - y^2 = a^2$  touch the parabola  $y^2 = 4ax$ . Prove that the locus of their middle points is the curve  $y^2(x - a) = x^3$ .
102. A line through the origin meets the circle  $x^2 + y^2 = a^2$  at  $P$  and the hyperbola  $x^2 - y^2 = a^2$  at  $Q$ . Prove that the locus of the point of intersection of the tangent at  $P$  to the circle with the tangent at  $Q$  to the hyperbola is the curve  $(a^4 + 4y^4)x^2 = a^6$ .
103. A variable straight line of slope 4 intersects the hyperbola  $xy = 1$  at two points. Find the locus of the point which divides the line segment between these two point in the ratio 1:2. {Ans.  $16x^2 + y^2 + 10xy - 2 = 0$ }
104. Find the locus of the midpoints of the chords of the circle  $x^2 + y^2 = 16$ , which are tangent to the hyperbola  $9x^2 - 16y^2 = 144$ . {Ans.  $(x^2 + y^2)^2 = 16x^2 - 9y^2$ }

105. The angle between a pair of tangents drawn from a point  $P$  to the parabola  $y^2 = 4ax$  is at  $45^\circ$ . Show that the locus of the point  $P$  is a hyperbola.
106. Prove that the locus of the point of intersection of the lines  $\sqrt{3}x - y - 4\sqrt{3}k = 0$  and  $\sqrt{3}kx + ky - 4\sqrt{3} = 0$  for different values of  $k$  is a hyperbola whose eccentricity is 2.
107.  $OA, OB$  are fixed straight lines,  $P$  is any point and  $PM, PN$  are the perpendiculars from  $P$  on  $OA, OB$ . Find the locus of  $P$  if the quadrilateral  $OMPN$  is of constant area. {Ans. Hyperbola}
108. Show that the locus of the foot of perpendicular drawn from the center of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , on any tangent to it is  $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$ .
109. A point  $P$  moves such that the tangents  $PT_1$  and  $PT_2$  from it to the hyperbola  $4x^2 - 9y^2 = 36$  are mutually perpendicular. Find the equation of the locus of  $P$ . {Ans.  $x^2 + y^2 = 5$ }
110. Show that the locus of the middle points of the portions of the tangents to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  intercepted between the axes is  $4x^2y^2 = a^2y^2 - b^2x^2$ .
111. The normal  $P$  to a hyperbola of eccentricity  $e$ , intersects its transverse and conjugate axes at  $Q$  and  $R$  respectively. Show that the locus of the middle point of  $QR$  is a hyperbola of eccentricity  $\frac{e}{\sqrt{e^2 - 1}}$ .
112. A tangent to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , cuts the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , in  $P$  and  $Q$ . Find the locus of the middle points of  $PQ$ . {Ans.  $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ }

### CATEGORY-18.12. ADDITIONAL QUESTIONS

# Mathematics for IIT-JEE

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*PART-V*  
*VECTOR AND THREE DIMENSIONAL GEOMETRY*

*CHAPTER-19*  
*VECTOR*

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## **CHAPTER-19**

### **VECTOR**

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#### **LIST OF THEORY SECTIONS**

- 19.1. Definition Of Vector And Its Properties
- 19.2. Multiplication Of A Vector By A Scalar
- 19.3. Addition And Subtraction Of Two (Or More) Vectors
- 19.4. Scalar (Dot) Product Of Two Vectors
- 19.5. Vector (Cross) Product Of Two Vectors
- 19.6. Scalar Triple Product Of Three Vectors
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#### **LIST OF QUESTION CATEGORIES**

- 19.1. Modulus, Equality, Scalar Multiplication, Addition And Subtraction Of Vectors
- 19.2. Direction Cosines Of Vectors
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- 19.4. Vector Bisecting Angle Between Two Vectors
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- 19.19. Shortest Distance Between Parallel And Skew Lines
- 19.20. Equation Of A Plane
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19.24. Geometrical Operations On Vectors

19.25. Additional Questions

## CHAPTER-19

### VECTOR

#### SECTION-19.1. DEFINITION OF VECTOR AND ITS PROPERTIES

##### 1. Scalar quantity

A quantity which is completely expressed by a single real number that gives its magnitude and is not related to any direction in space, is called a scalar quantity.

##### 2. Vector quantity

A quantity which is expressed by a real number, that gives its magnitude, and by a direction in space and follows law of parallelogram of addition, is called a vector quantity.

##### 3. Notation of vectors

Vector quantities are represented by either bold letters **a**, **b**,..... or  $\vec{a}$ ,  $\vec{b}$ ,..... or  $\bar{a}$ ,  $\bar{b}$ ,.....

##### 4. Representation of vectors

A vector is represented by a directed line segment. The length of the line segment is the magnitude of the vector and the direction of the line segment is the direction of the vector in space.

##### 5. Magnitude (Modulus) of a vector

The magnitude (modulus) of a vector  $\vec{a}$  is a non-negative real number and is denoted by  $|\vec{a}|$ .

##### 6. Equality of vectors

Two vectors  $\vec{a}$  and  $\vec{b}$  are said to be equal, denoted as  $\vec{a} = \vec{b}$ , iff they have same magnitude and same direction; otherwise they are said to be unequal, denoted as  $\vec{a} \neq \vec{b}$ .

##### 7. Unit vectors

- i. A vector whose modulus is unity, is called a unit vector. Unit vectors are denoted by  $\hat{a}$ ,  $\hat{b}$ ,.....
- ii.  $|\hat{a}| = 1$ .

##### 8. Zero (Null) vector

- i. A vector whose modulus is zero, is called a zero (null) vector. A zero vector can assume any direction. A zero vector is represented by  $\vec{0}$ .
- ii.  $|\vec{0}| = 0$ .

##### 9. Like and unlike vectors

Two (or more) vectors are said to be like when they have same direction and unlike when they have opposite direction.

##### 10. Collinear vectors

Two (or more) vectors having the same or opposite direction, are said to be collinear vectors.

##### 11. Coplanar vectors

Three (or more) vectors which lie on the same plane, are said to be coplanar vectors.

##### 12. Angle between two vectors

Angle between two vectors,  $\theta$ , is the angle between their directions such that  $0 \leq \theta \leq \pi$ .

#### SECTION-19.2. MULTIPLICATION OF A VECTOR BY A SCALAR

##### 1. Definition

- i. If  $m$  be a non-zero scalar and  $\vec{a}$  be a vector, then multiplication of  $\vec{a}$  with  $m$ , denoted by  $m\vec{a}$ , is defined as a vector whose magnitude is  $|m|$  times the magnitude of  $\vec{a}$  and whose direction is same as that of  $\vec{a}$  if



$m$  is positive and opposite to that of  $\vec{a}$  if  $m$  is negative. If  $m$  is zero, then  $0\vec{a} = \vec{0}$ . If  $\vec{a}$  is zero vector, then  $m\vec{0} = \vec{0}$ .

ii.  $(-1)\vec{a}$  is denoted as  $-\vec{a}$  and is called the negative of  $\vec{a}$ .  $-\vec{a}$  vector has the same magnitude as the vector  $\vec{a}$  but opposite direction. Therefore,  $|- \vec{a}| = |\vec{a}|$ .

iii.  $\hat{a} = \frac{1}{|\vec{a}|} \vec{a}$  is the unit vector in the direction of  $\vec{a}$ .

## 2. Properties

i.  $0\vec{a} = \vec{0}$ .

ii.  $m\vec{0} = \vec{0}$ .

iii.  $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$ .

iv.  $|m\vec{a}| = |m||\vec{a}|$ .

v.  $|- \vec{a}| = |\vec{a}|$ .

## 3. Theorem of Collinearity

If  $\vec{a}$  and  $\vec{b}$  are non-zero collinear vectors and  $\lambda$  is scalar, then vector  $\vec{b}$  can always be uniquely expressed

as  $\vec{b} = \lambda\vec{a}$ , where  $\lambda = \pm \frac{|\vec{b}|}{|\vec{a}|}$ , i.e.  $\vec{b} = \left( \pm \frac{|\vec{b}|}{|\vec{a}|} \right) \vec{a}$ . But if  $\vec{a}$  and  $\vec{b}$  are non-collinear vectors, then  $\vec{b} \neq \lambda\vec{a}$  for

any scalar  $\lambda$ .

## SECTION-19.3. ADDITION AND SUBTRACTION OF TWO (OR MORE) VECTORS

### 1. Definition

i. Sum of two (or more) vectors,  $\vec{a}$  and  $\vec{b}$ , denoted by  $\vec{a} + \vec{b}$ , is a vector determined by law of parallelogram, law of triangle and law of polygon of vectors.

ii. Subtraction of  $\vec{b}$  from  $\vec{a}$ , denoted as  $\vec{a} - \vec{b}$ , is defined as vector sum of  $\vec{a}$  and  $-\vec{b}$ .

### 2. Properties

i.  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

ii.  $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{c}) + \vec{b} = \vec{a} + \vec{b} + \vec{c}$

iii.  $\vec{a} + \vec{0} = \vec{a}$

iv.  $\vec{a} - \vec{a} = \vec{0}$

v.  $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

vi.  $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

### 3. Theorem of coplanarity and resolution of a vector w.r.t. system vectors

i. If  $\vec{a}$  and  $\vec{b}$  be any two non-collinear vectors, then any vector  $\vec{r}$  which is coplanar with  $\vec{a}$  and  $\vec{b}$ , can be uniquely expressed as  $\vec{r} = x\vec{a} + y\vec{b}$ , where  $x$  and  $y$  are unique scalars.  $x\vec{a}$  and  $y\vec{b}$  are called components of  $\vec{r}$  w.r.t. the system vectors  $\vec{a}$  and  $\vec{b}$ .

ii. If  $\vec{a}$  and  $\vec{b}$  be any two non-collinear vectors and  $\vec{r}$  is any vector non-coplanar with  $\vec{a}$  and  $\vec{b}$ , then  $\vec{r} \neq x\vec{a} + y\vec{b}$  for any scalars  $x$  and  $y$ .

iii. If  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are any three non-collinear, non-coplanar vectors, then any vector  $\vec{r}$ , can be uniquely

expressed as  $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ , where  $x, y$  and  $z$  are unique scalars.  $x\vec{a}$ ,  $y\vec{b}$  and  $z\vec{c}$  are called components of  $\vec{r}$  w.r.t. the system vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ .

#### 4. Theorem of equality of vectors in terms of components

- If  $\vec{a}$  and  $\vec{b}$  are any two non-collinear vectors and  $x$  and  $y$  are scalars, then  $x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = 0, y = 0$ .
- If  $\vec{a}$  and  $\vec{b}$  are any two non-collinear vectors and  $x_1, y_1, x_2, y_2$  are scalars, then  $x_1\vec{a} + y_1\vec{b} = x_2\vec{a} + y_2\vec{b} \Rightarrow x_1 = x_2, y_1 = y_2$ .
- If  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are any three non-collinear, non-coplanar vectors and  $x, y$  and  $z$  are scalars, then  $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x = 0, y = 0, z = 0$ .
- If  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are any three non-collinear, non-coplanar vectors and  $x_1, y_1, z_1, x_2, y_2, z_2$  are scalars, then  $x_1\vec{a} + y_1\vec{b} + z_1\vec{c} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c} \Rightarrow x_1 = x_2, y_1 = y_2, z_1 = z_2$ .

#### 5. $\hat{i}, \hat{j}, \hat{k}$ system

- $\hat{i}, \hat{j}, \hat{k}$  are mutually perpendicular, unit vectors forming a right-handed system.
- Any vector  $\vec{r}$ , can be uniquely expressed as  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$  or  $(x, y, z)$ .

#### 6. Modulus, equality, scalar multiplication, addition and subtraction of vectors in $\hat{i}, \hat{j}, \hat{k}$ system

- If  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ , then  $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$
- $x\hat{i} + y\hat{j} + z\hat{k} = \vec{0} \Rightarrow x = 0, y = 0, z = 0$ .  $x_1\hat{i} + y_1\hat{j} + z_1\hat{k} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k} \Rightarrow x_1 = x_2, y_1 = y_2, z_1 = z_2$ .
- If  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$  and  $m$  is a scalar, then  $m\vec{r} = (mx)\hat{i} + (my)\hat{j} + (mz)\hat{k}$ .
- If  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$  then  $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) \hat{i} + \left( \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) \hat{j} + \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \hat{k}$ .
- If  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$  then  $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$  and  $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$ .

#### 7. Direction cosines of a vector

- If  $\alpha, \beta, \gamma$  are the angles which a vector  $\vec{r}$  makes with the system vectors  $\hat{i}, \hat{j}, \hat{k}$ , then  $\cos \alpha, \cos \beta, \cos \gamma$  are known as the direction cosines of  $\vec{r}$  and are denoted by  $l, m, n$  respectively, i.e.  $l = \cos \alpha, m = \cos \beta, n = \cos \gamma$ .
- Direction cosines of vector  $\hat{i}$  are 1, 0, 0; direction cosines of vector  $\hat{j}$  are 0, 1, 0; direction cosines of vector  $\hat{k}$  are 0, 0, 1.
- If  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ , then  $\cos \alpha = \frac{x}{|\vec{r}|} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \cos \beta = \frac{y}{|\vec{r}|} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \cos \gamma = \frac{z}{|\vec{r}|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$  and  $\alpha = \cos^{-1} \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \beta = \cos^{-1} \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \gamma = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}$ .
- $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$  or  $l^2 + m^2 + n^2 = 1$ .

v.  $\vec{r} = (|\vec{r}|\cos\alpha)\hat{i} + (|\vec{r}|\cos\beta)\hat{j} + (|\vec{r}|\cos\gamma)\hat{k}$  and  $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$ .

vi. Like vectors have same direction cosines.

### 8. Linearly dependent and linearly independent vectors

- i. A vector  $\vec{a}$  is said to be a linear combination of vectors  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$  if there exists scalars  $x_1, x_2, \dots, x_n$ , such that
- $$\vec{a} = x_1\vec{r}_1 + x_2\vec{r}_2 + \dots + x_n\vec{r}_n.$$
- ii. A set of non-zero vectors  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$  is said to be linearly independent, if
- $$x_1\vec{r}_1 + x_2\vec{r}_2 + \dots + x_n\vec{r}_n = \vec{0} \Rightarrow x_1 = x_2 = \dots = x_n = 0.$$
- iii. A set of non-zero vectors  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$  is said to be linearly dependent, if there exists scalars  $x_1, x_2, \dots, x_n$  not all zero, such that
- $$x_1\vec{r}_1 + x_2\vec{r}_2 + \dots + x_n\vec{r}_n = \vec{0}.$$

### SECTION-19.4. SCALAR (DOT) PRODUCT OF TWO VECTORS

#### 1. Definition

Let  $\vec{a}$  and  $\vec{b}$  be two non-zero vectors and  $\theta$  be the angle between them, then the scalar product of  $\vec{a}$  and  $\vec{b}$ , denoted by  $\vec{a} \cdot \vec{b}$ , is defined as the scalar  $|\vec{a}||\vec{b}|\cos\theta$ . If  $\vec{a}$  or  $\vec{b}$  or both is a zero vector, then  $\vec{a} \cdot \vec{b}$  and is defined as scalar zero.

#### 2. Properties

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- $\vec{0} \cdot \vec{a} = 0$
- If  $\vec{a} \perp \vec{b}$ , then  $\vec{a} \cdot \vec{b} = 0$  (Test of perpendicularity)
- If  $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} = \vec{0}$  or  $\vec{b} = \vec{0}$  or  $\vec{a} \perp \vec{b}$
- If  $\theta$  is acute angle  $\Leftrightarrow \vec{a} \cdot \vec{b} > 0$ . If  $\theta$  is obtuse angle  $\Leftrightarrow \vec{a} \cdot \vec{b} < 0$ .
- $(m\vec{a}) \cdot (n\vec{b}) = (mn\vec{a}) \cdot \vec{b} = \vec{a} \cdot (mn\vec{b}) = mn(\vec{a} \cdot \vec{b})$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$
- $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$
- $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$
- $\hat{i} \cdot \hat{i} = 1 = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k}; \hat{i} \cdot \hat{j} = 0 = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k}$
- If  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$  then  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ .

#### 3. Angle between two vectors

i.  $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}}$

$$\Rightarrow \theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right) = \cos^{-1} \left( \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}} \right).$$

ii. If  $\vec{a} \perp \vec{b}$  then  $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$ .

#### 4. Vector bisecting angle between two vectors

A vector which bisects the angle between the vectors  $\vec{a}$  and  $\vec{b}$  is  $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$ . A vector which bisects the angle

between the vectors  $\vec{a}$  and  $-\vec{b}$  is  $\frac{\vec{a}}{|\vec{a}|} - \frac{\vec{b}}{|\vec{b}|}$ .

#### 5. Projection of a vector on a vector

Projection of vector  $\vec{a}$  on vector  $\vec{b}$  is denoted by  $\text{Proj}_{\vec{b}} \vec{a}$  and  $\text{Proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ .

### SECTION-19.5. VECTOR (CROSS) PRODUCT OF TWO VECTORS

#### 1. Definition

Let  $\vec{a}$  and  $\vec{b}$  be two non-zero, non-collinear vectors, then their vector product, denoted by  $\vec{a} \times \vec{b}$ , is defined as a vector whose magnitude is  $|\vec{a}| |\vec{b}| \sin \theta$ , where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ , and which is perpendicular to both  $\vec{a}$  and  $\vec{b}$  such that  $\vec{a}$ ,  $\vec{b}$  and  $\vec{a} \times \vec{b}$  forms a right-handed system. If  $\vec{a}$  or  $\vec{b}$  or both is  $\vec{0}$ , then  $\vec{a} \times \vec{b} = \vec{0}$ . If  $\vec{a}$  and  $\vec{b}$  are non-zero and collinear, then  $\vec{a} \times \vec{b} = \vec{0}$ .

#### 2. Properties

- $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ ;  $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ .
- $\vec{0} \times \vec{a} = \vec{0}$ .
- $\vec{a} \times \vec{a} = \vec{0}$ .
- If  $\vec{a}$  and  $\vec{b}$  are non-zero vectors then  $\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a}$  and  $\vec{b}$  are collinear vectors (Test of collinearity).
- $(m\vec{a}) \times (n\vec{b}) = m(n\vec{a} \times \vec{b}) = n(m\vec{a} \times \vec{b}) = mn(\vec{a} \times \vec{b})$ .
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ .
- $\hat{i} \times \hat{i} = \vec{0} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k}$ ;  $\hat{i} \times \hat{j} = \hat{k}$ ;  $\hat{j} \times \hat{k} = \hat{i}$ ;  $\hat{k} \times \hat{i} = \hat{j}$ .

$$\text{viii. If } \vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \text{ and } \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k} \text{ then } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

$$\text{ix. } \sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}.$$

$$\text{x. Unit vectors perpendicular to both } \vec{a} \text{ and } \vec{b} \text{ are } \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}.$$

### 3. Lagrange's identity

If  $\vec{a}$  and  $\vec{b}$  are any two vectors, then  $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$ .

## SECTION-19.6. SCALAR TRIPLE PRODUCT OF THREE VECTORS

### 1. Definition

- Let  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be three vectors, then scalar  $\vec{a} \cdot \vec{b} \times \vec{c}$  is called the scalar triple product of  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ .
- Let  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ ,  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$  and  $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ , then

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{a} \times \vec{b} \cdot \vec{c}.$$

- In scalar triple product, interchanging the positions of dot and cross does not change its value.  $\vec{a} \cdot \vec{b} \times \vec{c}$  or  $\vec{a} \times \vec{b} \cdot \vec{c}$  is denoted by  $[\vec{a}\vec{b}\vec{c}]$ .

### 2. Properties

- $[\vec{a}\vec{b}\vec{c}] = [\vec{b}\vec{c}\vec{a}] = [\vec{c}\vec{a}\vec{b}] = -[\vec{b}\vec{a}\vec{c}] = -[\vec{a}\vec{c}\vec{b}] = -[\vec{c}\vec{b}\vec{a}]$ .
- $[m\vec{a} \ \vec{b} \ \vec{c}] = m[\vec{a}\vec{b}\vec{c}]$ .
- $[\vec{a} + \vec{b} \ \vec{c} \ \vec{d}] = [\vec{a}\vec{c}\vec{d}] + [\vec{b}\vec{c}\vec{d}]$ .
- If any one of the vectors is a zero vector then  $[\vec{a}\vec{b}\vec{c}] = 0$ .
- If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are non-zero vectors but if any two of them are collinear then  $[\vec{a}\vec{b}\vec{c}] = 0$ .
- If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are non-zero, non-collinear vectors then  $[\vec{a}\vec{b}\vec{c}] = 0 \Leftrightarrow \vec{a}, \vec{b}, \vec{c}$  are coplanar (Test of coplanarity).
- If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are mutually perpendicular vectors, then if  $[\vec{a}\vec{b}\vec{c}] > 0 \Rightarrow$  Right-handed system & if  $[\vec{a}\vec{b}\vec{c}] < 0 \Rightarrow$  Left-handed system.
- $[\vec{a}\vec{b}\vec{c}][\vec{u}\vec{v}\vec{w}] = \begin{vmatrix} \vec{a} \cdot \vec{u} & \vec{b} \cdot \vec{u} & \vec{c} \cdot \vec{u} \\ \vec{a} \cdot \vec{v} & \vec{b} \cdot \vec{v} & \vec{c} \cdot \vec{v} \\ \vec{a} \cdot \vec{w} & \vec{b} \cdot \vec{w} & \vec{c} \cdot \vec{w} \end{vmatrix}$ .

## SECTION-19.7. VECTOR TRIPLE PRODUCT OF THREE VECTORS

### 1. Definition

- Let  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be three vectors, then the vector  $\vec{a} \times (\vec{b} \times \vec{c})$  or  $(\vec{a} \times \vec{b}) \times \vec{c}$  is called the vector triple product of  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ .
- The vector  $\vec{a} \times (\vec{b} \times \vec{c})$  is perpendicular to  $\vec{a}$  and lies in the plane of  $\vec{b}$  and  $\vec{c}$ .

### 2. Properties

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
- $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

## SECTION-19.8. RECIPROCAL SYSTEM OF VECTORS

### 1. Definition

Three non-coplanar vectors  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  and three non-coplanar vectors  $\vec{a}'$ ,  $\vec{b}'$ ,  $\vec{c}'$  form a reciprocal system of vectors if  $\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$  and  $\vec{a} \cdot \vec{b}' = \vec{a} \cdot \vec{c}' = \vec{b} \cdot \vec{a}' = \vec{b} \cdot \vec{c}' = \vec{c} \cdot \vec{a}' = \vec{c} \cdot \vec{b}' = 0$ .

## 2. Properties

- $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\vec{b}\vec{c}]}$ ,  $\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\vec{b}\vec{c}]}$ ,  $\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\vec{b}\vec{c}]}$
- $[\vec{a}'\vec{b}'\vec{c}'] = \frac{1}{[\vec{a}\vec{b}\vec{c}]}$
- $\hat{i}$ ,  $\hat{j}$ ,  $\hat{k}$  is its own reciprocal.

## SECTION-19.9. VECTOR EQUATIONS

If one or more vector equation(s) are given containing one or more unknown vector(s) and some known vectors and scalars, then unknown vector(s) can be determined in terms of known vectors and scalars.

## SECTION-19.10. POINTS

### 1. Representation of a point in three-dimensional space by position vector

If a point  $O$  is fixed as the origin and  $P$  is any point in space, then the vector  $\vec{OP}$  is called the position vector of the point  $P$  with respect to  $O$ .

### 2. Equation of a curve/ region

The equation of a curve/ region is the relation involving the position vector of every point on the curve/ region, and which holds for no other point except for those lying on the curve/ region.

### 3. Distance formula

- If  $\vec{a}$  is the p.v. of point  $A$  and  $\vec{b}$  is the p.v. of point  $B$ , then distance between the points  $A$  and  $B$  is  $|\vec{a} - \vec{b}|$ .
- Distance of the point  $A$  from the origin  $= |\vec{a}|$ .

### 4. Section formula

- Let  $A$  and  $B$  be two points with position vectors  $\vec{a}$  and  $\vec{b}$  respectively and let  $C$  be a point dividing  $AB$  internally in the ratio  $1 : \lambda$ , then the position vector of point  $C$  is given by  $\vec{OC} = \frac{\lambda\vec{a} + \vec{b}}{\lambda + 1}$ .
- If  $C$  is the mid-point of  $AB$ , then  $\vec{OC} = \frac{\vec{a} + \vec{b}}{2}$ .
- If  $C$  divides  $AB$  externally in the ratio  $1 : \lambda$ , then  $\vec{OC} = \frac{\lambda\vec{a} - \vec{b}}{\lambda - 1}$ .

### 5. Collinearity of three points

- Let  $A, B, C$  be three points with position vectors  $\vec{a}, \vec{b}, \vec{c}$  respectively, then points  $A, B, C$  are collinear iff vectors  $\vec{b} - \vec{a}$  and  $\vec{c} - \vec{a}$  are collinear i.e.  $\vec{b} - \vec{a} = \lambda(\vec{c} - \vec{a})$  for some scalar  $\lambda$  or  $(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \vec{0}$ .
- Points  $A, B, C$  are collinear iff there exists scalars  $x, y, z$  not all zero such that  $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ , where  $x + y + z = 0$ .

### 6. Coplanarity of four points

- Let  $A, B, C, D$  be four points with position vectors  $\vec{a}, \vec{b}, \vec{c}, \vec{d}$  respectively, then points  $A, B, C, D$  are

coplanar iff vectors  $\vec{b} - \vec{a}$ ,  $\vec{c} - \vec{a}$ ,  $\vec{d} - \vec{a}$  are coplanar i.e.  $\vec{b} - \vec{a} = \lambda(\vec{c} - \vec{a}) + \mu(\vec{d} - \vec{a})$  for some scalars  $\lambda$  and  $\mu$  or  $\begin{vmatrix} \vec{b} - \vec{a} & \vec{c} - \vec{a} & \vec{d} - \vec{a} \end{vmatrix} = 0$ .

- ii. Points  $A, B, C, D$  are coplanar iff there exists scalars  $x, y, z, u$  not all zero such that  $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$ , where  $x + y + z + u = 0$ .

### 7. Centroid of a triangle

Let  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be the position vectors of the vertices  $A, B, C$  of a triangle, then the position vector of the centroid  $\vec{OG}$  is given by  $\vec{OG} = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$ .

### 8. Centroid of a Tetrahedron

Let  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ ,  $\vec{d}$  be the position vectors of the vertices  $A, B, C, D$  of a tetrahedron, then the position vector of the centroid  $\vec{OG}$  is given by  $\vec{OG} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$ .

## SECTION-19.11. AREA AND VOLUME

### 1. Area of triangle, parallelogram and quadrilateral

- i. Area of  $\triangle ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$ .
- ii. Area of parallelogram  $ABCD = |\vec{AB} \times \vec{AD}| = \frac{1}{2} |\vec{AC} \times \vec{BD}|$ .
- iii. Area of quadrilateral  $ABCD = \frac{1}{2} |\vec{AC} \times \vec{BD}|$ .

### 2. Volume of three dimensional objects

- i. Volume of a Parallelopiped whose coterminous edges are  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c} = \left| [\vec{a} \ \vec{b} \ \vec{c}] \right|$
- ii. Volume of a Tetrahedron whose coterminous edges are  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c} = \frac{1}{6} \left| [\vec{a} \ \vec{b} \ \vec{c}] \right|$
- iii. Volume of a Triangular prism whose coterminous edges are  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c} = \frac{1}{2} \left| [\vec{a} \ \vec{b} \ \vec{c}] \right|$

## SECTION-19.12. STRAIGHT LINES

### 1. Equation of a line in various forms

- i. Vector equation of a line passing through a point having p.v.  $\vec{a}$  and parallel to gradient vector  $\vec{t}$  is  $\vec{r} = \vec{a} + \lambda \vec{t}$ , where  $\lambda$  is a scalar.
- ii. Vector equation of a line passing through two points having p.v.  $\vec{a}$  and  $\vec{b}$  is  $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ .  $\vec{b} - \vec{a}$  is a gradient vector of the line.

### 2. Distance of a point from a line

Distance of a point  $P$  having p.v.  $\vec{b}$  from the line  $\vec{r} = \vec{a} + \lambda \vec{t}$  is  $\frac{|(\vec{b} - \vec{a}) \times \vec{t}|}{|\vec{t}|}$ .

### 3. Foot of the perpendicular drawn from a given point on a given line

Let  $\vec{x}$  be the p.v. of foot of perpendicular drawn from a point having p.v.  $\vec{b}$  on a line  $\vec{r} = \vec{a} + \lambda \vec{t}$ , then

$$\left\{ \begin{array}{l} \vec{x} = \vec{a} + \alpha \vec{t} \\ (\vec{x} - \vec{b}) \cdot \vec{t} = 0 \end{array} \right\}.$$

Solve for scalar  $\alpha$  and find  $\vec{x}$ .

#### 4. Angle between two lines

i. Angle between two lines  $\vec{r} = \vec{a}_1 + \lambda \vec{t}_1$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{t}_2$  is the angle between their gradient vectors  $\vec{t}_1$  and  $\vec{t}_2$ .

ii. If  $\theta$  is the acute angle between the given lines, then

$$\cos \theta = \frac{|\vec{t}_1 \cdot \vec{t}_2|}{|\vec{t}_1| |\vec{t}_2|} \Rightarrow \theta = \cos^{-1} \left( \frac{|\vec{t}_1 \cdot \vec{t}_2|}{|\vec{t}_1| |\vec{t}_2|} \right).$$

iii. Lines are perpendicular if  $\vec{t}_1 \cdot \vec{t}_2 = 0$ .

iv. Lines are parallel if  $\vec{t}_1 = \alpha \vec{t}_2$ , for some scalar  $\alpha$ .

#### 5. Point of intersection of two lines

Let  $\vec{x}$  be the p.v. of point of intersection of two lines  $\vec{r} = \vec{a}_1 + \lambda \vec{t}_1$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{t}_2$ , then  $\left\{ \begin{array}{l} \vec{x} = \vec{a}_1 + \alpha \vec{t}_1 \\ \vec{x} = \vec{a}_2 + \beta \vec{t}_2 \end{array} \right\}$ .

Solve for scalars  $\alpha$  and  $\beta$ . If there is no value of  $\alpha$  and  $\beta$ , then the lines do not intersect. If there is value of  $\alpha$  and  $\beta$ , then lines intersect and find  $\vec{x}$ .

#### 6. Equation of angular bisectors of two intersecting lines

Equation of angular bisectors of two intersecting lines  $\vec{r} = \vec{a}_1 + \lambda \vec{t}_1$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{t}_2$  are

$$\vec{r} = \vec{a}_0 + \lambda \left( \frac{\vec{t}_1}{|\vec{t}_1|} \pm \frac{\vec{t}_2}{|\vec{t}_2|} \right), \text{ where } \vec{a}_0 \text{ is the p.v. of their point of intersection.}$$

#### 7. Shortest distance between two parallel lines

Shortest distance between two parallel lines  $\vec{r} = \vec{a}_1 + \lambda \vec{t}$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{t}$  is  $\frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{t}|}{|\vec{t}|}$ .

#### 8. Shortest distance between two skew lines

i. Two straight lines in space which are neither parallel nor intersecting are called skew lines, i.e. the skew lines are those lines which do not lie in the same plane.

ii. The shortest distance between the two skew lines is the length of their common perpendicular line segment.

iii. Let the two skew lines be  $\vec{r} = \vec{a}_1 + \lambda \vec{t}_1$  and  $\vec{r} = \vec{a}_2 + \lambda \vec{t}_2$ , then the shortest distance is the projection of

$$\vec{a}_2 - \vec{a}_1 \text{ on } \vec{t}_1 \times \vec{t}_2, \text{ which is } \frac{|(\vec{t}_1 \times \vec{t}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{t}_1 \times \vec{t}_2|} \text{ or } \frac{|[\vec{t}_1 \ \vec{t}_2 \ \vec{a}_2 - \vec{a}_1]|}{|\vec{t}_1 \times \vec{t}_2|}.$$

iv. If  $[\vec{t}_1 \ \vec{t}_2 \ \vec{a}_2 - \vec{a}_1] \neq 0$  then lines are skew, if  $= 0$  then lines are intersecting/parallel.

### SECTION-19.13. PLANES

#### 1. Equation of a plane in various forms

i. The vector equation of a plane passing through a point having position vector  $\vec{a}$  and normal vector  $\vec{n}$  is  $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$  or  $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$  or  $\vec{r} \cdot \hat{n} = \vec{a} \cdot \hat{n}$  (normal form), where  $\hat{n}$  is a normal unit vector.



- ii. Distance of the plane from origin,  $p = |\vec{a} \cdot \hat{n}|$ .
- iii. Therefore, the vector equation of planes having normal unit vector  $\hat{n}$  and a distance  $p$  from origin  $\vec{r} \cdot \hat{n} = \pm p$ .
- iv. The vector equation of a plane passing through a point having p.v.  $\vec{a}$  and parallel to  $\vec{b}$  and  $\vec{c}$  is  $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$  (parametric form), where  $\lambda$  and  $\mu$  are scalars, or  $[\vec{r} - \vec{a} \quad \vec{b} \quad \vec{c}] = 0$  (non-parametric form). A normal vector to the plane is  $\vec{n} = \vec{b} \times \vec{c}$ .
- v. The vector equation of a plane passing through three points  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  is  $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) + \mu(\vec{c} - \vec{a})$  (parametric form), where  $\lambda$  and  $\mu$  are scalars, or  $[\vec{r} - \vec{a} \quad \vec{b} - \vec{a} \quad \vec{c} - \vec{a}] = 0$  (non-parametric form). A normal vector to the plane is  $\vec{n} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$ .
- vi. To find the equation of the plane  $\vec{r} \cdot \hat{n} = p$  in parametric form, find any three points on the plane and find the parametric equation of the plane through these three points.

## 2. Distance of a point from a plane

The length of perpendicular from the point having p.v.  $\vec{b}$  to the plane  $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$  is  $\frac{|(\vec{b} - \vec{a}) \cdot \vec{n}|}{|\vec{n}|}$  or  $|(\vec{b} - \vec{a}) \cdot \hat{n}|$ .

## 3. Foot of perpendicular from a point on a plane

Let  $\vec{x}$  be the p.v. of the foot of perpendicular drawn from the point having p.v.  $\vec{b}$  on the plane

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0, \text{ then } \begin{cases} (\vec{x} - \vec{a}) \cdot \vec{n} = 0 \\ \vec{x} - \vec{b} = \lambda \vec{n} \end{cases}. \text{ Solve for } \vec{x}.$$

## 4. Angle between a line and a plane

- i. The angle between a line and a plane is the complement of the angle between the line and normal to the plane.
- ii. If  $\theta$  is the angle between the line  $\vec{r} = \vec{a} + \lambda\vec{t}$  and the plane  $(\vec{r} - \vec{b}) \cdot \vec{n} = 0$ , then

$$\sin \theta = \frac{|\vec{t} \cdot \vec{n}|}{|\vec{t}| |\vec{n}|} \Rightarrow \theta = \sin^{-1} \left( \frac{|\vec{t} \cdot \vec{n}|}{|\vec{t}| |\vec{n}|} \right).$$

- iii. If line is perpendicular to the plane then  $\vec{t}$  and  $\vec{n}$  are collinear i.e.  $\vec{t} = \alpha \vec{n}$ , for some scalar  $\alpha$ .
- iv. If line is parallel to the plane, then it is perpendicular to the normal to the plane i.e.  $\vec{t} \cdot \vec{n} = 0$ .

## 5. Condition for a line to lie in a plane

A line  $\vec{r} = \vec{a} + \lambda\vec{t}$  lies in the plane  $(\vec{r} - \vec{b}) \cdot \vec{n} = 0$  iff  $\vec{t} \cdot \vec{n} = 0$  and  $(\vec{a} - \vec{b}) \cdot \vec{n} = 0$ . If  $\vec{t} \cdot \vec{n} = 0$  but  $(\vec{a} - \vec{b}) \cdot \vec{n} \neq 0$ , then the line is parallel to the plane but not lying in the plane.

## 6. Condition of coplanarity of two lines and equation of the plane containing them

- i. Lines  $\vec{r} = \vec{a}_1 + \lambda\vec{t}_1$  and  $\vec{r} = \vec{a}_2 + \lambda\vec{t}_2$  are coplanar (intersecting/parallel) if  $[\vec{t}_1 \quad \vec{t}_2 \quad \vec{a}_2 - \vec{a}_1] = 0$ .
- ii. If lines are intersecting, then the equation of the plane containing them is  $\vec{r} = \vec{a}_1 + \lambda\vec{t}_1 + \mu\vec{t}_2$  or  $[\vec{r} - \vec{a}_1 \quad \vec{t}_1 \quad \vec{t}_2] = 0$ .
- iii. If lines are parallel, i.e.  $\vec{t}_1 = \alpha\vec{t}_2$  for some scalar  $\alpha$ , then the equation of the plane containing them is

$$\vec{r} = \vec{a}_1 + \lambda(\vec{a}_2 - \vec{a}_1) + \mu\vec{t}_1 \text{ or } [\vec{r} - \vec{a}_1 \quad \vec{a}_2 - \vec{a}_1 \quad \vec{t}_1] = 0.$$

### 7. Point of intersection of a line and a plane which are not parallel

- A line intersects a plane at only one point if the line is not parallel to the plane, i.e.  $\vec{t} \cdot \vec{n} \neq 0$ .
- Let  $\vec{x}$  be the p.v. of the point of intersection of the line  $\vec{r} = \vec{a} + \lambda\vec{t}$  and the plane  $(\vec{r} - \vec{b}) \cdot \vec{n} = 0$  ( $\vec{t} \cdot \vec{n} \neq 0$ ),

$$\text{then } \begin{cases} \vec{x} = \vec{a} + \lambda\vec{t} \\ (\vec{x} - \vec{b}) \cdot \vec{n} = 0 \end{cases} \text{ Solve for } \vec{x}.$$

### 8. Angle between two planes

- The angle between two planes is defined as the angle between their normals.
- The angle,  $\theta$ , between the planes  $(\vec{r} - \vec{a}) \cdot \vec{n}_1 = 0$  and  $(\vec{r} - \vec{b}) \cdot \vec{n}_2 = 0$  is given by

$$\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|} \Rightarrow \theta = \cos^{-1} \left( \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|} \right).$$

- The planes are perpendicular if  $\vec{n}_1$  and  $\vec{n}_2$  are perpendicular, i.e.  $\vec{n}_1 \cdot \vec{n}_2 = 0$ .
- The plane are parallel if  $\vec{n}_1$  and  $\vec{n}_2$  are parallel, i.e.  $\vec{n}_1 = \alpha \vec{n}_2$ , where  $\alpha$  is a scalar.

### 9. Line of intersection of two non-parallel planes

- Two non-parallel planes intersect along a line.
- Gradient vector of the line of intersection is perpendicular to the normal vectors of the planes, i.e. if  $\vec{n}_1$  and  $\vec{n}_2$  are the normal vectors of the planes then a gradient vector of the line of intersection is  $\vec{n}_1 \times \vec{n}_2$ .
- To find the equation of line of intersection, find a point which lies on both the planes, i.e. lies on the line of intersection. Gradient vector of the line of intersection is  $\vec{n}_1 \times \vec{n}_2$ . Find the equation of the line of intersection.

### 10. Equation of a plane passing through the intersection of two given planes

The equation of a plane passing through the intersection of the planes  $\vec{r} \cdot \vec{n}_1 = p_1$  and  $\vec{r} \cdot \vec{n}_2 = p_2$  is  $(\vec{r} \cdot \vec{n}_1 - p_1) + \lambda(\vec{r} \cdot \vec{n}_2 - p_2) = 0$  or  $\vec{r} \cdot (\vec{n}_1 + \lambda\vec{n}_2) = p_1 + \lambda p_2$ , where  $\lambda$  is any scalar parameter.

## EXERCISE-19

### CATEGORY-19.1. MODULUS, EQUALITY, SCALAR MULTIPLICATION, ADDITION AND SUBTRACTION OF VECTORS

- Given vectors  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$  and  $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ , find the following:-
  - $|\mathbf{a}|$ . {Ans.  $\sqrt{14}$ }
  - $|\mathbf{b}|$ . {Ans. 3}
  - $2\mathbf{a}$ . {Ans.  $4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ }
  - $-3\mathbf{b}$ . {Ans.  $-3\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$ }
  - $\mathbf{a} + \mathbf{b}$ . {Ans.  $3\mathbf{i} + \mathbf{j} + \mathbf{k}$ }
  - $\mathbf{a} - \mathbf{b}$ . {Ans.  $\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ }
  - $4\mathbf{a} + 5\mathbf{b}$ . {Ans.  $13\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ }
  - $|3\mathbf{a} - 2\mathbf{b}|$ . {Ans.  $3\sqrt{26}$ }
- If  $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$  and  $\vec{b} = 3\hat{i} + 6\hat{j} + 2\hat{k}$ , then find the vector in the direction of  $\vec{a}$  and having magnitude as  $|\vec{b}|$ . {Ans.  $\frac{7}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$ }
- If  $\vec{a}$  is a non-zero vector of modulus  $a$  and  $m$  is a non-zero scalar, then show that  $m\vec{a}$  is a unit vector if  $a = \frac{1}{|m|}$ .
- Show that the vector  $\cos\alpha \cos\beta \hat{i} + \cos\alpha \sin\beta \hat{j} + \sin\alpha \hat{k}$  is a unit vector.
- Resolve the vector  $\mathbf{d} = (1, 1, 1)$  into components with respect to three non-coplanar vectors  $\mathbf{a} = (1, 1, -2)$ ,  $\mathbf{b} = (1, -1, 0)$  and  $\mathbf{c} = (0, 2, 3)$ . {Ans.  $\mathbf{d} = \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b} + \frac{3}{5}\mathbf{c}$ }
- For what value of  $\alpha$  are the vectors  $\mathbf{a} = (2, 3, -4)$  and  $\mathbf{b} = (\alpha, -6, 8)$  parallel? {Ans.  $-4$ }
- For what value of  $\alpha$  are the vectors  $\mathbf{l} = (6, \alpha, -8)$  and  $\mathbf{m} = (-3, -1, 4)$  parallel? {Ans.  $2$ }
- Given three vectors  $\mathbf{a} = (2, 3, -5)$ ,  $\mathbf{b} = (3, 0, 1)$  and  $\mathbf{c} = (4, -3, 2)$ . Find the components and the magnitude of the vector  $\mathbf{d} = 3\mathbf{a} + \mathbf{b} - \mathbf{c}$ . {Ans.  $\mathbf{d} = (5, 12, -16)$ ,  $|\mathbf{d}| = 5\sqrt{17}$ }
- Find the vector  $\mathbf{b} = (x, y, z)$  which is collinear with the vector  $\mathbf{a} = (2\sqrt{2}, -1, 4)$  if  $|\mathbf{b}| = 10$ . {Ans.  $(4\sqrt{2}, -2, 8)$  or  $(-4\sqrt{2}, 2, -8)$ }
- Given three vectors  $\mathbf{a} = (3, -1)$ ,  $\mathbf{b} = (1, -2)$  and  $\mathbf{c} = (-1, 7)$ . Determine the resolution of the vector  $\mathbf{p} = \mathbf{a} + \mathbf{b} + \mathbf{c}$  into components with respect to the vectors  $\mathbf{a}$  and  $\mathbf{b}$ . {Ans.  $\mathbf{p} = 2\mathbf{a} - 3\mathbf{b}$ }
- Given three nonzero vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  each two of which are non-collinear. Find their sum if  $\mathbf{a} + \mathbf{b}$  is collinear with  $\mathbf{c}$  and  $\mathbf{b} + \mathbf{c}$  is collinear with  $\mathbf{a}$ . {Ans.  $\vec{0}$ }
- What is the unit vector parallel to  $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ ? What vector should be added to  $\mathbf{a}$  so that resultant is the unit vector  $\mathbf{i}$ ? {Ans.  $-2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ }
- The vectors  $\mathbf{a}$  and  $\mathbf{b}$  are non-collinear. Find for what value of  $x$ , the vectors  $\mathbf{c} = (x - 2)\mathbf{a} + \mathbf{b}$  and  $\mathbf{d} = (2x + 1)\mathbf{a} - \mathbf{b}$  are collinear? {Ans.  $x = \frac{1}{3}$ }

14. If  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are non-coplanar vectors, prove that the following vectors are coplanar:-  
 i.  $3\mathbf{a} - 7\mathbf{b} - 4\mathbf{c}, 3\mathbf{a} - 2\mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{b} + 2\mathbf{c}$ .  
 ii.  $5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c}, 7\mathbf{a} - 8\mathbf{b} + 9\mathbf{c}, 3\mathbf{a} + 20\mathbf{b} + 5\mathbf{c}$ .
15. If  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are non-coplanar vectors, check whether vectors  $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}, -2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}, \mathbf{a} - 3\mathbf{b} + 4\mathbf{c}$  are coplanar or not. {Ans. non-coplanar}
16. Find the constant  $\lambda$  so that the vectors  $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \mathbf{c} = 3\mathbf{i} + \lambda\mathbf{j} + 5\mathbf{k}$  are coplanar. {Ans.  $-4$ }
17. If the vectors  $\vec{a} + \lambda\vec{b} + 3\vec{c}, -2\vec{a} + 3\vec{b} - 4\vec{c}$  and  $\vec{a} - 3\vec{b} + 5\vec{c}$  are coplanar, then find the value of  $\lambda$ . {Ans.  $-2$ }
18. Find all the values of  $\lambda$  such that  $(x, y, z) \neq (0, 0, 0)$  and  $(\mathbf{i} + \mathbf{j} + 3\mathbf{k})x + (3\mathbf{i} - 3\mathbf{j} + \mathbf{k})y + (-4\mathbf{i} + 5\mathbf{j})z = \lambda(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ . {Ans.  $\lambda = 0, -1$ }
19. Given that the vectors  $\vec{a}$  and  $\vec{b}$  are non-collinear, find the values of  $x$  and  $y$  for which the vector equality  $2\vec{u} - \vec{v} = \vec{w}$  holds true if  $\vec{u} = x\vec{a} + 2y\vec{b}, \vec{v} = -2y\vec{a} + 3x\vec{b}, \vec{w} = 4\vec{a} - 2\vec{b}$ . {Ans.  $x = \frac{10}{7}, y = \frac{4}{7}$ }
20. A vector  $\mathbf{A}$  has components  $A_1, A_2, A_3$  in a  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  system. The  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  system is rotated about the  $\mathbf{k}$  vector through an angle  $\pi/2$ . Find the components of  $\mathbf{A}$  in new  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  system in terms of  $A_1, A_2, A_3$ . {Ans.  $A_2, -A_1, A_3$ }

### CATEGORY-19.2. DIRECTION COSINES OF VECTORS

21. Find the direction cosines of the following vectors:-  
 i.  $\hat{i} + \hat{j}$ . {Ans.  $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$ }
- ii.  $2\hat{i} + 3\hat{j} - \hat{k}$ . {Ans.  $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}$ }
22. The component of a vector on the system vectors are  $6, -3, 2$ . Find its modulus and direction cosines. {Ans.  $7, l = \frac{6}{7}, m = -\frac{3}{7}, n = \frac{2}{7}$ }
23. Find the angles at which the vector  $2\hat{i} - \hat{j} + 2\hat{k}$  is inclined to each of the system vectors. {Ans.  $\alpha = \cos^{-1} \frac{2}{3}, \beta = \pi - \cos^{-1} \frac{1}{3}, \gamma = \cos^{-1} \frac{2}{3}$ }
24. A vector  $\vec{r}$  is inclined to  $\mathbf{i}$  at  $45^\circ$  and  $\mathbf{j}$  at  $60^\circ$ . Find the angle at which  $\vec{r}$  is inclined to  $\mathbf{k}$ . {Ans.  $60^\circ$  or  $120^\circ$ }
25. A vector  $\vec{r}$  is inclined at equal angles to  $\mathbf{i}, \mathbf{j}$  and  $\mathbf{k}$ . If the magnitude of  $\vec{r}$  is 6 units, find  $\vec{r}$ . {Ans.  $\vec{r} = \pm 2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$ }
26. If a vector makes angles  $\alpha, \beta, \gamma$  with  $\mathbf{i}, \mathbf{j}$  and  $\mathbf{k}$  respectively, prove that  $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$ .

### CATEGORY-19.3. SCALAR (DOT) PRODUCT OF TWO VECTORS, ANGLE BETWEEN TWO VECTORS

27. Calculate the scalar product of the following vectors and angle between them:-  
 i.  $\mathbf{a} = (-1, 2, -2)$  and  $\mathbf{b} = (6, 3, -6)$ . {Ans.  $12, \cos^{-1} \frac{4}{9}$ }

- ii.  $\mathbf{a} = (2, 4, 1)$  and  $\mathbf{b} = (3, 5, 7)$ . {Ans.  $33, \cos^{-1} \frac{33}{\sqrt{1743}}$ }
- iii.  $\mathbf{a} = (-2, 3, 11)$  and  $\mathbf{b} = (5, 7, -4)$ . {Ans.  $-33, \cos^{-1} \left( -\frac{11}{\sqrt{1340}} \right)$ }
- iv.  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$  and  $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ . {Ans.  $-8, \cos^{-1} \left( -\frac{8}{\sqrt{174}} \right)$ }
- v.  $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$  and  $\mathbf{b} = \mathbf{i} - \mathbf{j} - 3\mathbf{k}$ . {Ans.  $90^\circ$ }
28. Find  $(2\mathbf{a} + 3\mathbf{b}) \cdot (4\mathbf{a} - 6\mathbf{b})$ , where  $\mathbf{a}, \mathbf{b}$  are mutually perpendicular unit vectors. {Ans.  $-10$ }
29. For what value of  $\alpha$  are the vectors  $\mathbf{a} = (1, \alpha, -2)$  and  $\mathbf{b} = (\alpha, 3, -4)$  mutually perpendicular? {Ans.  $-2$ }
30. If  $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$  and  $\vec{c} = 3\hat{i} + \hat{j}$ , then find  $t$  such that  $\vec{a} + t\vec{b}$  is perpendicular to  $\vec{c}$ . {Ans.  $8$ }
31. Being given that  $|\mathbf{a}| = 2$ ,  $|\mathbf{b}| = 5$  and angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $\frac{2\pi}{3}$  find the value of  $\alpha$  for which the vectors  $\mathbf{p} = \alpha\mathbf{a} + 17\mathbf{b}$  and  $\mathbf{q} = 3\mathbf{a} - \mathbf{b}$  are perpendicular. {Ans.  $40$ }
32. The vectors  $\mathbf{a}$  and  $\mathbf{b}$  make an angle of  $120^\circ$ ,  $|\mathbf{a}| = 3$  and  $|\mathbf{b}| = 5$ . Find  $|\mathbf{a} - \mathbf{b}|$ . {Ans.  $7$ }
33. If  $|\vec{a}| = 3$ ,  $|\vec{b}| = 4$  and  $|\vec{a} + \vec{b}| = 5$ , then find  $|\vec{a} - \vec{b}|$ . {Ans.  $5$ }
34. If  $|\vec{a}| = |\vec{b}|$  then show that  $\vec{a} + \vec{b}$  is perpendicular to  $\vec{a} - \vec{b}$ .
35. If  $\vec{a}$  and  $\vec{b}$  are unit vectors, then show that  $-1 \leq \vec{a} \cdot \vec{b} \leq 1$ .
36. Find the angle between the vectors  $\mathbf{a} - \mathbf{b}$  and  $\mathbf{a} + \mathbf{b}$  if  $\mathbf{a} = (1, 2, 1)$  and  $\mathbf{b} = (2, -1, 0)$ . {Ans.  $\cos^{-1} \frac{1}{11}$ }
37. If  $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$  and  $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ , then find the angle between the vectors  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$ . {Ans.  $90^\circ$ }
38. For what values of  $x$  are the vectors  $\mathbf{a} = (x, 3, 4)$  and  $\mathbf{b} = (5, 6, 3)$  perpendicular? {Ans.  $-6$ }
39. Given three vectors  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$ , prove that the vector  $(\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}$  is perpendicular to the vector  $\mathbf{c}$ .
40. Find the angle between the vectors  $2\mathbf{a}$  and  $\frac{1}{2}\mathbf{b}$  if  $\mathbf{a} = (-4, 2, 4)$  and  $\mathbf{b} = (\sqrt{2}, -\sqrt{2}, 0)$ . {Ans.  $\frac{3\pi}{4}$ }
41. The vectors  $\mathbf{a}$  and  $\mathbf{b}$  make an angle  $\varphi = 2\pi/3$ . Being given that  $|\mathbf{a}| = 3$  and  $|\mathbf{b}| = 4$ , calculate  $(3\mathbf{a} - 2\mathbf{b}) \cdot (\mathbf{a} + 2\mathbf{b})$ . {Ans.  $-61$ }
42. Find the vector  $\mathbf{a}$  which is collinear with the vector  $\mathbf{b} = (3, 6, 6)$  and satisfies the conditions  $\mathbf{a} \cdot \mathbf{b} = 27$ . {Ans.  $\mathbf{a} = (1, 2, 2)$ }
43. Find the vector  $\mathbf{a}$  which is collinear with the vector  $\mathbf{b} = (2, -1, 0)$  if  $\mathbf{a} \cdot \mathbf{b} = 10$ . {Ans.  $\mathbf{a} = (4, -2, 0)$ }
44. Find the vector  $\mathbf{x}$  which is collinear with the vector  $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$  and satisfies the condition  $\mathbf{x} \cdot \mathbf{a} = 3$ . {Ans.  $\mathbf{x} = \left(1, \frac{1}{2}, -\frac{1}{2}\right)$ }
45. Find the vector  $\mathbf{a}$  which is collinear with the vector  $\mathbf{b} = (1, -3, 1)$  and satisfies the condition  $\mathbf{a} \cdot \mathbf{b} = 22$ . {Ans.  $\mathbf{a} = (2, -6, 2)$ }
46. The vector  $\mathbf{b}$ , which is collinear with the vector  $\mathbf{a} = (8, -10, 13)$ , makes an acute angle with the unit vector

- k. Being given that  $|\mathbf{b}| = \sqrt{37}$ , find the vector  $\mathbf{b}$ . {Ans.  $\mathbf{b} = \left(\frac{8}{3}, -\frac{10}{3}, \frac{13}{3}\right)$ }
47. If the vector  $\vec{c}$  is perpendicular to the vectors  $\vec{a} = (2, -3, 1)$ ,  $\vec{b} = (1, -2, 3)$  and satisfies the condition  $\vec{c} \cdot (\hat{i} + 2\hat{j} - 7\hat{k}) = 10$ , then find  $\vec{c}$ . {Ans.  $7\hat{i} + 5\hat{j} + \hat{k}$ }
48. Find the vector  $\mathbf{b}$  being given that it satisfies the following conditions: the scalar product  $\mathbf{b} \cdot \mathbf{c} = 3$ , where  $\mathbf{c} = (1, -1, 2)$ , the vector  $\mathbf{b}$  is perpendicular to  $\mathbf{a}$  and  $\mathbf{d}$ , where  $\mathbf{a} = (-2, -1, 1)$  and  $\mathbf{d} = (3, 5, -2)$ . {Ans.  $\mathbf{b} = \left(\frac{9}{16}, \frac{3}{16}, \frac{21}{16}\right)$ }
49. Find the cosine of the angle between the vectors  $\mathbf{p}$  and  $\mathbf{q}$  which satisfy the system of equations  $\begin{cases} 2\mathbf{p} + \mathbf{q} = \mathbf{a} \\ \mathbf{p} + 2\mathbf{q} = \mathbf{b} \end{cases}$  if it is known that the vectors  $\mathbf{a}$  and  $\mathbf{b}$  are  $\mathbf{a} = (1, 1)$  and  $\mathbf{b} = (1, -1)$ . {Ans.  $-\frac{4}{5}$ }
50. The vertex  $\mathbf{x}$  satisfies the following conditions: (a)  $\mathbf{x}$  is collinear with the vector  $\mathbf{a} = 6\mathbf{i} - 8\mathbf{j} - 7.5\mathbf{k}$ ; (b)  $\mathbf{x}$  makes an acute angle with the unit vector  $\mathbf{k}$ ; (c)  $|\mathbf{x}| = 50$ . Find the vector  $\mathbf{x}$ . {Ans.  $(-24, 32, 30)$ }
51. Given two vectors  $\mathbf{a} = (-1, 1, 1)$  and  $\mathbf{b} = (2, 0, 1)$ . Find the vector  $\mathbf{x}$  if it is known that it is coplanar with the plane of the vectors  $\mathbf{a}$  and  $\mathbf{b}$ ; is perpendicular to the vector  $\mathbf{b}$  and  $\mathbf{a} \cdot \mathbf{x} = 7$ . {Ans.  $\left(-\frac{3}{2}, \frac{5}{2}, 3\right)$ }
52. Find a unit vector coplanar with  $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$  and  $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$  and perpendicular to  $\mathbf{i} + \mathbf{j} + \mathbf{k}$ . {Ans.  $\pm \frac{1}{\sqrt{2}}(\mathbf{j} - \mathbf{k})$ }
53. Given two vectors in space:  $\mathbf{a} = (1, 1, 2)$  and  $\mathbf{b} = (-1, 3, 1)$ . Find the unit vector which lies in the plane of the vectors  $\mathbf{a}$  and  $\mathbf{b}$  and makes an angle  $\alpha = \pi/4$  with the vector  $\mathbf{a}$ . {Ans.  $\left\{\left(1 \pm \sqrt{2}\right)\frac{\sqrt{3}}{2}, \left(5 \mp 7\sqrt{2}\right)\frac{\sqrt{3}}{30}, \left(10 \pm \sqrt{2}\right)\frac{\sqrt{3}}{30}\right\}$ }
54. What angle do the unit vectors  $\mathbf{a}$  and  $\mathbf{b}$  make if the vectors  $\mathbf{c} = \mathbf{a} + 2\mathbf{b}$  and  $\mathbf{d} = 5\mathbf{a} - 4\mathbf{b}$  are known to be mutually perpendicular? {Ans.  $\frac{\pi}{3}$ }
55. Find the value of the constant  $S$  such that the scalar product of the vector  $\mathbf{i} + \mathbf{j} + \mathbf{k}$  with the unit vector parallel to the sum of the vectors  $2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$  and  $S\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$  is equal to one. {Ans.  $S = 1$ }
56. Find  $\lambda$ , if the vectors  $2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{i} - 4\mathbf{j} + \lambda\mathbf{k}$  are mutually perpendicular. {Ans.  $\lambda = 2$ }
57. Determine the values of  $c$  so that for all real  $x$ , the vectors  $c\mathbf{x}\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$  and  $x\mathbf{i} + 2\mathbf{j} + 2cx\mathbf{k}$  make an obtuse angle with each other. {Ans.  $c \in \left(-\frac{4}{3}, 0\right]$ }
58. Find the vector  $\vec{a}$  which is coplanar with the vectors  $\hat{i}$  and  $\hat{j}$ , perpendicular to the vector  $\vec{b} = 4\hat{i} - 3\hat{j} + 5\hat{k}$  and  $|\vec{a}| = |\vec{b}|$ . {Ans.  $\sqrt{2}(3\hat{i} + 4\hat{j})$  or  $-\sqrt{2}(3\hat{i} + 4\hat{j})$ }
59. Find  $\lambda$  and  $\mu$  if the vectors  $\mathbf{a} = 3\mathbf{i} + \lambda\mathbf{j} - \mathbf{k}$  is perpendicular to the vector  $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mu\mathbf{k}$  and  $|\mathbf{a}| = |\mathbf{b}|$ . {Ans.  $\lambda = -\frac{31}{12}, \mu = \frac{41}{12}$ }
60. Find a vector of magnitude  $\sqrt{51}$  which makes equal angles with the vectors  $\mathbf{a} = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ ,  $\mathbf{b} = \frac{1}{5}(-4\mathbf{i} - 3\mathbf{k})$  and  $\mathbf{c} = \mathbf{j}$ . {Ans.  $\pm(5\mathbf{i} - \mathbf{j} - 5\mathbf{k})$ }

61. If  $|\mathbf{a}| = 3$ ,  $|\mathbf{b}| = 5$ ,  $|\mathbf{c}| = 7$  and  $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ . Prove that the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $\frac{\pi}{3}$ .
62. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are three vectors such that  $\vec{a} + \vec{b} + \vec{c} = \vec{0}$  and  $|\vec{a}| = 2$ ,  $|\vec{b}| = 3$ ,  $|\vec{c}| = 4$ , then find the value of  $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ . {Ans.  $-\frac{29}{2}$ }
63. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are three mutually perpendicular vectors each of magnitude unity, then find the value of  $|\vec{a} + \vec{b} + \vec{c}|$ . {Ans.  $\sqrt{3}$ }
64. If  $\vec{a}$  and  $\vec{b}$  are two unit vectors inclined at an angle  $\theta$  such that  $\vec{a} + \vec{b}$  is a unit vector, then find the value of  $\theta$ . {Ans.  $\frac{2\pi}{3}$ }
65. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are vectors such that  $\vec{a} \cdot \vec{b} = 0$  and  $\vec{a} + \vec{b} = \vec{c}$ , then show that  $|\vec{a}|^2 + |\vec{b}|^2 = |\vec{c}|^2$ .
66. If  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are non-collinear unit vectors, compute  $(2\mathbf{e}_1 - 5\mathbf{e}_2) \cdot (3\mathbf{e}_1 + \mathbf{e}_2)$  if  $|\mathbf{e}_1 + \mathbf{e}_2| = \sqrt{3}$ . {Ans.  $-\frac{11}{2}$ }
67. Let  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  be the vectors of lengths 3, 4, 5 respectively. If each one is perpendicular to the sum of other two, then find the magnitude of vector  $\mathbf{a} + \mathbf{b} + \mathbf{c}$ . {Ans.  $5\sqrt{2}$ }
68. If  $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$ , then show that  $\mathbf{a}$  and  $\mathbf{b}$  are perpendicular.
69. Prove that  $\left| \frac{\mathbf{a}}{a^2} - \frac{\mathbf{b}}{b^2} \right|^2 = \left| \frac{\mathbf{a} - \mathbf{b}}{ab} \right|^2$ .
70. If  $\mathbf{a}$  and  $\mathbf{b}$  are unit vectors and  $\theta$  is the angle between them, show that  $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\mathbf{a} - \mathbf{b}|$ .
71. If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are non coplanar vectors and  $\mathbf{n} \cdot \mathbf{a} = \mathbf{n} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{c} = 0$ , show that  $\mathbf{n}$  is a null vector.
72. If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are three mutually perpendicular vectors of equal magnitude, show that  $\mathbf{a} + \mathbf{b} + \mathbf{c}$  is equally inclined to  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  and also find the angle  $\theta$  which  $\mathbf{a} + \mathbf{b} + \mathbf{c}$  makes with any one of three given vectors. {Ans.  $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ }
73. Prove that  $\mathbf{r} = (\mathbf{r} \cdot \mathbf{i})\mathbf{i} + (\mathbf{r} \cdot \mathbf{j})\mathbf{j} + (\mathbf{r} \cdot \mathbf{k})\mathbf{k}$ .
74. If the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are coplanar and  $\mathbf{a}$ ,  $\mathbf{b}$  are non-collinear, then show that
- $$\begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} \end{vmatrix} \mathbf{c} = \begin{vmatrix} \mathbf{c} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{c} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} \end{vmatrix} \mathbf{a} + \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{a} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{b} \end{vmatrix} \mathbf{b} \text{ or } \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \end{vmatrix} = 0.$$

#### CATEGORY-19.4. VECTOR BISECTING ANGLE BETWEEN TWO VECTORS

75. Find a vector bisecting the angle between the vectors  $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  and  $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ . {Ans.  $3\mathbf{i} + \mathbf{k}$ }
76. The vector  $-\mathbf{i} + \mathbf{j} - \mathbf{k}$ , bisects the angle between the vectors  $\mathbf{c}$  and  $3\mathbf{i} + 4\mathbf{j}$ . Determine the unit vector along  $\mathbf{c}$ . {Ans.  $\frac{1}{15}(-11\mathbf{i} - 10\mathbf{j} - 2\mathbf{k})$ }

#### CATEGORY-19.5. PROJECTION OF A VECTOR

77. Find the projection of the vector  $\hat{i} - 2\hat{j} + \hat{k}$  on the vector  $4\hat{i} - 4\hat{j} + 7\hat{k}$ . {Ans.  $\frac{19}{9}$ }
78. Find the projection of  $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$  in the direction of vector  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ . What is the vector determined by the projection? {Ans.  $\frac{2}{\sqrt{14}}, \frac{1}{7}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ }
79. If  $\mathbf{a} = 4\mathbf{i} + 6\mathbf{j}$  and  $\mathbf{b} = 3\mathbf{j} + 4\mathbf{k}$ , find the vector form of the projection of  $\mathbf{a}$  on  $\mathbf{b}$ . {Ans.  $\frac{18}{25}(3\mathbf{j} + 4\mathbf{k})$ }
80. Given two vectors  $\mathbf{a} = (1, -1, 3)$  and  $\mathbf{b} = (3, -5, 6)$ . Calculate  $\text{Proj}_{(\mathbf{a}+\mathbf{b})}(\mathbf{a}-\mathbf{b})$ . {Ans.  $-\frac{22}{\sqrt{133}}$ }
81. Let  $\beta = 4\mathbf{i} + 3\mathbf{j}$  and  $\gamma$  be two vectors perpendicular to each other in the  $x-y$  plane. Find all the vectors in the same plane having projections 1 and 2 along  $\beta$  and  $\gamma$  respectively. {Ans.  $2\mathbf{i} - \mathbf{j}$  or  $\frac{1}{5}(-2\mathbf{i} + 11\mathbf{j})$ }

### CATEGORY-19.6. VECTOR (CROSS) PRODUCT OF TWO VECTORS

82. Given vectors  $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$  and  $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ , find  $\mathbf{a} \times \mathbf{b}$ . Also determine the sine of the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . {Ans.  $-3\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}, \sqrt{\frac{155}{156}}$ }
83. Find a unit vector perpendicular to both the vectors  $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  and  $\mathbf{b} = 12\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$ . {Ans.  $\pm \frac{1}{\sqrt{115}}(-5\mathbf{i} + 3\mathbf{j} - 9\mathbf{k})$ }
84. If  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ ,  $\mathbf{b} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$  and  $\mathbf{c} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ , show that  $\mathbf{a} \times \mathbf{b} = 7\mathbf{c}$  and  $\mathbf{b} \times \mathbf{c} = 7\mathbf{a}$ . Also prove that  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{0}$  in this case.
85.  $\mathbf{a}$  and  $\mathbf{b}$  are two vectors satisfying the condition  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ . Does it imply that one of the vectors  $\mathbf{a}$  or  $\mathbf{b}$  must be null vector? Give an example in support of your answer.
86. Prove that a vector of magnitude 9 perpendicular to both the vectors  $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$  and  $\mathbf{b} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$  is  $-3\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ .
87. The vector  $\mathbf{x}$ , which is perpendicular to the vectors  $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$  and  $\mathbf{b} = 18\mathbf{i} - 22\mathbf{j} - 5\mathbf{k}$ , makes an obtuse angle with the unit vector  $\mathbf{j}$ . Find its components if the length of the vector  $\mathbf{x}$  is 14. {Ans.  $(-4, -6, 12)$ }
88. Find the vector  $\mathbf{c}$  being given that it is perpendicular to the vectors  $\mathbf{a} = (2, 3, -1)$ ,  $\mathbf{b} = (1, -2, 3)$  and satisfies the condition  $\mathbf{c} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = -6$ . {Ans.  $\mathbf{c} = (-3, 3, 3)$ }
89. Find the cosine of the angle between the directions of the vectors  $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$  and  $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ . Also find a unit vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ . What is the sine of the angle between  $\mathbf{a}$  and  $\mathbf{b}$ ? {Ans.  $\frac{7}{3\sqrt{26}}, \frac{1}{\sqrt{185}}(7\mathbf{i} - 6\mathbf{j} - 10\mathbf{k}), \frac{1}{3}\sqrt{\frac{185}{26}}$ }
90. Show that the vectors  $\mathbf{a} = \frac{1}{7}(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$ ,  $\mathbf{b} = \frac{1}{7}(3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$  and  $\mathbf{c} = \frac{1}{7}(6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$  form an orthonormal triad. Also find two vectors of magnitude 3 each being normal to the plane containing the vectors  $\mathbf{a}$  and  $\mathbf{b}$ . {Ans.  $\pm \frac{3}{7}(6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ }
91. Given  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$  and  $\mathbf{c} = 3\mathbf{i} + \mathbf{j}$ , find the unit vector in the direction of the resultant of these vectors. Also find a vector  $\mathbf{r}$  which is normal to both  $\mathbf{a}$  and  $\mathbf{b}$ . What is the inclination of  $\mathbf{r}$  and  $\mathbf{c}$ .



$$\{\text{Ans. } \frac{1}{5\sqrt{2}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k}), (\mathbf{i}+\mathbf{j}-\mathbf{k}), \cos^{-1}\left(-\frac{4}{\sqrt{30}}\right)\}$$

92. If  $\vec{a} \cdot \vec{b} = |\vec{a} \times \vec{b}|$ , then find the angle between  $\vec{a}$  and  $\vec{b}$ . {Ans.  $45^\circ$ }
93. Vectors  $\vec{a}$  and  $\vec{b}$  are inclined at an angle  $\theta = 120^\circ$ . If  $|\vec{a}| = 1$ ,  $|\vec{b}| = 2$ , then find  $\left[(\vec{a} + 3\vec{b}) \times (3\vec{a} - \vec{b})\right]^2$ . {Ans. 300}
94. If  $(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 = 144$  and  $|\vec{a}| = 4$ , then find  $|\vec{b}|$ . {Ans. 3}
95. Prove that  $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$  or  $|\mathbf{a} \times \mathbf{b}|^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{vmatrix}$  and deduce the identity
- $$(m_1^2 + n_1^2 + p_1^2)(m_2^2 + n_2^2 + p_2^2) - (m_1 m_2 + n_1 n_2 + p_1 p_2)^2 = (m_1 n_2 - m_2 n_1)^2 + (n_1 p_2 - n_2 p_1)^2 + (p_1 m_2 - p_2 m_1)^2.$$
96. Prove that  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) + \mathbf{b} \times (\mathbf{c} + \mathbf{a}) + \mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{0}$ .
97. If  $\vec{a}$  and  $\vec{b}$  are two vectors such that  $\vec{a} \cdot \vec{b} = 0$  and  $\vec{a} \times \vec{b} = \vec{0}$ , then show that either  $\vec{a}$  or  $\vec{b}$  is a null vector.
98. If  $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$  and  $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ , show that  $\mathbf{a} - \mathbf{d}$  is parallel to  $\mathbf{b} - \mathbf{c}$ .
99. If  $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ , then prove that  $\mathbf{b}$  differs from  $\mathbf{c}$  by a vector which is parallel to  $\mathbf{a}$ .
100. If  $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{b} \neq \mathbf{0}$ , then show that  $\mathbf{a} - \mathbf{c} = k\mathbf{b}$ , where  $k$  is a scalar.
101. Interpret the equations: (i)  $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ , (ii)  $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$  and prove that if both the equations hold simultaneously, then  $\mathbf{b} = \mathbf{c}$  if  $\mathbf{a} \neq \mathbf{0}$ .

### CATEGORY-19.7. SCALAR TRIPLE PRODUCT

102. Find  $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{k} \times \hat{i}) + \hat{k} \cdot (\hat{i} \times \hat{j})$ . {Ans. 3}
103. If  $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$  and  $\vec{c} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ , find  $[\vec{a} \vec{b} \vec{c}]$ . {Ans. -2}
104. Find the constant  $\lambda$  so that the vectors  $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} + \lambda\mathbf{j} + \mathbf{k}$ ,  $\mathbf{c} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$  are coplanar. {Ans. -4}
105. If the vectors  $2\hat{i} - 3\hat{j} + 4\hat{k}$ ,  $\hat{i} + 2\hat{j} - \hat{k}$  and  $x\hat{i} - \hat{j} + 2\hat{k}$  are coplanar, then find  $x$ . {Ans.  $\frac{8}{5}$ }
106. If  $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ ,  $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$  and  $\vec{c} = 3\hat{i} + p\hat{j} + 5\hat{k}$  are coplanar, then find  $p$ . {Ans. -6}
107. If the vectors  $2\hat{i} - \hat{j} + \hat{k}$ ,  $\hat{i} + m\hat{j}$ ,  $\hat{i} + \hat{j} + \hat{k}$  are coplanar, then find the value of  $m$ . {Ans. -2}
108. If the vectors  $\vec{c}$ ,  $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$  and  $\vec{b} = \hat{j}$  are such that  $\vec{a}$ ,  $\vec{c}$  and  $\vec{b}$  form a right handed system, then find  $\vec{c}$ . {Ans.  $z\hat{i} - x\hat{k}$ }
109. If  $\vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b} = \vec{r} \cdot \vec{c} = 0$  for some non-zero vector  $\vec{r}$ , then find the value of  $[\vec{a} \vec{b} \vec{c}]$ . {Ans. 0}
110. Show that  $(\vec{a} + 2\vec{b} - \vec{c}) \cdot \{(\vec{a} - \vec{b}) \times (\vec{a} - \vec{b} - \vec{c})\} = 3[\vec{a} \vec{b} \vec{c}]$ .
111. If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  be three vectors, prove that  $[\mathbf{a} + \mathbf{b} \quad \mathbf{b} + \mathbf{c} \quad \mathbf{c} + \mathbf{a}] = 2[\mathbf{a} \mathbf{b} \mathbf{c}]$  and hence prove that the vectors  $\mathbf{a} + \mathbf{b}$ ,  $\mathbf{b} + \mathbf{c}$ ,  $\mathbf{c} + \mathbf{a}$  are coplanar if and only if  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are coplanar.
112. Prove that  $[\mathbf{a} - \mathbf{b} \quad \mathbf{b} - \mathbf{c} \quad \mathbf{c} - \mathbf{a}] = 0$ .
113. Express (a)  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  in terms of  $\mathbf{b} \times \mathbf{c}$ ,  $\mathbf{c} \times \mathbf{a}$ ,  $\mathbf{a} \times \mathbf{b}$ , and (b)  $\mathbf{b} \times \mathbf{c}$ ,  $\mathbf{c} \times \mathbf{a}$ ,  $\mathbf{a} \times \mathbf{b}$  in terms of  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ . {Ans.}

$$\vec{a} = \left[ \frac{\vec{a} \cdot \vec{a}}{\vec{a} \cdot \vec{b} \cdot \vec{c}} \right] (\vec{b} \times \vec{c}) + \left[ \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{b} \cdot \vec{c}} \right] (\vec{c} \times \vec{a}) + \left[ \frac{\vec{a} \cdot \vec{c}}{\vec{a} \cdot \vec{b} \cdot \vec{c}} \right] (\vec{a} \times \vec{b}) \}$$

**CATEGORY-19.8. VECTOR TRIPLE PRODUCT**

114. If  $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$  and  $\vec{c} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ , find  $\vec{a} \times (\vec{b} \times \vec{c})$  and  $(\vec{a} \times \vec{b}) \times \vec{c}$ . {Ans.  $-44\hat{i} + 26\hat{j} + 32\hat{k}$ ,  $-41\hat{i} + 22\hat{j} + 37\hat{k}$ }
115. Find  $\mathbf{i} \times (\mathbf{j} \times \mathbf{k})$ . {Ans.  $-\mathbf{k}$ }
116. Prove that  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$ .
117. Prove that  $\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k}) = 2\mathbf{a}$ .
118. Show that  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$  if and only if  $\mathbf{a}$  and  $\mathbf{c}$  are collinear or  $(\mathbf{a} \times \mathbf{c}) \times \mathbf{b} = \mathbf{0}$ .
119. If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{d}$  be four vectors, then prove that  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d}) = 0$ .
120. If the vectors  $\mathbf{b}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$  are not coplanar, then prove that  $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c})$  is parallel to  $\mathbf{a}$ .
121. Prove that  $[\mathbf{a} \times \mathbf{b} \quad \mathbf{b} \times \mathbf{c} \quad \mathbf{c} \times \mathbf{a}] = [\mathbf{abc}]^2$ . Hence show that the vectors  $\mathbf{a} \times \mathbf{b}$ ,  $\mathbf{b} \times \mathbf{c}$ ,  $\mathbf{c} \times \mathbf{a}$  are non-coplanar if and only if  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are non-coplanar.
122. If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  be three unit vectors such that  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{1}{2}\mathbf{b}$ . Find the angles which  $\mathbf{a}$  makes with  $\mathbf{b}$  and  $\mathbf{c}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  being non-parallel. {Ans.  $90^\circ, 60^\circ$ }

**CATEGORY-19.9. RECIPROCAL SYSTEM OF VECTORS**

123. Find a set of vectors reciprocal to the set  $-\hat{i} + \hat{j} + \hat{k}$ ,  $\hat{i} - \hat{j} + \hat{k}$ ,  $\hat{i} + \hat{j} + \hat{k}$ . {Ans.  $\frac{1}{2}(-\hat{i} + \hat{k})$ ,  $\frac{1}{2}(-\hat{j} + \hat{k})$ ,  $\frac{1}{2}(\hat{i} + \hat{j})$ }
124. Prove that the system of vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  is its own reciprocal.
125. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  and  $\vec{a}'$ ,  $\vec{b}'$ ,  $\vec{c}'$  are reciprocal system of vectors, then prove that
- $\vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' = \frac{\vec{a} + \vec{b} + \vec{c}}{[\vec{a} \vec{b} \vec{c}]}$ .
  - $\vec{a}' \cdot (\vec{a} + \vec{b}) + \vec{b}' \cdot (\vec{b} + \vec{c}) + \vec{c}' \cdot (\vec{c} + \vec{a}) = 3$ .
126. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be three non-coplanar vectors and  $\vec{a}'$ ,  $\vec{b}'$ ,  $\vec{c}'$  constitute the reciprocal system of vectors, then prove that
- $\vec{r} = (\vec{r} \cdot \vec{a}')\vec{a} + (\vec{r} \cdot \vec{b}')\vec{b} + (\vec{r} \cdot \vec{c}')\vec{c}$ .
  - $\vec{r} = (\vec{r} \cdot \vec{a})\vec{a}' + (\vec{r} \cdot \vec{b})\vec{b}' + (\vec{r} \cdot \vec{c})\vec{c}'$ .

**CATEGORY-19.10. VECTOR EQUATIONS**

127. If  $\vec{a} \cdot \hat{i} = \vec{a} \cdot (\hat{i} + \hat{j}) = \vec{a} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ , then find  $\vec{a}$ . {Ans.  $\hat{i}$ }
128. If  $\mathbf{a} \cdot \mathbf{b} \neq 0$ , find the vector  $\mathbf{r}$  which satisfies the equations  $(\mathbf{r} - \mathbf{c}) \times \mathbf{b} = \mathbf{0}$ ,  $\mathbf{r} \cdot \mathbf{a} = 0$ . {Ans.  $\mathbf{r} = \mathbf{c} - \left( \frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}} \right) \mathbf{b}$ }
129. Given that  $\vec{a} = (1, 1, 1)$ ,  $\vec{c} = (0, 1, -1)$  and  $\vec{a} \cdot \vec{b} = 3$ . If  $\vec{a} \times \vec{b} = \vec{c}$ , then find  $\vec{b}$ . {Ans.  $\left( \frac{5}{3}, \frac{2}{3}, \frac{2}{3} \right)$ }

130. Let  $\vec{a} = \hat{i} - \hat{j}$ ,  $\vec{b} = \hat{j} - \hat{k}$ ,  $\vec{c} = \hat{k} - \hat{i}$ . If  $\vec{d}$  is a unit vector such that  $\vec{a} \cdot \vec{d} = 0 = [\vec{b}\vec{c}\vec{d}]$ , then find  $\hat{d}$ . {Ans.  $\pm \frac{\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{6}}$ }
131. If  $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$  and  $\vec{r} \cdot \vec{a} = 0$  where  $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ ,  $\vec{b} = 3\hat{i} - \hat{j} + \hat{k}$  and  $\vec{c} = \hat{i} + \hat{j} + 3\hat{k}$ , then find  $\vec{r}$ . {Ans.  $2(-\hat{i} + \hat{j} + \hat{k})$ }
132. Let  $\mathbf{A} = 2\mathbf{i} + \mathbf{k}$ ,  $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{C} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$ . Find vector  $\mathbf{R}$  which satisfies  $\mathbf{R} \times \mathbf{B} = \mathbf{C} \times \mathbf{B}$  and  $\mathbf{R} \cdot \mathbf{A} = 0$ . {Ans.  $\mathbf{R} = -\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$ }
133. Solve the following simultaneous equations for vectors  $\mathbf{x}$  and  $\mathbf{y}$ :-  

$$\begin{cases} \mathbf{x} + \mathbf{y} = \mathbf{a} \\ \mathbf{x} \times \mathbf{y} = \mathbf{b} \\ \mathbf{x} \cdot \mathbf{a} = 1 \end{cases} \quad \left\{ \text{Ans. } \mathbf{x} = \frac{\mathbf{a} + (\mathbf{a} \times \mathbf{b})}{|\mathbf{a}|^2}, \mathbf{y} = \mathbf{a} - \frac{\mathbf{a} + (\mathbf{a} \times \mathbf{b})}{|\mathbf{a}|^2} \right\}$$
134. If  $c$  be a given non-zero scalar, and  $\mathbf{A}$  and  $\mathbf{B}$  be given non-zero vectors such that  $\mathbf{A} \perp \mathbf{B}$ , find the vector  $\mathbf{X}$  which satisfies the equations  $\mathbf{A} \cdot \mathbf{X} = c$  and  $\mathbf{A} \times \mathbf{X} = \mathbf{B}$ . {Ans.  $\mathbf{X} = \frac{c}{|\mathbf{A}|^2} \mathbf{A} - \frac{1}{|\mathbf{A}|^2} \mathbf{A} \times \mathbf{B}$ }
135. Given equations:  $\vec{r} \times \vec{a} = \vec{b}$ ,  $\vec{r} \times \vec{c} = \vec{d}$ . Obtain condition for these equations to be consistent, and then solve the equations. It is assumed that  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are non-coplanar. {Ans.  $\vec{b} \cdot \vec{c} = -\vec{d} \cdot \vec{a}$ ,  

$$\vec{r} = \frac{1}{[\vec{a}\vec{b}\vec{c}]} \left\{ -(\vec{b} \cdot \vec{d})\vec{a} + (\vec{a} \cdot \vec{d})\vec{b} + |\vec{b}|^2 \vec{c} \right\}$$
}

**CATEGORY-19.11. POINTS**

136. Given points  $A(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$  and  $B(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ , find  
 i. distance between the points  $A$  and  $B$ . {Ans.  $\sqrt{14}$ }  
 ii. mid-point of  $A$  and  $B$ . {Ans.  $\frac{3}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + 2\mathbf{k}$ }  
 iii. the point which divides  $AB$  in the ratio 2:3. {Ans.  $\frac{7}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} + \frac{11}{5}\mathbf{k}$ }
137. If the position vectors of  $P$  and  $Q$  are  $\hat{i} + 3\hat{j} - 7\hat{k}$  and  $5\hat{i} - 2\hat{j} + 4\hat{k}$  then find the cosine of the angle between  $\overrightarrow{PQ}$  and  $\hat{j}$ . {Ans.  $-\frac{5}{\sqrt{162}}$ }
138. If points  $A(60\hat{i} + 3\hat{j})$ ,  $B(40\hat{i} - 8\hat{j})$  and  $C(a\hat{i} - 52\hat{j})$  are collinear, then find  $a$ . {Ans. -40}
139. Prove that  $A(1,2,3)$ ,  $B(3,4,7)$ ,  $C(-3,-2,-5)$  are collinear and find the ratio in which  $B$  divides  $AC$ . {Ans. 1:3 externally}
140. If  $C$  is the middle point of  $AB$  and  $P$  is any point outside  $AB$ , then prove that  $\overrightarrow{PA} + \overrightarrow{PB} = 2\overrightarrow{PC}$ .
141. If  $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$ , prove that  $A$ ,  $B$ ,  $C$  are collinear.
142. If  $\vec{a}$ ,  $\vec{b}$  are the position vectors of  $A$  and  $B$  respectively  $C$  is a point on  $AB$  produced such that  $AC = 3AB$ , then find the position vector of  $C$ . {Ans.  $3\vec{b} - 2\vec{a}$ }
143. If  $P$ ,  $Q$ ,  $R$  are three points with respective position vectors  $\hat{i} + \hat{j}$ ,  $\hat{i} - \hat{j}$  and  $a\hat{i} + b\hat{j} + c\hat{k}$ . For what values

of  $a, b, c$ , the points  $P, Q, R$  are collinear. {Ans.  $a=1, c=0, b \in R$ }

144. If  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are non-coplanar vectors, prove that the following three points are collinear:-

i.  $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}, 2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}, -7\mathbf{b} + 10\mathbf{c}$ .

ii.  $\mathbf{a}, \mathbf{b}, 3\mathbf{a} - 2\mathbf{b}$ .

iii.  $-2\mathbf{a} + 3\mathbf{b} + 5\mathbf{c}, \mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, 7\mathbf{a} - \mathbf{c}$ .

iv.  $\mathbf{a} + \mathbf{b} + \mathbf{c}, 4\mathbf{a} + 3\mathbf{b}, 10\mathbf{a} + 7\mathbf{b} - 2\mathbf{c}$ .

145. If  $\mathbf{a}, \mathbf{b}$  are two non-collinear vectors, show that points  $l_1\mathbf{a} + m_1\mathbf{b}, l_2\mathbf{a} + m_2\mathbf{b}, l_3\mathbf{a} + m_3\mathbf{b}$  are collinear if

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

146. Prove that the following four points are coplanar:-

i.  $(4, 5, 1), (0, -1, -1), (3, 9, 4)$  and  $(-4, 4, 4)$ .

ii.  $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}, 6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$  and  $-2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ .

147. Check whether the four points  $4\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}, 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}, 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}, 5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$  are coplanar or not?

{Ans. non-coplanar}

148. If  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are non-coplanar vectors, prove that the four points as given below are coplanar:-

i.  $2\mathbf{a} + 3\mathbf{b} - \mathbf{c}, \mathbf{a} - 2\mathbf{b} + 3\mathbf{c}, 3\mathbf{a} + 4\mathbf{b} - 2\mathbf{c}, \mathbf{a} - 6\mathbf{b} + 6\mathbf{c}$ .

ii.  $6\mathbf{a} + 2\mathbf{b} - \mathbf{c}, 2\mathbf{a} - \mathbf{b} + 3\mathbf{c}, -\mathbf{a} + 2\mathbf{b} - 4\mathbf{c}, -12\mathbf{a} - \mathbf{b} - 3\mathbf{c}$ .

iii.  $6\mathbf{a} - 4\mathbf{b} + 10\mathbf{c}, -5\mathbf{a} + 3\mathbf{b} - 10\mathbf{c}, 4\mathbf{a} - 6\mathbf{b} - 10\mathbf{c}, 2\mathbf{b} + 10\mathbf{c}$ .

149. The position vectors of the points  $A, B, C$  and  $D$  are  $3\mathbf{i} - 2\mathbf{j} - \mathbf{k}, 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$  and  $4\mathbf{i} + 5\mathbf{j} + \lambda\mathbf{k}$  respectively. If the points  $A, B, C$  and  $D$  lie on a plane, find the value of  $\lambda$ . {Ans.  $-\frac{146}{17}$ }

150. Find the value of  $k$  for which the points  $A(1,0,3), B(-1,3,4), C(1,2,1)$  and  $D(k,2,5)$  are coplanar. {Ans. -1}

151. If  $P$  is a point in space such that  $OP = 12$  and  $\overrightarrow{OP}$  is inclined at angles of  $45^\circ$  and  $60^\circ$  with  $\hat{i}$  and  $\hat{j}$  respectively, then find the position vector of  $P$ . {Ans.  $6\sqrt{2}\hat{i} + 6\hat{j} \pm 6\hat{k}$ }

152. If the four points  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$  are coplanar, prove that  $[\mathbf{abc}] = [\mathbf{bcd}] + [\mathbf{cad}] + [\mathbf{abd}]$ .

153. If  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  be any three non-coplanar vectors, then prove that the points  $l_1\mathbf{a} + m_1\mathbf{b} + n_1\mathbf{c}, l_2\mathbf{a} + m_2\mathbf{b} + n_2\mathbf{c},$

$$l_3\mathbf{a} + m_3\mathbf{b} + n_3\mathbf{c} \text{ and } l_4\mathbf{a} + m_4\mathbf{b} + n_4\mathbf{c} \text{ are coplanar if } \begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

154. If  $G$  is the centroid of the triangle  $ABC$ , show that  $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{0}$  and conversely if

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{0}, \text{ then } G \text{ is the centroid of triangle } ABC.$$

155. Let  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots, \mathbf{r}_n$  be the position vectors of points  $P_1, P_2, P_3, \dots, P_n$  relative to an origin  $O$ . Show that if the vector equation  $a_1\mathbf{r}_1 + a_2\mathbf{r}_2 + \dots + a_n\mathbf{r}_n = \mathbf{0}$  holds, then a similar equation will also hold good with respect to any other origin  $P$  if  $a_1 + a_2 + \dots + a_n = 0$ .

**CATEGORY-19.12. AREA**

156. Show that the area of a triangle whose two adjacent sides are determined by the vector  $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ ,  $\mathbf{b} = -5\mathbf{i} + 7\mathbf{j}$  is  $20\frac{1}{2}$  square units.
157. Find the area of the triangle whose side vectors are  $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$  and  $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ . {Ans.  $\frac{\sqrt{83}}{2}$ }
158. Find the area of the triangle whose vertices are  $A(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ ,  $B(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$  and  $C(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ . {Ans.  $\frac{7}{2}$ }
159. Find the area of a triangle whose vertices are the points  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ . {Ans.  $\frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$ }
160. If  $A, B, C, D$  are any four points in space, prove that  $\left| \vec{AB} \times \vec{CD} + \vec{BC} \times \vec{AD} + \vec{CA} \times \vec{BD} \right| = 4$  (area of  $\triangle ABC$ ).
161. Prove that  $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2\mathbf{a} \times \mathbf{b}$  and interpret it.
162. Given that vectors  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  form a triangle such that  $\mathbf{A} = \mathbf{B} + \mathbf{C}$ . Find  $a, b, c, d$  such that the area of the triangle is  $5\sqrt{6}$  where  $\mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ ,  $\mathbf{B} = d\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ ,  $\mathbf{C} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ . {Ans.  $a = -8, b = 4, c = 2, d = -11; a = 8, b = 4, c = 2, d = 5$ }
163. Find the area of the parallelogram whose adjacent sides are  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ . {Ans.  $8\sqrt{3}$  sq. units }
164. Find the area of the parallelogram constructed on the vectors  $\vec{a} = \vec{p} + 2\vec{q}$  and  $\vec{b} = 2\vec{p} + \vec{q}$  where  $\vec{p}$  and  $\vec{q}$  are unit vectors forming an angle of  $30^\circ$ . {Ans.  $\frac{3}{2}$ }
165. Find the area of the parallelogram having diagonals  $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$  and  $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ . {Ans.  $5\sqrt{3}$  sq. units }
166. Find two unit vectors parallel to the diagonals of a parallelogram whose sides are  $2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$  and  $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ . Also find the area of the parallelogram. {Ans.  $\frac{1}{7}(3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}), \frac{1}{\sqrt{69}}(-\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}), 11\sqrt{5}$ }
167. If  $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = -\mathbf{i} + \mathbf{k}$ ,  $\mathbf{c} = 2\mathbf{j} - \mathbf{k}$  find the area of the parallelogram having diagonals  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{b} + \mathbf{c}$ . {Ans.  $\frac{\sqrt{21}}{2}$ }
168. Let  $A_r$  ( $r = 1, 2, 3, 4$ ) be the area of four faces of a tetrahedron. Let,  $\mathbf{n}_r$  be the outward drawn normals to the respective faces with magnitudes equal to corresponding areas. Prove that  $\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4 = \mathbf{0}$ .

**CATEGORY-19.13. VOLUME**

169. Find the volume of the parallelepiped whose edges are represented by the vectors:-  
i.  $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ ,  $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ ,  $\mathbf{c} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ . {Ans. 7}  
ii.  $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$ ,  $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ ,  $\mathbf{c} = 3\mathbf{i} - \mathbf{k}$ . {Ans. 4}
170. The volume of the parallelepiped whose edges are represented by  $-12\mathbf{i} + \lambda\mathbf{k}$ ,  $3\mathbf{j} - \mathbf{k}$ ,  $2\mathbf{i} + \mathbf{j} - 15\mathbf{k}$  is 546. Find the value of  $\lambda$ . {Ans. -3}
171. Three vectors  $\mathbf{a} = (12, 4, 3)$ ,  $\mathbf{b} = (8, -12, -9)$ ,  $\mathbf{c} = (33, -4, -24)$  define a parallelepiped. Evaluate the lengths of its edges, area of its faces and its volume. {Ans. 13, 17, 41, 220, 435, 455, 3696}
172. Find out the volume of a prism on triangular base, the three sides of the prism meeting on a vertex are given below:-

- i.  $\vec{OA} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ ,  $\vec{OB} = 12\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ ,  $\vec{OC} = 4\mathbf{i} + 12\mathbf{j} + 3\mathbf{k}$ . {Ans. 693.5}
- ii. Find the unit vector perpendicular to  $\vec{OA}$  and  $\vec{OB}$ . {Ans.  $\frac{1}{\sqrt{19345}}(-20\mathbf{i} + 132\mathbf{j} - 39\mathbf{k})$ }
- iii. Find the angle between  $\vec{OA}$  and  $\vec{OC}$ . {Ans.  $\cos^{-1} \frac{96}{169}$ }
- iv. Find the area of triangle  $OAB$ . {Ans.  $\frac{\sqrt{19345}}{2}$ }
173. Prove by vector method that the volume  $V$  of a tetrahedron in terms of the length of three coterminal edges and their mutual inclinations is  $V^2 = \frac{a^2 b^2 c^2}{36} \begin{vmatrix} 1 & \cos \phi & \cos \psi \\ \cos \phi & 1 & \cos \theta \\ \cos \psi & \cos \theta & 1 \end{vmatrix}$ .

#### CATEGORY-19.14. EQUATION OF A STRAIGHT LINE

174. Vertices of a triangle are  $A(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ ,  $B(\mathbf{i} - \mathbf{j} + \mathbf{k})$  and  $C(\mathbf{i} + \mathbf{j} + \mathbf{k})$ . Find the equation of sides and medians.

#### CATEGORY-19.15. POINT AND STRAIGHT LINE

175. Find the distance of the point  $B(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$  from the line which is passing through  $A(4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$  and which is parallel to the vector  $\vec{C} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ . Also find the foot of perpendicular from  $B$  on the line. {Ans.  $\sqrt{10}, 4\hat{i} + 2\hat{j} + 2\hat{k}$ }
176. Find the perpendicular distance of  $A(1, 4, -2)$  from  $BC$  where position vectors of  $B$  and  $C$  are respectively  $(2, 1, -2)$  and  $(0, -5, 1)$ . Also find the foot of perpendicular from  $A$  on the line  $BC$ . {Ans.  $\frac{3\sqrt{26}}{7}$ ,  $\frac{1}{49}(130\hat{i} + 145\hat{j} - 146\hat{k})$ }
177. The vertices of a triangle are  $A(4, 2, 3)$ ,  $B(1, 3, 1)$  and  $C(-5, 5, -2)$ . Use vectors to find the length of normal drawn from  $A$  to  $BC$ . Also find the foot of perpendicular from  $A$  on the line  $BC$ . {Ans.  $\frac{\sqrt{10}}{7}$ ,  $\frac{1}{49}(205\hat{i} + 95\hat{j} + 127\hat{k})$ }
178. Find the perpendicular distance of a corner of a unit cube from the diagonal not passing through it. {Ans.  $\frac{\sqrt{6}}{3}$ }
179. Prove that the perpendicular distance of the vertex  $A$  from the base of a  $\Delta ABC$  with  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  as position vectors of  $A, B, C$  respectively, is  $\frac{|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|}{|\mathbf{b} - \mathbf{c}|}$ .

#### CATEGORY-19.16. ANGLE BETWEEN TWO STRAIGHT LINES

180. The vectors  $3\hat{i} - 2\hat{j} + \hat{k}$ ,  $\hat{i} - 3\hat{j} + 5\hat{k}$  and  $2\hat{i} + \hat{j} - 4\hat{k}$  form the sides of a triangle. Show that this triangle is a right-angled triangle.
181. Given the vertices  $A(3, 2, -3)$ ,  $B(5, 1, -1)$  and  $C(1, -2, 1)$  of a triangle. Find its interior vertex angle  $A$ .  
{Ans.  $\cos^{-1} \frac{4}{9}$ }
182. The position vector of the points  $A, B, C, D$  are  $\mathbf{i} + \mathbf{j} + \mathbf{k}$ ,  $2\mathbf{i} + 3\mathbf{j}$ ,  $3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ ,  $\mathbf{k} - \mathbf{j}$  respectively. Show that lines  $AB$  and  $CD$  are parallel.
183. The position vectors of  $A, B, C, D$  with respect to the origin are respectively  $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ ,  $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ ,  $-5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ ,  $\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$ . Show that lines  $AB$  and  $CD$  are parallel.
184. Given four points  $A = (-2, -3, 8)$ ,  $B = (2, 1, 7)$ ,  $C = (1, 4, 5)$  and  $D = (-7, -4, 7)$ . Are the lines  $AB$  and  $CD$  parallel? {Ans. yes}
185. Let  $ABC$  be a triangle, the position vectors of whose vertices are respectively  $7\hat{j} + 10\hat{k}$ ,  $-\hat{i} + 6\hat{j} + 6\hat{k}$  and  $-4\hat{i} + 9\hat{j} + 6\hat{k}$ . Then show that the  $\triangle ABC$  is isosceles right-angled triangle.
186. Let  $\hat{\mathbf{a}}$  be a unit vector and  $\mathbf{b}$  be a non-zero vector not parallel to  $\hat{\mathbf{a}}$ . Find the angles of the triangle, two sides of which are represented by the vectors  $\sqrt{3}(\hat{\mathbf{a}} \times \mathbf{b})$  and  $\mathbf{b} - (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}}$ . {Ans.  $30^\circ, 60^\circ, 90^\circ$ }
187. Prove that the points  $A(-2, -3)$ ,  $B(-3, 1)$ ,  $C(7, 7)$  and  $D(3, 0)$  are the vertices of a trapezoid. Find the length of the median of the trapezoid. {Ans.  $\frac{3\sqrt{34}}{2}$ }
188. Given three successive vertices of a parallelogram :  $A(-3, -2, 0)$ ,  $B(3, -3, 1)$  and  $C(5, 0, 2)$ . Find its fourth vertex  $D$  and the angle between the lines  $AC$  and  $BD$ . {Ans.  $D = (-1, 1, 1)$ ,  $\frac{2\pi}{3}$ }
189. Prove that the points  $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ ,  $3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$  are the vertices of a right angled triangle. Find also the other two angles. {Ans.  $\cos^{-1} \sqrt{\frac{35}{41}}$ ,  $\cos^{-1} \sqrt{\frac{6}{41}}$ }
190.  $P, Q, R, S$  are the points  $\mathbf{i} - \mathbf{k}$ ,  $-\mathbf{i} + 2\mathbf{j}$ ,  $2\mathbf{i} - 3\mathbf{k}$  and  $3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$  respectively. Show that the projection of  $\overrightarrow{PQ}$  on  $\overrightarrow{RS}$  is equal to that of  $\overrightarrow{RS}$  on  $\overrightarrow{PQ}$  each being  $-\frac{4}{3}$ . Also find the cosine of their inclination. {Ans.  $-\frac{4}{9}$ }
191. A triangle is defined by its vertices  $A(2, 1, 2)$ ,  $B(1, 0, 0)$  and  $C(1 + \sqrt{3}, \sqrt{3}, -\sqrt{6})$ . Calculate the angles of the triangle and the length of the median  $m$  drawn to the side  $BC$ . {Ans.  $\cos A = \frac{\sqrt{3} + \sqrt{2} - 1}{\sqrt{9 + 2\sqrt{6} - 2\sqrt{3}}}$ ,  
 $\cos B = \frac{1 - \sqrt{2}}{\sqrt{6}}$ ,  $\cos C = \frac{2\sqrt{3} + \sqrt{2} - 1}{\sqrt{18 + 4\sqrt{6} - 4\sqrt{3}}}$ ,  $m = 2\sqrt{9 + 2\sqrt{6} - 2\sqrt{3}}$ }
192. Determine the lengths of the diagonals of a parallelogram constructed on the vectors  $\mathbf{a} = 2\mathbf{m} + \mathbf{n}$  and  $\mathbf{b} = \mathbf{m} - 2\mathbf{n}$ , where  $\mathbf{m}$  and  $\mathbf{n}$  are unit-vectors forming an angle of  $60^\circ$ . {Ans.  $\sqrt{7}, \sqrt{13}$ }
193. The position vectors of points  $A$  and  $B$  are  $6\hat{i} + 7\hat{j} + 4\hat{k}$  and  $-9\hat{j} + 2\hat{k}$  respectively. A line  $L_1$ , parallel to the vector  $3\hat{i} - \hat{j} + \hat{k}$ , passes through the point  $A$  and a line  $L_2$ , parallel to the vector  $-3\hat{i} + 2\hat{j} + 4\hat{k}$ , passes through the point  $B$ . Find the position vector of a point  $P$  on the line  $L_1$  and a point  $Q$  on the line

- $L_2$  such that line  $PQ$  is perpendicular to  $L_1$  and  $L_2$  both. {Ans.  $P(3,8,3)$ ,  $Q(-3,-7,6)$ }
194. In a triangle  $ABC$  the point  $A(-1, 2, 3)$  is the vertex of the right angle. Find the vertices  $B$  and  $C$  if it is known that  $B$  and  $C$  lie on the straight line  $MN$ , where  $M(-1, 3, 2)$ ,  $N(1, 1, 3)$  and  $\angle ABC = 30^\circ$ . {Ans.  $B\left(\frac{\pm 2\sqrt{3}-1}{3}, \frac{7 \mp 2\sqrt{3}}{3}, \frac{7 \pm \sqrt{3}}{3}\right)$ ,  $C\left(\frac{\mp 2\sqrt{3}-3}{9}, \frac{21 \pm 2\sqrt{3}}{9}, \frac{21 \mp \sqrt{3}}{9}\right)$ }
195. Prove that the angle between two diagonals of a cube is  $\cos^{-1}\left(\frac{1}{3}\right)$ .
196. A line makes angles  $\alpha, \beta, \gamma, \delta$  with the diagonals of a cube, prove that  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = 4/3$ .
197. Prove that in any triangle the line joining the mid-points of any two sides is parallel to the third side and half of its length.
198. Prove that the parallelogram whose diagonals are equal is a rectangle.
199. Prove that the diagonals of a rhombus are at right angles.
200. Prove that the straight line joining the mid-points of two non-parallel sides of a trapezium is parallel to the parallel sides and half of their sum.
201. Prove that the straight line joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and half of their difference.
202. Prove that the angle in a semi-circle is right angle.
203. Prove that in a tetrahedron if two pairs of opposite edges are perpendicular, the third pair is also perpendicular. Also show that sum of the squares on two opposite edges is the same for each pair for such a tetrahedron.
204. Prove that any two opposite edges of a regular tetrahedron are perpendicular.

### CATEGORY-19.17. POINT OF INTERSECTION OF STRAIGHT LINES

205. Find the point of intersection of the line joining the points  $6\hat{i} - 4\hat{j} + 4\hat{k}$ ,  $9\hat{i} - 6\hat{j} + 8\hat{k}$  and the line joining the points  $-\hat{i} - 2\hat{j} - 3\hat{k}$ ,  $\hat{i} + 2\hat{j} - 5\hat{k}$ . {Ans.  $-4\hat{k}$ }
206. Find the point of intersection of the lines  $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$  and  $\vec{r} = (\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} + \hat{k})$ . {Ans. Lines do not intersect}
207. The position vectors of the points  $P$  and  $Q$  are  $5\hat{i} + 7\hat{j} - 2\hat{k}$  and  $-3\hat{i} + 3\hat{j} + 6\hat{k}$  respectively. A line  $L_1$ , parallel to the vector  $3\hat{i} - \hat{j} + \hat{k}$ , passes through the point  $P$  and a line  $L_2$ , parallel to the vector  $-3\hat{i} + 2\hat{j} + 4\hat{k}$ , passes through the point  $Q$ . A third line parallel to the vector  $2\hat{i} + 7\hat{j} - 5\hat{k}$  intersects lines  $L_1$  and  $L_2$ . Find the position vector of the points of intersection. {Ans.  $2\hat{i} + 8\hat{j} - 3\hat{k}$ ,  $\hat{j} + 2\hat{k}$ }
208. In a triangle  $OAB$ ,  $E$  is the mid-point of  $OB$  and  $D$  is a point on  $AB$  such that  $AD:DB = 2:1$ . If  $OD$  and  $AE$  intersect at  $P$ , determine the ratio  $OP:PD$  using vector methods. {Ans.  $3:2$ }
209. In a triangle  $ABC$ ,  $D$  and  $E$  are points on  $BC$  and  $AC$  respectively, such that  $BD = 2DC$  and  $AE = 3EC$ . Let  $P$  be the point of intersection of  $AD$  and  $BE$ . Find  $BP/PE$  using vector methods. {Ans.  $\frac{8}{3}$ }
210. In a triangle  $ABC$ ,  $D$  divides  $BC$  in the ratio  $3:2$  and  $E$  divides  $CA$  in the ratio  $1:3$ . The lines  $AD$  and  $BE$  meet at  $H$  and  $CH$  meets  $AB$  in  $F$ . Find the ratio in which  $F$  divides  $AB$ . {Ans.  $2:1$ }
211. The median  $AD$  of a triangle  $ABC$  is bisected at  $E$  and  $BE$  is produced to meet  $AC$  in  $F$ , show by vector



method that  $EF = \frac{1}{4} BF$ .

212.  $ABC$  is a triangle,  $E$  and  $F$  are mid-points of  $AC$  and  $AB$  respectively.  $CP$  is drawn parallel to  $AB$  to meet  $BE$  produced in  $P$ . Show that  $\Delta FEP = \Delta FCE = \frac{1}{4} \Delta ABC$ .
213. The points  $D, E, F$  divide sides  $BC, CA, AB$  of a triangle  $ABC$  in the ratio 1:2. The pair of lines  $AD, BE; BE, CF; CF, AD$  meet at  $P, Q$  and  $R$  respectively. Prove that area of  $\Delta PQR = \frac{1}{7} \Delta ABC$ .
214. Prove that the medians of a triangle are concurrent and find the point of concurrency.
215. The lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.
216. The joins of the mid-points of the opposite edges of a tetrahedron intersect and bisect each other.

### CATEGORY-19.18. ANGULAR BISECTOR OF TWO STRAIGHT LINES

217. Find both the angular bisectors of the lines  $\vec{r} = (-\hat{i} - 2\hat{j} + 2\hat{k}) + \lambda(2\hat{i} + \hat{j} - \hat{k})$  and  $\vec{r} = (-3\hat{j} + 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ . {Ans.  $\vec{r} = (\hat{i} - \hat{j} + \hat{k}) + \lambda((2\sqrt{7} \pm \sqrt{3})\hat{i} + (\sqrt{7} \pm 2\sqrt{3})\hat{j} - (\sqrt{7} \pm 3\sqrt{3})\hat{k})$ }
218. Find the incentre of the triangle having vertices  $2\hat{i} - 3\hat{j} + \hat{k}$ ,  $\hat{i} - \hat{j} - \hat{k}$  and  $\hat{i} + \hat{j} + 3\hat{k}$ .
219. In a triangle  $ABC$ ,  $c = 6$ ,  $b = 8$  and the angle  $A = 90^\circ$ .  $AM$  and  $BN$  are the bisectors of the angles  $A$  and  $B$ . Find the angle between the lines  $AM$  and  $BN$ . {Ans.  $\cos^{-1} \frac{1}{\sqrt{10}}$ }
220. In parallelogram  $ABCD$  the internal bisectors of the consecutive angles  $B$  and  $C$  intersect at  $P$ . Use vector method to find  $\angle BPC$ . {Ans.  $90^\circ$ }
221. Prove that internal bisectors of the angles of a triangle are concurrent.

### CATEGORY-19.19. SHORTEST DISTANCE BETWEEN PARALLEL AND SKEW LINES

222. Find the shortest distance between the two lines  $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} - 3\hat{j} + \hat{k})$  and  $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \lambda(-2\hat{i} + 3\hat{j} - \hat{k})$ . {Ans.  $\sqrt{\frac{3}{14}}$ }
223. Find the shortest distance between the lines  $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$  and  $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$ . {Ans.  $\frac{\sqrt{101}}{3}$ }
224. Find the shortest distance between two lines, one joining the points  $(-1, 2, -3)$  and  $(-16, 6, 4)$  and the other line joining the points  $(1, -1, 3)$  and  $(4, 9, 7)$ . {Ans. 7 units}
225. Show that the shortest distance between a diagonal of a rectangular parallelepiped the lengths of whose three coterminal edges are  $a, b, c$  and the edges not meeting it are  $\frac{bc}{\sqrt{b^2 + c^2}}, \frac{ca}{\sqrt{c^2 + a^2}}, \frac{ab}{\sqrt{a^2 + b^2}}$ .

### CATEGORY-19.20. EQUATION OF A PLANE

226. Find the equation of the plane, through the point  $2\hat{i} + 3\hat{j} - \hat{k}$  and perpendicular to the vector  $3\hat{i} - 2\hat{j} - 2\hat{k}$ . Also find the perpendicular distance from the origin to this plane. {Ans.  $\vec{r} \cdot (3\hat{i} - 2\hat{j} - 2\hat{k}) = 2, \frac{2}{\sqrt{17}}$  units}

227. Find the equation of the plane passing through point  $\hat{i} + 2\hat{j} + 5\hat{k}$  and containing vectors  $5\hat{i} + 7\hat{j} + 9\hat{k}$  and  $3\hat{i} + 2\hat{j} - \hat{k}$ . Find its distance from origin. {Ans.  $\vec{r} = \hat{i} + 2\hat{j} + 5\hat{k} + \lambda(5\hat{i} + 7\hat{j} + 9\hat{k}) + \mu(3\hat{i} + 2\hat{j} - \hat{k})$  or  $\vec{r} \cdot (25\hat{i} - 32\hat{j} + 11\hat{k}) = 16, \frac{16}{\sqrt{1770}}$ }
228. Find the distance of the plane  $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} - 3\hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} - \hat{k})$  from origin. {Ans.  $\frac{9}{\sqrt{26}}$ }
229. Find the unit vector perpendicular to the plane passing through points  $P(\hat{i} - \hat{j} + 2\hat{k})$ ,  $Q(2\hat{i} - \hat{k})$  and  $R(2\hat{j} + \hat{k})$ . {Ans.  $\frac{1}{\sqrt{6}}(2\hat{i} + \hat{j} + \hat{k})$ }
230. Find the equation of the plane passing through three points  $-2\hat{i} + 6\hat{j} - 6\hat{k}$ ,  $-3\hat{i} + 10\hat{j} - 9\hat{k}$  and  $-5\hat{i} - 6\hat{k}$ . Also find its distance from origin. {Ans.  $\vec{r} \cdot (2\hat{i} - \hat{j} - 2\hat{k}) = 2, \frac{2}{3}$ }
231. The vertices of a triangle  $ABC$  are  $(4, 2, 3)$ ;  $(1, 3, 1)$  and  $(-5, 5, -2)$  respectively. Find the length of the perpendicular drawn from origin  $O$  to plane of  $\Delta ABC$ . {Ans.  $\sqrt{10}$ }
232. If  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are vectors from the origin to the points  $A, B, C$  show that  $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$  is perpendicular to the plane  $ABC$ .
233. Find the equation of the plane  $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 2$  in parametric form. {Ans.  $\mathbf{r} = 2\mathbf{i} + \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{k})$ }

#### CATEGORY-19.21. POINT AND PLANE

234. Find the equation of the plane passing through the point  $A(3, -2, 1)$  and perpendicular to the vector  $4\hat{i} + 7\hat{j} - 4\hat{k}$ . If  $PM$  be the perpendicular from  $P(1, 2, -1)$  to this plane, find its length. {Ans.  $\vec{r} \cdot (4\hat{i} + 7\hat{j} - 4\hat{k}) = -6, \frac{28}{9}$  units}
235. Let points  $P, Q, R$  have position vectors  $3\hat{i} - 2\hat{j} - \hat{k}$ ,  $\hat{i} + 3\hat{j} + 4\hat{k}$  and  $2\vec{i} + \vec{j} - 2\vec{k}$  respectively. Find the distance of  $P$  from the plane  $OQR$ . Also find the foot of perpendicular from  $P$  on the plane  $OQR$ . {Ans.  $3, \frac{1}{13}(11\hat{i} - 8\hat{j} - 2\hat{k})$ }
236. Find the distance of  $P(\mathbf{i} + \mathbf{j} + \mathbf{k})$  from the plane passing through the three points  $A(2\mathbf{i} + \mathbf{j} + \mathbf{k}), B(\mathbf{i} + 2\mathbf{j} + \mathbf{k}), C(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ . Also find the position vector of the foot of perpendicular from  $P$  on the plane. {Ans.  $\frac{1}{\sqrt{3}}, \frac{4}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ }
237. Show that the perpendicular distance of a point whose position vector is  $\vec{a}$  from the plane through three points with position vectors  $\vec{b}, \vec{c}, \vec{d}$  is  $\frac{[\vec{b}\vec{c}\vec{d}] + [\vec{c}\vec{a}\vec{d}] + [\vec{a}\vec{b}\vec{d}] - [\vec{a}\vec{b}\vec{c}]}{|\vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b}|}$ .

#### CATEGORY-19.22. LINE AND PLANE

238. Find the angle between the line  $\vec{r} = (\hat{i} - 2\hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} - \hat{k})$  and the plane  $\vec{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 3$ . {Ans.  $\sin^{-1} \frac{1}{6}$ }

239. Find the angle between the line  $\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(-\hat{i} + \hat{j} + \hat{k})$  and the plane  $\vec{r} \cdot (3\hat{i} + 2\hat{j} - \hat{k}) = 4$ . {Ans.  $\sin^{-1}\left(-\frac{2}{\sqrt{42}}\right)$ }
240. Check whether the line  $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$  lies in the plane  $\vec{r} \cdot (\hat{i} - \hat{j} - 2\hat{k}) = 5$  or not? {Ans. Yes}
241. Find the equation of the plane containing the lines  $\vec{r} = (6\hat{i} - 4\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 4\hat{k})$  and  $\vec{r} = (\hat{i} + 2\hat{j} - 5\hat{k}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ .
242. Find the equation of the plane containing the lines  $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$  and  $\vec{r} = (\hat{i} + \hat{j} - 2\hat{k}) + \lambda(2\hat{i} - 2\hat{j} + 2\hat{k})$ .
243. Find the point of intersection of the line  $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(\hat{i} + \hat{j} + 2\hat{k})$  and the plane  $\vec{r} \cdot (\hat{i} + 2\hat{j} + \hat{k}) = 1$ .
244. A straight line  $\vec{r} = \vec{a} + \lambda\vec{b}$  meets the plane  $\vec{r} \cdot \vec{n} = 0$  in  $P$ . Find the position vector of  $P$ . {Ans.  $\vec{a} - \frac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}} \vec{b}$ }
245. The points  $A, B, C, D$  and  $E$  are respectively  $A(1, 2, 1), B(2, 1, 2), C(0, -4, 4), D(2, -2, 2)$  and  $E(4, 1, 2)$ . The line  $AB$  cuts the plane  $CDE$  in the point  $P$ . Prove by vector method that the point  $P$  is  $(3, 0, 3)$ .
246. Find the vector equation of the straight line through the points  $\hat{i} - 2\hat{j} + \hat{k}$  and  $3\hat{k} - 2\hat{j}$ . Find also the point in which this line cuts the plane through the origin and the points  $4\hat{j}$  and  $2\hat{i} + \hat{k}$ . {Ans.  $\vec{r} = (1-t)\hat{i} - 2\hat{j} + (1+2t)\hat{k}, \frac{1}{5}(6\hat{i} - 10\hat{j} + 3\hat{k})$ }
247.  $ABC$  is a triangle whose vertices are  $\hat{i}, \hat{j}$  and  $\hat{k}$ . Find the distance of the point  $P(1, -2, 3)$  from the plane of the triangle  $ABC$ , measured along a line through  $P$  and parallel to the vector  $\hat{i} + \hat{j}$ . {Ans.  $\frac{1}{\sqrt{2}}$  units}
248. Find the image of the point  $2\hat{i} - 3\hat{j} + \hat{k}$  w.r.t the plane  $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 1$ . {Ans.  $-\frac{4}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{7}{3}\hat{k}$ }
249. Find the image of the line  $\vec{r} = (\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} + 2\hat{j} - \hat{k})$  w.r.t the plane  $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 1$ . {Ans.  $\vec{r} = \left(-\frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{1}{3}\hat{k}\right) + \lambda(8\hat{i} + 4\hat{j} - \hat{k})$ }
250. If a straight line is equally inclined to three coplanar straight lines, prove that it is perpendicular to their plane.

### CATEGORY-19.23. TWO OR MORE PLANES

251. Two planes are perpendicular to one another. One of them contains vectors  $\vec{a}$  and  $\vec{b}$  and the other contains vectors  $\vec{c}$  and  $\vec{d}$ , then find  $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ . {Ans. 0}
252. Find the angle between the planes  $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 2$  and  $\vec{r} = (\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} + 2\hat{j} - 3\hat{k})$ . {Ans.  $\cos^{-1} \sqrt{\frac{6}{83}}$ }
253. A non-zero vector  $\vec{a}$  is parallel to the line of intersection of the plane parallel to the vectors  $\hat{i}, \hat{i} + \hat{j}$  and

the plane parallel to the vectors  $\hat{i} - \hat{j}$ ,  $\hat{i} + \hat{k}$ . Find the angle between the vectors  $\vec{a}$  and  $\hat{i} - 2\hat{j} + 2\hat{k}$ . Also find the angle between the two given planes. {Ans.  $\frac{\pi}{4}$ ,  $\cos^{-1} \frac{2}{\sqrt{3}}$ }

254. Find the equation of line of intersection of the planes  $\vec{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 2$  and  $\vec{r} \cdot (\hat{i} + 2\hat{j} + \hat{k}) = 3$ . {Ans.  $\vec{r} = (\hat{i} + \hat{j}) + \lambda(3\hat{i} - \hat{j} - \hat{k})$ }
255. Obtain the plane containing the line  $\vec{r} = 2\hat{i} + \lambda(\hat{j} - \hat{k})$  and perpendicular to the plane  $\vec{r} \cdot (\hat{i} + \hat{k}) = 3$ . Also obtain the point where this plane meets the line  $\vec{r} = \lambda(2\hat{i} + 3\hat{j} + \hat{k})$ . {Ans.  $\vec{r} \cdot (\hat{i} - \hat{j} - \hat{k}) = 2$ ,  $(-2, -3, -1)$ }
256. Find the equation of the plane passing through the intersection of the planes  $\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 5$  and  $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$  and passing through the point  $(2, 1, -2)$ . {Ans.  $\vec{r} \cdot (3\hat{i} + 2\hat{j}) = 8$ }
257. Find the equation of the plane passing through the intersection of the planes  $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$  and  $\vec{r} \cdot (\hat{i} - \hat{j}) = -4$  and perpendicular to  $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = -8$ . {Ans.  $\vec{r} \cdot (-5\hat{i} + 2\hat{j} + 12\hat{k}) = 47$ }

#### CATEGORY-19.24. GEOMETRICAL OPERATIONS ON VECTORS

258. If  $\vec{a}$ ,  $\vec{b}$  are vectors forming consecutive sides of a regular hexagon  $ABCDEF$ , then find the vector representing side  $CD$ . {Ans.  $\vec{b} - \vec{a}$ }
259. If  $\vec{a}$ ,  $\vec{b}$  represent the diagonals of a rhombus, then show that  $\vec{a} \cdot \vec{b} = 0$ .
260. If  $ABCD$  is a rhombus whose diagonals cut at the origin  $O$ , then show that  $\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = \vec{0}$ .
261. Let  $D$ ,  $E$ ,  $F$  be the middle points of the sides  $BC$ ,  $CA$ ,  $AB$  respectively of a triangle  $ABC$ . Then show that  $\vec{AD} + \vec{BE} + \vec{CF} = \vec{0}$ .
262. In a trapezoid  $ABCD$  the vector  $\vec{BC} = \lambda \vec{AD}$ . Prove that the vector  $\vec{p} = \vec{AC} + \vec{BD}$  is collinear with  $\vec{AD}$  (and, hence, with  $\vec{BC}$ ) and find the coefficient  $\alpha$  in the notation  $\vec{p} = \alpha \vec{AD}$ . {Ans.  $\alpha = 1 + \lambda$ }
263. Given a parallelogram  $ABCD$  ( $AD \parallel BC$ ,  $AB \parallel CD$ ). A point  $K$  is chosen on the side  $AD$ , and a point  $L$  on the side  $AC$  such that  $|\vec{AK}| = \frac{|\vec{AD}|}{5}$  and  $|\vec{AL}| = \frac{|\vec{AC}|}{6}$ . Prove that the vectors  $\vec{KL}$  and  $\vec{BL}$  are collinear and find the proportionality factor  $\lambda$  in the notation  $\vec{KL} = \lambda \vec{BL}$ . {Ans.  $\lambda = -\frac{1}{5}$ }
264. A straight line is drawn through the vertex  $C$  of the square  $ABCD$  parallel to the diagonal  $BD$ , which cuts the straight line  $AD$  at a point  $E$ .  $Q$  is the point at which the diagonals of the square meet. Express the sum of the vectors  $\vec{AB}$  and  $\vec{CE}$  in terms of the vectors  $\vec{DC}$  and  $\vec{CQ}$ . {Ans.  $\vec{AB} + \vec{CE} = -(\vec{DC} + 2\vec{CQ})$ }
265. Assume that  $ABCD$  is a parallelogram, and  $AD \parallel BC$ ,  $K$  is the midpoint of  $BC$ ,  $L$  is the mid point of  $DC$ . Introduce the designations  $\vec{AK} = \vec{a}$ ,  $\vec{AL} = \vec{b}$  and express the vectors  $\vec{BD}$  and  $\vec{AC}$  in terms of  $\vec{a}$  and  $\vec{b}$ . {Ans.  $\vec{AC} = \frac{2}{3}\vec{a} + \frac{2}{3}\vec{b}$ ,  $\vec{BD} = -2\vec{a} + 2\vec{b}$ }
266. Being given vectors  $\vec{p}$  and  $\vec{q}$  on which a parallelogram is constructed, use them to express the vector which coincides with the altitude of the parallelogram, which is perpendicular to the side  $\vec{p}$ . {Ans.}

$$\pm \left\{ \mathbf{q} - \left( \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}|^2} \right) \mathbf{p} \right\}$$

267. Given a regular hexagon  $ABCDEF$ ,  $M$  is the midpoint of  $DE$ ,  $N$  is the midpoint of  $AM$ ,  $P$  is the midpoint of  $BC$ . Resolve the vector  $\vec{NP}$  into components with respect to the vectors  $\vec{AB}$  and  $\vec{AF}$ . {Ans.

$$\vec{NP} = \frac{3}{4} \vec{AB} - \frac{1}{2} \vec{AF} \}$$

268. If  $A_1, A_2, \dots, A_n$  are the vertices of regular plane polygon with  $n$  sides and  $O$  is its centre, show that

$$\sum_{i=1}^{n-1} \left( \vec{OA}_i \times \vec{OA}_{i+1} \right) = (1-n) \left( \vec{OA}_2 \times \vec{OA}_1 \right).$$

### CATEGORY-19.25. ADDITIONAL QUESTIONS

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# Mathematics for IIT-JEE

By **Er. Sanjiva Dayal**, *B.Tech. (I.I.T. Kanpur)*

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## *PART-V*

### *VECTOR AND THREE DIMENSIONAL GEOMETRY*

#### *CHAPTER-20*

### *THREE DIMENSIONAL COORDINATE GEOMETRY*

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## **CHAPTER-20**

### **THREE DIMENSIONAL COORDINATE GEOMETRY**

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#### **LIST OF THEORY SECTIONS**

- 20.1. Points
- 20.2. Area And Volume
- 20.3. Direction Cosines And Direction Ratios Of A Straight Line
- 20.4. Equation Of A Straight Line In Symmetrical Form
- 20.5. Points And Lines
- 20.6. Equation Of A Plane
- 20.7. Point(s) And Plane
- 20.8. Line(s) And Plane
- 20.9. Two (Or More) Planes
- 20.10. Equation Of A Line In Unsymmetrical Form

#### **LIST OF QUESTION CATEGORIES**

- 20.1. Points
- 20.2. Area And Volume
- 20.3. Direction Cosines And Direction Ratios
- 20.4. Equation Of A Line In Symmetrical Form
- 20.5. Angle Between Two Lines
- 20.6. Point Of Intersection Of Two Lines
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- 20.8. Shortest Distance Between Parallel And Skew Lines
- 20.9. Equation Of A Plane
- 20.10. Points And Plane
- 20.11. Angle Between A Line And A Plane
- 20.12. A Line Lying On A Plane
- 20.13. Coplanarity Of Two Lines And Plane Containing Them
- 20.14. Point Of Intersection Of A Line And A Plane
- 20.15. Angle Between Two Planes
- 20.16. Parallel Planes
- 20.17. Intersecting Planes
- 20.18. Planes Bisecting The Angle Between Two Planes
- 20.19. Equation Of A Line In Unsymmetrical Form
- 20.20. Locus Problems
- 20.21. Additional Questions



## CHAPTER-20

### THREE DIMENSIONAL COORDINATE GEOMETRY

#### SECTION-20.1. POINTS

##### 1. Representation of a point in three dimensional space by coordinates

- i. Coordinates of a point  $P$  in three dimensional space w.r.t.  $OXYZ$  axes is ordered triad of real numbers written as  $(x, y, z)$  such that the coordinates are the distances from the origin of the feet of the perpendiculars from the point  $P$  on the respective coordinate axes.
- ii. If system vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are along  $x$ -axis,  $y$ -axis,  $z$ -axis respectively, then vector  $\overrightarrow{OP} = (x, y, z)$  is the position vector of point  $P$ .

##### 2. Equation of a curve/ region

- i. The equation of a curve/ region is the relation which exists between the coordinates of every point on the curve/ region, and which holds for no other points except those lying on the curve/ region.
- ii. Equation of  $x$ -axis is  $y = z = 0$ ; equation of  $y$ -axis is  $x = z = 0$ ; equation of  $z$ -axis is  $x = y = 0$ .
- iii. Equation of  $x$ - $y$  plane is  $z = 0$ ; equation of  $y$ - $z$  plane is  $x = 0$ ; equation of  $x$ - $z$  plane is  $y = 0$ .

##### 3. Distance formula

- i. Distance between the points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- ii. Therefore, distance of the point  $P(x_1, y_1, z_1)$  from the origin  $O(0,0,0)$  is  $OP = \sqrt{x_1^2 + y_1^2 + z_1^2}$ .

##### 4. Section formula

- i. Coordinates of the point which divides the line segment joining points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$

$$\text{internally in the ratio } m_1 : m_2 \text{ are } \left( \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2} \right).$$

- ii. Coordinates of the mid-point of the line segment joining points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  are

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

- iii. Coordinates of the point which divides the line segment joining points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$

$$\text{externally in the ratio } m_1 : m_2 \text{ are } \left( \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2} \right).$$

##### 5. Condition for four points to be coplanar

$$\text{If } \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0, \text{ then the four points } (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) \text{ and } (x_4, y_4, z_4) \text{ are coplanar,}$$

if  $\neq 0$ , then points are non-coplanar.

##### 6. Centroid of a triangle

Centroid of the triangle with vertices  $A(x_1, y_1, z_1)$ ,  $B(x_2, y_2, z_2)$  and  $C(x_3, y_3, z_3)$  is

$$\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right).$$

### 7. Centroid of a Tetrahedron

Centroid of the tetrahedron with vertices  $A(x_1, y_1, z_1)$ ,  $B(x_2, y_2, z_2)$ ,  $C(x_3, y_3, z_3)$  and  $D(x_4, y_4, z_4)$  is

$$\left( \frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4} \right).$$

## SECTION-20.2. AREA AND VOLUME

### 1. Area of a triangle

Area of a triangle whose vertices are  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$  and  $(x_3, y_3, z_3)$  is

$$\Delta = \frac{1}{2} \sqrt{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2}.$$

### 2. Volume of a tetrahedron

Volume of a tetrahedron whose vertices are  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ ,  $(x_3, y_3, z_3)$  and  $(x_4, y_4, z_4)$  is

$$V = \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

## SECTION-20.3. DIRECTION COSINES AND DIRECTION RATIOS OF A STRAIGHT LINE

### 1. Direction cosines of a line

- If  $\alpha, \beta, \gamma$  are the angles which a particular direction of the line  $L$  makes with the positive  $x$ -axis,  $y$ -axis,  $z$ -axis respectively, then  $\cos \alpha$ ,  $\cos \beta$ ,  $\cos \gamma$  are known as the direction cosines of the line  $L$  and are generally denoted by  $l, m, n$  respectively i.e.  $l = \cos \alpha$ ,  $m = \cos \beta$ ,  $n = \cos \gamma$ .
- Clearly, the opposite direction of the line  $L$ , makes angles  $\pi - \alpha$ ,  $\pi - \beta$ ,  $\pi - \gamma$  with the positive  $x$ -axis,  $y$ -axis,  $z$ -axis respectively. Therefore, direction cosines of the line  $L$  are  $\cos(\pi - \alpha)$ ,  $\cos(\pi - \beta)$ ,  $\cos(\pi - \gamma)$  or  $-l$ ,  $-m$ ,  $-n$  also.
- $l^2 + m^2 + n^2 = 1$ .
- Parallel lines have same direction cosines.
- Unit vectors  $\hat{i} = (l, m, n)$  and  $-\hat{i} = (-l, -m, -n)$  are parallel to the line.
- Direction cosines of  $x$ -axis are  $1, 0, 0$  or  $-1, 0, 0$ ;  $y$ -axis are  $0, 1, 0$  or  $0, -1, 0$  and  $z$ -axis are  $0, 0, 1$  or  $0, 0, -1$ .

### 2. Direction ratios of a line

- If  $l, m, n$  are the direction cosines of a line  $L$ , then  $kl, km, kn$  ( $k \neq 0$ ) are the direction ratios of the line  $L$ .
- If  $a, b, c$  are direction ratios of the line  $L$  having direction cosines  $l, m, n$  then  $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$ .
- If  $a, b, c$  are the direction ratios of the line  $L$  then its direction cosines are  $\frac{a}{\sqrt{a^2 + b^2 + c^2}}$ ,

$$\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}} \text{ or } -\frac{a}{\sqrt{a^2 + b^2 + c^2}}, -\frac{b}{\sqrt{a^2 + b^2 + c^2}}, -\frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

iv. If  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  are the direction ratios of the same line or of parallel lines then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

v. Vector  $\vec{r} = (a, b, c)$  is parallel to the line.

### 3. Direction ratios of a line passing through two given points

i. Direction ratios of a line passing through two points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$  are  $x_2 - x_1, y_2 - y_1, z_2 - z_1$ .

ii. Vector  $\vec{r} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$  is parallel to the line.

## SECTION-20.4. EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM

### 1. Equation of a line passing through a given point and given direction ratios or direction cosines (Symmetrical form)

i. Equation of a straight line passing through a fixed point  $(x_1, y_1, z_1)$  and having direction ratios  $a, b, c$  or direction cosines  $l, m, n$  is  $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$  or  $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ .

ii. The coordinates of any point on the line  $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$  are  $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$ , where  $\lambda \in R$ .

iii. Equation of x-axis is  $\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$ ; equation of y-axis is  $\frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0}$ ; equation of z-axis is  $\frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1}$ .

### 2. Equation of a line passing through two given points

Equation of a line passing through two given points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

## SECTION-20.5. POINTS AND LINES

### 1. Angle between two lines

i. Let the direction ratios of the two lines be  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$ ; then the angle between them is

$$\theta = \cos^{-1} \left( \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right) = \cos^{-1} |l_1 l_2 + m_1 m_2 + n_1 n_2|, \text{ where } l_1, m_1, n_1 \text{ and } l_2, m_2, n_2$$

are their direction cosines.

ii. If  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$  or  $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \pm 1$ , then lines are parallel.

iii. If  $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$  or  $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ , then lines are perpendicular.

### 2. Point of intersection of two lines

- i. Let the two lines be  $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$  and  $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ .
- ii. Coordinates of a general point on these lines are  $(x_1 + a_1\lambda, y_1 + b_1\lambda, z_1 + c_1\lambda)$  and  $(x_2 + a_2\mu, y_2 + b_2\mu, z_2 + c_2\mu)$  respectively.
- iii. If these lines intersect then they have a common point i.e.  $\begin{cases} x_1 + a_1\lambda = x_2 + a_2\mu \\ y_1 + b_1\lambda = y_2 + b_2\mu \\ z_1 + c_1\lambda = z_2 + c_2\mu \end{cases}$ . If this system has unique solution in  $\lambda, \mu$  then lines intersect. If this system has no solution then lines do not intersect.
- iv. If the lines intersect, find  $\lambda$  (or  $\mu$ ) and substitute in the coordinates of general point to obtain the coordinates of point of intersection.
- 3. Foot of the perpendicular and perpendicular distance of a point from a line**
- i. Let the point  $P(x_2, y_2, z_2)$  and the line  $L \equiv \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ .
- ii. Let the foot of perpendicular from point  $P$  to line  $L$  be  $Q \equiv (x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$ . Then Direction ratios of  $PQ$  are  $x_1 + \lambda a - x_2, y_1 + \lambda b - y_2, z_1 + \lambda c - z_2$ .
- iii. Since  $PQ$  is perpendicular to the line  $L$ , therefore
- $$(x_1 + \lambda a - x_2)a + (y_1 + \lambda b - y_2)b + (z_1 + \lambda c - z_2)c = 0 \Rightarrow \lambda = \frac{a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1)}{a^2 + b^2 + c^2}.$$
- Substitute this value of  $\lambda$  to obtain coordinates of point  $Q$ . Now using distance formula find the perpendicular distance  $PQ$ .

**4. Shortest distance between two skew lines**

- i. The shortest distance between two skew lines  $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$  and  $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$  is given by

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - a_1c_2)^2 + (a_1b_2 - a_2b_1)^2}}.$$

- ii. If  $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0$  then lines are skew, if  $= 0$  then lines are intersecting/ parallel.

**SECTION-20.6. EQUATION OF A PLANE**

**1. Equation of the plane passing through a given point and given normal**

- i. Equation of the plane passing through a given point  $(x_1, y_1, z_1)$  and normal having direction ratios  $a, b, c$  is  $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ .
- ii. Equation of  $y$ - $z$  plane is  $x = 0$ ;  $z$ - $x$  plane is  $y = 0$  and  $x$ - $y$  plane is  $z = 0$ .

**2. General equation of a plane**

- i. General equation of a plane is  $ax + by + cz + d = 0$ , where  $a, b, c, d$  are real numbers such that  $a, b, c$  are

not all zero.  $a, b, c$  are the direction ratios of its normal.

ii. Three conditions are required to find the equation of a plane.

iii. A point  $\left(\lambda, \mu, \frac{-a\lambda - b\mu - d}{c}\right)$  lies on the plane  $\forall \lambda, \mu \in R$ .

### 3. Equation of a plane passing through three given points

i. Let the three given points be  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$  and  $(x_3, y_3, z_3)$ . Substitute these points in the general equation of the plane and find  $a, b, c, d$  by Cramer's rule.

ii. Equations of the plane can also be written as 
$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

### 4. Intercept form of a plane

i. The equation of a plane whose intercepts are  $a, b$  and  $c$  on  $x$ -axis,  $y$ -axis and  $z$ -axis respectively, is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

ii. Transforming the general equation  $Ax + By + Cz + D = 0$  in intercept form, we get

$$\frac{x}{\left(-\frac{D}{A}\right)} + \frac{y}{\left(-\frac{D}{B}\right)} + \frac{z}{\left(-\frac{D}{C}\right)} = 1. \text{ Therefore, intercepts are } -\frac{D}{A}, -\frac{D}{B}, -\frac{D}{C}.$$

### 5. Equation of the plane in normal form

i. If  $l, m, n$  are direction cosines of the normal to a given plane which is at a distance  $p$  from the origin, then the equation of the plane is  $lx + my + nz = \pm p$ .

ii. Transforming the general equation  $Ax + By + Cz + D = 0$  to normal form, we get

$$\pm \frac{A}{\sqrt{A^2 + B^2 + C^2}} x \pm \frac{B}{\sqrt{A^2 + B^2 + C^2}} y \pm \frac{C}{\sqrt{A^2 + B^2 + C^2}} z = \mp \frac{D}{\sqrt{A^2 + B^2 + C^2}}$$

Therefore,  $p = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$  and  $\frac{A}{\sqrt{A^2 + B^2 + C^2}}, \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \frac{C}{\sqrt{A^2 + B^2 + C^2}}$  are the direction cosines of the normal to the plane.

## SECTION-20.7. POINT(S) AND PLANE

### 1. Position of two points w.r.t. a plane

Two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  lie on the same or opposite sides of the plane  $ax + by + cz + d = 0$  if  $(ax_1 + by_1 + cz_1 + d)(ax_2 + by_2 + cz_2 + d)$  is  $>$  or  $<$  0.

### 2. Distance of a point from a plane

i. The length of perpendicular from a point  $P(x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$  is given by

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

ii. Therefore, length of perpendicular from origin is  $\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}.$

**SECTION-20.8. LINE(S) AND PLANE****1. Angle between a line and a plane**

- i. The angle between a line and a plane is the complement of the angle between the line and normal to the plane.
- ii. Angle between the line having direction ratios  $a, b, c$  and the plane  $a_1x + b_1y + c_1z + d_1 = 0$  is

$$\theta = \sin^{-1} \left( \frac{|aa_1 + bb_1 + cc_1|}{\sqrt{a^2 + b^2 + c^2} \sqrt{a_1^2 + b_1^2 + c_1^2}} \right).$$

- iii. If  $aa_1 + bb_1 + cc_1 = 0$ , then line is parallel to the plane.

- iv. If  $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$ , the line is perpendicular to the plane.

**2. Condition for a line to lie on a plane**

The line  $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$  lies in the plane  $a_1x + b_1y + c_1z + d_1 = 0$  if  $\begin{cases} a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0 \\ aa_1 + bb_1 + cc_1 = 0 \end{cases}$ .

**3. Condition for coplanarity of two lines and equation of the plane containing them**

- i. Lines  $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$  and  $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$  are coplanar (intersecting/ parallel), if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

- ii. If the lines are intersecting, then the equation of the plane containing them is  $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

$$\text{or } \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

**4. Point of intersection of a line and a plane which are not parallel**

- i. Let the line be  $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$  and the plane be  $a_1x + b_1y + c_1z + d_1 = 0$ .
- ii. Coordinates of any point on the line is  $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$ . If this point lies on the plane then  $a_1(x_1 + a\lambda) + b_1(y_1 + b\lambda) + c_1(z_1 + c\lambda) + d_1 = 0 \Rightarrow \lambda = -\frac{(a_1x_1 + b_1y_1 + c_1z_1 + d_1)}{aa_1 + bb_1 + cc_1}$ . Substitute this value of  $\lambda$  to obtain coordinates of the point of intersection.

**SECTION-20.9. TWO (OR MORE) PLANES****1. Angle between two planes**

- i. Angle between two planes is defined as the angle between their normals.
- ii. The angle between the two planes  $a_1x + b_1y + c_1z + d_1 = 0$  and  $a_2x + b_2y + c_2z + d_2 = 0$  is given by

$$\theta = \cos^{-1} \left( \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right).$$

iii. If  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ , then planes are parallel.

iv. If  $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$ , then planes are perpendicular.

## 2. Equation of a plane parallel to a given plane

Equation of a plane parallel to the plane  $ax + by + cz + d = 0$  is  $ax + by + cz + \lambda = 0$ .

## 3. Distance between parallel planes

Distance between two parallel planes  $ax + by + cz + d_1 = 0$  and  $ax + by + cz + d_2 = 0$  is  $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$ .

## 4. Equation of a plane passing through the intersection of two given planes

- The equation of a plane passing through the intersection of the planes  $a_1 x + b_1 y + c_1 z + d_1 = 0$  and  $a_2 x + b_2 y + c_2 z + d_2 = 0$  is  $(a_1 x + b_1 y + c_1 z + d_1) + \lambda(a_2 x + b_2 y + c_2 z + d_2) = 0$ , where  $\lambda$  is any real parameter.
- Equations of planes passing through  $x$ -axis is  $y + \lambda z = 0$ ;  $y$ -axis is  $z + \lambda x = 0$  and  $z$ -axis is  $x + \lambda y = 0$ .

## 5. Equation of planes bisecting the angle between two given planes

The equation of the planes bisecting the angles between the planes  $a_1 x + b_1 y + c_1 z + d_1 = 0$  and

$$a_2 x + b_2 y + c_2 z + d_2 = 0 \text{ are } \frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

## SECTION-20.10. EQUATION OF A LINE IN UNSYMMETRICAL FORM

### 1. Equation of a line as intersection of two planes (Unsymmetrical form)

- The equation of a line in unsymmetrical form is  $a_1 x + b_1 y + c_1 z + d_1 = 0$ ,  $a_2 x + b_2 y + c_2 z + d_2 = 0$ , where  $a_1 : b_1 : c_1 \neq a_2 : b_2 : c_2$ . It represents the line of intersection of the two planes  $a_1 x + b_1 y + c_1 z + d_1 = 0$  and  $a_2 x + b_2 y + c_2 z + d_2 = 0$ .

### 2. Finding equation of a line in symmetrical form from equation in unsymmetrical form

- Let  $a, b, c$  be the direction ratios of the line, then as the line lies in both the planes, it must be perpendicular to normals to these planes. Therefore,  $\begin{cases} aa_1 + bb_1 + cc_1 = 0 \\ aa_2 + bb_2 + cc_2 = 0 \end{cases}$ . Solve for  $a, b, c$ . Choose a point which lies on both the planes, i.e. lies on the line and find the equation of the line.
- To find a general point on this line, put  $x = \alpha$  in the equation of the planes and solve for  $y$  and  $z$  in terms of  $\alpha$ .

**EXERCISE-20****CATEGORY-20.1. POINTS**

1. Show that the points  $(0,7,10)$ ,  $(-1,6,6)$ , and  $(-4,9,6)$  form an isosceles right angled triangle.
2. Show that the points  $(1,2,3)$ ,  $(2,3,1)$  and  $(3,1,2)$  form an equilateral triangle.
3. Find the center of the sphere which passes through the points  $O(0,0,0)$ ,  $A(a,0,0)$ ,  $B(0,b,0)$  and  $C(0,0,c)$ .  
 {Ans.  $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ }
4. If  $P(-1,-1,-1)$ ,  $Q(1,3,2)$ ,  $R(5,11,8)$  are three points in a straight line, find the ratio in which  $Q$  divides  $PR$ .  
 {Ans. 1:2}
5. If  $P(3,2,-4)$ ,  $Q(5,4,-6)$  and  $R(9,8,-10)$  are collinear, then find the ratio in which  $R$  divides  $PQ$ . {Ans. 3:2 externally}
6. The mid-points of the sides of a triangle are  $(1,5,-1)$ ,  $(0,4,-2)$  and  $(2,3,4)$ . Find its vertices. {Ans.  $(1,2,3)$ ,  $(3,4,5)$ ,  $(-1,6,-7)$ }
7. Prove that the three points  $A$ ,  $B$  and  $C$  whose coordinates are  $(3,2,-4)$ ,  $(5,4,-6)$  and  $(9,8,-10)$  respectively are collinear.
8. Check whether that the points  $A$ ,  $B$  and  $C$  whose coordinates are  $(3,-2,4)$ ,  $(1,1,1)$  and  $(-1,4,2)$  respectively, are collinear or not.
9. Prove that the three points  $A(-2,3,5)$ ,  $B(1,2,3)$  and  $C(7,0,-1)$  are collinear. Also find the ratio in which  $C$  divides  $AB$ . {Ans. 3:2 externally}
10. Find the ratio in which the line joining the points  $(1,2,3)$  and  $(-3,4,-5)$  is divided by the  $x-y$  plane. Also find the coordinates of the point of division. {Ans. 3:5,  $\left(-\frac{1}{2}, \frac{11}{4}, 0\right)$ }

**CATEGORY-20.2. AREA AND VOLUME**

11. Find the area of the triangle whose vertices are  $A(1,2,3)$ ,  $B(2,-1,1)$  and  $C(1,2,-4)$ . {Ans.  $\frac{7\sqrt{10}}{2}$  sq. units}
12. Show that the area of a triangle whose vertices are the origin and the points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  is  

$$\frac{1}{2} \sqrt{(y_1 z_2 - y_2 z_1)^2 + (z_1 x_2 - z_2 x_1)^2 + (x_1 y_2 - x_2 y_1)^2}.$$

**CATEGORY-20.3. DIRECTION COSINES AND DIRECTION RATIOS**

13. Find the direction cosines of a line having direction ratios 1, -2, 3. {Ans.  $\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ }
14. Find the length of a segment of a line whose projections on the axes are 2, 3, 6. Also find the direction cosines of the line. {Ans. 7 units,  $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$ }
15. The projections of a directed line segment on the coordinate axes are 12, 4, 3. Find the DCs of the line.  
 {Ans.  $\frac{12}{13}, \frac{4}{13}, \frac{3}{13}$ }



16. If  $\alpha, \beta, \gamma$  are the angles which a directed line makes with the positive directions of the coordinate axes, then find the value of  $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$ . {Ans. 2}
17. Find the direction cosines of the line which is equally inclined to the axes. Find the total number of such lines. {Ans.  $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ ;  $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ ;  $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ ;  $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ ; 4 lines}
18. Find the direction cosines  $l, m, n$  of the two lines which are connected by the relations  $l + m + n = 0$  and  $mn - 2nl - 2lm = 0$ . {Ans.  $\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$ ;  $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}$ }
19. Find the direction cosines  $l, m, n$  of the two lines which are connected by the relations  $l - 5m + 3n = 0$  and  $7l^2 + 5m^2 - 3n^2 = 0$ . {Ans.  $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ ;  $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ }
20. Find the direction cosines of the sides and medians of the triangle  $ABC$  having vertices  $A \equiv (1, 3, 2)$ ,  $B \equiv (2, 1, 3)$  and  $C \equiv (3, 1, 2)$ .
21. Find the direction ratios and direction cosines of the diagonals of a unit cube having one vertex at origin and three edges along positive  $x, y, z$  axes.

#### CATEGORY-20.4. EQUATION OF A LINE IN SYMMETRICAL FORM

22. Vertices of a triangle are  $(2, 1, -1)$ ,  $(1, 2, 3)$  and  $(-1, 2, 1)$ . Find the equation of the sides and medians.
23. Check whether the line  $\frac{x-1}{2} = \frac{y+1}{4} = \frac{z+2}{3}$  passes through the point  $(2, 1, 3)$  or not?
24. Find the coordinates of the point lying on the line  $\frac{x-2}{3} = \frac{y+2}{4} = \frac{z-1}{2}$  having  $x$ -coordinate equal to 3.
25. Find the points in which the line  $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2}$  cuts the surface  $11x^2 - 5y^2 + z^2 = 0$ . {Ans.  $(1, 2, 3), (2, -3, 1)$ }

#### CATEGORY-20.5. ANGLE BETWEEN TWO LINES

26. If  $P, Q, R, S$  are four points with coordinates  $(2, 3, -1)$ ,  $(3, 5, -3)$ ,  $(1, 2, 3)$  and  $(3, 5, 7)$  respectively, prove that  $PQ$  is at right angles to  $RS$ .
27. Prove that the line joining the points  $(1, 2, 3)$  and  $(-1, -2, -3)$  is perpendicular to the line joining the points  $(-2, 1, 5)$  and  $(3, 3, 2)$ .
28. For what value of  $\lambda$ , the line joining the points  $P(2, 3, 4)$  and  $Q(1, -2, \lambda)$  is perpendicular to the line  $OP$ . {Ans.  $\frac{33}{4}$ }
29. If the vertices  $A, B$  and  $C$  of a triangle have coordinates  $(2, 3, 5)$ ,  $(-1, 3, 2)$  and  $(3, 5, -2)$  respectively, find the angles of the triangle  $ABC$ . {Ans.  $A = \cos^{-1} \frac{1}{\sqrt{3}}$ ,  $B = 90^\circ$ ,  $C = \cos^{-1} \sqrt{\frac{2}{3}}$ }
30. If  $P, Q, R, S$  are four points with coordinates  $(3, 4, 5)$ ,  $(4, 6, 3)$ ,  $(-1, 2, 4)$  and  $(1, 0, 5)$  respectively, then find the projection of  $PQ$  on  $RS$ . Also find the projection of  $RS$  on  $PQ$ . {Ans.  $\frac{4}{3}, \frac{4}{3}$ }

31. Find the direction ratios of a line which is perpendicular to the lines having direction ratios 2,-1,3 and 1,3,-2.
32. Find the value of  $\lambda$  for which the lines  $\frac{x-1}{1} = \frac{y-2}{\lambda} = \frac{z+1}{-1}$  and  $\frac{x+1}{-\lambda} = \frac{y+1}{2} = \frac{z-2}{1}$  are perpendicular to each other. {Ans. 1}
33. Show that the lines whose d.c.'s are given by  $l+m+n=0$  and  $2mn+3nl-5lm=0$  are at right angles.
34. Show that the lines whose direction cosines are given by the equations  $2l+2m-n=0$  and  $mn+nl+lm=0$  are at right angles.
35. Prove that the acute angle between the lines whose d.c.'s are given by the relations  $l+m+n=0$ ,  $l^2+m^2-n^2=0$  is  $\frac{\pi}{3}$ .
36. If a variable line in two adjacent positions has direction cosines  $l, m, n$  and  $l+\delta l, m+\delta m, n+\delta n$ , show that the small angle  $\delta\theta$  between the two positions is given by  $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$ .
37. Find the equations of the line through the point  $(x_1, y_1, z_1)$  at right angles to the lines  $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$  and  $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$ . {Ans.  $\frac{x-x_1}{m_1n_2-m_2n_1} = \frac{y-y_1}{n_1l_2-n_2l_1} = \frac{z-z_1}{l_1m_2-l_2m_1}$ }
38. If  $l_1, m_1, n_1; l_2, m_2, n_2$  and  $l_3, m_3, n_3$  are the direction cosines of three mutually perpendicular lines, then find the direction cosines of a line whose direction cosines are proportional to  $l_1+l_2+l_3, m_1+m_2+m_3, n_1+n_2+n_3$  and prove that this line is equally inclined to the given lines. {Ans.  $\frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}$ }
39. Show that the straight lines whose direction cosines are given by the equations  $al+bm+cn=0$  and  $ul^2+vm^2+wn^2=0$  are perpendicular if  $a^2(v+w)+b^2(u+w)+c^2(u+v)=0$  and parallel if  $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$ .
40. Prove that the straight lines whose direction cosines are given by the relations  $al+bm+cn=0$  and  $fmn+gnl+hlm=0$  are perpendicular if  $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$  and parallel if  $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$ .
41. Find the angle between two diagonals of a cube. {Ans.  $\cos^{-1} \frac{1}{3}$ }
42. A line makes angles  $\alpha, \beta, \gamma, \delta$  with the four diagonals of a cube, prove that  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}$ .
43. If the edges of a rectangular parallelopiped be  $a, b, c$ , show that the angle between the four diagonals are given by  $\cos^{-1} \left( \frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2} \right)$ .
44. If two pairs of opposite edges of a tetrahedron are perpendicular, then prove that the third pair is also perpendicular.
45. If in a tetrahedron  $OABC$ ,  $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$ , then show that its pairs of opposite

edges are at right angles.

46. If a pair of opposite edges of a tetrahedron be perpendicular then show that the distances between the middle points of the other two pairs of opposite edges are equal.

**CATEGORY-20.6. POINT OF INTERSECTION OF TWO LINES**

47. Check whether the lines  $\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2}$  and  $\frac{x-1}{1} = \frac{y-3}{-4} = \frac{z+2}{8}$  intersect or not. If they intersect, find their point of intersection. {Ans. intersect,  $(1, 3, -2)$ }
48. Prove that the lines  $\frac{x-a}{a'} = \frac{y-b}{b'} = \frac{z-c}{c'}$  and  $\frac{x-a'}{a} = \frac{y-b'}{b} = \frac{z-c'}{c}$  intersect and find the coordinates of the point of intersection. {Ans.  $(a+a', b+b', c+c')$ }

**CATEGORY-20.7. POINTS AND LINE**

49. Find the equation of the perpendicular from the point  $(1, 6, 3)$  to the line  $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ . Find also the coordinates of the foot of the perpendicular and perpendicular distance of the point from the line. {Ans.  $x-1=0, 2y+3z=21, (1, 3, 5), \sqrt{13}$ }
50. Find the equation of the perpendicular from the origin to the line  $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{1}$ . Find also the coordinates of the foot of the perpendicular and perpendicular distance of the point from the line. {Ans.  $\frac{x}{2} = \frac{y}{-1} = \frac{z}{-4}, \left(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}\right), \sqrt{\frac{7}{3}}$ }
51. Find the equation of the perpendicular from the point  $(3, -1, 11)$  to the line  $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ . Find also the coordinates of the foot of the perpendicular. Hence, find the length of the perpendicular. {Ans.  $\frac{x-3}{-1} = \frac{y+1}{6} = \frac{z-11}{-4}, (2, 5, 7), \sqrt{53}$ }
52. Find the distance of the point  $P(a, b, c)$  from the  $x$ -axis. {Ans.  $\sqrt{b^2 + c^2}$ }
53. Prove that the line through the points  $(a, b, c)$  and  $(a', b', c')$  passes through the origin if  $aa' + bb' + cc' = rr'$ , where  $r$  and  $r'$  are the distances of the points from the origin.

**CATEGORY-20.8. SHORTEST DISTANCE BETWEEN PARALLEL AND SKEW LINES**

54. Find the shortest distance between the lines  $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{4}$  and  $\frac{x+1}{2} = \frac{y}{3} = \frac{z-1}{4}$ . {Ans.  $\sqrt{\frac{3677}{29}}$ }
55. Find the shortest distance between the lines  $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$  and  $\frac{x+1}{3} = \frac{y+1}{-6} = \frac{z+1}{1}$ . {Ans.  $\frac{14}{\sqrt{5}}$ }
56. Find the shortest distance between the lines  $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$  and  $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ . Also find the equation of the line of shortest distance. {Ans.  $\frac{1}{\sqrt{6}}, \frac{3x-5}{3} = \frac{y-3}{-2} = \frac{3z-13}{3}$ }

57. Find the length and equations of the common perpendicular to the two lines  $\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2}$  and  $\frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{1}$ . {Ans. 9,  $\frac{x-1}{1} = \frac{y-3}{-4} = \frac{z+2}{8}$ }
58. Show that the distance between the lines  $x+a=2y=-12z$  and  $x=y+2a=6z-6a$  is  $2a$ .
59. Find the shortest distance between the lines  $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$  and  $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ . Find also the equation of the line of shortest distance and the points in which it meets the given lines. {Ans.  $3\sqrt{30}$ ,  $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$ ,  $(3,8,3)$ ,  $(-3,-7,6)$ }
60. Show that the shortest distances between the diagonals of a rectangular parallelopiped and the edges not meeting it are  $\frac{bc}{\sqrt{b^2+c^2}}$ ,  $\frac{ca}{\sqrt{c^2+a^2}}$ ,  $\frac{ab}{\sqrt{a^2+b^2}}$ , where  $a, b, c$  are the lengths of the edges.

### CATEGORY-20.9. EQUATION OF A PLANE

61. Given a plane  $2x+3y+4z+5=0$ . Find
- $x$ -intercept,  $y$ -intercept and  $z$ -intercept. {Ans.  $-\frac{5}{2}, -\frac{5}{3}, -\frac{5}{4}$ }
  - direction cosines of its normal. {Ans.  $\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$ }
  - its distance from origin. {Ans.  $\frac{5}{\sqrt{29}}$ }
62. Find the equation of the plane passing through the point  $(3,-3,1)$  and normal to the line joining the points  $(3,4,-1)$  and  $(2,-1,5)$ . {Ans.  $x+5y-6z+18=0$ }
63. The foot of the perpendicular from the origin to a plane is  $(4,-2,-5)$ . Find the equation of the plane. {Ans.  $4x-2y-5z=45$ }
64. Find the equation to the plane through  $P(2,3,-1)$  at right angles to  $OP$ . {Ans.  $2x+3y-z=14$ }
65. Find the equation of the plane perpendicular to the line segment from  $(-3,3,2)$  to  $(9,5,4)$  at the middle point of the segment. {Ans.  $6x+y+z=25$ }
66.  $O$  is the origin and  $A$  the point  $(a,b,c)$ . Find the equation of the plane through  $A$  and at right angles to  $OA$ . {Ans.  $ax+by+cz=a^2+b^2+c^2$ }
67. Find the equation of the plane passing through the points  $(2,2,-1)$ ,  $(3,4,2)$  and  $(7,0,6)$ . {Ans.  $5x+2y-3z=17$ }
68. Find the equation of the plane through the point  $(1,1,1)$  and containing the line joining the points  $(-2,-2,2)$  and  $(1,-1,2)$ . {Ans.  $x-3y-6z+8=0$ }
69. Show that the four points  $(-1,4,-3)$ ,  $(3,2,-5)$ ,  $(-3,8,-5)$  and  $(-3,2,1)$  are coplanar. Find the equation of the plane containing them. {Ans.  $x+y+z=0$ }
70. Show that the four points  $(0,-1,0)$ ,  $(2,1,-1)$ ,  $(1,1,1)$  and  $(3,3,0)$  are coplanar and hence show that the equation of the plane passing through these points is  $4x-3y+2z=3$ .

71. Find the area of the triangle included between the plane  $3x - 4y + z = 12$  and the coordinate planes. {Ans.  $6\sqrt{26}$  sq. units}
72. A plane meets the coordinate axes at  $A$ ,  $B$  and  $C$  such that the centroid of the triangle  $ABC$  is  $(1, -2, 3)$ . Find the equation of the plane. {Ans.  $6x - 3y + 2z = 18$ }
73. Find the equation of the plane passing through the point  $(1, 2, 1)$  and perpendicular to the line joining the points  $(1, 4, 2)$  and  $(2, 3, 5)$ . Find also the perpendicular distance of the origin from the plane. {Ans.  $x - y + 3z = 2, \frac{2}{\sqrt{11}}$ }
74. From a point  $P(x', y', z')$  a plane is drawn at right angles to  $OP$  to meet the coordinate axes at  $A$ ,  $B$  and  $C$ . Prove that the area of the triangle  $ABC$  is  $\frac{r^5}{(2x'y'z')}$ , where  $r$  is the length of  $OP$  and  $O$  is the origin.
75. A plane makes intercepts  $OA = a$ ,  $OB = b$  and  $OC = c$  respectively on the coordinate axes. Show that the area of the triangle  $ABC$  is  $\frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}$ .
76. A plane meets the coordinate axes in  $A$ ,  $B$  and  $C$  such that the centroid of the triangle  $ABC$  is the point  $(p, q, r)$ . Show that that equation of the plane is  $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$ .
77. Two systems of rectangular axes have the same origin. If a plane cuts them at distances  $a, b, c$  and  $a', b', c'$  respectively from the origin, show that  $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$ .

**CATEGORY-20.10. POINTS AND PLANE**

78. Find the distance of the point  $(1, 2, -3)$  from the plane  $2x + 3y - 4z + 2 = 0$ . {Ans.  $\frac{22}{\sqrt{29}}$ }
79. Find the ratio in which the join of  $A(2, 1, 5)$  and  $B(3, 4, 3)$  is divided by the plane  $2x + 2y - 2z = 1$ . Also find the coordinates of the point of division. {Ans.  $5:7, \left(\frac{29}{12}, \frac{9}{4}, \frac{25}{6}\right)$ }

**CATEGORY-20.11. ANGLE BETWEEN A LINE AND A PLANE**

80. Find the angle between the line  $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z+2}{4}$  and the plane  $2x - 3y + 5z + 1 = 0$ . {Ans.  $\sin^{-1} \frac{15}{\sqrt{1102}}$ }
81. Find the equation of the plane perpendicular to the line  $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z+1}{2}$  and passing through the point  $(2, 3, 1)$ . {Ans.  $x - y + 2z = 1$ }
82. Find the equation of the plane through the points  $(1, -2, 4)$  and  $(3, -4, 5)$  and parallel to the  $x$ -axis. {Ans.  $y + 2z - 6 = 0$ }
83. The equation of line  $L$  is  $\frac{x}{2} = \frac{y}{-3} = \frac{z}{6}$ . Through a point  $P(1, 2, 5)$ , a line  $PQ$  is drawn parallel to the plane

$3x + 4y + 5z = 0$  to meet line  $L$  in  $Q$ . Find the equation of line  $PQ$  and the coordinates of  $Q$ . {Ans.

$$\frac{x-1}{4} = \frac{y-2}{-13} = \frac{z-5}{8}, \left(3, -\frac{9}{2}, 9\right)\}$$

### CATEGORY-20.12. A LINE LYING ON A PLANE

84. Show that the equation of the plane containing the line  $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$  and the point  $(0, 7, -7)$  is  $x + y + z = 0$ .

85. Prove that the equation of the plane through the line  $\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+1}{2}$  and parallel to the line  $\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z+4}{5}$  is  $26x - 11y - 17z - 109 = 0$ .

86. Show that the plane through the point  $(\alpha, \beta, \gamma)$  and the line  $x = py + q = rz + s$  is given by

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+1 & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

### CATEGORY-20.13. COPLANARITY OF TWO LINES AND PLANE CONTAINING THEM

87. Find the equation of the plane which contains the two lines  $\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1}$  and  $\frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}$ . {Ans.  $8x + y - 26z + 6 = 0$ }

88. Prove that the lines  $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$  and  $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$  are coplanar. Find their point of intersection. Also find the equation of the plane in which they lie. {Ans.  $(-1, -1, -1)$ ,  $x - 2y + z - 2 = 0$ }

89. Prove that the lines  $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$  and  $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$  intersect, and find the coordinates of their point of intersection and the equation of the plane containing them. {Ans.  $(5, -7, 6)$ ,  $11x - 6y - 5z = 67$ }

90. Prove that the lines  $x = ay + b = cz + d$  and  $x = \alpha y + \beta = \gamma z + \delta$  are coplanar if  $(a\beta - b\alpha)(\gamma - c) - (c\delta - d\gamma)(\alpha - a) = 0$ .

91. Show that the lines  $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$  and  $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$  are coplanar and find the equation of the plane in which they lie. {Ans.  $x - 2y + z = 0$ }

92. Prove that the three lines drawn from a point with direction ratios  $1, -1, 1$ ;  $2, -3, 0$  and  $1, 0, 3$  are coplanar.

93. Show that the lines  $\frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma}$ ,  $\frac{x}{\left(\frac{\alpha}{a}\right)} = \frac{y}{\left(\frac{\beta}{b}\right)} = \frac{z}{\left(\frac{\gamma}{c}\right)}$  and  $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$  are coplanar, if  $a = b$  or  $b = c$

or  $c = a$ .

94. Prove that the lines  $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ ,  $\frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma}$  and  $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$  will be in one plane if

$$\frac{l}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0.$$

95. Prove that the three concurrent lines with direction cosines  $l_1, m_1, n_1$ ;  $l_2, m_2, n_2$  and  $l_3, m_3, n_3$  are coplanar,

$$\text{if } \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

96. The vertices of a triangle  $PQR$  are the points  $(-1, 2, -3)$ ,  $(5, 0, -6)$  and  $(0, 4, -1)$  in order. Find the direction ratios of the bisectors of the angle  $QPR$ . {Ans. 25, 8, 5}

97. If  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$  are the direction cosines of two concurrent lines, show that the direction cosine of two lines bisecting the angles between them are proportional to  $l_1 + l_2$ ,  $m_1 + m_2$ ,  $n_1 + n_2$  and  $l_1 - l_2$ ,  $m_1 - m_2$ ,  $n_1 - n_2$ .

98. The direction cosines of two lines are  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$  and  $\theta$  is the angle between them. Show that the direction cosines of the bisectors of the angle between them are  $\frac{l_1 + l_2}{2 \cos\left(\frac{\theta}{2}\right)}$ ,  $\frac{m_1 + m_2}{2 \cos\left(\frac{\theta}{2}\right)}$ ,  $\frac{n_1 + n_2}{2 \cos\left(\frac{\theta}{2}\right)}$  and

$$\frac{l_1 - l_2}{2 \sin\left(\frac{\theta}{2}\right)}, \frac{m_1 - m_2}{2 \sin\left(\frac{\theta}{2}\right)}, \frac{n_1 - n_2}{2 \sin\left(\frac{\theta}{2}\right)}.$$

#### CATEGORY-20.14. POINT OF INTERSECTION OF A LINE AND A PLANE

99. Show that the distance of the point of intersection of the line  $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$  and the plane  $x + y + z = 17$  from the point  $(3, 4, 5)$  is 3.

100. Find the distance of the point  $(1, 3, 4)$  from the plane  $2x - y + z = 3$  measured parallel to the line

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{-1}. \text{ {Ans. 0 units}}$$

101. Find the coordinates of the point where the line joining the points  $(2, -3, 1)$  and  $(3, -4, -5)$  cuts the plane  $2x + y + z = 7$ . {Ans.  $(1, -2, 7)$ }

102. Find the distance of the point  $(1, -2, 3)$  from the plane  $x - y + z = 5$  measured parallel to the line

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}. \text{ {Ans. 1}}$$

103. Find the distance of the point  $P(3, 8, 2)$  from the line  $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$  measured parallel to the plane  $3x + 2y - 2z + 17 = 0$ . {Ans. 7}

104. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane  $3x + 4y - 6z + 1 = 0$ . Find also the coordinates of the point on the line which is at the same distance from the foot of the perpendicular as the origin is. {Ans.  $\left(-\frac{3}{61}, -\frac{4}{61}, \frac{6}{61}\right)$ ,  $\left(-\frac{6}{61}, -\frac{8}{61}, \frac{12}{61}\right)$ }

105. Find the image of the point  $(1, 3, 4)$  in the plane  $2x - y + z + 3 = 0$ . {Ans.  $(-3, 5, 2)$ }

**CATEGORY-20.15. ANGLE BETWEEN TWO PLANES**

106. Find the equation of the plane through  $(1,0,-2)$  and perpendicular to each of the planes  $2x + y - z - 2 = 0$  and  $x - y - z - 3 = 0$ . {Ans.  $2x - y + 3z + 4 = 0$ }
107. Find the equation of the plane passing through the points  $(1,-1,2)$  and  $(2,-2,2)$  and which is perpendicular to the plane  $6x - 2y + 2z = 9$ . {Ans.  $x + y - 2z + 4 = 0$ }
108. Find the equation of the plane through the points  $(2,2,1)$  and  $(9,3,6)$  and perpendicular to the plane  $2x + 6y + 6z - 1 = 0$ . {Ans.  $3x + 4y - 5z = 9$ }
109. Find the angle between the planes  $2x - y + z = 7$  and  $x + y + 2z = 9$ . {Ans.  $\frac{\pi}{3}$ }
110. Show that the planes  $3x + 4y - 5z = 9$  and  $2x + 6y + 6z = 7$  are at right angles.
111. Find the equation of the plane through the points  $(1,-2,2)$ ,  $(-3,1,-2)$  and perpendicular to the plane  $x + 2y - 3z = 5$ . {Ans.  $x + 16y + 11z + 9 = 0$ }
112. Find the equation of the plane which contains the line  $x = \frac{y-3}{2} = \frac{z-5}{3}$  and which is perpendicular to the plane  $2x + 7y - 3z = 1$ . {Ans.  $9x - 3y - z + 14 = 0$ }
113. A plane meets a set of three mutually perpendicular planes in the sides of a triangle whose angles are  $A$ ,  $B$ , and  $C$  respectively. Show that the first plane makes with the other planes angles, the squares of whose cosines are  $\cot B \cot C$ ,  $\cot C \cot A$ ,  $\cot A \cot B$ .

**CATEGORY-20.16. PARALLEL PLANES**

114. Find the equation of the planes parallel to the plane  $x - 2y + 2z - 3 = 0$  which are at a unit distance from the point  $(1,2,3)$ . {Ans.  $x - 2y + 2z = 0$ ,  $x - 2y + 2z - 6 = 0$ }
115. Find the equation of the plane through the point  $(1,4,-2)$  and parallel to the plane  $-2x + y - 3z = 7$ . {Ans.  $2x - y + 3z + 8 = 0$ }
116. Find the distance between the parallel planes  $2x - 2y + z + 3 = 0$  and  $4x - 4y + 2z + 7 = 0$ . {Ans.  $\frac{1}{6}$  units}
117. Find the distance between the parallel planes  $2x - y + 3z - 4 = 0$  and  $6x - 3y + 9z + 13 = 0$ . {Ans.  $\frac{25}{3\sqrt{14}}$  units}
118. Find the equation of the plane through the point  $(1,3,2)$  and parallel to the plane  $3x - 2y + 2z + 33 = 0$ .  
Find the perpendicular distance of the point  $(3,3,2)$  from this plane. {Ans.  $3x - 2y + 2z - 1 = 0$ ,  $\frac{6}{\sqrt{17}}$ }
119. Find the equations of the planes parallel to the plane  $x + 2y - 2z + 8 = 0$  which are at a distance of 2 units from the point  $(2,1,1)$ . {Ans.  $x + 2y - 2z - 8 = 0$ ,  $x + 2y - 2z + 4 = 0$ }

**CATEGORY-20.17. INTERSECTING PLANES**

120. Find the equation of the plane which is perpendicular to the plane  $5x + 3y + 6z + 8 = 0$  and which contains the line of intersection of the planes  $x + 2y + 3z - 4 = 0$  and  $2x + y - z + 5 = 0$ . {Ans.  $51x + 15y - 50z + 173 = 0$ }



121. Find the equation of the plane through the line of intersection of the planes  $x + 2y + 3z + 5 = 0$ ,  $x - 3y + z + 6 = 0$  and passing through the origin. {Ans.  $x + 27y + 13z = 0$ }
122. Find the equation of the plane which contains the lines of intersection of the planes  $x + 2y + 3z - 4 = 0$  and  $2x + y - z + 5 = 0$  and which is perpendicular to the plane  $5x + 3y - 6z + 8 = 0$ . {Ans.  $33x + 45y + 50z - 41 = 0$ }
123. The planes  $3x - y + z + 1 = 0$ ,  $5x + y + 3z = 0$  intersect in the line  $L$ . Find the equation of the plane through the point  $(2, 1, 4)$  and perpendicular to  $L$ . {Ans.  $x + y - 2z + 5 = 0$ }
124. Find the equation of the plane through the line of intersection of the planes  $ax + by + cz + d = 0$  and  $\alpha x + \beta y + \gamma z + \delta = 0$  and parallel to  $x$ -axis. {Ans.  $(b\alpha - a\beta)y + (c\alpha - a\gamma)z + (d\alpha - a\delta) = 0$ }
125. Prove that the four planes  $my + nz = 0$ ,  $nz + lx = 0$ ,  $lx + my = 0$ ,  $lx + my + nz = p$  form a tetrahedron whose volume is  $\frac{2p^3}{3lmn}$ .
126. The plane  $lx + my = 0$  is rotated about its line of intersection with the plane  $z = 0$ , through the angle  $\alpha$ . Prove that the equation of the plane in its new position is  $lx + my \pm z\sqrt{l^2 + m^2} \tan \alpha = 0$ .

#### CATEGORY-20.18. PLANES BISECTING THE ANGLE BETWEEN TWO PLANES

127. Find the equations of the planes bisecting the angles between the planes  $3x - 6y + 2z + 5 = 0$  and  $4x - 12y + 3z = 3$ . Point out the plane which bisects the obtuse angle. {Ans.  $67x - 162y + 47z + 44 = 0$ ,  $11x + 6y + 5z + 86 = 0$ }
128. Find the equation of the plane which bisects the obtuse angle between the planes  $x + 2y + 2z = 9$  and  $4x - 3y + 12z + 13 = 0$ . {Ans.  $x + 35y - 10z = 156$ }
129. Find the bisector of the acute angle between the planes  $2x - y + 2z + 3 = 0$  and  $3x - 2y + 6z + 8 = 0$ . {Ans.  $23x - 13y + 32z + 45 = 0$ }
130. Find the equation of the plane bisecting the acute angle between the planes  $2x - y - 2z - 6 = 0$  and  $3x + 2y - 6z - 12 = 0$ . {Ans.  $23x - y - 32z - 78 = 0$ }

#### CATEGORY-20.19. EQUATION OF A LINE IN UNSYMMETRICAL FORM

131. Find the equations of the line  $3x + 2y - z - 4 = 0 = 4x + y - 2z + 3$  in symmetrical form. {Ans.  $\frac{x+2}{3} = \frac{y-5}{-2} = \frac{z}{5}$ }
132. Find the angle between the lines  $x + 2y - 2z = 11$ ,  $x - 2y + z = 9$  and  $\frac{x-3}{1} = \frac{y+5}{-3} = \frac{z-1}{2}$ . {Ans.  $\cos^{-1} \frac{1}{\sqrt{406}}$ }
133. Show that the lines  $3x + 2y + z = 5$ ,  $x + y - 2z = 3$  and  $2x - y - z = 0$ ,  $7x + 10y - 8z = 15$  are mutually perpendicular.
134. Show that the lines  $\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$  and  $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$  are coplanar. Also find their point of intersection and the equation of the plane in which they lie. {Ans.  $(2, 4, -3)$ ,  $45x - 17y + 25z + 53 = 0$ }

135. Show that the lines  $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$  and  $4x-3y+1=0=5x-3z+2$  are coplanar. Also find their point of intersection. {Ans.  $(-1, -1, -1)$ }
136. Find the length and position of the shortest distance between the lines  $\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}$  and  $5x-2y-3z+6=0=x-3y+2z-3$ . {Ans.  $\frac{17\sqrt{6}}{39}$ ,  $7x-2y-11z+20=0=13x-13z+24$ }
137. Show that the length of shortest distance between the line  $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z}{4}$ ;  $2x+3y-5z-6=0=3x-2y-z+3$  is  $\frac{97}{13\sqrt{6}}$ .
138. Find the equations of the line through the point  $(1, 2, 3)$  and parallel to the line  $x-y+2z-5=0$ ,  $3x+y+z-6=0$ . {Ans.  $\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$ }
139. Find the angle between the lines  $3x+2y+z=5$ ,  $x+y-2z=3$  and  $x+2y+z=8$ ,  $8x+12y+5z=0$ . {Ans.  $\cos^{-1} \frac{27}{5\sqrt{87}}$ }
140. Show that the following lines  $\frac{x-3}{3} = \frac{y-2}{-4} = z+1$  and  $x+2y+3z=0=2x+4y+3z+3$  are coplanar.
141. Find the equation of the plane containing the line  $\frac{y}{b} + \frac{z}{c} = 1$ ,  $x=0$  and parallel to the line  $\frac{x}{a} - \frac{z}{c} = 1$ ,  $y=0$ . {Ans.  $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$ }
142. Prove that the lines  $x=ay+b$ ,  $z=cy+d$  and  $x=a'y+b'$ ,  $z=c'y+d'$  are perpendicular if  $aa'+cc'+1=0$ .
143. Prove that the equation to the two planes inclined at an angle  $\alpha$  to the  $xy$ -plane and containing the line  $y=0, z\cos\beta = x\sin\beta$  is  $(x^2+y^2)\tan^2\beta + z^2 - 2zx\tan\beta = y^2\tan^2\alpha$ .
144. Show that the equation of the plane containing the line  $\frac{y}{b} + \frac{z}{c} = 1$ ,  $x=0$  and parallel to the line  $\frac{x}{a} - \frac{z}{c} = 1$ ,  $y=0$  is  $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$  and if  $2d$  is the shortest distance then show that  $d^{-2} = a^{-2} + b^{-2} + c^{-2}$ .
145. Find the equation of the plane through the lines  $ax+by+cz=0=a'x+b'y+c'z$  and  $\alpha x + \beta y + \gamma z = 0 = \alpha'x + \beta'y + \gamma'z$ . {Ans.  $\begin{vmatrix} x & y & z \\ bc' - b'c & ca' - c'a & ab' - a'b \\ \beta\gamma' - \beta'\gamma & \gamma\alpha' - \gamma'\alpha & \alpha\beta' - \alpha'\beta \end{vmatrix}$ }

### CATEGORY-20.20. LOCUS PROBLEMS

146.  $P$  is a variable point and the coordinates of two given points  $A$  and  $B$  are  $(-2, 2, 3)$  and  $(13, -3, 13)$  respectively. Find the locus of  $P$  if  $3PA = 2PB$ . {Ans.  $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$ }
147.  $A, B$  are the points  $(3, 4, 5)$  and  $(-1, 3, -7)$  respectively. A variable point  $P$  moves such that

i.  $PA = PB$ .

ii.  $PA^2 - PB^2 = 2k^2$ .

Find the locus of  $P$  in each of these cases. {Ans.  $8x + 2y + 24z + 9 = 0$ ,  $8x + 2y + 24z + 2k^2 = 0$ }

148. A variable plane is at a constant distance  $p$  from the origin  $O$  and meets axes in  $A, B$  and  $C$ . Show that the locus of centroid of the tetrahedron  $OABC$  is  $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$ .
149. A point  $P$  moves on the plane  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ , which is fixed. The plane through  $P$ , perpendicular to  $OP$  meets the coordinates axes in  $A, B$  and  $C$ . The planes through  $A, B$  and  $C$  parallel to  $yz, zx$  and  $xy$  planes intersect in  $Q$ . Prove that the locus of  $Q$  is  $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}$ .
150. A variable plane passes through a fixed point  $(\alpha, \beta, \gamma)$  and meets the axes of reference in  $A, B, C$ . Show that the locus of the point of intersection of the planes through  $A, B, C$  and parallel to the coordinate planes is  $\alpha x^{-1} + \beta y^{-1} + \gamma z^{-1} = 1$ .
151. A variable plane which remains at a constant distance  $3p$  from the origin cuts the coordinates axes at  $A, B, C$ . Show that the locus of the centroid of the triangle  $ABC$  is  $x^{-2} + y^{-2} + z^{-2} = p^{-2}$ .
152. A variable plane at a constant distance  $p$  from the origin meets the coordinates axes in  $A, B$  and  $C$ . Through  $A, B, C$  planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is  $x^{-2} + y^{-2} + z^{-2} = p^{-2}$ .
153. A variable plane makes intercepts on the coordinate axes the sum of whose squares is constant and equal to  $k^2$ . Show that the locus of the foot of the perpendicular from the origin to the plane is  $(x^{-2} + y^{-2} + z^{-2})(x^2 + y^2 + z^2)^2 = k^2$ .

### CATEGORY-20.21. ADDITIONAL QUESTIONS

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# Mathematics for IIT-JEE

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*PART-VI*  
*INTEGRAL CALCULUS*

*CHAPTER-21*  
*INDEFINITE INTEGRALS*

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## **CHAPTER-21**

### **INDEFINITE INTEGRALS**

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- 21.1. Definition Of Indefinite Integral
- 21.2. Basic And Standard Integrals
- 21.3. Methods Of Integration

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- 21.12. Integration Of Hyperbolic Functions
- 21.13. Integration Of Irrational Functions Containing  $\left(\frac{ax+b}{cx+d}\right)^{\frac{1}{n}}$
- 21.14. Integration Of Irrational Functions Of Type  $\int \sqrt{ax^2+bx+c} dx$  And  $\int (Ax+B)\sqrt{ax^2+bx+c} dx$
- 21.15. Integration Of Irrational Functions Of Type  $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}} dx$
- 21.16. Integration Of Irrational Functions Of Type  $\int \frac{dx}{(Ax+B)^n \sqrt{ax^2+bx+c}}$  And  $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$
- 21.17. Euler Substitutions
- 21.18. Application Of Methods Of Integration
- 21.19. Additional Questions

## CHAPTER-21

### INDEFINITE INTEGRALS

#### SECTION-21.1. DEFINITION OF INDEFINITE INTEGRAL

##### 1. Differential of a function

- i.  $dx$  denotes differential of  $x$ .  $df(x)$  denotes differential of function  $f(x)$ .
- ii. If  $\frac{df(x)}{dx} = F(x) \Rightarrow df(x) = F(x)dx$

##### 2. Definition of indefinite integral

- i. Integration, denoted by  $\int$ , is the inverse process of differential, i.e.  $\int df(x) = f(x)$ .
- ii. In general,  $\int df(x) = \int d(f(x) + C) = f(x) + C$ , where  $C$  is an arbitrary constant known as constant of integration.
- iii. If  $\frac{d}{dx} f(x) = F(x) \Rightarrow df(x) = F(x)dx \Rightarrow \int df(x) = \int F(x)dx \Rightarrow \int F(x)dx = f(x) + C$ . Therefore, integrating a function  $F(x)$  w.r.t.  $x$ , denoted by  $\int F(x)dx$ , means finding a function  $f(x)$  such that  $\frac{d}{dx} f(x) = F(x)$ .
- iv. Certain integrals, such as  $\int \sqrt{\sin x} dx$ , cannot be expressed in analytical form in terms of basic functions.

##### 3. Rules of integration

- i.  $\int df(x) = f(x) + C$
- ii.  $\int kf(x)dx = k \int f(x)dx$
- iii.  $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$

#### SECTION-21.2. BASIC AND STANDARD INTEGRALS

##### 1. Basic integrals

- i.  $\int x^a dx = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$
- ii.  $\int \frac{1}{x} dx = \ln|x| + C$
- iii.  $\int a^x dx = \frac{a^x}{\ln a} + C$
- iv.  $\int e^x dx = e^x + C$
- v.  $\int \cos x dx = \sin x + C$
- vi.  $\int \sin x dx = -\cos x + C$
- vii.  $\int \sec^2 x dx = \tan x + C$
- viii.  $\int \operatorname{cosec}^2 x dx = -\cot x + C$

- ix.  $\int \sec x \tan x dx = \sec x + C$   
 x.  $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$   
 xi.  $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C = -\cos^{-1} x + C$   
 xii.  $\int \frac{dx}{1+x^2} = \tan^{-1} x + C = -\cot^{-1} x + C$   
 xiii.  $\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + C = -\operatorname{cosec}^{-1} x + C$   
 xiv.  $\int \sinh x dx = \cosh x + C$   
 xv.  $\int \cosh x dx = \sinh x + C$   
 xvi.  $\int \operatorname{sech}^2 x dx = \tanh x + C$   
 xvii.  $\int \operatorname{cosech}^2 x dx = -\coth x + C$   
 xviii.  $\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$   
 xix.  $\int \operatorname{cosech} x \coth x dx = -\operatorname{cosech} x + C$

## 2. Standard integrals

- i.  $\int \tan x dx = \ln|\sec x| + C$   
 ii.  $\int \cot x dx = \ln|\sin x| + C$   
 iii.  $\int \sec x dx = \ln|\sec x + \tan x| + C$   
 iv.  $\int \operatorname{cosec} x dx = -\ln|\operatorname{cosec} x + \cot x| + C$   
 v.  $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$   
 vi.  $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$   
 vii.  $\int \sqrt{a^2+x^2} dx = \frac{x}{2} \sqrt{x^2+a^2} + \frac{a^2}{2} \ln|x+\sqrt{x^2+a^2}| + C$   
 viii.  $\int \sqrt{a^2-x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2} + C$   
 ix.  $\int \sqrt{x^2-a^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \ln|x+\sqrt{x^2-a^2}| + C$   
 x.  $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + C$   
 xi.  $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C$



$$\text{xii. } \int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left| x + \sqrt{a^2 + x^2} \right| + C$$

### 3. Direct integrals

- i. Direct integrals are those integrals which can be solved by comparing with Basic or Standard integrals.

## SECTION-21.3. METHODS OF INTEGRATION

### 1. Methods of integration of elementary trigonometric functions

- Integrals containing odd powers of  $\sin x$  &  $\cos x$
- Integrals containing even powers of  $\sin x$  &  $\cos x$
- Integrals containing integer powers  $\tan x$  &  $\cot x$ , Reduction formula
- Integrals containing even powers of  $\sec x$  &  $\operatorname{cosec} x$
- Change to one of the above types using trigonometric identities

### 2. Method of substitution

- Standard trigonometric substitutions

### 3. Integration by parts

- Recurrence of an integral
- Reduction formulae using integration by parts

### 4. Methods of integration of rational functions

- Integration of rational functions with linear denominator.
- Integration of rational functions with quadratic denominator.
- Integration of rational functions with denominator of degree more than two by Method of partial fraction.

### 5. Methods of integration of rational functions of $\sin x$ & $\cos x$

- Substitution  $\tan \frac{x}{2} = t$ .
- Integrals of type  $\int \frac{1}{a \sin x + b \cos x} dx$ .
- Integrals of type  $\int \frac{a \sin x + b \cos x}{p \sin x + q \cos x} dx$ .
- Integrals of type  $\int \frac{a \sin x + b \cos x + c}{p \sin x + q \cos x + r} dx$ .

### 6. Methods of integration of hyperbolic functions

- Hyperbolic identities
  - $\cosh^2 x - \sinh^2 x = 1$
  - $1 - \tanh^2 x = \operatorname{sech}^2 x$
  - $1 - \coth^2 x = -\operatorname{cosech}^2 x$
  - $\sinh 2x = 2 \sinh x \cosh x = \frac{2 \tanh x}{1 - \tanh^2 x}$
  - $\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$
  - $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$
- Hyperbolic functions are integrated using methods of integration of trigonometric functions.

iii. Substitution  $e^x = t$ .

## 7. Methods of integration of irrational functions

i. Integrals containing  $\left(\frac{ax+b}{cx+d}\right)^{\frac{1}{n}}$ .

ii. Integrals of type  $\int \sqrt{ax^2 + bx + c} dx$ .

iii. Integrals of type  $\int (Ax+B)\sqrt{ax^2 + bx + c} dx$ .

iv. Integrals of type  $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ .

v. Integrals of type  $\int \frac{Ax+B}{\sqrt{ax^2 + bx + c}} dx$ .

vi. Integrals of type  $\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}} dx, n \geq 2$ .

$$\text{Put } \int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}} dx = Q_{n-1}(x)\sqrt{ax^2 + bx + c} + \lambda \int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$

vii. Integrals of type  $\int \frac{dx}{(Ax+B)^n \sqrt{ax^2 + bx + c}}$ .

viii. Integrals of type  $\int \frac{dx}{(ax^2 + b)\sqrt{cx^2 + d}}$ .

ix. Other cases of integrals containing  $\sqrt{ax^2 + bx + c}$  to be done by standard trigonometric substitutions.

x. Integrals of the form  $\int R(x, \sqrt{ax^2 + bx + c}) dx$  can also be calculated by one of the following three Euler substitutions:-

a.  $\sqrt{ax^2 + bx + c} = t \pm \sqrt{ax}$  if  $a > 0$

b.  $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}$  if  $c > 0$

c.  $\sqrt{ax^2 + bx + c} = (x - \alpha)t$  if  $b^2 - 4ac > 0$  and  $\alpha$  is a real root of  $ax^2 + bx + c$ .

**EXERCISE-21****CATEGORY-21.1. DIRECT INTEGRALS**

1.  $\int \sin x d(\sin x)$ . {Ans.  $\frac{\sin^2 x}{2} + C$ }
2.  $\int \tan^3 x d(\tan x)$ . {Ans.  $\frac{\tan^4 x}{4} + C$ }
3.  $\int \frac{d(1+x^2)}{\sqrt{1+x^2}}$ . {Ans.  $2\sqrt{1+x^2} + C$ }
4.  $\int \cos 3x d(3x)$ . {Ans.  $\sin 3x + C$ }
5.  $\int \frac{d(1+\ln x)}{\cos^2(1+\ln x)}$ . {Ans.  $\tan(1+\ln x) + C$ }
6.  $\int \frac{d(1+x^2)}{1+x^2}$ . {Ans.  $\ln(1+x^2) + C$ }
7.  $\int \frac{d(\sin^{-1} x)}{\sin^{-1} x}$ . {Ans.  $\ln|\sin^{-1} x| + C$ }
8.  $\int e^{\sin x} d(\sin x)$ . {Ans.  $e^{\sin x} + C$ }
9.  $\int \frac{d\left(\frac{x}{3}\right)}{\sqrt{1-\left(\frac{x}{3}\right)^2}}$ . {Ans.  $\sin^{-1}\left(\frac{x}{3}\right) + C$ }
10.  $\int (x+1)^{15} dx$ . {Ans.  $\frac{(x+1)^{16}}{16} + C$ }
11.  $\int \frac{dx}{(2x-3)^5}$ . {Ans.  $-\frac{1}{8(2x-3)^4} + C$ }
12.  $\int \frac{dx}{(a+bx)^c}$  ( $c \neq 1$ ). {Ans.  $\frac{(a+bx)^{1-c}}{b(1-c)} + C$ }
13.  $\int \sqrt[5]{(8-3x)^6} dx$ . {Ans.  $-\frac{5}{33}(8-3x)^{\frac{11}{5}} + C$ }
14.  $\int \sqrt{8-2x} dx$ . {Ans.  $-\frac{\sqrt{(8-2x)^3}}{3} + C$ }
15.  $\int \frac{m}{\sqrt[3]{(a+bx)^2}} dx$ . {Ans.  $\frac{3m}{b}\sqrt[3]{a+bx} + C$ }
16.  $\int 2x\sqrt{x^2+1} dx$ . {Ans.  $\frac{2}{3}\sqrt{(x^2+1)^3} + C$ }
17.  $\int x\sqrt{1-x^2} dx$ . {Ans.  $-\frac{1}{3}\sqrt{(1-x^2)^3} + C$ }

18.  $\int x^2 \sqrt[5]{x^3 + 2} dx$ . {Ans.  $\frac{5}{18} \sqrt[5]{(x^3 + 2)^6} + C$ }
19.  $\int \frac{x dx}{\sqrt{x^2 + 1}}$ . {Ans.  $\sqrt{x^2 + 1} + C$ }
20.  $\int \frac{x^4 dx}{\sqrt{4 + x^5}}$ . {Ans.  $\frac{2}{5} \sqrt{4 + x^5} + C$ }
21.  $\int \frac{x^3 dx}{\sqrt[3]{x^4 + 1}}$ . {Ans.  $\frac{3}{8} \sqrt[3]{(x^4 + 1)^2} + C$ }
22.  $\int \frac{(6x - 5) dx}{2\sqrt{3x^2 - 5x + 6}}$ . {Ans.  $\sqrt{3x^2 - 5x + 6} + C$ }
23.  $\int \sin^3 x \cos x dx$ . {Ans.  $\frac{\sin^4 x}{4} + C$ }
24.  $\int \frac{\sin x dx}{\cos^2 x}$ . {Ans.  $\sec x + C$ }
25.  $\int \frac{\cos x dx}{\sqrt[3]{\sin^2 x}}$ . {Ans.  $3\sqrt[3]{\sin x} + C$ }
26.  $\int \cos^3 x \sin 2x dx$ . {Ans.  $-\frac{2}{5} \cos^5 x + C$ }
27.  $\int e^{3 \ln x} (x^4 + 1)^{-1} dx$ . {Ans.  $\frac{1}{4} \ln(x^4 + 1) + C$ }
28.  $\int \cos^3 x e^{\ln(\sin x)} dx$ . {Ans.  $-\frac{\cos^4 x}{4} + C$ }
29.  $\int \frac{\sqrt{\ln x}}{x} dx$ . {Ans.  $\frac{2}{3} \sqrt{(\ln x)^3} + C$ }
30.  $\int \frac{(\tan^{-1} x)^2 dx}{1 + x^2}$ . {Ans.  $\frac{(\tan^{-1} x)^3}{3} + C$ }
31.  $\int \frac{dx}{(\sin^{-1} x)^3 \sqrt{1 - x^2}}$ . {Ans.  $-\frac{1}{2(\sin^{-1} x)^2} + C$ }
32.  $\int \frac{dx}{\cos^2 x \sqrt{1 + \tan x}}$ . {Ans.  $2\sqrt{1 + \tan x} + C$ }
33.  $\int \cos 3x dx$ . {Ans.  $\frac{1}{3} \sin 3x + C$ }
34.  $\int (\cos x - \cos 2x) dx$ . {Ans.  $x \cos x - \frac{1}{2} \sin 2x + C$ }
35.  $\int \sin(2x - 3) dx$ . {Ans.  $-\frac{1}{2} \cos(2x - 3) + C$ }
36.  $\int \cos(1 - 2x) dx$ . {Ans.  $-\frac{1}{2} \sin(1 - 2x) + C$ }

37.  $\int \left[ \cos \left( 2x - \frac{\pi}{4} \right) \right]^{-2} dx$ . {Ans.  $\frac{1}{2} \tan \left( 2x - \frac{\pi}{4} \right) + C$ }
38.  $\int e^x (\sin e^x) dx$ . {Ans.  $-\cos(e^x) + C$ }
39.  $\int \frac{(2x-3)dx}{x^2-3x+8}$ . {Ans.  $\ln(x^2-3x+8) + C$ }
40.  $\int \frac{dx}{2x-1}$ . {Ans.  $\frac{1}{2} \ln|2x-1| + C$ }
41.  $\int \frac{dx}{cx+m}$ . {Ans.  $\frac{1}{c} \ln|cx+m| + C$ }
42.  $\int \frac{x dx}{x^2+1}$ . {Ans.  $\frac{1}{2} \ln(x^2+1) + C$ }
43.  $\int \frac{x^2 dx}{x^3+1}$ . {Ans.  $\frac{1}{3} \ln|x^3+1| + C$ }
44.  $\int \frac{e^x dx}{e^x+1}$ . {Ans.  $\ln(e^x+1) + C$ }
45.  $\int \frac{e^{2x} dx}{e^{2x}+a^2}$ . {Ans.  $\frac{1}{2} \ln(e^{2x}+a^2) + C$ }
46.  $\int \tan 3x dx$ . {Ans.  $-\frac{1}{3} \ln|\cos 3x| + C$ }
47.  $\int \cot(2x+1) dx$ . {Ans.  $\frac{1}{2} \ln|\sin(2x+1)| + C$ }
48.  $\int \frac{\sin 2x}{1+\cos^2 x} dx$ . {Ans.  $-\ln(1+\cos^2 x) + C$ }
49.  $\int \frac{dx}{x \ln x}$ . {Ans.  $\ln|\ln x| + C$ }
50.  $\int \frac{(\ln x)^m}{x} dx$ . {Ans.  $\frac{\ln^{m+1} x}{m+1} + C, m \neq -1; \ln|\ln x| + C, m = -1$ }
51.  $\int \frac{a^{\sqrt{x}}}{\sqrt{x}} dx$ . {Ans.  $\frac{2a^{\sqrt{x}}}{\ln a} + C$ }
52.  $\int e^{\sin x} \cos x dx$ . {Ans.  $e^{\sin x} + C$ }
53.  $\int a^{3x} dx$ . {Ans.  $\frac{a^{3x}}{3 \ln a} + C$ }
54.  $\int a^{-x} dx$ . {Ans.  $-\frac{1}{a^x \ln a} + C$ }
55.  $\int e^{-3x+1} dx$ . {Ans.  $-\frac{e^{1-3x}}{3} + C$ }
56.  $\int e^{x^2} x dx$ . {Ans.  $\frac{e^{x^2}}{2} + C$ }

57.  $\int e^{-x^3} x^2 dx$ . {Ans.  $-\frac{1}{3}e^{-x^3} + C$ }
58.  $\int \frac{dx}{\sqrt{1-25x^2}}$ . {Ans.  $\frac{1}{5}\sin^{-1} 5x + C$ }
59.  $\int \frac{dx}{1+9x^2}$ . {Ans.  $\frac{1}{3}\tan^{-1} 3x + C$ }
60.  $\int \frac{dx}{\sqrt{4-x^2}}$ . {Ans.  $\sin^{-1} \frac{x}{2} + C$ }
61.  $\int \frac{dx}{2x^2+9}$ . {Ans.  $\frac{1}{3\sqrt{2}}\tan^{-1} \frac{\sqrt{2}}{3}x + C$ }
62.  $\int \frac{dx}{\sqrt{4-9x^2}}$ . {Ans.  $\frac{1}{3}\sin^{-1} \frac{3x}{2} + C$ }
63.  $\int \frac{x dx}{x^4+1}$ . {Ans.  $\frac{\tan^{-1} x^2}{2} + C$ }
64.  $\int \frac{x dx}{\sqrt{a^2-x^4}}$ . {Ans.  $\frac{1}{2}\sin^{-1} \frac{x^2}{a} + C$ }
65.  $\int \frac{x^2 dx}{x^6+4}$ . {Ans.  $\frac{1}{6}\tan^{-1} \frac{x^3}{2} + C$ }
66.  $\int \frac{x^3 dx}{\sqrt{1-x^8}}$ . {Ans.  $\frac{1}{4}\sin^{-1} x^4 + C$ }
67.  $\int \frac{e^x dx}{e^{2x}+4}$ . {Ans.  $\frac{1}{2}\tan^{-1} \frac{e^x}{2} + C$ }
68.  $\int \frac{2^x dx}{\sqrt{1-4^x}}$ . {Ans.  $\frac{\sin^{-1} 2^x}{\ln 2} + C$ }
69.  $\int \frac{\cos \alpha d\alpha}{a^2+\sin^2 \alpha}$ . {Ans.  $\frac{1}{a}\tan^{-1}\left(\frac{\sin \alpha}{a}\right) + C$ }
70.  $\int \frac{e^{2x}-1}{e^x} dx$ . {Ans.  $e^x + e^{-x} + C$ }
71.  $\int (e^x+1)^3 dx$ . {Ans.  $\frac{1}{3}e^{3x} + \frac{3}{2}e^{2x} + 3e^x + x + C$ }
72.  $\int \frac{1+x}{\sqrt{1-x^2}} dx$ . {Ans.  $\sin^{-1} x - \sqrt{1-x^2} + C$ }
73.  $\int \frac{3x-1}{x^2+9} dx$ . {Ans.  $\frac{3}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1} \frac{x}{3} + C$ }
74.  $\int \sqrt{\frac{1-x}{1+x}} dx$ . {Ans.  $\sin^{-1} x + \sqrt{1-x^2} + C$ }
75.  $\int \frac{x(1-x^2)}{1+x^4} dx$ . {Ans.  $\frac{1}{2}\tan^{-1} x^2 - \frac{1}{4}\ln(x^4+1) + C$ }

76.  $\int \frac{1+x-x^2}{\sqrt{(1-x^2)^3}} dx$ . {Ans.  $\sin^{-1} x + \frac{1}{\sqrt{1-x^2}} + C$ }
77.  $\int \frac{dx}{(x+\sqrt{x^2-1})^2}$ . {Ans.  $\frac{2}{3} \left( x^3 - \sqrt{(x^2-1)^3} \right) - x + C$ }
78.  $\int \frac{2x - \sqrt{\sin^{-1} x}}{\sqrt{1-x^2}} dx$ . {Ans.  $-2\sqrt{1-x^2} - \frac{2}{3} \sqrt{(\sin^{-1} x)^3} + C$ }
79.  $\int \frac{x + (\cos^{-1} 3x)^2}{\sqrt{1-9x^2}} dx$ . {Ans.  $-\frac{1}{9} \left[ \sqrt{1-9x^2} + (\cos^{-1} 3x)^3 \right] + C$ }

**CATEGORY-21.2. INTEGRATION OF ELEMENTARY TRIGONOMETRIC FUNCTIONS**

80.  $\int \sin^3 x \cos^2 x dx$ . {Ans.  $\frac{1}{15} \cos^3 x (3 \cos^2 x - 5) + C$ }
81.  $\int \sin^4 x \cos^3 x dx$ . {Ans.  $\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$ }
82.  $\int \sin^3 x \cos^5 x dx$ . {Ans.  $-\frac{\cos^6 x}{6} - \frac{\cos^8 x}{8} + C$ }
83.  $\int \cos^3 x dx$ . {Ans.  $\sin x - \frac{\sin^3 x}{3} + C$ }
84.  $\int \sin^5 x dx$ . {Ans.  $-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$ }
85.  $\int \cos^2 x dx$ . {Ans.  $\frac{x}{2} + \frac{\sin 2x}{4} + C$ }
86.  $\int \sin^2 x dx$ . {Ans.  $\frac{x}{2} - \frac{\sin 2x}{4} + C$ }
87.  $\int \sin^4 x dx$ . {Ans.  $\frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$ }
88.  $\int \cos^6 x dx$ . {Ans.  $\frac{5}{16} x + \frac{1}{12} \sin 2x \left( \cos^4 x + \frac{5}{4} \cos^2 x + \frac{15}{8} \right) + C$ }
89.  $\int \sin^8 x dx$ . {Ans.  $\frac{35}{128} x - \frac{1}{4} \sin 2x + \frac{7}{128} \sin 4x + \frac{1}{24} \sin^3 2x + \frac{1}{1024} \sin 8x + C$ }
90.  $\int \cos^8 x dx$ .
91.  $\int \tan^5 x dx$ . {Ans.  $\frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln |\cos x| + C$ }
92.  $\int \tan^7 x dx$ . {Ans.  $\frac{\tan^6 x}{6} - \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} - \ln |\sec x| + C$ }
93.  $\int \tan^8 x dx$ . {Ans.  $\frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - \tan x + x + C$ }

94.  $\int \cot^4 x \, dx$ . {Ans.  $x - \frac{1}{3} \cot^3 x + \cot x + C$ }
95.  $\int \cot^7 x \, dx$ . {Ans.  $-\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \ln|\sin x| + C$ }
96.  $\int (\tan^2 x + \tan^4 x) \, dx$ . {Ans.  $\frac{1}{3} \tan^3 x + C$ }
97.  $\int \sec^4 x \, dx$ . {Ans.  $\tan x + \frac{1}{3} \tan^3 x + C$ }
98.  $\int \operatorname{cosec}^6 x \, dx$ . {Ans.  $-\cot x - \frac{2}{3} \cot^3 x - \frac{1}{5} \cot^5 x + C$ }
99.  $\int \cos x \sin 3x \, dx$ . {Ans.  $-\frac{1}{4} \left( \frac{\cos 4x}{2} + \cos 2x \right) + C$ }
100.  $\int \cos 2x \cos 3x \, dx$ . {Ans.  $\frac{1}{10} \sin 5x + \frac{1}{2} \sin x + C$ }
101.  $\int \sin 2x \sin 5x \, dx$ . {Ans.  $\frac{1}{6} \sin 3x - \frac{1}{14} \sin 7x + C$ }
102.  $\int \cos x \cos 2x \cos 3x \, dx$ . {Ans.  $\frac{1}{8} \left( 2x + \sin 2x + \frac{1}{2} \sin 4x + \frac{1}{3} \sin 6x \right) + C$ }
103.  $\int \frac{dx}{1 - \cos x}$ . {Ans.  $-\cot \frac{x}{2} + C$ }
104.  $\int \frac{dx}{1 + \sin x}$ . {Ans.  $\tan \left( \frac{x}{2} - \frac{\pi}{4} \right) + C$ }
105.  $\int \frac{1 - \cos x}{1 + \cos x} \, dx$ . {Ans.  $2 \tan \frac{x}{2} - x + C$ }
106.  $\int \frac{1 + \sin x}{1 - \sin x} \, dx$ . {Ans.  $2 \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) - x + C$ }

### CATEGORY-21.3. SUBSTITUTION

107.  $\int \frac{dx}{x\sqrt{x+1}}$ . {Ans.  $\ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + C$ }
108.  $\int \frac{x^5 dx}{\sqrt{a^3 - x^3}}$ . {Ans.  $-\frac{2}{9} \sqrt{a^2 - x^2} (2a^3 + x^3) + C$ }
109.  $\int \frac{x^5 dx}{(x^2 - 4)^2}$ . {Ans.  $\frac{x^2 - 4}{2} + \frac{8}{x^2 - 4} + 4 \ln|x^2 - 4| + C$ }
110.  $\int \frac{dx}{x^2 \sqrt{x^2 + a^2}}$ . {Ans.  $-\frac{\sqrt{x^2 + a^2}}{a^2 x} + C$ }
111.  $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}}$ . {Ans.  $\frac{a^2}{2} \sin^{-1} \frac{x}{a} - \frac{x}{2} \sqrt{a^2 - x^2} + C$ }



112.  $\int \frac{dx}{x\sqrt{x^2 - a^2}}$ . {Ans.  $-\frac{1}{a} \sin^{-1} \frac{a}{|x|} + C$ }
113.  $\int \frac{\sqrt{1+x^2}}{x^4} dx$ . {Ans.  $-\frac{\sqrt{(1+x^2)^3}}{3x^3} + C$ }
114.  $\int \frac{\sqrt{1-x^2}}{x^2} dx$ . {Ans.  $-\frac{\sqrt{1-x^2}}{x} - \sin^{-1} x + C$ }
115.  $\int \frac{dx}{\sqrt{(a^2 + x^2)^3}}$ . {Ans.  $\frac{x}{a^2 \sqrt{x^2 + a^2}} + C$ }
116.  $\int \frac{\sqrt{(9-x^2)^3}}{x^6} dx$ . {Ans.  $-\frac{\sqrt{(9-x^2)^5}}{45x^5} + C$ }
117.  $\int \frac{dx}{x^2 \sqrt{x^2 - 9}}$ . {Ans.  $\frac{\sqrt{x^2 - 9}}{9x} + C$ }
118.  $\int \frac{dx}{x\sqrt{1+x^2}}$ . {Ans.  $\ln \frac{|x|}{1+\sqrt{x^2+1}} + C$ }
119.  $\int \frac{dx}{\sqrt{(x^2 - a^2)^3}}$ . {Ans.  $-\frac{x}{a^2 \sqrt{x^2 - a^2}} + C$ }
120.  $\int x^2 \sqrt{4-x^2} dx$ . {Ans.  $\frac{x}{4}(x^2 - 2)\sqrt{4-x^2} + 2\sin^{-1} \frac{x}{2} + C$ }

#### CATEGORY-21.4. INTEGRATION BY PARTS

121.  $\int x \cos x dx$ . {Ans.  $x \sin x + \cos x + C$ }
122.  $\int x \sin 2x dx$ . {Ans.  $\frac{1}{4} \sin 2x - \frac{x}{2} \cos 2x + C$ }
123.  $\int x e^{-x} dx$ . {Ans.  $-e^{-x}(x+1) + C$ }
124.  $\int x 3^x dx$ . {Ans.  $\frac{3^x}{\ln^2 x} (x \ln 3 - 1) + C$ }
125.  $\int \ln x dx$ . {Ans.  $x \ln x - x + C$ }
126.  $\int x^n \ln x dx \quad (n \neq -1)$ . {Ans.  $\frac{x^{n+1}}{n+1} \left( \ln x - \frac{1}{n+1} \right) + C$ }
127.  $\int \tan^{-1} x dx$ . {Ans.  $x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$ }
128.  $\int x \tan^{-1} x dx$ . {Ans.  $\left( \frac{x^2+1}{2} \right) \tan^{-1} x - \frac{x}{2} + C$ }
129.  $\int \cos^{-1} x dx$ . {Ans.  $x \cos^{-1} x - \sqrt{1-x^2} + C$ }

130.  $\int \tan^{-1} \sqrt{x} dx$ . {Ans.  $x \tan^{-1} \sqrt{x} - \sqrt{x} + \tan^{-1} \sqrt{x} + C$ }
131.  $\int \frac{\sin^{-1} x}{\sqrt{x+1}} dx$ . {Ans.  $2\sqrt{x+1} \sin^{-1} x + 4\sqrt{1-x} + C$ }
132.  $\int x \tan^2 x dx$ . {Ans.  $x \tan x - \frac{x^2}{2} + \ln|\cos x| + C$ }
133.  $\int x \cos^2 x dx$ . {Ans.  $\frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x + C$ }
134.  $\int \frac{\ln x}{x^3} dx$ . {Ans.  $-\frac{1}{2x^2} \left( \ln x + \frac{1}{2} \right) + C$ }
135.  $\int \frac{x \tan^{-1} x}{\sqrt{1+x^2}} dx$ . {Ans.  $\sqrt{1+x^2} \tan^{-1} x - \ln(x + \sqrt{1+x^2}) + C$ }
136.  $\int \frac{\sin^{-1} \sqrt{x}}{\sqrt{1-x}} dx$ . {Ans.  $2(\sqrt{x} - \sqrt{1-x} \sin^{-1} \sqrt{x}) + C$ }
137.  $\int \ln(x^2 + 1) dx$ . {Ans.  $x \ln(x^2 + 1) - 2x + 2 \tan^{-1} x + C$ }
138.  $\int \frac{x^2 dx}{(1+x^2)^2}$ . {Ans.  $-\frac{x}{2(1+x^2)} + \frac{1}{2} \tan^{-1} x + C$ }
139.  $\int \frac{x^3 dx}{\sqrt{1+x^2}}$ . {Ans.  $x^2 \sqrt{1+x^2} - \frac{2}{3} \sqrt{(1+x^2)^3} + C$ }
140.  $\int x^2 \ln(1+x) dx$ . {Ans.  $\frac{(x^3+1)\ln(1+x)}{2} - \frac{x^3}{9} + \frac{x^2}{6} - \frac{x}{3} + C$ }
141.  $\int x^2 e^{-x} dx$ . {Ans.  $-e^{-x}(2+2x+x^2) + C$ }
142.  $\int x^3 e^x dx$ . {Ans.  $e^x(x^3 - 3x^2 + 6x - 6) + C$ }
143.  $\int x^2 a^x dx$ . {Ans.  $a^x \left( \frac{x^2}{\ln a} - \frac{2x}{\ln^2 a} + \frac{2}{\ln^3 a} \right) + C$ }
144.  $\int x^3 \sin x dx$ . {Ans.  $-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$ }
145.  $\int x^2 \cos^2 x dx$ . {Ans.  $\frac{x^3}{6} + \frac{x^2}{4} \sin 2x + \frac{x}{4} \cos 2x - \frac{1}{8} \sin 2x + C$ }
146.  $\int \ln^2 x dx$ . {Ans.  $x(\ln^2 x - 2 \ln x + 2) + C$ }
147.  $\int \frac{\ln^3 x}{x^2} dx$ . {Ans.  $-\frac{1}{x} (\ln^3 x + 3 \ln^2 x + 6 \ln x + 6) + C$ }
148.  $\int \frac{\ln^2 x}{\sqrt{x^5}} dx$ . {Ans.  $-\frac{8}{27\sqrt{x^3}} \left( \frac{9}{4} \ln^2 x + 3 \ln x + 2 \right) + C$ }
149.  $\int (\sin^{-1} x)^2 dx$ . {Ans.  $x(\sin^{-1} x)^2 + 2 \sin^{-1} x \sqrt{1-x^2} - 2x + C$ }

150.  $\int (\tan^{-1} x)^2 x dx$ . {Ans.  $\left(\frac{x^2+1}{2}\right)(\tan^{-1} x)^2 - x \tan^{-1} x + \frac{1}{2} \ln(1+x^2) + C$ }
151.  $\int e^x \sin x dx$ . {Ans.  $\frac{e^x(\sin x - \cos x)}{2} + C$ }
152.  $\int e^{3x}(\sin 2x - \cos 2x) dx$ . {Ans.  $\frac{e^{3x}}{13}(\sin 2x - 5 \cos 2x) + C$ }
153.  $\int e^{ax} \cos nx dx$ . {Ans.  $\frac{e^{ax}}{a^2 + n^2}(n \sin nx + a \cos nx) + C$ }
154.  $\int \sin(\ln x) dx$ . {Ans.  $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C$ }
155.  $\int \frac{x^2 dx}{\sqrt{1-x^2}}$ . {Ans.  $-\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x + C$ }
156.  $\int \sqrt{a^2 + x^2} dx$ . {Ans.  $\frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C$ }
157.  $\int \frac{x^2 e^x dx}{(x+2)^2}$ . {Ans.  $\left(\frac{x-2}{x+2}\right)e^x + C$ }
158.  $\int x^2 e^x \sin x dx$ . {Ans.  $\frac{1}{2}((x^2-1)\sin x - (x-1)^2 \cos x)e^x + C$ }

### CATEGORY-21.5. REDUCTION FORMULA

159. Derive reduction formula of  $I_n = \int \sin^n x dx$  and find  $\int \sin^{10} x dx$ .
160. Derive reduction formula of  $I_n = \int \cos^n x dx$  and find  $\int \cos^{11} x dx$ .
161. Derive reduction formula of  $I_n = \int \sec^n x dx$  and find  $\int \sec^5 x dx$ .
162. Derive reduction formula of  $I_n = \int \operatorname{cosec}^n x dx$  and find  $I_3$  and  $I_7$ .
163. Derive reduction formula of  $I_n = \int \frac{dx}{(x^2 + a^2)^n}$ .
164. Derive reduction formula of  $I_n = \int (a^2 - x^2)^n dx$ .
165. Derive reduction formula of  $I_n = \int (\ln x)^n dx$ .
166. Derive reduction formula of  $I_n = \int x^\alpha (\ln x)^n dx$  ( $\alpha \neq -1$ ).
167. Derive reduction formula of  $I_n = \int x^n e^x dx$ .
168. Derive reduction formula of  $I_n = \int e^{\alpha x} \sin^n x dx$ .
169. Derive reduction formula of  $I_n = \int \frac{x^n dx}{\sqrt{x^2 + a}}$ .
170. Derive reduction formula of  $I_{n,m} = \int \sin^n x \cos^m x dx$  and find  $I_{7,5}$  and  $I_{7,6}$ .

**CATEGORY-21.6. INTEGRATION OF RATIONAL FUNCTIONS WITH LINEAR DENOMINATOR**

171.  $\int \frac{dx}{2x+3}$ . {Ans.  $\frac{1}{2} \ln|2x+3| + C$ }
172.  $\int \frac{x}{x+4} dx$ . {Ans.  $x - 4 \ln|x+4| + C$ }
173.  $\int \frac{x}{2x+1} dx$ . {Ans.  $\frac{1}{2} \left( x - \frac{1}{2} \ln|2x+1| \right) + C$ }
174.  $\int \frac{3+x}{3-x} dx$ . {Ans.  $-x - 6 \ln|x-3| + C$ }
175.  $\int \frac{(2x-1)dx}{x-2}$ . {Ans.  $2x + 3 \ln|x-2| + C$ }
176.  $\int \frac{x+2}{2x-1} dx$ . {Ans.  $\frac{x}{2} + \frac{5}{4} \ln|2x-1| + C$ }
177.  $\int \frac{Ax}{a+bx} dx$ . {Ans.  $\frac{A}{b} \left( x - \frac{a}{b} \ln|bx+a| \right) + C$ }
178.  $\int \frac{x^2+x+2}{2x+3} dx$ . {Ans.  $\frac{x^2}{2} - \frac{x}{4} + \frac{11}{8} \ln|2x+3| + C$ }
179.  $\int \frac{x^3 dx}{x+1}$ . {Ans.  $\frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C$ }
180.  $\int \frac{x^4}{1-x} dx$ . {Ans.  $-\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} - x - \ln|x-1| + C$ }

**CATEGORY-21.7. INTEGRATION OF RATIONAL FUNCTIONS WITH QUADRATIC DENOMINATOR**

181.  $\int \frac{dx}{x(x-1)}$ . {Ans.  $\ln \left| \frac{x-1}{x} \right| + C$ }
182.  $\int \frac{dx}{x(x+1)}$ . {Ans.  $\ln \left| \frac{x}{x+1} \right| + C$ }
183.  $\int \frac{dx}{(x+1)(2x-3)}$ . {Ans.  $\frac{1}{5} \ln \left| \frac{2x-3}{x+1} \right| + C$ }
184.  $\int \frac{dx}{(a-x)(b-x)}$ . {Ans.  $\frac{1}{b-a} \ln \left| \frac{x-b}{x-a} \right| + C$ }
185.  $\int \frac{dx}{x^2-7x+10}$ . {Ans.  $\frac{1}{3} \ln \left| \frac{x-5}{x-2} \right| + C$ }
186.  $\int \frac{dx}{x^2+3x-10}$ . {Ans.  $\frac{1}{7} \ln \left| \frac{x-2}{x+5} \right| + C$ }
187.  $\int \frac{dx}{4x^2-9}$ . {Ans.  $\frac{1}{12} \ln \left| \frac{2x-3}{2x+3} \right| + C$ }

188.  $\int \frac{dx}{2-3x^2}$ . {Ans.  $\frac{1}{2\sqrt{6}} \ln \left| \frac{x\sqrt{3}+\sqrt{2}}{x\sqrt{3}-\sqrt{2}} \right| + C$ }
189.  $\int \frac{dx}{(x-1)^2+4}$ . {Ans.  $\frac{1}{2} \tan^{-1} \frac{x-1}{2} + C$ }
190.  $\int \frac{dx}{x^2+2x+3}$ . {Ans.  $\frac{1}{\sqrt{2}} \tan^{-1} \frac{x+1}{\sqrt{2}} + C$ }
191.  $\int \frac{dx}{x-x^2-2.5}$ . {Ans.  $\frac{2}{3} \tan^{-1} \frac{1-2x}{3} + C$ }
192.  $\int \frac{dx}{4x^2+4x+5}$ . {Ans.  $\frac{1}{4} \tan^{-1} \frac{2x+1}{2} + C$ }
193.  $\int \frac{x dx}{(x+1)(2x+1)}$ . {Ans.  $\ln \frac{|x+1|}{\sqrt{2x+1}} + C$ }
194.  $\int \frac{x dx}{2x^2-3x-2}$ . {Ans.  $\frac{1}{5} \ln((x-2)^2 \sqrt{2x+1}) + C$ }
195.  $\int \frac{(x+2)dx}{x^2+2x+2}$ . {Ans.  $\frac{1}{2} \ln(x^2+2x+2) + \tan^{-1}(x+1) + C$ }
196.  $\int \frac{(3x-1)dx}{4x^2-4x+17}$ . {Ans.  $\frac{3}{8} \left( \ln(4x^2-4x+17) + \frac{1}{6} \tan^{-1} \frac{2x-1}{4} \right) + C$ }
197.  $\int \frac{(x-2)dx}{x^2-7x+12}$ . {Ans.  $\ln \frac{(x-4)^2}{|x-3|} + C$ }
198.  $\int \frac{3-4x}{2x^2-3x+1} dx$ . {Ans.  $-\ln|2x^2-3x+1| + C$ }
199.  $\int \frac{(4-3x)dx}{5x^2+6x+18}$ . {Ans.  $\frac{29}{45} \tan^{-1} \frac{5x+3}{9} - \frac{3}{10} \ln(5x^2+6x+18) + C$ }
200.  $\int \frac{(1+x)^2}{x^2+1} dx$ . {Ans.  $x + \ln(x^2+1) + C$ }
201.  $\int \frac{x^2-1}{x^2+1} dx$ . {Ans.  $x - 2 \tan^{-1} x + C$ }
202.  $\int \frac{x^2+1}{x^2-1} dx$ . {Ans.  $x + \ln \left| \frac{x-1}{x+1} \right| + C$ }
203.  $\int \frac{x^3+x+1}{x^2+x+1} dx$ . {Ans.  $\frac{x^2}{2} - x + \frac{1}{2} \ln(x^2+x+1) + \sqrt{3} \tan^{-1} \left( \frac{2x+1}{\sqrt{3}} \right) + C$ }
204.  $\int \frac{x^4 dx}{x^2+1}$ . {Ans.  $\frac{x^3}{3} - x + \tan^{-1} x + C$ }

**CATEGORY-21.8. INTEGRATION OF RATIONAL FUNCTIONS WITH CUBIC DENOMINATOR**

205.  $\int \frac{dx}{6x^3-7x^2-3x}$ . {Ans.  $\frac{3}{11} \ln|3x+1| + \frac{2}{33} \ln|2x-3| - \frac{1}{3} \ln|x| + C$ }

206.  $\int \frac{dx}{x(x^2+1)}$ . {Ans.  $\ln \frac{|x|}{\sqrt{x^2+1}} + C$ }
207.  $\int \frac{dx}{1+x^3}$ . {Ans.  $\frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} + C$ }
208.  $\int \frac{32x dx}{(2x-1)(4x^2-16x+15)}$ . {Ans.  $\ln|2x-1| - 6\ln|2x-3| + 5\ln|2x-5| + C$ }
209.  $\int \frac{xdx}{x^3-1}$ . {Ans.  $\frac{1}{3} \ln \frac{|x-1|}{\sqrt{x^2+x+1}} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C$ }
210.  $\int \frac{xdx}{(x-1)^3}$ . {Ans.  $-\frac{1}{x-1} - \frac{1}{2(x-1)^2} + C$ }
211.  $\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} dx$ . {Ans.  $\ln \left| \frac{(x-1)^4(x-4)^5}{(x+3)^7} \right| + C$ }
212.  $\int \frac{(x^2-3x+2)dx}{x(x^2+2x+1)}$ . {Ans.  $\ln \left| \frac{x^2}{x+1} \right| + \frac{6}{x+1} + C$ }
213.  $\int \left( \frac{x+2}{x-1} \right)^2 \frac{dx}{x}$ . {Ans.  $4\ln|x| - 3\ln|x-1| - \frac{9}{x-1} + C$ }
214.  $\int \frac{x^2 dx}{x^3+5x^2+8x+4}$ . {Ans.  $\frac{4}{x+2} + \ln|x+1| + C$ }
215.  $\int \frac{(2x^2-3x-3)dx}{(x-1)(x^2-2x+5)}$ . {Ans.  $\ln \frac{\sqrt{(x^2-2x+5)^3}}{|x-1|} + \frac{1}{2} \tan^{-1} \frac{x-1}{2} + C$ }
216.  $\int \frac{x^3-1}{4x^3-x} dx$ . {Ans.  $\frac{x}{4} + \ln|x| - \frac{7}{16} \ln|2x-1| - \frac{9}{16} \ln|2x+1| + C$ }
217.  $\int \frac{(x^3+1)}{x^3-x^2} dx$ . {Ans.  $x + \frac{1}{x} + \ln \frac{(x-1)^2}{|x|} + C$ }
218.  $\int \frac{(x^4+1)dx}{x^3-x^2+x-1}$ . {Ans.  $\frac{(x+1)^2}{2} + \ln \frac{|x-1|}{\sqrt{x^2+1}} - \tan^{-1} x + C$ }
219.  $\int \frac{x^4 dx}{(x^2-1)(x+2)}$ . {Ans.  $\frac{x^2}{2} - 2x + \frac{1}{6} \ln \frac{|x-1|(x+2)^{32}}{|x+1|^3} + C$ }
220.  $\int \frac{x^5+x^4-8}{x^3-4x} dx$ . {Ans.  $\frac{x^3}{3} + \frac{x^2}{2} + 4x + \ln \left| \frac{x^2(x-2)^5}{(x+2)^3} \right| + C$ }

**CATEGORY-21.9. INTEGRATION OF RATIONAL FUNCTIONS WITH DENOMINATOR OF DEGREE FOUR**

221.  $\int \frac{dx}{x^4-x^2}$ . {Ans.  $\frac{1}{x} + \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C$ }

222.  $\int \frac{dx}{(x^2+1)(x^2+x)}$ . {Ans.  $\frac{1}{4} \ln \frac{x^4}{(x+1)^2(x^2+1)} - \frac{1}{2} \tan^{-1} x + C$ }
223.  $\int \frac{dx}{(x+1)^2(x^2+1)}$ . {Ans.  $\frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) - \frac{1}{2(x+1)} + C$ }
224.  $\int \frac{dx}{1+x^4}$ . {Ans.  $\frac{1}{4\sqrt{2}} \ln \frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{1-x^2} + C$ }
225.  $\int \frac{dx}{1-x^4}$ . {Ans.  $\frac{1}{4} \ln \left| \frac{x+1}{x-1} \right| + \frac{1}{2} \tan^{-1} x + C$ }
226.  $\int \frac{xdx}{x^4-3x^2+2}$ . {Ans.  $\frac{1}{2} \ln \left( \frac{x^2-2}{x^2-1} \right) + C$ }
227.  $\int \frac{(x^2-2x+3)dx}{(x-1)(x^3-4x^2+3x)}$ . {Ans.  $\frac{1}{x-1} + \ln \frac{\sqrt{(x-1)(x-3)}}{|x|} + C$ }
228.  $\int \frac{x^2 dx}{1-x^4}$ . {Ans.  $\frac{1}{4} \ln \left| \frac{x+1}{x-1} \right| - \frac{1}{2} \tan^{-1} x + C$ }
229.  $\int \frac{x^2-8x+7}{(x^2-3x-10)^2} dx$ . {Ans.  $\frac{8}{49(x-5)} - \frac{27}{49(x+2)} + \frac{30}{343} \ln \left| \frac{x-5}{x+2} \right| + C$ }
230.  $\int \frac{(2x^2-5)dx}{x^4-5x^2+6}$ . {Ans.  $\frac{1}{2\sqrt{2}} \ln \left| \frac{x-\sqrt{2}}{x+\sqrt{2}} \right| + \frac{1}{2\sqrt{3}} \ln \left| \frac{x-\sqrt{3}}{x+\sqrt{3}} \right| + C$ }
231.  $\int \frac{x^2 dx}{(x+2)^2(x+4)^2}$ . {Ans.  $2 \ln \left| \frac{x+4}{x+2} \right| - \frac{5x+12}{x^2+6x+8} + C$ }
232.  $\int \frac{x^3-6x^2+11x-5}{(x-2)^4} dx$ . {Ans.  $-\frac{1}{3(x-2)^3} + \frac{1}{2(x-2)^2} + C$ }
233.  $\int \frac{x^3-6x^2+9x+7}{(x-2)^3(x-5)} dx$ . {Ans.  $\frac{3}{2(x-2)^2} + \ln|x-5| + C$ }
234.  $\int \frac{(7x^3-9)dx}{x^4-5x^3+6x^2}$ . {Ans.  $\frac{3}{2x} - \frac{5}{4} \ln|x| + 20 \ln|x-3| - \frac{47}{4} \ln|x-2| + C$ }
235.  $\int \frac{(x^3-2x^2+4)dx}{x^2(x-2)^2}$ . {Ans.  $\frac{1}{4} \ln \left| \frac{x}{x-2} \right| - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{2(x-2)} + C$ }
236.  $\int \frac{(x^3-6)dx}{x^4+6x^2+8}$ . {Ans.  $\ln \frac{x^2+4}{\sqrt{x^2+2}} + \frac{3}{2} \tan^{-1} \frac{x}{2} - \frac{3}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + C$ }
237.  $\int \frac{1}{8} \left( \frac{x-1}{x+1} \right)^4 dx$ . {Ans.  $\frac{x}{8} - \ln|x+1| - \frac{9x^2+12x+5}{3(x+1)^2} + C$ }
238.  $\int \frac{x^5}{(x-1)^2(x^2-1)} dx$ . {Ans.  $\frac{(x+2)^2}{2} - \frac{1}{4(x-1)^2} - \frac{9}{4(x-1)} + \frac{31}{8} \ln|x-1| + \frac{1}{8} \ln|x+1| + C$ }

$$239. \int \frac{x^5 + 2x^3 + 4x + 4}{x^4 + 2x^3 + 2x^2} dx. \{ \text{Ans. } \frac{x^2}{2} - 2x - \frac{2}{x} + 2 \ln(x^2 + 2x + 2) - 2 \tan^{-1}(x+1) + C \}$$

$$240. \int \frac{x^5 dx}{x^4 + 4}. \{ \text{Ans. } \frac{x^2}{2} - \tan^{-1}\left(\frac{x^2}{2}\right) + C \}$$

**CATEGORY-21.10. INTEGRATION OF RATIONAL FUNCTIONS WITH DENOMINATOR OF DEGREE MORE THAN FOUR**

$$241. \int \frac{(3x^2 + x + 3)dx}{(x-1)^3(x^2+1)}. \{ \text{Ans. } \frac{1}{4} \left( \ln \frac{\sqrt{x^2+1}}{|x-1|} + \tan^{-1} x - \frac{7}{(x-1)^2} \right) + C \}$$

$$242. \int \frac{x^6 - 2x^4 + 3x^3 - 9x^2 + 4}{x^5 - 5x^3 + 4x} dx. \{ \text{Ans. } \frac{x^2}{2} + \ln \left| \frac{x(x-2)\sqrt{(x-1)(x+1)^3}}{x+2} \right| + C \}$$

$$243. \int \frac{dx}{x^6 + x^4}. \{ \text{Ans. } \tan^{-1} x + \frac{1}{x} - \frac{1}{3x^3} + C \}$$

$$244. \int \frac{3x^2 + 1}{(x^2 - 1)^3} dx. \{ \text{Ans. } -\frac{x}{(x^2 - 1)^2} + C \}$$

$$245. \int \frac{x^4 dx}{(1+x^2)^3}. \{ \text{Ans. } -\frac{x^3}{4(1+x^2)^2} - \frac{3x}{8(1+x^2)} + \frac{3 \tan^{-1} x}{8} + C \}$$

$$246. \int \frac{2x^5 - 3x^2}{1 + 3x^3 - x^6} dx. \{ \text{Ans. } -\frac{1}{3} \ln |1 + 3x^3 - x^6| + C \}$$

$$247. \int \frac{(3+x^2)^2 x^3 dx}{(1+x^2)^3}. \{ \text{Ans. } \frac{x^2}{2} + \frac{3}{2} \ln(1+x^2) + \frac{1}{(1+x^2)^2} + C \}$$

$$248. \int \frac{dx}{(x^4 - 1)^2}. \{ \text{Ans. } \frac{3}{8} \tan^{-1} x - \frac{x}{4(x^4 - 1)} - \frac{3}{16} \ln \left| \frac{x-1}{x+1} \right| + C \}$$

$$249. \int \frac{x^7 dx}{(1+x^4)^2}. \{ \text{Ans. } \frac{1}{4} \left( \ln(1+x^4) + \frac{1}{1+x^4} \right) + C \}$$

$$250. \int \frac{x^7 dx}{(1-x^2)^5}. \{ \text{Ans. } \frac{x^8}{8(1-x^2)^4} + C \}$$

$$251. \int \frac{x^3 dx}{(x-1)^{12}}. \{ \text{Ans. } -\frac{1}{8(x-1)^8} - \frac{1}{3(x-1)^9} - \frac{3}{10(x-1)^{10}} - \frac{1}{11(x-1)^{11}} + C \}$$

$$252. \int \frac{x^4 dx}{x^{15} - 1}. \{ \text{Ans. } \frac{1}{15} \left( \frac{1}{2} \ln \frac{(x^5 - 1)^2}{x^{10} + x^5 + 1} - \sqrt{3} \tan^{-1} \frac{2x^5 + 1}{\sqrt{3}} \right) + C \}$$

**CATEGORY-21.11. INTEGRATION OF RATIONAL FUNCTIONS OF  $\sin x$  AND  $\cos x$**

$$253. \int \frac{\cos 2x dx}{1 + \sin x \cos x}. \{ \text{Ans. } \ln(2 + \sin 2x) + C \}$$



254.  $\int \frac{1 - \sin x}{\cos x} dx$ . {Ans.  $\ln(1 + \sin x) + C$ }
255.  $\int \frac{\sin^3 x}{\cos x} dx$ . {Ans.  $\frac{\cos^2 x}{2} - \ln|\cos x| + C$ }
256.  $\int \frac{\cos^3 x dx}{\sin^4 x}$ . {Ans.  $\frac{1}{\sin x} - \frac{1}{3\sin^3 x} + C$ }
257.  $\int \frac{\sin^3 x}{\cos^4 x} dx$ . {Ans.  $\frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C$ }
258.  $\int \frac{dx}{\cos x \sin^3 x}$ . {Ans.  $\ln|\tan x| - \frac{1}{2\sin^2 x} + C$ }
259.  $\int \frac{\sin^4 x}{\cos^2 x} dx$ . {Ans.  $\tan x + \frac{1}{4}\sin 2x - \frac{3x}{2} + C$ }
260.  $\int \frac{dx}{\cos^3 x \sin^3 x}$ . {Ans.  $\frac{1}{2}(\tan^2 x - \cot^2 x) + 2\ln|\tan x| + C$ }
261.  $\int \frac{dx}{\sin^4 x \cos^4 x}$ . {Ans.  $\frac{(\tan^2 x - 1)(\tan^4 x + 10\tan^2 x + 1)}{3\tan^3 x} + C$ }
262.  $\int \frac{\sin x dx}{(1 - \cos x)^2}$ . {Ans.  $\frac{1}{\cos x - 1} + C$ }
263.  $\int \frac{\cos x dx}{(1 - \cos x)^2}$ . {Ans.  $\frac{1}{2}\cot \frac{x}{2} - \frac{1}{6}\cot^3 \frac{x}{2} + C$ }
264.  $\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x} dx$ . {Ans.  $\frac{1}{4}\ln\left|\frac{1 + \tan x}{1 - \tan x}\right| + \frac{1}{2}\sin x \cos x + C$ }
265.  $\int \frac{dx}{(\sin x + \cos x)^2}$ . {Ans.  $-\frac{1}{1 + \tan x} + C$ }
266.  $\int \frac{dx}{\sin x + \cos x}$ . {Ans.  $\frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{\pi}{8} + \frac{x}{2}\right)\right| + C$ }
267.  $\int \frac{dx}{a \cos x + b \sin x}$ . {Ans.  $\frac{1}{\sqrt{a^2 + b^2}}\ln\left|\tan\left(\frac{x + \tan^{-1} \frac{a}{b}}{2}\right)\right| + C$ }
268.  $\int \frac{dx}{\tan x \cos 2x}$ . {Ans.  $\ln \frac{|C \sin x|}{\sqrt{\cos 2x}}$ }
269.  $\int \frac{\cos^2 x dx}{\sin x \cos 3x}$ . {Ans.  $\ln \frac{|C \sin x|}{\sqrt{1 - 4\sin^2 x}}$ }
270.  $\int \frac{dx}{1 + \tan x}$ . {Ans.  $\frac{1}{2}(x + \ln|\sin x + \cos x|) + C$ }
271.  $\int \frac{dx}{5 - 3\cos x}$ . {Ans.  $\frac{1}{2}\tan^{-1}\left(2\tan \frac{x}{2}\right) + C$ }

272.  $\int \frac{dx}{5+4\sin x}$ . {Ans.  $\frac{2}{3} \tan^{-1} \frac{5 \tan \frac{x}{2} + 4}{3} + C$ }
273.  $\int \frac{2 - \sin x}{2 + \cos x} dx$ . {Ans.  $\ln(2 + \cos x) + \frac{4}{\sqrt{3}} \tan^{-1} \left( \frac{1}{\sqrt{3}} \tan \frac{x}{2} \right) + C$ }
274.  $\int \frac{\sin^2 x dx}{1 - \tan x}$ . {Ans.  $\frac{\cos x(\cos x - \sin x)}{4} - \frac{1}{4} \ln|\cos x - \sin x| + C$ }
275.  $\int \frac{dx}{4 + \tan x + 4 \cot x}$ . {Ans.  $\frac{4x}{25} - \frac{3}{25} \ln|\tan x + 2| + \frac{2}{5(\tan x + 2)} - \frac{3}{25} \ln|\cos x| + C$ }
276.  $\int \frac{dx}{(\sin x + 2 \sec x)^2}$ . {Ans.  $\frac{\cos 2x - 15}{15(4 + \sin 2x)} + \frac{4}{15\sqrt{15}} \sin^{-1} \left( \frac{4 \sin 2x + 1}{4 + \sin 2x} \right) + C$ }
277.  $\int \frac{dx}{5 - 4 \sin x + 3 \cos x}$ . {Ans.  $\frac{1}{2 - \tan \frac{x}{2}} + C$ }
278.  $\int \frac{dx}{4 - 3 \cos^2 x + 5 \sin^2 x}$ . {Ans.  $\frac{1}{3} \tan^{-1}(3 \tan x) + C$ }
279.  $\int \frac{dx}{1 + \sin^2 x}$ . {Ans.  $\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + C$ }
280.  $\int \frac{dx}{1 - \sin^4 x}$ . {Ans.  $\frac{1}{2} \tan x + \frac{1}{2\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + C$ }
281.  $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$ . {Ans.  $\frac{1}{ab} \tan^{-1} \left( \frac{a \tan x}{b} \right) + C$ }
282.  $\int \frac{dx}{\sin^2 x + \tan^2 x}$ . {Ans.  $-\frac{1}{2} \left( \cot x + \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{\tan x}{\sqrt{2}} \right) \right) + C$ }
283.  $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ . {Ans.  $\ln(\sin x + \cos x) + C$ }
284.  $\int \frac{\cos 2x}{\cos x} dx$ . {Ans.  $2 \sin x - \ln(\sec x + \tan x) + C$ }
285.  $\int \frac{\cos x dx}{\sin^3 x - \cos^3 x}$ . {Ans.  $\ln \frac{|\sqrt[3]{\tan x - 1}|}{\sqrt[6]{\tan^2 x + \tan x + 1}} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2 \tan x + 1}{\sqrt{3}} + C$ }
286.  $\int \frac{\tan x dx}{1 + \tan x + \tan^2 x}$ . {Ans.  $x - \frac{2}{\sqrt{3}} \tan^{-1} \frac{2 \tan x + 1}{\sqrt{3}} + C$ }
287.  $\int \frac{dx}{\sin^4 x + \cos^4 x}$ . {Ans.  $-\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \cot 2x) + C$ }
288.  $\int \frac{(x + \sin x) dx}{1 + \cos x}$ . {Ans.  $x \tan \frac{x}{2} + C$ }
289.  $\int \frac{dx}{\sin^5 x \cos^5 x}$ . {Ans.  $\frac{1}{4} (\tan^4 x - \cot^4 x) + 2(\tan^2 x - \cot^2 x) + 6 \ln|\tan x| + C$ }

290.  $\int \frac{\sin 2x \, dx}{\cos^4 x + \sin^4 x} \cdot \{ \text{Ans. } \tan^{-1}(\tan^2 x) + C \}$
291.  $\int \frac{dx}{1 + \sin x + \cos x} \cdot \{ \text{Ans. } \ln \left| 1 + \tan \frac{x}{2} \right| + C \}$
292.  $\int \frac{dx}{\sin 2x - 2 \sin x} \cdot \{ \text{Ans. } -\frac{1}{4} \ln \left| \tan \frac{x}{2} \right| + \frac{1}{8 \sin^2 \frac{x}{2}} + C \}$
293.  $\int \frac{dx}{1 + \cos^2 x} \cdot \{ \text{Ans. } \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{\tan x}{\sqrt{2}} \right) + C \}$
294.  $\int \frac{dx}{a^2 - b^2 \cos^2 x} \cdot \{ \text{Ans. } \frac{2}{b^2 \sin \left( 2 \cos^{-1} \frac{a}{b} \right)} \ln \left| \frac{\sin \left( \cos^{-1} \frac{a}{b} - x \right)}{\sin \left( \cos^{-1} \frac{a}{b} + x \right)} \right| + C \text{ if } a^2 < b^2 ;$   
 $\frac{1}{a^2 \sin \left( \cos^{-1} \frac{b}{a} \right)} \tan^{-1} \left( \frac{\tan x}{\sin \left( \cos^{-1} \frac{b}{a} \right)} \right) + C \text{ if } a^2 > b^2 \}$
295.  $\int \frac{(1 + \tan x) dx}{\sin 2x}.$
296.  $\int \frac{1 - \tan x}{1 + \tan x} dx \cdot \{ \text{Ans. } \ln |\sin x + \cos x| + C \}$
297.  $\int \frac{d\phi}{\sqrt{3} \cos \phi + \sin \phi} \cdot \{ \text{Ans. } \frac{1}{2} \ln \left| \tan \left( \frac{\phi}{2} + \frac{\pi}{6} \right) \right| + C \}$
298.  $\int \frac{\sin x \, dx}{1 + \sin x} \cdot \{ \text{Ans. } \sec x - \tan x + x + C \}$
299.  $\int \frac{\sin^2 x \cos x}{(1 + \sin^2 x)} dx \cdot \{ \text{Ans. } \sin x - \tan^{-1}(\sin x) + C \}$
300.  $\int \frac{\cos^2 3x}{\sin 3x} dx \cdot \{ \text{Ans. } \frac{1}{3} \left( \ln \left| \tan \frac{3x}{2} \right| + \cos 3x \right) + C \}$
301.  $\int \frac{dx}{1 - \sin 3x} \cdot \{ \text{Ans. } \frac{1}{3} \tan \left( \frac{\pi}{4} + \frac{3x}{2} \right) + C \}$
302.  $\int \frac{\sin 2x \, dx}{4 - \cos^2 2x} \cdot \{ \text{Ans. } -\frac{1}{8} \ln \left( \frac{2 + \cos 2x}{2 - \cos 2x} \right) + C \}$
303.  $\int (1 - \tan 3x)^2 dx \cdot \{ \text{Ans. } \frac{1}{3} (\tan 3x + \ln(\cos^2 3x)) + C \}$
304.  $\int \frac{d\phi}{\sin^2 \phi \cos^2 \phi} \cdot \{ \text{Ans. } -2 \cot 2\phi + C \}$

$$305. \int \frac{\cos 2x}{\cos^2 x} dx. \{ \text{Ans. } 2x - \tan x + C \}$$

$$306. \int \frac{\sin^4 x dx}{\cos^6 x}. \{ \text{Ans. } \frac{1}{5} \tan^5 x + C \}$$

**CATEGORY-21.12. INTEGRATION OF HYPERBOLIC FUNCTIONS**

$$307. \int \sinh^2 x dx. \{ \text{Ans. } \frac{\sinh x - \cosh x - x}{2} + C \}$$

$$308. \int \tanh^2 x dx. \{ \text{Ans. } x - \tanh x + C \}$$

$$309. \int \coth^2 x dx. \{ \text{Ans. } x - \coth x + C \}$$

$$310. \int \sinh^3 x dx. \{ \text{Ans. } \frac{1}{3} \cosh^3 x - \cosh x + C \}$$

$$311. \int \cosh^3 x dx. \{ \text{Ans. } \sinh x + \frac{1}{3} \sinh^3 x + C \}$$

$$312. \int \tanh^4 x dx. \{ \text{Ans. } x - \tanh x - \frac{1}{3} \tanh^3 x + C \}$$

$$313. \int \sinh^2 x \cosh^3 x dx. \{ \text{Ans. } \frac{1}{3} \sinh^3 x + \frac{1}{5} \sinh^5 x + C \}$$

$$314. \int \coth^5 x dx. \{ \text{Ans. } \ln|\sinh x| - \frac{1}{2} \coth^2 x - \frac{1}{4} \coth^4 x + C \}$$

$$315. \int \frac{dx}{\sinh x \cosh x}. \{ \text{Ans. } \ln|\tanh x| + C \}$$

$$316. \int \frac{dx}{\sinh x}. \{ \text{Ans. } \ln\left|\tanh \frac{x}{2}\right| + C \}$$

$$317. \int \frac{dx}{(1 + \cosh x)^2}. \{ \text{Ans. } \frac{1}{2} \tanh \frac{x}{2} - \frac{1}{6} \tanh^3 \frac{x}{2} + C \}$$

$$318. \int \frac{xdx}{\cosh^2 x}. \{ \text{Ans. } x \tanh x - \ln \cosh x + C \}$$

$$319. \int \frac{e^{2x} dx}{\sinh^4 x}. \{ \text{Ans. } -\frac{e^{3x}}{3 \sinh^3 x} + C \}$$

$$320. \int x^2 \sinh x dx. \{ \text{Ans. } x^2 \cosh x - 2x \sinh x + 2 \cosh x + C \}$$

$$321. \int \frac{e^x dx}{\cosh x + \sinh x}. \{ \text{Ans. } x + C \}$$

$$322. \int (\cosh^2 ax + \sinh^2 ax) dx. \{ \text{Ans. } \frac{1}{2a} \sinh 2ax + C \}$$

**CATEGORY-21.13. INTEGRATION OF IRRATIONAL FUNCTIONS CONTAINING  $\left(\frac{ax+b}{cx+d}\right)^{\frac{1}{n}}$**

$$323. \int \frac{dx}{x(\sqrt{x} + \sqrt[5]{x^2})}. \{ \text{Ans. } \ln \frac{x}{(1 + \sqrt[10]{x})^{10}} + \frac{10}{\sqrt[10]{x}} - \frac{5}{\sqrt[5]{x}} + \frac{10}{3\sqrt[10]{x^3}} - \frac{5}{2\sqrt[5]{x^2}} + C \}$$

$$324. \int \frac{dx}{\sqrt{x} + \sqrt[3]{x} + 2\sqrt[4]{x}}. \{ \text{Ans. } 2\sqrt{x} - 3\sqrt[3]{x} - 8\sqrt[4]{x} + 6\sqrt[6]{x} + 48\sqrt[12]{x} + 3\ln(1 + \sqrt[12]{x}) + \frac{33}{2}\ln(\sqrt[6]{x} - \sqrt[12]{x} + 2) - \frac{171}{\sqrt{7}}\tan^{-1}\frac{2\sqrt[12]{x}-1}{\sqrt{7}} + C \}$$

$$325. \int \frac{xdx}{(x+1)^{\frac{1}{2}} + (x+1)^{\frac{1}{3}}}. \{ \text{Ans. } 6\left(\frac{(x+1)^{\frac{3}{2}}}{9} - \frac{1}{8}(x+1)^{\frac{4}{3}} + \frac{1}{7}(x+1)^{\frac{7}{6}} - \frac{1}{6}(x+1) + \frac{1}{5}(x+1)^{\frac{5}{6}} - \frac{1}{4}(x+1)^{\frac{2}{3}}\right) + C \}$$

$$326. \int \sqrt{\frac{1+x}{1-x}} \frac{dx}{x}.$$

$$327. \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx. \{ \text{Ans. } (\sqrt{x}-2)\sqrt{1-x} - \sin^{-1}\sqrt{x} + C \}$$

$$328. \int \frac{x^2 + \sqrt{1+x}}{\sqrt[3]{1+x}} dx. \{ \text{Ans. } 6\sqrt[3]{(1+x)^2} \left( \frac{(1+x)^2}{16} - \frac{1+x}{5} + \frac{\sqrt{1+x}}{7} + \frac{1}{4} \right) + C \}$$

$$329. \int \sqrt[3]{\frac{1-x}{1+x}} \frac{dx}{x}. \{ \text{Ans. } \ln \frac{\left| \left( \frac{1-x}{1+x} \right)^{\frac{2}{3}} - 1 \right|}{\sqrt{\left( \frac{1-x}{1+x} \right)^{\frac{4}{3}} + \left( \frac{1-x}{1+x} \right)^{\frac{2}{3}} + 1}} + \sqrt{3} \tan^{-1} \frac{1 + 2\left( \frac{1-x}{1+x} \right)^{\frac{2}{3}}}{\sqrt{3}} + C \}$$

$$330. \int \frac{\sqrt{x} dx}{1+x^{\frac{2}{3}}}.$$

$$331. \int \frac{xdx}{\sqrt{2+4x}}. \{ \text{Ans. } \frac{\sqrt{2+4x}(x-1)}{6} + C \}$$

$$332. \int \frac{xdx}{\sqrt{1+2x}}. \{ \text{Ans. } x\sqrt{1+2x} - \frac{1}{3}\sqrt{(1+2x)^3} + C \}$$

$$333. \int x\sqrt{a+x} dx. \{ \text{Ans. } \frac{2}{15}(3x-2a)\sqrt{(a+x)^3} + C \}$$

$$334. \int \frac{\sqrt{x} dx}{\sqrt{2x+3}}. \{ \text{Ans. } \frac{1}{2}\sqrt{2x^2+3x} - \frac{3}{4\sqrt{2}}\ln\left(x + \frac{3}{4}\sqrt{x^2 + \frac{3x}{2}}\right) + C \}$$

$$335. \int \sqrt{\frac{a-x}{x-b}} dx. \{ \text{Ans. } \sqrt{(a-x)(x-b)} - (a-b)\tan^{-1}\sqrt{\frac{a-x}{x-b}} + C \}$$

$$336. \int \frac{\sqrt{x} dx}{\sqrt[4]{x^3+1}}. \{ \text{Ans. } \frac{4}{3}(\sqrt[4]{x^3} - \ln(\sqrt[4]{x^3+1})) + C \}$$

337.  $\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1} dx$ . {Ans.  $x+4\sqrt{x+1}+4\ln(\sqrt{x+1}-1)+C$ }
338.  $\int \frac{dx}{(2+x)\sqrt{1+x}}$ . {Ans.  $2\tan^{-1}\sqrt{1+x}+C$ }
339.  $\int \frac{\sqrt[3]{x}dx}{x(\sqrt{x}+\sqrt[3]{x})}$ . {Ans.  $\ln \frac{x}{(\sqrt[6]{x}+1)^6}+C$ }
340.  $\int \frac{dx}{(ax+b)\sqrt{x}}$ . {Ans.  $\frac{2}{\sqrt{ab}}\tan^{-1}\sqrt{\frac{ax}{b}}+C$ }
341.  $\int \frac{dx}{\sqrt{x}(x-1)}$ . {Ans.  $\ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| + C$ }
342.  $\int x\sqrt[3]{a+x}dx$ . {Ans.  $\frac{3(4x-3a)\sqrt[3]{(a+x)^4}}{28}+C$ }
343.  $\int \frac{x\sqrt{1+x}}{\sqrt{1-x}}dx$ . {Ans.  $\frac{1}{2}\sin^{-1}x - \frac{x+2}{2}\sqrt{1-x^2}+C$ }
344.  $\int \frac{dx}{x^3\sqrt{x-1}}$ . {Ans.  $\frac{\sqrt{x-1}(3x+2)}{4x^2}+C$ }
345.  $\int \sqrt{\frac{1-\sqrt[3]{x}}{1+\sqrt[3]{x}}} \frac{dx}{x}$ . {Ans.  $3\left(\ln|\sqrt[3]{x}| - \ln\left(1+\sqrt{1-\sqrt[3]{x^2}} - \sin^{-1}\sqrt[3]{x}\right)\right)+C$ }
346.  $\int \frac{dx}{x^3\sqrt{(1+x)^3}}$ . {Ans.  $\frac{15x^2+5x-2}{4x^2\sqrt{1+x}} + \frac{15}{8}\ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right|+C$ }
347.  $\int \frac{\sqrt{2x+1}}{x^2}dx$ . {Ans.  $\ln\left|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right| - \frac{\sqrt{2x+1}}{x}+C$ }

**CATEGORY-21.14. INTEGRATION OF IRRATIONAL FUNCTIONS OF TYPE  $\int \sqrt{ax^2+bx+cdx}$  AND**

$$\int (Ax+B)\sqrt{ax^2+bx+cdx}$$

348.  $\int \sqrt{x^2-2x-1}dx$ . {Ans.  $\frac{1}{2}(x-1)\sqrt{x^2-2x-1} - \ln|x-1+\sqrt{x^2-2x-1}|+C$ }
349.  $\int \sqrt{3x^2-3x+1}dx$ . {Ans.  $\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{3x^2-3x+1} + \frac{1}{8\sqrt{3}}\ln\left|\sqrt{3x^2-3x+1} + \frac{\sqrt{3}}{2}(2x-1)\right|+C$ }
350.  $\int \sqrt{1-4x-x^2}dx$ . {Ans.  $\frac{1}{2}\left((x+2)\sqrt{1-4x-x^2} + 5\sin^{-1}\frac{x+2}{\sqrt{5}}\right)+C$ }
351.  $\int (x+3)\sqrt{x^2+2x}dx$ .

**CATEGORY-21.15. INTEGRATION OF IRRATIONAL FUNCTIONS OF TYPE**  $\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}} dx$

352.  $\int \frac{dx}{\sqrt{1-(2x+3)^2}}$ . {Ans.  $\frac{1}{2} \sin^{-1}(2x+3) + C$ }
353.  $\int \frac{dx}{\sqrt{4x-3-x^2}}$ . {Ans.  $\sin^{-1}(x-2) + C$ }
354.  $\int \frac{dx}{\sqrt{8+6x-9x^2}}$ . {Ans.  $\frac{1}{3} \sin^{-1} \frac{3x-1}{3} + C$ }
355.  $\int \frac{dx}{\sqrt{2-6x-9x^2}}$ . {Ans.  $\frac{1}{3} \sin^{-1} \frac{3x+1}{\sqrt{3}} + C$ }
356.  $\int \frac{dx}{\sqrt{5-2x+x^2}}$ . {Ans.  $-\ln(1-x+\sqrt{5-2x+x^2}) + C$ }
357.  $\int \frac{dx}{\sqrt{9x^2-6x+2}}$ . {Ans.  $\frac{1}{3} \ln(3x-1+\sqrt{9x^2-6x+2}) + C$ }
358.  $\int \frac{dx}{\sqrt{12x-9x^2-2}}$ . {Ans.  $\frac{1}{3} \sin^{-1} \frac{3x-2}{\sqrt{2}} + C$ }
359.  $\int \frac{2x+3}{\sqrt{1+x^2}} dx$ . {Ans.  $2\sqrt{1+x^2} + 3\ln(x+\sqrt{1+x^2}) + C$ }
360.  $\int \frac{2x-1}{\sqrt{9x^2-4}} dx$ . {Ans.  $\frac{1}{9} (2\sqrt{9x^2-4} - 3\ln|3x+\sqrt{9x^2-4}|) + C$ }
361.  $\int \frac{(8x-11)dx}{\sqrt{5+2x-x^2}}$ . {Ans.  $-8\sqrt{5+2x-x^2} - 3\sin^{-1} \frac{x-1}{\sqrt{6}} + C$ }
362.  $\int \frac{(x-3)dx}{\sqrt{3-2x-x^2}}$ . {Ans.  $-\sqrt{3-2x-x^2} - 4\sin^{-1} \frac{x+1}{2} + C$ }
363.  $\int \frac{(3x-1)dx}{\sqrt{x^2+2x+2}}$ . {Ans.  $3\sqrt{x^2+2x+2} - 4\ln(x+1+\sqrt{x^2+2x+2}) + C$ }
364.  $\int \frac{2x+5}{\sqrt{9x^2+6x+2}} dx$ . {Ans.  $\frac{2}{9}\sqrt{9x^2+6x+2} + \frac{13}{9}\ln(3x+1+\sqrt{9x^2+6x+2}) + C$ }
365.  $\int \frac{(2-5x)dx}{\sqrt{4x^2+9x+1}}$ . {Ans.  $\frac{61}{16}\ln|8x+9+4\sqrt{4x^2+9x+1}| - \frac{5}{4}\sqrt{4x^2+9x+1} + C$ }
366.  $\int \frac{xdx}{\sqrt{3x^2-11x+2}}$ . {Ans.  $\frac{1}{3}\sqrt{3x^2-11x+2} + \frac{11}{6\sqrt{3}}\ln\left|x - \frac{11}{6} + \sqrt{x^2 - \frac{11}{3}x + \frac{2}{3}}\right| + C$ }
367.  $\int \frac{x^2 dx}{\sqrt{1-2x-x^2}}$ . {Ans.  $\frac{1}{2}(3-x)\sqrt{1-2x-x^2} + 2\sin^{-1} \frac{x+1}{\sqrt{2}} + C$ }
368.  $\int \frac{(2x^2-3x)dx}{\sqrt{x^2-2x+5}}$ . {Ans.  $x\sqrt{x^2-2x+5} - 5\ln(x-1+\sqrt{x^2-2x+5}) + C$ }

369.  $\int \frac{3x^2 - 5x}{\sqrt{3 - 2x - x^2}} dx$ . {Ans.  $-\frac{1}{2}(3x - 19)\sqrt{3 - 2x - x^2} + 14\sin^{-1} \frac{x+1}{2} + C$ }
370.  $\int \frac{3x^3 dx}{\sqrt{x^2 + 4x + 5}}$ . {Ans.  $(x^2 - 5x + 20)\sqrt{x^2 + 4x + 5} - 15\ln(x + 2 + \sqrt{x^2 + 4x + 5}) + C$ }
371.  $\int \frac{x^3 - x + 1}{\sqrt{x^2 + 2x + 2}} dx$ . {Ans.  $\left(\frac{1}{3}x^2 - \frac{5}{6}x + \frac{1}{6}\right)\sqrt{x^2 + 2x + 2} + \frac{5}{2}\ln(x + 1 + \sqrt{x^2 + 2x + 2}) + C$ }
372.  $\int \frac{3x^3 - 8x + 5}{\sqrt{x^2 - 4x - 7}} dx$ . {Ans.  $(x^2 + 5x + 36)\sqrt{x^2 - 4x - 7} + 112\ln|x - 2 + \sqrt{x^2 - 4x - 7}| + C$ }
373.  $\int \frac{3 + x^3}{\sqrt{2 + 2x^2}} dx$ . {Ans.  $\frac{1}{\sqrt{2}}\left(3\ln(x + \sqrt{1 + x^2}) + \frac{1}{3}(x^2 - 2)\sqrt{1 + x^2}\right) + C$ }
374.  $\int \frac{x^3 dx}{\sqrt{1 + 2x^2}}$ . {Ans.  $\frac{1}{6}(x^2 - 1)\sqrt{1 + 2x^2} + C$ }
375.  $\int \frac{x^4 dx}{\sqrt{x^2 + 1}}$ . {Ans.  $\left(\frac{x^3}{4} - \frac{3x}{8}\right)\sqrt{x^2 + 1} + \frac{3}{8}\ln(x + \sqrt{x^2 + 1}) + C$ }
376.  $\int \frac{x^4 dx}{\sqrt{x^2 + 4x + 5}}$ .

**CATEGORY-21.16. INTEGRATION OF IRRATIONAL FUNCTIONS OF TYPE**

- $\int \frac{dx}{(Ax + B)^n \sqrt{ax^2 + bx + c}}$  AND  $\int \frac{dx}{(ax^2 + b)\sqrt{cx^2 + d}}$
377.  $\int \frac{dx}{x\sqrt{x^2 + x + 1}}$ . {Ans.  $\ln \frac{|Cx|}{2 + x + 2\sqrt{x^2 + x + 1}}$ }
378.  $\int \frac{dx}{x\sqrt{x^2 + 4x - 4}}$ . {Ans.  $\frac{1}{2}\cos^{-1} \frac{2-x}{x\sqrt{2}} + C$ }
379.  $\int \frac{dx}{x\sqrt{x^2 + 2x - 1}}$ . {Ans.  $\sin^{-1} \frac{x-1}{x\sqrt{2}} + C$ }
380.  $\int \frac{dx}{x\sqrt{2 + x - x^2}}$ . {Ans.  $-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2 + x - x^2} + \sqrt{2}}{x}\right| + \frac{1}{2\sqrt{2}} + C$ }
381.  $\int \frac{dx}{(x-1)\sqrt{x^2 + x + 1}}$ . {Ans.  $-\frac{1}{\sqrt{3}}\ln\left|\frac{3 + 3x + 2\sqrt{3(x^2 + x + 1)}}{x-1}\right| + C$ }
382.  $\int \frac{dx}{(2x-3)\sqrt{4x - x^2}}$ . {Ans.  $-\frac{1}{\sqrt{15}}\ln\left|\frac{x+6 + \sqrt{60x - 15x^2}}{2x-3}\right| + C$ }
383.  $\int \frac{dx}{(x+1)^3 \sqrt{x^2 + 2x - 3}}$ .



$$384. \int \frac{dx}{x^4 \sqrt{x^2 - 3}}.$$

$$385. \int \frac{dx}{x^4 \sqrt{x^2 + 4}}. \{ \text{Ans. } \frac{\sqrt{4+x^2}(x^2-2)}{24x^3} + C \}$$

$$386. \int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}.$$

### CATEGORY-21.17. EULER SUBSTITUTIONS

$$387. \int \frac{dx}{1+\sqrt{x^2+2x+2}}. \{ \text{Ans. } \ln(x+1+\sqrt{x^2+2x+2}) + \frac{2}{x+2+\sqrt{x^2+2x+2}} + C \}$$

$$388. \int \frac{dx}{x+\sqrt{x^2-x+1}}. \{ \text{Ans. } 2\ln\left|\frac{\sqrt{x^2-x+1}+1}{x}\right| - \frac{1}{2}\ln\left|\frac{\sqrt{x^2-x+1}+1}{x}-1\right| + \frac{3}{\sqrt{x^2-x+1}+1} - \frac{3}{2}\ln\left|\frac{\sqrt{x^2-x+1}+1}{x}+1\right| + C \}$$

$$389. \int \frac{dx}{(1+x)\sqrt{1+x-x^2}}. \{ \text{Ans. } -2\tan^{-1}\left(\frac{\sqrt{1+x-x^2}+1+x}{x}\right) + C \}$$

$$390. \int \frac{xdx}{(\sqrt{7x-10-x^2})^3}. \{ \text{Ans. } -\frac{2}{9}\left(-\frac{5(x-2)}{\sqrt{7x-10-x^2}} + \frac{2\sqrt{7x-10-x^2}}{x-2}\right) + C \}$$

$$391. \int \frac{dx}{x-\sqrt{x^2+2x+4}}. \{ \text{Ans. } 2\ln|\sqrt{x^2+2x+4}-x| - \frac{3}{2(\sqrt{x^2+2x+4}-x-1)} - \frac{3}{2}\ln|\sqrt{x^2+2x+4}-x-1| + C \}$$

$$392. \int \frac{dx}{\sqrt{1-x^2}-1}. \{ \text{Ans. } \frac{1+\sqrt{1-x^2}}{x} + 2\tan^{-1}\sqrt{\frac{1+x}{1-x}} + C \}$$

$$393. \int \frac{dx}{\sqrt{(2x-x^2)^3}}. \{ \text{Ans. } \frac{x-1}{\sqrt{2x-x^2}} + C \}$$

$$394. \int \frac{(x+\sqrt{1+x^2})^{15} dx}{\sqrt{1+x^2}}. \{ \text{Ans. } \frac{(x+\sqrt{1+x^2})^{15}}{15} + C \}$$

### CATEGORY-21.18. APPLICATION OF METHODS OF INTEGRATION

$$395. \int \frac{\sqrt{1-x^3}}{x^2\sqrt{x}} dx. \{ \text{Ans. } -\frac{2}{3}\sqrt{\frac{1-x^3}{x^3}} - \frac{2}{3}\sin^{-1}\sqrt{x^3} + C \}$$

$$396. \int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}. \{ \text{Ans. } \frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}} + C \}$$

$$397. \int \sqrt{1+\sin x} dx. \{ \text{Ans. } \pm 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) + C \}$$

398.  $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$ . {Ans.  $2\sqrt{\tan x} + C$ }
399.  $\int \frac{\sqrt{\sin^3 2x}}{\sin^5 x} dx$ . {Ans.  $-\frac{4\sqrt{2}}{5} \sqrt{\cot^5 x} + C$ }
400.  $\int \frac{dx}{\sqrt[4]{\sin^3 x \cos^5 x}}$ . {Ans.  $4\sqrt[4]{\tan x} + C$ }
401.  $\int \frac{dx}{\sqrt{1 - \sin^4 x}}$ . {Ans.  $\frac{1}{\sqrt{2}} \ln(\sqrt{2} \tan x + \sqrt{1 + 2 \tan^2 x}) + C$ }
402.  $\int \sqrt{1 + \operatorname{cosec} x} dx$ . {Ans.  $2 \sin^{-1} \sqrt{\sin x} + C$ }
403.  $\int \frac{(\cos 2x - 3)dx}{\cos^4 x \sqrt{4 - \cot^2 x}}$ . {Ans.  $-\frac{1}{3} \tan x (2 + \tan^2 x) \sqrt{4 - \cot^2 x} + C$ }
404.  $\int \frac{dx}{\sin \frac{x}{2} \sqrt{\cos^3 \frac{x}{2}}}$ . {Ans.  $\frac{4}{\sqrt{\cot \frac{x}{2}}} + 2 \tan^{-1} \sqrt{\cos \frac{x}{2}} - \ln \frac{1 + \sqrt{\cos \frac{x}{2}}}{1 - \sqrt{\cos \frac{x}{2}}} + C$ }
405.  $\int \sqrt{\tan x} dx$ . {Ans.  $\frac{1}{\sqrt{2}} (\ln(\sin x + \cos x - \sqrt{\sin 2x}) + \sin^{-1}(\sin x - \cos x)) + C$ }
406.  $\int \frac{\sin^3 \alpha}{\sqrt{\cos \alpha}} d\alpha$ . {Ans.  $2\sqrt{\cos \alpha} \left( \frac{\cos^2 \alpha}{5} - 1 \right) + C$ }
407.  $\int \frac{x + \sin x}{1 + \cos x} dx$ . {Ans.  $x \tan \frac{x}{2} + C$ }
408.  $\int \frac{\sqrt{1 + x^2}}{2 + x^2} dx$ . {Ans.  $\frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2 + 2x^2} - x}{\sqrt{2 + 2x^2} + x} + \ln(x + \sqrt{x^2 + 1}) + C$ }
409.  $\int \frac{(x-1)dx}{x^2 \sqrt{2x^2 - 2x + 1}}$ . {Ans.  $\frac{\sqrt{2x^2 - 2x + 1}}{x} + C$ }
410.  $\int \frac{(2x+3)dx}{(x^2 + 2x + 3)\sqrt{x^2 + 2x + 4}}$ . {Ans.  $\ln \frac{\sqrt{x^2 + 2x + 4} - 1}{\sqrt{x^2 + 2x + 4} + 1} - \frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{2(x^2 + 2x + 4)}}{x + 1} + C$ }
411.  $\int \frac{dx}{x - \sqrt{x^2 - x + 1}}$ . {Ans.  $-\frac{3}{2(2x - 1 - 2\sqrt{x^2 - x + 1})} - \frac{3}{2} \ln |2x - 1 - 2\sqrt{x^2 - x + 1}| + 2 \ln |x - \sqrt{x^2 - x + 1}| + C$ }
412.  $\int \frac{dx}{x^2(x + \sqrt{1 + x^2})}$ . {Ans.  $\ln \left| \frac{x + \sqrt{x^2 + 1}}{x} \right| - \frac{\sqrt{1 + x^2}}{x} + C$ }
413.  $\int \frac{\sqrt{2x + x^2}}{x^2} dx$ . {Ans.  $\ln |x + 1 + \sqrt{x^2 + 2x}| - 2\sqrt{\frac{x+2}{x}} + C$ }
414.  $\int (1 + e^{3x})^2 e^{3x} dx$ . {Ans.  $\frac{1}{9} (1 + e^{3x})^3 + C$ }

415.  $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ . {Ans.  $2e^{\sqrt{x}} + C$ }
416.  $\int \frac{\sin x}{e^{\cos x}} dx$ . {Ans.  $e^{-\cos x} + C$ }
417.  $\int \sqrt{1-e^x} e^x dx$ . {Ans.  $-\frac{2}{3}(1-e^x)^{\frac{3}{2}} + C$ }
418.  $\int \left(2-3x^{\frac{4}{3}}\right)^{\frac{1}{5}} x^{\frac{1}{3}} dx$ . {Ans.  $-\frac{5}{24}\left(2-3x^{\frac{4}{3}}\right)^{\frac{6}{5}} + C$ }
419.  $\int \frac{dx}{e^x(3+e^{-x})}$ . {Ans.  $-\ln(3+e^{-x}) + C$ }
420.  $\int \frac{dx}{e^x \sqrt{1-e^{-2x}}}$ . {Ans.  $-\sin^{-1} e^{-x} + C$ }
421.  $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ . {Ans.  $2 \sin \sqrt{x} + C$ }
422.  $\int \frac{dx}{x\sqrt{3-\ln^2 x}}$ . {Ans.  $\sin^{-1} \frac{\ln x}{\sqrt{3}} + C$ }
423.  $\int \frac{\ln x dx}{x(1-\ln^2 x)}$ . {Ans.  $-\frac{1}{2} \ln|1-\ln^2 x| + C$ }
424.  $\int \frac{x^2-x+1}{\sqrt{(x^2+1)^3}} dx$ . {Ans.  $\frac{1}{\sqrt{x^2+1}} + \ln(x+\sqrt{x^2+1}) + C$ }
425.  $\int \frac{(\tan^{-1} x)^n}{1+x^2} dx$ . {Ans.  $\frac{(\tan^{-1} x)^{n+1}}{n+1} + C, n \neq -1; \ln|\tan^{-1} x| + C, n = -1$ }
426.  $\int \sqrt{\tan^3 x} \sec^4 x dx$ . {Ans.  $\frac{2}{45} \sqrt{\tan^5 x} (5 \tan^2 x + 9) + C$ }
427.  $\int (\sqrt{\sin x} + \cos x)^2 dx$ . {Ans.  $\frac{x}{2} + \frac{\sin 2x}{4} + \frac{4}{3} \sqrt{\sin^3 x} - \cos x + C$ }
428.  $\int a^{mx} b^{nx} dx$ . {Ans.  $\frac{a^{mx} b^{nx}}{m \ln a + n \ln b} + C$ }
429.  $\int x \sin x \cos x dx$ . {Ans.  $\frac{1}{8} \sin 2x - \frac{x}{4} \cos 2x + C$ }
430.  $\int x^2 \cos ax dx$ . {Ans.  $\frac{1}{\omega^2} ((\omega^2 x^2 - 2) \sin ax + 2ax \cos ax) + C$ }
431.  $\int e^{2x} x^3 dx$ . {Ans.  $e^{2x} \left( \frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) + C$ }
432.  $\int \frac{\ln(\cos x)}{\cos^2 x} dx$ . {Ans.  $\tan x \ln(\cos x) + \tan x - x + C$ }

433.  $\int \frac{\cot x}{\ln(\sin x)} dx$ . {Ans.  $\ln|\ln(\sin x)| + C$ }
434.  $\int \frac{dx}{e^x + 1}$ . {Ans.  $\ln \frac{e^x}{1+e^x} + C$ }
435.  $\int \frac{e^x - 1}{e^x + 1} dx$ . {Ans.  $2 \ln \left( e^{\frac{x}{2}} + e^{-\frac{x}{2}} \right) + C$ }
436.  $\int e^{e^x + x} dx$ . {Ans.  $e^{e^x} + C$ }
437.  $\int e^{2x^2 + \ln x} dx$ . {Ans.  $\frac{1}{4} e^{2x^2} + C$ }
438.  $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ . {Ans.  $x - \sqrt{1-x^2} \sin^{-1} x + C$ }
439.  $\int \frac{x \cos x}{\sin^3 x} dx$ . {Ans.  $-\frac{1}{2} \left( \frac{x}{\sin^2 x} + \cot x \right) + C$ }
440.  $\int e^x \sin^2 x dx$ . {Ans.  $\frac{e^x}{2} \left( 1 - \frac{2 \sin 2x + \cos 2x}{5} \right) + C$ }
441.  $\int \frac{\sqrt{1+\cos x}}{\sin x} dx$ . {Ans.  $\sqrt{2} \ln \left| \tan \frac{x}{4} \right| + C$ }
442.  $\int \frac{\ln(\ln x)}{x} dx$ . {Ans.  $\ln x \cdot \ln(\ln x) - \ln x + C$ }
443.  $\int x^3 e^{x^2} dx$ . {Ans.  $\frac{e^{x^2}(x^2-1)}{2} + C$ }
444.  $\int e^{-x^2} x^5 dx$ . {Ans.  $-\frac{1}{2} e^{-x^2} (x^4 + 2x^2 + 2) + C$ }
445.  $\int \frac{x^4 dx}{\sqrt{(1-x^2)^3}}$ . {Ans.  $-\frac{x(x^2-3)}{2\sqrt{1-x^2}} - \frac{3}{2} \sin^{-1} x + C$ }
446.  $\int \frac{\sqrt{(x^2-a^2)^5}}{x} dx$ . {Ans.  $\frac{1}{5} \sqrt{(x^2-a^2)^5} - \frac{a^2}{3} \sqrt{(x^2-a^2)^3} + a^4 \sqrt{x^2-a^2} + a^5 \sin^{-1} \frac{a}{|x|} + C$ }
447.  $\int \frac{\sqrt{x^2-8}}{x^4} dx$ . {Ans.  $\frac{\sqrt{(x^2-8)^3}}{24x^3} + C$ }
448.  $\int \frac{\sqrt{4+x^2}}{x^6} dx$ . {Ans.  $\frac{\sqrt{(4+x^2)^3}(x^2-6)}{120x^5} + C$ }
449.  $\int \frac{\sqrt{x^2+2x}}{x} dx$ . {Ans.  $\sqrt{x^2+2x} + \ln|x+1+\sqrt{x^2+2x}| + C$ }
450.  $\int \frac{\sqrt{1+x^8}}{x^{13}} dx$ . {Ans.  $-\frac{(1+x^8)^{\frac{3}{2}}}{12x^{12}} + C$ }

451.  $\int \frac{x dx}{(1-x^4)^{\frac{2}{3}}}$ . {Ans.  $\frac{x^2}{2\sqrt{1-x^4}} + C$ }
452.  $\int \frac{3x^2-1}{2x\sqrt{x}} \tan^{-1} x dx$ . {Ans.  $\frac{(x^2+1)\tan^{-1} x}{\sqrt{x}} - 2\sqrt{x} + C$ }
453.  $\int \frac{e^x(1+e^x)dx}{\sqrt{1-e^{2x}}}$ . {Ans.  $\sin^{-1} e^x - \sqrt{1-e^{2x}} + C$ }
454.  $\int \sqrt{e^x-1} dx$ . {Ans.  $2\sqrt{e^x-1} - 2\tan^{-1} \sqrt{e^x-1} + C$ }
455.  $\int \frac{\ln(x+1) - \ln x}{x(x+1)} dx$ . {Ans.  $-\frac{1}{2} \ln^2 \left(1 + \frac{1}{x}\right) + C$ }
456.  $\int \cos^{-1} \sqrt{\frac{x}{x+1}} dx$ . {Ans.  $x \cos^{-1} \sqrt{\frac{x}{x+1}} + \sqrt{x} - \tan^{-1} \sqrt{x} + C$ }
457.  $\int \ln(x + \sqrt{1+x^2}) dx$ . {Ans.  $x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C$ }
458.  $\int \sqrt[3]{\frac{\sin^2 x}{\cos^{14} x}} dx$ . {Ans.  $\frac{3}{55} \sqrt[3]{\tan^5 x} (5 \tan^2 x + 11) + C$ }
459.  $\int \frac{dx}{\cos^3 x \sqrt{\sin 2x}}$ . {Ans.  $\frac{\sqrt{2}}{5} (\tan^2 x + 5) \sqrt{\tan x} + C$ }
460.  $\int \frac{x dx}{x - \sqrt{x^2-1}}$ . {Ans.  $\frac{1}{3} \left( x^3 + \sqrt{(x^2-1)^3} \right) + C$ }
461.  $\int \frac{dx}{ae^{mx} + be^{-mx}}$ . {Ans.  $\frac{1}{m\sqrt{ab}} \tan^{-1} \left( e^{mx} \sqrt{\frac{a}{b}} \right) + C$ }
462.  $\int \frac{\ln(x+1)dx}{\sqrt{x+1}}$ . {Ans.  $2\sqrt{x+1}(\ln|x+1| - 2) + C$ }
463.  $\int (x^2 + 3x + 5) \cos 2x dx$ . {Ans.  $\left(\frac{x}{2} + \frac{3}{4}\right) \cos 2x + \left(\frac{x^2}{2} + \frac{3x}{2} + \frac{9}{4}\right) \sin 2x + C$ }
464.  $\int \tan^{-1}(1 + \sqrt{x}) dx$ . {Ans.  $x \tan^{-1}(1 + \sqrt{x}) - \sqrt{x} + \ln|x + 2\sqrt{x} + 2| + C$ }
465.  $\int \frac{\sin^{-1} x dx}{x^2}$ . {Ans.  $\ln \left| \frac{1 - \sqrt{1-x^2}}{x} \right| - \frac{\sin^{-1} x}{x} + C$ }
466.  $\int e^{\sqrt[3]{x}} dx$ . {Ans.  $3e^{\sqrt[3]{x}} \left( \sqrt[3]{x^2} - 2\sqrt[3]{x} + 2 \right) + C$ }
467.  $\int xe^{\sqrt[3]{x}} dx$ . {Ans.  $3e^{\sqrt[3]{x}} \left( \sqrt[3]{x^5} - 5\sqrt[3]{x^4} + 20x - 60\sqrt[3]{x^2} + 120\sqrt[3]{x} - 120 \right) + C$ }
468.  $\int (x^3 - 2x^2 + 5) e^{3x} dx$ . {Ans.  $e^{3x} \left( \frac{x^3}{3} - x^2 + \frac{2}{3}x + \frac{13}{9} \right) + C$ }
469.  $\int \sin \sqrt{x} dx$ . {Ans.  $2(\sin \sqrt{x} - \sqrt{x} \cos \sqrt{x}) + C$ }

470.  $\int \frac{dx}{x - \sqrt{x^2 - 1}}$ . {Ans.  $\frac{x^2}{2} + \frac{x}{2}\sqrt{x^2 - 1} - \frac{1}{2}\ln|x + \sqrt{x^2 - 1}| + C$ }
471.  $\int \frac{\sqrt{(1+x^2)^5}}{x^6} dx$ . {Ans.  $\ln(x + \sqrt{1+x^2}) - \frac{\sqrt{(1+x^2)^5}}{5x^5} - \frac{\sqrt{(1+x^2)^3}}{3x^3} - \frac{\sqrt{1+x^2}}{x} + C$ }
472.  $\int x \ln(1+x^3) dx$ . {Ans.  $\frac{x^2 \ln(1+x^3)}{2} - \frac{3}{4}x^2 + \frac{1}{4}\ln(x^2 - x + 1) - \frac{1}{2}\ln(x+1) + \frac{\sqrt{3}}{2}\tan^{-1}\frac{2x-1}{\sqrt{3}} + C$ }
473.  $\int \frac{(\ln x - 1)dx}{\ln^2 x}$ . {Ans.  $\frac{x}{\ln x} + C$ }
474.  $\int \frac{x \ln x}{\sqrt{(x^2 - 1)^3}} dx$ . {Ans.  $\tan^{-1}\sqrt{x^2 - 1} - \frac{\ln x}{\sqrt{x^2 - 1}} + C$ }
475.  $\int x^2 e^x \cos x dx$ . {Ans.  $\frac{e^{\frac{x}{2}}}{2}((x^2 - 1)\cos x + (x - 1)^2 \sin x) + C$ }
476.  $\int x e^{x^2}(x^2 + 1) dx$ . {Ans.  $\frac{x^2 e^{x^2}}{2} + C$ }
477.  $\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}}$ . {Ans.  $\frac{2}{3}\left(\frac{\tan^2 x - 3}{\sqrt{\tan x}}\right) + C$ }
478.  $\int \sqrt{\tan^2 x + 2} dx$ . {Ans.  $\tan^{-1}\frac{\tan x}{\sqrt{2 + \tan^2 x}} + \ln(\sqrt{2 + \tan^2 x} + \tan x) + C$ }
479.  $\int \frac{(x^2 - 1)dx}{x\sqrt{x^4 + 3x^2 + 1}}$ . {Ans.  $\ln \frac{x^2 + 1 + \sqrt{x^4 + 3x^2 + 1}}{x} + C$ }
480.  $\int \frac{x e^x dx}{(1+x)^2}$ . {Ans.  $\frac{e^x}{1+x} + C$ }
481.  $\int \frac{x e^x dx}{\sqrt{1+e^x}}$ . {Ans.  $2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - 2\ln \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} + C$ }
482.  $\int \frac{\tan^{-1} x dx}{x^4}$ . {Ans.  $\frac{1}{6}\ln \frac{1+x^2}{x^2} - \frac{\tan^{-1} x}{4} + \frac{x}{4(1+x^2)} + C$ }
483.  $\int \frac{x \tan^{-1} x}{(1+x^2)^2} dx$ . {Ans.  $-\frac{\tan^{-1} x}{2(1+x^2)} + \frac{\tan^{-1} x}{4} + \frac{x}{4(1+x^2)} + C$ }
484.  $\int \frac{\tan^{-1} x}{(1+x)^3} dx$ . {Ans.  $\frac{1}{4}\ln \frac{|x+1|}{\sqrt{x^2+1}} - \frac{\tan^{-1} x}{2(x+1)^2} - \frac{1}{4(x+1)} + C$ }
485.  $\int \frac{dx}{(1-2^x)^4}$ . {Ans.  $x - \log_2|1-2^x| + \frac{1}{\ln 2}\left(\frac{1}{1-2^x} + \frac{1}{2(1-2^x)^2} + \frac{1}{3(1-2^x)^3}\right) + C$ }
486.  $\int \frac{(e^{3x} + e^x)dx}{e^{4x} - e^{2x} + 1}$ . {Ans.  $\tan^{-1}(e^x - e^{-x}) + C$ }

487.  $\int \frac{dx}{\sqrt{1+e^x+e^{2x}}} \cdot \{ \text{Ans. } \ln \frac{1+e^x-\sqrt{1+e^x+e^{2x}}}{1-e^x+\sqrt{1+e^x+e^{2x}}} + C \}$
488.  $\int \frac{x^2-1}{x^2+1} \frac{dx}{\sqrt{1+x^4}} \cdot \{ \text{Ans. } \frac{1}{\sqrt{2}} \cos^{-1} \frac{x\sqrt{2}}{x^2+1} + C \}$
489.  $\int e^{\sin x} \frac{x \cos^3 x - \sin x}{\cos^2 x} dx \cdot \{ \text{Ans. } e^{\sin x} (x - \sec x) + C \}$
490.  $\int \sqrt{\tanh x} dx \cdot \{ \text{Ans. } \frac{1}{2} \ln \frac{1+\sqrt{\tanh x}}{1-\sqrt{\tanh x}} - \tan^{-1} \sqrt{\tanh x} + C \}$

### CATEGORY-21.19. ADDITIONAL QUESTIONS

491. If  $2f(x+y) = f(x)f(y) - f(x) - f(y) + 3 \forall x, y$  and  $f'(0) = 6$ , then find the function  $f(x)$ . {Ans.  $f(x) = 2e^{3x} + 1$ }
492. Let  $f$  and  $g$  be functions satisfying the following conditions:-
- $f(0) = 1$
  - $f'(x) = g(x)$
  - $g'(x) = f(x)$
  - $g(0) = 0$ .

Determine the function  $f(x)$ . {Ans.  $f(x) = \cosh x$ }

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# Mathematics for IIT-JEE

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*PART-VI*  
*INTEGRAL CALCULUS*

*CHAPTER-22*  
*DEFINITE INTEGRAL*

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## ***CHAPTER-22***

### ***DEFINITE INTEGRALS***

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#### **LIST OF THEORY SECTIONS**

- 22.1. Definition And Properties Of Definite Integrals
- 22.2. Theorems In Definite Integrals
- 22.3. Improper Integrals
- 22.4. Definite Integrals Containing Parameters
- 22.5. Estimating A Definite Integral
- 22.6. Definite Integral As Limit Of Sum Of Series
- 22.7. Average (Mean) Value Of A Function In An Interval

#### **LIST OF QUESTION CATEGORIES**

- 22.1. Direct Application Of Newton-Leibniz Formula
- 22.2. Substitution In Definite Integrals
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- 22.5. Definite Integral Of Piecewise Functions
- 22.6. Definite Integral Of Symmetric Functions
- 22.7. Definite Integral Of Even/ Odd Functions
- 22.8. Definite Integral Of Periodic Functions
- 22.9. Calculating Improper Integrals By First Principles
- 22.10. Reduction Formula In Improper Integrals
- 22.11. Calculating Improper Integrals By Properties
- 22.12. Definite Integral Containing Parameter
- 22.13. Definite Integral Of Piecewise Functions Containing Parameter
- 22.14. Estimating A Definite Integral
- 22.15. Definite Integral As Limit Of Sum Of Series
- 22.16. Average (Mean) Value Of A Function In An Interval
- 22.17. Additional Questions

## CHAPTER-22

### DEFINITE INTEGRALS

#### SECTION-22.1. DEFINITION AND PROPERTIES OF DEFINITE INTEGRALS

##### 1. Definite integral as area under the curve of a function

Numerical area under the curve of function  $f(x)$  between lines  $x = a$  and  $x = b$  is denoted by  $\int_a^b f(x)dx$ . By

$$\text{definition } \int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f\left(a + i\left(\frac{b-a}{n}\right)\right) \cdot \left(\frac{b-a}{n}\right).$$

##### 2. Proper and Improper integrals, Newton-Leibniz formula

If  $f(x)$  is continuous in  $[a, b]$  then  $\int_a^b f(x)dx$  is said to be a proper integral and

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \text{ (Newton-Leibniz formula), where } F(x) = \int f(x)dx \text{ and}$$

$F'(x) \equiv f(x) \forall x \in [a, b]$ ; otherwise  $\int_a^b f(x)dx$  is said to be improper integral and Newton-Leibniz formula is

not applicable and  $\int_a^b f(x)dx \neq F(b) - F(a)$ .

##### 3. Fundamental properties of definite integrals

$$\text{i. } \int_a^a f(x)dx = 0$$

$$\text{ii. } \int_a^b kf(x)dx = k \int_a^b f(x)dx$$

$$\text{iii. } \int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\text{iv. } \int_a^b f(x)dx = - \int_b^a f(x)dx$$

$$\text{v. } \int_a^b f(x)dx = \int_a^b f(t)dt$$

$$\text{vi. } \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

#### SECTION-22.2. THEOREMS IN DEFINITE INTEGRALS

##### 1. Substitution in definite integral

$$\int_a^b f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))\phi'(t)dt$$

where

- $x = \phi(t)$ ;  $dx = \phi'(t)dt$
- $\phi(\alpha) = a$  and  $\phi(\beta) = b$
- $\phi(t)$  is a one-one function in  $[\alpha, \beta]$
- $\phi(t)$  is differentiable in  $[\alpha, \beta]$ , otherwise  $\int_{\alpha}^{\beta} f(\phi(t))\phi'(t)dt$  will be an improper integral and Newton-Leibniz formula will not be applicable.

## 2. Integration by parts in definite integral

- If  $f(x)$  and  $g(x)$  are differentiable in  $[a, b]$  then  $\int_{x=a}^{x=b} f(x)dg(x) = [f(x)g(x)]_a^b - \int_{x=a}^{x=b} g(x)df(x)$ .
- Recurrance of an integral
- Reduction formula

## 3. Definite integral of symmetric functions

- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_0^a f(x)dx = 2 \int_0^{\frac{a}{2}} f(x)dx$ , if  $f(a-x) = f(x)$   
 $= 0$ , if  $f(a-x) = -f(x)$

## 4. Definite integral of even/odd functions

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx \text{ if } f(x) \text{ is an even function in } [-a, a]$$

$$= 0 \text{ if } f(x) \text{ is an odd function in } [-a, a]$$

## 5. Definite integral of periodic functions

If  $f(x)$  is a periodic function with period  $T$  then

- $\int_a^b f(x)dx = \int_{a+nT}^{b+nT} f(x)dx$ , where  $n$  is an integer
- $\int_0^{nT} f(x)dx = n \int_0^T f(x)dx$ , where  $n$  is an integer

### SECTION-22.3. IMPROPER INTEGRALS

#### 1. Types of Improper integrals

- If  $f(x)$  is continuous in  $[a, \infty)$  then  $\int_a^{\infty} f(x)dx = \lim_{\alpha \rightarrow \infty} \left[ \int_a^{\alpha} f(x)dx \right] = \lim_{\alpha \rightarrow \infty} [F(x)]_a^{\alpha} = \lim_{\alpha \rightarrow \infty} [F(\alpha) - F(a)]$ . If this

limit exists, then the improper integral is said to be convergent and the value of this limit is the value of the improper integral; and if this limit does not exist, then the improper integral is said to be divergent.

- ii. If  $f(x)$  is continuous in  $(-\infty, b]$  then  $\int_{-\infty}^b f(x)dx = \lim_{\alpha \rightarrow -\infty} \left[ \int_{\alpha}^b f(x)dx \right] = \lim_{\alpha \rightarrow -\infty} [F(x)]_{\alpha}^b = \lim_{\alpha \rightarrow -\infty} [F(b) - F(\alpha)]$ . If

this limit exists, then the improper integral is said to be convergent and the value of this limit is the value of the improper integral; and if this limit does not exist, then the improper integral is said to be divergent.

- iii. If  $f(x)$  is continuous in  $[a, b)$  and discontinuous at  $x = b$  then

$$\int_a^b f(x)dx = \lim_{\alpha \rightarrow b^-} \left[ \int_a^{\alpha} f(x)dx \right] = \lim_{\alpha \rightarrow b^-} [F(x)]_a^{\alpha} = \lim_{\alpha \rightarrow b^-} [F(\alpha) - F(a)].$$

If this limit exists, then the improper integral is said to be convergent and the value of this limit is the value of the improper integral; and if this limit does not exist, then the improper integral is said to be divergent.

- iv. If  $f(x)$  is continuous in  $(a, b]$  and discontinuous at  $x = a$  then

$$\int_a^b f(x)dx = \lim_{\alpha \rightarrow a^+} \left[ \int_{\alpha}^b f(x)dx \right] = \lim_{\alpha \rightarrow a^+} [F(x)]_{\alpha}^b = \lim_{\alpha \rightarrow a^+} [F(b) - F(\alpha)].$$

If this limit exists, then the improper integral is said to be convergent and the value of this limit is the value of the improper integral; and if this limit does not exist, then the improper integral is said to be divergent.

## 2. Substitution and Integration by parts in Improper integrals

## 3. Reduction formula in Improper integrals

## 4. Application of properties in Improper Integrals

### SECTION-22.4. DEFINITE INTEGRALS CONTAINING PARAMETERS

- $\frac{d}{dx} \int_a^x f(t)dt = f(x)$
- $\frac{d}{dx} \int_{\phi(x)}^{\phi(x)} f(t)dt = f(\phi(x)) \frac{d}{dx} \phi(x) - f(\phi(x)) \frac{d}{dx} \phi(x)$
- $\frac{d}{dx} \int_a^b f(x, t)dt = \int_a^b \frac{\partial}{\partial x} f(x, t)dt$
- $\frac{d}{dx} \int_{\phi(x)}^{\phi(x)} f(x, t)dt = \int_{\phi(x)}^{\phi(x)} \frac{\partial}{\partial x} f(x, t)dt + f(x, \phi(x)) \frac{d}{dx} \phi(x) - f(x, \phi(x)) \frac{d}{dx} \phi(x)$

### SECTION-22.5. ESTIMATING A DEFINITE INTEGRAL

- If  $f(x) \leq g(x)$  in  $[a, b]$  then  $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ .
- If  $L$  and  $G$  are the least and greatest values of a function  $f(x)$  defined on an interval  $[a, b]$  then  $L(b-a) \leq \int_a^b f(x)dx \leq G(b-a)$ .
- $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$

$$4. \left| \int_a^b f(x)g(x)dx \right| \leq \sqrt{\int_a^b f^2(x)dx} \sqrt{\int_a^b g^2(x)dx}$$

**SECTION-22.6. DEFINITE INTEGRAL AS LIMIT OF SUM OF SERIES**

$$1. \int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f\left(a + i\left(\frac{b-a}{n}\right)\right) \cdot \left(\frac{b-a}{n}\right)$$

**SECTION-22.7. AVERAGE (MEAN) VALUE OF A FUNCTION IN AN INTERVAL**

$$1. \text{ Average value of a function } f(x) \text{ in the interval } [a, b] = \frac{1}{(b-a)} \int_a^b f(x)dx.$$

## EXERCISE-22

### CATEGORY-22.1. DIRECT APPLICATION OF NEWTON-LEIBNIZ FORMULA

1.  $\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx$ . {Ans.  $\frac{\pi}{12}$ }
2.  $\int_0^{36} \frac{1}{2x+9} dx$ . {Ans.  $\ln 3$ }
3.  $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin 2x} dx$ . {Ans.  $\ln \sqrt{3}$ }
4.  $\int_0^1 \sqrt{1+x} dx$  {Ans.  $\frac{2}{3}(\sqrt{8}-1)$ }
5.  $\int_{-2}^{-1} \frac{dx}{(11+5x)^3}$ . {Ans.  $\frac{7}{72}$ }
6.  $\int_3^5 \frac{x^2}{x^2-4} dx$ . {Ans.  $2 + \ln\left(\frac{15}{7}\right)$ }
7.  $\int_2^{-13} \frac{dx}{\sqrt[5]{(3-x)^4}}$ . {Ans.  $-5(\sqrt[5]{16}-1)$ }
8.  $\int_4^9 \frac{y-1}{\sqrt{y}+1} dy$ . {Ans.  $\frac{23}{3}$ }
9.  $\int_0^{\frac{T}{2}} \sin\left(\frac{2\pi t}{T} - \phi_0\right) dt$ . {Ans.  $\frac{T}{\pi} \cos \phi_0$ }
10.  $\int_0^{16} \frac{dx}{\sqrt{x+9} - \sqrt{x}}$ . {Ans. 12}
11.  $\int_0^1 (e^x - 1)^4 e^x dx$  {Ans.  $\frac{(e-1)^5}{5}$ }
12.  $\int_0^{2a} \frac{3dx}{2b-x}$  ( $b > a > 0$ ). {Ans.  $3 \ln \frac{b}{b-a}$ }
13.  $\int_0^1 \frac{xdx}{(x^2+1)^2}$ . {Ans.  $\frac{1}{4}$ }
14.  $\int_1^{\sqrt[7]{2}} \frac{1}{x(2x^7+1)} dx$ . {Ans.  $\left(\frac{1}{7}\right) \ln\left(\frac{6}{5}\right)$ }

$$15. \int_1^e \frac{dx}{x\sqrt{1-(\ln x)^2}}. \{ \text{Ans. } \frac{\pi}{2} \}$$

$$16. \int_1^e \frac{1 + \log_{10} x}{x} dx. \{ \text{Ans. } 1 + \frac{1}{2} \log e \}$$

$$17. \int_1^2 \frac{e^{\frac{1}{x}} dx}{x^2}. \{ \text{Ans. } e - \sqrt{e} \}$$

$$18. \int_0^{\sqrt[n]{\frac{a}{2}}} \frac{x^{n-1} dx}{\sqrt{a^2 - x^{2n}}}. \{ \text{Ans. } \frac{\pi}{6n} \}$$

$$19. \int_1^{e^3} \frac{dx}{x\sqrt{1+\ln x}}. \{ \text{Ans. } 2 \}$$

$$20. \int_0^{\frac{\pi}{2}} \frac{1}{2 + \cos x} dx. \{ \text{Ans. } \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{1}{\sqrt{3}} \right) \}$$

$$21. \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^3 dx}{\left( \frac{5}{8} - x^4 \right)^{\frac{3}{2}}}. \{ \text{Ans. } \frac{4}{3} \}$$

$$22. \int_0^{\frac{a}{2}} \frac{a dx}{(x-a)(x-2a)}. \{ \text{Ans. } \ln \frac{3}{2} \}$$

$$23. \int_2^3 \frac{dx}{2x^2 + 3x - 2}. \{ \text{Ans. } \frac{1}{5} \ln \frac{4}{3} \}$$

$$24. \int_0^1 \frac{dx}{x^2 + 4x + 5}. \{ \text{Ans. } \tan^{-1} \frac{1}{7} \}$$

$$25. \int_1^2 \frac{dx}{x + x^3}. \{ \text{Ans. } \frac{1}{2} \ln \frac{8}{5} \}$$

$$26. \int_{-0.5}^1 \frac{dx}{\sqrt{8 + 2x - x^2}}. \{ \text{Ans. } \frac{\pi}{6} \}$$

$$27. \int_0^{\frac{\pi}{2}} \cos^5 x \cdot \sin 2x dx. \{ \text{Ans. } \frac{2}{7} \}$$

$$28. \int_0^{\frac{\pi}{\omega}} \sin^2(\omega x + \varphi_0) dx. \{ \text{Ans. } \frac{\pi}{2\omega} \}$$



29.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos^3 x \, dx}{\sqrt[3]{\sin x}}$ . {Ans.  $-\frac{1}{12}$ }
30.  $\int_{\alpha}^{\frac{\pi}{4}} \cot^4 \phi \, d\phi$ . {Ans.  $\frac{2}{3} + \frac{\pi}{4} - \alpha + \frac{\cot^3 \alpha}{3} - \cot \alpha$ }
31.  $\int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{\sin \frac{1}{x}}{x^2} \, dx$ . {Ans. 1}
32.  $\int_0^{\frac{\pi}{4}} \cos^7 2x \, dx$ . {Ans.  $\frac{8}{35}$ }
33.  $\int_0^1 \frac{1}{x^2 + 2x \cos \alpha + 1} \, dx$ . {Ans.  $\frac{\alpha}{2 \sin \alpha}$ }

### CATEGORY-22.2. SUBSTITUTION IN DEFINITE INTEGRALS

34.  $\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1} \, dx$ . {Ans.  $7 + 2 \ln 2$ }
35.  $\int_0^{\frac{\pi^2}{4}} \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx$ . {Ans. 2}
36.  $\int_0^1 \frac{x^3}{1+x^8} \, dx$ . {Ans.  $\frac{\pi}{16}$ }
37.  $\int_0^3 x\sqrt{1+x} \, dx$ . {Ans.  $\frac{9}{4}$ }
38.  $\int_0^1 \sqrt{x(1-x)} \, dx$ . {Ans.  $\frac{\pi}{8}$ }
39.  $\int_0^1 \frac{\sqrt{x} \, dx}{1+x}$ . {Ans.  $2 - \frac{\pi}{2}$ }
40.  $\int_3^8 \frac{x \, dx}{\sqrt{1+x}}$ . {Ans.  $\frac{32}{3}$ }
41.  $\int_0^1 \frac{x \, dx}{1+\sqrt{x}}$ . {Ans.  $\frac{5}{3} - 2 \ln 2$ }
42.  $\int_0^1 \frac{1}{(x^2+1)^{\frac{3}{2}}} \, dx$ . {Ans.  $\frac{1}{\sqrt{2}}$ }

43.  $\int_0^1 \frac{\sqrt{e^x} dx}{\sqrt{e^x + e^{-x}}} \cdot \{ \text{Ans. } \ln \frac{e+\sqrt{1+e^2}}{1+\sqrt{2}} \}$
44.  $\int_3^{29} \frac{\sqrt[3]{(x-2)^2} dx}{3+\sqrt[3]{(x-2)^2}} \cdot \{ \text{Ans. } 8 + \frac{3\sqrt{3}\pi}{2} \}$
45.  $\int_0^1 \frac{x^2 dx}{(1+x^2)^3} \cdot \{ \text{Ans. } \frac{\pi}{32} \}$
46.  $\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2} dx \cdot \{ \text{Ans. } \sqrt{2} - \frac{2}{\sqrt{3}} + \ln \frac{2+\sqrt{3}}{1+\sqrt{2}} \}$
47.  $\int_0^{\sqrt{7}} \frac{x^3 dx}{\sqrt[3]{a^2+x^2}} \cdot \{ \text{Ans. } \frac{141}{20} \}$
48.  $\int_{\frac{\sqrt{2}}{2}}^1 \frac{\sqrt{1-x^2}}{x^6} dx \cdot \{ \text{Ans. } \frac{8}{15} \}$
49.  $\int_1^2 \frac{\sqrt{x^2-1}}{x} dx \cdot \{ \text{Ans. } \sqrt{3} - \frac{\pi}{3} \}$
50.  $\int_{\sqrt{2}}^2 \frac{dx}{x^5 \sqrt{x^2-1}} \cdot \{ \text{Ans. } \frac{1}{32} \left( \pi + \frac{7\sqrt{3}}{2} - 8 \right) \}$
51.  $\int_0^{-\ln 2} \sqrt{1-e^{2x}} dx \cdot \{ \text{Ans. } \frac{\sqrt{3}}{2} + \ln(2-\sqrt{3}) \}$
52.  $\int_0^3 \frac{dx}{(x^2+3)^{\frac{5}{2}}} \cdot \{ \text{Ans. } \frac{\sqrt{3}}{24} \}$
53.  $\int_{2.5}^5 \frac{(\sqrt{25-x^2})^3}{x^4} dx \cdot \{ \text{Ans. } \frac{\pi}{3} \}$
54.  $\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{(2x^2+1)\sqrt{x^2+1}} \cdot \{ \text{Ans. } \tan^{-1} \frac{1}{2} \}$
55.  $\int_{\frac{\sqrt{3}}{8}}^{2\sqrt{2}} \frac{dx}{x\sqrt{(x^2-2)^5}} \cdot \{ \text{Ans. } \frac{\sqrt{6}}{27} + \frac{\pi\sqrt{2}}{48} \}$
56.  $\int_0^{\sqrt[5]{2}} \frac{x^9 dx}{(1+x^5)^3} \cdot \{ \text{Ans. } \frac{2}{45} \}$
57.  $\int_0^{\frac{1}{2}} \frac{x^3 dx}{x^2-3x+2} \cdot \{ \text{Ans. } 8\ln 3 - 15\ln 2 + \frac{13}{8} \}$

58.  $\int_0^{\sqrt[4]{2}} \frac{x^{15} dx}{(1+x^8)^{\frac{2}{5}}} \cdot \{ \text{Ans. } \frac{5}{192} (5 + 7\sqrt[5]{125}) \}$
59.  $\int_0^2 \frac{dx}{\sqrt{x+1} + \sqrt{(x+1)^3}} \cdot \{ \text{Ans. } \frac{\pi}{6} \}$
60.  $\int_0^1 \sqrt{2x+x^2} dx \cdot \{ \text{Ans. } \sqrt{3} - \frac{1}{2} \ln(2 + \sqrt{3}) \}$
61.  $\int_0^{\sqrt{3}} x^5 \sqrt{1+x^2} dx \cdot \{ \text{Ans. } \frac{848}{105} \}$
62.  $\int_0^{\ln 5} \frac{e^x \sqrt{e^x - 1}}{e^x + 3} dx \cdot \{ \text{Ans. } 4 - \pi \}$
63.  $\int_{\ln 3}^{\ln 4} \frac{e^x \sqrt{e^x - 3}}{e^x - 2} dx \cdot \{ \text{Ans. } \frac{4 - \pi}{2} \}$
64.  $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx \cdot \{ \text{Ans. } \frac{\pi}{2} - 1 \}$
65.  $\int_1^3 \frac{dx}{x\sqrt{x^2 + 5x + 1}} \cdot \{ \text{Ans. } \ln \frac{7+2\sqrt{7}}{9} \}$
66.  $\int_0^{\frac{\pi}{4}} \frac{x \sin x}{\cos^3 x} dx \cdot \{ \text{Ans. } \frac{\pi}{4} - \frac{1}{2} \}$
67.  $\int_0^{\frac{\pi}{2}} \frac{dx}{2 \cos x + 3} \cdot \{ \text{Ans. } \frac{2}{\sqrt{5}} \tan^{-1} \frac{1}{\sqrt{5}} \}$
68.  $\int_0^1 \frac{(3x+2)dx}{(x^2 + 4x + 1)^{\frac{5}{2}}} \cdot \{ \text{Ans. } \frac{19}{27} - \frac{5}{6\sqrt{6}} \}$
69.  $\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x dx}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \{ \text{Ans. } \frac{1}{(a^2 - b^2)} \ln \left| \frac{a}{b} \right| \}$
70. Prove  $\int_{-4}^{-5} e^{(x+5)^2} dx + 3 \int_{\frac{1}{3}}^{\frac{2}{3}} e^{9\left(x-\frac{2}{3}\right)^2} dx = 0$
71. Prove  $\int_a^b f(x) dx = (b-a) \int_0^1 f[(b-a)x + a] dx$

### CATEGORY-22.3. INTEGRATION BY PARTS IN DEFINITE INTEGRALS

72.  $\int_0^1 x e^{-x} dx$ . {Ans.  $1 - \frac{2}{e}$ }
73.  $\int_0^{\frac{\pi}{2}} x \cos x dx$ . {Ans.  $\frac{\pi}{2} - 1$ }
74.  $\int_0^1 x \tan^{-1} x dx$ . {Ans.  $\frac{\pi}{4} - \frac{1}{2}$ }
75.  $\int_1^e \ln x dx$ . {Ans. 1}
76.  $\int_1^4 e^{\sqrt{x}} dx$ . {Ans.  $2e^2$ }
77.  $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x dx}{\sin^2 x}$ . {Ans.  $\frac{\pi(9-4\sqrt{3})}{36} + \frac{1}{2} \ln \frac{3}{2}$ }
78.  $\int_0^{\pi} x^3 \sin x dx$ . {Ans.  $\pi^3 - 6\pi$ }
79.  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-x} \sin x dx$ . {Ans.  $-\frac{\sqrt{2}}{2} e^{-\frac{\pi}{4}}$ }
80.  $\int_1^2 x \log_2 x dx$ . {Ans.  $2 - \frac{3}{4 \ln 2}$ }
81.  $\int_0^{\frac{\pi^2}{4}} \sin \sqrt{x} dx$ . {Ans. 2}
82.  $\int_0^{e-1} \ln(x+1) dx$ . {Ans. 1}
83.  $\int_0^{\frac{\pi}{2}} e^{2x} \cos x dx$ . {Ans.  $\frac{e^{\pi}-2}{5}$ }
84.  $\int_1^2 e^x \left( \frac{1}{x} - \frac{1}{x^2} \right) dx$ . {Ans.  $e \left( \frac{e}{2} - 1 \right)$ }
85.  $\int_1^e \ln^3 x dx$ . {Ans.  $6 - 2e$ }
86.  $\int_0^1 (\sin^{-1} x)^4 dx$ . {Ans.  $\frac{\pi^4}{16} - 3\pi^2 + 24$ }

$$87. \int_1^{16} \tan^{-1} \sqrt{\sqrt{x}-1} dx. \{ \text{Ans. } \frac{16\pi}{3} - 2\sqrt{3} \}$$

#### CATEGORY-22.4. REDUCTION FORMULA IN DEFINITE INTEGRALS

88. Derive reduction formulas for computing the integrals  $\int_0^{\frac{\pi}{2}} \cos^n x dx$  and  $\int_0^{\frac{\pi}{2}} \sin^n x dx$  ( $n$  a positive integer or zero) and compute the integrals :

i.  $\int_0^{\frac{\pi}{2}} \sin^5 x dx$ ; {Ans.  $\frac{8}{15}$ }

ii.  $\int_0^{\frac{\pi}{2}} \cos^8 x dx$ ; {Ans.  $\frac{105\pi}{768}$ }

iii.  $\int_0^{\frac{\pi}{2}} \sin^{11} x dx$ ; {Ans.  $\frac{256}{693}$ }

89. Derive a reduction formula for computing the integral  $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$  ( $m$  and  $n$  positive integers or zero). Investigate particular cases of even and odd values of  $m$  and  $n$ . {Ans.  $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2} = \frac{n-1}{m+n} I_{m-2,n}$ }

90. Derive a reduction formula and compute the integral  $\int_{-1}^0 x^n e^x dx$  ( $n$  a positive integer).

91. Prove the reduction formula  $\int \frac{dx}{(1+x^2)^n} = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)} \int \frac{dx}{(1+x^2)^{n-1}}$  ( $n$  a positive integer) and with the aid of it compute the integral  $\int_0^1 \frac{dx}{(1+x^2)^4}$ . {Ans.  $\frac{11}{48} + \frac{5\pi}{64}$ }

92. Prove that if  $I_n = \int_1^e (\ln x)^n dx$ , then  $I_n = e - nI_{n-1}$  ( $n$  a positive integer).

93. Find the reduction formula for  $I_{m,n} = \int_0^1 x^m (\ln x)^n dx$ . {Ans.  $I_{m,n} = -\frac{n}{m+1} I_{m,n-1}$ }

94. Let  $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$  ( $n > 1$  and is an integer). Check to see that  $I_n + I_{n-2} = \frac{1}{n-1}$ .

95. Find  $\int_0^1 x^p (1-x)^q dx$  ( $p$  and  $q$  positive integers) {Ans.  $\frac{p!q!}{(p+q+1)!}$ }

#### CATEGORY-22.5. DEFINITE INTEGRAL OF PIECEWISE FUNCTIONS

96. If  $f(x) = x^2, \quad 0 \leq x < 1$   
 $\quad = \sqrt{x}, \quad 1 \leq x \leq 2$   
 find  $\int_0^2 f(x) dx$ . {Ans.  $\frac{4\sqrt{2}-1}{3}$ }
97. If  $f(x) = x, \quad x < 1,$   
 $\quad = x-1, \quad x \geq 1,$   
 then find  $\int_0^2 x^2 f(x) dx$ . {Ans.  $\frac{5}{3}$ }
98.  $\int_{-1}^2 |x| dx$  {Ans.  $\frac{5}{2}$ }
99.  $\int_0^2 |1-x| dx$  {Ans. 1}
100.  $\int_{-1}^1 |1-x| dx$ . {Ans. 2}
101.  $\int_{-1}^3 \frac{|x|}{x+5} dx$  {Ans.  $2 + 5 \ln\left(\frac{25}{32}\right)$ }
102.  $\int_{-2}^2 |2^x - 1| + |x-1| dx$  {Ans.  $5 + \left(\frac{9}{4}\right) \ln 2$ }
103.  $\int_{\frac{1}{e}}^e |\ln x| dx$ . {Ans.  $2\left(\frac{e-1}{e}\right)$ }
104.  $\int_{\frac{1}{2}}^2 |\log_{10} x| dx$ . {Ans.  $\frac{1}{2} \log_{10}\left(\frac{8}{e}\right)$ }
105.  $\int_0^2 \left| \cos\left(\frac{\pi x}{2}\right) \right| dx$ . {Ans.  $\frac{4}{\pi}$ }
106.  $\int_0^5 [x] dx$  {Ans. 10}
107.  $\int_0^2 x[x] dx$ . {Ans.  $\frac{3}{2}$ }
108.  $\int_{-2}^4 x[x] dx$ . {Ans.  $\frac{41}{2}$ }
109.  $\int_0^2 [x^2] dx$  {Ans.  $5 - \sqrt{2} - \sqrt{3}$ }

$$110. \int_0^{n^2} [\sqrt{x}] dx \quad \{\text{Ans. } \frac{n(n-1)(4n+1)}{6}\}$$

$$111. \int_0^{\pi} [2 \sin x] dx \quad \{\text{Ans. } \frac{2\pi}{3}\}$$

$$112. \int_{\pi}^{2\pi} [2 \sin x] dx \quad \{\text{Ans. } -\frac{5\pi}{3}\}$$

$$113. \int_{-1}^3 (|x-2| + [x]) dx \quad \{\text{Ans. } 7\}$$

$$114. \int_{-1}^2 \frac{|x|}{x} dx \quad \{\text{Ans. } 1\}$$

$$115. \int_a^b \frac{|x|}{x} dx \quad (a < b) \quad \{\text{Ans. } |b| - |a|\}$$

$$116. \int_{-2}^2 \min(\{x\}, -x - [-x]) dx \quad \{\text{Ans. } 1\}$$

### CATEGORY-22.6. DEFINITE INTEGRAL OF SYMMETRIC FUNCTIONS

$$117. \text{ Prove the validity of the equality } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$118. \text{ Prove the validity of the equality } \int_0^1 x^m (1-x)^n dx = \int_0^1 x^n (1-x)^m dx$$

$$119. \text{ Show that } \int_0^a (f(x) + f(-x)) dx = \int_{-a}^a f(x) dx$$

$$120. \text{ If } f(x) = f(a+b-x), \text{ then show that } \int_a^b xf(x) dx \text{ is equal to } \frac{1}{2}(a+b) \int_a^b f(x) dx.$$

$$121. \text{ Prove that } \int_0^{\frac{\pi}{2}} f(\cos x) dx = \int_0^{\frac{\pi}{2}} f(\sin x) dx. \text{ Apply the obtained result in computing the integrals}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx \text{ and } \int_0^{\frac{\pi}{2}} \sin^2 x dx. \quad \{\text{Ans. } \frac{\pi}{4}, \frac{\pi}{4}\}$$

$$122. \text{ Prove that } \int_0^{\pi} xf(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx = \frac{\pi}{2} \cdot 2 \int_0^{\frac{\pi}{2}} f(\sin x) dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx. \text{ Apply the obtained result to}$$

$$\text{the computation of the integral } \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx. \quad \{\text{Ans. } \frac{\pi^2}{4}\}$$

$$123. \int_0^{\pi} \sin^6 \frac{x}{2} dx \quad \{\text{Ans. } \frac{5\pi}{16}\}$$

$$124. \int_0^1 x(1-x)^{99} dx. \quad \{\text{Ans. } \frac{1}{10100}\}$$

$$125. \int_0^1 \sqrt{(1-x^2)^3} dx. \quad \{\text{Ans. } \frac{3\pi}{16}\}$$

$$126. \int_0^1 x^2 \sqrt{1-x^2} dx. \quad \{\text{Ans. } \frac{\pi}{16}\}$$

$$127. \int_0^{2\pi} \cos^{99} x dx. \quad \{\text{Ans. } 0\}$$

$$128. \text{ If } n \text{ is an odd positive integer, then find } \int_0^{2\pi} \cos^n x dx. \quad \{\text{Ans. } 0\}$$

$$129. \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx \quad \{\text{Ans. } \frac{\pi}{4}\}$$

$$130. \int_0^{\pi} x \sin x \cos^4 x dx. \quad \{\text{Ans. } \frac{\pi}{5}\}$$

$$131. \text{ Prove } \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{12}$$

$$132. \int_0^{2a} \frac{f(x)}{f(x) + f(2a-x)} dx. \quad \{\text{Ans. } a\}$$

$$133. \text{ Prove } \int_a^b \frac{f(x)}{f(x) + f(a+b-x)} dx = \frac{b-a}{2}$$

$$134. \int_0^{\frac{\pi}{2}} \frac{f(x)}{f(x) + f\left(\frac{\pi}{2} - x\right)} dx. \quad \{\text{Ans. } \frac{\pi}{4}\}$$

$$135. \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x} + \sqrt{x}} dx \quad \{\text{Ans. } \frac{1}{2}\}$$

$$136. \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\phi}{1 + \sin \phi} d\phi \quad \{\text{Ans. } \pi(\sqrt{2}-1)\}$$

$$137. \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx \quad \{\text{Ans. } \frac{1}{2}\}$$



$$138. \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx. \{ \text{Ans. } \frac{\pi}{4} \}$$

$$139. \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}} \{ \text{Ans. } \frac{\pi}{4} \}$$

$$140. \int_0^{\frac{\pi}{2n}} \frac{dx}{1 + \cot^n nx} \{ \text{Ans. } \frac{\pi}{4n} \}$$

$$141. \int_0^{\frac{\pi}{2}} \ln \left( \frac{4 + 3 \sin x}{4 + 3 \cos x} \right) dx. \{ \text{Ans. } 0 \}$$

$$142. \int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^3 x} dx. \{ \text{Ans. } \frac{\pi}{4} \}$$

$$143. \text{ Prove } \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{8} \ln 2$$

$$144. \int_0^{\pi} \frac{x dx}{1 + \cos \alpha \sin x} \quad (0 < \alpha < \pi) \{ \text{Ans. } \frac{\pi \alpha}{\sin \alpha} \}$$

$$145. \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x} dx. \{ \text{Ans. } 1 \}$$

$$146. \text{ Prove } \int_0^{\pi} \frac{x}{1 + \sin \alpha \sin x} dx = \frac{\pi}{\cos \alpha} \left( \frac{\pi}{2} - \alpha \right)$$

$$147. \text{ Prove } \int_0^{\frac{\pi}{4}} \ln(1 + \tan x) dx = \frac{\pi}{8} \ln 2$$

$$148. \text{ Prove } \int_0^{\frac{\pi}{2}} \frac{\cos x dx}{1 + \cos x + \sin x} = \frac{\pi}{4} - \frac{1}{2} \ln 2$$

$$149. \text{ Prove } \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx = 0$$

$$150. \text{ Prove } \int_0^{\frac{\pi}{2}} f(\sin 2x) \sin x dx = \int_0^{\frac{\pi}{2}} f(\sin 2x) \cos x dx = \sqrt{2} \int_0^{\frac{\pi}{4}} f(\cos 2x) \cos x dx.$$

$$151. \text{ Prove } \int_0^{\pi} \frac{x \sin 2x \sin \left( \frac{\pi}{2} \cos x \right)}{2x - \pi} dx = \frac{8}{\pi^2}$$

152. Prove  $\int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{\pi}{2} \cos x\right)}{2x - \pi} dx = \frac{8}{\pi}$

**CATEGORY-22.7. DEFINITE INTEGRAL OF EVEN/ODD FUNCTIONS**

153. Prove  $\int_{-a}^a \cos x \cdot f(x^2) dx = 2 \int_0^a \cos x \cdot f(x^2) dx$

154. Show that the numerical value of  $\int_{-2}^2 (px^3 + qx + s) dx$  depends only upon the constant  $s$ .

155.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{1 + \cos x}$ . {Ans. 2}

156.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \sin\left(2t - \frac{\pi}{4}\right) dt$ . {Ans.  $-\frac{\sqrt{2}}{3}$ }

157.  $\int_{-1}^1 \sin^{11} x dx$ .. {Ans. 0}

158.  $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} x^{10} \sin^9 x dx = 0$ .

159.  $\int_{-1}^1 \sin^3 x \cos^2 x dx$ . {Ans. 0}

160.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^2 x (\sin x + \cos x) dx$ . {Ans.  $\frac{4}{15}$ }

161.  $\int_{-\pi}^{\pi} (1 - x^2) \sin x \cos^2 x dx$ . {Ans. 0}

162.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1 + e^x} dx$ . {Ans. 1}

163.  $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{x \sin x}{\cos^2 x} dx$ . {Ans.  $2\left(\frac{2\pi}{3} - \ln \tan \frac{5\pi}{12}\right)$ }

164.  $\int_{-1}^1 \ln(x + \sqrt{x^2 + 1}) dx$ . {Ans. 0}

165.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{a - \sin \theta}{a + \sin \theta}\right) d\theta, a > 0. \{ \text{Ans. } 0 \}$
166.  $\int_{-1}^1 \frac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x} dx = 0.$
167.  $\int_{-1}^1 x|x| dx. \{ \text{Ans. } 0 \}$
168.  $\int_{-1}^1 e^{\cos x} dx = 2 \int_0^1 e^{\cos x} dx$
169.  $\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \ln \frac{1+x}{1-x} dx = 0.$
170.  $\int_{-a}^a \frac{x^2 dx}{\sqrt{a^2 + x^2}} \{ \text{Ans. } a^2 [\sqrt{2} - \ln(\sqrt{2} + 1)] \}$
171.  $\int_{-1}^1 \ln \frac{2-x}{2+x} dx \{ \text{Ans. } 0 \}$
172.  $\int_{-1}^1 [\sqrt{1+x+x^2} - \sqrt{1-x+x^2}] dx \{ \text{Ans. } 0 \}$
173.  $\int_{-a}^a x\sqrt{a^2 - x^2} dx \{ \text{Ans. } 0 \}$
174.  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x} dx \{ \text{Ans. } \frac{4}{3} \}$
175.  $\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2} - 2 dx \{ \text{Ans. } 4\ln\left(\frac{4}{3}\right) \}$
176.  $\int_{-1}^{\frac{3}{2}} |x \sin \pi x| dx \{ \text{Ans. } \frac{3}{\pi} + \frac{1}{\pi^2} \}$

### CATEGORY-22.8. DEFINITE INTEGRAL OF PERIODIC FUNCTIONS

177.  $\int_0^{100\pi} \sqrt{1 - \cos 2x} dx \{ \text{Ans. } 200\sqrt{2} \}$
178.  $\int_0^{\frac{32\pi}{3}} \sqrt{1 + \cos 2x} dx \{ \text{Ans. } \frac{44 - \sqrt{3}}{\sqrt{2}} \}$

179.  $\int_0^{50\pi} |\cos x| dx$ . {Ans. 100}
180.  $\int_0^{\frac{16\pi}{3}} |\sin x| dx$ . {Ans.  $\frac{21}{2}$ }
181.  $\int_0^{100} \{x\} dx$  {Ans. 50}
182.  $\int_0^{1000} e^{x-[x]} dx$ . {Ans.  $1000(e-1)$ }
183. If  $\int_0^{n\pi} f(\cos^2 x) dx = k \int_0^{\pi} f(\cos^2 x) dx$ , then find the value of  $k$ . {Ans.  $n$ }
184. Show that  $\int_0^{n\pi+\nu} |\sin x| dx = 2n+1 - \cos \nu$  where  $n$  is a positive integer and  $0 \leq \nu < \pi$ .
185. Show that if  $f(x)$  is a periodic function with the period  $T$ , then  $\int_a^{a+T} f(x) dx$  is independent of  $a$ .

### CATEGORY-22.9. CALCULATING IMPROPER INTEGRALS BY FIRST PRINCIPLES

186.  $\int_1^{\infty} \frac{dx}{x^4}$ . {Ans.  $\frac{1}{3}$ }
187.  $\int_1^{\infty} \frac{dx}{\sqrt{x}}$ . {Ans. diverges}
188.  $\int_0^{\infty} e^{-ax} dx$  ( $a > 0$ ). {Ans.  $\frac{1}{a}$ }
189.  $\int_0^{\infty} (a^{-x} - b^{-x}) dx$ . {Ans.  $\frac{1}{\ln a} - \frac{1}{\ln b}$ }
190.  $\int_2^{\infty} \frac{\ln x}{x} dx$ . {Ans. diverges}
191.  $\int_1^{\infty} \frac{dx}{x^2(x+1)}$ . {Ans.  $1 - \ln 2$ }
192.  $\int_0^{\infty} \frac{dx}{(1+x)^3}$ . {Ans.  $\frac{1}{2}$ }
193.  $\int_{\sqrt{2}}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$ . {Ans.  $\frac{\pi}{4}$ }
194.  $\int_1^{\infty} \frac{dx}{x\sqrt{1+x^2}}$ . {Ans.  $\ln(\sqrt{2}+1)$ }

195.  $\int_0^{\infty} x e^{-x^2} dx$ . {Ans.  $\frac{1}{2}$ }
196.  $\int_0^{\infty} x^3 e^{-x^2} dx$ . {Ans.  $\frac{1}{2}$ }
197.  $\int_0^{\infty} x \sin x dx$ . {Ans. diverges}
198.  $\int_0^{\infty} e^{-\sqrt{x}} dx$ . {Ans. 2}
199.  $\int_0^{\infty} e^{-x} \sin x dx$ . {Ans.  $\frac{1}{2}$ }
200.  $\int_0^{\infty} e^{-ax} \cos bx dx$ . {Ans.  $\frac{a}{a^2+b^2}$  if  $a > 0$ , diverges if  $a \leq 0$ }
201.  $\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx$ . {Ans.  $\frac{\pi}{4} + \frac{1}{2} \ln 2$ }
202.  $\int_0^{\infty} \frac{dx}{1+x^3}$ . {Ans.  $\frac{2\pi}{3\sqrt{3}}$ }
203.  $\int_1^{\infty} \frac{\sqrt{x}}{(1+x)^2} dx$ . {Ans.  $\frac{1}{2} + \frac{\pi}{4}$ }
204.  $\int_0^{\infty} \frac{x}{x^3+1} dx$ . {Ans.  $\frac{2\pi}{3\sqrt{3}}$ }
205.  $\int_1^{\infty} \frac{x^3+1}{x^4} dx$ . {Ans. diverges}
206.  $\int_0^{\infty} \frac{dx}{e^x x (\ln x)^2}$ . {Ans. 2}
207.  $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ . {Ans.  $\frac{\pi}{2}$ }
208.  $\int_0^1 \frac{dx}{1-x^2+2\sqrt{1-x^2}}$ . {Ans.  $\frac{\pi}{3\sqrt{3}}$ }
209.  $\int_{-1}^0 \frac{e^x}{x^3} dx$ . {Ans.  $-\frac{2}{e}$ }
210.  $\int_1^2 \frac{x dx}{\sqrt{x-1}}$ . {Ans.  $\frac{8}{3}$ }

211.  $\int_0^1 x \ln x \, dx$ . {Ans.  $-\frac{1}{4}$ }
212.  $\int_0^{\frac{1}{e}} \frac{dx}{x \ln^2 x}$ . {Ans. 1}
213.  $\int_1^2 \frac{dx}{x \ln x}$ . {Ans. diverges}
214.  $\int_1^e \frac{dx}{x \sqrt{\ln x}}$ . {Ans. 2}
215.  $\int_0^1 \frac{e^{\frac{1}{x}}}{x^3} dx$ . {Ans. diverges}
216.  $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$  ( $a < b$ ). {Ans.  $\pi$ }
217.  $\int_a^b \frac{x \, dx}{\sqrt{(x-a)(b-x)}}$  ( $a < b$ ). {Ans.  $\frac{\pi}{2}(a+b)$ }
218.  $\int_3^5 \frac{x^2 \, dx}{\sqrt{(x-3)(5-x)}}$ . {Ans.  $\frac{33\pi}{2}$ }
219.  $\int_{-1}^1 \frac{dx}{(2-x)\sqrt{1-x^2}}$ . {Ans.  $\frac{\pi}{\sqrt{3}}$ }
220.  $\int_{-1}^1 \frac{3x^2+2}{\sqrt[3]{x^2}} dx$ . {Ans.  $\frac{102}{7}$ }
221.  $\int_{-1}^1 \frac{x+1}{\sqrt[5]{x^3}} dx$ . {Ans.  $\frac{10}{7}$ }
222.  $\int_{-1}^1 \frac{x-1}{\sqrt[3]{x^5}} dx$ . {Ans. diverges}
223.  $\int_{-1}^1 \frac{\ln(2+\sqrt[3]{x})}{\sqrt[3]{x}} dx$ . {Ans.  $6 - \frac{9}{2} \ln 3$ }
224.  $\int_1^{\infty} \frac{dx}{x\sqrt{x-1}}$ . {Ans.  $\pi$ }
225.  $\int_{-1}^1 \ln \frac{1+x}{1-x} \frac{x^3 \, dx}{\sqrt{1-x^2}}$ . {Ans.  $\frac{5\pi}{3}$ }
226.  $\int_{-\infty}^{\infty} \frac{2x \, dx}{x^2+1}$ . {Ans. diverges}

227.  $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} \cdot \{ \text{Ans. } \pi \}$
228.  $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^2} \cdot \{ \text{Ans. } \frac{\pi}{2} \}$
229.  $\int_0^2 \frac{dx}{x^2 - 4x + 3} \cdot \{ \text{Ans. diverges} \}$
230.  $\int_0^{\infty} \frac{\tan^{-1}(x-1)dx}{\sqrt[3]{(x-1)^4}} \cdot \{ \text{Ans. } \frac{3+2\sqrt{3}}{4} \pi - \frac{3}{2} \ln 2 \}$

### CATEGORY-22.10. REDUCTION FORMULA IN IMPROPER INTEGRALS

231.  $\int_0^{\infty} \frac{dx}{(a^2 + x^2)^n} \quad (n \text{ a positive integer}). \{ \text{Ans. } \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} \frac{\pi}{2a^{2n-1}} \}$
232.  $\int_0^{\infty} x^n e^{-x} dx \quad (n \text{ a positive integer}). \{ \text{Ans. } n! \}$
233.  $\int_0^{\infty} x^{2n+1} e^{-x^2} dx \quad (n \text{ a positive integer}). \{ \text{Ans. } \frac{n!}{2} \}$
234.  $\int_0^1 (\ln x)^n dx \quad (n \text{ a positive integer}). \{ \text{Ans. } (-1)^n n! \}$
235.  $\int_0^1 \frac{x^m dx}{\sqrt{1-x^2}}$  for  $m$ : (a) even, (b) odd ( $m > 0$ ). {Ans. (a)  $\frac{(m-1)(m-3)\cdots 3 \cdot 1}{m(m-2)\cdots 4 \cdot 2} \frac{\pi}{2}$  (b)  $\frac{(m-1)(m-3)\cdots 4 \cdot 2}{m(m-2)\cdots 3 \cdot 1}$ }
236.  $\int_0^1 \frac{(1-x)^n}{\sqrt{x}} dx \quad (n \text{ a positive integer}). \{ \text{Ans. } 2 \frac{2n(2n-2)\cdots 4 \cdot 2}{(2n+1)(2n-1)\cdots 3 \cdot 1} \}$

### CATEGORY-22.11. CALCULATING IMPROPER INTEGRALS BY PROPERTIES

237.  $\int_1^{\infty} \frac{dx}{(x - \cos \alpha) \sqrt{x^2 - 1}} \quad (0 < \alpha < 2\pi). \{ \text{Ans. } \frac{\pi - \alpha}{\sin \alpha} \}$
238. Prove that  $\int_0^{\infty} \frac{dx}{1+x^4} = \int_0^{\infty} \frac{x^2 dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$
239. Prove that  $\int_0^{\infty} \frac{x \ln x}{(1+x^2)^2} dx = 0$
240. Compute the integral  $\int_1^{\infty} \frac{x^2 - 2}{x^3 \sqrt{x^2 - 1}} dx \{ \text{Ans. } 0 \}$
241.  $\int_0^{\frac{\pi}{2}} \ln \sin x dx. \{ \text{Ans. } -\frac{\pi}{2} \ln 2 \}$

$$242. \int_0^{\frac{\pi}{2}} \ln \sin 2x dx. \{ \text{Ans. } -\frac{\pi}{2} \ln 2 \}$$

$$243. \int_0^{\pi} x \ln(\sin x) dx. \{ \text{Ans. } -\frac{\pi^2}{2} \ln 2 \}$$

$$244. \int_0^{\frac{\pi}{2}} x \cot x dx. \{ \text{Ans. } \frac{\pi}{2} \ln 2 \}$$

$$245. \int_0^1 \frac{\sin^{-1} x}{x} dx. \{ \text{Ans. } \frac{\pi}{2} \ln 2 \}$$

$$246. \int_0^1 \frac{\ln x dx}{\sqrt{1-x^2}}. \{ \text{Ans. } -\frac{\pi}{2} \ln 2 \}$$

$$247. \text{ Prove } \int_{-1}^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = 2$$

$$248. \int_{-1}^1 \frac{x^2 \sin^{-1} x}{\sqrt{1-x^2}} dx \{ \text{Ans. } 0 \}$$

$$249. \text{ Prove } \int_0^{\frac{\pi}{2}} \sin 2x \ln \tan x dx = 0$$

$$250. \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx. \{ \text{Ans. } \frac{\pi^2}{4} \}$$

$$251. \int_0^{\pi} \frac{1}{a + b \cos x} dx. \{ \text{Ans. } \frac{\pi}{\sqrt{a^2 - b^2}} \}$$

$$252. \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx. \{ \text{Ans. } \frac{\pi}{4} \}$$

$$253. \int_0^{\pi} \frac{x dx}{1 + \cos^2 x} \{ \text{Ans. } \frac{\pi^2}{2\sqrt{2}} \}$$

$$254. \text{ If } n \text{ is even, then find } \int_0^{\pi} \frac{\sin nx}{\sin x} dx. \{ \text{Ans. } 0 \}$$

$$255. \text{ Prove } \int_0^{\infty} \frac{\ln\left(x + \frac{1}{x}\right)}{1 + x^2} dx = \pi \ln 2$$

$$256. \int_0^{2\pi} \frac{\sin^2 \theta}{a - b \cos \theta} d\theta, \quad a > b > 0 \{ \text{Ans. } \frac{2\pi}{b^2} (a - \sqrt{a^2 - b^2}) \}$$



257. Prove  $\int_0^1 \ln \sin\left(\frac{\pi}{2}x\right) dx = \ln \frac{1}{2}$

258. Prove  $\int_0^{\frac{\pi}{2}} \ln \tan x \, dx = \int_0^{\frac{\pi}{2}} \ln \cot x \, dx = 0$

259. Prove  $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \frac{\pi}{2}(\pi - 2)$

260.  $\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan x} dx$ . {Ans.  $\frac{\pi}{4}$ }

261.  $\int_0^{\pi} \frac{dx}{1 + 2 \sin^2 x}$  {Ans.  $\frac{\pi}{\sqrt{3}}$ }

262.  $\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$  {Ans.  $\frac{\pi^2}{16}$ }

263. Prove  $\int_0^{\infty} \frac{\ln(1+x^2)}{1+x^2} dx = \pi \ln 2$

264. Prove  $\int_0^{\pi} \ln(1 + \cos x) dx = -\pi \ln 2$

265. Prove  $\int_0^{\infty} \frac{x \, dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$

266.  $\int_0^{\frac{\pi}{2}} \frac{dx}{6 + \sin^2 x}$ . {Ans.  $\frac{\pi}{12} \sqrt{\frac{6}{7}}$ }

267. Show that  $\int_0^{\frac{\pi}{2}} \frac{|ab| dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2}$ , where  $a$  and  $b$  are any real numbers different from zero.

268. Prove  $\int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$

269. Prove  $\int_0^{\pi} \frac{x \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi^2}{4} \frac{(a^2 + b^2)}{a^3 b^3}$

### CATEGORY-22.12. DEFINITE INTEGRAL CONTAINING PARAMETER

270. Find the derivative  $\frac{dy}{dx}$  of the following functions:-

i.  $y = \int_0^x t \sin t \, dt$ . {Ans.  $x \sin x$ }

$$\text{ii. } y = \int_0^{x^2} \sqrt{1+t^2} dt. \{ \text{Ans. } 2x\sqrt{1+x^4} \}$$

$$\text{iii. } y = \int_{x^2}^{x^4} \sin \sqrt{t} dt. \{ \text{Ans. } 4x^3 \sin x^2 - 2x \sin x \}$$

$$\text{iv. } y = \int_{x^2}^{x^3} \ln t dt \quad (x > 0) \{ \text{Ans. } (9x^2 - 4x) \ln x \}$$

$$\text{v. } y = \int_{x^2}^{x^3} \frac{1}{\ln t} dt \quad (x > 0). \{ \text{Ans. } \frac{x(x-1)}{\ln x} \}$$

$$\text{vi. } y = \int_{\frac{1}{x}}^{\sqrt{x}} \cos(t^2) dt \quad (x > 0) \{ \text{Ans. } \frac{1}{x^2} \cos \frac{1}{x^2} + \frac{1}{2\sqrt{x}} \cos x \}$$

$$\text{vii. } y = \int_0^{2x} \frac{\sin t}{t} dt \{ \text{Ans. } \frac{\sin 2x}{x} \}$$

$$\text{viii. } y = \int_x^0 \sqrt{1+t^4} dt \{ \text{Ans. } -\sqrt{1+x^4} \}$$

$$\text{ix. } y = \int_1^{t^3} \sqrt{z} \ln z dz; x = \int_{\sqrt{t}}^3 z^2 \ln z dz \{ \text{Ans. } -36t^2 \sqrt{t} \}$$

$$\text{x. } \int_0^y e^{-t^2} dt + \int_0^{x^2} \sin^2 t dt = 0 \{ \text{Ans. } -2xe^{y^2} \sin^2 x^2 \}$$

$$\text{xi. } \int_0^y e^t dt + \int_0^x \sin t dt = 0 \{ \text{Ans. } -e^{-y} \sin x \}$$

$$\text{xii. } \int_{\frac{\pi}{2}}^x \sqrt{3-2\sin^2 z} dz + \int_0^y \cos t dt = 0 \{ \text{Ans. } -\frac{\sqrt{3-2\sin^2 x}}{\cos y} \}$$

$$\text{xiii. } x = \int_0^y \frac{1}{\sqrt{1+9u^2}} du. \{ \text{Ans. } \sqrt{1+9y^2} \}.$$

$$271. \text{ If } x = \int_0^y \frac{1}{\sqrt{1+4t^2}} dt, \text{ then find } \frac{d^2 y}{dx^2}. \{ \text{Ans. } 4y \}$$

272. Find the limits:

$$\text{i. } \lim_{x \rightarrow \infty} \frac{\int_0^x (\tan^{-1} x)^2 dx}{\sqrt{x^2+1}} \{ \text{Ans. } \frac{\pi^2}{4} \}$$

$$\text{ii. } \lim_{x \rightarrow 0} \frac{\int_0^{x^2} \cos t^2 dt}{x \sin x}. \{ \text{Ans. } 1 \}$$

- iii.  $\lim_{x \rightarrow \infty} \frac{\int_0^x e^{x^2} dx}{\int_0^x e^{2x^2} dx}$  {Ans. 0}
- iv.  $\lim_{x \rightarrow \infty} \frac{\left( \int_0^x e^x dx \right)^2}{\int_0^x e^{2x^2} dx}$  . {Ans. 0}
- v.  $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin \sqrt{x} dx}{x^3}$  {Ans.  $\frac{2}{3}$ }

273. If  $f(x)$  is continuous at  $x = 0$  and  $f(0) = 2$ , then find  $\lim_{x \rightarrow 0} \frac{\int_0^x f(x) dx}{x}$  . {Ans. 2}

274. Find the points of extremum of the following functions:

- i.  $f(x) = \int_0^x \frac{\sin t}{t} dt$ ,  $x > 0$ . {Ans.  $x = n\pi$  ( $n = 1, 2, \dots$ ) maxima when  $n$  is odd, minima when  $n$  is even}
- ii.  $f(x) = \int_1^x e^{-\frac{t^2}{2}} (1 - t^2) dt$  {Ans. maxima at  $x = 1$ , minima at  $x = -1$ }
- iii.  $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$  {Ans. maxima at  $x = \pm 1$ , minima at  $x = -2, 0, 2$ }

275. Solve the equation  $\int_{\sqrt{2}}^x \frac{dx}{x\sqrt{x^2 - 1}} = \frac{\pi}{12}$  . {Ans.  $x = 2$ }

276. Solve the equation  $\int_{\ln 2}^x \frac{dx}{\sqrt{e^x - 1}} = \frac{\pi}{6}$  . {Ans.  $x = \ln 4$ }

277. Find the greatest and least values of the function  $I(x) = \int_0^x \frac{2t + 1}{t^2 - 2t + 2} dt$  on the interval  $[-1, 1]$ . {Ans.

$$\frac{3\pi}{4} - \ln 2, \frac{3\pi}{4} + \ln \frac{13}{8} - 3 \tan^{-1} \frac{3}{2} \}$$

278. Find the point of extremum and the points of inflection on the graph of the function  $y = \int_0^x (t - 1)(t - 2)^2 dt$  .

{Ans. Minima at  $x = 1$ , points of inflection at  $x = 2, x = \frac{4}{3}$ }

279. Find the maxima of the function  $f(x) = \int_1^x \{2(t - 1)(t - 2)^3 + 3(t - 1)^2(t - 2)^2\} dt$  . {Ans.  $x = 1$ }

280. (a) Show that if  $f(t)$  is an odd function, then  $\int_a^x f(t)dt$  is an even function.

(b) Will  $\int_a^x f(t)dt$  be an odd function if  $f(t)$  is an even function? {Ans. No if  $a \neq 0$ , yes if  $a = 0$ }

281. Prove the validity of the equality  $\int_x^1 \frac{dt}{1+t^2} = \int_1^{\frac{1}{x}} \frac{dt}{1+t^2}$  ( $x > 0$ )

282. Prove the identity  $\int_{\frac{1}{e}}^{\tan x} \frac{t dt}{1+t^2} + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)} \equiv 1$

283. Prove the identity  $\int_0^{\sin^2 x} \sin^{-1} \sqrt{t} dt + \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} dt \equiv \frac{\pi}{4}$

284. It is known that  $f(x)$  is an odd function in the interval  $\left[-\frac{T}{2}, \frac{T}{2}\right]$  and has a period equal to  $T$ . Prove that

$\int_a^x f(t)dt$  is also a periodic function with the same period.

285. Evaluate  $\int_0^1 \frac{x^\alpha - 1}{\ln x} dx$ . {Ans.  $\ln|1 + \alpha|$ }

286. If  $y = \int_0^x f(t) \sin\{k(x-t)\} dt$ , then prove that  $\frac{d^2 y}{dx^2} + k^2 y = kf(x)$ .

### CATEGORY-22.13. DEFINITE INTEGRAL OF PIECEWISE FUNCTIONS CONTAINING PARAMETER

287. Given

$$\begin{aligned} f(x) &= x, & 0 \leq x \leq 1 \\ &= x^2, & 1 < x \leq 2 \\ &= 2^x, & x > 2. \end{aligned}$$

Find  $g(x) = \int_0^x f(x) dx, \quad x \geq 0$ .

288. If  $f(x) = \int_{-1}^x |t| dt$  for any  $x$ , then find  $f(x)$ .

$$\begin{aligned} \{\text{Ans. } f(x) &= \frac{1}{2}(x^2 - 1), & x < -1 \\ &= \frac{1}{2}(1 - x^2), & -1 \leq x < 0 \\ &= \frac{1}{2}(1 + x^2), & x \geq 0 \} \end{aligned}$$

289. If  $f(x) = \int_x^{x^2} |t| dt$  for any  $x$ , then find  $f(x)$ .

$$\{\text{Ans. } f(x) = \frac{x^2}{2}(1-x^2), \quad x < 0 \\ = \frac{x^2}{2}(x^2-1), \quad x \geq 0 \}$$

290. Find  $f(x) = \int_0^x \max(\sin x, \cos x) dx$ ,  $0 \leq x \leq \pi$ .

291. If  $x > 0$ , prove that  $\int_0^x [x] dx = \frac{1}{2}[x]([x]-1) + [x]\{x\}$ .

292. Find the greatest value of  $f(x) = \int_{-\frac{1}{2}}^x |t| dt$  on the interval  $\left[-\frac{1}{2}, \frac{1}{2}\right]$ . {Ans.  $\frac{1}{4}$ }

#### CATEGORY-22.14. ESTIMATING A DEFINITE INTEGRAL

293. Prove that  $\frac{\pi}{4} < \int_0^1 \frac{1}{\sqrt{1+x^4}} dx \leq 1$ .

294. Prove that  $\frac{5}{4} \leq \int_0^1 e^{x^2} dx \leq 2$ .

#### CATEGORY-22.15. DEFINITE INTEGRAL AS LIMIT OF SUM OF SERIES

295. Find  $\int_1^2 (x^2 - x) dx$  as limit of sum of series. {Ans.  $\frac{5}{6}$ }

296. Sum the series:-

i.  $\lim_{n \rightarrow \infty} \left[ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right]$  {Ans.  $\ln 2$ }

ii.  $\lim_{n \rightarrow \infty} \left[ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{5n} \right]$  {Ans.  $\ln 5$ }

iii.  $\lim_{n \rightarrow \infty} \left[ \frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}} \right]$  {Ans.  $\frac{\pi}{2}$ }

iv.  $\lim_{n \rightarrow \infty} \left[ \frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+(n-1)^2} \right]$  {Ans.  $\frac{\pi}{4}$ }

v.  $\lim_{n \rightarrow \infty} \left( \frac{1}{1+n^3} + \frac{4}{8+n^3} + \cdots + \frac{r^2}{r^3+n^3} + \cdots + \frac{1}{2n} \right)$  {Ans.  $\frac{1}{3} \ln 2$ }

vi.  $\lim_{n \rightarrow \infty} \left[ \frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \cdots + \frac{1}{n} \right]$  {Ans.  $\frac{\pi}{4} + \frac{1}{2} \ln 2$ }

- vii.  $\lim_{n \rightarrow \infty} \frac{1^{99} + 2^{99} + \dots + n^{99}}{n^{100}} \cdot \{ \text{Ans. } \frac{1}{100} \}$
- viii.  $\lim_{n \rightarrow \infty} \frac{1}{n} \left[ \sin^2 \frac{\pi}{2n} + \sin^2 \frac{2\pi}{2n} + \sin^2 \frac{3\pi}{2n} + \dots + \sin^2 \frac{n\pi}{2n} \right] \{ \text{Ans. } \frac{1}{2} \}$
- ix.  $\lim_{n \rightarrow \infty} \left[ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{an} \right]$  where  $a$  is a positive integer.  $\{ \text{Ans. } \ln a \}$
- x.  $\lim_{n \rightarrow \infty} \left[ \frac{1}{na} + \frac{1}{na+1} + \frac{1}{na+2} + \dots + \frac{1}{nb} \right] \cdot \{ \text{Ans. } \ln \left( \frac{b}{a} \right) \}$

297. Prove the following:-

- i.  $\lim_{n \rightarrow \infty} \left[ \frac{\sqrt{n+1} + \sqrt{n+2} + \dots + \sqrt{2n}}{n\sqrt{n}} \right] = \frac{2}{3}(2\sqrt{2} - 1).$
- ii.  $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \sqrt{\frac{n+r}{n-r}} = \frac{\pi}{2} + 1.$
- iii.  $\lim_{n \rightarrow \infty} \left[ \frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{1}{8n} \right] = \frac{3}{8}.$
- iv.  $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{\sqrt{n^2 + r^2}} = \ln(1 + \sqrt{2}).$
- v.  $\lim_{n \rightarrow \infty} \left[ \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n}}{n\sqrt{n}} \right] = \frac{2}{3}.$
- vi.  $\lim_{n \rightarrow \infty} \frac{1}{n} \left[ \sec^2 \frac{\pi}{4n} + \sec^2 \frac{2\pi}{4n} + \dots + \sec^2 \frac{n\pi}{4n} \right] = \frac{4}{\pi}.$
- vii.  $\lim_{n \rightarrow \infty} \frac{1}{n} \left[ \tan \frac{\pi}{4n} + \tan \frac{2\pi}{4n} + \dots + \tan \frac{n\pi}{4n} \right] = \frac{2}{\pi} \ln 2.$
298. Evaluate  $\lim_{n \rightarrow \infty} \left[ \left( 1 + \frac{1}{n^2} \right) \left( 1 + \frac{2^2}{n^2} \right) \dots \left( 1 + \frac{n^2}{n^2} \right) \right]^{1/n} \{ \text{Ans. } 2e^{\frac{\pi-4}{2}} \}$
299. Prove  $\lim_{n \rightarrow \infty} \frac{1}{n} [(n+1)(n+2) \dots (n+n)]^{1/n} = \frac{4}{e}.$
300. Evaluate  $\lim_{n \rightarrow \infty} \left[ \frac{n!}{n^n} \right]^{1/n} \{ \text{Ans. } \frac{1}{e} \}$

### CATEGORY-22.16. AVERAGE (MEAN) VALUE OF A FUNCTION IN AN INTERVAL

301. Compute the mean value of the function  $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$  on the interval  $[1, 4]$ .  $\{ \text{Ans. } \frac{20}{9} \}$
302. Compute the mean value of the function  $f(x) = \frac{1}{x^2 + x}$  on the interval  $\left[ 1, \frac{3}{2} \right]$ .  $\{ \text{Ans. } 2 \ln \frac{6}{5} \}$
303. Compute the mean value of the functions  $f(x) = \sin x$  and  $f(x) = \sin^2 x$  on the interval  $[0, \pi]$ .  $\{ \text{Ans. } \}$

$$\frac{2}{\pi}, \frac{1}{2} \}$$

304. Find the mean value of the function  $f(x) = \frac{2}{e^x + 1}$  on the interval  $[0, 2]$ . {Ans.  $2 + \ln \frac{2}{e^2 + 1}$ }

305. For what  $a$  the mean value of the function  $f(x) = \ln x$  on the interval  $[1, a]$  is equal to the average rate of change of the function in this interval? {Ans.  $a = e$ }

### CATEGORY-22.17. ADDITIONAL QUESTIONS

306. If  $\frac{d}{dx} f(x) = g(x)$  for  $a \leq x \leq b$ , then find  $\int_a^b f(x)g(x)dx$ . {Ans.  $\frac{[f(b)]^2 - [f(a)]^2}{2}$ }

307. If  $\int_0^{\frac{\pi}{3}} \frac{\cos x}{3 + 4 \sin x} dx = k \ln \left( \frac{3 + 2\sqrt{3}}{3} \right)$ , then find  $k$ . {Ans.  $\frac{1}{4}$ }

308. If  $f(x) = ae^{2x} + be^x + cx$  satisfies the conditions  $f(0) = -1$ ,  $f'(\ln 2) = 31$ ,  $\int_0^{\ln 4} (f(x) - cx)dx = \frac{39}{2}$ , then find  $a, b, c$ . {Ans.  $a = 5, b = -6, c = 3$ }

309. Find the values of  $a$  for which  $\int_0^a (3x^2 + 4x - 5)dx < a^3 - 2$ . {Ans.  $\frac{1}{2} < a < 2$ }

310. If  $\int_0^2 (x - \log_2 a)dx = 2 \log_2 \left( \frac{2}{a} \right)$ , then find the value of  $a$ . {Ans.  $a > 0$ }

311. If  $\int_1^a (a - 4x)dx \geq 6 - 5a, a > 1$ , then find  $a$ . {Ans. 2}

312. A cubic function  $f(x)$  vanishes at  $x = -2$  and has relative minimum/maximum at  $x = -1$  and  $x = \frac{1}{3}$  such that  $\int_{-1}^1 f(x)dx = \frac{14}{3}$ , then find  $f(x)$ . {Ans.  $x^3 + x^2 - x + 2$ }

313. If  $f(x) = A \sin \left( \frac{\pi x}{2} \right) + B$ ,  $f' \left( \frac{1}{2} \right) = \sqrt{2}$  and  $\int_0^1 f(x)dx = \frac{2A}{\pi}$  then find the value of the constants  $A$  and  $B$ .  
{Ans.  $A = \frac{4}{\pi}, B = 0$ }

314. If  $\int_{-1}^4 f(x)dx = 4$  and  $\int_2^4 (3 - f(x))dx = 7$ , then find the value of  $\int_2^{-1} f(x)dx$ . {Ans. -5}

315. Let  $f(x), x \geq 0$  be a non-negative continuous function and let  $F(x) = \int_0^x f(t)dt, x \geq 0$ . If for some  $c > 0$ ,  $f(x) \leq cF(x) \forall x \geq 0$ , then show that  $f(x) = 0 \forall x \geq 0$ .

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# Mathematics for IIT-JEE

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*PART-VI*  
*INTEGRAL CALCULUS*

*CHAPTER-23*  
*DIFFERENTIAL EQUATIONS*

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## ***CHAPTER-23***

### ***DIFFERENTIAL EQUATIONS***

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#### **LIST OF THEORY SECTIONS**

- 23.1. Definition, Order, Degree, Solution Of Differential Equation
- 23.2. First-Order Differential Equation Of Degree One
- 23.3. Differential Equation Of Higher Order And Higher Degree

#### **LIST OF QUESTION CATEGORIES**

- 23.1. Order And Degree Of A Differential Equation
- 23.2. Formation Of Differential Equation From Its General Solution
- 23.3. Variable Separable
- 23.4. Homogeneous Equations
- 23.5. Equations Reducible To Variable Separable And Homogeneous Form
- 23.6. Linear Differential Equations
- 23.7. Bernoulli's Equation
- 23.8. Miscellaneous First Order Differential Equations Of Degree One
- 23.9. First-Order Differential Equation Of Higher Degree
- 23.10. Second-Order Differential Equation
- 23.11. Additional Questions

## CHAPTER-23

### DIFFERENTIAL EQUATIONS

#### SECTION-23.1. DEFINITION, ORDER, DEGREE, SOLUTION OF DIFFERENTIAL EQUATION

##### 1. Differential equation

An equation containing two related variables and their derivatives is called a differential equation, i.e. a differential equation in two variables  $x$  and  $y$  is of the form  $f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right) = 0$ .

##### 2. Order of a differential equation

The order of a differential equation is the order of the highest order derivative appearing in the equation.

##### 3. Degree of a differential equation

The degree of a differential equation is the power of the highest order derivative, when derivatives are made free from radicals and fractions.

##### 4. Solution of a differential equation

- i. The solution of a differential equation is a relation between the two variables which satisfies the differential equation, i.e. the differential equation becomes an identity.
- ii. A differential equation has infinite solutions.

##### 5. General solution

- i. The solution which contains all the solutions of a differential equation, is called the general solution.
- ii. Number of arbitrary constants in a general solution is equal to the order of the differential equation.

##### 6. Particular solution

Solution obtained by giving particular values to the arbitrary constants in the general solution, is called a particular solution.

##### 7. Boundary conditions

An  $n^{\text{th}}$  order differential equation has  $n$  arbitrary constants in its general solution and therefore  $n$  conditions are required to get a particular solution, which are called boundary conditions. This particular solution satisfies the given differential equation and the given  $n$  conditions.

##### 8. Formation of differential equation from its general solution

Consider a general solution containing  $n$  arbitrary constants. Therefore, it must be the general solution of an  $n^{\text{th}}$  order differential equation. Differentiating the equation  $n$  times we get  $n$  additional equations.

Eliminating  $n$  arbitrary constants from these  $(n+1)$  equations, we obtain the required differential equation of order  $n$ .

#### SECTION-23.2. FIRST-ORDER DIFFERENTIAL EQUATION OF DEGREE ONE

##### 1. Differential equations in variable separable form

If a differential equation can be put in the form  $f(y)dy = g(x)dx$ , then the variables are separable and such equations can be solved by integrating both sides.

$$\int f(y)dy = \int g(x)dx$$

$$\Rightarrow F(y) = G(x) + C$$

##### 2. Substitution

- i. Given a differential equation  $f\left(x, y, \frac{dy}{dx}\right) = 0$ , substituting  $y = \phi(v) \Rightarrow \frac{dy}{dx} = \phi'(v) \cdot \frac{dv}{dx}$ , the differential

equation becomes  $f\left(x, v, \frac{dv}{dx}\right) = 0$ .

ii. Given a differential equation  $f\left(x, y, \frac{dy}{dx}\right) = 0$ , substituting  $x = \phi(u) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{1}{\phi'(u)}$ , the differential equation becomes  $f\left(u, y, \frac{dy}{du}\right) = 0$ .

iii. Given a differential equation  $f\left(x, y, \frac{dy}{dx}\right) = 0$ , substituting  $y = \phi(v)$  and  $x = \phi(u) \Rightarrow \frac{dy}{dx} = \frac{\phi'(v)}{\phi'(u)} \cdot \frac{dv}{du}$ , the differential equation becomes  $f\left(u, v, \frac{dv}{du}\right) = 0$ .

### 3. Homogeneous differential equation

If a differential equation can be put in the form  $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ , then it is said to be homogeneous and such

equations can be solved by substituting  $\frac{y}{x} = v \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$ , the equation becomes

$v + x \frac{dv}{dx} = f(v) \Rightarrow \frac{dv}{f(v) - v} = \frac{dx}{x}$ , which is variable separable.

### 4. Equation reducible to variable separable and homogeneous form

i. Equation of the form  $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$  can be reduced to variable separable or homogeneous form by substitution.

ii. Case I:- If  $\frac{a_2}{a_1} = \frac{b_2}{b_1} = k$ , then  $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} = \frac{a_1x + b_1y + c_1}{k(a_1x + b_1y) + c_2}$ .

Substituting  $a_1x + b_1y = v \Rightarrow \frac{dv}{dx} = a_1 + b_1 \frac{dy}{dx}$ , the equation becomes

$\frac{1}{b_1} \left( \frac{dv}{dx} - a_1 \right) = \frac{v + c_1}{kv + c_2} \Rightarrow \frac{dv}{dx} = a_1 + b_1 \left( \frac{v + c_1}{kv + c_2} \right)$ , which is variable separable.

iii. Case II:- If  $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$ , then substitute  $y = Y + k$  and  $x = X + h$ , where  $h$  and  $k$  are constants to be chosen

later. Now,  $\frac{dy}{dx} = \frac{dY}{dX}$  and the equation becomes  $\frac{dY}{dX} = \frac{(a_1X + b_1Y) + a_1h + b_1k + c_1}{(a_2X + b_2Y) + a_2h + b_2k + c_2}$ .

Now choose  $h, k$  such that

$\begin{cases} a_1h + b_1k + c_1 = 0 \\ a_2h + b_2k + c_2 = 0 \end{cases}$ , then the equation will be reduced to

$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$ , which is homogeneous differential equation.

### 5. Linear differential equations

i. If a differential equation can be put in the form

$\frac{dy}{dx} + P(x)y = Q(x)$ , then it is said to be a linear differential equation.

- ii. Find Integrating factor,  $I(x) = e^{\int P(x)dx}$
- iii. Multiplying both the sides by  $I(x)$ , we get

$$I(x)\frac{dy}{dx} + I(x)P(x)y = Q(x)I(x)$$

$$\Rightarrow \frac{d}{dx}(I(x)y) = Q(x)I(x)$$

$$\Rightarrow \int d(I(x)y) = \int Q(x)I(x)dx$$

$$\Rightarrow I(x)y = \int Q(x)I(x)dx + C$$

#### 6. Linear differential equation of the form $\frac{dx}{dy} + R(y)x = S(y)$

- i. Find Integrating factor,  $I(y) = e^{\int R(y)dy}$
- ii. Multiplying both the sides by  $I(y)$ , we get

$$I(y)\frac{dx}{dy} + I(y)R(y)x = S(y)I(y)$$

$$\Rightarrow \frac{d}{dy}(I(y)x) = S(y)I(y)$$

$$\Rightarrow \int d(I(y)x) = \int S(y)I(y)dy$$

$$\Rightarrow I(y)x = \int S(y)I(y)dy + C$$

#### 7. Equations reducible to Linear form (Bernoulli's differential equations)

- i. Differential equations of the type

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \text{ are Bernoulli's equations.}$$

- ii. Dividing both sides by  $y^n$ , we get

$$y^{-n}\frac{dy}{dx} + P(x)y^{-n+1} = Q(x)$$

- iii. Substituting  $y^{-n+1} = v \Rightarrow (-n+1)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$ , we get

$$\frac{1}{(-n+1)}\frac{dv}{dx} + P(x)v = Q(x)$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x), \text{ which is a linear differential equation.}$$

- iv. Bernoulli's equation may be of the form  $\frac{dx}{dy} + P(y)x = Q(y)x^n$ .

### SECTION-23.3. DIFFERENTIAL EQUATION OF HIGHER ORDER AND HIGHER DEGREE

#### 1. First-order differential equation of higher degree

- i. If the differential equation is a polynomial in  $\frac{dy}{dx}$  or solvable in  $\frac{dy}{dx}$ , then solve for  $\frac{dy}{dx}$  and put in the form  $\frac{dy}{dx} = \phi(x, y)$  and then solve as first-order differential equation of degree one.
- ii. If the differential equation can be put in the form  $y = \phi(p, x)$ , where  $p = \frac{dy}{dx}$ , then differentiating w.r.t.  $x$  we get  $p = \phi_1\left(\frac{dp}{dx}, p, x\right)$ , which is a first-order differential equation in  $p$  and  $x$ , and after solving we get its solution as  $\phi_2(p, x) = 0$ . Eliminating  $p$  from  $y = \phi(p, x)$  and  $\phi_2(p, x) = 0$ , we get the solution as  $\phi_3(x, y) = 0$ .
- iii. If the differential equation can be put in the form  $x = \phi(p, y)$ , where  $p = \frac{dy}{dx}$ , then differentiating w.r.t.  $y$  we get  $\frac{1}{p} = \phi_1\left(\frac{dp}{dy}, p, y\right)$ , which is a first-order differential equation in  $p$  and  $y$ , and after solving we get its solution as  $\phi_2(p, y) = 0$ . Eliminating  $p$  from  $x = \phi(p, y)$  and  $\phi_2(p, y) = 0$ , we get the solution as  $\phi_3(x, y) = 0$ .

## 2. Second-order differential equation

- i. If second-order differential equation is of the form  $\phi\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0$ , then substituting  $\frac{dy}{dx} = p$  and solving  $\phi\left(\frac{dp}{dx}, p, x\right) = 0$ , which is a first-order differential equation in  $p$  and  $x$ , we get its solution as  $\phi_1(p, x) = 0$ . Now solving  $\phi_1\left(\frac{dy}{dx}, x\right) = 0$ , we get the solution as  $\phi_2(x, y) = 0$ .
- ii. If second-order differential equation is of the form  $\phi\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0$ , then substituting  $\frac{dy}{dx} = p$  and  $\frac{d^2y}{dx^2} = p \frac{dp}{dy}$  and solving  $\phi\left(\frac{dp}{dy}, p, y\right) = 0$ , which is a first-order differential equation in  $p$  and  $y$ , we get its solution as  $\phi_1(p, y) = 0$ . Now solving  $\phi_1\left(\frac{dy}{dx}, y\right) = 0$ , we get the solution as  $\phi_2(x, y) = 0$ .

**EXERCISE-23****CATEGORY-23.1. ORDER AND DEGREE OF A DIFFERENTIAL EQUATION**

1. What is the order and degree of the differential equation  $\frac{d^3 y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$ . {Ans. 3, 1}
2. What is the order and degree of the differential equation  $x\left(\frac{d^3 y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^4 + y^2 = 0$ . {Ans. 3, 2}
3. What is the order and degree of the differential equation  $y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ . {Ans. 1, 2}
4. What is the order and degree of the differential equation  $y_3^{\frac{2}{3}} + 2 + 3y_2 + y_1 = 0$ . {Ans. 3, 2}
5. What is the order and degree of the differential equation  $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = k\frac{d^2 y}{dx^2}$ . {Ans. 2, 2}

**CATEGORY-23.2. FORMATION OF DIFFERENTIAL EQUATION FROM ITS GENERAL SOLUTION**

6. Find the differential equations whose solutions are  $y = A \cos(x + 3)$ ,  $A$  being constant. {Ans.  $\frac{dy}{dx} + y \tan(x + 3) = 0$ }
7. Find the differential equations whose solutions are  $y = x \sin(x + A)$ ,  $A$  being constant. {Ans.  $\left(x\frac{dy}{dx} - y\right)^2 + x^2 y^2 = x^4$ }
8. What is the degree of the differential equation satisfying  $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$ . {Ans. 1}
9. Find the differential equations of all circles passing through origin and having their centres on the  $x$ -axis.  
{Ans.  $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ }
10. Find the differential equations of all lines passing through origin  $y = mx$ . {Ans.  $y = x\frac{dy}{dx}$ }
11. Find the differential equations of all circles which have their centres on  $x$ -axis and have given radius  $a$ .  
{Ans.  $\pm\sqrt{a^2 - y^2} + y\frac{dy}{dx} = 0$ }
12. Prove that the differential equation of the family of parabolas  $y^2 = 4ax$  is  $2x\frac{dy}{dx} - y = 0$ .
13. Show that the differential equation of the family of circles of fixed radius  $r$  with centres on  $y$ -axis is  $(x^2 - r^2)\left(\frac{dy}{dx}\right)^2 + x^2 = 0$ .
14. Find the differential equation corresponding to the family of curves  $y = k(x - k)^2$  where  $k$  is an arbitrary

- constant. { Ans.  $\left(\frac{dy}{dx}\right)^3 - 4xy\frac{dy}{dx} + 8y^2 = 0$  }
15. From the differential equation of the circles represented by  $y^2 - 2ay + x^2 = a^2$ ,  $a$  being arbitrary constant.  
{ Ans.  $(2y^2 - x^2)\left(\frac{dy}{dx}\right)^2 + 4xy\frac{dy}{dx} + x^2 = 0$  }
16. Obtain the differential equation of which  $y = ax^2 + bx$  is the general solution,  $a, b$  being arbitrary constants. { Ans.  $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$  }
17. Find the differential equation whose solution is  $y = Ae^{2x} + Be^{-2x}$ . { Ans.  $\frac{d^2y}{dx^2} - 4y = 0$  }
18. Find the differential equation whose solution is  $y = A \sin 2x + B \cos 2x$ . { Ans.  $\frac{d^2y}{dx^2} + 4y = 0$  }
19. Find the differential equation whose solution is  $Ax^2 + By^2 = 1$ . { Ans.  $x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx}$  }
20. Find the differential equation whose solution is  $v = \frac{A}{r} + B$ . { Ans.  $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$  }
21. Find the differential equation of the family of curves  $xy = Ae^x + Be^{-x} + x^2$ . { Ans.  $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2$  }
22. Find the differential equation of the family of curves  $y = Ae^{3x} + Be^{5x}$  for different values of  $A$  and  $B$ . { Ans.  $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$  }
23. Find the differential equation of straight lines  $y = mx + c$ . { Ans.  $\frac{d^2y}{dx^2} = 0$  }
24. Find the differential equation having general solution  $y = a \sin x + b \cos x + x \sin x$ . { Ans.  $\frac{d^2y}{dx^2} + y = 2 \cos x$  }
25. Find the differential equations of the family of curves  $y = e^x(A \cos x + B \sin x)$  where  $A$  and  $B$  are arbitrary constants. { Ans.  $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$  }
26. Find the differential equation corresponding to  $y = ae^{2x} + be^{-3x} + ce^x$  where  $a, b, c$  are arbitrary constants.  
{ Ans.  $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$  }
27. Find the differential equation associated to the primitive  $y = a + be^{5x} + ce^{-7x}$ . { Ans.  $y_3 + 2y_2 - 35y_1 = 0$  }
28. Find the differential equation having general solution  $y = a \cos x + b \sin x + ce^{-x}$ .
29. Find the differential equation having general solution  $x^2 + y^2 + 2ax + 2by + c = 0$ , where  $a, b, c$  are



arbitrary constants. {Ans.  $\frac{d^3y}{dx^3} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right] = 3 \frac{dy}{dx} \left( \frac{d^2y}{dx^2} \right)^2$  }

### CATEGORY-23.3. VARIABLE SEPARABLE

30.  $\frac{dy}{dx} = \frac{x^2}{y^2}$ . {Ans.  $x^3 - y^3 = C$  }
31.  $(1 + y^2)dx + (1 + x^2)dy = 0$ . {Ans.  $x + y = C(1 - xy)$  }
32.  $(1 - x)dy - (3 + y)dx = 0$ . {Ans.  $(3 + y)(1 - x) = K$  }
33.  $(e^x + 1)ydy = (y + 1)e^x dx$  {Ans.  $k(y + 1)(e^x + 1) = e^y$  }
34.  $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$ . {Ans.  $-e^{-y} = e^x + \frac{1}{3}e^{x^3} + C$  }
35.  $x \cos^2 y dx = y \cos^2 x dy$ . {Ans.  $x \tan x - \ln \sec x = y \tan y - \ln \sec y + C$  }
36.  $(1 - x^2)(1 - y)dx = xy(1 + y)dy$ . {Ans.  $\ln x - \frac{x^2}{2} = -\frac{y^2}{2} - 2y - 2\ln(1 - y) + k$  }
37.  $(1 + x)y dx + (1 + y)x dy = 0$ . {Ans.  $\ln xy + (x + y) = C$  }
38.  $\frac{dy}{dx} + \sqrt{\frac{1 - y^2}{1 - x^2}} = 0$ . {Ans.  $\sin^{-1} y + \sin^{-1} x = C$  }
39.  $(x^2 - yx^2)\frac{dy}{dx} + (y^2 + xy^2) = 0$  {Ans.  $\ln \frac{x}{ky} = \frac{x + y}{xy}$  }
40.  $y \sec^2 x + (y + 7) \tan x \frac{dy}{dx} = 0$ . {Ans.  $y^7 \tan x = k e^{-y}$  }
41.  $xy \frac{dy}{dx} = \frac{1 + y^2}{1 + x^2} (1 + x + x^2)$ . {Ans.  $\frac{1}{2} \ln(1 + y^2) = \ln x + \tan^{-1} x + C$  }
42.  $\frac{dy}{dx} \tan y = \sin(x + y) + \sin(x - y)$  {Ans.  $\sec y = -2 \cos x + C$  }
43.  $\frac{dy}{dx} = \frac{1 + y^2}{(1 + x^2)xy}$ . {Ans.  $(1 + x^2)(1 + y^2) = kx^2$  }
44.  $\cos(x + y)dy = dx$ . {Ans.  $y - c = \tan\left(\frac{x + y}{2}\right)$  }
45.  $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ . {Ans.  $1 + \tan \frac{x + y}{2} = -\frac{2}{x + c}$  }
46.  $\frac{dy}{dx} = (4x + y + 1)^2$  {Ans.  $4x + y + 1 = 2 \tan(2x + k)$  }
47.  $\frac{dy}{dx} = (x + y)^2$  {Ans.  $x + y = \tan(x + c)$  }
48.  $\frac{dy}{dx} + 1 = e^{x+y}$ . {Ans.  $(x + c)e^{x+y} = -1$  }

49.  $\frac{dy}{dx} - x \tan(y-x) = 1$ . {Ans.  $\sin(y-x) = ke^{\frac{1}{2}x^2}$ }

50. If  $\frac{dy}{dx} = e^{-2y}$  and  $y = 0$  when  $x = 5$ , then find the value of  $x$  for  $y = 3$ . {Ans.  $\frac{e^6 + 9}{2}$ }

**CATEGORY-23.4. HOMOGENEOUS EQUATIONS**

51.  $(x+y)dx + (y-x)dy = 0$  {Ans.  $\tan^{-1} \frac{y}{x} - \frac{1}{2} \ln(x^2 + y^2) = C$ }

52.  $\frac{dy}{dx} = \frac{x-y}{x+y}$  {Ans.  $y^2 + 2xy - x^2 = k$ }

53.  $\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$  {Ans.  $(x+y)^3 = C(x-y)$ }

54.  $(x^2 + y^2) \frac{dy}{dx} = xy$ . {Ans.  $e^{\frac{x^2}{2y^2}} = ky$ }

55.  $(x^2 - y^2)dx + 2xy dy = 0$ . {Ans.  $x^2 + y^2 = kx$ }

56.  $2xy dy = (x^2 + y^2)dx$ . {Ans.  $x = k(x^2 - y^2)$ }

57.  $(x^2 - y^2)dx - xy dy = 0$ . {Ans.  $x^2(x^2 - 2y^2) = k$ }

58.  $(x^2 + xy)dy = (x^2 + y^2)dx$ . {Ans.  $k(y-x)^2 = xe^{-\frac{y}{x}}$ }

59.  $\frac{dy}{dx} = \frac{y^3 + 3x^2y}{x^3 + 3xy^2}$ . {Ans.  $xy = k^2(x^2 - y^2)^2$ }

60.  $(x^3 - 2y^3)dx + 3xy^2 dy = 0$ . {Ans.  $x^3 + y^3 = kx^2$ }

61.  $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy$ . {Ans.  $ky^3e^{\frac{x}{y}} = x^2$ }

62.  $x \frac{dy}{dx} = y + 2\sqrt{y^2 - x^2}$ . {Ans.  $y + \sqrt{y^2 - x^2} = kx^3$ }

63.  $(x\sqrt{x^2 + y^2} - y^2)dx + xy dy = 0$  {Ans.  $\sqrt{x^2 + y^2} = x \ln \frac{k}{x}$ }

64.  $x \frac{dy}{dx} = y - x \cos^2 \frac{y}{x}$ . {Ans.  $\tan \frac{y}{x} = \ln \frac{k}{x}$ }

65.  $\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x}$  {Ans.  $\sin(\frac{y}{x}) = kx$ }

66.  $\frac{dy}{dx} = \frac{y}{x} + \sin \frac{y}{x}$  {Ans.  $\tan \frac{y}{2x} = kx$ }

67.  $x \frac{dy}{dx} = y(\ln y - \ln x + 1)$  {Ans.  $y = xe^{kx}$ }

**CATEGORY-23.5. EQUATIONS REDUCIBLE TO VARIABLE SEPARABLE AND HOMOGENEOUS FORM**

68.  $\frac{dy}{dx} = \frac{2y + x - 1}{2x + 4y + 3}$ . {Ans.  $2y - x + \frac{5}{4} \ln(8y + 4x + 1) = C$ }

69.  $(2x + y + 1) dx + (4x + 2y - 1) dy = 0$ . {Ans.  $2y + x + \ln(2x + y - 1) = k$ }
70.  $\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$  when  $y = \frac{1}{3}$  at  $x = \frac{2}{3}$ . {Ans.  $y - x + \frac{1}{3} = \ln(x + y)$ }
71.  $\frac{dy}{dx} = \frac{2x + 2y - 2}{3x + y - 5}$ . {Ans.  $(2x + y - 3) = k(x - y - 3)^4$ }
72.  $\frac{dy}{dx} = \frac{x + 2y - 2}{2x + y - 3}$ . {Ans.  $(3x + 3y - 5) = k(x - y - 1)^3$ }
73.  $(2x + 3y - 5) dy + (3x + 2y - 5) dx = 0$ . {Ans.  $4xy + 3(x^2 + y^2) - 10(x + y) = k$ }
74.  $2(x - 3y + 1) \frac{dy}{dx} = 4x - 2y + 1$ . {Ans.  $2xy + 2y - 3y^2 - x - 2x^2 = k$ }
75.  $(2x - y + 1) dx + (2y - x - 1) dy = 0$ . {Ans.  $x^2 + y^2 - xy + (x - y) = C$ }
76.  $(x + 2y - 2) dx + (2x - y + 3) dy = 0$ . {Ans.  $x^2 - y^2 + 4xy - 4x + 6y = k$ }

### CATEGORY-23.6. LINEAR DIFFERENTIAL EQUATIONS

77.  $\frac{dy}{dx} + \frac{4x}{x^2 + 1} y = \frac{1}{(x^2 + 1)^3}$  {Ans.  $y(x^2 + 1)^2 = \tan^{-1} x + C$ }
78.  $2 \cos x \frac{dy}{dx} + 4y \sin x = \sin 2x$  {Ans.  $y \sec^2 x = \sec x + C$ }
79.  $x(x - 1) \frac{dy}{dx} - y = x^2(x - 1)^2$  {Ans.  $\frac{yx}{x - 1} = \frac{x^3}{3} + C$ }
80.  $\frac{ds}{dt} = -s + t$  {Ans.  $s = t - 1 + Ce^{-t}$ }
81.  $\frac{dv}{dt} + \frac{k}{m} v = -g$ . {Ans.  $v = ce^{\frac{-k}{m}t} - \frac{mg}{k}$ }
82.  $(x + y + 1) \frac{dy}{dx} = 1$  {Ans.  $x = ke^y - y - 2$ }
83.  $(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$  {Ans.  $2xe^{\tan^{-1} y} = e^{2 \tan^{-1} y} + k$ }
84.  $y^2 + \left(x - \frac{1}{y}\right) \frac{dy}{dx} = 0$  {Ans.  $x = \left(\frac{1}{y} + 1\right) + Ce^{\frac{1}{y}}$ }
85.  $x \cos x \frac{dy}{dx} + y(x \sin x + \cos x) = 1$  {Ans.  $yx \sec x = \tan x + C$ }
86.  $x \ln x \frac{dy}{dx} + y = 2 \ln x$  {Ans.  $y \ln x = (\ln x)^2 + C$ }
87.  $(2x - 10y^3) \frac{dy}{dx} + y = 0$  {Ans.  $x = 2y^3 + Cy^{-2}$ }
88.  $dx + xdy = e^{-y} \sec^2 y dy$  {Ans.  $xe^y = \tan y + C$ }
89.  $(1 + y + x^2 y) dx + (x + x^3) dy = 0$  {Ans.  $xy = -\tan^{-1} x + C$ }

90.  $(1-x^2)\frac{dy}{dx} - xy = \frac{1}{\sqrt{1-x^2}}$  {Ans.  $y\sqrt{1-x^2} = \ln\left(\frac{1+x}{1-x}\right) + C$ }
91.  $(x^2-1)\frac{dy}{dx} + 2(x+2)y = 2(x+1)$  {Ans.  $\frac{y(x-1)^3}{x+1} = \frac{(x+1)^2}{2} - 4(x+1) + 4\ln(x+1) + C$ }
92.  $(x + \tan y)dy = \sin 2y dx$  {Ans.  $x = \tan y + C\sqrt{\tan y}$ }
93.  $\sin x \frac{dy}{dx} + 2y + \sin x(1 + \cos x) = 0$  {Ans.  $y \tan^2 \frac{x}{2} = -x + \sin x + C$ }
94.  $\frac{dy}{dx} = \frac{x\sqrt{x^2-1} + y}{\sqrt{x^2-1}}$ . Given that  $y = 1$  when  $x = 1$ . {Ans.  $y[x - \sqrt{x^2-1}] = \frac{1}{3}\left[x^3 - (x^2-1)^{\frac{3}{2}}\right] + \frac{2}{3}$ }

### CATEGORY-23.7. BERNOULLI'S EQUATION

95.  $\frac{dy}{dx} + y = x^2 y^2$ . {Ans.  $\frac{1}{y} = x^2 + 2x + 2 - ce^x$ }
96.  $\frac{dy}{dx} + \frac{y}{x} = y^2$ . {Ans.  $y = \frac{1}{x(c - \ln x)}$ }
97.  $2\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$ . {Ans.  $x = y - cy\sqrt{x}$ }
98.  $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ . {Ans.  $\frac{1}{xy} = \frac{1}{2x^2} + c$ }
99.  $3\frac{dy}{dx} + \frac{2y}{x+1} = \frac{x^3}{y^2}$ . {Ans.  $y^3(1+x)^2 = \frac{1}{6}x^6 + \frac{2}{5}x^5 + \frac{1}{4}x^4 + C$ }
100.  $\frac{dy}{dx} + 2y \tan x = y^2$ . {Ans.  $\cos^2 x + \left(\frac{x}{2} + \frac{\sin 2x}{4}\right)y = cy$ }
101.  $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ . {Ans.  $x = e^y\left(cx^2 + \frac{1}{2}\right)$ }
102.  $\frac{dy}{dx} + y \cot x = y^2 \sin^2 x$ . {Ans.  $y = \frac{1}{\sin x(\cos x - c)}$ }
103.  $x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$  given  $y(0) = 1$ . {Ans.  $x^3 = 3y^3 \sin x$ }

### CATEGORY-23.8. MISCELLANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS OF DEGREE ONE

104.  $2xydy - (x^2 + y^2 + 1)dx = 0$ . {Ans.  $x^2 - y^2 - 1 + cx = 0$ }
105.  $(x^2 + y^2 + 2x)dx + 2ydy = 0$ . {Ans.  $e^x(x^2 + y^2) = c$ }
106.  $(x^2 - 2x + 2y^2)dx + 2xydy = 0$ . {Ans.  $x^2 y^2 = \frac{2}{3}x^3 - \frac{x^4}{4} + C$ }
107.  $\frac{dy}{dx} + yx = y^2 e^{\frac{x^2}{2}} \sin x$ . {Ans.  $ye^{\frac{x^2}{2}}(c - \cos x) + 1 = 0$ }

108.  $\frac{dy}{dx} + yx = y^2 e^{\frac{x^2}{2}} \ln x$ . {Ans.  $ye^{\frac{x^2}{2}}(x \ln x - x + c) + 1 = 0$ }
109.  $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ . {Ans.  $\sin y = (1+x)(e^x + c)$ }
110.  $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^3 x$ . {Ans.  $\sec x \sec y = \frac{x}{2} + \frac{\sin 2x}{4} + C$ }
111.  $\sec^2 y \frac{dy}{dx} + \tan y = x^3$ . {Ans.  $\tan y = x^3 - 3x^2 + 6x - 6 + Ce^{-x}$ }
112.  $x \frac{dy}{dx} + y \ln y = xye^x$ . {Ans.  $x \ln y = e^x(x-1) + C$ }
113.  $y(2xy + e^x)dx - e^x dy = 0$ . {Ans.  $y(x^2 + c) + e^x = 0$ }
114.  $\cos x dy = y(\sin x - y)dx$ . {Ans.  $\sec x = y(c + \tan x)$ }
115.  $\frac{dy}{dx} + y \cot x = y^2 \sin^2 x \cos^2 x$ . {Ans.  $\frac{\cos ecx}{y} = \frac{1}{3} \cos^3 x + C$ }
116.  $\frac{dy}{dx} + \frac{1}{x} \tan y = \frac{1}{x^2} \tan y \sin y$ . {Ans.  $\sin y(1 - 2Cx^2) = 2x$ }
117.  $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$ . {Ans.  $\sec y \sec x = \sin x + C$ }
118.  $\frac{dz}{dx} + \frac{z}{x} \ln z = \frac{z}{x^2} (\ln z)^2$ . {Ans.  $\frac{1}{x \ln z} = \frac{1}{2x^2} - C$ }
119.  $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ . {Ans.  $\tan y = \frac{1}{2}(x^2 - 1) + \frac{3}{2}e^{-x^2}$ }
120.  $(1+x^2)\frac{dy}{dx} - 4x^2 \cos^2 y + x \sin 2y = 0$ . {Ans.  $\tan y(1+x^2) = \frac{4}{3}x^3 + C$ }
121.  $\frac{dy}{dx} + \frac{1}{x} \sin 2y = x^3 \cos^2 y$ . {Ans.  $x^2 \tan y = \frac{1}{6}x^6 + C$ }
122.  $x^2 y dx - (x^3 + y^3)dy = 0$ . {Ans.  $e^{\frac{x^3}{3y^3}} = Cy$ }
123.  $(x^3 - y^3)dx + xy^2 dy = 0$ . {Ans.  $e^{\frac{y^3}{3x^3}} = \frac{k}{x}$ }
124.  $\frac{dy}{dx} + y \cos x = y^n \sin 2x$ . {Ans.  $\frac{1}{y^{n-1}} = 2 \sin x - \frac{2}{1-n} + Ce^{(n-1)\sin x}$ }
125.  $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2 y dy = 0$  {Ans.  $3y^2 = 2x^2 e^{\frac{1}{x^3}} + Cx^2$ }
126.  $(x - y^3) + 3xy^2 \frac{dy}{dx} = 0$  {Ans.  $y^3 = -x \ln x + Cx$ }
127.  $\frac{dy}{dx} = (\sin x - \sin y) \frac{\cos x}{\cos y}$ . {Ans.  $\sin y = \sin x - 1 + Ce^{-\sin x}$ }

128.  $\sin y \frac{dy}{dx} = \cos y(1 - x \cos y)$ . {Ans.  $\sec y = x + 1 + Ce^x$ }
129.  $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$ . {Ans.  $e^y = e^x - 1 + ce^{-e^x}$ }
130.  $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$ . {Ans.  $\frac{\sec^2 x}{y} = -\frac{1}{3} \tan^3 x + C$ }
131.  $(1 - x^2) \frac{dy}{dx} + xy = xy^2$ . {Ans.  $y = \frac{ky}{\sqrt{1-x^2}} + 1$ }
132.  $(1 - x^2) \frac{dy}{dx} + xy = a$ . {Ans.  $y = ax + C\sqrt{1-x^2}$ }
133.  $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$ . {Ans.  $\frac{x}{y} = -e^{x^3} + C$ }
134.  $\frac{dy}{dx} + (2x \tan^{-1} y - x^3)(1 + y^2) = 0$ . {Ans.  $2 \tan^{-1} y = x^2 - 1 + 2ce^{-x^2}$ }
135.  $2 \sin x \frac{dy}{dx} - y \cos x = xy^3e^x$ . {Ans.  $\frac{\sin x}{y^2} = (1-x)e^x + C$ }
136.  $(1 + x^2) \frac{dy}{dx} + xy = x^3y^3$ . {Ans.  $\frac{1}{y^2} = -(1+x^2)\ln(1+x^2) - 1 + C(1+x^2)$ }
137.  $2 \frac{dy}{dx} - y \sec x = y^3 \tan x$ . {Ans.  $\frac{\sec x + \tan x}{y^2} = x - \sec x - \tan x + C$ }
138.  $y \sin 2x dx - (1 + y^2 + \cos^2 x) dy = 0$ . {Ans.  $y^3 + 3y(1 + \cos^2 x) = C$ }

### CATEGORY-23.9. FIRST-ORDER DIFFERENTIAL EQUATION OF HIGHER DEGREE

139.  $x \left( \frac{dy}{dx} \right)^2 + (y - x) \frac{dy}{dx} - y = 0$ . {Ans.  $y = x + c_1; xy = c_2$ }
140.  $y = xy' + (y')^2$ . {Ans.  $y = cx + c^2; y = -\frac{x^2}{4}$  or  $(y - cx - c^2)(4y + x^2) = 0$ }
141.  $y = xy' - 3(y')^3$ . {Ans.  $y = cx - 3c^3; y^2 = \frac{4x^3}{81}$ }
142.  $y = xy' + \frac{1}{y'}$ . {Ans.  $y = cx + \frac{1}{c}; y^2 = 4x$ }
143.  $y = xy' + \sqrt{1 + (y')^2}$ . {Ans.  $y = cx + \sqrt{1 + c^2}; x^2 + y^2 = 1$ }
144.  $y = xy' + \sin y'$ . {Ans.  $y = cx + \sin c; y = x(\pi - \cos^{-1} x) + \sqrt{1 - x^2}$ }
145.  $xy' - y = \ln y'$ . {Ans.  $y = cx - \ln c; y = \ln x + 1$ }

### CATEGORY-23.10. SECOND-ORDER DIFFERENTIAL EQUATION

146.  $y'' = x + \sin x$ . {Ans.  $y = \frac{x^3}{6} - \sin x + c_1x + c_2$ }

$$147. y'' = \tan^{-1} x. \{ \text{Ans. } y = \frac{(x^2 - 1)\tan^{-1} x - x \ln(1 + x^2)}{2} + c_1 x + c_2 \}$$

$$148. y'' = \ln x. \{ \text{Ans. } y = \frac{x^2}{2} \left( \ln x - \frac{3}{2} \right) + c_1 x + c_2 \}$$

$$149. xy'' = y'. \{ \text{Ans. } y = c_1 x^2 + c_2 \}$$

$$150. y'' = y' + x. \{ \text{Ans. } y = c_1 e^x + c_2 - x - \frac{x^2}{2} \}$$

$$151. y'' = \frac{y'}{x} + x. \{ \text{Ans. } y = \frac{x^3}{3} + c_1 x^2 + c_2 \}$$

$$152. (1 + x^2)y'' + (y')^2 + 1 = 0. \{ \text{Ans. } y = (1 + c_1)^2 \ln|x + c_1| - c_1 x + c_2 \}$$

$$153. xy'' = y' \ln \frac{y'}{x}. \{ \text{Ans. } y = (c_1 x - c_1^2) e^{\frac{x}{c_1} + 1} + c_2 \}$$

$$154. 2xy'y'' = (y')^2 + 1. \{ \text{Ans. } y = \frac{2}{3c_1} (c_1 x - 1)^{\frac{3}{2}} + c_2 \}$$

$$155. y'' - 2 \cot x \cdot y' = \sin^3 x. \{ \text{Ans. } y = -\frac{\sin^3 x}{3} + c_1 \left( \frac{x}{2} - \frac{\sin 2x}{4} \right) \}$$

$$156. 1 + (y')^2 = 2yy''. \{ \text{Ans. } (x + c_2)^2 = 4c_1(y - c_1) \}$$

$$157. (y')^2 + 2yy'' = 0. \{ \text{Ans. } y = c_1(x + c_2)^{\frac{2}{3}} \}$$

$$158. y'' = \frac{1}{4\sqrt{y}}. \{ \text{Ans. } x = \frac{4}{3} (\sqrt{y} - 2c_1) \sqrt{\sqrt{y} + c_1} + c_2 \}$$

$$159. y'' + \frac{2}{1-y} (y')^2 = 0. \{ \text{Ans. } y = \frac{x + c_1}{x + c_2} \}$$

$$160. yy'' + (y')^2 = 1. \{ \text{Ans. } (x + c_2)^2 - y^2 = c_1 \}$$

$$161. yy'' = (y')^2. \{ \text{Ans. } y = c_1 e^{c_2 x} \}$$

$$162. 2yy'' - 3(y')^2 = 4y^2. \{ \text{Ans. } y = c_2 \sec^2(x + c_1) \}$$

$$163. y(1 - \ln y)y'' + (1 + \ln y)(y')^2 = 0. \{ \text{Ans. } y = e^{\frac{x+c_1}{x+c_2}} \}$$

$$164. xy'' - \frac{1}{4}(y'')^2 - y' = 0. \{ \text{Ans. } y = c_1 x^2 - c_1^2 x + c_2; y = \frac{x^3}{3} + c_3 \}$$

$$165. yy'y'' = (y')^3 + (y'')^2. \{ \text{Ans. } y = c_1 + c_2 e^{c_1 x}; y = \frac{4}{c_3 - x}; y = c_4 \}$$

### CATEGORY-23.11. ADDITIONAL QUESTIONS

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# Mathematics for IIT-JEE

By **Er. Sanjiva Dayal**, *B.Tech. (I.I.T. Kanpur)*

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*PART-VI*  
*INTEGRAL CALCULUS*

*CHAPTER-24*  
*APPLICATIONS OF CALCULUS*

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## ***CHAPTER-24***

### ***APPLICATIONS OF CALCULUS***

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#### **LIST OF THEORY SECTIONS**

- 24.1. Applications Of Derivatives In Geometrical Problems
- 24.2. Area Of Regions

#### **LIST OF QUESTION CATEGORIES**

- 24.1. Tangent, Normal And Other Properties Of A Given Curve Depending On Derivative At A Point
- 24.2. Applications Of Differential Equations In Finding Equations Of Curves
- 24.3. Least Distance Between Two Curves
- 24.4. Angle Of Intersection Of Two Curves
- 24.5. Rate Of Change Problems Of Physics, Geometry Etc.
- 24.6. Greatest & Least Value Problems Of Physics, Geometry Etc.
- 24.7. Area Of Regions
- 24.8. Additional Questions

## CHAPTER-24

### APPLICATIONS OF CALCULUS

#### SECTION-24.1. APPLICATIONS OF DERIVATIVES IN GEOMETRICAL PROBLEMS

##### 1. Properties of curves depending on derivative at $(x_0, y_0)$

- i. Slope of the tangent to the curve at  $(x_0, y_0) = \left( \frac{dy}{dx} \right)_{(x_0, y_0)}$ .
- ii. Slope of the normal to the curve at  $(x_0, y_0) = - \left( \frac{dx}{dy} \right)_{(x_0, y_0)}$ .
- iii. Equation of the tangent to the curve at  $(x_0, y_0)$  is  $y - y_0 = \left( \frac{dy}{dx} \right)_{(x_0, y_0)} (x - x_0)$ .
- iv. Equation of the normal to the curve at  $(x_0, y_0)$  is  $y - y_0 = - \left( \frac{dx}{dy} \right)_{(x_0, y_0)} (x - x_0)$ .
- v. The  $x$ -intercept of the tangent at  $(x_0, y_0) = x_0 - y_0 \left( \frac{dx}{dy} \right)_{(x_0, y_0)}$ .
- vi. The  $y$ -intercept of the tangent at  $(x_0, y_0) = y_0 - x_0 \left( \frac{dy}{dx} \right)_{(x_0, y_0)}$ .
- vii. The  $x$ -intercept of the normal at  $(x_0, y_0) = x_0 + y_0 \left( \frac{dy}{dx} \right)_{(x_0, y_0)}$ .
- viii. The  $y$ -intercept of the normal at  $(x_0, y_0) = y_0 + x_0 \left( \frac{dx}{dy} \right)_{(x_0, y_0)}$ .
- ix. The length of the tangent between  $(x_0, y_0)$  and the  $x$ -axis  $= |y_0| \sqrt{1 + \left( \frac{dx}{dy} \right)_{(x_0, y_0)}^2}$ .
- x. The length of the tangent between  $(x_0, y_0)$  and the  $y$ -axis  $= |x_0| \sqrt{1 + \left( \frac{dy}{dx} \right)_{(x_0, y_0)}^2}$ .
- xi. The length of the normal between  $(x_0, y_0)$  and the  $x$ -axis  $= |y_0| \sqrt{1 + \left( \frac{dy}{dx} \right)_{(x_0, y_0)}^2}$ .
- xii. The length of the normal between  $(x_0, y_0)$  and the  $y$ -axis  $= |x_0| \sqrt{1 + \left( \frac{dx}{dy} \right)_{(x_0, y_0)}^2}$ .
- xiii. Length of the sub-tangent at  $(x_0, y_0) = \left| y_0 \left( \frac{dx}{dy} \right)_{(x_0, y_0)} \right|$ .

xiv. Length of the sub-normal at  $(x_0, y_0) = \left| y_0 \left( \frac{dy}{dx} \right)_{(x_0, y_0)} \right|$ .

## 2. Properties of curves depending on derivative at $(x, y)$

i. Slope of the tangent to the curve at  $(x, y) = \frac{dy}{dx}$ .

ii. Slope of the normal to the curve at  $(x, y) = -\frac{dx}{dy}$ .

iii. Equation of the tangent to the curve at  $(x, y)$  is  $Y - y = \frac{dy}{dx}(X - x)$ .

iv. Equation of the normal to the curve at  $(x, y)$  is  $Y - y = -\frac{dx}{dy}(X - x)$ .

v. The  $x$ -intercept of the tangent at  $(x, y) = x - y \frac{dx}{dy}$ .

vi. The  $y$ -intercept of the tangent at  $(x, y) = y - x \frac{dy}{dx}$ .

vii. The  $x$ -intercept of the normal at  $(x, y) = x + y \frac{dy}{dx}$ .

viii. The  $y$ -intercept of the normal at  $(x, y) = y + x \frac{dx}{dy}$ .

ix. The length of the tangent between  $(x, y)$  and the  $x$ -axis  $= |y| \sqrt{1 + \left( \frac{dx}{dy} \right)^2}$ .

x. The length of the tangent between  $(x, y)$  and the  $y$ -axis  $= |x| \sqrt{1 + \left( \frac{dy}{dx} \right)^2}$ .

xi. The length of the normal between  $(x, y)$  and the  $x$ -axis  $= |y| \sqrt{1 + \left( \frac{dy}{dx} \right)^2}$ .

xii. The length of the normal between  $(x, y)$  and the  $y$ -axis  $= |x| \sqrt{1 + \left( \frac{dx}{dy} \right)^2}$ .

xiii. Length of the sub-tangent at  $(x, y) = \left| y \frac{dx}{dy} \right|$ .

xiv. Length of the sub-normal at  $(x, y) = \left| y \frac{dy}{dx} \right|$ .

## 3. Least distance between two non-intersecting curves

Least distance between two non-intersecting smooth curves is the length of shortest common normal.

## 4. Angle of intersection of two curves

i. Let the curves  $C_1 : f(x, y) = 0$  and  $C_2 : g(x, y) = 0$  intersect at the point  $(x_0, y_0)$ .

ii. Slope of tangent of curve  $C_1$  at  $(x_0, y_0) = m_1 = \left( \frac{dy}{dx} \right)_{(x_0, y_0)}$

iii. Slope of tangent of curve  $C_2$  at  $(x_0, y_0) = m_2 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$

iv. Angle of intersection  $= \theta = \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$ .

v. If  $m_1 m_2 = -1 \Rightarrow \theta = 90^\circ \Rightarrow$  curves intersect orthogonally.

vi. If  $m_1 = m_2 \Rightarrow \theta = 0^\circ \Rightarrow$  curves touch or cut with common tangent at the point of intersection.

### SECTION-24.2. AREA OF REGIONS

1. Area bounded by curves  $y = f(x)$  &  $y = g(x)$  and lines  $x = a$  &  $x = b$ , when  $f(x) > g(x) \forall x \in [a, b] = \int_a^b [f(x) - g(x)] dx$ .
2. Area bounded by curves  $x = \phi(y)$  &  $x = \varphi(y)$  and lines  $y = a$  &  $y = b$ , when  $\phi(y) > \varphi(y) \forall y \in [a, b] = \int_{y=a}^{y=b} [\phi(y) - \varphi(y)] dy$ .

**EXERCISE-24****CATEGORY-24.1. TANGENT, NORMAL AND OTHER PROPERTIES OF A GIVEN CURVE  
DEPENDING ON DERIVATIVE AT A POINT**

- Find the equation to the normal to the curve  $y = \sin x$  at  $(0,0)$ . {Ans.  $x + y = 0$ }
- Find the equation of the normal to the curve  $y = x + \sin x \cos x$  at  $x = \frac{\pi}{2}$ . {Ans.  $2x = \pi$ }
- Find the equation of the normal to the curve  $y = x(2-x)$  at the point  $(2,0)$ . {Ans.  $x - 2y = 2$ }
- Find the point on the curve  $y^2 = x$  where tangent makes  $45^\circ$  angle with  $x$ -axis. {Ans.  $\left(\frac{1}{4}, \frac{1}{2}\right)$ }
- At what point the slope of the tangent to the curve  $x^2 + y^2 - 2x - 3 = 0$  is zero?. {Ans.  $(1,2); (1,-2)$ }
- If the tangent to the curve  $xy + ax + by = 0$  at  $(1,1)$  is inclined at an angle  $\tan^{-1} 2$  with  $x$ -axis, then find  $a, b$ . {Ans.  $a = 1, b = -2$ }
- Find the value of  $t$  for which the tangent line to the curve  $x = t^2 - 1, y = t^2 - t$  is perpendicular to  $x$ -axis. {Ans.  $t = 0$ }
- Find the equation of tangent at those points where the curve  $y = x^2 - 3x + 2$  meets  $x$ -axis. {Ans.  $x + y - 1 = 0; x - y - 2 = 0$ }
- For the parabola  $y^2 = 4ax$ , find the ratio of the subtangent to the abscissa. {Ans.  $2:1$ }
- If the parametric equation of a curve given by  $x = e^t \cos t, y = e^t \sin t$ , then find the angle which the tangent to the curve at the point  $t = \frac{\pi}{4}$  makes with axis of  $x$ . {Ans.  $\frac{\pi}{2}$ }
- If  $y = 4x - 5$  is a tangent to the curve  $y^2 = px^3 + q$  at  $(2,3)$ , then find  $p, q$ . {Ans.  $p = 2, q = -7$ }
- Find the point where the curve  $y - e^{xy} + x = 0$  has a vertical tangent. {Ans.  $(1,0)$ }
- Show that the subtangent, ordinate and subnormal to the parabola  $y^2 = 4ax$  at a point (different from the origin) are in G.P..
- If the tangent to the curve  $x = at^2, y = 2at$  is perpendicular to  $x$ -axis, then find its point of contact. {Ans.  $(0,0)$ }
- Find the point where the tangent to the curve  $y = e^{2x}$  at the point  $(0,1)$  meets  $x$ -axis. {Ans.  $\left(-\frac{1}{2}, 0\right)$ }
- Find the equation of the tangent to the curve  $y = 1 - e^{\frac{x}{2}}$  at the point of intersection with the  $y$ -axis. {Ans.  $x + 2y = 0$ }
- Find the equation of the tangent to the curve  $x = t \cos t, y = t \sin t$  at the origin. {Ans.  $y = 0$ }
- If the tangent to the curve  $x = a(\theta + \sin \theta), y = a(1 + \cos \theta)$  at  $\theta = \frac{\pi}{3}$  makes an angle  $\alpha$  with  $x$ -axis, then find  $\alpha$ . {Ans.  $\frac{5\pi}{6}$ }
- Find the fixed point  $P$  on the curve  $y = x^2 - 4x + 5$  such that the tangent at  $P$  is perpendicular to the line  $x + 2y - 7 = 0$ . {Ans.  $(3,2)$ }

20. Find the point on the curve  $y = x^2 - 3x + 2$  where tangent is perpendicular to  $y = x$ . {Ans.  $(1,0)$ }
21. Find the slope of the tangent to the curve  $x = t^2 + 3t - 8$ ,  $y = 2t^2 - 2t - 5$  at point  $(2,1)$ . {Ans.  $\frac{6}{7}$ }
22. Find the equation of the normal to the curve  $x^3 + y^3 = 6xy$  at the point  $(3, 3)$ . {Ans.  $y = x$ }
23. Find the normal to  $y = x^3 - 3x$  which is parallel to  $2x + 18y = 9$ . {Ans.  $x + 9y = 20$ ,  $x + 9y = -20$ }
24. Determine the constant  $c$  such that the straight line joining the points  $(0, 3)$  and  $(5, -2)$  is tangent to the curve  $y = \frac{c}{x+1}$ . {Ans.  $c = 4$ }
25. Find the tangent and normal to the curve  $x = \frac{2at^2}{1+t^2}$ ,  $y = \frac{2at^3}{1+t^2}$  at the point for which  $t = \frac{1}{2}$ . {Ans.  $13x - 16y = 2a$ ,  $16x + 13y = 9a$ }
26. Find the tangents to  $y = (x^3 - 1)(x - 2)$  at the points where the curve cuts the  $x$ -axis. {Ans.  $y + 3x = 3$ ,  $y - 7x + 14 = 0$ }
27. Find the normal at point  $P$  for which  $\theta = \frac{\pi}{4}$  to the curve  $x = a \cos \theta + a \theta \sin \theta$ ,  $y = a \sin \theta - a \theta \cos \theta$ . {Ans.  $y + x = a\sqrt{2}$ }
28. Find the tangents and normal to the curve  $y(x-2)(x-3) - x + 7 = 0$  at the point where it cuts the axis of  $x$ . {Ans.  $x - 20y - 7 = 0$ ,  $20x + y - 140 = 0$ }
29. Find the normal to the curve  $y^2 = x^3$  at the point whose abscissa is  $x = 8$ . {Ans.  $x \pm 3\sqrt{2}y = 104$ }
30. Find the normal to the curve  $y = x^3 - 2x^2 + 4$  at the point where  $x = 2$ . {Ans.  $x + 4y = 18$ }
31. Find the normals to the points on the curve  $y = \frac{x}{(1-x^2)}$  where the tangent makes an angle of  $\frac{\pi}{4}$  with  $x$ -axis. {Ans.  $x + y = 0$ ,  $x + y = -\frac{\sqrt{3}}{2}$ ,  $x + y = \frac{\sqrt{3}}{2}$ }
32. Find the points on the curve  $9y^2 = x^3$  where normal to the curve makes equal intercepts with the axes. {Ans.  $\left(4, \frac{8}{3}\right)$ }
33. Find the equation to the tangent to  $x^3 = ay^2$  at  $(4am^2, 8am^3)$  and also the point in which the tangent cuts the curve again. {Ans.  $y = 3mx - 4am^3$ ,  $(am^2, -am^3)$ }
34. Show that the normal to the curve  $5x^5 - 10x^3 + x + 2y + 6 = 0$  at  $P(0, -3)$  meets the curve again at two points. Find the equation of tangents to the curve at these points. {Ans.  $y = 2x - 3$ }
35. Show that the tangent to the curve  $3xy^2 - 2x^2y = 1$  at  $(1, 1)$  meets the curve again at the point  $\left(-\frac{16}{5}, -\frac{1}{20}\right)$ .
36. Find the point on the curve  $y = x^4 - 6x^3 + 13x^2 - 10x + 5$  where the tangent is parallel to the line  $y = 2x$ . Show that two of these points have the same tangent. {Ans.  $(1, 3)$ ,  $(2, 5)$ ,  $\left(\frac{3}{2}, \frac{15}{6}\right)$ }
37. Find the points on the curve  $y = x^3$ , the tangents at which are inclined at an angle of  $60^\circ$  to  $x$ -axis. {Ans.  $\left(\frac{1}{3^4}, \frac{1}{3^4}\right)$ ,  $\left(-\frac{1}{3^4}, -\frac{1}{3^4}\right)$ }

38. Prove that all points on the curve  $y^2 = 4a \left[ x + a \sin \left( \frac{x}{a} \right) \right]$  at which the tangent is parallel to the  $x$ -axis lie on the parabola  $y^2 = 4ax$ .
39. The curve  $y = ax^3 + bx^2 + cx + 5$  touches the  $x$ -axis at  $P(-2, 0)$  and cuts the  $y$ -axis at a point  $Q$  where its gradient is 3. Find  $a, b, c$ . {Ans.  $a = -\frac{1}{2}, b = -\frac{3}{4}, c = 3$ }
40. Show that tangents to the Folium of Descartes  $x^3 + y^3 = 3axy$  at the point where it meets the parabola  $y^2 = ax$  are parallel to the axis of  $y$ . Also determine the point on the curve at which the tangent is parallel to  $x$ -axis. {Ans.  $\left( 2^{\frac{1}{3}}a, 2^{\frac{2}{3}}a \right)$ }
41. Find the equation of the normal to the curve  $x^3 + y^3 = 8xy$  at the point other than origin where it meets the curve  $y^2 = 4x$ . {Ans.  $y - x = 0$ }
42. Show that the line  $\frac{x}{a} + \frac{y}{b} = 1$  touches the curve  $y = be^{-\frac{x}{a}}$  at the point where the curve crosses the  $y$ -axis.
43. Show that the curve  $\left( \frac{x}{a} \right)^n + \left( \frac{y}{b} \right)^n = 2$  touches the straight line  $\frac{x}{a} + \frac{y}{b} = 2$  at that the point  $(a, b)$  whatever the value of  $n$  may be.
44. The rectangular coordinates of a point on the curve are given by  $x = 3\cos\theta - \cos^3\theta$ ,  $y = 3\sin\theta - \sin^3\theta$ . Find the equation of the normal at the point  $P$  where  $\theta = \frac{\pi}{4}$ . {Ans.  $y = x$ }
45. Tangents are drawn from origin to the curve  $y = \sin x$ . Prove that their points of contact lie on  $x^2 y^2 = x^2 - y^2$ .
46. Find all the tangents to the curve  $y = \cos(x + y)$ ,  $-\pi \leq x \leq \pi$  that are parallel to the line  $x + 2y = 0$ . {Ans.  $2x + 4y + 3\pi = 0, 2x + 4y - \pi = 0$ }
47. Find the equations of the tangents drawn to the curve  $y^2 - 2x^3 - 4y + 8 = 0$  from the point  $(1, 2)$ . {Ans.  $y - (2 + 2\sqrt{3}) = 2\sqrt{3}(x - 2), y - (2 - \sqrt{3}) = -2\sqrt{3}(x - 2)$ }
48. Find the equation of the straight line which is tangent at one point and normal at another point of the curve  $y = 8t^3 - 1, x = 4t^2 + 3$ . {Ans.  $27\sqrt{2}x - 27y - 89\sqrt{2} - 27 = 0, 27\sqrt{2}x + 27y - 89\sqrt{2} + 27 = 0$ }
49. Tangent at point  $P_1$  [other than  $(0, 0)$ ] on the curve  $y = x^3$  meets the curve again at  $P_2$ . The tangent  $P_2$  meets the curve at  $P_3$  and so on. Show that the abscissa of  $P_1, P_2, P_3, \dots, P_n$  form an G.P. Also find the ratio  $\frac{\text{area}\Delta P_1 P_2 P_3}{\text{area}\Delta P_2 P_3 P_4}$ . {Ans.  $\frac{1}{16}$ }
50. Find the equation of the normal to the curve  $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$  at  $x = 0$ . {Ans.  $x + y = 1$ }
51. Find the condition that the line  $Ax + By = 1$  may be a normal to the curve  $a^{n-1}y = x^n$ . {Ans.  $a^{n-1}B(B^2 + nA^2)^{n-1} = A^n n^n$ }
52. If  $ax + by = 1$  is a normal to the parabola  $y^2 = 4px$ , show that  $pa^3 + 2pab^2 = b^2$ .
53. In the curve  $x^a y^b = k^{a+b}$ , prove that the portion of the tangent intercepted between the coordinate axes is divided at its point of contact into segments which are in constant ratio (all the constants being positive).



54. Prove that the portion of the tangent to the curve  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$  which is intercepted between the axes is of constant length.
55. Prove that the sum of intercepts of the tangent to the curve  $\sqrt{x} + \sqrt{y} = \sqrt{a}$  upon the co-ordinate axes is of constant length.
56. If  $p_1$  and  $p_2$  be the lengths of perpendiculars from the origin on the tangent and normal to the curve  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$  respectively, prove that  $4p_1^2 + p_2^2 = a^2$ .
57. Find the abscissa of the point on the curve  $ay^2 = x^3$ , the normal at which cuts off equal intercepts from the axes. {Ans.  $\frac{4a}{9}$ }
58. Find the abscissa of the point on the curve  $xy = (c + x)^2$  the normal at which cuts off numerically equal intercepts from the axes of coordinates. {Ans.  $\pm \frac{c}{\sqrt{2}}$ }
59. In the curve  $x = a \left[ \cos t + \log \tan \frac{t}{2} \right]$ ,  $y = a \sin t$ , show that the portion of the tangent between the point of contact and the  $x$ -axis is of constant length.

#### **CATEGORY-24.2. APPLICATIONS OF DIFFERENTIAL EQUATIONS IN FINDING EQUATIONS OF CURVES**

60. The slope of the curve at any point is the reciprocal of twice the ordinate at the point. The curve also passes through the point (3, 4). Find the equation. {Ans.  $y^2 = x + 13$ }
61. Show that the equation of the curve whose slope at any point is equal to  $y + 2x$  and which passes through origin is  $y = 2(e^x - x - 1)$ .
62. The tangent at any point  $(x, y)$  of a curve makes an angle  $\tan^{-1}(2x + 3y)$  with  $x$ -axis. Find the equation of the curve if it passes through (1, 2).
63. Find the curve in which the slope of the tangent at any point equals the ratio of the abscissa to the ordinate of the point. {Ans. A rectangular hyperbola}
64. Find the equations of the curve for which sub-tangent is constant. {Ans.  $y = ke^{\frac{x}{a}}$ }
65. Find the equations of the curve for which sub-tangent varies as the abscissa. {Ans.  $cx = y^k$ }
66. Find the equations of the curve for which sub-normal is constant. {Ans.  $y^2 = 2ax + k$ }
67. Find the equations of the curve for which sub-normal is equal to abscissa. {Ans.  $x^2 - y^2 = a^2$ }
68. The tangent at any point  $P$  of a curve meets the axis of  $x$  in  $T$ . Find the curve for which  $OP = PT$ ,  $O$  being the origin. {Ans.  $xy = a^2$ }
69. A normal is drawn at a point  $P(x, y)$  of a curve. It meets  $x$ -axis at  $G$ . If  $PG$  is of constant length  $k$ , then show that the differential equation describing such curves is  $y \frac{dy}{dx} = \pm \sqrt{k^2 - y^2}$ . Find the equation of such a curve passing through  $(0, k)$ . {Ans.  $x^2 + y^2 = k^2$ }
70. Find the equation of family of curves for which the length of the normal at point  $P$  is equal to  $OP$ . {Ans.  $y^2 \pm x^2 = k^2$ }
71. Find the curve in which the sub-tangent is always bisected at the origin. {Ans.  $y^2 = kx$ }

72. Show that the curve for which the normal at every point passes through a fixed point is a circle.
73. Find the equation of a curve passing through  $\left(2, \frac{7}{2}\right)$  and having gradient  $1 - \frac{1}{x^2}$  at  $(x, y)$ . {Ans.  $xy = x^2 + x + 1$ }
74. Find the curve for which the intercept cut off by a tangent on  $x$ -axis is equal to four times the ordinate of the point of contact. {Ans.  $cy^4 = e^{-\frac{x}{y}}$ }
75. The normal  $PG$  to a curve meets the axis in  $G$ . If the distance of  $G$  from the origin is twice the abscissa of  $P$ , prove that the curve is a rectangular hyperbola. {Ans.  $y^2 = x^2 + C$ }
76. The normal at each point of a curve and the line from that point to the origin form an isosceles triangle with the base on the  $x$  axis. Find the equation of the curve. {Ans.  $x^2 - y^2 = a^2$ }
77. Find the curve which is such that portion of  $x$ -axis cut off between the origin and the tangent at any point is proportional to the ordinate of the point. {Ans.  $y = ae^{-\frac{x}{ky}}$ }
78. The tangent at a point  $P$  of a curve meets the axis of  $y$  in  $N$ . The parallel line through  $P$  to the axis of  $y$  meets the axis of  $x$  at  $M$ ;  $O$  is the origin. If the area of the triangle  $MON$  is constant, show that the curve is a hyperbola.
79. The normal at any point  $P$  of a curve cuts  $OX$  in  $G$  and  $N$  is the foot of the ordinate of  $P$ . If  $NG$  varies as the square of  $OP$ , find the curve. {Ans.  $y^2 = -x^2 - \frac{x}{k} - \frac{1}{2k^2} + 2Ce^{2kx}$ }
80. A curve is such that the length of perpendicular from origin on the tangent at any point  $P$  of the curve is equal to the abscissa of  $P$ . Prove that the differential equation of the curve is  $y^2 - 2xy \frac{dy}{dx} - x^2 = 0$  and hence find the curve. {Ans.  $x^2 + y^2 = Cx$ }
81. A particle  $P$  moves so that its velocities parallel to the axis of  $x$  and  $y$  are respectively  $-ky$  and  $kx$  where  $k$  is a constant different from zero. Find the path of the particle, if it passes through the point  $(3, 4)$ . {Ans.  $x^2 + y^2 = 25$ }

### CATEGORY-24.3. LEAST DISTANCE BETWEEN TWO CURVES

82. Find the least distance between the following curves:-

- $y^2 = x$  and  $y = x + 1$ . {Ans.  $\frac{3}{4\sqrt{2}}$ }
- $y = \ln x$  and  $y = x$ . {Ans.  $\frac{1}{\sqrt{2}}$ }

### CATEGORY-24.4. ANGLE OF INTERSECTION OF TWO CURVES

83. Find the angle of intersection of the following curves:-

- $y = x^2$  and  $6y = 7 - x^2$  {Ans.  $\tan^{-1} 7$ }
- $y = 4 - x^2$  and  $y = x^2$ . {Ans.  $\tan^{-1} \left( \frac{4\sqrt{2}}{7} \right)$ }
- $x^2 y = 1$  and  $y = x^2$  {Ans.  $\tan^{-1} \left( \frac{4}{3} \right)$ }

- iv.  $x^2 = 4ay$  and  $2y^2 = ax$  {Ans.  $\frac{\pi}{2}$  &  $\tan^{-1}\left(\frac{3}{5}\right)$ }
- v.  $x^2 + y^2 = a^2\sqrt{2}$  and  $x^2 - y^2 = a^2$  {Ans.  $\frac{\pi}{4}$ }
- vi.  $y^2 = x$  and  $x^2 = y$ . {Ans.  $\frac{\pi}{2}$ ,  $\tan^{-1}\left(\frac{3}{4}\right)$ }
- vii.  $xy = a^2$  and  $x^2 - y^2 = 2a^2$ . {Ans.  $90^\circ$ }
84. If the two curves  $y = a^x$  and  $y = b^x$  intersect at an angle  $\alpha$ , then show that  $\tan \alpha = \frac{\ln a - \ln b}{1 + \ln a \ln b}$ .
85. If the line  $y = x$  touches the curve  $y = x^2 + bx + c$  at a point  $(1,1)$  then find the value of  $b$  and  $c$ . {Ans.  $b = -1$ ,  $c = 1$ }
86. If the curve  $ay + x^2 = 7$  and  $x^3 = y$  cut orthogonally at  $(1,1)$ , then find the value of  $a$ . {Ans. 6}
87. If the curves  $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$  and  $y^3 = 16x$  intersect at right angles, show that  $a^2 = \frac{4}{3}$ .
88. If curve  $y = 1 - ax^2$  and  $y = x^2$  intersect orthogonally then find the value of  $a$ . {Ans.  $\frac{1}{3}$ }
89. Show that the curves  $x^3 - 3xy^2 = -2$  and  $3x^2y - y^3 = 2$  cut orthogonally.
90. Find the condition that the following conic may cut orthogonally:  
 $\frac{x^2}{a} + \frac{y^2}{b} = 1$  and  $\frac{x^2}{a'} + \frac{y^2}{b'} = 1$  {Ans.  $a - b = a' - b'$ }
91. Show that  $\frac{x^2}{a^2 + k_1} + \frac{y^2}{b^2 + k_1} = 1$  and  $\frac{x^2}{a^2 + k_2} + \frac{y^2}{b^2 + k_2} = 1$  intersect orthogonally.
92. Show that the curves  $y^2 = 4ax$  and  $ay^2 = 4x^3$  intersect each other at an angle of  $\tan^{-1} \frac{1}{2}$  and also if  $PG_1$  and  $PG_2$  be the normals to the two curves at common point of intersection (which is not origin) meeting the axis of  $x$  in  $G_1$  and  $G_2$ , then  $G_1G_2 = 4a$ .
93. Prove that the curves  $y = e^{-ax} \sin bx$  and  $y = e^{-ax}$  touch at the points for which  $bx = 2n\pi + \frac{\pi}{2}$ .
94. Show that the angle between the tangents at any point  $P$  and the line joining  $P$  to the origin  $O$  is the same at all points of the curve  $\ln(x^2 + y^2) = k \tan^{-1}\left(\frac{y}{x}\right)$ .
95. Show that the family of curves represented by  $\frac{dy}{dx} = \frac{x^2 + x + 1}{y^2 + y + 1}$  and the family represented by  $\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$  are orthogonal.

#### CATEGORY-24.5. RATE OF CHANGE PROBLEMS OF PHYSICS, GEOMETRY ETC.

96. A balloon, which always remains spherical, has a variable radius. If the radius of the balloon is increasing at a rate of  $\frac{1}{7}$  cm/sec, how fast is the volume increasing when the radius is  $\frac{7}{11}$  cm? [take  $\pi = \frac{22}{7}$ ] {Ans.  $\frac{8}{11}$  c.c./sec}
97. A spherical balloon is being inflated so that its volume increases uniformly at the rate of  $40 \text{ cm}^3/\text{min}$ . How

- fast is its surface area increasing when the radius is 8 cm? {Ans.  $10 \text{ cm}^2/\text{min}$ }
98. An inverted cone has a depth of 10 cm and base radius 5 cm. Water is poured into it at the rate of  $1\frac{1}{2}$  c.c. per minute. Find the rate at which the level of the water in the cone is rising when the depth is 4 cm. {Ans.  $\frac{3}{8\pi} \text{ cm/min}$ }
99. A man 1.80 meter high moves directly away from a lamp post 6 meters high at the rate of 5.544 km/hr. How fast does the length of his shadow change? {Ans. 2.376 km/hr}
100. A particle moves in a straight line with a velocity which varies as the square of the distance from a fixed point. Show that its acceleration varies as the cube of the distance.
101. An aeroplane travels  $s$  kilometer in  $t$  hours from its start, where  $s = 100t + 5t^2$ , find the velocity of the aeroplane at the end of 5 hours. {Ans. 150 km/hr}
102. The velocity of a body, in m/sec, in rectilinear motion is determined by the formula  $v = 3t + t^2$ . What acceleration will the body have 4 seconds after the start? {Ans.  $11 \text{ m/sec}^2$ }
103. The law of rectilinear motion of a body with a mass of 100 kg. is  $s = 2t^2 + 3t + 1$  ( $s$  is in meters and  $t$  in seconds). Determine the kinetic energy of the body 5 seconds after start. {Ans. 26450 Joules}
104. A body of mass 6 gram is in rectilinear motion according to law  $s = -1 + \log(t + 1) + (t + 1)^3$  ( $s$  is in centimeters and  $t$  in seconds). Find the kinetic energy of the body one second after it begins to move. {Ans.  $468\frac{3}{4}$  ergs.}
105. If the law of motion of a body is  $s = ae^t + be^{-t}$ , then prove that the acceleration is numerically equal to the distance covered.
106. The velocity of rectilinear motion of a body is proportional to the square root of the distances covered. Prove that the body moves with constant acceleration.
107. A raft is pulled to the bank by means of a rope, which is wound on a drum, at the rate of 3 meter per second. Determine the speed of the raft at the moment when it is 25 meter distance from the bank if the drum is situated on the bank 4 meter above the water level. {Ans. 3.03 m/sec}
108. A point is moving along the cubical parabola  $12y = x^3$ , which of the coordinates changes faster? {Ans. when  $-2 < x < 2$  abscissa,  $x = \pm 2$  same,  $x < -2$  or  $x > 2$  ordinate}
109. A point is moving along the parabola  $y^2 = 12x$  at the rate of 10 cm/sec. Find the component velocity parallel to the axes, when it is at the point (3, 6). {Ans.  $5\sqrt{2} \text{ cm/sec}$ }
110. A body of mass  $m$  falls from rest in a medium whose resistance force is proportional to the square of the velocity of the body, the proportionality constant being  $k$ . Find the dependence of the velocity on time. If  $m = 10 \text{ kg}$ ,  $k = 1 \text{ kg/m}$  &  $g = 10 \text{ m/sec}^2$ , find the terminal velocity and the time at which the body attains 99% of the terminal velocity. {Ans.  $v = \sqrt{\frac{mg}{k}} \left( \frac{e^{2t\sqrt{\frac{gk}{m}}} - 1}{e^{2t\sqrt{\frac{gk}{m}}} + 1} \right)$ , 10 m/sec, 2.65 sec}

#### CATEGORY-24.6. GREATEST & LEAST VALUE PROBLEMS OF PHYSICS, GEOMETRY ETC.

111. Find the maximum value of  $xy$  subject to  $x + y = 8$ . {Ans. 16}
112. Find the maximum slope of the curve  $y = -x^3 + 3x^2 + 2x - 27$ . {Ans. 5}
113. Break up the number 8 into two summands such that the sum of their cubes is the least possible. {Ans. 4 and 4}
114. What positive number added to its reciprocal yields the least possible sum? {Ans. 1}

115. Decompose the number 36 into two factors such that the sum of their squares is the least possible. {Ans. 6 and 6}
116. Find a positive number which exceeds its cube by greatest possible quantity. {Ans.  $\frac{1}{\sqrt{3}}$ }
117. Find the value of  $a$  so that the sum of the squares of the roots of the equation  $x^2 - (a-2)x - a + 1 = 0$  assume the least value. {Ans. 1}
118. Equal squares must be cut out of the corners of a square piece of card-board 18 cm by 18 cm in order to fold the cardboard along the dashed lines (Fig. 1) and make a box of the greatest possible capacity. What must the side of the cut-out square be? Solve this problem for a rectangular piece of cardboard 8 cm by 5 cm. {Ans. 3cm, 1cm}
119. A covered box of volume  $72 \text{ cm}^3$  and the base sides in a ratio of 1:2 is to be made. What must the lengths of all sides be so that the total surface area is the least possible? {Ans. 3, 6 and 4cm}
120. The volume of a regular triangular prism is  $v$ . What must the edge of the base be so that the total surface area is the least possible? {Ans.  $\sqrt[3]{4v}$ }
121. An open cylindrical tub has a given volume  $v$ . What must the radius of the base and the altitude of the cylinder be to yield the least possible surface area? {Ans.  $\sqrt[3]{\frac{v}{\pi}}, \sqrt[3]{\frac{v}{\pi}}$ }
122. Find the maximum area of the rectangle that can be inscribed in a circle of radius  $r$ . {Ans.  $2r^2$ }
123. Determine the relationship between the radius  $R$  and the altitude  $H$  of a cylinder which has the least possible total surface area at the given volume. {Ans.  $H = 2R$ }
124. A conic funnel with the generatrix 20 cm long must be made. What must the height of the funnel be to yield the greatest possible volume? {Ans.  $\frac{20}{\sqrt{3}}$  cm}
125. A sector with a central angle  $\alpha$  is cut out of a circle to make a cone. What value of the angle  $\alpha$  will yield the greatest possible volume of the cone? {Ans.  $2\pi\sqrt{\frac{2}{3}}$ }
126. The perimeter of an isosceles triangle is equal to  $2p$ . What must its sides be so that the volume of the solid generated by revolving the triangle about its base is the greatest possible? {Ans.  $\frac{3p}{4}, \frac{3p}{4}, \frac{p}{2}$ }
127. The perimeter of an isosceles triangle is equal to  $2p$ . What must its sides be so that the volume of the solid generated by revolving the triangle about the altitude dropped on the base is the greatest possible? {Ans.  $\frac{3p}{5}, \frac{3p}{5}, \frac{4p}{5}$ }
128. Determine the altitude of a cylinder of the greatest possible volume which can be inscribed in a sphere with a radius  $R$ . {Ans.  $\frac{2}{\sqrt{3}}R$ }
129. Determine the altitude of a cone with the greatest possible volume which can be inscribed in a sphere of a radius  $R$ . {Ans.  $\frac{4}{3}R$ }
130. A raindrop with an initial mass of  $m_0$  is falling under the force of gravity. The loss in mass, due to evaporation, is proportional to time (the proportionality factor is equal to  $k$ ). How many seconds will it take the kinetic energy to attain its greatest value after it begins to fall? What will its kinetic energy be? (Ignore the resistance of the air.) {Ans.  $\frac{2m_0}{3k}, \frac{2m_0^3 g^2}{27k^2}$ }
131. Fuel expenditures, in Rs./hr, for a steamship are proportional to the cube of its speed, in km/hr. As is known, at a speed of a 10 km/h fuel costs are 30 rupees per hour and other expenses (independent of the

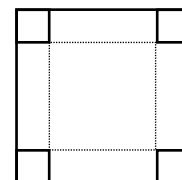


Fig. 1

- speed) amount to 480 rupees per hour. At what speed of the ship will the sum of expenses per kilometer be the smallest? What will the total sum of expenditures per hour be? {Ans. 20 km/hr, 720 rupees/hr}
132. Three points  $A$ ,  $B$  and  $C$  are situated so that the angle  $ABC$  is equal to  $60^\circ$ . A car leaves point  $A$ , while simultaneously a train leaves point  $B$ . The car travels toward  $B$  at a speed of 80 km/h, while the train travels towards  $C$  at 50 km/h. At what time (from the beginning of travel) is the distance between the train and the car the smallest if  $AB = 200$  km? {Ans.  $\frac{70}{43}$  hr}
133. A point is given on the circumference of a circle. Draw a chord  $BC$  parallel to the tangent at point  $A$  so that the area of the triangle  $ABC$  is the greatest. {Ans. Distance between chord  $BC$  and point  $A = \frac{3}{4} \times \text{diameter}$ }
134. Find the sides of a rectangle with the greatest possible perimeter inscribed in a semicircle of radius  $R$ . {Ans.  $\frac{4\sqrt{5}}{5}R, \frac{\sqrt{5}}{5}R$ }
135. In a given segment of a circle, inscribe a rectangle of the greatest possible area. {Ans. Height of rectangle  $= \frac{\sqrt{8R^2 + h^2} - 3h}{4}$ , where  $h$  is the distance of the mid-point of the chord subtending the arc of the segment from the center of the circle and  $R$  is the radius of the circle}
136. About a given cylinder, circumscribe a cone of the least possible volume (the planes of the bases of the cylinder and cone must coincide) {Ans. Radius of base circle of cone  $= 1.5 \times$  radius of cylinder}
137. Find the altitude of a right circular cone having the least volume circumscribed about a sphere of a radius  $R$ . {Ans.  $4R$ }
138. Find the angle at the vertex of an axial section of a cone with the least lateral surface circumscribed around a given sphere. {Ans.  $\approx 49^\circ$ }
139. What must the angle at the vertex of an isosceles triangle of a given area be such that the radius of the circle inscribed in this triangle is the greatest? {Ans.  $60^\circ$ }
140. Find the altitude of a cone of the least volume which can be drawn around a hemisphere of radius  $R$  (the centre of the base of the cone falls on the centre of the sphere). {Ans.  $\sqrt{3}R$ }
141. What must the altitude of a cone inscribed in a sphere of radius  $R$  be such that its lateral surface area is the greatest? {Ans.  $\frac{4}{3}R$ }
142. Prove that a conical tent of a given volume requires the least amount of a material when its height is  $\sqrt{2}$  times the radius of the base.
143. Through a given point  $P(1, 4)$  draw a straight line so that the sum of its positive intercepts on the coordinate axes is the smallest. {Ans.  $\frac{x}{3} + \frac{y}{6} = 1$ }
144. Find the sides of a rectangle with the greatest area which can be inscribed in the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . {Ans.  $a\sqrt{2}$  and  $b\sqrt{2}$ }
145. Find the ellipse having the least area which can be circumscribed about a given rectangle (the area of the ellipse with semiaxes  $a$  and  $b$  equals  $\pi ab$ ). {Ans. Area of ellipse  $= \frac{\pi}{2} \times$  area of rectangle}
146. At which point on the ellipse  $\frac{x^2}{8} + \frac{y^2}{18} = 1$  must a tangent be drawn such that the area of the triangle formed by this tangent and the coordinate axes is the smallest? {Ans.  $(2, 3)$ }
147. Two points  $A(1, 4)$  and  $B(3, 0)$  are given on the ellipse  $2x^2 + y^2 = 18$ . Find a third point  $C$  on the ellipse such that the area of the triangle  $ABC$  is the greatest. {Ans.  $(-\sqrt{6}, -\sqrt{6})$ }
148. A point is given on the axis of the parabola  $y^2 = 2px$  at a distance  $a$  from its vertex. Determine the

- abscissa  $x$  of the point on the curve closest to the given point. {Ans.  $x = a - p$  if  $a > p$ ,  $x = 0$  if  $a \leq p$ }
149. An iron band of width  $a$  must be bent into an open cylindrical groove (the cross section of the groove is in the form of an arc of the circular segment). Find the value of the central angle subtended by this arc which yields the greatest capacity of the groove. {Ans. Cross section is a semi-circle}
150. A log 20 m long has the form of a truncated cone whose ends have diameters of 2m and 1m. A beam of the square cross section and the greatest volume is to be cut from the log. The axis of the beam and the axis of the log must coincide. Find the dimensions of the beam. {Ans. length =  $\frac{40}{3}$  m, side of cross section =  $\frac{2\sqrt{2}}{3}$  m }
151. A series of trials resulted in  $n$  distinct values  $x_1, x_2, \dots, x_n$  of a quantity  $X$ . A value  $x$  is often used for  $X$  such that the sum of the squares of its deviations from  $x_1, x_2, \dots, x_n$  is the least possible. Find  $x$  which satisfies this requirement. {Ans.  $x = \frac{x_1 + x_2 + \dots + x_n}{n}$ }
152. A fishing boat is anchored 9 km away from the nearest point on shore. A messenger must be sent from the fishing boat to a camp, 15 km from the point on shore closest to the boat. If the messenger can walk at a speed of 5 km per hour and can row at 4 km per hour, then at what point on shore must he land in order to reach the camp in the shortest possible time? {Ans. 3 km from the camp}
153. A lantern must be hung directly above a circular plaza of radius  $R$ . At what height must it be installed to provide the best lighting for the road around the plaza? (The intensity of illumination of a surface is directly proportional to the cosine of the angle of incidence of the rays and inversely proportional to the square of the distance from the source of light.) {Ans.  $\frac{R}{\sqrt{2}}$ }
154. Two sources of light with candlepowers  $I_1$  and  $I_2$  are separated from one another by a length  $l$ . Find the least illuminated point. (The intensity of illumination of a surface is inversely proportional to the square of the distance from the source of light) {Ans.  $\frac{l\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$  from source  $I_1$ }
155. A picture 1.4m high is hung on a wall so that its lower edge is 1.8m above the line of the observer's eye. At what distance from the wall must the observer's stand to occupy a most favourable position for examining the picture, i.e. to ensure the greatest angle of view? {Ans. 2.4m}
156. A load of weight  $P$ , lying on a horizontal plane, must be displaced by a force  $F$  applied to it. The frictional force is directly proportional to the force pressing the load against the plane and is directed against the displacing force. The proportionality factor (coefficient of friction) is equal to  $k$ . At what angle  $\phi$  to the horizontal must the force  $F$  be applied so that its value turns out to be the least possible? Determine the least value of the displacing force. {Ans.  $\tan^{-1} k, \frac{kP}{\sqrt{1+k^2}}$ }
157. In a printed book the text must occupy  $S$  square centimeters of each page. The top and bottom margins must be  $a$  cm each, and the right- and left-handed margins  $b$  cm each. If we are interested only in saving the paper, then what must the size of the printed page be? {Ans.  $2b + \sqrt{\frac{Sb}{a}}$  and  $2a + \sqrt{\frac{Sa}{b}}$ }
158. A vertex of a parabola lies on a circle of radius  $R$ , its axis is directed along the diameter. What must the parameter of the parabola be so that the area of the segment bounded by the parabola and the chord (common with the circle) is the greatest? [The area of a symmetric parabolic segment is equal to two thirds of the product of its base (the chord) by the altitude (the distance from the vertex to the chord)] {Ans.  $\frac{R}{4}$ }
159. A cone of altitude  $H$  and the base radius  $R$  is cut by a plane parallel to the generatrix. What must the distance between the line of intersection (of this plane and the plane of the cone base) and the centre of cone base be so that the area of the section is the greatest? [The area of a symmetric parabolic segment is

equal to two thirds of the product of its base (the chord) by the altitude (the distance from the vertex to the chord)] {Ans.  $\frac{R}{2}$ }

160. For what point  $P$  of the parabola  $y^2 = 2px$  has the segment of the inner normal at  $P$  the smallest length? {Ans.  $(p, \pm p\sqrt{2})$ }
161. Show that a tangent to an ellipse whose segment intercepted by the axes is the shortest is divided, at the point of tangency, into two parts respectively equal to the semi-axes of the ellipse.
162. In an ellipse, prove that the distance between the centre and any normal does not exceed the difference between the semi-axes.
163. Given in a rectangular coordinate system  $xOy$ : a point  $(a, b)$  and a curve  $y = f(x)$ . Show that the distance between the fixed point  $(a, b)$  and a variable point  $(x, f(x))$  can reach an extremum only in the direction of the normal to the curve  $y = f(x)$ .
164. Let  $A(p^2, p)$ ,  $B(q^2, q)$  and  $C(r^2, r)$  be the vertices of a triangle  $ABC$ . A parallelogram  $AFDE$  is drawn with  $D, E, F$  on the line segments  $BC, CA, AB$  respectively. Find the maximum area of the parallelogram. {Ans.  $\frac{1}{4} |(p-q)(q-r)(r-p)|$ }
165. A semicircle of radius unity is drawn with line segment  $AB$  as diameter. Two externally touching circles having radii  $a$  and  $b$  are drawn inside the semicircle such that both these circles touch the semicircle internally and both these circles touch the diameter  $AB$  also. Find the maximum value of  $a + b$ . {Ans.  $2(\sqrt{2} - 1)$ }

#### CATEGORY-24.7. AREA OF REGIONS

166. Find the area bounded by the curve  $y = x^3$ ,  $y = 2$  and the ordinates  $x = -2$  and  $x = 1$ . {Ans.  $\frac{39}{4}$ }
167. Compute the area of the figure bounded by the parabolas  $y = x^2$  and  $y = \sqrt{x}$ . {Ans.  $\frac{1}{3}$ }
168. Find the area of the figure bounded by the curves  $y = e^x$ ,  $y = e^{-x}$  and the straight line  $x = 1$ . {Ans.  $e + \frac{1}{e} - 2$ }
169. Find the area of the figure bounded by the curves  $y = |x-1|$  and  $y = 3 - |x|$ . {Ans. 4}
170. Find the area bounded by the curves  $y^2 = 2x - x^2$  and the straight line  $y = -x$ . {Ans.  $\frac{2}{3}$ }
171. Find the area bounded by the curve  $y = x|x|$ ,  $x$ -axis and the ordinates  $x = 1$ ,  $x = -1$ . {Ans.  $\frac{2}{3}$ }
172. Find the area between the curve  $y = 4 + 3x - x^2$  and  $x$ -axis. {Ans.  $\frac{125}{6}$ }
173. Find the area of the region bounded by the curve  $y^2 = 4x$ , and  $y$ -axis and the line  $y = 3$ . {Ans.  $\frac{9}{4}$ }
174. If the area bounded by the curves  $y^2 = 4ax$  and  $y = mx$  is  $\frac{a^2}{3}$ , then find the value of  $m$ . {Ans. 2}
175. Find the area bounded by the curve  $y = x \sin x$  and  $x$ -axis between  $x = 0$  and  $x = 2\pi$ . {Ans.  $4\pi$ }
176. Find the smaller area enclosed by the circle  $x^2 + y^2 = 4$  and the line  $x + y = 2$ . {Ans.  $\pi - 2$ }
177. Find the area lying in the first quadrant and bounded by the curve  $y = x^3$  and the line  $y = 4x$ . {Ans. 4}



178. Find the area of the closed figure bounded by  $y = \frac{1}{\cos^2 x}$ ,  $x = 0$ ,  $y = 0$  and  $x = \frac{\pi}{4}$ . {Ans. 1}
179. Find the area of the closed figure bounded by  $x = -1$ ,  $x = 2$  and  
 $y = -x^2 + 2$ ,  $x \leq 1$   
 $= 2x - 1$ ,  $x > 1$   
 and the abscissa axis. {Ans.  $\frac{16}{3}$  sq. units}
180. Show that the area of the figure bounded by  $y = e^{x-1}$ ,  $y = 0$ ,  $x = 0$  and  $x = 2$  is greater than 2.
181. Find the positive value of the parameter 'a' for which the area of the figure bounded by  $y = \sin ax$ ,  $y = 0$ ,  
 $x = \frac{\pi}{a}$  and  $x = \frac{\pi}{3a}$  is 3. {Ans.  $\frac{1}{2}$ }
182. Find the area bounded by the curve  $y = \sin 2x$ ,  $y$ -axis and  $y = 1$ . {Ans.  $\frac{\pi}{4} - \frac{1}{2}$ }
183. Find the value of  $a$  for which the area between the curves  $y^2 = 4ax$  and  $x^2 = 4ay$  is 1 unit. {Ans.  $\frac{\sqrt{3}}{4}$ }
184. Find the area bounded by the loop of the curve  $ay^2 = x^2(a - x)$ . {Ans.  $\frac{8}{15}a^2$ }
185. Find the area bounded by the circle  $x^2 + y^2 = 16a^2$  and the parabola  $y^2 = 6ax$ . {Ans.  $\frac{4a^2}{3}(4\pi + \sqrt{3})$ ,  
 $\frac{4a^2}{3}(8\pi - \sqrt{3})$ }
186. Find the area bounded by the curves  $y = |x - 1|$ ,  $y = -1$  and  $|x| = 2$ . {Ans. }
187. Find the area cut off the parabola  $4y = 3x^2$  by the straight line  $2y = 3x + 12$ . {Ans. 27}
188. Find the area bounded by the parabola  $x^2 = 4y$  and the line  $x = 4y - 2$ . {Ans.  $\frac{9}{8}$ }
189. Find the area of the region for which  $0 < y < 3 - 2x - x^2$  and  $x > 0$ . {Ans.  $\frac{5}{3}$ }
190. Find the area between the curve  $y = 2x^4 - x^2$ , the  $x$ -axis and the ordinates of two minima of the curve.  
 {Ans.  $\frac{7}{120}$ }
191. Find the value of  $k$  for which the area of the figure bounded by the curve  $y = 8x^2 - x^5$ , the straight line  
 $x = 1$  and  $x = k$  and the  $x$ -axis is equal to  $\frac{16}{3}$ . {Ans.  $\sqrt[3]{8 - \sqrt{17}}$ }
192. Show that if  $A$  is the area between the curve  $y = \sin x$  and  $x$ -axis in the interval  $\left[0, \frac{\pi}{4}\right]$ , then in the same  
 interval, area between the curve  $y = \cos x$  and  $x$ -axis is  $1 - A$ .
193. Show that if  $A$  is the area lying between the curve  $y = \sin x$  and  $x$ -axis between  $x = 0$  and  $x = \frac{\pi}{2}$ , then  
 the area of the region between the curve  $y = \sin 2x$  and  $x$ -axis in the same interval is also  $A$ .

194. Find the area of the region lying between the line  $x - y + 2 = 0$  and the curve  $y = x^2$ . {Ans.  $\frac{10}{3}$ }
195. Find the area lying in the first quadrant and bounded by the circle  $x^2 + y^2 = 4$ , the line  $x = \sqrt{3}y$  and  $x$ -axis. {Ans.  $\frac{\pi}{3}$ }
196. Find the area bounded by the curves  $y = e^x$ ,  $y = e^{-x}$  and  $y = 2$ . {Ans.  $2\ln\left(\frac{4}{e}\right)$ }
197. In the interval  $\left[0, \frac{\pi}{2}\right]$ , find the area lying between the curves  $y = \tan x$ ,  $y = \cot x$  and  $x$ -axis. {Ans.  $\ln 2$ }
198. Find the area lying between the curves  $y^2 = 4x$  and  $y = 2x$ . {Ans.  $\frac{1}{3}$ }
199. Find the area of the figure bounded by the curve  $|y| = 1 - x^2$ . {Ans.  $\frac{8}{3}$ }
200. Compute the area of one of the curvilinear triangles bounded by the  $x$ -axis and the curves  $y = \sin x$  and  $y = \cos x$ . {Ans.  $2 - \sqrt{2}$ }
201. Compute the area of the figure bounded by the  $x$ -axis and the curves  $y = \sin^{-1} x$  and  $y = \cos^{-1} x$ . {Ans.  $\sqrt{2} - 1$ }
202. Compute the area of the figure bounded by the parabolas  $y = x^2$  and  $y = \frac{x^3}{3}$ . {Ans.  $\frac{9}{4}$ }
203. Compute the area of the figure contained between the curve  $y = \frac{1}{1+x^2}$  and the parabola  $y = \frac{x^2}{2}$ . {Ans.  $\frac{\pi}{2} - \frac{1}{3}$ }
204. Compute the area of the figure bounded by the curve  $y = \ln x$ , the  $y$ -axis and the straight lines  $y = \ln a$  and  $y = \ln b$ . {Ans.  $b - a$ }
205. Compute the area of the curvilinear triangle bounded by the  $y$ -axis and the curves  $y = \tan x$  and  $y = \frac{2}{3}\cos x$ . {Ans.  $\frac{1}{3} + \ln \frac{\sqrt{3}}{2}$ }
206. Compute the area bounded by the  $x$ -axis and the curve  $y = x - x^2\sqrt{x}$ . {Ans.  $\frac{3}{14}$ }
207. Compute the area of the figure bounded by the curve  $y = x(x-1)^2$  and the  $x$ -axis. {Ans.  $\frac{1}{12}$ }
208. Compute the area of the figure bounded by the curves  $y = \ln x$  and  $y = \ln^2 x$ . {Ans.  $3 - e$ }
209. Compute the area of the figure bounded by the curves  $y = \frac{\ln x}{4x}$  and  $y = x \ln x$ . {Ans.  $\frac{3-2\ln 2-2\ln^2 2}{16}$ }
210. Find the area of the figure bounded by the curve  $y = \sin^3 x + \cos^3 x$  and the segment of the  $x$ -axis joining two successive points of intersection of the curve with the  $x$ -axis. {Ans.  $\frac{5}{3}\sqrt{2}$ }
211. Find the area of the finite portion of the figure bounded by the curves  $y = 2x^2e^x$  and  $y = -x^3e^x$ . {Ans.  $\frac{18}{e^2} - 2$ }
212. Find the area bounded by the curve  $y = (x^2 + 2x)e^{-x}$  and the  $x$ -axis. {Ans. 4}
213. Compute the area of the figure contained between the parabola  $y = -x^2 + 4x - 3$  and the tangents to it at

the points  $(0, -3)$  and  $(3, 0)$ . {Ans.  $\frac{9}{4}$ }

214. Compute the area of the figure bounded by the parabola  $y^2 = 6x$  and a normal to it inclined at an angle of  $135^\circ$  to the  $x$ -axis. {Ans. 48}

215. Compute the area bounded by the curve  $y = e^{-x}(x^2 + 3x + 1) + e^2$ , the  $x$ -axis and two straight lines parallel to the  $y$ -axis drawn through the points of extremum of the function  $y$ . {Ans.  $\frac{3}{e}(e^3 - 4)$ }

216. The circle  $x^2 + y^2 = 8$  is divided into two parts by the parabola  $y = \frac{x^2}{2}$ . Find the areas of both parts.  
{Ans.  $2\pi + \frac{4}{3}, 6\pi - \frac{4}{3}$ }

217. Compute the area of the figure bounded by the curves  $y^2 = 2x + 1$  and  $x - y - 1 = 0$ . {Ans.  $\frac{16}{3}$ }

218. Compute the area of the figure bounded by the parabolas  $y^2 + 8x = 16$  and  $y^2 - 24x = 48$ . {Ans.  $\frac{32}{3}\sqrt{6}$ }

219. Compute the areas of the curvilinear figures formed by intersection of the ellipse  $\frac{x^2}{4} + y^2 = 1$  and the hyperbola  $\frac{x^2}{2} - y^2 = 1$ . {Ans.  $\pi - \frac{\sqrt{2}}{2}\ln 3 - 2\sin^{-1}\sqrt{\frac{2}{3}}, \sqrt{2}\ln 3 + 4\sin^{-1}\sqrt{\frac{2}{3}}$ }

220. Find the area of a loop of the line  $y^2 = x(x - 1)^2$ . {Ans.  $\frac{8}{15}$ }

221. Find the area of the figure enclosed by the curve  $y^2 = (1 - x^2)^3$ . {Ans.  $\frac{3\pi}{4}$ }

222. Find the area of the figure enclosed by the curve  $y^2 = x^2 - x^4$ . {Ans.  $\frac{4}{3}$ }

223. Compute the area of the figure bounded by two branches of the curve  $(y - x)^2 = x^5$  and the straight line  $x = 4$ . {Ans.  $73\frac{1}{7}$ }

224. Compute the area of the figure bounded by the curve  $(y - x - 2)^2 = 9x$ ,  $x$ -axis and  $y$ -axis. {Ans.  $\frac{1}{2}$ }

225. Find the area of the finite portion of the figure bounded by the curve  $x^2y^2 = 4(x - 1)$  and the straight line passing through its points of inflection. {Ans.  $8\left[\sqrt{1 + \frac{2}{\sqrt{3}}} - \tan^{-1}\sqrt{1 + \frac{2}{\sqrt{3}}}\right]$ }

226. Find the area of the figure bounded by the  $y$ -axis and the curve  $x = y^2(y - 1)$ . {Ans.  $\frac{1}{12}$ }

227. Find the area of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . {Ans.  $\pi ab$ }

228. An ellipse is cut out of a circle whose radius is  $a$ . The major axis of the ellipse coincides with one of the diameters of the circle, while the minor axis is equal to  $2b$ . Prove that the area of the remaining part equals that of the ellipse with the semiaxes  $a$  and  $a - b$ .

229. Find the area of the figure bounded by an arc of a hyperbola and its Latus rectum. {Ans.  $b\left(be - a\ln\left(e + \sqrt{e^2 - 1}\right)\right)$ }

230. The circle  $x^2 + y^2 = a^2$  is divided into three parts by the hyperbola  $x^2 - 2y^2 = \frac{a^2}{4}$ . Determine the areas of these parts. {Ans.  $a^2\left[\frac{\pi}{6} - \frac{\sqrt{2}}{8}\ln(\sqrt{3} + \sqrt{2})\right], a^2\left[\frac{2\pi}{3} + \frac{\sqrt{2}}{4}\ln(\sqrt{3} + \sqrt{2})\right]$ }

231. Find the area of the figure enclosed by the curve  $x^4 - ax^3 + a^2y^2 = 0$ . {Ans.  $\frac{\pi a^2}{8}$ }

232. Find the area enclosed by the curves  $3x^2 + 5y = 32$  and  $y = |x - 2|$ . {Ans.  $\frac{33}{2}$  sq. units}

233. Find the area of the region bounded by the curves  $y = x^2$ ,  $y = |2 - x^2|$  and  $y = 2$ . {Ans.  $\frac{20}{3} - 4\sqrt{2}$  sq. units}

234. Find the area given by  $x + y \leq 6$ ,  $x^2 + y^2 \leq 6y$  and  $y^2 \leq 8x$ . {Ans.  $\frac{1}{12}(27\pi - 2)$  sq. units}

**CATEGORY-24.8.    ADDITIONAL QUESTIONS**

# Mathematics for IIT-JEE

By **Er. Sanjiva Dayal,** *B.Tech. (I.I.T. Kanpur)*

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*PART-VII*  
*ALGEBRA*

*CHAPTER-25*  
*COMPLEX NUMBERS*

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## ***CHAPTER-25***

### ***COMPLEX NUMBERS***

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#### **LIST OF THEORY SECTIONS**

- 25.1. Definition Of Complex Numbers
- 25.2. Mathematical Operations On Complex Numbers In Algebraic Form
- 25.3. Argand Plane, Modulus, Argument
- 25.4. Polar Representation Of Complex Number
- 25.5. Eulerian Representation Of Complex Number
- 25.6. Power Of Complex Number
- 25.7. Equations In Complex Numbers
- 25.8. Proving Trigonometric Identities By Complex Numbers
- 25.9. Points
- 25.10. Straight Line
- 25.11. Circle
- 25.12. Conic Sections

#### **LIST OF QUESTION CATEGORIES**

- 25.1. Equality, Conjugate, Mathematical Operations On Complex Numbers In Algebraic Form
- 25.2. Modulus And Argument Of A Complex Number
- 25.3. Polar (Trigonometric) Representation Of Complex Number, De Moivre's Theorem
- 25.4. Eulerian Representation Of Complex Number
- 25.5. Integer Powers Of A Complex Number
- 25.6. Square Root Of A Complex Number In Algebraic Form
- 25.7. Roots Of A Complex Number In Polar/ Eulerian Form
- 25.8. Cube Roots Of Unity
- 25.9. Higher Roots Of Unity
- 25.10. Irrational And Non-Real Complex Powers Of A Complex Number
- 25.11. Logarithm Of A Complex Number
- 25.12. Equations In Complex Numbers
- 25.13. Complex Polynomials
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## CHAPTER-25

### COMPLEX NUMBERS

#### SECTION-25.1. DEFINITION OF COMPLEX NUMBERS

##### 1. Number system

- i. Natural numbers
- ii. Integers
- iii. Rational numbers
- iv. Irrational numbers
- v. Real numbers

##### 2. Definition of $i$

$i$  is a non-real complex number such that  $i^2 = -1$ .

##### 3. Integer powers of $i$

- i.  $i^0 = 1$
- ii.  $i^1 = i$
- iii.  $i^2 = -1$
- iv.  $i^3 = -i$
- v.  $i^4 = 1$

##### 4. Definition of Complex numbers

- i. If  $a, b$  are two real numbers, then a number of the form  $a + ib$  is called a complex number, generally denoted by  $z$ .
- ii.  $z = a + ib$  is called algebraic representation of the complex number  $z$ .
- iii. If a complex number  $z = a + ib$ , then  $a$  is called the real part of  $z$ , denoted by  $\text{Re}(z)$ , and  $b$  is called the imaginary part of  $z$ , denoted by  $\text{Im}(z)$ . Hence,  $\text{Re}(z) = a$  and  $\text{Im}(z) = b$ .
- iv. If  $\text{Im}(z) = 0$ , then the complex number is purely real. If  $\text{Re}(z) = 0$ , then the complex number is purely imaginary. 0 is purely real as well as purely imaginary number.
- v. If  $\text{Im}(z) \neq 0$ , then the complex number is non-real complex number.
- vi. Set of complex numbers is denoted by  $C$ . Set of complex numbers consists of real numbers and non-real complex numbers. Therefore,  $R \subset C$ .

##### 5. Equality of complex numbers

- i.  $a + ib = 0 \Rightarrow a = 0, b = 0$ .
- ii.  $a + ib = c + id \Rightarrow a = c, b = d$ .
- iii. 'Greater than' and 'Less than' relation is not defined for complex numbers.

##### 6. Conjugate of a complex number

If  $z = a + ib$  is a complex number, then its conjugate is denoted by  $\bar{z}$  and  $\bar{z} = a - ib$ .

#### SECTION-25.2. MATHEMATICAL OPERATIONS ON COMPLEX NUMBERS IN ALGEBRAIC FORM

##### 1. Basic mathematical operations on complex numbers in algebraic form

If  $z_1 = a_1 + ib_1$ ,  $z_2 = a_2 + ib_2$ , then

- i.  $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
- ii.  $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$

$$\text{iii. } z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$\text{iv. } \frac{z_1}{z_2} = \left( \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) + i \left( \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right)$$

## 2. Properties of conjugate

- i.  $\overline{\overline{z}} = z$
- ii.  $\bar{z} = z \Leftrightarrow z$  is purely real.
- iii.  $\bar{z} = -z \Leftrightarrow z$  is purely imaginary.
- iv.  $z + \bar{z} = 2\operatorname{Re}(z)$
- v.  $z - \bar{z} = 2i\operatorname{Im}(z)$
- vi.  $z\bar{z} = (\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2$
- vii.  $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- viii.  $\overline{z_1 + z_2 + z_3 + \dots} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \dots$
- ix.  $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
- x.  $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- xi.  $\overline{z_1 z_2 z_3 \dots} = \bar{z}_1 \bar{z}_2 \bar{z}_3 \dots$
- xii.  $\overline{\left( \frac{z_1}{z_2} \right)} = \frac{\bar{z}_1}{\bar{z}_2}$

## SECTION-25.3. ARGAND PLANE, MODULUS, ARGUMENT

### 1. Geometrical representation of complex numbers on Argand plane

- i. A complex number can be represented by a unique point on a plane which is known as the Argand plane w.r.t. real and imaginary axes.
- ii. If  $\operatorname{Re}(z) > 0$ ,  $\operatorname{Im}(z) > 0$  then  $z$  lies in 1<sup>st</sup> quadrant. If  $\operatorname{Re}(z) < 0$ ,  $\operatorname{Im}(z) > 0$  then  $z$  lies in 2<sup>nd</sup> quadrant. If  $\operatorname{Re}(z) < 0$ ,  $\operatorname{Im}(z) < 0$  then  $z$  lies in 3<sup>rd</sup> quadrant. If  $\operatorname{Re}(z) > 0$ ,  $\operatorname{Im}(z) < 0$  then  $z$  lies in 4<sup>th</sup> quadrant.

### 2. Position vector of a complex number

- i. Vector joining 0 to  $z$  on Argand plane is called position vector of complex number  $z$ , denoted by  $p.v.(z)$ .
- ii.  $p.v.(-z) = -p.v.(z)$ .
- iii.  $z$  and  $\bar{z}$  are symmetrical about real axis and  $p.v.(\bar{z})$  and  $p.v.(z)$  are symmetrical about real axis.

### 3. Geometrical addition and subtraction of complex numbers on Argand plane

- i.  $p.v.(z_1 + z_2) = p.v.(z_1) + p.v.(z_2)$ .
- ii.  $p.v.(z_1 - z_2) = p.v.(z_1) - p.v.(z_2)$ .

### 4. Modulus of a complex number

- i. Modulus of a complex number  $z$ , denoted by  $|z|$ , is its distance from 0 on the Argand plane.
- ii.  $|0| = 0$ .
- iii.  $|z|$  is a non-negative real number.
- iv.  $|z| = |p.v.(z)|$
- v. If  $z = a + ib$ , then  $r = |z| = \sqrt{a^2 + b^2}$ .



vi. If  $|z| = 1$ , then  $z$  is called a unimodular complex number.

### 5. Properties of modulus

- i.  $|\bar{z}| = |z| = |-z|$
- ii.  $z\bar{z} = |z|^2$
- iii.  $|z_1 z_2| = |z_1| |z_2|$
- iv.  $|z_1 z_2 \cdots z_n| = |z_1| |z_2| \cdots |z_n|$
- v.  $|z^n| = |z|^n$
- vi.  $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$
- vii.  $|z_1 + z_2| \leq |z_1| + |z_2|$  (Triangle inequality)
- viii.  $|z_1 - z_2| \geq ||z_1| - |z_2||$  (Triangle inequality)
- ix.  $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$
- x. If  $|z| = 1$ , then  $\bar{z} = \frac{1}{z}$ .

### 6. Argument (Amplitude) of a complex number

- i. Argument (Amplitude) of a complex number  $z$ , denoted by  $\arg(z)$  or  $\text{amp}(z)$ , is the angle which p.v.( $z$ ) makes with positive real axis.
- ii. Argument of 0 is not defined.
- iii. If  $\arg(z) = \theta$ , then Principal value of argument, denoted by  $\theta_p$ , is such that  $-\pi < \theta_p \leq \pi$ .
- iv. If  $\theta_p$  is the Principal value of argument then  $\theta = \theta_p + 2n\pi$  is also an argument.
- v. If  $z = a + ib$ , then  $\tan \theta = \frac{b}{a}$  and

$$\begin{aligned}\theta_p = \text{Principal value of } \arg(z) &= \tan^{-1}\left(\frac{b}{a}\right), \quad a > 0 \\ &= \pi + \tan^{-1}\left(\frac{b}{a}\right), \quad a < 0, b > 0 \\ &= -\pi + \tan^{-1}\left(\frac{b}{a}\right), \quad a < 0, b < 0.\end{aligned}$$

$$\text{vi. } \cos \theta = \frac{a}{\sqrt{a^2 + b^2}}, \sin \theta = \frac{b}{\sqrt{a^2 + b^2}}, \text{ where } \theta = \arg(z).$$

$$\text{vii. } a = r \cos \theta, b = r \sin \theta, \text{ where } \theta = \arg(z).$$

### 7. Properties of argument

- i.  $\arg(-z) = \arg(z) + \pi$
- ii.  $\arg(\bar{z}) = -\arg(z)$
- iii.  $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- iv.  $\arg(z_1 z_2 \cdots z_n) = \arg(z_1) + \arg(z_2) + \cdots + \arg(z_n)$

$$\text{v. } \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

### SECTION-25.4. POLAR REPRESENTATION OF COMPLEX NUMBER

#### 1. Polar (Trigonometric) representation of a complex number

- $z = a + ib = r(\cos\theta + i\sin\theta)$
- $\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$
- A complex number represented in algebraic form can be converted to polar form and vice versa.

#### 2. Mathematical operations on complex numbers in Polar form

- $r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) = (r_1 r_2)(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$
- $\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \left(\frac{r_1}{r_2}\right)(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$
- $r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \cdots = (r_1 r_2 \cdots)(\cos(\theta_1 + \theta_2 + \cdots) + i\sin(\theta_1 + \theta_2 + \cdots))$
- $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta, \quad n \in \mathbb{Q}$ . (De Moivre's theorem)
- $[r(\cos\theta \pm i\sin\theta)]^n = r^n(\cos n\theta \pm i\sin n\theta), \quad n \in \mathbb{Q}$ .

### SECTION-25.5. EULERIAN REPRESENTATION OF COMPLEX NUMBER

#### 1. Series

##### i. Sine series

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

##### ii. Cosine series

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

##### iii. Exponential series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

##### iv. Logarithmic series

$$\text{If } -1 < |z| \leq 1, \text{ then } \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

#### 2. Euler's identity

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

#### 3. Eulerian representation of a complex number

- $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ .
- $\bar{z} = re^{-i\theta}$
- A complex number represented in algebraic form can be converted to eulerian form and vice versa.

#### 4. Mathematical operations on complex numbers in Eulerian form

- $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

$$\text{iii. } (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) \dots = (r_1 r_2 \dots) e^{i(\theta_1 + \theta_2 + \dots)}$$

$$\text{iv. } (r e^{i\theta})^n = r^n e^{in\theta}$$

### 5. Geometrical multiplication and division of complex numbers on Argand plane

- i. Given  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$ , then  $p.v.(z_1 z_2)$  has length  $r_1 r_2$  and makes angle  $\theta_1 + \theta_2$  with real axis.
- ii.  $p.v.\left(\frac{z_1}{z_2}\right)$  has length  $\frac{r_1}{r_2}$  and makes angle  $\theta_1 - \theta_2$  with real axis.

## SECTION-25.6. POWER OF COMPLEX NUMBER

### 1. Integer power of a complex number

$$\text{i. } z^n = z \times z \times z \times \dots \times z \text{ } n \text{ times, } n \in N.$$

$$\text{ii. } z^{-n} = \frac{1}{z^n}, z \neq 0, n \in N.$$

$$\text{iii. } z^0 = 1, z \neq 0.$$

### 2. Square roots of a complex number in algebraic form

If  $z = a + ib$ , then the two square roots of  $z$  are:-

$$\text{i. } \sqrt{a + ib} = \pm \left( \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} + i \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}} \right), \text{ if } b > 0.$$

$$\text{ii. } \sqrt{a + ib} = \pm \left( \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} - i \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}} \right), \text{ if } b < 0.$$

### 3. Roots of a complex number in Polar/ Eulerian form

$$\text{i. If } z_0 = a + ib = r_0 (\cos \theta_0 + i \sin \theta_0) = r_0 e^{i\theta_0}, \text{ then the } n^{\text{th}} \text{ roots of } z_0 \text{ are}$$

$$\alpha_k = r_0^{\frac{1}{n}} e^{i \left( \frac{\theta_0 + 2k\pi}{n} \right)}, \quad k = 0, 1, 2, \dots, n-1.$$

$$\text{ii. If } \alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{n-1} \text{ are } n^{\text{th}} \text{ roots of a complex number } z_0, \text{ then } \alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{n-1} \text{ are } n \text{ roots of the polynomial } z^n - z_0 = 0.$$

$$\text{iii. } z^n - z_0 = (z - \alpha_0)(z - \alpha_1) \dots (z - \alpha_{n-1}).$$

$$\text{iv. } \alpha_0 + \alpha_1 + \dots + \alpha_{n-1} = 0.$$

### 7. Properties of $n$ th roots of unity

$$\text{i. The } n^{\text{th}} \text{ roots of unity are } \alpha_k = e^{i \frac{2k\pi}{n}} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k = 0, 1, \dots, n-1.$$

$$\text{ii. } \alpha_k^n = 1.$$

$$\text{iii. } \alpha_0 = 1 \text{ is always a root.}$$

$$\text{iv. } |\alpha_k| = 1, \text{ therefore roots lie on the unit circle.}$$

$$\text{v. Angular difference is same } = \frac{2\pi}{n}.$$

$$\text{vi. Roots are in G.P. with common ratio } = e^{i \frac{2\pi}{n}}.$$

$$\text{vii. } z^n - 1 = (z - \alpha_0)(z - \alpha_1) \dots (z - \alpha_{n-1})$$

$$\text{viii. } z^{n-1} + z^{n-2} + \dots + z^2 + z + 1 = (z - \alpha_1) \dots (z - \alpha_{n-1})$$

$$\text{ix. } \alpha_0 + \alpha_1 + \dots + \alpha_{n-1} = 0 \Rightarrow \begin{cases} 1 + \cos \frac{2\pi}{n} + \dots + \cos \frac{2(n-1)\pi}{n} = 0 \\ \sin \frac{2\pi}{n} + \dots + \sin \frac{2(n-1)\pi}{n} = 0 \end{cases}$$

$$\text{x. } \alpha_0 \alpha_1 \dots \alpha_{n-1} = 1, \quad n \text{ is odd} \\ = -1, \quad n \text{ is even}$$

xi. Non-real complex roots are in conjugate pairs.

### 8. Properties of cube roots of unity

i. The three cube roots of unity are

$$e^{i0} = \cos 0 + i \sin 0 = 1,$$

$$e^{i\frac{2\pi}{3}} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ and}$$

$$e^{i\frac{4\pi}{3}} = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

ii. The two non-real complex cube roots of unity, i.e.  $e^{i\frac{2\pi}{3}}$  and  $e^{i\frac{4\pi}{3}}$ , are denoted by  $\omega$  and  $\omega^2$  respectively.

$$\text{iii. } \omega^3 = 1$$

iv. 1 is a cube root of unity.

$$\text{v. } |\omega| = |\omega^2| = 1 \Rightarrow 1, \omega, \omega^2 \text{ lie on unit circle.}$$

$$\text{vi. Angular difference} = \frac{2\pi}{3}.$$

vii.  $1, \omega, \omega^2$  are in G.P.

$$\text{viii. } z^3 - 1 = (z - 1)(z - \omega)(z - \omega^2)$$

$$\text{ix. } z^2 + z + 1 = (z - \omega)(z - \omega^2)$$

$$\text{x. } 1 + \omega + \omega^2 = 0.$$

$$\text{xi. } 1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$$

$$\text{xii. } \overline{\omega} = \omega^2 \text{ and } \overline{\omega^2} = \omega$$

### 9. Irrational and non-real complex powers of a complex number

If  $z_1$  is a complex number and  $z_2$  is a non-real complex number then

$$z_1^{z_2} = (re^{i\theta})^{a+ib} = r^a r^{ib} e^{i\theta a} e^{-\theta b} = (r^a e^{-\theta b}) e^{i(\theta a + b \ln r)}, \text{ is one of the values. } z_1^{z_2} \text{ has infinite values for } \theta + 2n\pi.$$

### 10. Logarithm of a complex number

If  $z$  is a non-zero complex number and  $z = re^{i\theta}$ , then

$\ln z = \ln r + i\theta$  is one of the values.  $\ln z$  has infinite values for  $\theta + 2n\pi$ .

## SECTION-25.7. EQUATIONS IN COMPLEX NUMBERS

### 1. Solving equations in complex numbers

i. Some equations can be solved directly using same methods used for solving real equations.

- ii. Substitute  $z = x + iy$ , and get real equations in  $x, y$  and solve for  $x, y$ .
- iii. Substitute  $z = r(\cos\theta + i\sin\theta)$ , and get real equations in  $r, \theta$  and solve for  $r, \theta$ .
- iv. Substitute  $z = re^{i\theta}$ , and get real equations in  $r, \theta$  and solve for  $r, \theta$ .

### SECTION-25.8. PROVING TRIGONOMETRIC IDENTITIES BY COMPLEX NUMBERS

#### 1. Proving trigonometric identities by complex numbers

$$(\cos\theta + i\sin\theta)^n = \cos^n\theta + i \cdot {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta - i \cdot {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \dots$$

Therefore,

$$\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \dots$$

$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \dots$$

### SECTION-25.9. POINTS

#### 1. Representation of a point in two dimensional plane by complex number

- i. A point  $P$  in two dimensional Argand plane is represented by its corresponding complex number  $z$ , denoted by  $P(z)$ . Complex number  $z$  is called the affix of point  $P$ .
- ii. If the origin of coordinate system is at  $O$  and  $x$ -axis and  $y$ -axis are along real axis and imaginary axis respectively, then the coordinates of point  $P(x + iy)$  is  $(x, y)$ .

#### 2. Equation of a curve/ region

- i. The equation of a curve/ region is the relation involving the affix of every point on the curve/ region, and which holds for no other point except for those lying on the curve/ region.
- ii. Equation of real axis is  $\text{Im}(z) = 0$  or  $z - \bar{z} = 0$ ; equation of imaginary axis is  $\text{Re}(z) = 0$  or  $z + \bar{z} = 0$ .

#### 3. Converting complex equation to coordinate equation and vice versa

- i. For converting a complex equation of a curve  $f(z) = 0$ , substitute  $z = x + iy$  to get the coordinate equation of the curve as  $g(x, y) = 0$ .
- ii. For converting a coordinate equation of the curve  $g(x, y) = 0$ , substitute  $x = \frac{z + \bar{z}}{2}$  and  $y = \frac{z - \bar{z}}{2i}$  to get the complex equation of the curve as  $f(z) = 0$ .

#### 4. Distance formula

- i. Distance between the points  $P(z_1)$  and  $Q(z_2)$  is given by  $PQ = |z_2 - z_1|$ .
- ii. Therefore, distance of the point  $P(z_1)$  from origin is  $OP = |z_1|$ .

#### 5. Section formula

- i. Affix of the point which divides the line segment joining points  $P(z_1)$  and  $Q(z_2)$  internally in the ratio

$$m_1 : m_2 \text{ is } \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}.$$

- ii. Affix of mid-point of the line segment joining points  $P(z_1)$  and  $Q(z_2)$  is  $\frac{z_1 + z_2}{2}$ .

- iii. Affix of the point which divides the line segment joining points  $P(z_1)$  and  $Q(z_2)$  externally in the ratio

$$m_1 : m_2 \text{ is } \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}.$$

**6. Condition for three points to be collinear**

Three points  $P(z_1)$ ,  $Q(z_2)$ ,  $R(z_3)$  are collinear if  $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$  and non-collinear if  $\neq 0$ .

**7. Centroid of a triangle**

Centroid of the triangle with vertices  $A(z_1)$ ,  $B(z_2)$ ,  $C(z_3)$  is  $\frac{z_1 + z_2 + z_3}{3}$ .

**SECTION-25.10. STRAIGHT LINE****1. Equation of straight line in various forms**

i. Equation of straight line passing through point  $z_0$  and parallel to the position vector of  $z_1$  is  $z = z_0 + \lambda z_1$ ,  $\lambda \in R$  (Parametric form) or  $\bar{z}_1 z - z_1 \bar{z} + \bar{z}_0 z_1 - z_0 \bar{z}_1 = 0$  (Non-parametric form). This line makes angle  $\arg(z_1)$  with positive real axis. Slope of this line is  $\frac{\text{Im}(z_1)}{\text{Re}(z_1)} = \frac{z_1 - \bar{z}_1}{(z_1 + \bar{z}_1)i}$ .

ii. Equation of straight line passing through points  $z_1$  and  $z_2$  is  $z = z_1 + \lambda(z_2 - z_1)$ ,  $\lambda \in R$  (Parametric form) or  $(\bar{z}_1 - \bar{z}_2)z - (z_1 - z_2)\bar{z} + z_1 \bar{z}_2 - \bar{z}_1 z_2 = 0$  (Non-parametric form)

$$\text{or } \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

iii. General equation of a straight line is of the form  $\bar{a}z + a\bar{z} + b = 0$ , where  $a \in C$ ,  $b \in R$ .

$\bar{a}z + a\bar{z} + b = 0 \equiv z = -\frac{b}{2\bar{a}} + \lambda \frac{i}{a}$ . Therefore, this line passes through the point  $-\frac{b}{2\bar{a}}$  and is parallel to the p.v. of complex number  $\frac{i}{a}$  or  $ia$ .

**2. Equation of perpendicular bisector of a line segment**

Equation of perpendicular bisector of line segment joining points  $z_1$  and  $z_2$  is  $|z - z_1| = |z - z_2|$  or  $(\bar{z}_1 - \bar{z}_2)z + (z_1 - z_2)\bar{z} = |z_1|^2 - |z_2|^2$ .

**3. Distance of a point from a line**

i. The length of perpendicular from a point  $z_1$  on the line  $\bar{a}z + a\bar{z} + b = 0$  is  $\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$ .

ii. Length of perpendicular from origin on the line  $\bar{a}z + a\bar{z} + b = 0$  is  $\frac{|b|}{2|a|}$ .

**4. Angle between two lines**

If a line is parallel to the position vector of  $z_1$  and another line is parallel to the position vector of  $z_2$  then

the angle between them is  $\left| \arg\left(\frac{z_1}{z_2}\right) \right|$ .

**SECTION-25.11. CIRCLE****1. Equation of a circle in various forms**

i. Equation of a circle having centre at  $z_0$  and radius  $r$  is  $|z - z_0| = r$  or  $z\bar{z} - z_0\bar{z} - \bar{z}_0z + |z_0|^2 - r^2 = 0$ .

ii. General equation of a circle is  $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ ,  $a \in C$ ,  $b \in R$ ,  $|a|^2 > b$ , having centre  $-a$  and radius  $= \sqrt{|a|^2 - b}$ .

iii. Equation of a circle with  $z_1, z_2$  as end-points of diameter is

$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

$$\text{or } \arg\left(\frac{z - z_1}{z - z_2}\right) = \pm \frac{\pi}{2}$$

$$\text{or } \frac{z - z_1}{z - z_2} \text{ is purely imaginary}$$

$$\text{or } \frac{z - z_1}{z - z_2} = -\overline{\left(\frac{z - z_1}{z - z_2}\right)}$$

$$\text{or } (z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0.$$

iv. Equation of circle passing through three non-collinear points  $z_1, z_2, z_3$  is

$$\left(\frac{z - z_3}{z - z_1}\right)\left(\frac{z_2 - z_1}{z_2 - z_3}\right) \text{ is purely real}$$

$$\text{or } \left(\frac{z - z_3}{z - z_1}\right)\left(\frac{z_2 - z_1}{z_2 - z_3}\right) = \left(\frac{\bar{z} - \bar{z}_3}{\bar{z} - \bar{z}_1}\right)\left(\frac{\bar{z}_2 - \bar{z}_1}{\bar{z}_2 - \bar{z}_3}\right).$$

**2. Condition for four points being concyclic**

Four points  $z_1, z_2, z_3, z_4$  are concyclic if

$$\left(\frac{z_4 - z_3}{z_4 - z_1}\right)\left(\frac{z_2 - z_1}{z_2 - z_3}\right) \text{ is purely real, otherwise they are not concyclic.}$$

**SECTION-25.12. CONIC SECTIONS****1. General equation of a conic section**

Equation of a conic section having focus at  $z_0$ , directrix  $\bar{a}z + a\bar{z} + b = 0$  and eccentricity  $e$  is

$$|z - z_0| = e \frac{|\bar{a}z + a\bar{z} + b|}{2|a|}.$$

**2. Equation of an ellipse**

Equation of an ellipse having foci  $z_1, z_2$  and length of major axis  $2a$  is

$$|z - z_1| + |z - z_2| = 2a, (|z_1 - z_2| < 2a).$$

**3. Equation of a hyperbola**

Equation of an hyperbola having foci  $z_1, z_2$  and length of transverse axis  $2a$  is

$$|z - z_1| - |z - z_2| = 2a, (|z_1 - z_2| > 2a).$$

**EXERCISE-25****CATEGORY-25.1. EQUALITY, CONJUGATE, MATHEMATICAL OPERATIONS ON COMPLEX NUMBERS IN ALGEBRAIC FORM**

1. Simplify the following and put in algebraic form:-

i.  $i^{457}$  {Ans.  $i$ }

ii.  $(-i)^{4n+3}$  {Ans.  $i$ }

iii.  $\frac{1}{1+i}$  {Ans.  $\frac{1}{2} - \frac{1}{2}i$ }

iv.  $\frac{1-i}{1+i}$  {Ans.  $-i$ }

v.  $\frac{5+4i}{4+5i}$  {Ans.  $\frac{40}{41} - \frac{9}{41}i$ }

vi.  $\frac{(1+i)^2}{3-i}$  {Ans.  $-\frac{1}{5} + \frac{3}{5}i$ }

vii.  $\left(\frac{1-i}{1+i}\right)^2$  {Ans.  $-1$ }

viii.  $\left[\frac{2i}{(1+i)}\right]^2$  {Ans.  $2i$ }

ix.  $\frac{(1-i)^3}{1-i^3}$  {Ans.  $-2$ }

x.  $\left(\frac{1+i}{1-i}\right)^{4n+1}$  {Ans.  $i$ }

xi.  $\frac{1+2i}{1-(1-i)^2}$  {Ans.  $1$ }

xii.  $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$  {Ans.  $\frac{63}{25} - \frac{16}{25}i$ }

xiii.  $\frac{1+i}{1-i} - \frac{1-i}{1+i}$  {Ans.  $2i$ }

xiv.  $\frac{3-i}{2+i} + \frac{3+i}{2-i}$  {Ans.  $2$ }

xv.  $\frac{3}{1+i} - \frac{2}{2-i} + \frac{2}{1-i}$  {Ans.  $\frac{17}{10} - \frac{9}{10}i$ }

xvi.  $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$  {Ans.  $\frac{1}{4} + \frac{9}{4}i$ }

xvii.  $i^{57} + \frac{1}{i^{125}}$  {Ans.  $0$ }

xviii.  $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$  {Ans.  $-1$ }



xix.  $(1+i)^4 + (1-i)^4$ . {Ans. -8}

xx.  $i^2 + i^4 + i^6 + \dots (2n+1)$  terms. {Ans. -1}

xxi.  $\frac{(i^5 + i^6 + i^7 + i^8 + i^9)}{(1+i)}$ . {Ans.  $\frac{1}{2}(1+i)$ }

xxii.  $\frac{1+2i+3i^2}{1-2i+3i^2}$ . {Ans.  $-i$ }

xxiii.  $\frac{i^{4n+3} + (-i)^{8n-3}}{(-i)^{12n-1} - (i)^{2-16n}}$ ,  $n \in N$ . {Ans.  $-1-i$ }

xxiv.  $\frac{i^{4n+1} - i^{4n-1}}{2}$ . {Ans.  $i$ }

xxv.  $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ . {Ans. 0}

xxvi.  $\frac{(a+ib)^2}{(a-ib)} - \frac{(a-ib)^2}{(a+ib)}$  {Ans.  $\frac{2b(3a^2-b^2)}{a^2+b^2}i$ }

2. Find real values of  $x$  and  $y$  for which the following equations are satisfied:-

i.  $(1-i)x + (1+i)y = 1-3i$ , {Ans.  $x=2, y=-1$ }

ii.  $\frac{x-1}{3+i} + \frac{y-i}{3-i} = i$  {Ans.  $x = -\frac{17}{3}, y = \frac{19}{3}$ }

iii.  $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$  {Ans.  $x=3, y=-1$ }

iv.  $(x+iy)(2-3i) = 4+i$  {Ans.  $x = \frac{5}{13}, y = \frac{14}{13}$ }

v.  $(x+iy)(1+i) = 1-5i$ . {Ans.  $x=-2, y=-3$ }

vi.  $(1+i)y^2 + 6+i = (2+i)x$  {Ans.  $x=5, y=2$  or  $x=5, y=-2$ }

vii.  $2i^2 + 6i^3 + 3i^{16} - 6i^{19} + 4i^{25} = x+iy$ . {Ans.  $x=1, y=4$ }

viii.  $\sqrt{x^2 - 2x + 8} + (x+4)i = y(2+i)$  {Ans.  $x=-2, y=2$ }

ix.  $(x^4 + 2xi) - (3x^2 + yi) = (3+5i) + (1+2yi)$  {Ans.  $x=2, y=3$  or  $x=-2, y=\frac{1}{3}$ }

3. If  $z$  is a complex number such that  $z \neq 0$  and  $\operatorname{Re}(z) = 0$ , then show that  $\operatorname{Im}(z^2) = 0$ .

4. If  $x+iy = \frac{3+5i}{7-6i}$ , then find  $y$ . {Ans.  $\frac{53}{85}$ }

5. If  $z = \frac{1}{1-\cos\theta - i\sin\theta}$ , then find  $\operatorname{Re}(z)$ . {Ans.  $\frac{1}{2}$ }

6. If  $\frac{1}{1+\cos\theta + i\sin\theta} = x+iy$ , then find the value of  $x^2$ . {Ans.  $\frac{1}{2}$ }

7. If  $(x+iy)(p+iq) = (x^2 + y^2)i$ , then show that  $x=q, y=p$ .

8. If  $C^2 + S^2 = 1$ , then show that  $\frac{1+C+iS}{1+C-iS}$  is equal to  $C+iS$ .

9. If  $x = \frac{-5+i\sqrt{3}}{2}$ , then find the value of  $(x^2 + 5x)^2 + x(x+5)$ . {Ans. 42}

10. Find real  $\theta$  such that  $\frac{3+2i\sin\theta}{1-2i\sin\theta}$  is (i) real (ii) purely imaginary. {Ans.  $\theta = n\pi, \theta = n\pi \pm \frac{\pi}{3}$ }
11. Find the value of  $1+i^2+i^4+i^6+\dots+i^{2n}$ . {Ans. cannot be determined}
12. If  $\frac{1+x}{1-x} = \cos 2\theta + i \sin 2\theta$ , then find the value of  $x$ . {Ans.  $i \tan \theta$ }
13. Show that a real value of  $x$  will satisfy the equation  $\frac{1-ix}{1+ix} = a - ib$  if  $a^2 + b^2 = 1$  ( $a, b$  real).
14. If  $\frac{3}{2+\cos\theta+i\sin\theta} = x+iy$ , then show that  $(x-1)(x-3)$  equals  $-y^2$ .
15. For what real values of  $x$  and  $y$  are the numbers  $-3+ix^2y$  and  $x^2+y+4i$  conjugate complex? {Ans.  $(1,-4), (-1,-4)$ }
16. If the complex numbers  $\sin x + i \cos 2x$  and  $\cos x - i \sin 2x$  are conjugate to each other, then find  $x$ .
17. For what real values of  $x$  and  $y$  are the complex numbers  $x^2 - 7x + 9yi$  and  $y^2i + 20i - 12$  equal? {Ans.  $(3,4), (3,5), (4,4), (4,5)$ }
18.  $a^2 + b^2 + c^2 = 1, b+ic = (1+a)z$ , prove that  $\frac{a+ib}{1+c} = \frac{1+iz}{1-iz}$  where  $a, b, c$  are real numbers and  $z$  is a complex number.
19. If  $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ , then find  $A^2$ . {Ans.  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ }
20. If  $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , then show that  $AB + BA = O$ .
21. If  $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ , then show that  $(A+B)(A-B)$  is equal to  $A^2 - B^2$ .
22. If  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ , then show that  $(A+B)^2$  equals  $A^2 + B^2$ .
23. For a matrix  $A = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$ , show that  $A^3 + A = O$ .

### CATEGORY-25.2. MODULUS AND ARGUMENT OF A COMPLEX NUMBER

24. In which quadrant the complex number  $\left(\frac{1+2i}{1-i}\right)$  lies? {Ans. II quadrant}
25. Find the modulus of the following numbers:-
- i.  $z = \frac{(1-i\sqrt{3})(\cos\theta+i\sin\theta)}{2(1-i)(\cos\theta-i\sin\theta)}$ . {Ans.  $\frac{1}{\sqrt{2}}$ }
- ii.  $z = \frac{1}{(1-i)(2+3i)}$ . {Ans.  $\frac{1}{\sqrt{26}}$ }
- iii.  $z = \frac{1}{(2+3i)^2}$ . {Ans.  $\frac{1}{13}$ }

iv.  $z = 1 - \cos\theta + i\sin\theta$ . {Ans.  $2\left|\sin\frac{\theta}{2}\right|$ }

26. Find the argument of the following numbers:-

i.  $\frac{1-i\sqrt{3}}{1+i\sqrt{3}}$ . {Ans.  $\frac{4\pi}{3}$ }

ii.  $\frac{1+i\sqrt{3}}{\sqrt{3}+1}$ . {Ans.  $\frac{\pi}{3}$ }

iii.  $\frac{1+i\sqrt{3}}{\sqrt{3}+i}$ . {Ans.  $\frac{\pi}{6}$ }

iv.  $\frac{1}{i}$ . {Ans.  $-\frac{\pi}{2}$ }

v.  $\frac{1+7i}{(2-i)^2}$ . {Ans.  $\frac{3\pi}{4}$ }

vi.  $\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}$ . {Ans.  $\frac{\pi}{2}$ }

vii.  $\frac{1+2i}{1-(1-i)^2}$ . {Ans. 0}

27. Find the modulus and the principal value of the argument of the following numbers:-

i.  $1-i$  {Ans.  $\sqrt{2}, -\frac{\pi}{4}$ }

ii.  $-1-\sqrt{3}i$  {Ans.  $2, -\frac{2\pi}{3}$ }

iii.  $1+\sqrt{2}+i$  {Ans.  $\sqrt{4+2\sqrt{2}}, \frac{\pi}{8}$ }

28. Find the complex number in algebraic form if

i.  $|z| = \sqrt{2}, \arg(z) = \frac{3\pi}{4}$

ii.  $|z| = 2, \arg(z) = -\frac{2\pi}{3}$

iii.  $|z| = 3, \arg(z) = \frac{\pi}{12}$

iv.  $|z| = 4, \arg(z) = \frac{5\pi}{6}$ . {Ans.  $-2\sqrt{3}+2i$ }

29. If  $z \neq 0$  be a complex number and  $\arg(z) = \frac{\pi}{4}$ , then show that  $\operatorname{Re}(z) = \operatorname{Im}(z) > 0$ .

30. If a complex number  $z$  has argument  $\frac{\pi}{4}$  and modulus unity, then find the modulus of  $z^2 + z$ . {Ans.  $\sqrt{2+\sqrt{2}}$ }

31. Show that the amplitude of  $\frac{a+ib}{a-ib}$  is equal to  $\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$ .

32. If  $x+iy = (1+i)(1+2i)(1+3i)$ , then find the value of  $x^2 + y^2$ . {Ans. 100}

33. If  $z = (\cos\theta - i\sin\theta)^2 = x + iy$ , then find the value of  $x^2 + y^2$ . {Ans. 1}
34. If  $(1+i)(1+2i)(1+3i)\dots(1+ni) = x + iy$ , show that  $2.5.10\dots(1+n^2) = x^2 + y^2$ .
35. If  $(a_1 + ib_1)(a_2 + ib_2)\dots(a_n + ib_n) = A + iB$ , then find the value of  $(a_1^2 + b_1^2)(a_2^2 + b_2^2)\dots(a_n^2 + b_n^2)$ .  
{Ans.  $A^2 + B^2$ }
36. If  $(a_1 + ib_1)(a_2 + ib_2)\dots(a_n + ib_n) = A + iB$ , then find the value of  $\sum_{i=1}^n \tan^{-1}\left(\frac{b_i}{a_i}\right)$ . {Ans.  $\tan^{-1}\left(\frac{B}{A}\right)$ }
37. If  $a \in \mathbb{R}$ , then show that for any complex number  $(z+a)(\bar{z}+a)$  equals  $|z+a|^2$ .
38. If  $z_1$  and  $z_2$  are complex numbers, prove that  $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$  if and only if  $z_1\bar{z}_2$  is pure imaginary.
39. Demonstrate that the complex number  $x + iy$  whose modulus is unity,  $y \neq 0$ , can be represented as  
$$x + iy = \frac{a+i}{a-i} \text{ where } a \text{ is a real number.}$$
40. If  $|z_1| = |z_2| = \dots = |z_n| = 1$  prove that  $|z_1 + z_2 + \dots + z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|$ .
41. If  $\sqrt{3} + i = (a+ib)(c+id)$ , then show that  $\tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{d}{c}\right)$  has the value  $n\pi + \frac{\pi}{6}$ ,  $n \in \mathbb{Z}$ .
42. If  $z = 1 + i\sqrt{3}$ , then find the value of  $|\arg(z)| + |\arg(\bar{z})|$ . {Ans.  $\frac{2\pi}{3}$ }
43. If  $z_1, z_2$  and  $z_3, z_4$  are two pairs of conjugate complex numbers, then find  $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$ . {Ans. 0}
44. If  $|z_1| = |z_2|$  and  $\arg(z_1) + \arg(z_2) = 0$ , then show that  $z_1 = \bar{z}_2$ .
45. If  $\alpha$  and  $\beta$  are different complex numbers with  $|\beta| = 1$ , then show that  $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right|$  is equal to 1.
46. For any two complex numbers  $z_1, z_2$  and any two real numbers  $a$  and  $b$ , prove that  
$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2).$$
47. For any two non-zero complex numbers  $z_1$  and  $z_2$  if  $z_1\bar{z}_2 + \bar{z}_1z_2 = 0$ , then find the difference of amplitudes of  $z_1$  and  $z_2$ . {Ans.  $\frac{\pi}{2}$ }
48. Let  $z$  and  $w$  be two non-zero complex numbers such that  $|z| = |w|$  and  $\arg(z) + \arg(w) = \pi$ . Then show that  $z$  equals  $-\bar{w}$ .
49. Prove that  $\left| \frac{z_1 - z_2}{1 - \bar{z}_1z_2} \right| < 1$  if  $|z_1| < 1$  and  $|z_2| < 1$ .
50. Prove that following inequality:-  
$$\left[ (a_1 + a_2 + \dots + a_n)^2 + (b_1 + b_2 + \dots + b_n)^2 \right]^{\frac{1}{2}} < \sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \dots + \sqrt{a_n^2 + b_n^2}.$$
 where  $a_r, b_r$  ( $r = 1, 2, \dots, n$ ) are real.
51. Prove the following identities:-

- i.  $(x^2 + a^2)^4 = (x^4 - 6x^2a^2 + a^4)^2 + (4x^3a - 4xa^3)^2$ .  
 ii.  $(x^2 + a^2)^7 = (x^7 - 21x^5a^2 + 35x^3a^4 - 7xa^6)^2 + (7x^6a - 35x^4a^3 + 21x^2a^5 - a^7)^2$ .

**CATEGORY-25.3. POLAR (TRIGONOMETRIC) REPRESENTATION OF COMPLEX NUMBER, DE MOIVRE'S THEOREM**

52. Put the following numbers in trigonometrical form:-

- i. 1 {Ans.  $\cos 0 + i \sin 0$ }  
 ii.  $i$  {Ans.  $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ }  
 iii.  $-1$  {Ans.  $\cos \pi + i \sin \pi$ }  
 iv.  $-i$  {Ans.  $\cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2})$ }  
 v. 3 {Ans.  $3(\cos 0 + i \sin 0)$ }  
 vi.  $-5$  {Ans.  $5(\cos \pi + i \sin \pi)$ }  
 vii.  $6i$  {Ans.  $6(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$ }  
 viii.  $-2i$  {Ans.  $2(\cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2}))$ }  
 ix.  $1+i$  {Ans.  $\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ }  
 x.  $-1+i$  {Ans.  $\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$ }  
 xi.  $-\sqrt{3} + i$  {Ans.  $2(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$ }  
 xii.  $\frac{1+7i}{(2-i)^2}$  {Ans.  $\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$ }  
 xiii.  $2.5(\cos 300^\circ + i \sin 30^\circ)$  {Ans.  $\frac{5}{2\sqrt{2}}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ }  
 xiv.  $-5(\cos 40^\circ - i \sin 40^\circ)$  {Ans.  $5(\cos 140^\circ + i \sin 140^\circ)$ }  
 xv.  $\frac{(1+i)^{2n+1}}{(1-i)^{2n-1}}$  {Ans.  $2(\cos 0 + i \sin 0)$  if  $n$  is even,  $2(\cos \pi + i \sin \pi)$  if  $n$  is odd}  
 xvi.  $1 + i \tan \alpha$   $\left(-\pi < \alpha < \pi, \alpha \neq \pm \frac{\pi}{2}\right)$  {Ans.  $\sec \alpha (\cos \alpha + i \sin \alpha)$  when  $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ;  
 $(-\sec \alpha)[\cos(\pi + \alpha) + i \sin(\pi + \alpha)]$  when  $\alpha \in \left(-\pi, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ }

53. Put the following numbers in algebraic form:-

- i.  $\sqrt{3}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$  {Ans.  $\frac{3}{2} + \frac{\sqrt{3}}{2}i$ }  
 ii.  $\sqrt{2}\left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}\right)$  {Ans.  $\frac{\sqrt{5} + \sqrt{5}}{2} + i\left(\frac{\sqrt{5}-1}{2\sqrt{2}}\right)$ }  
 iii.  $2\left(\cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3}\right)$  {Ans.  $-1 + \sqrt{3}i$ }

54. Simplify the following and put in algebraic form:-

i.  $[(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)]^{-1}$  {Ans. 1}

ii.  $\frac{4(\cos 75^\circ + i \sin 75^\circ)}{0.4(\cos 30^\circ + i \sin 30^\circ)}$  {Ans.  $5\sqrt{2}(1+i)$ }

iii.  $\frac{1}{1 - \cos \theta + 2i \sin \theta}$  {Ans.  $\frac{1 - 2i \cot \frac{\theta}{2}}{5 + 3 \cos \theta}$ }

iv.  $\frac{(\cos x + i \sin x)(\cos y + i \sin y)}{(\cot u + i)(1 + i \tan v)}$  {Ans.  $\sin u \cos v [\cos(x + y - u - v) + i \sin(x + y - u - v)]$ }

v.  $\frac{(\cos 2\theta - i \sin 2\theta)^7 (\cos 3\theta + i \sin 3\theta)^{-5}}{(\cos 4\theta + i \sin 4\theta)^{12} (\cos 5\theta + i \sin 5\theta)^{-6}}$  {Ans.  $\cos 47\theta - i \sin 47\theta$ }

vi.  $(\cos 2\theta + i \sin 2\theta)^{-5} (\cos 3\theta - i \sin 3\theta)^6 (\sin \theta - i \cos \theta)^3$  {Ans.  $\sin 25\theta + i \cos 25\theta$ }

vii.  $(\sin \theta + i \cos \theta)^4$  {Ans.  $\cos 4\theta - i \sin 4\theta$ }

viii.  $(1+i)^5 + (1-i)^5$  {Ans. -8}

ix.  $\left\{ \frac{1 + \cos \frac{\pi}{8} + i \sin \frac{\pi}{8}}{1 + \cos \frac{\pi}{8} - i \sin \frac{\pi}{8}} \right\}^8$  {Ans. -1}

x.  $\frac{\left( \sin \frac{\pi}{8} + i \cos \frac{\pi}{8} \right)^8}{\left( \sin \frac{\pi}{8} - i \cos \frac{\pi}{8} \right)^8}$  {Ans. 1}

55. Prove the following:-

i.  $(\cos 60^\circ + i \sin 60^\circ)^6 = 1$ .

ii.  $[\sqrt{2}(\cos 56^\circ 15' + i \sin 56^\circ 15')]^8 = 16i$ .

iii.  $(\sin \theta + i \cos \theta)^n = \cos n\left(\frac{1}{2}\pi - \theta\right) + i \sin n\left(\frac{1}{2}\pi - \theta\right)$ .

iv.  $\left(\frac{1+i}{\sqrt{2}}\right)^8 + \left(\frac{1-i}{\sqrt{2}}\right)^8 = 2$ .

v.  $\left(\frac{1 + \cos \phi + i \sin \phi}{1 + \cos \phi - i \sin \phi}\right)^n = \cos n\phi + i \sin n\phi$ .

56. Find the modulus and argument of the complex number  $z_1 = z^2 - z$  if  $z = \cos \phi + i \sin \phi$ . {Ans.

when  $\sin \frac{\phi}{2} = 0$ , 0, not defined

when  $\sin \frac{\phi}{2} > 0$ ,  $2 \sin \frac{\phi}{2}$ ,  $\frac{\pi+3\phi}{2}$  }

when  $\sin \frac{\phi}{2} < 0$ ,  $-2 \sin \frac{\phi}{2}$ ,  $\frac{3\pi+3\phi}{2}$

57. Prove that  $\left(\frac{\sqrt{3}+i}{2}\right)^6 + \left(\frac{i-\sqrt{3}}{2}\right)^6 = -2$ .
58. Find the value of  $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^6 + \left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right)^6$ . {Ans. 2}
59. If  $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$ , then show that  $\text{Im}(z) = 0$ .
60. Simplify  $\frac{(1+i)^n}{(1-i)^{n-2}}$ . {Ans.  $2i^{n-1}$ }
61. Simplify  $(1+i)^8 + (1-i)^8$ . {Ans.  $2^5$ }
62. For  $n = 6k$ ,  $k \in \mathbb{Z}$ , find the value of  $\left(\frac{1-i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ . {Ans. 2}
63. If  $z = \cos\theta + i\sin\theta$ , then find  $|1+z+z^2|$ . {Ans.  $1+2\cos\theta$ }
64. If  $a = \cos\alpha + i\sin\alpha$ ,  $b = \cos\beta + i\sin\beta$ ,  $c = \cos\gamma + i\sin\gamma$ , then find  $abc + \frac{1}{abc}$ . {Ans.  $2\cos(\alpha + \beta + \gamma)$ }
65. If  $x = \cos\theta + i\sin\theta$ ,  $y = \cos\phi + i\sin\phi$ , then show that  $\frac{x-y}{x+y}$  is equal to  $i \tan \frac{\theta-\phi}{2}$ .
66. If  $a = \cos\alpha + i\sin\alpha$ ,  $b = \cos\beta + i\sin\beta$ ,  $c = \cos\gamma + i\sin\gamma$  and  $\frac{b}{c} + \frac{c}{a} + \frac{a}{b} = 1$ , then show that  $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta)$  is equal to 1.
67. Find the general value of  $\theta$  which satisfies the equation  $(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)(\cos 5\theta + i\sin 5\theta) \cdots (\cos(2n-1)\theta + i\sin(2n-1)\theta) = 1$ . {Ans.  $\frac{2r\pi}{n^2}$ }
68. If  $|z| = 1$ , prove that  $\frac{z-1}{z+1}$  ( $z \neq -1$ ) is a pure imaginary number. What will you conclude if  $z = 1$ ? If the number  $\frac{z-1}{z+1}$  is a pure imaginary, then prove that  $|z| = 1$ .
69. If  $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \cdots (\cos n\theta + i\sin n\theta) = 1$ , then find the value of  $\theta$ . {Ans.  $\frac{4m\pi}{n(n+1)}$ }
70. If  $x + \frac{1}{x} = 2\cos\frac{\pi}{10}$ , then find  $x^5 + \frac{1}{x^5}$ . {Ans. 0}
71. If  $\frac{1}{x} + x = 2\cos\theta$ , then find the value of  $x^n + \frac{1}{x^n}$ . {Ans.  $2\cos n\theta$ }
72. If  $x^2 - 2x\cos\theta + 1 = 0$ , then find the value of  $x^{2n} - 2x^n\cos n\theta + 1$ . {Ans. 0}
73. If  $z + z^{-1} = 1$ , then simplify  $z^{100} + z^{-100}$ . {Ans. -1}
74. If in polar form  $z_1 = (1, \alpha)$ ,  $z_2 = (1, \beta)$ ,  $z_3 = (1, \gamma)$  and  $z_1 + z_2 + z_3 = 0$ , then find  $z_1^{-1} + z_2^{-1} + z_3^{-1}$ . {Ans. 0}
75. If  $a = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$ , then find the value of  $\left(\frac{1+a}{2}\right)^{3n}$ . {Ans.  $\frac{(-1)^n}{2^{3n}}$ }

76. Find the sum of  $i - 2 - 3i + 4 \dots$  upto 100 terms. {Ans.  $50(1-i)$ }
77. If the first term and common ratio of a GP is  $\frac{1}{2}(\sqrt{3} + i)$ , then find the modulus of its  $n$ th term. {Ans. 1}
78. Let  $z_1 = 10 + 6i$  and  $z_2 = 4 + 6i$ . If  $z$  is a complex number such that the argument of  $\frac{z - z_1}{z - z_2}$  is  $\frac{\pi}{4}$ , then prove that  $|z - 7 - 9i| = 3\sqrt{2}$ .
79. Find the greatest value of the moduli of complex numbers  $z$ , satisfying the equation  $\left|z - \frac{4}{z}\right| = 2$ . {Ans.  $\sqrt{5} + 1$ }
80. It is known that  $\left|z + \frac{1}{z}\right| = a$  where  $z$  is a complex number. What are the greatest and the least possible values of  $|z|$ ? {Ans.  $\frac{a + \sqrt{a^2 + 4}}{2}, \frac{\sqrt{a^2 + 4} - a}{2}$ }

#### CATEGORY-25.4. EULERIAN REPRESENTATION OF COMPLEX NUMBER

81. Put the following numbers in Eulerian form:-
- $1$  {Ans.  $e^{i0}$ }
  - $i$  {Ans.  $e^{i\frac{\pi}{2}}$ }
  - $-1$  {Ans.  $e^{i\pi}$ }
  - $-i$  {Ans.  $e^{-i\frac{\pi}{2}}$ }
  - $1+i$  {Ans.  $\sqrt{2}e^{i\frac{\pi}{4}}$ }
  - $-1-i$  {Ans.  $\sqrt{2}e^{-i\frac{3\pi}{4}}$ }
82. Put the following numbers in algebraic form:-
- $3e^{i\frac{\pi}{2}}$  {Ans.  $3i$ }
  - $2e^{i\frac{\pi}{6}}$  {Ans.  $\sqrt{3} + i$ }
  - $e^{i7\pi}$  {Ans.  $-1$ }
83. If p.v.'s of  $z_1$  and  $z_2$  are perpendicular to each other, then show that  $\frac{z_1}{z_2}$  is purely imaginary.
84. If  $x_r = \cos \frac{\pi}{2^r} + i \sin \frac{\pi}{2^r}$ , prove that  $x_1 x_2 x_3 \dots$  ad inf.  $= -1$ .
85. If  $z_r = \cos \frac{\pi}{3^r} + i \sin \frac{\pi}{3^r}$ ,  $r = 1, 2, 3, \dots$  then prove that  $z_1 z_2 z_3 \dots \infty = i$ .
86.  $z_r = \cos \frac{r\pi}{10} + i \sin \frac{r\pi}{10}$  then  $z_1 z_2 z_3 z_4 = -1$ .
87. If  $2 \cos \theta = x + \frac{1}{x}$  and  $2 \cos \phi = y + \frac{1}{y}$  etc., then prove that



- i.  $xyz \dots + \frac{1}{xyz \dots} = 2 \cos(\theta + \phi + \dots)$
- ii.  $\frac{x}{y} + \frac{y}{x} = 2 \cos(\theta - \phi)$
- iii.  $x^m y^n + \frac{1}{x^m y^n} = 2 \cos(m\theta + n\phi)$
- iv.  $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2 \cos(m\theta - n\phi)$

### CATEGORY-25.5. INTEGER POWERS OF A COMPLEX NUMBER

88. Simplify:-

- i.  $(1+i)^{10}$ . {Ans.  $32i$ }
- ii.  $(1-i)^{-10}$ . {Ans.  $\frac{1}{32}i$ }
- iii.  $(\sqrt{3}+i)^5$ . {Ans.  $-16\sqrt{3}+16i$ }
- iv.  $(\sqrt{3}+i)^{10}$ . {Ans.  $512-512\sqrt{3}i$ }
- v.  $(\sqrt{3}+i)^{100}$ . {Ans.  $2^{99}-2^{99}\sqrt{3}i$ }
- vi.  $(1-\sqrt{3}i)^5$ . {Ans.  $16+16\sqrt{3}i$ }
89. If  $z = \left(\frac{1+i}{1-i}\right)$ , then find  $z^4$ . {Ans. 1}
90. If  $2\sqrt{2}z = (\sqrt{3}-1) + i(\sqrt{3}+1)$ , then find  $z^6$ . {Ans.  $i$ }
91. If  $x = \frac{3+5i}{2}$ , find the value of  $2x^3 + 2x^2 - 7x + 72$  and show that it will be unaltered if  $x = \frac{3-5i}{2}$ . {Ans. 4}
92. If  $x = -5+4i$  find the value of  $x^4 + 9x^3 + 35x^2 - x + 4$ . {Ans. -160}
93. For positive integers  $n_1, n_2$ , show that the value of the expression  $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$  is a real number.

### CATEGORY-25.6. SQUARE ROOT OF A COMPLEX NUMBER IN ALGEBRAIC FORM

94. Find the square root of the following numbers:-

- i.  $-4$  {Ans.  $\pm 2i$ }
- ii.  $i$  {Ans.  $\pm \frac{1}{\sqrt{2}}(1+i)$ }
- iii.  $-i$  {Ans.  $\pm \frac{1}{\sqrt{2}}(1-i)$ }
- iv.  $3+4i$  {Ans.  $\pm(2+i)$ }
- v.  $3-4i$  {Ans.  $\pm(2-i)$ }
- vi.  $5+12i$  {Ans.  $\pm(3+2i)$ }

- vii.  $1-i$  {Ans.  $\pm \left( \sqrt{\frac{\sqrt{2}+1}{2}} - i\sqrt{\frac{\sqrt{2}-1}{2}} \right)$ }
- viii.  $-5+12i$  {Ans.  $\pm(2+3i)$ }
- ix.  $-8-6i$  {Ans.  $\pm(1-3i)$ }
- x.  $-1+4\sqrt{5}i$  {Ans.  $\pm(2+\sqrt{5}i)$ }
- xi.  $1+4\sqrt{3}i$  {Ans.  $\pm(2+\sqrt{3}i)$ }
- xii.  $-7-24i$  {Ans.  $\pm(3-4i)$ }
95. Simplify the following and put in algebraic form:-
- i.  $\frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}}$ . {Ans.  $-\frac{3}{2}i, \frac{2}{3}i$ }
- ii.  $\left[ (\sqrt{3}+i)(\sqrt{3}-i) \right]^{\frac{3}{2}}$  {Ans.  $\pm \frac{1}{8}$ }
96. Simplify  $(\sqrt{-2})(\sqrt{-3})$ . {Ans.  $-\sqrt{6}$ }
97. If  $\sqrt{-7-24i} = x-iy$ , then find  $x^2 + y^2$ . {Ans. 25}
98. If  $x+iy = \sqrt{\frac{a+ib}{c+id}}$ , prove that  $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$ .
99. Find the modulus of  $\sqrt{2i} - \sqrt{-2i}$ . {Ans. 2}
100. If  $\sqrt{a+ib} = \pm(\alpha+i\beta)$ , then show that  $\sqrt{-a-ib}$  is equal to  $\pm(\beta-\alpha i)$ .
101. Prove that :  $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ . Interpret the result geometrically and deduce that  $|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|$ , all number involved being complex.
102. Prove that  $|z_1| + |z_2| = \left| \frac{1}{2}(z_1 + z_2) + \sqrt{z_1 z_2} \right| + \left| \frac{1}{2}(z_1 + z_2) - \sqrt{z_1 z_2} \right|$ .

### CATEGORY-25.7. ROOTS OF A COMPLEX NUMBER IN POLAR/ EULERIAN FORM

103. Find all the values of the given roots:-
- i.  $\sqrt{1-i}$ . {Ans.  $2^{\frac{1}{4}}e^{-i\frac{\pi}{8}}, 2^{\frac{1}{4}}e^{i\frac{7\pi}{8}}$ }
- ii.  $\sqrt[3]{i}$ . {Ans.  $\frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -i$ }
- iii.  $(2-2i)^{\frac{1}{3}}$ . {Ans.  $\sqrt{2}e^{-i\frac{\pi}{12}}, \sqrt{2}e^{i\frac{7\pi}{12}}, -1-i$ }
- iv.  $(-64)^{\frac{1}{4}}$ . {Ans.  $\pm 2 \pm 2i$ }
- v.  $(1-\sqrt{3}i)^{\frac{1}{4}}$ . {Ans.  $2^{\frac{1}{4}}e^{-i\frac{\pi}{12}}, 2^{\frac{1}{4}}e^{i\frac{11\pi}{12}}, 2^{\frac{1}{4}}e^{i\frac{23\pi}{12}}, 2^{\frac{1}{4}}e^{i\frac{35\pi}{12}}$ }
- vi.  $\sqrt[4]{i}$ . {Ans.  $e^{i\frac{\pi}{8}}, e^{i\frac{5\pi}{8}}, e^{i\frac{9\pi}{8}}, e^{i\frac{13\pi}{8}}$ }
104. Find the product of cube roots of  $-1$ . {Ans.  $-1$ }
105. Use De Moivre's theorem to solve the equation  $2\sqrt{2}x^4 = (\sqrt{3}-1) + i(\sqrt{3}+1)$ . {Ans.}

$$\cos\left(\frac{24k+5}{48}\right)\pi + i\sin\left(\frac{24k+5}{48}\right)\pi, k = 0, 1, 2, 3 \}$$

**CATEGORY-25.8. CUBE ROOTS OF UNITY**

106. If  $1, \omega, \omega^2$  are the three cube roots of unity, show that

- i.  $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$ .
- ii.  $(1 + \omega)^3 - (1 + \omega^2)^3 = 0$ .
- iii.  $(1 - \omega + \omega^2)^3 = (1 + \omega - \omega^2)^3 = -8$ .
- iv.  $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$ .
- v.  $(2 + 5\omega + 2\omega^2)^6 = (2 + 2\omega + 5\omega^2)^6 = 729$ .
- vi.  $(1 - \omega + \omega^2)^6 + (1 - \omega^2 + \omega)^6 = 128$ .
- vii.  $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) = 1$ .
- viii.  $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8) = 9$ .
- ix.  $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \dots$  to  $2n$  factors  $= 2^{2n}$
- x.  $x^3 + y^3 = (x + y)(\omega x + \omega^2 y)(\omega^2 x + \omega y)$ .
- xi.  $x^3 - y^3 = (x - y)(\omega x - \omega^2 y)(\omega^2 x - \omega y)$ .
- xii.  $(x + y)^2 + (x\omega + y\omega^2)^2 + (x\omega^2 + y\omega)^2 = 6xy$ .
- xiii.  $(a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^3 + b^3 + c^3 - 3abc$ .
- xiv.  $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = (2a - b - c)(2b - c - a)(2c - a - b) = 27abc$  if  $a + b + c = 0$ .

107. If  $\omega (\neq 1)$  be a cube root of unity and  $(1 + \omega)^7 = A + B\omega$ , then find  $A$  and  $B$ . {Ans. 1, 1}

108. If  $\omega$  is a complex cube root of unity, then find  $\frac{1}{1 + \omega} + \frac{1}{1 + \omega^2}$ . {Ans. 1}

109. If  $\omega$  is a complex cube root of unity, then find  $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)$ . {Ans. 9}

110. Simplify  $(3 + \omega + 3\omega^2)^4$ . {Ans.  $16\omega$ }

111. Simplify  $(2\omega + 5\omega^2)(5\omega + 2\omega^2)$ . {Ans. 19}

112. Find the value of the expression  $1 \cdot (2 - \omega)(2 - \omega)^2 + 2 \cdot (3 - \omega)(3 - \omega^2) + \dots + (n - 1)(n - \omega)(n - \omega^2)$ , where

$$\omega \text{ is an imaginary cube root of unity. \{Ans. } \left(\frac{n(n+1)}{2}\right)^2 - n \}$$

113. Find the value of the expression

$$\left(1 + \frac{1}{\omega}\right)\left(1 + \frac{1}{\omega^2}\right) + \left(2 + \frac{1}{\omega}\right)\left(2 + \frac{1}{\omega^2}\right) + \left(3 + \frac{1}{\omega}\right)\left(3 + \frac{1}{\omega^2}\right) + \dots + \left(n + \frac{1}{\omega}\right)\left(n + \frac{1}{\omega^2}\right), \text{ where } \omega \text{ is an imaginary cube root of unity. \{Ans. } \frac{n(n^2 + 2)}{3} \}$$

114. Find the value of the expression

$$2(1+\omega)(1+\omega^2)+3(2\omega+1)(2\omega^2+1)+4(3\omega+1)(3\omega^2+1)+\cdots+(n+1)(n\omega+1)(n\omega^2+1). \{\text{Ans. } \left(\frac{n(n+1)}{2}\right)^2 + n\}$$

115. If  $\omega$  is a complex cube root of unity, then find the value of  $\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} + \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}$ . {Ans. -1}

116. If  $1, \omega, \omega^2$  are the cube roots of unity, then evaluate  $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$ . {Ans. 0}

117. If  $\omega$  is a cube root of unity, then evaluate  $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$ . {Ans. 0}

118. Find the value of the determinant  $\begin{vmatrix} 1 & \omega^3 & \omega^5 \\ \omega^3 & 1 & \omega^4 \\ \omega^5 & \omega^4 & 1 \end{vmatrix}$ . {Ans. 3}

119. Prove that  $z^{14} + \frac{1}{z^{14}} = -1$ , where  $z$  is a root of the equation  $z + \frac{1}{z} = 1$ .

120. If  $\alpha$  and  $\beta$  are the complex cube roots of unity, show that  $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = 0$ .

121. If  $x = a + b$ ,  $y = a\alpha + b\beta$  and  $z = a\beta + b\alpha$  where  $\alpha$  and  $\beta$  are complex cube roots of unity, show that  $xyz = a^3 + b^3$ .

122. If  $x = a + b$ ,  $y = a\omega + b\omega^2$  and  $z = a\omega^2 + b\omega$  prove that  $x^3 + y^3 + z^3 = 3(a^3 + b^3)$ .

123. Simplify  $\left[\frac{-1+\sqrt{-3}}{2}\right]^{3n} + \left[\frac{-1-\sqrt{-3}}{2}\right]^{3n}$ . {Ans. 2}

124. Find the roots of the equation  $x^3 + 27 = 0$ . {Ans.  $-3, -3\omega, -3\omega^2$ }

125. Find the roots of the equation  $(x-1)^3 + 8 = 0$ . {Ans.  $-1, 1-2\omega, 1-2\omega^2$ }

126. Prove that:-

i.  $1 + \omega^n + \omega^{2n} = 0$ , when  $n$  is a positive integer but not a multiple of 3.

ii.  $1 + \omega^n + \omega^{2n} = 3$ , when  $n$  is a multiple of 3.

127. Given  $z_1 + z_2 + z_3 = A$ ,  $z_1 + z_2\omega + z_3\omega^2 = B$ ,  $z_1 + z_2\omega^2 + z_3\omega = C$ .

i. Express  $z_1, z_2, z_3$  in terms of  $A, B, C$ .

ii. Prove  $|A|^2 + |B|^2 + |C|^2 = 3(|z_1|^2 + |z_2|^2 + |z_3|^2)$ .

### CATEGORY-25.9. HIGHER ROOTS OF UNITY

128. Find all the six sixth roots of unity. Which of these are also cube roots of unity? {Ans.

$$e^{i\frac{k\pi}{3}}, k = 0, 1, 2, 3, 4, 5; 1, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}} \}$$

129. Find the seven seventh roots of unity and prove that sum of their  $n$ th powers always vanishes unless  $n$  be a

multiple of seven,  $n$  being an integer and then the sum is seven. {Ans.  $e^{i\frac{2k\pi}{7}}, k = 0, 1, 2, 3, 4, 5, 6$ }

130. Find the value of  $\sum_{k=1}^6 \left( \sin \frac{2\pi k}{7} - i \cos \frac{2\pi k}{7} \right)$ . {Ans.  $i$ }

131. If  $\beta$  is an imaginary root of the equation  $z^n - 1 = 0$ , prove that  $1 + \beta + \beta^2 + \dots + \beta^{n-1} = 0$ .

132. Find the  $n$ ,  $n$ th roots of unity and prove that the sum of their  $p$ th powers vanishes unless  $p$  be a multiple of  $n$ ,  $p$  being an integer, and that then the sum is  $n$ . {Ans.  $e^{i\frac{2k\pi}{n}}, k = 0, 1, 2, \dots, n-1$ }

### CATEGORY-25.10. IRRATIONAL AND NON-REAL COMPLEX POWERS OF A COMPLEX NUMBER

133. Simplify the following:-

i.  $2^i$  {Ans.  $e^{i \ln 2}$ }

ii.  $i^{\sqrt{2}}$  {Ans.  $e^{i\frac{\pi}{\sqrt{2}}}$ }

iii.  $i^i$  {Ans.  $e^{-\frac{\pi}{2}}$ }

iv.  $(1+i)^i$  {Ans.  $e^{-\frac{\pi}{4}} e^{\frac{i \ln 2}{2}}$ }

### CATEGORY-25.11. LOGARITHM OF A COMPLEX NUMBER

134. Find the following numbers in algebraic form:-

i.  $\ln(-1)$ . {Ans.  $\pi i$ }

ii.  $\ln i$ . {Ans.  $\frac{\pi}{2} i$ }

iii.  $\ln \omega$ . {Ans.  $\frac{2\pi}{3} i$ }

iv.  $\ln(\ln i)$ . {Ans.  $\ln \frac{\pi}{2} + i \frac{\pi}{2}$ }

135. Find the value of  $z$  satisfying the equation  $\ln z + \ln z^2 + \dots + \ln z^n = 0$ . {Ans.

$$\cos \frac{4m\pi}{n(n+1)} + i \sin \frac{4m\pi}{n(n+1)}, m = 1, 2, \dots \}$$

### CATEGORY-25.12. EQUATIONS IN COMPLEX NUMBERS

136. Find all complex numbers  $z$  which satisfy the following equations:-

i.  $\bar{z} = 2 - z$  {Ans.  $z = 1 + iy, y \in R$ }

ii.  $(2+i)z^2 - (5-i)z + 2 - 2i = 0$  {Ans.  $1-i, \frac{4}{5} - \frac{2}{5}i$ }

iii.  $z^2 + \bar{z} = 0$  {Ans.  $0, -1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$ }

iv.  $z^3 = \bar{z}$  {Ans.  $0, i, -i, 1, -1$ }

v.  $z^2 = (\bar{z})^3$  {Ans.  $0, 1, e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}}, e^{i\frac{6\pi}{5}}, e^{i\frac{8\pi}{5}}$ }

vi.  $|z| - iz = 1 - 2i$  {Ans.  $2 - \frac{3}{2}i$ }

vii.  $z^2 + |z| = 0$ . {Ans.  $0, i, -i$ }

137. If  $z + a|z + 1| + i = 0$  and  $a = \sqrt{2}$ , then find  $z$ . {Ans.  $-2 - i$ }
138. Find the integral solutions of the equations:-  
 i.  $(1 - i)^n = 2^n$  {Ans.  $n = 0$ }  
 ii.  $(1 + i)^n = (1 - i)^n$ . {Ans.  $n = 4k$ }
139. Find the complex numbers  $z$  which simultaneously satisfy the equations  $\left| \frac{z - 12}{z - 8i} \right| = \frac{5}{3}$  and  $\left| \frac{z - 4}{z - 8} \right| = 1$ . {Ans.  $6 + 8i, 6 + 17i$ }
140. Find the common roots of the equations  $z^3 + 2z^2 + 2z + 1 = 0$  and  $z^{1985} + z^{100} + 1 = 0$  {Ans.  $\omega, \omega^2$ }
141. What is the number of solutions of the system of equations  $\operatorname{Re}(z^2) = 0, |z| = 2$ . {Ans. 4}
142. For every real number  $c$ , find all the complex numbers  $z$  which satisfy the equation  $|z|^2 - 2iz + 2c(1 + i) = 0$ . {Ans.  
 $c \in (-\infty, -1 - \sqrt{2}) \cup (\sqrt{2} - 1, \infty), z \in \varnothing$   
 $c \in [-1 - \sqrt{2}] z = (-1 - \sqrt{2}) - i$   
 $c \in (-1 - \sqrt{2}, \sqrt{2} - 1), z = c + (-1 + \sqrt{1 - c^2 - 2c})i; z = c + (-1 - \sqrt{1 - c^2 - 2c})i$   
 $c \in [\sqrt{2} - 1] z = (\sqrt{2} - 1) - i$ }
143. For every real number  $c \geq 1$ , find all the complex numbers  $z$  that satisfy the equation  $z + c|z + 1| + i = 0$ .  
 {Ans.  
 $c = 1, z = -1 - i$   
 $1 < c < \sqrt{2}, z = \frac{-c^2 \pm c\sqrt{2 - c^2}}{c^2 - 1} - i$   
 $c = \sqrt{2}, z = -2 - i$   
 $c > \sqrt{2}, \varnothing$ }
144. Find the range of real number  $\alpha$  for which the equation  $z + \alpha|z - 1| + 2i = 0; z = x + iy$  has a solution and find the solution.  
 {Ans.  $\alpha \in \left[-\frac{\sqrt{5}}{2}, 1\right)$   
 when  $\alpha \in \left[-\frac{\sqrt{5}}{2}, -1\right), z = \frac{\alpha^2 \pm \alpha\sqrt{5 - 4\alpha^2}}{\alpha^2 - 1} - 2i$   
 when  $\alpha \in [-1], z = \frac{5}{2} - 2i$   
 when  $\alpha \in (-1, 1), z = \frac{\alpha^2 + \alpha\sqrt{5 - 4\alpha^2}}{\alpha^2 - 1} - 2i$ }
145. For every real number  $c \geq 0$ , find all the complex numbers  $z$  which satisfy the equation  $2|z| - 4cz + 1 + ic = 0$ . {Ans.  
 $0 \leq c \leq \frac{1}{2}, \varnothing$   
 $c > \frac{1}{2}, z = \frac{4c + \sqrt{4c^2 + 3}}{16c^2 - 4} + \frac{1}{4}i$ }

146. For every real number  $c > 0$ , find all complex numbers  $z$ , satisfying the equation  $z|z| + cz + i = 0$ .

### CATEGORY-25.13. COMPLEX POLYNOMIALS

147. Let  $\alpha$  and  $\beta$  be the roots of the equation  $x^2 + x + 1 = 0$ . Find the equation whose roots are  $\alpha^{19}, \beta^7$ . {Ans.

$$x^2 + x + 1 = 0 \}$$

148. If  $3 + 4i$  is root of the equation  $x^2 + px + q = 0$ , then find  $p, q$ . {Ans.  $p = -6, q = 25$ }

149. If  $2 + i\sqrt{3}$  is a root of the quadratic equation  $x^2 + ax + b = 0$ , where  $a, b \in R$ , then find the values of  $a$  and  $b$ . {Ans.  $-4, 7$ }

150. For the equation  $3x^2 + px + 3 = 0$ , if one of the roots is square of the other, then find  $p$ . {Ans.  $3, -6$ }

151. Prove that  $x^4 + 4 = (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$ .

152. Solve the equation  $x^4 - 4x^2 + 8x + 35 = 0$  having given that one root is  $2 + \sqrt{3}i$ . {Ans.  $2 \pm \sqrt{3}i, -2 \pm i$ }

153. Let  $\omega$  and  $\omega^2$  be the cube roots of unity, then find the roots of the polynomial 
$$\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}.$$

{Ans.  $0, 0, 0$ }

154. If  $f(z)$  be divided by  $z - i$  and  $z + i$ , the remainders are respectively  $i$  and  $1 + i$ . Determine the remainder when  $f(z)$  is divided by  $z^2 + 1$ . {Ans.  $\frac{i}{2}z + (i + \frac{1}{2})$ }

155. It is given that  $n$  is an odd integer greater than 3 but  $n$  is not a multiple of 3. Prove that  $x^3 + x^2 + x$  is a factor of  $(x + 1)^n - x^n - 1$ .

156. Show that the polynomial  $x^{4l} + x^{4m+1} + x^{4n+2} + x^{4p+3}$  is divisible by  $x^3 + x^2 + x + 1$  where  $l, m, n, p$  are positive integers.

157. Prove that  $(x + y)^n - x^n - y^n$  is divisible by  $xy(x + y)(x^2 + xy + y^2)$ , if  $n$  is odd but not a multiple of 3.

158. If  $a_1, a_2, \dots, a_n$  are the real roots of the equation  $x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0$  ( $p_1, p_2, \dots, p_n$  are real), then prove that  $(1 + a_1^2)(1 + a_2^2) \dots (1 + a_n^2) = (1 - p_2 + p_4 - \dots)^2 + (p_1 - p_3 + p_5 - \dots)^2$ .

159. If  $\alpha, \beta$  are the roots of the equation  $x^2 - 2x + 4 = 0$ , prove that  $\alpha^n + \beta^n = 2^{n+1} \cos(n\pi/3)$ .

160. If  $x^2 - 2x \cos \theta + 1 = 0$ , prove that  $x^{2n} - 2x^n \cos n\theta + 1 = 0$ .

161. Construct an equation whose roots are  $n$ th powers of the roots of the equation  $x^2 - 2x \cos \theta + 1 = 0$ . {Ans.  $x^2 - 2 \cos n\theta x + 1 = 0$ }

162. Prove that if  $\cos \alpha + i \sin \alpha$  is a solution of the equation  $x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0$ , then prove that  $p_1 \sin \alpha + p_2 \sin 2\alpha + \dots + p_n \sin n\alpha = 0$ . ( $p_1, p_2, \dots, p_n$  are real)

### CATEGORY-25.14. PROVING TRIGONOMETRIC IDENTITIES BY COMPLEX NUMBERS

163. If  $\cos \alpha + \cos \beta + \cos \gamma = \sin \alpha + \sin \beta + \sin \gamma = 0$ , prove that

i.  $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$ .

ii.  $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$ .

iii.  $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$ .

$$\text{iv. } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \frac{3}{2}.$$

**CATEGORY-25.15. LOCUS PLOTTING**

164. Locate the point representing the complex numbers  $z$  on the Argand diagram for which

- i.  $\arg z = \pi/3$
- ii.  $\pi/3 < \arg z \leq 3\pi/2$
- iii.  $|\pi - \arg z| < \pi/4$
- iv.  $\text{Im}(z^2) = 1$
- v.  $|z - 2i| = \text{Im}(z)$ .

**CATEGORY-25.16. POINTS**

165. Find the lengths of the segments connecting the points represented by the following pairs of numbers:-

- i.  $5, -3$  {Ans. 8}
- ii.  $3, -4i$  {Ans. 5}
- iii.  $-6i, 3i$  {Ans. 9}
- iv.  $-1 - i, 2 + 3i$  {Ans. 5}
- v.  $3 - 2i, 3 + 5i$ . {Ans. 7}

166. Find the points which divides the line segment joining points  $1 + i$  and  $2 + 3i$  in the ratio 2:3 internally and externally. {Ans.  $\frac{7}{5} + \frac{9}{5}i, -1 - 3i$ }

167. Show that the points  $z_1 = 1 + 2i, z_2 = 2 + 3i, z_3 = 3 + 4i$  are collinear. Find the ratio in which  $z_3$  divides the line segment joining points  $z_1$  and  $z_2$ . {Ans. 2:1 externally}

168. Show that the points  $z_1 = 1 - i, z_2 = 2 + i, z_3 = 1 + 5i$  are collinear. Find the ratio in which  $z_3$  divides the line segment joining points  $z_1$  and  $z_2$ . {Ans. 2:3 externally}

169. The three points  $z_1, z_2, z_3$  are connected by relation  $az_1 + bz_2 + cz_3 = 0$  and  $a + b + c = 0$ . Prove that three points are collinear.

170. The cube roots of unity when represented on the Argand diagram form the vertices of an equilateral triangle (a) true (b) False. {Ans. True}

171. Show that area of the triangle on the Argand diagram formed by the complex numbers  $z, iz$  and  $z + iz$  is  $\frac{1}{2}|z|^2$ .

172. Find the area of the triangle formed by points  $z_1 = 2 + 3i, z_2 = 3 + 7i, z_3 = 4 - i$ . {Ans. 6}

173. Find the area of the triangle formed by three complex numbers  $1 + i, i - 1, 2i$  in the Argand plane. {Ans. 1}

174. Show that the triangle formed by the points  $1, \frac{1+i}{\sqrt{2}}$  and  $i$  as vertices in the Argand diagram is isosceles.

175. If the complex numbers  $z_1 = a + i, z_2 = 1 + ib, z_3 = 0$  form an equilateral triangle ( $a, b$  are real numbers between 0 and 1), then find  $a, b$ . {Ans.  $a = 2 - \sqrt{3}, b = 2 - \sqrt{3}$ }

176. Show that the origin and the roots of the equation  $z^2 + pz + q = 0$  form an equilateral triangle if  $p^2 = 3q$ .



177. If  $z_1, z_2$  are two complex numbers such that  $\operatorname{Im}(z_1 + z_2) = 0$ ,  $\operatorname{Im}(z_1 z_2) = 0$ , then show that  $z_1 = \bar{z}_2$ .  
If  $z_1, z_2, z_3$  are affixes of the vertices  $A, B$  and  $C$  respectively of a triangle  $ABC$  having centroid at  $G$  such that  $z = 0$  is the mid point of  $AG$ , then show that  $4z_1 + z_2 + z_3 = 0$ .
178. If  $z_1, z_2, z_3, z_4$  are the affixes of the vertices of a parallelogram taken in order in Argand plane, then show that  $z_1 + z_3 = z_2 + z_4$ .
179. If  $z_1, z_2$  represent adjacent vertices of a regular polygon of  $n$  sides such that  $\frac{\operatorname{Im}(z_1)}{\operatorname{Re}(z_2)} = \sqrt{2} - 1$ . Then find  $n$ .
180. Show that the triangle whose vertices are the points represented by the complex numbers  $z_1, z_2, z_3$  on the Argand diagram is equilateral if and only if  $\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$  that is, if  $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ .
181. Complex numbers  $z_1, z_2, z_3$  are the vertices  $A, B, C$  respectively of an isosceles right angled triangle with right angle at  $C$  show that  $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$
182. If  $z_1^2 + z_2^2 \pm 2z_1 z_2 \cos \theta = 0$ , prove that the points represented by  $z_1, z_2$  and the origin form an isosceles triangle,  $\theta$  being real.
183.  $z_1^2 + z_2^2 + z_1 z_2 = 0$ , prove that the points represented by  $z_1, z_2$  and the origin form an isosceles triangle with vertical angle  $2\pi/3$ .
184. Let  $z_1$  and  $z_2$  be complex numbers such that  $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$ . Prove that the origin and the two points represented by  $z_1$  and  $z_2$  form vertices of an equilateral triangle.
185. Let the complex numbers  $z_1, z_2$  and  $z_3$  be the vertices of an equilateral triangle. Let  $z_0$  be the circumcentre of the triangle. Prove that  $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$ .
186. Show that the triangle with vertices at the points  $z_1, z_2$  and  $(1-i)z_1 + iz_2$  is right angled and isosceles.
187. Show that the triangle whose vertices are  $z_1, z_2, z_3$  and  $z'_1, z'_2, z'_3$  are directly similar if  $\begin{vmatrix} z_1 & z'_1 & 1 \\ z_2 & z'_2 & 1 \\ z_3 & z'_3 & 1 \end{vmatrix} = 0$ .
188. The complex numbers  $z_1, z_2, z_3$  are the vertices of a triangle. Find all the complex numbers  $z$  which makes the triangle into a parallelogram. {Ans.  $z = z_1 + z_2 - z_3$  or  $z = z_1 - z_2 + z_3$  or  $z = -z_1 + z_2 + z_3$ }
189.  $ABCD$  is a rhombus. Its diagonals  $AC$  and  $BD$  intersect at the point  $M$  and satisfy  $BD = 2AC$ . If the points  $D$  and  $M$  represent the complex numbers  $1 + i$  and  $2 - i$  respectively, then find complex number representing  $A$ . {Ans.  $3 - \frac{1}{2}i, 1 - \frac{3}{2}i$ }
190. The centre of a regular hexagon is  $i$ . One vertex is  $(2 + i)$ ,  $z$  is an adjacent vertex. Then find  $z$ . {Ans.  $i + i(1 \pm \sqrt{3})$ }
191. Find the vertices of a regular polygon of  $n$  sides if its centre is located at  $z = 0$  and one of its vertices  $z_1$  is known. {Ans.  $z_k = z_1 e^{i\frac{2(k-1)\pi}{n}}$ ,  $k = 2, 3, \dots, n$ }
192. Points  $z_1$  and  $z_2$  are adjacent vertices of a regular polygon of  $n$  sides. Find the vertex  $z_3$  adjacent to

$z_2(z_3 \neq z_1)$ . Also find the center of the polygon. {Ans.  $z_3 = z_2 + (z_2 - z_1)e^{i\frac{2\pi}{n}}$ ,  $\frac{z_2 - z_1 e^{i\frac{2\pi}{n}}}{1 - e^{i\frac{2\pi}{n}}}$ }

193. Assume that  $A_i (i = 1, 2, \dots, n)$  are the vertices of a  $n$ -sided regular polygon inscribed in a circle of radius unity. Find

i.  $A_1 A_2^2 + A_1 A_3^2 + \dots + A_1 A_n^2$ . {Ans.  $2n$ }

ii.  $A_1 A_2 \cdot A_1 A_3 \cdot \dots \cdot A_1 A_n$ . {Ans.  $n$ }

194. Let  $A_1, A_2, \dots, A_n$  be vertices of an  $n$ -sided regular polygon such that  $\frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4}$  Find the value of  $n$ . {Ans.  $n = 7$ }

### CATEGORY-25.17. STRAIGHT LINE

195. Locate the points representing the complex numbers  $z$  on the Argand diagram for which

i.  $iz - i\bar{z} + 1 = 0$

ii.  $(1+i)z + (1-i)\bar{z} + 2 = 0$

iii.  $(-1+2i)z + (1+2i)\bar{z} + i = 0$

iv.  $|z+i| = |z-2|$

v.  $|z-1| = |z-i|$ .

vi.  $|z-i| > |z+1|$

vii.  $|z-2| < |z-4|$ .

viii.  $\log_{0.3}|z-1| > \log_{0.3}|z-i|$ .

ix.  $\log_{1/2}|z-2| > \log_{1/2}|z|$

196. If  $a$  is real and  $|z-ai| = |z+ai|$ , then find the locus of  $z$ . {Ans. real axis}

197. If  $z$  is a complex number such that  $\left| \frac{z-5i}{z+5i} \right| = 1$ , then find the locus of  $z$ . {Ans. real axis}

198. Show that if  $z = x+iy$  and  $w = \frac{1-iz}{z-i}$ , then  $|w|=1$  implies that in the complex plane  $z$  lies on real axis.

199. If the imaginary part of  $\frac{2z+1}{iz+1}$  is  $-2$ , then prove that the locus of the point representing  $z$  in the complex plane is a straight line & find its equation. {Ans.  $(1-2i)z + (1+2i)\bar{z} = 4$ }

200. Find the distance of the point  $2+3i$  from the line  $iz - i\bar{z} + 1 = 0$  and also find the distance of this line from origin. {Ans.  $\frac{5}{2}, \frac{1}{2}$ }

201. Find the angle between the lines  $(1+i)z + (1-i)\bar{z} + 2 = 0$  and  $(-1+2i)z + (1+2i)\bar{z} + i = 0$  and also find their point of intersection. {Ans.  $\tan^{-1} \frac{1}{3}, \frac{1}{2} + \frac{3}{2}i$ }

202. Let  $\bar{b}z + b\bar{z} = c, b \neq 0$ , be a line in the complex plane, where  $\bar{b}$  is the complex conjugate of  $b$ . If a point  $z_1$  is the reflection of a point  $z_2$  through this line, then show that  $c = \bar{z}_1 b + z_2 \bar{b}$ .

**CATEGORY-25.18. CIRCLE**

203. Locate the point representing the complex numbers  $z$  on the Argand diagram for which

- i.  $|z - i| = 1$
- ii.  $|z - 3i| = 2$
- iii.  $|z| < 1$ ,
- iv.  $|z| \geq 3$ ,
- v.  $|z - 3| = 1$ ,
- vi.  $|z - i| < 1$ ,
- vii.  $|i - 1 - 2z| > 9$ ,
- viii.  $2 \leq |z + i| \leq 3$ ,
- ix.  $|z - 1|^2 + |z + 1|^2 = 4$ .
- x.  $z\bar{z} + (-1 + 2i)z - (1 + 2i)\bar{z} + 4 = 0$
- xi.  $\log_{\sqrt{3}} \frac{|z|^2 - |z| + 1}{2 + |z|} < 2$
- xii.  $\log_{\tan 30^\circ} \left( \frac{2|z|^2 + 2|z| - 3}{|z| + 1} \right) < -2$ .

204. If  $|z| = 3$ , what is the location of the points corresponding to

- i.  $z + 2$
- ii.  $z - 1 + i$ .
- iii.  $2 - z$
- iv.  $-1 + 3z$

205. If  $z_1, z_2, z_3, z_4$  are the affixes of four points in the Argand plane, and  $z$  is the affix of a point such that

$$|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|, \text{ then show that } z_1, z_2, z_3, z_4 \text{ are concyclic.}$$

206. If  $z$  is complex number such that  $\operatorname{Re}\left(\frac{1}{z}\right) = 2$ , then show that the locus of  $z$  is a circle.

207. If  $\operatorname{Re}\left(\frac{z - 8i}{z + 6}\right) = 0$ , then show that  $z$  lies on the curve  $x^2 + y^2 + 6x - 8y = 0$ .

208. Show that the locus of the points  $z$  satisfying the condition  $\arg\left(\frac{z - 1}{z + 1}\right) = \frac{\pi}{3}$  is a circle.

209. Find the equation of the circle having  $i$  and  $1 - i$  as diametric end-points. {Ans.  $2z\bar{z} - z - \bar{z} - 2 = 0$ }

210. Among the complex numbers  $z$  which satisfy the condition  $|z - 25i| \leq 15$ , find the number having the least positive argument. {Ans.  $12 + 16i$ }

211. Find the complex number having least modulus which satisfies the condition  $|z - 2 + 2i| = 1$ . {Ans.  $(2 - \frac{1}{\sqrt{2}})(1 - i)$ }

212. If  $z$  is the affix of  $P$  in the Argand diagram and  $P$  moves so that  $\frac{z - i}{z - 1}$  is always purely imaginary, then

prove that locus of  $P$  is a circle of center  $\left(\frac{1}{2}, \frac{1}{2}\right)$  & radius  $\frac{1}{\sqrt{2}}$ .

213. Determine the locus of the point  $z$  such that  $\frac{z^2}{z-1}$  is always real. {Ans. A line & a circle}
214. If  $c$  be real, prove that the equation  $|z-a|^2 + |z-b|^2 = c$  represents a circle & determine its center & radius. {Ans.  $\frac{a+b}{2}, \frac{1}{2}\sqrt{2c-|a-b|^2}$ }
215. If the imaginary part of the expression  $\frac{z-1}{e^{i\theta}} + \frac{e^{i\theta}}{z-1}$  be zero, then determine the locus of the point  $z$ . {Ans. A line & a circle}
216. Find the equation in complex variables of all the circles which are orthogonal to  $|z|=1$  and  $|z-1|=4$ .  
{Ans.  $|z+(7+if)| = \sqrt{48+f^2}, f \in R$ }
217. If  $|z_1|=|z_2|=|z_3|=1$  and  $z_1+z_2+z_3=0$  show that  $z_1, z_2, z_3$  are the vertices of an equilateral triangle inscribed in a unit circle.
218. If  $A$  &  $B$  represents the complex numbers  $6i$  and  $3$  respectively and a point  $P$  moves such that  $PA = 2PB$ , then prove that the locus of  $P$  is  $z\bar{z} = (4+2i)z + (4-2i)\bar{z}$ , which is a circle, and find its center & radius.  
{Ans.  $4-2i, \sqrt{20}$ }

### CATEGORY-25.19. CONIC SECTIONS

219. Locate the point representing the complex numbers  $z$  on the Argand diagram for which

- $|z-i| + |z+1| = 5$
- $|z+1| + |z-2| = 3$
- $|z+i| + |z+1| = 1$
- $|z-1| + |z+1| \leq 5$
- $|z-i| - |z+5i| = 5$
- $|z-i| - |z+5i| = 10$
- $2z\bar{z} - z^2 - \bar{z}^2 - 2z - 2\bar{z} = 0$

220. Find the equation of the parabola having focus  $1+i$  and directrix  $iz - i\bar{z} + 1 = 0$ . {Ans.

$$z\bar{z} + (-2+i)\bar{z} + (-2+i)z + \frac{7}{2} = 0 \}$$

### CATEGORY-25.20. ADDITIONAL QUESTIONS

221. If  $(x+iy)^{\frac{1}{3}} = a+ib$ , then show that  $\frac{x}{a} + \frac{y}{b}$  is equal to  $4(a^2 - b^2)$ .

# Mathematics for IIT-JEE

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*PART-VII*  
*ALGEBRA*

*CHAPTER-26*  
*BINOMIAL THEOREM*

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## ***CHAPTER-26***

### ***BINOMIAL THEOREM***

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#### **LIST OF THEORY SECTIONS**

- 26.1. Binomial Theorem And Its Characteristics
- 26.2. Greatest Term
- 26.3. Some Important Expansions And Standard Result
- 26.4. Relationship Between Binomial Coefficients
- 26.5. Multinomial Theorem

#### **LIST OF QUESTION CATEGORIES**

- 26.1. General Term
- 26.2. Term Containing Given Power Of  $x$
- 26.3. Middle Term
- 26.4. Greatest Term
- 26.5. Sum Of The Coefficients/ Terms
- 26.6. Calculations Using Binomial Theorem
- 26.7. Divisibility Problems Using Binomial Theorem
- 26.8. Standard Result  $(p + \sqrt{q})^n + (p - \sqrt{q})^n = \text{Even Integer}$
- 26.9. Relationships Between Binomial Coefficients
- 26.10. Multinomial Theorem
- 26.11. Additional Questions



**5. Characteristics of Binomial theorem**

- Total number of terms in the binomial expansion is  $n + 1$ .
- Sum of the indices of  $a$  and  $b$  in each term is  $n$ .
- Since  ${}^nC_r = {}^nC_{n-r}$ , the coefficient of terms equidistant from the beginning and the end are equal.
- The coefficient of  $(r + 1)^{th}$  term in the expansion is  ${}^nC_r$ .

**6. General term in the expansion of  $(a + b)^n$** 

The  $(r + 1)^{th}$  term, denoted by  $T_{r+1}$ , is  $T_{r+1} = {}^nC_r a^{n-r} b^r$ ,  $r = 0, 1, 2, \dots, n$ .

**7. Binomial expansion of  $(1 + x)^n$** 

$$\begin{aligned} \text{i. } (1 + x)^n &= {}^nC_0 a^n + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n \\ &= \sum_{r=0}^n {}^nC_r x^r \end{aligned}$$

$$\text{ii. General term } T_{r+1} = {}^nC_r x^r, r = 0, 1, 2, \dots, n.$$

**8. Middle term**

Since the binomial expansion of  $(a + b)^n$  has  $(n + 1)$  terms, so  $\left(\frac{n}{2} + 1\right)^{th}$  term is the middle term if  $n$  is even;

$\left(\frac{n+1}{2}\right)^{th}$  and  $\left(\frac{n+3}{2}\right)^{th}$  terms is the middle term if  $n$  is odd.

**SECTION-26.2. GREATEST TERM****1. Greatest term in the expansion of  $(1 + x)^n$** 

- $T_r \leq T_{r+1} \Rightarrow r \leq \frac{(n+1)x}{1+x}$ .
- Case 1:** If  $\frac{(n+1)x}{1+x}$  is an integer and let  $\frac{(n+1)x}{1+x} = m$ , then  $T_1 < T_2 < \dots < T_m \leq T_{m+1} > T_{m+1} > T_{m+2} \dots$ .  
Therefore,  $T_m$  and  $T_{m+1}$  are both equal and are greatest terms.
- Case 2:** If  $\frac{(n+1)x}{1+x}$  is not an integer and let  $\left[\frac{(n+1)x}{1+x}\right] = m$ , then  $T_1 < T_2 < \dots < T_m < T_{m+1} > T_{m+1} > T_{m+2} \dots$ . Therefore,  $T_{m+1}$  is the greatest term.

**SECTION-26.3. SOME IMPORTANT EXPANSIONS AND STANDARD RESULT****1. Some important expansions**

- $(a + b)^n + (a - b)^n = 2({}^nC_0 a^n + {}^nC_2 a^{n-2} b^2 + \dots)$
- $(a + b)^n - (a - b)^n = 2({}^nC_1 a^{n-1} b + {}^nC_3 a^{n-3} b^3 + \dots)$
- $(1 + x)^n + (1 - x)^n = 2({}^nC_0 + {}^nC_2 x^2 + \dots)$
- $(1 + x)^n - (1 - x)^n = 2({}^nC_1 x + {}^nC_3 x^3 + \dots)$

**2. Standard result**

If  $p$  is an integer and  $q$  is a non-negative integer, then



$$(p + \sqrt{q})^n + (p - \sqrt{q})^n = 2({}^nC_0 p^n + {}^nC_2 p^{n-2} q + \dots) \\ = \text{even integer.}$$

### SECTION-26.4. RELATIONSHIP BETWEEN BINOMIAL COEFFICIENTS

#### 1. Methods to prove relationship between binomial coefficients

- Put a suitable real value of  $x$  in the expansion  $(1+x)^n$ .
- Differentiate (once or more than one times) the expansion  $(1+x)^n$  and then put a suitable real value of  $x$ .
- Multiply/ divide the expansion  $(1+x)^n$  by  $x, x^2, \dots$ , then differentiate (once or more than one times) and then put a suitable real value of  $x$ .
- Integrate (once or more than one times) the expansion  $(1+x)^n$  and then put a suitable real value of  $x$ .
- Multiply/ divide the expansion  $(1+x)^n$  by  $x, x^2, \dots$ , then integrate (once or more than one times) and then put a suitable real value of  $x$ .
- Multiply two expansions  $(1+x)^n$  and  $\left(1 + \frac{1}{x}\right)^n$  and compare coefficient of same powers of  $x$ .
- Expansions  $(1-x)^n, (1+x^2)^n, (1+x)^{2n}$  etc. can be used in above methods.
- Put a suitable complex value of  $x$  in the above expansions and equate real and imaginary parts.

### SECTION-26.5. MULTINOMIAL THEOREM

#### 1. Multinomial theorem for positive integral index

If  $x_1, x_2, \dots, x_k$  are real/ complex numbers/ expressions, then for all  $n \in \mathbb{N}$

$$(x_1 + x_2 + x_3 + \dots + x_k)^n = \sum \frac{n!}{r_1! r_2! r_3! \dots r_k!} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}, \text{ where } r_1, r_2, \dots, r_k \text{ are non-negative integers and } r_1 + r_2 + \dots + r_k = n.$$

**EXERCISE-26****CATEGORY-26.1. GENERAL TERM**

- Determine the value of  $x$  in the expression  $(x + x^{\log x})^5$  if the third term in the expansion is 10,00,000.  
{Ans.  $x = 10, 10^{-\frac{5}{2}}$ }
- If the third term in the expansion of  $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$  is 1000, then find the value of  $x$ . {Ans. 100}
- If the 4<sup>th</sup> term in the expansion of  $\left(ax + \frac{1}{x}\right)^n$  is  $\frac{5}{2}$ , then find the values of  $a$  and  $n$ .
- Find the 14<sup>th</sup> term from the end in the expansion of  $(\sqrt{x} - \sqrt{y})^{17}$ . {Ans.  ${}^{17}C_4 x^{\frac{13}{2}} y^2$ }
- Find the positive value of  $a$  so that the coefficients of  $x^5$  and  $x^{15}$  are equal in the expansion of  $\left(x^2 + \frac{a}{x^3}\right)^{10}$ . {Ans.  $\frac{1}{2\sqrt{3}}$ }
- For what value of  $r$  the coefficients of  $(r-1)$ th and  $(2r+3)$ rd terms in the expansion of  $(1+x)^{15}$  are equal? {Ans.  $r = 5$ }
- If the coefficients of  $(2r+1)$ th term and  $(r+2)$ th term in the expansion of  $(1+x)^{43}$  are equal, find  $r$ . {Ans.  $r = 14$ }
- If the coefficients of  $(2r+4)$ th and  $(r-2)$ th terms in the expansion of  $(1+x)^{18}$  are equal, find  $r$ . {Ans.  $r = 6$ }
- Prove that the coefficient of  $(r+1)$ th term in the expansion of  $(1+x)^{n+1}$  is equal to the sum of the coefficients of  $r$ th and  $(r+1)$ th terms in the expansion of  $(1+x)^n$ .
- Which term in the expansion of the binomial  $\left[\sqrt[3]{\frac{a}{b}} + \sqrt{\frac{b}{3a}}\right]^{21}$  contains  $a$  and  $b$  to one and same power?  
{Ans. 10<sup>th</sup> term}
- If  $a_1, a_2, a_3, a_4$  are the coefficients of any four consecutive terms in the expansion of  $(1+x)^n$ , prove that  
$$\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}.$$
- If the 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup> and 6<sup>th</sup> terms in the expansion of  $(x+\alpha)^n$  be respectively  $a, b, c$  and  $d$ , prove that  
$$\frac{b^2 - ac}{c^2 - bd} = \frac{5a}{3c}.$$
- The 3<sup>rd</sup>, 4<sup>th</sup> and 5<sup>th</sup> terms in the expansion of  $(x+a)^n$  are respectively 84, 280 and 560, find the values of  $x, a$  and  $n$ . {Ans.  $x = 1, a = 2, n = 7$ }
- If the 2<sup>nd</sup>, 3<sup>rd</sup> and 4<sup>th</sup> terms in the expansion of  $(x+a)^n$  are 240, 720 and 1080 respectively, find  $x, a$  and  $n$ . {Ans.  $x = 2, a = 3, n = 5$ }
- If the coefficients of three consecutive terms in the expansion of  $(1+x)^n$  be 165, 330 and 462, find  $n$ .
- If the coefficients of second, third and fourth terms in the expansion of  $(1+x)^{2n}$  are in A.P., show that

$$2n^2 - 9n + 7 = 0.$$

17. The coefficients of second, third and fourth terms in the expansion of  $(1+x)^n$  are in A.P., find  $n$ . {Ans.  $n = 7$ }
18. Determine at what  $x$  the 6<sup>th</sup> term in the expansion of the binomial  $\left[ \sqrt{2^{\log(10-3^x)}} + \sqrt[5]{2^{(x-2)\log 3}} \right]^m$  is equal to 21 if it is known that the binomial coefficients of the 2<sup>nd</sup>, 3<sup>rd</sup> and 4<sup>th</sup> terms in the expansion represent respectively the first, third and fifth terms of an A.P. {Ans.  $x = 0, 2$ }
19. Find  $n$  in the binomial  $\left( \sqrt[3]{2} + \frac{1}{\sqrt[3]{3}} \right)^n$  if the ratio of 7<sup>th</sup> term from the beginning to 7<sup>th</sup> term from the end is  $1/6$ . {Ans.  $n = 9$ }
20. The coefficients of 5<sup>th</sup>, 6<sup>th</sup> and 7<sup>th</sup> terms in the expansion of  $(1+x)^n$  are in A.P., find  $n$ . {Ans.  $n = 7, 14$ }
21. In the expansion of  $(1+x)^n$  the binomial coefficients of three consecutive terms are respectively 220, 495 and 792, find the value of  $n$ . {Ans.  $n = 12$ }
22. If the coefficients of  $r$ th,  $(r+1)$ th and  $(r+2)$ th terms in the expansion of  $(1+x)^{14}$  are in A.P., find  $r$ . {Ans.  $r = 5, 9$ }
23. Given positive integers  $r > 1$ ,  $n > 2$  and the coefficients of  $(3r)$ th and  $(r+2)$ th terms in the binomial expansion of  $(1+x)^{2n}$  are equal, then show that  $n = 2r$ .
24. Find the two consecutive terms in the expansion of  $(3+2x)^{74}$  whose coefficients are equal. {Ans. 30<sup>th</sup> & 31<sup>st</sup>}

### CATEGORY-26.2. TERM CONTAINING GIVEN POWER OF $x$

25. Find the term independent of  $x$  in the expansion of

i.  $\left( 3x - \frac{2}{x^2} \right)^{15}$  {Ans.  $T_6 = -(3003)3^{10}2^5$ }

ii.  $\left( x - \frac{1}{3x^2} \right)^9$  {Ans.  $T_4 = -\frac{28}{9}$ }

iii.  $\left( \frac{3}{2}x^2 - \frac{1}{3x} \right)^9$  {Ans.  $T_7 = \frac{7}{18}$ }

iv.  $\left( x^2 - \frac{1}{3x} \right)^9$  {Ans.  $T_7 = \frac{28}{243}$ }

v.  $\left( \sqrt{\frac{x}{3}} + \frac{3}{2x^2} \right)^{10}$  {Ans.  $T_3 = \frac{5}{4}$ }

vi.  $\left( \frac{1}{2}x^{1/3} + x^{-1/5} \right)^8$  {Ans.  $T_6 = 7$ }

vii.  $\left( x^3 - \frac{3}{x^2} \right)^{15}$  {Ans.  $T_{10} = {}^{15}C_9(-3)^9$ }

- viii.  $\left(x + \frac{1}{x}\right)^{2n}$  {Ans.  $\frac{1.3.5 \cdots (2n-1)}{n!} 2^n$ }
- ix.  $(1+x+2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$  {Ans.  $\frac{17}{54}$ }
26. Given that  $p$  is a constant,  $x$  is a variable and the fourth term in the expansion of  $\left(px + \frac{1}{x}\right)^n$  is  $\frac{5}{2}$ . Evaluate  $n$  and  $p$  {Ans.  $n = 6, p = \frac{1}{2}$ }
27. Find the coefficient of  $x^{10}$  and  $x^9$  in the expansion of  $\left(2x^2 - \frac{1}{x}\right)^{20}$ . {Ans.  $\frac{20!}{10!10!} 2^{10}, 0$ }
28. Find the coefficient of  $x^4$  in the expansion of  $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ . {Ans.  $\frac{405}{256}$ }
29. Find the coefficient of  $x^9$  in the expansion of  $\left(x^2 - \frac{1}{3x}\right)^9$ . {Ans.  $-\frac{28}{9}$ }
30. Find the coefficient of  $x^5$  and  $x^{-15}$  in the expansion of  $\left(3x^2 - \frac{b}{3x^3}\right)^{10}$ . {Ans.  $-9720b^3, -\frac{40}{27}b^7$ }
31. Prove that the ratio of the coefficient of  $x^{10}$  in  $(1-x^2)^{10}$  and the term independent of  $x$  in  $\left(x - \frac{2}{x}\right)^{10}$  is 1:32.
32. Find the coefficient of  $x$  in the expansion of  $(1-2x^3+3x^5)\left[1+(1/x)\right]^8$ . {Ans. 154}
33. Show that the coefficient of  $x^5$  in the expansion of  $(1+x^2)^5(1+x)^4$  is 60.
34. Find the coefficient of  $x^7$  in  $\left(ax^2 + \frac{1}{bx}\right)^{11}$  and of  $x^{-7}$  in  $\left(ax - \frac{1}{bx^2}\right)^{11}$  and find the relation between  $a$  and  $b$  so that these coefficients are equal. {Ans.  ${}^{11}C_5 \frac{a^6}{b^5}, {}^{11}C_6 \frac{a^5}{b^6}, ab = 1$ }
35. Find the coefficient of  $x^r$  in the expansion of  $\left(x + \frac{1}{x}\right)^n$  if it occurs. {Ans.  ${}^nC_{\frac{n-r}{2}}$ }
36. If  $x^{4r}$  occurs in the expansion of  $\left(x + \frac{1}{x^2}\right)^{4n}$  prove that its coefficient is  $\frac{(4n)!}{\left[\frac{4}{3}(n-r)\right]!\left[\frac{4}{3}(2n+r)\right]!}$ .
37. If  $x^p$  occurs in the expansion of  $\left(x^2 + \frac{1}{x}\right)^{2n}$  prove that its coefficient is  $\frac{(2n)!}{\left[\frac{1}{3}(4n-p)\right]!\left[\frac{1}{3}(2n+p)\right]!}$ .
38. Prove that the coefficient of  $x^n$  in the expansion of  $(1+x)^{2n}$  is double the coefficient of  $x^n$  in the expansion of  $(1+x)^{2n-1}$ .
39. Show that the coefficients of  $x^p$  and  $x^q$  (where  $p$  and  $q$  are positive integers) in the expansion of  $(1+x)^{p+q}$  are equal.
40. Find the coefficient of  $x^4$  in the expansion of
- i.  $(1+x+x^2+x^3)^{11}$  {Ans. 990}

ii.  $(2 - x + 3x^2)^6$  {Ans. 3660}

iii.  $(1 + x - 2x^2)^6$

### CATEGORY-26.3. MIDDLE TERM

41. Find the number of terms and the middle term or terms in the expansion of

i.  $\left(x + \frac{1}{x}\right)^{10}$ . {Ans. 11,  ${}^{10}C_5$ }

ii.  $\left(\frac{a}{x} + bx\right)^{12}$  {Ans. 13,  $T_7 = 924a^6b^6$ }

iii.  $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$  {Ans. 11,  $T_6 = -252$ }

iv.  $\left(3x - \frac{x^3}{6}\right)^9$ . {Ans. 10,  $T_5 = \frac{189}{8}x^{17}$ ,  $T_6 = -\frac{21}{16}x^{19}$ }

42. Prove that the middle term in the expansion of  $(1+x)^{2n}$  is  $\frac{1.3.5 \cdots (2n-1)}{n!} 2^n x^n$ .

### CATEGORY-26.4. GREATEST TERM

43. If  $x = \frac{1}{3}$ , find the greatest term in the expansion of  $(1+4x)^8$ . {Ans.  $T_6 = 56\left(\frac{4}{3}\right)^5$ }

44. Find the greatest term in the expansion of  $(2+3x)^9$  when  $x = \frac{3}{2}$ . {Ans.  $T_7 = \frac{3^{13.7}}{2}$ }

45. Find the value of the greatest term in the expansion of  $\sqrt{3}\left(1 + \frac{1}{\sqrt{3}}\right)^{20}$ . {Ans.  $T_8 = \frac{25840}{9}$ }

46. Find numerically the greatest term in the expansion of  $(2+3x)^{12}$  when  $x = \frac{5}{6}$ . {Ans.  $T_8 = {}^{12}C_7 2^7 \left(\frac{5}{2}\right)^5$ }

47. Find numerically the greatest term in the expansion of  $(3-5x)^{15}$  when  $x = \frac{1}{5}$ . {Ans.  $T_4 = T_5 = 455 \times 3^{12}$ }

48. Given that the 4<sup>th</sup> term in the expansion of  $\left(2 + \frac{3}{8}x\right)^{10}$  has the maximum numerical value, find the range of value of  $x$  for which this will be true. {Ans.  $x \in \left[-\frac{64}{21}, -2\right]$ }

### CATEGORY-26.5. SUM OF THE COEFFICIENTS/ TERMS

49. Find the sum of the coefficients of the polynomial  $(1+x-3x^2)^{2143}$ . {Ans. -1}

50. Find out the sum of the coefficients in the expansion of the binomial  $(5p-4q)^n$ , where  $n$  is a +ive integer. {Ans. 1}

51. If the complete expansion of the expression  $(1+x-2x^2)^6$  is  $1 + a_1x + a_2x^2 + \cdots + a_{12}x^{12}$ , prove that  $a_2 + a_4 + a_6 + \cdots + a_{12} = 31$ .

52. If  $(1 - x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$ , then find the value of  $a_0 + a_2 + a_4 + \cdots + a_{2n}$ . {Ans.  $\frac{3^n + 1}{2}$ }
53. In the expansion of  $(x + a)^n$  if the sum of odd terms be  $P$  and sum of even terms be  $Q$ , prove that
- $P^2 - Q^2 = (x^2 - a^2)^n$ ,
  - $4PQ = (x + a)^{2n} - (x - a)^{2n}$ .

#### CATEGORY-26.6. CALCULATIONS USING BINOMIAL THEOREM

54. Evaluate  $\left[ x + \sqrt{x^2 - 1} \right]^6 + \left[ x - \sqrt{x^2 - 1} \right]^6$ .
55. Find the value of  $\frac{18^3 + 7^3 + 3 \cdot 18 \cdot 7 \cdot 25}{3^6 + 6 \cdot 243 \cdot 2 + 15 \cdot 81 \cdot 4 + 20 \cdot 27 \cdot 8 + 15 \cdot 9 \cdot 16 + 6 \cdot 3 \cdot 32 + 64}$ . {Ans. 1}
56. Use the binomial theorem to find  $999^3$ . {Ans. 997002999}
57. Evaluate  $(0.99)^{15}$  and  $(1.0025)^{10}$  correct to four decimal places. {Ans. 0.8605, 1.025}
58. Determine larger of  $99^{50} + 100^{50}$  and  $101^{50}$ . {Ans.  $101^{50}$ }

#### CATEGORY-26.7. DIVISIBILITY PROBLEMS USING BINOMIAL THEOREM

59. Show that  $(1 + x)^n - nx - 1$  is divisible by  $x^2$ .
60. Show that  $2^{3n} - 7n - 1$  is divisible by 49. Hence show that  $3^{3n+3} - 7n - 8$  is divisible by 49,  $n \in N$ .
61. Show that  $3^{2n+2} - 8n - 9$  is divisible by 64,  $n \in N$ .
62. Show that  $2^{4n} - 15n - 1$  is divisible by 225.
63. What is remainder when  $5^{99}$  is divided by 13? {Ans. 8}

#### CATEGORY-26.8. STANDARD RESULT $(p + \sqrt{q})^n + (p - \sqrt{q})^n = \text{EVEN INTEGER}$

64. Show that  $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6 = 198$ . Hence show that the integral part of  $(\sqrt{2} + 1)^6$  is 197.
65. If  $(2 + \sqrt{3})^n = I + f$  where  $I$  and  $n$  are +ive integers and  $0 < f < 1$ , show that  $I$  is an odd integer and  $(1 - f)(I + f) = 1$ .
66. If  $(5 + 2\sqrt{6})^n = I + f$  where  $I$  and  $n$  are +ive integers and  $f$  is a +ive fraction less than one, show that  $I$  is an odd integer and  $(I + f)(1 - f) = 1$ .
67. If  $n$  be any positive integer, show that the integral part of  $(7 + 4\sqrt{3})^n$  is an odd number. Also if  $(7 + 4\sqrt{3})^n = I + f$  where  $I$  is a +ive integer and  $f$  is a proper fraction, show that  $(1 - f)(I + f) = 1$ .
68. Show that the integer next above  $(\sqrt{3} + 1)^{2m}$  contains  $2^{m+1}$  as a factor.
69. If  $(6\sqrt{6} + 14)^{2n+1} = P$ , prove that the integral part of  $P$  is an even integer and  $PF = 20^{2n+1}$  where  $F$  is the fractional part of  $P$ .
70. Let  $R = (5\sqrt{5} + 11)^{2n+1}$  and  $f = R - [R]$  where  $[ ]$  denotes the greatest integer function, prove that

$$Rf = 4^{2n+1}.$$

**CATEGORY-26.9. RELATIONSHIPS BETWEEN BINOMIAL COEFFICIENTS**

$$71. C_1 + 2C_2 + 3C_3 + \dots + nC_n = n \cdot 2^{n-1}.$$

$$72. C_1 - 2C_2 + 3C_3 - 4C_4 + \dots = 0.$$

$$73. C_1 + 2C_2x + 3C_3x^2 + \dots + 2nC_{2n}x^{2n-1} = 2n(1+x)^{2n-1}, \text{ where } C_r = \frac{(2n)!}{r!(2n-r)!}, r = 0, 1, 2, 3, \dots, 2n.$$

$$74. C_1 + 2C_2x + 3C_3x^2 + \dots + nC_nx^{n-1} = n(1+x)^{n-1}.$$

$$75. 1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \dots + n^2 \cdot C_n = n(n+1) \cdot 2^{n-2}.$$

$$76. C_0 - 2^2C_1 + 3^2C_2 + \dots + (-1)^n(n+1)^2C_n = 0, n > 2.$$

$$77. a - {}^nC_1(a-1) + {}^nC_2(a-2) - \dots + (-1)^n(a-n) = 0.$$

$$78. C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n = 2^n + n2^{n-1} = (n+2)2^{n-1}.$$

$$79. C_0 + 3C_1 + 5C_2 + \dots + (2n+1)C_n = (n+1)2^n.$$

$$80. \frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \dots + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}.$$

$$81. \text{ If } (1+x)^{15} = C_0 + C_1x + C_2x^2 + \dots + C_{15}x^{15}, \text{ find the value of}$$

$$\text{i. } C_2 + 2C_3 + 3C_4 + \dots + 14C_{15}. \{ \text{Ans. } 219923 \}$$

$$\text{ii. } \frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \dots + 15\frac{C_{15}}{C_{14}} \{ \text{Ans. } 120 \}$$

$$82. \frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \frac{C_6}{7} + \dots = \frac{2^n}{n+1}.$$

$$83. C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}.$$

$$84. C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \dots = \frac{1}{n+1}.$$

$$85. 2C_0 + 2^2\frac{C_1}{2} + 2^3\frac{C_2}{3} + 2^4\frac{C_3}{4} + \dots + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{n+1}-1}{n+1}.$$

$$86. \frac{1}{1!(n-1)!} + \frac{1}{3!(n-3)!} + \frac{1}{5!(n-5)!} + \dots = \frac{2^{n-1}}{n!}.$$

$$87. 2C_0 + \frac{2^2}{2}C_1 + \frac{2^3}{3}C_2 + \dots + \frac{2^{11}}{11}C_{10} = \frac{3^{11}-1}{11}.$$

$$88. C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + \dots + \frac{C_n}{n+1}x^n = \frac{(1+x)^{n+1}-1}{(n+1)x}.$$

$$89. \frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \dots + \frac{C_n}{n+2} = \frac{1+n \cdot 2^{n+1}}{(n+1)(n+2)}.$$

$$90. C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{(2n)!}{n!n!}.$$

$$91. \text{ If } (1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n, \text{ then show that the sum of the products of the } C_i \text{'s taken two at}$$

a time represented by  $\sum \sum C_i C_j$  is equal to  $2^{2n-1} - \frac{(2n)!}{2 \cdot (n!)^2} \cdot (0 \leq i < j \leq n)$

$$92. \quad C_0 C_1 + C_1 C_2 + \dots + C_{n-1} C_n = \frac{(2n)!}{(n-1)!(n+1)!}.$$

$$93. \quad C_0 C_2 + C_1 C_3 + C_2 C_4 + \dots + C_{n-2} C_n = \frac{(2n)!}{(n-2)!(n+2)!}.$$

$$94. \quad C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \dots + C_{n-r} C_n = \frac{(2n)!}{(n-r)!(n+r)!}.$$

$$95. \quad C_0 C_n + C_1 C_{n-1} + \dots + C_n C_0 = \frac{(2n)!}{n!n!}.$$

96. Prove that according as  $n$  is odd or even the value of  $C_0^2 - C_1^2 + C_2^2 - \dots + (-1)^n (C_n)^2 = 0$  or

$$(-1)^{n/2} \frac{n!}{(n/2)!(n/2)!}.$$

$$97. \quad ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \dots + ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n.$$

98. If  $C_0, C_1, C_2, \dots, C_n$  denote the binomial coefficient in the expansion of  $(1+x)^n$ , then find the value of  $aC_0 + (a+b)C_1 + (a+2b)C_2 + \dots + (a+nb)C_n$ . {Ans.  $(2a+nb)2^{n-1}$ }

99. Evaluate  $C_0^2 - C_1^2 + C_2^2 - \dots + (-1)^n C_n^2$  for  $n = 10$  and  $n = 11$ . {Ans. -252, 0}

100. Prove that  ${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \dots + {}^nC_r$ . Where  $r < m, r < n$  and  $m, n, r$  are +ive integers.

101. Prove that  $C_1^2 - 2 \cdot C_2^2 + 3 \cdot C_3^2 - \dots - 2nC_{2n}^2 = (-1)^{n-1} nC_n$ .

102. In the usual notations prove that  $(C_0 + C_1)(C_1 + C_2) \dots (C_{n-1} + C_n) = \frac{(n+1)^n}{n!} C_1 C_2 C_3 \dots C_n$ .

103. If  $(1+x)^n = \sum_{r=0}^n C_r x^r$ , prove that  $\frac{2^2 \cdot C_0}{1 \cdot 2} + \frac{2^3 \cdot C_1}{2 \cdot 3} + \dots + \frac{2^{n+2} \cdot C_n}{(n+1)(n+2)} = \frac{3^{n+2} - 2n - 5}{(n+1)(n+2)}$ .

$$104. \quad \frac{C_1}{1} - \frac{C_2}{2} + \frac{C_3}{3} - \frac{C_4}{4} + \dots + \frac{(-1)^{n-1}}{n} C_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

105. Prove that  $\frac{C_0}{x} - \frac{C_1}{x+1} + \frac{C_2}{x+2} - \dots + (-1)^n \frac{C_n}{x+n} = \frac{n!}{x(x+1) \dots (x+n)}$  where  $n$  is a +ive integer and  $x$  is not a

negative integer. Hence prove that  $\frac{C_0}{1} - \frac{C_1}{4} + \frac{C_2}{7} - \dots + (-1)^n \frac{C_n}{3n+1} = \frac{3^n \cdot n!}{1 \cdot 4 \cdot 7 \dots (3n+1)}$ .

106. Find the sum of  $3^n C_0 - 8^n C_1 + 13^n C_2 - 18^n C_3 + \dots$  upto  $(n+1)$  terms. {Ans. 0}

107. If  $a_0, a_1, a_2, \dots, a_{2n}$  be the coefficients in the expansion of  $(1+x+x^2)^n$  in ascending powers of  $x$  prove that  $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots - a_{2n-1}^2 + a_{2n}^2 = a_n$ .

108. If  $n$  is a positive integer and  $C_k = {}^nC_k$ , find the value of  $\sum_{k=1}^n k^3 (C_k / C_{k-1})^2$ . {Ans.  $\frac{n(n+2)(n+1)^2}{12}$ }

109. If  $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$ , prove that

$$i. \quad C_0 - C_2 + C_4 - \dots = 2^{\frac{n}{2}} \cos\left(\frac{n\pi}{4}\right)$$



$$\text{ii. } C_1 - C_3 + C_5 - \dots = 2^{\frac{n}{2}} \sin\left(\frac{n\pi}{4}\right)$$

$$\text{iii. } C_0 + C_3 + C_6 + \dots = \frac{1}{3} \left( 2^n + \cos \frac{n\pi}{3} \right)$$

$$\text{iv. } C_0 + C_4 + C_8 + \dots = 2^{\frac{n-1}{2}} \cos \frac{n\pi}{4} + 2^{n-2}$$

110. If  $T_0, T_1, T_2, \dots, T_n$  represent the terms in the expansion of  $(x+a)^n$ , then

$$(T_0 - T_2 + T_4 - \dots)^2 + (T_1 - T_3 + T_5 - \dots)^2 = (x^2 + a^2)^n.$$

111. Prove that  $\sum_{r=1}^k (-3)^{r-1} {}^{3n}C_{2r-1} = 0$ , where  $k = \frac{3n}{2}$  and  $n$  is an even positive integer.

### CATEGORY-26.10. MULTINOMIAL THEOREM

112. Write the general term in the expansion of  $(2x + y + 3z)^{10}$ . Also find the coefficient of  $x^2 y^5 z^3$  in this

$$\text{expansion. \{Ans. } \frac{10!}{r_1! r_2! r_3!} 2^{r_1} 3^{r_3} x^{r_1} y^{r_2} z^{r_3}, r_1 + r_2 + r_3 = 10; 18 \times 7! \}$$

113. Find the coefficient of  $x^7$  in the expansion of  $(1 + 3x - 2x^3)^{10}$ . {Ans. 62640}

### CATEGORY-26.11. ADDITIONAL QUESTIONS

114. Find the number of terms in the expansion of  $(1 + 2x + x^2)^{20}$ , when expanded in descending powers of  $x$ .  
{Ans. 41}

115. Show that no three consecutive binomial coefficients can be in (i) G.P. (ii) H.P.

116. Find the coefficient of  $x^{50}$  in the expression  $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \dots + 1001x^{1000}$  {Ans.  ${}^{1002}C_{50}$ }

117. If  $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$  and  $a_k = 1$  for all  $k \geq n$ , then show that  $b_n = {}^{2n+1}C_{n+1}$

118. Given  $s_n = 1 + q + q^2 + \dots + q^n$ ,  $S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \dots + \left(\frac{q+1}{2}\right)^n$ ,  $q \neq 1$ . Prove that

$${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \dots + {}^{n+1}C_{n+1} s_n = 2^n S_n.$$

119. Find the sum of the series:  $\sum_{r=0}^n (-1)^r {}^nC_r \left[ \frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \dots \text{up to } m \text{ terms} \right]$ . {Ans.  $\frac{2^{mn}-1}{2^{mn}(2^n-1)}$ }

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# Mathematics for IIT-JEE

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*PART-VII*  
*ALGEBRA*

*CHAPTER-27*  
*PERMUTATIONS AND COMBINATIONS*

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## **CHAPTER-27**

### **PERMUTATIONS AND COMBINATIONS**

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#### **LIST OF THEORY SECTIONS**

- 27.1. Multiplication And Addition Theorems
- 27.2. Permutation
- 27.3. Combination
- 27.4. Selection
- 27.5. Grouping
- 27.6. Derangement

#### **LIST OF QUESTION CATEGORIES**

- 27.1. Multiplication And Addition Theorems
- 27.2. Permutations Of Distinct Objects
- 27.3. Permutations Of Objects Not All Distinct
- 27.4. Permutations When Objects Can Repeat
- 27.5. Circular Permutations
- 27.6. Combinations From Distinct Objects
- 27.7. Combinations From Objects Not All Distinct
- 27.8. Permutations And Combinations
- 27.9. Total Selections From Distinct Objects
- 27.10. Total Selections From Objects Not All Distinct
- 27.11. Grouping Of Distinct Objects
- 27.12. Grouping Of Identical Objects
- 27.13. Derangements
- 27.14. Application Of Methods
- 27.15. Properties Of  ${}^nP_r$  And  ${}^nC_r$
- 27.16. Additional Questions

## CHAPTER-27

### PERMUTATIONS AND COMBINATIONS

#### SECTION-27.1. MULTIPLICATION AND ADDITION THEOREMS

##### 1. Multiplication theorem for Associated events

If an event can occur in  $m$  ways, and with each way of its occurrence, another event can occur in  $n$  ways (associated events), then the two events can together occur in  $m \times n$  ways.

##### 2. Addition theorem for Mutually exclusive events

If an event can occur in  $m$  ways and another event can occur in  $n$  ways and both cannot occur simultaneously (mutually exclusive), then either of the events can occur in  $m + n$  ways.

##### 3. Tree diagrams

Tree diagrams are used to divide an event into associated and mutually exclusive events.

#### SECTION-27.2. PERMUTATION

##### 1. Permutations of distinct objects

i. Number of ways of permuting (arranging)  $n$  distinct objects in  $r$  positions  $= {}^n P_r = \frac{n!}{(n-r)!}$  ( $1 \leq r < n$ ).

ii. Number of ways of permuting  $n$  distinct objects in  $n$  positions  $= {}^n P_n = \frac{n!}{0!} = n!$  ( $0! = 1$ ).

##### 2. Permutations of objects not all distinct

i. Number of ways of permuting  $n$  objects in  $n$  positions when  $n_1$  objects are alike of one kind  
 $= \frac{n!}{n_1!}$  ( $n_1 < n$ ).

ii. Number of ways of permuting  $n$  objects in  $n$  positions when  $n_1$  objects are alike of one kind,  $n_2$  are alike of another kind, .....,  $n_k$  are alike of another kind  $= \frac{n!}{n_1! n_2! \dots n_k!}$  ( $n_1 + n_2 + \dots + n_k = n$ ).

##### 3. Permutations when objects can repeat

Number of ways of permuting  $n$  different objects in  $r$  positions with repetitions  $= n^r$ .

##### 4. Circular permutations

Number of ways of permuting  $n$  distinct objects around a circle

$= (n-1)!$  (Clockwise and anti-clockwise orders are distinct).

$= \frac{(n-1)!}{2}$  (Clockwise and anti-clockwise orders are not distinct).

#### SECTION-27.3. COMBINATION

##### 1. Combinations from distinct objects

Number of ways of selecting  $r$  objects from  $n$  distinct objects  $= {}^n C_r = \frac{n!}{(n-r)! r!}$  ( $1 \leq r \leq n$ ).

##### 2. Combinations from objects not all distinct

i. Number of ways of selecting  $r$  objects from  $m$  identical objects of one kind,  $p$  identical objects of another kind,  $q$  identical objects of another kind, .....

= Coefficient of  $x^r$  in  $(x^0 + x^1 + x^2 + \dots + x^m)(x^0 + x^1 + x^2 + \dots + x^p)(x^0 + x^1 + x^2 + \dots + x^q) \dots$   
(An object of a kind may not be included in selection)

= Coefficient of  $x^r$  in  $(x^1 + x^2 + \dots + x^m)(x^1 + x^2 + \dots + x^p)(x^1 + x^2 + \dots + x^q) \dots$   
(At least one object of each kind is included in selection)

ii.  $(1-x)^{-n} = 1 + {}^nC_1 x + {}^{n+1}C_2 x^2 + \dots + {}^{n+r-1}C_r x^r + \dots$

### SECTION-27.4. SELECTION

#### 1. Total selections from distinct objects

Number of ways of selecting zero, one or more objects from  $n$  distinct objects

$$= {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n - 1 \quad (\text{Empty selection not permitted}).$$

$$= {}^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n \quad (\text{Empty selection permitted}).$$

#### 2. Total selections from objects not all distinct

Number of ways of selecting zero, one or more objects from  $p$  identical objects of one kind,  $q$  identical objects of another kind,  $r$  identical objects of another kind, .....

$$= (p+1)(q+1)(r+1) \dots - 1 \quad (\text{Empty selection not permitted}).$$

$$= (p+1)(q+1)(r+1) \dots \quad (\text{Empty selection permitted}).$$

### SECTION-27.5. GROUPING

#### 1. Grouping of distinct objects

i. Number of ways of dividing  $n$  distinct objects into  $m$  unequal sized groups containing  $p, q, r, \dots$  objects ( $p \neq q \neq r \dots$ ) is

$$= \frac{n!}{p!q!r! \dots} \quad (\text{Groups not ordered}).$$

$$= \frac{n!}{p!q!r! \dots} \times m! \quad (\text{Groups ordered}).$$

ii. Number of ways of dividing  $n$  distinct objects into  $m$  equal sized groups containing  $p$  objects each ( $mp = n$ ) is

$$= \frac{n!}{(p!)^m} \cdot \frac{1}{m!} \quad (\text{Groups not ordered}).$$

$$= \frac{n!}{(p!)^m} \cdot \frac{1}{m!} \times m! = \frac{n!}{(p!)^m} \quad (\text{Groups ordered}).$$

iii. Number of ways of dividing  $n$  distinct objects into  $i$  groups containing  $p$  objects each,  $j$  groups containing  $q$  objects each, .....

$$= \frac{n!}{(p!)^i (q!)^j \dots i! j! \dots} \quad (\text{Groups not ordered}).$$

$$= \frac{n!}{(p!)^i (q!)^j \dots i! j! \dots} \times (i+j+\dots)! \quad (\text{Groups ordered}).$$

#### 2. Grouping of identical objects

Number of ways of dividing  $n$  identical objects into  $m$  ordered groups

$$= \text{Coefficient of } x^n \text{ in } (x^0 + x^1 + \dots + x^n)^m = \text{Coefficient of } x^n \text{ in } (1-x)^{-m} = {}^{n+m-1}C_{m-1}$$

(Empty group permitted)

$$= \text{Coefficient of } x^n \text{ in } (x^1 + \dots + x^n)^m = \text{Coefficient of } x^n \text{ in } x^m (1-x)^{-m} = {}^{n-1}C_{m-1}$$

(Empty group not permitted)

**SECTION-27.6. DERANGEMENT****1. Derangements**

If  $n$  distinct objects have  $n$  assigned positions, then the number of ways in which they can be deranged so that none of them occupies its assigned position, is

$$= n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right).$$

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**EXERCISE-27**

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**CATEGORY-27.1. MULTIPLICATION AND ADDITION THEOREMS**

1. In a class there are 10 boys and 8 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection? {Ans. 80}
2. In a class there are 10 boys and 8 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways the teacher can make this selection? {Ans. 18}
3. Two men enter a railway compartment having 6 seats unoccupied. In how many ways can they be seated? {Ans. 30}
4. There are 3 candidates for a Classical, 5 for a Mathematical, and 4 for a Natural Science scholarship.
  - i. In how many ways one scholarship can be awarded in each subject? {Ans. 60}
  - ii. In how many ways only one scholarship can be awarded? {Ans. 12}
5. A room has 6 doors. In how many ways can a man enter the room through one door and come out through a different door? {Ans. 30}
6. The flag of a newly formed forum is in the form of three blocks, each to be coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible? {Ans. 120}
7. A lock consists of three rings each marked with 12 different letters; find in how many ways it is possible to make an unsuccessful attempt to open the lock. {Ans. 1727}
8. Five persons entered the lift cabin on the ground of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin
  - i. at any one of the 7 floors. {Ans.  $7^5$ }
  - ii. at different floors. {Ans. 2520}
9. In a monthly test, the teacher decides that there will be three questions, one from each of Exercises 7, 8 and 9 of the textbook. If there are 12 questions in Exercise 7, 18 in Exercise 8 and 9 in Exercise 9, in how many ways can three questions be selected? {Ans. 1944}
10. A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of February calendars should it prepare to serve for all the possibilities in the future years? {Ans. 14}
11. How many words (with or without meaning) of three distinct letters of the English alphabets are there? {Ans. 15600}
12. There are 6 multiple choice questions in an examination. How many sequence of answers are possible, if the first three questions have 4 choices each and the next three have 5 each? {Ans. 8000}
13. Find the total number of ways of answering 5 objective type questions, each question having 4 choices. {Ans.  $4^5$ }
14. For a set of five true/false questions, no student has written all correct answers, and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible? {Ans. 31}
15. How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6? {Ans. 64}
16. How many numbers are there between 100 and 1000
  - i. in which all the digits are distinct? {Ans. 648}
  - ii. such that every digit is either 2 or 9? {Ans. 8}
  - iii. such 7 is in the unit's place? {Ans. 90}
  - iv. such that at least one of their digits is 7? {Ans. 252}



- v. which have exactly one of their digits as 7? {Ans. 225}
17. A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the cards? {Ans. 729}
18. Seven different objects must be divided among three people. In how many ways can it be done if one or two of them can get no objects. {Ans. 2187}
19. How many three-digit numbers more than 600 can be formed by using the digits 2, 3, 4, 6, 7. {Ans. 50}
20. A telegraph has five arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that can be made? {Ans. 1023}
21. How many numbers between 3000 and 4000 can be formed from the digits 3, 4, 5, 6, 7 and 8, no digit being repeated in any number? {Ans. 60}
22. How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7 and 8. {Ans. 25}
23. How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 if  
i. repetition of digits is not allowed? {Ans. 300}  
ii. repetition of digits is allowed? {Ans. 1080}
24. How many three letter words can be formed using  $a, b, c, d, e$  if  
i. repetition is not allowed? {Ans. 60}  
ii. repetition is allowed? {Ans. 125}
25. In how many ways can the following prizes be given away to a class of 30 students, first and second in Mathematics, first and second in Physics, first in Chemistry and first in English? {Ans. 681210000}
26. How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4 if  
i. repetition of digits is allowed? {Ans. 375}  
ii. repetition of digits is not allowed? {Ans. 72}
27. How many three digit odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 if  
i. the repetition of digits is not allowed? {Ans. 60}  
ii. the repetition of digits is allowed? {Ans. 108}

### CATEGORY-27.2. PERMUTATIONS OF DISTINCT OBJECTS

28. Write down all the permutations of the set of three letters  $A, B, C$ . {Ans. 6 permutations}
29. Write down all the permutations of the vowels  $A, E, I, O, U$  in English alphabets taking three at a time, and starting with  $A$ . {Ans. 12 permutations}
30. Write down all the permutations of letters  $A, B, C, D$  taking three at a time. {Ans. 24 permutations}
31. In how many ways three different rings can be worn in four fingers with at most one in each finger? {Ans. 24}
32. Seven athletes are participating in a race. In how many ways can the first three prizes be won? {Ans. 210}
33. How many different signals can be made by 5 flags from 8 flags of different colours? {Ans. 6720}
34. In how many ways can 6 persons stand in a queue? {Ans. 720}
35. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible? {Ans. 2880}
36. Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them? {Ans. 172800}
37. How many different signals can be given using any number of flags from 5 flags of different colours? {Ans. 325}
38. In an examination hall there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination with 16 students in each class. It is desired that in each row, all

- students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated? {Ans.  $2 \times 16! \times 16!$ }
39. How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed? {Ans. 60}
40. How many four digit numbers are there with distinct digits? {Ans. 4536}
41. Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time. {Ans. 93324}
42. In how many ways 7 pictures can be hung from 5 picture nails on a wall? {Ans. 2520}
43. How many words, with or without meaning, can be formed using all the letters of the word *EQUATION*, using each letter exactly once. {Ans. 8!}
44. How many 4-letter words, with or without meaning, can be formed out of the letters of the word *LOGARITHMS*, if repetition of letters is not allowed? {Ans. 5040}
45. In how many ways can the letters of the word *PENCIL* be arranged so that
- N* is always next to *E*? {Ans. 120}
  - N* and *E* are always together? {Ans. 240}
46. How many different words can be formed with the letters of the word *EQUATION* so that
- the words begin with *E*? {Ans. 7!}
  - the words begin with *E* and end with *N*? {Ans. 6!}
  - the words begin and end with a consonant? {Ans. 4320}
47. How many words can be formed from the letters of the word *TRIANGLE*? How many of these will begin with *T* and end with *E*? {Ans. 40320, 720}
48. In how many ways can the letters of the word *DELHI* be arranged so that the vowels occupy only even places? {Ans. 12}
49. How many words can be formed from the letters of the word *DAUGHTER* so that
- the vowels always come together? {Ans. 4320}
  - the vowels never come together? {Ans. 36000}
  - no two vowels are together? {Ans. 14400}
50. In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together? {Ans. 282240}
51. In how many ways can 5 children be arranged in a row such that
- two of them, Ram and Shyam, are always together? {Ans. 48}
  - two of them, Ram and Shyam, are never together? {Ans. 72}
52. When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the Principal, how many arrangements are possible? {Ans.  $1440 \times 18!$ }
53. A code word is to consist of two distinct English alphabets followed by two distinct numbers from 1 to 9. For example, CA23 is a code word. How many such code words are there? How many of them end with an even integer? {Ans. 46800, 20800}
54. The Principal wants to arrange 5 students on the platform such that the boy Salim occupies the second position and such that the girl Sita is always adjacent to the girl Rita. How many such arrangements are possible? {Ans. 8}
55. How many numbers between 400 and 1000 can be formed with the digits 0, 2, 3, 4, 5, 6 if no digit is repeated in the same number? {Ans. 60}
56. How many four digit numbers divisible by 4 can be made with the digits 1, 2, 3, 4, 5 if the repetition of

- digits is not allowed? {Ans. 24}
57. Find the number of ways in which 5 boys and 5 girls be seated in a row so that
- no two girls may sit together. {Ans.  $5! \times 6!$ }
  - all the girls sit together and all the boys sit together. {Ans.  $2 \times 5! \times 5!$ }
  - all the girls are never together. {Ans.  $10! - 5! \times 6!$ }
58. Five boys and five girls form a line with the boys and girls alternating. Find the number of ways of making the line. {Ans.  $2 \times 5! \times 5!$ }
59. In how many ways 5 boys and 3 girls can be seated in a row so that
- not all the girls sit side by side. {Ans. 36000}
  - no two girls are together? {Ans. 14400}
60. In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls sit together in a back row on adjacent seats? {Ans.  ${}^{14}P_{12}, {}^{11}P_9 \times 24$ }
61. A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular side and two on the other side. In how many ways can they be seated? {Ans.  ${}^8P_4 \times {}^8P_2 \times 10!$ }
62. The letters of the word *RANDOM* are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word *RANDOM*. {Ans. 614}
63. How many six digits numbers can be formed from the digits 1, 2, 3, 4, 5, 6 and 7 so that the digits should not repeat and the terminal digits should be even. {Ans. 720}
64. How many natural numbers smaller than  $10^4$  are there, in the decimal notation of which all the digits are different. {Ans. 5274}
65. How many five digit numbers, which do not contain identical digits, can be written by means of the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9. {Ans. 15120}
66. How many different four digit numbers can be written using each of the digits 1, 2, 3, 4, 5, 6, 7 and 8 only once so that each number should contain a unity. {Ans. 840}
67. Show that the number of permutations of  $n$  different things taken all at a time in which  $p$  particular things are never together is  $n! - (n - p + 1)!p!$ .
68. How many words can be formed out of the letters of the word *ARTICLE* so that vowels occupy the even places? {Ans. 144}
69. In how many other ways can the letters of the word *SIMPLETON* be arranged? {Ans.  $9! - 1$ }
70. How many natural numbers are there which are divisible by 4, whose decimal notation consists only of the digits 0, 1, 2, 3 and 5 which do not repeat in any of these numbers. {Ans. 49}

### CATEGORY-27.3. PERMUTATIONS OF OBJECTS NOT ALL DISTINCT

71. There are 3 copies each of 4 different books. In how many ways can they be arranged in a shelf? {Ans.  $\frac{12!}{(3!)^4}$ }
72. How many different words can be formed with the letters of the word *MISSISSIPPI*? {Ans. 34650}
73. How many permutations of the letters of the word *APPLE* are there? {Ans. 60}
74. How many words can be formed using the letter *A* thrice, the letter *B* twice and the letter *C* thrice? {Ans. 560}
75. Find the number of different permutations of the letters of the word *BANANA*. {Ans. 60}
76. How many different words can be formed with the letters of the word *HARYANA*. How many of these

- begin with  $H$  and end with  $N$ ? In how many of these  $H$  and  $N$  are together? {Ans. 840, 20, 240}
77. How many different words can be formed by using all the letters of the word *ALLAHABAD*? In how many of them, vowels occupy the even positions? In how many of them, both  $L$  do not come together? {Ans. 7560, 60, 5880}
78. How many arrangements can be made with the letters of the word *MATHEMATICS*? In how many of them vowels are together? How many of them begin with  $C$ ? How many of them begin with  $T$ ? {Ans. 4989600, 120960, 453600, 907200}
79. If all the letters of the word *AGAIN* be arranged as in a dictionary, what is the fiftieth word? {Ans. *NAAIG*}
80. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3? {Ans. 360}
81. If the different permutations of the word, *EXAMINATION* are listed as in a dictionary, how many items are there in the list before the first word starting with  $E$ ? {Ans. 907200}
82. Find the number of ways of arranging  $2m$  white and  $2n$  red counters in a straight line so that each arrangement is symmetrical to a central mark. {Ans.  $\frac{(m+n)!}{m!n!}$ }

#### CATEGORY-27.4. PERMUTATIONS WHEN OBJECTS CAN REPEAT

83. How many numbers of 3 digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated? {Ans. 125}
84. In how many ways 5 rings of different types can be worn in 4 fingers? {Ans.  $4^5$ }
85. Find the number of numbers of 5 digits that can be formed with the digits 0, 1, 2, 3, 4 if the digits can be repeated in the same number? {Ans. 2500}
86. In how many ways can 3 prizes be distributed among 4 boys, when
- no boy gets more than one prize? {Ans. 24}
  - a boy may get any number of prizes? {Ans. 64}
  - no boy gets all the prizes? {Ans. 60}
87. How many 4-digit numbers are there, when a digit may be repeated any number of times? {Ans. 9000}
88. In how many ways can 7 letters be posted in 5 letter boxes? {Ans.  $5^7$ }
89. Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. Determine the number of words which have at least one letter repeated? {Ans. 69760}
90. How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit? {Ans. 365}
91. How many four digit numbers, which are divisible by 4, can be formed from the digits 1, 2, 3, 4 and 5. {Ans. 125}
92. How many different four digit numbers can be written using the digit 1, 2, 3, 4, 5, 6, 7 and 8 so that each of them contains only one unity, if any other digit can occur several times in the notation of these numbers. {Ans. 1372}
93. How many different numbers, which are smaller than  $2 \times 10^8$ , can be written by means of the digits 1 and 2. {Ans. 766}

#### CATEGORY-27.5. CIRCULAR PERMUTATIONS

94. In how many ways can 8 students be seated in a (i) line (ii) circle? {Ans. 40320, 5040}
95. In how many ways can 5 persons be seated around a circular table? In how many of these arrangements will two particular persons be next to each other? {Ans. 24, 12}
96. If 20 persons were invited for a party, in how many ways can they and the host be seated at a circular

- table? In how many of these ways will two particular persons be seated on either side of the host? {Ans.  $20!, 2 \times 18!$ }
97. There are 20 persons among whom are two brothers. Find the number of ways in which we can arrange them around a circle so that there is exactly one person between the two brothers? {Ans.  $2 \times 18!$ }
98. In how many ways can a party of 4 men and 4 women be seated at a circular table so that no two women are adjacent? {Ans. 144}
99. In how many ways can 7 boys be seated at a round table so that two particular boys are  
i. next to each other {Ans. 240}  
ii. separated? {Ans. 480}
100. A round table conference is to be held between 20 delegates of 2 countries. In how many ways can they be seated if two particular delegates are (i) always together? (ii) never together? {Ans.  $2 \times 18!, 17 \times 18!$ }
101. There are 5 gentlemen and 4 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together? {Ans. 2880}
102. In how many ways can seven persons sit around a table so that all shall not have the same neighbours in any two arrangements? {Ans. 360}
103. Three boys and three girls are to be seated around a table in a circle. Among them, the boy  $X$  does not want any girl neighbour and the girl  $Y$  does not want any boy neighbour. How many such arrangements are possible? {Ans. 4}
104. Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated? {Ans. 288}
105. Find the number of ways in which 10 different beads can be arranged to form a necklace. {Ans.  $\frac{9!}{2}$ }
106. What is the number of ways of painting the faces of a cube with six different colours. {Ans. 30}

#### **CATEGORY-27.6. COMBINATIONS FROM DISTINCT OBJECTS**

107. List the different combinations formed of three letters  $A, B, C$  taken two at a time. {Ans. 3 combinations}
108. Write all combinations of four letters  $A, B, C, D$  taken two at a time. {Ans. 6 combinations}
109. From a class of 32 students, 4 are to be chosen for a competition. In how many ways can this be done? {Ans.  ${}^{32}C_4$ }
110. Three gentlemen and three ladies are candidates for two vacancies. A voter has to vote for two candidates. In how many ways can one cast his vote? {Ans. 15}
111. Find the number of ways in which 5 identical balls can be distributed among 10 boxes, if not more than one ball can go into a box. {Ans. 252}
112. If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party? {Ans. 66}
113. A question paper has two parts, Part  $A$  and Part  $B$ , each containing 10 questions. If the student has to choose 8 from Part  $A$  and 5 from Part  $B$ , in how many ways can he choose the questions? {Ans. 11340}
114. In how many ways a committee of 5 members can be selected from 6 men and 5 women, consisting of 3 men and 2 women? {Ans. 200}
115. In how many ways can a cricket eleven be chosen out of a batch of 15 players if  
i. there is no restriction on the selection? {Ans. 1365}  
ii. a particular player is always chosen? {Ans. 1001}  
iii. a particular player is never chosen? {Ans. 364}
116. A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if

- i. at least five women have to be included in a committee?
  - ii. the women are in a majority?
  - iii. the men are in majority? {Ans. 6062, 2702, 1134}
117. A person wishes to make up as many different parties as he can out of 20 friends, each party consisting of the same number. How many should he invite at a time? In how many of these would the same man be found? {Ans. 10,  ${}^{19}C_9$ }
118. Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included? {Ans. 25}
119. In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers? Assume that the team of 11 players requires 5 batsmen, 3 all-rounders, 2 bowlers and 1 wicket keeper. {Ans. 141120}
120. A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done, when
- i. at least two ladies are included? {Ans. 186}
  - ii. at most two ladies are included? {Ans. 186}
121. A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour? {Ans. 425}
122. For the post of 5 teachers, there are 23 applicants, 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made? {Ans. 11760}
123. How many triangles can be formed by joining the vertices of a hexagon? {Ans. 20}
124. How many diagonals are there in a polygon with  $n$  sides? {Ans.  $\frac{n(n-3)}{2}$ }
125. A polygon has 44 diagonals. Find the number of its sides. {Ans. 11}
126. If  $m$  parallel lines in plane are intersected by a family of  $n$  parallel lines. Find the number of parallelograms formed. {Ans.  $\frac{mn(m-1)(n-1)}{4}$ }
127. There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the
- i. number of straight lines obtained from the pairs of these points. {Ans. 40}
  - ii. number of triangles that can be formed with the vertices as these points. {Ans. 116}
128. The sides  $AB$ ,  $BC$ ,  $CA$  of a triangle  $ABC$  have 3, 4, and 5 interior points respectively on them. Find the number of triangles that can be constructed using these points as vertices. {Ans. 205}
129. In a decagon, find the number of
- i. diagonals. {Ans. 35}
  - ii. triangles formed. {Ans. 120}
130. Out of 18 points in a plane, no three are in the same straight line except five points which are collinear. How many straight lines and triangles can be formed by joining them? {Ans. 144, 806}
131. There are  $n$  points in a plane, no three of which are in the same straight line with the exception of  $p$ , which are all in the same straight line; find the number (i) of straight lines, (ii) of triangles which result from joining them. {Ans.  ${}^nC_2 - {}^pC_2 + 1$ ,  ${}^nC_3 - {}^pC_3$ }
132. There are  $n$  points in space, no four of which are in the same plane with the exception of  $p$  which are all in the same plane. Find how many planes there are each containing three of the points. {Ans.  ${}^nC_3 - {}^pC_3 + 1$ }
133. There are 12 points in a plane of which 5 are collinear. Find (i) the number of straight lines obtained by joining these points in pairs (ii) the number of triangles that can be formed with vertices at these points. {Ans. 57, 210}

134. In a plane there are 37 straight lines, of which 13 pass through the point  $A$  and 11 pass through the point  $B$ . Besides, no three lines pass through one point, no line passes through both points  $A$  and  $B$ , and no two are parallel. Find the number of points of intersection of the straight lines. {Ans. 535}
135. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen? {Ans. 817190}
136. A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Chemistry Part II, unless Chemistry Part I is also borrowed. In how many ways can he choose the three books to be borrowed? {Ans. 41}
137. A student is allowed to select at most  $n$  books from a collection of  $(2n + 1)$  books. If total number of ways in which he can select a book is 63, find the value of  $n$ . {Ans. 3}
138. In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus signs are together? {Ans. 56}
139. In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together? {Ans. 1540}
140. If in a chess tournament each contestant plays twice against each of the others and in all 90 games are played, then find the number of participants. {Ans. 10}
141. There were two women participating in a chess tournament. Every participant played two games with the other participants. The number of games that the men played between themselves proved to exceed by 66 the number of games that the men played with the women. How many participants were there in the tournament and how many games were played. {Ans. 13, 156}

#### **CATEGORY-27.7. COMBINATIONS FROM OBJECTS NOT ALL DISTINCT**

142. In how many ways 5 balls can be chosen from 5 red balls, 7 black balls and 3 white balls. {Ans. 18}
143. In how many ways 10 balls can be chosen from 10 red balls, 10 black balls, 10 white balls and 10 green balls. {Ans. 286}
144. In an examination, the maximum marks for each of the three papers are 50 each. Maximum marks for the fourth paper are 100. Find the number of ways in which the candidate can score 60% marks in the aggregate. {Ans. 110551}
145. A person goes in for an examination in which there are four papers with a maximum of  $m$  marks for each paper; find the number of ways of getting  $2m$  marks on the whole. {Ans.  $\frac{1}{3}(m+1)(2m^2 + 4m + 3)$ }

#### **CATEGORY-27.8. PERMUTATIONS AND COMBINATIONS**

146. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed? {Ans. 25200}
147. How many four-letter words can be formed using the letters of the word *FAILURE*, so that  
i.  $F$  is included in each word? {Ans. 480}  
ii.  $F$  is not included in any word? {Ans. 360}
148. How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word *EQUATION* so that the two consonants occur together? {Ans. 1440}
149. How many words can be formed by taking 4 letters at a time out of the letters of the word *MATHEMATICS*. {Ans. 2454}
150. There are six periods in each working day of a school. In how many ways can one arrange 5 subjects such that each subject is allowed at least one period? {Ans. 1800}

151. Eighteen guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three other on the other side. Determine the number of ways in which the seating arrangement can be made? {Ans.  $\frac{11!(9!)^2}{65!}$ }
152. How many four-letter words can be formed using the letter of the word *INEFFECTIVE*? {Ans. 1422}
153. How many different seven digit numbers can be written using only three digits, 1, 2 and 3 under the condition that the digit 2 occurs twice in each number. {Ans. 672}
154. How many four digit numbers are there whose decimal notation contains not more than two distinct digits. {Ans. 576}
155. How many different seven digit numbers are there the sum of whose digits is even. {Ans.  $45 \times 10^5$ }
156. How many six digit numbers contain exactly four different digits. {Ans. 294840}
157. How many different numbers, which are smaller than  $2 \times 10^8$  and are divisible by 3, can be written by means of the digits 0, 1 and 2. {Ans. 4373}
158. How many different six digit numbers are there whose three digits are even and three digits are odd. {Ans. 160000}
159. How many different six digit numbers are there whose exactly three digits are odd. {Ans. 281250}

**CATEGORY-27.9. TOTAL SELECTIONS FROM DISTINCT OBJECTS**

160. A man has 6 friends, in how many ways may he invite one or more of them to dinner? {Ans. 63}
161. We must form a bouquet from 18 different flowers so that it should contain not less than three flowers. How many different ways are there of forming such bouquet. {Ans. 261972}
162. There are 3 books on Maths, 4 on Physics and 5 on English. How many different collections can be made such that each collection consists of
- one book of each subject {Ans. 60}
  - at least one book of each subject {Ans. 3255}
  - at least one book of English. {Ans. 3968}
163. Given 5 different green dyes, four different blue dyes and three different red dyes, how many combinations of dyes can be chosen taking at least one green and one blue dye? {Ans. 3720}

**CATEGORY-27.10. TOTAL SELECTIONS FROM OBJECTS NOT ALL DISTINCT**

164. Find the number of ways of selecting one or more letters from the letters *AAAA BBB CDE*. {Ans. 159}
165. A fruit basket contains 4 oranges, 5 apples and 6 mangoes. In how many ways can a person make a selection of fruits from among the fruits in the basket? {Ans. 209}
166. Prove that there will be only 24 selections containing at least one red ball out of a bag containing 4 red and 5 black balls. Balls of the same colour are identical. {Ans. 24}
167. There are  $p$  copies each of  $n$  different books, find the number of different ways in which a non-empty selection can be made from them. {Ans.  $(p+1)^n - 1$ }
168. Find the total number of factors (excluding 1) of 2160. {Ans. 39}
169. Find the total number of proper factors of 7875. {Ans. 23}
170. How many divisors are there of the number 38808 exclusive of the divisor 1 and the number itself? (Ans. 70)
171. Find the number of divisors of 9600 including 1 and 9600. (Ans. 48)
172. Find the number of factors (excluding 1 and the expression itself) of the product of  $a^7 b^4 c^3 def$ , where  $a, b, c, d, e, f$  are all prime numbers. {Ans. 1278}



**CATEGORY-27.11. GROUPING OF DISTINCT OBJECTS**

173. Find the number of ways of dividing 15 things into groups of 8, 4 and 3 respectively. {Ans.  $\frac{15!}{8!4!3!}$ }
174. In how many ways can 10 balls be divided between three boys, one receiving 2, one receiving 3 and one receiving 5 balls? {Ans. 15120}
175. In how many ways can a pack of 52 cards be divided
- equally into four groups? {Ans.  $\frac{52!}{(13!)^4 4!}$ }
  - equally among four players in order? {Ans.  $\frac{52!}{(13!)^4}$ }
  - into 4 sets, three of them having 17 cards each and the fourth just one card? {Ans.  $\frac{52!}{(17!)^3 3!}$ }
  - among four players, three of them having 17 cards each and the fourth just one card? {Ans.  $\frac{52!}{(17!)^3 3!} \times 4!$ }
176. In how many ways can  $2n$  people be divided into  $n$  couples? {Ans.  $\frac{(2n)!}{2^n \times n!}$ }
177. At an election, three wards of a town are canvassed by 4, 5 and 8 men respectively. If 20 men volunteer in, how many ways can they be allotted to the different wards? {Ans. 349188400}

**CATEGORY-27.12. GROUPING OF IDENTICAL OBJECTS**

178. Find the total number of ways in which 30 mangoes can be distributed among 5 persons. {Ans.  ${}^{34}C_4$ }
179. In how many ways can 10 identical presents be distributed among 6 children so that each child gets at least one present. {Ans. 126}
180. Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all balls. {Ans. 6}
181. Find the number of ways in which 16 sovereigns can be distributed among three applicants such that each applicant does not receive less than 3 sovereigns. {Ans. 36}
182. Determine the total number of non-negative integral solutions of  $x_1 + x_2 + x_3 + x_4 = 100$ . {Ans.  ${}^{103}C_3$ }
183. How many integral solutions are there to  $x + y + z + t = 29$ , when  $x \geq 1$ ,  $y \geq 2$ ,  $z \geq 3$  and  $t \geq 0$ ? {Ans. 2600}
184. Find the number of non-negative integral solutions of  $x_1 + x_2 + x_3 + 4x_4 = 20$ . {Ans. 536}
185. How many integral solutions are there to the system of equations  $x_1 + x_2 + x_3 + x_4 + x_5 = 20$  and  $x_1 + x_2 + x_3 = 5$  when  $x_k \geq 0$ ? {Ans. 336}
186. Six white and six black balls of the same size are distributed among ten boxes so that there is at least one ball in each box. What is the number of different distributions of the balls. {Ans. 26250}

**CATEGORY-27.13. DERANGEMENTS**

187. There are 6 letters in 6 addressed envelopes. Find the number of ways in which all letters are put in the wrong envelopes. {Ans. 265}
188. There are four balls of different colours and four boxes of colours, same as those of the balls. Find the

number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour. {Ans. 9}

189. There are seven marked cages for seven animals. In how many ways can two animals enter their own cages but five animals enter wrong cages? {Ans. 924}

**CATEGORY-27.14. APPLICATION OF METHODS**

190. What is the number of arrangements which can be made using all the letters of the word *LAUGH*, if the vowels are adjacent? {Ans. 48}
191. How many different words can be formed with the letters of the word *ORDINATE* so that
- the vowels occupy odd places {Ans. 576}
  - beginning with *O* {Ans. 7!}
  - beginning with *O* and ending with *E*. {Ans. 6!}
192. How many different words can be formed from the letters of the word *INTERMEDIATE*? In how many of them, two vowels never come together? {Ans. 19958400, 151200}
193. Find the number of ways of arranging the letters *AAAAA, BBB, CCC, D, EE, F* in a row if the letters *C* are separated from one another. {Ans. 95135040}
194. In how many ways can the letters of the word *ARRANGE* be arranged so that
- the two *R*'s are never together {Ans. 900}
  - the two *A*'s are together but not two *R*'s {Ans. 240}
  - neither two *A*'s nor the two *R*'s are together. {Ans. 660}
195. A five letter word is to be formed such that the letters appearing in the odd positions are taken from the unrepeated letters of the word *MATHEMATICS* whereas the letters which occupy even places are taken from amongst the repeated letters. {Ans. 540}
196. Find the number of ways in which (a) selection (b) an arrangement of 4 letters can be made from the letters of the word *PROPORTION*. {Ans. 53, 758}
197. How many different permutations can be formed from the letters of the word *EXAMINATION* taken four at a time? {Ans. 2454}
198. How many different words can be formed out of the letters of the word *MORADABAD* taken four at a time? {Ans. 626}
199. Prove that the number of words which can be formed out of the letters *a, b, c, d, e, f* taken 3 together, each word containing one vowel at least is 96.
200. Eleven animals of a circus have to be placed in eleven cages one in each cage. If 4 of the cages are too small for 6 of the animals, find the number of ways of caging the animals. {Ans. 604800}
201. How many committees of 5 members each can be formed from 8 official and 4 non-official members in the following cases
- each consisting of 3 official and 2 non-official members. {Ans. 336}
  - each contains at least two non-official members. {Ans. 456}
  - a particular official member is never included. {Ans. 462}
  - a particular non-official member is always included. {Ans. 330}
202. There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. {Ans. 51300}
- Further find in how many of these committees
- a particular professor is included {Ans. 10260}
  - a particular student is included {Ans. 7695}
  - a particular students is excluded. {Ans. 43605}

203. From 6 gentleman and 4 ladies a committee of 5 is to be formed. In how many ways can this be done if
- the committee is to include at least one lady {Ans. 246}
  - There is no restriction about its formation {Ans. 252}
204. From 4 officers and 8 jawans in how many ways can 6 be chosen (i) to include exactly one officer (ii) to include at least one officer? {Ans. 224, 896}
205. A candidate is required to answer 6 out of 10 questions, which are divided into two groups each containing 5 questions and he is not permitted to attempt more than 4 from each group. In how many ways can he make up his choice? {Ans. 200}
206. A candidate is required to answer 7 out of 15 questions which are divided into three groups  $A, B, C$  each containing 4, 5, 6 questions respectively. He is required to select at least 2 questions from each group. In how many ways can he make up his choice? {Ans. 2700}
207. A candidate is required to answer 7 questions out of 12 questions, which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from the either group. In how many different ways can he chose the 7 questions? {Ans. 780}
208. A question paper consists of two sections having respectively 3 and 4 questions. The following note is given on the paper "It is not necessary to attempt all the questions. One question from each section is compulsory" In how many ways can a candidate select the questions? {Ans. 105}
209. In how many ways can clear and overcast days occur in one weak? {Ans. 128}
210. A committee of 6 is chosen from 10 men and 7 women so as to contain at least 3 men and 2 women. In how many different ways can this be done if two particular women refuse to serve on the same committee? {Ans. 8610, 7800}
211. A guard of 12 men is formed from a group of  $n$  soldiers in all possible ways. Find
- the number of times two particular soldiers  $A$  and  $B$  are together on guard. {Ans.  ${}^{n-2}C_{10}$ }
  - the number of times three particular soldiers  $C, D, E$  are together on guard. {Ans.  ${}^{n-3}C_9$ }
  - $n$  if it is found that  $A$  and  $B$  are three times as often together on guard as  $C, D, E$  are. {Ans. 32}
212. There are 16 vacancies for clerks in a certain office. 20 applications are received. In how many ways can the clerks be appointed? How many times may a particular candidate be selected? {Ans. 4845, 3876}
213. To fill 12 vacancies there are 25 candidates of which 5 are from scheduled castes. If 3 of the vacancies are reserved for scheduled caste candidates while the rest are open to all, find the number of ways in which the selection can be made. {Ans. 4974200}
214. Find the number of ways of selecting 10 clerks from 27 male and 17 female applicants if the selection is to consist of either all males or all females. {Ans.  ${}^{27}C_{10} + {}^{17}C_{10}$ }
215. All possible two-factor products are formed from the numbers 1, 2, ..., 100. What is the number of factors out of the total obtained which are multiple of 3? {Ans. 2739}
216. A box contains two white balls, three black balls and 4 red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw? {Ans. 64}
217. Out of 8 sailors on a boat, 3 can work only on one side and 2 only on the other side. In how many ways can the sailors be arranged on the boat? {Ans. 1728}
218. In how many ways can a lawn tennis mixed double be made up from seven married couples if no husband and wife play in the same set? {Ans. 420}
219. In how many ways can  $n$  men be seated around a round table when in no two ways can a man has the same neighbours? {Ans.  $\frac{(n-1)!}{2}$ }
220.  $m$  men and  $n$  women are to be seated in row so that no two women sit together. If  $m > n$ , then show that

the number of ways in which they can be seated is  $\frac{m!(m+1)!}{(m-n+1)!}$ .

221. Five boys and five girls form a line with the boys and girls alternating. Find the number of ways of making the line. In how many different ways could they form a circle such that the boys and girls alternate? {Ans.  $2(5!)^2$ , 2880 }
222. A businessman hosts a dinner to 21 guests. He is having 2 round tables, which can accommodate 15 and 6 persons each. In how many ways can he arrange the guests? {Ans.  ${}^{21}C_{15} \times 14! \times 5!$ }
223. Find the number of ways in which ten candidates  $A_1, A_2, A_3 \dots A_{10}$  can be ranked (i) if  $A_1$  and  $A_2$  are next to each other, and (ii)  $A_1$  is just above  $A_2$  (iii) if  $A_1$  is always above  $A_2$ . {Ans. 725760, 9!, 1814400 }
224. In a class of 10 students there are 3 girls,  $A, B, C$ . In how many different ways can they be arranged in a row such that no two of the three girls are consecutive? {Ans.  $336 \times 7!$ }
225. In how many ways can 21 identical white balls and 19 identical black balls be arranged in a row so that no two black balls be together? What will be the result if all the balls are considered to be different? {Ans.  $1540, \frac{1}{6} \cdot 21! 22!$ }
226. There are six students  $A, B, C, D, E, F$ .
- In how many ways can they be seated in a line so that  $C$  and  $D$  do not sit together? {Ans. 480}
  - In how many ways can a committee of four be formed so as always to include  $C$ ? {Ans. 10}
  - In how many ways can a committee of four be formed so as to always include  $C$  but exclude  $D$ ? {Ans. 4}
227. A family consists of a grand father, 6 sons and daughters and 4 grand children. They are to be seated in a row for dinner. The grand children wish to occupy the two seats at each end and the grand father refuses to have a grand child on either side of him. In how many ways can the seating arrangements be made for the dinner? {Ans. 86400}
228. There are 6 professors of whom two are from Science, 2 from Arts and the remaining two from Commerce. They have to stand in a line so that the two Science teachers, two arts teachers and also the two Commerce teachers are together. Find the number of ways in which they can do so. {Ans. 48}
229. Six X's have to be placed in the squares of the Fig. 1 such that each row contains at least one X. In how many different ways can this be done? {Ans. 26}
230. Eleven books consisting of 5 mathematics, 4 physics and 2 on Chemistry are placed on a shelf at random. Find the number of possible ways of arranging them on the assumption that the books on the same subject are all together. {Ans. 34560}
231. A man has 7 relatives, 4 of them are ladies and 3 gentlemen; his wife has 7 relatives, and 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives? {Ans. 485}
232. In how many ways the letters of the word 'PERSON' can be placed in the squares of the adjoining Fig. 2 so that no row remains empty? {Ans. 18720}
233. Out of 16 players of a cricket team. 4 are bowlers and 2 are wicket keepers. A team of 11 players is to be chosen so as to contain at least 3 bowlers and at least 1 wicket-keeper. In how ways can the team be selected? {Ans. 2472}
234. In how many ways can three letters be posted in four letters boxes in a village? If all the three letters are not posted in the same letter box, find the corresponding number of ways of posting. {Ans. 64, 60}
235. If there are  $n$  students and  $r$  prizes ( $r < n$ ), prove that they can be given away (i) in  $n^r$  ways when a student can receive any number of prizes (ii) in  $n^r - n$  ways when a student cannot receive all the prizes.

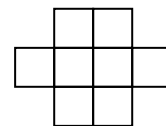


Fig. 1

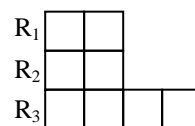


Fig. 2

236. There are 12 balls of which 4 are red, 3 black and 5 white. In how many ways can you arrange the balls so that no two white balls may occupy consecutive positions if
- balls of the same colour are identical, {Ans. 1960}
  - all balls are considered to be different. {Ans.  $\frac{7!8!}{3!}$ }
237. Show that the total number of permutations of  $n$  different things taken not more than  $r$  at a time, when each thing may be repeated any number of times is  $\frac{n(n^r - 1)}{n - 1}$ .
238. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five balls. In how many different ways can we place the balls so that no box remains empty? {Ans. 150}
239. Sixteen men compete with one another in a running, swimming and riding. How many prize lists could be made if there were altogether 6 prizes of different values, one for running, 2 for swimming and 3 for riding? {Ans. 12902400}
240. In how many ways can the letters of the word MULTIPLE be arranged in the following cases:
- without changing the order of the vowels {Ans. 3360}
  - keeping the position of the vowels fixed {Ans. 60}
  - without changing the relative order of vowels and consonants? {Ans. 360}
241. In how many ways can you select six coins out of 20 rupee pieces, 10 fifty paise pieces and 7 twenty five paise pieces? {Ans. 28}
242. There are  $n$  straight lines in a plane, no two of which are parallel, and no three pass through the same point. Their points of intersection are joined. Show that the number of fresh lines thus introduced is  $\frac{n(n-1)(n-2)(n-3)}{8}$ .
243. What is the rank of the word MOTHER when its letters are arranged as in a dictionary? {Ans. 309}
244. What is the total number of squares in a chessboard? {Ans. 204}
245. There are 5 eligible Punjabi grooms of which 3 know Bengali and 5 eligible Bengali grooms of which 2 know Punjabi. There are 5 eligible Punjabi brides and 5 eligible Bengali brides. If an eligible groom is agreeable to marry a girl of his community or knowing her language and brides have no choice, in how many ways 10 couples can be formed? {Ans. 144000}
246. Find the total number of functions from set  $A = \{a, b, c, d\}$  to set  $B = \{1, 2, 3\}$ . {Ans. 81}
247. Find the number of surjections from  $A = \{1, 2, \dots, n\}$ ,  $n \geq 2$  to  $B = \{a, b\}$ . {Ans.  $2^n - 2$ }

#### CATEGORY-27.15. PROPERTIES OF ${}^nP_r$ AND ${}^nC_r$

248. Prove with or without the use of the formula  ${}^nP_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$ .
249. Prove with or without the use of the formula  ${}^nP_r = n \cdot {}^{n-1}P_{r-1}$ .
250. If  ${}^{20}C_r = {}^{20}C_{r-10}$ , then find the value of  ${}^{18}C_r$ . {Ans. 816}
251. If  ${}^{20}C_r = {}^{20}C_{r+4}$ , then find the value of  ${}^rC_3$ . {Ans. 56}
252. If  ${}^{20}C_{r+1} = {}^{20}C_{r-1}$ , then find the value of  $r$ . {Ans. 10}
253. If  ${}^mC_1 = {}^nC_2$ , then show that  $2m = n(n-1)$ .
254. If  ${}^nC_{10} = {}^nC_{15}$ , find  ${}^{27}C_n$ . {Ans. 351}
255. If  ${}^nC_7 = {}^nC_4$ , find  $n$ . {Ans. 11}

256. If  ${}^{15}C_{3r} = {}^{15}C_{r+3}$ , find  $r$ . {Ans. 3}
257. If  ${}^nC_{12} = {}^nC_8$ , find  ${}^nC_{17}$  and  ${}^{22}C_n$ . {Ans. 1140, 231}
258. If  ${}^8C_r - {}^7C_3 = {}^7C_2$ , find  $r$ . {Ans. 3 or 5}
259. If  ${}^{15}C_r : {}^{15}C_{r-1} = 11 : 5$ , find  $r$ . {Ans. 5}
260. If  ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$ , find  $r$ . {Ans. 5}
261. If  ${}^{12}P_r = {}^{11}P_6 + 6 \cdot {}^{11}P_5$  then find the value of  $r$ . {Ans. 6}
262. If  ${}^nP_r = 720 {}^nC_r$ , then find the value of  $r$ . {Ans. 6}
263. If  ${}^nC_6 : {}^{n-3}C_3 = 33 : 4$ , find  $n$ . {Ans. 11}
264. If  ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$ , find  $r$ . {Ans. 41}
265. If  ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$ , find  $n$ . {Ans. 19}
266. If  ${}^{10}P_r = 604800$  and  ${}^{10}C_r = 120$ , find  $r$ . {Ans. 7}
267. If  ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$ , find  $n$ . {Ans. 4}
268. If  ${}^nP_r = {}^nP_{r+1}$  and  ${}^nC_r = {}^nC_{r-1}$ , find  $n$  and  $r$ . {Ans. 3, 2}
269. Prove  ${}^{4n}C_{2n} : {}^{2n}C_n = [1 \cdot 3 \cdot 5 \cdots (4n-1)] : [1 \cdot 3 \cdot 5 \cdots (2n-1)]^2$ .
270. Evaluate  ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$ . {Ans. 0}
271. Prove that  ${}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$ .
272. Prove that  ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$  if  $n > 7$ .
273. If  ${}^{n-1}C_6 + {}^{n-1}C_7 > {}^nC_6$ , then show that  $n > 13$ .
274. If  ${}^nC_{r-1} = 36$ ,  ${}^nC_r = 84$  and  ${}^nC_{r+1} = 126$ , find the values of  $n$  and  $r$ . {Ans. 9, 3}
275. If  ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$ , then find  $r$ . {Ans. 7}
276. Find the value of  ${}^{47}C_4 + \sum_{r=1}^5 ({}^{52-r}C_3)$ . {Ans.  ${}^{52}C_4$ }

### CATEGORY-27.16. ADDITIONAL QUESTIONS

# Mathematics for IIT-JEE

By **Er. Sanjiva Dayal**, *B.Tech. (I.I.T. Kanpur)*

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*PART-VII*  
*ALGEBRA*

*CHAPTER-28*  
*PROBABILITY*

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## ***CHAPTER-28***

## ***PROBABILITY***

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### **LIST OF THEORY SECTIONS**

- 28.1. Classical Definition Of Probability
- 28.2. Addition Theorems
- 28.3. Conditional Probability
- 28.4. Total Probability
- 28.5. Binomial Distribution
- 28.6. Expectation
- 28.7. Geometrical Definition Of Probability

### **LIST OF QUESTION CATEGORIES**

- 28.1. Experiment, Sample Space
- 28.2. Events
- 28.3. Classical Definition Of Probability, Complimentary Events
- 28.4. Addition Theorems, Mutually Exclusive Events, Exhaustive System Of Events
- 28.5. Conditional Probability
- 28.6. Multiplication Theorems
- 28.7. Independent Events
- 28.8. Applications Of Addition And Multiplication Theorems
- 28.9. Theorem Of Total Probability
- 28.10. Baye's Theorem
- 28.11. Binomial Distribution
- 28.12. Expectation
- 28.13. Geometrical Definition Of Probability
- 28.14. Application Of Methods
- 28.15. Additional Questions



## CHAPTER-28

### PROBABILITY

#### SECTION-28.1. CLASSICAL DEFINITION OF PROBABILITY

##### 1. Experiment

An operation which can produce some definite outcomes, is called an Experiment.

##### 2. Sample space

The set of all possible outcomes of an experiment is called the Sample space associated with it and is generally denoted by  $S$ .

##### 3. Trial

When an experiment is conducted, it is called a Trial. Each trial produces one definite outcome.

##### 4. Events

- i. A subset of the sample space is called an Event, generally denoted by  $A, B, C, \dots$
- ii. An event can be defined either by stating a subset of  $S$  or by stating a condition on the outcomes.
- iii. Each outcome of an experiment is called an Elementary event.
- iv. The union of two or more elementary events are called Compound events.
- v. If  $A$  is an event, then  $A \subseteq S$ .

##### 5. Favourable outcomes of an event

If  $A$  is an event, then those outcomes which belong to  $A$  are known as favourable outcomes of event  $A$ ; and those outcomes which does not belong to  $A$  are not favourable to the event  $A$ .

##### 6. Occurrence of an event

If the outcome of a trial belongs to event  $A$ , then it is said that event  $A$  has occurred in that trial otherwise it is said that event  $A$  has not occurred in that trial.

##### 7. Equally likely outcomes

Outcomes of an experiment are equally likely if there is no reason for an outcome to occur in preference to any other outcome.

##### 8. Classical definition of Probability

- i. If the outcomes of an experiment are equally likely then the probability of event  $A$  is defined as

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } S} = \frac{n(A)}{n(S)}$$

- ii. Because  $0 \leq n(A) \leq n(S)$ , therefore  $0 \leq P(A) \leq 1$ .
- iii. If  $P(A) > P(B)$ , then event  $A$  has more chance of occurring than event  $B$ .

##### 9. Empirical definition of probability

If there are  $N$  trials of an experiment and an event  $A$  occurs in  $n$  of these trials then  $P(A) = \lim_{N \rightarrow \infty} \frac{n}{N}$ .

##### 10. Certain event, impossible event, random event

- i. If  $A \equiv S$ , then event  $A$  is said to be a certain event and  $P(A) = 1$ .
- ii. If  $A \equiv \phi$ , then event  $A$  is said to be an impossible event and  $P(A) = 0$ .
- iii. If  $A \neq \phi$  &  $A \subset S$ , then event  $A$  is said to be a random event and  $0 < P(A) < 1$ .

##### 11. Complimentary events

- i. Complementary set of set  $A$ , i.e.  $\bar{A}$  is event  $\bar{A}$  and is said to be the complementary event of event  $A$ . Event  $\bar{A}$  denotes that event  $A$  does not occur.

- ii. Because  $n(\bar{A}) = n(S) - n(A)$ , therefore  
 $P(\bar{A}) = 1 - P(A)$  or  $P(A) + P(\bar{A}) = 1$  (Theorem of complimentary events).

### 12. Odds in favour of an event and odds against an event

If  $a$  is the number of outcomes favourable to the event  $A$  and  $b$  is the number of outcomes not favourable to the event  $A$ , then it is said that

- i. Odds in favour of event  $A$  are  $a : b$ , i.e.  $P(A) : P(\bar{A})$  or  $P(A) : 1 - P(A)$ .  
 ii. Odds against event  $A$  are  $b : a$ , i.e.  $P(\bar{A}) : P(A)$  or  $1 - P(A) : P(A)$ .

### 13. Properties of Sets

- i.  $(A \cup B) \cup C = A \cup (B \cup C)$ . (Associative Property)  
 ii.  $(A \cap B) \cap C = A \cap (B \cap C)$ . (Associative Property)  
 iii.  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ . (Distributive Property)  
 iv.  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ . (Distributive Property)  
 v.  $\overline{A \cup B} = \bar{A} \cap \bar{B}$ . (De Morgan's law)  
 vi.  $\overline{A \cap B} = \bar{A} \cup \bar{B}$ . (De Morgan's law)  
 vii.  $A - B = A \cap \bar{B}$ .  
 viii.  $A - B = A \Leftrightarrow A \cap B = \phi$ .  
 ix.  $(A - B) \cup B = A \cup B$ .  
 x.  $(A - B) \cap B = \phi$ .  
 xi.  $A \subseteq B \Leftrightarrow \bar{B} \subseteq \bar{A}$ .  
 xii.  $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ .

## SECTION-28.2. ADDITION THEOREMS

### 1. Addition theorems

- i. If  $A$  and  $B$  are any two events, then  

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 1 - P(\bar{A} \cap \bar{B}). \quad (\text{De Morgan's law})$$
- ii. If  $A, B, C$  are any three events, then  

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C}). \quad (\text{De Morgan's law})$$
- iii. If  $A, B, C, D, \dots$  are any events, then  

$$P(A \cup B \cup C \cup D \cup \dots)$$

$$= \sum P(A) - \sum P(A \cap B) + \sum P(A \cap B \cap C) + \dots + (-1)^{n+1} P(A \cap B \cap C \cap D \cap \dots)$$

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D} \cap \dots). \quad (\text{De Morgan's law})$$

### 2. Mutually exclusive events

- i. Events  $A$  and  $B$  are said to be mutually exclusive events if events  $A$  and  $B$  cannot occur together i.e.  
 $A \cap B = \phi$ .
- ii. Therefore, if  $A$  and  $B$  are mutually exclusive events, then  $P(A \cap B) = 0$ .

### 3. Addition theorems for mutually exclusive events

- i. If  $A$  and  $B$  are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B).$$

- ii. If  $A, B, C, D, \dots$  are mutually exclusive events, then

$$P(A \cup B \cup C \cup D \cup \dots) = P(A) + P(B) + P(C) + P(D) + \dots$$

#### 4. Exhaustive system of events

- i. If  $A_1, A_2, \dots, A_n$  are events such that  $A_1 \cup A_2 \cup \dots \cup A_n = S$ , then events  $A_1, A_2, \dots, A_n$  is said to be exhaustive system of events. In a trial, at least one of these events must occur.

- ii. If  $A_1, A_2, \dots, A_n$  are exhaustive system of events, then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = 1.$$

#### 5. Mutually exclusive and exhaustive system of events

If  $A, B, C, D, \dots$  are mutually exclusive and exhaustive system of events, then

$$P(A \cup B \cup C \cup D \cup \dots) = 1$$

$$\Rightarrow P(A) + P(B) + P(C) + P(D) + \dots = 1.$$

### SECTION-28.3. CONDITIONAL PROBABILITY

#### 1. Conditional probability

- i. The probability of occurrence of event  $A$  under the assumption that event  $B$  has occurred, is called the conditional probability of event  $A$  when  $B$  occurs and is denoted by  $P(A/B)$ .

$$\text{ii. } P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{P(A \cap B)}{P(B)}.$$

$$\text{iii. } P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{P(A \cap B)}{P(A)}.$$

- iii. If  $A$  and  $B$  are mutually exclusive events, then

$$P(A/B) = 0 = P(B/A).$$

#### 2. Multiplication theorems

- i. If  $A$  and  $B$  are any two events, then

$$P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B).$$

- ii. If  $A_1, A_2, \dots, A_n$  are any  $n$  events, then

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2/A_1) \cdot P(A_3/A_1 \cap A_2) \cdot \dots \cdot P(A_n/A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$

#### 3. Independent events

- i. Events are said to be independent if the occurrence or non-occurrence of one does not affect the probability of occurrence or non-occurrence of the other. Therefore, if  $A$  and  $B$  are independent events, then

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B).$$

- ii. If  $A$  and  $B$  are independent events then  $\bar{A}$  &  $B, A$  &  $\bar{B}, \bar{A}$  &  $\bar{B}$  are also independent events.

#### 4. Multiplication theorems for independent events

- i. If  $A$  and  $B$  are independent events, then

$$P(A \cap B) = P(A) \cdot P(B).$$

- ii. If  $A, B, C, D, \dots$  are independent events, then

$$P(A \cap B \cap C \cap D \cap \dots) = P(A) \cdot P(B) \cdot P(C) \cdot P(D) \cdot \dots$$

### 5. Addition theorems for independent events

- i. If  $A$  and  $B$  are independent events, then

$$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B) \\ = 1 - P(\bar{A}) \cdot P(\bar{B}). \quad (\text{De Morgan's law})$$

- ii. If  $A, B, C$  are independent events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(A) \cdot P(C) - P(B) \cdot P(C) + P(A) \cdot P(B) \cdot P(C) \\ = 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C}). \quad (\text{De Morgan's law})$$

- iii. If  $A, B, C, D, \dots$  are independent events, then

$$P(A \cup B \cup C \cup D \cup \dots) \\ = \sum P(A) - \sum P(A) \cdot P(B) + \sum P(A) \cdot P(B) \cdot P(C) + \dots + (-1)^{n+1} P(A) \cdot P(B) \cdot P(C) \cdot P(D) \cdot \dots \\ = 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C}) \cdot P(\bar{D}) \cdot \dots \quad (\text{De Morgan's law})$$

## SECTION-28.4. TOTAL PROBABILITY

### 1. Theorem of total probability

Let  $H_1, H_2, \dots, H_n$  be  $n$  mutually exclusive and exhaustive events and if  $A$  is any event which occurs with either  $H_1$  or  $H_2$  or  $\dots$  or  $H_n$ , then

$$P(A) = P(H_1) \cdot P\left(\frac{A}{H_1}\right) + P(H_2) \cdot P\left(\frac{A}{H_2}\right) + \dots + P(H_n) \cdot P\left(\frac{A}{H_n}\right) \\ = \sum_{i=1}^n P(H_i) \cdot P\left(\frac{A}{H_i}\right).$$

### 2. Baye's theorem

Let  $H_1, H_2, \dots, H_n$  be  $n$  mutually exclusive and exhaustive events and if  $A$  is any event which occurs with either  $H_1$  or  $H_2$  or  $\dots$  or  $H_n$ , then

$$P\left(\frac{H_j}{A}\right) = \frac{P(H_j) \cdot P\left(\frac{A}{H_j}\right)}{P(A)} = \frac{P(H_j) \cdot P\left(\frac{A}{H_j}\right)}{\sum_{i=1}^n P(H_i) \cdot P\left(\frac{A}{H_i}\right)}, \quad j = 1, 2, \dots, n.$$

## SECTION-28.5. BINOMIAL DISTRIBUTION

### 1. Binomial distribution

- i. Let  $P(A) = p$ , then  $P(\bar{A}) = 1 - p$ . If there are  $n$  successive independent trials of an experiment, then the probability that event  $A$  occurs exactly  $r$  times is  ${}^nC_r p^r (1-p)^{n-r}$ ,  $r = 0, 1, \dots, n$ .
- ii.  ${}^nC_r p^r (1-p)^{n-r}$  is the  $(r+1)^{\text{th}}$  term in the Binomial expansion of  $[p + (1-p)]^n$ .

## SECTION-28.6. EXPECTATION

### 1. Expectation

Let  $X$  be discrete random real variable which assumes the real values  $x_1, x_2, \dots, x_n$  with corresponding probabilities  $p_1, p_2, \dots, p_n$  respectively, then the mean value of  $X$  or the expectation of  $X$ , denoted by

$E(X)$ , is defined as  $E(X) = \sum_{i=1}^n x_i p_i$ , where  $\sum_{i=1}^n p_i = 1$ .

### SECTION-28.7. GEOMETRICAL DEFINITION OF PROBABILITY

#### 1. Geometrical definition of probability

If  $n(S)$  and  $n(A)$  are both infinite, then  $S$  and  $A$  may be represented geometrically by points in one dimension or two dimension or three dimension and the probability of  $A$  is defined as

$$P(A) = \frac{\text{Favourable length}}{\text{Total length}} \text{ or } \frac{\text{Favourable area}}{\text{Total area}} \text{ or } \frac{\text{Favourable volume}}{\text{Total volume}}.$$

## EXERCISE-28

### CATEGORY-28.1. EXPERIMENT, SAMPLE SPACE

- What is the sample space associated with the random experiment of tossing a coin? {Ans.  $S = \{H, T\}$ }
- Two coins are tossed together. What is the sample space of this experiment? {Ans.  $S = \{HH, HT, TH, TT\}$ }
- What is the sample space associated with the random experiment of throwing a die? {Ans.  $S = \{1, 2, 3, 4, 5, 6\}$ }
- Consider an experiment in which two dice are tossed. What is the sample space for this experiment? {Ans.  $S = \{(1,1), (1,2), \dots, (1,6), (2,1), \dots, (2,6), \dots, (6,1), \dots, (6,6)\}$ }
- From a bag containing 3 black and 2 white balls we draw two balls. If we denote the black balls as  $B_1, B_2, B_3$  and white balls as  $W_1, W_2$ , then what is the sample space associated with this experiment? {Ans.  $S = \{B_1W_1, B_1W_2, B_2W_1, B_2W_2, B_3W_1, B_3W_2, B_1B_2, B_1B_3, B_2B_3, W_1W_2\}$ }
- Consider the experiment in which a coin is tossed twice and if the second draw results in a tail, then a die is thrown. What is the sample space associated with this experiment? {Ans.  $S = \{HH, TH, HT1, HT2, HT3, HT4, HT5, HT6, TT1, TT2, TT3, TT4, TT5, TT6\}$ }
- A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 red 4 black balls; if it shows a tail, we throw a die. What is the sample space of this experiment? {Ans.  $S = \{HR_1, HR_2, HR_3, HB_1, HB_2, HB_3, HB_4, T1, T2, T3, T4, T5, T6\}$ }

### CATEGORY-28.2. EVENTS

- Consider the experiment of throwing a die. What are the elementary events of this experiment? {Ans.  $E_1 = \{1\}, E_2 = \{2\}, E_3 = \{3\}, E_4 = \{4\}, E_5 = \{5\}, E_6 = \{6\}$ }
- Consider the experiment in which two dice are tossed. Let  $A_1, A_2, A_3$  are events described as  
 $A_1$  = getting a doublet;  
 $A_2$  = getting 8 as the sum;  
 $A_3$  = getting an even number on first dice.  
 Write the sets  $A_1, A_2$  and  $A_3$ . {Ans.  $A_1 = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$ ;  
 $A_2 = \{(2,6), (6,2), (3,5), (5,3), (4,4)\}$  and  $A_3 = \{(2,1), (2,2), \dots, (2,6), (4,1), (4,2), \dots, (4,6), (6,1), (6,2), \dots, (6,6)\}$ }
- From a group of 2 men and 3 women, two persons are selected. Describe the sample space of the experiment. If  $E$  is the event in which one man and woman are selected, then which outcomes are favourable to  $E$ ? {Ans.  $S = \{M_1M_2, M_1W_1, M_1W_2, M_1W_3, M_2W_1, M_2W_2, M_2W_3, W_1W_2, W_1W_3, W_2W_3\}$ ;  
 $E = \{M_1W_1, M_1W_2, M_1W_3, M_2W_1, M_2W_2, M_2W_3\}$ }
- Consider the experiment of tossing three coins at a time. Let  $A_1, A_2, A_3$  are events described as  
 $A_1$  = getting two heads;  
 $A_2$  = number of heads exceeds the number of tails;  
 $A_3$  = getting at least one head.  
 Write the sets  $S$ , elementary events,  $A_1, A_2$  and  $A_3$ . {Ans.  $S = \{HHH, HHT, HTH, THH, TTH, HTT, THT, TTT\}$ ;  $E_1 = \{HHH\}, E_2 = \{HHT\}$  etc. are elementary

events;  $A_1 = \{HHT, HTH, THH\}$ ;  $A_2 = \{HHH, HHT, HTH, THH\}$ ;  
 $A_3 = \{HHH, HHT, HTH, THH, TTH, HTT, THT\}$

12. In a single throw of a die, the event  $A$  = getting a multiple of 3. How many outcomes are favourable outcomes of event  $A$ ? {Ans. 2}
13. In a single throw of a pair of dice, the event  $A$  = getting the sum as 9. How many outcomes are favourable outcomes of event  $A$ ? {Ans. 4}
14. Consider the random experiment of throwing an unbiased die. Let  $A$  be an event of getting an even number.
  - i. Suppose in a trial the outcome is 4. Can we say that  $A$  has occurred in this trial? {Ans. Yes}
  - ii. In another trial, the outcome is 3. Can we say that  $A$  has occurred in this trial? {Ans. No}
15. Suppose a die is thrown and the outcome of the trial is 4. Which of the following events have occurred and which have not occurred:-
  - i. Getting a number greater than or equal to 2, represented by the set  $\{2, 3, 4, 5, 6\}$ . {Ans. Occurred}
  - ii. Getting a number less than or equal to 5, represented by the set  $\{1, 2, 3, 4, 5\}$ . {Ans. Occurred}
  - iii. Getting an odd number, represented by the set  $\{1, 3, 5\}$ . {Ans. Not occurred}
  - iv. Getting a multiple of 3, represented by the set  $\{3, 6\}$ . {Ans. Not occurred}

### CATEGORY-28.3. CLASSICAL DEFINITION OF PROBABILITY, COMPLIMENTARY EVENTS

16. A die is rolled. What is the probability that a number 1 or 6 may appear? {Ans.  $\frac{1}{3}$ }
17. Two dice are thrown simultaneously. What is the probability of obtaining total score of 5?. {Ans.  $\frac{1}{9}$ }
18. Two dice are thrown simultaneously. What is the probability of obtaining total score of seven? {Ans.  $\frac{1}{6}$ }
19. What is the probability of getting a total of 10 in a single throw of two dice? {Ans.  $\frac{1}{12}$ }
20. Three identical dice are rolled. What is the probability that the same number will appear on each of them?. {Ans.  $\frac{1}{36}$ }
21. One card is drawn from a pack of 52 cards. What is the probability that it is the card of a king or spade? {Ans.  $\frac{4}{13}$ }
22. Consider the following events related to the experiment of throwing a die.  
 $A_1$  = getting a number less than 1;  
 $A_2$  = getting a number less than 8.  
 Which of these events is impossible event and which is certain event? {Ans.  $A_1$  is an impossible event and  $A_2$  is a certain event}
23. Six dice are thrown simultaneously. Find the probability that
  - i. all of them show the same number. {Ans.  $\frac{1}{6^5}$ }
  - ii. all of them show different number. {Ans.  $\frac{5}{324}$ }

- iii. exactly three of them show the same number and remaining three show different number. {Ans.  $\frac{25}{162}$ }
- iv. at least four of them show the same number. {Ans.  $\frac{406}{7776}$ }
24. A card is drawn at random from a pack of 100 cards numbered 1 to 100. What is the probability of drawing a number which is a square? {Ans.  $\frac{1}{10}$ }
25. The digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are written in random order to form a nine digit number. Find the probability that this number is divisible by 4. {Ans.  $\frac{2}{9}$ }
26. A four digit number is formed with the digits 1, 3, 4, 5 with no repetition. Find the chance that (i) the number is divisible by 5 and (ii) the number is odd. {Ans.  $\frac{1}{4}, \frac{3}{4}$ }
27. What is the probability that four S's come consecutively in the words formed by the letters of the word 'MISSISSIPPI'? {Ans.  $\frac{4}{165}$ }
28. Twelve balls are distributed among three boxes. What is the probability that the first box will contain 3 balls? {Ans.  $\frac{55 \times 2^{11}}{3^{12}}$ }
29. In shuffling a pack of 52 playing cards, four are accidentally dropped; find the chance that the missing cards should be one from each suit. {Ans.  $\frac{2197}{20825}$ }
30. Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose that each of them independently and with equal probability can leave the cabin at any floor beginning with the first. Find out the probability of all five persons leaving at different floors. {Ans.  $\frac{360}{7^4}$ }
31. Two persons each make a single throw with a pair of dice. Find the probability that the sum of numbers on two dice are equal for both persons. {Ans.  $\frac{73}{648}$ }
32. An urn contains 3 white and 5 black balls. One ball is drawn. What is the probability that it is black? {Ans.  $\frac{5}{8}$ }
33. From a pack of 52 cards, four cards are drawn. Find the chance that they will be the four honours of the same suite. {Ans.  $\frac{4}{270725}$ }
34. Three letters are written to different persons and addresses to three envelopes are also written. Letters were put in the envelopes without looking at the addresses. Find the probability that the letters go into right envelopes. {Ans.  $\frac{1}{6}$ }
35. In a hand at whist what is the chance that the 4 kings are held by a specified player? {Ans.  $\frac{11}{4165}$ }
36. Six boys and six girls sit in a row randomly. Find the probability that
- i. The six girls sit together. {Ans.  $\frac{1}{132}$ }
- ii. The boys and girls sit alternately. {Ans.  $\frac{1}{462}$ }
37. A has three tickets in a lottery containing 3 prizes and 9 blanks. B has two tickets in a lottery containing 2



- prizes and 6 blanks. Compare their chances of success. {Ans. 952 : 715}
38. In a multiple choice question there are four alternative answers, of which one or more are correct. A candidate will get marks in the question only if he ticks all the correct answers. The candidate decides to tick answers at random. If he is allowed upto three chances to answer the question, find the probability that he will get marks in the question. {Ans.  $\frac{1}{5}$ }
39. One card is drawn from each of two ordinary sets of 52 cards. Find the probability that one of them will be ace of hearts. {Ans.  $\frac{103}{2704}$ }
40. If two dice are thrown, what is the probability that at least one of the dice shows the number greater than 3? {Ans.  $\frac{3}{4}$ }
41. Two cards are drawn simultaneously from the same set. Find the probability that one of them will be the ace of hearts. {Ans.  $\frac{1}{26}$ }
42. In bridge game of playing cards, 4 players are distributed one card each by turn so that each player gets 13 cards. Find out the probability of a specified player getting a black ace and a king. {Ans.  $\frac{164502}{978775}$ }
43. If three squares are chosen at random on a chessboard, show that the chance that they should be in a diagonal line is  $\frac{7}{744}$ .
44. A five digit number is formed by the digits 1, 2, 3, 4, 5, without repetition. Find the probability that the number formed is divisible by 4. {Ans.  $\frac{1}{5}$ }
45. A ten digit number is formed using the digits from zero to nine, every digit being used exactly once. Find the probability that the number is divisible by four. {Ans.  $\frac{20}{81}$ }
46. What is the chance that a leap year selected at random will contain 53 Sundays? {Ans.  $\frac{2}{7}$ }
47. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. Find the chance of each. {Ans.  $\frac{1}{9}, \frac{1}{3}$ }
48. A lot of 100 bulbs from manufacturing process is known to contain 10 defective and 90 non-defective bulbs. If a sample of 8 bulbs is selected at random, what is the probability that
- the sample has 3 defective and 5 non-defective bulbs {Ans.  $\frac{413343}{12500000}$ }
  - the sample has at least one defective bulb? {Ans.  $\frac{56953279}{100000000}$ }
49. Three groups of children contain 3 girls and one boy, 2 girls and 2 boys, one girl and 3 boys. One child is selected at random from each group. Show that the chance that the three selected consists of 1 girl and 2 boys is  $\frac{13}{32}$ .
50. Out of 21 tickets marked with numbers from 1 to 21, three are drawn at random. Find the probability that the three numbers on them are in A.P. {Ans.  $\frac{10}{133}$ }
51. A coin whose faces are marked 3 and 5 is tossed 4 times. What is the probability that the sum of the numbers thrown being less than 15? {Ans.  $\frac{5}{16}$ }
52. Each coefficient in the equation  $ax^2 + bx + c = 0$  is determined by throwing an ordinary six faced die. Find the probability that the equation will have real roots. {Ans.  $\frac{43}{216}$ }
53. Out of  $(2n + 1)$  tickets consecutively numbered three are drawn at random. Find the probability that the numbers on them are in A.P. {Ans.  $\frac{3n}{4n^2 - 1}$ }
54. An even number of cards is drawn from a pack of 52 cards. Find the probability that half of these cards

- will be red and the other half black. {Ans.  $\frac{{}^{52}C_{26} - 1}{2^{51} - 1}$ }
55. There are  $n$  stations between two cities  $A$  and  $B$ . A train is to stop at three of these  $n$  stations. What is the probability that no two of these three stations are consecutive? {Ans.  $\frac{(n-3)(n-4)}{n(n-1)}$ }
56. If  $n$  biscuits are distributed among  $N$  beggars, find the chance that a particular beggar will get  $r (< n)$  biscuits. {Ans.  $\frac{{}^nC_r \times (N-1)^{n-r}}{N^n}$ }
57.  $M$  telegrams are distributed at random over  $N$  communication channels ( $N > M$ ). Find the probability that not more than one telegram will be sent over each channel. {Ans.  $\frac{{}^NP_M}{N^M}$ }
58.  $K$  balls are distributed at random and independently of one another among  $N$  cells which lie in a straight line ( $N > K$ ). Find the probability that they will occupy  $K$  adjacent cells. {Ans.  $\frac{(N-K+1)K!}{N^K}$ }
59. What is the probability that in a group of  $N$  people, at least two of them will have the same birthday? {Ans.  $1 - \frac{365!}{365^N \cdot (365-N)!}$ }
60. A box contains 2 fifty paise coins, 5 twenty five paise coins and a certain fixed number ( $N \geq 2$ ) of 10 and five paise coins. Five coins are taken out of the box at random. Find the probability that the total value of these 5 coins is less than one rupee and fifty paise. {Ans.  $1 - \frac{10(N+2)}{{}^{N+7}C_5}$ }
61.  $A$  is a set containing  $n$  elements. A subset  $P$  of  $A$  is chosen at random. A subset  $Q$  of  $A$  is also again chosen at random. Find the probability that
- $P \cap Q = \phi$ . {Ans.  $\left(\frac{3}{4}\right)^n$ }
  - $P = Q$ . {Ans.  $\frac{1}{2^n}$ }
  - $P \cup Q = A$  {Ans.  $\left(\frac{3}{4}\right)^n$ }
  - $P \cap Q$  contains just one element. {Ans.  $\frac{n}{3} \left(\frac{3}{4}\right)^n$ }
  - $P \cup Q$  contains just one element. {Ans.  $\frac{3n}{4^n}$ }
  - $Q$  is a subset of  $P$ . {Ans.  $\left(\frac{3}{4}\right)^n$ }
  - $P$  and  $Q$  have equal number of elements {Ans.  $\frac{{}^{2n}C_n}{4^n}$ }
  - $Q$  contains just one element more than  $P$  {Ans.  $\frac{{}^{2n}C_{n-1}}{4^n}$ }
62. If  $n$  different letters are randomly placed in  $n$  correctly addressed envelopes, find the probability that

- i. none of the letters is placed in correct envelope. {Ans.  $\sum_{i=2}^n \frac{(-1)^i}{i!}$ }
- ii. exactly  $r$  letters are placed in correct envelopes. {Ans.  $\frac{1}{r!} \cdot \sum_{i=2}^n \frac{(-1)^i}{i!}$ }

**CATEGORY-28.4. ADDITION THEOREMS, MUTUALLY EXCLUSIVE EVENTS, EXHAUSTIVE SYSTEM OF EVENTS**

63. Let  $A$ ,  $B$  and  $C$  are three arbitrary events. Find the expression for the events noted below, in the context of  $A$ ,  $B$  and  $C$ .
- Only  $A$  occurs {Ans.  $A \cap \bar{B} \cap \bar{C}$ }
  - Both  $A$  and  $B$ , but not  $C$  occur {Ans.  $A \cap B \cap \bar{C}$ }
  - All the three events occur {Ans.  $A \cap B \cap C$ }
  - At least one occurs {Ans.  $A \cup B \cup C$ }
  - At least two occur {Ans.  $(A \cap B) \cup (B \cap C) \cup (A \cap C)$ }
  - One and no more occurs {Ans.  $(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$ }
  - Two and no more occurs {Ans.  $(A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)$ }
  - None occurs {Ans.  $\bar{A} \cap \bar{B} \cap \bar{C}$  or  $\overline{A \cup B \cup C}$ }
  - Not more than two occur. {Ans.  $(A \cap B) \cup (B \cap C) \cup (A \cap C) - (A \cap B \cap C)$ }
64. If  $P(A) = \frac{1}{4}$ ,  $P(B) = \frac{1}{2}$ ,  $P(A \cup B) = \frac{5}{6}$ , then find  $P(A \cap B)$ . {Ans.  $\frac{1}{8}$ }
65. The probability that at least one of the events  $A$  and  $B$  occurs is 0.6. If  $A$  and  $B$  occur simultaneously with probability 0.2, then find  $P(\bar{A}) + P(\bar{B})$ . {Ans. 1.2}
66. A coin is tossed twice. Events  $E$  and  $F$  are defined as follows :  $E$  = heads on first toss,  $F$  = heads on second toss. Find the probability of  $E \cup F$  and  $E \cap F$ . {Ans.  $\frac{3}{4}$ ,  $\frac{1}{4}$ }
67. Consider the random experiment of throwing a die. Let  $A_1, A_2$  and  $A_3$  be three events given by  
 $A_1$  = getting an even number, represented by  $\{2, 4, 6\}$ ;  
 $A_2$  = getting an odd number, represented by  $\{1, 3, 5\}$  and  
 $A_3$  = getting a multiple of 3, represented by  $\{3, 6\}$ .  
 Which of these events are mutually exclusive? {Ans.  $A_1$  and  $A_2$ }
68. Consider the experiment of drawing a card from a well shuffled pack of 52 cards. Let  $A_1, A_2, A_3, A_4$  be four events defined as follows:-  
 $A_1$  = Card drawn is spades;  
 $A_2$  = Card drawn is clubs;  
 $A_3$  = Card drawn is hearts and  
 $A_4$  = Card is diamonds.  
 Whether  $A_1, A_2, A_3, A_4$  form a mutually exclusive and exhaustive system of events? {Ans. Yes}
69. An integer is chosen at random from the numbers ranging from 1 to 50. What is the probability that the integer chosen is a multiple of 2 or 3 or 10? {Ans.  $\frac{33}{50}$ }

70. Two cards are drawn from a pack of 52 cards. What is the probability that either both are red or both are kings? {Ans.  $\frac{55}{221}$ }
71. A basket contains 20 apples and 10 oranges out of which 5 apples and 3 oranges are defective. If a person takes out 2 at random what is the probability that either both are apples or both are good? {Ans.  $\frac{316}{435}$ }
72. There are three events  $A, B, C$  one of which must, and only one can happen; the odds are 8 to 3 against  $A$ , 5 to 2 against  $B$ . Find the odds against  $C$ . {Ans. 43 : 34}
73. A die is loaded so that the probability of face  $i$  is proportional to  $i, i = 1, 2, \dots, 6$ . What is the probability of an even number occurring when the die is rolled? {Ans.  $\frac{4}{7}$ }

### CATEGORY-28.5. CONDITIONAL PROBABILITY

74. Two coins are tossed.
- What is the conditional probability that two heads result, given that there is at least one head? {Ans.  $\frac{1}{3}$ }
  - What is the conditional probability that there is at least one head, given that two heads result? {Ans. 1}
75. A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once? {Ans.  $\frac{2}{5}$ }
76. Consider an experiment of throwing a pair of dice. Let  $A$  and  $B$  be the events given by  $A$  = the sum of points is 8;  $B$  = there is an even number on first die. Find  $P(A/B)$  and  $P(B/A)$ . {Ans.  $\frac{3}{18}, \frac{3}{5}$ }
77. A die is thrown three times.
- If the first throw is a four, find the chance that the sum of three numbers thrown is 15. {Ans.  $\frac{1}{18}$ }
  - If the sum of three numbers thrown is 15, find the chance that the first throw was a four. {Ans.  $\frac{1}{5}$ }
78. Suppose a bag contains 5 white and 4 red balls. Two balls are drawn from the bag one after the other without replacement. Consider the following events:-  
 $A$  = First ball is white;  
 $B$  = Second ball is red.  
 Find  $P(A/B)$  and  $P(B/A)$ . {Ans.  $\frac{5}{8}, \frac{1}{2}$ }
79. If  $A$  and  $B$  are two events such that  $P(A) = 0.5, P(B) = 0.6$  and  $P(A \cup B) = 0.8$ , find  $P(A/B)$  and  $P(B/A)$ . {Ans.  $\frac{1}{2}, \frac{3}{5}$ }
80. If  $A$  and  $B$  are two events such that  $P(A) = 0.3, P(B) = 0.6$  and  $P(B/A) = 0.5$ , find  $P(A/B)$  and  $P(A \cup B)$ . {Ans.  $\frac{1}{4}, \frac{3}{4}$ }
81. A die is rolled twice and the sum of the numbers appearing on them is observed to be 7. What is the conditional probability that the number 2 has appeared at least once? {Ans.  $\frac{1}{3}$ }

82. Two integers are selected at random from integers 1 through 11. If the sum is even, find the probability that both the numbers are odd. {Ans.  $\frac{3}{5}$ }
83. If  $P(\bar{A}) = 0.7$ ,  $P(B) = 0.7$  and  $P(B/A) = 0.5$ , then find  $P(A/B)$  and  $P(A \cup B)$ . {Ans.  $\frac{3}{14}, \frac{17}{20}$ }
84. A couple has 2 children. Find the probability that both are boys, if it is known that (i) one of the children is a boy; (ii) the older child is a boy. {Ans.  $\frac{1}{3}, \frac{1}{2}$ }
85. Consider a random experiment in which a coin is tossed and if the coin shows head it is tossed again but if it shows a tail then a die is tossed. If 8 possible outcomes are equally likely, find the probability that the die shows a number greater than 4 if it is known that the first throw of the coin results in a tail. {Ans.  $\frac{1}{3}$ }
86. A coin is tossed twice and the four possible outcomes are assumed to be equally likely. If  $A$  is the event, 'both head and tail have appeared', and  $B$  be the event, 'at most one tail is observed', find  $P(A)$ ,  $P(B)$ ,  $P(A/B)$  and  $P(B/A)$ . {Ans.  $\frac{1}{2}, \frac{3}{4}, \frac{2}{3}, 1$ }
87. A bag contains 3 red and 4 black balls and another bag has 4 red and 2 black balls. One bag is selected at random and from the selected bag a ball is drawn. Let  $A$  be the event that the first bag is selected,  $B$  be the event that the second bag is selected and  $C$  be the event that the ball drawn is red. Find  $P(A)$ ,  $P(B)$ ,  $P(C/A)$  and  $P(C/B)$ . {Ans.  $\frac{1}{2}, \frac{1}{2}, \frac{3}{7}, \frac{2}{3}$ }

#### CATEGORY-28.6. MULTIPLICATION THEOREMS

88. From a well shuffled pack of playing cards, two cards are drawn one by one without replacement. What is the probability that both are ace? {Ans.  $\frac{1}{221}$ }
89. A bag contains 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that first is white and second is black? {Ans.  $\frac{1}{4}$ }
90. Find the probability of drawing a diamond card in each of the two consecutive draws from a well shuffled pack of cards, if the card drawn is not replaced after the first draw. {Ans.  $\frac{1}{17}$ }
91. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one without replacement, find the probability of getting all white balls. {Ans.  $\frac{1}{969}$ }
92. A bag contains 19 tickets, numbered from 1 to 19. A ticket is drawn and then another ticket is drawn without replacement. Find the probability that both tickets will show even numbers. {Ans.  $\frac{4}{19}$ }
93. An urn contains 5 white and 8 black balls. Two successive drawings of three balls at a time are made such that the balls are not replaced before the second draw. Find the probability that the first draw gives 3 white balls and second draw gives 3 black balls. {Ans.  $\frac{7}{429}$ }
94. A box contains 2 black, 4 white and 3 red balls. One ball is drawn at random from the box, and kept aside.

From the remaining balls in the box, another ball is drawn at random and kept besides the first. This process is repeated till all the balls are drawn from the box. Find the probability that the balls drawn are in the sequence of 2 black, 4 white and 3 red. {Ans.  $\frac{1}{1260}$ }

95. Two balls are drawn from an urn containing 2 white, 3 red and 4 black balls one by one without replacement. What is the probability that at least one ball is red? {Ans.  $\frac{7}{12}$ }
96. A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one, and examined. The ones examined are not put back. What is the probability that 9<sup>th</sup> one examined is the last defective? {Ans.  $\frac{8}{195}$ }
97. A bag contains  $n$  white and  $n$  red balls. Pairs of balls are drawn without replacement until the bag is empty. Show that the probability that each pair consists of one white and one red ball is  $\frac{2^n}{2^n C_n}$ .
98. A bag contains  $n$  white and 3 black balls. Balls are drawn one by one without replacement till all the black balls are drawn. What is the probability that this procedure for drawing balls will come to an end at the  $r^{\text{th}}$  draw? {Ans.  $\frac{3(r-1)(r-2)}{(n+1)(n+2)(n+3)}$ }
99. Cards are dealt one by one from a well-shuffled pack of 52 cards until an ace appears. Show that the probability that exactly  $n$  cards are dealt before the first ace appear is  $\frac{4(51-n)(50-n)(49-n)}{52 \cdot 51 \cdot 50 \cdot 49}$ .

### CATEGORY-28.7. INDEPENDENT EVENTS

100. If  $A$  and  $B$  are independent events associated with a random experiment, then show that  
(i)  $\bar{A}, \bar{B}$  (ii)  $A, \bar{B}$  and (iii)  $\bar{A}, B$  are also independent events.
101. A coin is tossed thrice and all eight outcomes are equally likely.  
 $E$ : "The first throw results in head";  $F$ : "The last throw results in tail".  
Prove that Events  $E$  and  $F$  are independent.
102. A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events  $A, B, C$  are defined as  
 $A$ : "the first bulb is defective";  $B$ : "the second bulb is non-defective";  $C$ : "the two bulbs are both defective or both non-defective".  
Determine whether (i)  $A, B, C$  are pairwise independent, (ii)  $A, B, C$  are mutually independent. {Ans. Yes, No}
103. Two dice are tossed. If  $(m, n)$  denotes a typical sample point, find whether the following two events  $A$  and  $B$  are independent  $A = \{(m, n) : m + n = 11\}$ ,  $B = \{(m, n) : n \neq 5\}$ . {Ans. Dependent}
104. Assume that two coins are tossed, one a rupee and the other an eight anna piece. Let  $A$  be the event that rupee shows heads and  $B$  the event that the coins show different faces. Are  $A$  and  $B$  independent? {Ans. Yes}
105. An urn contains five balls alike in every respect save colour. If three of these balls are white and two are black and we draw two balls at random from this urn without replacing them. If  $A$  is the event that the first ball drawn is white and  $B$  the event that the second ball drawn is black, are  $A$  and  $B$  independent? {Ans. No}
106. An urn contains four tickets with numbers 112, 121, 211, 222 and one ticket is drawn. Let  $A_i$  ( $i = 1, 2, 3$ )

- be the event that the  $i^{\text{th}}$  digit of the number of tickets drawn is 1. Discuss the independence of the events  $A_1, A_2, A_3$ . {Ans. Pairwise independent}
107. A bag contains 10 white and 15 black balls. Two balls are drawn in succession with replacement. What is the probability that first is white and second is black? {Ans.  $\frac{6}{25}$ }
108. Find the probability of drawing a diamond card in each of the two consecutive draws from a well shuffled pack of cards, if the card drawn is replaced after the first draw. {Ans.  $\frac{1}{16}$ }
109. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one with replacement, find the probability of getting all white balls. {Ans.  $\frac{1}{256}$ }
110. An article manufactured by a company consists of two parts  $X$  and  $Y$ . In the process of manufacture of part  $X$ , 9 out of 104 parts may be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of the part  $Y$ . Calculate the probability that the assembled product will be defective. {Ans.  $\frac{55}{416}$ }
111. Given the probability that  $A$  can solve a problem is  $\frac{2}{3}$  and the probability that  $B$  can solve a problem is  $\frac{3}{5}$ , find the probability that (a) both will be able to solve the problem; (b) at least one of  $A$  and  $B$  will be able to solve the problem. {Ans.  $\frac{2}{5}, \frac{13}{15}$ }
112. A man and a woman appear in an interview for two vacancies in the same post. The probability of man's selection is  $\frac{1}{4}$  and that of the woman's selection is  $\frac{1}{3}$  what is the probability that
- both of them will be selected {Ans.  $\frac{1}{12}$ }
  - none of them will be selected {Ans.  $\frac{1}{2}$ }
  - only one of them will be selected. {Ans.  $\frac{5}{12}$ }
113. A problem in mathematics is given to the three students whose chances of solving it are respectively  $\frac{1}{2}, \frac{1}{3}$  and  $\frac{1}{4}$ . What is the probability that the problem will be solved? {Ans.  $\frac{3}{4}$ }
114. Three faces of a fair die are yellow, two faces red and one blue. The die is tossed three times. Find the probability that the colours, yellow, red and blue appear in the, first, second and the third tosses respectively. {Ans.  $\frac{1}{36}$ }
115. If four whole numbers taken at random are multiplied together, show that the chance that the last digit in the product is 1, 3, 7 or 9 is  $\frac{16}{625}$ .
116. The probability of an event happening in one trial of an experiment is 0.6. Three independent trials are made. What is the probability that the event happens at least once? {Ans. 0.936}
117. The probability that a person will hit a target in shooting practice is 0.3. If he shoots 10 times, find the probability that he hits the target. {Ans.  $1 - (0.7)^{10}$ }
118. An anti-aircraft gun can take a maximum of four shots at an enemy plane moving away from it. The probabilities of hitting the plane at the first, second, third and fourth shot are 0.4, 0.3, 0.2 and 0.1 respectively. What is the probability that the gun hits the plane? {Ans. 0.6976}
119. A worker attends three machines each of which operates independently of the other two. The probabilities of the event that the machines will not require operator's intervention during a shift are equal to  $P_1 = 0.4$ ,  $P_2 = 0.3$ ,  $P_3 = 0.2$ . Find the probability of the event that at least one machine will require worker's

intervention during a shift. {Ans. 0.976}

120. The probability of at least one double-six being thrown in  $n$  throws with two ordinary dice is greater than 99 percent. Calculate the least numerical value of  $n$ . {Ans. 164}
121. Let  $A$  and  $B$  two independent events such that the probability is  $\frac{1}{8}$  that they will occur simultaneously and  $\frac{3}{8}$  that neither of them will occur. Find  $P(A)$  and  $P(B)$ . {Ans.  $\frac{1}{2}, \frac{1}{4}$  or  $\frac{1}{4}, \frac{1}{2}$ }
122.  $A$  and  $B$  are two independent events. The probability that both  $A$  and  $B$  occurs is  $\frac{1}{6}$  and the probability that neither of them occurs is  $\frac{1}{3}$ . Find the probability of the occurrence of  $A$ . {Ans.  $\frac{1}{3}$  or  $\frac{1}{2}$ }

**CATEGORY-28.8. APPLICATIONS OF ADDITION AND MULTIPLICATION THEOREMS**

123. A bag contains 3 black and 2 red balls. One by one three balls are drawn without replacing them. Find the probability that the third ball is red. {Ans.  $\frac{2}{5}$ }
124. A bag contains 3 white, 3 black and 2 red balls. One by one, three balls are drawn with replacement. Find the probability that the third ball is red. {Ans.  $\frac{1}{4}$ }
125. There are three urns  $A$ ,  $B$  and  $C$ . Urn  $A$  contains 4 white balls and 5 blue balls. Urn  $B$  contains 4 white balls and 3 blue balls. Urn  $C$  contains 2 white balls and 4 blue balls. One ball is drawn from each of these urns. What is the probability that out of these three balls drawn, two are white balls and one is a blue ball? {Ans.  $\frac{64}{189}$ }
126. The odds that a book will be reviewed favourably by three independent critics are 5 to 2, 4 to 3, and 3 to 4 respectively. What is the probability that of three reviews a majority will be favourable? {Ans.  $\frac{209}{343}$ }
127.  $A$  speaks truth in 75 percent cases, and  $B$  in 80 percent of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact? {Ans. 35%}
128. Two persons  $A$  and  $B$  throw a die alternately till one of them gets a 'three' and wins the game. Find their respective probabilities of winning, if  $A$  begins. {Ans.  $\frac{6}{11}, \frac{5}{11}$ }
129.  $A$  and  $B$  throw alternately a pair of dice.  $A$  wins if he throws 6 before  $B$  throws 7 and  $B$  wins if he throws 7 before  $A$  throws 6. Find their respective chance of winning, if  $A$  begins. {Ans.  $\frac{30}{61}, \frac{31}{61}$ }
130. Three persons  $A$ ,  $B$ ,  $C$  throw a die in succession till one gets a 'six' and wins the game. Find their respective probabilities of winning, if  $A$  begins. {Ans.  $\frac{36}{91}, \frac{30}{91}, \frac{25}{91}$ }
131. Suppose the probability for  $A$  to win a game against  $B$  is 0.4. If  $A$  has an option of playing either a "best of three games" or a "best of 5 games" match against  $B$ , which option should  $A$  choose so that the probability of his winning the match is higher? (No game ends in a draw). {Ans. best of 3 games}
132. Two players play a game. In each of a set of games it is 2 to 1 in favour of the winner of the previous game. What is the chance that the player who wins the first game shall win three at least of the next four? {Ans.  $\frac{4}{9}$ }
133.  $A$ ,  $B$ ,  $C$  in order cut a pack of cards, replacing them after each cut, on the conditions that the first who cuts a spade shall win a prize. Find their respective chances. {Ans.  $\frac{16}{37}, \frac{12}{37}, \frac{9}{37}$ }
134. Three white balls and five black balls are placed in a bag and three men draw a ball in succession (the ball



- drawn not being replaced) until a white ball is drawn. Show that their respective chances are as 27 : 18 : 11.
135. If  $p$  is the probability that a man aged  $x$  years will die in a year, find the probability that out of  $n$  men  $A_1, A_2, \dots, A_n$  each aged  $x$  years,  $A_1$  will die in a year and will be the first to die. {Ans.  $\frac{1-(1-p)^n}{n}$ }
136. A coin is tossed  $(m+n)$  times,  $(m > n)$ . Show that
- that the probability of getting exactly  $m$  consecutive heads is  $\frac{n+3}{2^{m+2}}$ ;
  - the probability of at least  $m$  consecutive heads is  $\frac{n+2}{2^{m+1}}$ .
137. Two non-negative integers  $x$  and  $y$  are chosen at random with replacement. Find the probability that  $x^2 + y^2$  is divisible by 10. {Ans.  $\frac{9}{50}$ }

### CATEGORY-28.9. THEOREM OF TOTAL PROBABILITY

138. A bag contains 4 red and 3 black balls. A second bag contains 2 red and 4 black balls. One bag is selected at random. From the selected bag, one ball is drawn. Find the probability that the ball drawn is red. {Ans.  $\frac{19}{42}$ }
139. There are two bags, one of which contains three black and four white balls while the other contains four black and three white balls. A die is cast. If the face 1 or 3 turns up, a ball is taken from the first bag and if any other face turns up, a ball is chosen from the second bag. Find the probability of choosing a black ball. {Ans.  $\frac{11}{21}$ }
140. There are two bags. The first bag contains 5 white and 3 black balls and the second bag contains 3 white and 5 black balls. Two balls are drawn at random from the first bag and are put into the second bag without noticing their colours. Then two balls are drawn from the second bag. Find the probability that the balls are white and black. {Ans.  $\frac{673}{1260}$ }
141. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replaced into the urn, otherwise it is replaced along with another ball of the same colour. The process is repeated. Find the probability that the third ball drawn is black. {Ans.  $\frac{23}{30}$ }
142. In a bolt factory, machines  $A, B$  and  $C$  manufacture respectively 25%, 35% and 40% of the total bolts. Of their output, 5%, 4% and 2% are respectively defective bolts. A bolt is drawn at random from the product. What is the probability that the bolt drawn is defective? {Ans. 0.0345}
143. There are two groups of subjects one of which consists of 5 science subjects and 3 engineering subjects and the other consists of 3 science and 5 engineering subjects. An unbiased die is cast. If number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately. {Ans.  $\frac{13}{24}$ }
144. Three groups  $A, B$  and  $C$  are competing for positions on the Board of Directors of a company. The probabilities of their winning are 0.5, 0.3, 0.2 respectively. If the group  $A$  wins, the probability of introducing a new product is 0.7 and the corresponding probability for groups  $B$  and  $C$  are 0.6 and 0.5 respectively. Find the probability that the new product will be introduced. {Ans. 0.63}
145.  $A$  is one of 6 horses entered for a race, and is to be ridden by one of two jockeys  $B$  and  $C$ . It is 2 to 1 that  $B$

- rides  $A$ , in which case all the horses are equally likely to win. If  $C$  rides  $A$ , his chance is trebled. What are the odds against his winning? {Ans.  $13 : 5$ }
146. A lot contains 20 articles. The probability that the lot contains exactly 2 defective articles is 0.4 and the probability that the lot contains exactly 3 defective articles is 0.6. Articles are drawn at random one by one without replacement and tested till all the defective articles are found. What is the probability that the testing procedure ends at the twelfth testing? {Ans.  $\frac{99}{1900}$ }
147. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, 4, ..., 12 is picked and the number of the card is noted. What is the probability that the noted number is either 7 or 8? {Ans.  $\frac{193}{792}$ }
148. An urn contains,  $a$  white and  $b$  black balls and a second urn,  $c$  white and  $d$  black balls. One ball is transferred from the first urn into the second and one ball is then drawn from the second urn. What is the probability that it is a white ball? {Ans.  $\frac{ac+bc+a}{(a+b)(c+d+1)}$ }

### CATEGORY-28.10. BAYE'S THEOREM

149. In a bolt factory, machines  $A$ ,  $B$  and  $C$  manufacture respectively 25%, 35% and 40% of the total bolts. Of their output, 5%, 4% and 2% are respectively defective bolts. A bolt is drawn at random from the product. If the bolt drawn is found to be defective, what is the probability that it is manufactured by the (i). machine  $A$ ; (ii). machine  $B$ ; (iii). machine  $C$ ? {Ans.  $\frac{25}{69}, \frac{28}{69}, \frac{16}{69}$ }
150. A bag  $A$  contains 2 white and 3 red balls and a bag  $B$  contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag  $B$ . {Ans.  $\frac{25}{52}$ }
151. We have two boxes,  $B_1$  and  $B_2$ . Box  $B_1$  contains one red and one white marble. Box  $B_2$  contains three red marbles and one green marble. A box is selected by the toss of fair coin and one marble is drawn at random from the box selected. Given that a red marble is obtained, what is the probability that the marble was drawn from  $B_1$ . {Ans.  $\frac{2}{5}$ }
152. Suppose there are three urns containing 2 white and 3 black balls, 3 white and 2 black balls and 4 white and one black ball respectively. There is equal probability of each urn being chosen. One ball is drawn from an urn chosen at random. What is the probability that a white ball is drawn? If we are told that a white ball has been drawn, find the probability that it was drawn from the first urn. {Ans.  $\frac{3}{5}, \frac{2}{9}$ }
153. Three urns contain 6 red, 4 black; 4 red, 6 black; and 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that is drawn from the first urn. {Ans.  $\frac{2}{5}$ }
154. There are 3 bags, each containing 5 white balls and 3 black balls. Also there are 2 bags, each containing 2 white balls and 4 black balls. A white ball is drawn at random. Find the probability that this white ball is from a bag of the first group. {Ans.  $\frac{45}{61}$ }
155. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart. {Ans.  $\frac{11}{50}$ }

156. A pack of playing cards was found to contain only 51 cards. If the first 13 cards, which are examined, are all red, what is probability that the missing card is black? {Ans.  $\frac{2}{3}$ }
157. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter driver, car driver and a truck driver is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver? {Ans.  $\frac{1}{52}$ }
158. In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is  $\frac{1}{3}$  and the probability that he copies the answer is  $\frac{1}{6}$ . The probability that his answer is correct, given that he copied it, is  $\frac{1}{8}$ . Find the probability that he knew the answer to the question, given that he correctly answered it. {Ans.  $\frac{24}{29}$ }
159. A letter is known to have come either from *TATANAGAR* or *CALCUTTA*. On the envelope just two consecutive letters *TA* are visible. What is the probability that the letter has come from (i) Calcutta (ii) Tatanagar? {Ans.  $\frac{4}{11}, \frac{7}{11}$ }
160. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. {Ans.  $\frac{3}{8}$ }
161. An urn contains five balls. Two balls are drawn and are found to be white. What is the probability that all the balls are white? {Ans.  $\frac{1}{2}$ }
162. *A* and *B* are two independent witnesses (*i.e.* there is no collusion between them) in a case. The probability that *A* will speak the truth is  $x$  and the probability that *B* will speak the truth is  $y$ . *A* and *B* agree in a certain statement. Show that the probability that the statement is true is  $\frac{xy}{1-x-y+2xy}$ .

### CATEGORY-28.11. BINOMIAL DISTRIBUTION

163. What is the probability of getting exactly 2 tails in 6 tosses of a fair coin? {Ans.  $\frac{15}{64}$ }
164. Six coins are tossed simultaneously. Find the probability of getting at least 4 heads. {Ans.  $\frac{11}{32}$ }
165. In five throws with a single die what is the chance of throwing (i) three aces exactly, (ii) three aces at least. {Ans.  $\frac{125}{3888}, \frac{23}{648}$ }
166. If on an average 1 ship in every 10 is wrecked, find the chance that out of 5 ships expected 4 at least will arrive safely. {Ans.  $\frac{45927}{50000}$ }
167. A student is given a true-false exam with 10 questions. If he gets 8 or more correct answers he passes the exam. Given that he guesses at the answer to each question, compute the probability that he passes the exam. {Ans.  $\frac{7}{128}$ }
168. A man takes a step forward with the probability 0.4 and backwards with the probability 0.6. Find the probability that at the end of eleven steps he is just one step away from the starting point. {Ans. }

$$462(0.24)^5 \}$$

169. A coin is tossed  $n$  times. What is the chance that the head will present itself an odd number of times?

{Ans.  $\frac{1}{2}$ }

170. Numbers are selected at random, one at a time, from the two digit numbers 00, 01, 02, ..., 99 with replacement. An event  $E$  occurs if and only if the product of the two digits of a selected number is 18. If four numbers are selected, find the probability that the event  $E$  occurs at least 3 times. {Ans.  $\frac{97}{25^4}$ }

171. An urn contains 25 balls of which 10 balls bear a mark 'A' and the remaining 15 balls bear a mark 'B'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn this way, find the probability that

i. all will bear 'A' mark {Ans.  $\left(\frac{2}{5}\right)^6$ }

ii. not more than 2 will bear 'B' mark {Ans.  $7\left(\frac{2}{5}\right)^4$ }

iii. the number of balls with 'A' mark and 'B' mark will be equal {Ans.  $\frac{864}{3125}$ }

iv. at least one ball will bear 'B' mark {Ans.  $1 - \left(\frac{2}{5}\right)^6$ }

### CATEGORY-28.12. EXPECTATION

172. A die is tossed.  $X$  is the number appearing on the die. What is the expectation of  $X$ ? {Ans. 3.5}

173. A man plays a game of dice. If he throws six, he wins Rs. 90. If he throws three, four or five, he wins Rs. 20. If he throws one or two, he loses Rs. 30. What is the expectation of amount won by him? {Ans. 15}

### CATEGORY-28.13. GEOMETRICAL DEFINITION OF PROBABILITY

174. From each of two equal lines of length  $l$  a portion is cut off at random and removed. What is the chance that the sum of the remainders is less than  $l$ ? {Ans.  $\frac{1}{2}$ }

175. Three tangents are drawn at random to a given circle. Show that the odds are 3 to 1 against the circle being inscribed in the triangle formed by them.

### CATEGORY-28.14. APPLICATION OF METHODS

176. Out of  $3n$  consecutive integers, three are selected at random. Find the chance that their sum is divisible by 3 {Ans.  $\frac{3n^2 - 3n + 2}{(3n-1)(3n-2)}$ }

177. Two numbers are selected at random from 1, 2, 3, ..., 100 and multiplied. Find the probability correct to two places of decimals that the product thus obtained, is divisible by 3. {Ans.  $\frac{83}{150}$ }

178. If  $6n$  tickets numbered 0, 1, 2, ...,  $(6n-1)$  are placed in a bag, and three are drawn out, show that the chance that the sum of the numbers on them is equal to  $6n$  is  $\frac{3n}{(6n-1)(6n-2)}$ .

179. In a purse there are 10 coins, all 5 paise coins except one, which is a rupee. In another there are 10 coins all 5 paise coins. Nine coins are taken from the former and put into the latter and then nine coins are taken from the latter and put in to the former, find the chance that the rupee is still in the first purse. {Ans.  $\frac{10}{19}$ }

180. If  $m$  things are distributed among  $a$  men and  $b$  women, show that the chance that the number of things received by men is odd is  $\frac{1}{2} \cdot \frac{(b+a)^m - (b-a)^m}{(b+a)^m}$ .
181.  $A$  and  $B$  throw with 3 dice. If  $A$  throws 8, what is  $B$ 's chance of throwing a higher number? {Ans.  $\frac{20}{27}$ }
182. A person throws two dice, one the common cube and the other a regular tetrahedron, the number on the lowest face being taken in the case of the tetrahedron. What is the chance that the sum of the numbers thrown is not less than 5? {Ans.  $\frac{3}{4}$ }
183. Find the minimum number of tosses of a pair of dice so that the probability of getting the sum of the digits on the dice equal to 7 on at least one toss is greater than 0.95. {Ans. 17}
184. A seed merchant finds that 90% of his cucumber seeds germinate under standard conditions. He accordingly claims 90% germination when he sells them in packets of 10. Show that about one quarter of his customers will be entitled to complain that seeds in their packets do not reach the standard of germination. {Ans. 0.2646*N*}
185. A bag contains a certain number of balls, some of which are white. A ball is drawn and replaced, another is then drawn and replaced and so on. If  $p$  be the chance of drawing a white ball in a single trial, find the number of white balls that is most likely to have been drawn in  $n$  trials. For  $p = \frac{1}{3}$  and  $n = 12$ , calculate the number of white balls. {Ans.  $[np + p], 4$ }
186. If a fair coin is tossed 15 times, what is the probability of getting head as many times in the first ten throws as in the last five? {Ans.  $\frac{3003}{32768}$ }
187. Two dice are thrown together first and secondly three dice are thrown together. Find the probability that the total in the first throw is 4 or more and at the same time the total in the second throw is 6 or more. {Ans.  $\frac{1133}{1296}$ }
188. In a bag there are three tickets numbered 1, 2, 3. A ticket is drawn at random and put back and this done four times. Show that it is 41 to 40 that the sum of the numbers is drawn is even.
189. Two players  $A$  and  $B$  want respectively  $m$  and  $n$  points of winning a set of games. Their chances of winning a single game are  $p$  and  $q$  respectively where  $p + q = 1$ . The stake is to belong to the player who first makes up his set. Find the probabilities in favour of each player.
190. For two events  $A, B$  prove the following relations:-  
 i.  $P(\bar{A}/B) = 1 - P(A/B)$   
 ii.  $P(\bar{A} \cup \bar{B}) = 1 - P(A)P(B/A)$ .
191.  $A$  and  $B$  are two candidates seeking admission in I.I.T. The probability that  $A$  is selected is 0.5 and the probability that both  $A$  and  $B$  is at most 0.3. Is it possible that the probability of  $B$  getting selected is 0.9? {Ans. No}
192. Cards are drawn one-by-one at random from a well-shuffled full pack of 52 playing cards until 2 aces are obtained for the first time. If  $N$  is the number of cards required to be drawn, then show that  

$$P_r(N = n) = \frac{(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13} \text{ where } 2 \leq n \leq 50.$$
193. A person draws cards one-by-one from a pack. Show that the probability that exactly  $n$  cards are drawn until all the aces appear is  $\frac{4(n-1)(n-2)(n-3)}{52.51.50.49}$ .
194.  $A, B, C$  are events such that

$$P(A) = 0.3, P(B) = 0.4, P(C) = 0.8$$

$$P(AB) = 0.08, P(AC) = 0.28$$

$$P(ABC) = 0.09.$$

If  $P(A \cup B \cup C) \geq 0.75$ , then show that  $P(BC)$  lies in the interval  $0.23 \leq x \leq 0.48$ .

195. There are ten pairs of gloves in a bag. First  $A$  draws one glove from the bag, then  $B$  draws one glove, then  $A$  draws one glove and finally  $B$  draws one glove, drawn gloves being not replaced. Show that the chance of  $A$  drawing a pair is the same as that of  $B$  drawing a pair. Also find the probability that neither draws a pair. {Ans.  $\frac{290}{323}$ }
196. In a certain city two newspapers  $A$  and  $B$  are published. It is known that 25% of the city population reads  $A$  and 20% reads  $B$  while 8% reads both  $A$  and  $B$ . It is also known that 30% of those who read  $A$  but not  $B$  look into advertisements and 40% of those who read  $B$  but not  $A$  look advertisements while 50% of those who read both  $A$  and  $B$  look into advertisements. What is the percentage of the population who reads an advertisement? {Ans. 13.9%}
197. For any two events  $A$  and  $B$ , prove that  $P(\bar{A} \cap B) = P(B) - P(A \cap B)$ .
198. If  $B \subset A$ , then prove that
- $P(A \cap \bar{B}) = P(A) - P(B)$ .
  - $P(B) \leq P(A)$ .
199. For any two events  $A$  and  $B$ , prove that  $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$ .
200. For any two events  $A$  and  $B$ , prove that the probability that exactly one of  $A, B$  occurs is given by  $P(A) + P(B) - 2P(A \cap B) = P(A \cup B) - P(A \cap B)$ .
201. If  $A, B, C$  are three events, then prove that
- $P(\text{at least two of } A, B, C \text{ occur}) = P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$
  - $P(\text{Exactly two of } A, B, C \text{ occur}) = P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$
  - $P(\text{Exactly one of } A, B, C \text{ occurs}) = P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C)$ .

### CATEGORY-28.15. ADDITIONAL QUESTIONS