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Lobachevsky Geometry and Modern Nonlinear Problems

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Introduction

"... the difficulty of concepts increases as they approach the primary truths in nature ..."

N. I. Lobachevsky

Lobachevsky geometry: sources, philosophical significance, and its role in contemporary science

The aim of this book is to reveal the potential of Lobachevsky's geometry in the context of its emergence in various branches of current interest in contemporary science, first and foremost in nonlinear problems of mathematical physics. Looking "geometrically" at a wide circle of problems from the standpoint of Lobachevsky geometry allows one to apply in their study unified approaches that rest upon the methods of non-Euclidean hyperbolic geometry and its highly developed tools.

The discovery of non-Euclidean hyperbolic geometry by the great Russian mathematician Nikolai Ivanovich Lobachevsky, announced by him of the 12th of February, 1826, inaugurated an important historical stage in the development of mathematical thought as an axiomatically impeccably built new field of analytical knowledge. At the foundations of *Lobachevsky's geometry* lies a complete rethinking of the system of axioms of an intuitive geometry and the principles of its construction. Lobachevsky's geometry represented the crowning of attempts, undertaken over many centuries by thinkers of different historical periods, at establishing the correctness of Euclid's geometry that arose already at the dawn of our era.

The system of axioms of the "new" geometry proposed by Lobachevsky differs from the axioms of Euclidean geometry only through the formulation of Postulate V (the Axiom of Parallels). Let us give descriptive formulations of the corresponding variants of the *Axiom of Parallels*.

Euclid's Postulate V: *in a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.*

Lobachevsky's Axiom of Parallels: *through every point that does not lie on a given straight line there pass at least two distinct straight lines which lie in the same plane as the given straight line, and which do not intersect that straight line.*

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The axioms of Euclid's geometry that were not modified (19 axioms) form the content of the so-called *Absolute Geometry*, a uniqued fundamental component of the classical geometries.

Initially, the realization of what Lobachevsky's ideas mean did run into certain difficulties, the roots of which are in all probability hidden in the primary associative psychological perception of the notions and terminology it uses. For this reason we should mention at the outset that in Lobachevsky's geometry a "straight line" must be understood as a shortest (geodesic) line, i.e., a line along which the distance between any two points on it is minimal. At the same time, the notion of "*parallelism*" of two "straight lines" presumes only that they do not *intersect* and discards the familiar Euclidean property that two parallel straight lines are equidistant (lie at the same distance from one another). Thus, in the new non-Euclidean geometry there arises, it seems, a separation, of holding out classical notions and properties, interpretable "together" in Euclid's geometry.

The conceptual result of Lobachevsky's investigations is that Postulate V (or the Axiom of Parallels) is an independent (self-standing) assertion, which is not logically connected with the other adopted axioms. The possibility of "varying" the formulation of the "Axiom of Parallels" results in the emergence of "independent" geometries (the three known classical geometries: Euclidean, hyperbolic, and spherical). Of these, the hyperbolic geometry constructed by Lobachevsky has the special promising potential demanded by the modern scientific knowledge.

The new geometry, which rests on the introduced system of axioms, was referred to as an "imaginary" geometry already by Lobachevsky himself, who regarded it as a possible "theory of spatial relations".

The subsequent historical development of this theory confirmed objectively its depth and the fundamental prospects of its potential, as well as its definite influence on the development of such domains of knowledge as geometry in general, logic, differential equations, function theory, nonlinear problems of fundamental science, and so on. The path to recognition of the new mathematical theory did run, in particular, through achievements in the geometry of surfaces of negative curvature, the theory of functions of one complex variable, and the theory of partial differential equations.

In modern mathematical physics, the nonlinear modeling of *Lobachevsky geometry* shows up in such attributes of the aforementioned fields of knowledge as *solitons, Bäcklund transformations, pseudospherical surfaces, singularities, attractors, transcendents*, and so on. As it turns out, in the investigation of many actual nonlinear problems one can find a "unifying non-Euclidean common denominator".

Lobachevsky's works on the "theory of parallels" and their influence on the development of geometry

Noting the particular significance of N. I. Lobachevsky's geometric ideas and his contribution to the development of the foundations of the axiomatic structure of mathematical systems, we list below his works that founded the axioms of non-Euclidean hyperbolic geometry. The chronology of their public appearance establishes beyond doubt the priority of N. I. Lobachevsky, and subsequently of the Russian scientific school of geometry, in the development of the concepts of non-Euclidean hyperbolic geometry, in particular, of its relationships with other promising branches of mathematics and fundamental science.

I. "*Exposition succincte des principes de la géométrie avec une demonstration rigoureuse du theorem des paralléles*". *February 12, 1826*.

("*A concise exposition of the principles of geometry with a rigorous proof of the theorem on parallels*".)

This is the first public scientific announcement on the discovery of the new non-Euclidean geometry, made by N. I. Lobachevsky on the 12th of February 1826, as a report at the session of the Physical and Mathematical Section of Kazan university. The manuscript of the report was handed to three professors for safe keeping (however, the manuscript did not survive).

II. "*On the foundations of geometry*" (Russian). *Kazanskii Vestnik, 1829–1830.*

This is a systematic exposition of Lobachevsky's theory of parallel lines, the foundations of a new "imaginary" geometry. The work was published in separate parts over the period from February 1829 to August 1830 in the Kazanskii Vestnik (The Kazan Messenger).

In this study Lobachevsky discusses first how, in his understanding, one has to first establish and then logically develop the primary notions in geometry, and subsequently obtain propositions and theorems. Further, developing these ideas, Lobachevsky provides a systematic treatment (although in compressed form) of the *foundations of the theory of parallel straight lines*, "reaching" in this way the frontiers of analytic geometry: he finds the equations of straight lines and of the most important curves. The final part of the memoir is devoted to effective applications of the imaginary geometry to the calculation of simple and multiple definite integrals. It is precisely in the possible applications of his new theory that Lobachevsky always saw an additional confirmation of its truth and objectivity.

III. "*Imaginary geometry*" (Russian). *Uchenye Zapiski Kazanskogo Universiteta, 1835.*

IV. "*Application of imaginary geometry to certain integrals*" (Russian). *Uchenye Zapiski Kazanskogo Universiteta, 1836.*

In these works Lobachevsky provides a more detailed, and accordingly more accessible exposition of the ideas and results contained in the memoir [II]. In his treatment of the subject, Lobachevsky chooses here the opposite approach: starting from the relations that connect the sides and angles in a triangle in the *imaginary geometry*, he shows that these relations cannot lead to contradictory conclusions. Based on the relations used he obtains geometric properties of triangles and parallel straight lines. He also considers applications of the new geometry to calculus.

Soon after their publications, the works [III] and [IV] were also printed, with minor changes and additions, in French, in the well-known European mathematical

Journal Crelle—"Journal für die reine und angewandte Mathematik"; this made them more accessible to mathematical circles in Europe:

IIIa. "*Géométrie imaginaire*". *Journal für die reine und angewandte Mathematik, 1836.*

IVa. "*Application de la géométrie imaginaire à quelques integrals*". *Journal für die reine und angewandte Mathematik, 1837*

These articles were studied in detail by C. F. Gauss, the most prominent mathematician of the XIXth century, who also came very close to realizing that a *non-Euclidean geometry* exists, calling it in his works *anti-Euclidean*. However, Gauss expressed his high praise of Lobachevsky's results only in his private correspondence with mathematician colleagues.

V. "*New foundations of geometry with a complete theory of parallels*" (Russian). *Uchenye Zapiski Kazanskogo Universiteta, 1835–1838.*

This is the largest work of N. I. Lobachevsky, which sums up in detail, and in the necessary cases develops, the results of his earlier works. It is from this memoir that one can draw the most completely information on the global scientific, world-outlook and philosophical views of this great mathematician.

In this work the fundamental notions of geometry are discussed in detail: adjacency, cuts and the definition of the notion of point connected with them, lines, surfaces, and also the basic theorems on perpendicular straight lines and planes, relations in triangles, linear and angular measures, measuring of areas, and others. Starting from more general fundamental premises (compared with earlier works), a theory of parallel straight lines is constructed in detail. The fundamental equations of the *imaginary geometry* are introduced. As a whole, in this work Lobachevsky establishes the precise axiomatic foundations of geometry and defines the principles of its logical development, accompanying them with the corresponding foundational results in each of the fields he considered.

VI. "*Geometrische Untersuchungen zur Theorie der Parallellinien*". *Berlin, 1840*
("Geometric investigations on the theory of parallels").

The aim of this small, but, as it turned out, rather needed brochure, published in Berlin in 1840, was to present in an intuitive and visual manner all the fundamental ideas and results that constituted Lobachevsky's new non-Euclidean geometry. This aim was achieved; indeed, it is through this publication that the wide mathematical community (and first of all, the European one) was able to become acquainted and accept the ideas of the new geometry.

VII. "*Pangeometry*" (Russian). *Uchenye Zapiski Kazanskogo Imperatorskogo Universiteta* (*Scientific Memoirs of Kazan Imperial University*), *1855.*
(see: "*Pangeometry*", Edited and translated by Athanase Papadopoulos, Heritage of European Mathematics, European Mathematical Society Publishing House, Zürich, 2010.)

This is essentially a summarizing work on geometry, in which Lobachevsky, then already with the experience of a venerable mathematician, did collectively generalize and complete, all the result and ideas stated in his earlier works.

The name "Pangeometry" itself implies and understanding of geometry in its widest sense—as an *all-geometry*, which draws in all known (at that time) geometric representations on space structures.

In 1856 a French translation of this work appeared in a collection of scientific papers prepared for the 50th anniversary of Kazan University:

"*Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des parallèles*". *Uchenye Zapiski Kazanskogo Universiteta, 1856.*

The works [I]–[VII] constitute the geometric herritage of the prominent Russian mathematician N. I. Lobachevsky, which allowed to broaden the understanding of the very meaning of geometry as the science of the structure of space and, accordingly, of the principles of its construction and establishment. Lobachevsky's contribution to geometry became a fundament and a kind of standard that make possible the advancement of the mathematical world view as a whole.

In this connection let us mention the special role of the geometric investigations of B. Riemann. In his 1854 lectures "Über die Hypothesen welche der Geometrie zu Grunde liegen" ("On the hypotheses which lie at bases of geometry") Riemann formulated an original idea of mathematical space, the *manifold*, in his terminology. According to Riemann, geometry should be considered as a mathematical theory of continuous manifolds (different collections of homogeneous objects of, generally speaking, different nature). In his investigations Riemann develops a series of results on the intrinsic geometry of surfaces, a branch of geometry founded by C. F. Gauss in his treatise "Disquisitiones generales circa superficies curvas" (1827) ("General investigations of curved surfaces"). Intrisic geometry studies those properties of a surface that are connected with direct measurements *on* the surface. Riemann did effectively apply the notion of linear element, a metric introduced on a *manifold*.

The geometric theory treated by Riemann rests on three conceptual components, namely, the existence of the non-Euclidean *Lobachevsky geometry*, Gauss' achievements in the theory of intrinsic geometry of surfaces, and the notion of multi-dimensional space that took shape in mathematics at that time. An indisputable historical contribution of this research is the introduction of objects that are today known as Riemannian spaces—spaces that are characterized by their own curvature and which generalize our representations about Euclidean spaces, Lobachevsky's non-Euclidean hyperbolic spaces, and the spaces of elliptic geometry studied by Riemann himself. The problem, formulated in Riemann's work, *of understanding the origins of metric properties of spaces* became the harbinger of definite achievements in the general theory of relativity and, as will be shown in this book, remained of actual interest in problems of geometric interpretation of nonlinear differential equations of contemporary mathematical physics.

The fact that Lobachevsky singled out the *axiom of parallels* as an independent, "self-standing" axiomatic assertion showed that a certain revision, a renewed understanding and systematization of the axioms of geometry (axioms that lie at the foundation of absolute geometry), was needed. The solution of this fundamental mathematical problem and, thinking globally, deep philosophical question, was presented by the prominent mathematician David Hilbert at the crossroads of the XIX–XX centuries.

In his 1899 work "The foundations of geometry", Hilbert proposed a complete, separated into groups, system of axioms, which allows one, in the framework of modern geometry, to develop all ensuing "geometric constructions" and obtain the relations that connect them. At the basis of Hilbert's approach is the adoption of three primary systems of "things": "points", "straight lines", and "planes", the elements of which can be in certain relationships, ruled by terms such as "belongs", "are situated", "between", "parallel", "congruent", "continuous", and so on. The meaning of these very "things" (primary geometric "objects"), as well as of the "relations" that connect them, is completely defined by the the logic context of a complete set of stated axioms, divided into five groups: axioms of belonging (or connection, or incidence), axioms of order, axioms of congruence, the axiom of continuity, and the axiom of parallels. A detailed discussion of Hilbert's axiomatics is given in § 1.1.

It is important to note that already Hilbert himself did emphasize that as initial "things" one can take, in principle, elements of any nature, not necessarily rigidly associated with the usual stereotypes of our perception of space. For example, a "straight line" (thing) does not have to be a (Euclidean) straight line, and so on. What matters is that in the system of "things" used the full compatibility of the adopted system of axiomatic statements is preserved. This "geometric vision" of Hilbert harnesses the serious potential of the global understanding of geometry, as well as generalized principles of axiomatic construction of a mathematical theory.

Identifying the primary structural component— bricks—of the space being modeled and prescribing the types of rules that connect them is an initial problem of utmost importance in the process of creating a geometric theory. This is a primary complex of problems, each model solution of which further deepens our knowledge of the structure of real space and builds a "bridge" between Reality and the descriptive formalism that approximates it.

All these foundational problems occupied the thoughts of thinkers in all periods of history. The principles on their rigorous scientific resolution with the aim of building a geometric theory were extremely clearly formulated by the prominent Russian geometer N. V. Efimov: "*Geometry operates with notions that arise from experience as a result of a certain abstraction of the objects of real world, in which one pays attention to only certain properties of real objects*; *in rigorously logic arguments when one proves theorems one deals only with these properties of the objects—hence these properties must be mentioned in axioms and definitions*; *all the other properties, which we got used to imagine when we hear the words* "*point*", "*straight line*", "*plane*", *play no role whatsoever in logical constructions and should not be mentioned in the fundamental statements of geometry*".

Thus, Lobachehvsky's new non-Euclidean geometry became a kind of impulse to rethinking the bases and principles of the construction of modern geometry in general.

Recognition of non-Euclidean hyperbolic geometry and its philosophical significance

The unquestionable *priority* of N. I. Lobachevsky in the *discovery* of non-Euclidean hyperbolic geometry is established by his first public report *"Exposition succincte des principes de la géométrie avec une demonstration rigoureuse du theoreme des parallèles"*, made on the *12th of February, 1826*, at Kazan University. Furthermore, the fundamental contribution of this mathematical genius to the development of analytical foundations of the new geometry, the *Lobachevsky geometry*, is firmly established by a cycle of his scientific treatises, published over the subsequent 30 year period.[1] The first printed work of Lobachevsky was "On the foundations of geometry" (1829–30, [II]), which Lobachevsky himself called an extract from "Exposition".

This work has priority also over the scientific work of the prominent Hungarian mathematician János Bolyai, published in 1831 as an appendix "*Appendix, Scientam spatii absolute veram exhibens* ("Appendix Explaining the Absolutely True Science of Space independent of the truth or falsity of Euclid's axiom XI (which can never be decided a priori)"), which contains his results on the fundamental propositions of non-Euclidean geometry. However, the brilliant independent geometric ideas of János Bolyai were not destined to have significant continuation because of the following life conflict. At the beginning of 1832 Bolyai's work reached C. F. Gauss, who in a letter to his long-standing friend Farkas Bolyai (János' father) communicated that the results he did draw from "it Appendix" where a subject of his thoughts already for a long time and, essentially, were identical with the conclusions that he reached[2], concerning which now he cannot further undertake fast attempts to publication ("*To praise it would amount to praising myself. For the entire content of the work ... coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years*"). Later, in a letter to Gerling,[3] Gauss wrote about Bolyai's work: "I consider this young geometer, v. Bolyai, to be a genius of the first class ..." J. Bolyai, however took Gauss' judgement with prejudice, deciding that Gauss intended to take away the priority of his ideas. It is probably precisely J. Bolyai's prejudice to C. F. Gauss that became a kind of barrier to the further in-depth development of the geometric theory that he announced. No longer than a decade after, Bolyai was recognized as one of the prominent geometers of the first half of the XIXth century, and in 1902, with the occasion of the anniversary of 100 years from his birth, a prize bearing his name was established, laureates of which were later geometers like H. Poincaré (1905) and D. Hilbert (1910).

The advancement of the new non-Euclidean hyperbolic geometry is intimately related to the personality of Carl Friedrich Gauss, the greatest German mathematician, whose very deep mind and extraordinary mathematical insight allowed him to immediately understand and accept the objective existence and prospects of the geometry that was taking shape. This is confirmed by the aforementioned opinion-letter to Bolyai and the subsequent comments of this great

[1] Including translations into foreign languages by well-known European publishers.
[2] But not published by Gauss.
[3] V. S. Malakhovsky, "Selected Chapters on the History of Mathematics".

mathematician on the extensive work of N. I. Lobachevsky that he made in his private correspondence with colleagues in mathematics. Unfortunately, for reasons known only to Gauss himself, he did not feel that he could publicly discuss at extent this system of representations on the new non-Euclidean geometry which, beyond any doubt, emerged independently in his thinking, and, probably, was reflected in personal scientific notes, a fact witnessed not only by Gauss' own comments, but also by the the conclusions reached by the historians of mathematics of his time.

The translations of Lobachevsky's works of 1836 "Géométrie imaginaire" ([IIIa]) into French (1840) and "Geometric investigations on the theory of parallels" ([VI]) into German became accessible to Gauss. Becoming acquainted with Lobachevsky's investigations, Gauss expressed careful opinions about them, but only in private correspondence. An example is the following fragment from a letter of Gauss to his astronomer friend H. Schumacher (1846): "*You know that for 54 years now (even since 1792) I have held the same conviction (with some later enrichment, about which I don't want to comment here). I have found in Lobachevsky's work nothing that is new to me. In developing the subject, the author followed a road different from the one I took myself; Lobachevsky carried out the task in a masterly fashion and in a truly geometric spirit. I consider it a duty to call your attention to this work, since I have no doubt that it will give you a tremendous pleasure ...*"

Gauss played a special role in the geometric contributions of Lobachevsky achieving recognition, expressing (in the form of "personal communications") his authoritative opinion on the results about the new non-Euclidean geometry to a sufficiently wide circle of respected scientist of his time. It is due to Gauss' recommendation that in 1842 Lobachevsky was elected corresponding member of the Royal Society of Göttingen.

Thus, we see that at the source of the propositions of the new non-Euclidean geometry in the first half of the XIX-th century stood three giants of mathematics: N. I. Lobachevsky, J. Bolyai, and C. F. Gauss. However, the historical role Lobachevky played in this direction was special, since besides the titanical work at elaborating the new theory, he took upon himself the heavy burden of "adapting" it to the scientific and social communities, which in essence is always a main condition for the strengthening and advancement of any "revolutionary" body of knowledge.

The work of the Italian mathematician E. Beltrami "*Saggio di interpretazione della geometria non-euclidea*" (1868) represented the next stage in strengthening the position of the new non-Euclidean geometry; the results obtained therein allowed to bring Lobachevsky's geometry out of the category of "imaginary geometries" as a geometry that admits its own interpretation (though only partially) in the framework of the habitual Euclidean representations. Beltrami, studying the behavior of geodesics on the surface of the pseudosphere, established that the metric of the pseudosphere is identical in form with the metric of the Lobachevsky plane in a certain domain of it (more precisely, in a horodisc). That is, the conclusion was reached that on the pseudosphere, which is a surface in Euclidean space, there are realized all intrinsic-geometric laws of Lobachevsky's two-dimensional geometry (as applied to the indicated domain).

The final acceptance of non-Euclidean hyperbolic geometry by the scientific community came with the introduction of "virtual Euclidean representations (models)" for the *complete* Lobachevsky plane and is connected with the model interpretations of the two-dimensional Lobachevsky geometry proposed by F. Klein (in 1871) and H. Poincaré (in 1882). The Cayley-Klein model (the Klein model in the disc of the Euclidean plane which uses Cayley's projective metric) and the Poincaré model in the disc and in the half-plane (in the complex plane) are discussed in detail in § 1.2.

Speaking about the coming into life of Lobachevsky's geometry, it is necessary also to mention the works of the Russian mathematician F. Minding during the years 1838–1839 (see § 1.3) in which, in particular, he described all surfaces of revolution of constant negative curvature, namely, the *pseudosphere* and the surfaces know today as the *Minding bobbin* and *Minding top*, and obtained the form of the linear element for surfaces of this type. Interestingly, Minding himself noted the validity of the *formulas of trigonometry* on surfaces of constant negative curvature, *derivable* from the corresponding trigonometric formulas in spherical geometry by replacing the trigonometric functions involved by the "analogous" hyperbolic functions. Beltrami (in 1868) referred to these results of Minding when he analyzed the pseudosphere and emphasized that the aforementioned trigonometric relations are trigonometric formulas in Lobachevsky's geometry. Unfortunately, Minding himself did not pose the problem of connecting his results with the Lobachevsky geometry that was taking shape at that time. And Lobachevsky, by irony of fate, missed those issues of the scientific journal he regularly browsed that contained Minding's works, in which the first intuitive geometrical images of the new non-Euclidean geometry arose. What an extraordinary historical occurrence!

Historians of mathematics should also devote consideration to the personality of J. C. M. Bartels[4] and his "special mission of accompanying and supporting" the creators of contemporary non-Euclidean geometry. Already at the beging of his career of mathematician and pedagogue, Bartels became the teacher of the future king of mathematics C. F. Gauss at the Katherinenschule in Braunschweig. It is due to Bartels' efforts that the young Gauss received from the Duke of Braunschweig a scholarship to continue his education. In the 12-year period of his activity that followed (starting with 1808), Bartels served as a professor at the newly established Kazan University, and according to recollections of his contemporaries, in known situations he watched over and defended his capricious student Nikolai Ivanovich Lobachevsky. Finally, from 1820 on, Bartels taught and engaged in scientific research at the Dorpat (now Tartu) University, where he founded the Centre for Differential Geometry. Afterwards, at the end of the 30th (in the XIXth century), F. Minding, a professor also at Dorpat University, obtained important results on surfaces of revolution of constant negative curvature, on which a partial realization of non-Euclidean hyperbolic geometry takes place. It is amazing that no accounts or results are available that could shed light on Bartels' own about judgements about the new non-Euclidean geometry and on discussion with his colleagues on this theme. However, there is no doubt that under the influence exerted by this prominent mathematician his students acquired a high mathematical

[4] Johann Christian Martin Bartels was a German and Russian mathematician, a corresponding member of the St. Petersburg Academy of Science.

culture. The scale of Bartels' personality is also witnessed by the fact that for his exceptional contribution to science and education he was awarded the (practically inaccessible to scientists) high Russian government title of *secret adviser*.

An outstanding achievement of human thought can become part of the overall intellectual-spiritual heritage only when it reflects, to a certain extent, the demands of the scientific and cultural society of its time. This is equally true for Lobachevsky's geometry, as a theory that expands the boundaries of the mathematical ideas and philosophy of space. Undoubtedly, a historical factor concomitant with Lobachevsky's doctrine was the intellectual society in XIXth century Russia, divided at times by contradictions in world outlook, yet constantly preserving the need for a deep understanding of the meaning of existence, this being a characteristic trait of Russian national mentality.

Overall, in Russia the situation around *Lobachevsky's doctrine* turned out to be rather positive. This is demonstrated by fact Lobachevsky had the opportunity to regularly present parts of the theory he was developing in publications of Kazan University, who he led as rector starting in 1827, and to organize and participate in public debates. Nevertheless, there were also some negative instances, such as academician M. V. Ostrogradsky's rejection of the work "*On the foundations of geometry*", submitted to the Council of Kazan University in the Academy of Sciences. Also, in 1834, F. Bulgarin's well-known literature and general politics journal "Сын отечества" ("Son of the fatherland") published an extensive (anonymous) paper that "ridiculed" in a narrow-minded manner the doctrine of the new non-Euclidean geometry, as well as Lobachevsky himself. At the same time, though, one must speak also about the begining of the penetration of the ideas of the new geometric theory in European scientific circles and, generally, about the rise of the scientific interest in problems related to the direction of research under discussion, a confirmation of which is represented, for instance, by the construction of the "Minding surfaces", and so on.

In Russia, as time went on, something bigger than just recognition as a geometric discovery was awaiting Lobachevsky's theory: the fruits of the scientific investigations of Lobachevsky found reflection in the thinking of the most prominent Russian minds of the XIXth century and became integral part of the discussions of Russian intellectuals in their endless quest for understanding the universe. Opinions on non-Euclidean geometry can be found, for example, in the philosophical polemic of the main heroes of the novel "The Brothers Karamazov" by the great Russian writer and thinker F. M. Dostoyevsky:[5] "Yet there have been and still are geometricians and philosophers, and even some of the most distinguished, who doubt whether the whole universe, or to speak more widely, the whole of being, was only created in Euclid's geometry; they even dare to dream that two parallel lines, which according to Euclid can never meet on earth, may meet somewhere in infinity."[6]

Dostoyevsky, who had a serious basic mathematical education, sensed the subtleties of the circle of problems reached by the mathematical thought at the middle of the XIXth century, composed of questions on the foundation of mathematics, abstract problems of the geometry of space, and so on.

[5] See, e.g., "The Brothers Karamazov", Farrar, Straus and Giroux, 2002.

[6] See the translation at http://fyodordostoevsky.com/etexts/the_brothers_karamazov.txt

One can also speak of the particularly pronounced predisposition of the *Russian national culture* as a whole to comprehend, in particular, the new ideas of non-Euclidean geometry, which, it seems, were laid at the *roots of its civilization*. Important "facets", it would appear, of the unusual non-Euclidean geometry were, at the contemplative level of perception, imprinted in consciousness over a period of almost a thousand year of history of the new civilization in ancient Russia (Русь), which adopted the spiritual Christian principles of Byzantium and introduced in this inheritance the truly Slavic traditions and knowledge, expressed in *special forms* (shapes, images) that are not found in any other culture. Among such *forms*, for example, are the onion-shaped cupolas (domes) that crown the tops of innumerable Russian orthodox churches. The upper part of such an onion cupola, which extends in a harmonious way its central part (a sphere), rising towards the Sky, realizes a shape that from a contemporary analytic point of view belongs to hyperbolic geometry (it is the classical shape of a surface of revolution of negative curvature). It is natural to regard this part of the cupola as a model of a part of the upper sheet of the pseudosphere[7], a canonical surface that tends towards the point at infinity on the Absolute. Such an embodiment of the cupola shape in space can be traced through the Russian orthodox tradition starting from at least the first half of the XIIth century, and signifies the trinity of intuitive geometries accessible to human perception. To wit, starting with the indicated historical period, one can speak with confidence about the appearance of artificial forms that from the contemporary point of view belong to hyperbolic geometry or, in other words, about the results of the precise practical development of elements of non-Euclidean geometry. It is particularly remarkable that all that was mentioned above took place *more than five hundred years* before the discovery by I. Newton, G. Leibniz, and others of the differential and integral calculus, which lies at the foundations of the contemporary scientific and technological paradigm.

Side by side with the aforementioned contribution of N. I. Lobachevsky to the development of a global mathematical conception, we should address also the philosophical value of Lobachevsky's geometry as a theory that influences the development of various fields of knowledge. The general philosophical value of Lobachevsky's geometry can be described as follows. First, this geometric theory had a decisive role in the formation of the analytic conception of possible intuitive geometries (side by side with the Euclidean and spherical geometries) in the Euclidean space habitual to a human being (a passive observer). Figuratively speaking, *Lobachevsky's geometry became the third, crowning crystal in the triad of intuitive geometries*. Second, the geometric theory itself became a *tool* [8](rather than an *aim*, and, the more so, not a "scientific end in itself"), promoting the development of other fields of knowledge that lie at the foundation of contemporary philosophy and practice.

The stable growth in strength of a scientific theory over a long period of history is not possible without devoted followers, prominent scholars capable of developing its fundamental ideas. The author finds his duty to mention here a pleiad of eminent Russian scientists-mathematicians, "guardians of the space of

[7] Concerning the pseudosphere see § 1.3 and § 2.4.

[8] Already Lobachevsky himself actively applied the theory he developed to the calculation of complicated definite integrals, regarding this as an additional argument in favor of its truth.

Lobachevsky's ideas", the names of which are connected with the advancement and popularization of Lobachevsky's geometric doctrine in Russia and abroad over the last, more than 150 years. Here we should mention, among others, A. V. Vasil′ev,[9] A. P. Kotel′nikov, P. A. Shirokov, B. L. Laptev, A. P. Norden, V. F. Kagan, Yu. Yu. Nut, A. S. Smogorzhevskii, N. V. Efimov, and È. G. Poznyak. Special contributions to the development of Lobachevsky's geometry and its applications are due to the scientific geometrical schools of Kazan and Moscow universities.

Structure and contents of the book

The exposition in the book begins with the consideration of the key elements lying at the foundation of Lobachevsky's geometry, including its interpretations (models), and is carried out in a form adapted to the methods of modern geometry, function theory, and the theory of nonlinear differential equations. The central part of the book is devoted to problems connected with various aspects of the realization of hyperbolic geometry in Euclidean space, the study of pseudospherical surfaces, and the elaboration of effective geometric approaches to the study of certain nonlinear partial differential equations of mathematical physics, in particular, in the context of physical applications. The main text is organized into five chapters, preceded by the Introduction.

The first chapter is devoted to the foundations of Lobachevsky's geometry, consisting of three basic "ingredients": axiomatics, model interpretations, and the analysis of surfaces of revolution of constant negative curvature. These sections are structured with a view to the subsequent applications of the results presented in actual problems of mathematical physics. We also consider examples of C^1-regular surfaces of revolution with different signs of the curvature, which realize a harmonious combination of the classical intuitive geometries.

The second chapter deals with general problems connected with the realization of the two-dimensional Lobachevsky geometry in the three-dimensional Euclidean space $\mathbb{E}^3$. Here it is natural to interpret the Lobachevsky geometry as geometry of a two-dimensional Riemannian manifold of constant negative curvature. In this connection we introduce the fundamental systems of equations of the theory of surfaces in $\mathbb{E}^3$ and discuss specific features of the application of the tools presented to the analysis of surfaces of constant negative Gaussian curvature. In this chapter we consider such canonical geometric objects as the Beltrami pseudosphere and Chebyshev nets. We also examine D. Hilbert's results on the impossibility of realizing the complete Lobachevsky plane in the space $\mathbb{E}^3$. We mention the fundamental connection between surfaces of pseudospherical type and the sine-Gordon equation, a geometrically universal nonlinear partial differential equation. We give a brief survey of a number of fundamental results on isometric immersions of Riemannian metrics of negative curvature in Euclidean spaces.

The third chapter is devoted to geometric aspects of the sine-Gordon equation. We study the geometric notion of Bäcklund transformation for pseudospher-

[9] Special mention is due to the scientific-biographical works of A. V. Vasil′ev [13], which provide a detailed exposition of the life path of N. I. Lobachevsky and an analysis of his scientific achievements.

ical surfaces. At the same time we remark that the application of the method of Bäcklund transformations for the construction of exact solutions of nonlinear differential equations is one of the most effective approaches in mathematical physics. Special attention is given to the class of soliton solutions of the sine-Gordon equation and their geometric interpretation on the example of classical surfaces—the pseudosphere and the Dini surface—as well as to the study of the classes of two-soliton and breather pseudospherical surfaces. We investigate the Painlevé transcendental functions of the third kind, which form a special class of self-similar solutions of the sine-Gordon equation, the geometric interpretation of which in $\mathbb{E}^3$ is provided by Amsler's pseudospherical surface. Further, we study fundamental solvability questions for certain classical problems of mathematical physics, namely, the Darboux problem and the Cauchy problem for the sine-Gordon equation; we then use the results obtained to derive important geometric generalizations and consequences. In particular, we show how to construct solutions of the sine-Gordon equation on multi-sheeted surfaces. Moreover, based on the unique solvability of the Cauchy problem for the sine-Gordon equation presented in this chapter we prove a theorem on the *unique determinacy* of pseudospherical surfaces (the fact that a pseudospherical surface is uniquely determined by the corresponding initial data on its irregular singularities). We discuss classical questions connected with the Joachimsthal-Enneper surfaces, indicating the connection between these surfaces and classes of solutions of the sine-Gordon equation obtained by the method of separation of variables. The final section of the chapter is devoted to the fundamental connection that exists between the method of the inverse scattering transform and the theory of pseudospherical surfaces. This connection is expressed by the fact that the basic relations that arise in these two different branches of mathematics are structurally identical. On the whole, all the essential questions considered in Chapter 3 point to the presence of a significant geometric component connected with Lobachevsky's geometry in a wide spectrum of problems of topical problems of mathematical physics.

In Chapter 4 we present a geometric approach to the interpretation of certain nonlinear partial differential equations which connects them with special coordinate nets on the Lobachevsky plane Λ^2. We introduce the notion of the Lobachevsky class of partial differential equations (Λ^2-class), equations that admit the aforementioned interpretation. The resulting geometric concepts for nonlinear equations allow one to apply in their study the well developed tools and methods of non-Euclidean hyperbolic geometry. Many well-known nonlinear equations, among them the sine-Gordon, Korteweg-de Vries, Burgers, and Liouville equations, etc., which compose the Λ^2 class, are generated by their own coordinate nets on the Lobachevsky plane Λ^2. This makes it possible to investigate these equations by net (intrinsic-geometrical) methods that rest on Lobachevsky's geometry. Overall, the chapter is devoted to laying the foundations of the geometric concept of Λ^2-equations; in it we also discuss the prospects of applying geometric methods of hyperbolic geometry to the constructive analysis of differential equations.

In Chapter 5 we consider applications of the geometric formalism proposed in Chapter 4 for nonlinear differential equations to problems of theoretical physics and the theory of difference methods for the numerical integration of differential equations.

In the first part of the chapter we introduce the notion of *non-Euclidean phase spaces*, which are nonlinear analogs (with non-zero curvature) of the phase spaces of classical mechanics and statistical physics, and of the Minkowski space of the special theory of relativity.

The concept of non-Euclidean phase spaces rests on the principle of identity between the metric of the phase space and the metric generated by the model equation that describes the physical process under investigation. Due to the non-triviality of the curvature of non-Euclidean phase spaces, they exhibit singularities, which acquire the physical meaning of attractors and determine the behavior of regular phase trajectories. Non-Euclidean phase spaces represent a kind of "curvilinear (non-Euclidean) projection screens" on which the evolution of the physical process under consideration is displayed in regular manner. This in turns leads to the establishment of general principles governing the evolution of the corresponding physical systems. By the nature of the approaches employed, the material discussed belongs first and foremost to the methodology of mathematical physics.

In the second part of the chapter, based on the elaborated methodology of discrete coordinate nets on the Lobachevsky plane, we propose a geometric algorithm for the numerical integration of Λ^2-equations. The realization of such an approach is connected exclusively with the planimetric analysis (in the framework of hyperbolic geometry) of piecewise-geodesic discrete nets in the plane Λ^2 which in the limit go over into the smooth coordinate net that generated the Λ^2-equation under study. The implementation of the method is demonstrated on the example of the sine-Gordon equation; to construct the geometric algorithm for its numerical integration, one needs to study discrete rhombic Chebyshev nest on the plane Λ^2.

In the framework of the general geometric approach, the present monograph covers a rather wide spectrum of problems, starting with problems on the foundation of geometry and ending with methods for the integration of nonlinear partial differential equations of mathematical physics and the formulation of a number of general principles governing the evolution of physical systems. In the author's view, making such a diverse material accessible to the reader was possible only by varying the level of rigorousity of the exposition so that it reflects in each individual problem considered the established traditions and methodology of study.

The book is addressed to a wide circle of specialists in various fields of mathematics, physics, and science in general.

Chapter 1

Foundations of Lobachevsky geometry: axiomatics, models, images in Euclidean space

This first chapter is devoted to an exposition of the foundations of Lobachevsky geometry, formed by three classical components: axiomatics, model interpretations, and investigation of surfaces of constant negative curvature. The discussion of these parts is carried out keeping in mind what is required for their application to problems of contemporary mathematical physics.

1.1 Introduction to axiomatics

Constructing the foundations of geometry amounts to establishing a complete and, at the same time, sufficiently simple and consistent system of axioms (statements, the truth of which is accepted without proof), and the derivation from them, as logical consequences, of the key theorems of geometry. The principal requirements for the system of axioms are *completeness, minimality* of the collection of assertions involved, and their *consistency*. In this section we present, following the universally recognized work of D. Hilbert [17], the axiomatics adopted in modern geometry.

Hilbert's axiomatics starts by introducing three different systems of "things", primary geometric objects. The things of the first system are called *points*, those of the second (*straight*) *lines*, and those of the third, *planes*. The *points* are the elements of *linear geometry*; the *points and lines* are the elements of *plane geometry*; and the *points, lines, and planes* are the elements of *space geometry*. It is assumed that the *points, lines, and planes* are in certain *relations*, which are referred to by the words *"lies"*, "*between*", "*congruent*" (equal), "*parallel*", *continuous"*, and so on. The precise meaning of the terms that express *relationships* is specified by the content of the corresponding (groups of) axioms of geometry.

Let us list the axioms of geometry, dividing them into five groups.

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I. Axioms of belonging (or of incidence) (8 axioms).

II. Axioms of order (4 axioms).

III. Axioms of congruence (equality) (5 axioms).

IV. Axioms of continuity (2 axioms).

V. Axiom of parallels.

The axioms of groups I–IV (19 axioms) are shared by Euclidean geometry as well as by Lobachevsky geometry, and constitute the axiomatics of *Absolute Geometry*. Adding to them the Axiom of Parallels results in the complete system of axioms of either Euclidean geometry, or of Lobachevsky geometry. Let us now formulate the axioms, remarking that usage in axioms of the plural for geometric objects presumes that these objects are distinct (e.g., "two points" means "two distinct points").

Axioms

I. Axioms of belonging (incidence)

1. *For any two points A and B there exists a straight line a that passes through each of the points A and B* (Figure 1.1.1).

Figure 1.1.1

2. *For any two points A and B there exists no more than one straight line that passes through both A and B* (Figure 1.1.1).

3. *On each straight line there exist at least two points. There exist at least three points that do not lie on the same straight line.*

4. *For any three points A, B, C that do not lie on the same straight line there exists a plane α that contains each of the points A, B, C. For every plane there always exists a point which it contains.*

5. *For any three points A, B, C that do not lie on one and the same straight line there exists no more than one plane that passes through each of these three points.*

6. *If two points A, B of a straight line a lie in a plane α, then every point of a lies in the plane α.*

   (In this case one says: "the straight line a lies in the plane α", or "the plane α passes through the straight line a".)

7. *If two planes α and β have a point A in common, then they have at least one more point B in common.*

8. *There exist at least four points that do not lie in one plane.*

Following Hilbert [17] and the later classical works of V. F. Kagan [38] and N. V. Efimov [25] on the foundations of geometry, in the formulation of axioms we, while taking care to preserve the correct statement of the axioms, used for the terms expressing the relations between "things" the corresponding notions that are more customary in modern mathematics. Incidentally, these mathematical "synonyms" were given already by Hilbert himself. Thus, for example in Axioms I.1, I.2 we used: "*the line a passes through the points A and B*" instead of the equivalent "in meaning" as well as admissible formulation "*the straight line a is incident to each of the points A and B*". Furthermore, for example, instead of the possible statement "*the point A lies on the straight line a*" one used "*the point A is incident to the straight line a*". Also, expressions "*the straight lines a and b intersect in the point A*" and "*the straight lines a and b have a common point*" are equivalent, and so on.

Commenting upon the eight axioms of group I, which Hilbert referred to as axioms of *incidence*, let us point out that their "diversity of meaning" is deep and is aimed at optimizing the approach by which one derives their consequences. As an example, consider the first two axioms I.1 and I2, which in the standard modern courses on mathematics are replaced by a single (more "content-loaded") axiom: "*through any two distinct points there always passes a unique straight line*". This last formulation is undoubtedly correct, but to derive further geometric consequences, one in fact does not employ its full "meaning capacity"; rather, it is only applied partially, in accordance with the content of axioms I.1 and I.2.

Based on just the axioms I.1–I.8 of the first group, one can now, for example, prove the following theorems [17,25]:

Theorem 1. *Two straight lines that lie in one and the same plane have no more than one common point. Two planes either have no point in common, or they have a common straight line, on which all the common points of the two planes lie. A plane and a straight line that does not lie on it either have no common point, or have only one common point.*

Theorem 2. *Through a straight line and a point that does not lie on that straight line, as well as through two distinct straight lines with a common point, there always passes one and only one plane.*

II. Axioms of order

1. *If the point B lies between the point A and the point C, then A, B and C are distinct points of one straight line, and the point B also lies between the point C and the point A* (Figure 1.1.2).

2. *For any two points A and C, on the straight line AC there exists at least one point B such that the point C lies between the point A and the point B* (Figure 1.1.3).

3. *Of any three points on a straight line there exists no more than one that lies between the other two.*

Figure 1.1.2

Figure 1.1.3

Definition. On a straight line a consider two points A and B; the system of two points A and B is called a *segment* and is denoted by AB or BA. The points lying between A and B are called *points of the segment* AB (or *interior* points of the segment); the points A and B themselves are called the *endpoints* of the segment AB. All the remaining points of the line a are called the *external* points of the segment AB.

4. **Pasch's Axiom.** *Let A, B, and C be three points that do not lie on a straight line, and let a be a straight line in the plane ABC that does not pass through any of the points A, B, C. If the straight line a passes through one of the points of the segment AB, then it necessarily passes also either through a point of the segment AC, or through a point of the segment BC.* (Figure 1.1.4)..

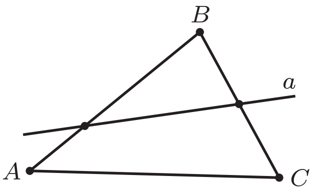

Figure 1.1.4

By their essence, the Axiom of group II define the notion of *bewteen*.

Based on this one can introduce an *order* for points on a straight line, in plane, or in space. The axioms of order were study in detail by the German mathematician M. Pasch [181].

The addition of the axioms of Group II to the axioms that we already considered allows one to obtain many important consequences [17, 25], among which we mention here the following examples:

Theorem 1. *Among any three points A, B, C lying on the same straight line there is one that lies between the two other.*

Theorem 2. *Between any two point of a straight line there exists infinitely many other points of the straight line.*

Theorem 3. *If the points C and D lie between the points A and B, then all the points of the segment CD belong to the segment AB.*

III. Axioms of congruence (equality)

The axioms of Group III will be formulated and commented upon simultaneously. This approach will allow the reader from the very beginning to follow the logic of the development of the content of the axiomatic statements of this group. The axioms in Group III define the notion of *congruence* (*equality*), and accordingly allow one to introduce the notion of *motion*.

To designate certain mutual relations that can hold between segments we use the term "*congruent*" (or "*equal*"). This kind of relation between segments is described by the axioms of Group III.

1. *If A and B are two points on the straight line a and A_1 is a point on another straight line a', then it is always possible to find a point B_1 on a given side of the straight line a' such that the segments AB and A_1B_1 are congruent* (*equal*) (Figure 1.1.5).

In particular, for the straight line a' one can also take the straight line a itself.

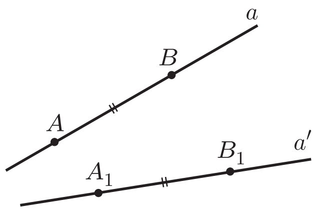

Figure 1.1.5

The congruence (equality) of the segments AB and A_1B_1 will be denoted by

$$AB = A_1B_1.$$

Axiom III.1 allows one to superpose equal segments.

We note that, according to the definition given above, a segment is given as a *system of two points A and B*, with nothing being said about the order in which they are positioned. Consequently, the following relations are equivalent in meaning:

$$AB = A_1B_1, \quad AB = B_1A_1, \quad BA = A_1B_1, \quad BA = B_1A_1.$$

2. *If both the segment A_1B_1 and the segment A_2B_2 are congruent to the segment AB, then the segments A_1B_1 and A_2B_2 are also congruent to each other.*

In other words, if two segments are congruent to a third segment, then they are congruent to one another.

3. *Let AB and BC be two segments on a straight line a that have no common interior points, and let A_1B_1 and B_1C_1 be two segments on a straight line a' that also have no common interior points. If*

$$AB = A_1B_1, \quad BC = B_1C_1,$$

then

$$AC = A_1C_1.$$

Axiom III.3 allows one to *add segments*.

To formulate the next two axioms of Group II we need to introduce the notion of an angle.

Definition A pair of half-lines (a system of two rays) ℓ and k that originate at one and the same point O and do not belong to one straight line is called an *angle*, and is denoted by $\angle(\ell, k)$ (Figure 1.1.6).

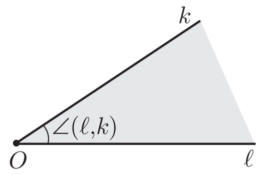

Figure 1.1.6

By *half-line* (or *ray*) with the origin at the point O one means the set of all points on a straight line that lie on the same side with respect to the point O. The rays ℓ and k are called the *sides* of the angle $\angle(\ell, k)$.

Angles can find themselves in a certain relation, termed "congruence" or equality, which is "governed" by Axioms III.4 and III.5.

4. *Suppose that on the plane α there is given an angle $\angle(\ell, k)$ and there is given a straight line a' in the same plane or some other plane α', and also that a side of the plane α' with respect to the straight line a' is chosen. Let ℓ' be a ray of the straight line a' starting from a point O'. Then in the plane α' there exists one and only one ray k' such that the angle $\angle(\ell', k')$ is congruent to the angle $\angle(\ell, k)$, and at the same time all interior points of the angle $\angle(\ell', k')$ lie on the chosen side with respect to the straight line a'.*

The congruence (equality) of angles is denoted by

$$\angle(\ell, k) = \angle(\ell', k').$$

Each angle is congruent to itself:

$$\angle(\ell, k) = \angle(k, \ell).$$

Axiom III.4 alows one to lay out angles: each angle can be placed, in a unique way, in a given plane, on a given side with respect to a given ray.

Before we formulate Axiom III.5 (the final axiom of Group III), let us clarify the notion of a triangle. By a *triangle* $\triangle ABC$ we mean a system of three segments, AB, BC, CA, which are called the *sides* of the triangle; the points A, B, C are called the *vertices* of the triangle.

5. *If for two triangles, $\triangle ABC$ and $\triangle A_1B_1C_1$, it holds that*

$$AB = A_1B_1, \quad AC = A_1C_1, \quad \angle BAC = \angle B_1A_1C_1,$$

then there also holds the equality (*congruence*)

$$\angle ABC = \angle A_1B_1C_1.$$

Remark. $\angle ABC$ denotes the angle with vertex B, on one side of which lies the point A, and on the other, the point C.

The first three axioms III.1–III.3 are *linear* axioms, because they concern only congruence of segments. Axiom III.4 defines the congruence of angles. Axiom III.5 connects congruence of segments as well as of angles. The last two axioms of Group III may be referred to as plane axioms, since they are assertions on geometric "objects" in the plane.

Using the axioms of Group III one introduces in geometry the notion of *motion*, as follows.

Consider two sets, Σ and Σ', between the points of which there is a one-to-one correspondence. (By set we mean a finite or infinite collection of points.) Any two points $A, B \in \Sigma$ define a segment AB, and the points $A', B' \in \Sigma'$ corresponding to them give a segment $A'B'$; we will say that the segments AB and $A'B'$ *correspond* to one another. If under the given one-to-one correspondence between Σ and Σ' any two corresponding segments are equal (congruent), then the *sets* Σ *and* Σ' and also said to be *equal* (congruent) In this case one say that the set Σ' is obtained by a *motion* of the set Σ, and conversely, Σ is obtained by a motion of Σ'.

The completion of the axiomatic system discussed by the axioms of congruence (Group III) makes it possible to obtain new wide classes of geometric consequences, which are considered in detail in, e.g., [25, 38].

IV. Axioms of continuity

1. **Archimedes' Axiom.** *Let AB and CD be two arbitrary segments; then on the straight line AB there exist a finite number of successively arranged points $A_1, A_2, A_3, \ldots, A_n$ such that the segments $AA_1, A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ are congruent to the segment CD and the point B lies between the points A and A_n* (Figure 1.1.7).

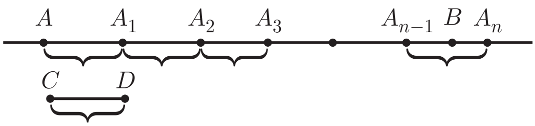

Figure 1.1.7

Axiom IV.1 is also called the *axiom of measure*. According to its meaning, the segment CD is a standard-of-length segment, a measurement *unit*, and the axiom asserts that it is possible "reach" any given point on a straight line and calculate the length of any segment.

2. **Cantor's Axiom.** *Suppose that on some straight line a there is an infinite system of segments $A_1B_1, A_2B_2, \ldots, A_nB_n, \ldots$, in which each successive segment is contained inside the preceding segment* (Figure 1.1.8). *Suppose that there is no segment that is contained inside all the segments of the given infinite system of segments. Then on the line a there exists a unique point M that lies inside all the segments $A_1B_1, A_2B_2, \ldots, A_nB_n, \ldots$ of the considered system.*

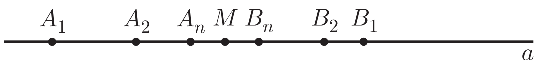

Figure 1.1.8

Axiom IV.2 given above expresses the well known Cantor principle of nested segments; it is precisely this formulation of the second axiom of Group IV that is used in the Efimov's system of axioms of geometry [25]. In Hilbert's work [17], the second axiom of Group IV is formulated as the axiom of *linear completeness* (this axiom is denoted below by IV.2*). Let us remark that the possible alternative usage of axioms IV.2 and IV.2* does not destroy the general consistent integrity of the adopted axiomatics; rather, it simply corresponds to an equivalent exposition of part IV.

2*. **Axiom of linear completeness** (according to Hilbert). *The points of a straight line form a system that does not admit any extension in which the linear order, the first axiom of congruence, and the Archimedes' axiom remain valid. That is, to such a system of points one cannot add points such that in the new system formed by the initial and the added points all axioms listed above will be valid.*

The axioms of the groups I through IV constitute the foundation of the so-called *Absolute Geometry*, a common component of Euclid's classical geometry and of the non-Euclidean hyperbolic geometry discovered by N. I. Lobachevsky. What makes these two geometries fundamentally different is the different content of the Axiom of Parallels. The Axiom of Parallels, the only axiom of Group V, has a meaning that is independent of the axioms of the first four groups.

V. Axiom of Parallels

To formulate of the axiom of group V, we first define the parallelism of straight lines.

Definition. Two straight lines that lie in the same same plane and have no point in common are said to be *parallel*.

The Axiom of Parallels that corresponds to the classical Euclidean geometry and expressed the famous Postulate V of Euclid reads as follows:

V. **Axiom of Parallels (Vth Postulate of Euclid).** *Let a be a straight line and A be a point that does not lie on the straight line a. Then in the plane determined by the straight line a and the point A there is a unique straight line that passes through A and does not intersect a* (Figure 1.1.9).

The Axiom of Parallels just formulated, together with the already introduced axioms of groups I–IV, form the well-known analytic geometry of Euclid in the Cartesian plane.

The revolutionary result of N. I. Lobachevsky's study of the foundations of geometry is that the Axiom of Parallels (the Group V axiom) is independent of

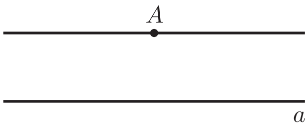

Figure 1.1.9

the axioms of the preceding four groups (I–IV). In other words, if to the axioms of the groups I–IV (the axioms of Absolute Geometry) one "adjoins", instead of Euclid's Postulate V, Lobachevsky's Axiom of Parallels (an assertion that is not compatible with the "Euclidean formulation"), then the resulting system of axioms leads to a new, logically correct notion of geometry, that of non-Euclidean hyperbolic geometry, or Lobachevsky geometry.

V*. **Axiom of Parallels (according to Lobachevsky).** *Let a be an arbitrary straight line and A be a point that does not lie on a. Then in the plane determined by the straight line a and the point A, there are at least two straight lines that pass through A and do not intersect the straight line a* (Figure 1.1.10).

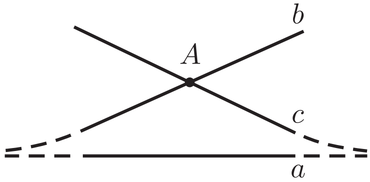

Figure 1.1.10

The content of Axiom V* suffices (if one considers the axioms of absolute geometry) for establishing that through the point A there pass infinitely many straight lines that do not intersect the straight line a.

We should emphasize that Figure 1.1.0 has an exclusively symbolic (conventional) character, illustrating, to the extent that one can possibly grasp by means of customary Euclidean images, the content of the axiom that lies at the foundation of non-Euclidean geometry. Axiom V* should be first and foremost regarded as a logical statement that establishes a special "relationship" between geometric objects ("things", in Hilbert's formulation). This is precisely the conclusion to which Lobachevsky arrived, proving that the performed modification of the Axiom of Parallels resulted in the inauguration of a new *geometric system*. Initially the acceptance of Lobachevsky's ideas did run into known difficulties, the roots of which lie most probably in the customary practical representations about the notions and terms used. It is therefore appropriate to state clearly here that in Lobachevsky's geometry a "straight line" is understood as a shortest line, i.e., a line such that the distance along it between two of its points is the smallest. The notion of "parallelism" of two "straight lines" refers only to the fact that they do not intersect, and does not incorporate the customary Euclidean property of two parallel straight lines being equidistant.

The first wide recognition of Lobachevsky's geometry is associated with the construction of its *models* which use images in Euclidean geometry, and which will be considered in § 1.2.

To complete the exposition of axiomatics, let us emphasize one more time that the axioms of Absolute Geometry (the groups I–IV), supplemented by Euclid's Postulate V, constitute the foundation classical Euclidean geometry, while in the case when they are alternatively supplemented by Lobachevsky's Axiom of Parallels they constitute the foundations of the non-Euclidean hyperbolic geometry.

1.2 Model interpretations of Lobachevsky's planimetry

The firm establishment of Lobachevsky's geometry as a geometric theory was made possible by the construction of intuitive interpretations of it in the framework of ordinary Euclidean geometry. To obtain a conventional "Euclidean picture" of Lobachevsky geometry we will consider in the Euclidean plane certain domains, in which Euclidean geometric objects (elements) and the operations performed on them are endowed with the *virtual* meaning of the elements and operations corresponding to them in the Lobachevsky planimetry. That is, to the "things" and the "relations" that connect them, which obey the axioms of Absolute Geometry (groups I–IV) and Lobachevsky's Axiom of Parallels (V) we associate intuitive Euclidean images, but endowed with a special meaning. Such an approach allows one to obtain a conventional Euclidean model representation of the Lobachevsky plane, and at the same time establish in a clear manner the correctness of the adopted system of axioms.

We will consider three classical interpretations (models) of the Lobachevsky plane: the Cayley-Klein interpretation, the Poincaré interpretation in the disc, and the Poincaré interpretation in the half-plane.[1] In the ensuing exposition the Lobachevsky plane will be also referred to briefly as the *plane* Λ^2.

1.2.1 The Cayley-Klein model

In 1871 F. Klein proposed a model of the Lobachevsky planimetry that uses the projective metric discovered previously by A. Cayley. This model, on the exposition of which we embark now, is called the *Cayley-Klein model* of the Lobachevsky geometry.

In the Euclidean plane $\mathbb{E}^2(x,y)$ consider the unit disc Ω, the interior of which we will, following Klein [44, 45] interpret as the Lobachevsky plane Λ^2:

$$\Lambda^2 = \{\Omega(x,y)\colon\ x^2+y^2<1\}.$$

According to this interpretation, each interior point of the disc Ω is an interior point of the plane Λ^2. The boundary ω of the disc Ω (the unit circle $\omega\colon\ x^2+y^2=1$) represents the set of points at infinity of the plane Λ^2, and is called the *absolute*.

[1] Together with our exposition, for a first acquaintance with the foundations of Lobachevsky's planimetry we refer the reader also to the graphic and content-rich introduction to this theme found in the books [54] and [129].

The role of the straight lines in the interpretation under consideration is played by the chords of the circle ω. The Euclidean images that in such an interpretation correspond to the "things" in the plane Λ^2 (points, straight lines, etc.), as depicted in Figure 1.2.1, obey the axioms of Absolute Geometry (the axioms of the groups I–IV), as well as Lobachevsky's Axioms of Parallels. Indeed, as Figure 1.2.1 makes clear, through a point M that does not lie on a given straight line ('chord") m there pass at least two straight lines b and c that are parallel to (do not intersect) m. This expresses the realization of Lobachevsky's Axiom of Parallels (Group V) in the model at hand.

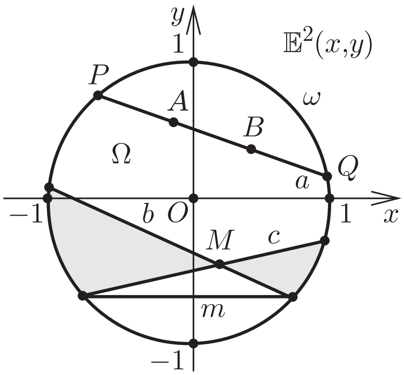

Figure 1.2.1

Note that the common points on the absolute of the chords b and m and of the chords c and m, respectively, are points at infinity, and must be interpreted not as points of intersection of the straight lines connected with them (the straight lines b and m, or the straight lines c and m) in the plane Λ^2, but as points indicating the common direction along which these straight lines tend to infinity. Hypothetically, the common points of chords on the absolute can also be considered as points at infinity, in (to) which parallel straight lines "intersect" (converge). Historically, a similar conception took shape in Leonardo da Vinci's genial work aimed at developing the doctrine of *perspective* in art. Subsequently, a similar approach was applied to lay down the bases of *projective geometry* by its founder G. Desargues.

In fact, on the plane Λ^2, through the point M there pass infinitely many straight lines parallel to m; all these straight lines lie in the "shaded" region in Figure 1.2.1 enclosed between the straight lines b and c. Overall, three different variants of mutual relationship are possible between straight lines in the plane Λ^2: 1) intersecting (for example, b and c); 2) parallel (b and c are parallel to m); 3) divergent.

In the Cayley-Klein model, to introduce a rule for measuring the distance between points in the plane Λ^2 one resorts to one of the key invariant relations of projective geometry, the *cross ratio of four points*[2] (henceforth referred to for brevity as *cross ratio*).

[2]This relation is also called *double ratio* or *anharmonic ratio*.

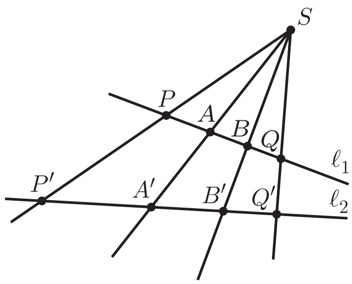

Figure 1.2.2

The cross ratio of four points that lie on the same straight line is defined as the fraction built from the ratios of the distances from each of the points of one pair to the points of the other pair. In general, one can form six similar ratios; they are all related to one another and each of them has the property that its value is invariant under projective transformations.

Let us explain the meaning of this last property. Consider two arbitrary straight lines, ℓ_1 and ℓ_2, and a point S (Figure 1.2.2), from which we draw rays that project segments of ℓ_1 onto the corresponding segments on ℓ_2.

Given four points P, A, B, Q lying on the straight line ℓ_1, we define their *cross ratio* by

$$[P, A, B, Q] \equiv \frac{BP}{BA} : \frac{QP}{QA}. \tag{1.2.1}$$

Under the central projection from the vertex S (Figure 1.2.2), the points P, A, B, Q of the straight line ℓ_1 are mapped into the points P', A', B', Q' of the straight line ℓ_2. The value of the cross ratio of the four points P, A, B, Q is also preserved for their projections, the points P', A', B', Q' (regardless on the choice of the second straight line ℓ_2):

$$[P, A, B, Q] = [P', A', B', Q']. \tag{1.2.2}$$

The proof of the invariance property (1.2.2) of the cross ration is readily carried out by computing the areas of the corresponding triangles with common vertex S (Figure 1.2.2). To this end, we need to express the lengths of the segments on ℓ_1 and ℓ_2 through the sines of the angles adjacent to the vertex S, i.e., through invariable quantities that are not connected with the choice of new straight lines that play the role of ℓ_2.

The invariance of the cross product plays a key role when one introduced the rule for measuring the length in the plane Λ^2.

Expression (1.2.1) involves the ratios of the distances from each of the points of the second pair $\{B, Q\}$ to the points of the first pair $\{P, A\}$. We note that, in general, in order to give the cross ratio via (1.2.1) the order in which the four points are arranged on the projective line (the straight line ℓ_1) is not important; what is essential is the order of the placement in the symbolic bracket $[P, A, B, Q]$. In other words, what matters is the placement of the four points in pairs inside the bracket itself, into the first and the second pair. The invariance property of the value of the cross ratio defined by formula (1.2.1) is preserved for each chosen variant of placement of the points in the symbolic bracket of the type $[P, A, B, Q]$. Moreover, the following useful rules for inner permutation in the bracket hold:

$$\begin{gathered}[P, A, B, Q] = [A, P, B, Q]^{-1} = [P, A, Q, B]^{-1}, \\ [P, A, B, Q] = 1 + [P, B, A, Q], \\ [P, A, B, Q] = [B, Q, P, A].\end{gathered} \tag{1.2.3}$$

To verify the rules (1.2.3), we must introduce on the straight line ℓ_1 (the projective line) some system of coordinates (a system of projective coordinates) and denote the coordinates of the points P, A, B, Q in question by t_1, t_2, t_3, t_4, respectively. Then the cross ratio can be written as

$$[\overset{1}{P}, \overset{2}{A}, \overset{3}{B}, \overset{4}{Q}] = \frac{t_3 - t_1}{t_3 - t_2} : \frac{t_4 - t_1}{t_4 - t_2}. \tag{1.2.4}$$

Applying formula (1.2.4) in the left- and right-hand sides of relations (1.2.3) establishes their validity.

The quantity $[P, A, B, Q]$, defined via (1.2.4), is preserved by linear-fractional transformations of the straight line (the transformations that preserve the value of the cross ratio of four points are called *projective* transformations).

Indeed, the linear-fractional transformation

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \alpha\delta - \beta\gamma \neq 0, \quad \alpha, \beta, \gamma, \delta = \text{const} \tag{1.2.5}$$

maps any point on a projective line with original coordinate t into the point with coordinate t':

$$t_1 \longrightarrow t_1', \quad t_2 \longrightarrow t_2', \quad t_3 \longrightarrow t_3', \quad t_4 \longrightarrow t_4'.$$

Therefore, the considered 4-tuple of points P, A, B, Q is mapped by transformation (1.2.5) into the new 4-tuple of points P', A', B', Q', with corresponding coordinates

$$t_1' = \frac{\alpha t_1 + \beta}{\gamma t_1 + \delta}; \quad t_2' = \frac{\alpha t_2 + \beta}{\gamma t_2 + \delta}; \quad t_3' = \frac{\alpha t_3 + \beta}{\gamma t_3 + \delta}; \quad t_4' = \frac{\alpha t_4 + \beta}{\gamma t_4 + \delta}. \tag{1.2.6}$$

If we now write the cross ratio (1.2.4) for the points P', A', B', Q', then using (1.2.6) it is not hard to verify that

$$[P', A', B', Q'] = [P, A, B, Q].$$

Thus, a linear-fractional transformation of the straight line preserves the value of the cross ratio of four points, and hence is *projective*.

In the Cayley-Klein model under discussion, the non-Euclidean distance $\rho(A, B)$ between two arbitrary points $A, B \in \Lambda^2$ of the Lobachevsky plane (Figure 1.2.1) is given in terms of the cross ratio of four points, by means of the formula

$$\rho(A, B) = \frac{1}{2} \left| \ln[A, B, P, Q] \right|, \tag{1.2.7}$$

where

$$[A, B, P, Q] = \frac{PA}{PB} : \frac{QA}{QB}. \tag{1.2.8}$$

In (1.2.7), the inner points A and B of the plane Λ^2 (Figure 1.2.1) form the first pair in the symbolic bracket, while the points at infinity P and Q of the straight line $AB \subset \Lambda^2$, which belong to the absolute ω, are placed, by definition, in the second pair. That is, in the calculation of the distance $\rho(A, B)$, in (1.2.8) one uses a somewhat different (not successive, like on the line) order of the points in the symbolic bracket compared to (1.2.1). It is clear that such a "correction" does not change the general properties of the cross ratio presented above.

The lengths of the segments PA, PB, QA, QB used in (1.2.8) are calculated in the ordinary, Euclidean way (these segments are parts of chords of the unit circle ω in the Euclidean plane $\mathbb{E}^2(x, y)$ centered at the origin of coordinates). The expression (1.2.8) in the argument of the logarithm appearing in (1.2.7) is obviously positive, but can take values smaller than 1; this explains the presence of the modulus in (1.2.7).

From (1.2.7) and (1.2.8) one derives directly the properties of the distance:

$$\begin{gathered} \rho(A, A) = 0, \\ \rho(A, B) = \rho(B, A), \\ \rho(A, B) \to \infty \quad \text{for} \quad B \to Q, \quad Q \in \omega, \end{gathered} \tag{1.2.9}$$

as well as the triangle inequality

$$\rho(A, B) + \rho(B, C) \geq \rho(A, C), \quad A, B, C \in \Lambda^2. \tag{1.2.10}$$

In (1.2.10) equality holds if the points A, B, C lie on the same straight line in the plane Λ^2 (the point B lies between A and C). The strict inequality in (1.2.10) holds for three points that do not lie on the same straight line.

The rule for measuring distance (1.2.7) introduced above yields a clear algorithm for calculating the distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the Cayley-Klein model (Figure 1.2.1). Indeed, this algorithm has the standard "Euclidean" character of a typical problem in analytic geometry and consists of the following steps: (1) derive the equation of the straight line a that passes through the two given points $A(x_1, y_1)$ and $B(x_2, y_2)$; (2) find the coordinates of the points P and Q where the straight line a and the absolute (the unit circle ω) intersect; (3) calculate the lengths of the segments figuring in the ratio (1.2.8); (4) apply the formula (1.2.7) to calculate the sought-for non-Euclidean distance $\rho(A, B)$ in the Lobachevsky plane.

If one implements the indicated algorithm in order to calculate the distance between two close points $A(x, y)$ and $B(x + \Delta x, y + \Delta y)$, then in the limit as $\Delta x \to 0$, $\Delta y \to 0$, the expression for the square of the distance $\rho(A(x, y), B(x + \Delta x, y + \Delta y))$ yields the form of the projective metric of the Lobachevsky plane:

$$ds^2 = \frac{(1 - y^2)dx^2 + 2xy\, dx\, dy + (1 - x^2)dy^2}{(1 - (x^2 + y^2))^2} \tag{1.2.11}$$

(*in the Cayley-Klein model*).

Essentially, the representation of the metric (1.2.11) is the realization in Cartesian coordinates of the general metric rule (1.2.7).

In Subsection 1.2.2 below, when we consider the Poincaré disc model, we will allow ourselves to pass, with all the details, from the original formula (1.2.7), which arises from projective geometry, to the metric of the Lobachevsky plane corresponding to the model chosen, related to the metric (1.2.11) given above. Here we merely remark that the metric form (1.2.11) is obtained by implementing the algorithm described above, by expanding the right-hand side of (1.2.7) in a Taylor series in the small values Δx and Δy, and finally passing to the limit to get dx and dy.

It is worth mentioning that the explicit form (1.2.11) of the Lobachevsky's plane metric allows one to calculate its curvature, which yields $K \equiv -1$. That is to say, the Lobachevsky plane is characterized by having a constant negative curvature at all its points. This general property of the curvature for the Lobachevsky plane Λ^2 plays a key role in the study of the Lobachevsky geometry as the geometry of a two-dimensional smooth Riemannian manifold of negative curvature, in particular, in the search for realizations of certain parts of the plane Λ^2 as surfaces of Gaussian curvature $K \equiv -1$ in the three-dimensional Euclidean space $\mathbb{E}^3$. The new geometric concepts introduced here will be considered in detail in Chapter 2.

We devote the final part of this subsection to the consideration (outside of the framework of the Cayley-Klein model) of such fundamental concepts of Lobachevsky geometry as the *angle of parallelism* and the *Lobachevsky function*. To this aim, as before, we consider in the Lobachevsky plane some straight line m and a point M not on it, as well as two straight lines b and c that pass through M parallel to m. We depict these straight lines conventionally in the plane Λ^2 (with no reference to any specific model) (Figure 1.2.3).

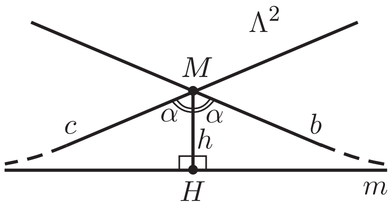

Figure 1.2.3

Draw through the point M a straight line that is perpendicular to the straight line m and intersect it at the point H (Figure 1.2.3). In other words, drop from the point M the perpendicular MH to m and denote it by $h \equiv MH$. Consider the angles that arise in this way. Recall that in our exposition the notion of angle appeared first when we introduced the axioms of Group III (Subsection 1.1.1). Note also that in the models of the Lobachevsky plane used (in particular, in the Cayley-Klein model), the "model Euclidean" angles do not necessarily coincide with the corresponding angles of the Lobachevsky planimetry. In Figure 1.2.3 equal angles and right angles are marked in the sense of the geometry of the plane Λ^2.

Suppose the straight lines b and c, with $b \parallel m$, $c \parallel m$, make at the point M equal angles with the perpendicular MH to m. The *angle of parallelism* α for the point M and the straight line m (Figure 1.2.3) is defined to be the smallest acute angle that the straight line b, as well as the straight line c (which are parallel to m), can make with the perpendicular MH. The angle of parallelism α is determined by the *distance* from the point M to the straight line m.

Let us prove that the distance MH from M to m determines the value of the angle of parallelism α, which decreases as the point M moves farther away from the straight line m.

To show this, together with m and M (Figure 1.2.4 *a*) we consider some other straight line m_1 and another point M_1, which lies at the same distance from m as the point M, i.e., $M_1H_1 = MH$ (Figure 1.2.4 *b*). We note, though, that Figure 1.2.4 *b*, like figures 1.1.10 and 1.2.3, is just a conventional symbolic way of depicting the properties of parallel straight lines.

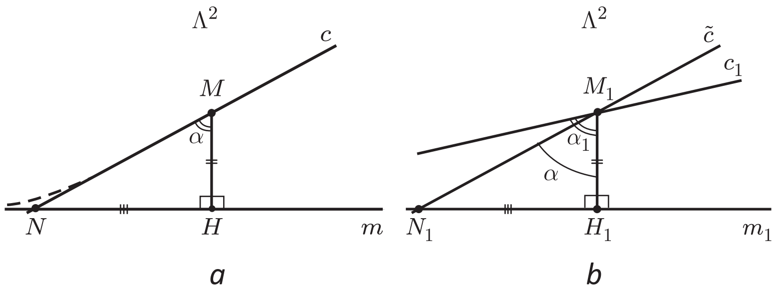

Figure 1.2.4

Let c be a line that passes through the point M and is parallel to the straight line m (Figure 1.2.4 *a*); also, let c_1 be a line that passes through the point M_1 and is parallel to the straight line m_1 (Figure 1.2.4 *b*). Denote by α and α_1 the angle of parallelism corresponding to the points M and M_1, respectively.

Suppose that the assertion we want to prove is not true, i.e., the angle of parallelism does not depend on the distance from the given point to the straight line. In other words, assume that to equal distances MH and M_1H_1 from M and M_1 to m and m_1, respectively, correspond different values of the angle of parallelism α and α_1. For definiteness, suppose that $\alpha < \alpha_1$.

Draw through the point M_1 a straight line $\widetilde{c}$, which makes an angle equal to α with the perpendicular M_1H_1 (Figure 1.2.4 *b*). Since, by assumption, the angle α is smaller than the angle α_1, $\widetilde{c}$ intersects m_1 at some point N_1. Then on m one can indicate a point N such that $NH = N_1H_1$. This yields two equal triangles:

$$\triangle MHN = \triangle M_1H_1N_1$$
$$\left(MH = M_1H_1, \quad NH = N_1H_1, \quad \angle MHN = \angle M_1H_1N_1 = \frac{\pi}{2}\right).$$

Consequently, $\angle NMH = \alpha$, and so the straight line c (parallel to the straight line m) and the straight line MN (which intersects the straight line m) must coincide, and we arrived at a contradiction. Thus, we proved that the angle of parallelism at an arbitrary point M with respect to a given straight line m is uniquely determined by the distance from M to m.

With the notation $x \equiv MH$ (i.e., by introducing a coordinate on MH), we can now write

$$\alpha = \Pi(x). \tag{1.2.12}$$

The function $\Pi(x)$ is called the *Lobachevsky function.* It describes the dependence of the *angle of parallelism* at a given point in the Lobachevsky plane Λ^2 on the distance x from this point to some given straight line. The Lobachevsky function (1.2.12) plays a fundamental role in non-Euclidean hyperbolic geometry.

Intuitive arguments similar to those used above (see, e.g., [25]) allow one to establish that the function $\Pi(x)$ is monotonically decreasing. In fact, the behavior of the Lobachevsky function can be further specified as follows:

$\Pi(x)$ *is defined for* $x > 0$, *is continuous, monotonically decreasing, and such that*

$$\begin{aligned} &\Pi(x) \to \frac{\pi}{2} \quad \text{as} \quad x \to 0, \\ &\Pi(x) \to 0 \quad \text{as} \quad x \to \infty. \end{aligned} \tag{1.2.13}$$

From (1.2.13) it follows that in "small domains" of the plane Lobachevsky's geometry is close to the Euclidean geometry, since indeed in such domains the angle of parallelism will be close to $\frac{\pi}{2}$ (the natural Euclidean value). On the whole, relation (1.2.12) itself expresses the dependence between angles and linear quantities in the Lobachevsky geometry.

Let us discuss a number of important consequences that result from considering the Lobachevsky function:

1) In *Lobachevsky geometry* the angular and linear quantities are mutually related (see formula (1.2.12)). In particular, in a triangle on the plane Λ^2 the angles and the sides determine one another: the angles give the lengths of the sides of the triangle, and vice versa. Consequently, in the Lobachevsky geometry triangles with equal corresponding angles are equal.

2) In *Euclidean geometry* there are absolute constant angle variables—quantities that can be recovered by geometric constructions that rely solely on the axioms. For example, "ruler-and-compass" constructions allow one, in an invariant manner, to reconstruct *right* angles at any point of the Euclidean plane. As for linear dimensions, in Euclidean geometry they can be copied by employing only a chosen scale (a standard segment).

In *Lobachevsky geometry*, side by side with absolute angular constants, there exist also absolute linear constants; this characteristic property of non-Euclidean hyperbolic geometry is expressed by relation (1.2.12). As an example of absolute linear dimension one can give the length of a segment d:

$$\Pi(d) = \frac{\pi}{4},$$

which is connected with the angle magnitude $\pi/4$ by means of the Lobachevsky function according to (1.2.12). In other words, in the Lobachevsky plane the absolute linear quantity d is connected in a one-to-one manner with the absolute angular quantity $\pi/4$.

The Lobachevsky function can be expressed in terms of elementary functions as (see [25])

$$\Pi(x) = 2\arctan\left(e^{-x/R}\right), \tag{1.2.14}$$

where $R = \text{const}$ is the radius of curvature of the space. Henceforth, with no loss of generality, we will put $R = 1$, which corresponds to the curvature value $K \equiv -1$ of the plane Λ^2.

The representation (1.2.14) for the Lobachevsky function $\Pi(x)$ opens the possibility of studying the Lobachevsky geometry analytically.

1.2.2 The Poincaré disc model of the Lobachevsky plane

In 1882 H. Poincaré, while developing a branch of the theory of functions of a complex variable dealing with *automorphic functions*, proposed an interpretation of the Lobachevsky plane in the disc. Automorphic functions are functions that are invariant under some group of linear-fractional transformations of the argument [19]. The interpretation proposed by Poincaré rests upon the fact that transformations of automorphic functions coincide with the transformations in the non-Euclidean hyperbolic geometry.

Following Poincaré, consider in the *complex plane* $\mathbb{C}$ the unit disc $\Omega = \{z = x + iy \in \mathbb{C},\ |z| \le 1\}$, the interior of which will be interpreted as the hyperbolic plane, and the boundary of which $\omega\colon\ x^2 + y^2 = 1$, i.e., the *absolute*, will represent the points at infinity on Λ^2. In Poincaré's interpretation the role of non-Euclidean straight lines is played by arcs of Euclidean circles that lean orthogonally on the absolute ω, as well as by the diameters of the disc (Figure 1.2.5). The angles between the indicated non-Euclidean straight lines are the usual Euclidean angle formed by the circles when they intersect. In what follows, the Poincaré disc model of the Lobachevsky plane will be denoted by $\Lambda^2(\Omega)$.

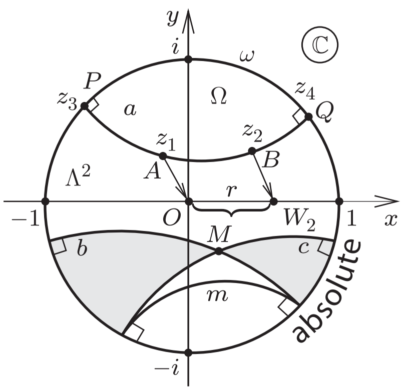

Figure 1.2.5

From the interpretation described (Figure 1.2.5) it is clear that through any two points $A, B \in \Lambda^2$ there always passes a unique straight line a, which has corresponding points at infinity $P, Q \in \omega$ on the absolute. In the present model Lobachevsky's Axiom of Parallels has a clear meaning (see Figure 1.2.5): through each point $M \in \Lambda^2$ that does not lie on a given straight line $m \subset \Lambda^2$ one can always draw at least two straight lines, b and c, that are parallel to m. Moreover, the other axioms, those of the groups I–IV (i.e., the axioms of Absolute Geometry) can also be illustrated in a sufficiently intuitively manner. As one can see in Figures 1.2.1 and 1.2.5, between the Cayley-Klein model and the Poincaré model there are certain analogies. However, it is important to mention here one essential difference: in contrast to the Cayley-Klein model considered earlier, the Poincaré model is *conformal*: in it the angles between non-Euclidean straight lines on the hyperbolic plane Λ^2 are identical to the angles formed by their Euclidean counterparts (arcs of circles orthogonal to the absolute) in the model itself; hence, these angles are preserved by conformal transformations in the disc.

The same way as in the preceding subsection, we introduce the non-Euclidean distance between two points in Λ^2 by means of the cross ratio of four points, via the formula

$$\rho(A, B) = \frac{1}{2}\big|\ln[z_1, z_2, z_3, z_4]\big| = \frac{1}{2}\left|\ln \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}\right|, \tag{1.2.15}$$

where z_1, z_2, z_3, z_4 are the complex numbers corresponding to the points A, B, P, Q, respectively (Figure 1.2.5)

It is readily verified that the distance ρ defined in (1.2.15) enjoys the properties (1.2.9) and (1.2.10).

According to Poincaré's interpretation, each point in the Lobachevsky plane Λ^2 is identified with the corresponding complex number, hence in the notation of the distance we will also use complex numbers:

$$\rho(A, B) \equiv \rho(z_1, z_2).$$

Let us calculate the distance $\rho(z_1, z_2)$. To this end we apply the standard linear-fractional transformation [105]

$$W(z) = e^{i\alpha} \cdot \frac{z - z_1}{\overline{z}_1 z - 1}, \tag{1.2.16}$$

(where i in the imaginary unit), which maps the disc Ω into itself, the point z_1 into the center O of the disc: $W(z_1) = O$, and the point z_2 into some point $W_2 = W(z_2)$ with $|W_2| < 1$. Moreover, $W(z_3) = -1$, $W(z_4) = 1$ (Figure 1.2.5).

The linear-fractional transformation, which in our treatment realizes a non-Euclidean translation on Λ^2, preserves the non-Euclidean straight lines, distances, and angles, and consequently one has

$$\rho(z_1, z_2) = \rho(O, W_2).$$

By suitably choosing the rotation angle α in (1.2.16) one can always ensure that the straight line OW_2 (the image of the straight line AB under the mapping (1.2.16)) coincides with the real axis x of the complex plane (see Figure 1.2.5). Denoting the length of the segment by $|OW_2| = r$, we use formula (1.2.15) to calculate the non-Euclidean distance $\rho(O, W_2)$ by means of the cross ratio of the points $\{-1, 0, r, 1\}$, which lie on the diameter of the disc Ω:

$$\rho(O, W_2) = \frac{1}{2}\big|\ln[0, r, -1, 1]\big| = \frac{1}{2}\ln\frac{1+r}{1-r}. \tag{1.2.17}$$

The value r in (1.2.17) is defined by

$$r = |W_2| = \left|\frac{z_2 - z_1}{\overline{z}_1 z_2 - 1}\right|. \tag{1.2.18}$$

Therefore, the non-Euclidean distance between two points A and B of the Lobachevsky plane Λ^2, which in the Poincaré model correspond to the complex numbers z_1 and z_2, is calculated by the formula

$$\rho(A, B) = \frac{1}{2}\ln\frac{1 + \left|\dfrac{z_2 - z_1}{\overline{z}_1 z_2 - 1}\right|}{1 - \left|\dfrac{z_2 - z_1}{\overline{z}_1 z_2 - 1}\right|}. \tag{1.2.19}$$

The formulas (1.2.17)–(1.2.19) obtained above allow one to address now the problem of calculating the length of a non-Euclidean curve on Λ^2 and the derivation of the explicit form of the metric of the Lobachevsky plane in the case of the Poincaré disc model.

As in Euclidean geometry, in Lobachevsky geometry the length of a curve is defined by approximating the curve by the length of a broken line inscribed in it, and subsequently passing to the limit by letting the length of the typical elementary segment in the broken line tend to zero.

Consider a curve $\ell \subset \Lambda^2$ and partition it in some way into elementary segments. Choose on ℓ two successive points z and $z + \Delta z$ of this partition. Now use (2.17)–(1.2.19) to compute the non-Euclidean distance $\Delta s \equiv \rho(z, z+\Delta z)$ between these points:

$$\Delta s = \frac{1}{2}\ln\frac{1+r^*}{1-r^*}, \quad \text{where} \quad r^* = \frac{|\Delta z|}{|1 - (z + \Delta z)\cdot\overline{z}|}. \tag{1.2.20}$$

Assuming that Δz is sufficiently small, we neglect it in the denominator of the expression (1.2.20) for r^*. Then since $z \cdot \overline{z} = |z|^2$, we get

$$r^* = \frac{|\Delta z|}{1 - |z|^2}. \tag{1.2.21}$$

Now using the Taylor series let us write the asymptotic representation, for $r^* \to 0$, of the function in the right-hand side of (1.2.20):

$$\frac{1}{2}\ln\frac{1+r^*}{1-r^*} = \frac{1}{2}\ln(1 + 2r^* + o(r^*)) = r^* + o(r^*) = r^*\left(1 + \frac{o(r^*)}{r^*}\right). \tag{1.2.22}$$

We thus get

$$\Delta s = \frac{|\Delta z|}{1-|z|^2} \cdot \left(1 + \frac{o(r^*)}{r^*}\right), \quad r^* \to 0. \tag{1.2.23}$$

Passing in (1.2.23), in the limit $r^* \to 0$, from Δz and Δs to $dz = dx + idy$ and ds, respectively, we arrive at the explicit form of the metric of Lobachevsky's plane (*in the Poincaré disc model*):

$$ds^2 = \frac{|dz|^2}{(1-|z|^2)^2} = \frac{dx^2+dy^2}{(1-(x^2+y^2))^2}. \tag{1.2.24}$$

Accordingly, the non-Euclidean length $\mathcal{L}$ of the curve $\ell \subset \Lambda^2$ is computed as

$$\mathcal{L} = \int_\ell \frac{dz}{1-|z|^2}. \tag{1.2.25}$$

Relations (1.2.24) and (1.2.25) are key metric relations in the Poincaré disc model of the Lobachevksy plane.

Next let us discuss the main geometric images—typical types of lines—in the Lobachevsky planimetry, using its Poincaré disc model.

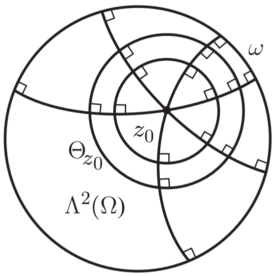

Figure 1.2.6

The non-Euclidean circle Θ_{z_0} with center at the point $z_0 = x_0 + iy_0$ and radius R is defined as the geometric locus of the points in the plane Λ^2 for which the non-Euclidean distance to the point z_0 is constant and equal to R:

$$\rho(z_0, z) = R, \quad z \in \Theta_{z_0}. \tag{1.2.26}$$

Using (1.2.17), we rewrite (1.2.26) as

$$\frac{1}{2} \ln \frac{1+r}{1-r} = R,$$

or

$$r = \tanh R. \tag{1.2.27}$$

Writing (1.2.27) in detail in the setting of the given definition yields (upon applying (1.2.18)) the relation that gives a non-Euclidean circle in Λ^2:

$$\left|\frac{z-z_0}{\overline{z}_0 z-1}\right| = \tanh R, \quad z = x+iy. \tag{1.2.28}$$

The subsequent transition to the variables x, y in (1.2.28) convinces us that in the Poincaré disc model the non-Euclidean circle Θ_{z_0} is represented by a Euclidean circle that lies inside the disc and does not touch the absolute (Figure 1.2.6).

Note that the non-Euclidean circles on Λ^2 with common center at an interior point z_0 are orthogonal to the system of non-Euclidean straight lines passing through z_0 (Figure 1.2.6).

Horocycle. To explain the notion of *horocycle* we perform the following geometric construction. In the Lobachevsky plane Λ^2 , considered in the Poincaré disc interpretation, consider a family of straight lines $KK_1, KK_2, \ldots, KK_7, \ldots,$ which emanate from the common point at infinity K on the absolute ω, and hence are parallel to one another (Figure 1.2.7).

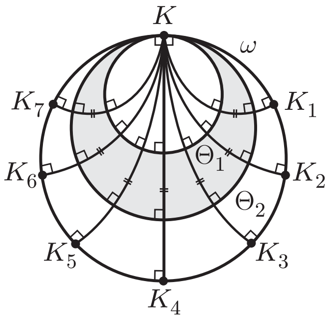

Figure 1.2.7

As another family of lines in Figure 1.2.7 we take the Euclidean circles Θ_1, Θ_2, which are contiguous to the absolute ω at the point K (i.e., K is a common point of the internally tangent Euclidean circles ω, Θ_1, Θ_2). It is clear that the circles Θ_1 and Θ_2 intersect orthogonally the non-Euclidean straight lines emanating from the point K. In other words, the "circles" of the type Θ_1 or Θ_2 are trajectories that intersect orthogonally the pencil of parallel straight lines emanating from the common point at infinity K on the absolute ω and also contain K.

The lines of type Θ_1, Θ_2 (Euclidean circles tangent to the absolute) in the Lobachevsky plane are called *horocycles*. A horocycle is also interpreted as a *limit circle* in the plane Λ^2, by which one means that in the Poincaré disc model the Euclidean image of a horocycle is the circle of largest radius among all concentric non-Euclidean circles with given center (Figure 1.2.6).

Horocycles have the following characteristic property: *the lengths of the segments of lines emanating for one and the same point at infinity $K \in \omega$ and enclosed between two arbitrary horocyles Θ_1 and Θ_2 (with the same point at infinity K on the absolute) are equal* (Figure 1.2.7).

To make things more transparent, this property will be proved in the following subsection, when we consider the Poincaré interpretation of Lobachevsky planimetry in the half-plane.

Another important property of horocycles is connected with the notion of rotation about a point at infinity in the plane Λ^2 [25]. By *rotation around the point at infinity* $K \in \omega$ we will understand a motion in the plane Λ^2 under which any straight line KK_i of the considered pencil of parallel straight lines is taken into another straight line KK_j of the same pencil; moreover, any point $A \in KK_i$ is taken into a point $A' \in KK_j$ in such a way that the segment AA' is a secant of equal inclination of the straight lines KK_i and KK_j (i.e., it makes with these straight lines equal inner same-side angles). In this sense, *horocycles* are *lines* in the Lobachevsky plane that are *invariant* under rotations of the corresponding point at infinity. Alternatively, a *horocycle* is the *geometric locus* in Λ^2 of the *tips of secants of equal inclination*, drawn from some point $A \in KK_i$ to all the other straight lines parallel to the KK_i in a given direction (lines that emanate from a common point at infinity $K \in \omega$).

Equidistants. Using the Poincaré disc interpretation $\Lambda^2(\Omega)$, let us describe one more type of lines in the plane Λ^2, the *equidistants*. Consider a non-Euclidean straight line $a \subset \Lambda^2$ with points at infinity $P, Q \in \omega$ on the absolute (Figure 1.2.8) (recall that a straight line a in the disc Ω is given by an arc PQ of Euclidean circle that is orthogonal to the absolute ω). Draw through the points $P, Q \in \omega$ another (not orthogonal to ω) Euclidean circle, and denote the arc of this circle in the disc Ω by c. It turns out that c is the geometric locus of the points in Λ^2 for which the distance to the given straight line a is constant. In other words, $c \subset \Lambda^2$ is a line that lies at constant distance (in the non-Euclidean sense) from a. Such lines in the Lobachevsky plane are called *equidistants*.

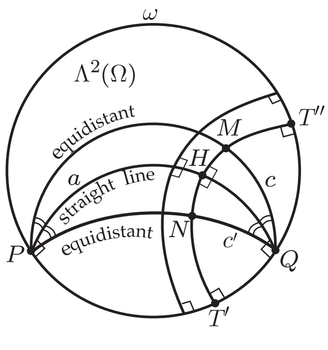

Figure 1.2.8

The distance from an arbitrary point M of the equidistant c to the given straight line a is the length of the perpendicular MH descending from M to a (base line). The segment MH is a piece of the straight line $T'T''$ that intersects orthogonally the straight line a at the point H (Figure1.2.8). One can reason in exactly the same way for the other, "symmetric" equidistant c' and a point $N \in c'$. In the $\Lambda^2(\Omega)$ interpretation, equidistants are represented by arcs of circles that lean on the absolute at the same points as the base straight line a, and at those points make equal angles with a (Figure 1.2.8).

The proof of properties of equidistant lines will be carried out in the next subsection by resorting to the Poincaré half-plane model of the plane Λ^2. Here we recall one more time that, given a straight line a, there always exist two distinct equidistants c and c' equally distanced from a, in opposite directions as one moves away from a. In the Poincaré disc interpretation, two equidistants c and c' that lie at equal distances from the straight line a have the property that the Euclidean arcs representing them make equal angles with the arc a.

Completing here the discussion of the Poincaré disc interpretation $\Lambda^2(\Omega)$, we formulate an important generalization:

Part of the Euclidean circles contained in the disc Ω that realizes, according to Poincaré, the Lobachevsky planimetry, give in the plane Λ^2 four characteristic types of lines:

1) *straight lines* (arcs of circle that are orthogonal to the absolute),[3]
2) *non-Euclidean circles* (interior circles with respect to the absolute),
3) *horocycles* (interior circles tangent to the absolute),
4) *equidistants* (arcs of circles that are not orthogonal to the absolute).

With the geometric images listed above in mind, let us examine their relationships with various types of straight lines in the Lobachevsky plane Λ^2. In Λ^2 there are three possible types of (pairs of) straight lines: 1) intersecting; 2) parallel; 3) divergent (two straight lines are said to be *divergent* if they have a common perpendicular, which realizes the smallest distance between the two straight lines, on both sides of which the straight lines diverge unboundedly).

To the three types of straight lines in the plane Λ^2 listed above there correspond pencils of three different types:

1) Pencil of the 1st kind (or simply "pencil"): the set of all straight lines that pass through a given point (the center of the pencil). A trajectory orthogonal to a pencil of the 1st kind is a *circle*.
2) Pencil of the 2nd kind (or *hyperpencil*): the set of all straight lines orthogonal to a given straight line (called the base, or center of the pencil). A trajectory orthogonal to a pencil of the 2nd kind is an *equidistant*.
3) Pencil of the 3rd kind: the set of all straight lines parallel in a given direction with a given line. A trajectory orthogonal to a pencil of the 3rd kind is, as we will see later, a *horocyle*, which can also be regarded as a circle with the center at infinity.

Another result that is important for the subsequent development of the subject is that all geometric images (types of straight lines) considered above in the plane Λ^2 are given by explicit equations in the Cartesian coordinates x, y (either by the equation of a circle, or by that of a straight line). This fact, in conjunction with the availability (also in explicit form) of the metric (1.2.24) of the Lobachevsky

[3]Straight lines can also be represented by diameters.

plane (in the same variables), allows one to apply in their study the tools and methods of Riemannian geometry, of the theory of curves and surfaces, and so on. Based on these, in Chapter 2 we will obtain important geometric characteristics of various geometric elements of Lobachevsky planimetry.

1.2.3 The Poincaré half-plane model of the Lobachevsky plane

The interpretation of the Lobachevsky plane "in the half-plane" proposed by Poincaré, and denoted here by $\Lambda^2(\Pi)$, is obtained from the "disc interpretation" considered in Subsection 1.2.2 by mapping the unit disc $\Omega = \{z = x + iy \in \mathbb{C},\ |z| < 1\}$ (Figure 1.2.5) conformally onto the upper half-plane $\Pi = \{w = u + iv \in \mathbb{W},\ \operatorname{Im} w > 0\}$. Such a mapping $w : \Omega \to \Pi$, $z \mapsto w(z)$, is realized by means of a linear-fractional transformation, for the construction of which it is convenient to associate beforehand three distinct points z', z'', z''' of the complex plane $\mathbb{C}$ with three distinct points w_1, w_2, w_3 in the "new" complex plane $\mathbb{W}$.

In particular, the association

$$z' = 1 \mapsto w_1 = 0, \quad z'' = i \mapsto w_2 = 1, \quad z''' = -1 \mapsto w_3 = \infty \tag{1.2.29}$$

defines the linear-fractional transformation

$$w(z) = i \cdot \frac{1 - z}{1 + z}; \quad \operatorname{Im} w > 0 \quad \text{for} \quad |z| < 1, \tag{1.2.30}$$

which maps the unit disc Ω into the upper half-plane Π. The mapping (1.2.30) lies at the foundation of the Poincaré half-plane interpretation of the plane Λ^2: in this case the complex half-plane Π plays the role of the Lobachevsky plane Λ^2, and the role of the absolute is played by a straight line, namely, the real u-axis (Figure 1.2.9).

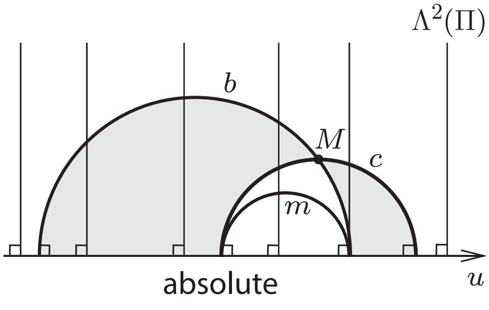

Figure 1.2.9

Under the transformation (1.2.30), which maps the "disc model" into the "half-plane model", the new images of the typical lines of the plane Λ^2 are the images of the lines with corresponding meaning in the disc model.

The linear-fractional transformation $w(z)$ maps circles and straight lines into circles and straight lines [95,105]. Therefore, in the upper half-plane Π the role of the straight lines in the plane Λ^2 is played by two families of lines: arcs of semicircles that are orthogonal to the absolute ($v = 0$), and Euclidean rays with the origin on the absolute and orthogonal to the absolute. As shown in Figure 1.2.9, in the present case Lobachevsky's axiom of parallels has a transparent interpretation, in which the straight lines m and b, c in Figure 1.2.9 have the same meaning as in Figure 1.2.5.

Let us give the general form of a linear-fractional transformation, defined by the a priori associated two triples of points $(z', z'', z''') \in \mathbb{C}$ and $(w_1, w_2, w_3) \in \mathbb{W}$ [105]:

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z'}{z - z''} : \frac{z''' - z'}{z''' - z''}. \tag{1.2.31}$$

Relation (1.2.31) allows enough freedom in the choice of a mapping between the planes $\mathbb{C}$ and $\mathbb{W}$, in particular, in the choice of a point on the absolute $\omega \subset \mathbb{C}$ that goes into the "infinity" in the half-plane Π (in the complex plane $\mathbb{W}$). The lines in the plane $\Lambda^2(\Omega)$ that originate at such a point have a special character specific to the mapping $w : \Omega \to \Pi$ and play a distinguished role in our considerations (see below).

In the case we are interested in, that of the mapping $w : \Omega \to \Pi$ of the disc onto the half-plane, the chosen points z', z'', z''' must lie on the absolute ω (the boundary of the disc Ω), while the corresponding points w_1, w_2, w_3 must also lie at infinity in the half-plane Π, in the sense of the Lobachevsky planimetry. That is to say, the points w_1, w_2, w_3 must lie on the absolute (the line $v = 0$) in Π; or, as already mentioned, it is possible that one of the points can be interpreted as the point at infinity.

We recall that when one effects the linear-fractional transformation it is important which of the points on the absolute ω of the complex plane $\mathbb{C}$ (Figure 1.2.5) is "designated" as the point at infinity (∞ or $i\infty$) in the new complex plane $\mathbb{W}$, because the lines on the plane Λ^2 (in the disc model) which originate at this particular point have properties that are specific to the mapping at hand.

Let us clarify the behavior of the singularities of the mapping $w : \Omega \to \Pi$, $z \mapsto w(z)$ for various types of lines on the plane Λ^2.

I. **Straight lines** (Figure 1.2.10 *a* and *b*).

Mapping:

$w(z)\colon\ \Omega \to \Pi$,

$w(A) = \infty$ (the point A is mapped into the point at infinity).

In the $\Lambda^2(\Omega)$ *model*:

straight lines are arcs of circles that lean orthogonally on the absolute (Figure 1.2.10 *a*).

In the $\Lambda^2(\Pi)$ *model*:

straight lines are:

1) arcs of circles (semicircles) that lean orthogonally on the absolute (the real axis $v = 0$) (Figure 1.2.10 *b*).

2) Euclidean rays with the origin on the absolute that are perpendicular on the absolute) (if such lines are images of straight lines in $\Lambda^2(\Omega)$ that originate at the point $A \in \omega$) (Figure 1.2.10 *b*).

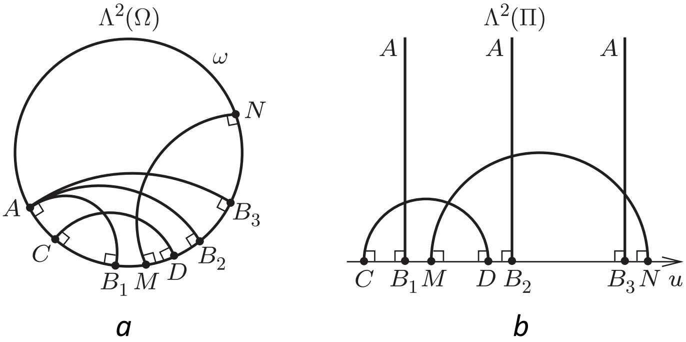

Figure 1.2.10

II. **Circles** (Figure 1.2.11 *a* and *b*).

Mapping :

$$w(z)\colon\ \Omega \to \Pi.$$

In the $\Lambda^2(\Omega)$ *model*:

circles are Euclidean circles interior with respect to the absolute ω (Figure 1.2.11 *a*).

In the $\Lambda^2(\Pi)$*model*:

circles are Euclidean circles in the upper half-plane Π (which are not tangent to the absolute, i.e., to the real axis) (Figure 1.2.11 *b*).

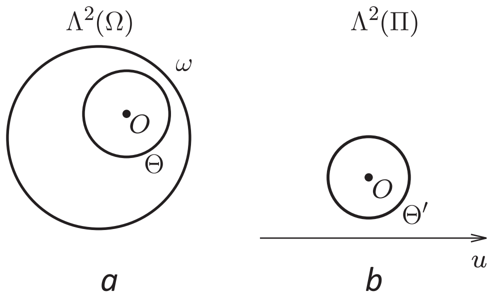

Figure 1.2.11

III. **Horocycles** (Figure 1.2.12 *a* and *b*).

Mapping:

$$w(z)\colon\ \Omega \to \Pi,$$

$w(K) = \infty$, the point $N' = \omega(N)$ lies on the u-line.

In the $\Lambda^2(\Omega)$ *model*:

horocycles are circles, interior with respect to the absolute ω, that are tangent to the absolute (Figure 1.2.12 *a*).

In the $\Lambda^2(\Pi)$ *model*:

horocycles are:

1) circles tangent to the real axis, i.e., to the absolute (Figure 1.2.12 *b*).
2) straight lines that are parallel to the absolute (if they are images of horocycles tangent to the absolute ω at the point K that is mapped by $w(z)$ into the point at infinity) (Figure 1.2.12 *b*).

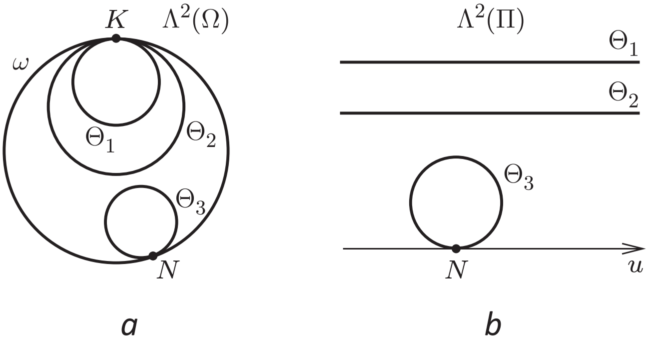

Figure 1.2.12

IV. **Equidistants** (Figure 1.2.13 *a* and *b*).

Mapping:

$w(z)\colon\ \Omega \to \Pi$,

$w(A) = \infty$.

In the $\Lambda^2(\Omega)$ *model*:

equidistants are arcs of circles leaning on the absolute ω at two common points A and B (they make equal angles with the base line (geodesic) that leans on the absolute at the same points A and B) (Figure 1.2.13 *a*).

In the $\Lambda^2(\Pi)$ *model*:

equidistants are:

1) arcs of circles that lean (not orthogonally) on the absolute (i.e., on the u-axis) at two common points C and D (Figure 1.2.13 *b*).
2) Euclidean rays that originate a common point B on the absolute and make equal angles with a straight line that has the same point B on the absolute (a Euclidean ray that originates the point B (Figure 1.2.13 *b*).

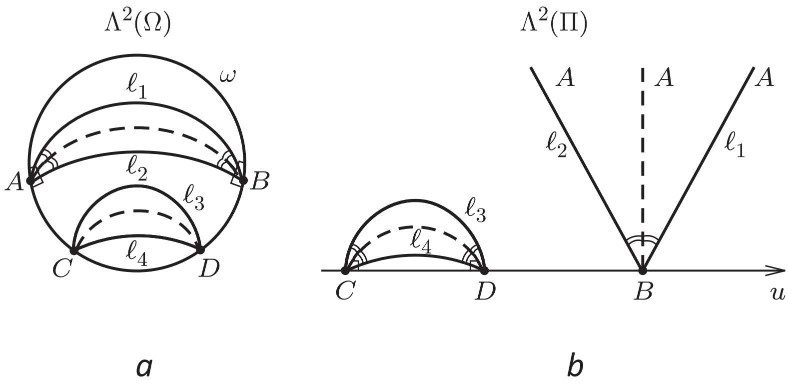

Figure 1.2.13

The mapping $w : \Lambda^2(\Omega) \to \Lambda^2(\Pi)$, $z \mapsto w(z)$, given by means of formula (1.2.30) (or in the general form (1.2.31)) is linear-fractional, and as such preserves the *cross ratio of four points* (see Subsection 1.2.1). Consequently, the distance between points in the Poincaré half-plane model can be defined in precisely the same way as we did in the previous models, via the formulas (1.2.7)–(1.2.8), or via (1.2.15), in terms of the cross ratio of four points.

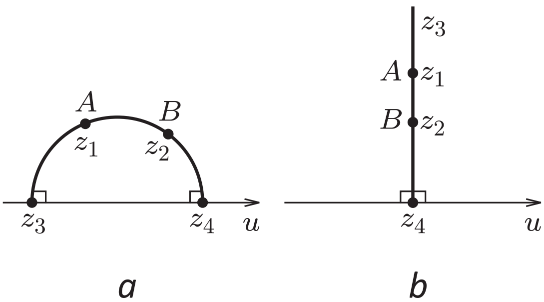

Figure 1.2.14

In the plane $\Lambda^2(\Pi)$ there are two possible representations of a straight line passing through two points A and B:

1) as a semi-circle that leans orthogonally on the absolute (Figure 1.2.14 *a*);

2) a ray perpendicular to the absolute (Figure 1.2.14 *b*).

In the first case (Figure 1.2.14 *a*), the distance $\rho(A, B)$ is calculated by precisely formula (1.2.15).

In the second case (Figure 1.2.14 *b*), when all four points $z_1 = u_1 + iv_1$, $z_2 = u_1 + iv_2$, $z_3 = u_1 + i\infty$, $z_4 = u_1$ $(v_1, v_2 > 0)$ lie on a ray that has the same direction as the positive imaginary axis, formula (1.2.15) simplifies considerably:

$$\rho(A, B) = \frac{1}{2}\left|\ln \frac{v_2}{v_1}\right|. \tag{1.2.32}$$

Next, using the Poincaré half-plane model, specifically, the depiction of typical lines on Λ^2 and formula (1.3.32) for the computation of distances, we prove the properties of horocycles and equidistants formulated in Subsection 1.2.2.

Let us show that the lengths of segments of pencils of straight lines that emanate form a common point at infinity $K \in \omega$ and lie between two arbitrary horocycles Θ_1 and Θ_2 (with the same point at infinity K on the absolute) are equal (Figure 1.2.7).

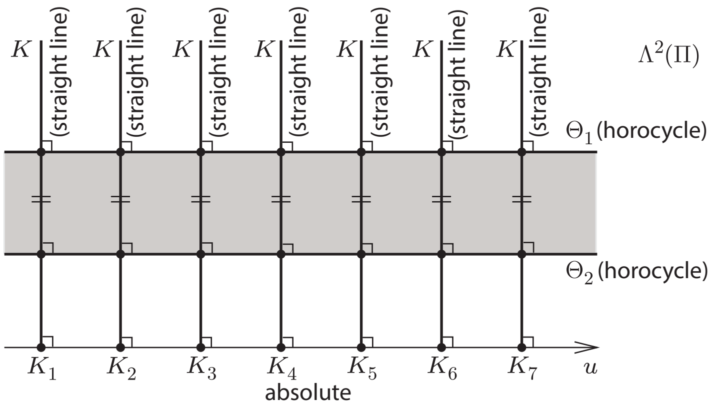

Figure 1.2.15

Indeed, consider a linear-fractional transformation $w : \Omega \to \Pi$ which maps the point $K \in \omega$ (Figure 1.2.7) into the point at infinity in the half-plane Π: $w(K) = \infty$. Then the straight lines $KK_1, KK_2, \ldots, KK_7, \ldots$ of the pencil under consideration are mapped into rays in the half-plane Π that are perpendicular to the absolute (Figure 1.2.15). Correspondingly, the horocycles Θ_1 Θ_2 are mapped into two straight lines that are parallel to the u-axis.

Figure 1.2.15 is the image of Figure 1.2.7 under the selected linear-fractional transformation. From the constructions shown in Figure 1.2.15 and the distance formula (1.2.32) it is clear that the straight-line segments we are interested in (segments in the shaded region) are *equal*.

Next, in the framework of Poincaré's half-plane model, we discuss the properties of equidistants. An equidistant is the geometric locus, in the plane Λ^2, of all points for which the non-Euclidean distance to a given straight line is constant (Figure 1.2.8).

Let us apply a linear-fractional transformation $w : \Omega \to \Pi$ that sends the point P (Figure 1.2.8) to infinity: $w(P) = i\infty$. Then in the half-plane Π the straight line PQ will be depicted by a ray QP that is perpendicular to the absolute (Figure 1.2.16).

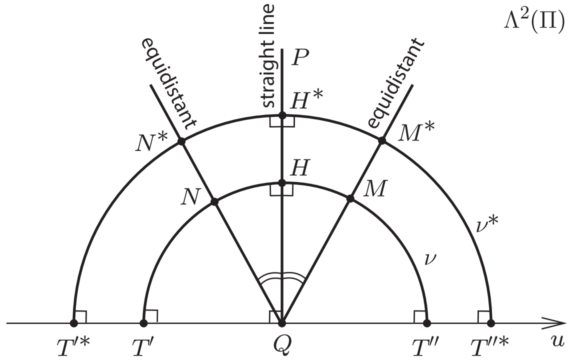

Figure 1.2.16

The straight line $T'T''$ (Figure 1.2.8), perpendicular to PQ, along which one measures the distance from the points of the equidistant QM to the base straight line PQ, is represented in the half-plane $\Lambda^2(\Pi)$ by a semi-circle ν centered at the point Q. Then for all points of the type $M, M^*, \ldots$ (or $N, N^*, \ldots$) (Figure 1.2.16), at which the equidistant in question intersects the straight lines of the type $T'T''$ (semi-circles $\nu, \nu^*, \ldots$ in the half-plane $\Lambda^2(\Pi)$), the value of the cross ratio of four points must be the same:

$$[H, M, T', T''] = [H^*, M^*, T'^*, T''^*].$$

Moreover, as we already mentioned, the value of the cross ratio is uniquely determined by the angles formed by the projection rays (in our case, the rays QT', QH, QM (or QN), and QT''); consequently, it will have the same value for all the points obtained by intersecting the indicated rays with semi-circles centered at the vertex of the projection, i.e., the point Q (Figure 1.2.16). It follows that in the half-plane $\Lambda^2(\Pi)$ the *equidistants* must be represented by *rays of equal inclination to the straight line* from which they lie at equal distance: the rays QM and QN are equidistants lying at the same distance from the straight line QP. If we now return to the $\Lambda^2(\Omega)$ interpretation, then to their images on $\Lambda^2(\Pi)$ will correspond arcs of circles with two common points on the absolute, which make equal angles with the given straight line (arcs of circles that lean orthogonally on the absolute in exactly the same points).

Let us give the form of the Lobachevsky plane metric that corresponds to the Poincaré half-plane model [25]:

$$ds^2 = \frac{1}{v^2}\left(du^2 + dv^2\right). \tag{1.2.33}$$

The metric form (1.2.33) can be obtained by using the same method as the one in Subsection 1.2.2. to obtain expression (1.2.24).

An important result of the introduction of models of the Lobachevsky plane Λ^2 is the derivation of the explicit expressions (1.2.11), (1.2.24), and (1.2.33) for the metric of Λ^2. The existence of the metric form allows one to apply the tools and methods of the theory of surfaces to the investigation of the spatial (Euclidean) realization of the Lobachevsky geometry. A key factor in such an approach is that the Lobachevsky plane's metric has *curvature* equal to $K \equiv -1$. (For a detailed treatment of the notion of *Gaussian curvature* of a metric, see Chapter 2.)

In addition, let us mention one of the important possible applications of the Poincaré half-plane model $\Lambda^2(\Pi)$. In the $\Lambda^2(\Pi)$ interpretation, the typical lines in Λ^2 (straight lines, equidistants, horocycles) are given, in particular, by very simple linear equations ("straight-line equations" in the variables u and v), and consequently can be quite conveniently studied by methods of the differential geometry of surfaces (surfaces carrying a metric of the form (1.2.33)), corresponding to the $\Lambda^2(\Pi)$ interpretation). As it turns out, a common "indicator" of the lines in the plane Λ^2 studied in the present section is that they have *constant geodesic curvature*. The notion of the geodesic curvature of a curve on a surface will be discussed in detail in § 2.7.

To end this section we wish to mention also the classical work of F. Klein [163] on the foundations of non-Euclidean geometry.

In the next section we will study the classical surfaces in the three-dimensional Euclidean space $\mathbb{E}^3$ on which the geometry of individual parts of the plane Λ^2 can be realized. These surfaces were obtained first by F. Minding (1839), practically simultaneously with Lobachevsky's studies. However, Minding's work had an independent aim of its own—the study of surfaces of revolution of constant Gaussian curvature. Only afterwards it was established that on Minding's surfaces of revolution with $K \equiv \text{const} < 0$ one can realize the geometry of "fragments" of the Lobachevsky plane.

1.3 Classical surfaces of revolution of constant negative curvature

1.3.1 F. Minding's investigation of surfaces of revolution

During the same time period that Lobachevsky's work originated from, F. Minding studied the surfaces of revolution of constant curvature. His works [175–177] became recognized as important applications of the theory of surfaces, which before his time was systematically treated by G. Monge in his classical handbook on differential geometry "Feuilles d'Analyse Appliquée à la Géométrie" ("Applications of Analysis to Geometry") (1807). Following Monge's approach, in which the Cartesian coordinates (x, y, z) of an arbitrary point on a surface in the Euclidean space $\mathbb{E}^3$ are connected by the implicit relation

$$\mathcal{F}(x, y, z) = 0, \tag{1.3.1}$$

or, more frequently, in the solved form

$$z = z(x, y), \tag{1.3.2}$$

Minding studied surfaces of revolution of *constant* curvature K, positive as well as negative.

For surfaces given in the space $\mathbb{E}^3$ in the Monge form (1.3.2), the curvature is given by the formula [40]

$$K = \frac{rt - s^2}{(1 + p^2 + q^2)^2}, \tag{1.3.3}$$

in which the following notations for partial derivatives are used:

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

To solve the differential equation (1.3.3) with respect to the function $z = z(x, y)$ means to describe, in the Euclidean space $\mathbb{E}^3$, all surfaces with a priori given curvature K according to their shape and position in space. In the general case, for an arbitrary curvature $K(x, y)$, equation (1.3.3) cannot be integrated exactly. Nevertheless, when the curvature of the surface is constant, important particular typical cases can be studied exhaustively.

F. Minding carefully studied surfaces of constant positive curvature, as well as surfaces of constant negative curvature. In what follows we will focus on the second case, viewing it as a direction that leads to examples of surfaces that realize in the Euclidean space $\mathbb{E}^3$ certain "fragments" of the two-dimensional Lobachevsky geometry.

Minding's method amount to searching for a form $x = \varphi(z)$, $x \geq 0$, of the meridian (the curve that is being rotated around the axis Oz) such that the curvature of the sought-for surface will be constant. Then, instead of equation (1.3.3), one can use the equation [39]

$$K = -\frac{\varphi''}{\varphi(1 + \varphi'^2)^2}. \tag{1.3.4}$$

In the case we are interested in the curvature is $K \equiv -1$, and equation (1.3.4) is recast as

$$\varphi = \frac{\varphi''}{(1 + \varphi'^2)^2}. \tag{1.3.5}$$

Equation (1.3.5) can be reduced (upon multiplying it by φ' and integrating the result) to the first-order equation

$$\varphi' = \pm\sqrt{\frac{\lambda + \varphi^2}{(1 - \lambda) - \varphi^2}}, \quad \lambda = \text{const.}$$

Since $x = \varphi(z)$ and $\dfrac{dx}{dz} = \varphi'(z)$, we can rewrite this last equation as

$$\frac{dz}{dx} = \pm\sqrt{\frac{(1 - \lambda) - x^2}{\lambda + x^2}}, \tag{1.3.6}$$

and accordingly obtain an elliptic integral, denoted here by $\mathfrak{M}(x)$, of the form

$$z \equiv \mathfrak{M}(x) = \pm \int\limits_{x_0}^{x} \sqrt{\frac{(1-\lambda)-x^2}{\lambda+x^2}}\,dx, \quad z(x_0)=0. \tag{1.3.7}$$

The analysis of the eliptic integral (1.3.7) yields the possible forms of the meridian and shows that there exist three different types of surfaces of revolution of negative curvature ($K \equiv -1$), corresponding to the three typical ranges of variation of the parameter λ.

1) $\lambda > 0$. As it follows from (1.3.7), in this case the admissible positive values of the parameter λ obey the condition

$$0 < \lambda < 1. \tag{1.3.8}$$

The domain of definition of the function $z(x)$ will be the interval

$$x \in [0, \sqrt{1-\lambda}]. \tag{1.3.9}$$

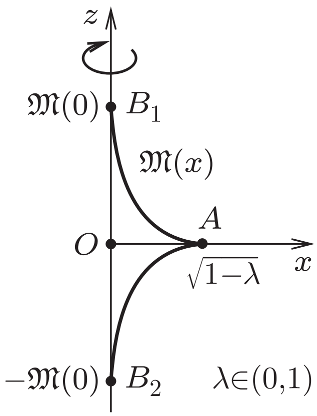

Figure 1.3.1

The qualitative shape of the meridian of revolution given by formula (1.3.7) and subject to conditions (1.3.8), (1.3.9) is shown in Figure 1.3.1: this curve is convex relative to the axis of revolution Oz and reaches the largest distance from Oz for $x_0 = \sqrt{1-\lambda}$: $z(x_0) = 0$ (in this point A the meridian of revolution is tangent to the axis Ox). At the points $B_1(0, \mathfrak{M}(0))$ and $B_2(0, -\mathfrak{M}(0))$, the meridian "merges" with the axis of revolution Oz.

In the case under consideration ($0 < \lambda < 1$, $x \in [0, \sqrt{1-\lambda}]$), the calculation of the integral (1.3.7) reduces to that of two elliptic integrals $\mathcal{F}_1(x,k)$ and $\mathcal{F}_2(x,k)$:

$$\mathfrak{M}(x) = \int_{x_0}^{x} \sqrt{\frac{x_0^2-t^2}{t^2+x_1^2}}\,dt = \mathcal{F}_1\left(\frac{\sqrt{x_0^2-x^2}}{x_0}, x_0\right) - \mathcal{F}_2\left(\frac{\sqrt{x_0^2-x^2}}{x_0}, x_0\right), \tag{1.3.10}$$

where

$$x_0 = \sqrt{1-\lambda}, \quad x_1 = \sqrt{\lambda},$$

$$\mathcal{F}_1(x,k) = \int_0^x \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt, \quad \mathcal{F}_2(x,k) = \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}.$$

When one rotates the meridian $\mathfrak{M}(x)$ around the axis Oz, the point A lying at the maximal distance from the axis Oz traces on the surface obtained a cuspidal edge (a circle), while the points B_1 and B_2 become "cusp (peak) points" of the surface. The points B_1 and B_2 also play the role of "suture" points of infinitely many copies of the surface ("tops").

Figure 1.3.2 shows, in a unified scale, three successive versions of the surface obtained for increasing (from left to right, with condition (1.3.8) enforced) values of the parameter λ. Such a surface was termed the *Minding top* (sometimes called also the Minding "*lampion*").

The generalized shape of a surface of Minding's "top" type is shown in Figure 1.3.3.

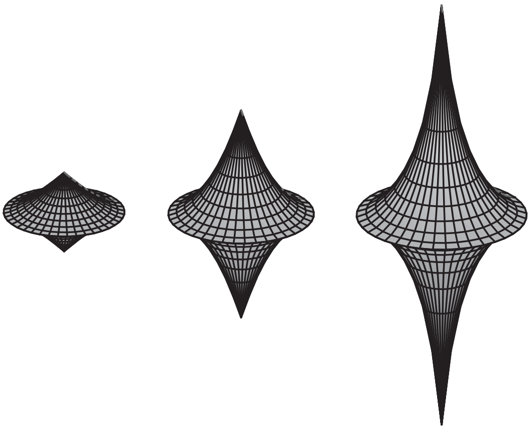

Figure 1.3.2

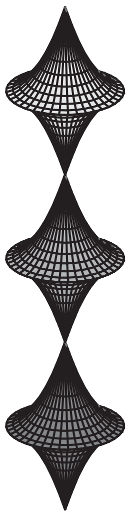

Figure 1.3.3

2) $\lambda < 0$. In this case we have

$$\frac{dz}{dx} = \pm\sqrt{\frac{(1+|\lambda|)-x^2}{x^2-|\lambda|}}, \quad \lambda < 0. \tag{1.3.11}$$

The right-hand side in (1.3.11) is defined for

$$x \in [\sqrt{|\lambda|}, \sqrt{1+|\lambda|}], \tag{1.3.12}$$

and one has

$$z'(\sqrt{|\lambda|}) = \infty, \quad z'(\sqrt{1+|\lambda|}) = 0.$$

From (1.3.12) it follows that the points A_1 and A_2 of the rotating meridian lying at the maximal distance from the axis Oz (Figure 1.3.4) trace on the surface (Figure 1.3.5) cuspidal edges (circles). The point C that lies at the minimal distance (equal to $\sqrt{|\lambda|}$) from the axis Oz is a point of smooth suture of two continuous parts of the meridian.

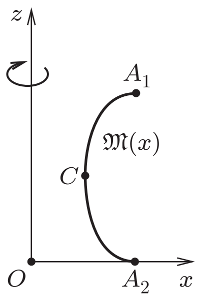

Figure 1.3.4

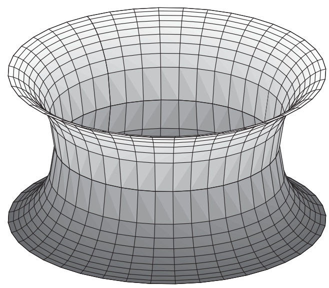

Figure 1.3.5

More precisely, in the present case the meridian is given by

$$z \equiv \mathfrak{M}(x) = \pm \int\limits_{\sqrt{|\lambda|}}^{x} \sqrt{\frac{(1+|\lambda|)-x^2}{x^2-|\lambda|}}dx, \quad \lambda < 0,$$

and it can be expressed in terms of the standard elliptic integrals as[4]

$$\begin{aligned}\mathfrak{M}(x) = \int_{x_1}^{x} \sqrt{\frac{x_0^2-t^2}{t^2-x_1^2}}dt = x_0 \cdot \left(\mathcal{F}_2\left(\frac{x_0\sqrt{x^2-x_1^2}}{x}, \frac{1}{x_0}\right) - \mathcal{F}_3\left(\frac{1}{x_0}\right)\right) \\ + \frac{x_1^2}{x_0} \cdot \left(\mathcal{F}_5\left(\frac{1}{x_0^2}, \frac{1}{x_0}\right) - \mathcal{F}_4\left(\frac{\sqrt{x^2-x_1^2}}{x}, \frac{1}{x_0^2}, \frac{1}{x_0}\right)\right), \quad (1.3.13)\end{aligned}$$

where

$$\begin{aligned} x_0 &= \sqrt{1+|\lambda|}, \quad x_1 = \sqrt{|\lambda|}, \\ \mathcal{F}_3(k) &= \mathcal{F}_2(1,k), \\ \mathcal{F}_4(x,\nu,k) &= \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}\sqrt{1-\nu t^2}}, \\ \mathcal{F}_5(\nu,k) &= \mathcal{F}_4(1,\nu,k). \end{aligned}$$

[4]Here we drop the expression with sign "+" for $\mathfrak{M}(x)$.

Strictly speaking, in Figure 1.3.4 we see not the meridian, given by relation (1.3.11) (or (1.3.13)), but the revolution curve obtained by smoothly joining two equal parts of this meridian at the point C. The corresponding classical surface of revolution obtained in this manner (Figure 1.3.5) is called the *Minding* "*bobbin*" (or "*spool*"). This surface cannot be continued regularly beyond its cuspidal edges (circles), but the latter can be considered as suture borders for distinct copies of "bobbins".

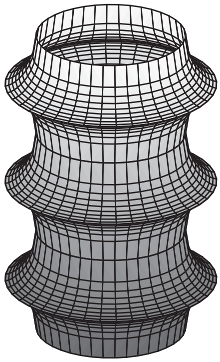

Figure 1.3.6

The generalized surface consisting of Minding "bobbins", which corresponds exactly to the formulas (1.3.13), is shown in Figure 1.3.6.

3) $\lambda = 0$. In this case the relation (1.3.6) becomes considerably simpler:

$$\frac{dz}{dx} = \pm\frac{\sqrt{1-x^2}}{x}. \tag{1.3.14}$$

The right-hand side of (1.3.14) is defined for

$$0 < x \leq 1.$$

The point $(x = 1,\ z = 0)$ is the point on the meridian lying farthest from the axis of revolution Oz. Since $z'(1) = 0$, the point $x = 1$ traces the cuspidal edge (circle). When $x \to 0$, the meridian curve tends asymptotically to the axis of revolution at infinity (for $z \to \infty$).

Relation (1.3.14) can be integrated exactly in terms of elementary functions. Indeed, the substitution

$$x = \sin t,$$

reduces (1.3.14) to the form

$$dz = \pm\frac{\cos^2 t}{\sin t}\,dt. \tag{1.3.15}$$

Observing that

$$x = 1 \quad \text{for} \quad t = \frac{\pi}{2},$$

$$z = 0 \quad \text{for} \quad x = 1,$$

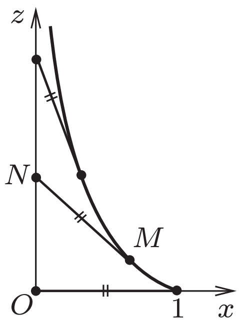

Figure 1.3.7

we obtain

$$z = \pm\left(\ln\tan\frac{t}{2} + \cos t\right), \quad x = \sin t,$$

or

$$z = \pm\left(\ln\frac{1-\sqrt{1-x^2}}{x} + \sqrt{1-x^2}\right), \quad x \in (0,1]. \tag{1.3.16}$$

The plane curve $z = z(x)$ defined by the formula (1.3.16) is called the *tractrix*[5] (Figure 1.3.7). The tractrix enjoys the following remarkable property: the length of the segment of tangent to any point of the tractrix, measured from that point to the point where the tangent intersects the axis Oz, is a constant quantity (see the segment MN in Figure 1.3.7). In the case we are dealing with here, in which the curvature $K \equiv -1$, the segment MN is of length 1. We remark that for the tractrix $z \to \infty$ as $x \to 0$, and $z(1) = 0$.

The surface obtained by rotating the tractrix around the axis Oz is called the *pseudosphere* (Figure 1.3.8). The pseudosphere plays in non-Euclidean geometry the same canonical role the sphere does in Euclidean geometry.

The arguments above make it clear that, in a certain sense, the pseudosphere ($\lambda = 0$) can be interpreted as the result of the limit transformation, as $\lambda \to 0$, of the surface of Minding's “bobbin”; in this case the “cusp (peak) points” of the “bobbin” become the points at infinity on the pseudosphere.

1.3.2 Surfaces of revolution of curvature $K \equiv -1$ and the corresponding domains in the plane Λ^2

Here we will establish an intuitive relationship between Lobachevsky's planimetry and the surfaces of constant negative curvature $K \equiv -1$, considered in Subsection 1.3.1. We consider the mapping of the aforementioned surfaces to the Lobachevsky plane Λ^2, with preservation of the metric. We will study each of the three types of surfaces separately.

[5] The tractrix was described for the first time by Huygens.

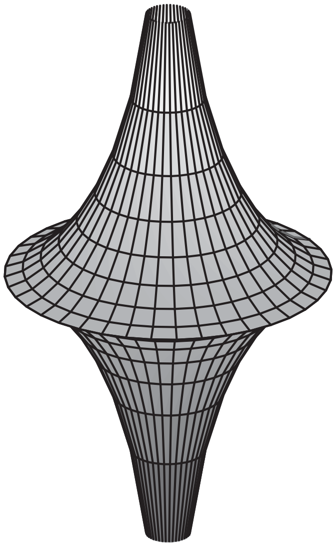

Figure 1.3.8

1) The Minding "top". Consider the "upper part" (Figure 1.3.9. a) of the piece of the Minding "top" (Figure 1.3.2) enclosed between two conical points. This part of the surface is bounded on one side by the cuspidal edge (circle), and on the other by the conical point C (the peak). The points on each of the circular parallels on the surface have the property that they lie at the same distance from the conical (peak) point C (Figure 1.3.9). Let us cut our surface along some meridian and hypothetically imagine that we superpose the available piece of surface (of curvature $K \equiv -1$) on the Lobachevsky plane Λ^2, i.e., we "unroll" the surface on the hyperbolic plane. We consider here the Poincaré interpretation of the Lobachevsky plane in the disc.

Then the "circular parallels" on the piece of surface with the "cut" are represented on the plane Λ^2 by arcs of concentric circles, the center of which on Λ^2 is the image of the conical point C of the surface (Figure 1.3.9 b). The meridians of the surface are represented on Λ^2 by rays (more precisely, by segments of rays) with origin at the point C that are orthogonal to the arcs of concentric circles. Thus, based on the system of parallels and meridians, transported from the surface (Figure 1.3.9 a) to the plane Λ^2 (Figure 1.3.9 b), we obtain on the Lobachevsky plane a domain—the preimage of the Minding "top" (with the same intrinsic geometric properties). In a certain sense, the domain shown in Figure 1.3.9 b, is the hyperbolic analogue of what we obtain when we unroll the ordinary cone on the Euclidean plane. By means of the correspondence between the parallels and meridians on the surface in $\mathbb{E}^3$ and on the Lobachevsky plane Λ^2 it is possible to map any domain on the surface under consideration onto a domain in the Lobachevsky plane Λ^2. Further, employing an infinite number of copies of

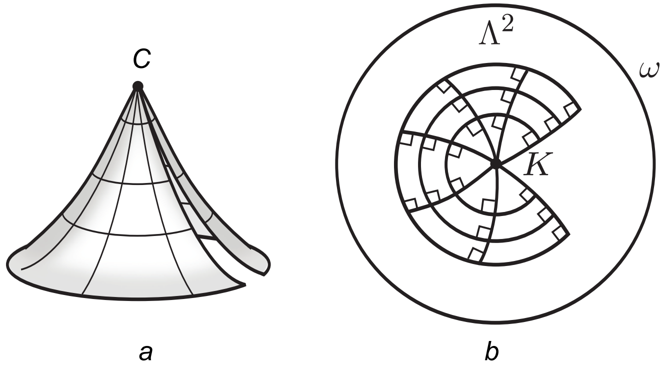

Figure 1.3.9

the domain obtained in Λ^2, and gluing them to one another successively along the corresponding borders of the cuts, one can obtain an infinite "winding" of their unrolled counterparts, which covers ("winds on") the surface of the Minding "top" infinitely many times. After some finite number of "windings", we obtain a double covering, and so on. The covering obtained in the way described above is the *universal covering* of the surface.

We remark also that the investigated surface itself can also be interpreted as the result of the realization in the Euclidean space $\mathbb{E}^3$ of the corresponding domain in the Lobachevsky plane endowed with its geometry. The problem of the realization of domains of the plane Λ^2 as surfaces in $\mathbb{E}^3$ that carry Lobachevsky's geometry (on the corresponding parts), reduces to the problem of obtaining isometric immersions of domains of the Lobachevsky plane in $\mathbb{E}^3$ (see Chapter 2).

2) The Minding "bobbin' and the equidistant strip. The unrolling of the surface of revolution of the Minding "bobbin" type on the plane Λ^2 with preservation of the metric is carried out in the same way as for the "top". Specifically, we cut the typical piece of surface (Figure 1.3.10 *a*) along some arbitrary meridian. Since the meridians are geodesics (shortest curves) on this surface, they are mapped in the plane Λ^2 into straight lines (more precisely, segments of divergent straight lines). Then the circular parallels on the surface are mapped on $\Lambda^2(\Omega)$ into a system of arcs of Euclidean circles that pass through two common points on the absolute, realizing in this way the property of equidistant lines (Subsection 1.2.2) that lie at equal distance from a "central" straight line.

In the resulting isometric "development" on $\Lambda^2(\Omega)$ (Figure 1.3.10 *b*), to the smallest circular parallel (the one of smallest diameter) on the surface (in the "most narrow part of the surface" (Figure 1.3.10 *a*)) will correspond the base line (geodesic line), which to make things intuitive is represented by a diameter. It is obvious that the cuspidal edges are mapped into equidistants that lie the farthest from the base line. Overall, the piece of Minding's "bobbin" shown in Figure 1.3.10 *a* is mapped into the shaded area in Figure 1.3.10 *b*, bounded by two equidistants and two straight lines (the "borders" of the meridian cut).

In much the same way one can construct other copies of the developed piece

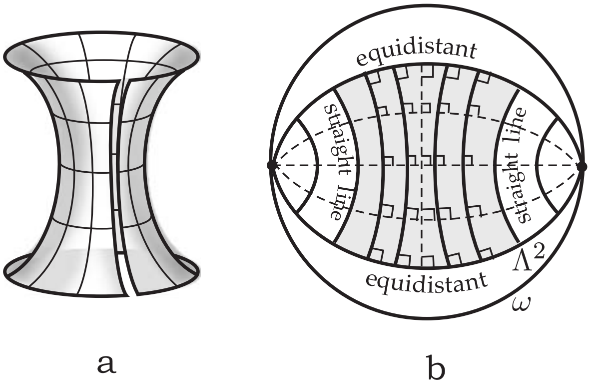

Figure 1.3.10

of surface and then glue them along their borders, i.e., along the images of the meridian cut. As a result, the equidistant strip is filled by successively glued domains (Figure 1.3.10 *b*), copies of the "shaded" domain (in the "hyperbolic sense"). However, in contrast to the preceding example of the Minding "top", the domain obtained in the Lobachevsky plane—the equidistant strip—cannot be covered ("exhausted") even by an infinite number of successively glued copies of the developed surface. In the opposite direction, the Minding "bobbin" can be wrapped up infinitely many times by the equidistant strip, its universal covering.

3) Pseudosphere and horodisc. The procedure for constructing the domain in $\Lambda^2(\Omega)$ corresponding to the pseudosphere differs from that in the case of the Minding "top" only by the fact that the center of the system of concentric circles in the plane Λ^2 that correspond to the circular parallels on the pseudosphere is the point at infinity in Λ^2, which lies on the absolute. Hence, to the circular parallels of the pseudosphere (Figure 1.3.11 *a*) there corresponds a system of horocycles with a common point on the absolute (Figure 1.3.11 *b*) (this point is the image of the point at infinity on the axis of revolution, to which the pseudosphere tends asymptotically).

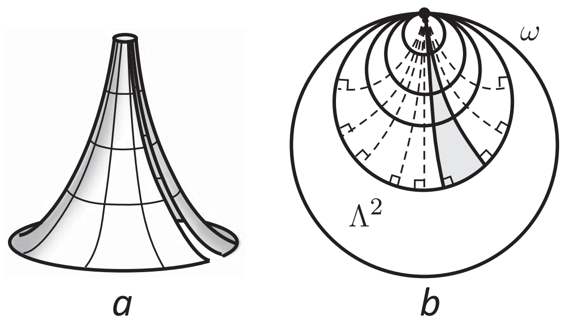

Figure 1.3.11

The cuspidal edge of the pseudosphere correspond to the "maximal" horocycle, i.e., the boundary of the domain obtained in Λ^2 (Figure 1.3.11 *b*).

The hyperbolic development of one copy of the upper sheet of the pseudosphere onto the plane $\Lambda^2(\Omega)$ is the domain "cut" from the system of horocyles by two straight lines which emanate from a common point they have with the horodiscs on the absolute and which correspond to the two boundaries of the meridian cut on the pseudosphere (the shaded area in Figure 1.3.11 *b*). To such a domain obtained in Λ^2 one can "glue" new identical copies, corresponding to the upper part of the pseudosphere under consideration (Figure 1.3.11 *a*); the boundaries of the "glued" domains are indicated by dots (Figure 1.3.11 *b*)). But like in the case of the Minding "bobbin", the domains indicated fail to exhaust the whole horodisc, which is the characteristic domain in the Lobachevsky plane for the pseudosphere. Nevertheless, the pseudosphere itself can be interpreted as a surface that is covered infinitely many times by the horodisc. In other words, the *horodisc* is the *universal covering of the pseudosphere*. We remark that for the complete pseudosphere (Figure 1.3.8), i.e., the surface consisting of the "lower" and the "upper" symmetric sheets, one needs to consider, for all the discussed domains on Λ^2, two identical copies.

1.3.3 C^1-regular surfaces of revolution, consisting of pieces of constant curvature of different signs

In this subsection we extend the classical series of surfaces of revolution of constant curvature by adding to the already provided list of pseudospherical ($K \equiv -1$) surfaces of revolution certain model *surfaces of revolutions*, "composed" of regularly contiguous "pieces", the curvature of which is *constant* and can be negative as well as null or positive: $K = -1, 0, +1$. Such surfaces realize a harmonious combination of the hyperbolic, Euclidean, and spherical geometry in the usual Euclidean space $\mathbb{E}^3$. From the analytical point of view, the task is to find in space $\mathbb{E}^3$ geometrical images and shapes in the form of C^1-regular surfaces[6] constructed from pieces with different signs of the Gaussian curvature.

The images of the examples of surfaces given below, which combine harmoniously all three geometries intuitively accessible to the human imagination, are inalienable attributes and symbols of cultural-historical and spiritual heritage accumulated over a millennium.

1) The "cupola" surface. The surfaces that play a canonical role in the spherical and the hyperbolic non-Euclidean geometries are respectively the sphere and the pseudosphere. To construct in $\mathbb{E}^3$ a smooth C^1-regular surface of "cupola" type, composed from parts of a sphere and of a pseudosphere, we need to find the points where the meridians of these standard surfaces can be joined smoothly.

The revolution meridian (profile) of the pseudosphere is the tractrix $z_{\mathrm{psph}}(x)$, given by formula (1.3.16); the meridian of the unit sphere (the "right" semi-circle) is given by the corresponding expression: $z_{\mathrm{sph}}(x) = \pm\sqrt{1-x^2}$, $x \in [0,1]$. The revolution profiles $z_{\mathrm{psph}}(z)$ and $z_{\mathrm{sph}}(z)$, considered in the upper half-plane ($z \geq 0$),

[6]The functions that describe such surfaces must have a continuous first-order derivative. Intuitively, this property ensures the visual smoothness of the surface.

can be smoothly joined when their derivatives are equal, which is achieved at the point $x_0 = \frac{\sqrt{2}}{2}$:

$$\left(\frac{dz_{\rm pshp}}{dx}\right)_{x=x_0} = \left(\frac{dz_{\rm sph}}{dx}\right)_{x=x_0} = -1, \quad x_0 = \frac{\sqrt{2}}{2}.$$

Note that to the profile $z_{\rm psph}(x)$, which lies in the upper half-plane ($z \geq 0$), there corresponds in the right-hand side of formula (1.3.16) the "minus" sign. Moreover, $z_{\rm psph}(x_0) \neq z_{\rm sph}(x_0)$, and so in order to compose a C^1-regular revolution meridian of the "cupola" surface, the tractrix profile must be shifted upward by the constant $C = \sqrt{2} + \ln(\sqrt{2} - 1)$. All this leads to the following explicit representation for the revolution meridian of a cupola-type surface:

$$z_{\rm cupola}(x) = \begin{cases} -\left(\ln \frac{1-\sqrt{1-x^2}}{x} + \sqrt{1-x^2}\right) + \sqrt{2} + \ln(\sqrt{2} - 1), & \text{if } x \in \left(0, \frac{\sqrt{2}}{2}\right], \\ \pm\sqrt{1-x^2}, & \text{if } x \in \left[\frac{\sqrt{2}}{2}, 1\right]. \end{cases} \tag{1.3.17}$$

The revolution profile (1.3.17) is shown in Figure 1.3.12. Its rotation yields a "cupola"-type surface, composed of regularly joined pieces of a pseudosphere (curvature $K \equiv -1$) and of a sphere (curvature $K \equiv +1$). The "jump" of the curvature on this surface takes place on the circle where the sphere and pseudosphere are joined; the cupola surface itself is however C^1-regular (visually continuous and smooth).

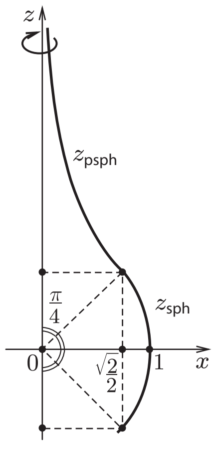

Figure 1.3.12. Shape of revolution of a "cupola"-type surface

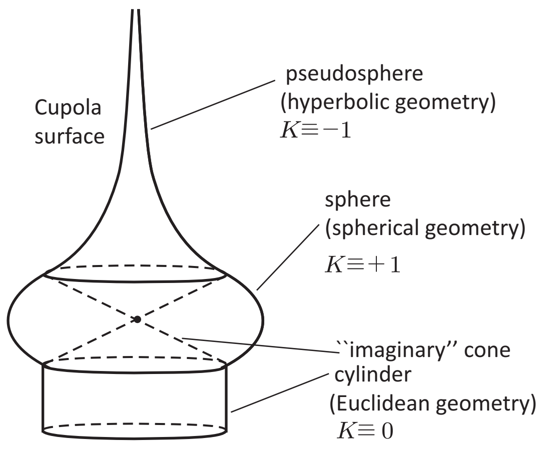

Figure 1.3.13. Model of a "cupola"-type surface: combination of canonical images of all three intuitive geometries. The drawing is done in the *golden ratio* scale

Figure 1.3.1 shows, in the "golden ratio" scale, the "cupola" surface under discussion, completed in its lower part by a part of a cylinder—a model of Euclidean geometry (of curvature $K \equiv 0$). On the whole, the geometric image of the surface, shown in Figure 1.3.13, includes images of the three intuitive classical geometries: the *pseudosphere* (Lobachevsky's hyperbolic geometry), the *sphere* (spherical geometry), and the *cylinder* with the imaginary "inner" *cone* (Euclidean geometry).

In the global philosophical view of the world, to the canonical forms that compose the *cupola* surface there correspond three successive *development categories* such as *formation* (cylinder, cone—Euclidean, "flat" geometry), *completion* (sphere—spherical geometry), and *convergence* (pseudosphere: convergence to the point at infinity on the absolute—hyperbolic geometry).

2) The "bell" surface. A second example of C^1-regular surface that combines harmoniously the images and shapes of the three basic geometries of constant curvature in $\mathbb{E}^3$ is the "bell"-type surface. The history of using bells as special sources of pure sound is thousands of years long. Here we remark that bells of various shapes realize constructively the "synthesis" in various versions of pieces of images of the three intuitive geometries: spherical, Euclidean, and hyperbolic.

The simplest model of a "bell"-shaped surface can be obtained, for example, by rotating the smooth profile consisting of a "quarter" of a circle (1), a rectilinear segment (2), and "half" of the revolution meridian (3) of the "bobbin" of constant negative curvature. The revolution meridian of the *bell surface*, given by the indicated expressions, is shown in Figure 1.3.14. The function $\mathfrak{M}(x)$ (the revolution meridian of the "bobbin" of constant negative curvature) is given by formula (1.3.13).

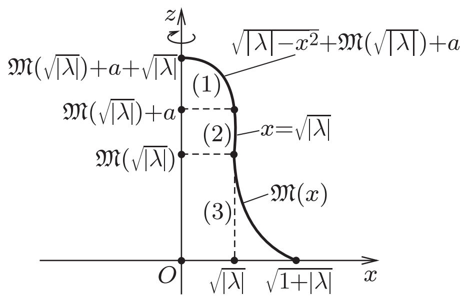

Figure 1.3.14. The revolution meridian of a "bell"-type surface, $|\lambda| \in (0, 1)$, $a = \text{const}$

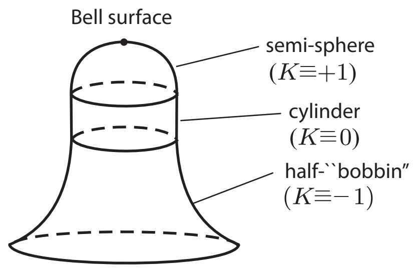

Figure 1.3.15. The simplest model of a "bell"-type surface

The rotation of the profile shown in Figure 1.3.14 produces in $\mathbb{E}^3$ the C^1-regular surface of a *bell* (Figure 1.3.15), consisting respectively (from top to bottom) of an "upper semi-sphere" ($K \equiv +1$), a part of a cylinder ($K \equiv 0$), and a "half-bobbin" ($K \equiv -1$).

It is rather natural to assume that if in the practical construction of bells one could approach forms of constant curvature, then one could generate "supplementary" sound nuances.

* * *

The examples considered above make it clear that on surfaces of revolution of constant negative curvature one cannot realize globally the geometry of the Lobachevsky plane: on such surfaces there always arise singularities—cuspidal edges, cusp (peak, spike) points—which can be interpreted as obstructions to the further regular extension of the surfaces beyond them. And conversely, the domains in Λ^2 corresponding to such surfaces are also far from exhausting the whole Lobachevsky plane. It turns out that this individual observation for surfaces of revolution is the manifestation of a global rule etablished by D. Hilbert in his work "Über Flächen von konstanter Krümmung" ("On surfaces of constant curvature"), published in 1901 [17]. Hilbert proved that every surface of constant negative curvature in $\mathbb{E}^3$ must have a singularity—an irregular edge (cuspidal edge), a cusp point, and so on, and the complete Lobachevsky plane cannot be realized in $\mathbb{E}^3$. Hilbert's results in this direction will be studied in detail in Chapter 2.

Following F. Klein's figurative example [45], we emphasize here an important property shared by Lobachevsky's hyperbolic geometry as well as by the Euclidean and the spherical geometries (generally, by the geometry of surfaces (spaces) of constant curvature). According to Klein's formulation, "tin-plate pieces" that cover exactly and without gaps a given surface of constant curvature admit ∞^3 motions along this surface such that at any time they cover the entire surface without gaps. Moreover, the tin-plate pieces themselves do not, in general, remain rigid, but deform in a certain way, without changing their intrinsic metric properties. This property plays a fundamental role in solving problems of covering space forms, realized by means of surfaces of constant curvature, by various geometric structural drawings, ornaments, and mosaics.

The study of the intrinsic-geometrical properties of surfaces of constant negative curvature and the associated problems of contemporary mathematical physics will be one of the central direction of investigation in the following chapters of the book.

Chapter 2

The problem of realizing the Lobachevsky geometry in Euclidean space

In this chapter we deal with general problems connected with the realization of the two-dimensional Lobachevsky geometry in the three-dimensional Euclidean space. In particular, we give an exposition of Lobachevsky planimetry as the geometry of a two-dimensional Riemannian manifold of constant negative curvature. We describe the apparatus of fundamental systems of equations of the theory of surfaces in $\mathbb{E}^3$ and discuss specifics of its application to the analysis of surfaces of constant negative Gaussian curvature. We also consider canonical geometric objects such as the Beltrami pseudosphere and Chebyshev nets, and we present D. Hilbert's result on the impossibility of a regular realization of the complete Lobachevsky plane in $\mathbb{E}^3$. We indicate fundamental connections that exist between the structure of pseudospherical surfaces and the sine-Gordon equation, one of the universal nonlinear partial differential equations. In the final section of the chapter we survey briefly a series of basic results on isometric immersions of Riemannian metrics of negative curvature in Euclidean space.

2.1 Lobachevsky planimetry as the geometry of a two-dimensional Riemannian manifold

2.1.1 The notion of Riemannian manifold

The establishment of a direct connection between Lobachevsky's geometry and surfaces of constant negative curvature goes back to 1868, when E. Beltrami [140, 141] carried out a detailed analysis of the *pseudosphere* surface[1] (§ 2.4 will be

[1] Although the pseudosphere was obtained for the first time in Minding's works as a surface of revolution of curvature $K \equiv -1$, Minding himself did not connect the surface he obtained with

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devoted to this investigation). Practically at the same time, B. Riemann, in his 1854 work "On the hypotheses which lie at the bases of geometry" ("Über die Hypothesen, welche der Geometrie zu Grunde liegen") advanced ideas that allowed a deeper comprehension of non-Euclidean geometry. In that work Riemann introduced the notion of *manifold* (*Mannigfaltigkeit*), thereby inaugurating the study of the intrinsic geometry of spaces as a separate discipline.

In his investigations Riemann considered a system (set) of n independent variables $\overline{x} = (x_1, \dots, x_n)$, each of which can assume arbitrary real values. Riemann calls the collection of all possible values of such a system an *n-dimensional manifold*. A *point* of the manifold is any set of fixed values $\overline{x}^* = (x_1^*, \dots, x_n^*)$. In this sense, the three-dimensional Euclidean space $\mathbb{E}^3$, for example, is a particular case of three-dimensional manifold.

On a manifold one can introduce a linear element (a metric) by means of a positive definite quadratic form

$$ds^2 = \sum_{i,j=1}^{n} a_{ij}(\overline{x}) dx_i dx_j, \quad a_{ij} = a_{ji}. \tag{2.1.1}$$

The linear element ds^2 thus introduced enables one to establish metric relations on the underlying manifold. If on the manifold one has a curve, given parametrically as

$$x_i = f_i(t), \quad i = 1, 2, \dots, n,$$

then the length of its arc, corresponding to the parameter range $t \in [t_1, t_2]$, is calculated as

$$\ell = \int_{t_1}^{t_2} \sqrt{\sum_{i,j=1}^{n} a_{ij}(\overline{x}) dx_i dx_j} = \int_{t_1}^{t_2} \sqrt{\sum_{i,j=1}^{n} a_{ij}(t) f_i'(t) f_j'(t)}\, dt.$$

The geodesics (shortest) curves on the manifolds are defined by requiring the length of an arc passing through two given points to be minimal, which is expressed by the variational condition

$$\delta \int_{t_1}^{t_2} \sqrt{\sum_{i,j=1}^{n} a_{ij} dx_i dx_j} = 0.$$

Before we turn to treating Lobachevsky's geometry in the framework of Riemannian geometry, let us formulate the general definition of a *manifold* that is suitable for the modern geometric representations [12, 24, 59, 66, 118].

We call *n-dimensional differentiable manifold* (or *smooth manifold*) $\mathcal{M}$ any set $\mathcal{M}$ of points that can be describes as follows.

$\mathcal{M}$ can be represented as the union of a finite or countable number of domains σ_k, in each of which one can introduce local coordinates x_k^α, $\alpha = 1, \dots, n$.

Lobachevsky's hyperbolic geometry.

The domains σ_k "composing" the manifold are called *coordinate neighborhoods* (or *charts*). A nonempty intersection $\sigma_k \cap \sigma_\ell$ of a pair of such domains in $\mathcal{M}$ is also a domain in Euclidean space, in which two local system of coordinates, (x_k^α) and (x_ℓ^α), exist simultaneously.

The differentiability condition for the manifold is that each of the two such local coordinate systems (x_ℓ^α) and (x_k^α) can be expressed in the entire domain $\sigma_k \cap \sigma_\ell$ through the other coordinate system in differentiable manner, via *transition maps*

$$\begin{aligned} x_\ell^\alpha &= x_\ell^\alpha(x_k^1, \dots, x_k^n), \\ x_k^\alpha &= x_k^\alpha(x_\ell^1, \dots, x_\ell^n), \end{aligned} \qquad \alpha = 1, \dots, n, \tag{2.1.2}$$

with nonzero transition Jacobian:

$$\det \left\| \frac{\partial x_\ell^\alpha}{\partial x_k^\beta} \right\| \neq 0.$$

The highest possible smoothness class of the transition maps (2.1.2) for all possible overlapping pairs of coordinate neighborhoods σ_k and σ_ℓ is called the *smoothness class* of the manifold $\mathcal{M}$ defined by the "atlas" $\{\sigma_k\}$.

Again, an obvious example of smooth (differentiable) manifold is the Euclidean space itself.

The treatment of two-dimensional Lobachevsky geometry in §§ 1.1 and 1.2 allows us, in accordance with the notion of manifold introduced above, to define the Lobachevsky plane Λ^2 as a two-dimensional smooth Riemannian manifold of constant negative curvature. The ensuing question of whether it is possible to realize the Lobachevsky plane Λ^2, or individual parts thereof, in the space $\mathbb{E}^3$ reduces to the problem of finding isometric immersions of them in $\mathbb{E}^3$. We will next formulate the isometric immersion problem.

2.1.2 The notion of isometric immersion

Let us pose the problem of finding in three-dimensional Euclidean space $\mathbb{E}^3$ a domain S that realizes the geometry of a given two-dimensional smooth manifold $\mathcal{M}_2$. In other words: Is it possible, and in which cases, to find in $\mathbb{E}^3$ some two-dimensional subset, the intrinsic geometry of which coincides with the geometry of the given two-dimensional smooth manifold? Moreover, the rule for calculating the distance between any two points of the sought-for domain $S \subset \mathbb{E}^3$, done by using the metric of the Euclidean space $\mathbb{E}^3$ (the ambient space), must correspond on S to precisely the metric (2.1.1) of the given manifold $\mathcal{M}_2$.

Generally, the problem of finding in some space $\mathbb{E}$ of a subset that has the same intrinsic-geometric properties as an a priori given smooth manifold $\mathcal{M}$ is referred to as the *problem of isometric immersion* of the manifold $\mathcal{M}$ in the space $\mathbb{E}$: $\mathcal{M} \xrightarrow{\text{isom}} \mathbb{E}$.

Clearly, in the formulation of the isometric immersion problem there is a high degree of arbitrariness in the choice of the original manifold, as well as in the choice of the ambient space.

In the framework of our basic question of whether it is possible to realize the Lobachevsky plane in Euclidean space we need to study the problem of the existence of a C^n-smooth map

$$f : D \to \mathbb{E}^3$$

from some domain[2] $D(u, v) \subset \mathbb{R}^2$, endowed with a metric of type (2.1.1), to the space $\mathbb{E}^3$.

The pair $\{D, f\}$ gives in $\mathbb{E}^3(x, y, z)$ (here x, y, z are the Cartesian coordinates in the space $\mathbb{E}^3$) a surface S:

$$S = f(D) \subset \mathbb{E}^3.$$

To each point of the surface S will correspond in $\mathbb{E}^3$ its Cartesian coordinates

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \tag{2.1.3}$$

where u, v are intrinsic coordinates on $S \subset \mathbb{E}^3$ that correspond to the coordinate parametrization of the original manifold $D(u, v)$.

Curves on the surface S are images of lines in the domain D under the map f.

The radius vector $\overrightarrow{r}$ of the surface $S \subset \mathbb{E}^3$ is defined as

$$\overrightarrow{r} = \{x(u, v), y(u, v), z(u, v)\}.$$

The ambient Euclidean space $\mathbb{E}^3$ induces on the surface S thus obtained a rule for measuring length:

$$ds^2 = dx^2 + dy^2 + dz^2,$$

which by using (2.1.3) leads to a Riemannian metric on S

$$ds^2 = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2, \tag{2.1.4}$$

which coincides with the metric (2.1.1) of the manifold we started with.

Suppose that for the metric (2.1.1) given on the manifold D (for $n = 2$) the map f gives in $\mathbb{E}^3$ a surface $S[\overrightarrow{r}(u, v)]$:

$$E = \overrightarrow{r_u}^2, \quad F = (\overrightarrow{r_u}, \overrightarrow{r_v}), \quad G = \overrightarrow{r_v}^2,$$

on which the metric (2.1.4), induced by the metric on the space $\mathbb{E}^3$, coincides with the metric one already has on D. Then one says that the map $f\colon\ D \xrightarrow{f} \mathbb{E}^3$ defines an *isometric immersion* of the manifold D with the given metric in the space $\mathbb{E}^3$. Such an isometric immersion of D in $\mathbb{E}^3$ will be denoted here by

$$D \stackrel{\text{isom}}{\longrightarrow} \mathbb{E}^3.$$

A classical question connected with Lobachevsky's hyperbolic geometry is: Which parts of the plane Λ^2 can be isometrically immersed in the three-dimensional Euclidean space $\mathbb{E}^3$? As we shall see in § 2.6, the complete Lobachevsky

[2]Here we put $u \equiv x_1, v \equiv x_2$

plane cannot be immersed in $\mathbb{E}^3$. More generally, one can ask what is the minimal dimension m such that the complete Lobachevsky plane can be isometrically immersed in $\mathbb{E}^m$? It has been shown that the plane Λ^2 can be immersed in the spaces $\mathbb{E}^6$ and $\mathbb{E}^5$ (see [74, 100, 146]); the question of the immersibility of Λ^2 in $\mathbb{E}^4$ remains open at this time and represents one of the unsolved fundamental problems of geometry.

The study of the problem of possible realization of two-dimensional metrics of constant negative curvature in the space $\mathbb{E}^3$ is directly connected with the analysis and integration of the fundamental equations of the theory of surfaces, to which we turn next.

2.2 Surfaces in $\mathbb{E}^3$ and their fundamental characteristics

2.2.1 The notion of surface in the space $\mathbb{E}^3$

In the classical theory of surfaces a primary notion is that of a *simple surface*.

Definition. A set S of points $M(x, y, z)$ in the Euclidean space $\mathbb{E}^3(x, y, z)$ whose Cartesian coordinates are given by relations of the form

$$x = \varphi(u, v), \quad y = \psi(u, v), \quad z = \chi(u, v), \quad u, v \in D, \tag{2.2.1}$$

where D is some simply-connected domain in the parameter (u, v)-plane, is called a *simple surface*, if to distinct pairs of values (u, v) correspond distinct points of S.

Definition. A simple surface S is said to be *smooth at the point* $P \in S$, if the tangent plane to S at P exists and some neigborhood of the point P projects on the tangent plane in a one-to-one manner.

A simple surface S is said to be *smooth* if the above one-to-oneness property holds at all its points and the tangent spaces to S vary continuously.

To describe a surface in space one uses, side by side with the functions (2.2.1), its radius vector

$$\overrightarrow{r}(u, v) = \varphi(u, v)\,\overrightarrow{i} + \psi(u, v)\,\overrightarrow{j} + \chi(u, v)\,\overrightarrow{k}, \tag{2.2.2}$$

where $\overrightarrow{i}$, $\overrightarrow{j}$, $\overrightarrow{k}$ are the unit direction vectors of the Cartesian coordinate axes in $\mathbb{E}^3$.

It is then clear that the surface S represents the geometric locus of points described by the tip of the radius vector $\overrightarrow{r}(u, v)$, $(u, v) \in D$ (Figure 2.2.1). Moreover, the smoothness property of the surface S formulated above is expressed by the condition of nontriviality of the vector product:

$$[\overrightarrow{r}_u \times \overrightarrow{r}_v] \neq 0, \tag{2.2.3}$$

where $\overrightarrow{r}_u$ and $\overrightarrow{r}_v$ are the partial derivatives of the radius vector $\overrightarrow{r}$ with respect to u and v.

The vectors $\overrightarrow{r}_u$ and $\overrightarrow{r}_v$—the tangent vectors to the u and v coordinate lines, respectively, on the surface S—uniquely determine the tangent plane at the point P.

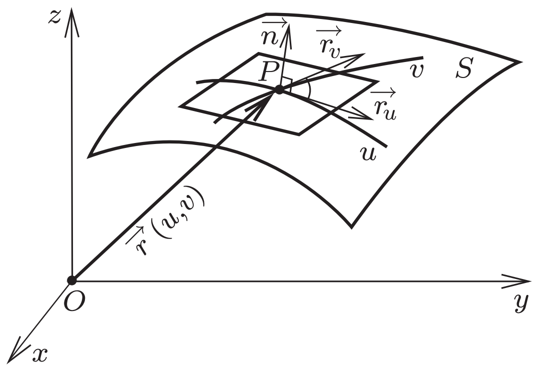


Figure 2.2.1

We say that the surface $S \subset \mathbb{E}^3$ is *smooth of class* C^n, $n \geq 1$ (or is C^n-*smooth*), if its radius vector satisfies the condition

$$\overrightarrow{r}(u,v) \in C^n(D).$$

The points at which the smoothness condition (2.2.3) is violated are called *singular*. On a surface such an individual point can be, for example, a cusp point; it is also possible that a continuous collection of singular points form an irregular edge of the surface (as a rule, a cuspidal edge). Such surfaces, given parametrically (see (2.2.1)), may self-intersect.

For this reason, in what follows we will, without restricting the generality of the surfaces studied, allow them to have self-intersections and irregular singularities, and then use for such "objects" of our investigations the general term of "*surface*".

Next we will discuss a number important general characteristics of surfaces in $\mathbb{E}^3$, referring the reader to the works [39, 40, 70, 81, 101] on the classical theory of surfaces, and remarking at the same time that the branch of geometry considered here can be quite appropriately understood as one of the beautiful practical applications of mathematical analysis [36].

2.2.2 First fundamental form of a surface

Consider in $\mathbb{E}^3$ a smooth surface S, given by its radius vector $\overrightarrow{r}(u,v)$.

The *first fundamental form of the surface* S is defined to be the square of the differential of the radius vector of S:

$$\mathrm{I}(u,v) = d\overrightarrow{r}^2 = (\overrightarrow{r}_u du + \overrightarrow{r}_v dv)^2, \tag{2.2.4}$$

or

$$\mathrm{I}(u,v) = \overrightarrow{r}_u{}^2 du^2 + 2(\overrightarrow{r}_u, \overrightarrow{r}_v) du dv + \overrightarrow{r}_v{}^2 dv^2, \tag{2.2.5}$$

where $(\overrightarrow{r}_u,\ \overrightarrow{r}_v)$ denotes the scalar (inner) product of the vectors $\overrightarrow{r}_u$ and $\overrightarrow{r}_v$.

The first fundamental form (2.5.5) is a quadratic form in the differentials du and dv and is always *positive definite*, thanks to the positivity of its leading principal minors (by Sylvester's criterion [35]):

1) $\vec{r_u}^2 > 0$,

2) $\vec{r_u}^2 \cdot \vec{r_v}^2 - (\vec{r_u}, \vec{r_v})^2 = ([\vec{r_u} \times \vec{r_v}])^2 > 0$.

The coefficients of the first fundamental form (2.2.5) are usually denoted by

$$E(u,v) = \vec{r_u}^2, \quad F(u,v) = (\vec{r_u}, \vec{r_v}), \quad G(u,v) = \vec{r_v}^2, \tag{2.2.6}$$

and then (2.2.5) is rewritten as

$$\mathrm{I}(u,v) = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2. \tag{2.2.7}$$

The first fundamental form of the surface can be used for various calculations [36, 101]: length of curves, angles between curves, areas of domains on the surface, and so on. For example, the length of a curve l on the surface S, given parametrically by

$$l\colon\ u = u(t), \quad v = v(t), \quad t \in [t_1, t_2]$$

or, respectively, by the radius vector

$$\vec{r}(t) = \vec{r}(u(t), v(t)), \quad t \in [t_1, t_2],$$

is calculated by the formula

$$l = \int\limits_{t_1}^{t_2} |\vec{r}\,'|dt = \int\limits_{t_1}^{t_2} \sqrt{E(u(t),v(t)) \cdot u'^2 + 2F(u(t),v(t)) \cdot u' \cdot v' + G(u(t),v(t))v'^2}dt.$$

Let us provide also the formula for calculating the area A_Σ of a domain Σ on the surface S for which the parameters (u,v) range in a domain D:

$$A_\Sigma = \iint_D \sqrt{EG - F^2}dudv.$$

If we choose some curve on the surface, then the restriction of the first fundamental form to that curve coincides with the square of the differential of its arc: $\vec{dr}^2 = ds^2$. The square of the linear element ds^2 is called the *metric of the surface*:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

The geometric properties of a surface that can be obtained based on only the first fundamental form (i.e., on the coefficients E, F, G), constitute what is called the *intrinsic geometry* of the surface.

If one pictures the surface as a flexible, inextensible pellicle, then when one bends it all intrinsic geometric properties are preserved, whereas its shape in space changes.

Surfaces that have identical intrinsic geometries are said to be *isometric*.

2.2.3 Second fundamental form of a surface

At each point of a C^2-smooth surface S one can consider the unit normal vector $\overrightarrow{n}$ (see Figure 2.2.1):

$$\overrightarrow{n}(u,v) = \frac{[\overrightarrow{r}_u \times \overrightarrow{r}_v]}{|[\overrightarrow{r}_u \times \overrightarrow{r}_v]|}, \quad |\overrightarrow{n}| = 1, \tag{2.2.8}$$

where $|[\overrightarrow{r}_u \times \overrightarrow{r}_v]| = \sqrt{EG - F^2}$.

Let us consider the second differential of the radius vector $\overrightarrow{r}(u,v)$ of the surface,

$$d^2\overrightarrow{r} = \overrightarrow{r}_{uu}du^2 + 2\overrightarrow{r}_{uv}dudv + \overrightarrow{r}_{vv}dv^2 + \overrightarrow{r}_u d^2u + \overrightarrow{r}_v d^2v,$$

and introduce the notion of the *second fundamental form of the surface* S as the scalar product of the vectors $d^2\overrightarrow{r}$ and $\overrightarrow{n}$:

$$\mathrm{II}(u,v) = (d^2\overrightarrow{r}, \overrightarrow{n}). \tag{2.2.9}$$

Using the expression of $d^2\overrightarrow{r}$ given above and the fact that the vector $\overrightarrow{n}$ is orthogonal to the vectors $\overrightarrow{r}_u$ and $\overrightarrow{r}_v$ (which span the tangent plane to the surface), i.e.,

$$(\overrightarrow{r}_u, \overrightarrow{n}) = 0, \quad (\overrightarrow{r}_v, \overrightarrow{n}) = 0, \tag{2.2.10}$$

we obtain the following representation of the second fundamental of the surface S, defined in (2.2.9):

$$\mathrm{II} = (\overrightarrow{r}_{uu}, \overrightarrow{n})du^2 + 2(\overrightarrow{r}_{uv}, \overrightarrow{n})dudv + (\overrightarrow{r}_{vv}, \overrightarrow{n})dv^2. \tag{2.2.11}$$

Denoting the coefficients of the second fundamental form by

$$L = (\overrightarrow{r}_{uu}, \overrightarrow{n}), \quad M = (\overrightarrow{r}_{uv}, \overrightarrow{n}), \quad N = (\overrightarrow{r}_{vv}, \overrightarrow{n}), \tag{2.2.12}$$

we recast (2.2.11) as

$$\mathrm{II}(u,v) = L(u,v)du^2 + 2M(u,v)dudv + N(u,v)dv^2. \tag{2.2.13}$$

The coefficients (2.2.12) can be also written in a different form (involving only the radius vector of the surface), to which one arrives by substituting the expression (2.2.8) of $\overrightarrow{n}$ in (2.2.12):

$$L = \frac{(\overrightarrow{r}_{uu}, \overrightarrow{r}_u, \overrightarrow{r}_v)}{|[\overrightarrow{r}_u \times \overrightarrow{r}_v]|}, \quad M = \frac{(\overrightarrow{r}_{uv}, \overrightarrow{r}_u, \overrightarrow{r}_v)}{|[\overrightarrow{r}_u \times \overrightarrow{r}_v]|}, \quad N = \frac{(\overrightarrow{r}_{vv}, \overrightarrow{r}_u, \overrightarrow{r}_v)}{|[\overrightarrow{r}_u \times \overrightarrow{r}_v]|}. \tag{2.2.14}$$

In (2.2.14) the parentheses denote the mixed products of the indicated vectors.

The form (2.2.14) of the coefficients L, M, N of the second fundamental form is convenient, for example, if we want to describe a surface in the space $\mathbb{E}^3(x,y,z)$, given in the Monge form: $z = z(x,y)$ (see Subsection 1.3.1). In this case (putting $u \equiv x$, $v \equiv y$) the radius vector of the surface is

$$\overrightarrow{r} = \overrightarrow{r}(x,y) = \{x, y, z(x,y)\}.$$

Correspondingly, the mixed products of the vectors in (2.2.14) are calculated as

$$(\overrightarrow{r}_{xx}, \overrightarrow{r}_x, \overrightarrow{r}_y) = \begin{vmatrix} 0 & 0 & z_{xx} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = z_{xx}, \quad (\overrightarrow{r}_{xy}, \overrightarrow{r}_x, \overrightarrow{r}_y) = \begin{vmatrix} 0 & 0 & z_{xy} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = z_{xy},$$

$$(\overrightarrow{r}_{yy}, \overrightarrow{r}_x, \overrightarrow{r}_y) = \begin{vmatrix} 0 & 0 & z_{yy} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = z_{yy}.$$

Now observing that

$$|[\overrightarrow{r}_x \times \overrightarrow{r}_y]| = \sqrt{1 + z_x^2 + z_y^2},$$

and substituting the expressions obtained in (2.2.14), we obtain the expressions of the coefficients L, M, N for a surface S given in Monge form:

$$L = \frac{z_{xx}}{\sqrt{1 + z_x^2 + z_y^2}}, \quad M = \frac{z_{xy}}{\sqrt{1 + z_x^2 + z_y^2}}, \quad N = \frac{z_{yy}}{\sqrt{1 + z_x^2 + z_y^2}}. \tag{2.2.15}$$

To conclude this subsection we indicate another way of computing the coefficients of the second fundamental form. To this end we consider the conditions (2.2.10) and differentiate each of these equalities with respect to u and v, taking into account the form (2.2.12) of the coefficients we are interested in.

In this way we obtain a new representation for the coefficients of the second fundamental form of a surface:

$$L = -(\overrightarrow{r}_u, \overrightarrow{n}_u), \quad M = -(\overrightarrow{r}_u, \overrightarrow{n}_v) = -(\overrightarrow{r}_v, \overrightarrow{n}_u), \quad N = -(\overrightarrow{r}_v, \overrightarrow{n}_v). \tag{2.2.16}$$

The expressions (2.2.16) obtained for the coefficients L, M, N correspond to the following equivalent definition of the second fundamental form:

$$\mathrm{II}(u, v) = -(d\overrightarrow{r}, d\overrightarrow{n}). \tag{2.2.17}$$

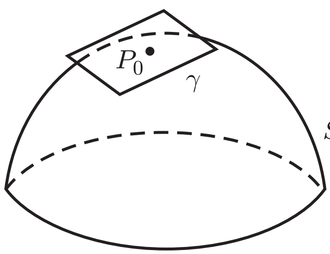


Figure 2.2.2

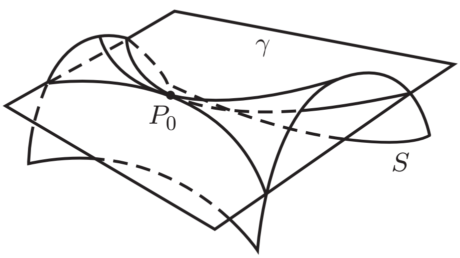


Figure 2.2.3

The second fundamental form carries information on the shape of the surface in space. Let us briefly explain the classification of points of a surface according to the sign of the determinant of the second fundamental form [80, 81].

A point P on a C^2-smooth surface S is called *elliptic* if the determinant of the second fundamental form of S at P is positive, i.e., $(LN - M^2)\mid_{P_0 \in S} > 0$. Intuitively this means that all the points of S that are sufficiently close to the point P lie on the same side with respect to the tangent plane γ to S at P (Figure 2.2.2).

The point $P_0 \in S$ is called *hyperbolic*, if $(LN - M^2)\mid_{P_0 \in S} < 0$. In this case in any neighborhood of P there are parts of the surface S that lie on different sides with respect to the tangent plane γ (Figure 2.2.3).

Finally, if $(LN - M^2)\mid_{P_0 \in S} = 0$, but at least one of the coefficients $L(P_0)$, $N(P_0)$ is not zero, then $P_0 \in S$ is called a *parabolic* point of the surface S. In this case there exists directions on S along which the second fundamental form vanishes, while in the other directions it has one and the same sign (Figure 2.2.4).

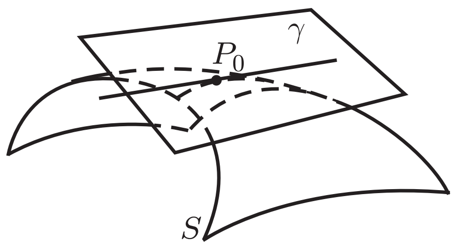


Figure 2.2.4

If $L(P_0) = N(P_0) = M(P_0) = 0$, then P is called a *flat point* (or a *geodesic point*) of the surface.

As these name indicate, the first and second forms are fundamental characteristics of a surface.

In the next section of the present chapter (§ 2.3), while studying the basic equations of the theory of surfaces, we will answer the following question: Under what conditions two given quadratic forms can play the role of the first and, respectively, the second fundamental form of some surface? The answer will be provided by Bonnet's theorem.

2.2.4 Gaussian curvature of a surface

In general, in geometry the term "curvature" is used to express characteristics of geometric shapes (objects) that measure how much they deviate from the the corresponding classical Euclidean shapes (objects). In this sense the *Gaussian curvature* of a surface in the space $\mathbb{E}^3$ can be interpreted as a measure of how much the surface deviates (is "deformed") from an Euclidean plane. Furthermore, it turns out that surfaces in $\mathbb{E}^3$ can be intuitively classified according to the sign of the Gaussian curvature.

Let us consider a normal section of the surface S at the point P, i.e., a section of the surface by a plane π that passes through the normal $\overrightarrow{n}(P)$ to S at P. This section is a plane curve $l \subset S$ that passes through the point P in the direction of the vector $\overrightarrow{l}$ tangent to it (Figure 2.2.5). Obviously, there are infinitely many such sections for every chosen point P.

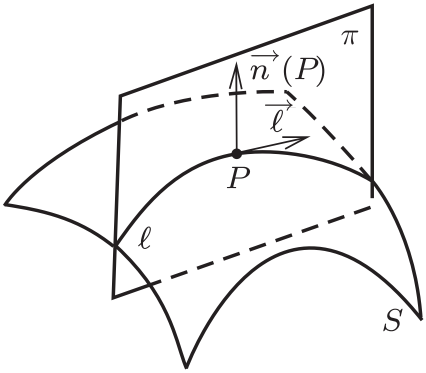


Figure 2.2.5

The *normal curvature* k_{n} of the surface S at the point P in the direction $\overrightarrow{l}$ is defined to be the curvature of the corresponding normal section, i.e., the curvature of the plane curve l. Recall that the curvature of a plane curve is the limit $k = \lim_{\Delta s \to 0} \frac{\Delta\vartheta}{\Delta s}$, where $\Delta\vartheta$ is the change in the angle of the position of the tangent to the curve and Δs is the distance along the curve between the points of tangency.

Let us define the *principal directions* on a surface at a given point. A direction on a regular surface is called *principal* if the normal curvature of the surface attains an extremum in that direction. At each point of a C^2-smooth surface there are no less than two distinct principal directions. The corresponding extremum values k_1 and k_2 of the normal curvature k_{n} in the principal directions are called the *principal curvatures* of the surface at the given point.

The *Gaussian curvature* K of a surface at a point is defined to be the product of its principal curvatures at that point:

$$K = k_1 \cdot k_2. \tag{2.2.18}$$

The Gaussian curvature can be expressed through the coefficients of the first and second fundamental forms of the surface (see § 2.3) as follows:

$$K = \frac{LN - M^2}{EG - F^2}. \tag{2.2.19}$$

Moreover, the Gaussian curvature K can be obtained using only the first fundamental form, a fact expressed by what is known as Gauss' Theorema Egregium.

In other words, the Gaussian curvature is an intrinsic geometric characteristic of a surface.

As one can see from (2.2.19), since the first fundamental form is positive definite, the sign of the Gaussian curvature K coincides with the sign of the determinant of the second fundamental form. Accordingly, at elliptic points (or in elliptic domains) of the surface, the curvature K is positive: $K > 0$. An example of elliptic surface is shown in Figure 2.2.2 (any section of such a "convex" surface by a plane cuts from the surface a "cap" with boundary).

If $K < 0$ on the entire surface, then the surface is hyperbolic (see, for example, Figure 2.2.3). Nontrivial examples of hyperbolic surfaces are the Minding surfaces of revolution considered in § 1.3: the "top" (Figure 1.3.2), the "bobbin" (Figure 1.3.5), and the pseudosphere (Figure 1.3.8).

2.3 Fundamental systems of equations in the theory of surfaces in $\mathbb{E}^3$

This section is devoted to the derivation and subsequent analysis of the fundamental relations that completely determine a surface in the space $\mathbb{E}^3$.

2.3.1 Derivational formulas

Consider in the space $\mathbb{E}^3$ a C^3-smooth surface S, given by its radius vector $\overrightarrow{r}(u, v)$, $(u, v) \in D \subset \mathbb{R}^2$. As we already indicated, the variables u and v play the role of intrinsic coordinates on the surface S: the families of lines $u = \text{const}$ and $v = \text{const}$ form a coordinate net on S.

Pick a point P in a regular part of the surface S, i.e., a point at which

$$W(P) = E(P)G(P) - F^2(P) \neq 0.$$

At each such point the vectors $\overrightarrow{r}_u$, $\overrightarrow{r}_v$, $\overrightarrow{n}$ (Figure 2.2.1) are linearly independent, and hence form a basis in $\mathbb{E}^3$.

Let us decompose the vectors $\overrightarrow{r}_{uu}$, $\overrightarrow{r}_{uv}$, $\overrightarrow{r}_{vv}$, $\overrightarrow{n}_u$, and $\overrightarrow{n}_v$, which appear in the definition of the first and second fundamental forms of the surface, with respect to the basis $\overrightarrow{r}_u$, $\overrightarrow{r}_v$, $\overrightarrow{n}$:

$$\overrightarrow{r}_{uu} = \Gamma_{11}^1 \overrightarrow{r}_u + \Gamma_{11}^2 \overrightarrow{r}_v + \lambda_{11} \overrightarrow{n}, \tag{2.3.1}$$

$$\overrightarrow{r}_{uv} = \Gamma_{12}^1 \overrightarrow{r}_u + \Gamma_{12}^2 \overrightarrow{r}_v + \lambda_{12} \overrightarrow{n}, \tag{2.3.2}$$

$$\overrightarrow{r}_{vv} = \Gamma_{22}^1 \overrightarrow{r}_u + \Gamma_{22}^2 \overrightarrow{r}_v + \lambda_{22} \overrightarrow{n}, \tag{2.3.3}$$

$$\overrightarrow{n}_u = \alpha_{11} \overrightarrow{r}_u + \alpha_{12} \overrightarrow{r}_v + \alpha_{10} \overrightarrow{n}, \tag{2.3.4}$$

$$\overrightarrow{n}_v = \alpha_{21} \overrightarrow{r}_u + \alpha_{22} \overrightarrow{r}_v + \alpha_{20} \overrightarrow{n}, \tag{2.3.5}$$

where Γ_{ij}^k, λ_{ij}, α_{ij} are the decomposition coefficients, which need to be determined.

Let us verify that all the decomposition coefficients in the relations (2.3.1)–(2.3.5) can be calculated in terms of only the coefficients of the first and second fundamental forms of the surface.

Multiplying each of the equalities (2.3.1)–(2.3.3) scalarly by the unit normal vector $\vec{n}$ ($\vec{n}^2 = 1$) and using (2.2.10) we obtain

$$\lambda_{11} = (\vec{r}_{uu}, \vec{n}) = L, \quad \lambda_{12} = (\vec{r}_{uv}, \vec{n}) = M, \quad \lambda_{22} = (\vec{r}_{vv}, \vec{n}) = N. \tag{2.3.6}$$

Next let us calculate the coefficients Γ^k_{ij}, $i, j, k = 1, 2$, known as the *Christoffel symbols*. To this end we consider the relation (2.3.1) and take its scalar product with the vectors $\vec{r}_u$ and $\vec{r}_v$, which yields the following two relations:

$$\begin{aligned} (\vec{r}_{uu}, \vec{r}_u) &= \Gamma^1_{11} \cdot E + \Gamma^2_{11} \cdot F, \\ (\vec{r}_{uu}, \vec{r}_v) &= \Gamma^1_{11} \cdot F + \Gamma^2_{11} \cdot G. \end{aligned} \tag{2.3.7}$$

The expressions in the left-hand side of (2.3.7) can be written as

$$\begin{aligned} (\vec{r}_{uu}, \vec{r}_u) &= \frac{1}{2}\left(\vec{r}_u^{\,2}\right)'_u = \frac{1}{2}E_u, \\ (\vec{r}_{uu}, \vec{r}_v) &= (\vec{r}_u, \vec{r}_v)'_u - (\vec{r}_u, \vec{r}_{uv}) = F_u - \frac{1}{2}E_v. \end{aligned}$$

Accordingly, the system (2.3.7) can be recast as a very simple linear inhomogeneous system in the unknowns Γ^1_{11} and Γ^2_{11}:

$$\begin{aligned} \Gamma^1_{11} \cdot E + \Gamma^2_{11} \cdot F &= \frac{1}{2}E_u, \\ \Gamma^1_{11} \cdot F + \Gamma^2_{11} \cdot G &= F_u - \frac{1}{2}E_v. \end{aligned} \tag{2.3.8}$$

The solution of (2.3.8) (under our assumption that $W = EG - F^2 \neq 0$) is

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2W}\left(E_u G - 2F_u F + E_v F\right), \\ \Gamma^2_{11} &= \frac{1}{2W}\left(-E_u F + 2F_u E - E_v E\right), \quad W = EG - F^2. \end{aligned} \tag{2.3.9}$$

Proceeding in an analogous manner with the equalities (2.3.2) and (2.3.3), we obtain formulas for the remaining Christoffel symbols Γ^k_{ij}:

$$\begin{aligned} \Gamma^1_{12} &= \frac{1}{2W}\left(E_v G - G_u F\right), \\ \Gamma^2_{12} &= \frac{1}{2W}\left(G_u E - E_v F\right), \end{aligned} \tag{2.3.10}$$

and

$$\begin{aligned} \Gamma^1_{22} &= \frac{1}{2W}\left(-G_u G + 2F_v G - G_v F\right), \\ \Gamma^2_{22} &= \frac{1}{2W}\left(G_v E - 2F_v F + G_u F\right). \end{aligned} \tag{2.3.11}$$

Therefore, expressions (2.3.9)–(2.3.11) and (2.3.6) provide the exact form of the coefficients appearing in the system (2.3.1)–(2.3.3).

Now let us find the coefficients α_{ij} that appear in the system (2.3.4),(2.3.5). To this end, we differentiate the equality $\vec{n}^2 = 1$, which yields

$$(\vec{n}, \vec{n}_u) = 0, \quad (\vec{n}, \vec{n}_v) = 0. \tag{2.3.12}$$

From (2.3.12) it follows that the vectors $\vec{n}_u$ and $\vec{n}_v$, with their common origin at the point P, lie in the tangent plane to the surface at P, and hence can be written as linear combinations of the vectors $\vec{r}_u$ and $\vec{r}_v$. Indeed, taking the scalar product of the equalities (2.3.4) and (2.3.5) with the unit vector $\vec{n}$ and using (2.2.10), we obtain

$$\alpha_{10} = \frac{1}{2}\left(\vec{n}^2\right)'_u = 0, \quad \alpha_{20} = \frac{1}{2}\left(\vec{n}^2\right)'_v = 0. \tag{2.3.13}$$

The system (2.3.4),(2.3.5) can be now rewritten in the “simplified” form

$$\begin{aligned} \vec{n}_u &= \alpha_{11}\vec{r}_u + \alpha_{12}\vec{r}_v, \\ \vec{n}_v &= \alpha_{21}\vec{r}_u + \alpha_{22}\vec{r}_v. \end{aligned} \tag{2.3.14}$$

Essentially, relations (2.3.14) effect the transition from the basis $\vec{r}_u, \vec{r}_v$ in the tangent space to the basis $\vec{n}_u, \vec{n}_v$. Take the first equality in (2.3.14) and multiply it scalarly by the vector $\vec{r}_u$, and then by the vector $\vec{r}_v$. Using (2.2.6) and (2.2.16), we obtain

$$-L = \alpha_{11}E + \alpha_{12}F, \quad -M = \alpha_{11}F + \alpha_{12}G. \tag{2.3.15}$$

The linear system (2.3.15) for α_{11} and α_{12} has the solution

$$\alpha_{11} = \frac{1}{W}(MF - LG), \quad \alpha_{12} = \frac{1}{W}(LF - ME). \tag{2.3.16}$$

Similar arguments yield the other two coefficients:

$$\alpha_{21} = \frac{1}{W}(NF - MG), \quad \alpha_{22} = \frac{1}{W}(MF - NE). \tag{2.3.17}$$

Thus, the system of equations (2.3.1)–(2.3.5) for the radius vector $\vec{r}(u,v)$ of the surface S and the unit normal vector $\vec{n}(u,v)$ is completely determined by the coefficients of the first and second fundamental forms of the surface. Moreover, the corresponding coefficients are given by the formulas (2.3.6), (2.3.9)–(2.3.11), (2.3.16) and (2.3.17).

The system of equations (2.3.1)–(2.3.5) is known as the system of *derivational formulas*.

The system of derivational formulas (2.3.1)–(2.3.55) can be written in matrix form, grouping the available equations into two triples, (2.3.1), (2.3.2), (2.3.4) and (2.3.2), (2.3.3), (2.3.5):

$$\begin{pmatrix} \vec{r}_u \\ \vec{r}_v \\ \vec{n} \end{pmatrix}_u = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & L \\ \Gamma^1_{12} & \Gamma^2_{12} & M \\ \alpha_{11} & \alpha_{12} & 0 \end{pmatrix} \begin{pmatrix} \vec{r}_u \\ \vec{r}_v \\ \vec{n} \end{pmatrix}, \tag{2.3.18}$$

$$\begin{pmatrix} \overrightarrow{r_u} \\ \overrightarrow{r_v} \\ \overrightarrow{n} \end{pmatrix}_v = \begin{pmatrix} \Gamma^1_{12} & \Gamma^2_{12} & M \\ \Gamma^1_{22} & \Gamma^2_{22} & N \\ \alpha_{21} & \alpha_{22} & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{r_u} \\ \overrightarrow{r_v} \\ \overrightarrow{n} \end{pmatrix}. \tag{2.3.19}$$

If we now denote the matrices involved by

$$R = \begin{pmatrix} \overrightarrow{r_u} \\ \overrightarrow{r_v} \\ \overrightarrow{n} \end{pmatrix}, \quad A = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & L \\ \Gamma^1_{12} & \Gamma^2_{12} & M \\ \alpha_{11} & \alpha_{12} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma^1_{12} & \Gamma^2_{12} & M \\ \Gamma^1_{22} & \Gamma^2_{22} & N \\ \alpha_{21} & \alpha_{22} & 0 \end{pmatrix},$$

we can rewrite the system in “compact form” as

$$R_u = AR, \tag{2.3.20}$$
$$R_v = BR. \tag{2.3.21}$$

In order for the system (2.3.20), (2.3.21) to be consistent (i.e., for the derivation formulas (2.3.1)–(2.3.5) to be consistent), the condition $(R_u)_v = (R_v)_u$ must be satisfied, which in the case of the system (2.3.20), (2.3.21) obtained above takes on the form

$$A_v - B_u + [A, B] = 0, \tag{2.3.22}$$

where $[A, B] = AB - BA$.

We must emphasize here that a relation of the form (2.3.22), which encodes fundamental connections in the theory of surfaces—a theory more than two centuries old—turned out to be intimately related to a series of important equations of modern mathematical physics. Thus, for example, structural relations of the form (2.3.22) arise in the implementation of the algorithms of the method of the inverse scattering transform, applied to the integration of nonlinear differential equations, and is connected, in particular, with the formulation of the Zakharov-Shabat modified scattering problem [1, 51]. The “geometric sources” of various modern nonlinear problems will be discussed in detail in Chapter 4. Here we mention only that the presence of nontrivial curvature in non-Eulidean geometry and the emergence of nonlinearities in various novel models in science are “facts” of identical nature.

2.3.2 The Peterson-Codazzi and Gauss equations. Bonnet’s theorem

The consistency (compatibility) condition (2.3.22) of the system of differentiation formulas (2.3.1)–(2.3.5) can be written in terms of components; in this case we first of all have nine equations that connect the corresponding coefficients of the 3×3-matrices appearing in (3.2.22):

1) $(\Gamma^1_{11})_v - (\Gamma^1_{12})_u + \Gamma^2_{11}\Gamma^1_{22} - \Gamma^1_{12}\Gamma^2_{12} + \alpha_{21}L - \alpha_{11}M = 0,$

2) $(\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2 + \alpha_{22}L - \alpha_{12}M = 0,$

3) $L_v - M_u + \Gamma^2_{11} \cdot N - \Gamma^1_{12} \cdot L + (\Gamma^1_{11} - \Gamma^2_{12})M = 0,$

4) $(\Gamma^1_{12})_v - (\Gamma^1_{22})_u + (\Gamma^1_{12})^2 + \Gamma^2_{12}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^1_{11} - \Gamma^2_{22}\Gamma^1_{12} + \alpha_{21}M - \alpha_{11}N = 0,$

5) $(\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11} + \alpha_{22}M - \alpha_{12}N = 0,$

6) $M_v - N_u + \Gamma^2_{12}N - \Gamma^1_{22}L + (\Gamma^1_{12} - \Gamma^2_{22})M = 0,$

7) $(\alpha_{11})_v - (\alpha_{21})_u + \alpha_{11}\Gamma^1_{12} + \alpha_{12}\Gamma^1_{22} - \alpha_{21}\Gamma^1_{11} - \alpha_{22}\Gamma^1_{12} = 0,$

8) $(\alpha_{12})_v - (\alpha_{22})_u + \alpha_{11}\Gamma^2_{12} + \alpha_{12}\Gamma^2_{22} - \alpha_{21}\Gamma^2_{11} - \alpha_{22}\Gamma^2_{12} = 0,$

9) $\alpha_{11}M + \alpha_{12}N - \alpha_{21}L - \alpha_{22}M = 0.$

A more detailed analysis of the nine equations listed above when we substitute in them the already obtained expressions for the coefficients Γ^k_{ij}, α_{ij} (formulas (2.3.9)–(2.3.11), (2.3.16), (2.3.17)) shows that in fact only three of these nine equations are independent. In particular, the last equation becomes an identity.

Leaving the direct verification of this fact to the diligent reader, we provide below the three fundamental equations (equations 3, 6, and 1) of the system, which express the relationships between the coefficients of the first and second fundamental forms of a surface in $\mathbb{E}^3$:

$$L_v + \Gamma^1_{11}M + \Gamma^2_{11}N = M_u + \Gamma^1_{12}L + \Gamma^2_{12}M, \tag{2.3.23}$$

$$M_v + \Gamma^1_{12}M + \Gamma^2_{12}N = N_u + \Gamma^1_{22}L + \Gamma^2_{22}M, \tag{2.3.24}$$

$$\frac{1}{F}\left((\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^1_{22}\right) = \frac{1}{F}(\alpha_{21}L - \alpha_{11}M) = \frac{LN - M^2}{EG - F^2} = K. \tag{2.3.25}$$

The system (2.3.23)–(2.3.25) can be written in expanded form as

$$2W(L_v - M_u) - (EN - 2FM + GL)(E_v - F_u) + \begin{vmatrix} E & E_u & L \\ F & F_u & M \\ G & G_u & N \end{vmatrix} = 0, \tag{2.3.26}$$

$$2W(M_v - N_u) - (EN - 2FM + GL)(F_v - G_u) + \begin{vmatrix} E & E_v & L \\ F & F_v & M \\ G & G_v & N \end{vmatrix} = 0, \tag{2.3.27}$$

(*Peterson-Codazzi equations*),

$$\begin{aligned} K &= \frac{LN - M^2}{EG - F^2} \\ &= -\frac{1}{4W^2}\begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} - \frac{1}{2\sqrt{W}}\left[\left(\frac{E_v - F_u}{\sqrt{W}}\right)_v - \left(\frac{F_v - G_u}{\sqrt{W}}\right)_u\right]. \end{aligned} \tag{2.3.28}$$

(*Gauss formula*).

The system (2.3.26)–(2.3.28) is called the *system of Peterson-Codazzi and Gauss equations*.

Formula (2.3.28) expresses Gauss's theorem (*Theorema Egregium*): *The Gaussian curvature of a C^3-smooth surface can be expressed in term of only the coefficients of the first fundamental form of the surface and their partial derivatives.*

The essential role played by the first and second fundamental forms of a surface in $\mathbb{E}^3$ is expressed by Bonnet's theorem.

Bonnet's Theorem *Let*

$$E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2 \tag{I}$$

and

$$L(u,v)du^2 + 2M(u,v)dudv + N(u,v)dv^2 \tag{II}$$

be two arbitrary quadratic forms, the first of which is positive definite (*i.e.*, $W = EG - F^2 > 0$). *Assume that the coefficients E, F, G, L, M, N are connected by the system of Peterson-Codazzi and Gauss equations.*

Then in the Euclidean space $\mathbb{E}^3$ there exists a unique (*up to position in space*) *surface for which the quadratic forms* (I) *and* (II) *are the first, and respectively the second fundamental form.*

Let us mention also that the Peterson-Codazzi and Gauss equations together with the derivational equations allow one to structure analytically an algorithm for solving the problem of isometric immersion of two-dimensional metrics in the three-dimensional Euclidean space (Subsection 2.1.2). The first step in constructing an immersion of a given two-dimensional metric is to determine the coefficients L, M, N of the second fundamental form from the given metric (the coefficients E, F, G) and is connected with the integration of the Peterson-Codazzi and Gauss equations. The second step consists in finding, starting from the sets $\{E, F, G\}$ and $\{L, M, N\}$ (obtained in the "preliminary" extrinsic geometric realization) and using the derivational equations, the radius vector $\overrightarrow{r}$ of the sought-for surface $S \subset \mathbb{E}^3$ and its unit normal vector $\overrightarrow{n}$.[3]

2.3.3 The Rozhdestvenskii-Poznyak system of equations in Riemann invariants

We next consider an important modification of the Peterson-Codazzi and Gauss equations in the case of negative Gaussian curvature $K = -k^2 < 0$. To do this we introduce new, "reduced" coefficients l, m, n of the second fundamental form, which are obtained by multiplying the ordinary coefficients L, M, N by $\dfrac{1}{\sqrt{W}}$, $W = EG - F^2 > 0$:

$$l = \frac{1}{\sqrt{W}} \cdot L, \quad m = \frac{1}{\sqrt{W}} \cdot M, \quad n = \frac{1}{\sqrt{W}} \cdot N. \tag{2.3.29}$$

[3] The scheme described does in no way "invalidate" a successfull search for an isometric immersion based on the primary relations

$$\overrightarrow{r_u}^2 = E(u,v), \quad \overrightarrow{r_u} \cdot \overrightarrow{r_v} = F(u,v), \quad \overrightarrow{r_v}^2 = G(u,v),$$

which retain their universal form also in the more general case of a higher-dimensional ambient space $\mathbb{E}^n$ with $n \geq 3$.

In terms of the functions l, m, n, the system of Peterson-Codazzi and Gauss equations (2.3.23)–(2.3.25) (or (2.3.26)–(2.3.28)) is recast as

$$l_v - m_u = -\Gamma^2_{22}\, l + 2\Gamma^2_{12}\, m - \Gamma^2_{11}\, n, \tag{2.3.30}$$

$$n_u - m_v = -\Gamma^1_{22}\, l + 2\Gamma^1_{12}\, m - \Gamma^1_{11}\, n, \tag{2.3.31}$$

$$ln - m^2 = -k^2. \tag{2.3.32}$$

Now let us introduce the functions $r = r(u, v)$ and $s = s(u, v)$ by

$$r = -\frac{m+k}{n}, \quad s = -\frac{m-k}{n}. \tag{2.3.33}$$

The choice (2.3.33) is "geometrically" dictated by the fact that in the case of the realization of a metric (2.2.7) of negative Gaussian curvature in the form of a regular surface in $\mathbb{E}^3$, the functions $r(u, v)$ and $s(u, v)$ will have the intuitive meaning of the angular coefficients of the images of the asymptotic lines on the surface in the (u, v)-parametric plane. Applying Gauss's formula (2.3.32) to (2.3.33), it is easy to find that

$$l = \frac{2krs}{s-r}, \quad m = -\frac{k(r+s)}{s-r}, \quad n = \frac{2k}{s-r}. \tag{2.3.34}$$

If we now substitute the expressions (2.3.34) for the unknowns l, m, n in the Peterson-Codazzi equations (2.3.30), (2.3.31), we arrive at a system of equations of the form (see [76]):

$$r_u + sr_v = A_0 + A_1 \cdot r + A_2 \cdot s + A_3 \cdot r^2 + A_4 \cdot rs + A_5 \cdot r^2 s, \tag{2.3.35}$$

$$s_u + rs_v = A_0 + A_1 \cdot s + A_2 \cdot r + A_3 \cdot s^2 + A_4 \cdot rs + A_5 \cdot rs^2, \tag{2.3.36}$$

for which the coefficients in the right-hand sides are given by the formulas

$$\begin{aligned} A_0 &= -\Gamma^2_{11}, \\ A_1 &= \Gamma^1_{11} - \Gamma^2_{12} + Q_u, \\ A_2 &= -\Gamma^2_{12} - Q_u, \\ A_3 &= \Gamma^1_{12} + Q_v, \\ A_4 &= -\Gamma^2_{22} + \Gamma^1_{12} - Q_v, \\ A_5 &= \Gamma^1_{22}, \\ Q &= \frac{1}{2}\ln k. \end{aligned} \tag{2.3.37}$$

The system (2.3.35), (2.3.36) is a system of quasilinear equations of hyperbolic type.[4] The expressions in the left-hand sides of the equations are the total

[4]Equations of this type, for example, are effectively studied in gas dynamics problems [97].

derivatives of the functions $r(u,v)$ and $s(u,v)$ along the *characteristics* of the system that are given by the equations

$$\frac{dv}{du} = r(u,v), \quad \frac{dv}{du} = s(u,v).$$

The functions $r(u,v)$ and $s(u,v)$ are called *Riemann invariants*.

The system in "Riemann invariants" was first proposed by B. L. Rozhdestvenskii [96] in the case when the metric is written in a semi-geodesic coordinate system. In the generic case (for arbitrary metrics of negative curvature) the system in its definitive form (2.3.35), (2.3.36) was obtained by E. G. Poznyak [76].

The system (2.3.35), (2.3.36) is called *the Rozhdestvenskii-Poznyak system of fundamental equations of the theory of surfaces of negative curvature in Riemann invariants*.

In general, the system in "Riemann invariants" is not equivalent to the system of Peterson-Codazzi and Gauss equations; however, when the condition $r(u,v) \neq s(u,v)$ is satisfied, the functions l, m, n can be uniquely recovered from the solutions of the system (2.3.35), (2.3.36) by means of the relations (2.3.34).

Notice also that the form of the system in "Riemann invariants" (2.3.35), (2.3.36) offers (thanks to the quasilinearity of the equations) additional possibilities in the study of the problem of isometric immersion of individual special domains of the Lobachevsky plane Λ^2 in the space $\mathbb{E}^3$ (see, e.g., [74, 125]).

2.3.4 Structure equations of a surface in $\mathbb{E}^3$

In the theory of surfaces, side by side with the approaches already described above, a method was developed that uses the moving frame technique and provides a description of surfaces in term of Cartan exterior forms [43, 55, 116, 117, 128]. Let us present briefly this method. We should emphasize that the approach described below reveals that its basic relations share certain features with the structure of a series of nonlinear differential equations integrable by the method of the inverse scattering transform (see § 3.9).

2.3.4.1. Linear differential forms: exterior product and exterior differential. Here we define two important operations for linear differential forms, necessary for deriving the structure equations of surfaces in $\mathbb{E}^3$.

Consider linear differential forms of first degree (or 1-*forms*)

$$\omega_1 = a_1 du + b_1 dv \qquad \omega_2 = a_2 du + b_2 dv.$$

The *exterior product* of the differential du by the differential dv is defined to be the symbol $du \wedge dv$, interpreted geometrically as the area of the oriented rectangle with sides du and dv. The following rules are assumed to hold:

$$du \wedge dv = -dv \wedge du,$$
$$du \wedge du = dv \wedge dv = 0.$$

The *exterior product* of two 1-forms ω_1 and ω_2 is defined as

$$\omega_1 \wedge \omega_2 = (a_1 du + b_1 dv) \wedge (a_2 du + b_2 dv) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} du \wedge dv. \tag{2.3.38}$$

The expression $\lambda(u, v) du \wedge dv$ appearing in (2.3.38) is called a 2-*form* (or *exterior differential form of degree* 2).

From (2.3.38) one directly derives the following properties of the exterior product:

1) $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$,
2) $\omega_1 \wedge (\omega_2 \pm \omega_3) = \omega_1 \wedge \omega_2 \pm \omega_1 \wedge \omega_3$,
3) $(\lambda\omega_1) \wedge \omega_2 = \omega_1 \wedge (\lambda\omega_2) = \lambda(\omega_1 \wedge \omega_2), \quad \lambda = \lambda(u, v)$.

The *exterior derivative* $d\omega$ of the 1-form

$$\omega = p(u, v)du + q(u, v)dv$$

is the 2-form defined as

$$d\omega = dp \wedge du + dq \wedge dv. \tag{2.3.39}$$

Since $dp = p_u du + p_v dv$ and $dq = q_u du + q_v dv$, applying the properties listed above we can verify that (2.3.39) can be rewritten as

$$d\omega = (q_u - p_v) du \wedge dv.$$

2.3.4.2. Structure equations of Euclidean space. Consider in the Euclidean space $\mathbb{E}^3$ a point A, given by the radius vector $\overrightarrow{R}$, and associate with it an orthonormal frame (a triple of orthonormal vectors) $\{A, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ (Figure 2.3.1):

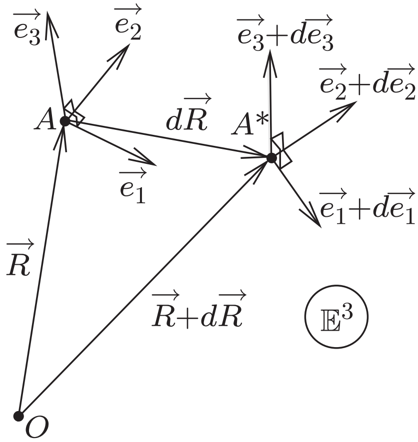


Figure 2.3.1

$$\vec{e}_j \vec{e}_k = \delta_{jk}, \quad \delta_{jk} = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \neq k, \end{cases} \qquad j, k = 1, 2, 3. \tag{2.3.40}$$

At a point A^* close to A, "separated" from A by $d\vec{R}$, consider another orthonormal frame,

$$\{A^*, \vec{e}_1 + d\vec{e}_1, \vec{e}_2 + d\vec{e}_2, \vec{e}_3 + d\vec{e}_3\}.$$

Choosing the initial frame $\{A, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ as a basis in $\mathbb{E}^3$, let us decompose the above vectors with respect to this frame:

$$d\vec{R} = \omega^j \vec{e}_j, \quad j = 1, 2, 3. \tag{2.3.41}$$

$$d\vec{e}_j = \omega_j^k \vec{e}_k, \quad j, k = 1, 2, 3. \tag{2.3.42}$$

We now pose the problem of finding the "decomposition coefficients" ω^j and ω_j^k in (2.3.41) and (2.3.42), with the understanding that they represent linear differential forms of the type $a(u, v)du + b(u, v)dv$, i.e., "1-forms".

We compute the exterior derivative of the left- and right-hand sides of equality (2.3.40), and then use the decomposition (2.3.42). This yields

$$\omega_j^k = -\omega_k^j, \tag{2.3.43}$$

and, correspondingly,

$$\omega_1^1 = \omega_2^2 = \omega_3^3 = 0. \tag{2.3.44}$$

Further, taking the exterior derivative of (2.3.41) we get

$$d^2\vec{R} = d\omega^j \wedge \vec{e}_j - \omega^j \wedge d\vec{e}_j = 0,$$

whence

$$d\omega^j = \omega^k \wedge \omega_k^j. \tag{2.3.45}$$

Similarly, from (2.3.42) we obtain

$$d\omega_j^k = \omega_j^l \wedge \omega_l^k. \tag{2.3.46}$$

The equations (2.3.45), (2.3.46) are called the *structure equations of the Euclidean space* $\mathbb{E}^3$.

Using the properties (2.3.43) and (2.3.44) of linear differential forms, we write the structure equations (2.3.45), (2.3.46) in expanded form:

$$\begin{aligned} d\omega^1 &= -\omega^2 \wedge \omega_1^2 - \omega^3 \wedge \omega_1^3, \\ d\omega^2 &= \omega^1 \wedge \omega_1^2 - \omega^3 \wedge \omega_2^3, \\ d\omega^3 &= \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3. \end{aligned} \tag{2.3.47}$$

$$\begin{aligned} d\omega_1^2 &= -\omega_1^3 \wedge \omega_2^3, \\ d\omega_2^3 &= -\omega_1^2 \wedge \omega_1^3, \\ d\omega_1^3 &= \omega_1^2 \wedge \omega_2^3. \end{aligned} \tag{2.3.48}$$

2.3.4.3. Structure equations of a surface in $\mathbb{E}^3$. To find the structure equations of some surface S in $\mathbb{E}^3$ we shall assume that the point A chosen earlier lies on S:

$$A \in S(\overrightarrow{r}) \subset \mathbb{E}^3, \quad \overrightarrow{r} = \overrightarrow{R}\,|_{A\in S}\,. \tag{2.3.49}$$

Essentially, (2.3.49) is a "constraint condition" on points of Euclidean space.

Let us place the "first two" vectors $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ of the frame $\{A, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ in the tangent space Π_A to the surface S at the point A: $\overrightarrow{e_1}, \overrightarrow{e_2} \in \Pi_A$. Usually, as $\overrightarrow{e_1}$ one takes the tangent vector to the coordinate line u on the surface:

$$\overrightarrow{e_1} = \frac{1}{\overrightarrow{r_u}}|\overrightarrow{r_u}|.$$

Obviously, the vector $d\overrightarrow{r} \equiv d\overrightarrow{R}$ satisfies the condition $d\overrightarrow{r} \in \Pi_A$. Therefore, equations (2.3.41) must not involve the last term with the coefficient ω^3, which "drives" the vector $d\overrightarrow{R}$ beyond the limits of the tangent plane Π_A, i.e.,

$$\omega^3 = 0 \quad \text{for } A \in S.$$

Consequently, the last equation in (2.3.47) can be written as

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0. \tag{2.3.50}$$

A relation of the type (2.3.50), which connects four 1-forms, can (according to Cartan's theorem [116]) hold only in the case when these 1-forms are linearly dependent, and then

$$\begin{aligned} \omega_1^3 &= a\,\omega^1 + b\,\omega^2, \\ \omega_2^3 &= b\,\omega^1 + c\,\omega^2. \end{aligned} \tag{2.3.51}$$

Substitution of (2.3.51) in the first equation of (2.3.48) yields

$$d\omega_1^2 = -K \cdot \omega^1 \wedge \omega^2,$$

where $K = ac - b^2$ is the Gaussian curvature of the surface.

Eventually we obtain three equations that connect the 1-forms ω^1, ω^2, ω_1^2:

$$\begin{aligned} d\omega^1 &= -\omega^2 \wedge \omega_1^2, \\ d\omega^2 &= \omega^1 \wedge \omega_1^2, \\ d\omega_1^2 &= -K \cdot \omega^1 \wedge \omega^2. \end{aligned} \tag{2.3.52}$$

The system (2.3.52) is called the *system of structure equations of a surface in the Euclidean space* $\mathbb{E}^3$.

The metric of the surface is determined by the 1-forms ω^1 and ω^2 as (see [55, 116]):

$$ds^2 = (\omega^1)^2 + (\omega^2)^2. \tag{2.3.53}$$

In § 3.9 we will study the connection between the system of equations (2.3.52) and a special system of equations that arises in the implementation of the method of the inverse scattering transform for the integration of nonlinear differential equations.

2.4 The Beltrami pseudosphere

In the context of our general treatment of questions connected with the possibility of realizing the Lobachevsky geometry in Euclidean space, central objects of interest are the surfaces of Gaussian curvature $K \equiv -1$. The "primordial" such surface, from which a whole class of surfaces that are now called *pseudospherical* originated, is the *pseudosphere* (see § 1.3). The pseudosphere was the first established geometric object in $\mathbb{E}^3$ that serves as the Euclidean image of non-Euclidean hyperbolic geometry, restricted to a subset. A contribution to this result belongs to E. Beltrami[5] [63, 140, 141] who, while investigating the behavior of special types of curves (first and foremost, geodesics) on the pseudosphere, proved that its metric is identical in form with the metric of the Lobachevsky plane, considered in a certain subset of it. This means that the hyperbolic planimetry is realized (though only partially) on the pseudosphere.

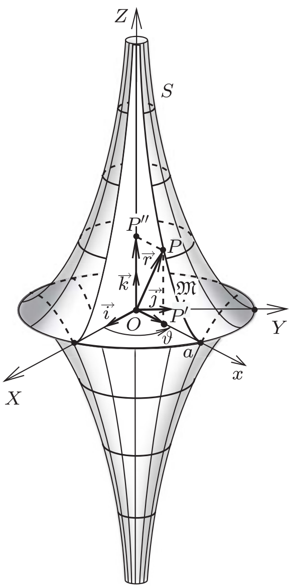


Figure 2.4.1

Let us present the ideas of Beltrami's classical study [140] published already in 1868, with the understanding that the clear awareness of a key geometric object—the pseudosphere—in conjunction with the apparatus of the theory of surfaces we introduced before (see § 2.3), enables us to enrich considerably our representations about the class of surfaces of constant negative curvature.[6]

[5]Beltrami's works became the primary fundamental basis for a series of research directions in contemporary geometry, such as, for example, the theory of geodesic mappings [174].

[6]See, for example, the Bäcklund transformations for pseudospherical surfaces (§ 3.1).

So, let us consider the pseudosphere S, i.e., the surface of revolution obtained by rotating the tractrix (see (1.3.16), § 1.3).

Concretely, suppose that in the space $\mathbb{E}^3$ there is given a Cartesian coordinate system $\{X, Y, Z\}$; let $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ denote the unit vectors of the coordinate axes. We will consider that the axis of revolution of the pseudosphere coincides with the axis OZ, and that the origin of coordinates, i.e., the point O, lies in the plane of the "big circle" of the pseudosphere (Figure 2.4.1), i.e., in the plane of its edge. Pick on our surface some meridian $\mathfrak{M}$, and then pick a point $P \in \mathfrak{M}$. Then the vector $\overrightarrow{r} = \overrightarrow{OP}$ is the radius vector of the pseudosphere.

Let us find the explicit form of the radius vector of the pseudosphere. Denote by P' and P'' the projections of the point $P \in \mathfrak{M} \subset S$ on the (x, y)-plane and on the axis OZ, respectively (Figure 2.4.1). Then

$$\overrightarrow{r} = \overrightarrow{OP} = \overrightarrow{OP}' + \overrightarrow{OP}''; \tag{2.4.1}$$

moreover,

$$\overrightarrow{OP}' = x \cdot (\cos\vartheta\, \overrightarrow{i} + \sin\vartheta\, \overrightarrow{j}\,),$$
$$\vartheta = (\widehat{\overrightarrow{OP}', \overrightarrow{i}}\,), \quad \overrightarrow{OP}'' = z\,\overrightarrow{k}.$$

The length $|\overrightarrow{OP}'| \equiv x$ of the vector $\overrightarrow{OP}'$ is equal to the distance $x(z)$ from some point P of the meridian $\mathfrak{M}$ to the axis of revolution OZ. Recall that the shape of the meridian $z(x)$ is given by the formula (1.3.16) we already obtained. Substituting the expression (1.3.16) in (2.4.1) we obtain

$$\overrightarrow{r} = \sin t \cdot (\cos\vartheta\, \overrightarrow{i} + \sin\vartheta\, \overrightarrow{j}\,) + \left(\ln\cot\frac{t}{2} - \cos t\right)\overrightarrow{k}, \tag{2.4.2}$$

where $x = \sin t$.

To simplify the ensuing calculations, we introduce the vector $\overrightarrow{\nu}$,

$$\overrightarrow{\nu} = \cos\vartheta\, \overrightarrow{i} + \sin\vartheta\, \overrightarrow{j},$$

which lies in the (x, y)-plane and obviously satisfies the conditions $\overrightarrow{\nu} \parallel \overrightarrow{OP}'$, $\overrightarrow{\nu} \perp \overrightarrow{k}$. Then we have

$$\overrightarrow{r} = x \cdot \overrightarrow{\nu} + z \cdot \overrightarrow{k}, \tag{2.4.3}$$

$$d\overrightarrow{r} = \overrightarrow{\nu}\, dx + x d\overrightarrow{\nu} + \overrightarrow{k}\, dz. \tag{2.4.4}$$

Since the vectors $\overrightarrow{\nu}$, $d\overrightarrow{\nu}$, and $\overrightarrow{k}$ are orthogonal, i.e.,

$$(\overrightarrow{\nu}, d\overrightarrow{\nu}) = 0, \quad (\overrightarrow{\nu}, \overrightarrow{k}\,) = 0, \quad (d\overrightarrow{\nu}, \overrightarrow{k}\,) = 0,$$

relation (2.4.4) yields

$$d\overrightarrow{r}^{\,2} = \overrightarrow{\nu}^{\,2} dx^2 + x^2 d\overrightarrow{\nu}^{\,2} + \overrightarrow{k}^{\,2} dz^2. \tag{2.4.5}$$

Next, using the equalities

$$\vec{\nu}^2 = 1, \quad d\vec{\nu}^2 = d\vartheta^2, \quad \vec{k}^2 = 1,$$

we simplify (2.4.5) to

$$d\vec{r}^2 = dx^2 + x^2 d\vartheta^2 + dz^2. \tag{2.4.6}$$

If we now insert in (2.4.6) the expression (1.3.16), which gives the explicit form of the tractrix (the meridian of revolution of the pseudosphere), we arrive at the first fundamental form of the pseudosphere:

$$ds^2 \equiv d\vec{r}^2 = \frac{dx^2}{x^2} + x^2 d\vartheta^2, \quad x \in (0, a], \ \vartheta \in [0, 2\pi). \tag{2.4.7}$$

The substitution $x = e^{-\sigma}$, $d\sigma = -dx/x$ allows us to recast (2.4.7) as

$$ds^2 = d\sigma^2 + e^{-2\sigma} d\vartheta^2. \tag{2.4.8}$$

The metric (2.4.8) is one of the possible representations of the metric of the Lobachevsky plane [25, 39]. Applying the Gauss formula to the metric (2.4.8), the reader will be able to independently verify that its Gaussian curvature is indeed equal to -1.

Let us mention that, side by side with (2.4.8), there is a more general result [25], which yields the metric form

$$ds^2 = du^2 + \cosh^2(\sqrt{-K} \cdot u) dv^2 \tag{2.4.9}$$

of a surface of constant negative Gaussian curvature K, written in semi-geodesic coordinates on the surface with a geodesic base line.

The formulas obtained in this section allow us to carry out various calculations related to the pseudosphere. In particular, there are interesting results that reveal certain similarities between the pseudosphere and the ordinary sphere.

For example, the total area of the pseudosphere is equal to $4\pi a^2$ (where a denotes the radius of the "big parallel", i.e., of the edge of the pseudosphere),[7] which corresponds precisely to the value of the area of the ordinary Euclidean sphere of radius a. The volume of the whole pseudosphere is equal to $\frac{2}{3}\pi a^3$, which is half the volume of the ordinary ball of radius a [40].

Overall, let us emphasize that in his works Beltrami did, in essence, build an analytic geometry on the pseudosphere which locally "turned out" to be the two-dimensional hyperbolic Lobachevsky geometry. An important achievement of Beltrami is the derivation of the equations of a geodesic line ("straight line" on the hyperbolic plane) by using special coordinates (known today as *Beltrami coordinates* [137]), in which a "straight line" is given by a *linear equation*.

Therefore, the analytic description of the *pseudosphere* consists of its representation as a surface of revolution of constant negative curvature (Subsection 1.3.1), Beltrami's results presented in this section, and the interpretation of the

[7] In this case the Gaussian curvature of the pseudosphere is calculated as $K = -1/a^2$.

pseudosphere as the surface whose universal covering space is a horodisc domain in the Lobachevsky plane Λ^2 (Subsection 1.3.2).

Historically, E. Beltrami's actual contribution to the study of the pseudosphere's surface found recognition in the fact that this surface is now known as the *Beltrami pseudosphere*.

2.5 Chebyshev nets

In his 1878 work "Sur la coupe des vetements" ("On cutting cloth") [119, 120, 194], the prominent Russian mathematician P. L. Chebyshev introduced a special class of nets of lines, which reflect the characteristic structure of cloth (women fabrics). Such nets, which are now known as *Chebyshev nets*, have the following characteristic property: *in any coordinate quadrilateral of such a net, the opposite sides are of equal lengths.* Later on, a generalization of this notion of net was widely developed in geometry [127]. Making note of the priority and significance of Chebyshev's contribution to the foundation of classical theory of nets, we devote the first part of this section to a detailed analysis of his work. As a whole, Chebyshev nets represent a canonical geometric structure which can be associated to a series of important nonlinear equations of mathematical physics (see Chapter 4).

2.5.1 On P. L. Chebyshev's work "Sur la coupe des vetements" ("On cutting cloth"). The Chebyshev equation

"Sur la coupe des vetements" was first presented by P. L. Chebyshev on August 28, 1878, as a report at the VII-th meeting of the French Association for the Advancement of Science. This work studies the general problem of covering the surface of a solid body by "pieces of a flat material", or, in other words, unrolling this surface on a plane (or constructing a template of it). Let us present now the main tenents of Chebyshev's investigations in the setting of the methods of modern geometry and mathematical physics [92], while preserving, whenever possible, Chebyshev's original terminology.

In "Sur la coupe ..." Chebyshev poses the problem of the *relation between the shape of the body and the shape of the pieces of cloth covering it*. For this purpose, the covering material (cloth) is represented as a net composed of two families of threads: warp (base) threads and weft (woof) threads.[8].

Chebyshev asks: "What is the nature of changes that the elements of cloth undergo when the cloth envelopes the body?" Here it is assumed that "in the first approximation, when the cloth is being bent to cover some body, nothing changes except for the inclination angles of the warp and weft threads, the length of the threads remaining the same" (see Figure 2.5.1 *a*: cloth (medium-sized cell), *b* covering cloth).

In order for the cloth threads, which on the covered surface are subject to extension, to remain in equilibrium, they must be of minimal length.

[8] In the weaving bussiness, weft threads are transverse threads which interlace the longitudinal warp threads that form the base

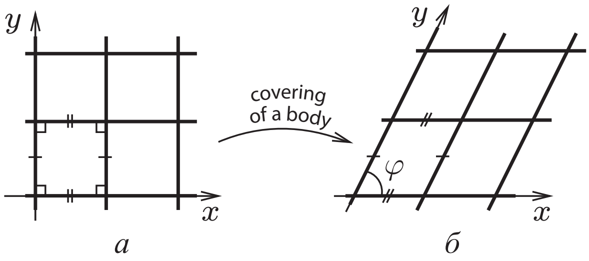


Figure 2.5.1

Let us write the square of the length element for the uncut cloth (i.e., the cloth lying on the plane) (Figure 2.5.1 *a*):

$$ds^2 = dx^2 + dy^2,$$

and for the cloth that covers the body:

$$ds^2 = dx^2 + 2\cos\varphi\, dxdy + dy^2. \tag{2.5.1}$$

We use the Gauss formula (2.3.28) to compute the curvature $K(x, y)$ of the body's surface (shell):

$$K(x,y)\sin^2\varphi = \frac{\partial^2 \cos\varphi}{\partial x \partial y} + \cos\varphi \frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}. \tag{2.5.2}$$

Relation (2.5.2) is a condition on the variation of the net angle $\varphi(x, y)$ of the covering net. As it follows from (2.5.1), in each cell of the net under consideration the opposite sides are equal (Figure 2.5.1 *b*).

Further in Chebyshev's work, it is proposed to pass to the most "appropriate" coordinates for the investigation at hand, the coordinates connected to the shortest lines on the surface. To this end, one writes the equation for the shortest lines (geodesics) for the metric (2.5.1):

$$\sin^2\varphi\,\frac{d^2y}{dx^2} + \frac{\partial\cos\varphi}{\partial x}\left(1 + \cos\varphi\,\frac{dy}{dx}\right) - \frac{\partial\cos\varphi}{\partial y}\left(\cos\varphi + \frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)^2 = 0. \tag{2.5.3}$$

The solution of (2.5.3) is, by assumption, the line $y = 0$, which implies that the following condition is satisfied:

$$\left.\left(\frac{\partial\cos\varphi}{\partial x}\right)\right|_{y=0} = 0.$$

That is to say, along the entire x-line on the covering shell the angle φ is constant and equal to $\pi/2$. In a similar way one establishes that $\varphi = \pi/2$ for $x = 0$.

Following Chebyshev, we expand $\cos\varphi$ in a series of powers of x and y:

$$\cos\varphi = xy(A_0 + A_1x + A_2y + \cdots) \tag{2.5.4}$$

(rigorously speaking, such an expansion is valid only in a neighborhood of zero).

Inserting (2.5.4) in equation (2.5.2), we obtain an expression for the shortest, with respect to the y-axis, lines on the surface (shell) of the covered body:

$$y = U - \left(\frac{K_0}{2}U + \frac{K_2}{4}U^2\right)x^2 - \frac{K_1}{6}Ux^3 + \cdots, \tag{2.5.5}$$
$$y = U, \quad \frac{dy}{dx} = 0 \ \text{ for } \ x = 0,$$

where

$$K_0 = A_0,\ K_1 = 2A_1,\ K_2 = 2A_2, \dots .$$

Thus, using the metric (2.5.1), we can now calculate the distance from an arbitrary point of the surface to the y-axis (the line $x = 0$):

$$s = \int_0^x (y'^2 + 1 + 2y'\cos\varphi)^{1/2} dx.$$

Inverting this last relation with respect to x, we finally obtain

$$\begin{aligned} x = x(s, U, K) = s + \frac{1}{6}\left(K_0^2U^2 + K_0K_2U^3 + \frac{1}{4}K_2^2U^4\right)s^3 \\ + \frac{1}{8}\left(K_0K_1U^2 + \frac{1}{2}K_1K_2U^3\right)s^4 + \cdots, \\ y = y(s, U, K) = U - \left(\frac{1}{2}K_0U + \frac{1}{4}K_2U^2\right)s^2 - \frac{1}{6}K_1Us^3 + \cdots . \end{aligned} \tag{2.5.6}$$

Relation (2.5.6) establishes a connection between the new, "suitable" for cloth cutting coordinates (s, U) (semi-geodesic coordinates with base y), and the original Chebyshev coordinates (x, y), associated with the covering cloth.

Chebyshev mentions in his work that, based on the derived formulas (2.5.6), one can find the curves along which the different pieces of fabric must be cut in order to assemble from them the shell of some given body. In doing so, two preliminary conditions must be satisfied: 1) the parts of the surface that are covered by the different pieces of fabric must be known, and 2) the positions of the base threads on the surface (the lines $x = 0$ and $y = 0$) must be known (determined). These cutting curves can be obtained by using the relations (2.5.6), which define the rectangular coordinates x, y of the points of the cloth pieces in their original plane for various values U and s given on the boundary of the corresponding parts of the body's surface.

It is known that in order to illustrate his ideas, Chebyshev, in his report, proposed a sufficiently precise minimal (consisting of only two parts) cloth-cutting template for the surface of the ball. Although in a subsequent published version of the report [149] no drawing of such a template is provided, it is traditionally considered that Chebyshev is indeed the author of the plane cut pattern of the surface of the ball that is used to produce tennis balls (see Figure 2.5.2: covering of the ball).

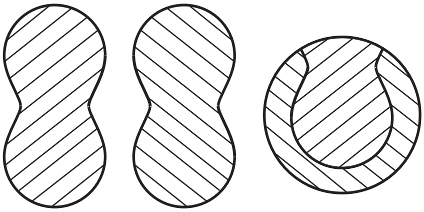


Figure 2.5.2

Attempting to evaluate the work "On cutting cloth", we note that the results obtained therein have a significance that goes beyond the framework of the problem posed originally. First of all, in this work Chebyshev obtained for the first time a nonlinear equation of the type (2.5.2) (known today as the *Chebyshev equation*),[9] which reduces to the form

$$\frac{\partial^2 \varphi}{\partial x \partial y} = -K(x,y) \cdot \sin\varphi \tag{2.5.7}$$

(*Chebyshev equation*).

In the case when $K \equiv -1$, the Chebyshev equation (2.5.7) becomes the nowadays well-known in applications sine-Gordon equation:

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \sin\varphi \tag{2.5.8}$$

(*sine-Gordon equation*).

The sine-Gordon equation (2.5.8) has a universal character. On the one hand, it describes regular isometric immersions of parts of the Lobachevsky plane Λ^2 in the Euclidean space $\mathbb{E}^3$ [74, 78, 80], and on the other, it is a model equation widely used in physics [8, 50, 51, 80].

Let us emphasize also that it is in the work of Chebyshev we are discussing that a connection between a differential equation (specifically, equation (2.5.7)) and a certain corresponding coordinate net (the Chebyshev net) was noted for the first time. This "observation" was further developed in contemporary studies of nonlinear equations of mathematical physics, which were also successfully associated with certain coordinate nets that generate them [77, 79, 185], a fact that laid the foundations of geometric approaches to the integration of such equations.

2.5.2 Geometry of Chebyshev nets. The Servant-Bianchi equations

Suppose that on a regular surface S there is given a system of coordinate lines u and v. Take an arbitrary point $P(u,v) \in S$ and draw through it the corresponding

[9] At roughly the same time an equation of the form (2.5.7) appeared also in the work of Hazzidakis [160], who obtained a formula for the computation of the area of a net quadrilateral of the net of asymptotic lines on a surface of constant negative curvature.

coordinate lines from each of the two families, until they intersect the initial lines $v = 0$ (u-line) and $u = 0$ (v-line) at the points P_1 and P_2, respectively (Figure 2.5.3).

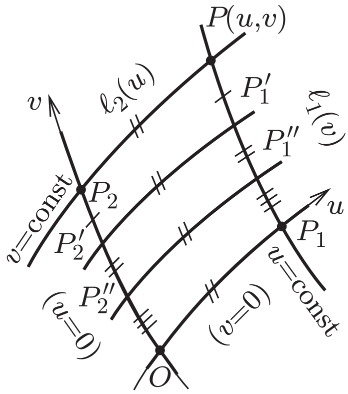


Figure 2.5.3

The segments PP_1 and PP_2 are parts of coordinate lines, on which the value of one of the coordinate parameters u or v are constant. The coordinates of the point $P(u, v)$ are determined by the coordinates of the points $P_1(u, 0)$ and $P_2(0, v)$.

Denote the lengths of the coordinate line segments PP_1 and PP_2 by

$$l_1 = PP_1, \quad l_2 = PP_2.$$

It is clear that in the general case the lengths l_1 and l_2 are functions of the position of the point $P(u, v)$ on the surface, i.e., $l_1 = l_1(u, v)$ and $l_2 = l_2(u, v)$.

We will be interested in the special (nontrivial) situation when

$$l_1 = l_1(v), \quad l_2 = l_2(u). \tag{2.5.9}$$

If some current point moves along the coordinate line PP_1 ($u = \text{const}$) (i.e., goes through the intermediate positions $P_1', P_1'', \ldots$), then the corresponding coordinate projection on the v-line ($u = 0$) goes through the intermediate points $P_2', P_2'', \ldots$). The lengths of the corresponding curvilinear segments will be equal:

$$P_2P = P_2'P_1' = P_2''P_1'' = \cdots = OP_1, \tag{2.5.10}$$

because

$$l_2(P) = l_2(P_1') = l_2(P_1'') = \cdots = l_2(P_2)$$

in view of (2.5.9) and the fact that the points $P, P_1', P_1'', \ldots, P_1$ lie on the same coordinate line $u = \text{const}$.

Equalities analogous to (2.5.10) can be derived by considering the motion of the current point along the coordinate lines of the other family:

$$P_2P_2' = PP_1', \quad P_2'P_2'' = P_1'P_1'', \quad P_2''O = P_1''P_1. \tag{2.5.11}$$

The relations (2.5.10), (2.5.11) obtained above express a key geometric property of the net of coordinate lines at hand: *in each net coordinate quadrilateral of such a net* [10] *the opposite sides are of equal lengths*. Such nets are called *Chebyshev nets*.[11]

On the whole, Chebyshev nets express the property of size uniformity of a curvilinear coordinate net.

Before we derive the form of the metric in the Chebyshev parametrization, let us rewrite a metric (2.2.7) of general form

$$ds^2 = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2$$

in a slightly modified form. To this end, recalling that the coefficients E and G are positive (see (2.2.6)), we introduce the new notations

$$E = A^2, \quad G = C^2.$$

Then the metric is written as

$$ds^2 = A^2du^2 + 2AC\cos\varphi\ dudv + C^2dv^2, \tag{2.5.12}$$

where $\cos\varphi = F/(AC)$.

From (2.5.12) it is clear that along the coordinate lines u ($v = \mathrm{const}$) and v ($u = \mathrm{const}$) the length changes as

$$\begin{aligned} ds_1 &= Adu \quad \text{on the line} \quad u\colon\ v = \mathrm{const}, \quad dv = 0,\\ ds_2 &= Cdv \quad \text{on the line} \quad v\colon\ u = \mathrm{const}, \quad du = 0. \end{aligned} \tag{2.5.13}$$

Returning to our exposition, let us remark that in a Chebyshev net the result of measuring the length s_1 in (2.5.13) along the line u does not depend on v, and hence in this case the coefficient A in (2.5.12) will depend only on u: $A = A(u)$. Similarly, $C = C(v)$. Therefore, for a Chebyshev net the square of the linear element (2.5.12) takes on the more precise form

$$ds^2 = A^2(u)du^2 + 2A(u)C(v)\cos\varphi(u,v)dudv + C^2(v)dv^2. \tag{2.5.14}$$

The substitution

$$\widetilde{u} = \int_{u_0}^{u} A(u)du, \quad \widetilde{v} = \int_{v_0}^{v} C(v)dv,$$

where (u_0, v_0) is some fixed point, reduces the metric (2.5.14) to the form

$$ds^2 = d\widetilde{u}^2 + 2\cos\varphi(\widetilde{u},\widetilde{v})d\widetilde{u}d\widetilde{v} + d\widetilde{v}^2. \tag{2.5.15}$$

Every metric can be reduced to the form (2.5.15) (in this case $\widetilde{E} = 1$, $\widetilde{F} = \cos\varphi$, $\widetilde{G} = 1$) by an admissible transition to the Chebyshev parametrization. And

[10] A net that satisfies condition (2.5.9).

[11] Alternative versions of the name found in the literature are Chebysheff, Chebyshov, Tchebychev, and Tchebycheff.

conversely, if a metric has the form (2.5.15), then the coordinates on the surface are *Chebyshev coordinates*. Henceforth the Chebyshev net of lines (u, v) will be denoted by $\mathrm{Cheb}(u, v)$. As a general notation for a net of lines (u, v) we will use the symbol $T(u, v)$.

Calculating the Christoffel symbols Γ^1_{12}, Γ^2_{12} of a *Chebyshev* metric (2.5.15) by means of formulas (2.3.10), it is not difficult to verify that

$$\Gamma^1_{12} = 0, \quad \Gamma^2_{12} = 0. \tag{2.5.16}$$

Equalities (2.5.16) express a characteristic Chebyshev net criterion, i.e., they are necessary and sufficient for a net to be a Chebyshev net. The necessity was established above, when we derived (2.5.16) from (2.5.15). Let us prove the sufficiency. So, suppose that the conditions (2.5.16) are satisfied. Then by (2.3.10) we have

$$\begin{aligned} \Gamma^1_{12} &= \frac{1}{2W}\,(E_v G - G_u F) = 0, \\ \Gamma^2_{12} &= \frac{1}{2W}\,(G_u E - E_v F) = 0. \end{aligned} \tag{2.5.17}$$

In essence, relations (2.5.17) represent a homogeneous system of linear equations for E_v and E_u that can have only the trivial solution:

$$E_v = 0, \quad G_u = 0. \tag{2.5.18}$$

This clearly means that

$$E = E(u), \quad G = G(v),$$

that is, the metric under consideration is of the form (2.5.14), which is reducible to the Chebyshev form (2.5.15). This establishes the sufficiency of equalities (2.5.16).

Let us study conditions under which on a surface it is possible to pass to a Chebyshev net. Consider a smooth surface S and on it some regular net $T_1(v_1, v_2) \subset S$.[12] Let $T_2(u_1, u_2) \subset S$ be another coordinate net on the same surface, so that

$$\begin{aligned} u_1 &= u_1(v_1, v_2), \\ u_2 &= u_2(v_1, v_2), \end{aligned} \tag{2.5.19}$$

with

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial v_1} & \dfrac{\partial u_1}{\partial v_2} \\ \dfrac{\partial u_2}{\partial v_1} & \dfrac{\partial u_2}{\partial v_2} \end{vmatrix} \neq 0.$$

(The last condition guarantees that the Jacobian of the transformation $(v_1, v_2) \mapsto (u_1, u_2)$) is not trivial.)

[12] In this setting of the problem it is convenient to denote the coordinate lines of different families of the net by the same letter, but with different indices.

Let us find under what conditions the change of coordinates (2.5.19) maps the initial coordinate net $T_1(v_1, v_2)$ into a new coordinate net $T_2(u_1, u_2)$ that is a *Chebyshev net*: $T_2(u_1, u_2) \equiv \mathrm{Cheb}(u_1, u_2)$. To this end we use the well-known rule of transformation of the Christoffel symbols when we pass from one coordinate system to another (see [40, 81]):

$$\widetilde{\Gamma}^{\gamma}_{\alpha\beta} = \left(\Gamma^k_{ij} \frac{\partial u_i}{\partial v_\alpha} \cdot \frac{\partial u_j}{\partial v_\beta} + \frac{\partial^2 u_k}{\partial v_\alpha \partial v_\beta} \right) \cdot \frac{\partial v_\gamma}{\partial u_k} \tag{2.5.20}$$

(here one sums over the repeated indices $i, j, k = 1, 2$).

If we assume that the new net is Chebyshev, then conditions (2.5.16) must be satisfied:

$$\mathrm{Cheb}(u_1, u_2): \quad \begin{array}{l} \widetilde{\Gamma}^1_{12} = 0, \\ \widetilde{\Gamma}^2_{12} = 0. \end{array} \tag{2.5.21}$$

Relations (2.5.21) represent, in view of (2.5.20), a homogeneous system of linear equations with respect to the quantities enclosed in parentheses in formula (2.5.20). Such a system has (provided that the condition on the Jacobian written below (2.5.19) is satisfied) only the trivial solution. Thus, we arrive at the system of equations

$$\begin{array}{l} \dfrac{\partial^2 u_1}{\partial v_1 \partial v_2} + \Gamma^1_{\alpha\beta} \cdot \dfrac{\partial u_\alpha}{\partial v_1} \cdot \dfrac{\partial u_\beta}{\partial v_2} = 0, \\ \dfrac{\partial^2 u_2}{\partial v_1 \partial v_2} + \Gamma^2_{\alpha\beta} \cdot \dfrac{\partial u_\alpha}{\partial v_1} \cdot \dfrac{\partial u_\beta}{\partial v_2} = 0, \end{array} \qquad \alpha, \beta = 1, 2, \tag{2.5.22}$$

which gives the relations effecting the transition from some given net $T_1(v_1, v_2)$ to a Chebyshev coordinate net $\mathrm{Cheb}(u_1, u_2)$:

$$T_1(v_1, v_2) \longmapsto \mathrm{Cheb}(u_1, u_2).$$

The system (2.5.22) is called the *system of Servant-Bianchi equations* [142, 192].

Let us make more precise the problem posed for equations (5.2.22) under the transition to a Chebyshev net.

Choose a point M on the surface S, and represent it in the two considered nets, $T_1(v_1, v_2)$ and $\mathrm{Cheb}(u_1, u_2)$:

$$M(v_1^0, v_2^0) \equiv M(u_1^0, u_2^0).$$

Recalling (2.5.19), we have

$$u_1^0 = u_1(v_1^0, v_2^0), \quad u_2^0 = u_2(v_1^0, v_2^0).$$

Let the initial data on the coordinate lines be defined as

$$\begin{array}{ll} u_1(v_1, v_2) \mid_{v_2 = v_2^0} & = f_1(v_1), \\ u_2(v_1, v_2) \mid_{v_2 = v_2^0} & = f_2(v_1). \end{array} \tag{2.5.23}$$

Obviously,

$$f_1(v_1^0) = u_1^0, \quad f_2(v_1^0) = u_2^0.$$

Similarly, let us write the initial conditions in terms of the functions $g_1(v_2)$ and $g_2(v_2)$:

$$\begin{aligned} u_1(v_1, v_2)\,|_{v_1=v_1^0} &= g_1(v_2), \\ u_2(v_1, v_2)\,|_{v_1=v_1^0} &= g_2(v_2), \\ g_1(v_2^0) = u_1^0, \quad g_2(v_2^0) &= u_2^0. \end{aligned} \tag{2.5.24}$$

The freedom in the choice of the functions f_1, f_2, g_1, and g_2 is restricted only by the natural condition (analogous to (2.5.19))

$$\frac{\partial f_1}{\partial v_1} \cdot \frac{\partial g_2}{\partial v_2} - \frac{\partial f_2}{\partial v_1} \cdot \frac{\partial g_1}{\partial v_2} \neq 0. \tag{2.5.25}$$

The system (5.2.22) under consideration can be reduced, by a linear change of variables, to a system of equations solved with respect to the highest-order derivatives, i.e., to a system in *normal* form [48, 113]. Consequently, to the system (2.5.22), supplemented by the conditions (2.5.23), (2.5.24), one can apply the classical theorems on existence and uniqueness of solutions from the theory of partial differential equations [46, 48].

From a geometric point of view, this means that *locally*, in the vicinity of the given regular point $M \in S$, one can always pass from an arbitrarily given regular coordinate net $T_1(v_1, v_2) \subset S$ to a Chebyshev net $\mathrm{Cheb}(u_1, u_2) \subset S$, and in fact in such a way that the initial coordinate lines u_1 and u_2 of the net $\mathrm{Cheb}(u_1, u_2)$ can be chosen with a considerable degree of freedom (the only constraint being condition (2.5.25)).

We can thus formulate the following theorem.

Theorem. *Suppose that through some point M on the regular part of the surface $S \subset \mathbb{E}^3$ one draws two different arbitrary* (*but such that* (2.5.25) *holds*) *curves on S. Then in a sufficiently small neighborhood of the point M there always exists a unique Chebyshev net, constructed on these curves.*

It is important to note that, overall, all the results present in this section concern surfaces with arbitrary Gaussian curvature.

Let us also mention several popular-science publications on the theme studied here [80, 89, 109] that the reader may find useful.

2.6 D. Hilbert's result on the impossibility of realizing the complete Lobachevsky plane Λ^2 in the space $\mathbb{E}^3$

By possible realization of the Lobachevsky plane Λ^2 in the Euclidean space $\mathbb{E}^3$ we mean the existence in $\mathbb{E}^3$ of a surface $S(\Lambda^2) \subset \mathbb{E}^3$ of constant negative Gaussian curvature on which the complete two-dimensional Lobachevsky geometry would be globally realized. According to the ideas of E. Beltrami, presented in his work

"Saggio di interpretazione della geometria non Euclidea" ("Essay on the interpretation of non-Euclidean geometry") [140], the role of the straight lines in the Lobachevsky plane must be played on the surface $S(\Lambda^2)$ by the shortest (geodesic) lines, and the "lengths" and "angles" on $S(\Lambda^2)$ must coincide with the corresponding "lengths" and "angles" in the Lobachevsky planimetry. Let us note that none of the classical pseudospherical surfaces (of curvature $K \equiv -1$) that we considered in § 1.3 can serve as an example of realization in $\mathbb{E}^3$ of the complete plane Λ^2. On such surfaces one can achieve only a "fragmentary" realization of the Lobachevsky plane Λ^2, reflecting the geometry of certain of its individual sub-domains (horodisc, equidistant strip, etc.), which cannot be extended beyond the singularities arising on the surface: irregular edges, peak (spike) points, and so on. Geometrically, the aforementioned "non-extendability" intuitively meant that it is impossible to continuously extend a surface of constant negative curvature beyond a singularity in such a way that the tangent plane to the surface will also vary in a continuous manner.

In the context of the problem we are discussing here, D. Hilbert posed in 1901 the following question: *Does there exists, in the Euclidean space $\mathbb{E}^3$, an analytic surface of constant negative curvature, free of singularities and regular everywhere?* In other words, does the complete Lobachevsky plane Λ^2 admit a regular realization in $\mathbb{E}^3$? The answer to this question, obtained by Hilbert himself, turned out to be negative, in the sense that such a realization is in principle not possible.

Let us present next Hilbert's ideas [17]. On a surface of constant negative curvature there always exist two distinct families of asymptotic lines, which form an asymptotic net on the surface. Furthermore, an asymptotic net of lines on a surface of curvature $K \equiv \text{const} < 0$ is always a *Chebyshev net*.[13] For this reason we will start under the assumption that on the surface $S(\Lambda^2) \subset \mathbb{E}^3$, with $K \equiv -1$, which supposedly realizes in regular manner in $\mathbb{E}^3$ the complete Lobachevsky plane Λ^2, there is a global Chebyshev net of asymptotic lines, $\text{Cheb}[S(\Lambda^2)]$. In this case the metric of the surface $S(\Lambda^2)$ takes on the form

$$ds^2 = du^2 + 2\cos\varphi(u,v)\,dudv + dv^2, \quad u,v \in \mathbb{R}^2, \tag{2.6.1}$$

and the net angle $\varphi(u,v)$ will satisfy the sine-Gordon equation

$$\frac{\partial^2\varphi}{\partial u \partial v} = \sin\varphi. \tag{2.6.2}$$

The regularity of the surface $S(\Lambda^2) \subset \mathbb{E}^3$ presumes the global existence on $S(\Lambda^2)$ of a *regular* net of asymptotic lines, i.e., a net in which the net angle $\varphi(u,v)$ must satisfy everywhere the condition

$$0 < \varphi(u,v) < \pi, \quad u,v \in \mathbb{R}^2. \tag{2.6.3}$$

Condition (2.6.3) must guarantee that the existing coordinate net $\text{Cheb}[S(\Lambda^2)]$ is not "degenerate", meaning that lines belonging to its different families are not tangent to one another.

---

[13] For more details, see Subsection 2.7.3.

In this connection, a fundamental result asserts that the sine-Gordon equation (2.6.2) does not admit regular solutions $\varphi(u,v)$ that are defined on the whole plane $\mathbb{R}^2(u,v)$ and satisfy the condition (2.6.3). Geometrically, this fact corresponds to the non-realizability of the complete plane Λ^2 in $\mathbb{E}^3$. Let us now turn now to the p r o o f of the assertion formulated above.

Suppose that, on the contrary, equation (2.6.2) has a solution that is defined on the whole parameter plane $\mathbb{R}^2(u,v)$ and satisfies (2.6.3). We will assume that the net angle function $\varphi(u,v)$ is defined and continuous for all $(u,v)\in\mathbb{R}^2$ and has continuous partial derivatives "in accordance to equation (2.6.2)".

Let us choose on the surface $S(\Lambda^2)$ an initial point $u_0=0,\ v_0=0$ such that in it

$$0<\varphi(0,0)<\pi, \tag{2.6.4}$$

$$\varphi_u(0,0)>0. \tag{2.6.5}$$

Such a choice of initial point is absolutely correct, because condition (2.6.4) is a particular case of (2.6.3), and the vanishing of the derivative φ_u in (2.6.5) would, in conjunction with (2.6.2), contradict condition (6.2.4) itself. (The alternative case $\varphi_u(0,0)<0$ in (2.6.5) can be treated in analogous manner.)

By the assumed condition (2.6.3),

$$\sin\varphi(u,v)>0,\quad \forall u,v\in\mathbb{R}^2,$$

whence,

$$\frac{\partial^2\varphi}{\partial u\partial v}=\frac{\partial}{\partial v}(\varphi_u)>0. \tag{2.6.6}$$

Inequality (2.6.6) means that the function φ_u is increasing in the parameter v (for any fixed value of u). Consequently,

$$\begin{gathered}0<\varphi_u(0,v_1)<\varphi_u(0,v),\\ 0<\varphi_u(u^*,v_1)<\varphi_u(u^*,v)\end{gathered} \tag{2.6.7}$$

for $v>v_1>0$ and fixed u^*.

Consider now in the plane $\mathbb{R}^2(u,v)$ the point $(0,v_1)$. Thanks to the condition $\varphi_u(0,v_1)>0$, one can assert that on the line $v=v_1=\text{const}$ (line of the Chebyshev net $\text{Cheb}(u,v)$) one can exhibit a segment $u\in[0,3a]$[14] on which the following inequality holds:

$$\varphi_u(u,v_1)>0,\quad u\in[0,3a]. \tag{2.6.8}$$

Let

$$m=\min_{u\in[0,3a]}\varphi_u(u,v_1),\quad m>0.$$

Let us divide the segment $[0,3a]$ into three equal parts. For the first and the third of them we write the classical Lagrange formula for the function $\varphi(u,v)$, for $v>v_1$ (Figure 2.6.1):

[14]The existence of such a segment, on which (2.6.8) holds is, in essence, a particular case of the more general condition that there exists a neighborhood of the point $(0,v_1)$ in which the derivative φ_u is strictly positive.

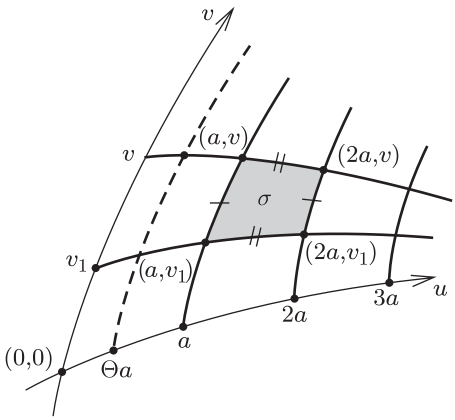


Figure 2.6.1

$$\begin{aligned}\varphi(a,v)-\varphi(0,v)&=\varphi_u(\Theta_1 a,v)\cdot a,\\ \varphi(3a,v)-\varphi(2a,v)&=\varphi_u(2a+\Theta_2 a,v)\cdot a,\end{aligned}\tag{2.6.9}$$

where $\Theta_1,\Theta_2\in(0,1)$. In what follows we will use for Θ_1 and Θ_2 the common general notation Θ.

By the second inequality in (2.6.7),

$$\varphi_u(\Theta a,v)>\varphi_u(\Theta a,v_1)\geq m.\tag{2.6.10}$$

From (2.6.9) and (2.6.10) it follows that

$$\begin{aligned}\varphi(a,v)-\varphi(0,v)&>ma,\\ \varphi(3a,v)-\varphi(2a,v)&>ma\quad\text{for } v>v_1.\end{aligned}$$

Hence,

$$\begin{aligned}\varphi(a,v)&>\varphi(0,v)+ma>ma,\\ \varphi(2a,v)&<\varphi(3a,v)-ma<\pi-ma\quad\text{for }\ v>v_1,\end{aligned}\tag{2.6.11}$$

where we have used condition (2.6.3)

As the second inequality in (2.6.7) shows, on the half-strip $\{u\in[0,3a], v>v_1\}$ the solution $\varphi(u,v)$ is monotonically increasing in u; in particular, it necessarily holds that

$$\varphi(a,v)\leq\varphi(u,v)\leq\varphi(2a,v),\quad u\in[a,2a],\quad v>v_1,$$

which upon using (2.6.11) is strengthened to

$$ma<\varphi(u,v)<\pi-ma,\quad m,a>0.\tag{2.6.12}$$

From (2.6.12) it follows that

$$\begin{gathered} \sin\varphi(u,v) > \sin(ma) \equiv M, \\ M = \text{const}, \quad 0 < M < 1. \end{gathered} \tag{2.6.13}$$

Thus, we derived the important estimate (2.6.13), which holds in the quadrilateral σ of the Chebyshev net with the vertices at the points (a, v_1), (a, v), $(2a, v)$, and $(2a, v_1)$ (Figure 2.6.1). Let us calculate the double integral over this domain of the left-hand and right-hand sides of the equation (2.6.2). This yields

$$\begin{aligned} \iint_\sigma \frac{\partial^2\varphi}{\partial u \partial v}\, dudv &= \int\limits_a^{2a} \int\limits_{v_1}^{v} \frac{\partial^2\varphi}{\partial u \partial v}\, dudv \\ &= (\varphi(2a,v) - \varphi(a,v)) - (\varphi(2a,v_1) - \varphi(a,v_1)) < \pi. \end{aligned} \tag{2.6.14}$$

To obtain the final estimate (2.6.14) we used the original condition (2.6.4) and the fact that $\varphi(u,v)$ is an increasing function of u.

$$\iint_\sigma \sin\varphi(u,v) dudv > M \iint_\sigma dudv = Ma(v - v_1). \tag{2.6.15}$$

The value of v in (2.6.15) can be always chosen so that the value of the integral will become larger than π.

Thus, the obtained estimates (2.6.14) and (2.6.15) lead to a contradiction. Therefore, there does not exist a solution $\varphi(u,v)$ of the sine-Gordon equation (2.6.2) that satisfies the condition (2.6.3) in the whole (u,v)-parameter plane. Geometrically this means that in the space $\mathbb{E}^3$ there is no analytic surface of constant negative curvature that has no singularities and is regular everywhere. In other words, according to Hilbert's result, *the geometry of the Lobachevsky plane* Λ^2 *cannot be globally realized on some regular analytic surface in the three-dimensional Euclidean space* $\mathbb{E}^3$.

Hilbert's classical result was subsequently developed in works of N. V. Efimov and E. G. Poznyak, in which it was generalized to the case of complete regular surfaces with a negative upper bound of the Gaussian curvature [26, 28]. N. V. Efimov also proved the nonimmersibility in $\mathbb{E}^3$ of the Lobachevsky *half-plane* [27].

2.7 Investigation of pseudospherical surfaces and the sine-Gordon equation

Hilbert's result given in § 2.6 means that in the Euclidean space $\mathbb{E}^3$ there is possible only a partial, "fragmentary" realization of the Lobachevsky planimetry on surfaces, which unavoidably have singularities: irregular edges (cuspidal edges), cusp points, etc. These singularities are "constituted" by precisely those points in which the solution of the sine-Gordon equation (the net angle) "crosses" values that are multiples of π. It is at these points that the Chebyshev net of asymptotic lines is "crushed": lines of different families of the net become tangent to one another, thereby forming envelopes of both families [70, 78], a prototype of irregular edge of a surface in $\mathbb{E}^3$. In this section we address various problems concerned with the structure of surfaces of constant negative curvature.

2.7.1 Curves in space. Frenet formulas

Consider a C^3-smooth curve $\mathcal{L}$ in the space $\mathbb{E}^3$, defined by its radius vector $\overrightarrow{R}$:

$$\overrightarrow{R}(t) = \{x(t), y(t), z(t)\}. \tag{2.7.1}$$

The vector $\overrightarrow{R}'(t)$ is the direction vector of the tangent to the curve $\mathcal{L}$. Recall that a straight line is said to be *tangent* to a curve at a given point is it realizes the limit position of a straight-line secant that passes through the given point ("point of tangency") and another, infinitesimally close to it point of the curve, when the later approaches the former.

Let introduce the (natural) parameter s for the given curve $\mathcal{L}$ by

$$s = \int\limits_{t_0}^{t} |\overrightarrow{R}'|dt; \tag{2.7.2}$$

s has the meaning of length, measured along the curve $\mathcal{L}$ itself.

Differentiating (2.7.2) we obtain

$$ds = |\overrightarrow{R}'|dt,$$

whence

$$(ds)^2 = (d\overrightarrow{R})^2, \tag{2.7.3}$$

or

$$\left(\frac{d\overrightarrow{R}}{ds}\right)^2 = 1.$$

That is to say, the vector

$$\overrightarrow{\tau}(s) \equiv \frac{d\overrightarrow{R}}{ds}, \quad \overrightarrow{\tau}^2(s) = 1 \tag{2.7.4}$$

is the *unit tangent vector* to the curve $\mathcal{L}$ at each of its points.

We introduce also the vector

$$\frac{d\overrightarrow{\tau}}{ds} = \frac{d^2\overrightarrow{R}}{ds^2},$$

which clearly characterizes the speed with which the tangent vector $\overrightarrow{\tau}(s)$ rotates as one "advances" along the curve $\mathcal{L}$; its length $k(s) = \left|d^2\overrightarrow{R}/ds^2\right|$ is the *curvature* of the curve $\mathcal{L}$. At the same time, it is convenient to use the unit vector $\overrightarrow{\nu}(s)$ given by

$$\overrightarrow{\nu}(s) \equiv \frac{1}{k(s)} \cdot \frac{d\overrightarrow{\tau}}{ds} = \frac{1}{k(s)} \cdot \frac{d^2\overrightarrow{R}}{ds^2}, \tag{2.7.5}$$

called the *principal normal vector* to the curve.

Differentiating the equality $\overrightarrow{\tau}^2 = (\overrightarrow{\tau}, \overrightarrow{\tau}) = 1$ (see (2.7.4)), we obtain

$$\left(\overrightarrow{\tau}(s), \frac{d\overrightarrow{\tau}}{ds}\right) = 0,$$

and in view of (2.7.5) we have

$$(\overrightarrow{\tau}(s), \overrightarrow{\nu}(s)) = 0. \tag{2.7.6}$$

The vanishing of the scalar product (2.7.6) means that the unit vectors $\overrightarrow{\tau}$ and $\overrightarrow{\nu}$ are orthogonal. We take the vector (cross) product these vectors to construct a new unit vector $\overrightarrow{\beta}(s)$ as

$$\overrightarrow{\beta} = [\overrightarrow{\tau}, \overrightarrow{\nu}]; \tag{2.7.7}$$

$\overrightarrow{\beta}$ is called the *binormal vector* to the curve.

The right-handed triple of orthonormal vectors $\overrightarrow{\tau}$, $\overrightarrow{\nu}$, $\overrightarrow{\beta}$ (see Figure 2.7.1) given by formulas (2.7.4), (2.7.5), and (2.7.6) is called the *fundamental trihedron* (*frame*), or *Frenet trihedron* (*frame*) of the space curve $\mathcal{L}$.[15]

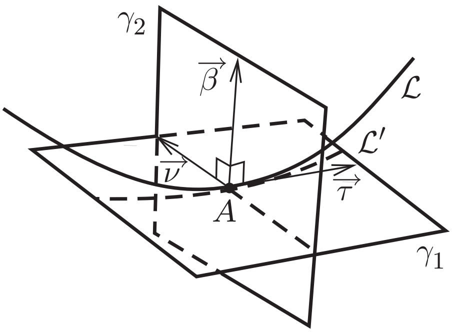


Figure 2.7.1

Consider the vectors $\overrightarrow{\tau}$, $\overrightarrow{\nu}$, and $\overrightarrow{\beta}$ in some point A on the curve $\mathcal{L}$. The plane γ_1 that passes through A and contains the vectors $\overrightarrow{\tau}$ and $\overrightarrow{\nu}$ is called the *osculating plane* of $\mathcal{L}$ at the point A. In view of the way our vectors were constructed, it is clear that $\overrightarrow{\beta}$ is orthogonal to the plane γ_1. The plane γ_2 that passes through the point A and contains the principal normal $\overrightarrow{\nu}$ and the binormal $\overrightarrow{\beta}$ is called the

[15] Sometimes called the "trihedron attached to the curve".

normal plane to the curve $\mathcal{L}$ (Figure 2.7.1); γ_2 is orthogonal to the tangent vector $\overrightarrow{\tau}$ to the curve $\mathcal{L}$. The mutual positions of the geometric objects under discussion is shown in Figure 2.7.1: $\overrightarrow{\tau}, \overrightarrow{\nu} \in \gamma_1$; $\overrightarrow{\beta} \perp \gamma_1$; $\overrightarrow{\nu}, \overrightarrow{\beta} \in \gamma_2$; $\overrightarrow{\tau} \perp \gamma_2$; $\gamma_1 \perp \gamma_2$. On this figure, the doted curve $\mathcal{L}'$ in the osculating plane γ_1 is the projection of the curve $\mathcal{L}$.

Differentiating (2.7.7) we obtain

$$\frac{d\overrightarrow{\beta}}{ds} = \left[\frac{d\overrightarrow{\tau}}{ds}, \overrightarrow{\nu}\right] + \left[\overrightarrow{\tau}, \frac{d\overrightarrow{\nu}}{ds}\right] = [k\overrightarrow{\nu}, \overrightarrow{\nu}] + \left[\overrightarrow{\tau}, \frac{d\overrightarrow{\nu}}{ds}\right] = \left[\overrightarrow{\tau}, \frac{d\overrightarrow{\nu}}{ds}\right]. \tag{2.7.8}$$

From (2.7.8) it follows that the vector $d\overrightarrow{\beta}/ds$ is orthogonal to the vector $\overrightarrow{\tau}$; moreover, $d\overrightarrow{\beta}/ds$ is orthogonal to $\overrightarrow{\beta}$ "by construction". Hence, $d\overrightarrow{\beta}/ds$ is orthogonal to the plane that contains $\overrightarrow{\tau}$ and $\overrightarrow{\beta}$, and accordingly is collinear with the vector

$$\frac{d\overrightarrow{\beta}}{ds} = -æ(s) \cdot \overrightarrow{\nu}(s). \tag{2.7.9}$$

The proportionality coefficient $æ(s)$ in (2.7.9) is called the *torsion* of the curve $\mathcal{L}$. It measures the speed at which the osculating plane rotates around the tangent straight line to the curve. If the curve is plane, its torsion is equal to zero.

A cyclic permutation of the orthonormal vectors in (2.7.7) yields

$$\overrightarrow{\nu} = [\overrightarrow{\beta}, \overrightarrow{\tau}]. \tag{2.7.10}$$

Differentiating in (2.7.10) and recalling (2.7.9) and (2.7.5), we obtain

$$\frac{d\overrightarrow{\nu}}{ds} = \left[\frac{d\overrightarrow{\beta}}{ds}, \overrightarrow{\tau}\right] + \left[\overrightarrow{\beta}, \frac{d\overrightarrow{\tau}}{ds}\right],$$

or

$$\frac{d\overrightarrow{\nu}}{ds} = æ(s)\overrightarrow{\beta}(s) - k(s)\overrightarrow{\tau}(s). \tag{2.7.11}$$

Let us add to the list of relations given above the equality

$$\frac{d\overrightarrow{\tau}}{ds} = k(s)\overrightarrow{\nu}(s), \tag{2.7.12}$$

which is a consequence of (2.7.5).

Equations (2.7.9), (2.7.11), and (2.7.12) form a system called the *Frenet formulas*: they describe how the vectors of the fundamental frame (trihedron) change as one moves along the curve $\mathcal{L}$ in $\mathbb{E}^3$.

It is convenient to write the Frenet formulas in matrix form:

$$\frac{d}{ds}\begin{pmatrix} \overrightarrow{\tau} \\ \overrightarrow{\nu} \\ \overrightarrow{\beta} \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & æ(s) \\ 0 & -æ(s) & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{\tau} \\ \overrightarrow{\nu} \\ \overrightarrow{\beta} \end{pmatrix}. \tag{2.7.13}$$

To finish this subsection, let us "summarize" the formulas for the important characteristics of the space curve $\mathcal{L}$ in $\mathbb{E}^3$, given in terms of its radius vector $\overrightarrow{R}(s)$ (where s is the natural parameter on the curve):

$$\begin{gathered}\overrightarrow{\tau} = \overrightarrow{R}'(s), \quad \overrightarrow{\nu} = \frac{\overrightarrow{R}''(s)}{|\overrightarrow{R}''(s)|}, \quad \overrightarrow{\beta} = \frac{[\overrightarrow{R}'(s), \overrightarrow{R}''(s)]}{|\overrightarrow{R}''(s)|}, \\ k = |\overrightarrow{R}''(s)|, \\ æ = \frac{(\overrightarrow{R}'(s), \overrightarrow{R}''(s), \overrightarrow{R}'''(s))}{k^2}.\end{gathered} \tag{2.7.14}$$

Let us list a number of references that are useful in the study of the questions considered here: [7, 64, 70, 81, 94].

In what follows, our main interest in the realm of space curves will be paid to the asymptotic lines and the cuspidal edges on pseudospherical surfaces.

2.7.2 Surface strip. Curvature of a curve on a surface

Consider in $\mathbb{E}^3$ a curve $\mathcal{L}$, given by its radius vector $\overrightarrow{R} = \overrightarrow{R}(s)$, where s is the natural parameter of the curve. Let $\overrightarrow{\mu}(s)$ be some unit vector given on $\mathcal{L}$, such that $\overrightarrow{\mu}(s)$ is orthogonal to the tangent vector $\overrightarrow{\tau}(s)$ to $\mathcal{L}$. Such a pair, consisting of a curve $\mathcal{L}$ and a vector $\overrightarrow{\mu}$ with the indicated property, will be called a *surface strip* and will be denoted by $\Pi(\mathcal{L}, \overrightarrow{\mu})$. Further, let us introduce the vector $\overrightarrow{n_{\mathrm{g}}}(s) = [\overrightarrow{\mu}(s), \overrightarrow{\tau}(s)]$, called the *geodesic normal vector* of the surface strip Π. The vectors $\overrightarrow{\tau}$, $\overrightarrow{n_{\mathrm{g}}}$, $\overrightarrow{\mu}$ form an orthonormal triple. Let us calculate their derivatives and decompose them with respect to the triple itself:

$$\begin{aligned}\frac{d\overrightarrow{\tau}}{ds} &= \alpha_{11}\overrightarrow{\tau} + \alpha_{12}\overrightarrow{n_{\mathrm{g}}} + \alpha_{13}\overrightarrow{\mu}, \\ \frac{d\overrightarrow{n_{\mathrm{g}}}}{ds} &= \alpha_{21}\overrightarrow{\tau} + \alpha_{22}\overrightarrow{n_{\mathrm{g}}} + \alpha_{23}\overrightarrow{\mu}, \\ \frac{d\overrightarrow{\mu}}{ds} &= \alpha_{31}\overrightarrow{\tau} + \alpha_{32}\overrightarrow{n_{\mathrm{g}}} + \alpha_{33}\overrightarrow{\mu}.\end{aligned} \tag{2.7.15}$$

The coefficients of the decomposition (2.7.15) can be determined by differentiating the equalities

$$\begin{gathered}\overrightarrow{\tau}^2 = 1, \quad \overrightarrow{n_{\mathrm{g}}}^2 = 1, \quad \overrightarrow{\mu}^2 = 1, \\ (\overrightarrow{\tau}, \overrightarrow{n_{\mathrm{g}}}) = 0, \quad (\overrightarrow{\mu}, \overrightarrow{n_{\mathrm{g}}}) = 0, \quad (\overrightarrow{\tau}, \overrightarrow{\mu}) = 0.\end{gathered}$$

This yields

$$\begin{gathered}\alpha_{11} = \alpha_{22} = \alpha_{33} = 0, \\ \alpha_{21} = -\alpha_{12}, \quad \alpha_{31} = -\alpha_{13}, \quad \alpha_{32} = -\alpha_{23}.\end{gathered} \tag{2.7.16}$$

For the nonzero coefficients in (2.7.16), which characterize the surface strip, we will use the following notation and terminology:

$k_{\mathrm{g}} \equiv \alpha_{12}$ is the *geodesic curvature* of the surface strip,

$k_n \equiv \alpha_{13}$ is the *normal curvature* of the surface strip,

$æ_{\mathrm{g}} \equiv \alpha_{23}$ is the *geodesic torsion* of the surface strip.

Using the notations introduced, the system (2.7.15) is rewritten as

$$\begin{aligned} \frac{d\vec{\tau}}{ds} &= k_{\mathrm{g}}\,\vec{n}_{\mathrm{g}} + k_{\mathrm{n}}\,\vec{\mu}, \\ \frac{d\vec{n}_{\mathrm{g}}}{ds} &= -k_{\mathrm{g}}\,\vec{\tau} + æ_{\mathrm{g}}\,\vec{\mu}, \\ \frac{d\vec{\mu}}{ds} &= -k_{\mathrm{n}}\,\vec{\tau} - æ_{\mathrm{g}}\,\vec{n}_{\mathrm{g}}. \end{aligned} \tag{2.7.17}$$

The system (2.7.17) is called the system of *Frenet differentiation formulas* for the surface strip. The vectors $\vec{\tau}$, $\vec{n}_{\mathrm{g}}$, $\vec{\mu}$ form the fundamental frame of the surface strip. The system (2.7.17) admits the convenient matrix representation

$$\frac{d}{ds}\begin{pmatrix} \vec{\tau} \\ \vec{n}_{\mathrm{g}} \\ \vec{\mu} \end{pmatrix} = \begin{pmatrix} 0 & k_{\mathrm{g}} & k_{\mathrm{n}} \\ -k_{\mathrm{g}} & 0 & æ_{\mathrm{g}} \\ -k_{\mathrm{n}} & -æ_{\mathrm{g}} & 0 \end{pmatrix}\begin{pmatrix} \vec{\tau} \\ \vec{n}_{\mathrm{g}} \\ \vec{\mu} \end{pmatrix}. \tag{2.7.18}$$

Notice that if the vector $\vec{\mu}$ coincides with the principal normal vector $\vec{\nu}$ to the curve $\mathcal{L}$ (see Subsection 2.7.1), then a comparison of the systems (2.7.17) and (2.7.12) shows that in this case $k_{\mathrm{g}} = 0$. A surface strip along which $k_{\mathrm{g}} = 0$ identically is called a *geodesic surface strip*.

If now the vector $\vec{\mu}$ coincides with the binormal vector $\vec{\beta}$, then from (2.7.9) and the third equation in (2.7.17) it readily follows that $k_{\mathrm{n}} = 0$. A surface strip along which $k_{\mathrm{n}} = 0$ identically is called an *asymptotic surface strip*.

Now let us give formulas for the calculation of the geometric quantities introduced above [7]:

$$\begin{aligned} k_{\mathrm{n}}(s) &= \left(\frac{d^2\vec{R}}{ds^2}, \vec{\mu}\right), \\ k_{\mathrm{g}}(s) &= \left(\frac{d\vec{R}}{ds}, \frac{d^2\vec{R}}{ds^2}, \vec{\mu}\right), \\ æ_{\mathrm{g}}(s) &= \left(\frac{d\vec{R}}{ds}, \vec{\mu}, \frac{d\vec{\mu}}{ds}\right). \end{aligned} \tag{2.7.19}$$

(The parentheses in (2.7.19) denote the mixed product of vectors.)

Note that the normal curvature $k_{\mathrm{n}}(s)$ of a curve that arose in our considerations has exactly the same meaning as the analogous notion introduced in Subsection 2.2.4 by means of intuitive geometric arguments.

Let us complete the exposition by giving an important classical formula. The normal curvature k_{n} of a curve that passes through a given point on the surface in the direction $(du : dv)$ [16] can be calculated by the formula [7, 81]

$$k_{\mathrm{n}} = \left(\frac{\mathrm{II}(u,v)}{\mathrm{I}(u,v)}\right)\bigg|_{(du:dv)} = \left(\frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}\right)\bigg|_{(du:dv)}. \tag{2.7.20}$$

The notion of surface strip will be used in Subsection 2.7.4 when we will consider E. G. Poznyak's theorem on pseudospherical surfaces. In this case, as curves (base lines) that specify the surface strip we will use the asymptotic lines on the surface.

2.7.3 The Chebyshev net of asymptotic lines on a pseudospherical surface

A direction $(du : dv)$ at a given point $P(u,v)$ on the surface $S(\overrightarrow{r}(u,v)) \subset \mathbb{E}^3$ is called *asymptotic* if in this direction the *normal curvature* k_{n} is equal to zero. From (2.7.20) it follows that the asymptotic directions $(du : dv)$ on a surface are given by the differential relation

$$L(u,v)du^2 + 2M(u,v)dudv + N(u,v)dv^2 = 0. \tag{2.7.21}$$

Further, (2.7.21) yields an important consequence for the ensuing analysis: at any hyperbolic point P of the surface (where $(LN - M^2)|_{P\in S} < 0$, see Subsection 2.2.3) there always exists two distinct asymptotic directions.

Recall that a line on a surface is called an *asymptotic line*, if at each of its points its direction is asymptotic. We list two important properties of asymptotic lines:

1) If a surface in $\mathbb{E}^3$ contains a *straight line*, then this straight line is an asymptotic line on the surface.

2) The tangent plane to the surface at any point of an asymptotic lines is the osculating plane of the surface at that point.

On a pseudospherical surface one can always introduce asymptotic coordinates (u,v). The coordinate directions are then given as $(du : 0)$ and $(0 : dv)$. Using these conditions in (2.7.21), we immediately obtain

$$L = 0, \quad \text{and} \quad N = 0. \tag{2.7.22}$$

Thus, if on a surface in $\mathbb{E}^3$ the intrinsic coordinates u, v become asymptotic coordinates, then its second fundamental form takes on the form

$$\mathrm{II}(u,v) = 2M(u,v)dudv, \tag{2.7.23}$$

where $M^2 = EG - F^2$.

[16] The parameters u, v retain their meaning from § 2.3.

Under the conditions (2.7.22) and (2.7.23), the Peterson-Codazzi equations reduce to the equalities

$$E_v G - G_u F = 0,$$
$$G_u E - E_v F = 0,$$

which represent exactly the criterion (2.5.17) for a net to be a Chebyshev net (vanishing of the Christoffel symbols: $\Gamma^1_{12} = 0,\ \Gamma^2_{12} = 0$).

We thus reached the following conclusion: an asymptotic coordinate net on a pseudospherical surface is a *Chebyshev net*. Accordingly, in asymptotic coordinates (u, v), the metric of a pseudospherical surface reads

$$ds^2 = du^2 + 2\cos z(u, v) du dv + dv^2, \tag{2.7.24}$$

where $z(u, v)$ is a solution of the sine-Gordon equation

$$z_{uv} = \sin z. \tag{2.7.25}$$

Now let us turn to the fundamental equations that define a pseudospherical surface in the space $\mathbb{E}^3$. To this end we write the Peterson-Codazzi and Gauss equations (2.3.23)–(2.3.25) for the Chebyshev metric (2.7.24) of Gaussian curvature $K \equiv -1$:

$$L_v - M_u - \frac{z_u}{\sin z} \cdot (N - \cos z \cdot M) = 0, \tag{2.7.26}$$
$$M_v - N_u + \frac{z_v}{\sin z} \cdot (L - \cos z \cdot M) = 0, \tag{2.7.27}$$
$$LN - M^2 = -\sin^2 z. \tag{2.7.28}$$

To the case when on the surface obtained in $\mathbb{E}^3$, equipped with the metric (2.7.24), the intrinsic coordinates u, v become asymptotic coordinates correspond the following solutions of the system (2.7.26)–(2.7.28):

$$L = 0, \quad M = \sin z, \quad N = 0. \tag{2.7.29}$$

In order to obtain the radius vector $\vec{r}(u, v)$ of the original pseudospherical surface in asymptotic Chebyshev coordinates with given net angle $z(u, v)$, it is necessary to integrate the system of differentiation formulas (2.3.2)–(2.3.5) (or (2.3.18), (2.3.19)), applied to the coefficients of the second fundamental form (2.7.29). In this case the system of differentiation formulas takes on the form

$$\vec{r}_{uu} = z_u \cdot \vec{n}_u, \tag{2.7.30}$$
$$\vec{r}_{uv} = \sin z \cdot \vec{n}, \tag{2.7.31}$$
$$\vec{r}_{vv} = z_v \cdot \vec{n}_v, \tag{2.7.32}$$
$$\vec{n}_u = \cot z \cdot \vec{r}_u - \frac{1}{\sin z} \cdot \vec{r}_v, \tag{2.7.33}$$
$$\vec{n}_v = -\frac{1}{\sin z} \cdot \vec{r}_u + \cot z \cdot \vec{r}_v. \tag{2.7.34}$$

The system (2.7.30)–(2.7.34) for the radius vector $\vec{r}(u,v)$ and the unit normal vector $\vec{n}(u,v)$ of the pseudospherical surface S, which realizes an isometric immersion in $\mathbb{E}^3$ of the Chebyshev metric (2.7.24) in asymptotic coordinates, will be referred to as the *fundamental system of equations for pseudospherical surfaces*.

Taking into account the mutual orientation in space of the vectors $\vec{r}_u$, $\vec{r}_v$, and $\vec{n}$, and also equalities (2.7.33) and (2.7.34), we write the following useful relations:

$$[\vec{r}_u, \vec{r}_v] = \sin z \cdot \vec{n}, \tag{2.7.35}$$

$$[\vec{r}_u, \vec{n}] = \cot z \cdot \vec{r}_u - \frac{1}{\sin z} \cdot \vec{r}_v, \tag{2.7.36}$$

$$[\vec{r}_v, \vec{n}] = \frac{1}{\sin z} \cdot \vec{r}_u - \cot z \cdot \vec{r}_v. \tag{2.7.37}$$

Let us calculate the geodesic curvature k_{g} and geodesic torsion $æ_{\mathrm{g}}$ of the asymptotic lines u and v on a pseudospherical surface (by definition, their normal curvature $k_{\mathrm{n}} = 0$).

To this end we pick some asymptotic line u, given by the condition $v = v^*$ (where v^* is a fixed value). It is clear that the radius vector $\vec{R}(u)$ of the asymptotic line u can be "extracted" from the radius vector $\vec{r}(u,v)$ of the surface S itself. Therefore,

$$\vec{R}(u) = \vec{r}(u,v^*), \quad \vec{R}_u(u) = \vec{r}_u(u,v^*), \quad \vec{R}_{uu}(u) = \vec{r}_{uu}(u,v^*). \tag{2.7.38}$$

Moreover, from the system (2.7.30)–(2.7.34) and relations (2.7.35)–(2.7.37) one obtains (assuming that $v = v^*$, with v^* fixed)

$$[\vec{R}_u, \vec{R}_{uu}] = [\vec{r}_u, \vec{r}_{uu}] = -z_u \vec{n}, \qquad (\vec{R}_u, \vec{R}_{uu}, \vec{n}) = -z_u. \tag{2.7.39}$$

In the preceding considerations the parameter u plays the role of the natural parameter s on the asymptotic line of the family selected. Hence, applying the second formula in (2.7.19) (in the present case $\vec{\mu} \equiv \vec{n}$), we obtain the geodesic curvature $k_{\mathrm{g}}^1(u)$ of the asymptotic line u:

$$k_{\mathrm{g}}^{(1)} = (\vec{R}_u, \vec{R}_{uu}, \vec{n}) = -z_u. \tag{2.7.40}$$

Similarly, we can apply the third formula in (2.7.19) allows us to calculate the geodesic torsion of the asymptotic line u:

$$æ_{\mathrm{g}}^{(1)} = 1. \tag{2.7.41}$$

(Note that in (2.7.40) and (2.7.41) the upper index "(1)" in the notations $k_{\mathrm{g}}^{(1)}$ and $æ_{\mathrm{g}}^{(1)}$ refer to the "first" family of asymptotic lines, the u-family.)

Totally analogous computations for the asymptotic lines v (of the second family) yield

$$k_{\mathrm{g}}^{(2)} = z_v, \quad æ_{\mathrm{g}}^{(2)} = -1. \tag{2.7.42}$$

Side by side with the arguments used above to derive the Servant-Bianchi system of equations (2.5.22), let us now present another geometric approach [74], which shows how two arbitrary nonintersecting lines on a pseudospherical surface can be included in a local Chebyshev net $\mathrm{Cheb}(u, v)$ with some initial net angle $z(u, v)$.

Let (u_0, v_0) be a point of intersection of the two given lines. Choose a neighborhood of (u_0, v_0) in which the lines have no other intersection. Denote the natural parameters on the lines, measured from the intersection point (u_0, v_0), by u and v. Denote the geodesic curvatures of the lines by $k_{\mathrm{g}}^{(1)}(u)$ and $k_{\mathrm{g}}^{(2)}(v)$, and the angle the lines make by $z(u_0, v_0)$. Suppose that on the surface there are introduced Chebyshev coordinates (u, v), with the angle $z(u, v)$ subject to determination from the condition that the first (second) line is given by the relation $v = v_0$ (respectively, $u = u_0$). The geodesic curvature of a line on the surface is a characteristic of the intrinsic geometry of the surface, and is given for the first and, respectively, the second line, by (2.7.40) and (2.7.42):

$$k_{\mathrm{g}}^{(1)}(u) = -z_u, \quad k_{\mathrm{g}}^{(2)}(v) = z_v.$$

Let us formulate the Darboux problem[17] for the determination of the net angle $z(u, v)$ of the locally reconstructible Chebyshev net:

$$\begin{aligned} z_{uv} &= \sin z, \\ z(u_0, v) &= z(u_0, v_0) + \int_{v_0}^{v} k_{\mathrm{g}}^{(2)}(\xi) d\xi, \\ z(u, v_0) &= z(u_0, v_0) - \int_{u_0}^{u} k_{\mathrm{g}}^{(1)}(\eta) d\eta. \end{aligned} \tag{2.7.43}$$

The Darboux problem (2.7.43) for the sine-Gordon equation has a unique solution in the domain considered (see Chapter 3), which enables us to construct a Chebyshev net with the corresponding net angle, namely, the solution of problem (2.7.43). It is important to note that the geometric uniqueness of the net is guaranteed by the fact that through any given point of the surface, in any given direction, there passes a unique line with that direction and with a prescribed geodesic curvature.

2.7.4 Pseudospherical surfaces and the sine-Gordon equation

D. Hilbert's result on the nonimmersibility of the complete Lobachevsky plane Λ^2 in the Euclidean space $\mathbb{E}^3$ (§ 2.6) and its stengthening by N. V. Efimov [27] to the nonimmersibility of the Lobachevsky half-plane in $\mathbb{E}^3$ placed natural "size restrictions" on the realizability of Lobachevsky planimetry in the three-dimensional Euclidean plane. In this connection there arises the general problem of finding the

[17] A detailed analysis of the Darboux problem for the sine-Gordon equation will be carried out in Chapter 3.

potential boundary of "realizability–non-realizability in $\mathbb{E}^3$" of geometrically typical domains of the Lobachevsky plane Λ^2. In the development of this topic an important role is played by a theorem of E. G. Poznyak [71] which establishes a constructive connection between pseudospherical surfaces (objects in $\mathbb{E}^3$, on which the geometry on corresponding parts of the Lobachevsky plane Λ^2 is realized) and regular solutions of the sine-Gordon equation. As we shall show in Chapter 3, the sine-Gordon equation has a rather wide spectrum of classes of regular solutions, which in turn makes it possible to obtain the corresponding geometric images in $\mathbb{E}^3$, i.e., the pseudospherical surfaces, which reflect various geometric properties of parts of the Lobachevsky plane that can be isometrically immersed in $\mathbb{E}^3$.

As we already mentioned, if in some domain $D(u,v) \subset \mathbb{R}^2$ the solution $z(u,v)$ of the sine-Gordon equation

$$z_{uv} = \sin z \tag{2.7.44}$$

satisfies the condition

$$0 < z < \pi, \tag{2.7.45}$$

then the domain $\Lambda_D \subset \Lambda^2$ corresponding to D in the Lobachevsky plane can be regularly and isometrically immersed in $\mathbb{E}^3$, and in fact in such a way that the u and v lines on the resulting surface in $\mathbb{E}^3$ form a net of asymptotic lines with net angle $z(u,v)$.

The fact noted above can to a certain extent be generalized to the case when the solution $z(u,v)$ of the sine-Gordon equation is not subject to the constraint (2.7.45). This is expressed by the theorem of E. G. Poznyak given below. Generally, when the constraint (2.7.45) is discarded, one can make the following a priori assumption: to each regular solution $z(u,v)$, $u,v \in \mathbb{R}^2$, there is associated a pseudospherical surface that has singularities which correspond to the values $z(u,v) = k\pi$, with k an integer.

Theorem 2.7.1 (E.G. Poznyak, [71])**.** *. Suppose the function $z = z(u,v) \in C^4$, defined on the whole plane $\mathbb{R}^2(u,v)$, is a solution of the sine-Gordon equation* (2.7.44)*. Then there exists a vector function $\overrightarrow{r} = \overrightarrow{r}(u,v) \in C^2$, defined on $\mathbb{R}^2(u,v)$, such that its graph in the domain $\{z : z \neq k\pi\}$ (k integer), represents a pseudospherical surface $S[z]$. Moreover, the coordinate lines u,v on this surface form a net of asymptotic lines with net angle $z(u,v)$.*

Note. To the level lines $z(u,v) = k\pi$ (where k is an integer) there correspond singularities of the pseudospherical surface $S[z]$, i.e., its irregular edges (cuspidal edges), cusp points, and so on. At these singularities the adjacent regular components of the pseudosperical surface "merge", and this occurs in such a way that the u and v lines of the *unified* Chebysehv net of asymptotic lines on $S[z(u,v)]$, $(u,v) \in \mathbb{R}^2$ preserve their smoothness everywhere (in particular, when one "crosses" a singularity). Intuitive examples that illustrate this are provided by the pseudosphere and the Minding "bobbin" and "top" (see Chapter 1).

Let us indicate the main ideas of the proof of Theorem 2.7.1.

Suppose that $z(u,v)$ is a solution of the sine-Gordon equation (2.7.44), defined on the whole plane $\mathbb{R}^2(u,v)$. Introduce the following two functions:

$$\widetilde{k}_{\mathrm{g}}^{(2)}(v) = z_v(0,v), \quad \widetilde{æ}_{\mathrm{g}}^{(2)}(v) = -1. \tag{2.7.46}$$

By (2.7.42), one can find an asymptotic strip (defined by an asymptotic line and its binormal vector, or, equivalently, by the "area element" tangent to it), the curvature and torsion of which coincide with the introduced functions (2.7.46). Denote this strip by $\widetilde{\Pi}^{(2)}$. Let $\vec{\tau}^{(2)}(v)$, $\vec{n}_{\mathrm{g}}^{(2)}(v)$, $\vec{\mu}^{(2)}(v)$ be the fundamental frame of the strip $\widetilde{\Pi}^{(2)}$, defined in each point of its base line (asymptotic line of the v-family). In each chosen point of the base line v, construct another frame of vectors, connected with the asymptotic line u of the other family, passing through the considered point (u, v). With the chosen line u we associate a second asymptotic strip $\widetilde{\Pi}^{(1)}$, and define the corresponding fundamental frame $\vec{\tau}^{(1)}$, $\vec{n}_g^{(1)}$, $\vec{\mu}^{(1)}$ as follows: the vector $\vec{\mu}^{(1)}$ in the given point coincides with $\vec{\mu}^{(2)}(v)$, the vector $\vec{\tau}^{(1)}$ lies in the "area element" tangent to the strip $\widetilde{\Pi}^{(2)}$ at the point (u, v) and makes an angle of $z(0, v)$ with the vector $\vec{\tau}^{(2)}(v)$, and finally $\vec{n}_{\mathrm{g}}^{(1)}$ is the vector product of the unit vectors $\vec{\mu}^{(1)}$ and $\vec{\tau}^{(1)}$ (i.e., $\vec{\tau}^{(1)}$, $\vec{n}_g^{(1)}$, $\vec{\mu}^{(1)}$ form a right-handed triple of vectors).

Now let us introduce the functions k_{g} and $æ_{\mathrm{g}}$, depending on the two parameters u and v, by

$$k_{\mathrm{g}}(u, v) = -z_u(u, v), \quad æ_{\mathrm{g}}(u, v) = 1. \tag{2.7.47}$$

By the Frenet formulas (2.7.9), (2.7.11) and (2.7.12), the functions $k_{\mathrm{g}}(u, v)$ and $æ_{\mathrm{g}}(u, v)$ figuring in (2.7.47) define, for each fixed value v, a unique asymptotic strip, on which they serve as the curvature and torsion of the base (asymptotic) line. Moreover, the fundamental frame of this base line is given by the vectors $\vec{\tau}^{(1)}$, $\vec{n}_{\mathrm{g}}^{(1)}$, $\vec{\mu}^{(1)}$ (for fixed v and $u = 0$).

The following assertion holds true [71]: the collection of base curves, constructed for all values of v, constitutes for the special parametrization u, v introduced above the graph of the sought-for function $\vec{r}(u, v)$, i.e., a pseudospherical surface. In other words, a pseudospherical surface can be "sewn" from asymptotic strips.

Let us justify this assertion. Suppose the vectors $\vec{\tau}(u, v)$, $\vec{n}_{\mathrm{g}}(u, v)$, and $\vec{\mu}(u, v)$ form the fundamental frame of strips with base lines u (or v).

Using the Frenet formulas (2.7.11), (2.7.12), (2.7.9) in conjunction with relations (2.7.47) and the fact that $\vec{\nu} \equiv \vec{n}_{\mathrm{g}}$, $\vec{\beta} \equiv \vec{\mu}$, we have

$$\frac{\partial \vec{\tau}}{\partial u} = -z_u \cdot \vec{\nu}, \tag{2.7.48}$$

$$\frac{\partial \vec{\nu}}{\partial u} = z_u \cdot \vec{\tau} + \vec{\beta}, \tag{2.7.49}$$

$$\frac{\partial \vec{\beta}}{\partial u} = -\vec{\nu}. \tag{2.7.50}$$

Let us differentiate the equalities (2.7.48)–(2.7.50) with respect to v. Since the function $z(u, v)$ satisfies the sine-Gordon equation (2.7.25), we obtain the following system of equations for the unknown functions $\dfrac{\partial \vec{\tau}}{\partial v}$, $\dfrac{\partial \vec{\nu}}{\partial v}$, and $\dfrac{\partial \vec{\beta}}{\partial v}$:

$$\left(\frac{\partial \overrightarrow{\tau}}{\partial v}\right)_u = -z_u \frac{\partial \overrightarrow{\nu}}{\partial v} - \sin z \cdot \overrightarrow{\nu}, \tag{2.7.51}$$

$$\left(\frac{\partial \overrightarrow{\nu}}{\partial v}\right)_u = z_u \frac{\partial \overrightarrow{\tau}}{\partial v} + \frac{\partial \overrightarrow{\beta}}{\partial v} + \sin z \cdot \overrightarrow{\tau}, \tag{2.7.52}$$

$$\left(\frac{\partial \overrightarrow{\beta}}{\partial v}\right)_u = -\frac{\partial \overrightarrow{\nu}}{\partial v}. \tag{2.7.53}$$

The system (2.7.51)–(2.7.53) can be integrated exactly, yielding the solution

$$\begin{aligned} \frac{\partial \overrightarrow{\tau}}{\partial v} &= \sin z \cdot \overrightarrow{\beta}, \\ \frac{\partial \overrightarrow{\nu}}{\partial v} &= -\cos z \cdot \overrightarrow{\beta}, \\ \frac{\partial \overrightarrow{\beta}}{\partial v} &= -\sin z \cdot \overrightarrow{\tau} + \cos z \cdot \overrightarrow{\nu}. \end{aligned} \tag{2.7.54}$$

Analyzing the obtained solution (2.7.54) and noting that $\overrightarrow{\tau} = \overrightarrow{r}_u$ (see (2.7.4)), we recast the first relation in (2.7.54) as

$$\frac{\partial}{\partial u}(\overrightarrow{r}_v) = \sin z \cdot \overrightarrow{\beta}. \tag{2.7.55}$$

The solution of equation (2.7.55) is given by the function

$$\overrightarrow{r}_v = \cos z \cdot \overrightarrow{\tau} + \sin z \cdot \overrightarrow{n}_{\mathrm{g}}. \tag{2.7.56}$$

Moreover, as we already noted,

$$\overrightarrow{r}_u = \overrightarrow{\tau}. \tag{2.7.57}$$

Since $\overrightarrow{\tau}$ and $\overrightarrow{n}$ are orthogonal unit vectors, from (2.7.56) and (2.7.57) it follows directly that

$$\overrightarrow{r}u^2 = 1, \quad (\overrightarrow{r}_u, \overrightarrow{r}_v) = \cos z, \quad \overrightarrow{r}_u^2 = 1,$$

or

$$E(u,v) = 1, \quad F(u,v) = \cos z(u,v), \quad G(u,v) = 1.$$

This shows that to the graph of the vector function $\overrightarrow{r}(u,v)$ there corresponds in $\mathbb{E}^3$ the following expression for the metric of the surface:

$$ds^2 = du^2 + 2\cos z(u,v)dudv + dv^2.$$

The metric of our surface in $\mathbb{E}^3$ with radius vector $\overrightarrow{r}(u,v)$ is identical to the metric, considered earlier in parts of the Lobachevsky plane, written with respect to the Chebyshev parametrization, under the condition that $z(u,v) \neq k\pi$, where k in an integer. In the case when $z = k\pi$ the metric degenerates (the coordinate lines u and v become tangent). This condition translates into "passage" through the irregular singularities of the surface (cuspidal edges, cusp points, and so on) and the "transition" to the next (adjacent) regular component of the surface. It is important to emphasize here that globally (as a system of curves), the Chebyshev coordinate net of asymptotic lines preserves its regularity on the whole pseudospherical surface (i.e., for all values of (u, v)).

The content of E. G. Poznyak's theorem considered here can formulated in a slightly different manner, as in Theorem 2.7.2 below [94].

In the plane $\mathbb{R}^2(u, v)$ we will consider some domain D, in which there is defined a solution $z(u, v) \in C^4(D)$ of the sine-Gordon equation. For D we will take either a rectangle $[u_1, u_2] \times [v_1, v_2]$, or a (finite or infinite) strip, or even the entire plane $\mathbb{R}^2(u, v)$. Suppose there are also given a point $(u_0, v_0) \in D$ and an orthogonal right-handed triple of vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$.

Theorem 2.7.2. *Under the conditions formulated above, in the domain $D(u, v)$ there exists a unique vector function $\vec{r}(u, v) \in C^3(D) : D \to \mathbb{E}^3$, which defines in $\mathbb{E}^3$ a pseudospherical surface (with radius vector $\vec{r}(u, v)$ and unit normal vector $\vec{n}(u, v)$), on which the coordinates (u, v) are Chebyshev asymptotic coordinates with the net angle $z(u, v)$. Moreover,*

$$\vec{r}_u(u_0, v_0) = \vec{e}_1, \tag{2.7.58}$$

$$\vec{r}_v(u_0, v_0) = \cos z(u_0, v_0) \cdot \vec{e}_1 + \sin z(u_0, v_0) \cdot \vec{e}_2, \tag{2.7.59}$$

$$\vec{n}(u_0, v_0) = \vec{e}_3. \tag{2.7.60}$$

To the level lines $z(u, v) = k\pi$ (with k an integer) there correspond irregular singularities of this surface.

Proof. The fundamental frame of the asymptotic line $v = 0$ is constructed as the unique solution of the system of Frenet equations

$$\frac{\partial}{\partial u}\begin{pmatrix} \vec{\tau}^{(v_0)} \\ \vec{\beta}^{(v_0)} \\ \vec{\nu}^{(v_0)} \end{pmatrix} = \begin{pmatrix} 0 & -z_u & 0 \\ z_u & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \vec{\tau}^{(v_0)} \\ \vec{\beta}^{(v_0)} \\ \vec{\nu}^{(v_0)} \end{pmatrix}. \tag{2.7.61}$$

Moreover, it is appropriate to define the triple of vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, indicated in the statement of the theorem as

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} \equiv \begin{pmatrix} \vec{\tau}^{(v_0)} \\ \vec{\beta}^{(v_0)} \\ \vec{\nu}^{(v_0)} \end{pmatrix}, \quad \vec{\nu} \equiv \vec{n}.$$

The radius vector $\vec{r}(u, v_0)$ of the asymptotic line $v = v_0$ is found by integrating the system (2.7.59):

$$\vec{r}(u, v_0) = \int_{u_0}^{u} \vec{\tau}^{(v_0)}(\xi)\, d\xi. \tag{2.7.62}$$

Let us fix this line. Through each of its point there passes an asymptotic line u of the other family, the fundamental frame of which is determined from the system

$$\frac{\partial}{\partial v}\begin{pmatrix}\vec{\tau}^{(u)}\\ \vec{\beta}^{(u)}\\ \vec{\nu}^{(u)}\end{pmatrix}=\begin{pmatrix}0 & z_v & 0\\ -z_v & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\vec{\tau}^{(u)}\\ \vec{\beta}^{(u)}\\ \vec{\nu}^{(u)}\end{pmatrix}, \tag{2.7.63}$$

with the initial conditions

$$\left.\begin{pmatrix}\vec{\tau}^{(u)}\\ \vec{\beta}^{(u)}\\ \vec{\nu}^{(u)}\end{pmatrix}\right|_{v=v_0}=\left(\begin{array}{cc|c}\cos z(u,v_0) & \sin z(u,v_0) & 0\\ -\sin z(u,v_0) & \cos z(u,v_0) & 0\\ \hline 0 & 0 & 1\end{array}\right)\begin{pmatrix}\vec{\tau}^{(v_0)}(u)\\ \vec{\beta}^{(v_0)}(u)\\ \vec{\nu}^{(v_0)}(u)\end{pmatrix}. \tag{2.7.64}$$

The isolated "corner minor" (matrix) in the right-hand side of (2.7.62) gives the rotation by angle $z(u,v_0)$ in the plane of the vectors $\vec{\tau}^{(v_0)}(u)$ and $\vec{\beta}^{(v_0)}(u)$. This rotation maps $\vec{\tau}^{(v_0)}(u)$ into $\vec{\tau}^{(u)}(v_0)$, and $\vec{\beta}^{(v_0)}(u)$ into $\vec{\beta}^{(u)}(v_0)$.

The radius vector

$$\vec{r}(u,v)=\vec{r}(u,v_0)+\int_{v_0}^{v}\vec{\tau}^{(u)}(\eta)\,d\eta \tag{2.7.65}$$

will define the sought-for pseudospherical surface. To show that this surface has singularities for $z(u,v)=k\pi$ (where k is an integer), it suffices to calculate the principal curvatures: $K_1=(\cot z)/2$, $K_2=-(\tan z)/2$. As $z\to k\pi$ one of the principal curvatures tends to zero, while the other tends to infinity (in this process the Gaussian curvature remains $K=K_1\cdot K_2=-1$). Theorem 2.7.2 is proved. □

2.7.5 Geodesic curvature and torsion of an irregular edge

In this subsection we will obtain formulas for the calculation of the main geometric characteristics of irregular edges (cuspidal edges), namely, their *geodesic curvature* and *torsion*.

Suppose we have a solution $z(u,v)\in C^4$ of the sine-Gordon equation (2.7.25). Then by E. G. Poznyak's theorem, to this solution there corresponds in $\mathbb{E}^3$ a pseudospherical surface $S[\vec{r}(u,v),z(u,v)]$, $\vec{r}\in C^3$, $\vec{n}\in C^2$. The preimages of the irregular edges on this surface S are the lines $v=f(u)$ given in the (u,v)-parametric plane by the condition

$$v=f(u),\quad z(u,v)=k\pi,\quad k \text{ an integer.} \tag{2.7.66}$$

In other words, the singularities on the pseudospherical surface are determined by the level lines $k\pi$ (with k an integer) of the function $z=z(u,v)$, i.e., of the solution of the sine-Gordon equation (2.7.25).

Let us derive a formula for the geodesic curvature k_{g} of the irregular edge of the pseudospherical surface, defined by condition (2.7.64). To this end we use a standard differential-geometric formula for the calculation of the geodesic curvature of a curve that is given parametrically as $u = u(t)$, $v = v(t)$ (where t is the parameter) and lies on a surface with a metric of general form (2.2.7):

$$k_{\mathrm{g}} = \frac{\sqrt{EG - F^2}}{(Eu'^2 + 2Fu'v' + Gv'^2)^{3/2}} \cdot \bigl|u''v' - v''u' + M_1 v' - M_2 u'\bigr|. \tag{2.7.67}$$

In the case we are interested in, of a metric of the form (2.7.24), formula (2.7.65) becomes

$$k_{\mathrm{g}} = \frac{|\sin z|}{(u'^2 + 2\cos z \cdot u'v' + v'^2)^{3/2}} \cdot \bigl|u''v' - v''u' + M_1 v' - M_2 u'\bigr| \tag{2.7.68}$$

(the "prime" in (2.7.66) denotes differentiation with respect to the parameter t).

In formulas (2.7.65) and (2.7.66) we used the notations

$$\begin{aligned} M_1 &= \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} v'^2, \\ M_2 &= \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u'v' + \Gamma^2_{22} v'^2. \end{aligned} \tag{2.7.69}$$

The Christoffel symbols (2.3.9)–(2.3.11) for the Chebyshev metric (2.7.24) are given by the expressions

$$\begin{aligned} \Gamma^1_{11} &= \frac{z_u \cos z}{\sin z}, & \Gamma^2_{11} &= -\frac{z_u}{\sin z}, \\ \Gamma^1_{22} &= -\frac{z_v}{\sin z}, & \Gamma^2_{22} &= \frac{z_v \cos z}{\sin z}, \\ \Gamma^1_{12} &= 0, & \Gamma^2_{12} &= 0. \end{aligned} \tag{2.7.70}$$

As the parameter t in (2.7.66) we take the already available variable u: $t \equiv u$. Then in view of (2.7.67) and (2.7.68), the representation (2.7.66) for the line (2.7.64) can be recast as

$$k_{\mathrm{g}}\big|_{z=k\pi} = \left|\frac{z_u + (-1)^k z_u v' - (-1)^k z_v v'^2 - z_v v'^3}{(1 + (-1)^k v')^3}\right|. \tag{2.7.71}$$

In (2.7.69) the "prime" already denotes differentiation with respect to u.

Using the classical theorem on the differentiation of implicit functions, we obtain for the derivative of the level line given by condition (2.7.64) the expression

$$v'(u) = -\frac{z_u}{z_v}. \tag{2.7.72}$$

Substituting (2.7.70) in (2.7.69) finally yields an expression for the geodesic curvature of an irregular edge of a pseudospherical surface:

$$k_{\mathrm{g}}\big|_{z=k\pi} = \left|\left(\frac{z_u z_v}{z_u - (-1)^k z_v}\right)\bigg|_{z=k\pi}\right|. \tag{2.7.73}$$

Formula (2.7.71) can be rewritten by explicitly isolating in it the expression $v = f(u)$ for the level line (see (2.7.64)):

$$k_{\mathrm{g}}\big|_{z=k\pi} = \left| \frac{z_u(u, f(u))}{(-1)^k + f'(u)} \right|. \tag{2.7.74}$$

Let us indicate also another approach for deriving the formula (2.7.72), based on the analysis of the fundamental equations (2.7.30)–(2.7.34) for pseudospherical surfaces. Namely, from (2.7.33) and (2.7.34) it follows, in particular, that on an irregular singularity of the surface (see condition (2.7.64))

$$(\vec{r}_u - (-1)^k \vec{r}_v)\big|_{z(u,v)=k\pi} = 0. \tag{2.7.75}$$

The radius vector $\vec{R}(u)$ of an irregular edge of the pseudospherical surface is obviously defined by the condition

$$\vec{R}(u) = \vec{r}(u, f(u)), \tag{2.7.76}$$

which "extracts" it from the radius vector of the surface.

From (2.7.74) we obtain, using (2.7.73), the formula

$$\vec{R}_u = (1 + (-1)^k \cdot f'(u)) \cdot \vec{r}_u(u, f(u)). \tag{2.7.77}$$

To compute the geodesic curvature and torsion of an irregular edge using the formulas (2.7.14) we need the corresponding "components", the final expressions of which is given below [94] (their derivation rests on the relations (2.7.73)–(2.7.75), (2.7.70), and on the fundamental equations (2.7.30)–(2.7.34)):

$$\begin{aligned} \vec{R}_{uu} = {} & \left(z_u \cot z - \frac{z_v}{\sin z} \cdot f'^2 \right) \cdot \vec{r}_u \\ & + \left(-\frac{z_u}{\sin z} + z_v \cot z \cdot f'^2 + f'' \right) \cdot \vec{r}_v + 2f' \cdot \sin z \cdot \vec{n}, \end{aligned} \tag{2.7.78}$$

$$\begin{aligned} [\vec{R}_u, \vec{R}_{uu}] &= (-z_u + (-1)^k z_v \cdot f'^2)(1 + (-1)^k f') \cdot \vec{n} \\ &= -(1 + (-1)^k f')^2 z_u \cdot \vec{n}, \end{aligned} \tag{2.7.79}$$

$$\vec{R}_{uuu} = -z_u(1 + (-1)^k \cdot f')(1 - (-1)^k \cdot f') \cdot \vec{n} + \cdots. \tag{2.7.80}$$

The calculation of the requisite mixed products of vectors yields, respectively,

$$(\vec{R}_u, \vec{R}_{uu}, \vec{n}) = -(1 + (-1)^k f')^2 \cdot z_u, \tag{2.7.81}$$

$$(\vec{R}_u, \vec{R}_{uu}, \vec{R}_{uuu}) = (1 + (-1)^k f')^3 \cdot (1 - (-1)^k f') z_u^2. \tag{2.7.82}$$

Substitution of the expressions given above in (2.7.14) and (2.7.19) confirms the already obtained formula (2.7.72) and yields the following explicit expression for the torsion of an irregular edge of a pseudospherical surface:

$$\text{æ} = \frac{1 - (-1)^k f'(u)}{1 + (-1)^k f'(u)} \tag{2.7.83}$$

(the level line $v = f(u)$ is found from the condition (2.7.64)).

2.7.6 Lines of constant geodesic curvature on the plane Λ^2

Let us study the geodesic curvature of the typical lines on the Lobachevsky plane Λ^2 that we considered in § 1.2: straight lines (geodesics), equidistants, horocycles, and non-Euclidean circles.

We work in the interpretation $\Lambda^2(\Pi)$ of the Lobachevsky plane in the half-plane (see Subsection 1.2.3) and the general formula (2.7.65) for the computation of the geodesic curvature of a curve on a surface with a given metric.

As it was established in Subsection 1.2.3, in the Poincaré half-plane model $\Lambda^2(\Pi)$ of the Lobachevsky plane Λ^2 the aforementioned typical lines on Λ^2 are *mapped* either into *straight lines*, or into *circles* (or pieces thereof) (see Figures 1.2.9–1.2.14). Let us calculate the *geodesic curvature* of the straight lines and of the circles, using the metric (1.2.33) on the Lobachevsky plane corresponding to $\Lambda^2(\Pi)$.

2.7.6.1. In the model $\Lambda^2(\Pi)$, suppose that the image of some line is given by the *straight line* equation

$$v = au + b, \quad v \geq 0; \quad a, b = \text{const}. \tag{2.7.84}$$

Let us calculate the geodesic curvature of the line (2.7.82) using the formula (2.7.65), applied for the metric (1.2.33):

$$ds^2 = \frac{1}{v^2}\big(du^2 + dv^2\big).$$

The calculation of the "intermediate components" in (2.7.65) yields:

$$\begin{aligned} &E = \frac{1}{v^2}, && F = 0, && G = \frac{1}{v^2}; && W = EG - F^2 = \frac{1}{v^4} \\ &E_u = 0, && F_u = 0, && G_u = 0, && \\ &E_v = -\frac{2}{v^3}, && F_v = 0, && G = -\frac{2}{v^3}. && \end{aligned} \tag{2.7.85}$$

$$\begin{aligned} &\Gamma^1_{11} = 0, && \Gamma^1_{12} = -\frac{1}{v}, && \Gamma^1_{22} = 0, \\ &\Gamma^2_{11} = \frac{1}{v}, && \Gamma^2_{12} = 0, && \Gamma^2_{22} = -\frac{1}{v}. \end{aligned} \tag{2.7.86}$$

Using in (2.7.82) the obvious parametrization $v = at+b$, $u = t$ $(v' = a,\ v'' = 0,\ u' = 1,\ u'' = 0)$, we obtain also the expressions of the type (2.7.67) for the present case:

$$M_1 = -\frac{2a}{v}, \quad M_2 = \frac{1-a^2}{v}. \tag{2.7.87}$$

Substituting expressions (2.7.83)–(2.7.85) in the formula (2.7.65), we obtain the value of the geodesic curvature of the line (2.7.82) on $\Lambda^2(\Pi)$:

$$k_{\mathrm{g}} = \frac{1}{\sqrt{1+a^2}}. \tag{2.7.88}$$

Let us analyze this last result.

a) If $a = 0$ (the straight line (2.7.82) is parallel to the u-axis), then $k_{\mathrm{g}} = 1$. This indicates that horocycles (represented on $\Lambda^2(\Pi)$ by straight lines parallel to the u-axis, see Figure 2.1.12) have *geodesic curvature* $k_{\mathrm{g}} = 1$.

b) If $|a| \to \infty$ (the straight line (2.7.82) takes its limiting position, orthogonal to the u-axis), then $k_{\mathrm{g}} \to 0$. This confirms the fact that *the geodesic curvature of straight lines* (rays on $\Lambda^2(\Pi)$) that are perpendicular on the u-axis *vanishes*: $k_{\mathrm{g}} = 0$.

c) If $|a| \in (0, +\infty)$ (the lines (2.7.82) are represented by slanted rays on $\Lambda^2(\Pi)$), then $0 < k_{\mathrm{g}} < 1$. This range of variation of the geodesic curvature characterizes the *equidistants*.

2.7.6.2. Let us carry out a similar investigation for the geodesic curvature of the geometric images represented in the Poincaré model $\Lambda^2(\Pi)$ by *circles or pieces thereof* (see Figures 1.2.9–1.2.14). We use the corresponding parametric presentation

$$u = R\cos t + a, \quad v = R\sin t + b; \quad v \geq 0, \tag{2.7.89}$$

where $a, b, R = \text{const}$, $b \geq 0$, $R > 0$.

To calculate the geodesic curvatures k_{g} of the lines given by equations (2.7.87) we us use formula (2.7.65). The values of the expressions (2.7.83) and (2.7.84) are preserved (they are the same as for the line (2.7.82) considered above); only M_1 and M_2 in (2.7.85) change, according to (2.7.87). We thus obtain for the geodesic curvature k_{g} of the line (2.7.87) the value

$$k_{\mathrm{g}} = \frac{b}{R}. \tag{2.7.90}$$

Let us analyze the expression (2.7.88):

1) If $b = 0$ (in this case the lines (2.7.87) are semi-circles that lean orthogonally on the absolute, i.e., *straight lines*, see Figure 1.2.10 *b*), then $k_{\mathrm{g}} = 0$.

2) If $b = R$, then on the half-plane $\Lambda^2(\Pi)$ we obtain circles that are tangent to the absolute (the u-axis), i.e., *horocycles*; their geodesic curvature is $k_{\mathrm{g}} = 1$ (Figure 1.2.12 *b*).

3) If $0 < b < R$, then on $\Lambda^2(\Pi)$ to the lines (2.7.87) correspond pieces (arcs) of circles that lean non-orthogonally to the absolute (the u-axis). These lines represent *equidistants*, and their geodesic curvature $k_{\mathrm{g}} \in (0, 1)$ (Figure 1.2.13 *b*).

4) Finally, for $b > R$ we have $k_{\mathrm{g}} > 1$. In this case to the lines (2.7.87) correspond circles that are entirely contained in the upper half-plane $\Lambda^2(\Pi)$, i.e., *non-Euclidean circles* of the plane Λ^2 (Figure 1.2.11 *b*).

We summarize the results obtained in Table 2.7.1, identifying the type of line in the Lobachevsky plane by the value of its geodesic curvature.

Table 2.7.1

Type of line of constant geodesic curvature on the Lobachevsky plane	Value of geodesic curvature
Geodesic lines (“*straight lines*”)	$k_{\mathrm{g}} = 0$
Equdistants	$0 < k_{\mathrm{g}} < 1$
Horocycles	$k_{\mathrm{g}} = 1$
Non-Euclidean circles	$k_{\mathrm{g}} > 1$

We supplement Table 2.7.1 by the intuitive Figure 2.7.2, which uses the disc model of the Lobachevsky plane.

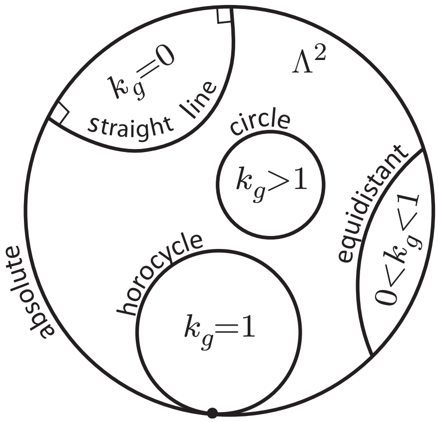


Figure 2.7.2

Relaxing rigorousity, we mention here the interesting analogy between the *geodesic curvature* k_{g} (an “indicator” of the type of line on the Lobachevsky plane) and the *eccentricity* e (“classification parameter” for 2nd order curves in the Euclidean plane in classical analytic geometry [34, 53]). The parameters k_{g} and e in the indicated branches of geometry have a unified grading of their values in the classification of characteristic types of lines: straight line on the plane Λ^2 ($k_{\mathrm{g}} = 0$) — Euclidean circle ($e = 0$); equidistant ($0 < k_{\mathrm{g}} < 1$) — ellipse ($0 < e < 1$); horocycle ($k_g = 1$) — parabola ($e = 1$); non-Euclidean circle ($k_g > 1$) — hyperbola ($e > 1$).

2.8 Isometric immersions of two-dimensional Riemannian metrics of negative curvature in Euclidean spaces

In this section we give a brief survey of studies connected with the realization in Euclidean space of two-dimensional metrics of (generally, non-constant) negative curvature. Hilbert's fundamental result on the impossibility of realizing the complete Lobachevsky plane Λ^2 in $\mathbb{E}^3$ served as the starting point for a wider re-thinking of the whole thematic of possible realization of two-dimensional metrics of negative curvature in Euclidean space. In this way an objective basis emerged for the enrichment of the "list" of modern directions of research, among which we distinguish the following:

– description (classification) of domains of the Lobachevsky plane Λ^2 (individual parts of it) that admit regular isometric immersion in the Euclidean space $\mathbb{E}^3$;

– study of the general problems of immersibility of two-dimensional Riemannian metrics of (non-constant) negative curvature in $\mathbb{E}^3$;

– study of problems concerned with obtaining regular isometric immersions of metrics of negative curvature in Euclidean spaces $\mathbb{E}^n$ of higher dimensions $n > 3$.

At this time the research directions listed above already became independent, highly developed branches of modern geometry. The fundamental development of various aspects of these branches is presented in many works, in particular, in [21, 22, 26, 27, 74, 75, 99, 122, 123, 168, 179]. In what follows we confine ourselves to a selective presentation of various separate basic results that reflect the conceptual evolution of the thematics.

2.8.1 Λ-type metrics

In 1961 N. V. Efimov and E. G. Poznyak [28] obtained a generalization of Hilbert's theorem on surfaces of negative curvature; specifically, they showed that a special type of metrics, the *Λ-type metrics* [18], with slowly varying curvature, do not admit regular isometric immersion in $\mathbb{E}^3$. By Λ-type metric one means a two-dimensional metric whose curvature is bounded above by a negative constant: $K \leq \text{const} < 0$.

The final answer to this question was given by N. V. Efimov (in 1963), who proved a theorem of fundamental importance on the nonexistence in the space $\mathbb{E}^3$ of a complete regular surface with *negative* supremum of the Gaussian curvature.

Theorem 2.8.1 (N.V. Efimov [26]). *On a regular surface with a complete metric the supremum of the Gaussian curvature cannot be smaller than zero.*

Clearly, this theorem includes the result asserting the nonexistence of a regular isometric immersion of a Λ-type metric in $\mathbb{E}^3$. On the other hand, based on it is totally justified to raise the following question: which *parts of a Λ-type metric* can be immersed in the space $\mathbb{E}^3$? We note that by "part" of a metric one usually means the corresponding part of the two-dimensional manifold on which this two-dimensional Riemannian metric is defined.

In the papers [75, 76] it is proved that if the curvature K of a Λ-type metric is a bounded $C^{2,1}$-function (in some system of semi-geodesic coordinates), then any *infinite strip* in this metric ("domain", or "part of the metric", situated between

[18] In the original paper the authors used the symbol Л.

two equidistants) can be regularly and isometrically immersed $\mathbb{E}^3$. Let us examine this result in more detail.

So, an infinite strip in a Λ-type metric is defined as the part of the metric situated between two equidistants l_1 and l_2. Given such a strip, let us choose in it as one of the families of coordinate lines (the u-lines) the geodesic lines that are orthogonal to the equidistants l_1 and l_2; the other family of coordinate lines (the v-lines) consists of the lines equidistant to the given curve l_1.

In the semi-geodesic coordinate system obtained in this way the metric takes the form

$$ds^2 = du^2 + B^2(u, v)dv^2. \tag{2.8.1}$$

Moreover, the domain in which the u and v coordinates range (the infinite strip) is defined as

$$\Pi_a = \{0 \le u \le a, \quad -\infty < v < +\infty\}.$$

We shall assume that in the strip Π_a the following conditions are satisfied:

1°. The function $B(u, v)$ is bounded and of class $C^{4,1}$ in Π_a.

2°. $\inf_{\Pi_a} B(u, v) > 0$.

3°. The curvature K of the metric (2.8.1) in Π_a is bounded from above by a negative constant: $K(\Pi_a) \le \text{const} < 0$.

Under the conditions 1°–3° the following theorem holds true.

Theorem 2.8.2 ([74]). *If 1°–3° hold, then the metric in the infinite strip Π_a can be immersed in the Euclidean space $\mathbb{E}^3$. The surface that realizes this metric is of class $C^{3,1}$.*

A detailed proof of this theorem is contained in [74].

An important consequence of Theorem 2.8.2 is that the $C^{4,1}$-metric of any geodesic disc in an arbitrary regular metric of negative curvature can be immersed in the Euclidean space $\mathbb{E}^3$. This is quite intuitive, since the metric of any geodesic disc can (under conditions 1°–3°) be included in the metric of an infinite strip with linear element (2.8.1) (figuratively speaking, a geodesic disc can always be "placed" inside some equidistant strip). An immersion of a geodesic disc of arbitrarily large radius and of an arbitrary variable negative curvature $K(u, v) \le \text{const} < 0$ was constructed by for the first time by E. G. Poznyak [75].

The question of minimizing the regularity requirements on the metric for the possible immersions in $\mathbb{E}^3$ under consideration was studied by E. V. Shikin [74, 123, 124]. He proposed an approach for obtaining immersions of metrics of negative curvature that is based on the investigation of the Darboux equation. In this connection we explain below the original general formulation of the corresponding problem.

Suppose that in the Euclidean space $\mathbb{E}^3(X, Y, Z)$ the surface that realizes the isometric immersion of a metric of curvature K is described by the parametric equations

$$X = X(u, v), \quad Y = Y(u, v), \quad Z = Z(u, v).$$

Then the function $Z(u,v)$ for the third Cartesian coordinate must satisfy the Darboux equation

$$rt - s^2 = K \cdot B^2 \cdot (1 - p^2 - q^2) + \frac{B_u^2}{B^2}\, q^2 - \Big(BB_u \cdot p - \frac{B_v}{B}\, q\Big) r - 2\,\frac{B_u}{B}\, qs, \quad (2.8.2)$$

where

$$p = Z_u, \quad q = Z_v, \quad r = Z_{uu}, \quad s = Z_{uv}, \quad t = Z_{vv}.$$

The Darboux equation (2.8.2) is written for the metric (2.8.1), considered in a semi-geodesic system of coordinates, but it preserves its structure with respect to the second derivatives of the function $Z(u,v)$ also when the original metric is of the most general form.

An analysis of the Darboux equation (2.8.2) that uses its reduction to a system of five quasilinear equations of special form (under the condition that the curvature is negative: $K < 0$), allowed one to obtain a series of results on immersion of metrics under relaxed requirements on their regularity. In this connection we state below the following important theorem proved by E. V. Shikin [74, 124].

Theorem 2.8.3. *Suppose that in the infinite strip* Π_a *there is given a metric of the type* (2.8.1) *of negative curvature* K.

Suppose further that:

1°. *The function* $B(u,v)$ *is bounded and of class* C^2 *in* Π_a, *and* $\inf_{\Pi_a} B(u,v) > 0$.

2°. *The curvature* $K(u,v)$ *is a bounded* C^1 *function in* Π_a, *bounded from above by a negative constant. Moreover, the function* $\operatorname{grad} K$ *is uniformly continuous in* Π_a.

Then the considered metric (2.8.1) *can be globally immersed in* $\mathbb{E}^3$. *Moreover, the surface that realizes this metric is of class* C^2.

2.8.2 Two classes of domains in the plane Λ^2 that are isometrically immersible in $\mathbb{E}^3$

In this subsection we draw the reader's attention to two types of results pertaining to the general task of describing typical domains of the Lobachevsky plane Λ^2 that admit regular isometric immersions in $\mathbb{E}^3$.

2.8.2.1. Infinite polygons on Λ^2. By *infinite polygon* on the Lobachevsky plane Λ^2 we mean a convex set obtained by intersecting a finite or countably infinite number of closed half-planes in Λ^2 whose boundaries have no common points [3, 73]. The boundary of any such infinite polygon is composed of a finite or countable number of straight lines of the plane Λ^2, called the *sides* of the polygon. To provide an intuitive representation of polygons on the plane Λ^2 it is convenient to use its Poincaré disc model (see Subsection 2.2.2).

Figure 2.8.1 depicts an infinite polygon (pentagon) in the plane Λ^2 with the vertices A_1, A_2, A_3, A_4, A_5, which are points at infinity lying on the absolute. Any two neighboring edges (straight lines in Λ^2, i.e., arcs of circles leaning orthogonally

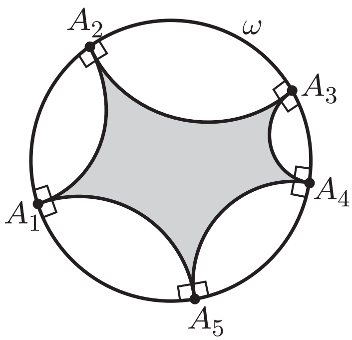


Figure 2.8.1

on the absolute ω) *converge* at infinity to (but do not intersect at) the *corresponding vertex*. It is due to this specific feature of their structure that such polygons started to be referred to as *infinite* polygons.

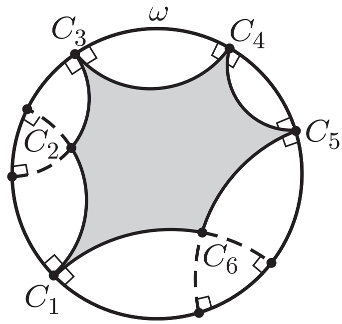


Figure 2.8.2

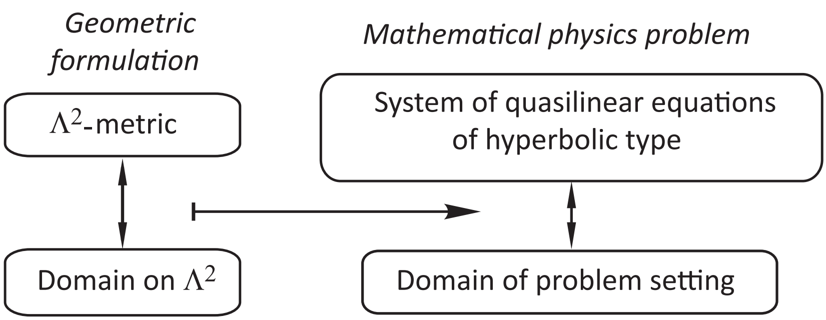
We shall also use the notion of *proper polygon* on the plane Λ^2 , defined as a polygon that can have as vertices both ordinary points and points at infinity of Λ^2. Such a proper polygon (like the infinite polygons) does not need to contain any half-plane of Λ^2. Figure 2.8.2 shows an example of a proper hexagon $C_1C_2C_3C_4C_5C_6 \subset \Lambda^2$, two vertices of which, C_2 and C_5, are ordinary points of Λ^2, while the remaining four vertices are points at infinity.

The problem of the immersibility of polygons of the plane Λ^2 in $\mathbb{E}^3$ was studied in detail by E. G. Poznyak in [73]. He found that all the polygons on Λ^2 with a finite number of vertices, as well as some kinds of polygons with a countable number of vertices, can be regularly and isometrically immersed in $\mathbb{E}^3$. Moreover, he showed that the immersible polygons can be covered by a regular Chebyshev net. The following theorem is established in [73].

Theorem 2.8.4. *Every proper polygon on the Lobachevsky plane Λ^2 admits a C^∞-smooth isometric immersion in the space $\mathbb{E}^3$.*

Let us point out here that the basic approach to establishing the isometric immersibility of a some or another domain (in particular, a polygon) of the plane Λ^2 in $\mathbb{E}^3$ reduces to the investigation of the solvability problem for the corresponding form of the system of fundamental equations that give the immersion of a Λ^2-metric[19] in $\mathbb{E}^3$ for the domain in question. To this end, in the works of E. G. Poznyak and E. V. Shikin [74, 76, 82, 123, 126] it was proposed to adopt as the main object of investigation the fundamental system of equations in Riemann invariants, i.e., a system of quasilinear equations of hyperbolic type. We presented such a system (2.3.35), (2.3.36) in § 2.3.

We should emphasize that as a geometric object, the domain in Λ^2 itself plays here, in a certain sense, a secondary role: from the point of hyperbolic planimetry its geometric structure is completely understood; such is, for instance, the geometry of an infinite polygon. What really matters is that the domain in Λ^2 considered is, essentially, the domain where a classical boundary value problem of mathematical physics [112] for the corresponding system of quasilinear equations of hyperbolic type (the Rozhdestveskii-Poznyak system of equations) is posed. The general scheme of the "geometry—mathematical physics" approach discussed here is described below.



Undoubtedly, the mathematical physics problem arising from the Lobachevsky geometry is a nontrivial object of investigation from the point of view of the general theory of differential equations [48, 97]. Hence, in order to partially simplify it, it is advisable to use a Λ^2-metric of special type—first of all, a metric written in a semi-geodesic system of coordinates. It is precisely such an approach that was used in the works [41, 73, 122, 126], in combination with other "refined" methods, to establish the immersibility in $\mathbb{E}^3$ of some typical domains of the Lobachevsky plane.

Let us metion here the work [60], in which the existence of infinite polygons in a two-dimensional metric of negative curvature is established, i.e., it is shown that the notion of an infinite polygon on the Lobachevsky plane can be generalized to the case of non-constant curvature (in this case a half-plane must be understood as the part of a complete metric bounded by a geodesic).

[19] Metric of curvature -1, given in some domain of the plane Λ^2.

2.8.2.2. Domains containing horodiscs. Another class of domains of the plane Λ^2 that can be immersed in $\mathbb{E}^3$ is studied in the works of E. V. Shikin, Zh. Kaidasov, and D. V. Tunitskii [41, 115]. In [41] it is proved that for any two horodiscs in Λ^2 there exists a convex domain that contains them and admits a regular isometric immersion in the space $\mathbb{E}^3$. The general form of such a domain is shown in Figure 2.8.3. Let us remark that, generally speaking, the method applied by the above authors to prove the existence of a solution to the fundamental system of equations in the domain under question can be modified to work also for the case of "analogous" domains in the Lobachevsky plane that cover several horodiscs.

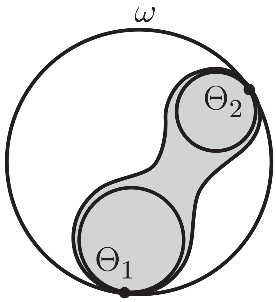


Figure 2.8.3

A special kind of generalization of the result on the immersion of a "domain with two horodiscs" is given in [115]. Therein the author introduces the notion of a *simple zone* Π_ω as a domain in the (ξ, η)-parameter plane defined by the conditions

$$|\xi| < \omega(\eta), \quad \eta \in (-\infty, +\infty),$$

where $\omega(\eta)$ is a positive continuous function given for all η.

The following result holds true [115].

Theorem 2.8.5. *A simple zone Π_ω, in which there is given a metric of the form*

$$ds^2 = d\xi^2 + B^2(\xi, \eta) d\eta^2,$$

and the conditions

$$B(0, \eta) = 1, \quad B_\xi(0, \eta) = 0, \quad K = -\frac{B_{\xi\xi}}{B} < 0,$$

are satisfied, admits an isometric immersion in $\mathbb{E}^3$ as a C^2-smooth surface.

Obviously,[20] Theorem 2.8.5 implies the existence in $\mathbb{E}^3$ of a C^2-smooth surface of constant negative curvature $K = \text{const} < 0$, on which two nonintersecting horodiscs are "placed".

[20] To make the presentation transparent, is it natural to use the Poincaré half-plane model of the plane Λ^2.

2.8.3 On the isometric immersions of the plane Λ^2 in the space $\mathbb{E}^n$ with $n > 3$

In view of the nonimmersibility of the plane Λ^2 in the Euclidean space $\mathbb{E}^3$ is is natural to ask: *What is the minimal dimension of an Euclidean space in which the Lobachevsky plane can be immersed?*

Here one thinks, first and foremost, about the Euclidean spaces $\mathbb{E}^4$, $\mathbb{E}^5$, and $\mathbb{E}^6$. We draw the reader's attention to a number of known results on this problem.

In 1955 D. Blanuša [146] constructed an explicit parametric representation of a *surface* (with no self-intersections) of *class* C^∞ in the space $\mathbb{E}^6$, the intrinsic geometry of which coincides with the planimetry of the complete Lobachevsky plane Λ^2. The metric of this surface coincides with the metric of the Lobachevsky plane:

$$ds^2 = du^2 + \cosh^2 u \, dv^2, \quad K \equiv -1.$$

We recall that an isometric realization of a metric by means of a surface without self-intersections is called an *embedding*. Thus, what Blanuša obtained is an embedding of the Lobachevsky plane in the Euclidean space $\mathbb{E}^6$. This results is one of the brilliant achievements in the theory of isometric immersions of two-dimensional manifolds in Euclidean spaces. Below we give an overview of this result.

The proposed construction employs two pairs of functions of a special form. The first pair is

$$\psi_1(u) = e^{2[(|u|+1)/2]+5}, \quad \psi_2(u) = e^{2[|u|/2]+6}. \tag{2.8.3}$$

(here the square brackets denote the "integer part"). The functions $\psi_1 > 0$, $\psi_2 > 0$ in (2.8.3) are piecewise-constant ("step") functions which are allowed to grow sufficiently fast as $|u| \to \infty$. At the same time, on their intervals of constancy ("steps") the derivatives of these functions vanish.

The second pair is formed by the functions

$$\begin{aligned} \varphi_1(u) &= \left(\frac{1}{A} \int_0^{u+1} \sin \pi\xi \cdot e^{-1/\sin^2 \pi\xi} d\xi \right)^{1/2}, \\ \varphi_2(u) &= \left(\frac{1}{A} \int_0^{u} \sin \pi\xi \cdot e^{-1/\sin^2 \pi\xi} d\xi \right)^{1/2}. \end{aligned} \tag{2.8.4}$$

The constant A in (2.8.4) is given by

$$A = \int_0^1 \sin \pi\xi \cdot e^{-1/\sin^2 \pi\xi} d\xi.$$

Note that the functions φ_1, φ_2 in (2.8.4) can have zeros of infinite order precisely at the discontinuity points of the corresponding function ψ_1 or ψ_2.

Together with (2.8.3) and (2.8.4), we introduce the functions

$$f_1(u) = \frac{\varphi_1(u)}{\psi_1(u)} \cdot \sinh u, \quad f_2(u) = \frac{\varphi_2(u)}{\psi_2(u)} \cdot \sinh u. \tag{2.8.5}$$

In terms of the functions (2.8.3)–(2.8.5) introduced above, the exact formulas for Blanuša's embedding of the Lobachevsky plane $\Lambda^2(u,v)$ in the Euclidean space $\mathbb{E}^6(X_1, X_2, X_3, X_4, X_5, X_6)$ read

$$\begin{aligned}
X_1 &= \int_0^u \sqrt{1 - f'^2_1(\xi) - f'^2_2(\xi)}\, d\xi, \\
X_2 &= v, \\
X_3 &= f_1(u)\cos(v\psi_1(u)), \\
X_4 &= f_1(u)\sin(v\psi_1(u)), \\
X_5 &= f_2(u)\cos(v\psi_2(u)), \\
X_6 &= f_2(u)\sin(v\psi_2(u)).
\end{aligned} \tag{2.8.6}$$

By (2.8.6), the linear element of the resulting surface $S(\Lambda^2) \subset \mathbb{E}^6$ can be written as:

$$ds^2 = dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 + dX_5^2 + dX_6^2 = du^2 + \cosh^2 u\, dv^2.$$

It is important to make clear that in the "functional constructions" used in (2.8.6) the jumps of the step functions ψ_1 and ψ_2 are "compensated" by the corresponding zeros of the functions φ_1 and φ_2, which ensures the requisite smoothness of the embedding (2.8.6).

E. R. Rozendorn [99, 100] proposed a refinement of the above method, which allowed him to establish the immersibility of the Lobachevsky plane Λ^2 in $\mathbb{E}^5$.

The question whether it is possible to realize the complete Lobachevsky plane Λ^2 in the four-dimensional Euclidean space $\mathbb{E}^4$ remains open and represents one of the puzzling unsolved problems of contemporary non-Euclidean geometry.[21]

To complete this section, we draw the reader's attention to various problems of current interest connected with the theme discussed here.

A special example of a Λ-type metric that is immersible in $\mathbb{E}^4$ was provided in [98] by using the so-called construction of a weakly irregular saddle of negative curvature. The paper [103] investigated the possibility of a special type of immersion of Λ^2 in $\mathbb{E}^4$ by using a metric of revolution.

Let us touch upon a more general topic: the problem of isometric immersion of the higher-dimensional Lobachevsky space Λ^p, $p \geq 3$, in a Euclidean space $\mathbb{E}^n$, $n > 3$. A local embedding of Λ^p in $\mathbb{E}^{2p-1}$ was constructed by F. Schur [191]. Moreover, in [49, 148] it was established that Λ^p cannot be locally immersed already in $\mathbb{E}^{2p-2}$. D. Blanuša was also the first to obtain an isometric embedding of Λ^p, $p \geq 3$, in $\mathbb{E}^\infty$,

[21]The intuitive take of many geometers on this question predicts that the answer will be affirmative.

as well as a series of embeddings of Λ^p in $\mathbb{E}^n$, with n finite [145, 146]. Deserving of special attention are the general "configurations" introduced by J. Nash and M. Gromov, applicable for the construction of isometric embeddings of a non-compact Riemannian manifold M^p in the spaces $\mathbb{E}^n$ [21, 159, 178]. Undoubtedly, interesting are also the results of Yu. A. Aminov on isometric immersions of domains of the n-dimensional Lobachevsky space in higher-dimensional Euclidean spaces [4, 5, 6]. In the context of our exposition we mention also the investigations of I. Kh. Sabitov [102] on isometric immersions of locally Euclidean metrics and the fundamental survey of A. A. Borisenko [11] on isometric immersions of space forms in Riemannian and pseudo-Riemannian spaces.

Chapter 3

The sine-Gordon equation: its geometry and applications of current interest

This chapter is devoted to geometrical aspects in the study of the sine-Gordon equation as a canonical (from the point of view of non-Euclidean hyperbolic geometry) nonlinear equation that has wide applications in contemporary mathematical physics. A far-reaching fact that enables the realization of diverse approaches to the investigation of problems connected with the sine-Gordon equation is the intimate association of this equation with surfaces of constant negative curvature, i.e., with pseudospherical surfaces.

The beginning of the chapter (§§ 3.1 and 3.2) is devoted to geometric (primary) concept of Bäcklund transformation for pseudospherical surfaces, developed further in essential manner in the theory of nonlinear differential equations. The application of the method of Bäcklund transformations to the construction of solutions of nonlinear equations is one of the effective approaches in modern mathematical physics. Special attention is devoted to the class of soliton solutions of the sine-Gordon equation and to the investigation of their geometric interpretation on the example of the classical pseudosphere and Dini surfaces, as well as to the class of two-soliton and breather pseudospherical surfaces (§§ 3.3 and 3.4). In § 3.5 we study the Painlevé transcendental functions of type III as functions that constitute a special class of self-similar solutions of the sine-Gordon equations whose geometric image in $\mathbb{E}^3$ is Amsler's pseudospherical surface. In §§ 3.6 and 3.7 we study fundamental problems concerning the solvability of certain classical problems of mathematical physics – the Darboux problem and the Cauchy problem – for the sine-Gordon equation, based on which we derive important geometric generalizations and consequences. In particular, we present the idea of constructing solutions of the sine-Gordon equation on multi-sheeted surfaces. Moreover, relying on the established unique solvability of the Cauchy problem for the sine-Gordon equation, we prove a theorem on the unique determinacy of pseudospherical surfaces (the fact that a pseudospherical surface is determined by the corresponding

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DOI 10.1007/978-3-319-05669-2_4,

initial data on its irregular singularities). Classical problems connected with the Joachimsthal–Enneper surfaces are discussed in § 3.8, where we exhibit a link between these surfaces and a class of solutions of the sine-Gordon equation that are obtained by the method of separation of variables. The final § 3.9 deals with the fundamental connection between the the method of the Inverse Scattering Problem (or Transform) (MIST) and the theory of pseudospherical surfaces, which is expressed by the fact that the basic "input" relations in MIST and the fundamental system of structure equations for pseudospherical surfaces in $\mathbb{E}^3$ are structurally identical. All together, the consideration of all principial problems in Chapter 3 points to the presence of a significant geometric component (in the context of Lobachevsky geometry) in a wide spectrum of problems of modern mathematical physics.

3.1 The Bäcklund transformation for pseudospherical surfaces

The modern concept of Bäcklund transformation has its origin in the classical theory of pseudospherical surfaces and emerged from the mathematically daring idea of L. Bianchi [143] concerning the possible existence of a specific link between pseudospherical surfaces, formulated in his doctoral dissertation of 1879. The general meaning of the proposed idea of transformation is as follows: *is it possible, given a pseudospherical surface, to construct a new, different pseudospherical surface in such a way that the construction algorithm will rely exclusively on "information" about the given surface?* The answer to this – rather optimistic, it would seem – question turned out to be affirmative. Even more, the proposed geometric idea was considerably developed and generalized already in 1883 by A. Bäcklund [137], and the transformation of surfaces itself became known as the *Bäcklund transformation*. From the point of view of modern methodological concepts, the idea of the Bäcklund transformation can be more widely interpreted as the possibility of distinguishing certain classes of mathematical (and not only) objects, which admit their own nontrivial " self-organization" – compositional generation (based on a superposition principle) of new similar objects that preserve (inherit) some special key criterion. Let us next discuss the original geometric content of the classical Bäcklund transformations, relying on the original works of Bianchi [142–144] and Bäcklund [137]; we also mention the modern monographs [154, 187, 195], in which the circle of problems considered here is considered in connection with various fields of contemporary mathematical physics.

3.1.1 Pseudospherical surfaces: basic relations

Suppose that in three-dimensional Euclidean space $\mathbb{E}^3$ there is given a pseudospherical surface S, described by its radius vector $\vec{r}(u, v)$. we will assume that the coordinates u and v are asymptotic coordinates on the surface S. Then, by the considerations in § 2.7.3, the square of the linear element on S will coincide with the metric (2.7.24) of the surface, written in the Chebyshev coordinatization, and the coefficients of the second fundamental form will be given by expressions (2.7.29).

Thus, for the given pseudospherical surface $S[\vec{r}(u,v)]$ one has the following basic relations:

$$\mathrm{I}(u,v) = du^2 + 2\cos\omega(u,v)dudv + dv^2, \tag{3.1.1}$$

$$\begin{aligned} E(u,v) &= \vec{r}_u^{\,2} = 1, \\ F(u,v) &= (\vec{r}_u, \vec{r}_v) = \cos\omega(u,v), \\ G(u,v) &= \vec{r}_v^{\,2} = 1, \end{aligned} \tag{3.1.2}$$

$$\mathrm{II}(u,v) = 2\sin\omega(u,v)dudv, \tag{3.1.3}$$

$$L = 0, \quad M = \sin\omega(u,v), \quad N = 0. \tag{3.1.4}$$

The net angle function $\omega(u,v)$ of the Chebyshev net (of asymptotic lines on $S[\vec{r}(u,v)]$) satisfies the sine-Gordon equation

$$\omega_{uu} = \sin\omega. \tag{3.1.5}$$

The radius vector $\vec{r}(u,v)$ and the unit normal vector $\vec{n}(u,v)$ to the surface $S[\vec{r}(u,v)]$ are given by the system of derivation formulas (2.7.30)–(2.7.34). Let us rewrite the system (2.7.30)–(2.7.34) in a convenient matrix form, denoting the net angle by ω:

$$\begin{pmatrix} \vec{r}_u \\ \vec{r}_v \\ \vec{n} \end{pmatrix}_u = \begin{pmatrix} \omega_u \cot\omega & -\dfrac{\omega_u}{\sin\omega} & 0 \\ 0 & 0 & \sin\omega \\ \cot\omega & -\dfrac{1}{\sin\omega} & 0 \end{pmatrix} \begin{pmatrix} \vec{r}_u \\ \vec{r}_v \\ \vec{n} \end{pmatrix}, \tag{3.1.6}$$

$$\begin{pmatrix} \vec{r}_u \\ \vec{r}_v \\ \vec{n} \end{pmatrix}_v = \begin{pmatrix} 0 & 0 & \sin\omega \\ -\dfrac{\omega_v}{\sin\omega} & \omega_v \cot\omega & 0 \\ -\dfrac{1}{\sin\omega} & \cot\omega & 0 \end{pmatrix} \begin{pmatrix} \vec{r}_u \\ \vec{r}_v \\ \vec{n} \end{pmatrix} \tag{3.1.7}$$

(The subscripts u and v in the left-hand side column vectors denote partial derivatives).

The system (3.1.6), (3.1.7) is the system (2.3.18), (2.3.19), written by using the Christoffel symbols (2.7.68) for a Chebyshev metric of the form (3.1.1). Recall that in the chosen coordinate parametrization of the surface, $\vec{r}_u$ and $\vec{r}_v$ are tangent vectors to the asymptotic (coordinate) lines u and v on the surface. Correspondingly, the unit normal vector $\vec{n}(u,v)$ is orthogonal to the tangent plane of the surface, which contains $\vec{r}_u$ and $\vec{r}_v$ (Figure 3.1.1). Along with the triple $\vec{r}_u$, $\vec{r}_v$, $\vec{n}$, let us introduce the new orthogonal triple of vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ (the trihedron, or frame) $\{\vec{A}, \vec{B}, \vec{C}\}$) as follows: $\vec{A}$ is just a renaming of the vector $\vec{r}_u$ (see Figure 3.1.1):

$$\vec{A} = \vec{r}_u. \tag{3.1.8}$$

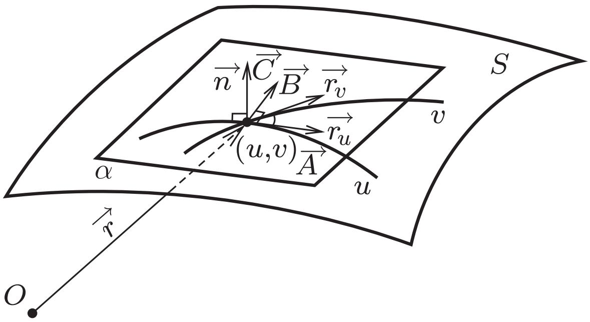


Figure 3.1.1

For $\vec{B}$ we take a unit vector in the tangent space space to the surface S at the given point (u,v) that is orthogonal to $\vec{A} = \vec{r}_u$: $\vec{B} \perp \vec{A}$.

It is readily verified that the vector $\vec{B}$ chosen in this way is given by the expression

$$\vec{B} = -\cot\omega\,\vec{r}_u + \frac{1}{\sin\omega}\vec{r}_v, \quad |\vec{B}| = 1. \tag{3.1.9}$$

Now it is clear that for the third vector $\vec{C}$ of the trihedron we need to take the already available unit normal $\vec{n}$:

$$\vec{C} = \vec{n}. \tag{3.1.10}$$

Essentially, to pass to the triple $\{\vec{A}, \vec{B}, \vec{C}\}$ we changed only one vector in the triple $\vec{r}_u$, $\vec{r}_v$, $\vec{n}$.

Thus, in each point of the pseudospherical surface $S[\vec{r}(u,v)]$ we defined, via relations (3.1.8)–(3.1.10), a trihedron (frame) $\{\vec{A}, \vec{B}, \vec{C}\}$. With respect to this trihedron the system (3.1.6), (3.1.7) can be written in the form

$$\begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix}_u = \begin{pmatrix} 0 & -\omega_u & 0 \\ \omega_u & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix}, \tag{3.1.11}$$

$$\begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix}_v = \begin{pmatrix} 0 & 0 & \sin\omega \\ 0 & 0 & -\cos\omega \\ -\sin\omega & \cos\omega & 0 \end{pmatrix} \begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix} \tag{3.1.12}$$

Note that as a result of passing to the new system the matrices in the right-hand sides of (3.1.11) and (3.1.12) acquired an asymmetric form.

The compatibility condition for the system (3.1.11), (3.1.12) (i.e., the equality of the mixed derivatives of the left-hand sides) is expressed by the sine-Gordon equation.

The preliminary arguments given above allow us to address now the main subject of this section, an algorithm for constructing pseudospherical surfaces with the aid of Bäcklund transformations.

3.1.2 Geometry of the Bäcklund transformation

Suppose that in the Euclidean space $\mathbb{E}^3$ there is given a pseudospherical surface $S[\vec{r}]$ (of Gaussian curvature $K = -1$), described by its radius vector $\vec{r}(u, v)$. Then the families of u- and v-lines yield the coordinate net of asymptotic lines on the surface S.

To obtain a new pseudospherical surface $S^*[\vec{r}^{\,*}]$ that is associated to S by means of a Bäcklund transformation, we will use the a priori connection between the radius vectors of the two surfaces:

$$\vec{r}^{\,*} = \vec{r} + a(\cos\beta\,\vec{A} + \sin\beta\,\vec{B}), \tag{3.1.13}$$

where $a = \text{const} > 0$ is some constant of the transformation.

Since the vectors $\vec{A}$ and $\vec{B}$ are orthonormal, the expression in parentheses in (3.1.13) is a unit vector lying in the tangent plane to the surface S at the current point $P(u, v)$. It is also clear from (3.1.13) that

$$|\vec{r}^{\,*} - \vec{r}| = a.$$

The function $\beta = \beta(u, v)$ has the meaning of an angle, and the conditions that it must satisfy will be specified below.

An important requirement on the Bäcklund transformation (3.1.13) is that the coordinates u, v on the new surface $S^*[\vec{r}^{\,*}(u, v)]$ be also asymptotic. In particular, this means that in the coordinates u, v the metric ds^{*2} of the surface S^* will be in the Chebyshev form, i.e., the following equalities must hold:

$$\begin{aligned} E^* &= (\vec{r}^{\,*}_u(u, v))^2 = 1, \\ F^* &= (\vec{r}^{\,*}_u(u, v), \vec{r}^{\,*}_v(u, v)) = \cos\omega^*(u, v), \\ G^* &= (\vec{r}^{\,*}_v(u, v))^2 = 1, \end{aligned} \tag{3.1.14}$$

Geometrically, the transition (3.1.13) from the surface $S[\vec{r}]$ to the surface $S^*[\vec{r}^{\,*}]$ means that to each point $P(u, v,) \in S[\vec{r}]$ one associates a point

$$P^*(u, v) \in S^*[\vec{r}^{\,*}]$$

in such a way that the vector

$$\overrightarrow{PP^*} = a(\cos\beta\,\vec{A} + \sin\beta\,\vec{B})$$

lies in both the tangent plane α to the surface S at the point P and the tangent plane α^* to the surface S^* at the point P^* (Figure 3.1.2). Note, however, that the planes α and α^* make some nonzero angle γ.

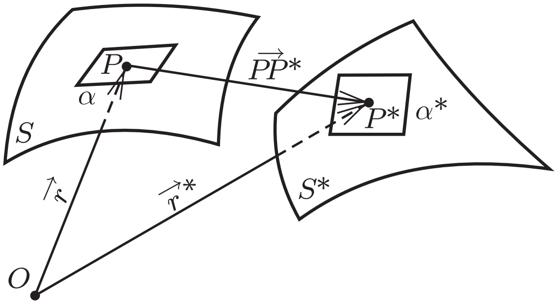


Figure 3.1.2

Let us find the conditions that the function $\beta(u,v)$ appearing in (3.1.13) must satisfy. To this end we use (3.1.13) to calculate the derivatives $\vec{r}_u^{\,*}$ and $\vec{r}_v^{\,*}$:

$$\begin{aligned}\vec{r}_u^{\,*} &= \vec{r}_u + a(\cos\beta\,\vec{A}_u + \sin\beta\,\vec{B}_u - \beta_u\sin\beta\,\vec{A} + \beta_u\cos\beta\,\vec{B}),\\ \vec{r}_v^{\,*} &= \vec{r}_v + a(\cos\beta\,\vec{A}_v + \sin\beta\,\vec{B}_v - \beta_v\sin\beta\,\vec{A} + \beta_v\cos\beta\,\vec{B}),\end{aligned} \tag{3.1.15}$$

If we use here the expressions for $\vec{A}_u$, $\vec{B}_u$, $\vec{A}_v$, $\vec{B}_v$ given by (3.1.11) and (3.1.12), together with the representations

$$\vec{r}_u = \vec{A}, \quad \vec{r}_v = \cos\omega\,\vec{A} + \sin\omega\,\vec{B},$$

which follow from (3.1.8) and (3.1.9), we obtain

$$\vec{r}_u^{\,*} = (1 + a\,(\omega_u - \beta_u)\sin\beta)\,\vec{A} - a\,(\omega_u - \beta_u)\cos\beta\,\vec{B} + a\sin\beta\,\vec{C}, \tag{3.1.16}$$

$$\vec{r}_v^{\,*} = (\cos\omega - a\beta_v\sin\beta)\,\vec{A} + (\sin\omega + a\beta_v\cos\beta)\,\vec{B} + a\sin(\omega-\beta)\vec{C}, \tag{3.1.17}$$

If we now subject expressions (3.1.15) to conditions (3.1.14) (and recall that the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, are orthogonal), we arrive at the system

$$\begin{aligned}\beta_u &= \omega_u + k\sin\beta,\\ \beta_v &= \frac{1}{k}\sin(\beta-\omega),\end{aligned} \tag{3.1.18}$$

where the coefficient k is given by

$$k = \frac{1}{a}\left(1 \pm \sqrt{1-a^2}\right),$$

or

$$a = \frac{2k}{1+k^2}. \tag{3.1.19}$$

Upon introducing the new function

$$W = \beta - \omega,$$

system (3.1.18) can be also recast in the "symmetric" form

$$\begin{aligned} W_u &= k \sin\beta, \\ \beta_v &= \frac{1}{k}\sin W. \end{aligned} \tag{3.1.20}$$

The system (3.1.20) (or, equivalently, the system (3.1.18)) expresses the necessary condition for the first and third equalities in (3.1.14) to hold. In this way, system (3.1.20) gives the sought-for function $\beta = \beta(u, v)$. Turning now to the second relation in (3.1.14), let us calculate the scalar product $(\overrightarrow{r}^*_u, \overrightarrow{r}^*_v)$ using (3.1.16), (3.1.17), and (3.1.20). We get

$$(\overrightarrow{r}^*_u, \overrightarrow{r}^*_v) = \cos(2\beta - \omega),$$

or, equivalently,

$$(\overrightarrow{r}^*_u, \overrightarrow{r}^*_v) = \cos\omega^*, \quad \text{where} \quad \omega^* = 2\beta - \omega. \tag{3.1.21}$$

We have thus verified that the metric ds^{*2} of the new (sought-for) pseudospherical surface $S^*[\overrightarrow{r}^*(u, v)]$ takes on the Chebyshev form

$$ds^{*2} = du^2 + 2\cos\omega^*(u, v)dudv + dv^2. \tag{3.1.22}$$

Consequently, the function $\omega^*(u, v)$ satisfies the sine-Gordon equation

$$\omega^*_{uv} = \sin\omega^*. \tag{3.1.23}$$

To prove that the resulting coordinate net u, v on the surface $S^*[\overrightarrow{r}^*(u, v)]$ is not only Chebyshev, but also asymptotic, we calculate the coefficients L^*, M^*, N^* of the second fundamental form of S^*, using the obtained expressions (3.1.16)–(3.1.18). First note that

$$\overrightarrow{n}^* = \frac{[\overrightarrow{r}^*_u \times \overrightarrow{r}^*_v]}{|[\overrightarrow{r}^*_u \times \overrightarrow{r}^*_v]|},$$

and so, in view of (3.1.16), (3.1.17) and (3.1.19),

$$\overrightarrow{n}^* = -\frac{2k}{1+k^2}\sin\beta \cdot \overrightarrow{A} + \frac{2k}{1+k^2}\cos\beta \cdot \overrightarrow{B} + \frac{1-k^2}{1+k^2} \cdot \overrightarrow{C}. \tag{3.1.24}$$

From (3.1.24) we immediately get

$$\overrightarrow{n}^*_u = -\frac{2k}{1+k^2}\sin\beta\cos\beta\cdot\overrightarrow{A} + \left(\frac{2k}{1+k^2}\cos^2\beta - 1\right)\cdot\overrightarrow{B} + \frac{1-k^2}{1+k^2}\cos\beta\cdot\overrightarrow{C}, \tag{3.1.25}$$

$$\begin{aligned} \overrightarrow{n}^*_v = &\left(-\frac{2k}{1+k^2}\sin\omega^* + \frac{k^2}{1+k^2}\sin\omega\right)\cdot\overrightarrow{A} \\ &+ \left(\frac{1}{1+k^2}\cos\omega^* - \frac{k^2}{1+k^2}\cos\omega\right)\cdot\overrightarrow{B} - \frac{2k}{1+k^2}\cos(\omega - \beta)\cdot\overrightarrow{C}. \end{aligned} \tag{3.1.26}$$

Now using the relations (3.1.16), (3.1.17), (3.1.25), and (3.1.26) obtained above for $\overrightarrow{r}_u^*$, $\overrightarrow{r}_v^*$, $\overrightarrow{n}_u^*$, and $\overrightarrow{n}_u^*$ to find the coefficients L^*, M^* and N^* for the "new" pseudospherical surface $S^*[\overrightarrow{r}^*]$, we finally find that, indeed, the following equalities hold:

$$\begin{aligned} L^* &= -(\overrightarrow{r}_u^*, \overrightarrow{n}_u^*) = 0, \\ M^* &= -(\overrightarrow{r}_u^*, \overrightarrow{n}_u^*) = \sin\omega^*, \\ N^* &= -(\overrightarrow{r}_v^*, \overrightarrow{n}_v^*) = 0. \end{aligned} \tag{3.1.27}$$

The obtained values of the coefficients (3.1.27) show that the resulting Chebyshev net of lines Cheb(u, v) on S^* is asymptotic, and its net angle $\omega^* = 2\beta - \omega$ satisfies the sine-Gordon equation (3.1.23).

Making the substitution

$$\beta = \frac{\omega^* + \omega}{2}, \quad W = \frac{\omega^* - \omega}{2}$$

in the system (3.1.20), we obtain the important differential relations

$$\begin{aligned} \left(\frac{\omega^* - \omega}{2}\right)_u &= k \cdot \sin\left(\frac{\omega^* + \omega}{2}\right), \\ \left(\frac{\omega^* - \omega}{2}\right)_v &= \frac{1}{k} \cdot \sin\left(\frac{\omega^* - \omega}{2}\right), \end{aligned} \qquad k = \text{const.} \tag{3.1.28}$$

The system (3.1.28) yields a Bäcklund transformation between the given (known) solution $\omega(u, v)$ and the corresponding new solution $\omega^*(u, v)$ of the sine-Gordon equation. The numerical parameter k is the transformation parameter. The functions $\omega(u, v)$ and $\omega^*(u, v)$ have the intuitive geometric meaning of net angles of the Chebyshev coordinate nets of asymptotic lines on the surfaces $S[\overrightarrow{r}(u, v)]$ and $S[\overrightarrow{r}^*(u, v)]$, respectively.

We remark that the numerical coefficient k used in (3.1.28) is a fixed parameter of the Bäcklund transformation. More precisely, if γ is the angle between the tangent planes α and α^* to the surface S and S^* at the points $P \in S$ and $P^* \in S^*$, respectively, which correspond under the Bäcklund transformation (see Figure 3.1.2), then

$$k = \tan\frac{\gamma}{2}. \tag{3.1.29}$$

In other terms, using the normals $\overrightarrow{n}$ and $\overrightarrow{n}^*$ to the surfaces S and S^* at the points P and P^*, respectively, we can also write

$$k = \frac{1}{a}\left(1 - (\overrightarrow{n}, \overrightarrow{n}^*)\right), \tag{3.1.30}$$

which is identical to (3.1.19). The considerations presented in this subsection allow us to formulate an algorithm which uses Bäcklund transformations for constructing pseudospherical surfaces.

Bäcklund transformation for pseudospherical surfaces. *Suppose the original pseudospherical surface $S[\overrightarrow{r}(u, v)]$ is given by the radius vector $\overrightarrow{r}(u, v)$ and corresponds to the solution $\omega(u, v)$ of the sine-Gordon equation ($\omega(u, v)$ is the net* *angle of the Chebyshev net of asymptotic lines* u, v *on* S). *Then the radius vector* $\vec{r}^{\,*}(u,v)$ *of the new pseudospherical surface* $S^*[\vec{r}^{\,*}(u,v)]$, *which corresponds to its own solution* $\omega^*(u,v)$ *of the sine-Gordon equation* ($\omega^*(u,v)$ *is the net angle of the Chebyshev net of asymptotic lines* u, v *on* S^*), *obtained by means of the system* (3.1.28), *is given by the formula*

$$\vec{r}^{\,*} = \vec{r} + \frac{2k}{1+k^2}\cdot\frac{1}{\sin\omega}\left[\sin\left(\frac{\omega-\omega^*}{2}\right)\vec{r}_u + \sin\left(\frac{\omega+\omega^*}{2}\right)\vec{r}_v\right]. \tag{3.1.31}$$

The constant numerical parameter k in formula (3.1.31), which defines the Bäcklund transformation for pseudospherical surfaces, coincides with the parameter which "figures" in the system (3.1.28) that gives the Bäcklund transformation for solutions of the sine-Gordon equation (net angles of Chebyshev nets of asymptotic lines on the corresponding surfaces).

The Bäcklund transformation, which according to (3.1.28) and (3.1.31) sends a solution ω (or the corresponding surface S) into the solution ω^* (respectively, the surface S^*), with transformation parameter k, is usually denoted by $\mathbb{B}_k$:[1]

$$\omega^* = \mathbb{B}_k(\omega), \quad S^* = \mathbb{B}_k(S),$$

or

$$[S^*, \omega^*] = \mathbb{B}_k[S, \omega].$$

The Bäcklund transformation for pseudospherical surfaces is shown schematically in Figure 3.1.3.

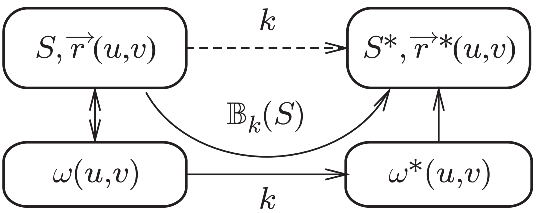


Figure 3.1.3

The actual construction of a new pseudospherical surface S^* by means of the Bäcklund transformation is carried out in two steps:

1) *Given a solution* $\omega(u,v)$ *of the sine-Gordon equation, one uses the system of nonlinear equations* (3.1.28) *to find a new solution* $\omega^*(u,v)$ *of the same equation*: $\omega^* = \mathbb{B}_k(\omega)$.

2) *One constructs the radius vector* $\vec{r}^{\,*}(u,v)$ *of the new surface* S^* *according to formula* (3.1.31): $S^* = \mathbb{B}_k(S)$.

The described implementation of the method of Bäcklund transformations will be exemplified in § 3.4 on various types of two-soliton pseudospherical surfaces. Ideas concerned with "transferring" the Bäcklund transformation to the case of the ambient space $\mathbb{E}^4$ are discussed in [134].

[1] This notation was introduced by Bianchi.

3.2 Soliton solutions of the sine-Gordon equation. The Lamb diagram

Let us analyze the system of equations (3.1.28), which "governs" the transformation of solutions of the sine-Gordon equation, and thus corresponds to the construction of pseudospherical surfaces by means of the Bäcklund transformations.

3.2.1 The Bianchi diagram

Assuming that the solution $\omega^*(u,v)$ of the system (3.1.28) exists, let us apply to the original given solution $\omega(u,v)$ of the sine-Gordon equation two Bäcklund transformations with different parameters k_1 and k_2:

$$\begin{aligned} \omega_1^{(k_1)} &= \mathbb{B}_{k_1}(\omega), \\ \omega_1^{(k_2)} &= \mathbb{B}_{k_2}(\omega). \end{aligned} \tag{3.2.1}$$

Now let us apply again, this time to the functions $\omega_1^{(k_1)}$ and $\omega_1^{(k_2)}$ from (3.1.21), the Bäcklund transformations with coefficient k_2 and k_1, respectively:

$$\begin{aligned} \omega_1^{(k_1,k_2)} &= \mathbb{B}_{k_2}(\omega^{(k_1)}) = \mathbb{B}_{k_2}\mathbb{B}_{k_1}(\omega), \\ \omega_1^{(k_2,k_1)} &= \mathbb{B}_{k_1}(\omega^{(k_2)}) = \mathbb{B}_{k_1}\mathbb{B}_{k_2}(\omega). \end{aligned} \tag{3.2.2}$$

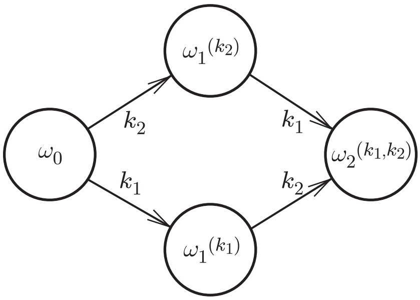


Figure 3.2.1

It is natural to assume that the successive application of the Bäcklund transformation enjoys the permutability (commutativity) property:

$$\mathbb{B}_{k_2}\mathbb{B}_{k_1}(\omega) = \mathbb{B}_{k_1}\mathbb{B}_{k_2}(\omega). \tag{3.2.3}$$

Indeed, the fact that relation (3.2.3) holds is the content of *Bianchi's permutability theorem* [142], which is valid for all solutions of the sine-Gordon equation that are connected to one another by a Bäcklund transformation (3.1.28). This theorem will be discussed a bit later.

Let us study the Bäcklund transformation (3.1.28) under the assumption that property (3.2.3) established by Bianchi holds. This formulation of the problem is illustrated by the classical Bianchi diagram shown in Figure 3.2.1.

To simplify the handling of the transformations, in the "superscript notation" of the solutions we will indicate only the index of the parameter, namely, we denote

$$\omega_1^{(k_1)} \equiv \omega_1^1, \quad \omega_1^{(k_2)} \equiv \omega_1^2, \quad \omega_2 \equiv \omega_2^{(k_1,k_2)} = \omega_2^{1,2} = \omega_2^{(2,1)}.$$

Choosing as "primer" ("starter") solution of the transformation (3.1.28) (and, correspondingly, of the Bianchi diagram (Figure 3.2.1)) some solution $\omega = \omega_0(u,v)$ of the sine-Gordon equation, let us write the first iteration of the Bäcklund transformation for the first equation in (3.1.28):

$$\begin{aligned} \omega_{1,u}^1 &= \omega_{0,u} + 2k_1 \cdot \sin\left(\frac{\omega_1^1 + \omega_0}{2}\right), \\ \omega_{1,u}^2 &= \omega_{0,u} + 2k_2 \cdot \sin\left(\frac{\omega_1^2 + \omega_0}{2}\right). \end{aligned} \tag{3.2.4}$$

(Here the lower "letter" index after the comma indicated the variable with respect to which one differentiates.)

Similarly, the second iteration of the Bäcklund transformation corresponding to the Bianchi diagram (Figure 3.2.1) yields

$$\begin{aligned} \omega_{2,u} &= \omega_{1,u}^1 + 2k_2 \cdot \sin\left(\frac{\omega_2 + \omega_1^1}{2}\right), \\ \omega_{2,u} &= \omega_{1,u}^2 + 2k_1 \cdot \sin\left(\frac{\omega_2 + \omega_1^2}{2}\right). \end{aligned} \tag{3.2.5}$$

If in (3.2.5) we subtract the second equation from the first, we obtain

$$\omega_{1,u}^1 - \omega_{1,u}^2 + 2k_2 \cdot \sin\left(\frac{\omega_2 + \omega_1^1}{2}\right) - 2k_1 \cdot \sin\left(\frac{\omega_2 + \omega_1^2}{2}\right) = 0. \tag{3.2.6}$$

Now inserting expressions (3.2.4) in (3.2.6) we get

$$\begin{aligned} 2k_1 &\left[\sin\left(\frac{\omega_1^1 + \omega_0}{2}\right) - \sin\left(\frac{\omega_2 + \omega_1^2}{2}\right)\right] \\ &\qquad - 2k_2 \left[\sin\left(\frac{\omega_1^2 + \omega_0}{2}\right) - \sin\left(\frac{\omega_2 + \omega_1^1}{2}\right)\right] = 0. \end{aligned} \tag{3.2.7}$$

Routine trigonometric transformations in (3.2.7) yield

$$\begin{aligned} &\left[4k_1 \cdot \sin\left(\frac{\omega_0 - \omega_2}{4} + \frac{\omega_1^1 - \omega_1^2}{4}\right) - 4k_2 \cdot \sin\left(\frac{\omega_0 - \omega_2}{4} - \frac{\omega_1^1 - \omega_1^2}{4}\right)\right] \\ &\qquad \times \cos\left(\frac{\omega_0 + \omega_2}{4} + \frac{\omega_1^1 + \omega_1^2}{4}\right) = 0. \end{aligned} \tag{3.2.8}$$

The expression in the square brackets in (3.2.8) vanishes provided that

$$\omega_2 = \omega_0 + 4\arctan\left(\frac{k_1+k_2}{k_1-k_2}\cdot\tan\left(\frac{\omega_1^1-\omega_2^1}{4}\right)\right). \tag{3.2.9}$$

The algebraic recursion formula (3.2.9) is called the *Bianchi formula*; it gives the Bäcklund transformation for constructing a new solution ω_2 of the sine-Gordon equation from the already known solutions ω_0 and ω_1^1, ω_1^2, which occupy the preceding "positions" in the Bianchi diagram (Figure 3.2.1).

Let us remark that the possible vanishing of the second factor in (3.2.8) gives a linear dependence of the functions ω_0, ω_1^1, ω_1^2, ω_2, which turns out to be incompatible with the form of the sought-for nonlinear transformation.

Now that we derived the recursion formula (3.2.9), the *permutability* property (3.2.3) of the Bäcklund transformation introduced above can be verified directly, by showing that for any solution $\omega_0(u,v)$ of the sine-Gordon equation, the obtained function $\omega_2(u,v)$ satisfies a system of equations of the type (3.1.28) in the index k_1, as well as in the index k_2, i.e., that $\omega_2 = \mathbb{B}_{k_1}\big(\omega_1^{(k_2)}\big) = \mathbb{B}_{k_2}\big(\omega_1^{(k_1)}\big)$ if $\omega_1^{(k_1)} = \mathbb{B}_{k_1}(\omega_0)$ and $\omega_1^{(k_2)} = \mathbb{B}_{k_2}(\omega_0)$. Historically, the proof of the permutability property was given in detail by Bianchi himself [142]. Let us mention here that the classical Bianchi formula can be generalized, in the framework of the method of the inverse scattering problem, by means of the n-fold Darboux transformation [136, 171].

The extended version of the diagram, which uses the necessary number of compatible Bianchi diagrams, is called in the current literature the *Lamb diagram*. The Lamb diagram (Figure 3.2.2) is used, in particular, to construct multi-soliton solutions of the sine-Gordon equation.

The Lamb diagram (Figure 3.2.2) illustrates the construction of solutions of the sine-Gordon equation in accordance with the recursion formula (3.2.9). Generally, in order to construct the solution ω_n at the nth step of the transformation (the nth layer of the Lamb diagram) it is necessary to know its "precursors" on the two preceding layers of the diagram, namely, the solutions $\omega_{n-1}^{(1,2)}$, $\omega_{n-1}^{(2,3)}$ on the $(n-1)$th layer, and the solutions $\omega_{n-2}^{(1)}$, $\omega_{n-2}^{(2)}\,\omega_{n-2}^{(3)}$ on the $(n-2)$th layer. In particular, for the first three layers the "configuration" of such solutions is depicted by the "shaded triangle" in Figure 3.2.2. The corresponding coefficients (parameters) of the Bäcklund transformations used in the process are indicated in the diagram.

Let us consider the algorithm for constructing solutions of the sine-Gordon equation by means of formula (3.2.9) and the Lamb diagram in the case when for the "primer" solutions ω_0 on the first layer one takes the "vacuum" (trivial) solution $\omega_0 \equiv 0$.

To find the solution ω_1 on the next (second) layer, we must solve in the present case the system of equations (3.1.28) (where now $\omega \equiv \omega_0 = 0$, $\omega^* \equiv \omega_1$):

$$\begin{aligned} \omega_{1,u} &= 2k\cdot\sin\frac{\omega_1}{2},\\ \omega_{1,v} &= \frac{2}{k}\cdot\sin\frac{\omega_1}{2}, \end{aligned} \qquad k = \text{const.} \tag{3.2.10}$$

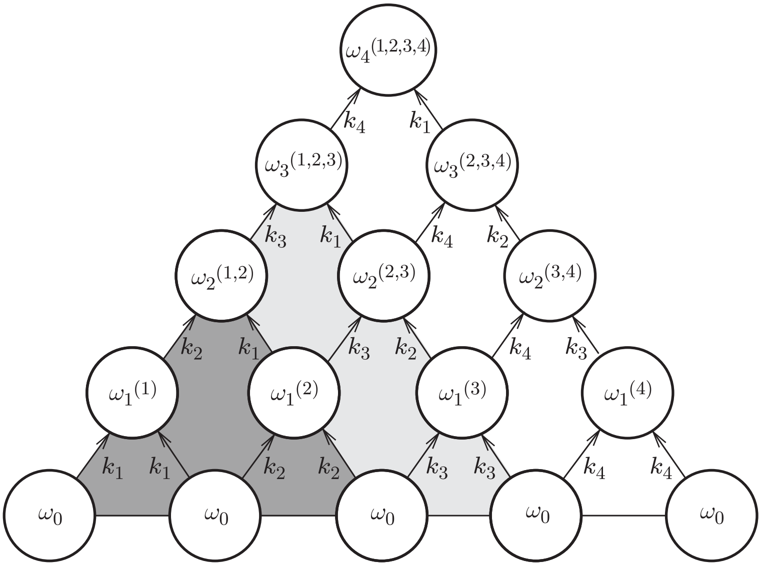


Figure 3.2.2

Standard integration of the system (3.2.10) yields the general form of the solutions that occupy the 2nd layer of the Lamb diagram:

$$\omega_1(u,v) = 4\arctan\left(e^{ku+v/k+b}\right), \qquad k, b = \text{const.} \tag{3.2.11}$$

The solutions $\omega_1(u,v)$ are called *one-soliton* solutions (from the term "solitary wave") or "kink-type" solutions; their physical meaning will be discussed at the end of this section.

The solution $\omega_0 \equiv 0$ and the already obtained solution $\omega_1(u,v)$ (see (3.2.1)) allow us to use formula (3.2.9) and, respectively, the Lamb diagram, for constructing the solution $\omega_2(u,v)$. As a result, the next (3rd) layer (Figure 3.2.2) is occupied by the two-soliton solutions $\omega_2(u,v)$:

$$\omega_2^{1,2}(u,v) = 4\arctan\left(\frac{k_1+k_2}{k_1-k_2}\cdot\tan\frac{\omega_1^1-\omega_1^2}{4}\right),$$

or

$$\omega_2^{1,2}(u,v) = 4\arctan\left(\frac{k_1+k_2}{k_1-k_2}\cdot\frac{e^{k_1u+v/k_1+b_1}-e^{k_2u+v/k_2+b_2}}{1+e^{(k_1+k_2)u+(1/k_1+1/k_2)v+(b_1+b_2)}}\right), \tag{3.2.12}$$

where k_1, k_2, b_1, b_2 are constants.

From the point of view of the physics of nonlinear waves, the two-soliton solutions $\omega_2^{1,2}(u,v)$ realize the *coupled state* of two one-soliton solutions ω_1^1 and ω_1^2, i.e., their nonlinear superposition, which asymptotically, for large values of time, decays into two solitons.

The already available solutions ω_0, ω_1, and ω_3 occupy the first three layers of the Lamb diagram. By successively applying the (algebraic) recursion formula (3.2.9), which in reference to the Lamb diagram we write in the general form

$$\omega_n = \omega_{n-2} + 4\arctan\left(\frac{k_i + k_j}{k_i - k_j} \cdot \tan\left(\frac{\omega^i_{n-1} - \omega^j_{n-1}}{4}\right)\right), \tag{3.2.13}$$

where k_i, k_j are constants, one can obtain the solution ω_n on any layer of the diagram. The transition to a new (next) layer is reflected in the Lamb diagram by shifting up by one step (layer) the shaded triangle in Figure 3.2.2. Altogether a solution $\omega_n(u,v)$ living on the $(n+1)$st layer of the Lamb diagram depends on n numerical parameters of type k_i. The solutions $\omega_n(u,v)$ of the sine-Gordon equation generated by formula (3.2.13) are called *multi-soliton* (and sometimes multi-solitonic or n-soliton) solutions [50,51,57,65].

3.2.2 Clairin's method

From an analytical point of view we can consider that the coefficients k_i in (3.2.13) are *complex*, and that the Bäcklund transformation (3.2.13) itself can be also obtained outside the framework of the geometric methodology, by seeking a hypothetical connection between different solutions of the sine-Gordon equation and their derivatives. Such an approach to the construction of transformations of solutions to nonlinear differential equations is known as *Clairin's method* [67, 187], and we present it next.

Let us consider the Bäcklund transformation on the example of the general Klein-Gordon equation

$$z_{uv} = \mathcal{F}(z). \tag{3.2.14}$$

Suppose that we know some solution $z(u,v)$ of equation (3.2.14). Let us try to find a new solution $\zeta(u,v)$, related to $z(u,v)$ by a system of equations of the form

$$\begin{aligned} \zeta_u &= P(\zeta, z, z_u), \\ \zeta_v &= Q(\zeta, z, z_u). \end{aligned} \tag{3.2.15}$$

Direct substitution of expressions (3.2.15) in (3.2.14) yields (taking into account that $\zeta_{uv} = \zeta_{vu} = \mathcal{F}(\zeta)$)

$$\begin{aligned} Q \cdot P_\zeta + z_v \cdot P_z + \mathcal{F}(z) \cdot P_{z_u} &= \mathcal{F}(\zeta), \\ P \cdot Q_\zeta + z_u \cdot Q_z + \mathcal{F}(z) \cdot Q_{z_v} &= \mathcal{F}(\zeta). \end{aligned} \tag{3.2.16}$$

Generally speaking, the system (3.2.16) may be incompatible, for instance, due to an incomplete set of arguments in the functions P and Q. Hence, it is necessary to study the compatibility of systems of the type (3.2.16). In [67] the compatibility condition of the system (3.2.16) – and with it the existence of a Bäcklund transformation of the form (3.2.15) for the Klein-Gordon equation (3.2.14) – were established. This condition has the form

$$\mathcal{F}'' + C \cdot \mathcal{F} = 0, \quad C = \text{const}. \tag{3.2.17}$$

Condition (3.2.17) means that for the Klein-Gordon equation (3.2.14) a Bäcklund transformation of the sought-form (3.2.15) exists for two forms of the right-hand side of that equation:

$$\mathcal{F} = A \cdot \sin\left(\sqrt{|C|} \cdot z\right) + B \cdot \cos\left(\sqrt{|C|} \cdot z\right),$$

or

$$\mathcal{F} = A \cdot \sinh\left(\sqrt{|C|} \cdot z\right) + B \cdot \cosh\left(\sqrt{|C|} \cdot z\right),$$

where $C = \text{const}$, $A, B = \text{const}$.

Let us examine in more detail the first, closer to us case, in which the Klein-Gordon equation becomes the sine-Gordon equation. Then we have

$$\begin{aligned} z_{uv} &= \sin z, \\ \zeta_{uv} &= \sin \zeta. \end{aligned}$$

In view of these relations, equations (3.2.16) become

$$\begin{aligned} Q \cdot P_\zeta + z_v \cdot P_z + \sin z \cdot P_{z_u} &= \sin \zeta, \\ P \cdot Q_\zeta + z_u \cdot Q_z + \sin z \cdot Q_{z_v} &= \sin \zeta. \end{aligned} \tag{3.2.18}$$

Differentiating the first (second) equation in this system with respect to z_v (respectively, z_u) we arrive, after routine transformations, at

$$P_{z_u z_v} = 0, \quad Q_{z_u z_v} = 0. \tag{3.2.19}$$

This implies that

$$\begin{aligned} P(\zeta, z, z_u) &= P_1(\zeta, z) + P_2(\zeta, z) \cdot z_u, \\ Q(\zeta, z, z_v) &= Q_1(\zeta, z) + Q_2(\zeta, z) \cdot z_v. \end{aligned} \tag{3.2.20}$$

The functions P_1, P_2, Q_1, Q_2 can be specified further. Indeed, let us substitute (3.2.20) in (3.2.18) and compare the coefficients of z_u, z_v and z_{uv} in the left- and right-hand sides of each of the equations. This yields

$$\begin{aligned} Q_1 P_{2,\zeta} &= 0, & P_1 Q_{2,\zeta} &= 0, \\ P_{2,z} + P_{2,\zeta} Q_2 &= 0, & Q_{2,z} + Q_{2,\zeta} P_2 &= 0, \\ P_{1,z} + P_{1,\zeta} Q_2 &= 0, & Q_{1,z} + Q_{1,\zeta} P_2 &= 0. \end{aligned} \tag{3.2.21}$$

From the first four equations in (3.2.21) it follows that

$$P_2 = a = \text{const}, \quad Q_2 = b = \text{const}, \tag{3.2.22}$$

thanks to which the last two equations in (3.2.21) become

$$\begin{aligned} b \cdot \frac{\partial P_1}{\partial \zeta} + \frac{\partial P_1}{\partial z} &= 0, \\ a \cdot \frac{\partial Q_1}{\partial \zeta} + \frac{\partial Q_1}{\partial z} &= 0. \end{aligned} \tag{3.2.23}$$

It is readily verified that the solutions of the obtained system (3.2.23) have the general form

$$\begin{aligned} P_1 &= f(\zeta - bz), \\ Q_1 &= g(\zeta - az). \end{aligned} \tag{3.2.24}$$

Next, substituting the functions P_1, Q_1, P_2, Q_2 thus found in (3.2.20), and then in (3.2.18), we obtain the system of equations

$$\begin{aligned} Q_1 \cdot \frac{\partial P_1}{\partial \zeta} &= \sin\zeta - a\sin z, \\ P_1 \cdot \frac{\partial Q_1}{\partial \zeta} &= \sin\zeta - b\sin z. \end{aligned} \tag{3.2.25}$$

Adding the two equations in (3.2.25) and integrating the result with respect to ζ we obtain

$$f(\zeta - bz)\cdot g(\zeta - az) = -2\cos\zeta - (a+b)\zeta\sin\zeta + C(z), \tag{3.2.26}$$

where $C(z)$ is some function that needs to be further specified.

A careful analysis of relation (3.2.26) shows that it will be satisfied for the following choice of the numerical parameters and functions involved:

$$\begin{gathered} a = 1, \quad b = -1, \quad C(z) = 2\cos z, \\ f(\zeta + z) = 2k\sin\frac{\zeta + z}{2}, \quad g(\zeta - z) = \frac{2}{k}\sin\frac{\zeta - z}{2}, \end{gathered} \tag{3.2.27}$$

where k is an arbitrary numerical parameter. Thus, we have determined the functions P_1 and Q_1 in (3.2.24). Next, substituting the functions P_1, Q_1, P_2, Q_2 found above (see (3.2.22), (3.2.24), (3.2.27)) in (3.2.20), and then in the system (3.2.15), we obtain for solutions of the sine-Gordon equation the transformation

$$\begin{aligned} \zeta_u &= z_u + 2k\sin\frac{\zeta + z}{2}, \\ \zeta_v &= -z_v + \frac{2}{k}\sin\frac{\zeta - z}{2}, \end{aligned} \tag{3.2.28}$$

which is precisely the Bäcklund transformation (3.1.28), introduced earlier when we considered transformations of pseudospherical surfaces.

At the same time, the system (3.2.28), which was derived by Clairin's method, has certain "advantages", since without loss of generality and without resorting to geometric images, it can be considered over the field of complex numbers. In this case, upon choosing in formula (3.212), which corresponds to a system of the type (3.2.28) (given over the field of complex numbers), complex-conjugate coefficients $k_1 = a + ib$, $k_2 = a - ib$ (we put $b_1 = b_2$ in (3.2.12)), we obtain the following solution of the sine-Gordon equation:

$$z(u,v) = 4\arctan\frac{a\cdot\sin\left(b\cdot u - \dfrac{b}{a^2+b^2}\cdot v\right)}{b\cdot\cosh\left(a\cdot u + \dfrac{a}{a^2+b^2}\cdot v\right)}, \tag{3.2.29}$$

called a *breather* solution. The solution (3.2.29) itself is not a complex-valued function. In § 3.4 we will construct a pseudospherical surface that provides an interpretation of a breather solution of type (3.2.29). Let us mention that in the modern theory of nonlinear differential equations there are sufficiently well developed methods that generalize Clairin's approach and are used to obtain hierarchies of solutions connected with certain a priori types of constraints (by analogy with (3.2.15)). In particular, modifications of such methods for wide classes of equations associated with the method of the inverse scattering problem and pseudospherical surfaces are quite strongly developed in work of K. Tenenblat and her colleagues [138, 150, 195].

3.2.3 The concept of soliton solution of a nonlinear equation

In this subsection we address the physical interpretation of solutions of the form (3.2.13) as "solitary waves", the interactions of which have certain special properties. Such solutions arise quite often in nonlinear models that describe various physical phenomena. In particular, the by-now familiar to us sine-Gordon equation is encountered in many physical models, but, as a rule, in the form of the nonlinear *wave* equation (with "wave-type" left-hand side)

$$z_{xx} - z_{tt} = \sin z. \tag{3.2.30}$$

The x and t in (3.2.20) usually play the role of a space and a time variable, respectively.

Let us note that the sine-Gordon equation (3.2.30) is obtained from the "standard" sine-Gordon equation (3.1.5) as a result of the change of variables

$$x = u + t, \quad t = u - v, \quad z = \omega. \tag{3.2.31}$$

From a geometrical point of view, the substitution (3.2.31) corresponds to the transition from the Chebyshev coordinate net on a pseudospherical surface to the orthogonal coordinate net associated to the lines of principal curvature of the surface (for example, in the case of the pseudosphere such an orthogonal net is formed by the meridians of revolution and the circular parallels).

Let us turn now to the concept of soliton. In the physics of nonlinear waves the term soliton is used for solitary waves that propagate at a constant speed and the profile of which is preserved in time. The most important is that this kind of waves interact in a special characteristic way, the only result of which is a phase shift of the interacting waves. Let us formalize these statements [50, 51, 107].

Consider the general form of a partial differential equation for a function $z(x,t)$ of two independent variables x and t:

$$L[z] = 0, \quad z = z(x,t). \tag{3.2.32}$$

A solution of equation (3.2.32) of the form

$$z = z(x,t) = z_{\mathrm{ST}}(\xi), \quad \xi = x - wt, \quad w = \mathrm{const}, \tag{3.2.33}$$

will be called a *stationary traveling wave*.

We call *solitary wave* $z_{\mathrm{ST}}(\xi)$ a *localized* traveling wave, that is, a wave (one-dimensional profile) z_{ST}, the transition of which from one stable limit state (for $\xi \to -\infty$) to another stable limit state (for $\xi \to +\infty$) (with, possibly, return after a perturbation of the initial state) is practically localized.

A typical example of solitary state is the solution

$$z^1_{\mathrm{KdV}}(x,t) = -\frac{c}{2}\left(\operatorname{sech}\left[\frac{\sqrt{c}}{2}(x-ct)\right]\right)^2, \quad c = \text{const} > 0, \tag{3.2.34}$$

of the well-known Korteweg-de Vries equation

$$z_t - 6zz_x + z_{xxx} = 0. \tag{3.2.35}$$

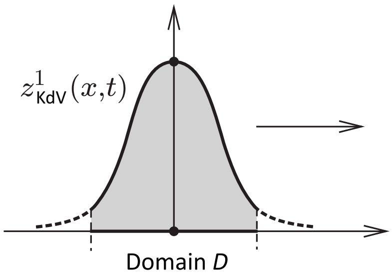


Figure 3.2.3

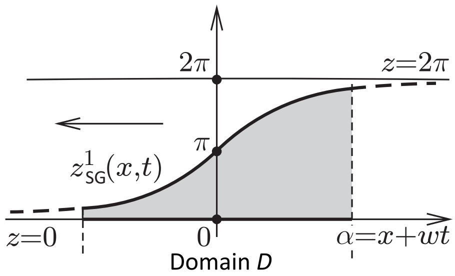


Figure 3.2.4

The solution (3.2.34) has a "bell-shaped" profile. In the context of the preceding discussion we should single out the one-soliton solutions $z^1_{\mathrm{SG}}(x,t)$ of the sine-Gordon equation (Figure 3.2.4):

$$z^1_{\mathrm{SG}}(x,t) = 4 \arctan e^{\alpha}, \quad \alpha = x + wt, \quad w = \text{const}. \tag{3.2.36}$$

Solutions of the form (3.2.36) (solutions with two distinct limit states 0 and 2π) are also called "*kink-type*" *solutions*.

The requirement of being "practically localized" allows for the existence an infinitesimal rapidly-decaying "tail" of an anomalous perturbation. In the framework of the physical methodology, this requirement is absolutely correctly replaced by the condition of "complete localization", i.e., one assumes that outside some bounded domain D the considered nonlinear perturbation vanishes.

The solutions of the Korteweg-de Vries (3.2.34) and sine-Gordon (3.2.36) equations shown in figures 3.2.3 and 3.2.4, respectively, have exponentially decaying "tails", and their infinitely-decaying corrections to the localized perturbation can be neglected.

A *soliton* is a solution of equation (3.2.32) in the form of a solitary stationary wave $z_{\mathrm{ST}}(x - wt)$, which upon interacting with other waves of the same kind preserves asymptotically its profile and speed.

More precisely, let $z(x, t)$ be a solution of an equation of type (3.2.32) that for "large negative" values of time is a coupling of N solitary waves:

$$z(x,t) \approx \sum_{j=1}^{N} z_{\mathrm{ST}}(\xi_j) \quad \text{as } t \to -\infty,$$

where $\xi_j = x - w_j t$, with w_j the speed of the jth solitary wave. Then these solitary wave $z_{\mathrm{ST}}(\xi_j)$ are *solitons*, if the only result of their interaction ("participation" in the coupled state) is a phase shift, i.e., for large values of time it holds that

$$z(x,t) \approx \sum_{j=1}^{N} z_{\mathrm{ST}}(\bar{\xi}_j) \quad \text{as } t \to +\infty,$$

where $\bar{\xi}_j = x - w_j t + \delta_j$, $\delta_j = \text{const}$.

The quantity δ_j defines the phase shift for the corresponding soliton.

To draw conclusions about soliton properties of nonlinear waves (as solutions of nonlinear differential equations) is possible only based on a analysis of their interaction. This analysis is carried out by investigating the asymptotic (for large values of the variable ξ_j) properties (states) of solutions, which are superpositions of simpler (interacting) nonlinear waves. A solution that represents a *composite state*, i.e., a superposition of several solitons is called a *multi-soliton solution*. For example, such solutions can be obtained by means of a Bäcklund transformation.

An asymptotic investigation of two-soliton solutions of the sine-Gordon equation is carried out in [62], where one can find also a proof of the stability of a one-soliton solution under small perturbations. This allows one to regard the one-soliton solution as an elementary solitonic object, i.e., as a primary wave entity that is not a composite state of other nonlinear waves that are solutions of the sine-Gordon equation.

The soliton is an exceptionally important concept in the physics of nonlinear phenomena. We give below a short list of physical problems in which the sine-Gordon equation arises as a model equation, and in which the aforementioned special features of wave objects, solitons and "kinks", acquire in each case its special meaning.

The sine-Gordon equation arises as a model equation in the description of the following physical phenomena, among others:

1. Propagation of ultra-short pulses in two-level resonant media [83, 110, 169, 172].
2. Dynamics of Bloch walls in ferromagnetic crystals [47].
3. Certain problems of nonlinear electrodynamics [84].
4. The Josephson effect [8].
5. Certain problems of the unified theory of elementary particles [107].
6. Propagation of oscillations in mechanical transmission lines [23, 107].
7. Propagation of dislocations in crystals [23, 47].

Among the references listed in the bibliography let us emphasize A. Scott's monograph [107], which presents of a sufficiently universal approach to the investigation of nonlinear waves, and among them, of the solutions of sine-Gordon equation. It uses as models chains (lines, lattices) consisting of sets of identical standard branches constructed from inductors and capacitors, which are widely used in radiophysics. Such chains are convenient objects of study from the "wave-theoretic" point of view, since they permit the realization of various dispersive and nonlinear characteristics of the physical systems under investigation. Let us mention also A. S. Davydov's monograph [23], in which the connection between the sine-Gordon equation and the specific features of various physical models is traced in a rather "refined" manner. Of particular interest is also the fundamental monograph of A. Barone and G. Paterno [8], in which a multi-plane analysis of various aspects of the Josephson effect is carried out.

3.3 Exact integration of the fundamental system of equations of pseudospherical surfaces in the case of one-soliton solutions of the sine-Gordon equation

As established in § 3.1, to construct new pseudospherical surfaces by means of the formula for the Bäcklund transformation (3.1.31) we need a "primer" pair (S,ω) $=$ ({ pseudospherical surface S},{solution $\omega(u,v)$ of the sine-Gordon equation}), which corresponds to the initial, "support layer" of the transformation, the scheme of which is shown in Figure 3.1.3. The Bäcklund transformation of *solutions* of the sine-Gordon equation accompanying the transformation of pseudospherical surfaces is given by formula (3.2.13), in which for the initial nontrivial solution one takes the one-soliton solution (3.2.11) of the sine-Gordon equation. Let us determine what pseudospherical surface can form a "pair" with the solution (3.2.11), in the sense that a corresponding Chebyshev metric of the type (3.1.1) is realized on it. In other words, we address the problem of integrating the fundamental system of equations (2.7.30)–(2.7.34) for pseudospherical surfaces in the case of one-soliton solutions $\omega_1(u,v)$ of the sine-Gordon equation.

3.3.1 Exact integration method. The Dini surface and the pseudosphere

In the setting of the problem formulated above, the fundamental system of equations for pseudospherical surfaces, (2.7.30)–(2.7.34), reads

$$\vec{r}_{uu} = \omega_{1u}\cdot\vec{n}_u, \tag{3.3.1}$$

$$\vec{r}_{uv} = \omega_{1uv}\cdot\vec{n}, \tag{3.3.2}$$

$$\vec{r}_{vv} = \omega_{1v}\cdot\vec{n}_v, \tag{3.3.3}$$

$$\vec{n}_u = \cot\omega_1\cdot\vec{r}_u - \frac{1}{\sin\omega_1}\cdot\vec{r}_v, \tag{3.3.4}$$

$$\vec{n}_v = \frac{1}{\sin\omega_1} \cdot \vec{r}_u + \cot\omega_1 \cdot \vec{r}_v. \tag{3.3.5}$$

Let us use the property of linear dependence of the partial derivatives of the one-soliton solutions $\omega_1(u, v)$ given by (3.2.11):[2]

$$\begin{aligned} \omega_{1u} &= k^2 \cdot \omega_{1v}, \\ \omega_{1uu} &= k^2 \cdot \omega_{1uv} = k^4 \omega_{1vv}. \end{aligned} \tag{3.3.6}$$

In view of properties (3.3.6), the system (3.3.1)–(3.3.3) can be reduced to

$$\vec{r}_u + k^2 \vec{r}_v = \omega_{1u} \cdot \vec{n} + \vec{C}_0, \quad \vec{C}_0 = \text{const}. \tag{3.3.7}$$

The left-hand side of equation (3.3.7) can also be expressed starting from (3.3.4) and (3.3.5), by considering those equations as a simple system for $\vec{r}_u$ and $\vec{r}_v$:

$$\vec{r}_u + k^2 \cdot \vec{r}_v = -\frac{k^2 + \cos\omega_1}{\sin\omega_1} \cdot \vec{n}_u - \frac{1 + k^2 \cdot \cos\omega_1}{\sin\omega_1} \cdot \vec{n}_v. \tag{3.3.8}$$

From relations (3.3.7) and (3.3.8) we obtain for the unit normal vector $\vec{n}$ to the sought-for surface the equation

$$\vec{n} + \frac{k^2 + \cos\omega_1}{\omega_{1u} \cdot \sin\omega_1} \cdot \vec{n}_u + \frac{1 + k^2 \cdot \cos\omega_1}{\omega_{1u} \cdot \sin\omega_1} \cdot \vec{n}_v = -\frac{1}{\omega_{1u}} \cdot \vec{C}_0. \tag{3.3.9}$$

Without loss of generality, we will look for a solution of equation (3.3.9) in the form

$$\vec{n}(u, v) = \vec{n}_0(\omega_1) + \vec{n}_1(u, v). \tag{3.3.10}$$

In representation (3.3.10) the isolated on purpose self-similar component $\vec{n}_0$ of the argument ω_1 allows one to eliminate the inhomogeneity in the equation, given by the second component $\vec{n}_1(u, v)$. As a result, substitution of expression (3.3.10) in (3.3.9) leads to an ordinary differential equation for $\vec{n}_0(\omega_1)$:

$$\frac{1 + 2k^2 \cdot \cos\omega_1 + k^4}{k^2 \cdot \sin\omega_1} \cdot (\vec{n}_0)'_{\omega_1} + \vec{n}_0 + \frac{\vec{C}_0}{2k \cdot \sin(\omega_1/2)} = 0. \tag{3.3.11}$$

and a homogeneous partial differential equation for $\vec{n}_1(u, v)$:

$$\vec{n}_1 + \frac{\delta}{\rho} \cdot \vec{n}_{1u} + \frac{\mu}{\rho} \cdot \vec{n}_{1v} = 0. \tag{3.3.12}$$

In the derivation of (3.3.11) and (3.3.12) we used the fact that

$$\omega_{1u} = 2k \cdot \sin(\omega_1/2).$$

[2] A similar property is enjoyed also by solutions of stationary traveling wave type, see § 3.3 below. The ideas of the exact integration approach presented here remain valid for such solutions.

We also adopted the notations

$$\rho = \omega_{1u} \cdot \sin\omega_1, \quad \delta = k^2 + \cos\omega_1, \quad \mu = 1 + k^2 \cdot \cos\omega_1.$$

Equation (3.3.11) can be integrated in the standard manner and has the general solution

$$\vec{n}_0(\omega_1) = \sqrt{1 + 2k^2\cos\omega_1 + k^4} \cdot \vec{C}_1 - \frac{2k\sin(\omega_1/2)}{(1+k^2)^2} \cdot \vec{C}_0, \quad \vec{C}_0, \vec{C}_1 = \text{const.} \tag{3.3.13}$$

Now let us turn to equation (3.3.12). We assume that the components of the sought-for vector function

$$\vec{n}_1(u,v) = \{n_{11}(u,v), n_{12}(u,v), n_{13}(u,v)\}$$

are of the form

$$n_{1j}(u,v) = \varphi_{1j}(\alpha) \cdot \sin\beta_j + \varphi_2(\alpha) \cdot \cos\beta_j, \quad j = 1,2,3, \tag{3.3.14}$$

where $\beta_j = a_{1j}u - a_{2j}v$, with $a_{1j}, a_{2j} = \text{const}$, and the functions $\varphi_{1j}, \varphi_{2j}$ are subject to determination.

With no loss of generality, in the argument $\alpha = ku + (v/k) + b$ of the solution of equation (3.3.11) we take $b = 0$. Also, we employ the useful relation

$$\alpha = \ln\left(\tan\frac{\omega_1}{4}\right).$$

To find the functions $\varphi_{1j}(\alpha)$ and $\varphi_{2j}(\alpha)$, we substitute the expression of the solution (3.3.14) in equation (3.3.12). We obtain

$$(\rho\varphi_{1j} + f_1\varphi_{1j}' + f_{2j}\varphi_{2j}) \cdot \sin\beta_j + (\rho\varphi_{2j} + f_1\varphi_{2j}' - f_{2j}\varphi_{1j}) \cdot \cos\beta_j = 0, \tag{3.3.15}$$

where

$$\left.\begin{aligned} f_1 &= k\delta + \frac{\mu}{k}, \\ f_2 &= -a_{1j}\delta + a_{2j}\mu \end{aligned}\right. \quad j = 1,2,3,$$

and the prime denotes differentiation with respect to α.

Equation (3.3.15) is equivalent to the system

$$f_1\varphi_{1j}' + \rho\varphi_{1j} + f_{2j}\varphi_{2j} = 0, \quad j = 1,2,3, \tag{3.3.16}$$

$$f_1\varphi_{2j}' + \rho\varphi_{2j} - f_{2j}\varphi_{1j} = 0, \quad j = 1,2,3. \tag{3.3.17}$$

Multiplying equations (3.3.16) and (3.3.17) by φ_{1j} and φ_{2j}, respectively, and adding the results, we arrive at the following ordinary differential equation for the function $\varphi_{1j}^2 + \varphi_{2j}^2$:

$$\frac{f_1}{2}(\varphi_{1j}^2 + \varphi_{2j}^2)' + \rho(\varphi_{1j}^2 + \varphi_{2j}^2) = 0,$$

the integration of which yields

$$\varphi_{1j}^2 + \varphi_{2j}^2 = G_1(\alpha), \tag{3.3.18}$$

where $G_1(\alpha) = A \cdot \exp\left(-\int (2\rho/f_1) d\alpha\right)$, $A = \text{const}$. If we use this last expression in the system (3.3.16), (3.3.17) , we obtain

$$\varphi_{1j}'^{\,2} + \varphi_{2j}'^{\,2} = G_{2j}(\alpha), \tag{3.3.19}$$

where the right-hand side is given by

$$G_{2j}(\alpha) = \frac{f_{2j}^2 - \rho^2}{f_1^2} G_1(\alpha) - \frac{\rho}{f_1} G_1'(\alpha).$$

The substitution

$$\begin{aligned} \eta_{1j} &= \varphi_{1j} + i\varphi_{2j}, \\ \eta_{2j} &= \varphi_{1j} - i\varphi_{2j}, \end{aligned} \tag{3.3.20}$$

where i is the imaginary unit, reduces the system (3.3.18), (3.3.19) to the compact form

$$\begin{aligned} \eta_{1j}\eta_{2j} &= G_1(\alpha), \\ \eta_{1j}'\eta_{2j}' &= G_2(\alpha), \end{aligned} \qquad j = 1, 2, 3. \tag{3.3.21}$$

Taking the logarithm of the first equation in (3.3.21), differentiating the resulting expression with respect to α, and then multiplying by η_{2j}'/η_{2j}, we obtain the following quadratic equation for the function η_{2j}'/η_{2j}:

$$\left(\frac{\eta_{2j}'}{\eta_{2j}}\right)^2 - \frac{G_1'}{G_1}\frac{\eta_{2j}'}{\eta_{2j}} + \frac{G_{2j}}{G_1} = 0.$$

Solving this equation, and hence the system (3.3.21), and using the formulas (3.3.20) to effect the reverse transition, we obtain for $\varphi_{1j}(\alpha)$ and $\varphi_{2j}(\alpha)$ the expressions

$$\varphi_{1j}(\alpha) = \exp\left(-\int \frac{\rho}{f_1} d\alpha\right) \left[P_j \cos\left(-\int \frac{f_{2j}}{f_1} d\alpha\right) + Q_j \sin\left(-\int \frac{f_{2j}}{f_1} d\alpha\right)\right], \tag{3.3.22}$$

$$\varphi_{2j}(\alpha) = \exp\left(-\int \frac{\rho}{f_1} d\alpha\right) \left[P_2 \sin\left(-\int \frac{f_{2j}}{f_1} d\alpha\right) - Q_j \cos\left(-\int \frac{f_{2j}}{f_1} d\alpha\right)\right], \tag{3.3.23}$$

where $P_j, Q_j = \text{const}$, $j = 1, 2, 3$.

Let us calculate the integrals appearing in (3.3.22) and (3.3.23). We have

$$\exp\left(-\int \frac{\rho}{f_1} d\alpha\right) = \sqrt{1 + 2k^2 \cos\omega_1 + k^4}.$$

The second integral, which appears in the argument of sin and cos, can be calculated in general form, but upon subsequent verification it turns out that the equation in question is verified only when $a_{1j} = 1$ and $a_{2j} = 1$ for $j = 1, 2, 3$. In this last case this integral becomes considerably simpler:

$$\int \frac{f_{2j}}{f_1} d\alpha = -\arctan\left(\frac{2k}{1+k^2}\cos\frac{\omega_1}{2}\right).$$

Substituting the calculated integrals in (3.3.22) and (3.3.23), we write

$$\begin{aligned} \vec{\varphi}_1(\alpha) &= (1-k^2)\cdot\vec{P} + 2k\sin\frac{\omega_1}{2}\cdot\vec{Q}, \\ \vec{\varphi}_2(\alpha) &= 2k\cos\frac{\omega_1}{2}\cdot\vec{P} - (1-k^2)\cdot\vec{Q}, \end{aligned} \tag{3.3.24}$$

where

$$\vec{P} = \{P_1, P_2, P_3\}, \quad \vec{Q} = \{Q_1, Q_2, Q_3\},$$
$$\vec{\varphi}_1(\alpha) = \{\varphi_{11}(\alpha), \varphi_{12}(\alpha), \varphi_{13}(\alpha)\}, \quad \vec{\varphi}_2(\alpha) = \{\varphi_{21}(\alpha), \varphi_{22}(\alpha), \varphi_{23}(\alpha)\}.$$

Therefore, the unit normal vector $\vec{n}(u,v)$ to the sought-for pseudospherical surface is given by expressions (3.3.10), (3.3.13), (3.3.14) and (3.3.24), up to the constant vectors $\vec{P}$ and $\vec{Q}$, which are determined from the normalization condition for the vector $\vec{n}(u,v)$.

Now let us determine the radius vector

$$\vec{r}(u,v) = \{r_1(u,v), r_2(u,v), r_3(u,v)\}$$

of the sought-for pseudospherical surface S_1 that corresponds to the solution $\omega_1(u,v)$ of the sine-Gordon equation. To this end we employ equation (3.3.7). Starting from the qualitative form of the vector function $\vec{n}(u,v)$ obtained above, it is natural to assume the following representation for the components $r_j(u,v)$:

$$r_j(u,v) = p_j(\alpha)\cdot\sin\beta_j + q_j(\alpha)\cdot\cos\beta_j + g_j(\alpha) + e_{1j}u + e_{2j}v, \quad j = 1, 2, 3. \tag{3.3.25}$$

The expressions (3.2.25) "involve" the components of the following vectors:

$$\vec{p}(\alpha) = \{p_1(\alpha), p_2(\alpha), p_3(\alpha)\}, \quad \vec{q}(\alpha) = \{q_1(\alpha), q_2(\alpha), q_3(\alpha)\},$$
$$\vec{g}(\alpha) = \{g_1(\alpha), g_2(\alpha), g_3(\alpha)\}, \quad \vec{e}_1, \vec{e}_2 = \text{const}.$$

Substitution of expression (3.3.25) in equation (3.3.7) splits it into a relation for the self-similar part of the solution (which depends on α):

$$\vec{g}\,'(\alpha) = \sin\frac{\omega_1}{2}\cdot\vec{n}_0(\omega_1) + \frac{1}{2k}\cdot\vec{C}_0, \tag{3.3.26}$$

and a system of differential equations for the vectors $\vec{p}(\alpha)$ and $\vec{q}(\alpha)$:

$$2k\cdot\vec{p}\,' - (1-k^2)\cdot\vec{q} = \omega_{1u}\cdot\varphi_1(\alpha), \tag{3.3.27}$$

$$2k \cdot \vec{q}\,' - (1-k^2) \cdot \vec{p} = \omega_{1u} \cdot \varphi_2(\alpha). \tag{3.3.28}$$

Solving the nonhomogeneous system (3.3.27), (3.3.28) by the method of variation of constants, we obtain

$$\begin{aligned} \vec{p}(\alpha) &= -\frac{2k}{\cosh\alpha} \cdot \vec{P}, \\ \vec{q}(\alpha) &= -\frac{2k}{\cosh\alpha} \cdot \vec{Q}. \end{aligned} \tag{3.3.29}$$

Now substitution of (3.3.26) and (3.3.39) in (3.3.25) yields the radius vector $\vec{r}(u,v)$ of the sought-for pseudospherical surface S_1 (up to constant vectors):

$$\vec{r}(u,v) = \frac{\sin(u-v)}{\cosh\alpha} \cdot \vec{B}_1 + \frac{\cos(u-v)}{\cosh\alpha} \cdot \vec{B}_2 + \tanh\alpha \cdot \vec{B}_3 + u \cdot \vec{e}_1 + v \cdot \vec{e}_2. \tag{3.3.30}$$

The precise values of the constant vectors $\vec{B}_1 = 2k\vec{P}$, $\vec{B}_2 = -2k\vec{Q}$, $\vec{B}_3$, $\vec{e}_1$ and $\vec{e}_2$ are found from the conditions

$$\vec{r}_u^{\,2} = 1, \quad (\vec{r}_u, \vec{r}_v) = \cos\omega_1, \quad \vec{r}_v^{\,2} = 1,$$

which determine the pseudospherical metric under consideration. As it turns out, the vectors $\vec{B}_1$, $\vec{B}_2$, $\vec{B}_3$ have the same length $2k/(1+k^2)$ and form in $\mathbb{E}^3$ an orthogonal frame, while $\vec{e}_1$ and $\vec{e}_2$ are equal unit vectors collinear with the vector $\vec{B}_3$:

$$\vec{e}_1 = \vec{e}_2, \quad |\vec{e}_1| = |\vec{e}_2| = 1, \quad \vec{e}_1 || \vec{B}_3, \quad \vec{e}_2 || \vec{B}_3.$$

The pseudospherical surface S_1 constructed according to (3.3.30) and corresponding to the one-soliton solution ω_1 of the sine-Gordon equation is a helical surface. This becomes clear if we orient the vectors $\vec{B}_1$, $\vec{B}_2$, and $\vec{B}_3$ along the Cartesian coordinate axes in $\mathbb{E}^3(x,y,z)$:

$$\vec{B}_1\left(-\frac{2k}{1+k^2}, 0, 0\right), \quad \vec{B}_2\left(0, \frac{2k}{1+k^2}, 0\right), \quad \vec{B}_3\left(0, 0, -\frac{2k}{1+k^2}\right),$$
$$\vec{e}_1(0,0,1), \quad \vec{e}_2(0,0,1).$$

Then the Cartesian presentation of the surface S_1 in $\mathbb{E}^3(x,y,z)$ reads (here u and v are asymptotic coordinates on the surface)

$$\begin{aligned} x &= -\frac{2k}{1+k^2} \cdot \frac{1}{\cosh(ku + k/v)} \cdot \sin(u-v), \\ y &= \frac{2k}{1+k^2} \cdot \frac{1}{\cosh(ku + k/v)} \cdot \cos(u-v), \\ z &= -\frac{2k}{1+k^2} \cdot \tanh(ku + k/v) + u + v \end{aligned} \tag{3.3.31}$$

(*Dini surface*).

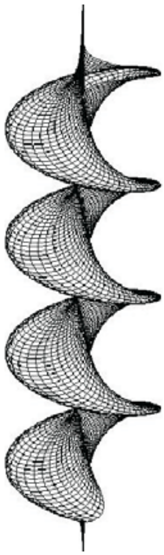


Figure 3.3.1

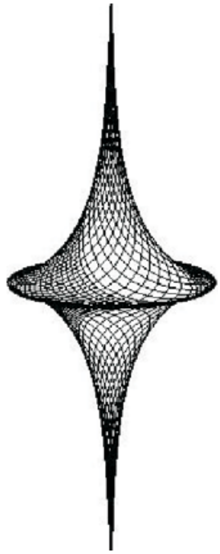


Figure 3.3.2

It is worth mentioning that the above equations (3.3.31) describe the well-known classical helical *Dini surface* of constant negative curvature -1 (see [142]), shown here in Figure 3.3.1.

In the case when the parameter k is equal to 1, equations (3.3.31) become the equations of the pseudosphere (see § 1.3) in asymptotic coordinates (Figure 3.3.2):

$$
\begin{aligned}
x &= -\frac{\sin(u-v)}{\cosh(u+v)}, \\
y &= \frac{\cos(u-v)}{\cosh(u+v)}, \\
z &= -\tanh(u+v) + u + v
\end{aligned}
\tag{3.3.32}
$$

(*pseudosphere*).

Thus, equations (3.3.31) and (3.3.32) define two types of surfaces: the *Dini surface* ($k \neq 1$) and the *pseudosphere* ($k = 1$), which serve as "primer" pairs for the Bäcklund transformation with the one-soliton solution $\omega_1(u, v)$ of the sine-Gordon equation for the corresponding value of the parameter k. Recall that the solution $\omega_1(u, v)$ (3.2.11) itself has the geometric meaning of the net angle of the coordinate net of asymptotic lines on the aforementioned pseudospherical surfaces.

3.3.2 Interpretation of the one-soliton solution in the plane Λ^2

In complete agreement with D. Hilbert's results that the plane Λ^2 cannot be immersed in $\mathbb{E}^3$ (§ 2.6) and E. G. Poznyak's theorem on pseudospherical surfaces

(Subsection 2.7.4), the Dini surface ($k \neq 1$) and the pseudosphere ($k = 1$) have irregular singularities, namely, an irregular cuspidal edge and the axis Oz that is asymptotically reachable at infinity (for the pseudosphere, Oz serves as the axis of revolution). By Theorem 2.7.1, the preimages of these singularities in the parametric (u, v)-plane are the level lines $\omega_1(u, v) = m\pi$ (with m an integer) of the one-soliton solution (3.2.11) of the sine-Gordon equation. Moreover, the indicated lines are also preimages of the boundaries of domains in the Lobachevsky plane Λ^2 that can be immersed in $\mathbb{E}^3$ precisely as the Dini surface or the pseudosphere; the domains in Λ^2 that *can be immersed* in $\mathbb{E}^3$ cannot be extended beyond these boundaries.

Let us clarify which domains in the plane Λ^2 correspond, in the sense just discussed, to the one-soliton solution $\omega_1(u, v)$. With the solution $\omega_1(u, v)$ given by (3.2.11): $\{(u, v) \in \mathbb{R}^2,\ \omega_1 \in (0, 2\pi)\}$, there are associated three levels $m\pi$ $(m = 0, 1, 2)$.

Let us study the level line

$$\omega_1(u, v) = \pi,$$

which according to (3.2.11) is given by the equation

$$v = -k^2 u. \tag{3.3.33}$$

Using (3.3.33), we calculate the geodesic curvature of $k_{\mathrm{g}}(\omega_1 = \pi)$ of this line using formula (2.7.72), which yields

$$k_{\mathrm{g}}(\omega_1 = \pi) = \frac{2k}{1 + k^2} = \text{const}. \tag{3.3.34}$$

For definiteness, in (3.2.11) we take $k > 0$.

It readily follows from (3.3.34) that

$$k_{\mathrm{g}}(\omega_1 = \pi) = 1 \quad \text{if } k = 1, \tag{3.3.35}$$

and

$$0 < k_{\mathrm{g}}(\omega_1 = \pi) < 1 \quad \text{if } k \in (0, 1) \cup (1, +\infty). \tag{3.3.36}$$

Therefore, in the case (3.3.35) (resp., (3.3.36)), to the level line (3.3.33) there corresponds in the Lobachevsky plane Λ^2 a *horocycle* (resp., an *equidistant*, also called a *hypercircle*).

An analogous analysis of the asymptotically “reachable” at infinity level lines $\omega_1(u, v) = 0$ and $\omega_1(u, v) = 2\pi$ shows that their geodesic curvature is equal to zero:

$$k_{\mathrm{g}}(\omega_1 = 0) = 0, \quad k_{\mathrm{g}}(\omega_1 = 2\pi) = 0. \tag{3.3.37}$$

Hence, on the plane Λ^2 these lines are *geodesics*.

Thus, we conclude that on the Lobachevsky plane Λ^2 to a one-soliton solution $\omega_1(u, v)$ there corresponds, in the sense described above, either a *horosdisc* $\vartheta(\omega_1)$ for $k = 1$ (a domain, the boundary of which is a horocycle), or (for $k \neq 1$) a *strip* $\Pi(\omega_1)$ bounded by two (asymptotically reachable) geodesic lines and an

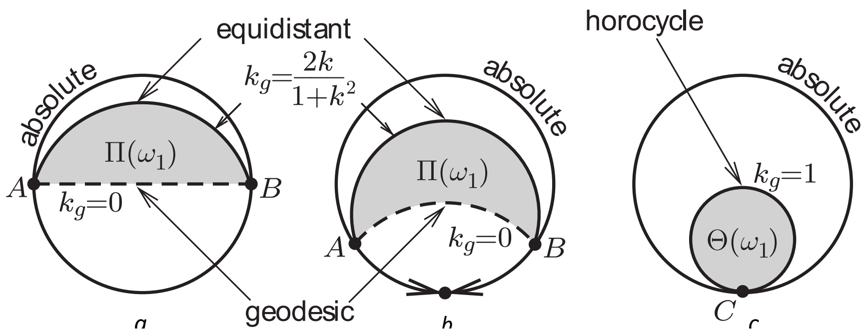


Figure 3.3.3

equidistant with geodesic curvature (3.3.34). The width of such a strip is constant and equals

$$\delta = \operatorname{arctanh} \frac{2k}{1+k^2}, \quad k \in (0,1) \cup (1,+\infty).$$

Strictly speaking, to the solution $\omega_1(u,v)$, which is given on the whole plane $\mathbb{R}^2(u,v)$, there correspond on Λ^2 two copies of each of the domains $\Pi(\omega_1)$ or $\vartheta(\omega_1)$, glued in a regular manner along their boundaries (i.e., along an equidistant or a horocycle, respectively), because in order to obtain a representation on Λ^2 we need to consider the two half-planes into which $\mathbb{R}^2(u,v)$ is divided by the level line (3.3.33). Moreover, the images u and v of the asymptotic lines on the surface pass in a regular manner from one copy of the corresponding domain in Λ^2 to the other copy. For example, in the space $\mathbb{E}^3$ to each copy of the horocycle there corresponds its own (lower, or upper) plane of the pseudosphere.

The domains of the plane Λ^2 discussed above are shown in Figure 3.3.3. Although the domains $\Pi(\omega_1)$ or $\vartheta(\omega_1)$ obtained in Λ^2 correspond to one and the same solution of the sine-Gordon equation (one-soliton solution) ,

$$\omega_1(u,v) = 4 \arctan \exp(ku + v/k),$$

geometrically there are essential differences: the strip $\Pi(\omega_1)$ (Figure 3.3.3 *a,b*) has two points at infinity on the absolute (the points A and B), whereas the horocycle has only one such point (the point C, Figure 3.3.3 *c*). The "hypothetical dynamics of the transition" of the domain $\Pi(\omega_1)$ into the domain $\vartheta(\omega_1)$ as $k \to 1$ ("merger" of the two points at infinity into one) is shown in Figure 3.3.3 *b*.

From the point of view of space geometry of the Dini surface and the pseudosphere in $\mathbb{E}^3$, the aforementioned passage to the limit $k \to 1$ appears more natural: it corresponds to "compressing" the cuspidal edge of the Dini surface along the axis Oz with constant twisting (see (2.7.8) and (3.3.33)):

$$æ(\omega_1 = \pi) = \frac{1-k^2}{1+k^2},$$

with subsequent limit "collapse" to a circle (zero torsion) – the edge of the pseudosphere.

Note that the above arguments concerning the "passage to the limit" can be useful in obtaining a geometric interpretation of nonlinear waves.

3.3.3 Solutions of stationary traveling wave type and their geometric realization

A key initial condition necessary for the implementation of the exact integration method proposed in Subsection 3.3.1 for the basic system of equations for pseudospherical surfaces (3.3.1)–(3.3.5) in the case of a one-soliton solution $\omega_1(u,v)$ of the sine-Gordon equation is the property (3.3.6) of *linear dependence* of the partial derivatives of the solution $\omega_1(u,v)$. It turns out that the sine-Gordon equation has also other solutions that enjoy this property, and so in order to provide their geometric interpretation as pseudospherical surfaces one can, generally speaking, apply the same method. We are thus dealing with solutions that depend on a linear self-similar argument $au+bv$:

$$\omega_{\mathrm{ST}} = \omega(au+bv), \quad a, b = \mathrm{const.} \tag{3.3.38}$$

The solutions (3.3.38) belong to the class of *stationary traveling waves* (see Subsection 3.2.3). In certain physical applications, such solutions are also called *single-phase solutions* [23].

We now turn to the classification of solutions of stationary traveling wave type (single-phase solutions). To this end we rewrite the sine-Gordon equation (3.1.5) in a form with a "wave-type" left-hand side:[3]

$$Z_{xx} - Z_{tt} = \sin Z, \tag{3.3.39}$$

which is obtained by means of the change of variables

$$x = u+v, \quad t = u-v, \quad \omega = Z.$$

With no loss of generality, we represent the stationary traveling wave solutions of equation (3.3.39) in the form

$$Z_{\mathrm{ST}}(x,t) = Z(\vartheta), \quad \vartheta = x - wt, \quad w = \mathrm{const}, \tag{3.3.40}$$

(here w denotes the speed of the stationary traveling wave).

Upon passing to the variable ϑ, equation (3.3.39) reduces to the ordinary differential equation

$$(1-w^2)\cdot Z''_{\vartheta\vartheta}(\vartheta) = \sin Z(\vartheta). \tag{3.3.41}$$

Multiplying (3.3.41) by Z'_ϑ and integrating the result, we reduce further the order of the equation:

$$(1-w^2)\cdot Z'^2_\vartheta = 2(E_0 - \cos Z), \quad E_0 = \mathrm{const.} \tag{3.3.42}$$

[3]This is the form in which the sine-Gordon equation is most frequently encountered in applications.

The solutions of equation (3.3.42) can be divided into two classes:

1) *magnetic-type* solutions (with $w^2 < 1$);

2) *electric-type* solutions (with $w^2 > 1$).

From the point of view of physics, the term magnetic-type (resp., electric-type) wave is connected with the fact that for $w^2 < 1$ (resp., $w^2 > 1$) one can use a Lorentz transformation to pass to a new reference frame, in which the single-phase wave under consideration becomes a function of the *space coordinate* (resp., *time*) only, and induces a *magnetic* (resp., *electric*) field.

Let us investigate the solutions of *magnetic type*. For $w^2 < 1$, it follows directly from (3.3.42) that

$$\int\limits_{Z(\vartheta_0)}^{Z(\vartheta)} \frac{dZ}{\sqrt{E_0 - \cos Z}} = \sqrt{2}\,\widetilde{\omega}\gamma\vartheta, \tag{3.3.43}$$

where $\gamma = \frac{1}{\sqrt{1-w^2}}$ and $\widetilde{\omega} = \pm 1$.

We consider equation (3.3.42) in the following three typical cases:

a) $E_0 = 1$ ($w^2 < 1$). Setting $E_0 = 1$ and $Z(\vartheta_0) = \pi$ in (3.3.43), we can verify that in this case equation (3.3.34) can be integrated exactly in elementary functions, yielding

$$Z(\vartheta) = 4\arctan\left[\exp\left(\widetilde{\omega}\gamma\vartheta\right)\right]. \tag{3.3.44}$$

The obtained solution (3.3.44) is recognized as the already familiar one-soliton solution of the sine-Gordon equation (provided we make the inverse change of variables $(x,t) \mapsto (u,v)$).

b) $E_0 > 1$ ($w^2 < 1$). Taking $Z(\vartheta_0) = \pi$, we recast (3.3.43) as

$$\int\limits_{\pi}^{Z(\vartheta)} \frac{dZ}{\sqrt{A + \sin^2(Z/2)}} = \sqrt{2}\,\widetilde{\omega}\gamma\vartheta, \tag{3.3.45}$$

where $A = (E_0 - 1)/2$, $0 \le A \le 1$.

If we introduce the new variable τ and parameters k and k_1 by the formulas

$$\tau = \cos\frac{Z}{2}, \quad A = \frac{k_1^2}{k^2}, \quad k_1^2 = 1 - k^2, \quad 0 \le k \le 1,$$

equation (3.3.45) reduces to

$$\int\limits_{0}^{\cos(Z/2)} \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = -\frac{1}{k}\widetilde{\omega}\gamma\vartheta. \tag{3.3.46}$$

To describe further the solutions, we use the Jacobi elliptic functions $\mathrm{sn}(y,k)$, $\mathrm{cn}(y,k)$, and $\mathrm{dn}(y,k)$, which depend on the variable y and the modulus k, with $0 \le k \le 1$. Each of the listed Jacobi elliptic function is defined by inverting the corresponding elliptic integral:

$$
\begin{aligned}
y &= \int_0^{\mathrm{sn}(y,k)} \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}},\\
y &= \int_0^{\mathrm{cn}(y,k)} \frac{d\tau}{\sqrt{(1-\tau^2)(k_1^2+k^2\tau^2)}},\\
y &= \int_0^{\mathrm{dn}(y,k)} \frac{d\tau}{\sqrt{(1-\tau^2)(\tau^2-k_1^2)}}.
\end{aligned}
\tag{3.3.47}
$$

The parameter $k_1 = \sqrt{1-k^2}$ is called the *complementary modulus* of the elliptic function.

In accordance with (3.3.47), equation (3.3.46) yields the solution corresponding to the case under consideration:

$$Z(\vartheta) = 2\widetilde{\omega}\arcsin[\mathrm{sn}(y,k)] = \pi, \quad y = \frac{1}{k}\gamma\vartheta. \tag{3.3.48}$$

c) $|E_0| < 1$ $(w^2 < 1)$. Completely analogous arguments lead to the following solution of equation (3.3.41):

$$Z(\vartheta) = 2\arcsin[\mathrm{dn}(y,k)] = \pi, \quad y = \gamma\vartheta. \tag{3.3.49}$$

The study of the solutions of *electric* type is qualitatively the same as that of the preceding case, so we only list the resulting solutions of this type:

a) $E_0 = -1$ $(w^2 > 1)$:

$$Z(\vartheta) = 4\arctan\left[\exp\left(\widetilde{\omega}\widetilde{\gamma}\vartheta\right)\right] - \pi, \tag{3.3.50}$$

where $\widetilde{\gamma} = 1/\sqrt{w^2-1}$;

b) $E_0 < -1$ $(w^2 > 1)$:

$$Z(\vartheta) = 2\widetilde{\omega}\arcsin[\mathrm{sn}(y,k)], \quad y = \widetilde{\gamma}\vartheta; \tag{3.3.51}$$

c) $|E_0| < 1$ $(w^2 > 1)$:

$$Z(\vartheta) = 2\arccos[\mathrm{dn}(y,k)], \quad y = \widetilde{\gamma}\vartheta. \tag{3.3.52}$$

Single-phase solutions are remarkable, in particular, because among them one finds unbounded solutions of the sine-Gordon equation; this fact suggests that in $\mathbb{E}^3$ can live pseudospherical surfaces with a countable number of irregular singularities. An example of unbounded solution is (3.3.48) (with periodicity accounted for); indeed, looking at the solution (3.3.48) ($E_0 > 1$, $w^2 < 1$) for $w = 1$ we can verify that its derivative is positive: $Z'_\vartheta > 0$. Therefore, this solution increases monotonically and unboundedly, because the right-hand side in (3.3.42) is periodic in Z.

In the case of the newly derived solutions (3.3.48)–(3.3.52), the method proposed in Subsection 3.3.1 for constructing the corresponding pseudospherical surface is in principle applicable. But due to the specificity of the solutions (3.3.48)–(3.3.52), namely, the fact that they are transcendental elliptic functions, the implementation of the method may involve quadratures of those special functions. For this reason, its complete implementation requires the study of additional properties of the Jacobi functions and is a problem interesting in its own right. In our exposition we confine ourselves to discussing the geometric interpretation of stationary traveling waves in the Lobachevsky plane Λ^2.

From a qualitative point of view, the study of the geometric interpretation of the solutions (3.3.48)–(3.3.52) repeats the analogous considerations for the one-soliton solution $\omega_1(u, v)$ in § 3.2. Moreover, we note that from the form of the solutions (3.3.48)–(3.3.52) it follows, upon using the inverse change of variables $(x, t) \mapsto (u, v)$ (see (3.3.39)), that in the (u, v)-parameter plane the level lines $Z = m\pi$ (m an integer) of the solutions under consideration are straight lines, the geometric curvature of which with respect to the Chebyshev metric of the form (3.1.1) can be calculated by formula (2.7.72) and turns out to be *constant* (for solutions of all types). This gives a type of lines, namely, the boundaries of domains in the plane Λ^2 that can be isometrically immersed in $\mathbb{E}^3$. An analysis of the domains in Λ^2 that arise in the geometric interpretation of the solutions of stationary traveling wave type (single-phase solutions) was carried out for the first time in [68]. Among the domains obtained therein one can list the *non-Euclidean annulus*, the *domain lying between two horocycles* with a common point at infinity, and the *equidistant strip*.

The inverse problem of recovering the solutions of the sine-Gordon equation (as the net angle of the Chebyshev net of asymptotic lines) from a given pseudospherical surface in $\mathbb{E}^3$ was studied in [20]. Therein it was established that the aforementioned net angles on the classical pseudospherical surfaces of revolution are "connected" with functions that are reducible to the form (3.3.44) and (3.3.48)–(3.3.52).

Therefore, one can definitely say that the pseudosphere, the Minding "bobbin" and "top", and the Dini surface provide exact "geometric images" of the class of single-phase solutions of the sine-Gordon equation.

In connection with the discussion above, let us also state also a theorem proved in [68]:

Suppose that in $\mathbb{E}^3$ there is given a pseudospherical surface $S[\overrightarrow{r}(u, v)]$, which according to E. G. Poznyak's theorem (*Theorem* 2.7.1) *corresponds to some regular solution $\omega(u, v)$ of the sine-Gordon equation defined in the whole plane $\mathbb{R}^2(u, v)$, in which one takes as coordinate lines the asymptotic lines u, v. Suppose that about* *this surface one knows that on it two arbitrary asymptotic lines of the same family can be superposed by a motion. Then the solution $\omega(u,v)$ (or, correspondingly, $Z(x,t)$), is of stationary traveling wave type.*

Conversely, to solutions of the type (3.3.40) *correspond pseudospherical surfaces on which any two asymptotic lines from the same family are congruent.*

Let us note that the property that any two asymptotic lines from the same family on a pseudospherical surface can be superposed exactly is a characteristic property of pseudospherical helical surfaces and surfaces of revolution.

3.4 Two-soliton pseudospherical surfaces

The implementation of the Bäcklund transformation algorithm (see Subsection 3.1.2) for the construction of new pseudospherical surfaces from already known ones is intimately connected with the accompanying Bäcklund transformation (3.1.28) for solutions of the sine-Gordon equation. Moreover, transition to any new (next) layer (or level) of the Bäcklund transformation and generation of a new pair "(pseudospherical surface, solution of the sine-Gordon equation)" starts by the construction of a "new" solution of the sine-Gordon equation according to the system (3.1.28), and only then is completed by producing the new pseudospherical surface itself according to the recipe (3.1.31) (see the scheme in Figure 3.1.3). The solvability of the system (3.1.28) is the primary and necessary condition for the subsequent "full deployment" of the geometric algorithm under consideration. As we already mentioned, in general the construction of new solutions ω^* of the system (3.1.28) from an arbitrarily given solution ω remains an unsolved problem.[4] However, the *hierarchy of multi-soliton solutions* (3.2.13) predicted in § 3.2 (a separate class of physically meaningful solutions of the sine-Gordon equation) *can be fully studied geometrically* in the framework of the Bäcklund transformation scheme. Indeed, in § 3.3 we provided the explicit formulas (3.3.31) and (3.3.32), giving the Dini surface and the pseudosphere, which correspond to the one-soliton solution ω_1 (3.2.11) of the sine-Gordon equation. That is, for the class of multi-soliton solutions we already have a completely occupied first ("priming") layer of the Bäcklund transformation (an explicitly given "surface–solution" pair: "$(S[\overrightarrow{r}(u,v),\omega_1],\omega_1(u,v))$". Further "advance" along the "chain" of multi-soliton solutions enables us to obtain, by the rule (3.1.31), the corresponding *multi-soliton pseudospherical surfaces*. The first iteration of the Bäcklund transformation in this approach is connected with the construction of *two-soliton pseudospherical surfaces*, which interpret geometrically the two-soliton solutions ω_2 of the sine-Gordon equation, given by formula (3.2.12) [93].

3.4.1 Geometric study of two-soliton solutions

We will classify two-soliton pseudospherical surfaces according to the character of the behavior of their irregular singularities (such singularities may manifest as *irregular edges* (*cuspidal edges*), *wedge points*, and so on) which, as we recall,

[4] Undoubtedly, to obtain certain preliminary results in the investigation of the system (3.1.28) one can use numerical methods for the integration of nonlinear differential equations.

according to D. Hilbert's result on the nonimmersibility of the Lobachevsky plane Λ^2 in $\mathbb{E}^3$ and its refinement for the case of pseudospherical surfaces provided by E. G. Poznyak's theorem (see §§ 2.6 and 2.7), correspond to level lines $z(u,v) = n\pi$ (with n an integer) of solutions of the sine-Gordon equation.[5] The two-soliton solution in question $z_2(u,v)$ of the form (3.2.12) has singularities that correspond to three possible values: $z_2(u,v) = 0$, $z_2(u,v) = \pi$, and $z_2(u,v) = -\pi$. Let us study the geometric characteristics of these singularities (their geodesic curvature and their torsion). To this end we pose the problem of describing the irregular singularities in parametric form.

Let us perform in (3.2.12) the change of variables[6]

$$\begin{aligned} &\left(u, v, \omega_2^{(1,2)}\right) \mapsto \left(x, t, z_2^{(1,2)}\right), \\ &\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ k_1 k_2 & k_1 k_2 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}, \\ &\omega_2^{(1,2)} \equiv z_2^{(1,2)}. \end{aligned} \tag{3.4.1}$$

(with no loss of generality we put $b_1 = 0$, $b_2 = 0$ in (3.2.12).)

As a result of transformation (3.4.1), the two-soliton solution (3.2.12) takes on the form

$$z_2^{(1,2)}(x,t) = -4 \arctan\left(\frac{k_1 + k_2}{k_1 - k_2} \cdot \frac{\sinh((k_1 - k_2)x)}{\cosh((k_1 + k_2)t)} \right). \tag{3.4.2}$$

The formula (3.4.2) of two-soliton solutions allows us immediately to find explicitly, in the variables x and t, the expressions for the level lines we are interested in. Passing back to the variables u, v via (3.4.1) and using expression (3.4.2) for the multi-soliton solution $z^{(1,2)}$ we arrive at the following parametric expressions for the level sets $z^{(1,2)} = n\pi$, $n = 0, \pm 1$:

1) *Level line* $z^{(1,2)} = 0$:

$$\begin{aligned} u(t) &= t, \\ v(t) &= k_1 k_2 t. \end{aligned} \tag{3.4.3}$$

.

2) *Level line* $z^{(1,2)} = \pm\pi$:

$$\begin{aligned} u(t) &= \pm \frac{1}{k_1 - k_2} \operatorname{arcsinh}\left(\frac{k_1 - k_2}{k_1 + k_2} \cdot \cosh((k_1 + k_2)t \right), \\ v(t) &= \mp \frac{k_1 k_2}{k_1 - k_2} \operatorname{arcsinh}\left(\frac{k_1 - k_2}{k_1 + k_2} \cdot \cosh((k_1 + k_2)t \right) + k_1 k_2 t. \end{aligned} \tag{3.4.4}$$

[5] In what follows for the solutions of the sine-Gordon equation we use the notation $\omega \equiv z$.

[6] The variables x, t introduced here differ from the analogous variables in the "wave-type" sine-Gordon equation (3.2.31) only through the scale factor $k_1 k_2$, and geometrically will correspond to curvature lines on the surface.

In the sequel we will need expressions for the derivatives z_u and z_v that appear in the formulas for geodesic curvature and torsion, in the variables x and t. To this end let us first write down the relation between the differentiation operators

$$\begin{pmatrix} \dfrac{\partial}{\partial u} \\ \dfrac{\partial}{\partial v} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ \dfrac{1}{k_1 k_2} & \dfrac{1}{k_1 k_2} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial t} \end{pmatrix}. \tag{3.4.5}$$

Applying (3.4.5) to calculate of the derivatives of the solution $z_2^{(1,2)}$ with respect to u and v, we obtain their expressions in the variables x and t:

$$(z_2^{(1,2)})_u = 2\frac{(k_1+k_2)\frac{\cosh((k_1-k_2)x)}{\cosh((k_1+k_2)t)} + \frac{(k_1+k_2)^2}{k_1-k_2} \cdot \frac{\sinh((k_1-k_2)x)\sinh((k_1+k_2)t)}{\cosh^2((k_1+k_2)t)}}{1+\left(\frac{k_1+k_2}{k_1-k_2} \cdot \frac{\sinh((k_1-k_2)x)}{\cosh((k_1+k_2)t)}\right)^2}, \tag{3.4.6}$$

$$(z_2^{(1,2)})_v = -\frac{2}{k_1 k_2}\frac{(k_1+k_2)\frac{\cosh((k_1-k_2)x)}{\cosh((k_1+k_2)t)} - \frac{(k_1+k_2)^2}{k_1-k_2} \cdot \frac{\sinh((k_1-k_2)x)\sinh((k_1+k_2)t)}{\cosh^2((k_1+k_2)t)}}{1+\left(\frac{k_1+k_2}{k_1-k_2} \cdot \frac{\sinh((k_1-k_2)x)}{\cosh((k_1+k_2)t)}\right)^2}. \tag{3.4.7}$$

Now let us return to the original task of calculating the geodesic curvature $k_{\mathrm{g}}(z_2 = n\pi)$ and the torsion $æ(z_2 = n\pi)$ of irregular edges of the pseudospherical surface $S[z_2]$ that provides the geometric interpretation of the two-soliton solution z_2 of the sine-Gordon equation.[7] We use the already available formulas (2.7.72) and (2.7.81), in the two possible cases:

1) *Level line* $z^{(1,2)} = 0$ $(n = 0)$:

$$k_{\mathrm{g}}(z_2 = 0) = \left.\frac{(z_2)_u \cdot (z_2)_v}{(z_2)_u - (z_2)_v}\right|_{z_2=0}, \tag{3.4.8}$$

$$æ(z_2 = 0) = \left.\frac{(z_2)_u + (z_2)_v}{(z_2)_u - (z_2)_v}\right|_{z_2=0}. \tag{3.4.9}$$

As follows from (3.4.2), the level line $z_2 = 0$ is given by the condition

$$\sinh((k_1 - k_2)x) = 0. \tag{3.4.10}$$

By (3.4.10), in the variables x, t the derivatives $(z_2)_u$ and $(z_2)_v$, calculated in (3.4.6) and (3.4.7) and figuring in expressions (3.4.8) and (3.4.9), take the following form:

$$\begin{aligned} (z_2^{(1,2)})_u\Big|_{z_2=0} &= \frac{2(k_1+k_2)}{\cosh((k+1+k_2)t)}, \\ (z_2^{(1,2)})_v\Big|_{z_2=0} &= -\frac{1}{k_1k_2}\frac{2(k_1+k_2)}{\cosh((k+1+k_2)t)}. \end{aligned} \tag{3.4.11}$$

[7] Here and in what follows, for the sake of simplicity, we will denote the two-soliton solution $z_2^{(1,2)}$, which includes the two parameters k_1 and k_2, also by z: $z_2^{(1,2)} \equiv z_2$.

The variable t will be regarded here as a parameter. Introducing instead of t the new parameter τ by

$$\tau = \frac{\cosh((k_1 + k_2)t)}{k_1 + k_2},$$

we recast (3.4.11) as

$$\begin{aligned} (z_2^{(1,2)})_u\Big|_{z_2=0} &= \frac{2}{\tau}, \\ (z_2^{(1,2)})_v\Big|_{z_2=0} &= -\frac{1}{k_1 k_2}\frac{2}{\tau}. \end{aligned} \tag{3.4.12}$$

Using (3.4.12) we finally obtain, by (3.4.8) and (3.4.9), the formulas

$$k_{\rm g}(z_2 = 0) = -\frac{1}{1 + k_1 k_2} \cdot \frac{2}{\tau}, \tag{3.4.13}$$

$$æ(z_2 = 0) = \frac{k_1 k_2 - 1}{k_1 k_2 + 1}. \tag{3.4.14}$$

2) *Level lines* $z_2^{(1,2)} = \pm\pi$ $(n = \pm 1)$.

The geodesic curvature and torsion of the corresponding irregular edges of the pseudospherical surface are calculated by means of formulas (2.7.72) and (2.7.81), and in the present case are given by the following expressions (in which for convenience we denote $z_2^{(1,2)} \equiv z_2$)

$$k_{\rm g}(z_2 = \pm\pi) = \frac{(z_2)_u \cdot (z_2)_v}{(z_2)_u + (z_2)_v}\bigg|_{z_2=\pm\pi}, \tag{3.4.15}$$

$$æ(z_2 = \pm\pi) = \frac{(z_2)_u - (z_2)_v}{(z_2)_u + (z_2)_v}\bigg|_{z_2=\pm\pi}. \tag{3.4.16}$$

Using (3.4.2), we write the relation that give the lines $z_2 = \pm\pi$:

$$\frac{k_1 + k_2}{k_1 - k_2} \cdot \frac{\sinh((k_1 - k_2)x)}{\cosh((k_1 + k_2)t)} = \mp 1. \tag{3.4.17}$$

Next, using the relations (3.4.6), (3.4.7) and (3.4.17), we calculate the functions $(z_2)_u\big|_{z_2=\pm\pi}$ and $(z_2)_v\big|_{z_2=\pm\pi}$, which figure in formulas (3.4.15) and (3.4.16), presenting the results in term of the parameter τ:

$$\begin{aligned} (z_2)_u\Big|_{z_2=\pm\pi} &= \frac{\sqrt{(k_1 - k_2)^2\tau^2 + 1} \mp \sqrt{(k_1 + k_2)^2\tau^2 - 1}}{\tau}, \\ (z_2)_v\Big|_{z_2=\pm\pi} &= -\frac{1}{k_1 k_2}\frac{\sqrt{(k_1 - k_2)^2\tau^2 + 1} \pm \sqrt{(k_1 + k_2)^2\tau^2 - 1}}{\tau}. \end{aligned}$$

Finally, substituting these expression in (3.4.15) and (3.4.16) we arrive at the formulas

$$k_{\mathrm{g}}(z_2 = \pm\pi) = \frac{2}{\tau}\frac{2k_1k_2\tau^2 - 1}{(k_1k_2-1)\sqrt{(k_1-k_2)^2\tau^2+1} \mp (k_1k_2+1)\sqrt{(k_1+k_2)^2\tau^2-1}}, \tag{3.4.18}$$

$$æ(z_2 = \pm\pi) = \frac{(k_1k_2+1)\sqrt{(k_1-k_2)^2\tau^2+1} \mp (k_1k_2-1)\sqrt{(k_1+k_2)^2\tau^2-1}}{(k_1k_2-1)\sqrt{(k_1-k_2)^2\tau^2+1} \mp (k_1k_2+1)\sqrt{(k_1+k_2)^2\tau^2-1}}. \tag{3.4.19}$$

The expressions (3.4.13), (3.4.14) and (3.4.18), (3.4.19) obtained above describe the basic characteristics (geodesic curvature and torsion) of the irregular edges of two-soliton pseudospherical surfaces. The explicit form of expressions (3.4.18) and (3.4.19) allows one to directly classify the surfaces $S[z_2]$ under study according to the specific structure of their singularities. "Indicators" of how the nature of the behavior of *irregular edges* changes qualitatively (transition from one typical "segment" to another) are provided by the critical points[8] τ_i of the functions in the right-hand sides of (3.4.18) and (3.4.19). Specifically, these points give intervals of sign constancy and, possibly, the singularities of the functions $k_{\mathrm{g}}(z_2 = \pm\pi)$ and $æ(z_2 = \pm\pi)$ in dependence on the mutual placement[9] on the parametric τ-axis, which in turn is determined by the values of the parameters k_1 and k_2 which figure in the soliton solution z_2.

The results of the detailed "technical analysis" of the behavior of singularities carried out above are presented in the following subsection in the summarizing Table 3.4.1 and reveal eight different variants of the structure of two-soliton pseudospherical surfaces. Here, in order to help the contemplative reader, we give the values for the critical points.

For the function $k_{\mathrm{g}}(z_2 = \pm\pi)$:

$$\tau_0 = 0,$$
$$\tau_{1,2} = \pm\sqrt{\frac{1+(k_1k_2)^2}{2k_1k_2(1+k_1^2)(1+k_2^2)}},$$
$$\tau_{3,4} = \pm\frac{1}{|k_1+k_2|},$$
$$\tau_{5,6} = \pm\sqrt{\frac{1}{2k_1k_2}}.$$

For the function $\tau(z_2 = \pm\pi)$:

$$\tau_{1,2} = \pm\sqrt{\frac{1+(k_1k_2)^2}{2k_1k_2(1+k_1^2)(1+k_2^2)}},$$

[8] The zeros of the numerators and denominators in (3.4.18) and (3.4.19).

[9] Assuming that the critical point itself does exist for the parameter values k_1 and k_2.

$$\tau_{3,4} = \pm\frac{1}{|k_1 + k_2|},$$
$$\tau_{5,6} = \pm\sqrt{\frac{1 + (k_1 k_2)^2}{2k_1 k_2(1 - k_1^2)(1 + k_2^2)}}.$$

3.4.2 "Gallery" of two-soliton pseudospherical surfaces

Let us analyze the space structure of the irregular edges of two-soliton pseudospherical surfaces as curves in the Euclidean space $\mathbb{E}^3(x, y, z)$, resorting to their geometric characteristics derived above.

We list below the general conclusions on the character of the behavior of irregular edges of a two-soliton pseudospherical surface:

- When $s \to \pm\infty$ (where s is a natural parameter, $s \sim \tau$), the edges of the surface wind asymptotically around the Oz-coordinate axis (have the character of a helical motion around Oz).
- When $s \to \pm\infty$ the edge $z_2 = 0$ approaches asymptotically the Oz-axis
- The edges $z_2 = \pi$ and $z_2 = -\pi$ have the property of spatial central symmetry.
- When $s \to \pm\infty$, the edges $z_2 = \pi$ and $z_2 = -\pi$ "degenerate" into helical lines that serve as the edges of the corresponding Dini surface with parameters k_1 (when $s \to +\infty$ and k_2 (when $s \to -\infty$), or vice versa.

We will classify the surfaces under consideration according to the possible mutual placement on the τ parametric axis of the critical points τ_i, i.e., the points where the sign of the geodesic curvature and torsion of the irregular edges on the surface changes (the "zeros" of the numerators and denominators in (3.4.13), (3.4.14), (3.4.18), and (3.4.19)).

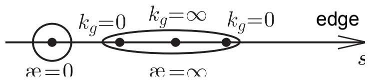
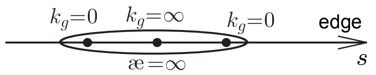
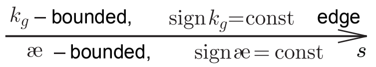
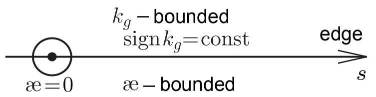
Here it is important to indicate the following geometrically typical cases.

1) If on an edge there is a point where $æ = 0$ (i.e., a point where the torsion changes sign), then at that point the edge changes the direction of its helical motion: right helical motion turn into left, or vice versa.

2) An edge may contain an arc with a cusp (turning) point, in which $k_{\mathrm{g}} = \infty$ and $æ = \infty$ simultaneously. On the two opposite sides with respect to such a point there are two point on the edge in which necessarily $k_{\mathrm{g}} = 0$.

In Table 3.4.1 we show four basic types of two-soliton pseudospherical surfaces with the important particular subcases indicated above. All the aforementioned specific features of the structure of edges of these surfaces are reflected by the corresponding position in the table. In turn, the table is illustrated by the "gallery" of two-soliton pseudospherical surfaces in figures 3.4.1–3.4.8, which includes all visually typical forms of surfaces with the accompanying configurations of irregular cuspidal edges.

Table 3.4.1. Two-soliton pseudospherical surfaces

<table>
<tr><th>Type</th><th>Conditions on parameters</th><th>Irregular edge</th><th>Mutual placement of typical points on the parameter axis</th><th>Subcase</th><th>Figure</th></tr>
<tr><td colspan="6">Solution z_2: "kink–kink" and "antikink–antikink" interaction ($k_1k_2 > 0$)</td></tr>
<tr><td rowspan="2">1</td><td rowspan="2">$k_1k_2 > 0$,
$(|k_1|-1)(|k_2|-1) > 0$</td><td>$z_2 = 0$</td><td>$æ = \mathrm{const} \neq 0$
$k_\mathrm{g} > 0$, k_g is bounded</td><td rowspan="2"></td><td rowspan="2">3.4.1</td></tr>
<tr><td>$z_2 = \pi$
($z_2 = -\pi$)</td><td></td></tr>
<tr><td rowspan="4">2</td><td rowspan="4">$k_1k_2 > 0$,
$(|k_1|-1)(|k_2|-1) \leqslant 0$</td><td rowspan="3">$z_2 = 0$</td><td rowspan="3">$æ = \mathrm{const}$</td><td>$|k_1| = 1$ ($|k_2| = 1$)</td><td>3.4.2</td></tr>
<tr><td>$k_1k_2 \neq 1, |k_1| \neq 1, |k_2| \neq 1$</td><td>3.4.3</td></tr>
<tr><td>$k_1k_2 = 1$</td><td>3.4.4</td></tr>
<tr><td>$z_2 = \pi$
($z_2 = -\pi$)</td><td></td><td></td><td>3.4.2–3.4.4</td></tr>
<tr><td colspan="6">Solution z_2: "kink–antikink" interaction ($k_1k_2 < 0$)</td></tr>
<tr><td rowspan="3">3</td><td rowspan="3">$k_1k_2 < 0$,
$(|k_1|-1)(|k_2|-1) \geqslant 0$</td><td rowspan="2">$z_2 = 0$</td><td rowspan="2">$æ = \mathrm{const} \neq 0$
$k_\mathrm{g} > 0$, k_g is bounded</td><td>$|k_1| \neq 1$ ($|k_2| \neq 1$)</td><td>3.4.5</td></tr>
<tr><td>$|k_1| = 1$ ($|k_2| = 1$)</td><td>3.4.6</td></tr>
<tr><td>$z_2 = \pi$
($z_2 = -\pi$)</td><td></td><td></td><td>3.4.5–3.4.6</td></tr>
<tr><td rowspan="3">4</td><td rowspan="3">$k_1k_2 < 0$,
$(|k_1|-1)(|k_2|-1) < 0$</td><td rowspan="2">$z_2 = 0$</td><td>k_g is bounded, $k_\mathrm{g} > 0$
$æ = \mathrm{const} \neq 0$</td><td>$k_1k_2 \neq -1$</td><td>3.4.7</td></tr>
<tr><td>$z_2 = 0$ degenerates into a "peak point"</td><td>$k_1k_2 = -1$</td><td>3.4.8</td></tr>
<tr><td>$z_2 = \pi$
($z_2 = -\pi$)</td><td></td><td></td><td>3.4.7–3.4.8</td></tr>
</table>

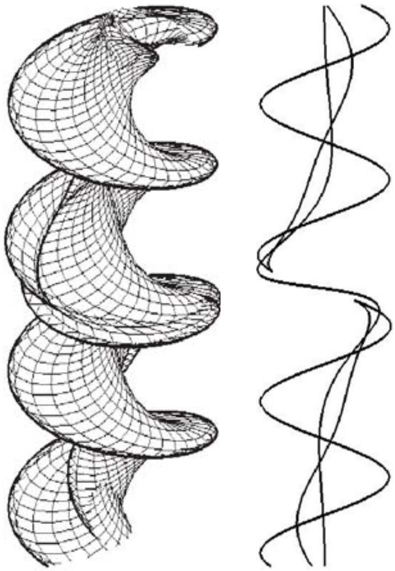


Figure 3.4.1

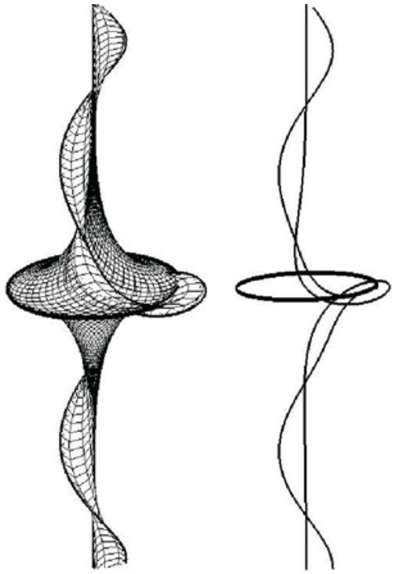


Figure 3.4.2

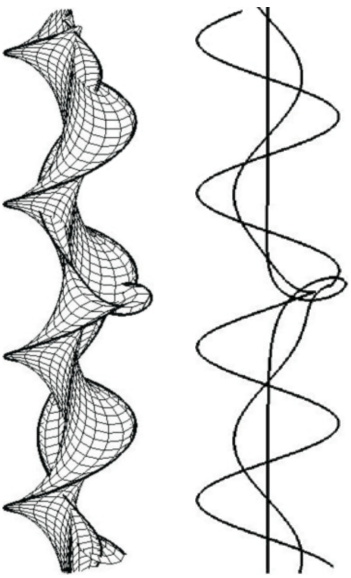


Figure 3.4.3

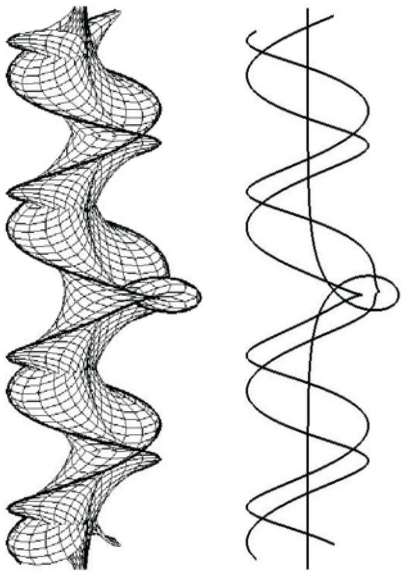


Figure 3.4.4

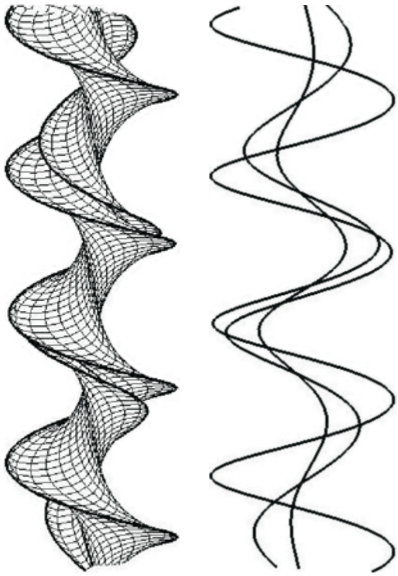


Figure 3.4.5

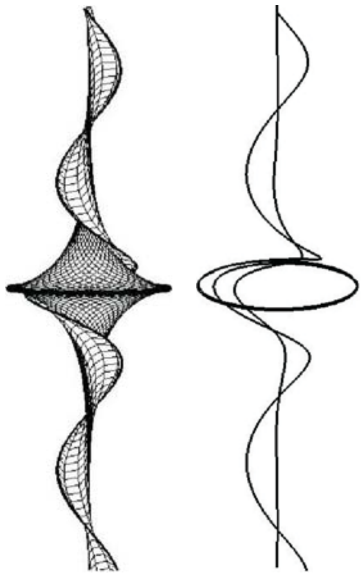


Figure 3.4.6

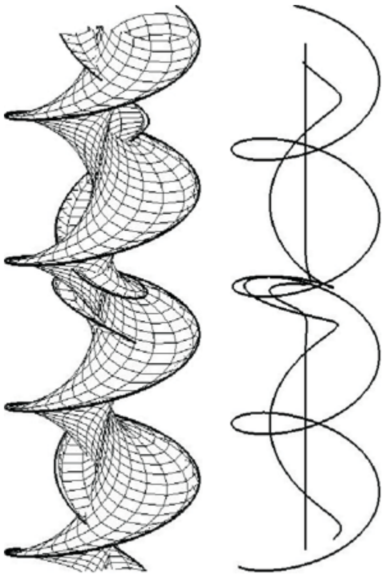


Figure 3.4.7

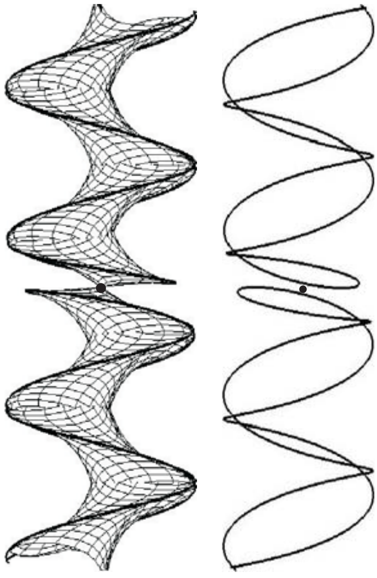


Figure 3.4.8

3.4.3 Breather pseudospherical surfaces

Special examples of pseudospherical surfaces are provided by the geometric interpretation of the breather solutions (3.2.29), which form a subclass of *periodic* solutions of the sine-Gordon equation, and which are obtained, together with the two-soliton solutions, by applying the Bäcklund transformation in the case when the transformation parameters k_1 and k_2 are complex-conjugate: $k_{1,2} = a \pm ib$, a, $b = \text{const}$. Without loss of generality, it is convenient to consider a solution of the form (3.2.29) under the assumption that $a^2+b^2 = 1$;[10] in this case the solution has the form

$$z^{\text{breath}}(u, v) = 4 \arctan \left[\frac{a}{b} \cdot \frac{\sin b(u-v)}{\cosh a(u+v)} \right]. \tag{3.4.20}$$

Passing in (3.2.40) from the variables (u, v) to the variables (x, t) via the formulas (3.2.31) enables us to talk about the periodicity of the breather solution z^{breath} with respect to t:

$$z^{\text{breath}}(x, t) = 4 \arctan \left[\frac{a}{b} \cdot \frac{\sin bt}{\cosh ax} \right]. \tag{3.4.21}$$

By Theorem 2.7.1, to each given solution z^{breath} of class C^4 there correspond in $\mathbb{E}^3$ (in the sense of the geometric interpretation adopted in § 2.7) a pseudospherical surface of class C^2 ($\overrightarrow{r} \in C^2$). To construct such breather pseudospherical surfaces $S[z^{\text{breath}}]$, we resort to the geometric Bäcklund transformation (3.1.31), in which we pass to the the curvature line coordinates (x, t). In this new parametrization the Bäcklund transformation takes the form

$$\overrightarrow{r}^* = \overrightarrow{r} + \sin \xi \cdot \left[\frac{\cos \vartheta^*}{\cos \vartheta} \cdot \overrightarrow{r}_x - \frac{\sin \vartheta^*}{\sin \vartheta} \cdot \overrightarrow{r}_t \right], \tag{3.4.22}$$

where $\vartheta = \omega/2$, $\vartheta^* = \omega^*/2$, $\sin \xi = 2k/(1+k^2)$.

Henceforth, in order to preserve the traditional notations for the solutions of the sine-Gordon equation that appear in different sections we will keep in mind that

$$z \equiv \omega \equiv 2\vartheta;$$

each of these notations "carries" its own meaning "load".

Like the transformation (3.1.31), the Bäcklund transformation (3.4.22) effects the transition from a pseudospherical surface $S[\overrightarrow{r}, \vartheta(x, t)]$ to a new pseudospherical surface $S^*[\overrightarrow{r}^*, \vartheta^*(x, t)]$ in the variables (x, t), and with the numerical parameter k:

$$S^* = \mathbb{B}_k(S).$$

Now let us construct the breather pseudospherical surface S^{breath} corresponding to the breather solution $z^{\text{breath}}(x, t)$, depending on two complex-conjugate parameters k_1 and k_2: $k_{1,2} = a \pm ib$. The breather solution $z^{\text{breath}}(x, t)$ can be obtained in two ways, by formally applying the Bäcklund transformation with the

[10]Under this assumption k_1 and k_2 lie on the unit circle in the complex plane.

complex parameter to the one-soliton solutions $z_1^{(k_1)}$ and $z_1^{(k_2)}$, each of which depends on the second complex parameter, the complex-conjugate of the parameter of the transformation:

$$\begin{aligned} z_{12}^{\text{breath}} &= \mathbb{B}_{k_2}(z_1^{(k_1)}),\\ z_{21}^{\text{breath}} &= \mathbb{B}_{k_1}(z_1^{(k_2)}). \end{aligned} \tag{3.4.23}$$

Thanks to the commutativity of the Bianchi diagram, the right-hand sides in (3.4.23) coincide, and so

$$z_{12}^{\text{breath}} = z_{21}^{\text{breath}} \equiv z^{\text{breath}}.$$

Analogous commutativity relations hold also for the *transformation of solutions* of the sine-Gordon equation *accompanying* the *transformation of pseudospherical surfaces*. Therefore, we can write

$$\begin{aligned} \vec{r}_{12} &= \mathbb{B}_{k_2}(\vec{r}_1^{(k_1)}),\\ \vec{r}_{21} &= \mathbb{B}_{k_1}(\vec{r}_1^{(k_2)}). \end{aligned} \tag{3.4.24}$$

In (3.4.24), $\vec{r}_1^{(k_1)}$ and $\vec{r}_1^{(k_2)}$ are the formal radius vectors of "hypothetic" surfaces that are given by formulas (3.3.31) and correspond to the one-soliton solutions $z_1(k_1)$ and $z_1(k_2)$, in which the numerical parameters k_1 and k_2 are complex-conjugate. Due to the commutativity of the Biachi diagram in the case of the Bäcklund transformation (3.4.22) for surfaces one has that

$$\vec{r}_{12} = \vec{r}_{21} \equiv \vec{r}^{\text{breath}}(x,t). \tag{3.4.25}$$

In view of (3.4.25), the radius vector $\vec{r}^{\text{breath}}$ of the pseudospherical surface $S^{\text{breath}}[\vec{r}^{\text{breath}}, z^{\text{breath}}]$ can be constructed as

$$\vec{r}^{\text{breath}} = \frac{\vec{r}_{12} + \vec{r}_{21}}{2}. \tag{3.4.26}$$

Let us write the expression (3.4.26) for $\vec{r}^{\text{breath}}$ in detail, using for this purpose the Bäcklund transformation in the form (3.4.22):

$$\begin{aligned} \vec{r}^{\text{breath}}(x,t) = \frac{1}{2}\Bigg[\vec{r}_1 + \vec{r}_2 &+ \cos\vartheta_{12}\cdot\left(\sin\xi_2\cdot\frac{\vec{r}_{1,x}}{\cos\vartheta_1} + \sin\xi_1\cdot\frac{\vec{r}_{2,x}}{\cos\vartheta_2}\right)\\ &- \sin\vartheta_{12}\cdot\left(\sin\xi_2\cdot\frac{\vec{r}_{1,t}}{\sin\vartheta_1} + \sin\xi_1\cdot\frac{\vec{r}_{2,t}}{\sin\vartheta_2}\right)\Bigg]. \end{aligned} \tag{3.4.27}$$

In (3.4.27) we used the following traditional notations (which have an important geometrical meaning), applied for the description of pseudospherical surfaces in curvature line coordinates:

$$\vartheta_i = \frac{\omega_1^{(k_i)}}{2} = \frac{z_1^{(k_i)}}{2}, \quad \vartheta_{12} = \frac{\omega_2^{(k_1,k_2)}}{2} = \frac{z_2^{(k_1,k_2)}}{2}, \quad \sin\xi_i = \frac{2k_i}{1+k_i^2}, \quad i=1,2.$$

The vector functions $\overrightarrow{r_i}$, $i = 1, 2$ are the radius vectors of the "one-soliton" pseudospherical surfaces, defined by means of formula (3.3.31) for the complex parameters k_1 and k_2.

It is instructive that, despite the presence of complex parameters k_1 and k_2 in the right-hand side of (3.4.27) (which appear in $\sin\xi_i$, ϑ_i and $\overrightarrow{r_i}$, $i = 1, 2$), the vector function $\overrightarrow{r}^{\,\text{breath}}(x, t)$ itself is real valued. To convince ourselves that this is true, it suffices to verify the following "intermediate" relations:[11]

1) $\overrightarrow{r_2} = (\overline{\overrightarrow{r_1}})$, i.e. $(\overrightarrow{r_1} + \overrightarrow{r_2})$ is *real*,
2) $\sin\xi_1 = \sin\xi_2 = \frac{1}{a}$ is *real*,
3) $\vartheta_2 = \overline{\vartheta_1}$,
4) $\left(\dfrac{\overrightarrow{r}_{1,\,x}}{\cos\vartheta_1} + \dfrac{\overrightarrow{r}_{2,\,x}}{\cos\vartheta_2}\right)$ is *real*,
5) $\left(\dfrac{\overrightarrow{r}_{1,\,y}}{\sin\vartheta_1} + \dfrac{\overrightarrow{r}_{2,\,y}}{\sin\vartheta_2}\right)$ is *real*.

Consequently, if in accordance with (3.4.27) we choose for the initial radius vectors $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ the "hypothetical Dini surface" representations (3.3.31) with respective parameters k_1 and k_2, we finally obtain the real-valued function $\overrightarrow{r}^{\,\text{breath}}(x, t)$, i.e., the radius vector of the sought-for pseudospherical surface:

$$\overrightarrow{r}^{\,\text{breath}}(x,t) = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} + \frac{2b}{a} \cdot \frac{\sin bt \cosh ax}{b^2\cosh^2 ax + a^2 \sin^2 bt} \cdot \begin{pmatrix} \sin t \\ -\cos t \\ 0 \end{pmatrix}$$
$$+ \frac{2b^2}{a} \cdot \frac{\cosh ax}{b^2\cosh^2 ax + a^2 \sin^2 bt} \begin{pmatrix} \cos t \cdot \cos bt \\ \sin t \cdot \cos bt \\ -\sinh ax \end{pmatrix}, \qquad (3.4.28)$$

where $a^2 + b^2 = 1$.

From the explicit expression (3.4.28) of the obtained surface it is not difficult to deduce that the curvature lines $t = \text{const}$ are plane curves in space. The breather pseudospherical surface itself belong to the class of Joachimsthal–Enneper surfaces, a general discussion of which will be made in § 3.8.

Essentially, the general form of expression (3.4.28) makes it clear that the class of breather pseudospherical surfaces is parametrically "regulated" by a single parameter b, with $b \in (0, 1)$. Specifically, to each rational value of b in $(0, 1)$ there corresponds a breather pseudospherical surface $S[\overrightarrow{r}^{\,\text{breath}}; b]$, which is periodic in t. If one takes $b = m/n$, where m and n are coprime, $m < n$, then the period of the breather solutions, and correspondingly, of the space structure of the surface, will be equal to $2\pi n/m$.

Figures 3.4.9–3.4.12 show examples of breather pseudospherical surfaces for different rational values of the parameter b. Overall, the explicit formula (3.4.28),

[11] The overbar denotes complex conjugation.

which describes the breather pseudospherical surfaces, provides a key to obtaining "fantastic variations" of pseudospherical surfaces.

Let us mention here some works of contemporary authors devoted to the investigation of various problems connected with pseudospherical surfaces and their applications: [89, 151, 164, 173, 187].

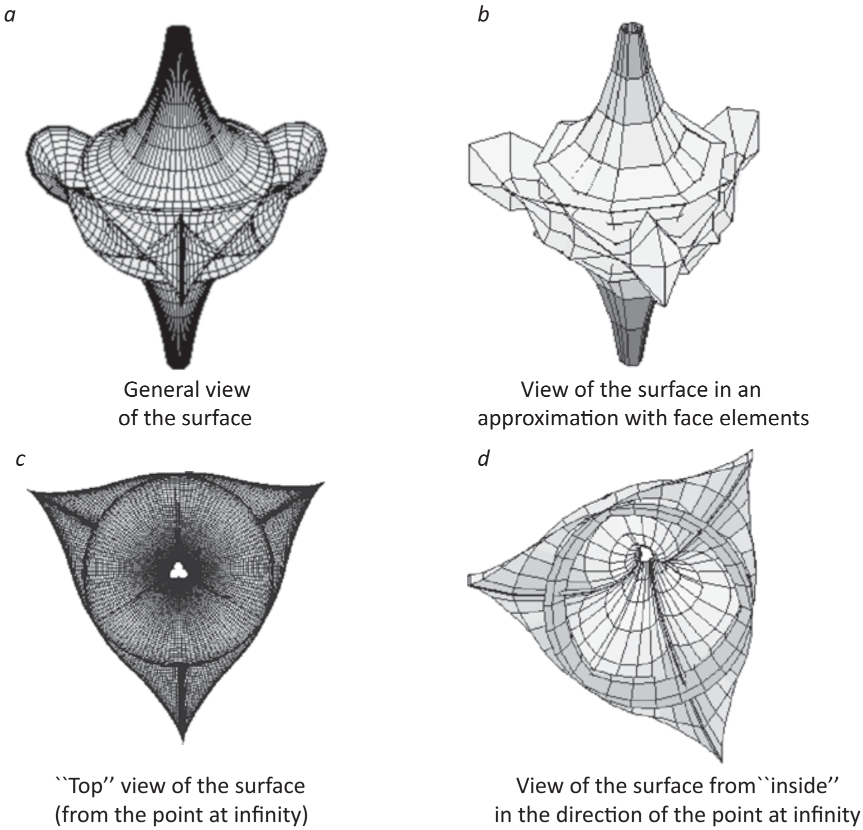


Figure 3.4.9. Breather surface; $b = \dfrac{3}{7}$.

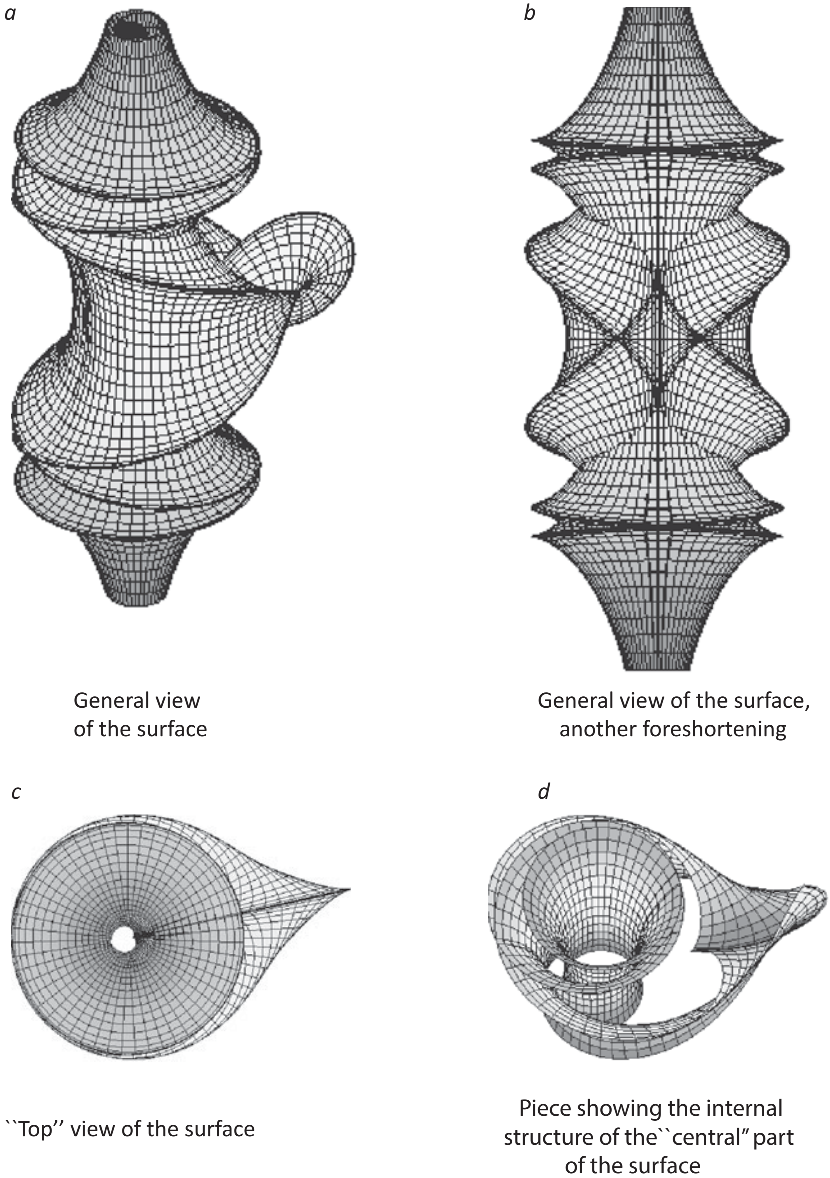


Figure 3.4.10. Breather surface; $b = \dfrac{1}{7}$.

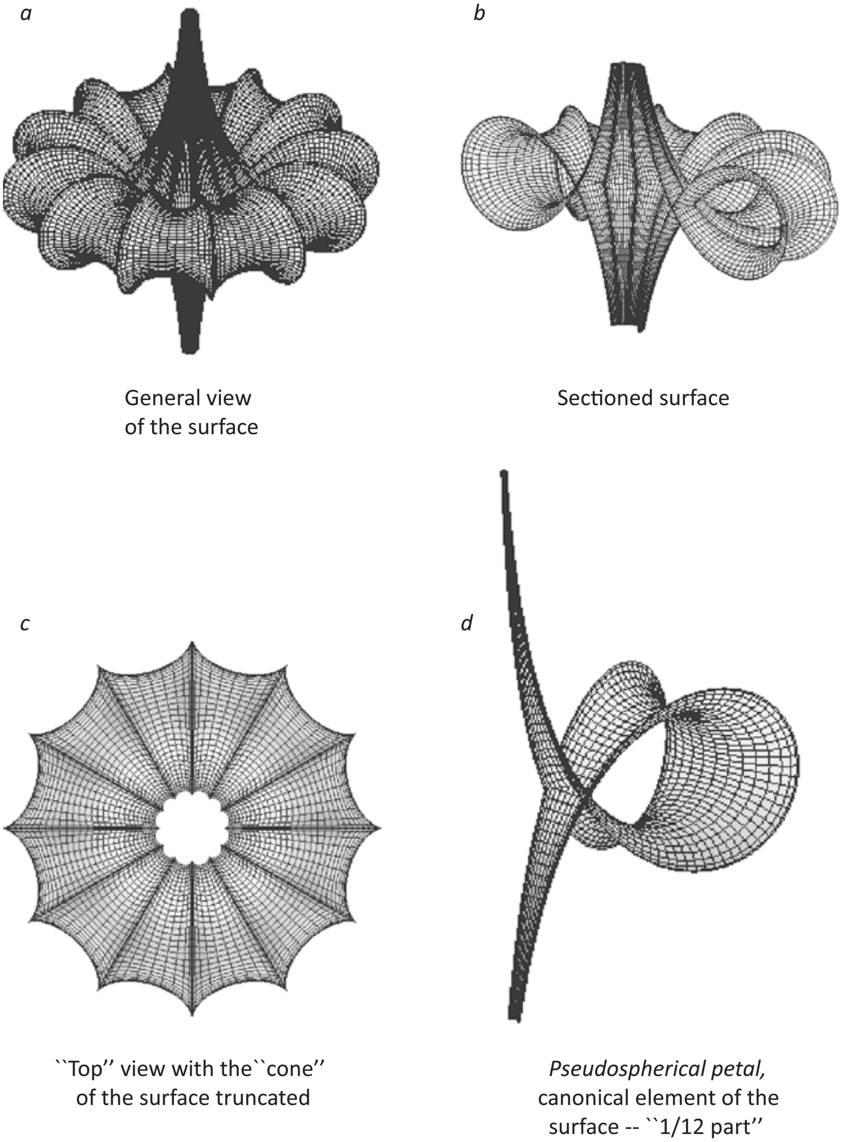
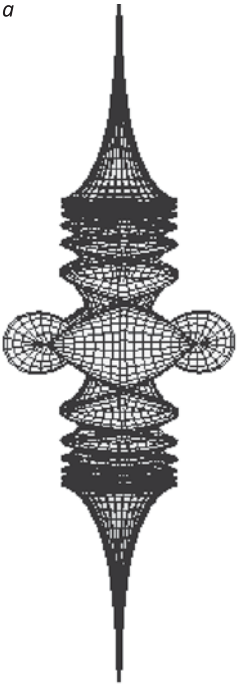


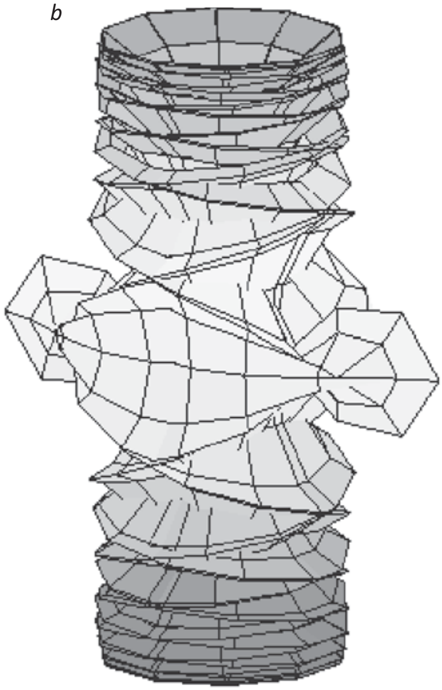
Figure 3.4.11. Breather surface; $b = \dfrac{6}{7}$.

a



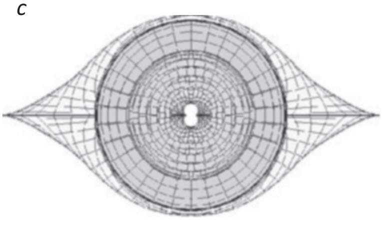
General view
of the surface

b



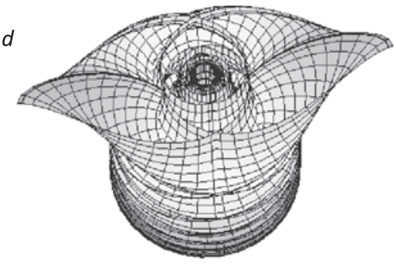
Structure of the ``central'' part of
the surface in an approximation
with face elements

c



``Top'' view of the surface
from the point at infinity

d



Piece showing the internal structure
of the ``central'' part of the surface

Figure 3.4.12. Breather surface; $b = \dfrac{1}{16}$.

3.5 The Amsler surface and Painlevé III transcendental functions

In this section we address the geometric interpretation of a class of self-similar solutions of the the sine-Gordon equation, which are *special functions* connected with the Painlevé III transcendental functions. The interest in such solutions is motivated, on one hand, by the fact that they are not solitons, and on the other, by the fact that for the moment no connections for them were revealed in the setting of the method of Bäcklund transformations. A typical *geometric representation* of a solution of this class from the point of view of Lobachevsky geometry is the *classical Amsler surface*.

3.5.1 The classical Amsler surface

Let us study solutions of the sine-Gordon equation

$$z_{uv} = \sin z, \tag{3.5.1}$$

which depend on the self-similar argument

$$t = uv. \tag{3.5.2}$$

Upon passing to the new variable (3.5.2), equation (3.5.1) becomes an ordinary differential equation for the unknown function $z(t)$, $t = uv$:

$$tz'' + z' = \sin z. \tag{3.5.3}$$

Note that if we now make the further complex changes of variables

$$w = e^{iz}, \tag{3.5.4}$$

where i is the imaginary unit, then equation (3.5.3) can be brought to the form

$$w'' - \frac{w'^2}{w} + \frac{2w' - w^2 + 1}{2t} = 0. \tag{3.5.5}$$

Equation (3.5.5) is one of the particular (but meaningful) cases of a more general ordinary differential equation, which gives the class of Painlevé III transcendental functions [2, 180]. A characteristic property of the indicated class of solutions is they have only *nonmovable critical points*, i.e., their *branch points* and *essential singular points* do not depend on the initial data that determine the solutions.

In 1955, M.-H. Amsler [135] published the results of his investigations of solutions of type $z(t)$ of the sine-Gordon equation in the framework of geometric analysis of pseudospherical surfaces that interpret a class of solutions of equation (3.5.1), namely, solutions that are related to the Painlevé III transcendental functions.

Amsler studied in detail the solution $z(t)$ of equation (3.5.3) that is regular at zero and satisfies the initial condition $z(0) = \pi/2$ and, correspondingly (see

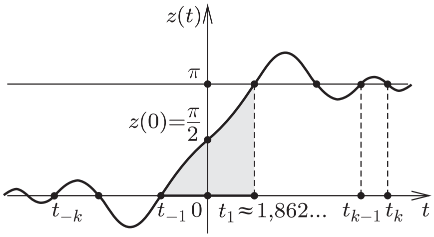


Figure 3.5.1

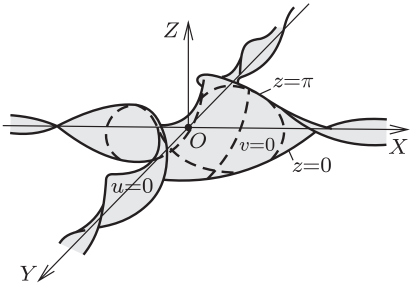


Figure 3.5.2

(3.5.3)), the condition $z'(0) = 1$. More precisely, in his work cited above, by applying numerical methods, Amsler investigated the *part of the solution* $z(t)$ between the first (closest to zero) value (attained for $t = t_{-1}$ (see Figure 3.5.1)) and the first (closest to zero) value $z = \pi$ (attained for $t = t_1$ (see Figure 3.5.1)). Numerically it was found that $t_1 = 1.862...$. Therefore, thanks to the symmetry of the problem under consideration, namely, $z(t) + z(-t) = \pi$, one also has $t_{-1} = -1.862...$. Thus, Amsler determined the *part of the solution* $z(t)$ contained between the two values closest to a multiple of π, which according to the theory of pseudospherical surfaces must correspond to a regular piece of the pseudospherical surface $S[\overrightarrow{r}(u, v), z(t), t = uv]$, bounded in space by two corresponding irregular cuspidal edges. This piece of the surface S ($t \in [-1.862..., 1.862...]$) was constructed by Amsler in [135] by the successive "plugging constructor" method from the asymptotic lines $v = \text{const}$. This regular piece of pseudospherical surface is depicted in Figure 3.5.2, and we will refer to it as the *classical Amsler surface*.

The main relation used in [135] to construct the surface under discussion is the equation for the normal vector $\overrightarrow{n}$ to the surface, considered on the asymptotic line $v = \text{const}$:

$$\overrightarrow{n}'' - z_u \cdot [\overrightarrow{n}', \overrightarrow{n}] + \overrightarrow{n} = 0. \tag{3.5.6}$$

(In equation (3.5.6), the derivatives are taken with respect to u.)

To obtain equation (3.5.6) we first need to recall the Frenet formulas for the surface strip (2.7.17) associated with the asymptotic line ($k_{\mathrm{g}} = -z_u$, $k_{\mathrm{n}} = 0$, $\tau_{\mathrm{g}} = 1$), and the relation $\overrightarrow{\tau} = [\overrightarrow{n}_{\mathrm{g}}, \overrightarrow{\mu}]$. Next, to construct the surface we use the formula $\overrightarrow{\tau} = -[\overrightarrow{\mu}', \overrightarrow{\mu}]$ to find the direction vector, from which by means of integration with respect to u we determine the radius vector of the asymptotic line (the parameter u is the natural coordinate on the asymptotic line $v = \text{const}$. To implement the "algorithm" just described, Amsler, in his investigations, did effectively use numerical methods for solving the differential equations involved.

The Amsler surface has an important special exceptional property which distinguishes it among all pseudospherical surfaces. Namely, it *completely includes* (*contains*) *two intersecting rectilinear generatrices* (*straight lines*), which coincide with the Cartesian coordinate axes OX and OY (see Figure 3.5.2).

Let us explain this property. Indeed, in the case of the considered self-similar solution $z(t)$, $t = uv$, of the sine-Gordon equation we turn to the fundamental system of equations for pseudospherical surfaces (2.7.30)–(2.7.34). Then taking into account the self-similarity type of the argument (3.5.2) for the radius vector $\overrightarrow{r}(u,v) \in C^3$ of the Amsler surface in the neighborhood of the set $\{(u,v)\colon\ uv = 0\}$, we can write

$$\begin{aligned} \overrightarrow{r}_{uu}(u,0) &= 0, \\ \overrightarrow{r}_{vv}(0,v) &= 0. \end{aligned} \tag{3.5.7}$$

Relations (3.5.7) express the equations (2.7.30), (2.7.32) of the fundamental system for the solution $z(t)$ with null initial data.

The integration of conditions (3.5.7) leads to the equations

$$\begin{aligned} \overrightarrow{r}(0,v) &= \overrightarrow{r}_v(0,0)\cdot v, \\ \overrightarrow{r}(u,0) &= \overrightarrow{r}_u(0,0)\cdot u, \end{aligned} \tag{3.5.8}$$

which give (since the vectors $\overrightarrow{r}_u(0,0)$ and $\overrightarrow{r}_v(0,0)$ are constant) two intersecting straight lines contained in the Amsler surface $S[\overrightarrow{r}(u,v),\ z(t),\ t = uv]$.

Note also that if a pseudospherical surface contains a straight line, then the latter is an asymptotic line on the surface. This clearly follows from the fact that the second differential $d^2\overrightarrow{r}$ vanishes in the direction $(du : dv)$ that gives the straight line. Consequently, the second fundamental form of the surface also vanishes in the direction $(du : dv)$: $\mathrm{II}(u,v) = (d^2\overrightarrow{r}, \overrightarrow{n}) = 0$. Therefore, the straight line present on the surface is an asymptotic line. With no loss of generality, we can consider that one of the intersecting straight lines on the Amsler surface is the asymptotic line $u = 0$, while the other is the asymptotic line $v = 0$. Note further that the geodesic curvature of an asymptotic line coincides with its total curvature and in the present case of "asymptotic straight lines" is equal to zero: $k_{\mathrm{g}}(u=0) = 0$, $k_{\mathrm{g}}(v=0) = 0$. As it was established in (2.7.40) and (2.7.42),

$$k_{\mathrm{g}}(u=0) = z_v(0,v), \quad k_{\mathrm{g}}(v=0) = -z_u(u,0),$$

and consequently

$$\begin{aligned} &z(0,v) = z(u,0) = z(0,0) = \text{const}, \\ &u \in (-\infty,+\infty), \quad v \in (-\infty,+\infty). \end{aligned} \tag{3.5.9}$$

Relations (3.5.9) specify, for the formulation of the Darboux problem for the self-similar solution $z(t)$ of argument (3.5.2), the conditions that guarantee the existence and uniqueness of this type of solution on the entire (u,v)-plane (for more details on the Darboux problem, see § 3.6).

Let us derive the equation for the radius vector of the Amsler surface. To this end we resort to the fundamental system of equations (2.7.30)–(2.7.34) of pseudospherical surfaces.

Preliminarily we compute

$$(z_u\overrightarrow{n})_v = z_{uv}\overrightarrow{n} + z_u\overrightarrow{n}_v = z_{uv}\overrightarrow{n} + \frac{v}{u}\,z_v\,\overrightarrow{n}_v, \tag{3.5.10}$$

where we used the property $uz_u = vz_v$ of the solution $z(t)$.

If we now use in (3.5.10) the equations (2.7.31) and (2.7.32), we get

$$\begin{aligned} z_{uv}\vec{n} + \frac{v}{u}\, z_v\, \vec{n}_v &= \vec{r}_{uv} + \frac{v}{u}\vec{r}_{vv} \\ &= \vec{r}_{uv} + \left(\frac{v}{u}\vec{r}_{vv} + \frac{1}{u}\vec{r}_v\right) - \frac{1}{u}\vec{r}_v = \left(\vec{r}_u + \frac{v}{u}\vec{r}_v - \frac{1}{u}\vec{r}\right)_v . \end{aligned} \tag{3.5.11}$$

In conclusion, relations (3.5.10) and (3.5.11) yield

$$(z_u\vec{n})_v = \left(\vec{r}_u + \frac{v}{u}\vec{r}_v - \frac{1}{u}\vec{r}\right)_v . \tag{3.5.12}$$

Integrating this last equality we get

$$z_u\vec{n} = \vec{r}_u + \frac{v}{u}\vec{r}_v - \frac{1}{u}\vec{r} + \vec{C}(u). \tag{3.5.13}$$

Similar arguments (repeating (3.5.10)–(3.5.13)) for the expression $(z_v\vec{n})_u$ and comparison of their result with (3.5.13) show that the resulting vector $\vec{C} \equiv 0$. Consequently, equation (3.5.13) can be recast in the more precise form

$$u \cdot z_u\vec{n} = u\vec{r}_u + v\vec{r}_v - \vec{r}. \tag{3.5.14}$$

Next, let us transform the left-hand side of (3.5.14) by using equation (2.7.31). In this way we arrive at the equation for the radius vector $\vec{r}(u,v)$ of the Amsler surface $S[\vec{r}(u,v),\ z(t),\ t = uv]$:

$$(tz')\vec{r}_{uv} - (u\vec{r}_u + v\vec{r}_v - \vec{r}) \cdot \sin z = 0, \quad t = uv. \tag{3.5.15}$$

Equation (3.5.15) represents the main relation to be used in a "direct" (chiefly *numerical*) investigation of the Amsler surface.

Among the results of the study of the classical Amsler surface we mention the work [161], which deals with a discrete analog, as well as the [147], in which the Amsler surface is considered in the context of the method of the inverse scattering problem. For a more detailed treatment of the problems considered here one can consult the papers [78, 94].

3.5.2 Asymptotic properties of self-similar solutions $z(t)$ and modeling of the complete Amsler surface

The classical Amsler surface considered above corresponds to a part of the solution $z(t)$ given on the segment $[t_{-1}, t_1]$ (see Figure 3.5.1), at the endpoints of which the closest "to one another" values 0 and π of the solution $z(t)$ are attained. Achieving a global geometric representation of the "complete Amsler surface", the surface corresponding to the solution $z(t)$, $t = uv$, on the *whole line* $t \in (-\infty, +\infty)$ (and, correspondingly, on the *whole real* (u,v)*-plane*) is connected with the study of the behavior of solutions of the form $z(t)$ as special functions, in particular, with the clarification of the properties of their asymptotic behavior at infinity. Let us address these questions.

Let us emphasize that the correctness of the study of properties of the solution $z(t)$, $t = uv$, on the whole (u, v)-plane is ensured by the already obtained corresponding formulation of the Darboux problem (3.5.9) for the solutions of the sine-Gordon equation of the type under consideration. The existence and uniqueness of solutions will be proved in a more general form in § 3.6.

Let us investigate the ordinary differential equation (3.5.3) by methods of stability theory [56, 121]. To this end we rewrite (3.5.3) as the equivalent system of first-order equations:

$$\frac{dy_1}{dt} = y_2, \tag{3.5.16}$$

$$\frac{dy_2}{dt} = -\frac{1}{t}\left(y_2 + \sin y_1\right), \tag{3.5.17}$$

where

$$y_1 = z(t) - \pi \equiv \widetilde{z}(t),$$
$$y_2 = \frac{d\widetilde{z}}{dt}.$$

Let us prove that the trivial solution $(y_1 = 0, y_2 = 0)$ of the system (3.5.16), (3.5.17) is stable. Our main investigation tool is *Lyapunov's second method* [56, 121]. Thus, to establish the stability of the trivial solution of the system under consideration it suffices to construct a positive definite Lyapunov function $V(t, y_1, y_2)$, the derivative of which with respect to the variable t by virtue of the system (3.5.16), (3.5.17),

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + y_2\frac{\partial V}{\partial y_1} - \frac{1}{t}\left(y_2 + \sin y_1\right)\frac{\partial V}{\partial y_2}, \tag{3.5.18}$$

has constant sign, opposite to the sign of V. (Concerning sign-definiteness and sign-constancy of functions, see, for instance, [56]).

Let us show that

$$V(t, y_1, y_2) = 2(1 - \cos y_1) + ty_2^2 \tag{3.5.19}$$

is a Lyapunov function for the system (3.5.16), (3.5.17) and determines the stability of its trivial solution.

Indeed, in the domain $S_0 = \{t \geq t_0 = 1,\ |y_j| \leq h,\ j = 1, 2;\ h > 0,\ h \text{ small}\}$ there holds the estimate

$$V(t, y_1, y_2) \geq W(y_1, y_2) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{y_1^{2n}}{(2n)!} + y_2^2 \geq 0, \tag{3.5.20}$$

which proves that the function $V(t, y_1, y_2)$ is positive definite. Substituting the expression (3.5.19) in the right-hand side of (3.5.18) we obtain

$$\frac{dV}{dt} = -y_2^2 \leq 0. \tag{3.5.21}$$

In view of the inequalities (3.5.20), (3.5.21), Lyapunov's theorem (see [56]) yields the stability of the trivial solution of the system (3.5.16), (3.5.17).

The stability property established above assumes the existence of solutions of equation (3.5.3) that are contained in an a priori given ϵ-strip about the solution $\widetilde{z}(t) \equiv 0$ (or $z \equiv \pi$); we denote such solutions by $\widetilde{z}_\varepsilon(t)$. (Completely analogously, the solutions $z = (2n+1)\pi$ are stable for $t > 0$, while the solutions $z = 2n\pi$ are stable for $t < 0$). The solutions $\widetilde{z}_\varepsilon(t)$ satisfy for all $\varepsilon > 0$ the condition $|\widetilde{z}_\varepsilon(t) - \pi| < \varepsilon$ for $t > 0$.

Let us note that a more "precise" analysis of the system (3.5.16), (3.5.17) (see [94]) allows one to state that the stability established above of the trivial solution is *asymptotic* (i.e., the solutions $\widetilde{z}_\varepsilon(t)$, which lie in the ε-strip, tend at infinity to the solution $\widetilde{z}(t) \equiv 0$). In connection with our discussion we introduce the following family of Lyapunov functions that generalize (3.5.19):

$$V_{m_1,m_2}(t, y_1, y_2) = 2\left(ft + \frac{m_2}{t^{m_1+1}}\right)(1 - \cos y_1) + \left(t + \frac{m_2}{t^{m_1}}\right) y_2^2, \tag{3.5.22}$$

where $m_1, m_2 = \text{const} \geq 0$.

From (3.5.22) and (3.5.19) it follows that

$$V(t, y_1, y_2) = V_{m_1,0}(t, y_1, y_2).$$

The general qualitative shape of the solution $z(t)$, obtained by numerical integration of equation (3.5.3), is shown in Figure 3.5.3, which illustrates in an intuitive manner the results obtained above.

Next, let us prove that the function $\widetilde{z}_\varepsilon(t)$ oscillates about zero for $t > 0$. To this end, we represent the equation (3.5.3), written for $\widetilde{z}_\varepsilon(t)$, in the linearized form

$$t\widetilde{z}_\varepsilon'' + \widetilde{z}_\varepsilon' + g_\varepsilon(t)\widetilde{z}_\varepsilon = 0, \tag{3.5.23}$$

where $g_\varepsilon(t) = (\sin \widetilde{z}_\varepsilon(t))/\widetilde{z}_\varepsilon(t)$.

Relation (3.5.23) is considered directly for some solution $\widetilde{z}_\varepsilon(t)$. Note that for each ε with $\varepsilon \geq \varepsilon_0 > 0$ one can always find a number $\sigma_1 = \sigma_1(\varepsilon) > 0$, such that the estimate

$$0 < \sigma_1 \leq g_\varepsilon(t) \leq 1 \tag{3.5.24}$$

holds. Let us write two auxiliary equations:

$$(tY_\sigma')' + \sigma \cdot Y_\sigma = 0, \tag{3.5.25}$$

where σ takes the corresponding values $\sigma = \sigma_1$ and $\sigma = 1$. Equations (3.5.25) are Bessel equations of order zero with respect to the variable $2\sqrt{\sigma t}$, the general solution of which $Y_1(t)$ (or $Y_{\sigma_1}(t)$) is a linear combination of the corresponding Bessel functions $J_0(2\sqrt{\sigma t})$ and Neumann functions $N_0(2\sqrt{\sigma t})$ [61, 112, 130].

Equation (3.5.23) and the two equations (3.5.25) differ only by the coefficients of the unknown function $\widetilde{z}_\varepsilon$ (or Y_σ) in the last left-hand side term. Hence, based on the *Sturm comparison theorem* for differential equations in selfadjoint form (see [113]), we conclude that the solution $\widetilde{z}_\varepsilon(t)$ behaves in the same way as the solutions $Y_1(t)$ and $Y_{\sigma_1}(t)$. Specifically, the solution $\widetilde{z}_\varepsilon(t)$ of equation (3.5.23) oscillate for

$t > 0$ together with the functions $Y_1(t)$ and $Y_{\sigma_1}(t)$, and the *zeroes* of the functions $Y_1(t)$, $\widetilde{z}_\varepsilon(t)$, and $Y_{\sigma_1}(t)$ *alternate*.

Therefore, we can speak about the oscillatory character of the behavior of all solutions $z(t)$ with $z(0) \in (0, 2\pi)$ about the level π for $t > 0$. In other words, the solution $z(t) \equiv \pi$ is asymptotically stable for $t > 0$ as well, with basin of "atraction" consisting of solutions with initial values $z(0) \in (0, 2\pi)$ (Figure 3.5.3 *b*), which have an oscillatory character.

Similarly, for $t < 0$ the solutions $z(t)$, $\pi < z(t) < \pi$, $t < 0$, oscillate about zero. Globally, the graph of the function $z(t)$ has the shape shown in Figure 3.5.3.

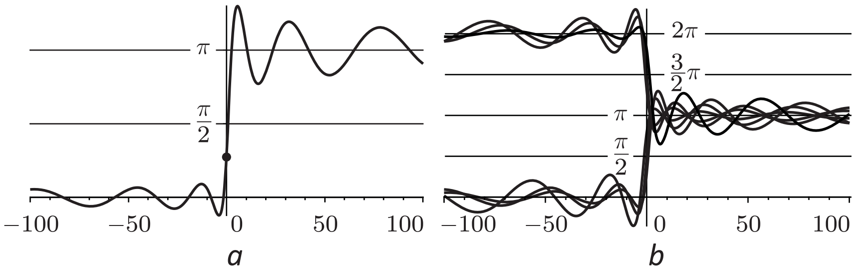


Figure 3.5.3

Let us summarize the results of the investigation of the part of the solution $z(t)$ to which must correspond the complete Amsler surface. The solution $z(t)$, $z(0) \in (0, 2\pi)$, tends asymptotically to π as $t \to +\infty$ and oscillates (about the level π). For $t < 0$ completely analogous properties hold concerning the level $z = 0$. These oscillations have the property that the "zeroes" of the function $z(t)$ "alternate" with the "zeroes" of the *Bessel equation of order zero* for the argument $2\sqrt{t}$ (for $t > 0$) (the general solution of the Bessel equation is represented as a linear combination $C_1 J_0(2\sqrt{t}) + C_2 N_0(2\sqrt{t})$ of Bessel and Neumann functions or, as a variant, of Hankel functions [61, 112, 130].

Figure 3.5.3 *a,b* shows the graphs of the solution $z(t)$, obtained by numerical integration of the equation (3.5.3) for various initial values $z(0)$ in the range $(0, 2\pi)$.

Now let us turn to the analysis of the complete Amsler surface $S[\overrightarrow{r}(u, v), z(t), t = uv]$, which corresponds to the solution $z(t)$, $t = uv$, of the sine-Gordon equation (3.5.3), defined on *the entire* (u, v)*-plane* (or, correspondingly, the entire line $t \in (-\infty, +\infty)$).

We recall that the classical Amsler surface already considered by us interprets geometrically the part of the self-similar solution $z(t) \in [0, \pi]$ of the sine-Gordon equation, defined on the segment $[t_{-1}, t_1]$ between the first value $z = 0$ at $t = t_{-1} < 0$ and the first value $z = \pi$ at $t = t_1 > 0$, for the chosen initial condition $z(0) = \pi/2$ (Figure 3.5.1). To the marked values $z = 0$ and $z = \pi$ (which are multiples of π) there correspond on the pseudospherical surface (Figure 3.5.2) cuspidal edges which represent "contracting helical curves" that "twist asymptotically" along the coordinate axes OX and OY.

If we now consider the solution $z(t)$, $t \in (-\infty, +\infty)$, studied by us, then this solution, oscillating, takes the value π for $t > 0$ countably many time, at the points $t = t_1, t_2, \ldots, t_k, \ldots$, as well as the value 0 for $t < 0$ countably many times, at the points $t = t_{-1}, t_{-2}, \ldots, t_{-k}, \ldots$ (Figure 3.5.2). Therefore, by Poznyak's theorem, the complete Amsler surface must have a *countable* number of irregular singularities, corresponding to the instants at which the solution $z(t)$, while oscillating, attains the values 0 and π. From (3.5.2) it is clear that the level lines $z(t, t = uv) = k\pi$, $k = 0, 1$, form a family of symmetric hyperbolas, as shown in Figure 3.5.4.

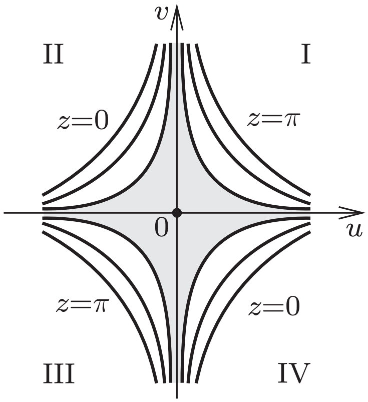


Figure 3.5.4

To the classical Amsler surface there corresponds in the (u, v)-parametric plane the central shaded domain (Figure 3.5.4). To each "successive" domain in the (u, v)-plane bounded by two successive hyperbola branches (simultaneously in the I, III or II, IV quadrants) will correspond a new regular part of the complete Amsler surface, glued to the already available surface along an irregular edge (the preimage of a hyperbola branch). Consequently, the complete Amsler surface consists of a countable system of regular domains which globablly form the whole pseudospherical surface under study, and which correspond to the respective countable system of segments $[t_1, t_2]$, $[t_2, t_3]$,…,$[t_{k-1}, t_k]$,… for $t > 0$, and $[t_{-1}, t_{-2}]$, $[t_{-2}, t_{-3}]$,…,$[t_{-k}, t_{-k-1}]$,… for $t < 0$, each of which is the domain of definition of the regular part of the complete Amsler surface corresponding to it.

Thus, we can propose the following general *model* representation of the complete Amsler surface: this surface is constructed from the classical Amsler surface by successively and regularly gluing to the latter, along irregular "helical" edges, new "regular shrinking helical strips" that correspond to the aforementioned system of segments (this "gluing" behaves like an "inner twisting"). A possible version of "inner twisting" and the corresponding construction of the complete Amsler surface in space are shown in Figure 3.5.5, which depicts a model cross-section of the Amsler surface by a plane perpendicular on the axis OX axis (or axis OY). The arcs shown are the "traces" of the countably many regular "twisting helical surface strips", glued to one another along irregular cuspidal edges (the "traces" of the irregular edges in Figure 3.5.5 serve as the "cusp points" of the coupling of arcs). As on "advances" along the coordinate axis OX, the "cross-section drawing" (Figure 3.5.5) rotates.

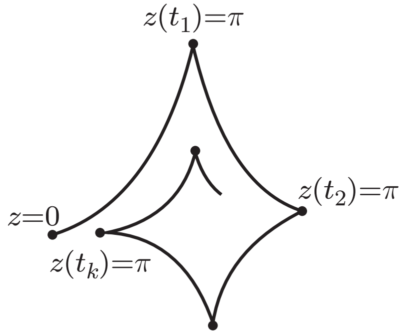


Figure 3.5.5

In connection with the consideration of the Amsler surface let us mention the work of E. V. Maevskii [52, 94], in which asymptotic methods are developed for analyzing the solutions of equation (3.5.3) and of equations of types close to them, as well as for obtaining asymptotic expansions of the radius vector of the Amsler surface in the vicinity of cuspidal edges.

3.5.3 Nonlinear equations and the Painlevé test

The sine-Gordon equation (3.5.3) under investigation is a special "point of contact" of the geometrical theme discussed here, namely, the *theory of pseudospherical surfaces*, with problems concerned with certain classes of special functions, namely the *Painlevé transcendents*, defined by a set of six ordinary differential equations (ODEs) which share the property that their solutions have no *movable critical points*. Determining and classifying special types of such second-order ODE's is undoubtedly a meritorious contribution of P. Painlevé and his colleagues; their results turned out to be of crucial value in many branches of contemporary mathematics and physics, and continue to do so. Later in the book we will discuss the connection of Painlevé transcendents with other important and inherent concepts of the modern theory of nonlinear differential equations, such as pseudospherical metrics, Bäcklund transformations, solitons, integrability by the method of the inverse scattering transform, and others. We begin by explaining the notion of Painlevé transcendental functions [1, 2, 19, 180]

Painlevé's investigations revolved around the second-order equation for the unknown function $\omega = \omega(z)$ of general form

$$\omega'' = \mathcal{F}(z, \omega, \omega'), \tag{3.5.26}$$

the right-hand side of which is a rational function in the arguments ω and ω' and a locally analytic function in the variable z. Painlevé and his students discovered that among the equations of this class there are 50 equations with the property that their solutions *have no movable critical points* (in the case when they are present). This property became known as the *Painlevé property* , and the corresponding equations are called *P-type equations*, or *Painlevé equations*. All P-type equations either admit their own explicit integration, or can be reduced, via transformations, to one of 6 canonical types of Painlevé equations.

The corresponding types of equations are usually denoted by P_{I}, P_{II}, P_{III}, P_{IV}, P_{V}, and P_{VI}. The customary form in which the P-type equations are written is a follows (see [1, 19, 180]):

$$
\begin{aligned}
P_{\mathrm{I}}\colon\quad & \frac{d^2\omega}{dz^2} = 6\omega^2 + z,\\
P_{\mathrm{II}}\colon\quad & \frac{d^2\omega}{dz^2} = z\omega + 2\omega^2 + \alpha,\\
P_{\mathrm{III}}\colon\quad & \frac{d^2\omega}{dz^2} = \frac{1}{\omega}\left(\frac{d\omega}{dz}\right)^2 - \frac{1}{z}\frac{d\omega}{dz} + \frac{1}{z}\left(\alpha\omega^2 + \beta\right) + \gamma\omega^3 + \frac{\delta}{\omega},\\
P_{\mathrm{IV}}\colon\quad & \frac{d^2\omega}{dz^2} = \frac{1}{2\omega}\left(\frac{d\omega}{dz}\right)^2 + \frac{3\omega^3}{2} + 4z\omega^2 + 2\left(z^2 - \alpha\right) + \frac{\beta}{\omega},\\
P_{\mathrm{V}}\colon\quad & \frac{d^2\omega}{dz^2} = \left\{\frac{1}{2\omega} + \frac{1}{\omega - 1}\right\}\left(\frac{d\omega}{dz}\right)^2 - \frac{1}{z}\frac{d\omega}{dz}\\
& \qquad + \frac{(\omega-1)^2}{z^2}\left(\alpha\omega + \frac{\beta}{\omega}\right) + \frac{\gamma m}{z} + \frac{\delta\omega(\omega+1)}{\omega - 1},\\
P_{\mathrm{VI}}\colon\quad & \frac{d^2\omega}{dz^2} = \frac{1}{2}\left\{\frac{1}{\omega} + \frac{1}{\omega-1} + \frac{1}{\omega - z}\right\}\left(\frac{d\omega}{dz}\right)^2 - \left\{\frac{1}{z} + \frac{1}{z-1} + \frac{1}{\omega - z}\right\}\frac{d\omega}{dz}\\
& \qquad + \frac{\omega(\omega-1)(\omega-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{\omega^2} + \frac{\gamma(z-1)}{(\omega-1)^2} + \frac{\delta z(z-1)}{(\omega - z)^2}\right\}.
\end{aligned}
\tag{3.5.27}
$$

Equations (3.5.27) cannot be integrated in elementary functions; their solutions form individual classes of *special functions*, called *Painlevé transcendental functions* (of the corresponding genus, from I to VI), or *Painlevé transcendents*.

Let us remark that, in particular, if in the third of equations (3.5.27) we choose for the constant coefficients the values $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $\gamma = 0$, and $\delta = 0$, then the P_{III} equation itself (for the Painlevé functions of genus III) reduces to the already familiar equation (3.5.5) investigated in the present section.

Overall, the presence of nonmovable critical points in an equation plays an important role in the study of the mathematical and applied problems associated with it. As an example, we must refer here the well-know work of S. Kovalevskaya on the theory of motion of a rigid body with a fixed point in a gravity field [1, 19]. The main idea of Kovalevskaya's investigations was to bring (by means of appropriate transformations) the defining parameters of the problems (models) to those admissible values for which the model corresponding to the problem is described by equations without movable critical points. In these cases, the equations under study admit their own exact integration. It is noteworthy that equations (3.5.27) can be regarded as a foundation ("standard series") for establishing general properties of other, connected with them, partial differential equations. It is precisely with this kind of specificity of equations of P-type that the *conjecture*, formulated in [133], that the nonlinear equations integrable by the method of the inverse scattering transform (MIST) posses the Painlevé property, is related.

The method of the inverse scattering transform [1, 42, 51, 62] (see also § 3.9) is currently one of the most effective approaches for constructing exact solutions of nonlinear partial differential equations. At the same time, the possibility of applying it in each concrete case (for an equation of interest) is associated with the nontrivial construction ("*guessing*") of "primer" ("starter") relations that specify the direct scattering problem (the first step of the method), and so every piece of a priori (even indirect) "information" on whether is it potentially possible to implement MIST is very important.

The conjecture on the integrability of equations arising in this connection became known as the *Painlevé test.*

Painlevé Test. *A nonlinear partial differential equation is integrable by the method of the inverse scattering transform if and only if it can be reduced* (*by means of a suitable change of variables*) *to one of the ordinary differential equations of P-type* (3.5.27).

Let us clarify what the formulated test means on the following examples.

1) *The sine-Gordon equation.* The sine-Gordon equation is integrable by MIST [31, 51, 131] a widely known class of solutions obtained by MIST is formed by the multi-soliton solutions (see § 3.2).

   The equation passes the Painlevé test: the substitution (3.5.2), (3.5.4) reduces the sine-Gordon equation (3.5.1) to equation (3.5.5), the P_{III}-type equation (3.5.27), for the third Painlevé transcendental function.

2) *Modified Korteweg-de Vries equation* (MKdV). The MKdV equation

$$z_v - 6z^2 z_u + z_{uuu} = 0,$$

   which is integrable by MIST [1, 42, 51], is reduced by the change of variables

$$(u, v, z) \to (t, \omega), \quad z(u, v) = (3v)^{-2/3}\omega(t), \quad t = (3v)^{-1/3}u,$$

   to an ODE of the form

$$\omega''' - 6\omega^2\omega' - (t\omega)' = 0, \quad \omega = \omega(t),$$

   which in turn leads, via order reduction, to the P_{II}-equation

$$\omega'' = 2\omega^2 + t\omega + \alpha, \quad \alpha = \text{const.}$$

3) *Boussinesq equation.* Consider the Boussinesq equation,

$$z_{vv} - z_{uu} = \frac{1}{2}\left(z^2\right)_{uu} + \frac{1}{4}\, z_{uuuu},$$

which is integrable by MIST [1, 29]. Seeking solutions in the form of travelling waves

$$z(u,v) = \omega(t), \quad t = u - av, \quad a = \mathrm{const}$$

leads to an ODE for the unknown function $\omega(t)$:

$$(1-a^2)\omega'' + \frac{1}{2}\big(\omega^2\big)'' + \frac{1}{4}\,\omega'''' = 0,$$

Integrating twice the last equation yields (for $a = \pm 1$) two equations of the form

$$\omega'' + 2\omega^2 + \alpha = 0, \quad \alpha = \mathrm{const},$$
$$\omega'' + 2\omega^2 + t = 0.$$

The second of these equations is of the Painlevé P_{I} type.

Currently a sufficiently rich list of nonlinear equations that confirm the validity of the Painlevé test is known [1, 69, 165]. Reference [1] contains number of key ideas concerning the justification of the Painlevé conjecture. For many equations of contemporary mathematical physics it is completely natural to consider the "Painlevé conjecture" in the light of its intimate connection with other inalienable "attributes" of nonlinear theory, such as Bäcklund transformations, solitons, pseudospherical metrics, and so on. The geometrical foundations ("roots") of these relationships will be discussed in Chapter 4, in the framework of the geometric concept of differential equations of Lobachevsky class.

3.6 The Darboux problem for the sine-Gordon equation

This section will be devoted to the consideration of the Darboux problem for the sine-Gordon equation (a problem with data on the characteristics of the differential equation),[12] and to the investigation of two important classes of solutions for this equation, dictated by the special character of the formulation of the Darboux problem for them.

3.6.1 The classical Darboux problem

The Darboux problem for the sine-Gordon equation is posed as follows:

$$z_{uv} = \sin z, \tag{3.6.1}$$

$$\begin{aligned} z(u,0) &= \varphi(u), \\ z(0,v) &= \psi(v), \\ \varphi(0) &= \psi(0). \end{aligned} \tag{3.6.2}$$

The functions $\varphi(u)$ and $\psi(v)$ in (3.6.2), which specify the values of the sought-for solution on the coordinate axes (which here are the characteristics of equation

[12] In works on differential equations, this type of problems is also refered to as the "Goursat problem".

(3.6.1)), are defined for $u \geq 0$, $v \geq 0$. The third condition in (3.6.2) expresses their agreement at zero.

The classical Darboux problem for the sine-Gordon equation in the form (3.6.1) was first studied by Bianchi in his fundamental lecture notes on differential geometry [142]. Let us formulate the result of Bianchi, which established the *existence* and *uniqueness* of the solution to the Darboux problem (3.6.1), (3.6.2), in the following theorem.

Theorem 3.6.1. *Suppose there are given functions*

$$\varphi(u) \in C^n[0,U], \quad \psi \in C^n[0,V], \quad n \geq 2,$$

which satisfy the compatibility condition at zero: $\varphi(0) = \psi(0)$. *Then the Darboux problem* (3.6.1), (3.6.2) *has a solution*

$$z(u,v) \in C^n([0,U] \times [0,V]),$$

and this solution is unique. The solution $z(u,v)$ *can be constructed as the limit of a uniformly convergent sequence* $\{z_k(u,v)\}$, *given by the recursion relations*

$$\begin{aligned} z_0(u,v) &= 0, \\ z_1(u,v) &= \varphi(u) + \psi(v) - \varphi(0), \end{aligned} \tag{3.6.3}$$

$$z_{k+1}(u,v) = z_1(u,v) + \int_0^u \int_0^v \sin z_k(\zeta,\eta)\, d\zeta d\eta. \tag{3.6.4}$$

Moreover, the solution $z(u,v)$ *obeys the estimate*

$$\begin{gathered} |z(u,v)| \leq Ce^{uv}, \\ C = \max_{[0,U]\times[0,V]} \{|\varphi(u)| + |\psi(v)| + |\varphi(0)|\}. \end{gathered} \tag{3.6.5}$$

Proof. To prove Theorem 3.6.1, we rewrite the Darboux problem (3.6.1), (3.6.2) in the integral form

$$z(u,v) = \varphi(u) + \psi(v) - \varphi(0) + \int_0^u \int_0^v \sin z(\zeta,\eta)\, d\zeta d\eta. \tag{3.6.6}$$

The integral equation (3.6.6) is equivalent to the Darboux problem (3.6.1), (3.6.2) in the following sense: Let $z(u,v)$ be a function that is defined and continuous in the coordinate rectangle $[0,U] \times [0,V]$ under consideration (Figure 3.6.1), and is a solution of equation (3.6.6). Then $z(u,v)$ has a continuous mixed derivative $z_{uv}(u,v)$ and satisfies the Darboux problem (3.6.1), (3.6.2). In the opposite direction, any solution of the classical Darboux problem (3.6.1), (3.6.2) that is continuous in the rectangle $[0,U] \times [0,V]$ satisfies the integral equation (3.6.6).

We will construct a solution of the integral equation (3.6.6) (and, correspondingly, of the Darboux problem (3.6.1), (3.6.2)) by the method of successive

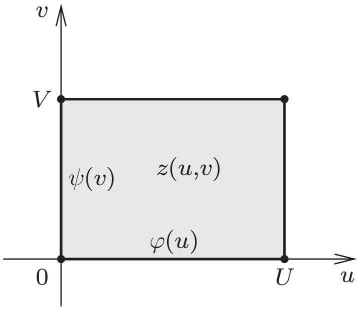


Figure 3.6.1

approximations. The successive iterations are introduced via the recursion relations (3.6.4). Then the solution $z(u, v)$ itself admits the series representation

$$z(u, v) = \sum_{k=0}^{\infty} (z_{k+1} - z_k). \tag{3.6.7}$$

Let us establish the convergence of series (3.6.7). To this end, based on (3.6.4), we derive the following estimates:

$$\begin{aligned}
|z_2 - z_1| &= \left| \int_0^u \int_0^v \sin z_1 d\zeta d\eta \right| \le \int_0^u \int_0^v |\sin z_1| d\zeta d\eta \le \frac{uv}{(1!)^2}, \\
|z_3 - z_2| &= \left| \int_0^u \int_0^v 2 \sin \frac{z_2 - z_1}{2} \cdot \cos \frac{z_2 + z_1}{2} d\zeta d\eta \right| \\
&\le \int_0^u \int_0^v 2 \left| \sin \frac{z_2 - z_1}{2} \right| d\zeta d\eta \le \int_0^u \int_0^v |z_2 - z_1| d\zeta d\eta \le \frac{(uv)^2}{(2!)^2}, \\
&\vdots \\
|z_{k+1} - z_k| &\le \int_0^u \int_0^v 2 \left| \sin \frac{z_k - z_{k-1}}{2} \right| d\zeta d\eta \le \int_0^u \int_0^v |z_k - z_{k-1}| \le \frac{(uv)^k}{(k!)^2}.
\end{aligned} \tag{3.6.8}$$

From estimate (3.6.8) [13] one concludes that the sequence of iterates $\{z_k(u, v)\}$ converges uniformly and, by (3.6.7), its *limit* as $k \to \infty$ is a *solution* of the integral equation (3.6.6).

From relations (3.6.4) we see that, if $z_1(u, v) \in C^n([0, U] \times [0, V])$ and $z_k(u, v) \in C^n([0, U] \times [0, V])$, then $z_{k+1}(u, v) \in C^n([0, U] \times [0, V])$. Hence, in

[13] The rigorous justification of estimates of the general form (3.6.8) is done by induction.

view of the uniform convergence of the series (3.6.7), the smoothness class of the solution $z(u,v)$ of the posed Darboux problem coincides with that of the function $z_1(u,v)$, which in turn is determined by the initial values of the problem.

Note also that to the integral inequality

$$z(u,v) \leq C + \int_0^u \int_0^v I(\zeta,\eta) z(\zeta,\eta) d\zeta d\eta, \quad I(\zeta,\eta) \equiv 1,$$

which is an obvious consequence of (3.6.6), one can apply directly the Wendroff inequality (estimate) [10]

$$|z(u,v)| \leq C \exp\left\{ \int_0^u \int_0^v I(\zeta,\eta) d\zeta d\eta \right\},$$

which yields precisely the estimate (3.6.5) for the solution $z(u,v)$ of the integral equation (3.6.6) and of the Darboux problem (3.6.1), (3.6.2) corresponding to it.

It now remains to show that the solution $z(u,v)$ obtained by the recipe (3.6.3), (3.6.4), (3.6.5) is unique. Let $\nu(u,v)$ be another solution, different from $z(u,v)$, of the integral equation (3.6.6). Write (3.6.6) for $\nu(u,v)$ and subtract from it (3.6.4). We obtain

$$|\nu - z_0| \leq M, \quad M = \text{const} > 0,$$

and correspondingly

$$\begin{aligned} |\nu - z_{k+1}| &= \left| \int_0^u \int_0^v 2 \sin \frac{\nu - z_k}{2} \cdot \cos \frac{\nu + z_k}{2} d\zeta d\eta \right| \\ &\leq \int_0^u \int_0^v 2 \left| \sin \frac{\nu - z_k}{2} \right| d\zeta d\eta \leq \int_0^u \int_0^v |\nu - z_k| d\zeta d\eta. \end{aligned}$$

Hence, the following estimate holds for all k:

$$|\nu - z_k| \leq M \frac{(uv)^k}{(k!)^2}.$$

This shows that the difference $|\nu - z|$ is arbitrarily small, i.e., the functions $z(u,v)$ and $\nu(u,v)$ coincide. Theorem 3.6.1 is proved. □

It is clear that the nature of the arguments used to prove Theorem 3.6.1 is such that the result on the existence and uniqueness of the solution to the Darboux problem (3.6.1), (3.6.2) can be "transferred" to the first coordinate quadrant as $U, V \to +\infty$, as well as to the whole (u,v)-plane: $u \in (-\infty, +\infty), v \in (-\infty, +\infty)$.

The existence and uniqueness of the solution to the Darboux problem (3.6.1), (3.6.2) *on the whole plane* gives, in essence, unlimited possibilities of "modeling" various classes of solutions of the sine-Gordon equations by varying the initial data in the formulation of the Darboux problem.

For instance, the class of *soliton* solutions of the sine-Gordon equation (see § 3.2), defined by the general formula (3.2.13), can obviously be obtained now as the family of solutions of the Darboux problem (3.6.1), (3.6.2), with the system of initial data extracted for $u = 0$ and $v = 0$ from the multi-soliton solution (3.2.13) (of course, in this case we are dealing already with a kind of inverse problem).

As it was already established (see § 3.5), the Painlevé-III transcendental functions, which correspond to the Amsler surface, "arise" as solutions of the Darboux problem (3.6.1), (3.6.2) with *constant* initial conditions (3.5.9).

The high degree of freedom in the choice of initial conditions of the Darboux problem is nevertheless insufficient for avoiding to impose certain fundamental requirements on the solutions of the sine-Gordon equation. Thus, the what would seem an "appealing attempt" at constructing *small* solutions (i.e., solutions with a small range of values, among them, solutions with a range smaller than π) of the sine-Gordon equation by choosing arbitrarily small initial data in the Darboux problem does not yield "spectacular" results. Specifically, the solutions produced in this manner will not be "small", but will change at a sufficiently high rate, "passing" through values that are multiples of π, thereby confirming Hilbert's result on the nonimmersibility of the Lobachevsky plane Λ^2 in the three-dimensional Euclidean space $\mathbb{E}^3$ (see § 2.6). Nonetheless, the construction of "small" solutions of the sine-Gordon equation is a significant independent problem, and will be considered in the next subsection.

To end this subsection we remark that an exposition of the basic ideas concerning the realization of the method of successive approximation in the case of hyperbolic equations of sufficiently general form can be found in [114].

3.6.2 The Darboux problem with small initial data

This subsection is devoted to the Darboux problem for the sine-Gordon equation with small initial data. The solutions of this problem are usually referred to as "small" solutions of the sine-Gordon equation. It is quite natural to expect that the solutions of this class will define their own special type of pseudospherical surfaces, the geometric characteristics of which include a small parameter. The idea of distinguishing this class of solutions is due E. G. Poznyak [72]. In that work an asymptotic representation of small solutions of the sine-Gordon equation in powers of the small parameter was obtained. On the whole, the *small parameter method* proves to be quite productive in the study of various types of differential equations (see, e.g., [14]).

Consider the following formulation of the Darboux problem for the sine-Gordon equation with small initial data, "controlled" by a small parameter ε:

$$z_{uv} = \sin z. \tag{3.6.9}$$

$$\begin{aligned} z(u,0) &= \varepsilon\varphi(u), \\ z(0,v) &= \varepsilon\psi(v), \\ \varphi(0) &= \psi(0), \end{aligned} \tag{3.6.10}$$

where $\varepsilon = \text{const}$, $0 < \varepsilon \ll 1$.

The functions in (3.6.10) are assumed to satisfy the minimal smoothness requirements:

$$\varphi(u) \in C^1(-\infty, +\infty), \quad \psi(v) \in C^1(-\infty, +\infty).$$

By *small solutions* $z(u, v, \varepsilon)$ of the sine-Gordon equation (3.6.9) for the given functions $\varphi(u)$ and $\psi(v)$ we mean the collection of all solutions of the Darboux problem (3.6.9), (3.6.10) for all values of the *small parameter* ε.

In accordance with the results of the preceding subsection, the Darboux problem (3.6.9), (3.6.10) has a solution $z(u, v, \varepsilon) \in C^1$ that is defined in the entire (u, v)-plane and has a continuous partial derivative z_{uv}.

By (3.6.5), the solution $z(u, v, \varepsilon)$ obeys the estimate

$$|z(u, v, \varepsilon)| < \varepsilon \cdot C_0 e^{|uv|}, \tag{3.6.11}$$

where $C_0 = \max\{|\varphi(u)| + |\psi(v)| + |\varphi(0)|\}$.

Let us obtain, following [72], an asymptotic representation of the solution $z(u, v, \varepsilon)$ of the Darboux problem (3.6.9), (3.6.10) in powers of the small parameter ε.

To this end we introduce the new function $\overset{\circ}{z}(u, v, \varepsilon)$ by

$$z(u, v, \varepsilon) = \varepsilon \overset{\circ}{z}(u, v, \varepsilon). \tag{3.6.12}$$

After the substitution of (3.6.12), the Darboux problem (3.6.9), (3.6.10) can be recast as

$$\overset{\circ}{z}_{uv} = \overset{\circ}{z} + \varepsilon^2 f(\overset{\circ}{z}, \varepsilon), \tag{3.6.13}$$

$$\begin{aligned} \overset{\circ}{z}(u, 0) &= \varphi(u), \\ \overset{\circ}{z}(0, v) &= \psi(v), \end{aligned} \tag{3.6.14}$$

where the function $f(\overset{\circ}{z}, \varepsilon)$ is defined by

$$f(\overset{\circ}{z}, \varepsilon) = \begin{cases} \dfrac{1}{\varepsilon^3}\left(\sin(\varepsilon \overset{\circ}{z}) - \varepsilon \overset{\circ}{z}\right), & \text{if } \varepsilon \neq 0, \\ -\dfrac{1}{6}\left(\overset{\circ}{z}(u, v, 0)\right)^3, & \text{if } \varepsilon = 0. \end{cases} \tag{3.6.15}$$

Let us write equation (3.6.13) separately as

$$\overset{\circ}{z}_{uv} - \overset{\circ}{z} = \varepsilon^2 f(\overset{\circ}{z}, \varepsilon) \tag{3.6.16}$$

and compare it with an equation that is well known in the theory of differential equations, namely, the *telegraph equation* [114]

$$\overset{\circ}{z}_{uv} + a \overset{\circ}{z} = f(u, v), \quad a = \text{const}. \tag{3.6.17}$$

For a known right-hand side (on a known solution), equation (3.6.16) coincides with the telegraph equation (3.6.17) (for $a = -1$). Consequently, the solution of the Darboux problem under investigation can be constructed by Riemann's method [114], applied to the telegraph linear equation. In this case (for $a = -1$), the Riemann function for the Darboux problem (3.6.13), (3.6.14) can be written in explicit form [72, 114]

$$U(u,v,\xi,\eta) = \sum_{n=0}^{\infty} \frac{(\xi-u)^n(\eta-v)^n}{(n!)^2}, \tag{3.6.18}$$

and the solution itself of the problem in question admits the representation

$$\begin{aligned} \mathring{z}\,(u,v,\varepsilon) &= \varphi(0)\sum_{n=0}^{\infty}\frac{u^n v^n}{(n!)^2} + \int_0^u \varphi'(\xi)\sum_{n=0}^{\infty}\frac{(\xi-u)^n\cdot(-v)^n}{(n!)^2}\,d\xi \\ &+ \int_0^v \psi'(\eta)\sum_{n=0}^{\infty}\frac{(-1)^n u^n\cdot(\eta-v)^n}{(n!)^2}\,d\eta \\ &+ \varepsilon^2\int_0^u\int_0^v f(\mathring{z}\,(\xi,\eta,\varepsilon),\varepsilon)\sum_{n=0}^{\infty}\frac{(\xi-u)^n\cdot(\eta-v)^n}{(n!)^2}\,d\xi d\eta. \end{aligned} \tag{3.6.19}$$

Let us estimate the last term in (3.6.19), applying to it the mean value theorem and using the majorant estimate

$$|f(\mathring{z},\varepsilon)| \le \frac{C_0^3 e^{3|uv|}}{6},$$

which follows from the structure of the function $f(\mathring{z},\varepsilon)$ (3.6.15), and the inequality (3.6.11). This yields

$$\begin{aligned} &\left|\varepsilon^2\int_0^u\int_0^v f(\mathring{z}\,(\xi,\eta,\varepsilon),\varepsilon)\sum_{n=0}^{\infty}\frac{(\xi-u)^n\cdot(\eta-v)^n}{(n!)^2}\,d\xi d\eta\right| \\ &\qquad = \varepsilon^2\left|f(\mathring{z}\,(\xi^*,\eta^*,\varepsilon),\varepsilon)\sum_{n=0}^{\infty}\frac{u^{n+1}\cdot v^{n+1}}{((n+1)!)^2}\right| \leqslant \varepsilon^2\,\frac{C_0^3 e^{4|uv|}}{6}, \end{aligned} \tag{3.6.20}$$

where $\xi^* \in (0,u)$, $\eta^* \in (0,v)$.

Hence, by (3.6.20), the term being estimated is of order $O(\varepsilon^2)$.

If we now return to the solution $z(u,v,\varepsilon)$ of the original Darboux problem (3.6.9), (3.6.10) using formula (3.6.12), then (3.6.19) and (3.6.20) finally yield

$$z(u,v,\varepsilon) = \varepsilon \cdot \left[\varphi(0) \sum_{n=0}^{\infty} \frac{u^n v^n}{(n!)^2} + \int_0^u \varphi'(\xi) \sum_{n=0}^{\infty} \frac{(\xi - u)^n \cdot (-v)^n}{(n!)^2}\, d\xi \right.$$
$$\left. + \int_0^v \psi'(\eta) \sum_{n=0}^{\infty} \frac{(-u)^n \cdot (\eta - v)^n}{(n!)^2}\, d\eta \right] + O(\varepsilon^3). \quad (3.6.21)$$

The functions $\varphi(u)$ and $\psi(v)$ in (3.6.21) satisfy the requirements

$$\varphi(u) \in \overline{C}^1[0,+\infty), \quad \psi(v) \in \overline{C}^1[0,+\infty)$$

(they are bounded functions of class C^1).

As one can see from (3.6.21), regardless of the fact that one can a priori choose an arbitrarily small (fixed) value of the parameter ε in the initial data (3.6.10), the solution $z(u,v,\varepsilon)$ itself of the Darboux problem under consideration varies (grows, decays) to a high degree, and thus $z(u,v,\varepsilon)$ can "cross" value levels that are multiples of π. A transparent illustrative example of this fact is provided by the solutions with small initial data $z(u,0) = z(0,v) = \varepsilon$, connected with the Amsler surface (see § 3.5). Such solution do, for any $\varepsilon \ll 1$, exhibit oscillatory behavior about the levels 0 and π. Overall, the fact that the range of variation of "small" solutions $z(u,v,\varepsilon)$ must exceed the value π is in complete agreement with Hilbert's result on the nonimmersibility of the complete plane Λ^2 in $\mathbb{E}^3$ and, correspondingly, with failure of condition (2.6.3) for the solutions of the sine-Gordon equation (see § 2.6).

3.6.3 Solutions of the sine-Gordon equation on multi-sheeted surfaces

In this subsection we use the results on the classical Darboux problem to present a method of "modeling" an essentially new class of solutions of the sine-Gordon equation, namely, *solutions* given on *multi-sheeted surfaces*.

In the integral formulation (3.6.6) of the Darboux problem (3.6.1), (3.6.2), considered in the coordinate rectangle $[0,U] \times [0,V]$, we pass to polar coordinates:

$$u = \rho \cos\varphi, \quad v = \rho \sin\varphi, \quad (3.6.22)$$

where $\rho \in [0,\rho^*]$, $\rho^* = \sqrt{U^2 + V^2}$, and $\varphi \in [0,\pi/2]$.[14]

This recasts the integral equation (3.6.6) (in polar coordinates) as

[14] Henceforth, φ will denote the polar angle.

$$
\begin{aligned}
z(\rho\cos\varphi, \rho\sin\varphi) = \bar{\psi}(\rho) &+ \int\limits_0^{\rho\sin\varphi} \widetilde{\rho}\, d\widetilde{\rho} \int\limits_0^{\pi/2} \sin z(\widetilde{\rho}\cos\widetilde{\varphi}, \widetilde{\rho}\sin\widetilde{\varphi})\, d\widetilde{\varphi} \\
&+ \int\limits_{\rho\sin\varphi}^{\rho\cos\varphi} \widetilde{\rho}\, d\widetilde{\rho} \int\limits_0^{\arcsin(\rho/\widetilde{\rho}\sin\varphi)} \sin z(\widetilde{\rho}\cos\widetilde{\varphi}, \widetilde{\rho}\sin\widetilde{\varphi})\, d\widetilde{\varphi} \\
&+ \int\limits_{\rho\cos\varphi}^{\rho} \widetilde{\rho}\, d\widetilde{\rho} \int\limits_{\arccos(\rho/\widetilde{\rho}\cos\varphi)}^{\arcsin(\rho/\widetilde{\rho}\sin\varphi)} \sin z(\widetilde{\rho}\cos\widetilde{\varphi}, \widetilde{\rho}\sin\widetilde{\varphi})\, d\widetilde{\varphi}, \quad (3.6.23)
\end{aligned}
$$

where

$$
\bar{\psi} = \begin{cases} \psi_m(\rho), & \text{if} \quad \varphi = \frac{\pi}{2}\, m, \quad m = 0, 1. \\ 0, & \text{if} \quad \varphi \neq \frac{\pi}{2}\, m, \quad 0 < \varphi < \frac{\pi}{2}, \end{cases} \tag{3.6.24}
$$

The functions ψ_0 and ψ_1 in (3.6.24) correspond to the functions φ and respectively ψ in (3.6.2).

To simplify, we denote the right-hand side of (3.6.23) by $J(z, \rho, \varphi)$; then the integral equation (3.6.23) takes on the compact form

$$
z(\rho, \varphi) = J(z, \rho, \varphi). \tag{3.6.25}
$$

As in Subsection 3.6.1, to prove the existence and uniqueness of the solution to problem (3.6.23), (3.6.24) (or (3.6.25), (3.6.24)), we apply the method of successive approximations, defining the successive iterations via the recursion relation

$$
z_{n+1}(\rho, \varphi) = J(z_n(\rho, \varphi), \rho, \varphi), \quad n = 0, 1, 2, \dots \tag{3.6.26}
$$

For the initial iteration in (3.6.26) we take $z_0 \equiv 0$, and then we estimate the modulus of the difference of two iterations in the standard manner:

$$
\begin{aligned}
|z_1 - z_0| &\le \frac{\pi}{2}, \\
|z_2 - z_1| &\le \int\limits_0^{\rho\sin\varphi} \widetilde{\rho}\, d\widetilde{\rho} \int\limits_0^{\pi/2} d\widetilde{\varphi} + \int\limits_{\rho\sin\varphi}^{\rho\cos\varphi} \widetilde{\rho}\, d\widetilde{\rho} \int\limits_0^{\arcsin(\rho/\widetilde{\rho}\sin\varphi)} d\widetilde{\varphi} \\
&\quad + \int\limits_{\rho\cos\varphi}^{\rho} \widetilde{\rho}\, d\widetilde{\rho} \int\limits_{\arccos(\rho/\widetilde{\rho}\cos\varphi)}^{\arcsin(\rho/\widetilde{\rho}\sin\varphi)} d\widetilde{\varphi} \le \frac{3\pi}{2} \cdot \frac{\rho^2}{1 \cdot 2}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
|z_3 - z_2| &\le \left(\frac{3\pi}{2}\right)^2 \frac{\rho^3}{3!}, \\
&\vdots \\
|z_{n+1} - z_n| &\le \left(\frac{3\pi}{2}\right)^n \frac{\rho^n}{n!}.
\end{aligned} \tag{3.6.27}
$$

The obtained estimate (3.6.27) establishes the uniform convergence, as $n \to \infty$, of the sequence $\{z_n(\rho, \varphi)\}$ to the solution $z(\rho, \varphi)$ of problem (3.6.23), (3.6.24), and simultaneously justifies the transition to polar coordinates in the Darboux problem under consideration.

When $\rho \to \infty$, the domain where the solution $z(\rho, \varphi)$ of the Darboux problem expands to the first quadrant Q_1 of the plane:

$$Q_1 = \left\{\rho \in [0, +\infty),\ \varphi \in \left[0, \frac{\pi}{2}\right]\right\}.$$

A natural generalization of the result obtained is its "transfer" to an arbitrary quadrant Q_{m+1}, with

$$Q_{m+1} = \left\{\rho \in [0, +\infty),\ \varphi \in \left[\frac{\pi}{2} m,\ \frac{\pi}{2}(m+1)\right],\ m = 0, 1, 2, \ldots\right\}.$$

Thus, we can talk about the well-posedness (in the sense of the existence and uniqueness of the solution) of the unified (basic) Darboux problem for the sine-Gordon equation in an arbitrary quadrant Q_{s+1}:

$$D(z(\rho, \varphi), \rho, \varphi) \equiv z(\rho, \varphi) - J(z(\rho, \varphi), \rho, \varphi) = 0, \tag{3.6.28}$$

$$\begin{aligned} z\left(\rho, \frac{\pi}{2} m\right) &= \psi_m(\rho), \\ \psi_s(0) &= \psi_{s+1}(0), \quad m = s, s+1. \end{aligned} \tag{3.6.29}$$

The Darboux problem (3.6.28), (3.6.29), considered in some selected quadrant,[15] represents a universal fundamental problem, which can serve as a basis for "constructing" new types of solutions of the sine-Gordon equation on multi-sheeted surfaces.

Let us now address the "modeling" of solutions of the sine-Gordon equation defined on multi-sheeted domains. "Locally" such solutions will satisfy an equation of the type (3.6.1).

1) *Solving the* sine-*Gordon equation on multi-sheeted surfaces with one branch point.* Let us consider a multi-sheeted surface Ω_1 with a single branch point O, which coincides with the origin of the original (u, v)-plane. The surface Ω_1 is constructed in the standard way by "gluing" a certain (possibly infinite) number of copies (sheets) of Euclidean coordinate planes along the corresponding boundaries of cuts made along the coordinate ray Ou, $u \in [0, +\infty)$. The conventional depiction of a multi-sheeted surface Ω_1 is shown in Figure 3.6.2.

Starting from the universal setting of the basic Darboux problem (3.6.28), (3.6.29) in polar coordinates, we formulate its generalized setting

[15] Clearly, the arguments used above allow one to correctly solve a Darboux problem of the type (3.6.28), (3.6.29) for domains of other type, for instance, for a "sector" with the vertex at the origin of coordinates and angle smaller than π, and so on.

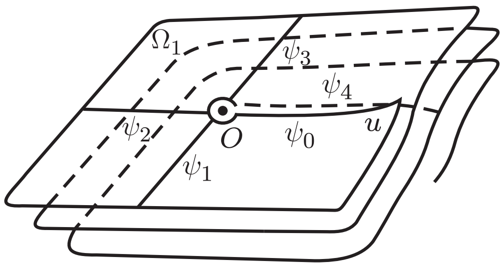


Figure 3.6.2

for the multi-sheeted domain Ω_1:

$$D(z(\rho,\varphi),\rho,\varphi)=0, \tag{3.6.30}$$

$$z\Big(\rho,\frac{\pi}{2}\,m\Big)=\psi_m(\rho),\quad \psi_i(0)=\psi_j(0),\quad m,i,j=0,1,2,\ldots, \tag{3.6.31}$$

$$\psi_m^{(k)}(0)=\psi_{m+2}^{(k)}(0),\quad k=0,1,2,\ldots. \tag{3.6.32}$$

Problem (3.6.30)–(3.6.32) is a *composition* of compatible Darboux problems of the form (3.6.28), (3.6.29), which together “exhaust” the entire multi-sheeted domain Ω_1. The corresponding initial data are specified by the set of functions ψ_0, ψ_1, ψ_2, ... $\psi_m,\ldots$, which agree at zero. The supplementary conditions (3.6.32) on the directional derivatives ensure the required C^k-smoothness of the solution. Thanks to the solvability of the basic problem of type (3.6.28), (3.6.29) posed in Ω_1, the problem (3.6.30)–(3.6.32) is uniquely solvable in the class of functions $z\in C^k(\Omega_1)$. The resulting generalized solution $z(\Omega_1)$ of the Darboux problem (3.6.30)–(3.6.32) coincides “locally” (within each sheet of the surface Ω_1, except at the branch point) with some solution z of the classical sine-Gordon equation (3.6.1).

2) *Example of setting of the Darboux problem on a multi-sheeted surface with a finite number of branch points.* Next let us consider the structure of a multi-sheeted surface Ω_N with a finite number N of branch points $\{A_j,\ j=1,\ldots,N\}$. Pick (and label by 0) some sheet L_0 of the surface Ω_N. On the sheet L_0 (a copy of the Euclidean plane) one can always indicate a straight line $l\subset L_0$ with direction vector $\overrightarrow{l}$, on which all branch points $A_1,\ldots,A_N$ project in a one-to-one manner (Figure 3.6.3). Suppose the gluing of the successive sheets of the multi-sheeted surface Ω_N is done along the boundaries of the cuts made along rays with the origin at the branch points $A_1,\ldots,A_N$, rays that are orthogonal to the straight line l and are directed to the same side relative to l. The “sheet-to-sheet” transition is effected for all sheets in one and the same oriented manner: “the left boundary” of the cut on one sheet is glued to the “right boundary” of the corresponding (i.e., referring to the same branch point) cut on the next sheet. The general principle guiding the construction of the multi-sheeted surface Ω_N is illustrated in Figure 3.6.3.

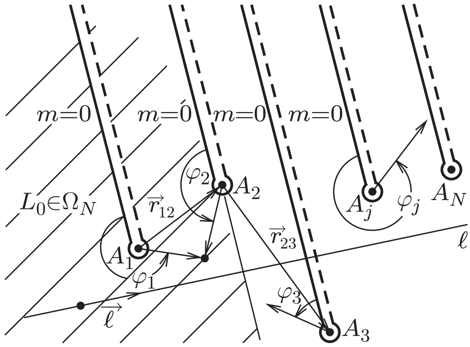


Figure 3.6.3

Let us formulate the Darboux problem for the sine-Gordon equation on the multi-sheeted surface Ω_N. To this end we introduce, for each $j = 1, \ldots, N$, "local" polar coordinates (ρ_j, φ_j) centered at the branch point A_j. For the sake of definiteness, the angles φ_j will be measured in counterclockwise direction. Also, we will consider that the "left" boundaries of cuts lie in the considered sheet L_0, while the "right" boundaries lie in the next glued sheet. The positions of the branch points $\{A_j,\ j = 1, \ldots, N\}$ will be specified by the successive set of vectors $\{\overrightarrow{r}_{j,j+1}\}$ that connect them.

These "preparations" allow us to formulate a connected "chain" of Darboux problems, which yield a regular generalized solution of the sine-Gordon equation on Ω_N.

The first link in the "chain" is the Darboux problem for the domain attached to the first branch point A_1, bounded "to the left" by the straight line containing the cut emanating from the branch point A_2 (the hatched domain in Figure 3.6.3).

This problem for a part of the solution, $z_{A_1}(\rho_1, \varphi_1)$ (in the domain corresponding to the branch point A_1), is posed as follows:

$$D(z_{A_1}(\rho_1, \varphi_1), \rho_1, \varphi_1) = 0, \tag{3.6.33}$$

$$z_{A_1}\left(\rho_1, \frac{\pi}{2}\, m\right) = \left[\begin{array}{ll} \psi^1_m(\rho_1), & \text{if } \rho_1 \in [0, +\infty), \quad m = 0, 1, 2, \\ \psi^1_3(\rho_1), & \text{if } \rho_1 \in [0, |\overrightarrow{r}_{12}| \cdot (\widehat{\overrightarrow{r}_{12}, \overrightarrow{l}})], \quad m = 3. \end{array}\right. \tag{3.6.34}$$

$$\left.\begin{array}{c} \psi^1_p(0) = \psi^1_q(0), \quad p, q = 0, 1, 2, 3, \\ \left.\dfrac{\partial \psi^1_\alpha}{\partial \overrightarrow{\rho}_1}\right|_{\rho_1=0} = \left.\dfrac{\partial \psi^1_{\alpha+2}}{\partial \overrightarrow{\rho}_1}\right|_{\rho_1=0}, \\ \left.\dfrac{\partial^2 \psi^1_\alpha}{\partial \overrightarrow{\rho}_1^{\,2}}\right|_{\rho_1=0} = \left.\dfrac{\partial^2 \psi^1_{\alpha+2}}{\partial \overrightarrow{\rho}_1^{\,2}}\right|_{\rho_1=0}. \end{array}\right. \quad \alpha = 0, 1. \tag{3.6.35}$$

The upper index in the functions ψ_k^j corresponds to the ordering index j of the branch point A_j. Thus, for example, the functions ψ_1^1, ψ_3^1 in (3.6.35) correspond to the setting of the problem associated with the first branch point A_1. Conditions (3.6.35) guarantee the C^2-regularity of the solution z_{A_1}.

The formulation (3.6.33)–(3.6.35) includes the setting of four consistent standard (basic) Darboux problems: two problems in quadrants and two problems in a half-strip, for which the existence and uniqueness of the solution is guaranteed by the well-posedness of the basic Darboux problem of type (3.6.30)–(3.6.32).

The next step is to successively and regularly "glue" ("left–to–right") the solution $z_{A_1}(\rho_1, \varphi_1)$ obtained above the solutions $z_{A_2}, \dots, z_{A_N}$ of the Darboux problems corresponding to the respective branch points $A_2, \dots, A_N$, i.e., the regular conjugation transition, to the right, to domains along the straight line l (Figure 3.6.3).

The transition alluded to above is given by the recurrently inter-related chain of Darboux problems of the form

$$D(z_{A_j}(\rho_j, \varphi_j), \rho_j, \varphi_j) = 0, \tag{3.6.36}$$

$$z_{A_j}\left(\rho_j, \frac{\pi}{2} m\right) = \begin{cases} z_{A_{j-1}}(\rho_{j-1}, \varphi_{j-1}), & m = 0, 1, 2, \\ \psi_3^j(\rho_j)\Big|_{\vec{\rho}_j \equiv \vec{r}_{j,j+1}}, & m = 3, \end{cases} \tag{3.6.37}$$

$$\begin{gathered} \psi_m^j(\rho_j)\Big|_{\rho_j = 0} = z_{A_{j-1}}(\rho_{j-1}, \varphi_{j-1})\Big|_{\vec{\rho}_{j-1} = \vec{r}_{j-1,j}}, \\ \left(\frac{\partial z_{A_{j-1}}}{\partial \vec{l}} - \frac{\partial z_{A_j}}{\partial \vec{l}}\right)\Bigg|_{\rho_j = 0} = 0, \\ \left(\frac{\partial^2 z_{A_{j-1}}}{\partial \vec{l}^{\,2}} - \frac{\partial^2 z_{A_j}}{\partial \vec{l}^{\,2}}\right)\Bigg|_{\rho_j = 0} = 0. \end{gathered} \tag{3.6.38}$$

Problem (3.6.36)–(3.6.38) is posed in the domain corresponding to an arbitrary "intermediate" branch point A_j.

Thus, we presented an algorithm for constructing a generalized solution $z(\Omega_N)$ of the sine-Gordon equation on the multi-sheeted surface Ω_N, expressed as a recursive chain of regular, consistent basic Darboux problems for the available domains (half-strips, sectors, quadrants, etc.) on Ω_N.

3) *On the Darboux problem on a surface with an infinite* (*countable*) *number of branch points*. In the method for constructing solutions $z(\Omega_N)$ of the sine-Gordon equation on a multi-sheeted surface Ω_N considered above, the fact that the finite set of branch points can be "ordered" plays an essential role. Indeed, it is this ability of ordering the branch points that allows one to extend the algorithm presented to the case of a multi-sheeted surface with an infinite (countable) number of branch points $\{A_j\}$, $j \in \mathbb{N}$. Such a multi-sheeted surface will be denoted by $\widetilde{\Omega}_\infty$.

An example of surface $\widetilde{\Omega}_\infty$ to which one can "extend" the algorithm worked out above for the recursive setting of the Darboux problem for the sine-Gordon equation is the multi-sheeted surface with the following distribution of branch points on any of its sheets: the branch points $\{A_j\}$ lie on the unit semi-circle (Figure 3.6.4) and are given in polar coordinates by

$$A_j: \;\; \rho_j = 1, \quad \varphi_j = \frac{\pi}{j}, \quad j \in \mathbb{N}.$$

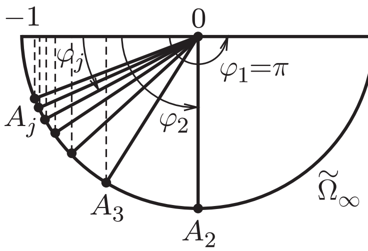


Figure 3.6.4

It is obvious that on the multi-sheeted surface $\widetilde{\Omega}_\infty$ with the sheet structure shown in Figure 3.6.4 one can give a generalized solution $z(\widetilde{\Omega}_\infty)$ of the sine-Gordon equation as the limiting solution, as $j \to \infty$, of the recursive and consistent chain of corresponding Darboux problems.

To finish the present section, we state a general proposition on the structure of pseudospherical surfaces $S[\Omega] \subset \mathbb{E}^3$ that correspond to generalized solutions $z(\Omega)$ of the sine-Gordon equation on multi-sheeted surfaces Ω. Such surfaces necessarily contain solitary (localized) singular points, corresponding to the branch points of the multi-sheeted surfaces Ω (the domains of definition of the solutions $z(\Omega)$). In neighborhoods of such singular points in space, the structure of the generalized pseudosperical surfaces $S[\Omega]$ does probably have the character of a "multi-sheeted twisting" that converges to the singular point.[16]

3.7 The Cauchy problem for the sine-Gordon equation. Unique determinacy of pseudospherical surfaces

In this section we establish the solvability (existence and uniqueness of the solution) of the Cauchy problem for the sine-Gordon equation, which has a fundamental importance for the theory of pseudospherical surfaces: a geometric application of this result is the theorem asserting that pseudospherical surfaces are uniquely determined by their irregular cuspidal edges (singularities), which will be proved in the second part of this section.

[16]Similar to a *flower bud*.

3.7.1 The Cauchy problem for the sine-Gordon equation: existence and uniqueness of the solution

Let us formulate the Cauchy problem for the sine-Gordon equation, with the initial values of the unknown function $z(u,v)$ and of its derivative $z_u(u,v)$ specified on a curve l, given by the equation $v = f(u)$:

$$\begin{gathered} z_{uv} = \sin z, \\ z(u, f(u)) = \mu(u), \\ z_u(u, f(u)) = \nu(u). \end{gathered} \tag{3.7.1}$$

We shall assume that $f(u) \in C^n$, $n \geq 2$, and $f'(u)$ has constant sign on the segment $[u_1, u_2]$, which guarantees the existence of the inverse function $f^{-1}(v) \in C^n$. For definiteness, we will assume that the function $f(u)$ is decreasing on $[u_1, u_2]$, i.e., $f'(u) < 0$ for $u \in [u_1, u_2]$. We denote $v_1 = f(u_2)$, $v_2 = f(u_1)$, and consider the rectangle $\Pi = [u_1, u_2] \times [v_1, v_2]$.

The following existence and uniqueness theorem for the solution of the Cauchy problem (3.7.1) holds true.

Theorem 3.7.1. *Let $\mu(u) \in C^n[u_1, u_2]$ and $\nu(u) \in C^{n-1}[u_1, u_2]$. Then under the above assumptions on the function $f(u)$, in the rectangle $\Pi = [u_1, u_2] \times [v_1, v_2]$ (see Figure (3.7.1)) there exists one, and only one, solution $z(u,v) \in C^n(\Pi)$ of the Cauchy problem* (3.7.1) *for the sine-Gordon equation.*

Proof. As in the case of the Darboux problem (§ 3.6), the solution of the Cauchy problem (3.7.1) will be constructed as the limit of a uniformly convergent sequence $\{z_k(u,v)\}$, given by recursion relations introduced via the integral equation equivalent to the Cauchy problem (3.7.1).

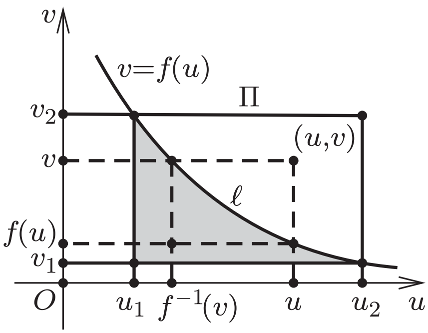


Figure 3.7.1

In the case of the rectangle Π under consideration, which consists of two curvilinear triangles with the common boundary l (Figure 3.7.1), the Cauchy problem (3.7.1) reduces to the integral equation

$$z(u,v) = \mu(f^{-1}(v)) - \int_u^{f^{-1}(v)} \nu(s)ds + \int_{f^{-1}(v)}^{u} ds \int_{f(s)}^{v} \sin z(s,t)dt. \tag{3.7.2}$$

Equation (3.7.2) is obtained by integrating twice the sine-Gordon equation, with the initial conditions accounted for. Indeed, the "first integration" of the sine-Gordon equation with respect to the variable t from $f(s)$ to v, with the last condition in (3.7.1) accounted for, yields

$$z_s(s,v) = \nu(s) + \int\limits_{f(s)}^{v} \sin z(s,t)dt.$$

The subsequent integration of this equation with respect to s from $f^{-1}(v)$ to u yields indeed the equation (3.7.2), equivalent to the original Cauchy problem (3.7.1).

Turning now to the integral equation (3.7.2), let us introduce the iterative sequence $\{z_k(u,v)\}$ by the recursion formula

$$z_{k+1}(u,v) = \mu(f^{-1}(v)) - \int\limits_{u}^{f^{-1}(v)} \nu(s)ds + \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(s)}^{v} \sin z_k(s,t)dt, \tag{3.7.3}$$

and take $z_0 \equiv 0$ as the initial iteration.

Then for the first iteration $z_1(u,v)$ we obtain

$$z_1 = \mu(f^{-1}(v)) - \int\limits_{u}^{f^{-1}(v)} \nu(s)ds.$$

Let us show that the sequence $\{z_k(u,v)\}$ constructed via (3.7.3) converges uniformly as $k \to \infty$ to the solution $z(u,v)$ of the Cauchy problem (3.7.1). To this end we derive estimates for the modulus of the difference of two successive iterations.

We have

$$|z_2(u,v) - z_1(u,v)| = \left| \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(s)}^{v} (\sin z_1(s,t) - \sin z_0(s,t))dt \right|$$

$$\leq \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(s)}^{v} |\sin z_1(s,t)|dt \leq \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(s)}^{v} dt = \int\limits_{f^{-1}(v)}^{u} (v - f(s))ds. \tag{3.7.4}$$

Looking at Figure 3.7.1, consider in the rectangle Π "the top domain"[17] lying above the curve l. Since $f(u)$ is monotonically decreasing, the integrand in the right-hand side of (3.7.4) is bounded from above by $(v - f(u))$ for $s \in [f^{-1}(v), u]$,

[17] The considerations for the "bottom domain" are analogous.

$u \in [u_1 u_2]$. Consequently,

$$|z_2(u,v) - z_1(u,v)| \leq \int\limits_{f^{-1}(v)}^{u} (v - f(s))ds \leq \int\limits_{f^{-1}(v)}^{u} (v - f(u))ds$$
$$= (u - f^{-1}(v))(v - f(u)) = \frac{(u - f^{-1}(v))(v - f(u))}{(1!)^2}. \quad (3.7.5)$$

Now let us estimate the modulus of the difference of two successive iterates:

$$|z_{k+1}(u,v) - z_k(u,v)| = \left| \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(s)}^{v} (\sin z_k(s,t) - \sin z_{k-1}(s,t))dt \right|$$
$$\leq \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(s)}^{v} |\sin z_k(s,t) - \sin z_{k-1}(s,t)|dt$$
$$= \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(s)}^{v} \left| 2 \sin \frac{z_k(s,t) - z_{k-1}(s,t)}{2} \cos \frac{z_k(s,t) + z_{k-1}(s,t)}{2} \right| dt$$
$$\leq \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(s)}^{v} |z_k(s,t) - z_{k-1}(s,t)|dt \leq \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(u)}^{v} |z_k(s,t) - z_{k-1}(s,t)|dt.$$
$$(3.7.6)$$

In (3.7.6) we used the fact that $|\sin x| \leq |x|$, $|\cos x| \leq 1$, as well as the monotonicity of the function $f(u)$.

The general estimate is obtained by induction. Thus, let us assume that for z_k and z_{k-1}

$$|z_k(u,v) - z_{k-1}(u,v)| \leq \frac{(u - f^{-1}(v))^{k-1}(v - f(u))^{k-1}}{((k-1)!)^2}. \quad (3.7.7)$$

Then by (3.7.6) we obtain

$$|z_{k+1}(u,v) - z_k(u,v)| \leq \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(u)}^{v} \frac{(s - f^{-1}(t))^{k-1}(t - f(s))^{k-1}}{((k-1)!)^2} dt$$
$$\leq \int\limits_{f^{-1}(v)}^{u} ds \int\limits_{f(u)}^{v} \frac{(s - f^{-1}(v))^{k-1}(t - f(u))^{k-1}}{((k-1)!)^2} dt \leq \frac{(u - f^{-1}(v))^k (v - f(u))^k}{(k!)^2}.$$
$$(3.7.8)$$

The upper estimate (3.7.8) thus obtained establishes the uniform convergence of the series

$$z_0 + \sum_{k=1}^{\infty} (z_k - z_{k-1})$$

on any compact domain lying above the curve l, and hence also the convergence of the sequence $\{z_k(u,v)\}$ to the solution $z(u,v)$ of the posed Cauchy problem (3.7.1).

Let us investigate the smoothness of the obtained solution $z = \lim_{k\to\infty} z_k$. Under the assumptions of the theorem, it is clear that $z_1(u,v) \in C^n(\Pi)$. Next, let us differentiate twice with respect to u and v in (3.7.3):

$$\begin{aligned} \frac{\partial^2 z_{k+1}}{\partial u^2} &= \frac{\partial^2 z_1}{\partial u^2} - f'(u) \cdot \sin z_k(u,v), \\ \frac{\partial^2 z_{k+1}}{\partial u \partial v} &= \sin z_k(u,v), \\ \frac{\partial^2 z_{k+1}}{\partial v^2} &= \frac{\partial^2 z_1}{\partial v^2} - \frac{1}{f'(u) \cdot f^{-1}(v)} \cdot \sin z_k(u,v). \end{aligned} \tag{3.7.9}$$

The equalities (3.7.9) show that the smoothness class is preserved for each successive iteration in the recursion relation (3.7.3). Consequently, the solution $z(u,v)$ of the Cauchy problem lies in $C^n(\Pi)$.

Next, let us establish the uniqueness of the solution $z(u,v)$. Suppose that $\zeta(u,v) \in C^n(\Pi)$ is another solution of the Cauchy problem and, correspondingly, of the integral equation (3.7.2). Subtracting (3.7.3) from (3.7.2), written for ζ, we have[18]

$$|\zeta - z_1| \leqslant M, \quad M = \text{const} > 0,$$

and, generally,

$$|\zeta - z_{k+1}| \leq \int_{f^{-1}(v)}^{u} ds \int_{f(s)}^{v} |\zeta - z_k| dt.$$

Hence, the following estimate holds for all k:

$$|\zeta - z_k| \leq M \cdot \frac{(u - f^{-1}(v))^k \cdot (v - f(u))^k}{(k!)^2}. \tag{3.7.10}$$

By (3.7.10), when we let $k \to \infty$, the difference $|\zeta - z|$ becomes arbitrarily small, i.e., the solutions $z(u,v)$ and $\zeta(u,v)$ coincide. Therefore, the solution $z(u,v)$ of the Cauchy problem (3.7.1) is unique.

All the arguments used above in the proof for the case of the "top triangle" in the rectangle Π can be similarly applied to the "bottom triangle" lying under the curve l (Figure 3.7.1). Therefore, the assertion of the theorem holds true for the entire rectangle Π. Theorem 3.7.1 is proved. □

[18] In view of the continuity of the functions ζ and z_1.

Based on Theorem 3.7.1, we can quite obviously talk about the *existence and uniqueness of the solution* of the Cauchy problem *in the whole* (u, v)*-plane* in the case when the curve l is given on the whole line $u \in (-\infty, +\infty)$, i.e., l is parametrized as $v = f(u)$, with $f \in C^n(-\infty, +\infty)$. In this case the proof amounts to examining the "chain" of Cauchy problems, each "link" of which corresponds to its own interval of monotonicity of the function $v = f(u)$. Transparent examples of this are the formulations of the Cauchy problems whose solutions are the one-soliton solution $z_1(u, v)$ and the two-soliton solution $z_2(u, v)$ of the sine-Gordon equation; in these cases, it is convenient to take for the lines of "*type l*" the *level lines* of the solutions, $z_1(u, v) = \pi$ and $z_2(u, v) = 0$, which are *straight lines*.

3.7.2 Theorem on unique determinacy of pseudospherical surfaces

The solvability of the Cauchy problem for the sine-Gordon equation established in Subsection 3.7.1 has an important geometric application. Namely, Theorem 3.7.1 proved above, which establishes the existence and uniqueness of the solution to the Cauchy problem (3.7.1), enables us to assert that pseudospherical surfaces are *uniquely determined* by their "irregular" edges (cuspidal edges).

Indeed, every solution $z(u, v)$ of the sine-Gordon equation does unavoidably reach the values $z = n\pi$, with n an integer, to which, according to Poznyak's theorem (Theorem 2.7.1), correspond singularities (irregular edges and so on) of the pseudospherical surface $S[z]$. The curvature and torsion of an individual *edge* of the surface, regarded as a *curve in space*, can be uniquely calculated from the equation $v = f(u)$ of its preimage in the (u, v)-parametric plane, defined by the condition $z(u, v) = n\pi$, with n an integer, and from the derivative $z_u(u, f(u))$ (see (2.7.72) and (2.7.81)). This correspondence is one-to-one, since conversely, from a given edge we can, according to (2.7.72) and (2.7.81), find $v = f(u)\colon\ z(u, f(u)) = n\pi$ and $z_u(u, f(u))$, expressions that give the initial data for the uniquely solvable Cauchy problem of the type (3.7.1). Therefore, from the resulting solution of the sine-Gordon equation one recovers the corresponding piece of the pseudospherical surface $S[z]$.

Let us formulate the discussed property that a pseudospherical surface is determined by its irregular edges as the following unique determinacy result.

Theorem 3.7.2 (Unique determinacy of pseudospherical surfaces). *Suppose there is given a space curve $L \subset \mathbb{E}^3$, specified by its radius vector $\overrightarrow{R}(s)$ (s being the natural parameter), and characterized by its curvature $k(s)$ and torsion $æ(s)$, $æ \neq \pm 1$. Then to each piece of the curve L, given by $s \in [s_1, s_2]$, on which $k(s) > 0$, one can uniquely associate a rectangle $\Pi = [u(s_1), u(s_2)] \times [v(s_1), v(s_2)]$ in the (u, v)-parametric plane, and correspondingly two pieces of a pseudospherical surface in the space $\mathbb{E}^3$, given on Π, for which L is a cuspidal edge.*

Remark. Figuratively speaking, for each regular curve $L \subset \mathbb{E}^3$ one can always (under the assumptions of Theorem 3.7.2), indicate in a unique manner a pseudospherical surface abutting to it, for which L is a cuspidal edge.

Proof of Theorem 3.7.2. Using the results of § 2.7, we resort to the formulas (2.7.72) and (2.7.81) in order to "translate" the geometric characteristics $k(s)$ and $æ(s)$ of the curve $L \subset \mathbb{E}^3$ into initial data for the solution $z(u, v)$ of the sine-Gordon equation (the net angle of the Chebyshev net of asymptotic lines on a pseudospherical surface) that this solution takes on an irregular edge of the pseudospherical surface:

$$\begin{aligned} f'(u) = (-1)^n \cdot \frac{1-æ(s)}{1+æ(s)}: \quad (u, f(u)) = n\pi, \quad n \text{ an integer}, \\ z_u(u, f(u)) = -\frac{2k_{\mathrm{g}}(s)}{1+æ(s)}, \\ \frac{ds}{du} = \pm\frac{1}{1+æ(s)}. \end{aligned} \tag{3.7.11}$$

In the right-hand sides of (3.7.11) we use the substitution $s = s(u)$. Also in (3.7.11), we consider that the preimage of the curve L in the (u, v)-parametric plane is given by the equation $v = f(u)$: $z(u, f(u)) = n\pi$, n an integer.

In essence, expressions (3.7.11) give the initial data for the solution of the Cauchy problem (3.7.1), arising from the geometric interpretation of solutions of the sine-Gordon equation. Let us clarify the resulting "specific geometric features" of the problem under consideration. The curvature k of the curve $L \subset \mathbb{E}^3$ (irregular edge) can differ from the geodesic curvature k_{g} of the same curve regarded as a curve on the corresponding pseudospherical surface only by sign: $k_{\mathrm{g}} = \pm k$. Consequently, k_{g} and $z_u(u, f(u))$ are uniquely determined, up to sign, by the data $k(s)$ and $æ(s)$. A possible change of sign in the initial data $z_u(u, f(u))$ and $z(u, f(u)) = n\pi$ in the Cauchy problem leads to a change of sign of the solution $z(u, v)$ itself. Geometrically, this is connected with changing (into the opposite) the direction of the normal vector and the direction in which one measures the coordinates u and v on the asymptotic lines.

The sign in the equality $k_{\mathrm{g}} = \pm k$ will be taken so as to preserve the C^3-smoothness required for the function $z_u(u, f(u))$, which together with this choice is ensured by the assumed smoothness of the functions $k(s(u))$ and $æ(s(u))$. Since, by the assumptions of the theorem, the torsion $æ(s)$ does not take the values ± 1, $f'(u)$ (see 3.7.11) has constant sign. For definiteness (for fixed n), we shall assume that $f'(u) < 0$. Then the sign in the expression for the derivative ds/du is chosen, for instance, so that the derivative will be positive. When all the listed conditions are satisfied, one can always indicate on the curve L a piece on which the coordinate u varies in some segment $[u_1, u_2]$ (in general, such a segment can be infinite).

Put $v_1 = f(u_1)$, $v_2 = f(u_2)$ and consider the rectangle $\Pi = [u_1, u_2] \times [v_1, v_2]$. By Theorem 3.7.1 , the Cauchy problem with the initial data (3.7.11) has in Π a unique solution $z(u, v) \in C^4(\Pi)$. The requisite C^4-smoothness of the solution is guaranteed by the corresponding requirement that the initial data be C^3-smooth.

Let us construct the pseudospherical surface $S[z]$ corresponding to the obtained solution $z(u, v) \in C^4(\Pi)$. To this end, we take some point $s_0 = s(u_1)$ on L such that $k(s_0) \neq 0$, $æ(s_0) \neq 0$, and $k(s_0)$, $æ(s_0)$ are finite. Denote this point by A_0. Let $\overrightarrow{e_1}$ denote the tangent vector to the curve L at A_0, oriented in the direction of increase of the coordinate u. Further, let $\overrightarrow{e_3}$ denote the vector that coincides at A_0 with the binormal vector $\overrightarrow{b}(A_0)$ to the curve L. We also introduce the vector $\overrightarrow{e_2}$, finally forming the right-handed triple of vectors $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$.

With these "preparations" we are now ready to use the precise statement of Theorem 2.7.2, according to which in the rectangle Π there exists a unique vector-valued function $\overrightarrow{r}(u,v) \in C^3(\Pi)$, which defines in $\mathbb{E}^3$ a pseudospherical surface $S[z(\Pi)]$ for which the curve L serves as cuspidal edge. Theorem 3.7.2 is proved. $\square$

The mapping of the rectangle Π in the (u,v)-parametric plane (Figure 3.7.1) into a piece of the pseudospherical surface $S[z]$ illustrating Theorem 3.7.2 is shown in Figure 3.7.2.

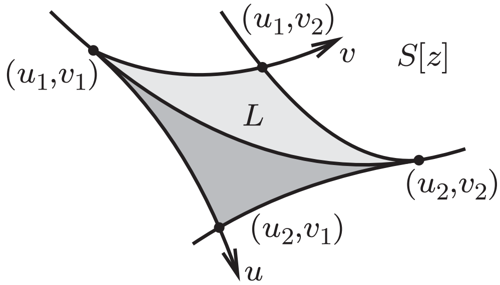


Figure 3.7.2

By picking out the rectangle Π in the proof of Theorem 3.7.2 our considerations acquire, in a certain sense, a "universal" character. Specifically, the rectangle Π is a selected domain, in which for definiteness, without diminishing the generality of our approach, we assume that the set of properties of the functions $k(s)$, $æ(s)$, $v = f(u)$ and their derivatives that ensure the solvability in Π of the Cauchy problem of the type (3.7.1) under conditions (3.7.11) are satisfied. A possible modified set of the indicated properties (corresponding, for example, to an adjacent piece of the curve L) is connected, first of all, with a change of the sign of the functions involved and does not affect the result on the solvability of the Cauchy problem in other, "adjacent" rectangles of Π type. Overall, such an approach enables one to state more generally that for every space curve L with curvature $k(s)$ and torsion $æ \neq \pm 1$ such that the C^4-smoothness of the solution of the corresponding Cauchy problem is guaranteed, there exists one, and only one pseudospherical surface $S[L]$ for which L is an irregular cuspidal edge.

In other words, *every pseudospherical surface can be uniquely recovered from its irregular*[19] *cuspidal edge* (*regular space curve*).

For a transparent explanation we draw the reader's attention to the "skeleton" consisting of the sets of space curves (Figures 3.4.1.*b*–3.4.6*b*) that play the role of cuspidal edges of two-soliton pseudospherical surfaces. For each skeleton of this type, the corresponding two-soliton pseudospherical surface "stretches" to cover it in a unique way.

[19]Irregular in the sense that the pseudospherical metric is degenerate on it

3.8 Method of separation of variables. Joachimsthal-Enneper surfaces

3.8.1 Standard separation of variables for the sine-Gordon equation

Let us describe a class of solutions of the sine-Gordon equation that are obtained by the methods of separation of variables. We take the sine-Gordon equation in the form with "wave-type" left-hand side,

$$z_{xx} - z_{tt} = \sin z, \tag{3.8.1}$$

which is most frequently encountered in physical applications. The equation $z_{uv} = \sin z$ is reduced to the form (3.8.1) by the change of variables

$$u = \frac{1}{a} \cdot \left(\frac{x+t}{2}\right), \quad v = a \cdot \left(\frac{x-t}{2}\right), \quad a = \text{const.} \tag{3.8.2}$$

The geometrical meaning of the transformation (3.8.2) will be discussed in Subsection 3.8.2.

We shall seek the solutions of equation (3.8.1) in the form

$$z(x,t) = 4\arctan\left[\frac{X(x)}{T(t)}\right]. \tag{3.8.3}$$

Substituting expression (3.8.3) in (3.8.1), we obtain the equation

$$(X^2+T^2)\left(\frac{X''}{X} + \frac{T''}{T}\right) - 2(X')^2 - 2(T')^2 = T^2 - X^2. \tag{3.8.4}$$

Here the primes denote differentiation of the involved functions $X(x)$ and $T(t)$ of one variable with respect to their arguments.

Passing to differential consequences of relation (3.8.4) allows us to separate variables as (see [51]):

$$\frac{1}{XX'}\left(\frac{X''}{X}\right)' = -\frac{1}{TT'}\left(\frac{T''}{T}\right)' = -4k^2, \quad k = \text{const.} \tag{3.8.5}$$

Each of the equations in (3.8.5) can be reduced to an equation of two orders lower:

$$\begin{aligned} (X')^2 &= -k^2X^4 + \mu_1 X^2 + \nu_1, \quad \mu_1, \nu_1 = \text{const}, \\ (T')^2 &= k^2T^4 + \mu_2 T^2 + \nu_2, \quad \mu_2, \nu_2 = \text{const}. \end{aligned} \tag{3.8.6}$$

The integration constants in (3.8.6) are related by the conditions

$$\begin{aligned} \mu_1 - \mu_2 &= 1, \\ \nu_1 + \nu_2 &= 0. \end{aligned}$$

With no loss of generality we can put $\mu_1 = m^2$ and $\nu_1 = n^2$. Let us rewrite equations (3.8.6), passing to elliptic integrals[20]

$$\begin{aligned} \pm \int \frac{dX}{\sqrt{-k^2X^4 + m^2X^2 + n^2}} &= x, \\ \pm \int \frac{dT}{\sqrt{k^2T^4 + (m^2-1)T^2 - n^2}} &= t. \end{aligned} \tag{3.8.7}$$

Therefore, the unknown functions $X(x)$ and $T(t)$ will be determined as the inverses of the elliptic integrals figuring in the left-hand sides of (3.8.7). Generally, the functions $X(x)$ and $T(t)$ that are defined by the relations (3.8.7) and give the sought-for solution $z(x,t)$ (3.8.3) of the sine-Gordon equation are *transcendental*. Nevertheless,we will now give examples showing that, for special choices of the constants $\{k, m, n\}$ the corresponding solutions $z(x,t)$ of the form (3.8.3) can be expressed in terms of elementary functions. Moreover, among these solutions we will recognize some that are identical with some of the solutions we considered earlier.

Example 1. $k = 0$, $m > 1$, $n = 0$. For this choice of the constants, the integrals in (3.8.7) can be calculated in terms of elementary functions. In particular, the system (3.8.7) yields

$$X = b_1 e^{\pm mx}, \quad T = b_2 e^{\pm\sqrt{m^2-1}\cdot t}, \quad b_1, b_2 = \text{const}.$$

For these functions $X(x)$ and $T(t)$, the solution of the sine-Gordon equation (according to (3.8.3)) takes on the form

$$z(x,t) = 4\arctan\left[b \cdot \exp\left(\pm\frac{x \pm w\cdot t}{\sqrt{1-w^2}}\right)\right], \tag{3.8.8}$$

where $w = \sqrt{m^2-1}/m = \text{const}$ and $b = b_1/b_2$.

From the point of view of wave physics, a solution of the form (3.8.3) describes the motion of a one-dimensional "*shelf*"-type profile with speed w. It is quite obvious that for a suitable choice of the constant $b = b(w)$, the inverse change of variables $(x,t) \mapsto (u,v)$ brings the solution (3.8.8) to the form of the *one-soliton* solution of the type (3.2.11), studied in the first part of this chapter.

Example 2. $k = 0$, $m > 1$, $n \neq 0$. For this choice of the constant parameters involved, system (3.8.6) can again be integrated in elementary functions:

$$\begin{aligned} X &= \pm\frac{n}{m}\cdot\sinh(mx + c_1), \\ T &= \frac{n}{\sqrt{m^2-1}}\cdot\cosh(\sqrt{m^2-1}\cdot t + c_2), \quad c_1, c_2 = \text{const}. \end{aligned}$$

[20] The reader should be aware that a certain mutual relationship exists between the integrals (3.8.7) and the Jacobi special functions (see § 3.3).

In accordance with (3.8.3), this yields the following solution of the sine-Gordon equation (3.8.1):

$$z(x,t) = \pm 4 \arctan \left[\frac{\sqrt{m^2-1}}{m} \cdot \frac{\sinh(mx+c_1)}{\cosh(\sqrt{m^2-1}\cdot t + c_2)} \right]. \tag{3.8.9}$$

Let us study the "wave nature" of the solution (3.8.9). For definiteness, choose in the right-hand side of (3.8.9) the sign "+" and assume that $c_1 = c_2 = 0$. Let us verify that the solution (3.8.9) represents a *bound state* ("interaction–collision") of two solitons.

Based on the form of the solution (3.8.9), we derive its asymptotics:

1) $x \to -\infty$, $t \to -\infty$ (*distant past*):

$$z(x,t) \mapsto -4 \arctan \left[w \cdot e^{-m(x-wt)} \right], \quad w = \frac{\sqrt{m^2-1}}{m}.$$

The asymptotic representation thus obtained is a "shelf"-type profile that grows monotonically in the range from -2π to 0 and moves in the positive direction of x.

2) $x \to +\infty$, $t \to -\infty$:

$$z(x,t) \mapsto 4 \arctan \left[w \cdot e^{-m(x+wt)} \right].$$

This expression describes a "shelf" profile that grows monotonically from 0 to 2π and moves in the negative direction of x.

The two "shelf" profiles obtained (one-dimensional profiles of type (3.211)) collide (interact) at $t = 0$ and for the subsequent "positive times" acquire the following asymptotic representations:

3) $x \to -\infty$, $t \to +\infty$:

$$z(x,t) \mapsto -4 \arctan \left[w \cdot e^{-m(x+wt)} \right].$$

4) $x \to +\infty$, $t \to +\infty$ (*distant future*):

$$z(x,t) \mapsto 4 \arctan \left[w \cdot e^{m(x-wt)} \right].$$

These asymptotics unequivocally indicate that (3.8.9) is a *two-soliton* solution of the sine-Gordon equation.[21] It realizes the state of nonlinear superposition of two one-soliton solutions. Since the solution (3.8.9) varies from-2π to 2π, in physics it also referred to as 4π-*pulse*.

[21] Concerning the soliton properties of solutions, see § 3.2.

If one takes two arbitrary constants $k_1 = \text{const} > 0$, $k_2 = \text{const} > 0$ and one defines the constant m involved by

$$m = \frac{k_1 + k_2}{2\sqrt{k_1 k_2}},$$

then upon performing the inverse change of coordinates $(x, t) \mapsto (u, v)$ by formulas (3.6.2) with constant $a = \sqrt{k_1 k_2}$, it is readily verified that the solution (3.8.9) goes into the two-soliton solution (3.2.12) considered in § 3.2, written in the coordinates (u, v) of the Chebyshev net.

Example 3. $k \neq 0$, $-1 < m < 1$, $n = 0$. In this case the system (3.8.7) admits integration in elementary functions, which results in a solution of the sine-Gordon equation (3.8.1) of the form

$$z(x, t) = -4 \arctan\left[\frac{m}{\sqrt{1 - m^2}} \cdot \frac{\sin(\sqrt{1 - m^2}x + c_1)}{\cosh(mt + c_2)}\right]. \tag{3.8.10}$$

This is a breather solution of the sine-Gordon equation, analogous to the solution (3.2.29) obtained in § 3.2. Formally, the solution (3.8.10) can be derived also from the solution (3.8.9) by means of the substitution $\sqrt{m^2 - 1} = i\sqrt{1 - m^2}$, where i is the imaginary unit.

3.8.2 Joachimsthal-Enneper surfaces

Apparently, the method of separation of variables was first implemented for the sine-Gordon equation by R. Steuerwald [193], in connection with his investigation of a special class of surfaces that he referred to as *Enneper surfaces*.[22] In earlier geometric works, the surfaces in this category were referred to as *Joachimsthal surfaces*. For this reason, in order to give the deserved priority to Joachimsthal's investigations and at the same time recognize the fundamental contribution of Steuerwald's results, these surfaces, on the study of properties of which we embark now, will be referred to as *Joachimsthal-Enneper surfaces*.

Thus, by *Joachimsthal-Enneper surface* in the Euclidean space $\mathbb{E}^3$ we mean here a surface for which one family of curvature lines consists of plane curves, which lie in planes that pass through a general common fixed axis l. The other family of curvature lines consists of "spherical lines", namely, lines on spheres whose centers lie on the axis l. The Joachimsthal-Enneper surface itself intersects each such sphere along a curvature line at a 90° angle [58, 94, 127, 152].

Alternatively, one terms Joachimsthal-Enneper surface a surface for which one family of curvature lines lies in the *planes of one pencil* (with common axis l). Moreover (see [127]), such surfaces are characterized as surfaces formed by orthogonal trajectories that are curvature lines of a one-parameter family of spheres with the centers on the same straight line (the straight line l).

[22] These Enneper surfaces should be distinguished from the well-known *minimal Enneper surface*.

In the Cartesian system of coordinates of the Euclidean space $\mathbb{E}^3(X, Y, Z)$, all possible Joachimsthal-Enneper surfaces can be described in general form by quadratures, up to two arbitrary functions $R(x)$ and $V(t)$ [58, 94]:

$$\begin{aligned} X &= \frac{R(x) \cdot \sin t}{\cosh \tau}, \\ Y &= \frac{R(x) \cdot \cos t}{\cosh \tau}, \\ Z &= x - R(x) \cdot \tanh \tau, \end{aligned} \tag{3.8.11}$$

where $\tau(x, t) = \displaystyle\int \frac{dx}{R(x)} + V(t)$.

Now let us discuss the geometric content of the method of separation of variables for the sine-Gordon equation associated with the Joachimsthal-Enneper pseudospherical surfaces. The transition (3.8.2) from (u, v) to (x, t) for $a = 1$ on the pseudospherical surface $S[\vartheta]$, $\vartheta = z/2$, amounts to choosing as new coordinates the *curvature lines*, which form on $S[\vartheta]$ two families of orthogonal lines, with respect to which the fundamental quadratic forms of the surface $S[\vartheta]$ take on the form

$$\begin{aligned} \mathrm{I}(x, t) &= \cos^2 \vartheta dx^2 + \sin^2 \vartheta dt^2, \\ \mathrm{II}(x, t) &= \sin \vartheta \cos \vartheta \cdot (dx^2 - dt^2). \end{aligned} \tag{3.8.12}$$

The function $\vartheta = \vartheta(x, t)$ in (3.8.12) satisfies a "wave"-type sine-Gordon equation (3.8.1):

$$\frac{\partial^2 \vartheta}{\partial x^2} - \frac{\partial^2 \vartheta}{\partial t^2} = \sin \vartheta \cdot \cos \vartheta. \tag{3.8.13}$$

The solution $\vartheta = \vartheta(x, t)$ has the geometric meaning of the angle formed at an arbitrary point of the surface S by the positive direction of the coordinate line (curvature line) $t = \text{const}$ and the positive direction of the asymptotic line $x - t = \text{const}$ (or $v = \text{const}$).

Let us give a number of useful relations connected with curvature lines.

In the curvature line coordinates, the derivational formulas are conveniently written with respect to the orthonormal triplet of vectors $\{\vec{e}_1, \vec{e}_2, \vec{n}\}$:

$$\vec{e}_1 = \frac{\vec{r}_x}{|\vec{r}_x|}, \quad \vec{e}_2 = \frac{\vec{r}_t}{|\vec{r}_t|}, \quad \vec{n},$$

the *basic thrihedron* of the surface.

In this case

$$\vec{r}_x = \cos \vartheta \cdot \vec{e}_1, \quad \vec{r}_t = \sin \vartheta \cdot \vec{e}_2.$$

Then the corresponding derivational formulas (2.3.18) and (2.3.19) read

$$\begin{pmatrix} \overrightarrow{e_1} \\ \overrightarrow{e_2} \\ \overrightarrow{n} \end{pmatrix}_x = \begin{pmatrix} 0 & \vartheta_t & -\sin\vartheta \\ -\vartheta_t & 0 & 0 \\ \sin\vartheta & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \overrightarrow{e_1} \\ \overrightarrow{e_2} \\ \overrightarrow{n} \end{pmatrix},$$
$$\begin{pmatrix} \overrightarrow{e_1} \\ \overrightarrow{e_2} \\ \overrightarrow{n} \end{pmatrix}_t = \begin{pmatrix} 0 & \vartheta_x & 0 \\ -\vartheta_x & 0 & \cos\vartheta \\ 0 & -\cos\vartheta & 0 \end{pmatrix} \cdot \begin{pmatrix} \overrightarrow{e_1} \\ \overrightarrow{e_2} \\ \overrightarrow{n} \end{pmatrix}. \tag{3.8.14}$$

The compatibility condition of the matrix equations (3.8.14) is expressed by the sine-Gordon equation (3.8.13).

Now let us obtain expressions for the curvature and torsion of the curvature lines. The direction vector of the line (x), given by the condition $t = t_0$, is the vector $\overrightarrow{e_1}$, and the geodesic normal vector is $\overrightarrow{e_2}$. Comparing formulas (3.8.14) with the Frenet formulas (2.7.18) for a surface strip, we conclude that the geodesic torsion of the curvature line is equal to zero (which is a general property of the curvature lines of any surface), while the curvatures of the curvature line (x) under consideration are given by the formulas

$$k_{\mathrm{g}} = \frac{\vartheta_t}{\cos\vartheta}, \quad k_{\mathrm{n}} = -\tan\vartheta \qquad \text{(for the line } (x)\text{)}. \tag{3.8.15}$$

Analogous formulas hold for the line (t):

$$k_{\mathrm{g}} = -\frac{\vartheta_x}{\sin\vartheta}, \quad k_{\mathrm{n}} = \cot\vartheta \qquad \text{(for the line } (t)\text{)}. \tag{3.8.16}$$

R. Steuerwald [193] carried out a fundamental investigation of those pseudospherical surfaces (which he called Enneper surfaces), a characteristic property of which is the general form (3.8.17) of the corresponding solution $\vartheta(x,t)$ of the sine-Gordon equation (3.8.13) (which has the aforementioned geometric meaning of a special angle on $S[\vartheta]$):

$$\tan\frac{\vartheta}{2} = e^{A(x)+B(t)}, \tag{3.8.17}$$

which allows us to separate variables in equation (3.8.13).

From the expression (3.8.17) of the function $\vartheta = z/2$ it follows (see (3.8.15) and (3.8.16)) that the geodesic curvature for an arbitrarily chosen curvature line (x) is equal to $B'/\sinh(A+B)$ and, accordingly, is proportional to its normal curvature $1/\sinh(A+B)$. Therefore, the angle between the normal to the line (x) and the normal to the surface $S[\vartheta]$ is constant on each line (x) (see § 2.7). Since the geodesic torsion of a curvature line vanishes, the torsion of the line (x), regarded as a space curve, also vanishes, and hence every line (x) is a plane curve. These conclusions are in complete agreement with the definition of a Joachimsthal surface.[23]

[23] Strictly speaking, the fact that the planes containing the line (x) form a pencil (*all pass through one line*) requires additional (somewhat tedious) justification.

Let us briefly reproduce the main steps of the method of separation of variables for finding the solution in the form (3.8.17). Substitution of expression (3.8.17) in equation (3.8.13) yields

$$(A'' - B'') \cdot \cosh(A+B) + (1 + A'^2 - B'^2) \cdot \sinh(A+B) = 0. \tag{3.8.18}$$

Differentiating equation (3.8.18) separately with respect to x and to t and subsequently eliminating the functions $\cosh(A+B)$ and $\sinh(A+B)$ from the two resulting equalities, we arrive at

$$\begin{aligned} &A'[(A''^2 - B''^2) + (1 + A'^2 - B'^2)^2] + A'''[1 + B'^2 - A'^2] = 0, \\ &B'[(A''^2 - B''^2) - (1 + A'^2 - B'^2)^2] + B'''[1 + B'^2 - A'^2] = 0. \end{aligned} \tag{3.8.19}$$

Further, passing to differential consequences of (3.8.19), i.e., differentiating with respect to x and t, we get the following equations for $A(x)$ and $B(t)$:

$$\begin{aligned} A'A'''' - A''A''' - 4A'^3A'' = 0, \\ B'B'''' - B''B''' - 4B'^3B'' = 0. \end{aligned} \tag{3.8.20}$$

The order of the system (3.8.20) can be reduced twice, yielding

$$\begin{aligned} &A''^2 = (A'^2 - a^2) \cdot (A'^2 - b^2), \\ &B''^2 = (B'^2 + 1 - a^2) \cdot (B'^2 + 1 - b^2), \quad a, b = \text{const}. \end{aligned} \tag{3.8.21}$$

It is clear that, in general, the system (3.8.21) can be solved in terms of elliptic integrals. Specifically, the solution $\vartheta(x,t)$ in the form (3.8.17) is given (see [94, 193]), in the general case (for $A' \neq \text{const}$, $B' \neq \text{const}$), in terms of elliptic functions as

$$\vartheta(x,t) = 4 \arctan \left(\frac{\left(\mathrm{cn}\left(bx; \frac{a}{b}\right) + b \cdot \mathrm{dn}\left(bx; \frac{a}{b}\right)\right) \mathrm{cn}\left(\sqrt{1-b^2}\,t; \sqrt{\frac{a^2-b^2}{1-b^2}}\right)}{\sqrt{1-a^2} - \sqrt{1-b^2}\,\mathrm{dn}\left(\sqrt{1-b^2}\,t; \sqrt{\frac{a^2-b^2}{1-b^2}}\right)} \right), \tag{3.8.22}$$

where $\mathrm{cn}(x,k)$ and $\mathrm{dn}(x,k)$ are the Jacobi elliptic functions. In Steuerwald's work [193], this solution was constructed in terms of the Weierstrass σ-function.

Let us remark that in a number of cases the general solution (3.8.22) obtained above admits a representation in elementary functions. In particular, for $a^2 = b^2$ (3.8.21) reduces to the system

$$\begin{aligned} A'' &= A'^2 - a^2; \\ B'' &= B'^2 + 1 - a^2, \end{aligned}$$

which after integration leads to the already considered case of the *two-soliton* (or *breather*) solution.

For all solutions of the sine-Gordon equation (3.8.13), "selected" in the general form (3.8.19), we formulate and prove below a theorem that relates them geometrically, in a one-to-one manner, with the pseudospherical Joachimsthal-Enneper surfaces. An analogue of this theorem appeared first in the works of G. Darboux [152].

Theorem 3.8.1. *The pseudospherical surface $S[\vartheta] \subset \mathbb{E}^3(X,Y,Z)$ is a Joa-chimsthal-Enneper surface if and only if the solution $\vartheta(x,t)$ of the sine-Gordon equation* (3.8.13) *to which it is associated is of the form given by* (3.8.17) (*or, correspondingly, is given by the general formula* (3.8.22)). *Moreover, the radius vector of such a surface in the Cartesian coordinate system $\{X,Y,Z\}$ is given by*

$$\vec{r}(x,t) = \{\rho(x,t)\cos\varphi(t),\ \rho(x,t)\sin\varphi(t),\ h(x,t)\}, \tag{3.8.23}$$

where

$$\begin{aligned} \rho &= \frac{1}{ab}\sqrt{B'^2+1}\cdot\sin\frac{\vartheta}{2}, \\ \varphi &= ab\int\frac{dt}{B'^2+1}, \\ h &= \frac{1}{ab}\left(A'\cdot\cos\frac{\vartheta}{2}+\int A'^2dx\right), \qquad a,b=\text{const.} \end{aligned} \tag{3.8.24}$$

The position of the centers of the spheres involved in the definition of a Joachimsthal-Enneper surface is given by the vector $\vec{\nu} = \{0,0,\nu(x)\}$, where

$$\nu(x) = \frac{1}{ab}\left(-\frac{A''}{A'}+\int A'^2dx\right),$$

and the radii of the spheres are given by $R(x) = 1/|A'(x)|$.

If $A'(x_0) = 0$, then the corresponding "sphere" degenerates into a plane, given by the equation

$$Z = \frac{1}{ab}\int A'^2dx.$$

Proof. 1) Suppose the pseudospherical surface $S[\vartheta]$ is a Joachimsthal-Enneper surface (in the sense of the definition given at the beginning of this subsection). Let us show that the corresponding (according to Theorem 2.7.1) solution $\vartheta(x,t)$ of the sine-Gordon equation (3.8.13) has the structure (3.8.17).

Indeed, each line (t) lies on a corresponding sphere of radius $R = R(x_0)$, which intersects the surface $S[\vartheta]$ under consideration at a *right angle*, precisely along this line. Let us show that in this case the geodesic curvature of the line (t), equal to $k_{\mathrm{g}} = -\vartheta_x/\sin\vartheta$, is constant. Denote the angle between the principal normal to the line (t) and the normal to the surface by α. Then the angle between the normal to the sphere and the principal normal to the line (t) is $\alpha+(\pi/2)$. Further, since the normal curvature $k_{\mathrm{n}}^{(R)}$ of any curve on the sphere of radius R is equal to $k_{\mathrm{n}}^{(R)} = 1/R$, and since any curve on the sphere is a curvature line, it follows, by well-known formulas of classical differential geometry, that

$$\frac{1}{R} = k_{\mathrm{n}}^{(R)} = k\cos\left(\alpha+\frac{\pi}{2}\right) = -k\sin\alpha = -k_{\mathrm{g}}, \tag{3.8.25}$$

i.e.,

$$k_{\mathrm{g}} = -\frac{1}{R} = \text{const.}$$

Therefore, the geodesic curvature of the line (t) is constant. This allows us, by using formulas (3.8.15) and (3.8.16) and simple integration, to verify that in the case of a Joachimsthal-Enneper surface the solution $\vartheta(x,t)$ of the sine-Gordon equation has the general form (3.8.17).

2) Conversely, assume now that the solution $\vartheta(x,t)$ is given by expression (3.8.17), and let us show that the corresponding pseudospherical surface $S[\vartheta]$ with radius vector $\vec{r}(x,t)$ is a Joachimsthal-Enneper surface. Indeed, under our assumption the geodesic curvature of the line (t) is constant and equal to $k_{\mathrm{g}} = -A'(x_0)$.

Let us introduce the vector $\vec{\nu}$ by

$$\vec{\nu} \equiv \vec{r}(x,t) - \frac{1}{A'(x)} \cdot \vec{e}_1(x,t).$$

Then $\vec{\nu}$ depends only on x (its derivative with respect to t vanishes!). Consequently, the line (t) lies on the sphere of radius $R = 1/|A'(x_0)|$.

Let ψ denote the angle between the normal to the sphere and the normal to the surface. Then the torsion of the line (t) is $æ = -\dot{\alpha}$ (here we used the fact that $æ_{\mathrm{g}} = 0$). At the same time, $æ = -\dot{\alpha} - \dot{\psi}$, because the angle between the normal to the sphere and the principal normal of the line (t) is equal to $\alpha + \psi$, and $æ_{\mathrm{g}}^{(R)} = 0$.

Hence,

$$\dot{\psi} = 0,$$

so the angle ψ is constant.

Let us write the relations analogous to (3.8.25):

$$\frac{1}{R} = k_{\mathrm{n}}^{(R)} = k\cos(\alpha + \psi) = k_{\mathrm{n}}\cos\psi - k_{\mathrm{g}}\sin\psi. \tag{3.8.26}$$

It is clear that for constant ψ the equalities in (3.8.26) can hold only if

$$\psi = \frac{\pi}{2} \quad \text{or} \quad \psi = \frac{3\pi}{2}$$

(depending on the sign of $A'(x_0)$). This means that the sphere under consideration intersects our pseudospherical surface at a *right* angle.

To analyze the behavior of the vector $\vec{\nu}$, we calculate its derivative $\vec{\nu}_x$:

$$\vec{\nu}_x = \frac{1}{A'^2} \cdot [(A'' + A'^2\cos\vartheta) \cdot \vec{e}_1 - A'B' \cdot \sin\vartheta \cdot \vec{e}_2 + A' \cdot \sin\vartheta \cdot \vec{n}].$$

Next, using the formula

$$\cos\vartheta = \frac{A'' - B''}{1 - (A'^2 - B'^2)}$$

and equation (3.8.21), we get

$$B'' + (B'^2 + 1)\cdot\cos\vartheta = A'' + A'^2\cdot\cos\vartheta, \tag{3.8.27}$$

$$(A'' + A'^2\cdot\cos\vartheta)^2 + (B'^2+1)A'^2\cdot\sin\vartheta = a^2b^2. \tag{3.8.28}$$

The relations thus obtained lead to the important conclusion that $|\overrightarrow{\nu}_x| = ab/A'^2$, which allows us to find the derivative with respect to x of the unit vector:

$$\frac{\overrightarrow{\nu}_x}{|\overrightarrow{\nu}_x|} = \frac{1}{ab}[(A'' + A'^2\cos\vartheta)\cdot\overrightarrow{e}_1 - (A'B'\sin\vartheta)\cdot\overrightarrow{e}_2 + (A'\sin\vartheta)\cdot\overrightarrow{n}], \tag{3.8.29}$$

which turns out to be equal to zero. Therefore, the *unit vector* (3.8.29) (the tangent vector to the line of centers of the spheres that contain the line (t)) is *constant*. This means that the line of centers of sphere itself is a *straight line*. Thus, we have established that the pseudospherical surface $S[\vartheta]$ under consideration is a Joachimsthal-Enneper surface.

3) Let us present an approach for deriving the expressions (3.8.23), (3.8.24) for the radius vector $\overrightarrow{r}(x,t)$ of the Joachimsthal-Enneper pseudospherical surface $S[\vartheta]$. With no loss of generality, we take the constant unit vector (3.8.29) as the unit basis vector $\{0,0,1\}$ of the Cartesian coordinate system in the Euclidean space $\mathbb{E}^3(X,Y,Z)$. Consider some point $M\in S[\vartheta]$, specified by the radius vector $\overrightarrow{r}(x,t)$. Then M lies on the sphere of radius $R(x) = 1/|A'(x)|$ centered at the point C given by the vector $\overrightarrow{\nu}(x)$. Denote by M' the projection of M on the coordinate axis OZ. For the coordinate $h(x,t) = OM'$ we find that

$$h_x = \frac{1}{ab}\left(A'' + A'^2\cdot\cos\vartheta\right)\cdot\cos\vartheta,$$
$$h_t = -\frac{A'B'}{ab}\cdot\sin^2\vartheta.$$

Since

$$(\cos\vartheta)_x = -A'\cdot\sin^2\vartheta,$$
$$(\cos\vartheta)_t = -B'\cdot\sin^2\vartheta,$$

we get

$$h = \frac{1}{ab}\int\left(A''\cos\vartheta + A'^2\cos^2\vartheta\right)dx = \frac{1}{ab}\left(A'\cos\vartheta + \int A'^2 dx\right). \tag{3.8.30}$$

Further,

$$CM' = (\overrightarrow{r} - \overrightarrow{\nu})\cdot\frac{\overrightarrow{\nu}_x}{|\overrightarrow{\nu}_x|} = \frac{1}{ab}\cdot\frac{1}{A'}\left(A'' + A'^2\cos\vartheta\right).$$

Hence, using relation (3.2.28) we arrive at the expression

$$\rho^2 = R^2 - |CM'|^2 = \frac{1}{A'^2}\left(1 - \frac{1}{ab}\left(A'' + A'^2\cdot\cos\vartheta\right)^2\right) = \frac{1+B'^2}{a^2b^2}\sin^2\vartheta. \tag{3.8.31}$$

Simultaneously, we get the value of $\nu = OC = OM' - CM'$:

$$\nu = \frac{1}{ab}\left(-\frac{A''}{A'} + \int A'^2 dx\right). \tag{3.8.32}$$

The expression for the angle φ (or for $d\varphi$) in (3.8.24) is obtained from the metric relation

$$\cos^2\vartheta\, dx^2 + \sin^2\vartheta\, dt^2 = dh^2 + d\rho^2 + \rho^2 d\varphi^2$$

upon substituting in it the differentials dh and $d\rho$:

$$dh = \frac{1}{ab}\left[(A'' + A'^2\cos\vartheta)\cos\vartheta dx - A'B'\sin^2\vartheta dt\right],$$

$$d\rho = \frac{1}{ab}\left[\frac{B'}{\sqrt{1+B'^2}}\left(B''+(1+B'^2)\cos\vartheta\right)\sin\vartheta dt + A'\sqrt{1+B'^2}\sin\vartheta\cos\vartheta dx\right]$$

and subsequently using (3.8.27) and (3.8.28). Theorem 3.8.1 is proved. □

To conclude this section, let us mention again that a major contribution to the study of the Joachimsthal-Enneper surfaces considered here was made by R. Steuerwald [193]. He studied the Joachimsthal pseudospherical surfaces, which he called Enneper surfaces because the idea of deriving a formula for the radius vector of these surfaces is indeed due to A. Enneper [155] and H. Dobriner [153]. Also investigated were the surfaces obtained from them by the geometric Bäcklund transformation. For instance, in the particular case when $a = b$,

$$A' = -a\cdot\tanh(ax), \quad B' = -\sqrt{1-a^2}\cdot\tan\left(\sqrt{1-a^2}\cdot t\right)$$

one obtains the following (particular) breather solution of the sine-Gordon equation:

$$\vartheta(x,t) = 4\arctan\frac{a\cos(\sqrt{1-a^2}\cdot t)}{\sqrt{1-a^2}\cdot\cosh(ax)}.$$

In modern studies, the geometry of this solutions was considered in detail by J. J. Klein [164].

3.9 The system of structure equations of pseudospherical surfaces and the method of the inverse scattering transform

In this section we want to draw the reader's attention to the deep *geometric roots* of the Method of the Inverse Scattering Transform (MIST), a powerful modern approach to the integration of nonlinear partial differential equations [1, 30, 42, 51, 62, 111], roots that manifest themselves in the fact that the basic "priming" relations in MIST and the system of structure equations of pseudospherical surfaces (see § 2.3) are identical. *The identity in structure* of the important key relations that arise in both the implementation of MIST and in considering the problem of the realizability of the Lobachevsky geometry in the Euclidean space $\mathbb{E}^3$, points at the fundamental primary value of these two mathematical branches, which emerged more then a century apart in time.

3.9.1 The Method of the Inverse Scattering Transform: "priming" relations and applications

Apparently, the first result that started the development of the MIST is the 1965 discovery by N. Zabusky and M. Kruskal [196] of solitary wave-form solutions – *solitons* – for the well-known nonlinear evolutionary Korteweg-de Vries equation. In 1967 a group of researchers (C. S. Gardner, J. M. Green, M. D. Kruskal, and R. M. Miura) proposed a method for the integration of nonlinear partial differential equations based on the application of ideas from the direct and inverse scattering problem [158]. Then in 1968 this approach was essentially generalized and algorithmically formalized by P. Lax [170] (who introduced the nowadays well-known operator L-A pair, the *Lax pair*). In 1971 the method of integrating equations in question was effectively developed by V. E. Zakharov and A. B. Shabat [32] in order to obtain solutions of a rather wide collection of nonlinear partial differential equations, among them, the nonlinear Schrödinger equation. Based on the aforementioned ideas and results, in the years 1973–74 Ablowitz, Kaup, Newell and Segur [131–133] finally formalized a scheme (known as the AKNS scheme) for the application of the inverse scattering transform (IST) to the integration of nonlinear equations of mathematical physics.

In general terms, at the foundations of MIST lies the idea of associating to the *nonlinear partial differential equation* under study a *linear system* of differential equations that represents the correct formulation of the direct problem of scattering on some corresponding potential. This transition from a "nonlinear equation to a linear system" allows one to apply for the integration of the former the well developed arsenal of methods of scattering theory (methods of the direct and inverse problems) [30, 42, 62]. Of crucial importance in this approach is the fact that the nonlinear partial differential equation that is being integrated expresses exactly the compatibility (consistency) condition of the system that gives the direct scattering problem. Effecting the "transition" from the nonlinear equation to the associated linear system is some kind of an *art* and is connected with a certain intuitive search, which in case of success allows one to speak of the potential solvability of the original nonlinear equation.

Let us present formally what in principle is the "starting" stage of MIST. In the general case of the AKNS scheme in MIST, the direct scattering problem is formulated for a linear system of the form

$$\begin{aligned} \overrightarrow{v}_x &= X \cdot \overrightarrow{v}, \\ \overrightarrow{v}_t &= T \cdot \overrightarrow{v}. \end{aligned} \tag{3.9.1}$$

for the n-dimensional vector function $\overrightarrow{v}$. Here X and T are $n \times n$ matrices.

The compatibility condition for the system (3.9.1), $(\overrightarrow{v_x})_t = (\overrightarrow{v_t})_x$, leads to the important relation

$$X_t - T_x + [X,T] = 0, \tag{3.9.2}$$

where the commutator defined as $[X,T] = XT - TX$.

For a given matrix X in (3.9.2) one can, in principle, propose (strictly speaking, not always in an obvious way) a certain procedure for constructing a matrix T such that the resulting relation (3.9.2) takes on the form of the (integrable) nonlinear partial differential equation we are interested in. Here the fact that equation (3.9.2) is not trivial is guaranteed in MIST by the time-independence of a certain (spectral) parameter ξ that must appear in the operator X: $\xi_t = 0$. Furthermore, the complete solution of the nonlinear equation (3.9.2), with which the formulation of the original problem (3.9.1) is associated, can be constructed only in the case in which the the scattering problem corresponding to the equation under study is completely solvable [1, 30, 62].

The fact that a given nonlinear differential equation can possibly be integrated by means of MIST assumes that a special linear system (direct scattering problem (3.9.1)) is associated with this equation, and this is done in such a way that the *integrable equation* itself expresses the *compatibility condition* of that system (condition of the type (3.9.2)).

As an example, let us consider a widely used version of the linear scattering problem, namely, the modified *Zakharov-Shabat* problem, in the case of a two-dimensional vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$:

$$\begin{aligned} v_{1x} &= -i\xi v_1 + q(x,t)\cdot v_2, \\ v_{2x} &= r(x,t)\cdot v_1 + i\xi v_2, \end{aligned} \tag{3.9.3}$$

(here ξ is a spectral parameter and i is the imaginary unit), with general dependence on time:

$$\begin{aligned} v_{1t} &= A(x,t)v_1 + B(x,t)v_2, \\ v_{2t} &= C(x,t)v_1 + D(x,t)v_2. \end{aligned} \tag{3.9.4}$$

The functions A, B, C, D in (3.9.4) (which do not depend on v_1 and v_2) are subject to determination.

The compatibility conditions for equations (3.9.3) and (3.9.4),

$$\left(\frac{\partial}{\partial x} v_j\right)_t = \left(\frac{\partial}{\partial t} v_j\right)_x, \quad j = 1, 2,$$

lead to the following system of equations for A, B, C, D:

$$\begin{aligned} A_x &= q\cdot C - r\cdot B, \\ B_x + 2ix\cdot B &= q_t - 2A\cdot q, \\ C_x - 2ix\cdot C &= r_t - 2A\cdot r, \\ A &= -D. \end{aligned} \tag{3.9.5}$$

Let us solve the system (3.9.5) with respect to A, B, C as functions of r and q. To this end we can, for example, represent the sought-for coefficients in (3.9.4) by polynomials in the spectral parameter ξ. If then for A, B, C we restrict to terms of order at most two (i.e., up to ξ^2):

$$\begin{aligned} A &= A_0 + A_1\xi + A_2\xi^2, \\ B &= B_0 + B_1\xi + B_2\xi^2, \\ C &= C_0 + C_1\xi + C_2\xi^2, \end{aligned} \tag{3.9.6}$$

then it is routine to find from (3.9.4) the coefficients $A_l, B_l, C_l,\quad l = 1, 2, 3$, as functions of q and r. Here a *necessary condition* for A, B, C of the form (3.9.6) to be *solutions* of the system (3.9.5) is that the following two relations be satisfied:

$$\begin{aligned} q_{xx} &= q_t + 2q^2 r, \\ -r_{xx} &= r_t - 2qr^2. \end{aligned} \tag{3.9.7}$$

Thus, the system (3.9.7) expresses the compatibility condition of problem (3.9.3), (3.9.4) under the assumption that the functions A, B, C and D figuring in (3.9.4) have the structure (3.9.6).

One should mention here, in general, that it is *possible* to suitably *choose* q and r in (3.9.7) as functions that depend in a special manner on some new unknown function $u(x, t)$ and its partial derivatives, so that the system (3.9.7) will be reduced to an equivalent single partial differential equation for u,[24]

$$\mathcal{F}[u(x, t)] = 0. \tag{3.9.8}$$

In other words, the compatibility condition of the original problem (3.9.3) can be expressed by a single (as a rule, nonlinear) equation. The extended meaning of this statement will be made completely clear by the content of Chapter 4 (through the consideration of the concept of Λ^2-representation for differential equations). In the examples given below one can assume that $q \equiv u(x, t)$.

Consider the system (3.9.7). Upon choosing $r = \mp q^*$, (3.9.7) reduces to a single (unique) equation, the *nonlinear Schrödinger equation*:

$$iq_t = q_{xx} \pm 2q^* q^2.$$

If in the representation (3.9.6) of A, B, C one takes polynomials in ξ of degree up to and including order 3 (i.e., ξ^3), then arguments similar to those given above lead to the well-known *Korteweg-de Vries equation* (KdV) [1]

$$q_t + 6qq_x + q_{xxx} = 0$$

(for the choice $r = -1$), and the *modified Korteweg-de Vries equation* (MKdV):

$$q_t \pm 6q^2 q_x + q_{xxx} = 0$$

[24] Symbolically, the left-hand side in (3.9.8) represents a generalized differential operator.

(for the choice $r = \mp q^*$).

Let us provide another example. For the choice

$$A = \frac{a(x,t)}{\xi}, \quad B = \frac{b(x,t)}{\xi}, \quad C = \frac{c(x,t)}{\xi} \tag{3.9.9}$$

fulfillment of the compatibility conditions (3.9.5) for the problem (3.9.3), (3.9.4) at hand leads to the relations

$$a_x = \frac{1}{2}\left(qr\right)_t, \quad q_{xt} = -4iaq, \quad r_{xt} = -4iar. \tag{3.9.10}$$

If, for example, we take

$$a = \frac{i}{4}\cos u(x,t), \quad b = c = \frac{i}{4}\sin u(x,t), \quad q = -r = -\frac{u_x}{2}, \tag{3.9.11}$$

in (3.9.10), then we obtain the well-known sine-Gordon equation

$$u_{xt} = \sin u. \tag{3.9.12}$$

The choice

$$a = \frac{i}{4}\cosh u(x,t), \quad b = -c = \frac{i}{4}\sinh u(x,t), \quad q = r = \frac{u_x}{2} \tag{3.9.13}$$

results in the sinh-Gordon equation

$$u_{xt} = \sinh u. \tag{3.9.14}$$

Thus, we listed above examples of reductions of the system (3.9.3), (3.9.4) that gives the direct scattering problem, to various nonlinear equations of the type (3.9.8), which serve as the corresponding compatibility condition. Side by side we can also consider the inverse algorithmic scheme: going from a differential equation (3.9.8) to the setting of the scattering problem (3.9.3), (3.9.4). It is exactly this (second) formulation of the problem that lies at the foundations of the implementation of of MIST. The first main problem that arises in this way in MIST is: given a nonlinear equation (3.9.8), how to recover the linear problem (3.9.3), (3.9.4)? Equivalently, how from a given nonlinear differential equation (3.9.8) to construct the corresponding matrices X and T that are used in (3.9.3) and (3.9.1), respectively? Achieving success in this problem is essentially an art. And the known practical realizations of the algorithmic scheme "nonlinear equation $\longmapsto$ direct scattering problem" for equations such as sine-Gordon, Korteweg-de Vries, nonlinear Schrödinger, and other, given in numerous works [30, 32, 111, 131, 170], established the methodological foundations of MIST.

It is amazing that the methodology discussed above, which lies at the foundations of the apparatus of *modern* MIST, shares *unified* deep *roots* with the *classical theory of pseudospherical surfaces*. Moreover, the "geometric view" of nonlinear partial differential equations from the positions of non-Euclidean hyperbolic geometry allows one to use in their study well developed method of various branches of geometry. Below, in the second part of this section, we deal with the explicit connection "MIST – pseudospherical surfaces".

3.9.2 Pseudospherical surfaces and MIST

A paradoxical "observation", due to R. Sasaki [188,189] which provides a geometric foundation for the modern MIST, is that the system of type (3.8.9) (compatibility condition in the AKNS approach) is *structurally indentical* to the system of structure equations (2.3.52) for pseudospherical surfaces in $\mathbb{E}^3$.

Briefly, Sasaki's result looks absolutely simple and intuitive: if for the 1-forms ω^1, ω^2, ω_1^2 in the system of equations (2.3.52) we choose the expressions

$$\begin{aligned} \omega^1 &= (r+q)dx + (C+B)dt, \\ \omega^2 &= \eta dx + 2Adt, \quad \eta = -2ix, \\ \omega_1^2 &= (r-q)dx + (C-B)dt, \end{aligned} \tag{3.9.15}$$

then the system (3.5.52) of structure equations of pseudospherical surfaces of Gaussian curvature $K \equiv -1$ becomes identical with the system (3.5.9), which expresses the compatibility condition of the direct scattering problem in the AKNS approach in MIST. That is to say, for a correctly formulated "starter" problem in MIST it is always possible to construct 1-forms ω^1, ω^2, ω_1^2 of a pseudospherical surface.

Sasaki accompanied his general result by examples of such well-known nonlinear partial differential equations as sine-Gordon, Korteweg-de Vries, modified Korteweg-de Vries, and others. Let us present these examples in a somewhat modified form, by using a constant spectral parameter η: $\eta = \text{const}$.

Example 1. *The sine-Gordon equation.* The choice of 1-forms

$$\begin{aligned} \omega^1 &= \frac{1}{\eta}\sin u\, dt, \\ \omega^2 &= \eta dx + \frac{1}{\eta}\cos u\, dt, \\ \omega_1^2 &= u_x dx \end{aligned} \tag{3.9.16}$$

reduces the system of structure equations (2.3.52), and correspondingly, the compatibility conditions (3.9.5) (with (3.9.15) kept in mind), to the sine-*Gordon equation*

$$u_{xt} = \sin u$$

and corresponds to the choice of basic functions (3.9.9) and (3.9.11) in MIST.

Example 2. *The Korteweg-de Vries equation.* If we consider the 1-forms

$$\begin{aligned} \omega^1 &= (1-u)dx + (-u_{xx} + \eta u_x - \eta^2 u - 2u^2 + \eta^2 + 2u)dt, \\ \omega^2 &= \eta\, dx + (\eta^3 + 2\eta u - 2u_x)dt, \\ \omega_1^2 &= -(1+u)dx + (-u_{xx} + \eta u_x - \eta^2 u - 2u^2 - \eta^2 - 2u)dt, \end{aligned} \tag{3.9.17}$$

then similarly, the fulfillment of the system of structure equations for pseudospherical surfaces (for $K \equiv -1$), leads to the *Korteweg-de Vries equation*

$$u_t = u_{xxx} + 6uu_x.$$

Example 3. *Modified Korteweg-de Vries equation.* The set of 1-forms

$$\begin{aligned}
\omega^1 &= -\eta u_x dt, \\
\omega^2 &= \eta\, dx + \left(\frac{1}{2}\,\eta u^2 + \eta^3\right) dt, \\
\omega_1^2 &= u\, dx + \left(u_{xx} + \frac{1}{2}\,u^3 + \eta^2 u\right) dt
\end{aligned} \tag{3.9.18}$$

leads, in the framework of the geometrical interpretation under consideration, to the *modified Korteweg-de Vries equation* (*MK-dV equation*):

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x.$$

At the present time sufficiently wide classes of differential equations amenable to the above interpretation, and hence, presumably integrable by means of MIST, are known. In this connection we should mention, first of all, a series of works by K. Tenenblat and her colleagues [139, 149, 150, 162, 195] in which criteria for the validity of the geometric interpretation under consideration are established for nonlinear partial differential equations of various types. The most completely studied in the cited works is the geometric interpretation of *evolutionary* equations (equations of parabolic type) in the context of their integrability in the framework of the AKNS approach. In these cases the "geometric formulation" of the problem at hand (the problem of finding the corresponding 1-forms of a pseudospherical surface) is specified, as a rule, by a special choice of the 1-form ω^2 with a constant coefficient (parameter) η in front of dx:

$$\omega^2 = \eta\, dx + \cdots, \quad \eta = \text{const.}$$

In Chapter 4 we will present a generalized geometric "point of view" on these kind of problems in the framework of the geometric concept of Λ^2-equations, which are generated by special coordinate nets on the Lobachevsky plane Λ^2.

Thus, if a certain nonlinear equation of type (3.9.8) one is interested in admits its own integration by MIST (with a chosen starter system (3.9.3), (3.9.4)), the for any regular solution $u(x,t)$ of this equation one can construct 1-forms ω^1, ω^2, ω_1^2 that are associated with a pseudospherical surface. In other words, for each solution of such an equation one can write a *pseudospherical metric*

$$ds^2 = (\omega^1[u])^2 + (\omega^2[u])^2, \quad K \equiv -1.$$

Furthermore, it turns out that for wide classes of nonlinear equations one can associate pseudospherical metrics also outside of the framework of MIST. Moreover, the very formulation of the direct scattering problem of the type (3.9.3), (3.9.4) for the equation under study can be done based on only its possible interpretation in the setting of the methodology of Lobachevsky geometry (theory of pseudospherical surfaces). Such a geometric approach to the study of nonlinear differential equations will be considered in Chapter 4.

Chapter 4

Lobachevsky geometry and nonlinear equations of mathematical physics

In this chapter we present a geometric approach to the interpretation of nonlinear partial differential equations which connects them with special coordinate nets on the Lobachevsky plane Λ^2. We introduce the class of Lobachevsky differential equations (Λ^2-class), which admit the aforementioned interpretation. The development of this geometric approach to nonlinear equations of contemporary mathematical physics enables us to apply in their study the rather well developed apparatus and methods of non-Euclidean hyperbolic geometry. Many known nonlinear equations, in particular, the sine-Gordon, Korteweg-de Vries, Burgers, Liouville, and other equations, which form the Λ^2-class, are generated by their own coordinate nets on the Lobachevsky plane Λ^2. This allows us to study the equations by means of net (intrinsic-geometrical) methods on the basis of Lobachevsky geometry. Overall, Chapter 4 is devoted to the application of geometric methods of hyperbolic geometry to the constructive investigation of equations of Λ^2-class.

4.1 The Lobachevsky class of equations of mathematical physics

In this section we introduce the notion of the Lobachevsky class of differential equations, which enables us to give to many nonlinear equations of contemporary mathematical physics a universal "net-type" geometric interpretation, based on Lobachevsky's non-Euclidean hyperbolic geometry [77, 79, 183–185]. Such an approach opens avenues for the application of tools and methods of non-Euclidean geometry to the study of partial differential equations of various types.

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4.1.1 The Gauss formula as a generalized differential equation

Let us consider in the parameter (x,t)-plane the quadratic differential form

$$ds^2 = E[u(x,t)]dx^2 + 2F[u(x,t)]dxdt + G[u(x,t)]dt^2, \tag{4.1.1}$$

whose coefficients,

$$E = E[u(x,t)], \quad F = F[u(x,t)], \quad G = G[u(x,t)], \tag{4.1.2}$$

depend on some unknown function $u(x,t)$ and its partial derivatives with respect to x and t.

Let us calculate the "curvature of the quadratic form" (4.1.1), using the Gauss formula (2.3.28):

$$\begin{aligned} K = &-\frac{1}{4W^2[u]} \cdot \det \begin{bmatrix} E[u] & (E[u])_x & (E[u])_t \\ F[u] & (F[u])_x & (F[u])_t \\ G[u] & (G[u])_x & (G[u])_t \end{bmatrix} \\ &- \frac{1}{2\sqrt{W[u]}} \left\{ \frac{\partial}{\partial t} \left(\frac{(E[u])_t - (F[u])_x}{\sqrt{W[u]}} \right) - \frac{\partial}{\partial x} \left(\frac{(F[u])_t - (G[u])_x}{\sqrt{W[u]}} \right) \right\}, \end{aligned} \tag{4.1.3}$$

where $W[u] = E[u] \cdot G[u] - F^2[u]$.

The right-hand side of (4.1.3) is the familiar (for the *given form of the coefficients* (4.1.2)) expression of the curvature K in terms of the coefficients $E[u]$, $F[u]$, $G[u]$ and their partial derivatives with respect to x and t (of order up to and including two).

If we assume that the curvature is an *a priori given* function $K = K(x,t)$, then the resulting relation (4.1.3) can be interpreted as a differential equation for $u(x,t)$:

$$\mathcal{F}[u(x,t)] = 0. \tag{4.1.4}$$

And conversely, if $u(x,t)$ is a solution of the differential equation (4.1.4), the quadratic form (4.1.1) defines in the parameter (x,t)-plane a metric with the square of the linear element given by (4.1.1) and with the given curvature $K(x,t)$. Thus, one can say that the metric (4.1.1) (or the differential form (4.1.1)) with its a priori prescribed curvature $K(x,t)$) *generates* (via (4.1.3)) the differential equation (4.1.4) for the function $u(x,t)$.

The equations generated in the aforementioned sense for the a priori choice of the constant negative curvature $K(x,t) \equiv -1$ (the case of the Lobachevsky plane Λ^2) will be called Λ^2-*equations*. The class of differential equations formed by the Λ^2-equation will be referred to as the *Lobachevsky class* (or the Λ^2-*class*).

In the more general case, when the curvature function $K = K(x,t)$ is arbitrary, we will say that the corresponding differential equation (an equation generated by a metric of variable curvature) belongs to the *G-class* (*the Gauss class*); such equations will be referred to as *G-equations*.

Let us clarify the geometric interpretation of equations introduced above on a number of examples of known nonlinear equations of mathematical physics.

Example 1. Consider the quadratic form (Chebyshev net metric):

$$ds^2 = dx^2 + 2\cos u(x,t)dxdt + dt^2. \tag{4.1.5}$$

In this case the coefficients are

$$E[u] = 1, \quad F[u] = \cos u(x,t), \quad G[u] = 1.$$

Calculating the curvature $K(x,t)$ of the form (4.1.5) by the Gauss formula (4.1.3) we get

$$\begin{aligned} K(x,t) = &-\frac{1}{4\sin^4 u}\det\begin{bmatrix} 1 & 0 & 0 \\ \cos u & -u_x \sin u & -u_t \sin u \\ 1 & 0 & 0 \end{bmatrix} \\ &-\frac{1}{2\sin u}\left\{\frac{\partial}{\partial t}\left[\frac{u_x \sin u}{\sin u}\right] + \frac{\partial}{\partial x}\left[\frac{u_t \sin u}{\sin u}\right]\right\}, \end{aligned}$$

and so we arrive at the following G-equation:

$$u_{xt} = -K(x,t)\sin u(x,t) \tag{4.1.6}$$

(the *Chebyshev equation*).

Equation (4.1.6) is the already familiar to us (see § 2.5) equation that "governs" the variation of the net angle of the Chebyshev net of lines for the given curvature $K(x,t)$.

When $K \equiv -1$, (4.1.6) becomes the sine-Gordon equation[1]

$$u_{xt} = \sin u. \tag{4.1.7}$$

Example 2. Let us take a metric of the form

$$ds^2 = \eta^2 dx^2 + 2\eta\left(\frac{1}{2}\,\eta u^2 + \eta^3\right)dxdt + \left[\eta^2 u_x^2 + \left(\frac{1}{2}\,\eta u^2 + \eta^3\right)^2\right]dt^2, \tag{4.1.8}$$

where $\eta = \text{const}$. In this case

$$E[u] = \eta^2, \quad F[u] = \eta\left(\frac{1}{2}\,\eta u^2 + \eta^3\right),$$
$$G[u] = \eta^2 u_x^2 + \left(\frac{1}{2}\,\eta u^2 + \eta^3\right)^2.$$

Setting $K \equiv -1$ (i.e., working in the *Lobachevsky plane* Λ^2), the Gauss formula (4.1.3) yields the Λ^2-equation

$$u_t = \frac{3}{2}\,u^2 u_x + u_{xxx} \tag{4.1.9}$$

[1]In this chapter, following the mathematical physics traditions, we write the sought-for solution of the differential equation in question as $u = u(x,t)$, where x and t are the independent variables.

(the *modified Korteweg-de Vries equation*).

Hence, the modified Korteweg-de Vries equation (MKdV) (4.1.9) is also defined by a coordinate net on the Lobachevsky plane (given by the form (4.1.8) of the metric). It is is natural to call such a net an MKdV-net.

Example 3. For the metric

$$ds^2 = \frac{e^u}{2}(dx^2 + dt^2), \tag{4.1.10}$$

with the coefficients

$$E[u] = \frac{e^u}{2}, \quad F[u] = 0, \quad G[u] = \frac{e^u}{2}$$

we obtain for $K \equiv -1$ the equation

$$\Delta_2 u = e^u, \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} \tag{4.1.11}$$

(the *elliptic Liouville equation*).

If $u(x,t)$ is a solution of equation (4.1.11), then in accordance with (4.1.1), on the Lobachevsky plane there arises a net $\{(x,t)\}$ (the *Liouville net*) with the linear element (4.1.10), namely, the *isothermal* coordinate net.

The examples given above show how differential equations can be generated by metrics of a special form. As we will see later, many "concrete" nonlinear equations of mathematical physics belong to the Λ^2-class, i.e., are generated by pseudospherical metrics (metrics of curvature $K \equiv -1$). In general, the condition that the curvature of the generating metric is constant, $K \equiv$ const, is important, since in this case the curvature acquires the special meaning of an *invariant*, i.e., *it is preserved by transformations* generated by nets on two-dimensional smooth manifolds $\mathcal{M}_2$, connected with the realization of geometric algorithms for the integration of equations.

We should remark also that the geometric interpretation of equations introduced above, together with its clear geometric content is *universal*, since it "exhaust" all possible types in the standard classification of differential equations (as this was demonstrated on examples of hyperbolic, parabolic and elliptic equations, respectively).

It is also important to note that the *nonlinearity* in the "geometrically" derived equations of mathematical physics is *primarily a result of the nontriviality of the curvature* of the generating metric, as well as of the nonlinearity of its discriminant W.

The membership of equations in the Λ^2-class assumes that they possess certain general properties of geometric origin, the discussion of which we begin in the next subsection.

To finish the present subsection, we make an observation connected with the theory of nets [127]: Giving on the two-dimensional manifold $\mathcal{M}_2$ a metric of the type (4.1.1),

$$ds^2 = g_{ij}[u]dx^i dx^j, \quad g_{ij}[u] = \begin{pmatrix} E[u] & F[u] \\ F[u] & G[u] \end{pmatrix}, \tag{4.1.12}$$

is equivalent to giving on $\mathcal{M}_2$ a smooth tensor field (g_{ij}) of type $\binom{0}{2}$ that has the symmetry property

$$g_{ij} = g_{ji}$$

and is positive definite.

Every nondegenerate symmetric tensor g_{ij} gives rise to a net of lines on $\mathcal{M}_2$, the directing pseudovectors (tangent vectors to the one-parameter families of lines) of which, v_j and w_j, are solutions of the equation[2] (see [127])

$$g_{ij}x^i x^j = 0.$$

The specification of two fields of independent vectors v_j and w_j defines on $\mathcal{M}_2$ a two-parametric net of coorodinate lines $\{(x,t)\}$, $x \equiv x^1$, $t \equiv x^2$.

Therefore, it is totally correct to assert that a differential equation of the type (4.1.4) is generated not only by the metric (4.1.1) corresponding to it, but also by its "geometric preimage", the coordinate net on the two-dimensional smooth manifold $\mathcal{M}_2$ (and, in particular, on the Lobachevsky plane Λ^2).

4.1.2 Local equivalence of solutions of Λ^2-equations

Membership of equations in the Λ^2-class assumes that they have a general intrinsic-geometrical nature. In this subsection we give a theorem on the transformation of local solutions of Λ^2-equations which establishes their *local equivalence* [77, 79, 185].

Theorem 4.1.1 (Local equivalence of Λ^2-equations). *Suppose two different analytic differential equations belong to the Λ^2-class. Then from a local analytic solution of one of these equations one can always construct a local analytic solution of the other, and conversely.*

In the case where one of the Λ^2-equations in Theorem 4.1.1 is the sine-Gordon equation, the content of this the theorem is concretized in Theorem 4.1.2.

Theorem 4.1.2. *Suppose an analytic equation of type* (4.1.4) *belongs to the* Λ^2*-class. Then for any local analytic solution* $u(x,t)$ *of this equation one can always construct a local analytic solution* $z(\widetilde{x},\widetilde{t})$ *of the sine-Gordon equation*

$$z_{\widetilde{x}\widetilde{t}} = \sin z(\widetilde{x},\widetilde{t}), \quad z = z(\widetilde{x},\widetilde{t})$$

by means of the formula

$$\cos z = \left[\frac{\partial f_1}{\partial \widetilde{x}}\,\frac{\partial f_1}{\partial \widetilde{t}}\,E[u(x,t)] + \left(\frac{\partial f_1}{\partial \widetilde{x}}\,\frac{\partial f_2}{\partial \widetilde{t}} + \frac{\partial f_1}{\partial \widetilde{t}}\,\frac{\partial f_2}{\partial \widetilde{x}}\right)F[u(x,t)] \right.$$
$$\left.\left. + \frac{\partial f_2}{\partial \widetilde{x}}\,\frac{\partial f_2}{\partial \widetilde{t}}\,G[u(x,t)]\right]\right|_{\substack{x = f_1(\widetilde{x},\widetilde{t})\\ t = f_2(\widetilde{x},\widetilde{t})}}, \tag{4.1.13}$$

[2]This equation gives the pseudovectors of the net, i.e., specifies the ratios x^1/x^2.

where $E[u], F[u], G[u]$ *are the coefficients of the pseudospherical metric that generates equation* (4.1.4)*.*

The functions f_1 *and* f_2 *appearing in* (4.1.13) *satisfy the system*

$$\begin{aligned} &\frac{\partial^2 f_1}{\partial\widetilde{x}\partial\widetilde{t}} + \Gamma^1_{\alpha\beta}\frac{\partial f_\alpha}{\partial\widetilde{x}}\frac{\partial f_\beta}{\partial\widetilde{t}} = 0, \\ &\frac{\partial^2 f_2}{\partial\widetilde{x}\partial\widetilde{t}} + \Gamma^2_{\alpha\beta}\frac{\partial f_\alpha}{\partial\widetilde{x}}\frac{\partial f_\beta}{\partial\widetilde{t}} = 0, \end{aligned} \qquad (\alpha,\ \beta = 1, 2), \tag{4.1.14}$$

where $\Gamma^1_{\alpha\beta}$*,* $\Gamma^2_{\alpha\beta}$ *are the Christoffel symbols of the pseudospherical metric that generates the* Λ^2*-equation* (4.1.4)*, written in the variables* $x \equiv f_1$*,* $t \equiv f_2$ *(i.e.,* $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}(f_1, f_2)$*,* $\alpha, \beta, \gamma = 1, 2$*).*

Remark. The transformations established in theorems 4.1.1 and 4.1.2 are connected exclusively with a change of the independent variables and geometrically correspond to passing from one coordinate net to another in the plane Λ^2.

The proof of theorems 4.1.1 and 4.1.2 is prepared by § 2.5, which treats in detail the properties of Chebyshev nets and the conditions for passing to these nets in a regular domain on a surface, as well as by the methodology of Λ^2-equations introduced in Subsection 4.1.1. Hence, without repeating the arguments that we already used in the construction of Chebyshev nets, in the proof of the theorems given here the main attention is paid to the specifics of the corresponding algorithm in the case we are interested in, when the original given two-dimensional net is the net associated with a metric that generates a Λ^2-equation.

Proof of Theorem 4.1.2. Consider an Λ^2-equation of the type (4.1.4), as in the formulation of Theorem 4.1.2. Then this equation is generated by its corresponding metric

$$(ds^2)_1 = E[u]dx^2 + 2F[u]dxdt + G[u]dt^2, \quad K \equiv -1. \tag{4.1.15}$$

Let us determine whether it is possible to reduce the metric $(ds^2)_1$ to the Chebyshev metric

$$(ds^2)_2 = d\widetilde{x}^2 + 2\cos z(\widetilde{x}, \widetilde{t})d\widetilde{x}d\widetilde{t} + d\widetilde{t}^2, \quad K \equiv -1, \tag{4.1.16}$$

i.e., whether it is possible to pass from the existing net $T(x, t)$ that generates equation (4.1.4) to the Chebyshev net $\mathrm{Cheb}(\widetilde{x}, \widetilde{t})$.

Suppose that such a transition

$$T((x, t); (ds^2)_1) \longmapsto \mathrm{Cheb}((\widetilde{x}, \widetilde{t}); (ds^2)_2) \tag{4.1.17}$$

is effected on the plane Λ^2 by means of the transformation

$$x = x(\widetilde{x}, \widetilde{t}), \quad t = t(\widetilde{x}, \widetilde{t}), \tag{4.1.18}$$

and its correctness is guaranteed by the condition

$$\frac{D(x, t)}{D(\widetilde{x}, \widetilde{t})} = \frac{\partial x}{\partial\widetilde{x}}\frac{\partial t}{\partial\widetilde{t}} - \frac{\partial x}{\partial\widetilde{t}}\frac{\partial t}{\partial\widetilde{x}} \neq 0. \tag{4.1.19}$$

Let us determine the conditions on the transformation (4.1.18), (4.1.19), under which it maps the net $T(x,t)$ into the Chebyshev net $\mathrm{Cheb}(\widetilde{x},\widetilde{t})$. In § 2.5 it was established that a criterion for a net to be a Chebyshev net is the vanishing of the corresponding two Christoffel symbol (see (2.5.16)), i.e., for the net $\mathrm{Cheb}(\widetilde{x},\widetilde{t})$ it holds that

$$\widetilde{\Gamma}^1_{12}=0, \quad \widetilde{\Gamma}^2_{12}=0. \tag{4.1.20}$$

As we have shown, conditions of the type (4.1.20) lead to the Servant-Bianchi system (2.5.22). Let us write this system for our case (for agreement with the notation of § 2.5, we re-denote (x,t) by (x^1,x^2) and $(\widetilde{x},\widetilde{t})$ by (y^1,y^2); also, $(x,t)\equiv(v_1,v_2)$ and $(\widetilde{x},\widetilde{t})\equiv(u_1,u_2)$, see (2.5.22)):

$$\begin{aligned}\Gamma^1_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^2}\frac{\partial x^\beta}{\partial y^1}+\frac{\partial^2 x^1}{\partial y^2\partial y^1}&=0,\\ \Gamma^2_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^2}\frac{\partial x^\beta}{\partial y^1}+\frac{\partial^2 x^2}{\partial y^2\partial y^1}&=0.\end{aligned} \tag{4.1.21}$$

The existence of a solution

$$x_1=f_1(y_1,y_2), \quad x_2=f_2(y_1,y_2) \tag{4.1.22}$$

of the system (4.1.21) means that it is possible to reduce the metric $(ds^2)_1$ (4.1.15) to the form $(ds^2)_2$ (4.1.16). In general, equations (4.1.21) establish the existence of a (virtual, in a certain sense) Chebyshev net on an arbitrary two-dimensional smooth manifold $\mathcal{M}_2$ and the degree of arbitrariness with which such a set is determined.

Now let us address the question of the unique determinacy of the transition (4.1.22) to a Chebyshev net.

Let x_1°, x_2° be some fixed values of the variables x_1, x_2 (and, accordingly, of some selected point $A(x_1^\circ,x_2^\circ)\in\mathcal{M}_2$ (or, in particular, $A(x_1^\circ,x_2^\circ)\in\Lambda^2$). Let us pick arbitrary values y_1°, y_2° that correspond in the new variables to x_1°, x_2° (coordinates of the Chebyshev net $\mathrm{Cheb}(y_1,y_2)$). In other words, in agreement with (4.1.22), we require that

$$x_1^\circ=f_1(y_1^\circ,y_2^\circ), \quad x_2^\circ=f_2(y_1^\circ,y_2^\circ). \tag{4.1.23}$$

Let $g_1(y_1)$ and $g_2(y_1)$ denote the functions that the sought-for functions $f_1(y_1,y_2)$ and $f_2(y_1,y_2)$ become when we set $y_2=y_2^\circ$:

$$f_1(y_1,y_2^\circ)=g_1(y_1), \quad f_2(y_1,y_2^\circ)=g_2(y_1).^3 \tag{4.1.24}$$

By (4.1.23), the functions g_1 and g_2 satisfy the conditions

$$g_1(y_1^\circ)=x_1^\circ, \quad g_2(y_1^\circ)=x_2^\circ. \tag{4.1.25}$$

In much the same way, let us introduce the functions $h_1(y_2)$ and $h_2(y_2)$:

$$f_1(y_1^\circ,y_2)=h_1(y_2), \quad f_2(y_1^\circ,y_2)=h_2(y_2), \tag{4.1.26}$$

[3] Obviously, the functions g_1 and g_2 can be given in a sufficiently arbitrary manner.

$$h_1(y_2^\circ) = x_1^\circ, \quad h_2(y_2^\circ) = x_2^\circ. \tag{4.1.27}$$

The freedom in the choice of the functions $g_1(y_1)$, $g_2(y_1)$, $h_1(y_2)$, $h_2(y_2)$ is restricted only by the natural condition

$$\frac{dg_1}{dy_1}\frac{dh_2}{dy_2} - \frac{dg_2}{dy_1}\frac{dh_1}{dy_2} \neq 0, \tag{4.1.28}$$

the geometric meaning of which will be made clear below.

Further, the substitution

$$y_1 = w_1 + w_2, \quad y_2 = w_1 - w_2 \tag{4.1.29}$$

brings (4.1.21) to the form of a *normal system* of second-order partial differential equations (a system solved with respect to the highest-order derivatives):

$$\begin{aligned} \frac{\partial^2 x_1}{\partial w_1^2} &= P[w_1, w_2], \\ \frac{\partial^2 x_2}{\partial w_2^2} &= Q[w_1, w_2]. \end{aligned} \tag{4.1.30}$$

Thanks to assumption, made in the theorems 4.1.1 and 4.1.2, that the functions $u(x,t)$ (the sought-for solutions of an equation of type (4.1.4)) are *analytic*, the Christoffel symbols $\Gamma^1_{\alpha\beta}$, $\Gamma^2_{\alpha\beta}$, as well as the resulting “right-hand sides” in (4.1.30), that is, the functions $P[w_1, w_2]$ and $Q[w_1, w_2]$, will also be analytic functions.

Thus, the system (4.1.30) with the initial data (4.1.23)–(4.1.27) (written in the variables w_1 and w_2) satisfies the conditions of the *Cauchy-Kovalevskaya theorem for a normal system of differential equations* [46]. By the Cauchy-Kovalevskaya theorem, the posed problem (4.1.30), (4.1.23)–(4.1.27) is always uniquely locally solvable, i.e., has a unique solution in a neighborhood of the chosen point (w_1°, w_2°):

$$y_1^\circ = w_1^\circ + w_2^\circ, \quad y_2^\circ = w_1^\circ - w_2^\circ.$$

Turning now to the variables y_1 and y_2, we conclude that in some neighborhood ω_A of the point $A(x_1^\circ, x_2^\circ) \in \Lambda^2$ there exists a unique solution (4.1.22) of the system (4.1.21) with the given initial conditions (4.1.23)–(4.1.27).

The arguments above can be interpreted geometrically as follows: the equations

$$x_1 = g_1(y_1), \quad x_2 = g_2(y_1)$$

define on Λ^2 a line that passes through the point $A(x_1^\circ, x_2^\circ)$ and represents in the new parametrization the line $y_2 = y_2^\circ$. Correspondingly, the equations

$$x_1 = h_1(y_2), \quad x_2 = h_2(y_2)$$

give the coordinate line $y_1 = y_1^\circ$ of the new net $\mathrm{Cheb}(y_1, y_2)$ that passes through the point A. Two such lines can be chosen arbitrarily, with the natural constraint that they must not be tangent to one another at the point A. (This requirement is ensured by fulfillment of condition (4.1.28).)

Thus, the solution (4.1.22) of the system (4.1.21) with the initial conditions (4.1.23)–(4.1.27), exists in some neighborhood ω_A and gives the transformation $T(x,t) \to \mathrm{Cheb}(\widetilde{x},\widetilde{t})$, which leads to the Chebyshev net of coordinate lines on Λ^2 (and, in general, on $\mathcal{M}_2$). This result has the following geometric explanation: if through the point $A \in \mathcal{M}_2$ ($A \in \Lambda^2$) one draws two intersecting (but not tangent to one another) lines l_1 and l_2, then in a sufficiently small neighborhood ω_A of A there exists a uniquely determined Chebyshev net in which l_1 and l_2 are included.

Substitution of the already obtained solution (4.1.22) in the metric (4.1.15) (keeping in mind the transformations performed above) reduces it to the form (4.1.16). Comparing the coefficients of the metric (4.1.15) that we reduced to the form (4.1.16) with the coefficients of the (original) metric (4.1.16) itself, we obtain the formula (4.1.13) for the construction of solutions of the sine-Gordon equation. Theorem 4.1.2 is proven. □

Let us make a number of comments.

Comment 4.1.1. The arbitrariness in the choice of the initial data (4.1.22)–(4.1.27) (with condition (4.1.28) in force) enables us to construct an infinite family $\{z\}$ of solutions of the sine-Gordon equation for each given solution u of the given Λ^2-equation of the type (4.1.14). Now choosing the same "base" generators for the net Cheb in the formulation of the problems for two different Λ^2-equations,

$$\mathcal{F}_1[u_1] = 0, \quad \mathcal{F}_2[u_2] = 0$$

performing the transitions

$$T_1 \longmapsto \mathrm{Cheb}, \quad T_2 \longmapsto \mathrm{Cheb},$$

and then applying Theorem 4.1.2, we arrive to a solution z of the sine-Gordon equation

$$z = \Omega_1[u_1] = \Omega_2[u_2],$$

that is shared by the two Λ^2-equations.

In view of the analyticity of the solutions u_1 and u_2 (for the corresponding Λ^2-equations), the relations obtained above imply their local equivalence, which is precisely what Theorem 4.1.1 establishes.

Comment 4.1.2. The method that we used in the proof of Theorem 4.1.1, of passing to the Chebyshev net (choosing the Chebyshev net as a *universal connecting* object) has a general character and, generally speaking, is not related to the curvature of the manifold $\mathcal{M}_2$ under consideration. Hence, if in the case of an arbitrary curvature $K = K(x,t)$ we argue in much the same way as in the proof of Theorem 4.1.2, we can obtain an analog of the transformations (4.1.13), (4.1.14) for the variable-curvature case. However, in this last case the curvature K no longer retains the meaning of an invariant of the transformation, and consequently in the formulation of Theorem 4.1.3 we need to "replace" the sine-Gordon equation by the Chebyshev equation.

Theorem 4.1.3. *For each local analytic solution $u(x,t)$ of any analytic equation generated by a metric of the type* (4.1.1) *of curvature $K(x,t)$ (G-equation), one can always construct a local analytic solution of the Chebyshev equation*

$$z_{\widetilde{x}\widetilde{t}} = -K \cdot \sin z(\widetilde{x}, \widetilde{t})$$

by means of relations (4.1.13), (4.1.14)*, with the function z in them understood as a solution of the Chebyshev equation with the coefficient $K = K(f_1(\widetilde{x}, \widetilde{t}), f_2(\widetilde{x}, \widetilde{t}))$.*

Comment 4.1.3. The transformation established above for the solutions of the Λ^2- and G-equations has a local character. This is due, on the one hand, to the local character of the Cauchy-Kovalevskaya theorem applied, and on the other, to the problem of choosing a local Chebyshev net that is completely included in the global Chebyshev set "on the entire" $\mathcal{M}_2$.

The search for a possible transformation of nonlocal solutions should be connected to the search for a universal geometric object, defined "globally"on $\mathcal{M}_2$, or on the entire surface S that realizes the isometric immersion $\mathcal{M}_2 \overset{\text{isom}}{\longmapsto} \mathbb{E}^3$. In the case of pseudospherical surfaces as such an object it is appropriate to take the net of asymptotic lines (which is a Chebyshev net), given on entire surface S.

To construct a net of asymptotic lines on S we need to consider the problem of isometric immersion of of the generating metric of the form (4.1.1) in the space $\mathbb{E}^3$. Namely, given the coefficients $E[u], F[u], G[u]$, the task is to find the coefficients $L[u]$, $M[u]$, and $N[u]$ of the second fundamental form of the surface. This in turn is connected with the integration of the system of fundamental equations of the theory of surfaces in $\mathbb{E}^3$ (the Peterson-Codazzi and Gauss equations):

$$\begin{gathered}
(L[u])_t + \Gamma^1_{11} M[u] + \Gamma^2_{11} N[u] = (M[u])_x + \Gamma^1_{12} L[u] + \Gamma^2_{12} M[u], \\
(M[u])_t + \Gamma^1_{12} M[u] + \Gamma^2_{12} N[u] = (N[u])_x + \Gamma^1_{22} L[u] + \Gamma^2_{22} M[u], \\
\frac{L[u]N[u] - M^2[u]}{E[u]G[u] - F^2[u]} = K(x,t).
\end{gathered} \tag{4.1.31}$$

The vanishing condition for the second fundamental form $\mathrm{II}(u,v)$ of the surface,

$$\mathrm{II}(u,v) = L[u]dx^2 + 2M[u]dxdt + N[u]dt^2 = 0$$

yields in a unique manner the transition from the variables (x,t) in the Λ^2-equation to the asymptotic Chebyshev coordinate set (x_a, t_a) on S determined by the sine-Gordon equation. Therefore, in this case one can talk about obtaining a "global analogue" of the transformation (4.1.18), which enables us to make the transition to the "global" Chebyshev net $\mathrm{Cheb}(x_a, t_a)$ of asymptotic lines on the entire surface S. Finding an exact solution of the system (4.1.31) is equivalent to obtaining a "*global*" *analogue of the substitution* (4.1.18), thanks to which the transformation (4.1.13), (4.1.14) acquires a "global" character.

Comment 4.1.4 (On correctness criteria for the application of approximate methods for constructing of solutions of the Λ^2- and G-equations)**.** In general, the construction of an exact nonlocal solution of the problem (4.1.13), (4.1.14), (4.1.23)–(4.1.27) has a transcendental character. For this reason we resort to possible criteria for verifying the correctness of the results obtained by the application of numerical methods.

Let $z^* = z^*(y_1, y_2)$ be an approximate solution of the Chebyshev equation (or of the sine-Gordon equation, respectively, when $K \equiv -1$). Given the function z^*, we extract its initial values

$$\begin{aligned} z^*(0, y_2) &= f_1^*(y_2), \\ z^*(y_1, 0) &= f_2^*(y_1), \\ f_1^*(0) &\simeq f_2^*(0). \end{aligned}$$

Next, from the initial data $f_1^*(y_2)$ and $f_2^*(y_1)$ we recover the "exact" solution $z(y_1, y_2)$ corresponding to them by means of successive approximations for the Chebyshev equation, written in the integral form (see § 3.6):

$$\begin{aligned} z_{m+1}(y_1, y_2) = {} & f_1^*(y_2) + f_2^*(y_1) - f_1^*(0) \\ & + \int_0^{y_1}\int_0^{y_2} [-K(y_1, y_2)] \sin z_m(y_1, y_2) dy_1 dy_2. \end{aligned} \tag{4.1.32}$$

Under the assumption that the curvature is bounded, i.e.,

$$|K(y_1, y_2)| \leq K_0, \quad K_0 = \text{const} > 0,$$

and choosing as the initial iteration in (4.1.32) $z_0 \equiv 0$, it is not hard to estimate the modulus of the difference of two successive approximations as

$$|z_{m+1} - z_m| \leq (K_0)^m \frac{(y_1 y_2)^m}{(m!)^2},$$

which established the convergence of the sequence $\{z_m\}$:

$$\{z_m(y_1, y_2)\} \to z, \quad m \to \infty.$$

The coincidence, within the limits of the admissible accuracy ("residual") δ, of the solutions z and z^*:

$$z \simeq z^* + \delta,$$

represent the *correctness criterion* for the numerical algorithm that is being implemented.

In addition to this, one can use for verification the relations obtained simultaneously with formula (4.1.13) and stipulated by the intrinsic geometry of the Chebyshev net:

$$\left(E[z^*] \cdot (f_{1_{y_1}})^2 + 2F[z^*] \cdot f_{1_{y_1}} f_{2_{y_1}} + G[z^*] \cdot (f_{2_{y_1}})^2\right)\Big|_{\substack{x = f_1(y_1, y_2), \\ t = f_2(y_1, y_2)}} = 1,$$

$$\left(E[z^*] \cdot (f_{1_{y_2}})^2 + 2F[z^*] \cdot f_{1_{y_2}} f_{2_{y_2}} + G[z^*] \cdot (f_{2_{y_2}})^2\right)\Big|_{\substack{x = f_1(y_1, y_2) \\ t = f_2(y_1, y_2)}} = 1.$$

4.2 The generalized third-order Λ^2-equation. A method for recovering the structure of generating metrics

The recipe introduced in § 4.1 for generating a differential equation (Λ^2-equation) of the type (4.1.4) from a two-dimensional pseudospherical metric of the form (4.1.1) by means of the Gauss formula (4.1.3) presumes that it yields a "final" Λ^2-equation whose order is two units higher that the order of the metric one starts with. (By the order of the metric (4.1.1) we will mean the largest order of the derivatives of the unknown function $u(x,t)$ appearing in the coefficients $E[u(x,t)]$, $F[u(x,t)]$, and $G[u(x,t)]$ of the metric).

In this section we obtain a *generalized third-order* Λ^2*-equation* (generated by a corresponding pseudospherical metric (4.1.1) of *first order*). This equation will include as partial realizations all possible Λ^2-equations of order up to and including three (among them, for example, the nonlinear evolution equations of mathematical physics that we considered earlier, as well as other equations). Moreover, the obtained generalized equation will serve as a "support" in the elaboration of algorithms for recovering generating pseudospherical metrics for the nonlinear equations under investigation. Overall, the method proposed here offers a fundamentally new "geometric" way of "priming" the method of the inverse scattering transform (setting the "primer" problem of the form (3.9.3), (3.9.4)) *based on the obtained metric that generates* the equation.

4.2.1 The generalized third-order Λ^2-equation

Let us turn now to the direct derivation of the generalized third-order Λ^2-equation. We assume that the coefficients of the quadratic differential form (4.1.1) are of the form

$$E = E(u, u_x), \quad F = F(u, u_x), \quad G = G(u, u_x), \tag{4.2.1}$$

and insert them in the Gauss formula (4.1.3).

For coefficients of the form (4.2.1) the determinant appearing in formula (4.1.3) (in the first right-hand side term) takes on the form

$$\begin{aligned}
&\det\begin{pmatrix} E[u] & (E[u])_x & (E[u])_t \\ F[u] & (F[u])_x & (F[u])_t \\ G[u] & (G[u])_x & (G[u])_t \end{pmatrix} \\
&= \det\begin{pmatrix} E & (E_u u_x + E_{u_x} u_{xx}) & (E_u u_t + E_{u_x} u_{xt}) \\ F & (F_u u_x + F_{u_x} u_{xx}) & (F_u u_t + F_{u_x} u_{xt}) \\ G & (G_u u_x + G_{u_x} u_{xx}) & (G_u u_t + G_{u_x} u_{xt}) \end{pmatrix} \\
&= \det\begin{pmatrix} E & E_u & E_{u_x} \\ F & F_u & F_{u_x} \\ G & G_u & G_{u_x} \end{pmatrix} \cdot \det\begin{pmatrix} u_x & u_t \\ u_{xx} & u_{xt} \end{pmatrix}.
\end{aligned} \tag{4.2.2}$$

In our case, for the coefficients (4.2.1), the second term in the right-hand side of (4.1.3) becomes

$$\frac{1}{2\sqrt{W}}\left\{\frac{\partial}{\partial t}\left(\frac{(E[u])_t-(F[u])_x}{\sqrt{W[u]}}\right)-\frac{\partial}{\partial x}\left(\frac{(F[u])_t-(G[u])_x}{\sqrt{W[u]}}\right)\right\}$$
$$=\frac{1}{4W^2}\left\{2W(E_{tt}-G_{xx})+(F_t-G_x)W_x-(E_t-F_x)W_t\right\}. \tag{4.2.3}$$

The "components" figuring in relations (4.2.2) and (4.2.3) are given by

$$\begin{aligned}
&\text{(a)}\quad E_x=E_uu_x+E_{u_x}u_{xx}, && E_t=E_uu_t+E_{u_x}u_{xt},\\
&\text{(b)}\quad G_x=G_uu_x+G_{u_x}u_{xx}, && G_t=G_uu_t+G_{u_x}u_{xt},\\
&\text{(c)}\quad F_x=F_uu_x+F_{u_x}u_{xx}, && F_t=F_uu_t+F_{u_x}u_{xt},
\end{aligned} \tag{4.2.4}$$

$$\begin{aligned}
&\text{(a)}\quad E_{tt}=E_{uu}u_t^2+2E_{uu_x}u_tu_{xt}+E_uu_{tt}+E_{u_xu_x}u_{xt}^2+E_{u_x}u_{xtt},\\
&\text{(b)}\quad G_{xx}=G_{uu}u_x^2+2G_{uu_x}u_xu_{xx}+G_uu_{xx}+G_{u_xu_x}u_{xx}^2+G_{u_x}u_{xxx},
\end{aligned} \tag{4.2.5}$$

$$\begin{aligned}
&\text{(a)}\quad E_tW_x=E_uW_uu_xu_t+E_uW_{u_x}u_tu_{xx}+E_{u_x}W_uu_xu_{xt}\\
&\qquad\qquad\qquad +E_{u_x}W_{u_x}u_{xt}u_{xx},\\
&\text{(b)}\quad G_xW_x=G_uW_uu_x^2+G_uW_{u_x}u_xu_{xx}+G_{u_x}W_uu_xu_{xx}\\
&\qquad\qquad\qquad +G_{u_x}W_{u_x}u_{xx}^2,\\
&\text{(c)}\quad E_tW_t=E_uW_uu_t^2+E_uW_{u_x}u_tu_{xt}+E_{u_x}W_uu_tu_{xt}\\
&\qquad\qquad\qquad +E_{u_x}W_{u_x}u_{xx}u_{xt},\\
&\text{(d)}\quad F_xW_t=F_uW_uu_xu_t+F_uW_{u_x}u_xu_{xt}+F_{u_x}W_uu_tu_{xx}\\
&\qquad\qquad\qquad +F_{u_x}W_{u_x}u_{xt}u_{xx}.
\end{aligned} \tag{4.2.6}$$

Substitution of expressions (4.2.4)–(4.2.6) in relations (4.2.2) and (4.2.3) (i.e., essentially, in the Gauss formula (4.1.3)) allow us to *interpret the Gauss formula* as a partial differential *equation* for the unknown function $u(x,t)$, which appears in the generating metric of the form (4.2.1). Hence, we arrive at a generalized Gauss equation of the third order, generated by a first-order metric of arbitrary Gaussian curvature $K(x,t)$:

$$\sum_{\alpha,\beta,\gamma=1}^{2}a_{\alpha\beta\gamma}u_{\alpha\beta\gamma}+\sum_{\alpha,\beta,\gamma,\delta=1}^{2}a_{\alpha\beta,\gamma\delta}u_{\alpha\beta}u_{\gamma\delta}+\sum_{\alpha,\beta,\gamma=1}^{2}b_{\alpha,\beta\gamma}u_\alpha u_{\beta\gamma}$$
$$+\sum_{\alpha,\beta=1}^{2}c_{\alpha,\beta}u_\alpha u_\beta+\sum_{\alpha,\beta=1}^{2}d_{\alpha\beta}u_{\alpha\beta}=-4K(x,t)\cdot W^2 \tag{4.2.7}$$

(*generalized third-order G-equation*).

Each of the indices α, β, γ, and δ in (4.2.7) can take only two values: 1 or 2. An index attached to the function $u(x,t)$ denotes the derivative with respect to the corresponding variable "x" $\equiv$ "1", "t" $\equiv$ "2"; for example, $u_1\equiv u_x$, $u_{12}\equiv u_{xt}$, and so on. All nontrivial (non-zero) coefficients of the generalized equation (4.2.7) are given below in Table 4.2.1.

In the expressions listed in the table we use the notation

$$D \equiv \begin{vmatrix} E & E_u & E_{u_x} \\ F & F_u & F_{u_x} \\ G & G_u & G_{u_x} \end{vmatrix}.$$

Table 4.2.1

u_{xxx}	a_{111}	$2WG_{u_x}$
u_{xxt}	a_{112}	$-4WF_{u_x}$
u_{xtt}	a_{122}	$2WE_{u_x}$
u_{xx}^2	$a_{11,11}$	$2WG_{u_xu_x} - W_{u_x}G_{u_x}$
$u_{xx}u_{xt}$	$a_{11,12}$	$2(W_{u_x}F_{u_x} - 2WF_{u_xu_x})$
u_{xt}^2	$a_{12,12}$	$2WE_{u_xu_x} - W_{u_x}E_{u_x}$
u_xu_{xx}	$b_{1,11}$	$4WG_{uu_x} - G_uW_{u_x} - W_uG_{u_x}$
u_xu_{xt}	$b_{1,12}$	$D + F_{u_x}W_u + W_{u_x}F_u - 4WF_{uu_x}$
u_tu_{xx}	$b_{2,11}$	$F_uW_{u_x} + W_uF_{u_x} - D - 4WF_{uu_x}$
u_tu_{xt}	$b_{2,12}$	$4WE_{uu_x} - E_uW_{u_x} - W_uE_{u_x}$
u_x^2	$c_{1,1}$	$2WG_{uu} - W_uG_u$
u_xu_t	$c_{1,2}$	$2(W_uF_u - 2WF_{uu})$
u_t^2	$c_{2,2}$	$2WE_{uu} - W_uE_u$
u_{xx}	d_{11}	$2WG_u$
u_{xt}	d_{12}	$-4WF_u$
u_{tt}	b_{22}	$2WE_u$

The obtained equation (4.2.7) with the functional coefficients given in Table 4.2.1 is the generalized third-order Gauss equation (G-equation). In the geometrically characteristic case $K(x,t) \equiv -1$ (Lobachevsky plane), equation (4.2.7) becomes the generalized third-order Λ^2-equation; below we will focus on precisely this last equation.

4.2.2 The method of structural reconstruction of the generating metrics for Λ^2-equations

Let us formulate a *general algorithm of structural reconstruction* of the generating Λ^2 metric for nonlinear $(1+1)$-equations[4] and exemplify it in detail to construct a pseudospherical metric for the modified Korteweg-de Vries equation.

The study of the problem of deriving, for a given differential equation, a *geometric interpretation* (namely, given the equation, find the corresponding Λ^2-metric that generated it) is connected with subjecting equation (4.2.7) to additional constraints, which characterize the structure of the equation under study. Derivatives of the type $\{u_{0,n}\}$, defined in the sought-for metric for all solutions of the equation under study, are taken with respect to the independent variables.

[4]In a $(1+1)$-equation the unknown function depends on *one* space variable x and *one* time variable t.

This enables us to associate to each such term containing $u_{0,n}$ the components with the corresponding terms of the initial equation. This leads to a *system of relations for the coefficients of the sought-for metric*. The derivatives of the form $u_{m,0}$, $m = 1, 2$, are replaced by expressions determined by the form of the equation under study (for instance, $u_t = \mathcal{F}[u]$ or $u_{xt} = \mathcal{F}[u]$).

As promised, we will next implement in detail the method of reconstruction of a generating pseudospherical metric in the case of the modified Korteweg-de Vries equation.

Example. *Construction of a generating Λ^2-metric for the modified Korteweg-de Vries equation* (MKdV equation). We consider the MKdV equation, well known in mathematical physics:

$$u_t = \frac{3}{2}\, u^2 u_x + u_{xxx}. \tag{4.2.8}$$

Under the assumption that the pseudospherical metric that generates equation (4.2.8) is a first-order metric with the coefficients (4.2.1),

$$ds^2 = E(u, u_x)dx^2 + 2F(u, u_x)dxdt + G(u, u_x)dt^2,$$

let us find under what (detailed) conditions on the coefficients (4.2.1) of this metric the resulting generalized equation (4.2.7) is precisely the MKdV equation.

Here it is natural to interpret the equation (4.2.8) itself as a *constraint* on the unknown function $u = u(x, t)$ and its derivatives.

To begin with, let us write several differential consequences of equation (4.2.8) that will be needed later in order to perform certain manipulations in the generalized equation (4.2.7):

$$\begin{aligned}
u_t &= \frac{3}{2}u^2 u_x + u_{xxx},\\
u_{xt} &= 3uu_x^2 + \frac{3}{2}u^2 u_{xx} + u_{xxxx},\\
u_{xxt} &= 3u_x^3 + 9uu_x u_{xx} + \frac{3}{2}u^2 u_{xxx} + u_{xxxxx},\\
u_{xxxt} &= 18u_x^2 u_{xx} + 9uu_{xx}^2 + 12uu_x u_{xxx} + \frac{3}{2}u^2 u_{xxxx} + u_{xxxxxx},\\
u_{tt} &= 9u^3u_x^2 + \frac{9}{4}u^4 u_{xx} + 18u_x^2 u_{xx} + 9uu_{xx}^2\\
&\quad + 15uu_x u_{xxx} + 3u^2 u_{xxxx} + u_{xxxxxx},\\
u_{xtt} &= 27u^2u_x^3 + 27u^3 u_x u_{xx} + 45u_x u_{xx}^2 + \frac{9}{4}u^4 u_{xxx}\\
&\quad + 33u_x^2 u_{xxx} + 33uu_{xx}u_{xxx} + 21uu_x u_{xxxxx} + u_{xxxxxxx}.
\end{aligned} \tag{4.2.9}$$

In the case of the MKdV equation and its consequences (4.2.9) considered here, the generalized Λ^2-equation (4.2.7) (for $K \equiv -1$) reduces to a differential equations that contains only derivatives of the unknown function $u(x,t)$ with respect to x of order up to and including 7:

$$4W^2 = 2WE_{u_x}\Big(27u^3u_x u_{xx} + \frac{9}{4}u^4 u_{xxx} + 27u^2u_x^3 + 21uu_x u_{xxxx}$$

$$
\begin{aligned}
&+\ 33u_x^2u_{xxx} + 45u_{xx}^2u_x + 33uu_{xx}u_{xxx} + u_{xxxxxxx}\Big) \\
&-\ 4WF_{u_x}\left(\frac{3}{2}u^2u_{xxx} + 9uu_xu_{xx} + 3u_x^3 + u_{xxxxx}\right) \\
&+\ 2Wu_{xxx}G_{u_x} + u_xu_{xx}(4WG_{uu_x} - G_uW_{u_x} - W_uG_{u_x}) \\
&+\ u_x^2(2WG_{uu} - W_uG_u) + 2Wu_{xx}G_u + u_{xx}^2(2WG_{u_xu_x} - W_{u_x}G_{u_x}) \\
&+\ 2u_{xx}\Big(\frac{3}{2}\,u^2u_{xx} + 3uu_x^2 + u_{xxxx}\Big)(W_{u_x}F_{u_x} - 2WF_{u_xu_x}) \\
&+\Big(\frac{3}{2}u^2u_x + u_{xxx}\Big)\Big(\frac{3}{2}u^2u_{xx} + 3uu_x^2 + u_{xxxx}\Big)(4WE_{uu_x} - E_uW_{u_x} - W_uE_{u_x}) \\
&+\Big(\frac{3}{2}\,u^2u_x + u_{xxx}\Big)^2(2WE_{uu} - W_uE_u) \qquad\qquad (4.2.10) \\
&+\ u_{xx}\Big(\frac{3}{2}u^2u_x + u_{xxx}\Big)(F_uW_{u_x} + W_uF_{u_x} - D - 4WF_{uu_x}) + 2WE_u\times \\
&\times\Big(\frac{9}{4}u^4u_{xx} + 9u^3u_x^2 + 3u^2u_{xxxx} + 15uu_xu_{xxx} + 18u_{xx}u_x^2 + 9uu_{xx}^2 + u_{xxxxxx}\Big) \\
&+\Big(\frac{3}{2}u^2u_{xx} + 3uu_x^2 + u_{xxxx}\Big)^2(2WE_{u_xu_x} - W_{u_x}E_{u_x}) \\
&+\ u_x\Big(\frac{3}{2}u^2u_{xx} + 3uu_x^2 + u_{xxxx}\Big)(D + F_{u_x}W_u + W_{u_x}F_u - 4WF_{uu_x}) \\
&+ 2u_x\Big(\frac{3}{2}u^2u_x + u_{xxx}\Big)(W_uF_u - 2WF_{uu}) - 4WF_u\Big(\frac{3}{2}u^2u_{xx} + 3uu_x^2 + u_{xxxx}\Big).
\end{aligned}
$$

The next step in the implementation of the reconstruction algorithm consists in "ordering" expression (4.2.1) according to groups of terms in front of the derivatives $u_{xxxxxxx}$, u_{xxxxxx}, …, u_{xxx}, … (in order of decrease of the order of differentiation). We note again that the indicated derivatives (defined on each solution u of the MKdV equation) acquire here the meaning of independent "variables".

The first ordered term, which includes the 7-th order derivative, has the form

$$2W \cdot E_{u_x} \cdot u_{xxxxxxx} + \cdots\ ; \tag{4.2.11}$$

Since relation (4.2.10) means that equation (4.2.7) holds identically on all solutions of the MKdV equation (with the constraint (4.2.9) accounted for in (4.2.7)), all "functional coefficients" in front of the derivatives of the unknown functions u in (4.2.10) must be equal to zero. An examination of the first three ordered terms, in front of the derivatives of u with respect to x of order 7, 6, and 5 in (4.2.10) leads, in conjunction with (4.2.11), to the system

$$
\begin{aligned}
2WE_{u_x} &= 0, \\
2WE_u &= 0, \\
-4WF_{u_x} &= 0.
\end{aligned} \tag{4.2.12}
$$

From (4.2.12) we obtain (under the natural assumption that $W \neq 0$):

$$E = \eta^2 = \text{const}, \quad F = F(u). \tag{4.2.13}$$

Expression (4.2.13) is the first result on the path of finding the precise form of the coefficients of the generating metric. At the same time, it allows us to simplify considerably the form of equation (4.2.10), to

$$\begin{aligned}
4W^2 = {} & 2Wu_{xxx}G_{u_x} + u_x u_{xx}(4WG_{uu_x} - G_u W_{u_x} - W_u G_{u_x}) \\
& + u_{xx}\Big(\frac{3}{2}u^2 u_x + u_{xxx}\Big)(F_u W_{u_x} - D) \\
& + u_x^2(2WG_{uu} - W_u G_u) + 2Wu_{xx}G_u + u_{xx}^2(2WGu_x u_x - W_{u_x}G_{u_x}) \\
& + u_x\Big(\frac{3}{2}u^2 u_{xx} + 3uu_x^2 + u_{xxxx}\Big)(D + W_{u_x}F_u) \\
& + 2u_x\Big(\frac{3}{2}u^2 u_x + u_{xxx}\Big)(W_u F_u - 2WF_{uu}) \\
& - 4WF_u\Big(\frac{3}{2}u^2 u_{xx} + 3uu_x^2 + u_{xxxx}\Big); \qquad (4.2.14)
\end{aligned}$$

moreover,

$$D = \eta^2 G_{u_x} F_u, \quad W_u = \eta^2 G_u - 2FF_u, \quad W_{u_x} = \eta^2 G_{u_x}.$$

Continuing the implementation of the algorithm, let us write the conditions expressing the "vanishing" of the coefficients in front of the derivatives u_{xxxx} and u_{xxx} in (4.2.14):

for u_{xxxx}:

$$2F_u \cdot (\eta^2 G_{u_x} \cdot u_x - 2W) = 0, \qquad (4.2.15)$$

for u_{xxx}:

$$\eta^2 u_x G_u F_u - 2u_x(FF_u^2 + WF_{uu}) + WG_{u_x} = 0. \qquad (4.2.16)$$

It is readily verified that the equality $F_u = 0$ cannot be a consequence of relation (4.2.15), since otherwise (recalling (4.2.13) and (4.2.16)) all coefficients of the generating metric of the type (4.2.1) would be constant.

Thus, (4.3.15) yields

$$W = \frac{1}{2}\, a^2 G_{u_x} u_x. \qquad (4.2.17)$$

Accordingly, equation (4.2.16) becomes

$$2\eta^2 u_x G_u F_u - 4u_x FF_u^2 - 2\eta^2 u_x G_{u_x} F_{uu} + \eta^2 G_{u_x}^2 = 0. \qquad (4.2.18)$$

At this iteration step of the algorithm, if one takes (4.2.17) and (4.2.18) into account, the generalized Λ^2-equation (equation (4.2.7) $\to$ (4.2.10) $\to$ (4.2.14)) can be simplified further to

$$\begin{aligned}
2\eta^2 G_{u_x}^2 u_x^2 = {} & u_{xx}\big(u_x(3u_x G_{u_x} G_{uu_x} - G_u(G_{u_x} + G_{u_x u_x} u_x)) \\
& + \frac{3}{2}u^2 u_x(F_u(G_{u_x} + G_{u_x u_x} u_x) - 2G_{u_x} F_u) + 2u_x G_{u_x} G_u) \\
& - u_{xx}^2 G_{u_x}(u_x G_{u_x u_x} - G_{u_x}) \\
& + u_x^3(2G_{u_x} G_{uu} - G_{u_x u} G_u + 3u^2(G_{u_x u} F_u - 2G_{u_x} F_{uu})). \qquad (4.2.19)
\end{aligned}$$

Continuing the algorithmic scheme, now already for equation (4.2.19), we write the "vanishing coefficients" in the remaining terms:

for u_{xx}:

$$G_{u_x}(u_x G_{u_x u_x} - G_{u_x}) = 0. \tag{4.2.20}$$

Since $W \neq 0$, relation (4.2.17) shows that in (4.2.20) we cannot have $G_{u_x} = 0$. Setting the expression inside the parentheses in (4.2.20) equal to zero, one can readily get that

$$G = \lambda(u) u_x^2 + f(u). \tag{4.2.21}$$

Moreover, for u_{xx}:

$$3u_x^2 G_{u_x} G_{u u_x} - u_x G_u G_{u_x} - u_x^2 G_u G_{u_x u_x} + \frac{3}{2} u^2 u_x F_u G_{u_x} \\ + \frac{3}{2} u^2 u_x^2 F_u G_{u_x u_x} - 3u^2 u_x G_{u_x} F_u + 2u_x G_{u_x} G_u = 0. \tag{4.2.22}$$

Using (4.2.21), equation (4.2.21) can be simplified considerably to

$$\lambda \lambda_u u_x^4 = 0, \quad \lambda = \text{const},$$

and so

$$W = \lambda \eta^2 u_x^2, \quad G_{u_x} = 2\lambda u_x.$$

The results obtained to this point allow us to rewrite equation (4.2.16) in the compact form

$$g_1(u) \cdot u_x + g_2(u) \cdot u_x^2 = 0, \tag{4.2.23}$$

where

$$g_1(u) = 2\eta^2 G_u F_u - 4F F_u^2,$$
$$g_2(u) = -4\lambda \eta^2 F_{uu} + 4\lambda^2 \eta^2.$$

Since the coefficients (4.2.23) must vanish: $g_1(u) = 0$ and $g_2(u) = 0$, it holds that

$$\begin{aligned} \eta^2 G_u &= 2F F_u \\ F_{uu} &= \lambda. \end{aligned} \tag{4.2.24}$$

The second equation in (4.2.24) immediately yields

$$F = F(u) = \frac{\lambda}{2} u^2 + C_1 u + C_2, \quad C_1, C_2 = \text{const}. \tag{4.2.25}$$

Integration of the first equation in (4.2.24) gives

$$G = \frac{1}{\eta^2} F^2 + C, \quad C = C(u_x). \tag{4.2.26}$$

From the calculation of the already obtained determinant of the metric,

$$W = EG - F^2 = \lambda \eta^2 u_x^2$$

we obtain, using the coefficients E, F, G given by the expressions (4.2.13), (4.2.25), (4.2.26),

$$C(u_x) = \lambda u_x^2, \quad \lambda = \eta^2 = \text{const.} \tag{4.2.27}$$

Substituting the coefficients E, F, G (4.2.25), (4.2.26), with relation (4.2.27) accounted for, in the generalized third-order Λ^2-equation (4.2.19), transformed to the form

$$2\lambda\eta^4 = 2\left(\lambda\Big(\frac{\lambda}{2}u^2 + C_1 u + C_2\Big) + (\lambda u + C_1)^2\right) - 3\lambda u^2\eta^2,$$

and subsequently comparing the coefficients of like powers of the function u, we obtain the exact values of the constants involved:

$$C_1 = 0, \quad C_2 = \eta^4.$$

Putting all together, we finally obtain the exact explicit representation for the coefficients of the sought-for generating metric:

$$E = \eta^2, \quad F = \eta^2\Big(\frac{u^2}{2} + \eta^2\Big), \quad G = \eta^2 u_x^2 + \eta^2\Big(\frac{u^2}{2} + \eta^2\Big)^2, \tag{4.2.28}$$

and consequently the pseudospherical metric itself that generates the modified Korteweg-de Vries equation (4.2.8):

$$ds^2 = \eta^2 dx^2 + 2\eta^2\Big(\frac{u^2}{2} + \eta^2\Big)dxdt + \left[\eta^2 u_x^2 + \eta^2\Big(\frac{u^2}{2} + \eta^2\Big)^2\right]dt^2. \tag{4.2.29}$$

Thus, we fully implemented the algorithm of the *method of structural reconstruction* of the generating pseudospherical metric for the modified Korteweg-de Vries equation. Overall, the question whether the proposed algorithm is applicable to a given nonlinear equation is directly connected with the *compatibility* (or *consistency*) *problem* , as well as with the *explicit solvability of the system of equations*, obtained on the basis of the generalized third-order Λ^2 equation, which expresses the vanishing of all the "functional coefficients" in the equation of the type (4.2.7) (in the equation (4.2.10) in each concrete case).

Let us now formulate the general scheme of the algorithm of the *method of structural reconstruction* of the generating metric for a nonlinear third-order differential equation:

1. Reduce the Λ^2-equation (4.2.7), with the differential consequences of the equation under study accounted for, to a relation whose terms are arranged according to the order of the derivatives of the unknown function $u(x,t)$. (In the example considered above, that was equation (4.2.10).)

2. Derive the system of differential equations for the coefficients of the sought-for generating metric, $E(u, u_x)$, $F(u, u_x)$, $G(u, u_x)$, from the condition that all the "functional cofficients" in front of the terms with the derivatives of the unknown function u of different orders vanish.

3. Investigate of the compatibility of the aforementioned system of differential relations. Construct exact solutions of this system.

4.3 Orthogonal nets and the nonlinear equations they generate

As one can see from the discussion above (see § 4.2), given some nonlinear equation, the recovery of its generating Λ^2- or G-metric takes a rather large amount of work. For that reason, one of the approaches that allows one, to a certain extent, to "optimize" the problem of associating to Λ^2- and G-equations the Λ^2- and G-metrics that generate them, consists in cleverly describing those equations that are generated by two-dimensional metrics that have certain specific geometric properties, namely, metrics associated with certain classes of coordinate nets on two-dimensional smooth manifolds that have intuitive geometric features. As it turns out, such nets define a considerable number of nonlinear equations of current interest in mathematical physics.

A rich class of metrics that generate a sufficient number of well-known nonlinear equations is associated with the *orthogonal nets*. Such nets are given by the condition that the second coefficient of the metric of type (4.1.1) vanishes:

$$F[u(x,t)] \equiv 0, \quad (x,t) \in \mathbb{R}^2. \tag{4.3.1}$$

Accordingly, the metric itself, written in the orthogonal coordinate system, reads

$$ds^2 = E[u]dx^2 + G[u]dt^2. \tag{4.3.2}$$

Let us study the problem of finding the G-equations generated by metrics of the form (4.3.2) (the curvature $K(x,t)$ is assumed to be arbitrary).

Setting

$$E[u] = a^2[u], \quad G[u] = b^2[u],$$

(and then $W = a^2[u]b^2[u] > 0$), we rewrite the metric (4.3.2) as

$$ds^2 = a^2[u]dx^2 + b^2[u]dt^2. \tag{4.3.3}$$

Let us substitute (4.3.3) in the Gauss formula (4.1.3). This yields the equation

$$\left\{\left(\frac{a_u}{b}\, u_t\right)_t + \left(\frac{b_u}{a}\, u_x\right)_x\right\} = -2K(x,t) \cdot W^{1/2}. \tag{4.3.4}$$

It is convenient to recast (4.3.4) as

$$\left[\left(\frac{b_u}{a}\right)u_{xx} + \left(\frac{a_u}{b}\right)u_{tt}\right] + \left[\left(\frac{b_u}{a}\right)_u u_x^2 + \left(\frac{a_u}{b}\right)_u u_t^2\right] = -2K(x,t) \cdot W^{1/2}. \tag{4.3.5}$$

Equation (4.3.5) is the general G-equation generated by metrics of the form (4.3.3), written in an orthogonal net parametrization. Let us determine under what conditions on $a[u]$ and $b[u]$ the left-hand side of (4.3.5) expresses the action of one of the standard operators of mathematical physics: the Laplace operator, the wave operator, etc.

1) Equation (4.3.5) will be *elliptic* if, in particular, its left-hand side represents the Laplacian of the function u, which is the case whenever the following system of conditions are satisfied:

$$\begin{cases} \dfrac{a_u}{b} = \eta, \\ \dfrac{b_u}{a} = \eta, \end{cases} \qquad \eta = \text{const.} \tag{4.3.6}$$

Notice that fulfillment of conditions (4.3.6) automatically implies that the terms inside the second pair of brackets in the left-hand side of (4.3.5) vanish.

Integrating the system (4.3.6), we find for $a[u]$ and $b[u]$ the expressions

$$a[u] = A_1 \cdot e^{\eta u} + A_2 \cdot e^{-\eta u},$$
$$b[u] = A_1 \cdot e^{\eta u} - A_2 \cdot e^{-\eta u}, \quad A_1, A_2 = \text{const.}$$

Therefore, if conditions (4.3.6) are satisfied, then the metric (4.3.3) takes on the form

$$ds^2 = \left(A_1 \cdot e^{\eta u} + A_2 \cdot e^{-\eta u}\right)^2 dx^2 + \left(A_1 \cdot e^{\eta u} - A_2 \cdot e^{-\eta u}\right)^2 dt^2. \tag{4.3.7}$$

The metric (4.3.7) thus obtained, written in orthogonal coordinates, generates a general elliptic G-equation of the form

$$\Delta_2 u = -\frac{1}{\eta} \cdot K(x,t) \cdot \left(A_1^2 \cdot e^{2\eta u} - A_2^2 \cdot e^{-2\eta u}\right), \tag{4.3.8}$$

where $\Delta_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial t^2}$ is the Laplace operator.

By suitably choosing the constants A_1 and A_2 appropriately we can obtain as particular cases of the general equation well-known nonlinear equations encountered in mathematical physics. Let us give such examples of metrics and the equations they generate.

a) $A_1 = \frac{1}{\sqrt{2}}, \;\; A_2 = 0, \;\; \eta = \frac{1}{2}$.

Generating metric:

$$ds^2 = \frac{e^u}{2}\, dx^2 + \frac{e^u}{2}\, dt^2. \tag{4.3.9}$$

G-equation generated – the *elliptic Liouville equation*:

$$\Delta_2 u = -K(x,t) \cdot e^u. \tag{4.3.10}$$

When $K \equiv -1$ (the case of the Lobachevsky plane Λ^2) we obtain an important subcase of equation (4.3.10):

$$\Delta_2 u = e^u. \tag{4.3.11}$$

b) $A_1 = A_2 = \frac{1}{2}$, $\eta = \frac{1}{2}$.
Generating metric:

$$ds^2 = \cosh^2 \frac{u}{2}\, dx^2 + \sinh^2 \frac{u}{2}\, dt^2. \tag{4.3.12}$$

The G-equation corresponding to the metric (4.3.12):

$$\Delta_2 u = -K(x,t) \cdot \sinh u, \tag{4.3.13}$$

and its "Λ^2-analogue", the elliptic sinh-Gordon equation:

$$\Delta_2 u = \sinh u. \tag{4.3.14}$$

2) Now let us study the *hyperbolic* G-equations, which are "included" in (4.3.5) and are generated by a metric of the general form (4.3.3). Indeed, if the conditions

$$\begin{cases} \dfrac{b_u}{a} = \eta, \\ \dfrac{a_u}{b} = -\eta, \end{cases} \qquad \eta = \text{const.} \tag{4.3.15}$$

are satisfied, then in the left-hand side of (4.3.5) one obtains the Laplace operator.

The system (4.3.15) has the solutions

$$\begin{aligned} a[u] &= C_1 \sin \eta u - C_2 \cos \eta u, \\ b[u] &= -C_2 \sin \eta u - C_1 \cos \eta u, \end{aligned} \qquad \eta = \text{const}, \quad C_1, C_2 = \text{const.} \tag{4.3.16}$$

Using (4.3.16), let us write the generating metric of general form (4.3.3) for the case at hand:

$$ds^2 = (C_1 \sin \eta u - C_2 \cos \eta u)^2\, dx^2 + (C_2 \sin \eta u + C_1 \cos \eta u)^2\, dt^2. \tag{4.3.17}$$

The metric (4.3.17) generates the general hyperbolic G-equation

$$u_{xx} - u_{tt} = -K \cdot [C_1 C_2 \cdot \cos(2\eta u) - (C_1^2 - C_2^2) \cdot \sin \eta u \cdot \cos \eta u]. \tag{4.3.18}$$

Upon choosing for the constant parameters in (4.3.18) the values

$$C_1 = 0, \quad C_2 = 1, \quad \eta = \frac{1}{2},$$

we obtain the classical Chebyshev equation (see § 2.5):

$$U_{xx} - U_{tt} = -K(x,t) \sin U, \quad U = 2u, \tag{4.3.19}$$

in the variables x, t, relative to an orthogonal coordinate system.

Let us give additional examples that demonstrates how orthogonal coordinate nets can be applied in the analysis of nonlinear equations.

Let us consider the metric (4.3.2) with coefficients of the form

$$E = E(u_x), \quad G = G(u), \quad \text{under the condition} \quad K \equiv -1.$$

Here are two examples.

a) Taking the pseudospherical metric

$$ds^2 = u_x^2 dx^2 + \sinh^2 u \, dt^2 \tag{4.3.20}$$

as the generating metric yields as Λ^2-equation the hyperbolic cosh-Gordon equation

$$u_{xt} = \cosh u. \tag{4.3.21}$$

b) The pseudospherical metric

$$ds^2 = u_x^2 dx^2 + \cosh^2 u \, dt^2 \tag{4.3.22}$$

generates the hyperbolic Λ^2-equation called the sinh-Gordon equation,

$$u_{xt} = \sinh u. \tag{4.3.23}$$

The fact that is possible to associate nonlinear equations to orthogonal generating nets on the Lobachevsky plane Λ^2 enables one to propose geometric algorythms for their integration. Such methods are treated in the next section.

4.4 Net methods for constructing solutions of Λ^2-equations

The geometric interpretation of differential equations presented in this chapter assigns to each Λ^2-equation a pseudospherical metric that generates it (or a generating coordinate net on the Lobachevsky plane Λ^2). This geometric "view" allows one to pass from the investigation of the equations themselves to the analysis of their geometric preimages – the generating coordinates nets, and thus to enlist in the study of equations the tools of non-Euclidean differential geometry. In the realization of this approach it is expedient to use sufficiently well studied integrable Λ^2-equations (for example, the sine-Gordon equation) and the corresponding coordinate nets as *canonical* (supporting) information for constructing transformations that connect them with geometric objects (nets) that characterize other equations under study. A classical example of canonical (supporting) net is the "Chebyshev" net. As we will show below, an important role is played also by the semigeodesic net, used to construct transformations between solutions of elliptic equations.

It is important to emphasize that the transformations obtained connect solutions of various Λ^2-equations and arise "*at the level*" of the transformation of the *preimages* of the equations studied – the generating nets on the Lobachevsky plane Λ^2, and they do not "touch upon" the equations themselves. That is to say, the transformations obtained are the result of transformations between various generating nets on Λ^2 and the associated transformation of solutions, but not of transformations of the equations. Here the constant negative *curvature* $K \equiv -1$ of the generating pseudospherical metrics has the *meaning of an invariant* of the transformations performed. The diagram in Figure 4.4.1 explains the general algorithm and the sequence of links in of the net approach to the construction of solutions of Λ^2-equations.

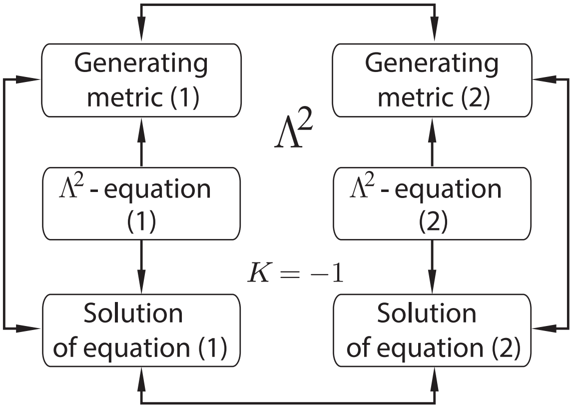

Figure 4.4.1

4.4.1 On mutual transformations of solutions of the Laplace equation and the elliptic Liouville equation

In this subsection we obtain exact explicit formulas for the construction of exact solutions of the elliptic Liouville equation [77, 90]

$$\Delta_2 u = e^u, \quad u = u(x,t) \tag{4.4.1}$$

from solutions of the Laplace equation

$$\Delta_2 v = 0, \quad v = v(x,t). \tag{4.4.2}$$

To construct solutions of the Λ^2-equation (4.1.1) we involve another (auxiliary) Λ^2-equation, namely

$$y_{\tau\tau} - y = 0, \quad y = y(\tau), \tag{4.4.3}$$

i.e., the ordinary differential equation generated by the pseudospherical metric

$$ds^2 = y^2(\tau) d\chi^2 + d\tau^2, \quad K(x,t) \equiv -1, \tag{4.4.4}$$

which plays the role of the supporting metric in our approach.

Recall that the Liouville equation (4.4.1) itself is generated by a Λ^2-metric of the form (see § 4.1)

$$ds^2 = \frac{e^u}{2}(dx^2 + dt^2). \tag{4.4.5}$$

The metric (4.4.5) generating the Liouville equation (4.4.1) is associated with the *isothermal* coordinate net on the Lobachevsky plane Λ^2, while the metric (4.4.4) that generates the Laplace equation (4.4.2) is associated with the *semigeodesic* coordinate net on Λ^2.

In the plane Λ^2, let us pass from the *semigeodesic* coordinate net $T^{\mathrm{sg}}(\chi,\tau)$ to the *isothermal* net $T^{\mathrm{is}}(x,t)$ (the Liouville net) via

$$\begin{aligned} w(x,t) &= \chi, \\ v(x,t) &= \int \frac{d\tau}{y(\tau)}. \end{aligned} \tag{4.4.6}$$

Substitution of (4.4.6) in the metric (4.4.4) reduces the latter to a metric (4.4.5), provided the following conditions are satisfied:

$$\begin{aligned} &v_x^2 + w_x^2 = v_t^2 + w_t^2, \\ &v_x v_t + w_x w_t = 0. \end{aligned} \tag{4.4.7}$$

Then the solution $u(x,t)$ of the Liouville equation (4.4.1) is given by the formula

$$u(x,t) = \ln\left[2y^2(\tau(x,t)) \cdot (v_x^2 + w_x^2)\right]. \tag{4.4.8}$$

It is easy to see that the system (4.4.7) connects two arbitrary *harmonically conjugate* functions $v(x,t)$ and $w(x,t)$, which satisfy the classical Cauchy-Riemann conditions [105]

$$\begin{aligned} v_x &= w_t, \\ v_t &= -w_x, \end{aligned} \tag{4.4.9}$$

and hence also the Laplace equation:

$$\begin{aligned} \Delta_2 v &= 0, \\ \Delta_2 w &= 0. \end{aligned} \tag{4.4.10}$$

Let us turn now to the construction of a solution $u(x,t)$ of equation (4.4.1) by means of formula (4.4.8). To this end, using the general solution

$$y(\tau) = C_1 e^{\tau} + C_2 e^{-\tau}, \quad C_1, C_2 = \mathrm{const} \tag{4.4.11}$$

of equation (4.4.3), we write the metric (4.4.4):

$$ds^2 = (C_1 e^{\tau} + C_2 e^{-\tau})^2 d\chi^2 + d\tau^2 \tag{4.4.12}$$

Now let us substitute the solution (4.4.11) in the second relation in (4.4.6). This yields the representation

$$\tau = \tau(v(x,t)),$$

which is necessary for (4.4.8).

Depending on the signs of the constants C_1 and C_2 chosen in the solution (4.4.11), the second relation in (4.4.6) yields three possible variants:

$$\begin{aligned} &1) \quad y^2(\tau(v)) = \frac{1}{v^2}, \\ &2) \quad y^2(\tau(v)) = \frac{1}{\sinh^2 v}, \\ &3) \quad y^2(\tau(v)) = \frac{1}{\sin^2 v}. \end{aligned} \tag{4.4.13}$$

Formula (4.4.8) in conjunction with (4.4.13) yields three formulas for constructing solutions of the elliptic Liouville equation (4.4.1) from an arbitrary solution $v(x,t)$, $v(x,t) \not\equiv \text{const}$, of the Laplace equation (4.4.2) [77, 90]:

$$
\begin{aligned}
u(x,t) &= \ln\left[\frac{2(v_x^2+v_t^2)}{v^2}\right],\\
u(x,t) &= \ln\left[\frac{2(v_x^2+v_t^2)}{\sinh^2 v}\right],\\
u(x,t) &= \ln\left[\frac{2(v_x^2+v_t^2)}{\sin^2 v}\right].
\end{aligned}
\tag{4.4.14}
$$

It goes without saying that the validity of the geometrically derived transformations (4.4.14) can be verified by their direct substitution in the Liouville equation (4.4.1). To this end, the following assertion proves useful.

If $\overset{(k)}{v}(x,t) \not\equiv \text{const}$ *is a solution of the Laplace equation* (4.4.2), *then the function* $\overset{(k+1)}{v}(x,t)$, *defined as*

$$
\overset{(k+1)}{v}(x,t) = \ln\left(\overset{(k)}{v}_x{}^2 + \overset{(k)}{v}_t{}^2\right),
\tag{4.4.15}
$$

is also a solution of the Laplace equation (4.4.2).

Formula (4.4.15) expresses a transformation (or self-transformation) for the Laplace equation that is analogous to the Bäcklund transformation. The transformation (4.4.15) is the natural result of applying the obtained transformation (4.4.14) to the Laplace and Liouville equations.

From the point of view of the theory of functions of a complex variables, the result obtained above implies that, *given any analytic function* $f(z) = v(x,t) + iw(x,t)$, *one can always construct* (*by means of formulas* (4.4.14)) *solutions of the three types of the elliptic Liouville equation.*

Let us give the "gradient" form of the solutions $u(x,t)$ in (4.4.14):

$$
\begin{aligned}
u(x,t) &= \ln\left[2\big(\operatorname{grad}(\ln v)\big)^2\right],\\
u(x,t) &= \ln\left[2\Big(\operatorname{grad}\Big(\ln\Big(\tanh\frac{v}{2}\Big)\Big)\Big)^2\right],\\
u(x,t) &= \ln\left[2\Big(\operatorname{grad}\Big(\ln\Big(\tan\frac{v}{2}\Big)\Big)\Big)^2\right].
\end{aligned}
\tag{4.4.16}
$$

For work connected with the study of the Liouville equation (4.4.1) we refer the reader also to [15].

4.4.2 On the equation $\Delta_2 u^* = e^{-u^*}$

Side by side with the Liouville equation (4.4.1), in applications [16, 33] one encounters also the equation of close form

$$\Delta_2 u^* = e^{-u^*}, \tag{4.4.17}$$

which is taken by the simple "reflection" $u^{**} = -u^*$ into the equation

$$\Delta_2 u^{**} = -e^{u^*}. \tag{4.4.18}$$

Like equation (4.4.1), equation (4.4.18) can be interpreted as a relation that generates a metric of the form (4.4.5), but in the case of an a priori given constant *positive* curvature $K \equiv +1$.[5]

The construction of solutions of equation (4.4.18) will be carried out by the general geometric algorithm discussed in Subsection 4.4.1. Namely, to construct the solution $u^{**}(x,t)$ of (4.4.18) we take as supporting metric the metric (4.4.4), but with prescribed constant positive curvature $K \equiv +1$. Then such a metric will generate, instead of (4.4.3), the related auxiliary equation

$$(y^{**})_{\tau\tau} + y^{**} = 0, \quad y^{**} = y^{**}(\tau). \tag{4.4.19}$$

Let us use the substitution (4.4.6) to pass from the metric (4.4.4) (the semi-geodesic net $T^{\mathrm{sg}}(\chi,\tau)$, curvature $K \equiv +1$) to the metric (4.4.5) (respectively, the isothermal net $T^{\mathrm{is}}(x,t)$, curvature $K \equiv +1$).

Starting from the general solution of the equation (4.4.19),

$$y^{**}(\tau) = C_1 \sin\tau + C_2 \cos\tau, \quad C_1, C_2 = \mathrm{const}, \tag{4.4.20}$$

we make the transition

$$T^{\mathrm{sg}}(\chi,\tau) \longmapsto T^{\mathrm{is}}(x,t).$$

Note that the relations (4.4.7) retain their form also in the case of curvature $K \equiv +1$ (up to the transformation of $y(\tau)$ into $y^{**}(\tau)$). Moreover, the function $[y^{**}(\tau)]^2$ is defined in terms of the solution y^{**} of equation (4.4.17), via the second relation in (4.4.6), as

$$[y^{**}(\tau(v))]^2 = \frac{1}{\cosh^2 v}.$$

Substituting this expression in (4.4.8) we finally construct the solution $u^*(x,t)$ (or the solution $u^{**}(x,t)$) from the solution $v(x,t)$ of the Laplace equation as

$$u^*(x,t) = \ln\left[\frac{\cosh^2 v}{2(v_x^2 + v_t^2)}\right]. \tag{4.4.21}$$

We will next discuss some important related issues arising in the study of the equation of Liouville type (4.4.1), (4.4.17), (4.4.18) at hand and the derived transformations (4.4.14)–(4.4.16) and (4.4.21).

[5] The Gaussian curvature $K \equiv +1$ is an "indicator" of spherical geometry.

4.4.3 Some applications connected with equations of Liouville type

1) *Centrally-symmetric metrics*. The well-known theoretical physics problem[6] of finding centrally-symmetric forms of two-dimensional metrics of constant curvature is connected with the search for "radial" solutions $u(r)$, $r = \sqrt{x^2 + y^2}$, of the Liouville equation (4.4.1) (for $K \equiv \text{const} < 0$) and of equation (4.4.17) (for $K \equiv \text{const} > 0$). The transformations (4.4.14) and (4.4.21) established above indicate that the search for such metrics relies on finding fundamental solutions $v(r)$ of the Laplace equation (4.4.2). Therefore, one can assert that for $K \equiv \text{const} < 0$ there exists three forms of centrally-symmetric metrics, while for $K \equiv \text{const} > 0$ there is only one such metric. It is interesting to note that the Bäcklund self-transformation (4.4.15) for the Laplace equation is the identity transformation on the "radial" solutions $v(r)$ of this equation.

2) *On problems of combustion theory.* The mathematical modeling of a number of problems of combustion theory, such as thermal explosion, forced auto-ignition, and others (which consider the thermal action of the surrounding medium on the reaction domain Ω) is connected with the study of initial-boundary value problems for the heat balance equation [16, 33]

$$\frac{\partial \vartheta}{\partial t} = \frac{1}{\delta} \Delta_2 \vartheta + e^{\vartheta},$$

where the quantity ϑ represents the temperature field in Ω. In particular, the fundamental problem of stationary theory (for $\vartheta_t \equiv 0$), which is "governed" by the Liouville-type equation of

$$\Delta_2 \vartheta_{\mathrm{ST}} + \delta e^{\vartheta_{\mathrm{ST}}} = 0,$$

is the investigation of the critical conditions, under which the problem under study is no longer solvable in the natural class of regular functions, which from the physical point of view corresponds to a forced explosion or auto-ignition (i.e., to *a discontinuity* (jump) of the solution ϑ_{ST}).

In this connection we remark that the relations (4.4.14)–(4.4.16) and (4.4.21) discussed above leave unchanged the domain Ω in which the problem for the Liouville-type equation (4.4.1), (4.4.17) and the corresponding problem for the Laplace equation (4.4.2) (with the corresponding nonlinear boundary conditions) are posed. For this reason, the possible singularities of the solution ϑ_{ST} come from the singularities of the right-hand sides in (4.4.14)–(4.4.16), (4.4.21). For example, the solution ϑ_{ST}, computed by means of the third formula in (4.4.14), is regular in the domain

$$\Omega_0 \colon \; k\pi < v(x,t) < (k+1)\pi, \quad k \text{ an integer}.$$

That is to say, there are geometric constraints on the configuration of the domain $\Omega : \;\; \Omega = \Omega_0$ that must be satisfied in order for the evolution of the process to be regular. This agrees with the known results of physical

[6] Encountered, first of all, in the general theory of relativity.

investigations [16]. Moreover, the blow-up regime $|\theta_{\mathrm{ST}}| > M$, for all $M > 0$, corresponds exactly to the degeneration of the metric (4.4.5) that generates the Liouville-type equation when the discriminant $W[\theta]$ vanishes: $W[\theta] = 0$, and to the singularities that arise in the Liouville net on $\mathcal{M}_2$ ($K \equiv \pm 1$).

3) *The multidimensional Liouville equation.* A formal generalization of the structure of the transformations (4.4.14), (4.4.21) allows us to guess a class of self-similar solutions (of a linear argument) for the multidimensional Liouville-type equation:

$$\Delta_n u = e^u, \tag{4.4.22}$$

$$\Delta_n \widetilde{u} = e^{-\widetilde{u}}, \tag{4.4.23}$$

where $\Delta_n = \dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$, $\bar{x} = (x_1, \ldots, x_n)$.

The solutions of this class are given as follows:

for equation (4.4.22):

$$\begin{aligned} u(\bar{x}) &= \ln\left(\frac{2}{\alpha^2(\bar{x})}\right), \\ u(\bar{x}) &= \ln\left(\frac{2}{\sinh^2 \alpha(\bar{x})}\right), \\ u(\bar{x}) &= \ln\left(\frac{2}{\sin^2 \alpha(\bar{x})}\right); \end{aligned} \tag{4.4.24}$$

for equation (4.4.23):

$$\widetilde{u}(\bar{x}) = \ln\left(\frac{\cosh^2 \alpha(\bar{x})}{2}\right), \tag{4.4.25}$$

where $\alpha(\bar{x}) = a_1 x_1 + \cdots + a_n x_n$, $\sum\limits_{i=1}^{n} a_i^2 = 1$.

4.4.4 Example of “net-based” construction of “kink” type solutions of the sine-Gordon equation

Let us construct, applying the net method, a solution $u(x, t)$ of the sine-Gordon equation (4.1.7). The symmetry transformation

$$(x, t) \mapsto (x, -t)$$

takes (4.1.7) into an equation of the form

$$\bar{u}_{xt} = -\sin \bar{u}, \tag{4.4.26}$$

with

$$\bar{u}(x,t) = u(x,-t),$$
$$u(x,t) = \bar{u}(x,-t).$$

Equations (4.1.7) and (4.4.26) represent particular realizations of the Chebyshev equation (4.1.6) that is generated by the metric of the Chebyshev net. Specifically, equation (4.1.7) is generated by a pseudospherical metric of the form (4.1.5) (curvature $K \equiv -1$), while equation (4.4.26) is generated by a metric of the same form (4.1.5), but with an a priori prescribed constant positive curvature $K \equiv +1$.

To construct a solution $\bar{u}(x,t)$ of the equation (4.4.26) we turn to the auxiliary metric of curvature $K \equiv +1$, written in the semigeodesic coordinates (χ, τ):

$$ds^2 = (y^{**})^2(\tau) d\chi^2 + d\tau^2, \quad K(x,t) \equiv +1. \tag{4.4.27}$$

The metric (4.4.27) generates again equation (4.4.19), which has a general solution of the form

$$y^{**}(\tau) = A_1 \sin\tau + A_2 \cos\tau, \quad A_1, A_2 = \text{const}. \tag{4.4.28}$$

Setting $A_1 = 0$ and $A_2 = 1$ in (4.4.28), we select the particular solution

$$Y^{**}(\tau) = \cos\tau$$

and rewrite with it the metric (4.4.27):

$$ds^2 = \cos^2\tau d\chi^2 + d\tau^2. \tag{4.4.29}$$

The quadratic form (4.4.29) with curvature $K \equiv +1$ is reduced to a metric of the form (4.1.5), written in the coordinates of the Chebyshev net $\mathrm{Cheb}(x,t)$ of the same curvature, by means of the substitution

$$\begin{aligned} x + t &= \chi, \\ x - t &= \int \frac{d\tau}{\sin\tau}. \end{aligned} \tag{4.4.30}$$

In this way we arrive at the metric

$$ds^2 = dx^2 + 2\cos 2\tau(x,t) dx dt + dt^2. \tag{4.4.31}$$

Comparing (4.4.31) with the classical Chebyshev metric (4.1.5), we find the solution $\bar{u}(x,t)$ of equation (4.4.26):

$$\bar{u}(x,t) = 2\tau(x,t). \tag{4.4.32}$$

The function $\tau(x,t)$ is calculated from the second relation in (4.4.30):

$$\tau(x,t) = 2\arctan e^{x-t}. \tag{4.4.33}$$

Correspondingly, turning to the original solution $u(x,t)$ of the sine-Gordon equation and using (4.4.32), (4.4.31), and (4.4.26), we obtain from (4.4.33) the expression

$$u(x,t) = 4\arctan e^{x+t}. \tag{4.4.34}$$

The solution (4.4.34) is a “kink”-type solution or one-soliton solution of the form (3.2.11) (of unit amplitude).

The examples given above show how the method of mutual transformation of nets on manifolds of constant curvature can be used to construct exact solutions of nonlinear differential equations.

4.5 Geometric generalizations of a series of model equations of mathematical physics

In this section we provide a list of G-equations that *generalize* a series of important – from the point of view of mathematical physics and applications – *nonlinear equations*, together with the metrics that generate them. Usually, partial differential equations are generalized by increasing the dimension of the differential operators they involve (Laplacians, d’Alembertians and so on), which essentially means that one considers physical models of higher dimensions. In our treatment here, the generalization of known $(1+1)$-equations will be done by means of introducing in the “process of generating” the equation (see § 4.1) an arbitrary curvature $K(x,t)$, which will be a priori prescribed for the generating metric. Such an approach allows us to preserve the form of the generating metric for the resulting G-equation (the same metric as for the original Λ^2-equation), and hence preserve the very type of the generating coordinate net on $\mathcal{M}_2$ associated with this equation. Overall, the approach relies on the application of unified methods of geometric investigation to the Λ^2-equation at hand (a nonlinear equation with constant coefficients), as well as to its generalization, the G-equation (a generalized analog with functional coefficients). On the other hand, the presence of an“additional” functional coefficient in the G-equation enables us, in the construction of the corresponding models, to exploit supplementary properties of the physical processes under study “governed” by that equation.

We next list a number of physically important *generalized equations* of contemporary mathematical physics and the metrics (of arbitrary curvature $K(x,t)$) that generate them. For each metric we indicated the type of the generating coordinate net – the unified geometric preimage of the Λ^2-equation and of the generalized G-equation corrresponding to it.

I. *Chebyshev equation* (*generalized sine-Gordon equation*):

$$u_{xt} = -K(x,t)\sin u(x,t),$$

generating metric:

$$ds^2 = dx^2 + 2\cos u(x,t)dxdt + dt^2$$

(*Chebyshev net*).

II. *Generalized Korteweg-de Vries equation* (KdV *G-equation*):

$$u_t = u_x + (1 + K(x,t) + 6u)u_x + u_{xxx},$$

generalized metric:

$$
\begin{aligned}
ds^2 &= [(1-u)^2+\eta^2]dx^2 \\
&\quad + 2[(1-u)(-u_{xx}+\eta u_x-\eta^2 u-2u^2+\eta^2+2u)+\eta(\eta^3+2\eta u-2u_x)]dxdt \\
&\quad + [(-u_{xx}+\eta u_x-\eta^2 u-2u^2+\eta^2+2u)^2+(\eta^3+2\eta u-2u_x)^2]dt^2, \;\; \eta=\text{const.}
\end{aligned}
$$

III. *Generalized modified Korteweg-de Vries equation* (MKdV *G-equation*):

$$u_t = \Big(1+K(x,t)+\frac{3}{2}\,u^2\Big)u_x + u_{xxx},$$

generating metric:

$$ds^2 = \eta^2 dx^2 + 2\eta\Big(\eta\frac{u^2}{2}+\eta^3\Big)dxdt + \Big[\eta^2 u_x^2 + \Big(\eta\frac{u^2}{2}+\eta^3\Big)^2\Big]dt^2, \quad \eta=\text{const}$$

(MKdV *net*).

IV. *Generalized Burgers equation* (*Burgers G-equation*):

$$u_t = (1+K(x,t)+u)\cdot u_x + u_{xx},$$

generating metric:

$$
\begin{aligned}
ds^2 = \Big(\frac{u^2}{4}+\eta^2\Big)dx^2 + 2\left[\eta^2\frac{u}{2}+\frac{u}{4}\Big(\frac{u^2}{2}+u_x\Big)\right]dxdt \\
+\left[\Big(\frac{u^2}{4}+\frac{u_x}{2}\Big)^2+\eta^2\frac{u^2}{4}\right]dt^2, \quad \eta=\text{const}
\end{aligned}
$$

(*Burgers net*).

V. *Generalized Liouville equation* (*G-Liouville equation*):

a) elliptic:

$$\Delta_2 u = -K(x,t)\,e^u,$$

generating metric:

$$ds^2 = \frac{e^u}{2}\,(dx^2+dt^2)$$

(*elliptic Liouville net – isothermal coordinate net*).

b) hyperbolic:

$$u_{xt} = -K(x,t)\,e^u,$$

generating metric:

$$ds^2 = (u_x^2+\eta^2)dx^2 + 2\eta e^u dxdt + e^{2u}dt^2$$

(*hyperbolic Liouville set*).

VI. *Generalized* sinh-*Gordon equation* (sinh-*Gordon G-equation*):

a) elliptic:

$$\Delta_2 u = -K(x,t)\sinh u,$$

generating metric:

$$ds^2 = \cosh^2\frac{u}{2}\,dx^2 + \sinh^2\frac{u}{2}\,dt^2.$$

b) hyperbolic:

$$u_{xt} = -K(x,t)\sinh u,$$

generating metric:

$$ds^2 = (u_x^2 + \eta^2)dx^2 + 2\eta\cosh u\,dxdt + \cosh^2 u\,dt^2.$$

VII. *Generalized equation generated by a "semi-geodesic" metric*:

$$y_{xx} + K(x,t)y(x) = 0,$$

generating metric:

$$ds^2 = dx^2 + y^2(x)dt^2$$

(*semi-geodesic coordinate net*).

The geometric class of the equations listed above awaits addition of new model equations of mathematical physics together with the generating metrics recovered for them.

Chapter 5

Non-Euclidean phase spaces. Discrete nets on the Lobachevsky plane and numerical integration algorithms for Λ^2-equations

In this chapter we apply the geometric Gaussian formalism for nonlinear equations of theoretical physics presented in Chapter 4 to the theory of difference methods for the numerical integration of differential equations. The first part of the chapter (§§ 5.1. and 5.2) is devoted to introducing the concept of *non-Euclidean phase spaces*, which are nonlinear analogs (with nontrivial curvature) of the phase spaces of classical mechanics, statistical physics, and of the Minkowski space of the special theory of relativity.

At the foundation of the concept of non-Euclidean phase spaces lies the principle of identical correspondence of the metric of the phase space and the metric generated by a model G- or Λ^2-equation. The fact that non-Euclidean phase spaces have nontrivial curvature gives rise in them of singularities, which acquire the physical meaning of attractors and which determine the behavior of regular trajectories. This in turn allows us to formulate general principles governing the evolution of physical systems described by G- and Λ^2-equations. Figuratively speaking, the phase spaces we have in mind are a kind of "curvilinear (non-Euclidean) projector screens", on which the evolution of the physical process under study is displayed in a regular manner. In view of the specific features of the approaches used therein, the material of the first part of this chapter can be said to originate in the methods of theoretical physics.

In the second part of the chapter we propose, based on the elaboration of the technique of discrete coordinate nets on the Lobachevsky plane, a geometric algorithm for the numerical integration of Λ^2-equations. The implementation of

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this approach is connected exclusively with the planimetric analysis (in the setting of hyperbolic geometry) of the aforementioned piecewise-geodesic discrete nets on the Λ^2-plane, which in the limit pass into a smooth coordinate net that generates the Λ^2-equation under study. The sough-for solution of the problem at hand is computed as the corresponding characteristic of the limit net. The implementation of the method is exemplified on the sine-Gordon equation, for which the algorithm of numerical integration requires the study of a *discrete rhombic Chebyshev net* on the Λ^2-plane.

5.1 Non-Euclidean phase spaces. General principles of the evolution of physical systems

5.1.1 Introductory remarks

In classical mechanics, statistical physics, special theory of relativity, and other branches of physics, it turned out that in order to provide a transparent, intuitive representation of the evolution of the modeled systems and the dynamics of physical processes it is effective to use geometric representations of the parametric plane, phase space, space-time, and so on [18, 106].

These already traditional representations rest upon the idea of using as "projection screens" "*flat objects*" in the Euclidean space $\mathbb{E}^N$ and the pseudo-Euclidean space $\mathbb{E}^{L,1}_{L+1}$ (or, more precisely, zero-curvature spaces, or individual "fragments" thereof).

Let us sketch the main "steps" of the general methodology discussed here. The "time evolution" of a physical process Π is specified in the phase space Φ by a vector field $\overrightarrow{a}(\Phi)$. Namely, to each state $D(q_1,\dots,q_m)$ of the physical system under investigation (i.e., to each finite set of generalized variables q_i) one associates a point Q in phase space:

$$D(q_1,\dots,q_m)\mapsto Q\in\Phi,\quad \dim\Phi=m.$$

Accordingly, to a change in the state of the system will correspond a motion of the point Q along some curve l: $Q\in l\subset\Phi$, called *phase trajectory*. The tangent vector $\overrightarrow{a}$ to the curve l characterizes the speed at which the state of the system changes.

The classical methodology of phase spaces used in theoretical physics is connected with the consideration of ordinary (Euclidean) spaces, in which the phase trajectories are "placed". In this section we will introduce the notion of *non-Euclidean phase spaces* (N-EPhS) as spaces that encode the metric nature of the differential equations governing the physical systems under investigation. The non-triviality of the curvature of N-EPhS presupposes that they exhibit singularities, which in turn determine the character of the behavior and structure of a phase trajectory: the phase space is divided by the singularities it contains into a collection of regular "pieces" (domains of the phase space), onto each of which there is mapped one of the variants of autonomous evolution of the physical phenomenon.

5.1.2 The notion of non-Euclidean phase space

The notion of non-Euclidean phase spaces is based on the *principle asserting that the metric of the phase space and the metric generating the G- or Λ^2-equation that describes the physical process under investigation coincide* ([85, 86, 91]).

Indeed, suppose that the physical process Π is "governed" by some G-equation of the type (4.1.4):

$$\Pi\colon\ \mathcal{F}[u] = 0, \tag{5.1.1}$$

and moreover

$$(ds_*^2) \xmapsto{G} \{\mathcal{F}[u] = 0\}, \tag{5.1.2}$$

i.e., the metric ds_*^2 generates the G-equation (5.1.1).

Then in order to describe the system (5.1.1), (5.1.2) we choose a phase space Φ that carries the metric (ds_*^2) (5.1.2):

$$\Pi\colon\ \Phi = \Phi[ds_*^2]. \tag{5.1.3}$$

It is clear that the geometric character of the behavior of the phase trajectory $l(\Pi) \subset \Phi[ds_*^2]$, which describes the evolution of the physical process, will be then described in the framework of the *intrinsic* geometry of the phase space $\Phi[ds_*^2]$.

Let us consider the notion of non-Euclidean phase spaces in application to models of physical phenomena described by G- and Λ^2-equations of mathematical physics (see Chapter 4). By their physical meaning, the G- and Λ^2-equations are $(1+1)$-equations of the type (4.1.4) for a function $u(x,t)$, i.e., equations in which the independent variables are "separated" according to their meaning into the "space coordinate" x and the "time" t. Because of this, the non-Euclidean phase spaces discussed below (which have, as we mentioned before, a generalized meaning) should i be more precisely and intuitively interpreted as "non-Euclidean parametric projection screens" (with singularities), on which there is a globally defined regular parametric coordinate net $T(x,t)$ (where x is the space coordinate and t is time). We will use such "projection screens" to regularly display the dynamics of the physical phenomena under study.

If we consider a physical process Π described by a Λ^2-equation, then the role of the non-Euclidean phase spaces (parametric projection screens) will be played by pseudospherical surfaces in $\mathbb{E}^3$. Let us study this in detail.

We address the analysis of the structure of N-EPhS of the type $\Phi[ds_*^2]$, using their transparent representation as isometrically immersed smooth manifolds $\mathcal{M}_2$ in the space $\mathbb{E}^3$, i.e., as a special type of two-dimensional surfaces S in Euclidean space.

Recall that if some surface S, endowed with a metric (5.1.2), is given in $\mathbb{E}^3$ by the radius vector $\overrightarrow{r}(x,t)$ ((x,t) are intrinsic coordinates on S), then for a global regular coordinate net $T(x,t) \subset S$ to be correctly defined on S it is necessary that it be nondegenerate. The nondegeneracy property of a net manifests in the "independence" of its coordinate lines x and t, and is geometrically given in $\mathbb{E}^3$ by the condition that they be not tangent to one another:

$$[\overrightarrow{r_x} \times \overrightarrow{r_t}] \neq 0 \tag{5.1.4}$$

($\overrightarrow{r_x}$ and $\overrightarrow{r_t}$ are the tangent vectors to the families of coordinates lines on S).

The points M of the surface $S \subset \mathbb{E}^3$, where the opposite of condition (5.1.4) holds, i.e.,

$$[\overrightarrow{r}_x \times \overrightarrow{r}_t] = 0, \tag{5.1.5}$$

are (if they exists) called the *singular points* (*singularities*) of the surface. We denote the set of such points by

$$\{M\} = \text{edge}\{\Phi[ds_*^2]\}.$$

We also regard as singular the points $M \in S$ at which the function $\overrightarrow{r}(x,t)$ fails to have at least one of its derivatives $\overrightarrow{r}_x$, $\overrightarrow{r}_t$. "Irregularities" can also be "stretched", for instance, they can be irregular edges, to which there correspond on $\mathcal{M}_2$ *envelopes* of lines of tangency points of the coordinate lines.

As it was established in § 2.2, the singularities of the surface $S[\overrightarrow{r}(x,t)]$, identified by condition (5.1.5), are determined by the vanishing of the discriminant of the metric:

$$W[u] = E[u] \cdot G[u] - F^2[u] = 0. \tag{5.1.6}$$

The degeneracy condition (5.1.6) of the metric ds_*^2 gives the set of singular points of the surface (of the two-dimensional non-Euclidean phase space) $\Phi[ds_*^2]$:

$$\text{edge}\{\Phi[ds_*^2]\} \sim \{E[u]G[u] - F^2[u] = 0\}. \tag{5.1.7}$$

Now let us discuss the geometric interpretation of the physical phenomena described by G-equations of the type (4.1.4).

Let the surface $\Phi[ds_*^2]$ be the *phase surface* (non-Euclidean parametric "plane") in $\mathbb{E}^3$, corresponding to the formulation of problem (5.1.1)–(5.1.3). Assuming that "globally", on the entire surface Φ, there exists a regular net $T(x,t)$ of lines x, t, we take $T(x,t)$ as coordinate net. If $\Phi[ds_*^2]$ is a pseudospherical surface, then it is natural to take as regular coordinate net on it the Chebyshev net $\text{Cheb}(x,t)$. It is important that the coordinate lines x and t of the net $\text{Cheb}(x,t)$ *retain* their *regularity* as space curves during the "transition" from one regular part of the surface $\Phi[ds_*^2]$ to another regular part *through the singularities* (*irregular edges*), contiguous to them.

Further, to each possible state $D(q_1, q_2)$ of the physical system under study (in our case $q_1 = x$, $q_2 = t$) there corresponds on the phase surface $\Phi[ds_*^2]$ one, and only one point $P(x,t) \in \Phi[ds_*^2]$. Also, to the time evolution of the physical process Π (i.e., to the change of the state of the physical system) there corresponds a *motion of point* P on $\Phi[ds_*^2]$ along some curve l, called *phase trajectory*. For example, the analogue of the phase trajectory in the special theory of relativity is the *world line* in the Minkowski space. In the representation we are considering, the phase trajectory l is also a space curve in $\mathbb{E}^3$ with radius vector $\overrightarrow{r}_l$.

In view of their physical meaning (which endows them with content), *phase trajectories* (regarded as curves on manifolds and surfaces, or as curves in space), which arise in a big variety of branches of science, are *regular* lines (curves). This fundamental property arises from the following natural fundamental requirements:

1) the time evolution of the physical process must be regular (all parameters that reflect the state of the physical system must vary continuously);

2) the primary requirement that the cause-effect relation must hold in the observed (in particular, modeled) physical phenomenon.

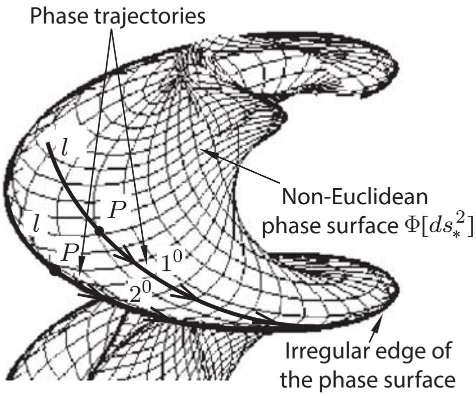


Figure 5.1.1. Two possible qualitatively different variants of the behavior (positioning) of the phase trajectory l on the non-Euclidean phase surface $\Phi[ds_*^2]$

For this reason, the concept of non-Euclidean phase spaces must necessarily rest upon the *principle of regularity of phase trajectories*.

The coordinate projection, on the t-line ("time axis"), of the "current point" that traces the phase trajectory

$$P(x,t) \in l \subset \Phi[ds_*^2]$$

grows monotonically as the process Π evolves, and consequently the phase trajectory l unavoidably approaches an irregular edge of the phase surface (Figure 5.1.1). However, the phase trajectory l is not allowed to intersect an irregular edge, because that would destroy the regularity of the radius vector $\overrightarrow{r_l}$, which specifies the phase trajectory in space. Note that "crossing" a singular edge in a regular manner is analytically possible only "along the coordinate lines x or t"; however, the phase trajectory cannot (even locally) coincide with a coordinate line, because this would violate the determinacy (univoqueness property) of the cause-effect relation of the observed process. Therefore, when the phase trajectory approaches an irregular edge on the phase surface, it asymptotically becomes infinitesimally close to that edge (variant 1^0 in Figure 5.1.1). If, however, the point P "falls" from the very beginning on an edge of the phase surface (due to the initial conditions taken for the given process), then the phase trajectory l traced by its motion will at all subsequent moments of time coincide with precisely that edge (variant 2^0 in Figure 5.1.1), maintaining its regularity.

Therefore, one can distinguish two possible qualitatively different variants of "behavior" of the phase trajectory l on the non-Euclidean phase surface $\Phi[ds_*^2]$:

1^0. *The entire phase trajectory l lies in the regular part of the phase surface $\Phi[ds_*^2]$, and as "time grows" approaches asymptotically an irregular edge.*

The coordinate projection of the "current point" $P \in l$ on the coordinate line t ("time axis") grows monotonically (Figure 5.1.1, variant 1^0).

2^0. *The phase trajectory l coincides with an irregular edge of the phase surface* $\Phi[ds_*^2]$ (Figure 5.1.1, variant 2^0).

The properties of the geometric behavior of the phase trajectory on the phase surfaces determine the general principle of the evolution of physical systems, which will be formulated in the next subsection.

5.1.3 General evolution principle for physical systems described by G-equations

In this subsection we formulate a general evolution principle for physical systems governed by G-equations. We discuss the properties of "observable" quantities u^* that in a real experiment correspond to analytical solutions $u(x, t)$ of model G-equations of mathematical physics. We exhibit invariant states of physical systems, which are characterized by minimal loss of energy. We also list such states for a number of well-known nonlinear equations arising in applications.

To the two variants of geometric "behavior" of the phase trajectory displayed above there correspond in a one-to-one manner two essentially different types of evolution of the physical phenomenon under study. These laws are expressed by the following evolution principle.

Principle 5.1.1 (*General evolution principle for physical systems described by G-equations*)**.** *Suppose that some physical process* Π *is described by the G-equation* (4.1.4)*, generated by the metric ds_*^2 in* (4.1.1)*:*

$$\Pi\colon\ (ds_*^2) \overset{G}{\longmapsto} \{\mathcal{F}[u] = 0\}.$$

Then Π *evolves according to one of the following possible "scenarios"*:

1) *If at the initial moment of time $t = t_0$ the condition*

$$\left(E[u] \cdot G[u] - F^2[u]\right)\big|_{t=t_0} = 0 \tag{5.1.8}$$

is satisfied, then it will be satisfied at all subsequent moments of time:

$$\Pi\colon\ \left(E[u] \cdot G[u] - F^2[u]\right) = 0 \quad \textit{for} \quad t \in (t_0, +\infty). \tag{5.1.9}$$

2) *If at the initial moment of time $t = t_0$*

$$\left(E[u] \cdot G[u] - F^2[u]\right)\big|_{t=t_0} \neq 0, \tag{5.1.10}$$

then as time varies the physical system will asymptotically and monotonically stabilize to the state (5.1.8)*:*

$$\Pi\colon\ \left(E[u] \cdot G[u] - F^2[u]\right) \to 0 \quad \textit{as} \quad t \to +\infty. \tag{5.1.11}$$

The state of the system described in Assertion 1) of Principle 5.1.1 will be called *invariant* state of the system.

Note that *Assertion* 1 of Principle 5.1.1 corresponds to variant 2^0 (Figure 5.1.1) of behavior of the phase trajectory on the NEPhS, while *Assertion* 2 corresponds to variant 1^0 (Figure 5.1.1).

Before we turn to examples of physical phenomena, let us explain the meaning of some of the notions used in the methodology of non-Euclidean phase spaces.

The specifics of the interpretation of a physical process are connected with the preliminary choice of a system $\{u^*\}$ of quantities that can be observed in an experiment (or in a thought experiment). Note that, in general, differences may exists between an analytical solution $u(x,t)$ of a model G-equation and the corresponding real, experimentally observable quantity u^*. Such differences between u^* and $u(x,t)$ may be connected with the presence of an additional constraint

$$x_{\text{obs}} = g(t) \tag{5.1.12}$$

on the solution of the G-equation, imposed by the observer, which arises due to the need of a clear representation of the process Π, the specifics of the experiment being performed, etc. Generally speaking, the appearance of a constraint of the form (5.1.12) in some concrete physical phenomenon has its own specific roots in the particular features of the mathematical model used, and will be traced back to them it separately in each of the examples considered below. Here we only mention that the presence of a constraint (5.1.12) is often explained by the need to select an "affixment", i.e., a characteristic reference point of the studied (observed) object. After such a selection is made, the observation is carried out is some distinguished sufficiently small domain (we assume that we are dealing with a point-like model). For instance, to observe the propagation of a localized bell-shaped pulse (profile), typical for solitary waves, it is advisable to choose as affixment the peak point of the pulse (Figure 5.1.2). In this case the meaning of the constraint of type (5.1.12) is that it represents the law of motion of the pulse's peak. The indicated choice is also convenient because the peak point of the pulse is connected with its amplitude, which usually needs also to be analyzed. The introduction of an "affixment" is a standard tool used in the construction of dynamical models of physical processes.[1]

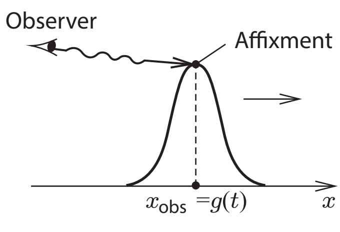


Figure 5.1.2. Choice of an affixment for carrying out the observations

[1] One can also consider a set (system) of "affixments", to which on the NEPhS will correspond a certain associated collection of phase trajectories.

The presence in an experiment (trial) of a constraint (5.1.12) for the solution $u(x,t)$ of the model equation at hand is by its nature analogous to the formulation and treatment of a conditional extremum problem in classical analysis, where we are interested in the behavior of a function of several variables in a "slice" defined by the constraint.

In a real experiment one "measures" an observable quantity u^*:

$$u^* = u(x,t)\big|_{(x,t)\in l}; \quad l\colon\ x_{\text{obs}} = g(t). \tag{5.1.13}$$

In general, problems of experimental observability in phenomena referred to in Principle 5.1.1 are very closely related with the problem of isometric immersion of two-dimensional smooth manifolds and, in particular, parts of the Lobachevsky plane Λ^2, in the Euclidean space $\mathbb{E}^3$. That is, such phenomena are in a certain sense related, through their deep content, to the laws of non-Euclidean geometry.

Now let us go back to analyzing what Principle 5.1.1 says. A state of the system defined by conditions (5.1.8) and (5.1.9) will be called, as indicated above, an *invariant* state, and will be denoted by

$$\operatorname{inv}\{\Pi, \mathcal{F}[u(x,t)] = 0\}.$$

Obviously,

$$\operatorname{inv}\{\Pi, \mathcal{F}[u] = 0\} \sim \operatorname{edge}\{\Phi[ds_*^2], \mathcal{F}[u] = 0\} \sim \{E[u]G[u] - F^2[u] = 0\}. \tag{5.1.14}$$

In Table 5.1.1. we list the invariant states of physical systems described by a number of well-known nonlinear Λ^2-equations.

By Principle 5.1.1, the corresponding physical systems either are in one of their invariant states, or stabilize to it. Assertion 1 of Principle 5.1.1 can be interpreted as a conservation law of the observed physical quantity.

Let us illustrate Principle 5.1.1 on the example of propagation of waves on shallow water [51]. Consider waves on the surface of a fluid, assuming that their maximal disturbance amplitude α is small compared with the depth of the fluid h: $\varepsilon = \frac{\alpha}{h} \ll 1$, and their disturbance length (wave length) λ_0 is large with respect to h: $\delta = \frac{h}{\lambda_0} \ll 1$. The investigation of such a model by methods of perturbation theory (with the role of small parameters played here by ε and δ) leads to the Korteweg-de Vries equation [51], in which the unknown function u has the meaning of the wave amplitude. In accordance with (5.1.14), the invariant states of this system are defined as

$$\operatorname{inv}\{\Pi,\ u_t + 6uu_x + u_{xxx} = 0\} \sim \{u_x^* = 0\}.$$

Consequently, by Assertion 2) of Principle 5.1.1, the process of propagation of shallow water waves must stabilize to the state $u_x^* = 0$ (i.e., $u^* = \text{const}$), which corresponds to the damping of such waves, observed in practice. Assertion 1) of Principle 5.1.1 describes in this model the rest (unperturbed) state of the fluid's surface.

A wide spectrum of phenomena are governed by the sine-Gordon equation. Below, in § 5.2, for this Λ^2-equation we will formulate the *$n\pi$-invariance principle*, which concretizes the content of Principle 5.1.1 and describes in a unified "geometric" way a large class of phenomena of different physical nature.

Table 5.1.1. Λ^2-equations and invariant states of physical systems

Λ^2-equation	invariant state of physical system, observable quantity u^*
1. sine-Gordon equation $u_{xt} = \sin u$	$u^* = n\pi,\ n$ an integer
2. Korteweg-de Vries equation $u_t + 6uu_x + u_{xxx} = 0$	$u_x^* = 0$
3. Modified Korteweg-de Vries equation $u_t + \frac{3}{2}u^2 u_x + u_{xxx} = 0$	$u_x^* = 0$
4. Burgers equation $u_t + uu_x + u_{xx} = 0$	$u_x^* = 0$
5. Hyperbolic Liouville equation $u_{xt} = e^u$	$u_x^* = 0$
6. Elliptic Liouville equation $\Delta_2 u = e^u$	$u^* \to -\infty$
7. Hyperbolic sinh-Gordon equation $u_{xt} = \sinh u$	$u_x^* = 0$
8. Elliptic sinh-Gordon equation $\Delta_2 u = \sinh u$	$u^* = 0$

5.2 The $n\pi$-Invariance Principle. The sine-Gordon equation as a model equation in physics

In this section we formulate the evolutionary *$n\pi$-Invariance Principle* for physical phenomena described by the sine-Gordon equation, and then examine on examples of physical processes the rules that follow from the "action" of the fundamental statements of this principle [80, 85, 86].

5.2.1 The $n\pi$-Invariance Principle

The generalized content of Principle 5.1.1 is transparently and effectively realized for phenomena described by the sine-Gordon equation; in this case Principle 5.1.1 is re-expressed as the Principle 5.2.1 below, the evolutionary $n\pi$-Invariance Principle [86].[2]

Principle 5.2.1 (*$n\pi$-Invariance Principle*)**.** Suppose some physical process Π is modeled by the sine-Gordon equation. Then the *observable physical quantity* u^*, corresponding to the solution u of the sine-Gordon equation, obeys the following rules:

[2]Specifically, the $n\pi$-Invariance Principle and the "geometric" consideration of the corresponding physical processes preceded the formulation of the general evolutionary Principle 5.1.1.

I. *The values $u^* = n\pi$ (with n an integer) are invariants of the physical process* Π.

   In other words, if at the initial moment of time $t = t_0$ the observable quantity has the value $u^* = n\pi$, then it maintains that value at all subsequent moments of time ($t > t_0$).

II. *If at some moment of time $u^* \neq n\pi$ (with n an integer), then as time passes the observable quantity tends asymptotically and monotonically to a value that is a multiple of π, and does it in such a way that its variation during the course of the entire process is less than π.*

According to Assertion I of Principle 5.2.1, u^* can assume the value $n\pi$ only in the case when the observed quantity u^* is identically equal to $n\pi$ for the entire duration of the process (*state of $n\pi$-invariance*). The values $u^* = n\pi$ (with n an integer) are *quantized* energetically-stable *states* of the system.

From the positions of theoretical physics, Assertion II of Principle 5.2.1 means the "monotone stabilization" of the physical system to an energetically stable state.

The general qualitative character of the variation of the observable quantity u^* in accordance with Principle 5.2.1 is displayed in Figure 5.2.1.

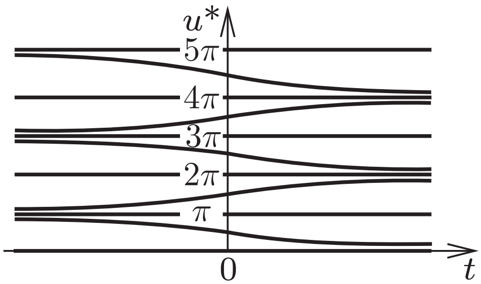


Figure 5.2.1. General qualitative character of the variation of the observable quantity u^* in phenomena described by the sine-Gordon equation

Before we embark on a detailed consideration of examples of physical phenomena, emphasizing the *universality* of the Principle 5.2.1 formulated above, let us list a number of physical laws that correspond to the $n\pi$-Invariance Principle.

In well-known phenomena the state of *$n\pi$-invariance* (Assertion I of Principle 5.2.1) is expressed by the following laws: 1) The self-induced transparency effect, which arises in the propagation of ultrashort pulses in two-level resonant media (propagation through a medium without loss of energy of pulses with profiles whose area is a multiple of π) [83, 169]. 2) Equilibrium positions of atoms in crystal lattices [23, 47]. 3) Meson vacuum states [182]. 4) Topologically invariant states of elementary particles [156]. 5) Stable orientation states of the magnetization vector in a ferromagnetic material with respect to an external magnetic field [157]. Overall, the state of $n\pi$-invariance realizes a stable equilibrium state of the physical system.

Assertion II of Principle 5.2.1 "unifies" such physical laws as: 1) Law of area change of an ultrashort pulse ("area theorem") [110, 169]. 2) Instability of intermediate positions of atoms in crystals, discrete character of dislocations [47]. 3) Relaxation of excited states of elementary particles to a vacuum state [182]. 4) The damping of current in a Josephson junction [8]. 5) The rotation of the magnetization vector in a "180° Bloch wall" [157].

We now turn to the consideration of physical processes described by the sine-Gordon equation and, accordingly, "governed" by the $n\pi$-Invariance Principle (Principle 5.2.1). In a certain sense such phenomena admit their own interpretation "through the prism" of hyperbolic non-Euclidean geometry. The main result of such a geometric "view" is to exhibit the quantized $n\pi$-states (states of $n\pi$-invariance) of physical systems, which reflect in fundamental manner the impossibility of realizing the complete Lobachevsky plane Λ^2 in the Euclidean space $\mathbb{E}^3$.

5.2.2 Bloch wall dynamics in ferromagnetic materials

Let us consider a ferromagnetic crystal that has one axis of easy magnetization, oriented along the x_3-axis of the Cartesian coordinate system x_1, x_2, x_3 (Figure 5.2.2). Under the action of an external magnetic fields $\overrightarrow{H}$ directed along the easy magnetization axis, the ferromagnetic crystal acquires an ordered domain structure (a domain is a region of the ferromagnetic material in all points of which the magnetization vector $\overrightarrow{J}$ is aligned in the same direction [23, 157]).

A specific feature of the influence of a field $\overrightarrow{H}$ directed along the easy magnetization axis x_3 is that at each point of the domain the vector $\overrightarrow{J}$ becomes parallel to the x_3-axis and thus acquires a unique orientation within the domain. Globally, in the crystal the magnetization vector can take two possible (opposite) orientations. Two neighboring domains (domains with opposite orientations of the magnetization vectors) are separated by a so-called *Bloch wall*, a thin layer (of $\sim 200\ \mathring{A}$ width), inside of which the magnetization vector $\overrightarrow{J}$ rotates to pass from one stationary state to the other (Figure 5.2.2). The change (rotation) of the vector $\overrightarrow{J}$ "in the width" of the domain wall when one passes between neighboring domains takes place in a continuous fashion.

Under the action of the external magnetic field $\overrightarrow{H}$, each domain tends to change its own configuration (as a result of the re-orientation of the magnetization vectors), and accordingly there is a tendency to motion (displacement) of the Bloch wall as a transition region between domains. The domains in which the magnetic moments are in an energetically more advantageous position in the magnetic field $\overrightarrow{H}$ tend to increase their volume at the expense of the domains in which the magnetic moments have a less advantageous direction (i.e., they deviate significantly from the direction of the field $\overrightarrow{H}$). The displacement of magnetic walls is accompanied by the rotation of the magnetic moments inside the walls themselves (Figure 5.2.2).

A Bloch wall can be regarded as a local structure, inside which the transition from one direction of the magnetization vector $\overrightarrow{J}$ to the other (opposite) direction

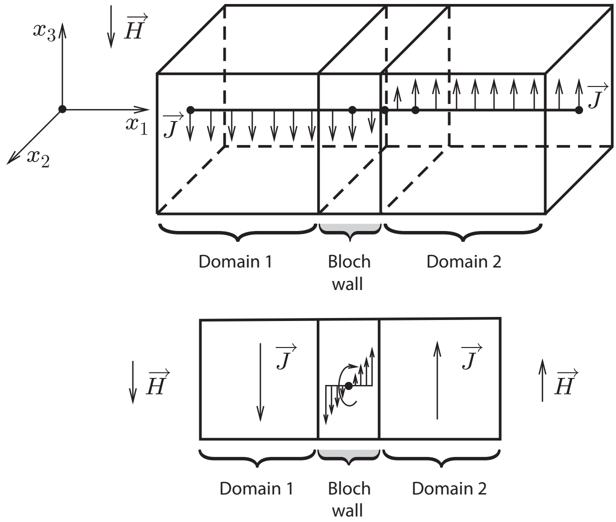


Figure 5.2.2. Orientation of the magnetization vector in the "domain–Bloch wall–domain system"

is governed by the principle of minimum energy. To describe the dynamics of a Bloch wall, it is convenient to use as characteristic parameter of the process the angle $\Theta(x_1, t)$ that the projection of the vector $\vec{J}(x_1)$ on the (x_2, x_3)-plane makes with the x_3-axis. Figure 5.2.3 shows the "instantaneous" position of the magnetization vector $\vec{J}$ at an interior point (x_1^0, x_2^0, x_3^0) of the ferromagnetic crystal. The computation of the Bloch wall dynamics, carried out on the basis of a general variational principle, leads to a sine-Gordon equation of the form (see [157])

$$\frac{\partial^2 \Theta}{\partial x_1^2} - \frac{\partial^2 \Theta}{\partial t^2} = \sin \Theta. \tag{5.2.1}$$

The variation of the angle $\Theta(x_1, t)$ in (5.2.1) depends on the exchange energy (the first term in the left-hand side) and the energy of anisotropy (the right-hand side).

Let us address now the geometric interpretation of this phenomenon. The instantaneous distribution of the vector field $\vec{J}$ in the ferromagnetic material (Figure 5.2.3) will be characterized by the angle Θ, which varies monotonically from 0 to π (within the width of the Bloch wall); this law corresponds exactly to Assertion II of Principle 5.2.1. In the case where the angle satisfies

$$\Theta(\tilde{D})\big|_{t=t_0} = 0 \quad \big(\text{or} \;\; \Theta(\tilde{D})\big|_{t=t_0} = \pi\big), \tag{5.2.2}$$

the inner structure in the corresponding localized region of the ferromagnet (the domain $\tilde{D}$) does not change. In the model considered here the condition (5.2.2)

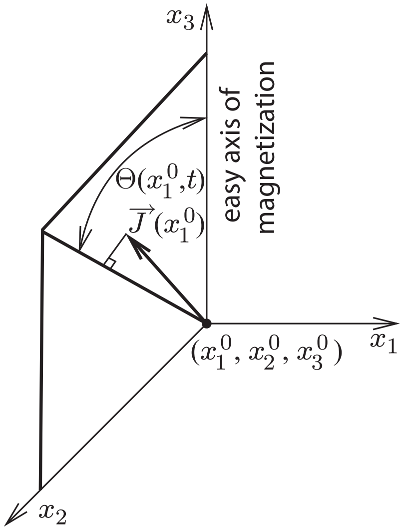


Figure 5.2.3

determines the state of $n\pi$-invariance. The domain-separating wall with the aforementioned character of the behavior of the magnetization vector $\vec{J}$ is a typical object of the physics of domain walls, called a "180°-*Bloch wall*"; when one passes through such a wall the magnetization vector $\vec{J}$ rotates by 180° (by π).

Therefore, the character of the variation of the magnetization vector $\vec{J}$ in the vicinity of a Bloch wall is completely regulated by the assertions of Principle 5.2.1 and amounts to the following:

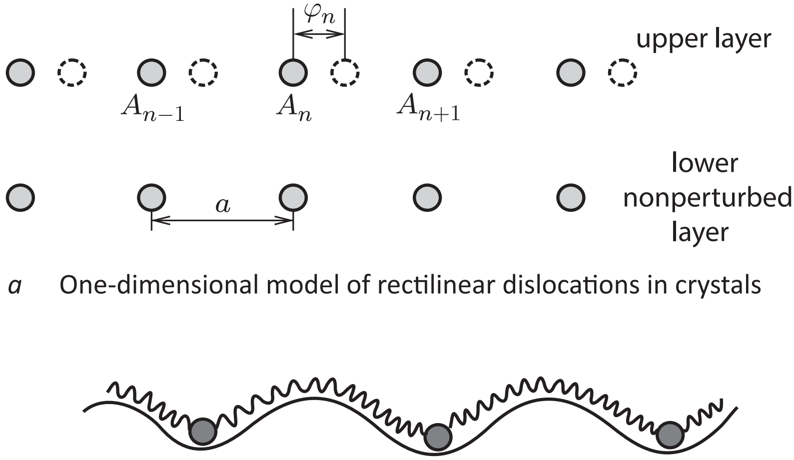
1) Outside a Bloch wall the magnetization vector $\vec{J}$ is aligned with the external magnetic field (two opposite directions are allowed).

2) Within a Bloch wall the vector $\vec{J}$ rotates in such a way that the monotone variation of its orientation angle Θ does not exceed π.

5.2.3 Dislocations in crystals

Dislocations (linear defects in a crystal structure) represent the break in the regularity of the crystal lattice in directions in which two domains of the crystal that are displaced relative to one another are contiguous [23, 47].

For a rectilinear boundary dislocation, the action of the undisplaced (lower) part of the crystal on the displaced atoms of the upper layer distributed along the dislocation axis (Figure 5.2.4) can be described by means of a Hamiltonian which includes the interaction in the form of a periodic potential. As a model of dislocation we consider a chain of identical atoms $\{A_n\}$ of mass m, shifted by a respective distance φ_n relative to the equilibrium positions of the atoms in the



a One-dimensional model of rectilinear dislocations in crystals

b Mechanical simulator of the model of dislocations in crystals

Figure 5.2.4

crystal, which are represented by a second (lower) chain of atoms with the period a of the crystal lattice (Figure 5.2.4, *a*, *b*).

The Hamiltonian of the system considered is given by

$$H = \sum \frac{(m\dot{\varphi}_n^2 + \kappa(\varphi_n - \varphi_{n-1})^2 + U(\varphi_n))}{2}, \quad \kappa = \text{const}, \tag{5.2.3}$$

where φ_n denotes the local deviation of the nth atom of the upper chain from the equilibrium state, and $U(\varphi_n)$ is the potential with which the nonperturbed (lower) chain acts on the nth atom of the upper chain

$$U(\varphi_n) = U_0 \left[1 - \cos\left(\frac{2\pi\varphi_n}{a}\right)\right].$$

The Hamiltonian equations for (5.2.3) read

$$m\,\frac{d^2\varphi_n}{dt^2} + \kappa(2\varphi_n - \varphi_{n-1} - \varphi_{n+1}) + \frac{U_0\pi}{a}\sin\left(\frac{2\pi\varphi_n}{a}\right) = 0. \tag{5.2.4}$$

Now let us pass in (5.2.4) to the continual approximation:

$$\varphi_n(t) \mapsto \varphi(x,t),$$

and simultaneously introduce the function

$$u(x,t) = \frac{2\pi\varphi_n}{a}. \tag{5.2.5}$$

This leads to the sine-Gordon equation [23, 47]

$$u_{xx} - \frac{1}{c_0^2} u_{tt} = \frac{1}{\lambda_0^2} \sin u, \tag{5.2.6}$$

where c_0 and λ_0 are certain characteristic constants that arise in the model.

Let us analyze the dislocation phenomenon in crystals, described by equation (5.2.6), in the context of the Principle 5.2.1 formulated above. If at the initial time $t = t_0$ the observable quantity u^* satisfies

$$u^*(t = t_0) = k\pi, \quad k \text{ an integer}, \tag{5.2.7}$$

then this value of u^*, a multiple of π, is preserved at all subsequent moments of time. Condition (5.2.7) corresponds to the fact that in the present case the magnitude of the deviation φ_n of the n-th atom from the equilibrium position (see (5.2.5)) can only take multiples of a as values. To this corresponds the nonperturbed uniform discrete distribution of atoms in the crystal lattice. Such a stable set realizes a state of $n\pi$-invariance (Assertion I of Principle 5.2.1).

The tendency

$$u^* \to k\pi \quad \text{as } t \to \infty,$$

which follows Assertion II of Principle 5.2.1, corresponds to the (limit state of) stabilization of the crystal lattice:

$$\varphi_n \to \frac{k}{2} \cdot a \quad \text{as} \quad t \to \infty. \tag{5.2.8}$$

By (5.2.8), the crystal structure stabilizes asymptotically (in time) to the original equilibrium state. “Weak” local perturbations of the equilibrium state of atoms stabilize to the original state, whereas “large” perturbations can lead to a “chain-like” replacement of atoms in the upper chain of atoms (by a discrete successive “shift”), which incidentally keeps the general structure of their arrangement unchanged.

5.2.4 Propagation of ultrashort pulses in two-level resonant media

In a two-level resonant medium atoms can be in two energy states — the bottom (ground) state, with energy E_1, and the top one, with energy E_2 ($E_2 > E_1$) (Figure 5.2.5). Such a two-level system is in the modern theory of resonant interaction of radiation with matter one of the fundamental models describing real quantum objects (atoms, molecules), and with its help one can elaborate rather complete representations for many problem of quantum electronics, nonlinear optics, laser spectroscopy [83, 169].

The propagation of *ultrashort* electromagnetic pulses of 10^{-9}–10^{-12} sec duration in a two-level resonance medium has a number of specific peculiarities: models based on linear dispersion theory (for small intensities), or on radiative transfer equations for velocities (non-coherent interaction) are not applicable to it. A typical special feature of the pulse propagation process under investigation is that relaxation phenomena (collisions, spontaneous radiation) do not manage

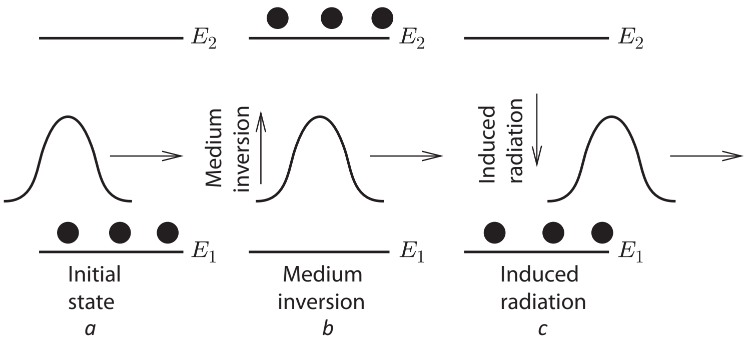


Figure 5.2.5. Propagation of an ultrashort pulse in a two-level resonant medium

to destroy the *phase memory* of the two-level *system*, and consequently the polarization of the medium becomes a nonlinear function of the amplitude and phase of the propagating ultrashort pulse.

When an ultrashort pulse propagates in a two-level resonant medium, the atoms of the medium that lie in a lower energy state E_1 (Figure 5.2.5 *a*) pass, under the action of the leading edge of the pulse, into a higher state with energy E_2, as a result of which the medium becomes completely inverted (Figure 5.2.5 *b*). Under the action of the remaining part of the pulse, the atoms that passed in the upper (unstable) energy state are induced to radiate, and the energy they acquired before is returned to the propagating pulse (Figure 5.2.5 *c*). Consequently, the energy injected in the two-level quantum system is returned, as a result of which when the pulse exits the system its initial shape and intensity are restored. Such a phenomenon can occur because the duration of the considered pulse is shorter than the relaxation time, hence the inversion of the medium and the subsequent induced radiation occur faster than the relaxation processes that could have destroyed the coherence of the interaction.

To model the process of propagation of an ultrashort pulse in a two-level resonant medium we must write the *Maxwell equations* to describe the electromagnetic pulse and consider a quantum (two-level) ensemble of atoms to describe the medium.

Indeed, under the action of an electromagnetic wave

$$E(x,t) = \mathcal{E}(x,t)\cos(\omega t - kx)$$

passing through a medium, the latter acquires the specific polarization

$$P(x,t) = \mathcal{P}(x,t)\sin(\omega t - kx).$$

The state of such a physical system is determined by the system of equations [83, 169]

$$\frac{\partial \mathcal{E}}{\partial x} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} = a\mathcal{P}, \quad a = \frac{2\pi\omega}{\hbar c} n_0 d^2,$$
$$\frac{\partial \mathcal{P}}{\partial t} + \frac{1}{T_1}\mathcal{P} = \mathcal{E}\mathcal{N},$$
$$\frac{\partial \mathcal{N}}{\partial t} + \frac{1}{T_2}\mathcal{N} = -\mathcal{E}\mathcal{P},$$

where n_0 is the density of atoms, $\mathcal{N}$ is the normalized density of excited atoms, and T_1 and T_2 are the longitudinal and transversal relaxations times of the excited states. If the duration τ_0 of the pulse satisfies

$$\tau_0 \ll T_1, \quad \tau_0 \ll T_2$$

(the so-called case of *absence of phase modulation*), then the system of equations given above simplifies to

$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{E}\mathcal{N},$$
$$\frac{\partial \mathcal{N}}{\partial t} = -\mathcal{E}\mathcal{N},$$
$$\frac{\partial \mathcal{E}}{\partial x} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} = a\mathcal{P},$$

and the substitution

$$\varphi(x,t) = \int_{-\infty}^{t} \mathcal{E}(x,t)dt$$

reduces it to the sine-Gordon equation for the function $\varphi(x,t)$ [83, 169]:

$$\varphi_{xt} = \sin\varphi.$$

At the same time, if we introduce the quantity

$$\vartheta(x) = \int_{-\infty}^{\infty} \mathcal{E}(x,t)dt,$$

which is called the *pulse area* and approximates $\varphi(x,t)$,

$$\varphi \simeq \vartheta, \quad t \in (t_*, \infty),$$

then the following relation, which is known as the "area theorem", holds true [83, 110]:

$$\frac{d\vartheta}{dx} = -\mathcal{K}\sin\vartheta, \quad \mathcal{K} = \text{const}. \tag{5.2.9}$$

In accordance with the "area theorem", when $t \to \infty$, the area ϑ of the ultrashort pulse tends asymptotically to values that are multiples of π, whenever $\vartheta(t = t_0) \neq n\pi$ (n an integer). Also according to this theorem, if at the initial moment of time

t_0 it holds that $\vartheta(t = t_0) = n\pi$ (the initial value of the area is a multiple of π), then the same remains true during the entire subsequent evolution of the process:

$$\vartheta(t > t_0) = n\pi.$$

In experiments the pulse area ϑ is the observable variable: $u^* = \vartheta$. The character of the variation of $\vartheta = u^*$ by virtue of equation (5.2.9) is identical with that stated by the general laws of variation of an observable quantity, shown in Figure 5.2.1 for phenomena governed by the $n\pi$-Invariance Principle (Principle 5.2.1).

The conservation of area of ultrashort pulses was discovered experimentally [83, 172], and became known as the *self-induced transparency effect* — the phenomenon of passage of an ultrashort pulse through a resonant medium without loss of energy. Within the setting of the NEPhS concept, the area theorem (for $u^* \equiv \vartheta$) is explained theoretically precisely by Principle 5.2.1.

To complete this section, we re-emphasize the "geometric component" of the physical laws discussed above. The dependence depicted in Figure 5.2.1 completely and exactly reflects the contents of Principle 5.2.1 for the laws governing the change of an observable quantity u^* in all the phenomena described by the sine-Gordon equation. Geometrically, such a law is a natural expression of the fact that the whole Lobachevsky plane Λ^2 cannot be immersed in the Euclidean space $\mathbb{E}^3$ — the space that "hosts" the model of the physical process itself, governed by the sine-Gordon equation — a canonical relation in non-Euclidean hyperbolic geometry. Unavoidable attributes of admissible isometric immersions of individual domains of the plane Λ^2 in $\mathbb{E}^3$ are the singularities on the resulting pseudospherical surfaces, which correspond to the $n\pi$ level lines (with n an integer) of solutions of the sine-Gordon equation and play the role of special stationary (quantized) states of the modeled physical systems.

5.3 Discrete nets on the Lobachevsky plane and an algorithm for the numerical integration of Λ^2-equations

In this section we present a geometric approach to the elaboration of numerical algorithms for integrating nonlinear equations of mathematical physics, based on the construction and subsequent analysis, by methods of non-Euclidean geometry, of discrete (difference) net analogs of the problems under study on manifolds of non-zero curvature (first and foremost, on the Lobachevsky plane) [88]. The arsenal of methods of the approach developed here relies on the concept of Λ^2-representation for partial differential equations, which associates these equations with metrics of constant negative curvature. Based on the study of a discrete rhombic Chebyshev net on the Lobachevsky plane, we present a direct geometric algorithm for integrating the Darboux problem for the sine-Gordon equation. We also discuss various issues that arise in the context of the problems considered here.

The notion of partial differential equations of Lobachevsky class (Λ^2-class) introduced in the works [77, 185, 186] brings together the equations that admit a unified geometric interpretation in the setting of two-dimensional Lobachevsky geometry: every equation of Λ^2-class (Λ^2-equation) is generated by a corresponding pseudospherical metric. In this section we present a geometric method for the numerical integration of Λ^2-equations that is based on the modeling of discrete nets on the Lobachevsky plane Λ^2 associated with the pseudospherical metrics that generate the equations under study. The realization of the geometric algorithms presented here is connected exclusively with the planimetric analysis of piecewise-geodesic nets on the hyperbolic plane. On the example of the sine-Gordon equation we examine in detail the implementation of the algorithm, which is connected with the study of a *discrete rhombic Chebyshev net* on the plane Λ^2.

It is important to point out that the difference (discrete) approximation obtained below for the given problem is, generally speaking, derived directly from the geometry of the corresponding generating coordinate nets on a non-Euclidean smooth manifold and its construction does not require the application of the standard stencils [104] for the approximation of derivatives used in the theory of difference schemes on the standard Euclidean plane. The geometric approach to the integration of nonlinear equations presented here indicates that it is possible to develop a general theory of difference methods that is based on the construction and study of *discrete analogs of problems* of mathematical physics on *manifolds of non-zero curvature*.

5.3.1 Λ^2-representation of equations and a general scheme for the geometric construction of algorithms for their numerical integration

We begin this subsection by briefly recalling the basic positions of the theory of Λ^2-representations that are needed below (see also Chapter 4).

Suppose we are given a differential equation

$$f[u(x,t)] = 0, \quad u(x,t) \in C^n(\mathbb{R}^2(x,t)), \tag{5.3.1}$$

that belongs to the Lobachevsky class, i.e., is associated to a certain generating pseudospherical Λ^2-metric $ds^2[u(x,t)]$ (metric of Gaussian curvature $K \equiv -1$), defined on each solution of this equation:

$$ds^2 = E[u(x,t)]dx^2 + 2F[u(x,t)]dxdt + G[u(x,t)]dt^2. \tag{5.3.2}$$

We write the Λ^2-representation of equation (5.3.1) (the set of coefficients of the Λ^2-metric that generates it) as

$$\Lambda^2[f[u(x,t)] = 0] \equiv \{E[u(x,t)], F[u(x,t)], G[u(x,t)]\}.$$

The fact that the equation possesses the above geometric interpretation (Λ^2-representation) allows us to pass from the investigation of equation (5.3.1) itself to the analysis of its geometric image, the coordinate net $T(x,t) \subset \Lambda$ associated with the metric (5.3.2). We remark that if the net $T(x,t) \subset \Lambda^2$, which through the metric $ds^2[u(x,t)]$ carries the characteristics of equation (5.3.1), represents on Λ^2 an independent geometric object, which can be studied exclusively in the framework of Lobachevsky planimetry. Then the solution $u(x,t)$ of the original equation is constructed as a geometric characteristic of the generating net $T(x,t)$.

Generally, the proposed geometric approach to the integration of Λ^2-equations is implemented according to the following scheme:

i) Construct, for the given differential equation (5.3.1), a metric of the type (5.3.2) that generates it (i.e., find a Λ^2-representation).[3]

ii) Choose the key geometric characteristics of the generating coordinate net $T(x,t)$, which give the solution of the original equation (5.3.1) (choose the k-characteristics).[4]

iii) For the net $T(x,t)\subset\Lambda^2$ introduce a discrete analogue $T^{\mathrm{d}}\subset\Lambda^2$, i.e., a discrete net that "inherits" the k-characteristics of the original net.

iv) Derive algorithmic (recurrent) net relations in the net T^{d}: compute a discrete analog u^{d} of the solution $u(x,t)$ at the nodes of the net T^{d}.

v) Investigate the convergence of the "discrete" algorithm obtained: prove the convergence of the sequence of solutions $\{u^{\mathrm{d}}\}$ of the discrete problem to the solution u of the original Λ^2-equation when the typical linear dimension a of the cell of the discrete net becomes infinitesimally small:

$$u^{\mathrm{d}}\to u,\quad T^{\mathrm{d}}\to T\quad\text{when}\ \ a\to 0.$$

5.3.2 Discrete rhombic Chebyshev net. The "discrete Darboux problem" for the sine-Gordon equation

Let us now implement the general geometric approach to the numerical integration of Λ^2-equations on the example of the sine-Gordon equation

$$u_{xt}=\sin u. \tag{5.3.3}$$

With the sine-Gordon equation there is associated the Λ^2-metric of the Chebyshev net that generates it:

$$ds^2=dx^2+2\cos u\,dxdt+dt^2. \tag{5.3.4}$$

The metric (5.3.4) is connected on the plane Λ^2 with the Chebyshev coordinate net $\mathrm{Cheb}(x,t)$. A characteristic property of the Chebyshev net is that the lengths of opposite sides in an arbitrary coordinate quadrilateral are equal (see § 2.5). We choose this property as a k-characteristic of the net $\mathrm{Cheb}(x,t)$. Recall that the solution $u(x,t)$ of the equation (5.3.3) has the meaning of the net angle of the Chebyshev net $\mathrm{Cheb}(x,t)$.

Let us also reproduce here the standard formulation of the Darboux problem (3.6.1), (3.6.2) for the sine-Gordon equation, considered in detail in § 3.6:

$$\begin{gathered}u_{xt}=\sin u,\\ u(x,0)=\varphi(x),\quad u(0,t)=\psi(t),\quad \varphi(0)=\psi(0).\end{gathered} \tag{5.3.5}$$

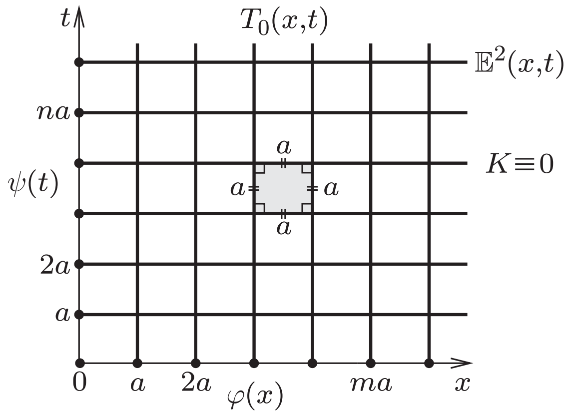


Figure 5.3.1. Formulation of the Darboux problem in the Euclidean plane $\mathbb{E}^2$

Figure 5.3.1 explains how the formulation of the Darboux problem (5.3.5) is interpreted in the parameter plane $\mathbb{E}^2(x,t)$ (the Euclidean plane); here $T_0(x,t)$ denotes the uniform net on $\mathbb{E}^2(x,t)$. The functions $\varphi(x)$ and $\psi(t)$ can be considered as initial data for the sought-for solution $u(x,t)$ on the characteristics of the sine-Gordon equation. Or, geometrically, as initial values of the net angle of the net $\mathrm{Cheb}(x,t)$ on its generators $(x:0)$ and $(0:t)$, which satisfy at zero the conjugation condition $\varphi^{(k)}(0)=\psi^{(k)}(0)$, condition that guarantees the requisite smoothness of the solution of the posed Darboux problem.

When we transfer (via the Λ^2-representation for the sine-Gordon equation) the analysis from the Euclidean plane $\mathbb{E}^2(x,t)$ to the Lobachevsky plane $\Lambda^2(x,t)$:

$$\left\{T_0(x,t),\mathbb{E}^2(x,t)\right\}\longrightarrow\left\{\mathrm{Cheb}(x,t),\Lambda^2(x,t),ds^2[u(x,t)]\right\},$$

the uniform Cartesian coordinate net $T_0(x,t)\subset\mathbb{E}^2$ is mapped into the Chebyshev net $\mathrm{Cheb}(x,t)\subset\Lambda^2$ (Figure 5.3.2).

Let us study on Λ^2 the net $T^{\mathrm{d}}(a)\subset\Lambda^2$, the discrete analog of the net $\mathrm{Cheb}(x,t)$ composed of rhombuses $R(a)$ of side a (the edges of the rhombuses are segments of geodesics on Λ^2). The fact that for any a the net $T^{\mathrm{d}}(a)$ (a set composed of rhombuses) on Λ^2 exists follows from the possibility of recovering the net from the initial data: the geometric construction of such a net from the data of the Darboux problem (5.3.5) at each step amounts ro the standard planimetric construction on Λ^2 of a point (the vertex of the rhombus), equidistant from the two vertices of the already available adjacent edges of the rhombus $R(a)$ (see, e.g., [108]). Therefore, the object on Λ^2 we need to study in connection with problem

[3]For example, for the modified Korteweg-de Vries equation the algorithm for recovering the generating metric is considered in detail in § 5.4.

[4]For example, in the case of the sine-Gordon equation as k-characteristic one choose the net angle, which always coincides with a solution of the equation.

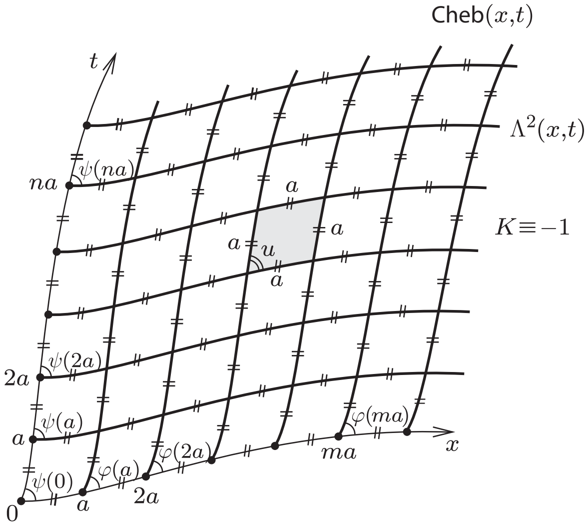


Figure 5.3.2. Formulation of the Darboux problem in the Lobachevsky plane Λ^2

(5.3.5) is the discrete net $T^d(a)$ consisting of two families of piecewise-geodesic broken lines with generatrices $l^x(a)$ and $l^t(a)$ (Figure 5.3.3).

Our algorithmic geometric method for constructing solutions of the Darboux problem (5.3.5) will be connected with finding the net angle $u(x,t)$ of the net $\mathrm{Cheb}(x,t)$, obtained as the limit value as $a \to 0$ of the discretely given function z_{mn}, defined at the nodes of type (m,n) of the discrete rhombic net T^{d}; the net T^{d} is given initially by the broken-line (piecewise-geodesic) generatrices $l^x(a)$ and $l^t(a)$ (in the discrete net the angles will be denoted by z). Therefore, the construction of the solution of the Darboux problem (5.3.5) reduces to the consideration of the corresponding exclusively planimetric problem on the hyperbolic plane. The problem (5.3.5) itself (the problem of determining the net angles of a regular Chebyshev net) is restated in terms of the discrete net $T^{\mathrm{d}}(a)$ as follows:

$$T^{\mathrm{d}}(a) = \begin{cases} z_{m,0} = \varphi(ma), \\ z_{0,n} = \psi(na), \quad m,n = 0,1,2,\ldots, \\ z_{0,0} = \varphi(0) = \psi(0). \end{cases} \tag{5.3.6}$$

In what follows we will investigate geometrically the discrete net $T^{\mathrm{d}}(a)$ that corresponds to the Darboux problem (5.3.5)

5.3.3 Recursion relations for the net angle of the discrete rhombic Chebyshev net

Let us consider the fragment of discrete (piecewise-geodesic) net $T^{\mathrm{d}}(a)$ (Figure 5.3.4) consisting of four rhombuses $R_{kl}(a)$ (the indices in the notation of a rhombus

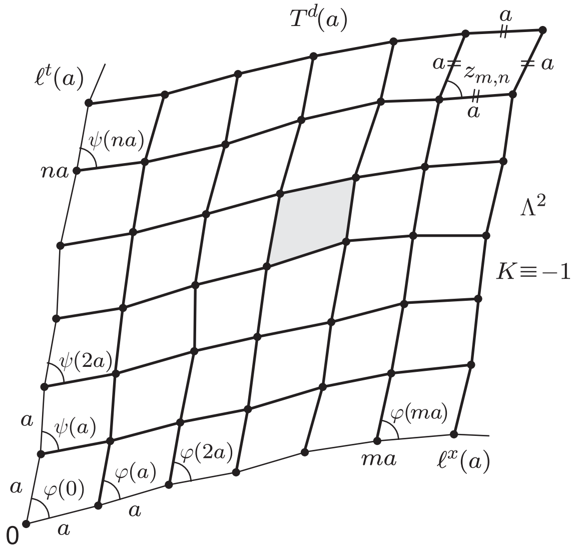


Figure 5.3.3. Piecewise-geodesic rhombic net on the plane Λ^2

correspond to the smallest values of the indices of its vertices) that meet at the $(m+1,\ n+1)$th node $A_{m+1,n+1}$. We pose the problem of computing recursively the angles $z_{k,l}$ of the net $T^{\mathrm{d}}(a)$ from their initial values on the broken-line generatrices $l^x(a)$ and $l^t(a)$.

On the Lobachevsky plane, the magnitude of each full angle is equal to 2π, hence

$$z_{m+1,n+1} = 2\pi - (\gamma_1 + \gamma_2 + \gamma_3), \tag{5.3.7}$$

where $\gamma_1 = \angle A_{m+1,n+2}A_{m+1,n+1}A_{m,n+1}$, $\gamma_2 = \angle A_{m,n+1}A_{m+1,n+1}A_{m+1,n}$, $\gamma_3 = \angle A_{m+1,n}A_{m+1,n+1}A_{m+2,n+1}$.

Let us calculate the angles γ_1 and γ_3 referring to the rhombuses $R_{m,n+1}$ and $R_{m+1,n}$, respectively. To this end let us state the cosine and sine theorems (laws) for an arbitrary geodesic triangle in the hyperbolic plane Λ^2, with edges a, b, and c, and opposite angles α, β, and γ [108]:

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \, \cos\alpha, \tag{5.3.8}$$

and

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}. \tag{5.3.9}$$

Opposite interior angles in an arbitrary rhombus $R_{kl}(a)$ on Λ^2 are equal, and the diagonals of the rhombus are the bisectors of its interior angles. Consequently, $\gamma_2 = z_{m,n}$.

Let us apply successively the law of cosines (5.3.8) and law of sines (5.3.9) to the triangle $\triangle A_{m+1,n}A_{m+1,n+1}A_{m+2,n} \subset R_{m+1,n}$:

$$\cosh |A_{m+1,n+1}A_{m+2,n}| = \cosh^2 a - \sinh^2 a \, \cos z_{m+1,n}, \tag{5.3.10}$$

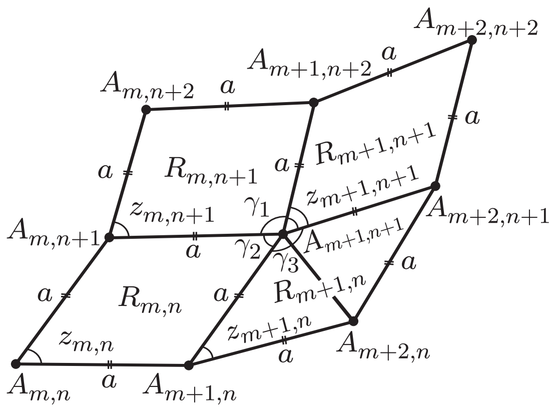


Figure 5.3.4. Characteristic fragment of the discrete rhombic net on Λ^2, consisting or four "adjacent" rhombuses with a common vertex

$$\frac{\sin(\angle A_{m+1,n}A_{m+1,n+1}A_{m+2,n})}{\sinh a} = \frac{\sin z_{m+1,n}}{\sinh |A_{m+1,n+1}A_{m+2,n}|}. \tag{5.3.11}$$

From (5.3.10) and (5.3.11) we find that

$$\sin(\angle A_{m+1,n}A_{m+1,n+1}A_{m+2,n}) = \frac{\sin z_{m+1,n} \sinh a}{[(\cosh^2 a - \sinh^2 a \, \cos z_{m+1,n})^2 - 1]^{1/2}}.$$

Using in what follows the notation Ω_{ij}:

$$\sin \Omega_{ij}(z_{ij}, a) = \frac{\sin z_{i,j} \sinh a}{[(\cosh^2 a - \sinh^2 a \, \cos z_{ij})^2 - 1]^{1/2}},$$

we write the angles γ_1 and γ_3 as

$$\begin{aligned} \gamma_1 &= 2\Omega_{m,n+1}, \\ \gamma_3 &= 2\Omega_{m+1,n}. \end{aligned} \tag{5.3.12}$$

Returning to formula (5.3.7), we derive for the net angles in the net $T^{\mathrm{d}}(a)$ the the recursion relation

$$z_{m+1,n+1} = 2\pi - (z_{m,n} + 2\Omega_{m+1,n} + 2\Omega_{m,n+1}). \tag{5.3.13}$$

Formula (5.3.13) is the basic recursion relation for computing the values of the net angle in a discrete rhombic Chebyshev net on the Lobachevsky plane.

5.3.4 Convergence of the algorithm

The relation (5.3.13) obtained above defines recursively the dependence of the angles of the discrete net $T^{\mathrm{d}}(a)$. Let us prove the convergence of the proposed algorithm for constructing solutions of problem (5.3.5), (5.3.6), i.e., the convergence, as $a \to 0$, of the sequence $\{z_{m,n}(a)\}$ to the exact solution u of the sine-Gordon equation (5.3.3). Note that the recursion relation (5.3.13), which gives the values $z_{m,n}(a)$ in the nodes of the net $T^{\mathrm{d}}(a)$, i.e., the nodes of a stencil that lies entirely on the two-dimensional manifold Λ^2 of Gaussian curvature $K \equiv -1$, can be interpreted as a difference scheme. Therefore, according to the general theory of difference schemes [104], the convergence of the geometric algorithm constructed above can be proved by establishing the approximation and stability properties of the difference analogues (5.3.13) of problem (5.3.5)

5.3.4.1. Order of approximation. Let us estimate the order of approximation of the difference scheme (5.3.13), obtained via a geometric analysis of the discrete net $T^{\mathrm{d}}(a)$ on the Lobachevsky plane Λ^2. To this end, in the difference scheme (5.3.13) we are using, we replace the discrete function $z_{m,n}(a)$ by the value of the exact solution[5] $u(x,t)$ of the sine-Gordon equation at the (m,n) node and compute the error with which the original equation (5.3.3) is satisfied.

Let us insert in the left-hand side of (5.3.13) the value of the solution $u(x,t)$ of the sine-Gordon equation (5.3.3) in the node $((m+1),(n+1))$, in the form of a Taylor series expansion in powers of a in the node (m,n), up to terms of order $O(a^5)$ inclusively (we denote the values of the function u in the node (m,n) by $u^* = u_{m,n}$):

$$\begin{aligned} u_{m+1,n+1} = {} & u^* + u^*_x a + u^*_t a + \frac{1}{2}\, u^*_{xx} a^2 + \frac{1}{2}\, u^*_{tt} a^2 + u^*_{xt} a^2 + \frac{1}{6}\, u^*_{xxx} a^3 \\ & + \frac{1}{6}\, u^*_{ttt} a^3 + \frac{1}{2}\, u^*_{xxt} a^3 + \frac{1}{2}\, u^*_{xtt} a^3 + \frac{1}{24}\, u^*_{xxxx} a^4 + \frac{1}{24}\, u^*_{tttt} a^4 \\ & + \frac{1}{6}\, u^*_{xxxt} a^4 + \frac{1}{4}\, u^*_{xxtt} a^4 + \frac{1}{6}\, u^*_{xttt} a^4 + O(a^5). \end{aligned} \tag{5.3.14}$$

Let us state the properties of the functions $\Omega_{m+1,n}$ and $\Omega_{m,n+1}$ (which hereafter will be denoted by the unique symbol Ω: $\Omega = \Omega_{m+1,n}$ or $\Omega = \Omega_{m,n+1}$) that we need in order to analyze the right-hand side of (5.3.14).

The function $\Omega(z,a)$ is even in the argument a, and so

$$\left.\frac{\partial^{2p+1}\Omega}{\partial a^{2p+1}}\right|_{a=0} = 0, \quad p = 1, 2, \dots .$$

At the same time,

$$\frac{\partial \Omega(z,0)}{\partial z} = -\frac{1}{2}, \quad \frac{\partial^l \Omega(z,0)}{\partial z^l} \equiv 0, \quad l = 2, 3, \dots .$$

Using the properties indicated above of the derivatives of the function Ω, we obtain the following representations for $\Omega_{m+1,n}$ and $\Omega_{m,n+1}$, which appear in the right-hand side of equation (5.3.13), by substituting in the latter the exact

[5] $u(x,t) \equiv z(x,t)$.

solution of the sine-Gordon equation:

$$1)\quad \Omega(u_{m+1,n},a) = \Omega(u^*,0) + \Omega_z(u^*,0)h_1 + \frac{1}{2}\,\Omega_{aa}(u^*,0)a^2 + \frac{1}{2}\,\Omega_{zaa}(u^*,0)h_1a^2 + \frac{1}{4}\,\Omega_{zzaa}(u^*,0)h_1^2a^2 + \frac{1}{24}\,\Omega_{aaaa}(u^*,0)a^4 + O(a^5), \quad (5.3.15)$$

where

$$h_1 = u_x^*a + \frac{1}{2}\,u_{xx}^*a^2 + \frac{1}{6}\,u_{xxx}^*a^3 + \frac{1}{24}\,u_{xxxx}^*a^4 + O(a^5);$$

$$2)\quad \Omega(u_{m,n+1},a) = \Omega(u^*,0) + \Omega_z(u^*,0)h_2 + \frac{1}{2}\,\Omega_{aa}(u^*,0)a^2 + \frac{1}{2}\,\Omega_{zaa}(u^*,0)h_2a^2 + \frac{1}{4}\,\Omega_{zzaa}(u^*,0)h_2^2a^2 + \frac{1}{24}\,\Omega_{aaaa}(u^*,0)a^4 + O(a^5), \quad (5.3.16)$$

where

$$h_2 = u_t^*a + \frac{1}{2}\,u_{tt}^*a^2 + \frac{1}{6}\,u_{ttt}^*a^3 + \frac{1}{24}\,u_{tttt}^*a^4 + O(a^5).$$

Computation of the derivatives of the function Ω that appear in (5.3.15) and (5.3.16) yields

$$\begin{gathered}\Omega_z\big|_{a=0} = -\tfrac{1}{2}, \quad \Omega_a\big|_{a=0} = 0, \quad \Omega_{aa}\big|_{a=0} = -\tfrac{1}{2}\sin z,\\ \Omega_{zaa}\big|_{a=0} = \tfrac{1}{2}\cos z, \quad \Omega_{zzaa}\big|_{a=0} = \tfrac{1}{2}\sin z,\\ \Omega_{aaa}\big|_{a=0} = 0, \quad \Omega_{aaaa}\big|_{a=0} = -\tfrac{1}{2}\sin z\Big(1 - 6\sin^2\tfrac{z}{2}\Big).\end{gathered} \quad (5.3.17)$$

Now substituting expressions (5.3.15)–(5.3.17) in the left- and right-hand sides of relation (5.3.13) and using the differential consequences

$$\begin{gathered}u_{xxt} = u_x\cos u, \quad u_{ttx} = u_t\cos u,\\ u_{xxxt} = u_{xx}\cos u - u_x^2\sin u, \quad u_{tttx} = u_{tt}\cos u - u_t^2\sin u,\end{gathered}$$

of the sine-Gordon equation (5.3.3), we obtain

$$\begin{aligned}&u_{xt}^* - \sin u^*\\ &= \bigg\{\frac{1}{4}\cos u^*(u_{xx}^* + u_{tt}^*) - \frac{1}{4}\sin u^*\big(u_x^{*2} + u_t^{*2}\big) + \frac{1}{48}\sin z\Big(1 - 6\sin^2\frac{u^*}{2}\Big)\\ &\quad - \frac{1}{6}\big[\cos u^*(u_{xx}^* + u_{tt}^*) - \sin u^*(u_x^{*2} + u_t^{*2})\big]\\ &\quad - \frac{1}{4}(\sin u^*\cos u^* - u_x^*u_t^*\sin u^*)\bigg\}a^2 + O(a^3).\end{aligned} \quad (5.3.18)$$

Note that the terms of order $O(a^0)$ that arise in the derivation of (5.3.18) vanish identically:

$$2u^* - 2\pi + 4\Omega(u^*,0) \equiv 0.$$

Therefore, the order of approximation of the difference scheme (5.3.13) investigated here yields the estimate

$$u^*_{xt} - \sin u^* = O(a^2), \tag{5.3.19}$$

which establishes that *the approximation of the sine-Gordon equation by its difference analogue* (5.3.13) *is of second order*.

5.3.4.2. Stability of the difference analogues of the Darboux problem. Let us prove the stability of the difference analogue of the Darboux problem (5.3.5):

$$\begin{aligned} z_{m+1,n+1} &= 2\pi - (z_{m,n} + 2\Omega_{m+1,n} + 2\Omega_{m,n+1}), \\ z_{m,0} &= \varphi(ma), \quad z_{0,n} = \psi(na), \quad \varphi(0) = \psi(0). \end{aligned} \tag{5.3.20}$$

Side by side with problem (5.3.20) we formulate the corresponding perturbed problem

$$\begin{aligned} \overline{z}_{m+1,n+1} &= 2\pi - (\overline{z}_{m,n} + 2\overline{\Omega}_{m+1,n} + 2\overline{\Omega}_{m,n+1}) + a^2 Y_{m,n}(a), \\ \overline{z}_{m,0} &= \varphi(ma), \quad \overline{z}_{0,n} = \psi(na), \quad \varphi(0) = \psi(0), \end{aligned} \tag{5.3.21}$$

where $\overline{\Omega}_{i,j} = \Omega_{i,j}(\overline{z}_{ij}, a)$.

Problems (5.3.20), (5.3.21) are considered in the domain

$$D \subset \Lambda^2 \colon\ D = l^x[0, B_1] \times l^t[0, B_2],$$

specified by the broken-line generatrices $l^x[0, B_1]$ and $l^t[0, B_2]$ (Figure 5.3.3), with the respective lengths B_1 and B_2. (The length of each of the aforementioned generatrices is the sum of the lengths of its links , which are segments of geodesics (shortest curves) on Λ^2.)

To prove the stability of the original difference problem we establish the existence of constants M_1, M_2, such that, for any sufficiently small linear size a of the discrete net $T^{\mathrm{d}}(a)$, every perturbation

$$Y_{m,n}(a) \colon\ \|Y(a)\| \le M_1$$

will obey the estimate

$$\|\overline{z} - z\| \le M_2 \cdot \|Y(a)\|. \tag{5.3.22}$$

Here the norm of the discrete function $q_{m,n}(a)$ in the net $T^{\mathrm{d}}(a)$ is defined in the standard manner, as the uniform Chebyshev norm:

$$\|q(a)\| = \max_{A_{m,n} \in T^{\mathrm{d}}(a) \subset D} |q_{m,n}(a)|.$$

Upon introducing the typical function $Q_{i,j}(z_{i,j}, a)$ by

$$Q_{i,j}(z_{i,j}, a) = \pi - 2\Omega_{i,j}(z_{i,j}, a) - z_{i,j},$$

it is not difficult to verify that the recursion relations (5.3.20) and (5.3.21) can be recast as

$$z_{m+1,n+1} = Q(z_{m+1,n}, a) - Q(z_{m,n+1}, a) + z_{m+1,n} + z_{m,n+1} - z_{m,n}, \tag{5.3.23}$$

and respectively

$$\overline{z}_{m+1,n+1} = Q(\overline{z}_{m+1,n}, a) - Q(\overline{z}_{m,n+1}, a) + \overline{z}_{m+1,n} + \overline{z}_{m,n+1} - \overline{z}_{m,n} + a^2 Y_{m,n}(a). \tag{5.3.24}$$

Further, let $\delta z_{m,n}$ denote the difference between the solution of problem (5.3.21) and that of the original problem (5.3.20):

$$\delta z_{m,n} = \overline{z}_{m,n} - z_{m,n}.$$

Subtracting from the relations of problem (5.3.21) the corresponding relations of problem (5.3.20), we obtain

$$\delta z_{m+1,n+1} = Q(\overline{z}_{m+1,n}, a) - Q(z_{m+1,n}, a) + Q(\overline{z}_{m,n+1}, a) - Q(z_{m,n+1}, a) + \delta z_{m+1,n} + \delta z_{m,n+1} - \delta z_{m,n} + a^2 Y_{m,n}(a), \tag{5.3.25}$$

$$\delta z_{m,0} = 0, \quad \delta z_{0,n} = 0. \tag{5.3.26}$$

Let us transform the difference

$$\Delta Q = Q(\overline{z}_{i,j}, a) - Q(z_{i,j}, a)$$

appearing in (5.3.25) by means of the Lagrange formula:

$$\Delta Q = \frac{\partial Q}{\partial z}(z^0, a)(\overline{z} - z) = -\left[1 + \frac{1}{2}\frac{\partial \Omega}{\partial z}(z^0, a)\right](\overline{z} - z), \tag{5.3.27}$$

where $z^0 \in [\overline{z}, z]$.

Using in (5.3.27) the form of Ω, we write

$$\Delta Q = R(z^0, a)(\overline{z} - z),$$

where

$$R(z^0, a) = -\frac{1 - \cosh a + \sinh^2 a \cdot \sin^2(z^0/2)}{1 + \sinh^2 a \cdot \sin^2(z^0/2)}. \tag{5.3.28}$$

In accordance with (5.3.28), the function $R(z^0, a)$ obeys the estimates

$$\cosh a - (1 + \sinh^2 a) \le R(z^0, a) \le \cosh a - 1. \tag{5.3.29}$$

If we now use in (5.3.29) the small-a asymptotic expansions

$$\cosh a = 1 + \frac{1}{2}a^2 + O(a^4), \quad \sinh a = a + O(a^3), \quad \sinh^2 a = a^2 + O(a^4),$$

we get

$$|R(z^0, a)| \le \frac{1}{2}(a^2 + \nu a^4), \quad \nu = \text{const} \ge 0. \tag{5.3.30}$$

Next, let us return to relation (5.3.25), recasting it as

$$(\delta z_{m+1,n+1} - \delta z_{m+1,n}) - (\delta z_{m,n+1} - \delta z_{m,n})$$
$$= R(z^0_{m+1,n}, a)\delta z_{m+1,n} + R(z^0_{m,n+1}, a)\delta z_{m,n+1} + a^2 Y_{m,n}(a). \tag{5.3.31}$$

At the same time, the initial conditions satisfy the relations

$$\delta z_{0,n+1} - \delta z_{0,n} = 0.$$

Given an index n, we fix some value

$$n = n^* \in \left\{0, 1, \ldots, \left[\frac{B_2}{a}\right]\right\} \quad \text{(where } [\,\cdot\,] \text{ denotes the integer part)},$$

and then sum in (5.3.31) with respect to the first index, so that m takes values from 0 to m^*, where $m^* \in \left\{0, 1, \ldots, \left[\frac{B_1}{a}\right]\right\}$. Then

$$\delta z_{m^*+1,n^*+1} - \delta z_{m^*+1,n^*}$$
$$= \sum_{m=0}^{m^*} \left[R(z^0_{m+1,n^*}, a)\delta z_{m+1,n^*} + R(z^0_{m,n^*+1}, a)\delta z_{m,n^*+1} + a^2 Y_{m,n}(a)\right]. \tag{5.3.32}$$

Recalling (5.3.30), we estimate the absolute magnitude of the left-hand side of (5.3.32):

$$|\delta z_{m^*+1,n^*+1} - \delta z_{m^*+1,n^*}|$$
$$\leq (m^* + 1)(a^2 + \nu a^4)\|\delta z\|_{m^*+n^*+1} + a^2(m^* + 1)\|Y(a)\|. \tag{5.3.33}$$

In (5.3.33) we use the auxiliary norm

$$\|q(a)\|_N = \max_{A_{m,n} \in T^{\mathrm{d}}(a) \subset D:\ m+n \leq N} |q_{m,n}(a)|.$$

If now in the left-hand side of (5.3.33) we use a property of the modulus as well as the inequality $|\cdot| \leq \|\cdot\|_N$, we obtain

$$|\delta z_{m^*+1,n^*+1}| \leq (m^* + 1)(a^2 + \nu a^4)\|\delta z\|_{m^*+n^*+1}$$
$$+ \|\delta z\|_{m^*+n^*+1} + a^2(m^* + 1)\|Y(a)\|. \tag{5.3.34}$$

Let us introduce the numerical parameter $N = m^* + n^* + 1$. Note that inequalities of the type (5.3.34) hold also for all other values of the indices m, n with $m + n + 1 \leq N$, and that either the norm $\|\delta z\|_{N+1}$ is attained on one of the moduli $|\delta z_{m+1,n+1}|$ for $m + n + 2 = N + 1$ (in which case the left-hand side of (5.3.34) can be replaced by $\|\delta z\|_{N+1}$), or there exists values $m = \overline{m}$ and $n = \overline{n}$, with $\overline{m} + \overline{n} + 1 \leq N$, for which

$$|\delta z_{\overline{m}+1,\overline{n}+1}| = \|\delta z\|_{N+1},$$

and correspondingly

$$|\delta z_{\overline{m}+1,\overline{n}+1}| \leq (\overline{m}+1)(a^2+\nu a^4)\|\delta z\|_{\overline{m}+\overline{n}+1} + \|\delta z\|_{\overline{m}+\overline{n}+1} + a^2(\overline{m}+1)\|Y(a)\|. \tag{5.3.35}$$

Since $\|\cdot\|_{\overline{m}+\overline{n}+1} \leq \|\cdot\|_N$ and $(\overline{m}+1)a \leq B_1$, the arguments given for (5.3.34) and (5.3.35) lead to the final recursive estimate

$$\|\delta z\|_{N+1} \leq B_1(a+\nu a^3)\|\delta z\|_N + \|\delta z\|_N + aB_1\|Y(a)\|, \tag{5.3.36}$$

where $N = 1, 2, \ldots, m+n+1$ or $N \in \left\{1, 2, \ldots, \left[\dfrac{B_1}{a}\right] + \left[\dfrac{B_2}{a}\right] - 1\right\}$.

To continue, let us recast (5.3.36) as

$$\|\delta z\|_N \leq \rho_N, \tag{5.3.37}$$

where

$$\rho_{N+1} = B_1(a+\nu a^3)\rho_N + \rho_N + aB_1\|Y(a)\|. \tag{5.3.38}$$

Since

$$\|\delta z\|_0 = \|\delta z\|_1 = 0,$$

the corresponding initial condition is

$$\rho_0 = 0. \tag{5.3.39}$$

Starting with (5.3.39), we compute ρ_{N+1} by formula (5.3.38):

$$\rho_{N+1} = B_1 a\|Y(a)\| \cdot \sum_{\rho=0}^{N} \left[1 + B_1(a+\nu a^3)\right]^{N-\rho}. \tag{5.3.40}$$

The power terms in the sum (5.3.40) can be estimated from above by the corresponding exponentials ($t^k \leq e^{kt}$, $k \geq 0$, $t > 0$), hence

$$\begin{aligned} \rho_{N+1} &\leq B_1 a\|Y(a)\| \sum_{\rho=0}^{N} \exp\left[(N-\rho)aB_1(1+\nu a^2)\right] \\ &\leq \left\{ aB_1 \exp\left[B_1(B_1+B_2)(1+\nu a^2)\right] \sum_{\rho=0}^{N} \exp\left[-\rho a(1+\nu a^2)B_1\right] \right\} \cdot \|Y(a)\|. \end{aligned} \tag{5.3.41}$$

The sum in the right-hand side of (5.3.41) can be interpreted as the lower Darboux sum (with the length of the elementary segment in the partition of the segment $[0, N]$ equal to 1), and therefore can be estimated from above by the corresponding definite integral:

$$\sum_{\rho=0}^{N} \exp\left[-\rho a(1+\nu a^2)B_1\right] \leq 1 + \int_0^N \exp\left[-aB_1(1+\nu a^2)\rho\right] d\rho,$$

the computation of which sharpens the estimate (5.4.31) to

$$\rho_{N+1} \leq M\|Y(a)\|, \tag{5.3.42}$$

where $M = (1 + B_1^2)\exp[B_1(B_1 + B_2)(1 + \nu B_1^2)] = \text{const}$.

From (5.3.37) and (5.3.42) it follows that

$$\|\delta z\|_{N+1} \leq M \cdot \|Y(a)\|, \quad M = \text{const}. \tag{5.3.43}$$

Inequality (5.4.43) holds for all values of the parameter N in its range $\left\{1, 2, \ldots, \left[\frac{B_1}{a}\right] + \left[\frac{B_2}{a}\right] - 1\right\}$, in particular, also for $N = \overline{N} = \left[\frac{B_1}{a}\right] + \left[\frac{B_2}{a}\right] - 1$.

Moreover,

$$\|\delta z\|_{\overline{N}+1} = \|\delta z\|,$$

which in view of (5.3.43) yields the final estimate

$$\|\delta z\| \leq M\|Y(a)\|, \quad M = \text{const}. \tag{5.3.44}$$

This establishes the stability of the difference (discrete) analog (5.3.20) of the Darboux problem (5.3.5) for the sine-Gordon equation under investigation.

5.3.5 Convergence of the algorithm. General problems of the approach

The a priori estimates (5.3.19) (5.3.44) found above for the approximation error and the stability of the difference scheme (5.3.20) under study establish its convergence, i.e., the solution $z_{m,n}(a)$ of problem (5.3.20) converges as $a \to 0$ to the exact solution u of the Darboux problem (5.3.5) for the sine-Gordon equation. This result, together with the well-posedness of our difference problem confirm also that one is right to apply the proposed geometric approach, based on the concept of Λ^2-representation for nonlinear equations [77], for the elaboration of geometric methods of their numerical integration. From the point of view of geometry, to the convergence of the algorithm considered here corresponds the "smoothing process" of the discrete net $T^{\mathrm{d}}(a)$ under question as $a \to 0$ and its passage, in the limit, into a regular smooth Chebyshev net, namely the net that generates (according to the theory of Λ^2-representations) the sine-Gordon equation under study.

The investigation carried out above concerned the rhombic Chebyshev net (the regular net $\mathrm{Cheb}(x, t)$ and the discrete net $T^{\mathrm{d}}(a)$). However, the arguments used and the results obtained can be carried over to the case of the parallelogram Chebyshev net $T^{\mathrm{d}}(a, b)$ (a and b denote the characteristic linear dimensions of the elementary coordinate cell).

Overall, the modeling of parallelogram-type discrete geometric constructions (on the Lobachevsky plane Λ^2, as well as in the space $\mathbb{E}^3$) is quite effective in the study (in particular, for the approximation) of pseudospherical surfaces. In this connection we mention the paper [190], in which to approximate a pseudospherical surface and the Chebyshev net of asymptotic lines on it one uses a modeled parallelogram lattice of special type (P-lattice), formed by two families of broken lines. In this way a complete analogy between the introduced discrete analogs of the geometric characteristic of a pseudospherical surface and their classical prototypes is achieved: the constant negative value of the "discrete Gaussian curvature", the constant (respectively, of opposite sign) "discrete torsions" of the broken lines of each of the families in the P-lattice, the qualitative similarity between the spherical map of the p-lattice and the spherical map of the Chebyshev net on the pseudospherical surface, and so on.

Let us mention also R. Koch's works [166, 167], in which geometric "constructions" close to the one considered here were studied.

We finish by listing a number of problems that arise in the context of the geometric approach treated here to the construction of algorithms for the numerical integration of Λ^2-equations:

1) Recover, from a given nonlinear equation, the net in the Lobachevsky plane Λ^2 that generates the equation; the correct determination of the characteristic properties of such a net (k-characteristics) that are preserved for the discrete net introduced.

2) Establish, based on the concept of Λ^2-equations, geometric criteria that a priori guarantee the approximation, stability, conservativeness, and other potentially important properties of the emerging difference analogs of the problems studied.

3) Generally, find a geometric classification of differential operators, and also of their discrete analogs, associated with coordinate nets on two-dimensional smooth manifolds of constant curvature.

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