*SCHAUM'S OUTLINE OF*

# THEORY AND PROBLEMS

of

# ENGINEERING ECONOMICS

by

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# Preface

Despite remarkable technological advances during the past several decades, most major engineering decisions are based on economic considerations — a situation that is unlikely to change in the years ahead. Hence the importance of economic principles to all undergraduate engineering students, regardless of their particular disciplinary interests.

This Schaum's Outline contains a clear and concise review of the principles of engineering economics, together with a large number of solved problems. Most chapters also contain a list of supplementary problems, which the readers may solve themselves. Thus, readers receive an exposure to the theory, as well as an opportunity to become actively involved in the application of this theory to typical (though simple) problem situations.

The book is designed to complement a standard undergraduate course in engineering economics. The first five chapters consider the mathematics of compound interest, emphasizing the time value of money. Chapters 6 through 9 discuss the application of this material in various decision-making criteria, and Chapter 10 deals with equipment replacement and retirement decisions. Chapter 11 considers the important topics of depreciation and taxes, and their impact on the decision-making process. Finally, Chapter 12 presents a realistic economic feasibility study.

The four appendixes to the book contain tables of various compound interest factors. Such tables continue to be useful, even in an era of electronic calculators and personal computers.

JOSÉ A. SEPULVEDA
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# Chapter 1

# Basic Concepts

## 1.1 INTEREST

***Interest*** is a fee that is charged for the use of someone else's money. The size of the fee will depend upon the total amount of money borrowed and the length of time over which it is borrowed.

**Example 1.1** An engineer wishes to borrow \$20 000 in order to start his own business. A bank will lend him the money provided he agrees to repay \$920 per month for two years. How much interest is he being charged?

The total amount of money that will be paid to the bank is $24 \times \$920 = \$22\,080$. Since the original loan is only \$20 000, the amount of interest is $\$22\,080 - \$20\,000 = \$2080$.

Whenever money is borrowed or invested, one party acts as the lender and another party as the borrower. The lender is the owner of the money, and the borrower pays interest to the lender for the use of the lender's money. For example, when money is deposited in a savings account, the depositor is the lender and the bank is the borrower. The bank therefore pays interest for the use of the depositor's money. (The bank will then assume the role of the lender, by loaning this money to another borrower, at a higher interest rate.)

## 1.2 INTEREST RATE

If a given amount of money is borrowed for a specified period of time (typically, one year), a certain percentage of the money is charged as interest. This percentage is called the ***interest rate.***

**Example 1.2** (a) A student deposits \$1000 in a savings account that pays interest at the rate of 6% per year. How much money will the student have after one year? (b) An investor makes a loan of \$5000, to be repaid in one lump sum at the end of one year. What annual interest rate corresponds to a lump-sum payment of **\$5425?**

(a) The student will have his original \$1000, plus an interest payment of $0.06 \times \$1000 = \$60$. Thus, the student will have accumulated a total of \$1060 after one year. (Notice that the interest rate is expressed as a decimal when carrying out the calculation.)

(b) The total amount of interest paid is $\$5425 - \$5000 = \$425$. Hence the annual interest rate is

$$\frac{\$425}{\$5000} \times 100\% = 8.5\%$$

Interest rates are usually influenced by the prevailing economic conditions, as well as the degree of risk associated with each particular loan.

## 1.3 SIMPLE INTEREST

***Simple interest*** is defined as a fixed percentage of the ***principal*** (the amount of money borrowed), multiplied by the life of the loan. Thus,

$$I = niP \tag{1.1}$$

where $I \equiv$ total amount of simple interest
$n \equiv$ life of the loan
$i \equiv$ interest rate (expressed as a decimal)
$P \equiv$ principal

It is understood that $n$ and $i$ refer to the same unit of time (e.g., the year).

Normally, when a simple interest loan is made, nothing is repaid until the end of the loan period; then, both the principal and the accumulated interest are repaid. The total amount due can be expressed as

$$F = P + I = P(1 + ni) \tag{1.2}$$

**Example 1.3** A student borrows $3000 from his uncle in order to finish school. His uncle agrees to charge him simple interest at the rate of $5\frac{1}{2}\%$ per year. Suppose the student waits two years and then repays the entire loan. How much will he have to repay?

By (*1.2*), F = $3000[1 + (2)(0.055)] = $3330.

## 1.4 COMPOUND INTEREST

When interest is compounded, the total time period is subdivided into several interest periods (e.g., one year, three months, one month). Interest is credited at the end of each interest period, and is allowed to accumulate from one interest period to the next.

During a given interest period, the current interest is determined as a percentage of the total amount owed (i.e., the principal plus the previously accumulated interest). Thus, for the first interest period, the interest is determined as

$$I_1 = iP$$

and the total amount accumulated is

$$F_1 = P + I_1 = P + iP = P(1 + i)$$

For the second interest period, the interest is determined as

$$I_2 = iF_1 = i(1 + i)P$$

and the total amount accumulated is

$$F_2 = P + I_1 + I_2 = P + iP + i(1 + i)P = P(1 + i)^2$$

For the third interest period,

$$I_3 = i(1 + i)^2P \qquad F_3 = P(1 + i)^3$$

and so on. In general, if there are n interest periods, we have (dropping the subscript):

$$F = P(1 + i)^n \tag{1.3}$$

which is the so-called law of compound interest. Notice that F, the total amount of money accumulated, increases exponentially with n, the time measured in interest periods.

**Example 1.4** A student deposits $1000 in a savings account that pays interest at the rate of 6% per year, compounded annually. If all of the money is allowed to accumulate, how much will the student have after 12 years? Compare this with the amount that would have accumulated if simple interest had been paid.

By (*1.3*),

$$F = \$1000(1 + 0.06)^{12} = \$2012.20$$

Thus, the student's original investment will have more than doubled over the 12-year period.

If simple interest had been paid, the total amount that would have accumulated is determined by (1.2) as

$$F = \$1000[1 + (12)(0.06)] = \$1720.00$$

## 1.5 THE TIME VALUE OF MONEY

Since money has the ability to earn interest, its value increases with time. For instance, $100 today is equivalent to

$$F = \$100(1 + 0.07)^5 = \$140.26$$

five years from now if the interest rate is 7% per year, compounded annually. We say that the ***future worth*** of **$100** is **$140.26** if $i = 7\%$ (per year) and $n = 5$ (years).

Since money increases in value as we move from the present to the future, it must decrease in value as we move from the future to the present. Thus, the ***present worth*** of **$140.26** is **$100** if $i = 7\%$ (per year) and $n = 5$ (years).

**Example 1.5** A student who will inherit $5000 in three years has a savings account that pays $5\frac{1}{2}\%$ per year, compounded annually. What is the present worth of the student's inheritance?

Equation (1.3) may be solved for P, given the value of F:

$$P = \frac{F}{(1+i)^n} = \frac{\$5000}{(1+0.055)^3} = \$4258.07$$

The present worth of $5000 is $4258.07 if $i = 5\frac{1}{2}\%$, compounded annually, and $n = 3$.

## 1.6 INFLATION

National economies frequently experience ***inflation***, in which the cost of goods and services increases from one year to the next. Normally, inflationary increases are expressed in terms of percentages which are compounded annually. Thus, if the present cost of a commodity is PC, its future cost, FC, will be

$$FC = PC(1+\lambda)^n \tag{1.4}$$

where $\lambda \equiv$ annual inflation rate (expressed as a decimal)
$n \equiv$ number of years

**Example 1.6** An economy is experiencing inflation at the rate of 6% per year. An item presently costs $100. If the 6% inflation rate continues, what will be the price of this item in five years?

By (1.4), $FC = \$100(1+0.06)^5 = \$133.82$.

In an inflationary economy, the value (buying power) of money decreases as costs increase. Thus,

$$\frac{F}{P} = \frac{PC}{FC} = \frac{1}{(1+\lambda)^n}$$

or

$$F = \frac{P}{(1+\lambda)^n} \tag{1.5}$$

where $F$ is the future worth, measured in today's dollars, of a present amount P.

**Example 1.7** An economy is experiencing inflation at an annual rate of 6%. If this continues, what will $100 be worth five years from now, in terms of today's dollars?

From (1.5),

$$F = \frac{\$100}{(1+0.06)^5} = \$74.73$$

Thus $100 in five years will be worth only $74.73 in terms of today's dollars. Stated differently, in five years $100 will be required to purchase the same commodity that can now be purchased for $74.73.

If interest is being compounded at the same time that inflation is occurring, then the future worth can be determined by combining (1.3) and (1.5):

$$F = \frac{P(1+i)^n}{(1+\lambda)^n} = P\left(\frac{1+i}{1+\lambda}\right)^n$$

or, defining the ***composite interest rate,***

$$\theta \equiv \frac{i-\lambda}{1+\lambda} \tag{1.6}$$

we have

$$F = P(1+\theta)^n \tag{1.7}$$

Observe that $\theta$ may be negative.

**Example 1.7** An engineer has received $10 000 from his employer for a patent disclosure. He has decided to invest the money in a 15-year savings certificate that pays 8% per year, compounded annually. What will be the final value of his investment, in terms of today's dollars, if inflation continues at the rate of 6% per year?

A composite interest rate can be determined from (1.6):

$$\theta = \frac{0.08-0.06}{1+0.06} = 0.0189$$

Substituting this value into (*1.7*), we obtain

$$F = \$10\,000(1+0.0189)^{15} = \$13\,242.61$$

(If more significant figures are included in the value for $\theta$, the future value $13 236.35 is obtained.)

## 1.7 TAXES

In most situations, the interest that is received from an investment will be subject to taxation. Suppose that the interest is taxed at a rate $t$, and that the period of taxation is the same as the interest period (e.g., one year). Then the tax for each period will be $T = tiP$, so that the net return to the investor (after taxes) will be

$$I' = I - T = (1-t)iP \tag{1.8}$$

If the effects of taxation and inflation are both included in a compound interest calculation, (1.7) may still be used to relate present and future values, provided the composite interest rate is redefined as

$$\theta \equiv \frac{(1-t)i-\lambda}{1+\lambda} \tag{1.9}$$

**Example 1.8** Refer to Example 1.7. Suppose the engineer is in the 32% tax bracket, and is likely to remain there throughout the lifetime of the certificate. If inflation continues at the rate of 6% per year, what will be the value of his investment, in terms of today's dollars, when the certificate matures?

Let us assume that the engineer is able to invest the entire $10 000 in a savings certificate and that the 32% tax bracket includes all federal, state, and local taxes. By (*1.9*),

$$\theta = \frac{(1-0.32)(0.08)-0.06}{1+0.06} = -0.00528$$

and (1.7) then gives

$$F = \$10\,000(1-0.00528)^{15} = \$9236.61$$

Because of the combined effects of inflation and taxation, $\theta$ is negative, and the engineer ends up with ***less real purchasing power*** after 15 years than he has today. (To make matters worse, the engineer will most likely have to pay taxes on the original $10 000, substantially reducing the amount of money available for investment.)

The subject of taxation is considered in much greater detail in Chapter 11.

## 1.8 CASH FLOWS

A cash flow is the difference between total cash receipts (inflows) and total cash disbursements (*outflows*) for a given period of time (typically, one year). Cash flows are very important in engineering economics because they form the basis for evaluating projects, equipment, and investment alternatives.

The easiest way to visualize a cash flow is through a cash flow diagram, in which the individual cash flows are represented as vertical arrows along a horizontal time scale. Positive cash flows (net inflows) are represented by upward-pointing arrows, and negative cash flows (net outflows) by downward-pointing arrows; the length of an arrow is proportional to the magnitude of the corresponding cash flow. Each cash flow is assumed to occur at the end of the respective time period.

**Example** 1.9 A company plans to invest $500 000 to manufacture a new product. The sale of this product is expected to provide a net income of $70 000 a year for 10 years, beginning at the end of the first year. Figure 1-1 is the cash flow diagram for this proposed project. Notice that the initial $500 000 investment is represented by a downward-pointing arrow located at the end of year 0 (i.e., at the beginning of year 1). Each annual net income ($70 000) is indicated by an upward-pointing arrow located at the end of the corresponding year.

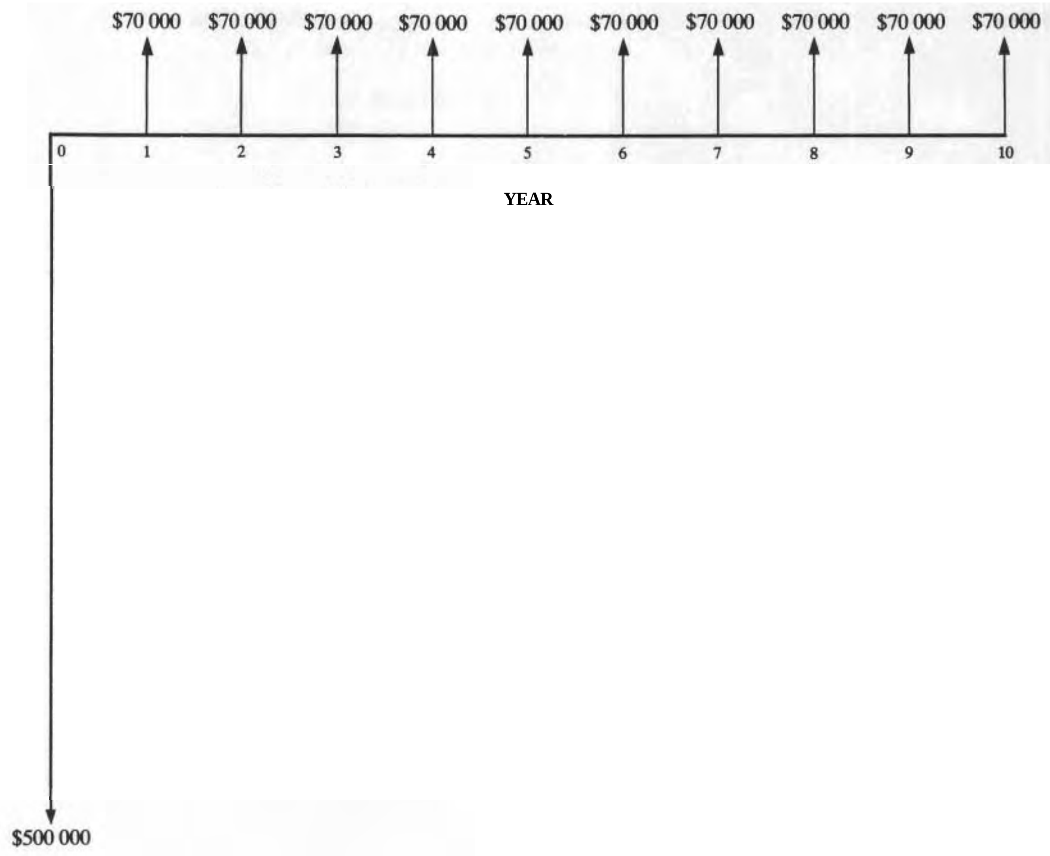


**Fig.** *1-1*

In a lender-borrower situation, an inflow for the one is an outflow for the other. Hence, the cash flow diagram for the lender will be the mirror image in the time line of the cash flow diagram for the borrower.

## Solved Problems

**1.1** The ABC Company deposited \$100 000 in a bank account on June 15 and withdrew a total of \$115 000 exactly one year later. Compute: *(a)* the ***interest*** which the ABC Company received from the \$100 000 investment, and (b) the annual ***interest rate*** which the ABC Company was paid.

(*a*) $$I = \$115\,000 - \$100\,000 = \$15\,000$$

(*b*) $$i = \frac{\$15\,000/\text{year}}{\$100\,000} \times 100\% = 15\% \text{ per year}$$

**1.2** What is the annual rate of simple interest if \$265 is earned in four months on an investment of \$15 000?

From (*1.2*),

$$\begin{aligned} F &= P(1+ni) \\ \$15\,265 &= \$15\,000\left(1+\frac{4}{12}i\right) \\ \$15\,265 &= \$15\,000 + \$5000i \\ \frac{\$265}{\$5000} &= i \\ i &= 5.3\% \end{aligned}$$

**1.3** Determine the principal that would have to be invested to provide \$200 of simple interest income at the end of two years if the annual interest rate is 9%

By (*1.1*),

$$\begin{aligned} I &= niP \\ \$200 &= 2(0.09)P \\ P &= \$1111.11 \end{aligned}$$

**1.4** Compare the interest earned from an investment of \$1000 for 15 years at 10% per annum simple interest, with the amount of interest that could be earned if these funds were invested for 15 years at 10% per year, compounded annually.

The simple interest is given by (*1.1*) as $I = (15)(0.10)(\$1000) = \$1500$. From (*1.3*),

$$\begin{aligned} I = F - P = P(1+i)^n - P &= \$1000(1+0.10)^{15} - \$1000 \\ &= \$1000(4.17725) - \$1000 = \$3177.25 \end{aligned}$$

or more than double the amount earned using simple interest.

**1.5** At what annual interest rate is \$500 one year ago equivalent to \$600 today?

From (*1.3*),

$$\$600 = \$500(1+i) \qquad \text{or} \qquad i = 20\%$$

**1.6** Suppose that the interest rate is 10% per year, compounded annually. What is the minimum amount of money that would have to be invested for a two-year period in order to earn \$300 in interest?

From (*1.3*),

$$P = \frac{F}{(1+i)^n}$$

$$P = \frac{P + \$300}{(1+0.10)^2}$$

$$1.21P = P + \$300$$

$$P = \$1428.57$$

**1.7** How long would it take for an investor to double his money at 10% interest per year, compounded annually?

By (*1.3*),

$$2P = P(1+0.10)^n$$

$$2 = (1.10)^n$$

$$n = \frac{\log 2}{\log 1.10} = 7.27 \text{ years}$$

Actually, since the interest is compounded only at the end of each year, the investor would have to wait 8 years.

**1.8** Suppose that a man lends \$1000 for four years at 12% per year simple interest. At the end of the four years, he invests the entire amount which he then has for 10 years at 8% interest per year, compounded annually. How much money will he have at the end of the 14-year period?

From (1.2) and (*1.3*),

$$F = P(1+n_1i_1)(1+i_2)^{n_2} = \$1000[1+(4)(0.12)](1+0.08)^{10}$$
$$= \$1000(1.48)(2.15892) = \$3195.21$$

**1.9** Let the inflation rate be 6% per year. If a person deposits \$50 000 in a bank account at 9% per annum simple interest for 10 years, will this effectively protect the purchasing power of the original principal?

The answer is not obvious, for the inflation rate, though the smaller, is compounded. From (*1.2*), the principal will grow to:

$$F = \$50\,000[1+(10)(0.09)] = \$95\,000$$

By (*1.5*), the inflation will reduce the "real" purchasing power of these funds to

$$F = \frac{\$95\,000}{(1+0.06)^{10}} = \$53\,047.50 > \$50\,000$$

Thus, the purchasing power of the investment will be protected, and a small amount of interest will be earned.

**1.10** Rework Problem 1.9, which is changed as follows: the individual is in the 35% tax bracket and pays taxes on all the interest received; the \$50 000 is invested at 9% per year, compounded annually.

Now (1.9) gives

$$\theta = \frac{(1-0.35)(0.09) - 0.06}{1+0.06} = \frac{-0.00150}{1.06} < 0$$

With $\theta$ negative, (1.7) implies $F < P$: the investment is not protected, and there will be a (small) loss.

**1.11** An individual wants to have $2000 at the end of three years. How much would the individual have to invest at a 10% per year interest rate, compounded annually, in order to obtain a net of $2000 after paying a $250 early withdrawal fee at the end of the third year? Draw a cash flow diagram for the individual.

By *(1.3)*,

$$\$2250 = P(1 + 0.10)^3 \qquad \text{or} \qquad P = \$1690.46$$

The cash flow diagram for the individual is given in Fig. 1-2.

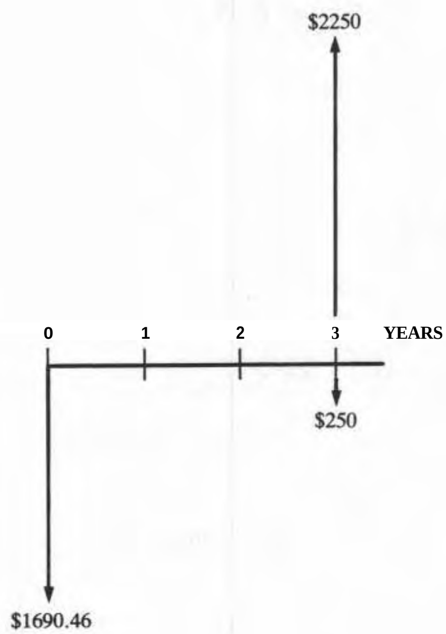


**Fig. 1-2**

**Table 1-1**

| End of Year | Savings | Withdrawals | Cash Flows |
|---|---|---|---|
| 0 | $600 | $ 0 | −$600 |
| 1 | | 300 | +300 |
| 2 | 500 | 300 | −200 |
| 3 | | 300 | +300 |
| 4 | 500 | | −500 |
| 5 | | 350 | +350 |
| 6 | 400 | | −400 |
| 7 | 400 | 350 | − 50 |
| 8 | 400 | | −400 |
| 9 | 400 | 350 | − 50 |
| 10 | 400 | | −400 |

**1.12** Suppose that you have a savings plan covering the next ten years, according to which you put aside $600 today, $500 at the end of every other year for the next five years, and $400 at the end of each year for the remaining five years. As part of this plan, you expect to withdraw $300 at the end of every year for the first **3** years, and $350 at the end of every other year thereafter. (a) Tabulate your cash flows. (b) Draw your cash flow diagram.

(a) See Table 1-1.

(b) See Fig. 1-3.

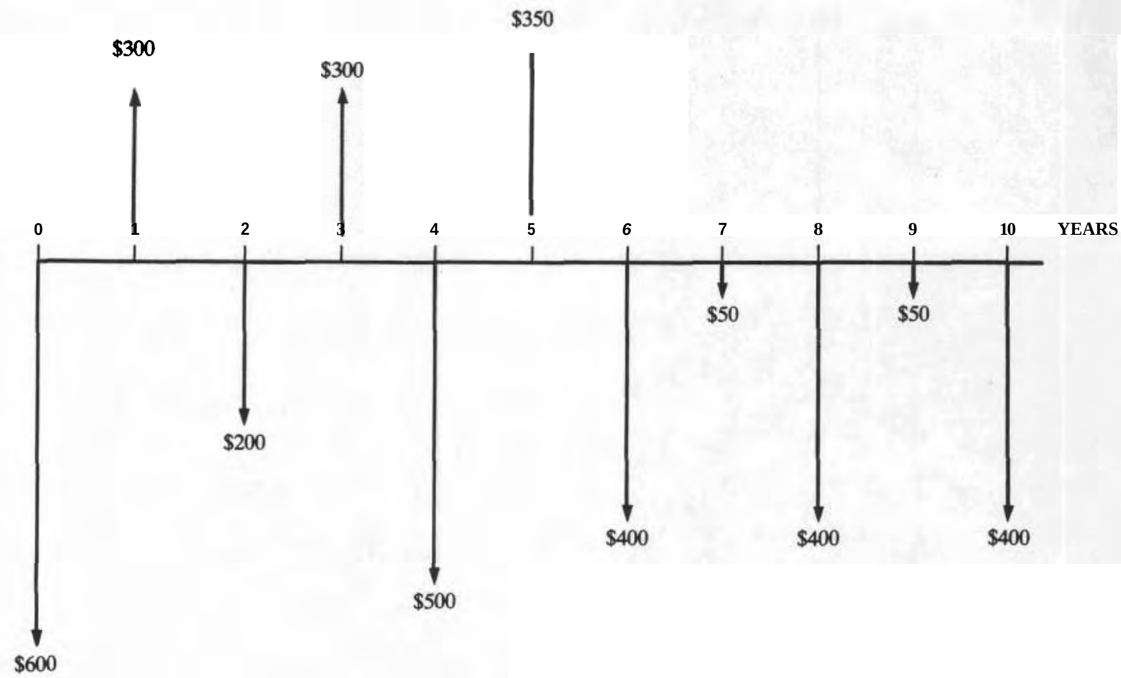


Fig. 1-3

**1.13** Under your six-year savings plan, you deposit \$1000 now, and \$1000 at the end of the fourth year, in a bank account that earns 8% per year, compounded annually. You withdraw all your accumulated interest at the end of the second year, and the further interest plus principal at the end of the sixth year. ***(a)*** Tabulate the cash flows and the balance in your investment account. ***(b)*** Draw a cash flow diagram for ***the bank.*** ***(c)*** Compute the penalty (suffered by you) for the early withdrawal of interest at the end of the second year.

***(a)*** **See Table** 1-2.

**Table 1-2**

| End of Year | Deposits | Withdrawals | Account Balance for First \$1000 | Account Balance for Second \$1000 | Account Balance Total | Cash Flows |
|---|---|---|---|---|---|---|
| 0 | \$1000 | \$ 0 | \$1000.00 | --- | \$1000.00 | −\$1000.00 |
| 1 | 0 | 0 | 1080.00 | --- | 1080.00 | 0 |
| 2 | 0 | 166.40 | 1000.00 | --- | 1000.00 | +166.40 |
| 3 | 0 | 0 | 1080.00 | --- | 1080.00 | 0 |
| 4 | 1000 | 0 | 1166.40 | \$1000.00 | 2166.40 | − 1000.00 |
| 5 | 0 | 0 | 1259.71 | 1080.00 | 2339.71 | 0 |
| 6 | 0 | 2526.88 | 1360.48 | 1166.40 | 2526.88 | + 2526.88 |
| | | | | | NET CASH now | +\$693.28 |

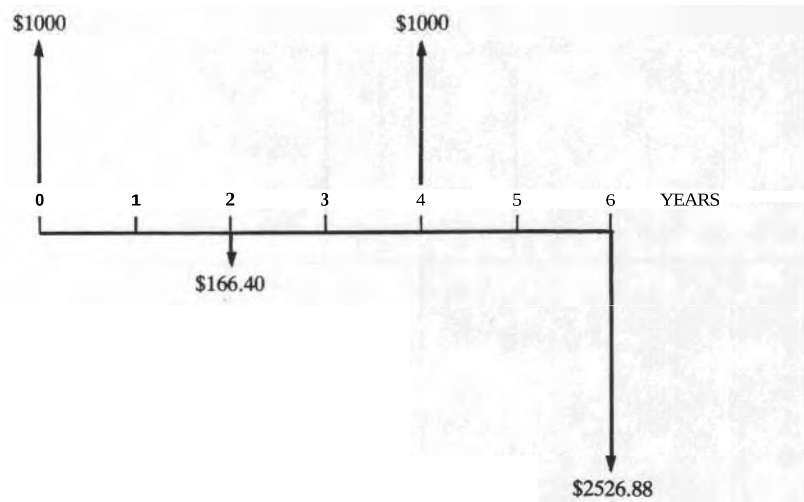


**Fig. 1-4**

(b) See Fig. 1-4.

(c) If the accumulated interest had not been withdrawn at the end of year 2, the $1000 invested at the start of the plan would have grown to

$$F = P(1+i)^n = \$1000(1.08)^6 = \$1586.87$$

and the total available for withdrawal at the end of year 6 would have been

$$\$1586.87 + \$1166.40 = \$2753.27$$

Hence, the net cash flow would have been $753.27, or $59.99 more.

## Supplementary Problems

**1.14** How much interest would be due at the end of one year on a loan of $10000 if the interest is 12% per year? ***Ans.*** $1200

**1.15** What is the annual interest rate on a $1000 loan in which all interest is paid at the end of the year, and a total of $1125 must be repaid at the end of the year? ***Ans.*** 12.5%

**1.16** If $300 is earned in three months on an investment of $12000, what is the annual rate of simple interest? ***Ans.*** 10%

**1.17** How long will it take for an investment of $5000 to grow to $7500, if it earns 10% simple interest per year? ***Ans.*** 5 years

**1.18** Find the principal of a loan in which the interest rate is $1\frac{1}{2}\%$ per month, payable monthly, and in which the borrower has just made the first monthly interest payment of $50. ***Ans.*** $3333.33

**1.19** Find the principal, if the principal plus interest at the end of one and one-half years is $3360 for a simple interest rate of 8% per annum. ***Ans.*** $3000

**1.20** Which is more desirable: investing $2000 at 6% per year compound interest for three years, or investing $2000 at 7% per year simple interest for three years? ***Ans.*** Simple interest is superior by $38.

**1.21** At what rate of interest, compounded annually, will an investment triple itself in *(a)* 8 years? *(b)* 10 years? *(c)* 12 years? ***Ans.*** *(a)* 14.7%; *(b)* 11.6%; *(c)* 9.6%

**1.22** What rate of interest, compounded annually, will result in the receipt of $15 938.48 if $10 000 is invested for 8 years? ***Ans.*** 6%

**1.23** How much money will be required four years from today to repay a $2000 loan that is made today *(a)* at 8% interest, compounded annually? *(b)* at 8% simple interest? ***Ans.*** *(a)* $2720.97; *(b)* $2640.00

**1.24** How many years will be required for an investment of $3000 to increase to $4081.47 at an interest rate of 8% per year, compounded annually? ***Ans.*** 4 years

**1.25** What is the present value of $10 000 to be received 20 years from now, if the principal is invested at 8% per year, compounded annually? ***Ans.*** $2145.48

**1.26** How many years will it take for an investment to double, if the interest rate is 8% per year, compounded annually? ***Ans.*** 9 years

**1.27** A person lends $2000 for five years at 10% per annum simple interest; then the entire proceeds are invested for 10 years at 9% per year, compounded annually. How much money will the person have at the end of the entire 15-year period? ***Ans.*** $7102.09

**1.28** Suppose that a person invests $3000 at 10% per year, compounded annually, for 8 years. *(a)* Will this effectively protect the purchasing power of the original principal, given an annual inflation rate of 8%? *(b)* If so, by how much? ***Ans.*** *(a)* yes; *(b)* $474.34

**1.29** Let the person in Problem 1.28 be in the 45% tax bracket and pay taxes on all the interest received. (a) Will the after-tax purchasing power of the original principal be protected? *(b)* Why?
***Ans.*** *(a)* no; *(b)* $512.57 in purchasing power will be lost

**1.30** What amount of money is equivalent to receiving $5000 two years from today, if interest is compounded quarterly at the rate of $2\frac{1}{2}$% per quarter? ***Ans.*** $4103.73

**1.31** On the first day of the year, a man deposits $1000 in a bank at 8% per year, compounded annually. He withdraws $80.00 at the end of the first year, $90.00 at the end of the second year, and the remaining balance at the end of the third year. *(a)* How much does he withdraw at the end of the third year? *(b)* What is his net cash flow? *(c)* How much better off, in terms of net cash flow, would he have been if he had not made the withdrawals at the ends of years one and two? ***Ans.*** *(a)* $1069.19; *(b)* $239.19; *(c)* $20.51

# Chapter 2

# Annual Compounding

## 2.1 SINGLE-PAYMENT, COMPOUND-AMOUNT FACTOR

Suppose that a given sum of money, $P$, earns interest at a rate $i$, compounded annually. We have already seen (Section 1.4) that the total amount of money, $F$, which will have accumulated from an investment of $P$ dollars after $n$ years is given by $F = P(1+i)^n$. The ratio

$$F/P = (1+i)^n \tag{2.1}$$

is called the single-payment, compound-amount factor. Numerical values of this factor may be calculated from (2.1) or obtained from compound interest tables such as those shown in Appendix A.

A fuller notation, $(F/P, i\%, n)$, is helpful when setting up the solution to a compound interest problem.

**Example 2.1** A student deposits $1000 in a savings account that pays interest at the rate of 6% per year, compounded annually. If all of the money is allowed to accumulate, how much money will the student have after 12 years?

We wish to solve for $F$, given $P$, $i$, and $n$. Thus,

$$F = P \times (F/P, i\%, n) = \$1000(F/P, 6\%, 12) = \$1000(2.0122) = \$2012.20$$

where the factor $(F/P, 6\%, 12)$ was evaluated from Appendix A.

## 2.2 SINGLE-PAYMENT, PRESENT-WORTH FACTOR

The single-payment, present-worth factor is the reciprocal of the single-payment, compound-amount factor:

$$P/F = (F/P)^{-1} = (1+i)^{-n} \tag{2.2}$$

The expanded notation for this quantity is $(P/F, i\%, n)$. Numerical values for the single-payment, present-worth factor can be obtained directly from (2.2) or from a set of tables such as those given in Appendix A.

**Example 2.2** A certain sum of money will be deposited in a savings account that pays interest at the rate of 6% per year, compounded annually. If all of the money is allowed to accumulate, how much must be deposited initially so that $5000 will have accumulated after 10 years?

We wish to solve for $P$, given $F$, $i$, and $n$. Thus,

$$P = F \times (P/F, i\%, n) = \$5000(P/F, 6\%, 10) = \$5000(0.5584) = \$2792.00$$

where Appendix A gives $(F/P, 6\%, 10)^{-1} = (1.7908)^{-1} = 0.5584$.

## 2.3 UNIFORM-SERIES, COMPOUND-AMOUNT FACTOR

Let equal amounts of money, $A$, be deposited in a savings account (or placed in some other interest-bearing investment) at the end of each year, as indicated in Fig. 2-1. If the money earns interest at a rate $i$, compounded annually, how much money will have accumulated after $n$ years?

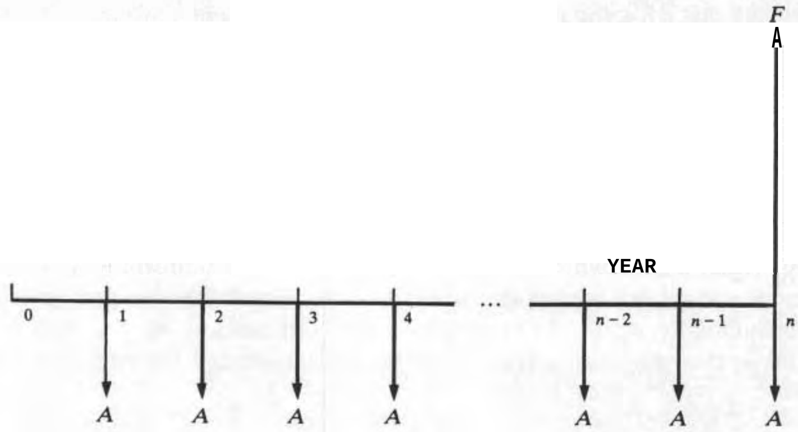


**Fig. 2-1**

To answer this question, we note that after $n$ years, the first year's deposit will have increased in value to

$$F_1 = A(1+i)^{n-1}$$

Similarly, the second year's deposit will have increased in value to

$$F_2 = A(1+i)^{n-2}$$

and so on. The total amount accumulated will thus be the sum of a geometric progression:

$$\begin{aligned} F &= F_1 + F_2 + \cdots + F_n \\ &= A(1+i)^{n-1} + A(1+i)^{n-2} + \cdots + A \\ &= A[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + 1] \\ &= A\,\frac{(1+i)^n - 1}{i} \end{aligned}$$

The ratio

$$F/A = \frac{(1+i)^n - 1}{i} \tag{2.3}$$

is called the ***uniform-series, compound-amount factor.*** Numerical values of this factor can be obtained directly, using *(2.3)* in conjunction with an electronic calculator, or from a set of compound interest tables such as those given in Appendix A. The extended notation $(F/A, i\%, n)$ is helpful when solving compound interest problems involving a uniform series.

**Example 2.3** A student plans to deposit $600 each year in a savings account, over a period of 10 years. If the bank pays 6% per year, compounded annually, how much money will have accumulated at the end of the 10-year period?

$$F = A \times (F/A, i\%, n) = \$600(F/A, 6\%, 10) = \$600(13.1808) = \$7908.48$$

## 2.4 UNIFORM-SERIES, SINKING-FUND FACTOR

The ***uniform-series, sinking-fund factor*** is the reciprocal of the uniform-series, compound-amount factor:

$$A/F = (F/A)^{-1} = \frac{1}{(1+i)^n - 1} \tag{2.4}$$

This quantity has the extended notation $(A/F, i\%, n)$.

**Example 2.4** Suppose that a fixed sum of money, A, will be deposited in a savings account at the end of each year for 20 years. If the bank pays 6% per year, compounded annually, find A such that a total of $50 000 will be accumulated at the end of the 20-year period.

$$A = F \times (A/F, i\%, n) = \$50\,000(A/F, 6\%, 20) = \$50\,000(0.02718) = \$1359$$

## 2.5 UNIFORM-SERIES, CAPITAL-RECOVERY FACTOR

Let us now consider a somewhat different situation involving uniform annual payments. Suppose that a given sum of money, $P$, is deposited in a savings account where it earns interest at a rate $i$ per year, compounded annually. At the end of each year a fixed amount, A, is withdrawn (Fig. 2-2). How large should A be so that the bank account will just be depleted at the end of n years?

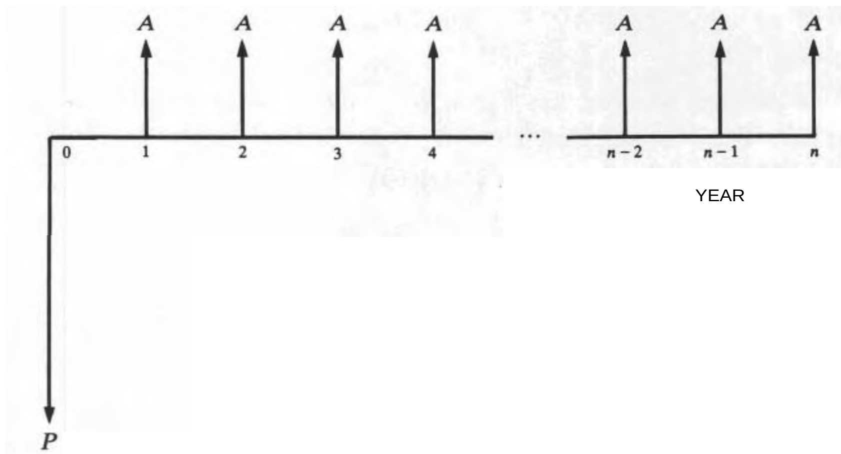


Fig. 2-2

We can make use of previously defined factors to solve this problem, since

$$A \equiv P \times (AIF) \times (FIP) \tag{2.5}$$

Substituting (2.4) and (2.1) into (2.5), we obtain

$$A = P\left[\frac{i}{(1+i)^n - 1}\right](1+i)^n = P\,\frac{i(1+i)^n}{(1+i)^n - 1}$$

The ratio

$$A/P = \frac{i(1+i)^n}{(1+i)^n - 1} = \frac{i}{1-(1+i)^{-n}} \tag{2.6}$$

is called the uniform-series, capital-recovery factor. Numerical values of this factor can be computed using (2.6) and an electronic calculator, or they can be obtained from a set of compound interest tables such as those given in Appendix A. Symbolically, the uniform-series, capital-recovery factor is written as $(A/P, i\%, n)$.

**Example 2.5** An engineer who is about to retire has accumulated $50 000 in a savings account that pays 6% per year, compounded annually. Suppose that the engineer wishes to withdraw a fixed sum of money at the end of each year for 10 years. What is the maximum amount that can be withdrawn?

$$A = P \times (A/P, i\%, n) = \$50\,000(A/P, 6\%, 10) = \$50\,000(0.1359) = \$6795$$

## 2.6 UNIFORM-SERIES, PRESENT-WORTH FACTOR

The ***uniform-series, present-worth*** factor is the reciprocal of the uniform-series, capital-recovery factor:

$$P/A = (A/P)^{-1} = \frac{(1+i)^n - 1}{i(1+i)^n} = \frac{1-(1+i)^{-n}}{i} \tag{2.7}$$

The extended notation is $(P/A, i\%, n)$.

**Example 2.6** An engineer who is planning his retirement has decided that he will have to withdraw \$10 000 from his savings account at the end of each year. How much money must the engineer have in the bank at the start of his retirement, if his money earns 6% per year, compounded annually, and he is planning a 12-year retirement (i.e., 12 annual withdrawals)?

$$P = A \times (P/A, i\%, n) = \$10\,000(P/A, 6\%, 12) = \$10\,000(8.3839) = \$83\,839$$

## 2.7 GRADIENT SERIES FACTOR

***A gradient series*** is a series of annual payments in which each payment is greater than the previous one by a constant amount, G. Let us develop the future worth of a gradient series by visualizing it in terms of its component parts, as in Fig. 2-3. Each level constitutes a uniform series, to which (2.3) applies; thus,

$$\begin{aligned} F &= A_0(F/A, i\%, n) + G(F/A, i\%, n-1) + G(F/A, i\%, n-2) + \cdots + G(F/A, i\%, 1) \\ &= A_0\frac{(1+i)^n - 1}{i} + G\frac{(1+i)^{n-1} - 1}{i} + G\frac{(1+i)^{n-2} - 1}{i} + \cdots + G\frac{(1+i) - 1}{i} \end{aligned}$$

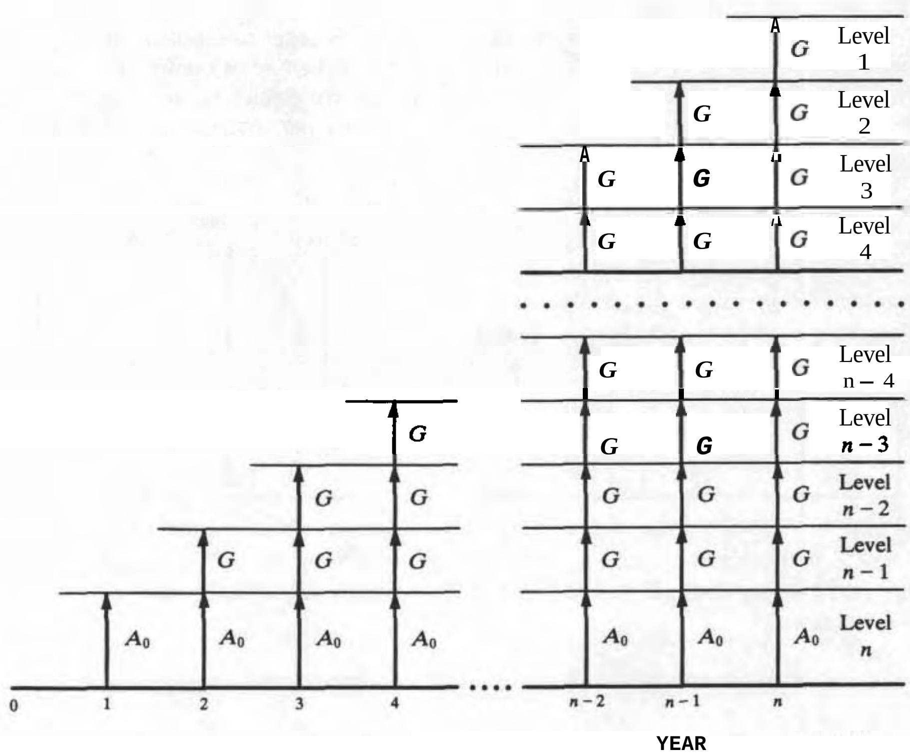


Fig. 2-3

The last $n-1$ terms on the right may be rearranged to give

$$F = A_0 \frac{(1+i)^n - 1}{i} + \frac{G}{i}[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1] - \frac{nG}{i}$$

and, summing the geometric progression, we obtain

$$F = A_0 \frac{(1+i)^n - 1}{i} + \frac{G}{i}\left[\frac{(1+i)^n - 1}{i}\right] - \frac{nG}{i} \tag{2.8}$$

Now consider a series of $n$ uniform annual payments, $A_0 + A$, where $A_0$ has the same value as in the above series of gradients and where $A$ is determined such that the future worth of this uniform series is the same as that given by **(2.8).** Thus,

$$F = (A_0 + A)\left[\frac{(1+i)^n - 1}{i}\right] = \left(A_0 + \frac{G}{i}\right)\left[\frac{(1+i)^n - 1}{i}\right] - \frac{nG}{i}$$

from which

$$A = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

or

$$A/G = \frac{1}{i} - \frac{n}{(1+i)^n - 1} \tag{2.9}$$

Equation **(2.9)** permits direct calculation of the **gradient series factor,** $(A/G, i\%, n)$. Alternatively, since

$$(A/G, i\%, n) = \frac{1}{i} - \frac{n}{i}(A/F, i\%, n) \tag{2.10}$$

the factor can be evaluated from a table of $(A/F, i\%, n)$.

**Example 2.7** An engineer is planning for a 15-year retirement. In order to supplement his pension and offset the anticipated effects of inflation, he intends to withdraw \$5000 at the end of the first year, and to increase the withdrawal by \$1000 at the end of each successive year (Fig. 2-4). How much money must the engineer have in his savings account at the start of his retirement, if money earns 6% per year, compounded annually?

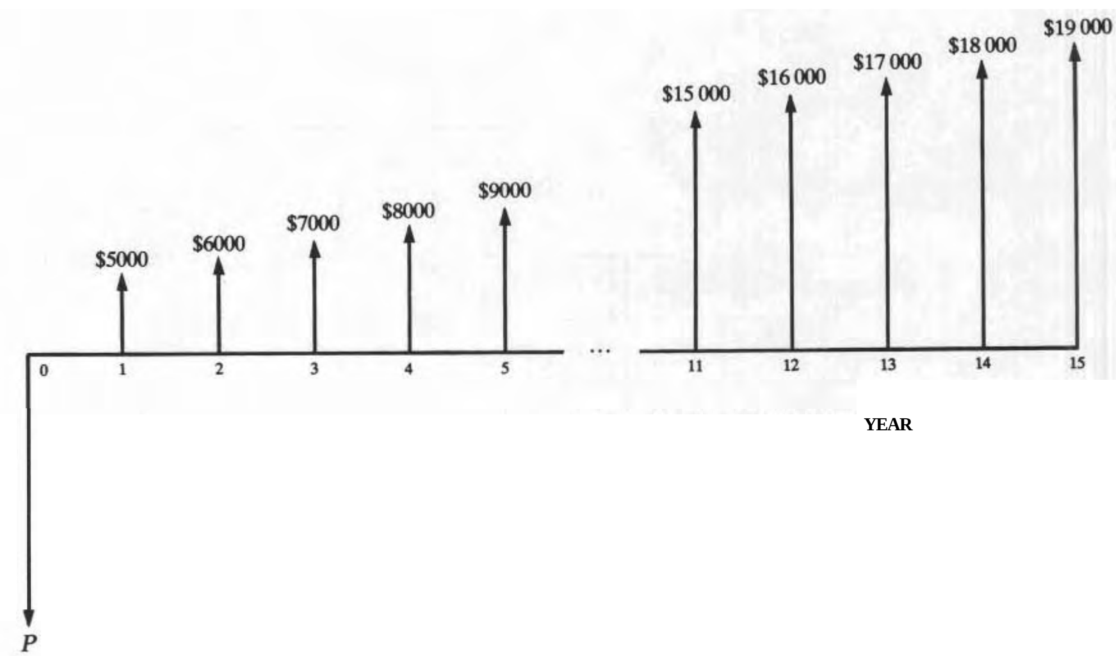


**Fig. 2-4**

We want to obtain the value of $P$, given the values of $A_0$, $G$, $i$, and $n$. We first obtain a series of uniform withdrawals $A'$ equivalent to the series of gradients.

$$A = G \times (A/G, i\%, n) = \$1000(A/G, 6\%, 15) = \$1000(5.9260) = \$5926$$

$$A' = A_0 + A = \$5000 + \$5926 = \$10\,926$$

We can now calculate $P$ as follows:

$$P = A' \times (P/A', i\%, n) = \$10\,926(P/A, 6\%, 15) = \$10\,926(9.7123) = \$106\,116.59$$

A more concise, and the preferable, way to solve this problem is to write

$$P = A_0 \times (P/A, 6\%, 15) + G \times (A/G, 6\%, 15) \times (P/A, 6\%, 15) \qquad (2.11)$$
$$= \$5000(9.7123) + \$1000(5.9260)(9.7123) = \$106\,116.59$$

The gradient series factor can also be used with a decreasing series of gradients.

**Example 2.8** How much money must initially be deposited in a savings account paying 5% per year, compounded annually, to provide for ten annual withdrawals that start at \$6000 and decrease by \$500 each year?

In this case, $A_0 = \$6000$ and $G = -\$500$. Thus,

$$P = \$6000(P/A, 5\%, 10) - \$500(A/G, 5\%, 10)(P/A, 5\%, 10)$$
$$= \$6000(7.7217) - \$500(4.0991)(7.7217) = \$30\,504.19$$

# Solved Problems

**2.1** A woman deposits \$2000 in a savings account that pays interest at 8% per year, compounded annually. If all the money is allowed to accumulate, how much will she have at the end of (a) 10 years? (b) 15 years?

(a) $$F = P \times (F/P, 8\%, 10) = \$2000(2.1589) = \$4317.80$$

(b) $$F = P \times (F/P, 8\%, 15) = \$2000(3.1722) = \$6344.40$$

Alternatively,

$$F = \$4317.80(F/P, 8\%, 5) = \$4317.80(1.4693) = \$6344.14$$

**2.2** How much money must be deposited in a savings account so that \$5500 can be withdrawn 12 years hence, if the interest rate is 9% per year, compounded annually, and if all the interest is allowed to accumulate?

$$P = F \times (P/F, 9\%, 12) = F \times (F/P, 9\%, 12)^{-1}$$
$$= \$5500(F/P, 9\%, 12)^{-1} = \$5500(2.8127)^{-1} = \$1955.42$$

**2.3** Repeat Problem 2.2 for an interest rate of $7\frac{1}{2}\%$ per year, compounded annually.

$$P = F \times (P/F, 7\tfrac{1}{2}\%, 12)$$

Appendix A does not include tabular entries for $i = 7\frac{1}{2}\%$ per year; therefore we will make direct use of (2.2).

$$P = \frac{\$5500}{(1+0.075)^{12}} = \frac{\$5500}{2.3818} = \$2309.20$$

**2.4** Suppose that a person deposits \$500 in a savings account at the end of each year, starting now, for the next 12 years. If the bank pays 8% per year, compounded annually, how much money will accumulate by the end of the 12-year period?

$$F = \$500 \times (F/A, 8\%, 12) = \$500(18.9771) = \$9488.55$$

**2.5** Repeat Problem 2.4 for an interest rate of $6\frac{1}{4}\%$ per year, compounded annually.

$$F = \$500(F/A, 6\tfrac{1}{4}\%, 12) = \$500\left[\frac{(1+0.0625)^{12}-1}{0.0625}\right] = \$500(17.1182) = \$8559.12$$

**2.6** How much money must be deposited at the end of each year in a savings account that pays 9% per year, compounded annually, in order to have a total of \$10 000 at the end of 14 years?

$$A = F \times (A/F, 9\%, 14) = \$10\,000(F/A, 9\%, 14)^{-1} = \$10\,000(26.0192)^{-1} = \$384.33$$

**2.7** A man has deposited \$50 000 in a retirement income plan with a local bank. This bank pays 9% per year, compounded annually, on such deposits. What is the maximum amount the man can withdraw at the end of each year and still have the funds last for 12 years?

From Appendix A,

$$A = \$50\,000(A/P, 9\%, 12) = \$50\,000(0.13965) = \$6982.50$$

**2.8** Repeat Problem 2.7 for an interest rate of $8\frac{3}{4}\%$ per year, compounded annually.

This problem must be solved by direct use of (2.6).

$$A = \$50\,000(A/P, 8\tfrac{3}{4}\%, 12)$$

$$= \$50\,000\left[\frac{0.0875(1+0.0875)^{12}}{(1+0.0875)^{12}-1}\right] = \$50\,000(0.1379) = \$6894.84$$

**2.9** Mr. Smith is planning his retirement. He has decided that he needs to withdraw \$12000 per year from his bank account to supplement his other income from Social Security and a private pension plan. How much money should he plan to have in the bank at the start of his retirement, if the bank pays 10% per year, compounded annually, and if he wants money *to* last for a 12-year retirement period?

$$P = A \times (P/A, 10\%, 12) = A \times (A/P, 10\%, 12)^{-1} = \$12\,000(0.14676)^{-1} = \$81\,766.15$$

**2.10** Mr. Doe is trying to decide whether to put his money in the XYZ Bank or the ABC Bank. The XYZ Bank pays 6% per annum interest, compounded annually; the ABC Bank pays 5% per annum interest, compounded quarterly. Mr. Doe expects to keep his money in the bank for 5 years. Which bank should he select?

From Appendix A,

for XYZ: $(F/P, 6\%, 5) = 1.3382$

for ABC: $(F/P, 1\frac{1}{4}\%, 20) = 1.2820$

He should choose XYZ, which offers the greater return per dollar.

**2.11** If, in Problem 2.10, Mr. Doe plans to keep his money in the bank for 10 years, is XYZ still the best choice?

for XYZ: $(F/P, 6\%, 10) = 1.7908$

for ABC: $(F/P, 1\frac{1}{4}\%, 40) = 1.6436$

It is; in fact, the advantage of XYZ increases with the longer time period.

**2.12** Mr. Franklin wants to save for a new sports car that he expects will cost **\$38** 000 four and one-half years from now. How much money will he have to save each year and deposit in a savings account that pays $6\frac{1}{4}\%$ per year, compounded annually, to buy the car in four and one-half years?

Since the interest is compounded only once a year, Mr. Franklin will have to accumulate the entire $38000 during the first four years. Therefore,

$$A = \$38\,000(A/F, 6\tfrac{1}{4}\%, 4) = \$38\,000\left[\frac{0.0625}{(1+0.0625)^4 - 1}\right]$$
$$= \$38\,000(0.2277) = \$8654.32 \text{ per year for four years}$$

**2.13** In Problem 2.12, suppose that Mr. Franklin makes a deposit at the ***beginning*** of each year, rather than at the end. How much money must be deposited each year?

There will now be five deposits, each of size $A$. At the end of 4 years, the first deposit will have accumulated to

$$A \times (F/P, 6\tfrac{1}{4}\%, 4)$$

and the next four deposits to

$$A \times (F/A, 6\tfrac{1}{4}\%, 4)$$

Hence

$$\$38\,000 = A[(F/P, 6\tfrac{1}{4}\%, 4) + (F/A, 6\tfrac{1}{4}\%, 4)]$$
$$= A(1.2745 + 4.3909) = 5.6654A$$

Solving, $A = \$6707.38$.

**2.14** A father wants to set aside money for his 5-year-old son's future college education. Money can be deposited in a bank account that pays 8% per year, compounded annually. What equal deposits should be made by the father, on his son's 6th through 17th birthdays, in order to provide $5000 on the son's 18th, 19th, 20th, and 21st birthdays?

On the son's 17th birthday, the deposits must have accumulated to

$$P = \$5000(P/A, 8\%, 4)$$
$$= \$5000(A/P, 8\%, 4)^{-1} = \$5000(0.30192)^{-1} = \$16\,560.68$$

Thus, the deposit size, A, must satisfy

$$\$16\,560.68 = A\,(F/A, 8\%, 12)$$
$$\$16\,560.68 = A\,(18.9771)$$
$$A = \$872.67$$

**2.15** Dr. Anderson plans to make a series of gradient-type withdrawals from her savings account over a 10-year period, beginning at the end of the second year. What equal annual withdrawals would be equivalent to a withdrawal of $1000 at the end of the second year, $2000 at the end of the third year,. . . , $9000 at the end of the 10th year, if the bank pays 9% per year, compounded annually?

In the notation of Section 2.7, $A_0 = 0$, $G = \$1000$. Hence, from Appendix A,

$$A = \$1000(A/G, 9\%, 10) = \$1000(3.7978) = \$3797.80$$

**2.16** Mr. Jones is planning a 20-year retirement; he wants to withdraw $6000 at the end of the first year, and then to increase the withdrawals by $800 each year to offset inflation. How much money should he have in his savings account at the start of his retirement, if the bank pays 9% per year, compounded annually, on his savings?

Using an analog of (2.11) and the tables in Appendix A,

$$P = A_0(P/A, 9\%, 20) + G\,(A/G, 9\%, 20)\,(P/A, 9\%, 20)$$
$$= A_0(A/P, 9\%, 20)^{-1} + G\,(A/G, 9\%, 20)\,(A/P, 9\%, 20)^{-1}$$
$$= \$6000(0.10955)^{-1} + \$800(6.7674)\,(0.10955)^{-1} = \$104\,189.14$$

**2.17** The ABD Company is building a new plant, whose equipment maintenance costs are expected to be \$500 the first year, \$150 the second year, \$200 the third year, \$250 the fourth year, etc., increasing by \$50 per year through the 10th year. The plant is expected to have a 10-year life. Assuming the interest rate is 8%, compounded annually, how much should the company plan to set aside now in order to pay for the maintenance?

The cash flow diagram is given in Fig. 2-5. First we compute the present worth at the end of year 1 [see (2.11)]:

$$\begin{aligned} P' &= \$500 + \$150(P/A, 8\%, 9) + \$50(A/G, 8\%, 9)\,(P/A, 8\%, 9) \\ &= \$500 + [\$150 + \$50(A/G, 8\%, 9)]\,(A/P, 8\%, 9)^{-1} \\ &= \$500 + [\$150 + \$50(3.4910)]\,(0.16008)^{-1} = \$2527.42 \end{aligned}$$

The present worth at the end of year 0 is thus

$$P = P'\,(P/F, 8\%, 1) = \$2527.42(1.0800)^{-1} = \$2340.20$$

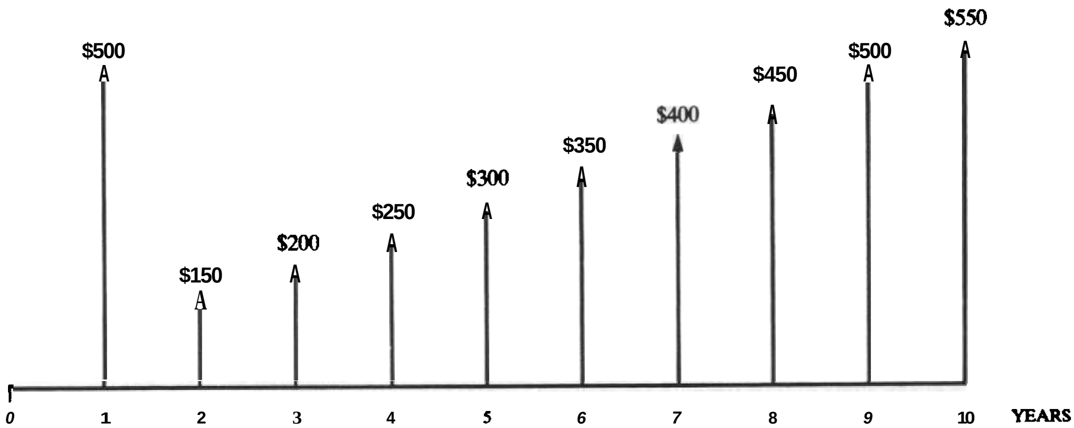


**Fig. 2-5**

**2.18** Slick Oil Company is considering the purchase of a new machine that will last 5 years and cost \$50 000; maintenance will cost \$6000 the first year, decreasing by \$1000 each year to \$2000 the fifth year. If the interest rate is 8% per year, compounded annually, how much money should the company set aside for this machine?

Proceeding as in Problem 2.16, and including the purchase price,

$$\begin{aligned} P &= \$50\,000 + \$6000(A/P, 8\%, 5)^{-1} - \$1000(A/G, 8\%, 5)\,(A/P, 8\%, 5)^{-1} \\ &= \$50\,000 + [\$6000 - \$1000(1.8465)]\,(0.25046)^{-1} = \$66\,583.49 \end{aligned}$$

**2.19** Mr. Holzman estimates that the maintenance cost of a new car will be \$75 the first year, and will increase by \$50 each subsequent year. He plans to keep the car for 6 years. He wants to know how much money to deposit in a bank account at the time he purchases the car, in order to cover these maintenance costs. His bank pays $5\frac{1}{2}\%$ per year, compounded annually, on savings deposits.

$$\begin{aligned} P &= \$75(P/A, 5\tfrac{1}{2}\%, 6) + \$50(P/G, 5\tfrac{1}{2}\%, 6) \\ &= [\$75 + \$50(A/G, 5\tfrac{1}{2}\%, 6)]\,(P/A, 5\tfrac{1}{2}\%, 6) \end{aligned}$$

The $P/A$ and $A/G$ factors must be evaluated directly from (2.7) and (2.9). Thus,

$$(P/A, 5\tfrac{1}{2}\%, 6) = \frac{(1+0.055)^6 - 1}{0.055(1+0.055)^6} = 4.9955$$

$$(A/G, 5\tfrac{1}{2}\%, 6) = \frac{1}{0.055} - \frac{6}{(1+0.055)^6 - 1} = 2.3441$$

and

$$P = [\$75 + \$50(2.3441)]\,(4.9955) = \$960.17$$

## Supplementary Problems

**2.20** An investment plan pays 15% per year, compounded annually. How much would have to be invested every year so that $40000 will be accumulated by the end of 10 years? ***Ans.*** $1970.08

**2.21** Repeat Problem 2.20 for an interest rate of $13\frac{1}{2}\%$ per year, compounded annually. ***Ans.*** $2119.48

**2.22** Mr. Doe borrowed $1000 from his bank at 8% per year, compounded annually. He can (i) repay the $1000 together with the interest at the end of 3 years, or (ii) pay the interest at the end of each year and repay the $1000 at the end of 3 years. By how much is (ii) better than (i)?
***Ans.*** $259.71 – $240.00 = $19.71

**2.23** Suppose that $2000 is invested now, $2500 two years from now, and $1200 four years from now, all at 8% per year, compounded annually. What will be the total amount 10 years from now? ***Ans.*** $10 849.42

**2.24** Repeat Problem 2.23 for an interest rate of $7\frac{3}{4}\%$ per year, compounded annually. ***Ans.*** $10 639.21

**2.25** On the day his son was born, a father decided to establish a fund for his son's college education. The father wants the son to be able to withdraw $4000 from the fund on his 18th birthday, again on his 19th birthday, again on his 20th birthday, and again on his 21st birthday. If the fund earns interest at 9% per year, compounded annually, how much should the father deposit at the end of each year, up through the 17th year? ***Ans.*** $350.49

**2.26** Repeat Problem 2.25 for an interest rate of $8\frac{1}{2}\%$ per year, compounded annually. ***Ans.*** $370.95

**2.27** A new machine is expected to cost $6000 and have a life of 5 years. Maintenance costs will be $1500 the first year, $1700 the second year, $1900 the third year, $2100 the fourth year, and $2300 the fifth year. To pay for the machine, how much should be budgeted and deposited in a fund that earns (a) 9% per year, compounded annually? (b) $10\frac{1}{2}\%$ per year, compounded annually?
***Ans.*** (a) $13 256.69; (*b*) $12 962.59

**2.28** The ABC Company has contracted to make the following payments: $10 000 immediately; $1000 at the end of year 1; $1500 at the end of year 2; $2000 at the end of year 3; $2500 at the end of year 4; $3000 at the end of year 5. What fixed amount of money should the company plan to set aside each year, at 8% interest per year, compounded annually, in order to make the above payments? ***Ans.*** $4427.82

**2.29** Suppose that someone deposits $2500 in a savings account at the end of each year for the next 15 years. How much money will the person have by the end of the 15th year if the bank pays (a) 8%, (b) $6\frac{3}{4}\%$, per year, compounded annually? ***Ans.*** (a) $67 880.28; (b) $61 626.00

**2.30** Mr. Jones has deposited his life savings of $70 000 in a retirement income plan with a local bank. The bank pays (a) 10%, (b) $11\frac{1}{4}\%$, per year, compounded annually, on such deposits. What is the maximum fixed amount Mr. Jones can withdraw at the end of each year and still have the funds last for 15 years?
***Ans.*** (a) $9203.16; (b) $9869.27

**2.31** Mr. White is planning to take early retirement. He has decided that he needs $15 000 per year to live on, for the first 5 years of his retirement; after that, his Social Security and other pension plans will provide him an adequate retirement income. How much money must he have in the bank for his 5-year early retirement period, if the bank pays (a) 10%, (b) $9\frac{1}{4}\%$, per year, compounded annually, on the funds? ***Ans.*** (a) $56 861.80; (b) $57 968.26

**2.32** Ms. Frank is planning for a 25-year retirement period and wishes to withdraw a portion of her savings at the end of each year. She plans to withdraw $10 000 at the end of the first year, and then to increase the amount of the withdrawal by $1000 each year, to offset inflation. How much money should she have in her savings account at the start of the retirement period, if the bank pays (a) 9%, (b) $7\frac{1}{2}\%$, per year, compounded annually? ***Ans.*** (a) $175 152.28; (b) $205 435.72

**2.33** How much money would have to be saved at (a) 8%, (b) $8\frac{1}{4}$%, per year, compounded annually, each year for the next 10 years if $50 000 is needed at the end of the 10th year?
***Ans.*** (a) $3451.47; (b) $3410.71

**2.34** A freshman college student, who owns a car, plans to buy a motorcycle. The student expects to save an increasing amount of money on travel every year he is in college, as he will make less use of his car each year. How much money should he plan to save and put in the bank from his job this summer, in order to pay his travel costs for his remaining 3 years of college? Assume that the bank pays 8% per year, compounded annually, and that his travel costs will be $900 the first year, $700 the second year, and $500 the third year. Ans. $1830.39

**2.35** A father wants to set aside money for his 8-year-old daughter's future education, by making monthly deposits to a bank account that pays 8% per year, compounded annually. What equal monthly deposits must the father make—the first 1 month after her 9th birthday and the last on her 17th birthday—in order for her to withdraw $4000 on each of her next four birthdays (the 18th through the 21st)?
***Ans.*** **$103.80**

**2.36** Suppose $5000 is deposited in a savings account that pays interest at 8% per year, compounded annually. If no withdrawals are made, how long will it take to accumulate $12 000?
Ans. 12 years (actually, the amount accumulated at the end of 12 years will be $12 590.85)

**2.37** Repeat Problem 2.36 for an interest rate of $6\frac{1}{2}$% per year, compounded annually. Ans. 14 years

**2.38** Suppose that $1000 is deposited in the bank at the end of each year. How long will it take to accumulate $20 000 if the interest rate is 6% per year, compounded annually?
Ans. 14 years (actually, $21 015.07 will have accumulated at the end of 14 years)

**2.39** Repeat Problem 2.38 for an interest rate of $5\frac{1}{4}$% per year, compounded annually. Ans. 15 years

**2.40** Ms. Brown deposits $750 in a savings account at the beginning of each year, starting now, for the next 10 years. If the bank pays (a) 7%, (b) $5\frac{3}{4}$%, per year, compounded annually, how much money will Ms. Brown have accumulated by the end of the 10th year? Ans. (a) $11 087.70; (b) $10 332.09

# Chapter 3

# Algebraic Relationships and Solution Procedures

## 3.1 RELATIONSHIPS BETWEEN INTEREST FACTORS

The following relationships are sometimes helpful in interest calculations, particularly when interest tables are used.

$$(F/P, i\%, n) = (F/P, i\%, n_1) \times (F/P, i\%, n_2) \qquad (n = n_1 + n_2) \tag{3.1}$$

$$(P/F, i\%, n) = (P/F, i\%, n_1) \times (P/F, i\%, n_2) \qquad (n = n_1 + n_2) \tag{3.2}$$

$$(F/A, i\%, n) = (F/A, i\%, n_1) + (F/P, i\%, n_1) + (F/P, i\%, n_1 + 1) + \cdots + (F/P, i\%, n - 1) \qquad (n > n_1) \tag{3.3}$$

$$(P/A, i\%, n) = (P/A, i\%, n_1) + (P/F, i\%, n_1 + 1) + (P/F, i\%, n_1 + 2) + \cdots + (P/F, i\%, n) \qquad (n > n_1) \tag{3.4}$$

$$(A/P, i\%, n) = (AIF, i\%, n) + i \tag{3.5}$$

$$\lim_{n\to\infty} (AIP, i\%, n) = i \tag{3.6}$$

**Example 3.1** Determine the value of $(F/P, 8\%, 37)$, using the tables presented in Appendix A.

The numerical value of $(F/P, 8\%, 37)$ is not included in the tables; however, from (3.1),

$$(F/P, 8\%, 37) = (FIP, 8\%, 30) \times (F/P, 8\%, 7) = 10.0627 \times 1.7138 = 17.2455$$

**Example 3.2** Determine the value of $(F/A, 8\%, 37)$, using the tables presented in Appendix A.

The numerical value of $(F/A, 8\%, 37)$ is not included in the tables; however, from (3.3) and (3.1),

$$\begin{aligned}(F/A, 8\%, 37) &= (F/A, 8\%, 35) + (F/P, 8\%, 35) + (F/P, 8\%, 36)\\ &= (F/A, 8\%, 35) + (F/P, 8\%, 35) + (F/P, 8\%, 35)(F/P, 8\%, 1)\end{aligned}$$

All of the right-hand terms can now be evaluated using the tables in Appendix **A.** Thus,

$$(F/A, 8\%, 37) = 172.3168 + 14.7853 + (14.7853)(1.08) = 203.0702$$

**Example 3.3** Estimate the value of $(A/P, 5\frac{1}{2}\%, 63)$.

From (3.6), $(A/P, 5\frac{1}{2}\%, 63) \approx 0.055$. (The correct value, to four decimal places, is 0.0570.)

In addition to the above relationships, there are a number of others that have been discussed in Chapter 2. Specifically, the reader is reminded of the ***reciprocal relationships***

$$(P/F, i\%, n) = (F/P, i\%, n)^{-1}$$
$$(A/F, i\%, n) = (F/A, i\%, n)^{-1}$$
$$(AIP, i\%, n) = (PIA, i\%, n)^{-1}$$

and various ***product relationships*** such as

$$(A/P, i\%, n) = (AIF, i\%, n) \times (F/P, i\%, n)$$
$$(FIG, i\%, n) = (F/A, i\%, n) \times (AIG, i\%, n)$$

## 3.2 LINEAR INTERPOLATION

If a required compound interest factor falls between two tabulated values, it may be desirable or necessary to use linear interpolation to approximate that factor. Let $(x_1, y_1)$ and $(x_2, y_2)$ be known tabulated points. We wish to determine the value of $y$ corresponding to some given value $x$, where $x_1 < x < x_2$. Then, by direct proportionality, we can write

$$\frac{y - y_1}{x - x_1} \approx \frac{y_2 - y_1}{x_2 - x_1}$$

Solving for $y$,

$$y \approx y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{3.7}$$

or, as more commonly applied in practice,

$$y \approx y_1 + \frac{x - x_1}{x_2 - x_1}(y_2 - y_1) \tag{3.8}$$

**Example 3.4** Approximate $(F/P, 8\%, 37)$, using the tabulated values in Appendix A and linear interpolation.

From Appendix A, $(F/P, 8\%, 35) = 14.7853$ and $(F/P, 8\%, 40) = 21.7245$. Therefore,

$$\begin{aligned}(F/P, 8\%, 37) &\approx (F/P, 8\%, 35) + \frac{37-35}{40-35}[(F/P, 8\%, 40) - (F/P, 8\%, 35)] \\ &= 14.7853 + \frac{2}{5}(21.7245 - 14.7853) = 17.5610\end{aligned}$$

The correct answer, from (*2.1*), is 17.2456.

**Example 3.5** Approximate $(A/P, 8\%, 37)$, using the tabulated values in Appendix A and linear interpolation.

From Appendix A, $(A/P, 8\%, 35) = 0.08580$ and $(A/P, 8\%, 40) = 0.08386$. Therefore,

$$\begin{aligned}(A/P, 8\%, 37) &\approx 0.08580 + \frac{37-35}{40-35}(0.08386 - 0.08580) \\ &= 0.08580 + (0.4)(-0.0019) = 0.08500\end{aligned}$$

The correct answer, from (*2.6*), is 0.08492.

**Example 3.6** Approximate $(F/A, 6\tfrac{3}{4}\%, 15)$ by interpolating linearly between the tabulated values given in Appendix A.

From Appendix A, $(F/A, 6\%, 15) = 23.2760$ and $(F/A, 7\%, 15) = 25.1290$ (note that these values come from two different tables). Hence,

$$\begin{aligned}(F/A, 6\tfrac{3}{4}\%, 15) &\approx (F/A, 6\%, 15) + \frac{6.75 - 6.00}{7.00 - 6.00}[(F/A, 7\%, 15) - (F/A, 6\%, 15)] \\ &= 23.2760 + (0.75)(25.1290 - 23.2760) = 24.6658\end{aligned}$$

The correct answer, from (*2.3*), is 24.6504.

## 3.3 UNKNOWN NUMBER OF YEARS

Sometimes we require the number of years, n, that corresponds to a given compound interest factor and a given annual interest rate, $i$. Solving the appropriate formulas from Chapter 2 by logarithms, we obtain the results displayed in Table 3-1. Formula (2.9) cannot be solved exactly for n, given $A/G$ and $i$; in this case, a graphical solution or linear interpolation is employed.

**Table 3-1**

| Factor | Number of Years |
|---|---|
| $F/P$ or $P/F$ | $n=\dfrac{\log (F/P)}{\log (1+i)}=\dfrac{-\log (P/F)}{\log (1+i)}$ |
| $F/A$ or $A/F$ | $n=\dfrac{\log [1+i(F/A)]}{\log (1+i)}=\dfrac{\log \left(1+\dfrac{i}{A/F}\right)}{\log (1+i)}$ |
| $P/A$ or $A/P$ | $n=\dfrac{-\log [1-i(P/A)]}{\log (1+i)}=\dfrac{-\log \left(1-\dfrac{i}{A/P}\right)}{\log (1+i)}$ |

**Example 3.7** How many years will be required for a given sum of money to triple, if it is deposited in a bank account that pays 6% per year, compounded annually?

We require $n$, given $F/P=3$ and $i=0.06$. From Table 3-1,

$$n=\frac{\log 3}{\log (1+0.06)}=18.85$$

But, since the interest is compounded only at the end of each year, the calculated value for $n$ must be an integer. Hence $n=19$ (corresponding to $F/P=3.0256$) is the correct solution.

**Example 3.8** Determine the value of $n$ corresponding to $A/F=0.01$, if $i=7\frac{1}{2}\%$ per year, compounded annually.

From Table 3-1,

$$n=\frac{\log \left(1+\dfrac{0.075}{0.01}\right)}{\log (1+0.075)}=\frac{\log 8.5}{\log 1.075}=29.59$$

The required number of years is therefore 30.

**Example 3.9** Determine the value of $n$ corresponding to $A/G=11.6$ and $i=7\%$, compounded annually, using the following known points $(n, A/G)$ (all at $i=7\%$): (35, 10.6687), (40, 11.4234), (45, 12.0360).

From the graph of the data, Fig. 3-1, we can see that the value of $n$ corresponding to $A/G=11.6$ is approximately 41.4.

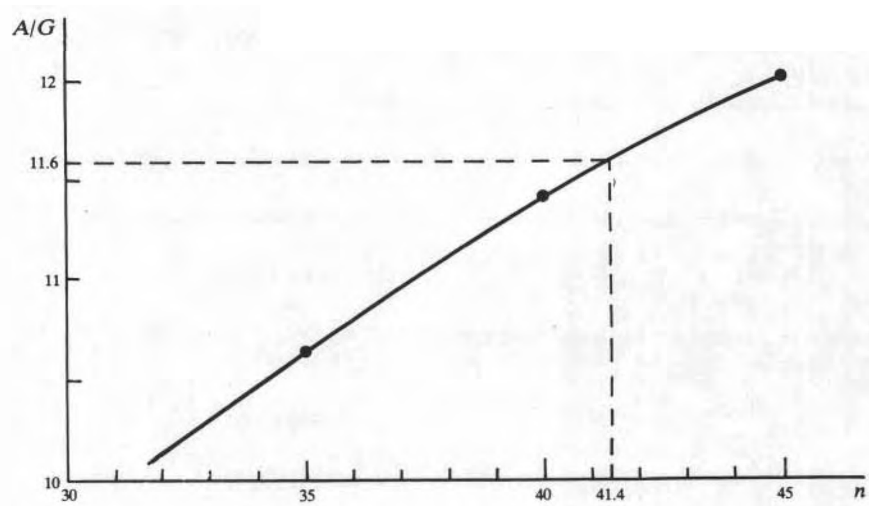


**Fig. 3-1**

The value of $n$ can also be approximated by interpolating linearly between the second two data points:

$$n \approx 40 + \frac{11.6000 - 11.4234}{12.0360 - 11.4234}(45 - 40) = 41.44$$

Since the interest is compounded annually, $n$ can only take on integer values. Thus, $n = 41$ comes closest to satisfying the given condition that $A/G = 11.6$.

## 3.4 UNKNOWN INTEREST RATE

Another situation that frequently arises is the need to solve for the annual interest rate that corresponds to a given compound interest factor and a specified number of years. If the given compound interest factor is either $F/P$ or $P/F$, then the value of i can be obtained explicitly as

$$i = (F/P)^{1/n} - 1 = (P/F)^{-1/n} - 1 \tag{3.9}$$

When one of the remaining five interest factors is specified, however, it is necessary to utilize a numerical or graphical solution procedure, or to interpolate linearly between tabulated values. If the given interest factor is not tabulated for the specified value of n, double interpolation will be required.

**Example 3.10** Determine the value of i corresponding to $F/A = 1000$ and $n = 38$, using the known values displayed in Table 3-2.

**Table 3-2**

| $i$ | $n$ | $F/A$ |
|---|---|---|
| 12% | 35 | 431.6635 |
| | 40 | 767.0914 |
| | 45 | 1358.2300 |
| 15% | 35 | 881.1702 |
| | 40 | 1779.0904 |
| | 45 | 3585.1286 |

We first obtain an interpolated value of $F/A$ for i = 12% and $n$ = 38:

$$(F/A, 12\%, 38) \approx 431.6635 + \frac{38 - 35}{40 - 35}(767.0914 - 431.6635) = 632.9202$$

Then we obtain an interpolated value of $F/A$ for i = 15% and $n$ = 38:

$$(F/A, 15\%, 38) \approx 881.1702 + \frac{38 - 35}{40 - 35}(1779.0904 - 881.1702) = 1419.9223$$

We can now interpolate between these calculated values to obtain the desired interest rate:

$$i \approx 12 + \frac{1000.0000 - 632.9202}{1419.9223 - 632.9202}(15 - 12) = 13.40\%$$

A more accurate result can be obtained if the factors $(F/A, 12\%, 38)$ and $(F/A, 15\%, 38)$ are evaluated directly from (2.3). Thus,

$$(F/A, 12\%, 38) = \frac{(1 + 0.12)^{38} - 1}{0.12} = 609.8305$$

$$(F/A, 15\%, 38) = \frac{(1 + 0.15)^{38} - 1}{0.15} = 1343.6222$$

and now linear interpolation gives:

$$i \approx 12 + \frac{1000.0000 - 609.8305}{1343.6222 - 609.8305}(15 - 12) = 13.60\%$$

This latter method is preferable, provided a calculator is available to carry out the exponentiation.

## Solved Problems

**3.1** Evaluate (a) $(FIP, 10\%, 44)$, (b) $(A/F, 10\%, 37)$.

(a) $$(F/P, 10\%, 44) = (F/P, 10\%, 40) \times (F/P, 10\%, 4) = (45.2593)(1.4641) = 66.2641$$

(b) By (3.3) and (3.1),

$$\begin{aligned}(F/A, 10\%, 37) &= (F/A, 10\%, 35) + (F/P, 10\%, 35) + [(F/P, 10\%, 35) \times (F/P, 10\%, 1)] \\ &= 271.0244 + 28.1024 + (28.1024)(1.1000) = 330.039\end{aligned}$$

Hence, $(A/F, 10\%, 37) = 11330.039 = 0.00303$.

**3.2** Estimate (a) $(AIP, 8\%, 100)$, (b) $(PIA, 20\%, 50)$.

(a) By (3.6), $(A/P, 8\%, 100) \approx 0.08$ (Appendix A gives the correct value as 0.08004).

(b) $$(P/A, 20\%, 50) = (A/P, 20\%, 50)^{-1} \approx (0.20)^{-1} = 5$$

The correct value is 4.9995.

**3.3** Find $(A/F, i\%, n)$ from (a) $(FIP, i\%, n)$, (b) $(P/F, i\%, n)$, (c) $(A/P, i\%, n)$.

(a) $$(AIF, i\%, n) = \frac{i}{(F/P, i\%, n) - 1}$$

(b) $$(A/F, i\%, n) = \frac{i(P/F, i\%, n)}{1 - (P/F, i\%, n)}$$

(c) $$(A/F, i\%, n) = (A/P, i\%, n) - i$$

**3.4** Derive (3.1) without reference to (2.1).

Investing \$1 for $n_1$ years at $i\%$ annual compound interest, and then investing the accumulated amount for a further $n_2$ years at the same rate, must be equivalent to investing \$1 for $n_1 + n_2$ years at $i\%$.

**3.5** Derive (3.3) without reference to (2.3) or (2.1).

As shown in Fig. 3-2, a uniform series of $n$ payments of \$1 may be considered as composed of two parts: the first $n - n_1$ payments, whose total future worth at the end of year $n$ is

$$(F/P, i\%, n - 1) + (F/P, i\%, n - 2) + \cdots + (F/P, i\%, n_1) \tag{1}$$

and the uniform series of the remaining $n_1$ payments, whose future worth is

$$(F/A, i\%, n_1) \tag{2}$$

The sum of (1) and (2) must be the future worth of the entire series, $(FIA, i\%, n)$.

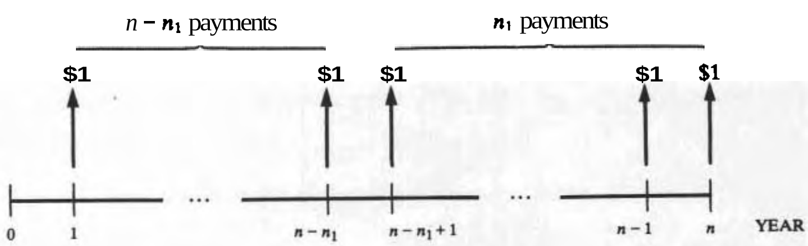


**Fig. 3-2**

It is obvious from Fig. 3-2 that the sum (1) above is equal to

$$(F/A, i\%, n - n_1)(FIP, i\%, n_1)$$

Hence we have the following, more compact version of (3.3):

$$(F/A, i\%, n) = (F/A, i\%, n_1) + (F/A, i\%, n - n_1)(F/P, i\%, n_1) \qquad (n > n_1) \qquad (3.10)$$

**3.6** Rework Problem **3.1(*b*)**, using (3.10).

$$(F/A, 10\%, 37) = (F/A, 10\%, 35) + (F/A, 10\%, 2)(F/P, 10\%, 35)$$
$$= 271.0244 + (2.1000)(28.1024) = 330.039$$

as before.

**3.7** Use linear interpolation and the tables given in Appendix A to evaluate **(a)** $(P/F, 6\%, 52)$, **(b)** $(P/A, 8\tfrac{1}{4}\%, 10)$, ***(c)*** $(A/G, 13\%, 48)$.

(a) First interpolate for $(F/P, 6\%, 52)$, then take the reciprocal.

$$(F/P, 6\%, 52) \approx 18.4202 + \frac{52-50}{55-50}(24.6503 - 18.4204) = 20.9122$$

$$(P/F, 6\%, 52) \approx (20.9122)^{-1} = 0.04782$$

(*b*) First interpolate for $(AIP, 8\tfrac{1}{4}\%, 10)$, then take the reciprocal.

$$(A/P, 8\tfrac{1}{4}\%, 10) \approx 0.14903 + \frac{8.25 - 8.00}{9.00 - 8.00}(0.15582 - 0.14903) = 0.15073$$

$$(P/A, 8\tfrac{1}{4}\%, 10) \approx (0.15073)^{-1} = 6.6345$$

*(c)* First interpolate on $n$ to obtain $(A/G, 12\%, 48)$ and $(A/G, 15\%, 48)$; then interpolate on i to obtain $(A/G, 13\%, 48)$.

$$(A/G, 12\%, 48) \approx 8.0572 + \frac{48-45}{50-45}(8.1597 - 8.0572) = 8.1187$$

$$(AIG, 15\%, 48) \approx 6.5830 + \frac{48-45}{50-45}(6.6205 - 6.5830) = 6.6055$$

$$(A/G, 13\%, 48) \approx 8.1187 + \frac{13-12}{15-12}(6.6055 - 8.1187) = 7.6143$$

**3.8** How many years will be required for a sum of money to quadruple, if it is deposited in a bank account that pays $6\tfrac{3}{4}\%$ per year, compounded annually?

$$n = \frac{\log 4}{\log(1 + 0.0675)} = \frac{0.60206}{0.02837} = 21.22 \text{ years}$$

or 22 years, if $n$ must be an integer.

**3.9** Determine the value of n corresponding to $P/A = 12$, if $i = 8\%$ per year, compounded annually.

$$n = \frac{-\log[1-(0.08)(12)]}{\log(1+0.08)} = \frac{1.39794}{0.03342} = 41.82$$

or 42, if $n$ must be an integer.

**3.10** Use linear interpolation to determine the value of n corresponding to $A/G = 5.4000$ and $i = 8\%$ per year, compounded annually.

From the tables in Appendix A,

$$(A/G, 8\%, 14) = 5.2731 \qquad (A/G, 8\%, 15) = 5.5945$$

Then

$$n \approx 14 + \frac{5.4000 - 5.2731}{5.5945 - 5.2731}(15 - 14) = 14.39 \text{ years}$$

**3.11** A bank will return $2345 on a 10-year certificate of deposit that originally cost $1000. What interest rate, compounded annually, is the bank paying?

$$i = \left(\frac{\$2345}{\$1000}\right)^{1/10} - 1 = 1.0899 - 1 = 8.89\%$$

**3.12** By interpolation, determine the value of i corresponding to $A/P = 0.15000$ and $n = 12$.

From the tables in Appendix A, we find that for $n = 12$:

| $i$ | $A/P$ |
|---|---|
| 10% | 0.14676 |
| 12% | 0.16144 |

Therefore, for $A/P = 0.15000$,

$$i \approx 10\% + \frac{0.15000 - 0.14676}{0.16144 - 0.14676}(12\% - 10\%) = 10.44\%$$

## Supplementary Problems

**3.13** Evaluate, using (3.1) and Appendix A, (a) $(F/P, 10\%, 41)$, (b) $(F/P, 12\%, 43)$, (c) $(F/P, 6\%, 57)$.
*Ans.* (a) 49.7852; (b) 130.7299; (c) 27.6971

**3.14** Evaluate, using (3.3) or (3.10) and the tables in Appendix A, (a) $(F/A, 10\%, 41)$, (b) $(F/A, 12\%, 43)$, (c) $(F/A, 6\%, 57)$. *Ans.* (a) 487.8518; (b) 1081.0826; (c) 444.9517

**3.15** Evaluate, using (3.4) and Appendix A, (a) $(P/A, 5\%, 52)$, (b) $(P/A, 9\%, 39)$, (c) $(P/A, 12\%, 43)$.
*Ans.* (a) 18.4181; (b) 10.7255; (c) 8.2696

**3.16** Evaluate, using (3.5) and the tabular values of $(A/P, i\%, n)$ given in Appendix A, (a) $(A/F, 4\%, 20)$, (b) $(A/F, 15\%, 8)$, (c) $(A/F, 12\%, 40)$. *Ans.* (a) 0.03358; (b) 0.07285; (c) 0.00130

**3.17** Estimate the following factors, using (3.6), and compare with the correct values as given by (2.6): (a) $(A/P, 8\%, 80)$, (b) $(A/P, 4\tfrac{3}{4}\%, 120)$, (c) $(A/P, 19.08\%, 100)$.
*Ans.* (a) 0.08 (correct value, 0.0802); (b) 0.0475 (correct value, 0.0477); (b) 0.1908 (correct value, 0.1908)

**3.18** From the interest tables in Appendix A, determine the value of each of the following compound interest factors using linear interpolation. Compare each value with the exact answer obtained from the appropriate formula in Chapter 2. (a) $(F/P, 7\frac{1}{2}\%, 12)$, (b) $(F/P, 12\%, 47)$, (c) $(P/F, 5\frac{1}{4}\%, 20)$, (d) (FIA, 4%, 38), (e) (FIA, $4\frac{3}{4}\%$, 38), (f) (AIF, $11\frac{1}{4}\%$, 30), (g) $(A/P, 5\%, 57)$, (h) $(A/P, 7\frac{1}{4}\%, 57)$, (i) $(P/A, 10\frac{3}{4}\%, 30)$, (j) $(A/G, 8\frac{1}{4}\%, 15)$, (k) $(A/G, 13\frac{3}{4}\%, 39)$.

Ans. (a) 2.3852 (exact: 2.3818); (b) 213.9934 (205.7061); (c) 0.3606 (0.3594); (d) 86.4762 (85.9703); (e) 92.0092 (101.7364); (*f*) 0.004869 (0.00479); (g) 0.05333 (0.05330); (h) 0.07391 (0.07387); (i) 8.9125 (8.8675); (*j*) 5.5545 (5.5541); (k) 7.0591 (7.0146)

**3.19** How many years (a whole number) will be required for a sum of money to double if the interest rate is (a) 10%, (b) 12%, (c) $13\frac{1}{2}\%$, per year, compounded annually?

Ans. (a) 8 years; (b) 7 years; (c) 6 years

**3.20** How many years (a whole number) will be required to accumulate \$10 000 if \$500 is deposited at the end of each year, and interest is payable at $6\frac{3}{4}\%$ per year, compounded annually? Ans. 14 years

**3.21** A person has \$80 000 in a savings account that earns interest at $7\frac{1}{2}\%$ per year, compounded annually. If the person withdraws \$12000 at the end of each year, after how many years (a whole number) will the savings be exhausted? Ans. 10 years

**3.22** Use linear interpolation to determine the value of $n$ corresponding to $A/G = 6.0000$, if $i = 9\%$ per year, compounded annually. Ans. 16.99

**3.23** Repeat Problem 3.22 for an interest rate of 11% per year, compounded annually. *Ans.* 18.84

**3.24** A man has entered into a contract in which he has agreed to lend \$1000 to a friend, and the friend has agreed to repay him \$1060.90 two years later. What annual compound rate of interest is the man receiving on his \$1000? *Ans.* 3%

**3.25** The First National Bank advertises it will pay \$3869.70 in cash at the end of 20 years to anyone who deposits \$1000. Federal Savings, a competitor, advertises that it pays 10% per year, compounded annually, on all deposits left one year or more. Which bank is paying the higher interest rate, and by how much? Ans. Federal Savings, by 3%

**3.26** A person who is about to retire has accumulated \$100 000 in a savings account. Suppose that the person withdraws \$8195.23 from the savings account at the end of each year for 20 years, at which time the account is totally depleted. What is the interest rate, based upon annual compounding?

Ans. $5\frac{1}{4}\%$ per year

**3.27** A person is considering entering into an agreement with an investment company to deposit \$1000 into a special account at the end of each year for 15 years. At the end of the period, the person would be able to withdraw a lump sum of \$28 800. At what rate would the person earn interest, if the interest was compounded annually? Ans. $8\frac{3}{4}\%$ per year

**3.28** A person is considering entering into an agreement with an investment company to deposit \$1000 into a special account at the end of the first year, \$1100 at the end of the second year, etc., increasing by \$100 each year. At the end of 15 years the person would be able to withdraw a lump sum of \$36 000. At what rate would the person earn interest, if the interest was compounded annually?

Ans. 5.54% per year

# Chapter 4

# Discrete, Periodic Compounding

## 4.1 NOMINAL AND EFFECTIVE INTEREST RATES

Many financial transactions require that interest be compounded more often than once a year (**e.g.**, quarterly, monthly, daily, etc.). In such situations, there are two expressions for the interest rate. The nominal interest rate, $r$, is expressed on an annual basis; this is the rate that is normally quoted when describing an interest-bearing transaction. The effective interest rate, i, is the rate that corresponds to the actual interest period. The effective interest rate is obtained by dividing the nominal interest rate by m, the number of interest periods per year:

$$i = \frac{r}{m} \tag{4.1}$$

**Example 4.1** A bank claims to pay interest to its depositors at the rate of 6% per year, compounded quarterly. What are the nominal and effective interest rates?

The nominal interest rate is $r = 6\%$. Since there are four interest periods per year, the effective interest rate is

$$i = \frac{6\%}{4} = 1.5\% \text{ per quarter}$$

## 4.2 WHEN INTEREST PERIODS COINCIDE WITH PAYMENT PERIODS

When the interest periods and the payment periods coincide, it is possible to make direct use both of the compound interest formulas developed in Chapter 2 and the compound interest tables presented in Appendix A, provided the interest rate, i, is taken to be the effective interest rate for that interest period. Moreover, the number of years, n, must be replaced by the total number of interest periods, mn.

**Example 4.2** An engineer plans to borrow $3000 from his company credit union, to be repaid in 24 equal monthly installments. The credit union charges interest at the rate of 1% per month on the unpaid balance. How much money must the engineer repay each month?

This problem can be solved by direct application of *(2.6)*, since the interest charges and the uniform payments are both determined on a monthly basis:

$$A = P \times (A/P, 1\%, mn) = \$3000\,(A/P, 1\%, 24)$$
$$= \$3000\,\frac{0.01(1+0.01)^{24}}{(1+0.01)^{24}-1} = \$141.22$$

We conclude that the engineer must repay $141.22 at the end of every month for 24 months.

Alternatively, Appendix A gives $(A/P, 1\%, 24) = 0.04707$, whence

$$A = \$3000(0.04707) = \$141.21 \text{ per month}$$

If, as in the case of commercial loans, the nominal interest rate is specified, the compound interest formulas of Chapter 2 and/or Appendix A can still be used, with i replaced by $r/m$, and n by mn.

**EXAMPLE 4.3** An engineer wishes to purchase an *$80 000* home by making a down payment of *$20 000* and borrowing the remaining $60 *000,* which he will repay on a monthly basis over the next *30* years. If the bank charges interest at the rate of $9\frac{1}{2}\%$ per year, compounded monthly, how much money must the engineer repay each month?

Again applying *(2.6)*,

$$A = P \times (A/P, r\%/m, mn)$$
$$= \$60\,000 \frac{(0.095/12)(1+0.095/12)^{(12)(30)}}{(1+0.095/12)^{(12)(30)}-1} = \$504.51$$

It is interesting to note that the total amount of money which will be repaid to the bank is

$$\$504.51 \times 360 = \$181\,623.60$$

or ***three times*** the amount of the original loan.

## 4.3 WHEN INTEREST PERIODS ARE SMALLER THAN PAYMENT PERIODS

If the interest periods are smaller than the payment periods, then the interest may be compounded several times between payments. One way to handle problems of this type is to determine the effective interest rate for the given interest period, and then treat each payment separately.

**Example 4.4** An engineer deposits *$1000* in a savings account at the end of each year. If the bank pays interest at the rate of 6% per year, compounded quarterly, how much money will have accumulated in the account after *5* years?

The effective interest rate is $i = 6\%/4 = 1.5\%$ per quarter; the first deposit accumulates for *16* quarters; etc.

$$F = \$1000(F/P, 1.5\%, 16) + \$1000(F/P, 1.5\%, 12)$$
$$+ \$1000(F/P, 1.5\%, 8) + \$1000(F/P, 1.5\%, 4) + \$1000(F/P, 1.5\%, 0)$$

The ***F/P*** factors can be obtained from either (2.1) or Appendix A.

$$F = \$1000(1.2690) + \$1000(1.1956) + \$1000(1.1265) + \$1000(1.0614) + \$1000(1.0000)$$
$$= \$5652.50$$

Another procedure, which is usually more convenient, is to calculate an effective interest rate for the given ***payment*** period, and then to proceed as though the interest periods and the payment periods coincided. This effective interest rate can be determined as

$$i = \left(1 + \frac{r}{\alpha}\right)^{\alpha} - 1 \qquad (4.2)$$

where $\alpha$ represents the number of interest periods per payment period and $r$ is the nominal interest rate for that payment period. If the payment period is one year, then $\alpha = m$, and we obtain the following expression for the ***effective annual interest rate:***

$$i = \left(1 + \frac{r}{m}\right)^{m} - 1 \qquad (4.3)$$

**Example 4.5** Rework Example *4.4* by using an effective annual interest rate.

Here, $r = 6\%$ and $\alpha = m = 4$, so that, by *(4.3)*,

$$i = \left(1 + \frac{0.06}{4}\right)^{4} - 1 = 0.06136$$

We can now apply ***(2.3)*** to obtain

$$F = \$1000(F/A, 6.136\%, 5) = \$1000 \frac{(1+0.06136)^5 - 1}{0.06136} = \$5652.40$$

which agrees with Example *4.4* to within roundoff errors.

Appendix B contains a tabulation of effective annual interest rates corresponding to various nominal interest rates. This table may be used in place of (*4.3*), if desired.

## 4.4 WHEN INTEREST PERIODS ARE LARGER THAN PAYMENT PERIODS

If the interest periods are larger than the payment periods, some of the payments may not have been deposited for an entire interest period. Such payments do not earn any interest during that interest period. In other words, interest is earned only by those payments that have been deposited or invested for the entire interest period.

Situations of this type can be treated in the following manner:

1. Consider all ***deposits*** that were made during the interest period to have been made at the ***end*** of the interest period (and therefore to have earned no interest during that interest period).
2. Consider all ***withdrawals*** that were made during the interest period to have been made at the ***beginning*** of the interest period (again earning no interest).
3. Then proceed as though the interest periods and the payment periods coincided.

**Example** 4.6 A person has $4000 in a savings account at the beginning of a calendar year; the bank pays interest at 6% per year, compounded quarterly. Table 4-1 shows the transactions carried out during the calendar year; the second column gives the effective dates according to rules 1 and 2 above. To find the balance in the account at the end of the calendar year, we calculate the effective interest rate, $6\%/4 = 1.5\%$ per quarter. Then, lumping the amounts at the effective dates and applying (*2.1*), we obtain

$$\begin{aligned} F &= (\$4000 - \$175)(F/P, 1.5\%, 4) + (\$1200 - \$1800)(F/P, 1.5\%, 3) \\ &\qquad + (\$180 - \$800)(F/P, 1.5\%, 2) + (\$1600 - \$1100)(F/P, 1.5\%, 1) + \$2300 \\ &= \$3825(1.0614) - \$600(1.0457) - \$620(1.0302) + \$500(1.0150) + \$2300 \\ &= \$5601.21 \end{aligned}$$

**Table 4-1**

| Date | Effective Date | Deposit | Withdrawal |
|---|---|---|---|
| Jan. 10 | Jan. 1 | | $ 175 |
| Feb. 20 | Mar. 31 | $1200 | |
| Apr. 12 | Apr. 1 | | 1500 |
| May 5 | June 30 | 65 | |
| May 13 | June 30 | 115 | |
| May 24 | Apr. 1 | | 50 |
| June 21 | Apr. 1 | | 250 |
| Aug. 10 | Sept. 30 | 1600 | |
| Sept. 12 | July 1 | | 800 |
| Nov. 27 | Oct. 1 | | 350 |
| Dec. 17 | Dec. 31 | 2300 | |
| Dec. 29 | Oct. 1 | | 750 |

## Solved Problems

**4.1** A bank advertises that it pays interest at the rate of 10% per year, compounded quarterly. What effective interest rate is the bank paying?

$$r = 10\% \qquad i = \frac{10\%}{4} = 2.5\% \text{ per quarter}$$

**4.2** An engineer has just borrowed $8000 from a local bank, at the rate of 1% per month on the unpaid balance. His contract states that he must repay the loan in 35 equal monthly installments. How much money must he repay each month?

$$A = \$8000(A/P, 1\%, 35) = \$8000(0.03400) = \$272.00$$

**4.3** A bank pays interest at the rate of 6% per year, compounded monthly. If a person deposits $2500 in a savings account at the bank, how much money will accumulate by the end of 2 years?

Equation (*2.1*), with the appropriate substitutions, gives

$$F = \$2500\left(1 + \frac{0.06}{12}\right)^{(12)(2)} = \$2500(1.005)^{24} = \$2817.89$$

Alternatively, using the tables in Appendix A, we have

$$F = \$2500(F/P, 0.5\%, 24) = \$2500(1.1272) = \$2818.00$$

**4.4** A man plans to buy a $150 000 house. He wants to make a down payment of $30 000 and to take out a 30-year mortgage for the remaining $120 000, at 10% per year, compounded monthly. How much must he repay each month?

Equation (*2.6*), with the appropriate substitutions, gives

$$A = \$120\,000\,\frac{(0.10/12)(1 + 0.10/12)^{(12)(30)}}{(1 + 0.10/12)^{(12)(30)} - 1} = \$120\,000\,\frac{0.16531}{18.8374} = \$1053.08$$

The solution can also be approximated by use of (*3.6*):

$$A = \$120\,000(A/P, 10\%/12, 360) \approx \$120\,000(0.10/12) = \$1000$$

**4.5** A man plans to save $1000 a month for the next 20 years, at 10% per year, compounded monthly. How much money will he have at the end of 20 years?

Equation (*2.3*), with the appropriate substitutions, gives

$$F = \$1000\,\frac{(1 + 0.10/12)^{(12)(20)} - 1}{0.10/12} = \$1000\,\frac{6.32807}{0.0083333} = \$759\,371.43$$

**4.6** Repeat Problem 4.5 using quarterly compounding.

$$F = \$1000\,\frac{(1 + 0.10/4)^{(4)(20)} - 1}{0.10/4} = \$1000\,\frac{6.20957}{0.025} = \$248\,382.80$$

Note that this is much less than the value obtained using monthly compounding.

**4.7** What is the present value of a stream of monthly payments of $500 each over 10 years, if the interest rate is 10% per annum, compounded monthly?

Equation (*2.7*), with the appropriate substitutions, gives

$$P = \$500\,\frac{(1+0.10/12)^{(12)(10)}-1}{(0.10/12)(1+0.10/12)^{(12)(10)}} = \$500\,\frac{1.70704}{(0.0083333)(2.70704)} = \$37\,835.72$$

**4.8** Repeat Problem 4.7 using daily compounding. For computational simplicity, assume 30 days in each month (many banks do this).

Here, $r = 10\%/12 = 0.00833333$ and $\alpha = 30$; hence by (*4.2*), the effective monthly interest rate is

$$i = \left(1 + \frac{0.00833333}{30}\right)^{30} - 1 = 0.0083670$$

Now use (*2.7*), as before.

$$P = \$500\,\frac{(1+0.0083670)^{(12)(10)}-1}{(0.0083670)(1+0.0083670)^{(12)(10)}} = \$500\,\frac{1.717909}{(0.0083670)(2.717909)} = \$37\,771.61$$

**4.9** How much money must be deposited in a savings account each month to accumulate $10 000 at the end of 5 years, if the bank pays interest at the rate of 6% per year, compounded (a) monthly? (*b*) semiannually? (*c*) quarterly? (d) daily?

In each case use (*2.4*), with the appropriate substitutions.

(*a*)
$$A = \$10\,000\,\frac{0.06112}{(1+0.06/12)^{(12)(5)}-1} = \$10\,000\,\frac{0.005}{0.34885} = \$143.33 \text{ per month}$$

(*b*)
$$A = \$10\,000\,\frac{0.0612}{(1+0.06/2)^{(2)(5)}-1} = \$10\,000\,\frac{0.03}{0.34392} = \$872.30 \text{ every 6 months}$$

or $872.30/6 = $145.38 per month.

(*c*)
$$A = \$10\,000\,\frac{0.0614}{(1+0.06/4)^{(4)(5)}-1} = \$10\,000\,\frac{0.015}{0.34686} = \$432.45 \text{ per quarter}$$

or $432.4513 = $144.15 per month.

(*d*)
$$r = \frac{6\%}{12} = 0.005 \qquad \alpha = 30$$

$$i = \left(1 + \frac{0.005}{30}\right)^{30} - 1 = 0.0050121$$

$$A = \$10\,000\,\frac{0.0050121}{(1+0.0050121)^{(12)(5)}-1} = \$10\,000\,\frac{0.0050121}{0.34982} = \$143.28 \text{ per month}$$

**4.10** Using a suitable version of (*2.9*), evaluate $A/G$ for $r = 12\%$, compounded monthly, and $n = 2$ years. Compare the result with the tabulated value in Appendix A.

$$A/G = \frac{m}{r} - \frac{mn}{(1+r/m)^{mn}-1}$$
$$= \frac{12}{0.12} - \frac{(12)(2)}{(1+0.12/12)^{(12)(2)}-1} = 100 - 88.9763 = 11.0237$$

Appendix A gives $(A/G, 1\%, 24) = 11.0237$.

**4.11** Mrs. Carter deposits $100 in the bank at the end of each month. If the bank pays (a) 6% per year, (*b*) 7% per year, compounded monthly, how much money will she have accumulated at the end of 5 years?

(*a*) The effective monthly interest rate is $6\%/12 = 0.5\%$. There will be a total of $5 \times 12 = 60$ monthly payments. Hence, using Appendix A,

$$F = \$100(F/A, 0.5\%, 60) = \$100(69.7700) = \$6977.00$$

(*b*) The effective monthly interest rate is $7\%/12 = 0.583333\%$. As the tabulated value of $(F/A, 0.583333\%, 60)$ is not readily available, we interpolate linearly between $(F/A, 0.5\%, 60)$ and $(F/A, 0.75\%, 60)$.

$$(F/A, 0.583333\%, 60) \approx 69.7700 + \frac{0.583333 - 0.5}{0.75 - 0.5}(75.4241 - 69.7700) = 71.6547$$

and

$$F \approx \$100(71.6547) = \$7165.47$$

Another (and more accurate) way to solve this problem is to apply (2.3):

$$F \approx \$100\,\frac{(1 + 0.00583333)^{60} - 1}{0.00583333} = \$100\,\frac{0.417625}{0.00583333} = \$7159.29$$

**4.12** In Problem 4.11, suppose that Mrs. Carter deposits \$100 a month during the first year, \$110 a month during the second year, \$120 a month during the third year, etc. How much will have accumulated at the end of 5 years if the interest rate is 6% per year, compounded monthly?

Treating each year separately,

$$\begin{aligned} F = {} & \$100(F/A, 0.5\%, 12)(F/P, 0.5\%, 48) + \$110(F/A, 0.5\%, 12)(F/P, 0.5\%, 36) \\ & + \$120(F/A, 0.5\%, 12)(F/P, 0.5\%, 24) + \$130(F/A, 0.5\%, 12)(F/P, 0.5\%, 12) \\ & + \$140(F/A, 0.5\%, 12) \end{aligned}$$

The required numerical values can be obtained from Appendix A (using interpolation in some cases), or from (2.1) and (2.3).

$$\begin{aligned} F &= [\$100(1.2705) + \$110(1.1967) + \$120(1.1272) + \$130(1.0617) + \$140](12.3356) \\ &= (\$671.972)(12.3356) = \$8289.18 \end{aligned}$$

## Supplementary Problems

**4.13** What is the effective annual interest rate if the nominal interest rate is 6%, compounded monthly?
*Ans.* 6.1678% per year

**4.14** How many years will be required for a sum of money to double, if the annual interest rate is 10%, compounded quarterly? *Ans.* $n = 5.86$ years (6 years, if $n$ must be an integer)

**4.15** Mr. Smith plans to deposit money in a bank that pays 10% interest per year, compounded daily. What effective rate of interest will he receive (*a*) yearly? (*b*) semiannually?
*Ans.* (*a*) 10.515%; (*b*) 5.0625%

**4.16** A bank pays interest at the rate of 12% per year, compounded monthly. If a man deposits \$3000 in the bank and leaves it for 5 years, how much money will accumulate, according to (*a*) (2.1)? (b) Appendix A? ***Ans.*** (*a*) \$5450.09; (*b*) \$5450.10

**4.17** A person deposits \$2000 in a savings account. If all of the money is allowed to accumulate, how much will the person have at the end of 5 years, given a nominal interest rate of 6%, compounded (*a*) annually? (*b*) quarterly? (*c*) monthly? (d) daily? Use the tables in Appendix A to obtain the answers whenever possible. *Ans.* (*a*) \$2676.40; (*b*) \$2693.80; (*c*) \$2697.80; (*d*) \$2699.65

**4.18** What amount of money is equivalent to receiving $8000 three years from today, if the interest rate is 8% per year, compounded semiannually? ***Ans.*** $6322.52

**4.19** A bank pays 6% interest per year, compounded quarterly. To what amount will a $5000 deposit grow if left in that bank for 10 years? ***Ans.*** $9070.09

**4.20** Repeat Problem 4.19 for annual compounding. ***Ans.*** $8954.24

**4.21** Calculate the amount of money that you would have in your savings account at the end of 12 months if you made the following deposits:

| End of Month | 1 | 3 | 6 | 7 | 8 | 11 |
|---|---|---|---|---|---|---|
| Deposit, $ | 200 | 90 | 70 | 75 | 85 | 70 |

Assume that the bank pays 6% interest per year, compounded semiannually, and that it pays simple interest on any interperiod deposits. ***Ans.*** $611.73

**4.22** Calculate the balance in Mr. Warren's account at the end of the year, if he deposits $100 each at the ends of months 1 and 6, and $200 each at the ends of months 7 and 9. His bank pays 8% per year, compounded quarterly, and simple interest on the interperiod deposits. ***Ans.*** $622.30

**4.23** Mr. Smith plans to deposit $8000 in a savings account at the end of each year for 5 years. The bank pays interest at the rate of 12% per year, compounded quarterly, on such a plan. Calculate how much money Mr. Smith can expect to withdraw at the end of 5 years, (a) by the method of Example 4.4, (b) by use of (4.3). ***Ans.*** (a) $51 382.40; (b) $51 394.73

**4.24** Suppose that $2000 is invested now, $2500 two years from now, and $1200 four years from now, all at 8% per year, compounded quarterly. What will be the total amount 10 years from now?
***Ans.*** $11 057.33

**4.25** A savings bank offers $1000 certificates of deposit. Each certificate can be redeemed for $2000 after $8\frac{1}{2}$ years. What is the nominal annual interest rate if the interest is compounded monthly? ***Ans.*** 8.182%

**4.26** What is the effective annual interest rate for the certificates of Problem 4.25? ***Ans.*** 8.496%

**4.27** Frank is trying to determine whether or not he can afford to borrow $10000 for 2 years. The bank charges 1% per month on the unpaid balance. Frank wants to repay the loan in 24 equal monthly installments, but feels he cannot pay more than $450 per month. Can he afford the loan?
***Ans.*** No (he would have to repay $470.70 per month)

**4.28** What will be the monthly payment on a 30-year, $100 000 mortgage loan, where the interest rate is 12% per year, (a) compounded monthly? (b) compounded daily? ***Ans.*** (a) $1028.61; *(b)* $1033.25

**4.29** What is the answer to Problem 4.28 under the approximation $\lim_{n\to\infty} (A/P, i, n) \approx i$? ***Ans.*** $1000

**4.30** Mrs. Jones plans to save $750 a month for the next 10 years, at 10% per year, compounded monthly. How much money will she have at the end of 10 years? ***Ans.*** $153 633.38

**4.31** What is the present value of a series of monthly payments of $300 each over 12 years, if the interest rate is 9% per year, compounded monthly? ***Ans.*** $26 361.29

**4.32** How much money must be deposited in a savings account each month to accumulate $12 000 at the end of 5 years, if the bank pays interest at the rate of 6% per year, compounded monthly? ***Ans.*** $171.99

**4.33** Compute the amount of the monthly deposits Mr. Jones must make for the next 5 years in order for him to accumulate $10 000 at the end of 5 years, at the nominal rate of 6% per year, compounded daily.
*Ans.* $143.28

**4.34** A series of quarterly payments of $1000 for 25 years is economically equivalent to what present sum, if the quarterly payments are invested at an annual rate of 8%, compounded quarterly?
*Ans.* $43 103.45

**4.35** Using a suitable form of (*2.9*), evaluate $A/G$ for $r = 12\%$, compounded monthly, and $n = 5$ years. Compare the result to the tabulated value in Appendix A. *Ans.* 26.5333 (same in Appendix A)

**4.36** An investment plan pays 15% per year, compounded monthly. How much would have to be invested every year so that $40 000 would be accumulated by the end of 10 years? *Ans.* $1869.13

**4.37** On the day of his son's birth, a father decided to establish a fund for the boy's college education. The father wants the son to be able to withdraw $4000 from the fund on his 18th birthday, again on his 19th birthday, again on his 20th birthday, and again on his 21st birthday. If the fund earns interest at 9% per year, compounded quarterly, how much should the father deposit at the end of each year, up through the 17th year? Compare with the result obtained for annual compounding (Problem 2.25).
*Ans.* $338.41

**4.38** Solve Problem 4.37 again, assuming now that the money is deposited in the bank at the beginning, rather than the end, of each year. *Ans.* $309.59

**4.39** A new machine is expected to cost $6000 and have a life of 5 years. Maintenance costs will be $1500 the first year, $1700 the second year, $1900 the third year, $2200 the fourth year, and $2300 the fifth year. How much should be deposited in a fund that earns 9% per year, compounded monthly, in order to pay for this machine? *Ans.* $13 180

**4.40** Suppose that a person deposits $2500 in a savings account at the end of each year for the next 15 years. If the bank pays (a) 8% per year, (b) $8\frac{1}{2}\%$ per year, compounded daily, how much money will the person have by the end of the 15th year? *Ans.* (a) $69 609; (b) $72 649

**4.41** Jones has deposited his life savings of $70 000 in a retirement income plan with a local bank. The bank pays (a) 10% per year, (b) 11.25% per year, compounded quarterly, on such deposits. What is the maximum fixed amount Jones can withdraw at the end of each year and still have the funds last for 15 years?
*Ans.* (a) $9404.33; (b) $10 132.00

**4.42** Ann White is planning to take early retirement. She has decided that she needs $15000 a year for the first 5 years of retirement; after that, Social Security and other pension plans will provide her with adequate retirement income. How much money will she need to have in the bank at the start of the 5-year period, if the bank pays (a) 10% per year, (b) 6.75% per year, compounded monthly? Compare the result in (a) with Problem 2.31(*a*). *Ans.* (*a*) $56 184; (b) $61 565

**4.43** Mr. Frank is planning for a 25-year retirement period, during which he wants to withdraw a portion of his savings at the end of each year. He plans to withdraw $10 000 at the end of the first year, and to then increase the amount of the withdrawal by $1000 each year (to offset inflation). How much money should he have in his savings account at the start of his retirement period in order to achieve these goals, if the bank pays 9% per year, compounded quarterly? Compare with Problem 2.32(*a*). *Ans.* $169 740

**4.44** Repeat Problem 4.43 for an interest rate of $8\frac{1}{2}\%$ per year, compounded quarterly. How significant is the $\frac{1}{2}\%$ difference? *Ans.* $179 267 (the $\frac{1}{2}\%$ difference requires almost $10 000 additional)

**4.45** The cost to maintain a new car is estimated to be $75 the first year, and to increase by $12 each year thereafter. How much money should be set aside for maintenance, if the car is to be kept 6 years and if the money which is set aside earns interest at the rate of 5% per year, compounded monthly?
*Ans.* $522.15

**4.46** A father wants to set aside money for his 8-year-old son's college education, by making annual deposits to a bank account in his son's name that pays 8% per annum, compounded quarterly. What equal deposits must the father make on the son's 9th through 17th birthdays, in order for the son to be able to withdraw $4000 on each of his four birthdays from the 18th to the 21st? *Ans.* $1013.76

**4.47** A savings account earns interest at the rate of $6\frac{3}{4}\%$ per year, compounded quarterly. How much money must initially be placed in the account to provide for fifteen end-of-year withdrawls, if the first withdrawal is $2000 and each subsequent withdrawal increases by $350? *Ans.* $36 792

**4.48** A bank offers its customers a Christmas Club account in which they deposit $25 a week for 39 weeks, starting in February. At the end of the period (mid-November), each customer can withdraw $1000. What is the nominal annual interest rate, assuming monthly compounding? (Hint: The 39 weeks compose nine interest periods, at the ends of which the lumped deposits are $100, $100, $125, . . .) *Ans.* 7.72%

**4.49** Mr. Williams deposits $200 in the bank at the end of each quarter. If the bank pays 6% per year, compounded quarterly, how much money will Mr. Williams have accumulated at the end of 12 years? *Ans.* $13913

**4.50** Repeat Problem 4.49 for a nominal interest rate of $6\frac{1}{2}\%$ per year, compounded monthly. *Ans.* $14373

**4.51** Repeat Problem 4.49 for the case where the money is deposited at the beginning of each quarter. *Ans.* $14 122

**4.52** An engineering student borrows $4000 to pay tuition for his senior year. Payments are to be made in 36 equal monthly installments, to begin the first month after graduation. How much money must the student repay each month, if he is graduated 9 months after taking out the loan and if the interest rate is 10% per year, compounded (*a*) monthly? (*b*) quarterly? (*c*) daily?
*Ans.* (*a*) $139.08; (*b*) $139.98; (*c*) $138.64

**4.53** A recent engineering graduate intends to purchase a new car. He plans to pay $2000 down and to finance the balance over a 4-year period. The maximum amount that he can repay each month is $200. What is the most expensive car that he can afford, assuming an interest rate of 12% per year, compounded monthly? *Ans.* $9595

**4.54** Suppose that the engineering graduate of Problem 4.53 can afford to repay $200 a month during the first year, $225 a month during the second year, $250 a month during the third year, and $275 a month during the fourth year. What is the most expensive car he can afford, assuming he pays $2000 down and the interest rate is 12% per year, compounded monthly? *Ans.* $10 878

**4.55** Repeat Problem 4.54 for an interest rate of $10\frac{3}{4}\%$ per year, compounded daily. *Ans.* $11 094

**4.56** A young couple are saving money in order to make a down payment on a house 6 years from now. Suppose that they save $150 a month during the first year, $165 a month during the second year, and so on, the amount increasing by $15 a month in each successive year. What is the most expensive house that they will be able to purchase at the end of the 6-year period, if they pay 25% down? Assume that their savings earns 7% per year, compounded quarterly. *Ans.* $65 327

**4.57** An engineer plans to borrow $10 000 to open his own consulting business. He must repay $215 a month for 5 years. What is the nominal annual interest rate, based on monthly compounding?
*Ans.* 10.51%

# Chapter 5

# Continuous Compounding

## 5.1 NOMINAL AND EFFECTIVE INTEREST RATES

Continuous compounding can be thought of as a limiting case of the multiple-compounding situation of Section 4.3. Holding the nominal annual interest rate fixed at $r$ and letting the number of interest periods become infinite, while the length of each interest period becomes infinitesimally small, we obtain from (4.3)

$$i = \lim_{m \to \infty} \left[ \left(1 + \frac{r}{m}\right)^m - 1 \right] = e^r - 1 \tag{5.1}$$

as the expression for the effective annual interest rate in continuous compounding.

**Example 5.1** A savings bank is selling long-term savings certificates that pay interest at the rate of $7\frac{1}{2}\%$ per year, compounded continuously. The bank claims that the actual annual yield of these certificates is 7.79%. What does this mean?

The nominal interest rate is $7\frac{1}{2}\%$. Since the interest is compounded continuously, the effective annual interest rate is given by (5.1) as

$$i = e^{0.075} - 1 = 0.077884 \approx 7.79\%$$

Formula (5.1) is very convenient, provided a calculator is available to carry out the exponentiation. Tabulated values of the effective annual interest rate may be used instead; see Appendix B.

## 5.2 DISCRETE PAYMENTS

If interest is compounded continuously but payments are made annually, we can still use the formulas of Chapter 2 for the various compound interest factors, provided i is given by (5.1). Thus:

$$F/P = e^{rn} \tag{5.2}$$

$$P/F = e^{-rn} \tag{5.3}$$

$$F/A = \frac{e^{rn} - 1}{e^r - 1} \tag{5.4}$$

$$A/F = \frac{e^r - 1}{e^{rn} - 1} \tag{5.5}$$

$$A/P = \frac{e^r - 1}{1 - e^{-rn}} \tag{5.6}$$

$$P/A = \frac{1 - e^{-rn}}{e^r - 1} \tag{5.7}$$

$$A/G = \frac{1}{e^r - 1} - \frac{n}{e^{rn} - 1} \tag{5.8}$$

where n represents the number of years, as before. These factors are denoted $[F/P, r\%, n]$, $[P/F, r\%, n]$, etc. (Notice the use of square brackets rather than parentheses, and reference to the nominal interest rate, to indicate continuous compounding.) The continuous compound interest factors can be evaluated directly from the above formulas, or they can be obtained from the tables presented in Appendix C.

**Example 5.2** A savings bank offers long-term savings certificates at $7\frac{1}{2}\%$ per year, compounded continuously. If a 10-year certificate costs $1000, what will be its value at maturity? Compare with the value that would be obtained if the interest were compounded annually rather than continuously.

From (5.2),

$$F = P \times [F/P, r\%, n] = \$1000\, e^{(0.075)(10)} = \$2117.00$$

This problem can also be solved using Appendix C. Since a table is not available for a nominal interest rate of $7\frac{1}{2}\%$ per year, however, it will be necessary to interpolate between the 7% and 8% values.

$$[F/P, 7\%, 10] = 2.0138 \qquad [F/P, 8\%, 10] = 2.2255$$

and

$$[F/P, 7\tfrac{1}{2}\%, 10] \approx 2.0138 + \frac{7.5-7.0}{8.0-7.0}(2.2255-2.0138) = 2.1197$$

The future worth of the savings certificate can now be obtained as

$$F \approx \$1000(2.1197) = \$2119.70$$

If the interest were compounded annually rather than continuously, the future worth would be

$$F = \$1000(1+0.075)^{10} = \$2061.00$$

or $56 less than the amount that is obtained with continuous compounding.

**Example 5.3** A savings account earns interest at the rate of 6% per year, compounded continuously. How much money must initially be placed in the account to provide for twenty end-of-year withdrawals, if the first withdrawal is $1000 and each subsequent withdrawal increases by $200?

The solution may be formulated after (2.11):

$$P = \$1000[P/A, 6\%, 20] + \$200[A/G, 6\%, 20]\,[P/A, 6\%, 20]$$

From (5.7) and (5.8),

$$[P/A, 6\%, 20] = \frac{1-e^{-(0.06)(20)}}{e^{0.06}-1} = 11.3009$$

$$[A/G, 6\%, 20] = \frac{1}{e^{0.06}-1} - \frac{20}{e^{(0.06)(20)}-1} = 7.5514$$

(these values can also be obtained from Appendix C); hence,

$$P = \$1000(11.3009) + \$200(7.5514)(11.3009) = \$28\,368.42$$

If interest is compounded continuously but payments are made p times a year, formulas (5.2) through (5.8) remain valid with $r$ replaced by $r/p$ and with n replaced by np. [These substitutions do not, of course, alter the forms of (5.2) and (5.3).]

**Example 5.4** A person borrows $5000 for 3 years, to be repaid in 36 equal monthly installments. The interest rate is 10% per year, compounded continuously. How much money must be repaid at the end of each month?

Calculating [A/P, 10%/12, 36] by (5.6), we have

$$A = \$5000\,\frac{e^{0.10/12}-1}{1-e^{-0.30}} = \$161.43$$

**Example 5.5** A bank offers its customers a Christmas Club account, in which they deposit $12.61 a week for 39 weeks, starting in mid-February. At the end of 39 weeks (mid-November), each customer will have accumulated $500, which can be withdrawn to pay for gifts and other seasonal expenses. What is the nominal interest rate, assuming continuous compounding?

We know that F/A = $500/$12.61 = 39.6511. Let us attempt to choose an interest rate that will yield this value when substituted into (5.4), which in this case takes the form

$$F/A = \frac{e^{rn}-1}{e^{r/p}-1}$$

with n = 0.75 year and p = 52 payment periods per year:

$$\text{for } r = 3\% \qquad F/A = \frac{e^{(0.03)(0.75)} - 1}{e^{0.03/52} - 1} = 39.4308$$

$$\text{for } r = 4\% \qquad F/A = \frac{e^{(0.04)(0.75)} - 1}{e^{0.04/52} - 1} = 39.5757$$

$$\text{for } r = 5\% \qquad F/A = \frac{e^{(0.05)(0.75)} - 1}{e^{0.05/52} - 1} = 39.7214$$

The desired value, $F/A = 39.6511$, lies somewhere between the 4% and the 5% values. Thus, using linear interpolation,

$$r \approx 4 + \frac{39.6511 - 39.5757}{39.7214 - 39.5757}(5-4) = 4.52\%$$

## 5.3 CONTINUOUS PAYMENTS

For p payments of $A$ per year and continuous compounding, we have, as in Example 5.5,

$$F = A\frac{e^{rn} - 1}{e^{r/p} - 1} = \frac{\bar{A}(e^{rn} - 1)}{p(e^{r/p} - 1)}$$

where $\bar{A} \equiv A/p^{-1}$ is the average rate of payment over a payment period. With $\bar{A}$ held fixed, the denominator above approaches $r$ as $p \to \infty$; and we obtain for ***continuous payments at rate*** $\bar{A}$ (dollars per unit time)

$$F/\bar{A} = \frac{e^{rn} - 1}{r} \tag{5.9}$$

In like manner, we find:

$$\bar{A}/F = \frac{r}{e^{rn} - 1} \tag{5.10}$$

$$\bar{A}/P = \frac{re^{rn}}{e^{rn} - 1} \tag{5.11}$$

$$P/\bar{A} = \frac{e^{rn} - 1}{re^{rn}} \tag{5.12}$$

The notation $[F/\bar{A}, r\%, n]$, etc., is used for these factors.

**Example 5.6** At what rate must funds be continuously added to a savings account in order to accumulate $10 000 in 15 years, if interest is paid at 5% per year, compounded continuously?

By (5.10),

$$\bar{A} = \$10\,000\,\frac{0.05}{e^{(0.05)(15)} - 1} = \$447.63 \text{ per year}$$

that is, $447.63 must flow uniformly into the account each year.

It is interesting to compare this result to a series of uniform, end-of-year payments, with interest compounded continuously as above. The amount of each such payment is given by (5.5) as

$$A = \$10\,000[A/F, 5\%, 15] = \$10\,000(0.0459) = \$459 \text{ per year}$$

Thus, an additional $11.37 would be required each year if the payments were made annually rather than continuously.

It is convenient to determine the end-of-year payment $A$ equivalent to continuous payments at rate $\bar{A}$, under continuous compounding. If (5.4) and (5.9) are to give the same value of $F$,

$$F = A\frac{e^{rn} - 1}{e^{r} - 1} = \bar{A}\frac{e^{rn} - 1}{r}$$

and so

$$A/\bar{A} = \frac{e^{r} - 1}{r} \tag{5.13}$$

This factor, written symbolically as $[A/\bar{A}, r\%]$, is tabulated in Appendix D.

**Example** 5.7 Rework Example 5.6, using the concept of the equivalent yearly payment.

The solution can be formulated as

$$\bar{A} = F[A/F, 5\%, 15]\,[\bar{A}/A, 5\%] = \frac{F\,[A/F, 5\%, 15]}{[A/\bar{A}, 5\%]}$$

From Appendixes C and D, $[A/F, 5\%, 15] = 0.0459$ and $[A/\bar{A}, 5\%] = 1.0254$. Thus,

$$\bar{A} = \frac{(\$10\,000)(0.0459)}{1.0254} = \$447.63 \text{ per year}$$

## Solved Problems

**5.1** What effective annual interest rate corresponds to a nominal interest rate of 10% per year, compounded continuously?

$$i = e^{0.10} - 1 = 0.105171 \approx 10.52\%$$

(This result could also have been obtained from Appendix B.)

**5.2** Determine the nominal interest rate corresponding to an effective interest rate of 10% per year, compounded continuously.

Solving (5.1) by natural logarithms,

$$r = \ln(1 + i) = \ln 1.10 = 0.0953$$

or 9.53%.

**5.3** How much money must be deposited in a savings account so that \$5500 can be withdrawn 12 years hence, if the interest rate is 9% per year, compounded continuously, and if all the interest is allowed to accumulate? Compare the answer with the result obtained for annual compounding, in Problem 2.2.

$$P = \$5500[P/F, 9\%, 12] = \$5500[F/P, 9\%, 12]^{-1} = \$5500(2.9447)^{-1} = \$1867.76$$

Comparing with Problem 2.2, we see that a savings of \$87.66 is realized by continuous compounding.

**5.4** How much money must be deposited at the end of each year in a savings account that pays 9% per year, compounded continuously, to have a total of \$10 000 at the end of 14 years? Compare the answer with the result obtained for annual compounding, in Problem 2.6.

$$A = \$10\,000[A/F, 9\%, 14] = \$10\,000[F/A, 9\%, 14]^{-1} = \$10\,000(26.8165)^{-1} = \$372.90$$

If the interest were compounded annually rather than continuously, the yearly deposit would have to be \$11.43 greater.

**5.5** Mr. Smith is planning his retirement. He has decided that he will need \$12 000 per year to live on, in addition to his other retirement income from Social Security and a private pension plan. How much money should he plan to have in the bank at the start of his retirement, if the bank pays 10% per year, compounded continuously, and if Mr. Smith wants to make 12 annual withdrawals of \$12 000 each?

$$P = A\,[P/A, 10\%, 12] = A\,[A/P, 10\%, 12]^{-1} = (\$12\,000)(0.15050)^{-1} = \$79\,734.22$$

In Problem 2.9 we saw that the required amount of money would be \$81 766.15 if the interest were compounded annually rather than continuously. Hence, the continuous compounding results in a savings of over \$2000.

**5.6** Ms. Brown deposits \$1000 in the bank at the end of the first year, \$1200 at the end of the second year, etc., continuing to increase the amount by \$200 a year, for 20 years. If the bank pays 7% per year, compounded continuously, how much money will have accumulated at the end of 20 years?

Writing $A' = \$1000 + A$, we have, from Section 2.7,

$$F = A'[F/A', 7\%, 20] = \{\$1000 + G[A/G, 7\%, 20]\}[F/A', 7\%, 20]$$

Substituting $G = \$200$ and using Appendix C,

$$F = [\$1000 + \$200(7.2453)](42.1359) = \$103\,193.35$$

**5.7** Calculate the factors $[F/A, 7\%, 20]$ and $[A/G, 7\%, 20]$ used in Problem 5.6.

By (5.4),

$$[F/A, 7\%, 20] = \frac{e^{(0.07)(20)} - 1}{e^{0.07} - 1} = \frac{4.055200 - 1}{1.072508 - 1} = 42.1359$$

and, by (5.8),

$$[A/G, 7\%, 20] = \frac{1}{e^{0.07} - 1} - \frac{20}{e^{(0.07)(20)} - 1} = \frac{1}{1.072508 - 1} - \frac{20}{4.055200 - 1} = 7.2454$$

**5.8** Mrs. Carter deposits \$100 in the bank at the end of each month. If the bank pays (a) 6% per year, (b) 7% per year, compounded continuously, how much money will she have accumulated at the end of 5 years? (Compare Problem 4.11.)

(a) The nominal monthly interest rate is $6\%/12 = 0.5\%$. There will be a total of $5 \times 12 = 60$ monthly payments. Hence,

$$F = \$100[F/A, 0.5\%, 60]$$

From Appendix C, $[F/A, 0.5\%, 60] = 69.7970$; therefore,

$$F = \$100(69.7970) = \$6979.70$$

(b) The nominal monthly interest rate is $7\%/12 = 0.583333\%$. As a tabulated value of $[F/A, 0.583333\%, 60]$ is not available, we interpolate linearly between $[F/A, 0.5\%, 60]$ and $[F/A, 0.75\%, 60]$:

$$[F/A, 0.583333\%, 60] \approx 69.7970 + \frac{0.583333 - 0.5}{0.75 - 0.5}(75.4912 - 69.7970) = 71.6951$$

The desired solution is then $F \approx \$100(71.6951) = \$7169.51$.

A more accurate procedure would be to use (5.4), with $r$ replaced by $r/12$:

$$F = \$100\frac{e^{(0.07)(5)} - 1}{e^{0.07/12} - 1} = \$100\frac{1.419068 - 1}{1.005850 - 1} = \$7163.56$$

**5.9** In Problem 5.8, suppose Mrs. Carter deposits \$100 a month during the first year, \$110 a month during the second year, \$120 a month during the third year, etc. How much will have accumulated at the end of 5 years if the interest rate is 6% per year, compounded continuously? (Compare Problem 4.12.)

Proceeding as in Problem 4.12, we have:

$$\begin{aligned} F = \$100[F/A, 0.5\%, 12][F/P, 6\%, 4] &+ \$110[F/A, 0.5\%, 12][F/P, 6\%, 3] \\ &+ \$120[F/A, 0.5\%, 12][F/P, 6\%, 2] + \$130[F/A, 0.5\%, 12][F/P, 6\%, 1] \\ &+ \$140[F/A, 0.5\%, 12] \end{aligned}$$

The required numerical values can be obtained from Appendix C. Thus,

$$F = [\$100(1.2712) + \$110(1.1972) + \$120(1.1275) + \$130(1.0618) + \$140](12.3364)$$
$$= (\$672.146)(12.3364) = \$8291.86$$

A neater solution procedure (which might also have been applied in Problem 4.12) is to consider the deposits as constituting a five-year gradient series, with

$$A_0 = \$100[F/A, 0.5\%, 12] \qquad \text{and} \qquad G = \$10[F/A, 0.5\%, 12]$$

Thus, as in Problem 5.6,

$$F = \{\$100[F/A, 0.5\%, 12] + \$10[F/A, 0.5\%, 12]\,[A/G, 6\%, 5]\}\,[F/A', 6\%, 5]$$
$$= \{\$100 + \$10[A/G, 6\%, 5]\}\,[F/A, 0.5\%, 12]\,[F/A', 6\%, 5]$$
$$= [\$100 + \$10(1.8802)](12.3364)(5.6578) = \$8291.87$$

**5.10** Suppose that $2000 is deposited each year, on a continuous basis, into a savings account that pays 6% per year, compounded continuously. How much money will have accumulated after 12 years?

With $\bar{A} = \$2000$ per year, (5.9) gives

$$F = \bar{A}\,\frac{e^{rn} - 1}{r} = \$2000\,\frac{e^{(0.06)(12)} - 1}{0.06} = \$2000(17.5739) = \$35\,147.77$$

Alternatively, using the tabular values in Appendixes C and D,

$$F = \$2000[A/\bar{A}, 6\%]\,[F/A, 6\%, 12] = \$2000(1.030609)(17.0519) = \$35\,147.68$$

## Supplementary Problems

**5.11** What effective annual interest rate corresponds to a nominal interest rate of 15% per year, compounded continuously? *Ans.* 16.18%

**5.12** What nominal interest rate corresponds to an effective interest rate of 12% per year, compounded continuously? *Ans.* 11.33%

**5.13** Determine the effective annual interest rate corresponding to a nominal interest rate of $8\frac{1}{2}\%$ per year, if the interest is compounded (*a*) quarterly, (*b*) monthly, (*c*) daily, (*d*) continuously.
*Ans.* (*a*) 8.77%; (*b*) 8.84%; (*c*) 8.871%; (*d*) 8.872%

**5.14** An investment plan pays 15% per year, compounded continuously. How much would have to be invested at the end of each year so that $40 000 will be accumulated by the end of 10 years? Compare with the result obtained for annual compounding (Problem 2.20). *Ans.* $1859.26

**5.15** Suppose that $2000 is invested now, $2500 is invested two years from now, and $1200 is invested four years from now, all at 8% per year, compounded continuously. What will be the total amount 10 years from now? *Ans.* $11 131.57

**5.16** Repeat Problem 2.25 for continuous compounding. *Ans.* $334.23

**5.17** Repeat Problem 2.27(*a*) for continuous compounding. *Ans.* $13 173.00

**5.18** Repeat Problem 2.29(*a*) for continuous compounding. *Ans.* $69 642.25

**5.19** Repeat Problem 2.30 for continuous compounding. *Ans.* (*a*) $9476.60; (*b*) $10 226.83

5.20 Repeat Problem 2.31(*a*) for continuous compounding. ***Ans.*** ***$56 118.82***

5.21 Repeat Problem 2.32(*a*) for continuous compounding. ***Ans.*** ***$167 884.49***

5.22 Repeat Problem *2.32* for continuous compounding at $8\frac{1}{2}\%$. How significant is the $\frac{1}{2}\%$ difference vis-a-vis Problem 5.21? ***Ans.*** ***$177 478.94***

5.23 The cost of maintaining a new car is estimated to be ***$75*** the first year and to increase by ***$12*** each year thereafter. How much money should be set aside for maintenance, if the car is to be kept ***6*** years and if the money which is set aside earns interest at the rate of ***5%*** per year, compounded continuously? (Compare Problem ***4.45.)*** ***Ans.*** ***$521.95***

5.24 Rework Problem ***4.46*** for continuous compounding. ***Ans.*** ***$1001.16***

5.25 Find the monthly payment on a 30-year, ***$100 000*** mortgage loan, where the interest rate is ***12%*** per year, compounded continuously. ***Ans.*** ***$1033.25*** [cf. Problem 4.28(*b*)]

5.26 Repeat Problem ***4.30*** for continuous compounding. ***Ans.*** ***$154 001.91***

5.27 Repeat Problem ***4.31*** for continuous compounding. ***Ans.*** ***$26 317.24***

5.28 Repeat Problem ***4.32*** for continuous compounding. ***Ans.*** ***$171.93***

5.29 Mr. Smith plans to purchase a new ***$10 000*** automobile. He wants to borrow all the money for the car, and repay it in equal monthly installments over a 4-year period. The nominal interest rate is ***11%*** per year, compounded continuously. What will be Mr. Smith's monthly payment? ***Ans.*** ***$258.70***

5.30 Repeat Problem 4.33 for continuous compounding. ***Ans.*** ***$143.27***

5.31 **A** savings bank offers ***$1000*** certificates of deposit. Each certificate can be redeemed for ***$2000*** after $8\frac{1}{2}$ years. What are ***(a)*** the nominal, ***(b)*** the effective, annual interest rate, if the interest is compounded continuously? ***Ans.*** ***(a) 8.155%; (b) 8.496%***

5.32 A savings account earns interest at the rate of $6\frac{3}{4}\%$ per year, compounded continuously. How much money must initially be placed in the account to provide for ***15*** end-of-year withdrawals if the first withdrawal is ***$2000*** and each subsequent withdrawal increases by $350? ***Ans.*** ***$36 620.17***

5.33 Repeat Problem ***4.48*** for continuous compounding. ***Ans.*** ***6.90%***

5.34 Repeat Problem ***4.49*** for continuous compounding. ***Ans.*** ***$13 953.93***

5.35 Repeat Problem ***4.50*** for continuous compounding. ***Ans.*** ***$14 423.37***

5.36 Repeat Problem ***4.51*** for continuous compounding. ***Ans.*** ***$14 163.24***

5.37 **A** savings account pays $5\frac{1}{2}\%$ per year, compounded continuously. How much money must be deposited at the end of each month in order to accumulate ***$10 000*** at the end of ***7*** years? ***Ans.*** ***$97.82***

5.38 Repeat Problem ***4.52*** for continuous compounding. ***Ans.*** ***$139.21***

5.39 Repeat Problem ***4.53*** for continuous compounding. ***Ans.*** ***$9586.27***

5.40 Repeat Problem ***4.54*** for continuous compounding. ***Ans.*** ***$10 867.03***

5.41 Repeat Problem ***4.54*** for continuous compounding at $10\frac{3}{4}\%$ per annum. ***Ans.*** ***$11 094.03***

5.42 Repeat Problem ***4.57*** for continuous compounding. ***Ans.*** ***$65 824.16***

**5.43** Repeat Problem 4.57 for continuous compounding at $6\frac{1}{2}\%$ per year. *Ans.* $64 878.08

**5.44** An engineer borrows $10 000 to buy a personal computer. He must repay $218.94 a month for 5 years. What is the nominal annual interest rate, based upon continuous compounding? *Ans.* 11%

**5.45** A consulting firm has a continuous cash inflow of $5 million a year. If this money is accumulated in an account that earns (a) 15% per year, (b) 13.5% per year, compounded continuously, how much money will have accumulated after 7 years? *Ans.* (a) $61 921 704; (b) $58 252 347

**5.46** A company has set aside $10 million to promote a new product. The money is to be spent continuously over a 3-year period (during which it is assumed that the sales of the product will offset expenses). If the $10 million is placed in an account that earns (a) 12% per year, (b) 10.75% per year, compounded continuously, what is the maximum rate at which money can be withdrawn during the 3-year period? *Ans.* (a) $3 969 256 per year; (b) $3 899 674 per year

**5.47** In Problem 5.46, how much money must be placed in the account if $4 million is to be spent on a continuous basis each year, and the interest rate is 10% per year, compounded continuously? *Ans.* $10 367 271

# Chapter 6

# Equivalence

## 6.1 ECONOMIC EQUIVALENCE

In economic analysis, "equivalence" means "the state of being equal in value." The concept is primarily applied in the comparison of different cash flows. As we know from earlier chapters, money changes value with time; therefore, one of the main factors when considering equivalence is to determine at which point(s) in time the money transactions occur. A second factor is the specific amounts of money involved in the transactions. Finally, the interest rate at which the equivalence is evaluated must also be considered.

**Example 6.1** Bob, an engineering student, has just received his salary for a summer job. After living expenses and entertainment, he has left *\$1000,* which he plans to save for a down payment on a new car. His father wants to borrow Bob's *\$1000,* and promises to return *\$1060* one year from now. According to his father, that is what Bob would receive if he put the money in his bank savings account, which pays an effective annual interest rate of 6%. What should Bob do?

If Bob's only alternatives are lending the money or depositing it in his current savings account, both courses of action are indeed equivalent. That is, either would provide Bob, one year from now, with *\$1060* in return for his decision to forego using his *\$1000* today. Given this equivalence, Bob's decision would be based on factors external to engineering economics (e.g., the degree to which he trusts his father).

However, if Bob had a different savings option—say, a savings certificate with a guaranteed 9% annual yield—the equivalent value of his assets one year from now would be *\$1090.* In this case, the lending and savings alternatives are no longer equivalent, and Bob has the problem of explaining this to his father.

In Example 6.1, $(F/P, 6\%, 1)$ served as the *equivalencing factor.* In general, all the compounding and discounting factors presented in earlier chapters are equivalencing factors.

Equivalence is not always directly apparent. Cash flows that have very different structures (i.e., different amounts being transacted at different points in time) may be equivalent at a certain interest rate.

**Example 6.2** A company which produces and markets microcomputers has just introduced a new line which is expected to sell for *\$10000* per system. Owing to market conditions, the company is being forced to offer financial incentives to potential customers. The company has decided to charge an interest rate of 10%, compounded yearly, and to give customers three options.

***Option 1:*** Pay in four equal yearly installments of

$$A = \$10\,000(A/P, 10\%, 4) = \$10\,000(0.3155) = \$3155$$

***Option 2:*** Pay the interest each year, and the principal (and interest) at the end of the fourth year. This means paying *\$1000 (\$10 000 × 0.10)* at the end of years *1,* 2, and *3,* and *\$11 000* at the end of year *4.*

***Option 3:*** Make a single payment of

$$F = \$10\,000(F/P, 10\%, 4) = \$10\,000(1.4641) = \$14\,641$$

at the end of year *4.*

Which option is best for a customer? for the company?

As summarized in Table *6-1,* the three payment plans offered a customer are quite different in structure. However, if *10%* is the "appropriate" interest rate for his economic evaluations, all three plans are equivalent: each provides him with a microcomputer worth *\$10 000* and gives him *4* years to repay at a *10%* interest rate, compounded yearly. From the company's point of view, similar reasoning applies: at 10% interest, all plans are equivalent because all result in the sale of a *\$10 000* item, and the money is recovered over a 4-year period.

**Table 6-1**

| End of Year | Payment | | |
|---|---|---|---|
| | Option 1 | Option 2 | Option 3 |
| 1 | $3155 | $1000 | $ 0 |
| 2 | 3155 | 1 000 | 0 |
| 3 | 3155 | 1 000 | 0 |
| 4 | 3155 | 11 000 | 14 641 |

Notice that different cash flows are equivalent if they have the same value at some point in time.

**Example 6.3** Are the financing plans of Example 6.2 still equivalent if the evaluation is made at the end of year 4?

Yes; at the end of year 4, all plans have an equivalent value of $14641 (up to roundoff errors) when the same interest rate (10%) is used to make the evaluations.

***Option 1 (equal payments):***

$$F = \$3155(F/A, 10\%, 4) = \$3155(4.641) = \$14\,642$$

***Option 2 (amortization of interest):***

$$F = \$1000(F/A, 10\%, 4) + \$10\,000 = \$14\,641$$

More generally, we can say that options 1 and 3 must be equivalent at the end of year 4, since they are obviously equivalent at the end of year 0; and that options 2 and 3 are equivalent, on the basis of the relation

$$(F/P, i\%, n) = i\,(F/A, i\%, n) + 1$$

derived in Problem 3.3(*a*). Thus, all three options are equivalent, to the customer and to the company, ***provided the two parties use the same interest rate.***

**Example 6.4** The company in Example 6.2 still uses interest rate $i = 10\%$, and therefore still offers the same three payment plans. The customer, however, calculates interest at rate $i'$, so that the values (costs) to him of the three options at the end of year 0 are as given in Table 6-2; for instance, for $i' = 12\%$, the value of option 2 is

$$P = \$1000(P/A, 12\%, 4) + \$10\,000(P/F, 12\%, 4)$$
$$= \$1000(3.0374) + \$10\,000(0.6355) = \$9392$$

**Table 6-2**

| Customer's Rate, $i'$ | Present Value | | |
|---|---|---|---|
| | Option 1 | Option 2 | Option 3 |
| 8% | $10 450 | $10 662 | $10 760 |
| 10% | 10 000 | 10 000 | 10 000 |
| 12% | 9 583 | 9 392 | 9 304 |

Note that when $i' \neq 10\%$, the options are no longer equivalent to the customer, and, more important, different interest rates may lead to different decisions. Consider, for example, a customer who has the money ($10000) on hand, but who knows that he can put that money in a savings account which pays 12% effective interest, compounded yearly. The best strategy for this customer would be to take option 3 and pay $14 641 four years from now. Since this amount is equivalent to only $9304 at the rate he has saved his money, he would end up with a net saving of $10 000 − $9304 = $696 (year 0 money). On the other hand, if a customer is used to paying only 8% for loans, his best alternative (assuming he cannot pay cash or borrow at that rate to buy the computer) is option 1, the least costly at the interest rate he normally uses to make his economic evaluations.

## 6.2 THE COST OF CAPITAL

From Example 6.4 it is seen that the relative evaluation of cash flows depends critically on the "appropriate" or "pertinent" interest rate used in the calculations. Unfortunately, the interest rate that determines the time value of money is not usually known, nor is it easy to determine. It stands to reason, though, that if money is to be invested in a project, the project's cash flow equivalent value should be calculated at an interest rate that exceeds the rate incurred in raising the initial capital. The extra percentage points are justified in terms of risks associated with the specific project and with long-term commitment of funds, and in terms of a profit margin required to get involved in the economic activity. Thus, a mining company considering diversification into plastics would use a higher interest rate to evaluate such a project than it would use for a new mining project, because of the risk of entering a new venture with unknown market factors and for which no experience is available.

There are several means for a company to raise money for a project. It may borrow from a bank at a specified interest rate; it may reinvest profits from other projects instead of distributing them to the owners or shareholders; it may sell stock, thereby increasing the number of owners (and decreasing the equity of each stockholder); and it may borrow from the public through the issue of bonds. Almost always, a combination of methods is employed, and one way to measure the cost of capital is to calculate a weighted average of the costs of funds acquired from all sources.

## 6.3 STOCK VALUATION

Stock represents a share of ownership in a company. Its equivalent-value calculation presents practical difficulties of estimating future dividends and selling price, which are affected not only by the company's performance but also by the overall situation of the economy and of the stock market.

**Example** 6.5 ABC Corporation's stock, which currently sells for ***$50*** per share, has been paying a ***$3*** annual dividend per share and increasing in value at an average rate of 5% per year, over the last ***5*** years. It is expected that the company's stock will maintain this performance over the next ***5*** years. ***(a)*** What is the company's cost of the capital raised through the selling of this stock? ***(b)*** Is this stock a good buy for an investor who expects a ***9%*** return on his investments?

***(a)*** From the company's point of view, it will receive ***$50*** per share today and would have to pay

$$\$50(1.05)^5 = \$63.81$$

to buy it back ***5*** years from now. In addition, it must pay a yearly dividend of ***$3*** per share. The equation of value at time 0 for this cash flow is therefore

$$\$50 = \$3(P/A, i\%, 5) + \$63.81(P/F, i\%, 5)$$

where $i\%$ is the cost of capital for money raised through the sale of this stock, assuming the forecast dividends and selling price are accurate. The solution for $i$ must be found by a trial-and-error approach:

$$\text{for } i = 10\% \qquad \$3(3.7907) + \$63.81(0.62092) = \$50.99$$
$$\text{for } i = 11\% \qquad \$3(3.6958) + \$63.81(0.59345) = \$48.96$$

Thus, by linear interpolation,

$$i \approx 10\% + (11\% - 10\%)\frac{50.99 - 50.00}{50.99 - 48.96} = 10.49\%$$

***(b)*** For a customer who wants to make 9% on his investments, the value of his expected receipts from this share is given by

$$P = \$3(P/A, 9\%, 5) + \$63.81(P/F, 9\%, 5) = \$3(3.8896) + \$63.81(0.64993) = \$53.14$$

Since the expected value of the share exceeds the asked price, it is expedient for him to buy it; if he does so, he should not only recover his ***$50*** investment and the 9% he expects yearly, but he should gain an extra ***$3.14*** (year 0 money) in the transaction.

Alternatively, since we infer from *(a)* that the company expects the stock to yield 10.49% to an investor, the investor ought to buy it, if he requires only 9% and if he agrees with the company as to the stock's future performance.

## 6.4 BOND VALUATION

***A bond*** is an economic instrument which has a ***face value*** guaranteed to be paid to the bondholder by the issuing company when the instrument reaches maturity. In addition, the bondholder usually receives periodic dividends at a specified interest rate. Bonds are transacted on the market, and their value depends on the size and timing of the dividends, the duration before maturity, and the rate of return desired by the bond purchaser. The company's cost of the capital raised through bonds will depend on their acceptability to the public.

**Example** 6.6 ABC Corporation has decided to sell $1000 bonds which will pay semiannual dividends of $20 (2% per period) and will mature in 5 years. The bonds are sold at $830, but after brokers' fees and other expenses the company ends up receiving $760. *(a)* What is the company's cost of the capital raised through the sale of these bonds? (b) Is the bond a good buy for an investor who expects a 9% return on his investments?

*(a)* From the company's point of view, it will receive $760 per bond today and will have to pay $1000 (the face value) 5 years hence, plus a $20 semiannual dividend. The equation of value at time 0 for this cash flow is

$$\$760 = \$20(P/A, i\%, 10) + \$1000(P/F, i\%, 10)$$

where 10 periods are used because dividends are paid twice a year and the bond matures in 5 years, and where $i\%$ is the cost of capital, effective per 6-month period. Solving by trial and error:

$$\begin{array}{ll} \textit{for } i = 5\% & \$20(7.7216) + \$1000(0.61392) = \$768.35 \\ \textit{for } i = 6\% & \$20(8.1108) + \$1000(0.55840) = \$720.62 \end{array}$$

and linear interpolation gives

$$i \approx 5\% + \frac{768.35 - 760.00}{768.35 - 720.62}(6\% - 5\%) = 5.17\%$$

The company's cost of capital for money raised through the sale of these bonds is, on a yearly basis, given by (4.3) as

$$i = (1.0517)^2 - 1 = 10.62\%$$

*(b)* From the investor's point of view, he will pay $830 today to receive $20 every 6 months and $1000 in 5 years. He expects an effective rate of 9% a year on his investments, or

$$\sqrt{1.09} - 1 = 4.40\%$$

per 6-month period. Hence the equivalent value at time 0 of his expected receipts is

$$\begin{aligned} P &= \$20(P/A, 4.40\%, 10) + \$1000(P/F, 4.40\%, 10) \\ &= \$20\,\frac{1 - (1.0440)^{-10}}{0.0440} + \$1000(1.0440)^{-10} = \$809.12 \end{aligned}$$

This value represents the maximum amount the investor can bid for this bond, if he requires an effective annual yield of 9% on his investments. Since the market value today ($830) exceeds this maximum amount, he should look for another business opportunity which could give him his required 9% return.

Notice that we could ***not*** conclude from *(a)* that the investor could realize 10.62% (>9%); for, in effect, part of that 10.62% goes to the brokers.

## 6.5 MINIMUM ATTRACTIVE RATE OF RETURN

If, as is often the case, the interest rate at which a project should be evaluated is not known, a ***target rate, cut-off rate,*** or ***valuation rate*** will be used. This rate is also called the ***minimum attractive rate of return*** (abbreviated MARR). While dependent on general company policy, the MARR may also be project specific, and will normally increase with the risk attending the project. It will certainly be higher than the cost of raising capital for the project, estimated as described in Section 6.2.

**Example 6.7** In order to finance a $100 000 project, ABC Corporation has decided to raise $20 000 through the sale of stock, as described in Example 6.5, and $30 000 through the issuance of bonds, as described in Example 6.6. For the balance, $10000 will be borrowed from a bank at an annual interest rate of 12% and $40 000 will be reinvested from last year's profits. (a) What is the project's cost of capital? (b) What MARR should be used to evaluate this project?

(a) For the four sources of capital, we have:

$$\textit{stock} \qquad \text{weight} = \frac{20\,000}{100\,000} \qquad \text{cost} = 10.49\%$$

$$\textit{bonds} \qquad \text{weight} = \frac{30\,000}{100\,000} \qquad \text{cost} = 10.62\%$$

$$\textit{loan} \qquad \text{weight} = \frac{10\,000}{100\,000} \qquad \text{cost} = 12.00\%$$

$$\textit{reinvestment} \qquad \text{weight} = \frac{40\,000}{100\,000} \qquad \text{cost} = 10.49\%$$

and so

$$\text{weighted average cost} = \sum_{\substack{\text{all}\\ \text{sources}}} (\text{source weight})\,(\text{source cost})$$

$$= (0.2)\,(10.49) + (0.3)\,(10.62) + (0.1)\,(12.00) + (0.4)\,(10.49) = 10.68\%$$

Note that the cost of capital for reinvested profits was assumed to be equal to the cost of common stock: this is a minimal value, based on the consideration that stockholders are being denied dividends from the retained earnings.

(b) The MARR for this project must exceed 10.68%.

The MARR will be treated in further detail in Chapter 9.

## 6.6 FAIR MARKET VALUE

The concept of equivalence, as applied in the foregoing examples, may be used to determine the actual cost of a loan, the maximum amount a person or company can bid on a desired property or equipment, and, in general, in the determination of the "fair market value" of an asset.

**Example 6.8** An engineering firm has turned to Friendly Shark, Inc., to borrow $30000 needed for a short-term (2-year) project, attracted by an advertisement announcing an interest rate of 12% per year. Friendly Shark's loan statement indicates the following:

| | |
|---|---|
| Interest: ($30 000) (1% per month) (24 months) = | $ 7 200 |
| Loan | 30 000 |
| Total | $37 200 |
| Monthly installment = $37 200/24 = $1550 | |

What is the actual cost of borrowing money from Friendly Shark, Inc.?

The engineering firm receives $30 000 immediately and must pay back $1550 per month over a 24-month period. The monthly interest rate i which makes these flows equivalent satisfies

$$\$30\,000 = \$1550(P/A, i\%, 24) \qquad \text{or} \qquad (P/A, i\%, 24) = 19.355$$

Now, $(P/A, 1.5\%, 24) = 20.030$ and $(P/A, 2.0\%, 24) = 18.914$. Hence, by interpolation,

$$i \approx 1.5\% + \frac{20.030 - 19.355}{20.030 - 18.914}(2\% - 1.5\%) = 1.80\%$$

or, on an annual basis,

$$i \approx (1.0180)^{12} - 1 = 23.87\%$$

which is rather different from the advertised rate.

**Example 6.9** A house is being advertised for sale by the owner. An investor estimates that the property could be rented out for $600 per month. Taxes and minor maintenance expenses are estimated at $1200 per year. The house has been recently remodeled and the tenant should have to pay all utilities. The investor thinks he could sell the house for $85 000 after 5 years. What is the largest amount that the investor can offer for the property if his MARR is 12%, compounded monthly?

The equivalent value at year 0 of the expected receipts and disbursements is given by

$$P = \$600(P/A, i_m\%, 60) + \$85\,000(P/F, i_y\%, 5) - \$1200(P/A, i_y\%, 5)$$

where $i_m = 1\% \equiv$ effective monthly MARR
$i_y = (1.01)^{12} - 1 = 12.68\%$ = effective yearly MARR

Now, $(P/A, 1\%, 60) = 44.955$ (from Appendix A), and

$$(P/F, 12.68\%, 5) = (1.1268)^{-5} = 0.5505$$

$$(P/A, 12.68\%, 5) = \frac{1-(P/F, 12.68\%, 5)}{0.1268} = 3.5449$$

Therefore,

$$P = \$600(44.955) + \$85\,000(0.5505) - \$1200(3.5449) = \$69\,511.62$$

The investor must buy the house for $69 511.62 (or less) to get an effective rate of 1% per month (or more) on his investment.

## Solved Problems

**6.1** Is the receipt of $4000 annually for 10 years equivalent to the receipt of $5000 annually for 8 years, if the interest rate is 8% per year, compounded annually?

$$P_1 = \$4000(P/A, 8\%, 10) = \$4000(0.14903)^{-1} = \$26\,840.23$$

$$P_2 = \$5000(P/A, 8\%, 8) = \$5000(0.17401)^{-1} = \$28\,733.98$$

The flows are not equivalent; the receipt of $5000 for 8 years gives the larger present value.

**6.2** If the interest rate is 8% per year, compounded annually, what is the equivalent present value of $10 000 (*a*) 1 year from today? (*b*) 5 years from today?

(*a*) $$P = \$10\,000(P/F, 8\%, 1) = \$10\,000(1.0800)^{-1} = \$9259.26$$

(*b*) $$P = \$10\,000(P/F, 85\%, 5) = \$10\,000(1.4693)^{-1} = \$6805.96$$

**6.3** What is the equivalent future value of $1000 annually for the next 9 years, if the interest rate is 8% per year, compounded annually?

$$F = \$1000(F/A, 8\%, 9) = \$1000(12.4876) = \$12\,487.60$$

**6.4** A man can invest $150 000 for 12 years in a business venture and expect to receive $6000 per year in return. If his MARR is 7% per year, compounded annually, would this investment be satisfactory?

The yearly return $A$ necessary to achieve this MARR is given by

$$\$150\,000 = A(P/A, 7\%, 12) \qquad \text{or} \qquad A = \$150\,000(0.12590) = \$18\,885.00 > \$6000$$

The investment would not be satisfactory.

**6.5** What amount of money is equivalent to receiving \$8000 three years from today, if the interest rate is 8% per year, compounded semiannually?

$$P = \frac{F}{\left(1+\frac{r}{m}\right)^{mn}} = \frac{\$8000}{\left(1+\frac{0.08}{2}\right)^{(2)(3)}} = \$6322.52$$

**6.6** A bank pays 6% interest per year, compounded *(a)* quarterly, *(b)* annually. A \$5000 deposit will grow to what amount if left in that bank for 2 years?

*(a)* $$F = \$5000(1.015)^8 = \$5632.46$$

*(b)* $$F = \$5000(1.06)^2 = \$5618.00$$

**6.7** What is the equivalent present value of the following series of payments: \$5000 the first year, \$5500 the second year, and \$6000 the third year? The interest rate is 8%, compounded annually.

$$\begin{aligned} P &= [\$5000 + \$500(A/G, 8\%, 3)]\,(P/A, 8\%, 3) \\ &= [\$5000 + \$500(0.9487)]\,(0.38803)^{-1} = \$14\,108.06 \end{aligned}$$

**6.8** What single amount at the end of the fourth year is equivalent to a uniform annual series of \$3000 per year for 10 years, if the interest rate is 10% per year, compounded annually?

Find the present value of the series and then move it ahead 4 years:

$$F = \$3000(P/A, 10\%, 10)(F/P, 10\%, 4) = \$3000(0.16275)^{-1}(1.4641) = \$26\,988.02$$

**6.9** A series of 10 annual payments of \$2000 is equivalent to two equal payments, one at the end of 15 years and the other at the end of 20 years. The interest rate is 8%, compounded annually. What is the amount of the two equal payments?

$$\begin{aligned} \$2000(P/A, 8\%, 10) &= X\,[(P/F, 8\%, 15) + (P/F, 8\%, 20)] \\ \$2000(0.14903)^{-1} &= X\,[(3.1722)^{-1} + (4.6610)^{-1}] \\ X &= \frac{\$13\,420.12}{0.31524 + 0.21455} = \$25\,331.08 \end{aligned}$$

**6.10** A new corporate bond is being offered in the market for \$930. The bond has a face value of \$1000 and matures in 10 years. The issuing corporation promises to pay \$70 in interest every year. (a) Should an investor requiring an 8% return on investment buy this bond? *(b)* What is the company's cost of the capital raised through this bond issue if the stockbroker's fee is \$15 per bond sold?

*(a)* At the investor's rate, the present value of the bond is

$$\begin{aligned} P &= \$70(P/A, 8\%, 10) + \$1000(P/F, 8\%, 10) \\ &= \$70(6.710) + \$1000(2.1589)^{-1} = \$932.89 > \$930 \end{aligned}$$

The investor should buy the bond.

*(b)* $$\$915 = \$70(P/A, i\%, 10) + \$1000(P/F, i\%, 10)$$

$$\begin{aligned} &\textit{for } i = 8\% \quad && \text{R.H.S.} = \$932.89 \\ &\textit{for } i = 9\% \quad && \text{R.H.S.} = \$70(6.418) + \$1000(0.4224) = \$871.66 \end{aligned}$$

Interpolating, $$i \approx 8\% + \frac{932.89 - 915}{932.89 - 871.66}(9\% - 8\%) = 8.29\%$$

The company's cost of capital from this bond issue is about 8.29%, compounded yearly.

**6.11** Is it expedient for Thomas, who requires an 8% return on his investments, to buy an artificial Christmas tree? The tree costs $45 and is expected to last eight years. The alternative is to keep purchasing natural trees, which now cost $8 (Christmas of year 0) and are expected to increase $1 in price each coming year.

The present value at 8% of a stream of payments of $9 one year from now, $10 in two years, etc., up to $16 in eight years, is:

$$P = [\$9 + \$1(A/G, 8\%, 8)]\,(P/A, 8\%, 8) = [\$9 + \$1(3.099)]\,(5.747) = \$69.53$$

This amount, plus this year's natural tree cost ($8), greatly exceeds the artificial tree's price; Thomas should buy the artificial tree.

**6.12** A mine is for sale. A mining 'engineer estimates that, at current production levels, the mine will yield an annual net income of $80000 for 15 years, after which the mineral will be exhausted. If an investor's MARR is 15%, what is the maximum amount he can bid on this property?

$$P = \$80\,000(P/A, 15\%, 15) = \$80\,000(5.847) = \$467\,760$$

**6.13** How long must a temporary warehouse last to be a desirable investment if it costs $16 000 to build, has annual maintenance and operating costs of $360, provides storage space valued at $3600 per year, and if the company MARR is 10%?

The yearly net income is $3600 – $360 = $3240, and so the warehouse must last at least $n$ years, where

$$(A/P, 10\%, n) = \frac{\$3240}{\$16\,000} - 0.2025$$

Solving by Table 3-1, or by scanning the table of $A/P$ in Appendix A, we find that n = 8 years.

**6.14** For what value of X are the following two cash flows equivalent at a 10% interest rate?

| End of Year | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Flow A, $ | −10 000 | 5000 | 5000 | 5000 | 5000 |
| Flow B, $ | $X$ | 3500 | 4500 | 5500 | 6500 |

Equating

$$P_A = -\$10\,000 + \$5000(P/A, 10\%, 4)$$

and

$$P_B = X + [\$3500 + \$1000(A/G, 10\%, 4)]\,(P/A, 10\%, 4)$$

we obtain

$$\begin{aligned} X &= -\$10\,000 + [\$1500 - \$1000(A/G, 10\%, 4)]\,(P/A, 10\%, 4) \\ &= -\$10\,000 + [\$1500 - \$1000(1.3812)]\,(0.31547)^{-1} \\ &= -\$9623 \end{aligned}$$

## Supplementary Problems

**6.15** What series of equal annual payments is economically equivalent to the investment of a present amount of \$5000 for 5 years at 12%, compounded annually? ***Ans.*** \$1387.03

**6.16** What single amount at the end of the fifth year is equivalent to a uniform annual series of \$2000 per year for 10 years, if the interest rate is 10%, compounded annually? ***Ans.*** \$19 791.09

**6.17** A series of 12 annual payments of \$2000 is equivalent to three equal payments, one each at the end of 12 years, 15 years, and 20 years. The interest rate is 10%, compound annually. What is the amount of the three equal payments? ***Ans.*** \$19 284.55

**6.18** What is the equivalent present value of the following series of payments: \$7000 the first year, \$6500 the second year, \$6000 the third year, \$5500 the fourth year, and \$5000 the fifth year? The interest rate is 10%, compounded annually. ***Ans.*** \$23 104.44

**6.19** Is a series of 100 equal quarterly payments of \$800 equivalent to a present amount of \$35000 if the interest rate is 8% per year, compounded quarterly?
***Ans.*** No; it is equivalent to exactly \$34 482.76.

**6.20** Machine X will produce cost savings of \$5000 per year for four years; machine Y will produce cost savings of \$4000 per year for five years. If the interest rate is 10%, compound annually, are these two machines economically equivalent in terms of the present value of their cost savings?
***Ans.*** No: $P_X = \$15\,849.37$ and $P_Y = \$15\,163.00$.

**6.21** Find the uniform annual series of seven payments that would be equivalent to the following gradient series: \$500 initially, with a \$50 increment per year, for a total of seven years. The interest rate is 12%, compound annually. ***Ans.*** \$627.58

**6.22** A woman can invest \$100 000 for 15 years in a bank and expect to receive a yearly return of \$10 000. The woman's objective is to earn 12% per year, compounded annually, on her investments. Is this objective met by the bank plan? ***Ans.*** No; the woman requires a return of \$14 682 per year.

**6.23** Machine A will save \$5000 per year for 6 years; machine B will save \$6000 per year for 5 years. If the interest rate is 10%, compounded annually, do these two machines have equal future values at the end of the sixth year? ***Ans.*** No: $F_A = \$38\,578$ and $F_B = \$36\,630.60 \times 1.10 = \$40\,293.66$.

**6.24** The XYZ Bank advertises it will pay \$3869.70 in cash at the end of 20 years to anyone who deposits \$1000; the ABC Bank states that it pays 10% per year, compounded annually, on all deposits left one year or more. Which bank is paying the higher interest rate, and by how much?
***Ans.*** ABC; XYZ is paying only 7% per year, compounded annually.

**6.25** A series of quarterly payments of \$1000 for 25 years is economically equivalent to what present sum, if the quarterly payments are invested at an annual rate of 8%, compounded quarterly?
***Ans.*** \$43 103.45

**6.26** What series of equal annual payments is economically equivalent to the investment of a present amount of \$5000 for 5 years at 12%, compounded annually? ***Ans.*** \$1387.03

**6.27** A promissory note has outstanding payments of \$650 at the end of each of the next five years. What market price would be paid for this note by an investor who requires a 12% yield on his investments, compounded quarterly? ***Ans.*** \$2311.47

**6.28** A loan of \$750 is to be repaid in 18 equal monthly installments, computed as follows:

| | |
|---|---|
| Loan | \$750 |
| Interest at 1% per month for 18 months | 144 |
| Credit check and processing fee | 60 |
| | \$954 |

Monthly payment: \$954118 = \$53

What (a) nominal and (b) effective annual interest rates are being charged?
***Ans.*** *(a)* 32.15% ; (b) 37.34%

# Chapter 7

# PW, FW, EUAS/EUAC

This chapter treats several valuation methods which are useful in deciding among economic alternatives. Its sequel, Chapter 8, is devoted to techniques that are primarily used for analyzing proposed capital investments.

## 7.1 PRESENT WORTH

The present worth (PW) or present value (PV) of a given series of cash flows is the equivalent value of the cash flows at the end of year 0 (i.e., at the beginning of year 1). For the case of annual compounding,

$$\begin{aligned} \text{PW} &= \text{CF}_0 + \text{CF}_1 (P/F, i\%, 1) + \text{CF}_2 (P/F, i\%, 2) + \cdots + \text{CF}_n (P/F, i\%, n) \\ &\equiv \sum_{j=0}^{n} \text{CF}_j (P/F, i\%, j) \end{aligned} \tag{7.1}$$

Here, $\text{CF}_j$ is the (positive or negative) cash flow for the jth year, $(P/F, i\%, j) = (1+i)^{-j}$, and $n$ is the total number of years.

**Example 7.1** Determine the present worth of the following series of cash flows, based on an interest rate of 12% per year, compounded annually: $0 (end of year 0), $1000 (1), $2000 (2), $3000 (3), $4000 (4), $4000 (5), $4000 (6).

By (*7.1*),

$$\begin{aligned} PW &= \$1000(P/F, 12\%, 1) + \$2000(P/F, 12\%, 2) + \$3000(P/F, 12\%, 3) \\ &\quad + \$4000[(P/F, 12\%, 4) + (P/F, 12\%, 5) + (P/F, 12\%, 6)] \\ &= \$1000(0.8929) + \$2000(0.7972) + \$3000(0.7118) + \$4000(1.7096) = \$11\,460.70 \end{aligned}$$

Alternatively, since the cash flows compose a gradient series followed by a uniform series,

$$\begin{aligned} PW &= [\$1000 + \$1000(A/G, 12\%, 4)](P/A, 12\%, 4) + \$4000(P/A, 12\%, 2)(P/F, 12\%, 4) \\ &= [\$1000 + \$1000(1.3589)](3.0374) + \$4000(1.6901)(0.6355) \\ &= \$7164.92 + \$4296.23 = \$11\,461.15 \end{aligned}$$

Figure *7-1* diagrams this latter solution (which agrees with the former up to roundoff errors).

| *End of Year* | *Cash Flow* | *Present worth* |
|---|---|---|
| 0 | 0 | $7164.92 + $4296.23 = $11 461.15 |
| 1 | $1000 | |
| 2 | 2000 | |
| 3 | 3000 | |
| 4 | 4000 | 6760.40 |
| 5 | 4000 | |
| 6 | 4000 | |

**Fig. 7-1**

Equation (7.1) is often applied to problems involving an initial cash outflow followed by a series of cash inflows; i.e., $CF_0 < 0$ and $CF_j > 0$ $(j > 0)$. In such cases, the PW is renamed the net present worth (NPW) or the net present value (NPV). Clearly the NPW is a monotonically decreasing function of the interest rate i (because, as i increases, the positive flows—and only these—are increasingly discounted). Applications of the NPW will be given in Chapter 8.

## 7.2 FUTURE WORTH

Given a series of cash flows as in Section 7.1, the future worth (FW) of the series is its equivalent value at the end of year n. Assuming annual compounding,

$$\begin{aligned} \mathrm{FW} &= \mathrm{CF}_0\,(F/P, i\%, n) + \mathrm{CF}_1\,(F/P, i\%, n-1) + \mathrm{CF}_2\,(F/P, i\%, n-2) + \cdots + \mathrm{CF}_{n-1}\,(F/P, i\%, 1) + \mathrm{CF}_n \\ &\equiv \sum_{j=0}^{n} \mathrm{CF}_j\,(F/P, i\%, n-j) \qquad (7.2) \end{aligned}$$

From the theory of Chapter 2, the future worth is related to the present worth via

$$\mathrm{FW} = \mathrm{PW} \times (F/P, i\%, n) \qquad (7.3)$$

**Example 7.2** **Determine the future worth of the cash flows given in Example 7.1, based on an interest rate of 12% per year, compounded annually, using (a) (7.2), (b) (7.3).**

(a)

$$\begin{aligned} \mathrm{FW} &= \$1000(F/P, 12\%, 5) + \$2000(F/P, 12\%, 4) + \$3000(F/P, 12\%, 3) \\ &\quad + \$4000[(F/P, 12\%, 2) + (F/P, 12\%, 1) + 1] \\ &= \$1000(1.7623) + \$2000(1.5735) + \$3000(1.4049) + \$4000(3.3744) = \$22\,621.60 \end{aligned}$$

(b)

$$\mathrm{FW} = \$11\,460.70\,(F/P, 12\%, 6) = \$11\,460.70\,(1.9738) = \$22\,621.13$$

The variation of FW with i, in the special case $CF_0 < 0$ and $CF_j > 0$ $(j > 0)$, is a little more complicated than the analogous variation of PW. If $|CF_0|$ is very small in comparison to the positive flows, then the future value of the positive flows—and along with it the FW—will increase as $i$ increases. However, if $|CF_0|$ is sufficiently large (as it will be, in practical applications), FW will be a monotonically decreasing function of i, like PW.

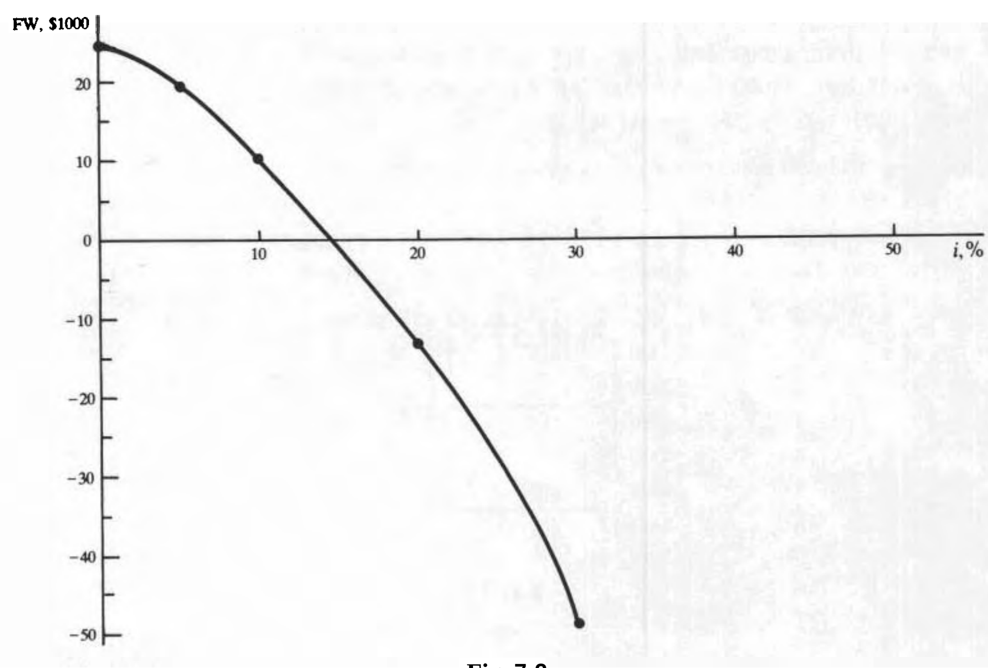


Fig. 7-2

**Example 7.3** Given the series

| End of Year | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Cash Flow, $1000 | −50 | 15 | 15 | 15 | 15 | 15 |

determine the future worth of the series at annual interest rates 0%, 5%, 10%, 20%, 30%, and 50%. Graph your results.

Using the equation

$$FW = -\$50\,000(F/P, i\%, 5) + \$15\,000(F/A, i\%, 5)$$

instead of (*7.2*), we calculate the following points (*i*, *FW*): (0%, $25 000), (5%, $19 070), (10%, $11 051), (20%, −$12 792), (30%, −$49 998), (50%, −$181 870). These points are plotted to give the curve of Fig. 7-2. It is seen that FW rapidly decreases with *i*, becoming zero at $i \approx 15\%$ (more precisely, at $i = 15.26\%$). In view of (*7.3*), *PW* must vanish at this same interest rate.

## 7.3 EQUIVALENT UNIFORM ANNUAL SERIES

The *equivalent uniform annual series* (EUAS) is obtained by converting the equivalent value (at a specified time, usually the present) of a given set of cash flows into a series of uniform annual payments. Thus, if interest is compounded annually, we can write

$$\text{EUAS} = \text{PW} \times (A/P, i\%, n) \tag{7.4}$$

or, substituting (*7.1*),

$$\text{EUAS} = \left[\sum_{j=0}^{n} \text{CF}_j\,(P/F, i\%, j)\right] \times (A/P, i\%, n) \tag{7.5}$$

The EUAS is widely used for analyzing decision alternatives.

**Example 7.4** The *EUAS* is particularly convenient when studying a repeated cycle of cash flows, as exemplified in Fig. 7-3. Applying (7.5) to the first 3-year cycle, we have, assuming $i = 9\%$:

$$\begin{aligned} EUAS &= [-\$600 + \$400(P/F, 9\%, 1) + \$300(P/F, 9\%, 2) + \$500(P/F, 9\%, 3)](A/P, 9\%, 3) \\ &= [-\$600 + \$400(0.9174) + \$300(0.8417) + \$500(0.7722)](0.3951) \\ &= \$160.24 \text{ per year for 3 years} \end{aligned}$$

| ***End of Year*** | ***Disbursements (Cash Outflows)*** | ***Receipts (Cash Inflows)*** |
|---|---|---|
| 0 | $600 | |
| 1 | | $400 |
| 2 | | 300 |
| 3 | 600 | 500 |
| 4 | | 400 |
| 5 | | 300 |
| 6 | 600 | 500 |
| 7 | | 400 |
| 8 | | 300 |
| 9 | 600 | 500 |
| 10 | | 400 |
| 11 | | 300 |
| 12 | | 500 |

**Fig. 7-3**

Because each cycle has this same EUAS (relative to its starting year), the EUAS for the entire series is \$160.24 per year for 12 years.

In the important special case $CF_0 < 0$ and $CF_j > 0$ $(j > 0)$, it can be shown that the EUAS decreases, in almost linear fashion, as i increases. Furthermore, as is shown by *(7.4)*, it becomes zero at the same value of i for which PW (and FW) becomes zero.

If many or all of the cash flows are negative (i.e., are costs), it may be convenient to deal with the negative of the EUAS; we call this quantity the equivalent uniform annual cost (EUAC). It is clear that, in calculating the EUAS or EUAC, we may neglect any constant yearly cash flow (e.g., a fixed annual maintenance charge), and simply add in that constant amount at the end.

## 7.4 CAPITAL RECOVERY

Let us apply the notion of EUAS/EUAC to an asset whose series of cash flows consists of just two terms: an original cost, $P$, and an (actual or estimated) salvage value, SV, at the end of n years. For this series,

$$\text{PW} = -P + \text{SV}\,(P/F, i\%, n)$$

and (7.4) gives

$$\begin{aligned}\text{EUAC} = -\text{EUAS} &= [P - \text{SV}\,(P/F, i\%, n)]\,(A/P, i\%, n)\\ &= P\,(A/P, i\%, n) - \text{SV}\,(A/F, i\%, n)\\ &= (P - \text{SV})(A/P, i\%, n) + i\,\text{SV} \qquad (7.6)\end{aligned}$$

in which (3.5) was used in the last step. Because here the EUAC represents the difference between the annualized cost of the asset and the annualized salvage value, it is retitled the capital recovery (cost) and resymbolized CR.

**Example** 7.5 A machine which costs \$50 000 when new has a 10-year lifetime and a salvage value equal to 10% of its original value. Determine the capital recovery, based upon an interest rate of 8% per year, compounded annually.

From (*7.6*),

$$\begin{aligned}\text{CR} &= (\$50\,000 - \$5000)(A/P, 8\%, 10) + 0.08(\$5000)\\ &= \$45\,000(0.1490) + \$5000(0.08) = \$7105 \text{ per year}\end{aligned}$$

This value represents the annualized net cost of the machine.

Unlike the EUAS/EUAC in general, the CR does not take into account operating or maintenance expenses associated with the asset.

## 7.5 CAPITALIZED EQUIVALENT

Suppose that a given sum of money, P, earns interest at an annual rate i. If the interest is withdrawn at the end of each year but the principal is left intact, then a perpetual series of uniform annual payments will be obtained, the amount of each payment being

$$A = iP \qquad (7.7)$$

[(*7.7*) also follows from (*3.6*).]

Looking at matters the other way round, we call P the capitalized equivalent (CE) of the perpetual annual payments A, and write

$$\text{CE} = \frac{A}{i} \qquad (7.8)$$

## Solved Problems

**7.1** Determine the present worth of the following cash flows, based on an interest rate of *(a)* 10% per year, *(b)* 15% per year, compounded annually. Explain the results.

| End of Year | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Cash Flow, $1000 | 3 | 6 | 4 | 1 | 7 | 5 |

*(a)*

$$\begin{aligned}\text{PW} &= \$3000 + \$6000(P/F, 10\%, 1) + \$4000(P/F, 10\%, 2)\\ &\quad + \$1000(P/F, 10\%, 3) + \$7000(P/F, 10\%, 4) + \$5000(P/F, 10\%, 5)\\ &= \$3000 + \$6000(0.9091) + \$4000(0.8264)\\ &\quad + \$1000(0.7513) + \$7000(0.6830) + \$5000(0.6209)\\ &= \$20\,397.00\end{aligned}$$

*(b)*

$$\begin{aligned}\text{PW} &= \$3000 + \$6000(0.8696) + \$4000(0.7561)\\ &\quad + \$1000(0.6575) + \$7000(0.5718) + \$5000(0.4972)\\ &= \$18\,388.10\end{aligned}$$

The PW at $i = 15\%$ is smaller than the PW at $i = 10\%$; at the higher interest rate, a smaller present sum can generate the given series of ***positive*** payments.

**7.2** Compute the corresponding future worths of the cash flows in Problem 7.1. Explain the results.

Since the present worths are known, it is simplest to use *(7.3)*.

*(a)* $$\text{FW} = (\$20\,397.00)(F/P, 10\%, 5) = (\$20\,397.00)(1.6105) = \$32\,849.37$$

*(b)* $$\text{FW} = (\$18\,388.10)(F/P, 15\%, 5) = (\$18\,388.10)(2.0114) = \$36\,985.82$$

At the higher interest rate, the ***positive*** payments accumulate to a higher future value.

**7.3** Compute the present worth of the following cash flows at *(a)* $i = 6\%$ per year, *(b)* $i = 15\%$ per year, compounded annually. Explain the results.

| End of Year | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Cash Flow, $1000 | −40 | 12 | 12 | 12 | 12 |

*(a)* $$\text{PW} = -\$40\,000 + \$12\,000(P/A, 6\%, 4) = -\$40\,000 + \$12\,000(0.28859)^{-1} = +\$1581.48$$

*(b)* $$\text{PW} = -\$40\,000 + \$12\,000(P/A, 15\%, 4) = -\$40\,000 + \$12\,000(0.35027)^{-1} = -\$5740.71$$

In accordance with the discussion in Section *7.1*, the PW declines as the interest rate increases.

**7.4** Compute the future worth and the equivalent uniform annual series value for the cash flows in Problem 7.3. Explain the results.

Use *(7.3)* and (7.4).

*(a)*

$$\begin{aligned}\text{FW} &= (\$1581.38)(F/P, 6\%, 4) = (\$1581.48)(1.2625) = +\$1996.62\\ \text{EUAS} &= (\$1581.48)(A/P, 6\%, 4) = (\$1581.48)(0.28859) = +\$456.40\end{aligned}$$

(*b*)

$$FW = (-\$5740.71)(F/P, 15\%, 4) = (-\$5740.71)(1.7490) = -\$10\,040.50$$
$$EUAS = (-\$5740.71)(A/P, 15\%, 4) = (-\$5740.71)(0.35027) = -\$2010.80$$

**The behavior discussed in Sections 7.2 and 7.3 is exhibited here: both FW and EUAS decrease as $i$ increases.**

**7.5** An engineer is thinking of starting a part-time consulting business next September 5, on his 40th birthday. He expects the business will require an initial cash outlay of \$5000, to come from his savings, and will cost \$500 per year to operate; the business ought to generate \$2000 per year in cash receipts. During the 20 years that he expects to operate the business, he plans to deposit the annual net proceeds in a bank each year, at an interest rate of 8% per year, compounded annually. When he retires, on his 60th birthday, the engineer expects to invest whatever proceeds plus interest he then has from the business in a long-term savings plan that pays 10% per year, compounded annually. What is the maximum amount he could withdraw from the savings plan each year during his retirement and still have the funds last 15 years?

**The net proceeds from the business will be $\$2000 - \$500 = \$1500$ per year. Therefore, at the end of 20 years, the engineer will have**

$$FW = \$1500(F/A, 8\%, 20) - \$5000(F/P, 8\%, 20)$$
$$= \$1500(45.7620) - \$5000(4.6610) = \$45\,338$$

**The maximum annual amount he could withdraw is therefore**

$$A = \$45\,338(A/P, 10\%, 15) = \$45\,338(0.13147) = \$5960.59$$

**7.6** Let $i = 15\%$ per year, compounded annually. Determine the present worth of the following cash flows:

| End of Year | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Cash Flow, \$1000 | −10 | 2 | 2 | 6 | 6 |

**Analyze the last two flows as $\$2000 + \$4000$. Then,**

$$PW = -\$10\,000 + \$2000(P/A, 15\%, 4) + \$4000(P/A, 15\%, 2)(P/F, 15\%, 2)$$
$$= -\$10\,000 + \$2000(0.35027)^{-1} + \$4000(0.61512)^{-1}(1.3225)^{-1} = \$626.93$$

**7.7** Determine the EUAS for the repeated cycle of disbursements and receipts shown in Fig. 7-4, if the interest rate is 10% per year, compounded annually.

$$EUAS = [-\$500 + \$200(P/F, 10\%, 1) + \$150(P/F, 10\%, 2)$$
$$+ \$300(P/F, 10\%, 3) + \$400(P/F, 10\%, 4)]\,(A/P, 10\%, 4)$$
$$= [-\$500 + \$200(1.1000)^{-1} + \$150(1.2100)^{-1} + \$300(1.3310)^{-1} + \$400(1.4641)^{-1}](0.31547)$$
$$= \$96.03 \text{ per year for } 16 \text{ years}$$

**7.8** A machine costs \$40000 to purchase and \$10000 per year to operate. The machine has no salvage value, and a 10-year life. If $i = 10\%$ per year, compounded annually, what is the equivalent uniform annual cost of the machine?

**The EUAC is given by (7.4) or (7.5), with costs counted as positive, together with the fixed operating cost.**

$$EUAC = \$40\,000(A/P, 10\%, 10) + \$10\,000 = \$40\,000(0.16275) + \$10\,000 = \$16\,510$$

| *End of Year* | *Disbursements* | *Receipts* |
|---|---|---|
| 0 | $500 | |
| 1 | | $200 |
| 2 | | 150 |
| 3 | | 300 |
| 4 | 500 | 400 |
| 5 | | 200 |
| 6 | | 150 |
| 7 | | 300 |
| 8 | 500 | 400 |
| 9 | | 200 |
| 10 | | 150 |
| 11 | | 300 |
| 12 | 500 | 400 |
| 13 | | 200 |
| 14 | | 150 |
| 15 | | 300 |
| 16 | | 400 |

**Fig. 7-4**

**7.9** A new bridge with a 100-year life is expected to have an initial cost of $20 million. This bridge must be resurfaced every five years, at a cost of $1 million. The annual inspection and operating costs are estimated to be $50 000. Determine the present-worth cost of the bridge using the capitalized equivalent approach (i.e., take the life of the bridge as infinite). The interest rate is 10% per year, compounded annually.

The present worth of the nonrecurring cost is simply $P_1 = \$20$ million. The recurring $1 million cost is equivalent to

$$A_1 = (\$1\,000\,000)(A/F, 10\%, 5) = (\$1\,000\,000)(6.1051)^{-1} = \$163\,797 \text{ per year}$$

Thus, there are two annual costs, $A_1 = \$163\,797$ and $A_2 = \$50\,000$; their combined capitalized equivalent is

$$\text{CE} = \frac{\$163\,797 + \$50\,000}{0.10} = \$2\,137\,970$$

and the total present-worth cost is

$$P_1 + CE = \$20\,000\,000 + \$2\,137\,970 = \$22\,137\,970$$

**7.10** Determine the approximate size of the annual payment needed to retire $70 000 000 in bonds issued by a city to build a dam. The bonds must be repaid over a 50-year period, and they earn interest at an annual rate of 6%, compounded annually.

To the extent that 50 years may be taken as infinite,

$$A = CE \times i = (\$70\,000\,000)(0.06) = \$4\,200\,000$$

**7.11** A machine that cost $30 000 new has an 8-year life and a salvage value equal to 10% of its original cost. The annual maintenance cost of this machine is $1000 the first year, with an increase of $200 each year hereafter; the annual operating cost is $800 per year. Determine the EUAC of this machine if the interest rate is 10% per year, compounded annually.

The EUAC is the sum of three terms: the CR given by (7.6); the EUAC for the $200 gradient; and the fixed annual cost, $1000 + $800 = $1800.

$$\text{EUAC} = (\$30000 - \$3000)(A/P, 10\%, 8) + (0.10)(\$3000) + \$200(A/G, 10\%, 8) + \$1800$$
$$= \$27\,000(0.18744) + \$300 + \$200(3.0045) + \$1800 = \$7761.78$$

## Supplementary Problems

**7.12** Find the present worth of the machine of Problem 7.8.
*Ans.* PW = −$101 443.93 (the negative value indicates a cash outflow or cost)

**7.13** Rework Problem 7.8, using annual interest rates of (a) 5%, (b) 15%, and (c) 20%, compounded annually. (d) Comment on the results.
*Ans.* (a) EUAC = $15 180; (b) EUAC = $17790; (c) EUAC = $19 541; (d) the EUAC increases with $i$, since all costs are positive and since the value of $(A/P, i\%, 10)$ increases as $i$ increases.

**7.14** A machine costs $30 000 to purchase and $12 000 per year to operate. The machine has a 10-year life and no salvage value. Determine the EUAC of this machine at annual interest rates of (a) 5%, (b) 15%, and (c) 20%, compounded annually. *Ans.* (a) $15 885.00; (b) $17 977.50; (c) $19 155.60

**7.15** A used machine costs $20 000 to purchase. It has an annual maintenance cost of $20 000, a salvage value of $5000, and a 10-year life. If the interest rate is 10% per year, compounded annually, what is the present-worth *cost* of the machine? *Ans.* $140 960.12

**7.16** Rework Problem 7.6 for $i = 8\%$ per year, compounded annually. *Ans.* $2739.71

**7.17** Use (7.2) to determine the future worth of the cash flows in Problem 7.6, for $i = 8\%$ per year, compounded annually. *Ans.* $3727.20

**7.18** Calculate by (7.2) the future worth of the cash flows of Problem 7.6, using (a) $i = 15\%$ per year, (b) $i = 8\%$ per year, compounded annually. *Ans.* (a) $1096.50; (b) $3727.38

**7.19** Compute the future-worth cost of the machine of Problem 7.15. *Ans.* $365 608.25

**7.20** Compute the EUAC of the machine of Problem 7.15, using the result of Problem 7.15.
*Ans.* $22 941.26 per year

**7.21** Compute the capital recovery for the machine of Problem, 7.15, using (7.6). Compare with the answer to Problem 7.20. *Ans.* $2941.25 = $22 941.26 – $20 000

**7.22** Determine the present worth of the following series of cash flows, given $i = 15\%$: $CF_0 = -\$10\,000$, $CF_1 = CF_2 = CF_3 = CF_4 = \$5000$, $CF_5 = -\$2000$, $CF_6 = \$3000$. *Ans.* $6566.15

**7.23** Calculate the present worth of the following cash flows, when $i = 10\%$ per year, compounded annually.

| End of Year | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Cash Flow, $1000 | −10 | 4 | 4 | 4 | 4 |

*Ans.* $2679.49

**7.24** Calculate the present worth of the following cash flows, if the interest rate is 12% per year, compounded annually.

| End of Year | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Cash Flow, $1000 | −15 | 5 | 6 | 2 | 4 |

*Ans.* −$1786.86

**7.25** A machine which costs $100 000 when new has a lifetime of 15 years and a salvage value equal to 20% of its original cost. Determine the capital recovery for this machine, if the interest rate is 10% per year, compounded annually. *Ans.* $12 517.60 per year

**7.26** Repeat Problem 7.25, using a salvage value equal to 5% of the machine's original cost.
*Ans.* $12 989.56 per year

**7.27** Determine the capital recovery for the machine in Problem 7.25, if the interest rate is 15% per year, compounded annually, and total operating and maintenance costs are $2000 per year.
*Ans.* $16681.60 per year

**7.28** Determine the amount of money required to generate an infinite number of annual payments of $5000 each, if the interest rate is 10% per year, *(a)* compounded annually, *(b)* compounded continuously.
*Ans.* *(a)* $50 000; *(b)* $47 528.52

**7.29** Mr. Diamond expects to invest $1000 per year for each of the next 20 years in an investment plan that pays 10% per year, compounded annually. At the end of the 20th year, he expects to withdraw the balance in his investment plan and deposit it in a savings account. This savings account pays 6% per year, compounded *monthly.* Mr. Diamond wants to withdraw a fixed amount from this savings account each month, for a total of five years. How large may this fixed amount be? *Ans.* $1107.13

**7.30** Repeat Problem 7.7 for an interest rate of 15% per year, compounded annually.
*Ans.* $74.71 per year for 16 years

**7.31** Costs of $10 000, $20 000, and $23 000 are incurred at the ends of three successive years. Find the EUAC for $k$ repetitions of the cycle, if *(a)* $i = 10\%$ per year, *(b)* $i = 15\%$ per year, compounded annually.
*Ans.* *(a)* $17 250.55 per year for $3k$ years; *(b)* $21 322.10 per year for $3k$ years

**7.32** A flood-control dam with a 100-year life has an initial cost of $15 million. The gates in the dam must be replaced every five years, at a cost of $2 million. If the interest rate is 8% per year, compounded annually, what is the capitalized equivalent of the annual cost of the dam? *Ans.* $4 261 412

**7.33** What is the total present worth of the dam described in Problem 7.32? *Ans.* $19 261 412

**7.34** Assuming that money earns 10% a year, which would be the better arrangement for leasing an electron microscope: *(1)* paying a deposit of $100 000, to be returned at the end of the lease period; or *(2)* paying $10 000 a year for as long as the device is kept?
*Ans.* For (1), *(7.6)* gives EUAC $= iP = (0.10)(\$100\,000) = \$10\,000$, and so the two plans are equivalent.

# Chapter 8

# Net Present Value, Rate of Return, Payback Period, Benefit-Cost Ratio

This chapter continues the ideas developed in Chapter 7, particularly as they are applied in deciding among alternative capital investments.

## 8.1 NET PRESENT VALUE

The definition, (*7.1*), of the NPV is repeated here:

$$\text{NPV} = -|\text{CF}_0| + \sum_{j=1}^{n} \text{CF}_j\,(P/F, i\%, j) \tag{8.1}$$

in which the notation emphasizes our assumption that the initial cash flow, $\text{CF}_0$, is negative (a capital outlay). No assumption is made concerning the signs of the remaining $\text{CF}_j$, although often these terms will all be positive (revenues). In the special case $\text{CF}_j = A$ $(j = 1, 2, \ldots, n)$, (8.1) becomes, in view of (*3.4*),

$$\text{NPV} = -|\text{CF}_0| + A\,(P/A, i\%, n) \tag{8.2}$$

as it must. Another name for the NPV is the discounted cash *flow*, or DCF.

From (8.1) it is seen that the NPV is positive when and only when the total value of the returns $\text{CF}_j$ (in year 0 dollars) exceeds the amount invested, $|\text{CF}_0|$ (year 0 dollars); that is to say, when and only when the original amount, earning compound interest at rate i for n years, would be insufficient to generate the returns. For a proposed investment to be economically acceptable, the NPV must be positive or, at worst, zero (in which case the investment of $|\text{CF}_0|$ would just suffice to yield the revenues $\text{CF}_j$).

**Example** 8.1 The cash flows associated with a milling machine are $\text{CF}_0 = -\$50\,000$. $\text{CF}_j = \$15\,000$ $(j = 1, \ldots, 5)$. Use *(8.2)* to determine the economic acceptability of this machine at interest rates of (a) 10%, (*b*) 15%, and (c) 20% per year, all compounded annually.

(*a*) $\text{NPV} = -\$50\,000 + \$15\,000(P/A, 10\%, 5) = -\$50\,000 + \$15\,000(0.26380)^{-1} = \$6861.26$

(*b*) $\text{NPV} = -\$50\,000 + \$15\,000(P/A, 15\%, 5) = -\$50\,000 + \$15\,000(0.29832)^{-1} = \$281.58$

(*c*) $\text{NPV} = -\$50\,000 + \$15\,000(P/A, 20\%, 5) = -\$50\,000 + \$15\,000(0.33438)^{-1} = -\$5140.85$

The machine is seen to be an economically acceptable investment when the interest rate is **10%**, and (barely) when the interest rate is **15%**. It is not economically justifiable to buy the machine if the interest rate is *20%*.

## 8.2 RATE OF RETURN

The rate of return (ROR) for a series of cash flows is that particular value, i*, of the interest rate for which the NPV vanishes. Thus, if we plot the NPV as a function of i, using (8.1) or (*8.2*), the curve will cross the i-axis at i*. Alternatively, we could find, by trial and error, i-values for which the NPV is slightly positive and slightly negative, and interpolate linearly between them for i*. If a more accurate approximation for i* is required, the Newton-Raphson iteration method or another numerical technique can be used to solve (8.1) or (8.2) for i, with the left side replaced by zero.

**Example 8.2** Find the ROR for the machine of Example *8.1.*

By linear interpolation between the results of Example 8.1(*b*) and (c):

| $i$ | NPV |
|---|---|
| 15% | \$281.58 |
| $i^*$ | 0 |
| 20% | −\$5140.85 |

$$i^* \approx 15\% + \frac{0 - \$281.58}{-\$5140.85 - \$281.58}(20\% - 15\%) = 15.26\%$$

### Case of a Single Sign-Reversal

From Section 7.1, we know that when $CF_0 < 0$ and $CF_j > 0$ ($j > 0$)—that is, when there is just one reversal of sign in the sequence $CF_0, CF_1, CF_2, \ldots, CF_n$—the NPV is a monotone decreasing function of i, and so i* is uniquely determined. Moreover, at this unique ROR, the FW and EUAS are zero. (Compare Example 8.2 with Example 7.3 and Fig. 7-2.)

### Case of Multiple Sign-Reversals

When the sequence $CF_0, CF_1, CF_2, \ldots, CF_n$ shows more than one reversal of sign, it is possible that NPV = 0 for several values of the interest rate; there could thus be several rates of return.

**Example 8.3** For the series of cash flows

| End of Year | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Cash Flow, \$1000 | −3 | 0 | 6 | 6 | 0 | −10 |

determine the NPV at annual interest rates 0%, 5%, 10%, 20%, 30%, 50%, and 70%. From a graph of the results, find the rate(s) of return.

For the given flows,

$$\text{NPV} = -\$3000 + \$6000(P/A, i\%, 2)(P/F, i\%, 1) - \$10\,000(P/F, i\%, 5)$$

and evaluation at the specified interest rates gives the points

| $i$, % | 0 | 5 | 10 | 20 | 30 | 50 | 70 |
|---|---|---|---|---|---|---|---|
| NPV, \$ | −1000 | −210 | 257 | 620 | 588 | 128 | −407 |

which are plotted in Fig. *8-1.* It is seen that there are two rates of return in this case, $i^* \approx 7\%$ and $i^* \approx 54\%$.

An upper bound on the number of (positive) rates of return may be obtained by deriving from (8.1) the polynomial equation

$$CF_0x^n + CF_1x^{n-1} + \ldots + CF_{n-1}x + CF_n = 0 \qquad (8.3)$$

for $x \equiv 1 + i^*$. According to Descartes' rule of signs, the number of positive real roots $x$ cannot exceed the number of sign changes in the series of *coefficients* $CF_0, CF_1, \ldots, CF_n$. Now, a positive $x$ might correspond to a negative i*; therefore, the number of sign changes is a fortiori an upper limit on the number of i*-values. In particular, if there are no sign changes, there is no ROR for the given flows.

**Example 8.4** There are two sign reversals in the cash flows of Example *8.3,* and two values of $i^*$ were found. In this case, the upper bound is actually attained.

If multiple i*-values exist, it is usually better to abandon the ROR method and instead to. investigate the sign of the NPV for various assumed values of the interest rate.

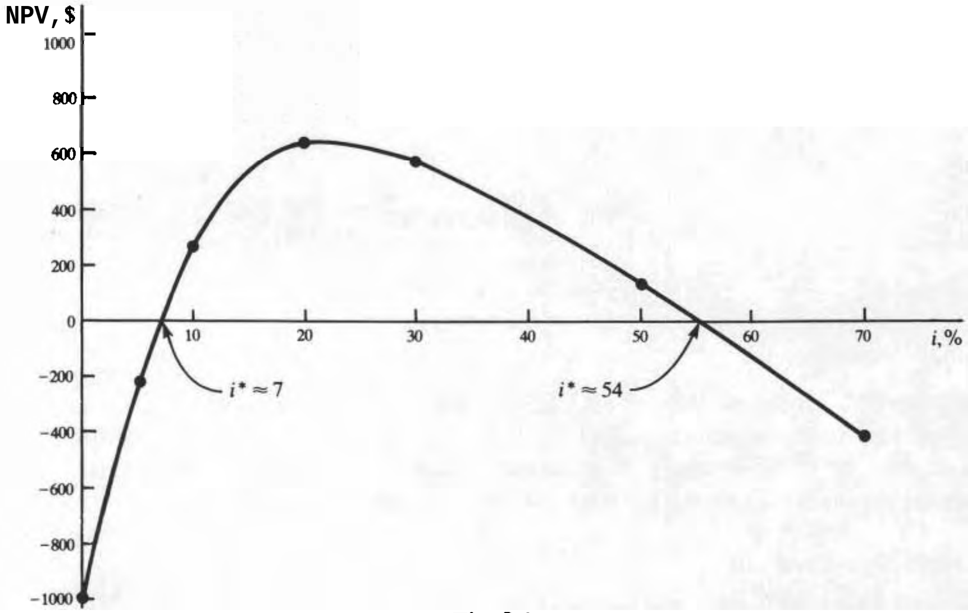


**Fig. 8-1**

## 8.3 PAYBACK PERIOD

The payback period (PBP) is the time required for an initial investment to be recovered, neglecting the time value of money. Thus, if $|CF_0|$ represents the initial investment and $CF_j$ is the net cash inflow for the jth year ($j = 1, 2, \ldots, n$), the payback period satisfies

$$|CF_0| = \sum_{j=1}^{PBP} CF_j \tag{8.4}$$

If the yearly cash inflows are equal, or if an average value is used, then (8.4) simplifies to

$$PBP = \frac{|CF_0|}{YCF} \tag{8.5}$$

where YCF represents the (average) yearly cash inflow.

**Example 8.5** Determine the payback period for a proposed investment as follows:

| End of Year | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Cash Flow, $1000 | –50 | 10 | 12 | 15 | 18 | 20 |

The sum of the first three yearly cash inflows, $37 000, is less than the initial investment, $50 000; but the sum of the first four yearly cash inflows, $55 000, exceeds the initial investment. Hence the payback period will be somewhere between 3 and 4 years. Linear interpolation yields

$$PBP \approx 3 + \frac{\$50\,000 - \$37\,000}{\$55\,000 - \$37\,000}(4-3) = 3.72 \text{ years}$$

Because it ignores the time value of money, the payback method should not be used in place of the other methods discussed above. On the other hand, the payback method is valuable for a secondary analysis, when the NPV or ROR is used as the primary method. As will be further discussed in Chapter 9, there are many practical examples where an investment is sought with a high rate of return and a sufficiently short payback period.

**Example 8.6** Determine the payback period and the net present value for each proposal in Table 8-1, using an interest rate of 10% per year, compounded annually. Which proposal is best?

**Table 8-1**

| End of Year | Cash Flows Proposal A | Proposal B | Proposal C |
|---|---|---|---|
| 0 | −$75 000 | −$75 000 | −$75 000 |
| 1 | 25 000 | 20 000 | 0 |
| 2 | 25 000 | 25 000 | 0 |
| 3 | 25 000 | 30 000 | 0 |
| 4 | 25 000 | 35 000 | $130 000 |

Proposals A and B each have a 3-year payback period; however, proposal A has an NPV of $4248, while proposal B has an NPV of $10 289. Proposal C has an NPV of $13 792, but it has a 3.58-year payback period (assuming the $130 000 to be evenly spread over the fourth year). In summary:

| Proposal | NPV, $ | PBP, years |
|---|---|---|
| A | 4 248 | 3.00 |
| B | 10 289 | 3.00 |
| C | 13 792 | 3.58 |

Since proposal A is inferior to proposal B, it can be eliminated from further consideration. Proposal C is economically superior to proposal B, but its longer payback period might constrain the decision maker to choose proposal B instead (e.g., if the firm were cash-poor and could not afford to wait until the end of the third year to receive any cash inflows). Thus, the choice of the best investment alternative may involve a trade-off among two or more objectives, with the PBP providing important secondary information.

## 8.4 BENEFIT-COST RATIO

The ***benefit-cost ratio* (BCR)** is often used to assess the value of a municipal project in relation to its cost; it is defined as

$$\mathrm{BCR} \equiv \frac{B - D}{C} \tag{8.6}$$

where B represents the equivalent value of the benefits associated with the project, D represents the equivalent value of the disbenefits, and C represents the project's net cost. Similarly, the ***net benefit value*** (NBV) is defined as

$$\mathrm{NBV} \equiv B - D - C \tag{8.7}$$

For a project to be desirable, $\mathrm{BCR} > 1$ or $\mathrm{NBV} > 0$. This rule must be applied with caution, however, since benefit quantification is usually not very precise and since the distinction between disbenefits and costs is somewhat conjectural. The BCR may vary considerably depending on whether the disbenefits are included in the numerator, or are classified as a cost and included in the denominator. If questions arise about the classification of disbenefits, it is better to use the NBV approach, because (8.7) gives the same value irrespective of how the disbenefits are classified.

Either present worth (the NPV), future worth, or the EUAS approach may be used to evaluate B, D, and C, provided the same method be used for all three terms.

**Example 8.7** A large city is located close to a major seaport. It has been proposed that a new superhighway be built between the city and the seaport, running parallel to the present congested, two-lane highway. A group of consulting engineers has estimated that the new highway will provide the following direct benefits: *(1)* additional commerce between the city and the seaport, having a value of $50 million per year; *(2)* future economic growth within the region over a 10-year period, resulting in an increase of $5 million per year in commercial activity, beginning in the second year; *(3)* a reduction in highway accidents, resulting in a direct savings of approximately $0.8 million per year. On the other hand, the following disadvantages or disbenefits are associated with the new highway: (i) the destruction of valuable farmland that currently contributes $1.3 million per year to the regional economy; (ii) a decrease in commercial activity along the present highway, resulting in a loss of $0.7 million per year. Assess the desirability of the proposed superhighway, based on a construction cost of $280 million and a yearly maintenance cost of $1.5 million. Assume a lifetime of 30 years and an interest rate of 7%, compounded annually.

Over the entire 30-year period, the yearly net benefits, $B - D$, are given by the EUAS method as

$$\begin{aligned} B - D &= \$50 + \$5(A/G, 7\%, 10)(P/A, 7\%, 10)(A/P, 7\%, 30) + \$0.8 - (\$1.3 + \$0.7) \\ &= \$60.0 \text{ million} \end{aligned}$$

Similarly, the yearly costs are given by

$$C = \$280(A/P, 7\%, 30) + \$1.5 = \$24.1 \text{ million}$$

and the benefit-cost ratio is

$$\text{BCR} = \frac{\$60.0}{\$24.1} = 2.49$$

Since BCR > 1, the proposed highway is considered to be desirable.

## Solved Problems

**8.1** Compute the net present value (NPV) of the following cash flows:

| End of Year | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Cash Flow, $1000 | −6 | 4 | 2 | −3 | −2 | 3 |

The interest rate is 15% per year, compounded annually.

By (8.1),

$$\begin{aligned} \text{NPV} &= -\$6000 + \$4000(P/F, 15\%, 1) + \$2000(P/F, 15\%, 2) \\ &\quad - \$3000(P/F, 15\%, 3) - \$2000(P/F, 15\%, 4) + \$3000(P/F, 15\%, 5) \\ &= -\$6000 + \$4000(1.1500)^{-1} + \$2000(1.3225)^{-1} \\ &\quad - \$3000(1.5209)^{-1} - \$2000(1.7490)^{-1} + \$3000(2.0114)^{-1} \\ &= -\$2633.99 \end{aligned}$$

**8.2** A new plant to produce steel tubing requires an initial investment of $10 million. It is expected that after three years of operation an additional investment of $5 million will be required; and after six years of operation, another investment of $3 million. Annual operating costs will be $3 million and annual revenues will be $8 million. The life of the plant is 10 years. If the interest rate is 15% per year, compounded annually, what is the NPV of this plant?

The data imply a level cash flow of

$$-\$3\,000\,000 + \$8\,000\,000 = +\$5\,000\,000$$

per year in years 1 through 10, plus flows of $-\$5\,000\,000$ and $-\$3\,000\,000$ in years 3 and 6, respectively. Hence, in units of \$1 million,

$$\begin{aligned} \text{NPV} &= -10 + 5(P/A, 15\%, 10) - 5(P/F, 15\%, 3) - 3(P/F, 15\%, 6) \\ &= -10 + 5(0.19925)^{-1} - 5(1.5209)^{-1} - 3(2.3131)^{-1} \\ &= 10.5096 \end{aligned}$$

or \$10 509 600.

**8.3** The XYZ Company is contemplating the purchase of a new milling machine. The purchase price of the new machine is \$60 000 and its annual operating cost is \$2 675.40. The machine has a life of seven years, and it is expected to generate \$15 000 in revenues in each year of its life, What is the net present value of the investment in this machine if the interest rate is (a) 8% per year, (b) 10% per year, (c) 12% per year, compounded annually? Interpret your results.

In years 1 through 7, the net annual cash flow is

$$\$15\,000 - \$2675.40 = \$12\,324.60$$

(*a*)

$$\begin{aligned} \text{NPV} &= -\$60\,000 + \$12\,324.60(P/A, 8\%, 7) \\ &= -\$60\,000 + \$12\,324.60(0.19207)^{-1} = \$4167.23 \end{aligned}$$

(*b*)

$$\begin{aligned} \text{NPV} &= -\$60\,000 + \$12\,324.60(P/A, 10\%, 7) \\ &= -\$60\,000 + \$12\,324.60(0.20541)^{-1} = \$0 \end{aligned}$$

(*c*)

$$\begin{aligned} \text{NPV} &= -\$60\,000 + \$12\,324.60(P/A, 12\%, 7) \\ &= -\$60\,000 + \$12\,234.60(0.21912)^{-1} = -\$3754.11 \end{aligned}$$

When the interest rate is less than 10%, the present worth of the annual cash flows of \$12 324.60 for 7 years is greater than the \$60 000 investment; hence, the NPV is a positive number. When the interest rate is 10%, the present worth of the annual cash flows is just equal to the \$60000 investment, and NPV = 0. When the interest rate is greater than 10%, the present worth of the annual cash flows is less than the investment, and the NPV is negative. Thus, when the interest rate is above 10%, it would not be economical to purchase the milling machine.

**8.4** Determine the rate of return (ROR) for the machine of Problem 8.3.

By definition, the ROR is the interest rate at which NPV = 0; thus, by Problem 8.3(*b*), ROR = 10%. (For the given cash flows, we know that the ROR is unique.)

**8.5** Find the ROR for cash flows of $-\$50\,000$ in year 0 and $+\$16\,719$ each year in years 1–5.

From (*8.2*),

$$0 = -\$50\,000 + \$16\,719(P/A, i^*\%, 5)$$

$$(A/P, i^*\%, 5) = \frac{\$16\,719}{\$50\,000} = 0.33438$$

Locating the combination n = 5, $A/P = 0.33438$ in Appendix A, we see that $i^* = 20\%$.

**8.6** Solve Problem 8.5 by trial and error, using (8.2).

***Try*** $i = 15\%$

$$\text{NPV} = -\$50\,000 + \$16\,719(P/A, 15\%, 5) = -\$50\,000 + \$16\,719(0.29832)^{-1} = +\$6043.85$$

The given cash flows are such as to make the NPV monotonically decreasing in $i$; hence $i^*$, the value at which the NPV vanishes, must be greater than 15%.

*Try* $i = 25\%$

$$\text{NPV} = -\$50\,000 + \$16\,719(P/A, 25\%, 5) = -\$50\,000 + \$16\,719(0.37185)^{-1} = -\$5038.32$$

Since the NPV now is negative, it must be that $15\% < i^* < 25\%$. Try a rate that is halfway between these two rates:

*Try* $i = 20\%$

$$\text{NPV} = -\$50\,000 + \$16\,719(P/A, 20\%, 5) = -\$50\,000 + \$16\,719(0.33438)^{-1} = \$0$$

Hence, $i^* = 20\%$.

**8.7** Solve Problem 8.5 by linear interpolation between $i = 15\%$ and $i = 25\%$.

Using the NPV-values calculated in Problem 8.6,

$$i^* \approx 15\% + \frac{0 - \$6043.85}{-\$5038.32 - \$6043.85}(25\% - 15\%) = 20.45\%$$

The value 20.45% is slightly in error because we have used *linear* interpolation over a relatively wide range of values (from 15% to 25%), whereas (8.2) is a nonlinear equation.

**8.8** Compute the ROR for the following cash flows:

| End of Year | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Cash Flow, $1000 | −50 | 30 | −1 | 30 |

Descartes' rule tells us to expect no more than three positive values for $i^*$. Adding and subtracting \$31 000 from $CF_2$, we obtain

$$0 = -\$50\,000 + \$30\,000(P/A, i^*\%, 3) - \$31\,000(P/F, i^*\%, 2)$$

As a first approximation, neglect the last term on the right:

$$(A/P, i^*\%, 3) = \frac{\$30\,000}{\$50\,000} = 0.60000$$

From the tables in Appendix A, we see that:

| $i$ | $(A/P, i\%, 3)$ |
|---|---|
| 30% | 0.55063 |
| 40% | 0.62936 |

Hence, for the approximate equation, $30\% < i^* < 40\%$. Restoring the neglected term should pull $i^*$ closer to 30%; hence, try $i = 30\%$:

$$\text{NPV} = -\$50\,000 + \$30\,000(0.55063)^{-1} - \$31\,000(1.6900)^{-1} = -\$13\,860.15$$

Under the previous approximation, $\text{NPV} = -\$31\,000(P/F, i^*\%, 2) = -\$18\,000$, and under the present approximation, $\text{NPV} = -\$13\,860.15$; we conclude that $i^* < 30\%$. Try $i = 8\%$:

$$\text{NPV} = -\$50\,000 + \$30\,000(0.38803)^{-1} - \$31\,000(1.1664)^{-1} = +\$736.11$$

Hence, $i^* > 8\%$. Try $i = 10\%$:

$$\text{NPV} = -\$50\,000 + \$30\,000(0.40211)^{-1} - \$31\,000(1.2100)^{-1} = -\$1013.38$$

Hence, $8\% < i^* < 10\%$. Try $i = 9\%$:

$$NPV = -\$50\,000 + \$30\,000(0.39505)^{-1} - \$31\,000(1.1881)^{-1} = -\$152.33$$

Hence, $8\% < i^* < 9\%$. By linear interpolation,

$$i^* \approx 8\% + \frac{0 - \$736.11}{-\$152.33 - \$736.11}(9\% - 8\%) = 8.83\%$$

It is not difficult to show that there are no other positive values—in fact, no other real values—of $i^*$ besides $i^* \approx 8.83\%$.

**8.9** Determine the payback period (PBP) for the cash flows of (a) Problem 8.5, (b) Problem 8.8. (c) Comment on the results in the light of the corresponding ROR-values.

(a)
$$PBP = \frac{\$50\,000}{\$16\,719 \text{ per year}} = 2.99 \text{ years}$$

(b) We have $CF_1 + CF_2 = \$29\,000$, $CF_1 + CF_2 + CF_3 = \$59\,000$; hence,

$$PBP \approx 2 + \frac{\$50\,000 - \$29\,000}{\$59\,000 - \$29\,000}(3 - 2) = 2.7 \text{ years}$$

(c) The cash flows of Problem 8.5 have a *longer* payback period, but a *higher* rate of return, than those of Problem 8.8. We might say that the PBP and the ROR give opposite rankings of the two investments. However, we must always bear in mind that the PBP takes no account of interest.

**8.10** In Example 8.7, assume that the affected farmers are lobbying for relocation payments and subsidies to compensate them for the lost farmland. Estimates of "equitable" payments vary widely, but one number being considered is an annual payment of $3.25 million per year to the farmers over the 30-year period. Would the project still be desirable, with this new disbenefit?

$$BCR = \frac{\$60.0 - \$3.25}{\$24.1} = 2.35$$

Since $BCR > 1$, the project is still desirable.

**8.11** A public works project is proposed that has total present-worth benefits of $75 million and total present-worth costs of $55 million. In deliberating this proposal, some members of the town council have suggested that the project has a total present-worth disbenefit of $15 million; other members feel the $15 million should be treated as a cost. How should the proposal be evaluated?

In this particular case, it makes very little difference whether the $15 million is classified as a disbenefit or as a cost. If the $15 million is treated as a disbenefit,

$$BCR = \frac{\$75 - \$15}{\$55} = 1.09$$

If it is treated as a cost,

$$BCR = \frac{\$75}{\$55 + \$15} = 1.07$$

However, to eliminate any confusion, (8.7) should be used:

$$NBV = \$75 - \$15 - \$55 = \$5 \text{ million}$$

Since $NBV > 0$, the project is economically acceptable.

**8.12** The ABC Company is considering the purchase of a new sanding machine; machine models A and B are available. Both models have a five-year life, and their cash flows are as given in Table 8-2. The interest rate is 10% per year, compounded annually. Which model should ABC buy?

**Table 8-2**

| End of Year | Cash Flows | |
|---|---|---|
| | Model A | Model **B** |
| 0 | −$30 000 | −$30 000 |
| 1 | 10 000 | 30 000 |
| 2 | 10 000 | 5 000 |
| 3 | 10 000 | 3 000 |
| 4 | 10 000 | 2 000 |

By the net present value method:

$$\text{NPV}_A = -\$30\,000 + \$10\,000(P/A, 10\%, 4) = -\$30\,000 + \$10\,000(0.31547)^{-1} = \$1698.74$$

$$\text{NPV}_B = -\$30\,000 + \$30\,000(P/F, 10\%, 1) + \$5000(P/F, 10\%, 2) + \$3000(P/F, 10\%, 3) + \$2000(P/F, 10\%, 4) = \$5024.93$$

By the payback period method:

$$\text{PBP}_A = \frac{\$30\,000}{\$10\,000 \text{ per year}} = 3 \text{ years} \qquad \text{PBP}_B = 1 \text{ year}$$

By the rate of return method:

***for model A***

$$0 = -\$30\,000 + \$10\,000(P/A, i^*\%, 4)$$
$$(A/P, i^*\%, 4) = 0.33333$$

and by linear interpolation in Appendix **A**,

$$i^* \approx 12\% + \frac{0.33333 - 0.32923}{0.35027 - 0.33333}(15\% - 12\%) = 12.24\%$$

***for model B***

$$0 = -\$30\,000 + \$30\,000(P/F, i^*\%, 1) + \$5000(P/F, i^*\%, 2) + \$3000(P/F, i^*\%, 3) + \$2000(P/F, i^*\%, 4)$$

By trial and error:

| $i^*$ | NPV |
|---|---|
| 20% | $1172.84 |
| 25% | −$435.48 |

and by linear interpolation,

$$i^* \approx 20\% + \frac{\$0 - \$1172.84}{-\$435.48 - \$1172.84}(25\% - 20\%) = 23.65\%$$

It is seen that all three methods rate model **B** above model **A; ABC** should purchase model **B.**

## Supplementary Problems

**8.13** A new plant to produce tractor gears requires an initial investment of \$10 million. It is expected that a supplemental investment of \$4 million will be needed every 3 years to update the plant. The plant is expected to start producing gears 2 years after the initial investment is made (at the start of the third year). Revenues of \$5 million per year are expected to begin to flow at the start of the fourth year. Annual operating and maintenance costs are expected to be \$2 million per year. The plant has a 15-year life. List the annual cash flows.

*Ans.* $CF_0 = -\$10\,000\,000$, $CF_1 = CF_2 = 0$, $CF_3 = -\$6\,000\,000$, $CF_4 = CF_5 = CF_7 = CF_8 = CF_{10} = CF_{11} = CF_{13} = CF_{14} = \$3\,000\,000$, $CF_6 = CF_9 = CF_{12} = CF_{15} = -\$1\,000\,000$

**8.14** What is the NPV of the plant in Problem 8.13 if the interest rate is 10% per year, compounded annually? *Ans.* –\$5 336 645.33

**8.15** Is the plant described in Problems 8.13 and 8.14 an economically acceptable investment?
*Ans.* No, because the NPV is negative.

**8.16** A different plant from the one described in Problem 8.13 can be built for an initial investment of \$13 million and no supplemental investments. All other data are the same as in Problems 8.13 and 8.14. (a) Compute the net present value. (b) Is this plant an economically acceptable investment?
*Ans.* (a) +\$855 708.47; (b) yes

**8.17** Is the investment described in Problem 8.16 still economically acceptable if the interest rate is 15% per year, compounded annually? Use the net present value method.
*Ans.* No: NPV = $-\$3\,624\,238.52 < 0$.

**8.18** Compute the NPV of an investment with $CF_0 = -\$50\,000$ and $CF_j = +\$12\,000$ $(j = 1,\ldots,6)$ if the annual interest rate, compounded annually, is (a) 8%, (b) 10%, (c) 12%, (d) 15%. (e) Interpret the results.
*Ans.* (a) \$5473.37; (b) \$2262.53; (c) –\$663.98; (d) –\$4586.74. (e) The investment is not economically acceptable when the interest rate is 12% or greater, in which case the present worth of the cash flows is less than the (present worth of the) investment.

**8.19** Is the conclusion of Problem 8.15 changed if the interest rate is 5% per year, compounded annually? *Ans.* No: NPV = –\$1 928 607.02.

**8.20** What is the NPV of the investment described in Problem 8.13 if the interest rate is 3% per year, compounded annually? *Ans.* \$48 465.06

**8.21** What can be said about the ROR of the plant of Problem 8.13, in view of the results of Problems 8.19 and 8.20? *Ans.* There is at least one value of $i^*$ between 3% and 5%.

**8.22** Approximate the ROR for Problem 8.18 by interpolation between the results of Problem 8.18(*b*) and (c). *Ans.* $i^* \approx 10.454\%$

**8.23** Compute the payback period for the investment of (a) Problem 8.3, (*b*) Problem 8.16, (*c*) Problem 8.18. *Ans.* (a) PBP = 11 years; (b) PBP = 8 years; (c) PBP = 4.17 years

**8.24** What is the NPV of the plant described in Problem 8.16, if the interest rate is 12% per year, compounded annually? *Ans.* –\$1 195 881.54

**8.25** What is the ROR of the plant described in Problem 8.16? Solve by interpolation, using the NPV data from Problems 8.16 and 8.24. *Ans.* 10.834%

**8.26** What can be said about the ROR for a set of positive cash flows?
*Ans.* Since the NPV is positive for every positive $i$, no ROR exists.

**8.27** Rework Problem 8.12 if the interest rate is (a) 15% per year, (b) 25% per year, compounded annually. Ans. (a) $NPV_A = -\$1450.60$, $NPV_B = \$2983.17$ (buy model B). (b) Neither model is economically acceptable; both have negative net present values.

**8.28** A new highway has been proposed to join two cities, at a total construction cost of \$700000000. The new highway has a 20-year life. It would render obsolete the current railroad system connecting the two cities, which would be dismantled at a cost of \$100 000 000. This would put the 4000 railroad employees out of work; they would each be paid \$6000 per year in damages, for a total of 20 years. The railroad would require a \$1 000000 annual maintenance program if it is kept. The railroad property has an assessed valuation of \$30000000; the property would be purchased at that figure and used as the new roadbed. The highway is estimated to yield, in taxes on the trucks using it, \$0.005 per ton-mile more than the railroad; a total of 500 million annual ton-miles of use is expected. It is also estimated that the general tax revenues would increase by \$10 000 each year because of the new highway. On the other hand, it is estimated that the new highway would cost \$2 000 000 per year to maintain. What is the BCR for the new highway, relative to the current railroad, assuming the project would be financed by 7% per year interest-bearing bonds, with the interest compounded annually? What is the NBV? [Hint: The annual cost for 20 years is given by

$$(\$700\,000\,000 + \$100\,000\,000 + \$30\,000\,000)(A/P, 7\%, 20) + (\$2\,000\,000 - \$1\,000\,000) + (4000)(\$6000)$$

and the corresponding annual benefit is $(500\,000\,000)(\$0.005) + \$10\,000(A/G, 7\%, 20)$.]
Ans. BCR = 0.0249, NBV = −\$100 770 537.00

**8.29** Should the highway in Problem 8.28 be built? Why?
Ans. No, because the BCR is less than 1.0 and the NBV is negative, relative to keeping the railroad.

**8.30** A state agency is contemplating giving a total of \$5 000 000 in grants to various universities. These grants would be paid out in installments of \$500 000 per year over a 10-year period. The grants would enable low-skilled persons to be retrained for new jobs, with a resulting benefit of \$5000 per year in increased income for each of 1000 persons in the regional labor force, over the next 10 years. These state grants would enable the universities to obtain a total of \$1 000000 in matching federal funds for their operations, thereby reducing by \$1 000000 the total amount of state funds which would normally be required by the universities. However, the grants would require 10 persons to be added to the university staffs at an average annual salary of \$20 000 each, which would have to be paid out of state funds. The current annual interest rate is 10% on funds of this type. Is the retraining program an economical investment for the state, on a present-worth basis? Compute both the present-worth BCR and the present-worth NBV. Ans. BCR = 8.33, NBV = +\$27 035 330.26; yes, it is economical.

**8.31** Does the answer to Problem 8.30 change if the BCR and NBV are computed in terms of annualized costs? Ans. BCR = 8.33, NBV = +\$4 400 000; again the investment is economical.

**8.32** A state agency is contemplating building a new 5630-acre industrial park. The property can be acquired at a cost of \$1000 per acre. Roadways and other improvements are estimated to cost a total of \$1000 per acre, with these costs spread evenly over the next 10 years. The few residents currently on the property will be moved out over the next three years; the displacement costs are estimated at \$1 000 000 this year, \$500 000 next year, and \$200 000 the year after. The park is expected to provide the state with new tax revenues of \$5 000 000 per year, starting five years from now, with an increase of \$2 000 000 per year thereafter. The state has decided to evaluate this project on the basis of a 10-year lifetime. The funds for the project will be borrowed at an interest rate of 7% per year, compounded annually. (a) What is the present-worth NBV of the project? (b) What is the present-worth BCR? (c) Is the project economically justifiable? Ans. (a) \$14 401 566; (b) 2.30; (c) yes

# Chapter 9

# Choosing Among Investment Alternatives

## 9.1 SETTING THE MARR

In Chapter 6 we introduced the MARR as the smallest yield rate at which a proposed investment would be acceptable. How is this cut-off rate arrived at?

1. The MARR may be set equal to the interest rate that is available at a local savings bank or other institution. The MARR then becomes the "opportunity cost of money," in that it measures the opportunity lost from not placing money in the bank.
2. For most businesses, the savings bank rate would be lower than their usual overall rate of return on investment. Thus, the MARR is sometimes set equal to the firm's current average return on total investment.
3. The MARR may be purposely set higher than either the bank rate on savings or the firm's current return on investment. It may be set according to the firm's long-range profit goals, so as to achieve a desired future growth rate; it may be set at a high ,level to encourage the search for more profitable new ventures; it may be chosen large to offset the high degree of risk attached to the investment.

Under option 1 and (within the law of averages) under option 2, the MARR is an ***attainable*** rate; i.e., there are investment alternatives that actually achieve that rate. However, under option 3, the MARR is a ***target*** rate, with no guaranteed means of realization.

**Example 9.1** The ABC Company is currently earning an average before-tax return of 25% on its total investment. The board of directors of ABC is considering three proposals as given in Table 9-1.

**Table 9-1**

| | Cash Flows | | |
|---|---|---|---|
| End of Year | Proposal A | Proposal B | Proposal C |
| 0 | −$40 000 | −$60 000 | −$50 000 |
| 1 | 18 000 | 25 000 | 27 000 |
| 2 | 18 000 | 25 000 | 27 000 |
| 3 | 18 000 | 25 000 | 27 000 |
| 4 | 18 000 | 25 000 | 27 000 |

Proposal A is for a new machine that will replace one of their older, worn-out pieces of equipment; this machine is vital to ABC's production. Proposal B is for a plant expansion. Proposal C is for an addition to ABC's product line. There is a high probability that this product could fail in the marketplace, resulting in the loss of most of the $50 000 initial investment. The board feel that they would need at least a 40% rate of return on this project to compensate for its additional riskiness. Which of these three proposals are acceptable?

Compute net present values, using the attainable MARR of *25%* for proposals A and B, and the target MARR of 40% for proposal C:

$$\mathrm{NPV}_A = -\$40\,000 + \$18\,000(P/A, 25\%, 4) = +\$2508.97$$
$$\mathrm{NPV}_B = -\$60\,000 + \$25\,000(P/A, 25\%, 4) = -\$959.76$$
$$\mathrm{NPV}_C = -\$50\,000 + \$27\,000(P/A, 40\%, 4) = -\$71.19$$

Only proposal A is acceptable ($\mathrm{NPV}_A > 0$).

The ROR method leads to the same conclusion: linear interpolation in Appendix A gives

$$i_A^* \approx 28.75\% \qquad i_B^* \approx 24.07\% \qquad i_C^* \approx 39.99\%$$

and only $i_A^*$ exceeds the associated MARR.

Because $\mathrm{NPV}_B$ and $\mathrm{NPV}_C$, though negative, are small in magnitude (causing $i_B^*$ and $i_C^*$ to be just under their associated MARRs), the ultimate decisions concerning proposals B and C may have to be made on the basis of other considerations, such as ABC's long-term product strategies and the company's ability to raise capital. If capital is scarce, proposal A must be given the highest priority, since ABC's continued profits appear to depend on that piece of machinery.

**Example** 9.2 The XYZ Company has $50 million which can be invested in proposal A ($i_A^* = 17\%$) or in proposal B ($i_B^* = 29\%$); or else it can exercise the do-nothing alternative and invest the $50 million in modernizing current operations. A target MARR of 35% has been established by XYZ's management to achieve their long-range plans and strategies. The XYZ Company currently earns an average of 25% on its total investment in plant and equipment, some of which is very old. Which alternative should XYZ pursue?

For the do-nothing alternative, $i^* = 25\%$; thus, none of the three alternatives meets the desired 35% MARR. If the company is serious about the 35% MARR, then additional alternatives should be sought. Proposal B, which has an ROR slightly better than the current average rate of return on total investment, would clearly enhance the company's average rate of return. In addition, given that some of the company's plant and equipment is "very old," using the available $50 million for refurbishing this old plant and equipment might also improve the company's average rate of return. One reasonable strategy would be to spend part of the $50 million on new plant and equipment, and then to search for higher-profit proposals (e.g., a 35% ROR) on which to spend the balance.

## 9.2 PROJECT SELECTION AND BUDGET ALLOCATION

Determining the best way to allocate a given budget among several competing projects is a commonly encountered problem, because often there are more worthwhile project proposals and ideas than can be funded with the available monies. The solution principle is to evaluate each project in terms of present worth or some similar measure, and to choose that set of projects for which the sum of the measures is a maximum, subject to the budget constraint.

### Independent Projects

Two or more proposals or projects are independent when the acceptance or rejection of any one of them does not entail the acceptance or rejection of any other. For instance, a proposal to air-condition the company offices and a proposal to undertake an advertising campaign for a new product would usually be considered independent.

For independent projects, the following selection algorithm will always maximize the financial return on the available monies.

**Step 1** Compute i* for each project.

**Step 2** Eliminate any project whose i*-value is less than the MARR (if no MARR exists, omit this step).

**Step 3** Arrange the surviving proposals from step 2 in descending order of i*-value.

**Step 4** Select proposals from the top of this list downward, until an additional selection would exceed the available funds or the budget.

If, as often will be the case, some funds remain at the end of step 4, there are three options: (i) if one or more of the remaining projects is divisible into subprojects, then these subprojects may be funded, using the above algorithm, until the available funds are exhausted; (ii) the remaining funds may be invested in the do-nothing alternative, at the MARR (for an attainable MARR) or at some rate less than the MARR (for a target MARR); (iii) the remaining funds are simply "left over."

**Example** 9.3 The BK Company is considering five proposals for new equipment, as indicated in Table 9-2. Each piece of equipment has a life of 100 years. Treating that period as infinite, the ROR will be the interest rate at which $I$ is the capitalized equivalent of the perpetual series of payments $R$; hence, (7.8) gives the third row of Table 9-2. The BK Company has established a MARR of 11% and has a budget of \$325 000. Which **proposal(s)** should the company select?

**Table 9-2**

| | Proposal 1 | Proposal 2 | Proposal 3 | Proposal 4 | Proposal 5 |
|---|---|---|---|---|---|
| Annual Revenue, $R$ | \$5 000 | \$6000 | \$25000 | \$16000 | \$20 000 |
| Investment, $I$ | \$60 000 | \$50 000 | \$100 000 | \$100 000 | \$100 000 |
| $i^* \approx R/I$ | $8\frac{1}{3}$% | 12% | 25% | 16% | 20% |

Using the selection algorithm, we obtain the following list:

| | $i^*$ | Investment | |
|---|---|---|---|
| Proposal 3 | 25% | \$100 000 | |
| Proposal 5 | 20% | \$100 000 | |
| Proposal 4 | 16% | \$100 000 | |
| ---------------------- | | | **Budget cut-off** |
| Proposal 2 | 12% | \$50 000 | |
| ---------------------- | | | **MARR cut-off** |
| Proposal 1 | $8\frac{1}{3}$% | \$60 000 | |

Proposal 2 is acceptable from the standpoint of the MARR criterion, but **insufficient** funds are available to include it. Thus, proposals 3, 5, and 4 are selected, and \$25 000 is left unspent from the \$325 000 budget.

## Mutually Exclusive Projects

A set of projects are mutually exclusive if at most one of them may be accepted. It is thus a question of picking the single economically best project (or of rejecting them all).

For mutually exclusive projects, the selection algorithm given below will always yield the maximum total return on the total amount invested. First, we shall need some terminology. Let $I$ denote the investment cost of a project, and $R$ the measure of revenues @resent worth, EUAS, etc.) from the project. We shall say that project 1 dominates project 2 if $I_1 \leq I_2$ and $R_1 \geq R_2$. Clearly, a dominated project can never be the best of a mutually exclusive set. Further, for any two projects—a standard and a ***challenger*—define** the incremental rate of return of the challenger as

$$\Delta i^* \equiv \frac{R_{\text{challenger}} - R_{\text{standard}}}{I_{\text{challenger}} - I_{\text{standard}}}$$

**Step 1** Eliminate any project whose investment exceeds the budget.

**Step 2** Arrange the surviving projects in ascending order of investment (break any investment-ties arbitrarily). Now eliminate any project that is dominated by another project; the candidates that remain will be in ascending order both of investment and of return. Compute i* for each candidate.

**Step 3** Eliminate from further consideration any candidate having i* < MARR.

**Step 4** From the surviving candidates, select as the standard that candidate which has the smallest investment.

**Step 5** Compute the incremental rate of return of the challenger that immediately succeeds the standard in the list of candidates.

**Step 6** If $\Delta i^* \leq$ MARR, eliminate this challenger from further consideration and repeat step 5 for the next challenger; if $\Delta i^* >$ MARR, eliminate the old standard from further consideration, replace it with this challenger as the new standard, and repeat step 5.

**Step 7** Select the one surviving candidate: it is the best alternative.

**Example 9.4** The KLN Company is attempting to determine the economically best size of processor machine for their facilities. The six alternative machine sizes which are feasible are as given in Table 9-3. Each machine has a life of 100 years and no salvage value, so that $i^* \approx R/I$, as in Example 9.3. The company has a total capital budget of \$350 000 and a MARR of 15%. Which machine should they buy?

**Table 9-3**

| Size of Machine | Annual Revenue, $R$ | Investment, $I$ | $i^*$ |
|---|---|---|---|
| Economy | \$ 7 200 | \$60000 | 12% |
| Regular | 25 000 | 100 000 | 25% |
| Super | 36 000 | 200 000 | 18% |
| **Delux** | 45 000 | 220 000 | 20.45% |
| Bulk | 50 000 | 300000 | 16.67% |
| Extended | 52 000 | 385000 | 20.5% |

The Extended machine is unacceptable according to step 1 of the selection algorithm, and the Economy machine is unacceptable according to step 3. The application of steps 5 and 6 to the four surviving candidates is shown in Table 9-4; step 7 gives Delux as the winner.

**Table 9-4**

| Comparisons | $i^*$ | Steps 5 and 6 |
|---|---|---|
| Regular vs. Super | **25%** vs. 18% | Standard #1 vs. Challenger #1: $\Delta i^* = \frac{\Delta R}{\Delta I} = \frac{\$36\,000 - \$25\,000}{\$200\,000 - \$100\,000} = \frac{\$11\,000}{\$100\,000} = 11\% <$ MARR<br>**Decision:** reject challenger #1 and repeat step 5 |
| Regular **vs. Delux** | 25% **vs.** 20.45% | Standard #1 vs. Challenger #2: $\Delta i^* = \frac{\$20\,000}{\$120\,000} = 16.7\% >$ MARR<br>**Decision:** replace standard #1 (Regular) with challenger #2 (Delux) and repeat step 5 |
| **Delux vs.** Bulk | 20.45% **vs.** 16.67% | Standard #2 vs. Challenger #3: $\Delta i^* = \frac{\$5\,000}{\$80\,000} = 6.25\% <$ MARR<br>**Decision:** reject challenger #3 |

Let us examine the logic of the selection algorithm, on the assumption that the company can realize 15% (the MARR) by implementing the do-nothing alternative. Consider the first comparison in Table 9-4. Super costs $\Delta I = \$100\,000$ more than Regular, and yields $\Delta R = \$11\,000$ more per year. If the company chose Super, it would, in effect, be making $11000 a year on a $100000 investment; that is, it would be investing at rate $\Delta i^* = 11\%$, whereas it could be earning MARR = 15%. Choosing Super would thus entail an opportunity loss of

$$(15\% - 11\%)(\$100\,000) = \$4000 \text{ per year}$$

Or, looked at in a slightly different way, if the Regular machine is purchased, at a saving of $100 000, the company will earn $25 000 a year on the machine, plus 15% × $100 000 = $15 000 a year on the do-nothing alternative. This is a total annual return of $40 000 on a total investment of $200 000. The same total investment in the Super machine will earn only $36 000 a year.

In this first comparison, it so happens that the economically superior machine has the larger i*-value. Note however, that the eventual winner, Delux, has a smaller i*-value than Regular. As we have seen, when purchase prices differ, a mere comparison of i*-values is not decisive; one must also consider what will be done with any funds left over from the purchase of the cheaper machine.

### Other Interrelationships Between Projects

A project may be contingent upon some other project, in the sense that the acceptance or rejection of one may result in the corresponding acceptance or rejection of the other. For example, the purchase of a new computer storage disk may be contingent on the purchase of a new computer.

Some projects may be joint, in that either both are accepted or both are rejected. For instance, though they may be purchased separately, the tractor and trailer of a rig for highway hauling of steel are normally joint items.

Some projects may be financially interdependent; i.e., approval of one exhausts the available funds and thus precludes approval of the other.

Joint projects should be treated as a single, total project or investment. Contingent projects should be evaluated both jointly and separately. The basic investment should first be evaluated alone. The contingent items should then be brought in and their effect on the total investment evaluated. Financial interdependences are usually resolved by considering the ***irreducibles***, those factors to which a dollar value cannot be attached.

**Example 9.5** Which project(s) in Table 9-5 should be approved, if the budget is $150 000 and the MARR is 15%?

**Table 9-5**

| Project | Investment | $i^*$ |
|---|---|---|
| A | $100000 | 20% |
| B | 50000 | 20% |
| C | 50000 | 20% |
| D | 50000 | 20% |
| E | 150000 | 20% |

This is an instance of financial interdependence. Because of the $150 000 budget, selecting project A and either B, C, or D precludes selecting any others; selecting E precludes selecting any others; and selecting B and C and D precludes selecting any others. Thus, there are five alternatives (investment portfolios), each representing a total investment of $150 000 and each with an ROR of 20%. The choice among them must be made on the basis of the intrinsic characteristics of the projects, the need for a diversified portfolio, and other irreducible factors.

## 9.3 THE REINVESTMENT FALLACY

It is implicitly assumed in the NPV, EUAS/EUAC, and ROR methods that any cash inflows generated by an investment are reinvested, at the rates MARR, MARR, and i*, respectively. If such an assumption does not hold, and if the assets being compared have unequal service lives, fallacious results may be obtained.

**Example** 9.6 Consider two competing projects, for which MARR = 16%:

| | $C$ | $A$ | $n$ |
|---|---|---|---|
| Project A | $100 000 | $23 000 | 9 |
| Project B | 100 000 | 35 000 | 4 |

Here, C is the initial investment, A is the annual net cash inflow, and n is the service life of the asset. The ROR method yields:

***project A***

$$0 = -\$100\,000 + \$23\,000\,(P/A, i^*, 9)$$

$$(A/P, i^*, 9) = 0.23$$

$$i^* \approx 17.7\% \text{ (by interpolation in Appendix A)}$$

***project B***

$$0 = -\$100\,000 + \$35\,000(P/A, i^*, 4)$$

$$(A/P, i^*, 4) = 0.35$$

$$i^* \approx 15\%$$

Hence, according to the MARR, project A is acceptable and project B is not.

However, suppose that the cash flows can be reinvested at 25%, compounded annually. Thus, the $23 000 annual cash inflows from project A are actually equivalent to a future value

$$\$23\,000(F/A, 25\%, 9) = \$593\,400$$

nine years hence, and the annual cash inflows from project B are actually equivalent to a future value

$$\$35\,000(F/A, 25\%, 4)\,(F/P, 25\%, 5) = \$615\,900$$

nine years hence. Thus (the initial investments being equal) project B is actually the preferred alternative.

The reinvestment fallacy can be avoided if the MARR is set at the reinvestment rate (whose value, however, may be very difficult to predict) and if future values based on this MARR are compared, as in Example 9.6. As for the matter of unequal lives, it can sometimes be ignored, and the NPV, EUAC, or ROR method applied notwithstanding. For other situations, a replacement method which assumes that each asset, at the end of its useful life, is replaced with a new asset identical in kind, may be more appropriate. This method will be employed in Chapter 10.

## Solved Problems

**9.1** The management of the Conway Corporation is considering five alternative new-product proposals that their employees have submitted to them:

| Proposal | Rate of Return |
|---|---|
| Fryer | 49% |
| Box Loader | 26% |
| Conveyor | 19.5% |
| Planer | 23% |
| Cutter-Loader | 26.5% |

Conway currently enjoys an average return of 26% on total investment. Which proposal(s) is (are) acceptable?

Using the current average return on total investment as the MARR, only Fryer and Cutter-Loader are ***strictly*** acceptable. If the rule is: ROR $\geq$ MARR, then Box Loader is also acceptable.

**9.2** Rework Problem **9.1** if **Conway** desire a future average return of **31%.**

Only Fryer is acceptable under the target MARR.

**9.3** The Wyandot Company currently earns an average rate of return of **30%** on its total investment. The board of directors of Wyandot is considering the three proposals whose cash flows are specified in Table **9-6.** Which **proposal(s)** is (are) acceptable, if the board of directors has set a target MARR of **25%**? Make an NPV calculation.

**Table *9-6***

| End of Year | Proposal A | Proposal **B** | Proposal C |
|---|---|---|---|
| 0 | −$50 000 | −$75 000 | −$100 000 |
| 1 | 15 000 | 30 000 | 35 000 |
| 2 | 15000 | 30 000 | 35 000 |
| 3 | 15 000 | 30 000 | 35 000 |
| 4 | 15 000 | 30 000 | 35 000 |

$$\text{NPV}_A = -\$50\,000 + \$15\,000\,(P/A, 25\%, 4) = -\$14\,575.85$$
$$\text{NPV}_B = -\$75\,000 + \$30\,000\,(P/A, 25\%, 4) = -\$4151.71$$
$$\text{NPV}_C = -\$100\,000 + \$35\,000\,(P/A, 25\%, 4) = -\$17\,344.00$$

Since all three net present values are negative, none of the projects is acceptable.

**9.4** Solve Problem **9.3** by an ROR calculation.

***proposal A*** $\quad (A/P, i_A^*, 4) = \text{———} = 0.30000$

$7\% < i_A^* < 8\%$ (from Appendix A)

***proposal B*** $\quad (A/P, i_B^*, 4) = \text{———} = 0.40000$

$20\% < i_B^* < 25\%$ (from Appendix A)

***proposal C*** $\quad (A/P, i_C^*, 4) = \dfrac{\$35\,000}{\$100\,000} = 0.35000$

$12\% < i_C^* < 15\%$ (from Appendix A)

All three rates of return are smaller than the target MARR (25%), and so none of the projects is acceptable.

**9.5** With reference to Problems **9.3** and **9.4,** how would Wyandot's average return on investment be affected by the acceptance of proposal B?

The effect would depend on the amount of Wyandot's total invested capital. Suppose Wyandot to be a very small firm, whose total investment before accepting proposal B is $225 000. Since $i_B^*$, the rate of return of proposal B, is 21.85% (by linear interpolation in Appendix A), the average rate of return after acceptance of proposal B would be

$$\frac{(30\%)(\$225\,000) + (21.85\%)(\$75\,000)}{\$300\,000} = 27.96\%$$

or a decrease of about 2%. However, if Wyandot were a larger company, with, say, \$925 000 in invested capital, then the new average rate would be

$$\frac{(30\%)(\$925\,000) + (21.85\%)(\$75\,000)}{\$1\,000\,000} = 29.39\%$$

a decrease of only $\frac{6}{10}\%$.

**9.6** Refer to Problem 9.3. Suppose that if proposal B were rejected, the competition would be likely to introduce a new product that would cut Wyandot's market share in half, and hence cause them to suffer a 50% reduction in current profits. Should proposal B be accepted or rejected under these circumstances?

We have seen that acceptance of proposal B is unprofitable per se. However, rejection could only prove more unprofitable (unless Wyandot's profits are minuscule to begin with). Thus, proposal B should be accepted, as "the lesser of two evils."

**9.7** The Clearwater Company has a budget of \$500000 which can be spent on the five independent projects of Table 9-7. If MARR = 20%, how should the budget be allocated?

**Table 9-7**

| Project Number | $i^*$ | Total Project Cost |
|---|---|---|
| 1 | 29.1% | \$150 000 |
| 2 | 10.5% | 50000 |
| 3 | 21.5% | 200000 |
| 4 | 19.5% | 75000 |
| 5 | 23.2% | 25 000 |

The selection algorithm for independent projects gives:

**Step 2** Eliminate projects 2 and 4.

**Step 3** Select

| | |
|---|---|
| Project 1 | \$150 000 |
| Project 5 | 25 000 |
| Project 3 | 200 000 |
| TOTAL | \$375 000 |

with \$500 000 – \$375 000 = \$125 000 unspent.

**9.8** Rework Problem 9.7 if (a) MARR = 25%, (*b*) MARR = 19%, (c) MARR = 18% and capital is rationed at \$400 000.

(a) Only project 1 can be funded; \$350 000 remains unspent.

(*b*) Only project 2 is unacceptable, and \$450 000 is spent as follows:

| | |
|---|---|
| Project 1 | \$150 000 |
| Project 3 | 200 000 |
| Project 4 | 75 000 |
| Project 5 | 25 000 |
| TOTAL | \$450 000 |

(c) The MARR eliminates project 2, and the ranking becomes:

| | Project Cost | Cumulative Amount Spent |
|---|---|---|
| Project 1 | $150 000 | $150 000 |
| Project 5 | 25 000 | 175 000 |
| Project 3 | 200 000 | 375 000 |
| Project 4 | 75 000 | |

Unless it is divisible, project 4 cannot be funded: its inclusion would exceed the $400000 budget constraint.

**9.9** Grampian Manufacturing Company is attempting to determine the "best"-sized milling machine for their production shop. Five alternative sizes are available, as given in Table 9-8. Grampian has a budget of $250 000, and MARR = 15%. Which size machine should they purchase? Assume that n = 100 years and that the ultimate salvage value is zero for each machine.

**Table 9-8**

| Size | Annual Revenue | Initial Cost | i* |
|---|---|---|---|
| Economy | $5000 | '$50000 | 10% |
| Regular | 25000 | 100000 | 25% |
| Super | 36000 | 200000 | 18% |
| Delux | 45000 | 220000 | 20.45% |
| Super Delux | 50 000 | 300 000 | 16.67% |

This situation involves mutually exclusive projects. The selection algorithm (Section 9.2) gives:

**Step 1** Eliminate Super Delux.'

**Step** 2 See Table 9-8.

**Step** 3 Eliminate Economy.

**Steps** 4 through **6** Compare Super against Regular:

$$\Delta i^* = \frac{36\,000 - 25\,000}{200\,000 - 100\,000} = 11\% < \text{MARR}$$

hence, eliminate Super. Compare Delux against Regular:

$$\Delta i^* = \frac{45\,000 - 25\,000}{220\,000 - 100\,000} = 16.67\% > \text{MARR}$$

hence, Delux becomes the new standard.

**Step 7** Select Delux.

In this case, $30 000 will be left over from the original $250 000 budget.

**9.10** Rework Problem 9.9 for MARR = 20%.

Now only Regular and Delux survive step 1 of the algorithm. From step 5, with Delux as the challenger,

$$\Delta i^* = 16.67 < \text{MARR}$$

Hence, Regular is selected, and $100 000 is spent.

**9.11** Rework Problem 9.9 for a budget of \$200 000.

In this case, Delux and Super Delux are eliminated in step 1 of the algorithm, and Economy in step 3. Then, only Regular and Super remain; from Problem 9.9, Regular wins.

**9.12** The CCC Corporation is weighing the purchase of a computer. The basic machine costs \$200 000; the costs of various peripheral equipment are:

| | |
|---|---|
| Slow Printer | \$ 10 000 |
| Fast Printer | 20 000 |
| Low-Resolution Video Display | 2 000 |
| High-Resolution Video Display | 5 000 |
| Disk Drives (each) | 2 000 |
| Remote-User Ports (each) | 2 000 |
| Software | 100 000 |
| Special Disks | 1 000 |

The job that the computer would perform is now being done by hand, by five persons, at an annual salary and overhead cost of \$100 000. Though these people would all be replaced by the computer, the purchase of the computer would necessitate the hiring of two programmer-operators and one mathematician, at a total salary and overhead cost of \$80 000. Classify the investment decisions and proposals involved in this situation.

The peripheral equipment are all contingent projects-contingent on the purchase of the basic computer. The slow and fast printers are mutually exclusive, as are the high- and low-resolution videos. Some type of printer and/or video, some software, some disks, and one or more disk drives would seem to be mandatory: they are joint proposals with the computer. The do-nothing alternative (continuing with the handicraft technology of five persons) and the computer purchase alternative are mutually exclusive projects.

**9.13** If, in Problem 9.12, CCC has only \$210 000 available for the computer system, exclusive of software, does this introduce any financial interdependences?

Yes. If \$10 000 is spent on the slow printer, then no other peripherals can be purchased. If \$10 000 is spent on some combination of peripherals (**e.g.**, special disks plus one remote-user port plus one disk drive plus one high-resolution video), then no other items can be purchased beyond the basic computer.

**9.14** For the data of Problem 9.12 and Table 9-9, and for MARR = 15%, should the computer be purchased? What peripherals should be purchased?

**Table 9-9**

| Comparison | $\Delta i^*$ |
|---|---|
| Do-Nothing vs. Basic Computer plus Software | 16% |
| Slow vs. Fast Printer | 8% |
| Low- vs. High-Resolution Video Computer | 13% |
| Computer with One Disk Drive vs. Computer with Two Disk Drives | 12% |
| Zero vs. Two Remote Ports | 21% |

The basic computer plus software, one slow printer, one disk drive, one low-resolution video, and two remote ports should be purchased, since $\Delta i^* >$ MARR for these items. The high-resolution video may be justifiable on the basis of irreducibles, such as operator eye-fatigue, since its $\Delta i^*$ is not that far away from the established MARR.

**9.15** ABC Company have decided to automate certain of their procedures by installing a computer system. The cash flows for two competing systems are given in Table 9-10; both systems have a five-year life and zero salvage value. If the MARR is 15%, which system should ABC purchase?

**Table 9-10**

| End of Year | System 1 | System 2 |
|---|---|---|
| 0 | −$50 000 | −$75 000 |
| 1 | 22 000 | 24 000 |
| 2 | 22 000 | 24 000 |
| 3 | 22 000 | 24 000 |
| 4 | 22 000 | 24 000 |
| 5 | 22 000 | 24 000 |

In the case of equal lifetimes, the NPV (or the EUAS) is a linear function of initial cost and annual revenue; hence we compute

$$\begin{aligned}\Delta NPV &\equiv NPV_2 - NPV_1 \\ &= [-\$75000 - (-\$50\,000)] + (\$24000 - \$22\,000)(P/A, 15\%, 5) \\ &= -\$18\,295.79\end{aligned}$$

As $\Delta NPV < 0$, system 2 is economically inferior to system 1; system 1 should be purchased.

**9.16** In a situation like that of Problem 9.12, the directors of a firm are trying to decide whether to buy computer system A, to buy computer system B, or to stay with the current manual technology. Advise them, given a MARR of 15%, a planning horizon of 4 years, and costs as in Table 9-11.

**Table 9-11**

| | Manual | System A | System B |
|---|---|---|---|
| **Equipment** | | | |
| Computer | | $200 000 | $200 000 |
| Printer | | 20 000 | 10 000 |
| Video | | 5 000 | 2 000 |
| Disk Drives | | 4 000 | 4 000 |
| Remote Ports | | 2 000 | 0 |
| Disks | | 5 000 | 2 000 |
| | | TOTAL $236000 | TOTAL $218000 |
| **Software** | | 100 000 | 50 000 |
| **Annual Manpower** | $100 000 | 80 000 | 40 000 |
| **Annual Overhead** | 50 000 | 20 000 | 40 000 |

We compare either computer system to the manual technology by the difference method of Problem 9.15, choosing present-worth cost as economic parameter. For system B versus manual,

$$\text{APW} = \$268\,000 + (\$80\,000 - \$150\,000)(P/A, 15\%, 4)$$
$$= \$68\,154.17 > 0$$

and for system A versus manual,

$$\text{APW} = \$336\,000 + (\$100\,000 - \$150\,000)(P/A, 15\%, 4)$$
$$= \$193\,252.98 > 0$$

The strict conclusion is that the current manual technology should be retained. However, a consideration of the irreducibles (e.g., improved output quality when the job is done by computer) might make system B more attractive.

**9.17** Illustrate the reinvestment fallacy by supposing that, in Problem 9.15, the revenues from system 2 could be reinvested at 40% in years 3 through 5.

At the end of five years, the net future worth of system 1 is:

$$\text{FW}_1 = -\$50\,000(F/P, 15\%, 5) + \$22\,000(F/A, 15\%, 5)$$
$$= -\$50\,000(2.0114) + \$22\,000(6.7424) = \$47\,762.80$$

and the net future worth of system 2 is (draw a time diagram):

$$\text{FW}_2 = -\$75\,000(F/P, 15\%, 5) + \$24\,000(F/P, 15\%, 1)(F/P, 40\%, 3) + \$24\,000(F/A, 40\%, 4)$$
$$= -\$75\,000(2.0114) + \$24\,000(1.15)(2.7440) + \$24\,000(7.1040)$$
$$= \$95\,375.40$$

The effect of the reinvestment is to reverse the conclusion of Problem 9.15: now, system 2 is the better.

## Supplementary Problems

**9.18** The executives of the XYZ Company are considering the three independent proposals whose cash flows are given in Table 9-12, The MARR is 10%. Evaluate these three proposals by the NPV method.

**Table 9-12**

| End of Year | Proposal A | Proposal B | Proposal C |
|---|---|---|---|
| 0 | −$50 000 | −$80 000 | −$150 000 |
| 1 | 16 461.50 | 28 021.60 | 45 000 |
| 2 | 16 461.50 | 28 021.60 | 45 000 |
| 3 | 16 461.50 | 28.021.60 | 45 000 |
| 4 | 16 461.50 | 28 021.60 | 45 000 |

*Ans.* $\text{NPV}_A = \$2180.87$, $\text{NPV}_B = \$8824.93$, $\text{NPV}_C = -\$7355.69$; proposal C is unacceptable.

**9.19** Rework Problem 9.18 for MARR = 15%.
*Ans.* $\text{NPV}_A = -\$3003.40$, $\text{NPV}_B = 0$, $\text{NPV}_C = -\$21\,527.68$; proposals A and C are unacceptable and proposal B is barely acceptable.

**9.20** Evaluate the three proposals in Problem 9.18 using the ROR method and linear interpolation.
*Ans.* $i_A^* = 12\%$, $i_B^* = 15\%$, $i_C^* = 7.71\%$; since $i_C^* <$ MARR, proposal C is unacceptable.

**9.21** The capital budgeting committee of the ABC Company is contemplating five independent proposals for projects to be included in the forthcoming year's budget; their cash flows are given in Table 9-13. The ABC Company has established a MARR of 20%. Assuming that capital is not rationed, which projects should the company select and what is the total investment required? Use the ROR method.

**Table 9-13**

| End of Year | Project 1 | Project 2 | Project 3 | Project 4 | Project 5 |
|---|---|---|---|---|---|
| 0 | −$100 000 | −$200 000 | −$150 000 | −$80 000 | −$300 000 |
| 1 | 35 027 | 77 258 | 63 516 | 32000 | 98 769 |
| 2 | 35 027 | 77 258 | 63 516 | 32000 | 98 769 |
| 3 | 35 027 | 77 258 | 63 516 | 32000 | 98 769 |
| 4 | 35 027 | 77 258 | 63 516 | 32000 | 98 769 |

*Ans.* $i_1^* = 15\%$, $i_2^* = 20\%$, $i_3^* = 25\%$, $i_4^* = 21.85\%$ (by interpolation), $i_5^* = 12\%$. Projects 2, 3, and 4 should be selected. at a total investment of $430 000.

**9.22** Would the results of Problem 9.21 change if (a) MARR = 10%? (b) MARR = 13% and capital is rationed at $430 000? *Ans.* (a) no; (b) no

**9.23** How would the results of Problem 9.21 change if the MARR is 13%, the budget constraint is $480 000, and all the projects are divisible into smaller projects?
Ans. Select projects 2, 3, and 4, and spend $50 000 on project 1.

**9.24** How would the results of Problem 9.23 change if the MARR is 16% and none of the projects are divisible? *Am.* Select projects 2, 3, and 4, and leave $50 000 unspent.

**9.25** Would the results of Problem 9.24 change if all the projects were divisible? *Ans.* no

**9.26** The executives of the XYZ Company are attempting to determine the economically best process-control computer to purchase for one of their production lines. The choice has been narrowed to the five mutually exclusive alternatives whose cash flows are presented in Table 9-14. If capital is not rationed and the MARR is 7%, which computer should the company purchase? *Ans.* C or *D*

**Table 9-14**

| End of Year | Computer A | Computer B | Computer C | Computer D | Computer E |
|---|---|---|---|---|---|
| 0 | −$20 000 | −$30 000 | −$28 000 | −$35 000 | −$25 000 |
| 1 | 6 309.40 | 9 057.60 | 9 218.44 | 11 523.05 | 7 886.75 |
| 2 | 6 309.40 | 9 057.60 | 9 218.44 | 11 523.05 | 7 886.75 |
| 3 | 6 309.40 | 9 057.60 | 9 218.44 | 11 523.05 | 7 886.75 |
| 4 | 6 309.40 | 9 057.60 | 9 218.44 | 11 523.05 | 7 886.75 |

**9.27** Will the result of Problem 9.26 change if the MARR is (a) 10%? (b) 11%? (c) 13%?
*Ans.* (a) no; (b) no; (c) yes (none of the alternatives is acceptable)

**9.28** Rework Problem 9.26 if capital is rationed at (a) $30 000, (b) $28 000, (c) $25 000.
*Am.* (a) C; (b) C; (c) E

**9.29** For what combinations of capital rationing and MARR values in Problem 9.26 would computer B be preferred to computer C? ***Ans.*** None: C dominates B (Section 9.2).

**9.30** The executives of the ABC Company are trying to select the most economical feeder machine. Cash flows for the six available models are shown in Table 9-15. The MARR is 8%. (*a*) Compute the ROR for each alternative model. (*b*) Determine which model should be purchased.
***Ans.*** (*a*) $i_A^* = 10\%$, $i_B^* = 25\%$, $i_C^* = 18\%$, $i_D^* = 20.5\%$, $i_E^* = 19.5\%$, $i_F^* = 20.5\%$; (*b*) F

**Table 9-15**

| End of Year | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 0 | −$50 000 | −$100 000 | −$200 000 | −$220 000 | −$250 000 | −$380 000 |
| 1 | 13 190 | 37185 | 63991.20 | 74 386.40 | 82 693.50 | 128 485.60 |
| 2 | 13 190 | 37185 | 63991.20 | 74 386.40 | 82 693.50 | 128 485.60 |
| 3 | 13 190 | 37 185 | 63 991.20 | 74 386.40 | 82 693.50 | 128 485.60 |
| 4 | 13 190 | 37 185 | 63 991.20 | 74 386.40 | 82 693.50 | 128 485.60 |
| 5 | 13 190 | 37 185 | 63 991.20 | 74 386.40 | 82 693.50 | 128 485.60 |

**9.31** Which model should be purchased in Problem 9.30, if there is a budget constraint of (*a*) $280 000? (*b*) $230 000? (*c*) $210 000? ***Ans.*** (*a*) E; (*b*) D; (*c*) C

**9.32** How would the result of Problem 9.30(*b*) change if the MARR is (*a*) 9%? (*b*) 10%? (*c*) 12%? (*d*) 15%? (*e*) 20%? ***Ans.*** (*a*)–(*e*) no change

**9.33** How would the result of Problem 9.30(*b*) change if MARR = 15% and the budget constraint is (*a*) $210 000? (*b*) $280 000? ***Ans.*** (*a*) B (the only choice); (*b*) D

**9.34** The KJL Company is contemplating five independent projects, with cash flows as in Table 9-16. The MARR is 12% and the budget constraint is $200000. (*a*) Compute the rate of return for each project. (*b*) What is the optimum portfolio, if the minimum desired payback period (Section 8.3) is two years?
***Ans.*** (*a*) $i_A^* = i_B^* = i_C^* = i_D^* = i_E^* = 15\%$. (*b*) $PBP_A = PBP_B = PBP_C = 2.28$ years, $PBP_D = 2.055$ years $PBP_E = 2.0$ years; the only acceptable choice is project E.

**Table 9-16**

| End of Year | Project A | Project B | Project C | Project D | Project E |
|---|---|---|---|---|---|
| 0 | −$100 000 | −$50 000 | −$75 000 | −$60 000 | −$95 000 |
| 1 | 43798 | 21 899 | 38 848.50 | 30000 | 50 000 |
| 2 | 43 798 | 21 899 | 38 848.50 | 29000 | 45000 |
| 3 | 43798 | 21 899 | 38 848.50 | 18228 | 26 609 |

**9.35** How would the result of Problem 9.34(*b*) change if (*a*) the minimum desired payback period is 2.2 years? (*b*) the minimum desired payback period is 2.2 years and the budget constraint is $150 000?
***Ans.*** (*a*) D, or E, or D and E; (*b*) D or E

**9.36** Two alternative cleaning machines are being considered as replacements for an older, worn-out cleaner. The cash flows for the two mutually exclusive alternatives are presented in Table **9-17;** the MARR is **10%.** ***(a)*** Compute the present-worth difference of value (APW) between the two machines. ***(b)*** Compute the incremental rate of return (Ai*) on the investment difference between the two machines. (c) Which machine should be purchased? ***Ans.*** ***(a)*** **$816.33;** ***(b)*** **37.9%;** ***(c)*** machine #2

**Table 9-17**

| End of Year | Machine #1 | Machine #2 |
|---|---|---|
| **0** | **−$20 000** | **−$28 000** |
| **1** | **4 864.60** | **8 419.88** |
| **2** | **4 864.60** | **8 419.88** |
| **3** | **4 864.60** | **8 419.88** |
| **4** | **4 864.60** | **8 419.88** |
| **5** | **4 864.60** | **8 419.88** |
| **6** | **4 864.60** | **8 419.88** |

# Chapter 10

# Equipment Replacement and Retirement

## 10.1 RETIREMENT AND REPLACEMENT DECISIONS

The decision to replace equipment or to retire it (to take the equipment out of service without replacing it) can be motivated by the physical impairment of the equipment, its obsolescence, or external economic conditions. Retirement and replacement decisions should always be based on economics rather than on whether or not the equipment has reached the end of its physical service life. A piece of equipment may have many years of service life remaining beyond the point at which it has become uneconomical to operate it.

All past investments and expenses connected with the equipment are ***sunk costs,*** which do not enter into a **retirement/replacement** decision. ***Only current and future costs and investments are relevant.***

## 10.2 ECONOMIC LIFE OF AN ASSET

As operating equipment ages, the usual pattern is for its capital costs to decline while its operating costs rise. When summed, these two cost functions often result in a cost function that is generally U-shaped. Ideally, the equipment should be retired at the lowest point on this total cost function.

**Example 10.1** A machine has an initial cost of \$10 000. Being a special-purpose custom-built unit, it can only be resold as scrap at \$500, no matter what its age. The machine has a 10-year service life. The annual operating costs are \$2000 for each of the first two years, with an increase of \$600 per year thereafter. The MARR is 10%. When is the optimum time to retire the machine?

The cost curve for this type of problem is the net EUAC, evaluated at the MARR, as a function of time. From Section 7.4, we know that the net EUAC will be made up of two components: the capital recovery cost,

$$\mathrm{CR}(j) = (\$10\,000 - \$500)\,(A/P, 10\%, j) + (0.10)(\$500) \tag{1}$$

and the equivalent annualized operating cost,

$$A(j) = \$1400 + \$600(A/G, 10\%, j) + \$600(P/F, 10\%, 1)\,(A/P, 10\%, j) \tag{2}$$

In deriving (*1*), the constancy of the salvage value was used; in deriving (*2*), the gradient series was extended backwards by writing the first year's cost as \$1400 + \$600. In both expressions, $j$ is the time, in years; thus, to keep the machine for 4 years would cost the company CR(4) + A(4) per year.

Substituting $j = 1, 2, \ldots, 10$ in (*1*) and (*2*), we generate Table 10-1; the points are plotted in Fig. 10-1. The data show that the machine should be retired at the end of 7 years.

The time interval for which the EUAC of an asset is smallest (7 years, for the machine of Example 10.1) is called the ***economic life*** of the asset. As we have seen, the economic life is given analytically as that value $j^*$ at which

$$\mathrm{EUAC}(j) = \mathrm{CR}(j) + A(j)$$

is minimized.

The concept of economic life may also furnish the basis for a replacement decision, particularly when service lives are not precisely known, when salvage values in each year are not known, or when the equipment obsolesces rapidly.

**Table 10-1**

| Years of Service Life, $j$ | $CR(j)$ | $A(j)$ | $EUAC(j) = CR(j) + A(j)$ |
|---|---|---|---|
| 1 | $10 500.00 | $2 000.00 | $12 500.00 |
| 2 | 5 523.81 | 2000.00 | 7 523.81 |
| 3 | 3 870.05 | 2181.24 | 6 051.29 |
| 4 | 3 046.97 | 2400.77 | 5 447.74 |
| 5 | 2 556.10 | 2629.99 | 5 186.09 |
| 6 | 2 231.30 | 2859.40 | 5 090.70 |
| 7 | 2 001.40 | 3085.07 | 5 086.47 = min. |
| 8 | 1 830.68 | 3304.85 | 5 135.53 |
| 9 | 1 699.58 | 3518.12 | 5 217.70 |
| 10 | 1 596.13 | 3724.16 | 5 320.29 |

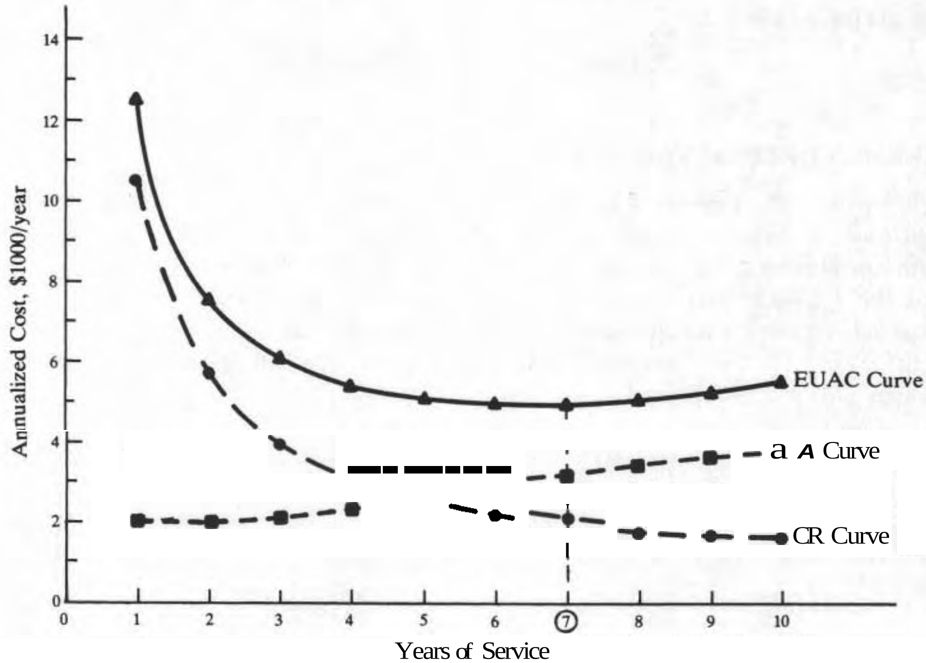


**Fig. 10-1**

**Example** 10.2 The XYZ Company purchased a very specialized machine three years ago for $25000. This machine is not readily salable and is assumed to have a zero salvage value. Operating costs are expected to be $10 000 next year, and to increase by $800 per year thereafter. The company has an opportunity to replace the existing machine with another specialized one that will cost $12 000. This machine has no salvage value, a useful life of 10 years, and operating costs of $5000 in the first year, with an annual increase of $1200 thereafter. If the MARR is 15%, should the company replace the old machine with the new one?

For the new machine:

| Year, $j$ | $\text{EUAC}(j) = \$12\,000(A/P, 15\%, j) + \$5000 + \$1200(A/G, 15\%, j)$ |
|---|---|
| 1 | \$18 800.00 |
| 2 | 12 940.52 |
| 3 | 11 344.16 |
| 4 | 10 793.96 |
| $5 = j^*$ | 10 647.20 |
| 6 | 10 687.52 |
| ... | (increases) |

The economic life of the new machine is thus 5 years, with a corresponding EUAC of \$10 647.20.

For the old machine,

$$\text{EUAC}(j) = A(j) = \$10\,000 + \$800(A/G, 15\%, j)$$

which is a strictly increasing function of $j$. Hence, $j^* = 1$ year, with a corresponding EUAC of \$10 000.

The old machine should be kept for its economic life of one more year, since its EUAC of \$10 000 is less than the EUAC of \$10 647.20 for the new machine. At the end of that year, the analysis should be repeated and updated to take any new information into account. If there is no new information, and the above data are still valid, then at the end of next year the old machine should be replaced by the new one, because, at that time, the EUAC for the old machine will be

$$\$10\,000 + \$800 = \$10\,800 > \$10\,647.20$$

## 10.3 RETIREMENT/REPLACEMENT ECONOMICS

The decision to retire a piece of equipment is seldom taken without replacing that equipment. Thus, in most cases, a joint retirernentlreplacement decision is made.

The optimum retirementlreplacement point is that point where the EUAC curve of the old machine and the EUAC curve of the new machine intersect. Thus, if in Example **10.1** a new replacement machine were available whose EUAC was **\$5500** at five years and **\$4000** at all years beyond the fifth, then the old machine should be retired at the end of the fifth year, and replaced with the cheaper new machine.

**Example** 10.3 The XYZ Company owns a **4-year-old** pump that originally cost \$3000. For the past four years the operating and maintenance costs of this pump have been:

| Year of Service, k | Operating and Maintenance Costs, $C_k$ |
|---|---|
| 1 | \$ 90 |
| 2 | 180 |
| 3 | 560 |
| 4 | 950 |

The company originally planned to keep this pump 8 years. If the pump is retained, the expected future operating and maintenance costs will be:

| Year of Service, k | Operating and Maintenance Costs, $C_k$ |
|---|---|
| 5 | \$1125 |
| 6 | 1500 |
| 7 | 1700 |
| 8 | 2000 |

The pump could be sold today as a used pump for $SV_0 = \$1200$. It is expected that the pump could be sold a year from now for \$900; two years from now for \$800; and afterwards for \$500. A new, energy-saving pump, with expected service life of 8 years, has just become available for $P = \$4000$. Its costs are as follows:

| *Year of Service, j* | *Operating and Maintenance Costs, $C'_j$* |
|---|---|
| 1 | $ 40 |
| 2 | 80 |
| 3 | 260 |
| 4 | 450 |
| 5 | 625 |
| 6 | 1000 |
| 7 | 1200 |
| 8 | 1500 |

It is estimated that if this new pump is purchased it could be sold one year later for **$3100,** two years later for **$2000,** three years later for **$1500,** four years later for **$1000,** and thereafter for **$900.** Given a MARR of 15%, should the **XYZ** Company replace the old machine now, or at some later time?

In computing the EUAC curve of the old pump, the sunk costs of years **1** through **4** are disregarded. Thus, $k = 5$ becomes $j = 1$, and we have:

$$\text{EUAC}(j) = \text{CR}(j) + A(j) \qquad (j = 1, \ldots, 4) \tag{1}$$

where
$$\text{CR}(j) = (\text{SV}_0 - \text{SV}_j)(A/P, 15\%, j) + (0.15)\text{SV}_j$$
$$A(j) = [C_1(P/F, 15\%, 1) + \cdots + C_j(P/F, 15\%, j)](A/P, 15\%, j)$$

Similarly, for the new pump,

$$\text{EUAC}'(j) = \text{CR}'(j) + A'(j) \qquad (j = 1, \ldots, 8) \tag{2}$$

where
$$\text{CR}'(j) = (P - \text{SV}'_j)(A/P, 15\%, j) + (0.15)\text{SV}'_j$$
$$A'(j) = [C'_1(P/F, 15\%, 1) + \cdots + C'_j(P/F, 15\%, j)](A/P, 15\%, j)$$

Evaluating (1) for $j = 1, \ldots, 4$ and (2) for $j = 1, \ldots, 8$, we generate Table **10-2.** It is seen that all the EUAC-values for the new pump are below the lowest EUAC-value for the old pump. Hence, the old pump should immediately be replaced by the new one. Even in the worst case, where the new pump is kept for only two years, its EUAC would still be lower than that for keeping the old pump for even one more year. The reader should note the oscillation of the EUAC curve for the new pump; the curve is not of the simple form shown in Fig. **10-1.**

**Table 10-2**

| Years From Now, j | EUAC($j$) | EUAC$'$($j$) |
|---|---|---|
| 1 | $1605.00 | $1540.00 |
| 2 | 1663.90 | 1588.84 |
| 3 | 1796.30 | 1436.55 |
| 4 | 1852.18 | 1384.18 |
| 5 | | 1402.41 |
| 6 | | 1371.86 |
| 7 | | 1368.52 |
| 8 | | 1387.94 |

The question of how long the new pump should be kept is readily answered from its EUAC data: its economic life is **7** years. Of course, the unexpected advent of some new technology could alter these plans, just as the availability of the new pump altered the XYZ Company's original plans.

## Comparative Use Value

It is sometimes useful to compare **EUAC's** via the ***comparative use value*** (CUV) of the current equipment relative to the replacement equipment. Let subscripts 1 and 2 pertain to the current and replacement machines, respectively; after $n_1$ and $n_2$ years of service, their annualized operating and maintenance costs are $A_1(n_1)$ and $A_2(n_2)$, and their salvage values are $SV_1$ and $SV_2$. Then, the CUV is given by

$$\begin{aligned}(\text{CUV} - SV_1)(A/P, i\%, n_1) + i\, SV_1 + A_1(n_1) \\ = (P_2 - SV_2)(A/P, i\%, n_2) + i\, SV_2 + A_2(n_2)\end{aligned} \tag{10.1}$$

where $P_2$ is the initial cost of machine 2. By *(10.1)*, the CUV is the putative market price of machine 1 that makes $\text{EUAC}_1(n_1) = \text{EUAC}_2(n_2)$. If the actual market or current salvage value, $P_1$, is known, (10.1) can be solved algebraically to yield

$$\text{CUV} = P_1 + [\text{EUAC}_2(n_2) - \text{EUAC}_1(n_1)]\,(P/A, i\%, n_1) \tag{10.2}$$

That is, the CUV of the current machine with respect to a replacement machine is the market value of the current machine plus the present value of all annual cost savings realized over the service period of the current machine.

It is evident from (10.2) that $\text{CUV} > P_1$ if and only if $\text{EUAC}_2(n_2) > \text{EUAC}_1(n_1)$; in this case, and only then, the current machine should be kept.

**Example 10.4** Compute the **CUV** for keeping the old pump of Example 10.3 another four years, as against buying the new pump and keeping it seven years.

Substituting the numerical values from Example 10.3 and Appendix A into *(10.2)*, we find

$$\begin{aligned}\text{CUV} &= \$1200 + (\$1368.52 - \$1852.18)(0.35027)^{-1} \\ &= \$1200 - \$1381 = -\$181\end{aligned}$$

The CUV is less than the current salvage value by $1381, which amount represents the present-worth loss that would be incurred in keeping the old pump four more years.

## Present-Worth Method for Equal Service Periods

If machines 1 and 2 have present-worth *costs* $PW_1$ and $PW_2$, and service periods $n_1 = n_2 = n$, then

$$PW_1 = \text{EUAC}_1(n)\,(P/A, i\%, n) \qquad PW_2 = \text{EUAC}_2(n)\,(P/A, i\%, n)$$

and so

$$PW_1 - PW_2 = [\text{EUAC}_1(n) - \text{EUAC}_2(n)]\,(P/A, i\%, n)$$

Thus, a comparison of **EUAC's** may be replaced by a comparison of **PW's** (although usually there is no computational advantage in so doing).

## Short-Study-Period Method for Unequal Service Periods

In the case $n_1 \neq n_2$, we might compare the costs of a series of machines 1 and a series of machines 2 over the least common multiple of $n_1$ and $n_2$; this approach will be illustrated in Section 10.4. However, if we do not want to assume a continual substitution of machines in kind, we may restrict the study period to the smallest of $n_1$, $n_2$, and the forecasting horizon. Thus, only known data are included and tenuous estimates are ruled out. In this short-study-period approach, any "unused" values or costs are distributed back over the study period.

**Example 10.5** The ABC Company is contemplating replacing its current milling machine with an improved machine. Data are as shown in Table 10-3. The executives at the ABC Company do not feel that any estimates beyond 10 years are accurate for decision making purposes. If the MARR is 15%, should the company replace the current machine with the new improved one?

The length of the study period is the minimum of 15, 10, and 10 years; i.e., 10 years. The present-worth *cost* of 15 more years of service from the current machine is

$$\begin{aligned}PW_1 &= \$500 - \$100(P/F, 15\%, 15) + \$1000(P/A, 15\%, 15) \\ &= \$500 - \$100(8.1371)^{-1} + \$1000(0.17102)^{-1} = \$6334.99\end{aligned}$$

**Table 10-3**

| | Salvage Value | | Original Cost | Annual Cost | Service Life |
|---|---|---|---|---|---|
| | Now | End of Life | | | |
| Current Machine | $500 | $100 | $5000 | $1000 | 15 more years |
| Improved Machine | — | 500 | 7000 | 375 | 10 years |

Annualizing this over the 10-year study period gives

$$EUAC_1(10) = (\$6334.99)\,(A/P, 15\%, 10) = (\$6334.99)\,(0.19925) = \$1262.25$$

For the improved machine:

$$\begin{aligned} EUAC_2(10) &= (\$7000 - \$500)\,(A/P, 15\%, 10) + (0.15)\,(\$500) + \$375 \\ &= \$6500(0.19925) + \$75 + \$375 = \$1745.13 \end{aligned}$$

Thus, keeping the current machine is $\$1745.13 - \$1262.25 = \$482.88$ cheaper (per year for 10 years) than replacing it with the improved machine.

**Example 10.6** Suppose that in Example 10.5 the company executives feel they cannot accurately forecast annual costs beyond five years into the future. All other data remain the same. Using a five-year study period, we obtain for the current machine:

$$EUAC_1(5) = (\$6334.99)\,(A/P, 15\%, 5) = \$1889.85$$

and for the improved machine:

$$EUAC_2(5) = (\$7000 - \$500)\,(A/P, 15\%, 5) + (0.15)\,(\$500) + \$375 = \$2389.08$$

The numbers are different, but the decision is the same as before: it is cheaper to keep the current machine. The difference between the alternatives in the 5-year study is $\$2389.08 - \$1889.85 = \$499.23$, which is slightly larger than the $482.88 difference found in the 10-year study of Example 10.5. In this case, using the shorter study period did not reduce the amount of discrimination between the alternatives.

## 10.4 REPLACEMENT ASSUMPTION FOR UNEQUAL-LIVED ASSETS

Equipment investment decisions frequently involve the comparison of assets with unequal lives. In most businesses, a piece of equipment will be replaced with a like one at the end of its useful life, in order for the firm to continue operating. When this is the case, the cash flows of the unequal-lived assets should be estimated into the future until the least common multiple of their individual useful lives has been reached. The EUAS method can then be readily applied.

**Example 10.7** A company can purchase either of two alternative machines, A and B, with the following characteristics:

| | $P$ | $A$ | $n$ |
|---|---|---|---|
| Machine A | $15 000 | $6000 | 5 |
| Machine B | 9 000 | 3000 | 3 |

Here, $P$ is the initial outlay, $A$ is the annual net cash flow, and $n$ is the useful life of the machine. It is assumed that when either machine is at the end of its life, a similar replacement machine will be purchased. Which machine should be purchased, under a MARR of 10%?

Machines A and B will first reach a common multiple of their individual useful lives at the end of fifteen years. The cash flows for this fifteen-year period are displayed in Table 10-4. Note the replacements of machine A by exact replicas at the end of five and ten years; and of machine B, at the end of three, six, nine, and twelve years. (Both machines are also replaced at the end of fifteen years, but those costs belong to the next study period.)

**Table 10-4**

| End of Year | Machine A | Machine B |
|---|---|---|
| 0 | −$15 000 | −$9000 |
| 1 | 6 000 | 3000 |
| 2 | 6000 | 3000 |
| 3 | 6 000 | 3000 − 9000 |
| 4 | 6 000 | 3000 |
| 5 | 6000−15000 | 3000 |
| 6 | 6 000 | 3000 − 9000 |
| 7 | 6000 | 3000 |
| 8 | 6000 | 3000 |
| 9 | 6000 | 3000 − 9000 |
| 10 | 6000−15000 | 3000 |
| 11 | 6000 | 3000 |
| 12 | 6000 | 3000 − 9000 |
| 13 | 6 000 | 3000 |
| 14 | 6000 | 3000 |
| 15 | 6000 | 3000 |

Proceeding as in Example 7.4, we compute:

$$\begin{aligned}\text{EUAS}_A &= -\$15\,000(A/P, 10\%, 5) + \$6000 \\ &= \$2043 \text{ for fifteen years} \\ \text{EUAS}_B &= -\$9000(A/P, 10\%, 3) + \$3000 \\ &= -\$619 \text{ for fifteen years}\end{aligned}$$

The conclusion is that machine B is unacceptable, and that machine A should be purchased.

# Solved Problems

**10.1** Mr. Jones bought a new car in September 1981 for $7800. He paid $2400 down, and financed the balance with a loan at 18% nominal interest to be repaid in 35 monthly installments of $199.42 each. It is understood that if Mr. Jones fails to make a payment, the car will be repossessed by the loan company and, though Mr. Jones will owe nothing, he will lose all money already paid. He can also pay the outstanding balance of the loan at any time. Twelve months after the transaction, Mr. Jones's balance is $3855. Thanks to a good deal, Mr. Jones has this amount and is going to pay off the loan. At this point, Mr. Jones's brother offers him an essentially identical car (they bought the same model the same day, and the use and maintenance of both cars have been very similar) for $3500. Mrs. Jones would prefer to keep their current car "because we have already spent almost $4800 on it." What is the sensible decision to make?

This is a clear and very common instance of a sunk cost: Mrs. Jones is wrong in her analysis. While it is true that they have invested a large amount of money so far, it is also true that this money will not be recovered, no matter what decision is taken. The sole question is whether Mr. Jones should acquire a car for \$3855 or acquire a car for \$3500. Provided the two cars are physically equivalent, the answer is obvious.

**10.2** A machine can be sold now for \$15 000; if kept for another year, its salvage value will decline to \$13 000. The operating expenses for this year are expected to be \$30 000. A new machine is available for \$50 000, with expected operating expenses of \$18 000 for the first year, increasing by \$1000 a year because of deterioration. It is believed that after 5 years new technology would make replacement necessary; the new machine's salvage value at that time is estimated to be \$20 000. The MARR is 20%. Should the new machine be acquired?

By (*10.1*),

$$(\text{CUV} - \$13\,000)(A/P, 20\%, 1) + (0.20)(\$13\,000) + \$30\,000$$
$$= (\$50\,000 - \$20\,000)(A/P, 20\%, 5) + (0.20)(\$20\,000) + \$18\,000 + \$1000(A/G, 20\%, 5)$$

whence CUV = \$13 893.25. Since the comparative use value of the old machine is smaller than its net market value (\$15 000), the machine should be replaced.

**10.3** A contractor can purchase a used machine for \$1000. The market value of the machine is expected to decrease \$70 the first year and \$60 per year, the second and third years. Operating disbursement is estimated at \$8000 the first year and is expected to increase by \$175 a year thereafter. An alternative is to buy a new machine costing \$10 000. It is believed that the salvage value of this machine will decrease by 15% each year over a maximum service life of 20 years. The operating expenses are estimated at \$6050 the first year and are expected to increase by \$135 a year after that. If the MARR is 10%, which machine should be bought, and when should that machine be replaced? (Assume that the used machine is unique, but that new machines are always available.)

For the used machine:

$$\text{EUAC}_1(1) = (\$1000 - \$930)(A/P, 10\%, 1) + (0.10)(\$930) + \$8000 = \$8170.00$$
$$\text{EUAC}_1(2) = (\$1000 - \$870)(A/P, 10\%, 2) + (0.10)(\$870) + \$8000 + \$175(A/G, 10\%, 2) = \$8245.24$$
$$\text{EUAC}_1(3) = (\$1000 - \$810)(A/P, 10\%, 3) + (0.10)(\$810) + \$8000 + \$175(A/G, 10\%, 3) = \$8321.30$$

For the new machine:

$$\text{EUAC}_2(1) = [\$10\,000 - (0.85)(\$10\,000)](A/P, 10\%, 1) + (0.10)(0.85)(\$10\,000) + \$6050 = \$8550.00$$
$$\text{EUAC}_2(2) = [\$10\,000 - (0.85)^2(\$10\,000)](A/P, 10\%, 2) + (0.10)(0.85)^2(\$10\,000) + \$6050 + \$135(A/G, 10\%, 2) = \$8435.71$$
$$\text{EUAC}_2(3) = \$8342.20$$
$$\text{EUAC}_2(4) = \$8206.37$$
$$\text{EUAC}_2(5) = \$8435.11$$

The used machine should be bought and kept for one year (its economic life). At that time, the new machine should be bought and kept for four years (its economic life).

**10.4** A machine costs \$10 000 and is expected to have scrap value \$1500 whenever it is retired. The operating disbursements for the first year are expected to be \$1500 and they will then increase \$400 per year, as a result of deterioration. If the MARR is 15%, determine the machine's economic life.

For $j = 1, 2, \ldots,$

$$CR(j) = (\$10\,000 - \$1500)(A/P, 15\%, j) + (0.15)(\$1500)$$
$$A(j) = \$1500 + \$400(A/G, 15\%, j)$$
$$EUAC(j) = CR(j) + A(j)$$

**The evaluations, Table *10-5*, give an economic life of *8* years.**

**Table 10-5**

| Years of Service, $j$ | $CR(j)$ | $A(j)$ | $EUAC(j)$ |
|---|---|---|---|
| 1 | \$10 000.00 | \$1500.00 | \$11 500.00 |
| 2 | 5 453.52 | 1686.05 | 7 139.57 |
| 3 | 3 947.83 | 1862.85 | 5 810.68 |
| 4 | 3 207.30 | 2030.52 | 5 232.82 |
| 5 | 2 760.72 | 2189.12 | 4 949.84 |
| 6 | 2 471.04 | 2338.88 | 4 809.92 |
| 7 | 2 268.06 | 2479.96 | 4 748.02 |
| 8 | 2 119.23 | 2612.52 | 4 731.75 |
| 9 | 2 006.35 | 2736.88 | 4 743.23 |
| 10 | 1 918.63 | 2853.28 | 4 771.91 |

**10.5** A plant is considering buying a second-hand machine to use as stand-by equipment. The machine costs ***\$3000*** and has an economic life of ***10*** years, at which time its salvage value is ***\$600;*** expected annual operating costs are ***\$100.*** Without a stand-by machine, the plant would have to shut down an average of seven days a year at a cost of ***\$50*** per day. If the **MARR** is **10%,** is it expedient to buy the stand-by machine?

$$EUAC_{\text{shut-down}} = (7)(\$50) = \$350$$
$$EUAC_{\text{stand-by}} = (\$3000 - \$600)(A/P, 10\%, 10) + (0.10)(\$600) + \$100 = \$550$$

**The stand-by machine should not be purchased.**

**10.6** XYZ Company is considering replacing a machine. The new improved machine will cost ***\$16000*** installed; it will have an estimated service life of 8 years and ***\$3000*** salvage value. It is estimated that operating expenses will average ***\$1000*** a year. The present machine was purchased for ***\$20 000*** four years ago and is ,estimated to have 8 more years of service life, at the end of which its salvage value will be ***\$2000.*** Operating costs are ***\$1800*** per year. If replaced now, it can presumably be sold for ***\$5000.*** Using a **MARR** of **15%,** determine whether to replace the existing machine.

**Compare present-worth costs over the next *8* years.**

$$PW_{\text{new}} = \$16000 + \$1000(P/A, 15\%, 8) - \$3000(P/F, 15\%, 8) = \$19\,507$$
$$PW_{\text{existing}} = \$5000 + \$1800(P/A, 15\%, 8) - \$2000(P/F, 15\%, 8) = \$12\,423$$

**Do not replace the machine.**

**10.7** Consider the replacement situation indicated in Table ***10-6.*** If estimates beyond ***8*** years are unreliable and if the **MARR** is **15%,** decide whether it is expedient to replace the current equipment.

**The study period is limited by the current equipment and the forecast horizon to *8* years.**

$$EUAC_{\text{current}}(8) = (\$8000 - \$1000)(A/P, 15\%, 8) + (0.15)(\$1000) + \$3600 = \$5310$$

**Table 10-6**

| | Salvage Value | | Original Cost | Annual Cost | Service Life |
|---|---|---|---|---|---|
| | Now | End of Life | | | |
| Current Equipment | $8000 | $1000 | $17 000 | $3600 | 8 more years |
| Replacement Candidate | | 6000 | 19 000 | 800 | 15 years |

For the replacement candidate, the present-worth cost is given by

$$\mathrm{PW}_{\text{replacement}} = \$19\,000 + \$800(P/A,\,15\%,\,15) - \$6000(P/F,\,15\%,\,15) = \$22\,940$$

Annualizing over the 8-year period,

$$\mathrm{EUAC}_{\text{replacement}}(8) = \$22\,940(A/P,\,15\%,\,8) = \$5112$$

Replace the current equipment.

**10.8** To keep an existing machine going for a number of years, an extensive (and expensive: $4000) overhaul is needed. Maintenance is expected to be $2000 annually for the next 2 years and to increase by $1000 per year after that. The machine has no present or future salvage value. An alternative machine costs $8000 and, owing to its specialized nature, it has no salvage value after it is installed. Maintenance expenses are expected to be $1000 the first year, increasing by $500 per year in subsequent years. If the MARR is 15%, determine the best course of action.

For the defender, writing the first year's maintenance as $1000 + $1000,

$$\mathrm{EUAC}_1(j) = \$4000(A/P,\,15\%,\,j) + \$1000(P/F,\,15\%,\,1)(A/P,\,15\%,\,j) + \$1000 + \$1000(A/G,\,15\%,\,j)$$

and for the challenger,

$$\mathrm{EUAC}_2(j) = \$8000(A/P,\,15\%,\,j) + \$1000 + \$500(A/G,\,15\%,\,j)$$

From the evaluations in Table 10-7, we see that the economic lives are $j_1^* = 4$ years and $j_2^* = 7$ years. It follows that the best course of action is to overhaul the current equipment and keep it for four years. Then (provided an analysis indicates that things remain as expected), buy the new equipment and keep it for seven years.

**Table 10-7**

| $j$ | $\mathrm{EUAC}_1(j)$ | $\mathrm{EUAC}_2(j)$ |
|---|---|---|
| 1 | $6600 | $10 200 |
| 2 | 4460 | 6 154 |
| 3 | 4040 | 4 957 |
| 4 | 4032 | 4 465 |
| 5 | 4175 | 4 248 |
| 6 | | 4 163 |
| 7 | | 4 148 |
| 8 | | 4 173 |

**10.9** It is necessary to pump twice as much water as can be handled by the existing small pump, which is now 5 years old. This pump can be sold now for $1200 or kept for 5 years, after which it will have zero salvage value. Operating expenses are $3000 per year. If the pump is kept, a similar one must be purchased for $3500, with operating costs of $2500 per year; its salvage values after 5 and 10 years are the same as for the original pump. A large pump, equal in capacity to the two small pumps, costs $6000, with operating expenses of $4500 per year. New machines have economic lives of 10 years, and zero salvage values at that date. Analyze the situation, if the MARR is 10%.

There are 3 possible alternatives: (1) to replace the present pump by a new, large pump; (2) to buy a new small pump now and every fifth year hereafter; (3) to buy a new small pump now, and after 5 years to sell it and install a single large pump. We calculate the present-worth costs for a 10-year study period.

*plan 1* $PW_1 = \$6000 + \$4500(P/A, 10\%, 10) = \$33\,651$

*plan 2*
$$PW_2 = \$1200 + \$3000(P/A, 10\%, 5) + \$3500 + \$2500(P/A, 10\%, 10)$$
$$+ [\$3500(A/P, 10\%, 10) + \$2500](P/A, 10\%, 5)(P/F, 10\%, 5)$$
$$= \$38\,658$$

*plan 3*
$$PW_3 = \$1200 + \$3000(P/A, 10\%, 5) + \$3500 + \$2500(P/A, 10\%, 5) - \$1200(P/F, 10\%, 5)$$
$$+ [\$6000(A/P, 10\%, 10) + \$4500](P/A, 10\%, 5)(P/F, 10\%, 5)$$
$$= \$37\,694$$

Note, in the calculations for plans 2 and 3, how certain costs are spread out over ten years and then are partially brought back into the study period.

The conclusion is that plan 1 is best, with plan 3 being slightly superior to plan 2.

**10.10** A company decides to automate a process by installing a machine that costs \$8000 and is expected to save \$2500 per year. Discuss the wisdom of this decision, if the economic life of the machine is 10 years, at which time it has a \$2000 salvage value, and if the MARR is 15%.

The present worth of the decision is (costs counted negative):

$$PW = -\$8000 + \$2000(P/F, 15\%, 10) + \$2500(P/A, 15\%, 10) = +\$5041.38$$

that is, the company can expect net savings of \$5041.38 (today's dollars) over the next 10 years.

## Supplementary Problems

**10.11** Solve Problem 10.10 for an economic life of 3 years.
*Ans.* $PW = -\$976.94$ (the company should not automate)

**10.12** Table 10-8 shows the expected annual operating costs and salvage values for a machine whose initial cost is \$20 000. Find the economic life of the machine, if the MARR is 20%. *Ans.* 6 years

**Table 10-8**

| Year of Service | Salvage Value at End of Year | Operating Cost for Year |
|---|---|---|
| 1 | \$10 000 | \$ 2 000 |
| 2 | 9 000 | 3 000 |
| 3 | 8 000 | 4 000 |
| 4 | 7 000 | 5 000 |
| 5 | 6 000 | 6 000 |
| 6 | 5 000 | 7 000 |
| 7 | 4 000 | 8 000 |
| 8 | 3 000 | 9 000 |
| 9 | 2 000 | 10 000 |
| 10 | 1 000 | 11 000 |

**10.13** For a MARR of 15%, find the economic life of the new machine in Problem 10.3. *Ans.* 15 years

**10.14** Repeat Problem 10.7 for a forecast horizon of 5 years.
*Ans.* $EUAC_{current}(5) = \$5838$, $EUAC_{replacement}(5) = \$5843$; therefore, keep the current equipment. (Note that this is a conservative criterion, since it tends to preserve the existing situation when the challenger's edge consists in savings in the distant future, the estimation of which must be unreliable.)

**10.15** For the situation described in Problem 10.5, determine the number of down-days per year that would justify the acquisition of the stand-by machine. *Ans.* 12.01 (thus, 13)

**10.16** A word-processor was bought two years ago for $22 000. At the time, the machine was expected to last six years and to have operating costs of $7200 the first year, increasing by $300 per year thereafter. The salvage value at the end of the sixth year is assumed to be zero. Another company is presently offering a competitive machine for $16 000; they will give $10 000 for the one in use as trade-in value. Although the book value of the old machine is $14 167, this offer of $10000 is thought to be fair, since the technical obsolescence of the current system would make it very difficult to get a better offer. For this reason, it is believed that the salvage value of the current machine will decrease by $2500 per year over the next four years. The new machine is expected to last four years and to have operating expenses of $6500 per year. After four years, its salvage value will be zero. If the MARR is 15%, should the machine be changed? If so, when?
*Ans.* The old machine's current market value (salvage value) is $10 000; its economic life is found as 4 more years, with $EUAC_{old}(4) = \$11\,700.59$. The new machine also has an economic life of 4 years, with $EUAC_{new}(4) = \$12\,104.32$. Thus, the old machine should be kept for four more years.

**10.17** Refer to Table 10-9. If the MARR is 15%, should the current machine be replaced?
*Ans.* No (based on a 10-year study period)

**Table 10-9**

| | Salvage Value | | Original Cost | Annual Cost | Service Life |
|---|---|---|---|---|---|
| | Now | End of Life | | | |
| Current Machine | $14 000 | $1000 | $50 000 | $10 000 | 10 more years |
| Improved Machine | | 5000 | 45 000 | 3750 | 15 years |

**10.18** A pump costing $18000 is expected to have operating disbursements of $6500 the first year. The machine's resale value is expected to decline by 15% a year, while its operating expenses are expected to increase by $500 a year. If the MARR is 20%, determine the economic life and the corresponding annual equivalent cost. *Ans.* 8 years, $EUAC(8) = \$12\,181.50$

**10.19** In a replacement analysis, data for the challenger are as follows:

Initial cost (installed): $12 000

Maximum service life: 8 years

Operating expenses: none, the first three years; $2000 the fourth and fifth years; increasing by $2500 per year after the fifth year

Salvage value: zero at all times

The MARR is 10%. Tabulate the EUAC of the challenger and infer its economic life.
*Ans.* See Table 10-10.

**10.20** Is it expedient to replace the current machine of Table 10-11, if the MARR for this type of study is 15%?
*Ans.* No: CUV = $1088 > $1000

**Table 10-10**

| $j$ | EUAC($j$) |
|---|---|
| 1 | $13 200 |
| 2 | 6 914 |
| 3 | 4 825 |
| 4 | 4 390 |
| 5 | 4 071 |
| $6 = j^*$ | 3 706 |
| 7 | 3 843 |
| 8 | 4 162 |

**Table 10-11**

| | Current Machine | Challenger |
|---|---|---|
| Service Life | 3 more years | 7 years |
| Original Cost | | $5500 |
| Salvage Value | $1000 now,<br>$200 after 3 years | $500 after 7 years |
| Annual Cost | $1300 | $600 |

**10.21** Compute the CUV of the new machine in Problem 10.13, if the old machine is kept two years and the new one is kept ten years. ***Am.*** CUV = $9448 (<$10 000).

**10.22** The data in Table 10-12 pertain to the average standard-size 1979-model automobile, purchased for $6263. Assuming that 12% is a good estimate of the pertinent interest rate, determine the economic life of such an automobile. ***Am.*** 10 years

**Table 10-12**

| Years of Service | Operating Cost | Salvage Value |
|---|---|---|
| 1 | $2178 | $4503 |
| 2 | 1880 | 3582 |
| 3 | 2080 | 2881 |
| 4 | 1554 | 2255 |
| 5 | 2166 | 1723 |
| 6 | 1923 | 1285 |
| 7 | 2379 | 890 |
| 8 | 1476 | 564 |
| 9 | 1604 | 251 |
| 10 | 1078 | 0 |

**10.23** A 4-year-old die-casting machine, of market value $3500, is 50% too small for future production needs. A new machine with identical production capacity costs $5000 installed. Both machines are expected to have economic lives of 6 years from this date. Salvage values at that date will be $1000 for the new, and $700 for the old, machine. Annual operating expenses for the new and old machines are expected to be $3500 and $4000, respectively. A double-capacity machine is also available; its installed cost is $12000, with a salvage value of $2000 at the end of its 6-year economic life. Operating costs are expected to be $6000 per year. If the MARR is 10%, which machine should be purchased?
Ans. $\text{EUAC}_{\text{new,small}}$ = $9231, $\text{EUAC}_{\text{large}}$ = $8496; buy the large machine.

**10.24** A certain 5-year-old machine has a salvage value of $1200 if sold today, and of $400 if sold 5 years from now. Its operating expenses are $800 per year. A new improved machine is available for $2400, has expected operating expenses of $500 per year, and has a salvage value of $1000 at the end of its 5-year economic life. There is also the possibility of overhauling the old machine at a cost of $600, which would increase the salvage value in 5 years by $200 and reduce the operating expenses by $200 per year. If the MARR is 10%, which course of action should be taken?
***Ans.*** $\text{EUAC}_{\text{old}}$ = $1051, $\text{EUAC}_{\text{new}}$ = $969, $\text{EUAC}_{\text{overhaul}}$ = $977; buy the new machine.

# Chapter 11

# Depreciation and Taxes

## 11.1 DEFINITIONS

***Depreciation*** is a way of accounting for the cost of an asset when income is determined for tax purposes. The cost, including any delivery or installation charges, is treated as a prepayment for future services; and depreciation consists in amortizing this prepayment over the period of use of the asset.

The ***annual depreciation*** is the amount of the asset's cost that is charged off in a given year; the total of the annual depreciations to date is the ***accumulated depreciation.*** The ***salvage value*** or ***scrap value*** of an asset is the estimated proceeds that will be realized from its sale or disposition when it is retired. Under federal tax law, the ***net salvage value*** is either zero or the salvage value minus the cost of removing the asset from the premises, whichever is greater. The ***adjusted cost*** of an asset is its original cost less its net salvage value.

The ***useful life,*** over which an asset is depreciated, may not be the same as its service life, physical life, economic life, market life, etc. The U.S. government publishes guidelines for most equipment, showing the ranges of useful lives allowed in tax computations.

### Depreciable And Nondepreciable Assets

The current tax laws permit only assets with a useful life or more than one year to be depreciated. Depreciation is allowed only for assets used in a business, trade or profession, or held for the production of income. Personal property, such as a family residence or automobile used for pleasure, is not depreciable; however, that portion of an automobile or other property which is used in business may be depreciable. Depreciation is not permitted on land (or on its upkeep), even when it is used for business purposes or income generation. However, buildings and equipment which occupy that land are depreciable if used in a business or to generate income. Inventories of goods used in a business, other stock in trade, and short-term assets that will be consumed during a normal year's operation of the business are not depreciable.

### Computation Methods

The four traditional methods of computing an asset's depreciation from its cost, useful life, and salvage value will be presented in Sections 11.2–11.5. A newer method is treated in Section **11.11.**

## 11.2 STRAIGHT-LINE METHOD

For an asset with useful life $n$ years, the annual depreciation in year $j$ is

$$\text{SD} \equiv \frac{\text{adjusted cost}}{n} \qquad (j = 1, 2, \ldots, n) \tag{11.1}$$

a constant independent of $j$; this corresponds to the constant annual depreciation ***rate*** $r_s \equiv 100\%/n$. The accumulated depreciation at the end of year $j$ is simply

$$\text{ASD}_j \equiv j \times \text{SD} \tag{11.2}$$

and the ***book value*** of the asset at the end of year $j$ is defined as

$$\mathrm{SB}_j \equiv (\text{original cost}) - \mathrm{ASD}_j \tag{11.3}$$

In particular, for $j = n$, (11.3) gives

$$\mathrm{SB}_n = (\text{original cost}) - (\text{adjusted cost}) = \text{net salvage value}$$

**Example 11.1** A new machine costs \$160 000, has a useful life of 10 years, and can be sold for \$15 000 at the end of its useful life. It is expected that \$5000 will be spent to dismantle and remove the machine at the end of its useful life. Determine the straight-line depreciation schedule for this machine.

Here, the adjusted cost is \$160 000 – (\$15 000 – \$5000) = \$150 000, and the rate of depreciation is 10% per year. Applying *(11.1)*, *(11.2)*, and *(11.3)*, we generate Table 11-1. At the end of year 10, the sale of the asset for \$15 000 will remove the \$10 000 book value from the firm's accounting records.

**Table 11-1**

| Year, $j$ | Depreciation Charge (10%) for Year, SD | Accumulated Depreciation, $\mathrm{ASD}_j$ | Book Value at End of Year, $\mathrm{SB}_j$ |
|---|---|---|---|
| 1 | \$15 000 | \$ 15 000 | \$145 000 |
| 2 | 15 000 | 30 000 | 130 000 |
| 3 | 15 000 | 45 000 | 115 000 |
| 4 | 15 000 | 60 000 | 100 000 |
| 5 | 15 000 | 75 000 | 85 000 |
| 6 | 15 000 | 90 000 | 70 000 |
| 7 | 15 000 | 105 000 | 55 000 |
| 8 | 15 000 | 120 000 | 40 000 |
| 9 | 15 000 | 135 000 | 25 000 |
| 10 | 15 000 | 150 000 | 10 000 |

## 11.3 DECLINING-BALANCE METHOD

In this method, the annual depreciation in year $j$ is computed as a fixed fraction of the asset's book value at the end of year $j - 1$:

$$\mathrm{DD}_j = \frac{r_d}{100\%} \times \mathrm{DB}_{j-1} \qquad (j = 1, 2, \ldots, n) \tag{11.4}$$

where $r_d$ is the declining-balance annual depreciation rate, in percent, and $\mathrm{DB}_0 \equiv$ original cost. The accumulated depreciation at the end of year $j$ is

$$\mathrm{ADD}_j \equiv \sum_{k=1}^{j} \mathrm{DD}_k \tag{11.5}$$

and the book value at the end of year $j$ is

$$\mathrm{DB}_j \equiv \mathrm{DB}_0 - \mathrm{ADD}_j \tag{11.6}$$

Equations *(11.4)*, *(11.5)*, and (11.6) imply the recursion formula

$$\mathrm{DB}_j = \left(1 - \frac{r_d}{100\%}\right)\mathrm{DB}_{j-1} \qquad (j = 1, 2, \ldots, n)$$

which may be solved to give the following explicit expressions ($k = 1, 2, \ldots, n$):

$$\mathrm{DB}_k = \left(1 - \frac{r_d}{100\%}\right)^k \mathrm{DB}_0$$

$$\mathrm{DD}_k = \mathrm{DB}_{k-1} - \mathrm{DB}_k = \left(1 - \frac{r_d}{100\%}\right)^{k-1} \left(\frac{r_d}{100\%}\right) \mathrm{DB}_0 \tag{11.7}$$

$$\mathrm{ADD}_k = \mathrm{DB}_0 - \mathrm{DB}_k = \left[1 - \left(1 - \frac{r_d}{100\%}\right)^k\right] \mathrm{DB}_0$$

We see that the depreciation amount (and also the book value) decreases geometrically with time. Thus, the declining-balance method results in a larger share of the depreciation being charged during the earlier years of the asset's life. In contrast to the straight-line method, it is an ***accelerated*** depreciation method.

Also unlike the straight-line method, the declining-balance method does not ***automatically*** take account of the net salvage value of the asset. Thus, according to the first equation *(11.7)*, the book value steadily decreases through positive values, and could very well become smaller than the net salvage value. Salvage value is included in the method by fiat: Federal tax law forbids the application of the method past the point at which $\mathrm{ADD}_k$ becomes greater than the adjusted cost of the asset—which is precisely the point at which $\mathrm{DB}_k$ becomes smaller than the net salvage value.

**Example** 11.2 Apply the double-declining-balance method (i.e., $r_d = 2r_s = 200\%/n$) to (a) the machine of Example *11.1;* (b) the machine of Example *11.1,* with the net salvage value changed to *$30 000.*

In either case, construct the depreciation schedule by applying *(11.4)*, with $r_d = 20\%$ and $\mathrm{DB}_0 = \$160\,000$, for as long as is permitted.

*(a)* See Table *11-2.* Here, the accumulated depreciation never reaches *$150000,* the adjusted cost of the machine. The excess of the final book value over the net salvage value,

$$\$17\,179.87 - \$10\,000 = \$7179.87$$

will presumably be deducted from the firm's income as a capital loss, upon sale of the machine.

**Table 11-2**

| Year, $j$ | Depreciation Charge *(20%)* for Year, $\mathrm{DD}_j$ | Accumulated Depreciation, $\mathrm{ADD}_j$ | Book Value at End of Year, $\mathrm{DB}_j$ |
|---|---|---|---|
| 1 | $32 000 | $ 32 000 | $128 000 |
| 2 | 25 600 | 57 600 | 102 400 |
| 3 | 20 480 | 78 080 | 81 920 |
| 4 | 16 384 | 94 464 | 65 536 |
| 5 | 13 107.20 | 107 571.20 | 52 428.80 |
| 6 | 10 485.76 | 118 056.96 | 41 943.04 |
| 7 | 8 388.61 | 126 445.57 | 33 554.43 |
| 8 | 6 710.88 | 133 156.45 | 26 843.55 |
| 9 | 5 368.71 | 138 525.16 | 21 474.84 |
| 10 | 4 294.96 | 142 820.12 | 17 179.87 |

(b) See Table *11-3.* Entries for years *1* through 7 are computed in normal fashion. The depreciation charge for the 8th year becomes *$3554.43;* any larger amount would cause the accumulated depreciation to exceed the legal maximum of

$$\$160\,000 - \$30\,000 = \$130\,000$$

No depreciation can be taken in years 9 and *10.* A total of *$30 000* in book value, equal to the machine's net salvage value, remains undepreciated. The sale of the machine for its salvage value will remove this *$30 000* from the firm's accounting records.

**Table 11-3**

| Year | Depreciation Charge for Year | Accumulated Depreciation | Book Value at End of Year |
|---|---|---|---|
| 1 | $ 32000 | $ 32 000 | $128 000 |
| 2 | 25 600 | 57 600 | 102 400 |
| 3 | 20 480 | 78 080 | 81 920 |
| 4 | 16 384 | 94 464 | 65 536 |
| 5 | 13 107.20 | 107 571.20 | 52 428.80 |
| 6 | 10 485.76 | 118 056.96 | 41 943.04 |
| 7 | 8 388.61 | 126 445.57 | 33 554.43 |
| 8 | 3 554.43 | 130 000 | 30 000 |
| 9 | 0 | 130 000 | 30 000 |
| 10 | 0 | 130 000 | 30 000 |

## 11.4 SUM-OF-YEARS'-DIGITS METHOD

The sum of years, SY, for an asset with useful life $n$ years is

$$\mathrm{SY} \equiv \sum_{j=1}^{n} j = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{11.8}$$

In the sum-of-years'-digits method, the annual depreciation in year $j$ is given by

$$\mathbf{SYD}_j = \frac{n+1-j}{\mathrm{SY}} \times (\text{adjusted cost}) \qquad (j = 1, 2, \ldots, n) \tag{11.9}$$

whence the accumulated depreciation at the end of year $j$ is given by

$$\mathbf{ASYD}_j = \frac{jn - [j(j-1)/2]}{\mathrm{SY}} \times (\text{adjusted cost}) \tag{11.10}$$

The book value at the end of year $j$ is defined in the usual way:

$$\mathbf{SYB}_j \equiv (\text{original cost}) - \mathrm{ASYD}_j \tag{11.11}$$

Because $\mathrm{ASYD}_n$ = adjusted cost, $\mathbf{SYB}_n$ = net salvage value, as in the straight-line method.

**Table 11-4**

| Year, I | Depreciation Charge for Year, $\mathbf{SYD}_j$ | Accumulated Depreciation, $\mathbf{ASYD}_j$ | Book Value at End of Year, $\mathbf{SYB}_j$ |
|---|---|---|---|
| 1 | $27 272.73 | $ 27 272.73 | $132 727.27 |
| 2 | 24 545.45 | 51 818.18 | 108 181.82 |
| 3 | 21 818.18 | 73 636.36 | 86 363.64 |
| 4 | 19 090.91 | 92 727.27 | 67 272.73 |
| 5 | 16 363.64 | 109 090.91 | 50 909.09 |
| 6 | 13 636.36 | 122 727.27 | 37 272.73 |
| 7 | 10 909.09 | 133 636.36 | 26 363.64 |
| 8 | 8 181.82 | 141 818.18 | 18 181.82 |
| 9 | 5 454.55 | 147 272.73 | 12 727.27 |
| 10 | 2 727.27 | 150 000.00 | 10 000.00 |

According to *(11.8)*, a different fraction is applied each year to the adjusted cost of the asset to obtain the annual depreciation. The denominator of this fraction is the total of the digits representing the years of the estimated useful life of the asset; the numerator changes each year, so as to represent the number of years of useful asset life remaining at the start of that year. Currently, the tax laws prohibit the sum-of-years'-digits method from being used on any property for which the double-declining-balance method is prohibited.

**Example** 11.3 Apply the sum-of-years'-digits method to the machine of Example 11.1.

By *(11.8)*, SY = (10)(11)/2 = 55, and the adjusted cost of the machine is $150 000. Repeated application of (11.9) generates Table 11-4.

## 11.5 SINKING-FUND METHOD

This method depreciates an asset as if the firm were to make a series of equal annual deposits (a sinking fund) whose value at the end of the asset's useful life just equaled the cost of replacing the asset. Writing

$A' \equiv$ sinking-fund deposit

$C \equiv$ (purchase price of replacement asset) – (net salvage value of current asset)

$n \equiv$ useful life of current asset

$i \equiv$ annual interest rate

we have: $A' = C\,(A/F, i\%, n)$. The amount in the sinking fund at the end of year $j$ ($j = 1, 2, \ldots, n$) is identified with the accumulated depreciation to date; thus,

$$\mathrm{ASFD}_j \equiv A'\,(F/A, i\%, j) = C\,(A/F, i\%, n)\,(F/A, i\%, j) \tag{11.12}$$

and the depreciation amount in year $j$ is

$$\begin{aligned}\mathrm{SFD}_j &= \mathrm{ASFD}_j - \mathrm{ASFD}_{j-1}\\ &= C\,(A/F, i\%, n)\,[(F/A, i\%, j) - (F/A, i\%, j-1)]\\ &= C\,(A/F, i\%, n)\,(1+i)^{j-1}\end{aligned} \tag{11.13}$$

As usual, the book value is defined as

$$\mathrm{SFB}_j \equiv \text{(original cost)} - \mathrm{ASFD}_j \tag{11.14}$$

**Table 11-5**

| Year, $j$ | Depreciation Charge for Year, $\mathrm{SFD}_j$ | Accumulated Depreciation, $\mathrm{ASFD}_j$ | Book Value at End of Year, $\mathrm{SFB}_j$ |
|---|---|---|---|
| 1 | $ 7387.81 | $ 7387.81 | $152 612.19 |
| 2 | 8 495.98 | 15 883.79 | 144 116.21 |
| 3 | 9 770.38 | 25 654.17 | 134 345.83 |
| 4 | 11 235.93 | 36 890.10 | 123 109.90 |
| 5 | 12 921.32 | 49 811.43 | 110 188.57 |
| 6 | 14 859.52 | 64 670.95 | 95 329.05 |
| 7 | 17 088.45 | 81 759.40 | 78 240.60 |
| 8 | 19 651.72 | 101 411.12 | 58 588.88 |
| 9 | 22 599.48 | 124 010.60 | 35 989.40 |
| 10 | 25 989.40 | 150 000.00 | 10 000.00 |

It is seen from (11.13) that the annual depreciation amount ***increases*** geometrically with time—just the opposite of the declining-balance method. As a matter of tax law, the sinking-fund method may be used only when the replacement asset will have the same original cost as the current asset, in which case $C$ = adjusted cost of current asset. (Otherwise, the firm could take a total depreciation allowance in excess of the current asset's adjusted cost, and this is not allowed.)

**Example 11.4** Apply the sinking-fund method to the machine of Example 11.1, given i = 15%.
By repeated application of (*11.13*), starting with

$$SFD_1 = \$150\,000(A/F, 15\%, 10) = \$7387.81$$

Table 11-5 is generated.

## 11.6 GROUP AND COMPOSITE DEPRECIATION

The methods discussed in Sections 11.2–11.5 are all ***unit depreciation methods*** in that they apply to a single item, asset, or unit. When there are many like items, a ***group depreciation*** becomes convenient, whereby a single annual depreciation figure is computed for the ensemble, using the sum of the items' original costs and the sum of their salvage values. The group's useful life is the average of the lives of all the items. Any of the unit depreciation methods can be used to compute the group depreciation.

**Example 11.5** The ABC Company purchased five cutters with useful lives of 5, 6, 7, 8, and 10 years; the cutters cost \$10000, \$12000, \$13 000, \$14000, and \$16000, respectively. The salvage value of each cutter is estimated to be \$500. Compute the annual group depreciation charge, using the straight-line method.

$$\text{group adjusted cost} = (\$10\,000 + \cdots + \$16\,000) - 5(\$500) = \$62\,500$$

$$\text{group useful life} = \frac{5+6+7+8+10}{5} = 7.2 \text{ years}$$

$$\text{group SD} = \frac{\$62\,500}{7.2} = \$8680.56$$

When a mixed collection of assets is subdivided into groups according to useful life, one can perform a ***composite depreciation.*** First, an annual depreciation charge is calculated for each life group by use of a group depreciation method. The sum of these charges then gives the composite annual depreciation. A composite life, ***n***, can now be ***defined*** as the total of all adjusted costs, divided by the composite depreciation amount. The corresponding composite depreciation rate is defined as 100%/*n*.

**Example 11.6** Rework Example 11.5 by composite depreciation.
Applying the straight-line method to each life group, which in this case consists of a single item, we obtain:

$$SD_{5\text{-group}} = \frac{\$10\,000 - \$500}{5} = \$1900.00$$

$$SD_{6\text{-group}} = \frac{\$12\,000 - \$500}{6} = \$1916.67$$

$$SD_{7\text{-group}} = \frac{\$13\,000 - \$500}{7} = \$1785.71$$

$$SD_{8\text{-group}} = \frac{\$14\,000 - \$500}{8} = \$1687.50$$

$$SD_{10\text{-group}} = \frac{\$16\,000 - \$500}{10} = \$1550.00$$

Then,

$$\text{composite SD} = \$1900.00 + \cdots + \$1550.00 = \$8839.88$$

$$\text{composite life} = \frac{62\,500}{8839.88} = 7.07 \text{ years}$$

$$\text{composite depreciation rate} = \frac{100\%}{7.07} = 14.14\%$$

Comparing with Example 11.5, we see that, under composite depreciation, a little more is charged off each year for a slightly smaller number of years.

The composite depreciation rate (or useful life) is restricted by guidelines issued by the U.S. Treasury Department. Variations from these guidelines must be individually approved by the Internal Revenue Service.

## 11.7 ADDITIONAL FIRST-YEAR DEPRECIATION; INVESTMENT TAX CREDIT

The U.S. government from time to time seeks to encourage new capital investments by business, to stimulate the economy. Two incentives have been used, either of which may be enacted into law when the federal government feels stimuli are needed, and withdrawn when it feels stimuli are not needed.

The ***additional first-year depreciation*** provision allows an additional percentage depreciation deduction during the year in which the asset was purchased. The percentage is based on the original cost of the asset, and is in addition to any regular depreciation. The ***investment tax credit*** provision allows a business to reduce its annual income tax by some stated percentage of the original cost of any assets purchased during that year.

**Example** 11.7 The SSG Company spent $1000000 for new equipment on January 1 of this year. The equipment has a useful life of 10 years, zero salvage value, and is depreciated by the straight-line method. Additional first-year depreciation of **20%** and an investment tax credit of 7% apply. Compute the total first-year depreciation, depreciation for other years, and the first-year investment tax credit for the SSG Company.

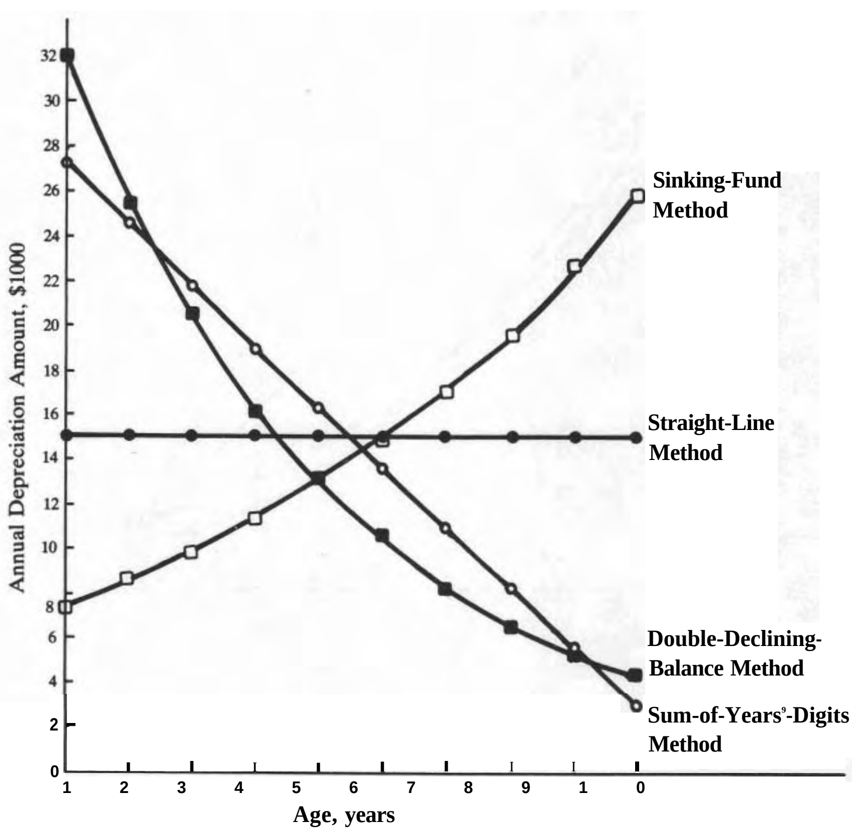


**Fig. 11-1**

We have
$$\text{SD} = \frac{\$1\,000\,000}{10} = \$100\,000$$

This year, the depreciation charge will be

$$S_1 = \$100\,000 + (0.20)(\$1\,000\,000) = \$300\,000$$

Since 3 SD is charged off in year 1, the annual depreciation will be SD in years 2 through 8, and zero in years 9 and 10. In addition, the SSG Company can reduce this year's tax bill by

$$(0.07)(\$1\,000\,000) = \$70\,000$$

## 11.8 COMPARISON OF DEPRECIATION METHODS

Figures 11-1 and 11-2 are plots of the annual depreciation charges and book values from Tables 11-1, 11-2, 11-4, and 11-5. Figure 11-1 makes manifest what we have said about the four traditional depreciation schemes: the sum-of-years'-digits method and the declining-balance methods are ***accelerated*** (heaviest depreciation in earlier years); the sinking-fund method is ***decelerated*** (heaviest depreciation in later years); and the straight-line method is neither accelerated nor decelerated. Notice, in Fig. 11-2, that the book values generated by the straight-line method are intermediate between those for the accelerated and the decelerated methods. This holds true in general.

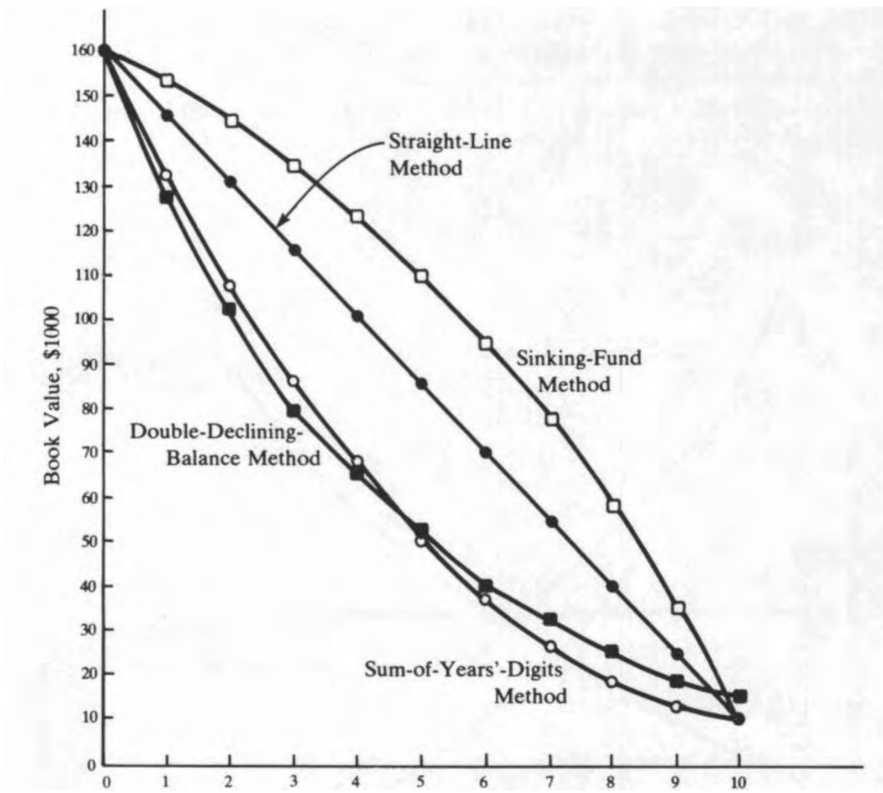


Fig. 11-2

## 11.9 BUSINESS NET INCOME AND TAXES

In the U.S., most corporations are subject to a two-step income tax, characterized by a ***base tax rate*** and a ***surtax rate.*** The base tax rate, $r_t$, is currently 22% of net taxable income, **I**, where

$$I = \text{(gross receipts and sales)} - \text{(bad debts)} - \text{(cost of goods sold)}$$
$$- \text{(wages + salaries)} - \text{(interest + rent)} - \text{depreciation} \tag{11.15}$$

The surtax rate, $s_t$, is presently 26%; it applies only to income in excess of \$25 000. Total corporate income tax is thus given by

$$T = Ir_t + (I - \$25\,000)s_t \tag{11.16}$$

where it is assumed that $I \geq \$25\,000$. For most corporations, $I \gg \$25\,000$, so that

$$T \approx t_r I \tag{11.17}$$

where $t_r = r_t + s_t = 48\%$.

Unincorporated businesses are generally taxed at the individual tax rate(s) of the owner(s).

**Example 11.8** The KJL Corporation received \$10 000 000 from the sales of their products during the current year. A total of \$1000 of these sales was never actually collected and was accounted for as bad debts. The company spent \$3 000 000 in the production and warehousing of their products during the current year. A total of \$1 000 000 was spent for wages and salaries, \$500 000 was paid out in interest on long-term loans, \$700 000 was spent for rental of space and equipment, and \$600000 depreciation was charged off. Compute the KJL Corporation's income tax bill for the current year, if the base tax rate is 22% and the surtax rate is 26%.

From (11.15):

| | | | |
|---|---|---|---|
| Gross Receipts and Sales | = | \$10 000 000 | |
| Less: Bad Debts | = | 1 000 | |
| Gross Income | = | \$ 9 999 000 | |
| Period Costs: | | | |
| Cost of Goods Sold | = | | \$3 000 000 |
| Wages and Salaries | = | | 1 000 000 |
| Interest | = | | 500 000 |
| Rent | = | | 700 000 |
| Less: Total Period Costs | = | 5 200 000 | |
| Net Income | = | \$ 4 799 000 | |
| Less: Depreciation | = | 600 000 | |
| Net Taxable Income | = | \$ 4 199 000 | |

$$T = (\$4\,199\,000)(0.22) + (\$4\,199\,000 - \$25\,000)(0.26) = \$2\,009\,020$$

[or, by (*11.17*), $T \approx 0.48 \times \$4\,199\,000 = \$2\,015\,520$].

Capital gains (losses) occur when an asset is sold for more (less) than its book value. Under current U.S. tax laws, if an asset has been held more than six months, it is a long-term asset; otherwise, it is a short-term asset. Long-term capital gains and losses are aggregated separately from short-term capital gains and losses. If the long-term aggregate is positive (a capital gain), it is taxed at only 30% (not at $t_r = 48\%$). If the long-term aggregate is negative (a capital loss), this loss may be carried forward and spread arbitrarily over the next five years, as an offset to any capital gains during those years. If the short-term aggregate is positive (a capital gain), it is taxed as regular income, at $t_r$; if it is negative (a capital loss), it is treated like a long-term capital loss.

**Example 11.9** Assume that in the current year the KJL Corporation sells a machine, which it has used for several years, for \$600000. The machine originally cost \$500000 and has been depreciated under the double-declining-balance method; the current book value is \$300 000. The corporation also had short-term capital losses of \$50 000 and short-term capital gains of \$20 000. Compute the KJL Corporation's income tax bill, using the other information in Example 11.8.

Sale of the machine results in a ***long-term capital gain*** of

$$\$600\,000 - \$500\,000 = \$100\,000$$

plus ***ordinary income*** in the amount

$$(\$600\,000 - \$300\,000) - \$100\,000 = \$200\,000$$

In addition, there is an aggregate short-term capital loss of $30 000. KJL will therefore have to pay taxes of

$$\$2\,009\,020 + (0.48)(\$200\,000) + (0.30)(\$100\,000) = \$2\,135\,020$$

but may claim a credit of $30 000 against future capital gains.

## 11.10 COMPARATIVE EFFECTS OF DEPRECIATION METHODS ON INCOME TAXES

Depreciation is deducted as an expense of doing business. Thus, by lowering net income, depreciation lowers income taxes. Specifically, if the normal tax rate is $t_r$, then the ***depreciation tax shield*** in year $j$, or amount of taxes saved in that year because depreciation is taken, is

$$S_j = t_r D_j \tag{11.18}$$

where $D_j$ is the depreciation charge for that year.

Over the life of an asset, the total amount of depreciation tax shield will be the same under the straight-line, sum-of-years'-digits, and sinking-fund methods (since each of these methods charges off the entire adjusted cost of the asset). The declining-balance method will often yield a slightly smaller depreciation tax shield over the life of the asset, which, however, will be more or less compensated for by the capital loss suffered when the asset is sold at a salvage value below its book value. Thus, all four methods will give approximately the same ***total*** tax shield over the useful life of the asset. However, because of the time value of money, accelerated depreciation methods (declining-balance and sum-of-years'-digits) will yield a larger present-worth net income after taxes. This is because the accelerated methods provide a larger tax shield in the earlier years of the asset's life; and the earlier the savings, the less they are discounted in calculating the present worth.

**Example 11.10** Assume that the tax rate is 52% and the (before-tax) net income is $100 000 per year, before depreciation. Compare the effects of the four depreciation methods from (*a*) Example 11.1, (*b*) Example 11.2(*a*), (*c*) Example 11.3, (*d*) Example 11.4. Assume MARR = 15%.

(*a*)
annual taxes = (0.52)($100 000 − $15 000) = $44 200
total taxes for 10 years = $442 000
present worth of 10 years' taxes = $44 200($P/A$, 15%, 10) = $221 831.87

(*b*) The tax shield from the capital loss (at maximum rate) is $7179.87 (0.30) = $2153.96.

**Table 11-6**

| Year | Net Taxable Income, 1 | Taxes, 0.521 | Present Worth of Taxes |
|---|---|---|---|
| 1 | $100 000 − $32 000 = $68 000.00 | $35 360.00 | $30 747.83 |
| 2 | 100 000 − 25 600 = 74 400.00 | 38 688.00 | 29 253.69 |
| 3 | 100 000 − 20 480 = 79 520.00 | 41 350.40 | 27 188.56 |
| 4 | 100 000 − 16 384 = 83 616.00 | 43 480.32 | 24 860.01 |
| 5 | 100 000 − 13 107.20 = 86 892.80 | 45 184.26 | 22 464.56 |
| 6 | 100 000 − 10 485.76 = 89 514.24 | 46 547.41 | 20 123.73 |
| 7 | 100 000 − 8388.61 = 91 611.39 | 47 637.92 | 17 908.86 |
| 8 | 100 000 − 6710.88 = 93 289.12 | 48 510.34 | 15 858.12 |
| 9 | 100 000 − 5368.71 = 94 631.29 | 49 208.27 | 13 988.06 |
| 10 | 100 000 − 4294.96 = 95 705.04 | 49 766.62 | 12 301.55 |
| | TOTALS | $445 735.54 | $214 694.97 |

Then, from Table 11-6,

$$\text{adjusted total taxes for 10 years} = \$445\,735.54 - \$2153.96 = \$443\,581.58$$

$$\text{present worth of 10 years' taxes} = \$214\,694.97$$

$$\text{adjusted present worth of 10 years' taxes} = \$214\,694.97 - \$2153.96(P/F, 15\%, 10) = \$214\,162.54$$

*(c)* The 10 years' taxes and their present worth are given by the totals in Table 11-7.

**Table 11-7**

| Year | Net Taxable Income, 1 | Taxes, 0.521 | Present Worth of Taxes |
|---|---|---|---|
| 1 | \$100 000 – \$27 272.73 = \$72 727.27 | \$ 37 818.18 | \$ 32 885.37 |
| 2 | 100 000 – 24 545.45 = 75 454.55 | 39 236.37 | 29 668.33 |
| 3 | 100 000 – 21 818.18 = 78 181.82 | 40 654.55 | 26 731.02 |
| 4 | 100000 – 19090.91 = 80 909.09 | 42 072.73 | 24 055.31 |
| 5 | 100 000 – 16 363.64 = 83 636.36 | 43 490.91 | 21 622.67 |
| 6 | 100 000 – 13 636.36 = 86 363.64 | 44 909.09 | 19 415.44 |
| 7 | 100 000 – 10 909.09 = 89 090.91 | 46 327.27 | 17 416.14 |
| 8 | 100 000 – 8181.82 = 91 818.18 | 47 745.45 | 15 608.07 |
| 9 | 100 000 – 5454.55 = 94 545.45 | 49 163.63 | 13 975.37 |
| 10 | 100 000 – 2727.27 = 97 272.73 | 50 581.82 | 12 503.05 |
| | TOTALS | \$442 000.00 | \$213 880.77 |

(d) The 10 years' taxes and their present worth are given by the totals in Table 11-8.

**Table 11-8**

| Year | Net Taxable Income, 1 | Taxes, 0.521 | Present Worth of Taxes |
|---|---|---|---|
| 1 | \$100 000 – \$ 7387.81 = \$92 612.19 | \$ 48 158.34 | \$ 41 876.82 |
| 2 | 100 000 – 8495.98 = 91 504.02 | 47 582.09 | 35 978.90 |
| 3 | 100 000 – 9770.38 = 90 229.62 | 46 919.40 | 30 850.27 |
| 4 | 100 000 – 11 235.93 = 88 764.06 | 46 157.31 | 26 390.59 |
| 5 | 100 000 – 12 921.32 = 87 078.67 | 45 280.91 | 22 512.62 |
| 6 | 100 000 – 14 859.52 = 85 140.48 | 44 273.05 | 19 140.46 |
| 7 | 100 000 – 17 088.45 = 82 911.55 | 43 114.00 | 16 208.15 |
| 8 | 100 000 – 19 651.72 = 80 348.28 | 41 781.10 | 13 658.32 |
| 9 | 100 000 – 22 599.48 = 77 400.52 | 40 248.27 | 11 441.07 |
| 10 | 100 000 – 25 989.40 = 74 010.60 | 38 485.51 | 9 513.03 |
| | TOTALS | \$441 999.98 | \$227 570.23 |

Comparing the above results, we see that the total taxes for 10 years are \$442 000 under each depreciation method (with a deviation of \$1581.58 for the double-declining-balance method). However, on a present-worth basis we have:

| *Method* | *Present-Worth Tax Advantage (+) or Disadvantage (–) Relative to Straight-Line Method* |
|---|---|
| Double-Declining-Balance | +$7669.33 |
| Sum-of-Years'-Digits | +$7951.18 |
| Sinking-Fund | –$5738.36 |

## 11.11 THE ACCELERATED COST RECOVERY SYSTEM

The Accelerated Cost Recovery System (ACRS) is a depreciation method recently instituted by the Internal Revenue Service. It is mandatory for most tangible assets placed in service after December 31, 1980. Its main features are that salvage value is not relevant and that the useful life of the asset is limited to 3, 5, 10, or 15 years. The IRS publishes depreciation scales for each class life. Thus, the percentages for three-year property placed in service during 1982 are 25% for the first year, 38% for the second year, and 37% for the third year. This class life includes assets with a useful life of 4 years or less, such as automobiles, small trucks, and some manufacturing tools. Items used in research and experimentation are also included in this category.

Five-year property includes office furniture, some storage facilities, and, in general, all property that is not three-, ten-, or fifteen-year. The percentages are 15% for the first year; 22% for the second year; and 21% for the third, fourth, and fifth year.

Ten-year property includes assets with a useful life of less than 12.5 years. The percentages are 8% for the first year, 14% for the second year, 12% for the third year, 10% for each of years four through six, and 9% for each of years seven through ten.

Assets with a useful life of more than 12.5 years are designated as fifteen-year property. There is one rate structure for low-income housing and another for all other fifteen-year property. Percentages for the fifteen-year asset also depend on the month the property was placed in service; e.g., for an asset (not low-income housing) placed in service in April, the percentages are:

| Year | 1 | 2 | 3 | 4 | 5 | 6–10 | 11–15 | 16 |
|---|---|---|---|---|---|---|---|---|
| % | 9 | 11 | 9 | 8 | 7 | 6 | 5 | 1 |

**Example 11.11** Apply the ACRS to the asset of Example 11.1.

The calculations for this ten-year asset are given in Table 11-9.

**Table 11-9**

| Year | Depreciation Rate for Year | Depreciation Charge for Year | Accumulated Depreciation | Book. Value at End of Year |
|---|---|---|---|---|
| 1 | 8% | $12 800 | $ 12800 | $147 200 |
| 2 | 14 | 22 400 | 35 200 | 124 800 |
| 3 | 12 | 19 200 | 54 400 | 105 600 |
| 4 | 10 | 16 000 | 70 400 | 89 600 |
| 5 | 10 | 16 000 | 86 400 | 73 600 |
| 6 | 10 | 16 000 | 102 400 | 57 600 |
| 7 | 9 | 14 400 | 116 800 | 43 200 |
| 8 | 9 | 14 400 | 131 200 | 28 800 |
| 9 | 9 | 14 400 | 145 600 | 14 400 |
| 10 | 9 | 14 400 | 160 000 | 0 |

**Example** 11.12 Assume again, as in Example 11.10, and tax rate of 52% and an annual income of $100 000 before depreciation and taxes. Compare the effect of the ACRS (using Table 11-9) with those of the four traditional methods, as found in Example 11-10.

**Table 11-10**

| Year | Net Taxable Income, I | Taxes, 0.521 | Present Worth of Taxes (i = 15%) |
|---|---|---|---|
| 1 | $100 000 – $12 800 = $87 200 | $45 344 | $39 429.57 |
| 2 | 100 000 – 22 400 = 77 600 | 40 352 | 30 511.91 |
| 3 | 100 000 – 19 200 = 80 800 | 42 016 | 27 626.20 |
| 4 | 100 000 – 16 000 = 84 000 | 43 680 | 24 974.18 |
| 5 | 100 000 – 16 000 = 84 000 | 43 680 | 21 716.68 |
| 6 | 100 000 – 16 000 = 84 000 | 43 680 | 18 884.07 |
| 7 | 100 000 – 14 400 = 85 600 | 44 512 | 16 733.71 |
| 8 | 100 000 – 14 400 = 85 600 | 44 512 | 14 551.05 |
| 9 | 100 000 – 14 400 = 85 600 | 44 512 | 12 653.09 |
| 10 | 100 000 – 14 400 = 85 600 | 44 512 | 11 002.09 |
| | TOTALS | $436 800 | $218 082.55 |

See Table 11-10. The comparison made in Example 11.10 may now be extended as follows:

| *Method* | *Present-Worth Tax Advantage* (+) *or Disadvantage* (–) *Relative to Straight-Line Method* |
|---|---|
| Double-Declining-Balance | +$7669.33 |
| Sum-of-Years'-Digits | +$7951.18 |
| Sinking-Fund | –$5738.36 |
| ACRS | +$3748.72 |

It is seen that, in this case, the ACRS is better than straight-line depreciation but not as good as the two traditional accelerated methods. However, had the useful life of the asset been 12 years, the ACRS would have been best (the asset would still be ten-year property, but the traditional methods would have to be applied over the whole 12 years).

## 11.12 CHOICE OF DEPRECIATION METHOD

As indicated, a taxpayer would have to present excellent arguments (based on facts, not opinions) to be allowed to use a method other than ACRS for tangible assets. For intangible assets (franchises, designs, drawings, copyrights, patterns, subscription lists, customer lists, etc.), the traditional methods can still be applied, singly or in combination, provided the useful life can be projected with reasonable accuracy.

As can be seen from Fig. 11-2, the use of the double-declining-balance during the early years, combined with a switch to the sum-of-years'-digits in later years, may provide the greatest possible tax shield, for this combination gives the largest possible accumulated depreciation charge (smallest possible book value) over the life of the asset. However, this combination may result in an accumulated depreciation at the end of the useful life of the asset which exceeds the asset's adjusted cost—a violation of tax regulations. Moreover, the current tax laws specifically prohibit certain switches in depreciation method without the prior approval of the Internal Revenue Service.

## 11.13 DEPRECIATION AND CASH now

Depreciation, as an accounting charge against income, is not itself a cash flow. However, it does influence the amount of income tax paid, which is a (negative) cash flow. For year j, let us write:

$$\begin{aligned} \text{BTCF}_j &\equiv \text{before-tax net cash flow} \\ \text{ATCF}_j &\equiv \text{after-tax net cash flow} \\ I_j &\equiv \text{net taxable income} \\ \text{ATI}_j &\equiv \text{after-tax net income} \\ T_j &\equiv \text{income tax} \\ D_j &\equiv \text{depreciation charge} \end{aligned}$$

Then, $\text{ATCF}_j = \text{BTCF}_j - T_j$. But, by *(11.15)*,

$$\text{ATI}_j = I_j - T_j = (\text{BTCF}_j - D_j) - T_j$$

Consequently,

$$\text{ATCF}_j = \text{ATI}_j + D_j \tag{11.19}$$

that is to say, the after-tax cash flow for the year is the sum of the after-tax net income for the year and the depreciation charge for the year.

## 11.14 BEFORE AND AFTER-TAX ECONOMIC ANALYSES

Economic analyses should generally be made on an after-tax basis, unless it is clear that tax considerations are irrelevant. We have seen that depreciation can cause the before-tax and after-tax pictures to differ. Deduction of interest paid on borrowed money will have a similar effect. Perhaps most significant is the fact that businesses are judged (by analysts and investors) on the basis of their after-tax performance.

**Example 11.13** The ABC Company is planning to buy a new pump. The pump costs *$50000,* and has a 10-year life and zero salvage value. The pump will increase the company's net income before taxes by *$12 000* in each of the *10* years. The company's tax rate is 51%. What is the ROR on the pump?

We make three different analyses, which give three different results.

***Before-*Tax**

$$0 = -\$50\,000 + \$12\,000(P/A, i^*\%, 10) \qquad \text{whence} \qquad i^* = 20.2\%$$

***After-*Tax, *No Depreciation***

With $D_j = 0$, *(11.19)* gives $\text{ATCF}_j = (0.49)(\$12\,000) = \$5880$; thus,

$$0 = -\$50\,000 + \$5880(P/A, i^*\%, 10) \qquad \text{whence} \qquad i^* = 3\%$$

***After-*Tax, *Straight-Line Depreciation***

With $D_j = \$50\,000/10 = \$5000$, *(11.19)* gives

$$\text{ATCF}_j = (0.49)(\$12\,000 - \$5000) + \$5000 = \$8430$$

and so

$$0 = -\$50\,000 + \$8430(P/A, i^*\%, 10) \qquad \text{whence} \qquad i^* = 10.9\%$$

**Example 11.14** The ABC Company (Example *11.13)* can purchase an alternative "Superpump" that costs *$100 000* and generates an annual increase in the company's net income before taxes of *$23 852.* This pump also has a 10-year life and zero salvage value; however, a special provision in the tax laws permits depreciation of this pump over a 5-year period. Which pump should the company buy?

For the Superpump, we make two analyses:

***Before-Tax***

$$0 = -\$100\,000 + \$23\,852(P/A, i^*, 10) \qquad \text{whence} \qquad i^* = 20\%$$

***After-Tax, Straight-Line Depreciation***

For $j = 1, 2, \ldots, 5$:

$$D_j = \frac{\$100\,000}{5} = \$20\,000$$

$$\text{ATCF}_j = (0.49)(\$23\,852 - \$20\,000) + \$20\,000 = \$21\,887.48$$

while for $j = 6, 7, \ldots, 10$:

$$D_j = 0$$

$$\text{ATCF}_j = (0.49)(\$23\,852) = \$11\,687.48$$

$$0 = -\$100\,000 + \$21\,887.48(P/A, i^*\%, 5) + \$11\,687.48(P/A, i^*\%, 5)(\text{PIF}, i^*, 5)$$

which yields $i^* = 12.8\%$.

Comparing the above results with those of Example 11.3, we conclude that the Superpump is not to be preferred on a before-tax basis, but is definitely to be preferred on an after-tax basis. In such cases, the after-tax picture is always the correct one.

## Solved Problems

**11.1** A company's tax rate is 52%. To improve labor relations, the company has decided to donate $1000 000 to its labor union to build a sports arena for the use of union members and the general public. (a) If the gift is ruled tax deductible, what is the actual cost to the company? (b) If the gift is ruled nondeductible, what is the actual cost to the company, and how does it account for the gift? *(c)* If the labor union is a tax-exempt corporation, to what extent is the general public (through the government) subsidizing the arena?

(*a*)

$$\text{tax savings} = (0.52)(\$1\,000\,000) = \$520\,000$$
$$\text{actual cost to the company} = \$480\,000$$

(b) $1 000 000; nondeductible expense.

*(c)* If the donation is ruled tax deductible, the general public would be footing the company's tax savings, $520 000. (If the labor union were not tax-exempt, it would have to pay taxes on the gift, in which case the public's subsidy would amount to $520 000 minus the union's taxes.)

**11.2** A computer system can be purchased for $18 000. The operating costs will be $10 000 per year, and the useful life is expected to be 5 years, with $5000 salvage value at that time. The present annual sales volume should increase by $16000 as a result of acquiring the computer system. The company's tax rate is 50%. (a) Depreciate the asset by the straight-line method. (b) Compute annual taxes, annual cash flows after taxes, and after-tax ROR for the investment.

(a) SD = ($18 000 – $5000)/5 = $2600; see Table 11-11.

**Table 11-11**

| Year | Depreciation Charge for Year | Accumulated Depreciation | Book Value at End of Year |
|---|---|---|---|
| 1 | $2600 | $2600 | $15 400 |
| 2 | 2600 | 5 200 | 12 800 |
| 3 | 2600 | 7 800 | 10 200 |
| 4 | 2600 | 10 400 | 7 600 |
| 5 | 2600 | 13 000 | 5 000 |

(b) The annual net cash flow before taxes is \$16 000 – \$10 000 = \$6000; see Table 11-12.

**Table 11-12**

| | Year 1 | Year 2 | Year 3 | Year 4 | Year 5 |
|---|---|---|---|---|---|
| BTCF | \$6000 | \$6000 | \$6000 | \$6000 | \$6000 |
| Depreciation | 2600 | 2600 | 2600 | 2600 | 2600 |
| Net Taxable Income | 3400 | 3400 | 3400 | 3400 | 3400 |
| Tax (@ 50%) | 1700 | 1700 | 1700 | 1700 | 1700 |
| ATCF | 4300 | 4300 | 4300 | 4300 | 4300 |

The after-tax ROR may be found by equating the after-tax PW to zero (see Chapter 7):

$$0 = -\$18\,000 + \$4300(P/A, i^*\%, 5) + \$5000(P/F, i^*\%, 5)$$

or $i^* = 12.6\%$.

**11.3** Rework Problem 11.2 if the IRS rules that the equipment's ***tax*** life is eight years, with \$2000 salvage value at that date.

(a) SD = (\$18 000 – \$2000)/8 = \$2000; see Table 11-13.

**Table 11-13**

| Year | Depreciation Charge for Year | Accumulated Depreciation | Book Value at End of Year |
|---|---|---|---|
| 1 | \$2000 | \$ 2 000 | \$16 000 |
| 2 | 2000 | 4 000 | 14 000 |
| 3 | 2000 | 6 000 | 12 000 |
| 4 | 2000 | 8 000 | 10 000 |
| 5 | 2000 | 10 000 | 8 000 |

The useful life, or depreciation period, remains 5 years. At that time the computer system would presumably be sold—for \$3000 less than its book value. Thus the company would take a long-term capital loss of \$3000 and carry it forward to offset long-term capital gains over the next five years.

(b) See Table 11-14.

**Table 11-14**

| | Year 1 | Year 2 | Year 3 | Year 4 | Year5 |
|---|---|---|---|---|---|
| BTCF | \$6000 | \$6000 | \$6000 | \$6000 | \$6000 |
| Depreciation | 2000 | 2000 | 2000 | 2000 | 2000 |
| Net Taxable Income | 4000 | 4000 | 4000 | 4000 | 4000 |
| Tax (@50%) | 2000 | 2000 | 2000 | 2000 | 2000 |
| ATCF | 4000 | 4000 | 4000 | 4000 | 4000 |

Assuming that the company takes its tax credit [see *(a)*] in year 6, it will save $(0.30)(\$3000) = \$900$ in long-term capital gains taxes in that year. Hence, the after-tax ROR is given by

$$0 = -\$18\,000 + \$4000(P/A, i^*\%, 5) + \$5000(P/F, i^*\%, 5) + \$900(P/F, i^*\%, 6)$$

or $i^* = 11.4\%$.

**11.4** Rework Problem 11.2 using the double-declining-balance depreciation method.

*(a)* With $r_d = 2r_s = 200\%/5 = 40\%$, we generate Table 11-15.

**Table 11-15**

| Year | Depreciation Charge for Year | Accumulated Depreciation | Book Value at End of Year |
|---|---|---|---|
| 1 | $7200 | $ 7 200 | $10 800 |
| 2 | 4320 | 11 520 | 6 480 |
| 3 | 1480 | 13 000 | 5 000 |
| 4 | 0 | 13 000 | 5 000 |
| 5 | 0 | 13 000 | 5 000 |

**Observe** that the third-year values had to be adjusted so that the accumulated depreciation would not exceed the maximum set by the IRS (the adjusted cost). No depreciation can be taken in years 4 and 5, and the book value of $5000 remains undepreciated.

(b) See Table 11-16.

**Table 11-16**

| | Year 1 | Year 2 | Year 3 | Year 4 | Year 5 |
|---|---|---|---|---|---|
| BTCF | $6000 | $6000 | $6000 | $6000 | $6000 |
| Depreciation | 7200 | 4320 | 1480 | 0 | 0 |
| Net Taxable Income | – 1200 | 1680 | 4520 | 6000 | 6000 |
| Tax (@50%) | –600 | 840 | 2260 | 3000 | 3000 |
| ATCF | 6600 | 5160 | 3740 | 3000 | 3000 |

For the after-tax ROR:

$$\begin{aligned} 0 &= -\$18\,000 + \$6600(P/F, i^*\%, 1) + \$5160(P/F, i^*\%, 2) \\ &\quad + [\$3740 + \$3000(P/A, i^*\%, 2)](P/F, i^*\%, 3) + \$5000(P/F, i^*\%, 5) \end{aligned}$$

or $i^* = 13.6\%$.

**11.5** Rework Problem 11.2 using the sum-of-years'-digits method of depreciation.

*(a)* $SY = (5)(6)/2 = 15$; see Table 11-17.

**Table 11-17**

| Year | Depreciation Charge for Year | Accumulated Depreciation | Book Value at End of Year |
|---|---|---|---|
| 1 | $(\frac{5}{15})(\$13\,000) = \$4333.33$ | $ 4333.33 | $13 666.67 |
| 2 | $(\frac{4}{15})(\ 13\,000) = \ 3466.67$ | 7800.00 | 10 200.00 |
| 3 | $(\frac{3}{15})(\ 13\,000) = \ 2600.00$ | 10 400.00 | 7600.00 |
| 4 | $(\frac{2}{15})(\ 13\,000) = \ 1733.33$ | 12 133.33 | 5866.67 |
| 5 | $(\frac{1}{15})(\ 13\,000) = \ 866.67$ | 13 000.00 | 5000.00 |

**Table 11-18**

| | Year 1 | Year 2 | Year 3 | Year 4 | Year 5 |
|---|---|---|---|---|---|
| BTCF | $6000.00 | $6000.00 | $6000 | $6000.00 | $6000.00 |
| Depreciation | 4333.33 | 3466.67 | 2600 | 1733.33 | 866.67 |
| Net Taxable Income | 1666.67 | 2533.33 | 3400 | 4266.67 | 5133.33 |
| Tax (@50%) | 833.33 | 1266.67 | 1700 | 2133.33 | 2566.67 |
| ATCF | 5166.67 | 4733.33 | 4300 | 3866.67 | 3433.33 |

(b) See Table 11-18.

In writing the equation for the after-tax ROR, we note that the depreciation charges form a gradient series, with

$$G = -\frac{1}{15}(\$13\,000) = -\$866.67$$

Hence, the $\text{ATCF}_j$ also form a gradient series, with

$$G' = (0.50)(-\$866.67) = -\$433.33$$

and we have:

$$0 = -\$18\,000 + [\$5166.67 - \$433.33(A/G, i^*\%, 5)]\,(P/A, i^*\%, 5) + \$5000(P/F, i^*\%, 5)$$

or $i^* = 14.5\%$.

**11.6** Rework Problem 11.2(*a*) using the sinking-fund method of depreciation and a before-tax MARR of *12%.*

From (*11.13*), with C = adjusted cost = $13 000,

$$\text{SFD}_j = \$13\,000(A/F, 12\%, 5)(1.12)^{j-1} = (\$2046.33)(1.12)^{j-1}$$

and we obtain Table 11-19.

**Table 11-19**

| Year | Depreciation Charge for Year | Accumulated Depreciation | Book Value at End of Year |
|---|---|---|---|
| 1 | $2046.33 | $ 2046.33 | $15 953.67 |
| 2 | 2291.89 | 4338.21 | 13 661.79 |
| 3 | 2566.91 | 6905.12 | 11 094.88 |
| 4 | 2874.94 | 9780.07 | 8219.93 |
| 5 | 3219.93 | 13 000.00 | 5000.00 |

**11.7** Refer to Problems 11.2 and 11.6. Is the computer system a viable proposition if the company's after-tax MARR is 12%?

**Table 11-20**

| | Year 1 | Year 2 | Year 3 | Year 4 | Year 5 |
|---|---|---|---|---|---|
| BTCF | $6000.00 | $6000.00 | $6000.00 | $6000.00 | $6000.00 |
| Depreciation | 2046.33 | 2291.89 | 2566.91 | 2874.94 | 3219.93 |
| Net Taxable Income | 3953.67 | 3708.11 | 3433.09 | 3125.06 | 2780.07 |
| Tax (@ 50%) | 1976.84 | 1854.06 | 1716.55 | 1562.53 | 1390.04 |
| ATCF | 4023.16 | 4145.95 | 4283.45 | 4437.47 | 4609.96 |

The after-tax cash flows are computed in Table 11-20. Then, at MARR = 12%,

$$\begin{aligned} PW &= -\$18\,000 + \$4023.16(P/F, 12\%, 1) + \$4145.95(P/F, 12\%, 2) + \$4283.45(P/F, 12\%, 3) \\ &\quad + \$4437.47(P/F, 12\%, 4) + \$4609.96(P/F, 12\%, 5) + \$5000(P/F, 12\%, 5) \\ &= \$219.18 \end{aligned}$$

From this, we conclude that the investment meets the after-tax MARR and pays an extra $219.18 (in today's money) over the five-year useful life.

**11.8** Rework Problem 11.2 using the ACRS. Assume that the IRS classifies the asset as five-year property.

(a) See Table 11-21.

**Table 11-21**

| Year | Depreciation Rate for Year | Depreciation Charge for Year | Accumulated Depreciation | Book Value at End of Year |
|---|---|---|---|---|
| 1 | 15% | $2700 | $ 2700 | $15 300 |
| 2 | 22 | 3960 | 6 660 | 11 340 |
| 3 | 21 | 3780 | 10 440 | 7 560 |
| 4 | 21 | 3780 | 14 220 | 3 780 |
| 5 | 21 | 3780 | 18 000 | 0 |

(b) See Table 11-22.

**Table 11-22**

| | Year 1 | Year 2 | Year 3 | Year 4 | Year 5 |
|---|---|---|---|---|---|
| BTCF | $6000 | $6000 | $6000 | $6000 | $6000 |
| Depreciation | 2700 | 3960 | 3780 | 3780 | 3780 |
| Net Taxable Income | 3300 | 2040 | 2220 | 2220 | 2220 |
| Tax (@50%) | 1650 | 1020 | 1110 | 1110 | 1110 |
| ATCF | 4350 | 4980 | 4890 | 4890 | 4890 |

Sale of the equipment for its salvage value at the end of year 5 produces a long-term capital gain, of which the value after taxes is

$$(1-0.30)(\$5000) = \$3500$$

Hence, the equation for the after-tax ROR is

$$0 = -\$18\,000 + \$4350(P/F, i^*\%, 1) + \$4980(P/F, i^*\%, 2) + \$4890(P/A, i^*\%, 3)(P/F, i^*\%, 2) + \$3500(P/F, i^*\%, 5)$$

giving $i^* = 14.45\%$.

**11.9** Rework Problem 11.8 on the assumption that the asset is reclassified as three-year property.

*(a)* See Table 11-23.

**Table 11-23**

| Year | Depreciation Rate for Year | Depreciation Charge for Year | Accumulated Depreciation | Book Value at End of Year |
|---|---|---|---|---|
| 1 | 25% | $4500 | $ 4 500 | $13 500 |
| 2 | 38 | 6840 | 11 340 | 6 660 |
| 3 | 37 | 6660 | 18 000 | 0 |
| 4 | 0 | 0 | 18 000 | 0 |
| 5 | 0 | 0 | 18 000 | 0 |

*(b)* See Table 11-24.

**Table 11-24**

| | Year 1 | Year 2 | Year 3 | Year 4 | Year 5 |
|---|---|---|---|---|---|
| BTCF | $6000 | $6000 | $6000 | $6000 | $6000 |
| Depreciation | 4500 | 6840 | 6660 | 0 | 0 |
| Net Taxable Income | 1500 | −840 | −660 | 6000 | 6000 |
| Tax (@50%) | 750 | −420 | −330 | 3000 | 3000 |
| ATCF | 5250 | 6420 | 6330 | 3000 | 3000 |

$$0 = -\$18\,000 + \$5250(P/F, i^*\%, 1) + \$6420(P/F, i^*\%, 2) + \$6330(P/F, i^*\%, 3) + \$3000(P/A, i^*\%, 2)(P/F, i^*\%, 3) + \$3500(P/F, i^*\%, 5)$$

whence $i^* = 16.27\%$. Note the benefit to the taxpayer when the asset's cost is depreciated over a period shorter than the useful life.

## Supplementary Problems

**11.10** For the asset of Problem 11.2, *(a)* determine the ROR of the project before taxes and *(b)* recommend a depreciation method on the basis of Problems 11.2*(b)*, 11.4*(b)*, 11.5*(b)*, 11.7, and 11.8*(b)*.
*Ans.* *(a)* 24.4%; *(b)* sum-of-years'-digits (if allowed by the IRS, which is doubtful)

**11.11** A car rental agency has bought three economy-size, four medium-size, and two full-size cars; see Table 11-25. Using composite depreciation, compute the annual straight-line depreciation charge and the (composite) life for this collection of cars. ***Ans.*** \$12 100, 324/121 = 2.678 years

**Table 11-25**

| | Economy | Medium | Full |
|---|---|---|---|
| Initial Cost (each) | \$6200 | \$8200 | \$13 000 |
| Service Life | 2 years | 3 years | 3 years |
| Salvage Value (each) | \$3600 | \$4900 | \$7300 |

**11.12** Rework Problem 11.11 using group depreciation. ***Ans.*** \$12 150, 8/3 = 2.667 years

**11.13** Rework Problem 10.16, assuming a Cyear tax life remaining for the current machine and a Cyear tax life for the new one. Straight-line depreciation is used, and an after-tax MARR of 15% is applicable. The tax rate is 50%. ***Ans.*** after-tax $\text{EUAC}_{\text{old}}$ = \$6565, after-tax $\text{EUAC}_{\text{new}}$ = \$7640; keep old machine.

**11.14** Would the decision made in Problem 10.5 change if the second-hand machine could be depreciated in 10 years by the sum-of-years'-digits method and the company's tax rate is 52%? The after-tax MARR is 10%. ***Ans.*** after-tax $\text{EUAC}_{\text{shut-down}}$ = \$168, after-tax $\text{EUAC}_{\text{stand-by}}$ = \$407; decision unchanged.

**11.15** Rework Problem 10.15, using an after-tax MARR of 10% and straight-line depreciation. The company's tax rate is 52%. ***Ans.*** 18 days

**11.16** Would the economic life of the challenger in Problem 10.19 change if the sum-of-years'-digits depreciation method is used, the tax life is 8 years, the tax rate is 52%, and the after-tax MARR is 10%?
***Ans.*** no [after-tax EUAC(6) = \$23211

**11.17** Perform an after-tax analysis for the situation described in Problem 10.20. Assume that the service life and the tax life are identical, that straight-line depreciation is used, that the after-tax MARR is 10%, and that the company's tax rate is 50%. ***Ans.*** after-tax CUV > \$1000; keep current machine.

**11.18** A machine's current book value is \$600. The machine cost \$1300 three years ago. Operating expenses have been \$380 per year, and the machine could last for three more years. Because of a breakthrough in design, a replacement machine which would save \$300 per year sells for \$1000 and has an expected service life of eight years. The scrap value of either machine at any time after installation is \$100. If the IRS allows straight-line depreciation over a six-year period for this type of machinery and if an after-tax MARR of 12% is acceptable, should the current machine be changed? The company's tax rate is 52%.
***Ans.*** after-tax $\text{EUAC}_{\text{current}}$ = \$90, after-tax $\text{EUAC}_{\text{replacement}}$ = \$167; keep current machine.

**11.19** An income-producing asset costs \$60 000, has an estimated useful life of 7 years, has no salvage value after installation, and is expected to produce annual net savings of \$15 000. The company's tax rate is 52%. Compute (a) the before-tax ROR, and (b) the after-tax ROR under straight-line depreciation.
***Ans.*** (a) 16.3%; (*b*) 8.6%

**11.20** For the situation described in Problem 11.19, compute the after-tax ROR when sum-of-years'-digits depreciation is charged. ***Ans.*** 9.7%

**11.21** Fifty percent of the asset described in Problem 11.19 was financed from capital borrowed at 7%. This loan is to be repaid at the end of the seventh year, but interest is due on the principal at the end of each year. Rework Problem 11.20.
***Ans.*** 26.0% (notice the big difference made by tax-deductible interest and the delay of seven years in half of the investment)

**11.22** Would the result of Problem 10.23 change under straight-line depreciation, a 35% tax rate, and an after-tax MARR of 10%? Assume that the current machine is fully depreciated. ***Ans.*** no

**11.23** Rework Problem 10.24 using straight-line depreciation, a 50% tax rate, and an after-tax MARR of 10%. Assume that salvage values are book values and that overhaul expenditures are depreciable.
***Ans.*** Now the best action is to overhaul the old machine.

**11.24** An underwater camera is purchased for $1000; it has an expected life of 12 years, at the end of which the estimated salvage value is $730. Using straight-line depreciation, find the book value of the camera at the end of 8 years. ***Ans.*** $420

**11.25** An engineer is being transferred to another state and must vacate her house. The house, bought for $40 000 eight years ago, can be sold now for $60 000; the property is free of debt. If the house is sold, she will have to pay a 15% capital gains tax. The engineer is also considering leasing the house for five years, receiving $7200 annual rental. In this case, she estimates an annual disbursement of $1800 for taxes, insurance, and maintenance. She would also be allowed $1200 per year depreciation on her tax return (in addition to her cash disbursements); her rental income would be taxed at 30%. (Note that houses cannot be depreciated during the years used as the owner's personal residence.) Determine her after-tax ROR if she decides to lease the property and if the property shall be worth $64000 at the end of the lease. ***Ans.*** 8%

**11.26** A truck was bought 10 years ago for $70 000; its current salvage value is $14 000. It is believed that it can last 5 more years, at which time its salvage value will be $8000. Its operating expenses amount to $14 000 per year and they are expected to remain at that level for the next 5 years. The truck is currently being depreciated by the straight-line method, using a 15-year life and estimated salvage value of $10 000. A new truck can be purchased for $65 000. It will have yearly operating costs of $9000 and would last 20 years; salvage value after 20 years is estimated at $15 000. Again, straight-line depreciation would be used. Supposing that tax rates are 50% on income and 15% on capital gains (or losses), and that the company's after-tax MARR is 10%, should it buy the new truck?
***Ans.*** No: after-tax $EUAC_{old}$ = $7967, after-tax $EUAC_{new}$ = $10 623

**11.27** A $60 000 asset will be depreciated by the straight-line method over a six-year period. No salvage value is expected. If the company's tax rate is 50%, what would be the present-worth advantage of using the sum-of-years'-digits method, given a 10% after-tax MARR? ***Ans.*** $1184

**11.28** Rework Problem 11.26 using the ACRS for the new truck, classified as a 10-year asset. The current truck is still depreciated by the straight-line method.
***Ans.*** No: after-tax $EUAC_{old}$ = $7967, after-tax $EUAC_{new}$ = $9481

**11.29** Rework Problem 11.27 using the ACRS, with a five-year life, instead of the sum-of-years'-digits method.
***Ans.*** −$3217 (i.e., straight-line depreciation is better in this case)

# Chapter 12

# Preparing and Presenting an Economic Feasibility Study

## 12.1 INTRODUCTION

This chapter attempts to bridge the gap between the specific analytic techniques-equivalent uniform annual series, rate of return, etc.—and the "wide-angle" considerations that determine (or should determine) investment decisions in the real world. Our frame of reference will be the presentation of a feasibility analysis to a lending institution.

**Example 12.1** It will be assumed throughout this chapter that the project under study can be characterized as "marginal" with respect to the overall economic environment. Discuss the need for such an assumption and give examples of marginal projects.

A project is "marginal" if it will not significantly alter the economic environment. A dry-cleaner outlet in a major shopping center, a small die-casting factory in an industrial park, a new boutique in a fashion mall, are typical examples. Although undeniably important to their promoters, these ventures will not effect major changes in the economic patterns of the communities where they are to be implemented. For such ventures, clear-cut decisions may be derived from an engineering economic analysis.

By contrast, projects which represent a structural investment for the community (which would, for example, significantly alter the unemployment level or the gross regional product) would need in-depth study in every aspect and a thorough sensitivity analysis of each assumption underlying the feasibility study. Most likely, final decisions would be largely political in nature. Two such nonmarginal projects were Walt Disney World, which changed central Florida from a depressed rural area to a tourist capital, and the Trans-Alaska Oil Pipeline.

The sections that follow will discuss, one at a time, the main components of the feasibility study.

## 12.2 BACKGROUND INFORMATION

This section of the report should consist of:

1. A brief summary of the project, covering the nature of the venture, location, expected site, life, overall capital costs, financing, and return on investment
2. A description of the promoting individuals or institution: name(s), addresses (both legal and of proposed facility location), and institution characteristics (capital, number of shares of stock, principal shareholders, etc.)

**Example 12.2** Why do most lending institutions require background information of the above type?

The lender will charge interest; nevertheless, he is risking his money, and before committing himself, understandably wants to gauge the risk as accurately as possible. Complete details about the board and executive officers of the promoting company, any partnerships or relationships to other companies, insurance available, and—most important—a history of past and current projects, are all commonly required. For industries currently in operation, data about their capacity, production level, productivity indicators, sales, labor force, salary structure, overhead, inventory turnover ratio, and other items (which may include some seemingly unrelated to the project itself) may be in order. Sometimes the lender requires an organizational chart, as well as details about production planning and control, quality control, labor relations, and financial situation (end of year balances for the last few years).

## 12.3 MARKET STUDY

In this section the report describes what is to be produced, and where and to whom it is going to be sold. An analysis should be made of:

1. The product—description, brand or name, quality standards to be met, characteristics, and utilization. Subproducts (if any)—description, utilization (if it is not going to be marketed).
2. The market--estimated demand for the final product. If the product can serve as raw material or intermediate product for some other article (**e.g.,** a microprocessor to be used in an electronic toy), an estimate of potential uses and corresponding demands is in order. An analysis of complementary and competitive products, as well as their current and expected availabilities, should also be included.

**Example** 12.3 Discuss the importance of a thorough, realistic market study. What should be included in it, and what should be its ultimate objective? Show a typical summary output of this phase of the analysis.

A number of otherwise carefully planned ventures have failed because of an unjustified belief that the market will buy whatever can be produced. An excessive capacity is then built to exploit the good idea—so good, in fact, that it attracts immediate competition, and the long-term market share turns out to be lower than predicted. (Case in point: some fast-food retailers.)

A marketing study should provide information about: volume of similar products sold in the target region over the last, say, five years; local production; imports from, and exports to, other areas; consumption. It should give a quantification (and causes) of any unmet demand. An analysis of main producers is a must: location, market share, other areas served by them. Also, an analysis of main consumers: where they are located, historical consumption, uses of our product.

The goal of the market study is to provide the basis for a forecast of demand for the product in the first (say) five years after the operations start up. Not just a single number, but a range of values should be sought. A ***pessimistic,*** an ***optimistic,*** and a ***most probable value*** would be invaluable in a sensitivity analysis, provided the technical bases of the forecast are sound. A description of the techniques used in forecasting, as well as of the assumptions and data bases used in the analysis, should be included in an appendix to the report.

Figure 12-1 suggests the summary format.

| | Year 1 (units) | Year 2 (units) | Year 3 (units) |
|---|---|---|---|
| 1. Estimate of Regional Production | | | |
| 2. Imports | | | |
| 3. Exports | | | |
| 4. Estimated Demand (1 + 2 – 3) | | | |
| **With respect to** the **company** | | | |
| 5. Estimated Production | | | |
| 6. Imports | | | |
| 7. Exports | | | |
| 8. Regional Demand Met (5 + 6 – 7) | | | |

**Fig. 12-1**

3. Raw materials analysis. [Too often this phase of the study is replaced by an assumption that whatever is required will be there when (and in the amount) needed.]

Example 12.4 Discuss the potential dangers of omitting a raw materials analysis. Describe what such an analysis would yield. Give an example of obvious need for this type of study.

Unreliable providers may cause shortages, or they may force operators to keep excessive raw materials inventories lest they expose themselves to production stoppages. A raw materials analysis should include a study of prices (in the national and international markets), with considerations of transportation, insurance, tariffs, quality and delivery reliability of provider included for the latter and normal commercialization channels. If the project size is large, its potential effect on the raw materials market should be also considered. The availability of alternative providers, or even of alternative materials, may also turn out to be an important factor.

An extreme instance of the necessity for raw materials analysis was provided by the oil crisis of the 1970s.

## 12.4 PROJECT ENGINEERING

This section makes an analysis of available technologies for the process. A brief technical/economic comparison should be made and a justification of the selected technology provided. A statement of the potential consequences of this selection, a comparison with the "normal" competitor, as well as with the "state of the art" competitor, are desirable. Opinions from outside consultants, if available, regarding the selection of technology are also recommended.

**Example** 12.5 Describe the contents of the typical project engineering segment of the report. Comment on the effect of technology, size, start-up, and major equipment. Illustrate a typical summary output of this phase of the analysis.

The project engineering section of the economic feasibility report should include a thorough description of:

***Fabrication process.*** No detail plans are needed; a flow diagram with durations of the stages, capacities, yields, and material and energy balances is normally sufficient for most manufacturing industries.

***Project size.*** The planned production capacities must be forecast, indicating expected dates to attain them. Operating conditions (shifts per day, days per year) must be indicated. The relationship between this production plan and the market and raw materials studies should be included. An analysis of the market-share penetration must be provided. A justification of size from the point of view of the selected technology, financing limitations, plant location, seasonality factors, market restrictions, etc., is important. Analyses of size versus production costs, break-even point, resources needed to compensate for operating deficits over the start-up period, and impact of unforeseen slumps in sales are also required. A note should be made if future expansions to adapt to enlarged market share and product acceptance are envisioned.

***Location.*** This extremely important factor is frequently lost in the shuffle. It is evident that remoteness of a location influences the pool of manpower available for the project, as well as the level of investment needed to provide housing, transportation, energy, water, sewage and tailings treatment, etc. These same factors should be considered for any location. Special consideration should be given to potential benefits related to the project's location, such as tax breaks, availability of cheap transportation, surrounding market, population, local regulations controlling noise and pollution, etc. The main factors influencing the selection of the project site should be discussed in detail, and an analysis of potential alternative sites, if any, should also be included.

***Physical means of production.*** The plant site must be specified (total and build surfaces). Buildings required must also be detailed. Areas should be classified as direct production, ancillary facilities, administrative, warehouses, etc. If some of the buildings are already available and require little renovation, that should be noted. It is also very important to specify any earth moving, roads, docks, railroad connections, fences, etc., which should be needed as part of the site preparation. Note that electric substations, water treatment plant, and the like, are normally considered auxiliary facilities (see below).

***Major equipment.*** Separate lists are usually made of domestic and imported equipment. The units are listed in order of decreasing value, until the total value of the two lists represents 60 to 70 percent of the entire capital investment. Numbers of the various units, their technical characteristics, theoretical capacities, prices (FOB; for imported equipment, extra charges such as transportation, in-transit insurance, tariffs, etc., must be detailed), and manufacturers are all indicated. Prices quoted should reflect actual market values, if at all possible. Pro forma invoices are of great help in documenting this phase of the analysis.

***Auxiliary facilities*** (for direct production)—electric energy, gas, fuel, compressed air, water treatment plant, internal communications, internal transportation, sewage, tailings treatment plant, etc. If applicable, flow diagrams and material and/or energy balances must be provided. A global estimate of capital expenditures (as a function of plant size, if applicable) should also be included.

***Service facilities.*** These include plant security, medical facilities, dining room, and other personnel-related facilities. A global estimate of capital expenditures (as a function of plant size, if applicable) is required.

***Raw materials and supplies.*** For each production level, an estimate must be made of quantity, quality requirements, annual consumption, availability, unit price, and consumption per unit of product. For electric energy, it is customary to indicate installed power, processes at constant and at variable load, and maximum illumination-related load. An estimate of size and value must be given for all raw materials and for in-production or finished-product inventories expected for the process.

***Transportation expenses—for*** raw materials, fuels, and intermediate and finished products. If contracts are to be awarded to third parties, their availability and maximum requirements (at start-up and in the long run), as well as prices, must be included.

***Manpower requirements.*** Estimate of personnel required at different levels in each unit, including labor, direct supervisory personnel, indirect supervisory personnel, service people (labs, maintenance, security), administration, marketing, and management. Detail wage and salary structures, as well as fringe benefit charges. Any contractual obligations should also be included. Any need for specially trained personnel, training facilities, start-up consultants, etc., should also be reported.

The project engineering phase of the analysis should produce a summary bar diagram describing target completion date for all major project components, as well as a plan for project implementation until "normal" operation is achieved. Such a bar chart is illustrated in Fig. 12-2.

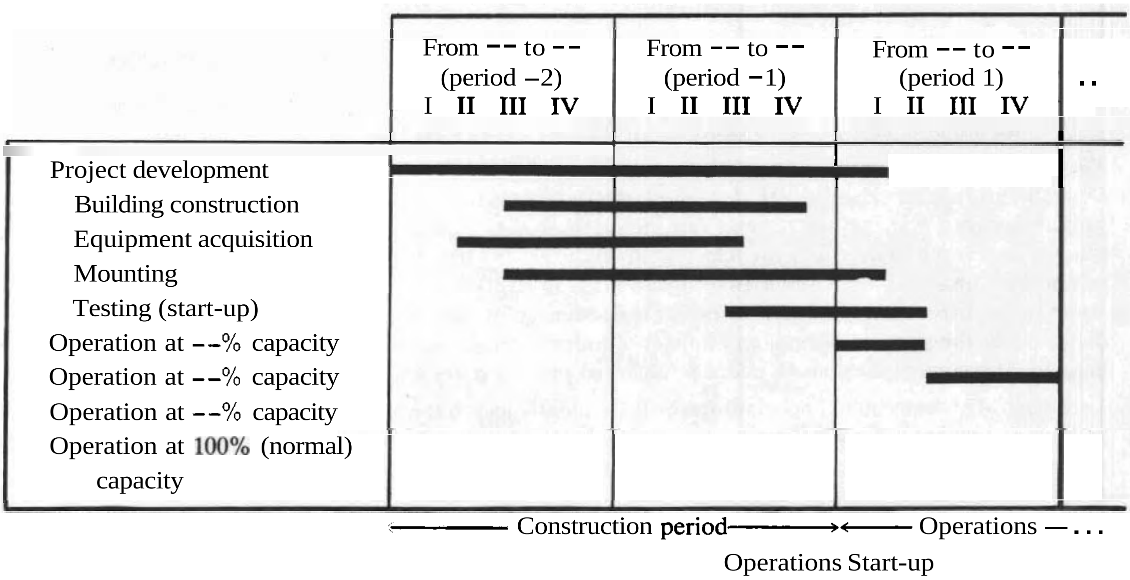


**Fig. 12-2**

## 12.5 COST ESTIMATION

This section of the economic feasibility report falls into two main parts: the analysis of *investment costs* (or *first costs)* and the analysis of *operating costs.*

**Example 12.6** Itemize the components of the investment costs for a typical industrial project.

The investment costs include all expenditures related to the implementation of the project, from its conception until start-up. A distinction is usually made between *depreciable* and *nondepreciable* investment costs.

***Fixed capital costs.*** (Depreciable, except for land purchasing). These include costs of: residential buildings (offices, cafeteria, etc.); industrial buildings (including warehouses); roads, energy lines, railroads, and other infrastructure; hauling, loading, and unloading equipment; industrial machinery and equipment; spare parts (includes maintenance, repair, and standby equipment); ancillary facilities (electrical substation, laboratories, transformers, fire extinguishers, etc.); office furniture. Usually only buildings and main equipment and facilities are estimated accurately. Spare parts, furniture, etc., are often estimated as a percentage of those items.

***Operating capital costs.*** This is the amount of money required to start up the project and keep it working. The operating capital costs usually increase until the project reaches the level of normal operation; then they stay on a plateau throughout the project's lifetime, and are recovered in the final year of operation. They cannot be depreciated. They include: cash (to pay salaries, to cover emergencies, and—sometimes—to help in operating process); circulating capital (accounts receivable minus accounts payable); stocks and inventories (general merchandise, finished or intermediate or secondary products, raw materials, in-transit material, packages, consumption materials, etc.); material handling (loading and transportation and unloading from and to the warehouses, cost of inventory control and insurance, protection of inventories, etc.).

The analysis of first cost should produce a table such as Table 12-1 (for each alternative size of project considered).

**Table 12-1**

| *Item* | $ |
|---|---|
| Feasibility studies | |
| Land | |
| Land preparation | |
| Construction & installation (buildings) | |
| Equipment (incl. insurance, transportation, installation & start-up) | |
| For the main plant | |
| For ancillary facilities | |
| For construction | |
| Patents and royalties | |
| Supervision | |
| Consultantships (legal and engineering) | |
| Start-up costs | |
| Subtotal | |
| Contingency costs (% of subtotal) | |
| Operating capital | |
| TOTAL INVESTMENT COST | $ |

Typical errors in the analysis of investment costs are: (i) underestimation of transportation, installation, and start-up costs; (ii) underestimation of time needed to construct the project; (iii) underestimation of the operating capital; (iv) underestimation of the time needed to test-run equipment and to reach the level of normal operation; and (most common) (v) omission of a sensitivity study of project size. Keep in mind that first costs, being incurred in year 0, are not discounted. Thus, too-large estimates of investment costs can kill a project's feasibility a lot faster than any overestimation of operating expenses.

**Example** 12.7 Itemize the components of the operating costs for a typical industrial project.

***Direct costs:*** raw material (includes cost of handling); direct materials (explosives, catalysts, grinding balls, packages, etc.); direct labor (including direct supervision)—salaries, fringe benefits, overtime, etc.; direct-production utilities (energy, fuel, lubricants, steam, water, etc.).

***Indirect costs:*** indirect labor (salaries, fringe benefits, overtime, etc., for general supervision, maintenance, general engineering, security, plant protection, quality control, laboratories); indirect materials (e.g., lab reagents); other indirect costs (health clinic, recreation and eating facilities, transportation of personnel, communications, lights, cleaning, etc.); employee benefits (child-care center, gymnasium, etc.).

***Overhead costs:*** administrative costs (salaries of managerial personnel, secretaries, legal and engineering staff; rent, office cleaning, office materials, reproduction, etc.); fixed charges (taxes, insurance); selling expenses (salesmen, commissions, travel, market surveys, entertainment of clients, displays, sales space, etc.); research and development; financing charges (interest and loan payments); bad debts; contributions (not in excess of 5% of taxable income); losses by fire, theft, etc., not covered by insurance.

The analysis of the operating costs should produce a table such as Table 12-2 (for each alternative size of project considered).

**Table 12-2**

| *Item* | *Year 1* | *Year2 ··· Yearn* |
|---|---|---|
| Direct costs | | |
| Labor | | |
| Material | | |
| Indirect costs | | |
| Labor | | |
| Material | | |
| Other | | |
| Overhead costs | | |
| Administrative costs | | |
| Selling costs | | |
| Other | | |
| TOTAL OPERATING COST ($) | | |

The most common error in the analysis of operating costs is the wrong estimation of plant utilization (i.e., designing with overcapacity), which strongly affects direct costs.

## 12.6 ESTIMATION OF REVENUES

A projection of annual income has to be made, with consideration of: (i) sales of main and secondary products; (ii) services provided; (iii) recuperation of operating capital (typically occurs at the end of year $n$, when stocks and inventories are depleted, circulating capital is settled, and material handling stops).

The most common error in this phase of the analysis is to assume that all the production will be sold. Sometimes market conditions will not permit this to happen; or the product quality may fluctuate, and sizable quantities may have to be recycled or scrapped.

Another common error, from a company point of view, is to forget that revenue consists only in actual *incremental* receipts. For example, if an ice cream distributor is evaluating a new line of products, actual net receipts attributable to this product is given by the expected sales less the loss in sales of current products because of customers' shifting preferences. In the extreme case, the public may not be spending more money for the company's products (although the new product is experiencing brisk sales); rather, they have ceased buying some of the old products and are using that money to buy the new one!

## 12.7 FINANCING

Once the cost and revenue analyses are ready, (11.15) and (11.19) may be used to prepare a summary table of yearly cash flows (see Table 12-3). Then a specific performance indicator, such as after-tax PW or after-tax ROR, can be computed. In fact, it is recommended that both these indicators be calculated. For, on the one hand, if the company has a good estimate of its (after-tax) MARR, then the PW corresponding to that MARR will represent to the company its net increase in assets as a result of the project once the initial investment (and all operating costs) have been recovered and the MARR realized. On the other hand, financial institutions prefer the ROR, since it is independent of the company's MARR and is useful in choosing among independent projects that are similar in size and duration (see Section 9.2).

**Table 12-3**

| | |
|---|---|
| (1) | Sales |
| (2) | Services provided |
| (3) | Recovered operating capital |
| (4) | TOTAL **RECEIPTS** |
| (5) | Less: Direct labor |
| (6) | Less: Direct material |
| (7) | Less: Maintenance |
| (8) | Less: Indirect material |
| (9) | Less: Indirect labor |
| (10) | Less: Overhead |
| (11) | Less: Other expenses |
| (12) | Net income before taxes |
| (13) | Less: Depreciation |
| (14) | Net taxable income |
| (15) | Less: Taxes [rate × (14)] |
| (16) | After-tax net income |
| (17) | Depreciation |
| (18) | After-tax net cash flow |

It should be noted that, generally, not one but two economic evaluations must be presented to a lending institution. One of them analyzes the project as if it were going to be completely financed by the owner. The results of this analysis reflect the project's potential of success (and its ability to generate enough revenue to pay the loan). The second analysis takes account of the size of the loan and of the corresponding schedule of payments. This analysis furnishes the owner with a better representation of the economic potential of the project as it is actually intended to be financed. Both sorts of evaluation are carried out using the techniques of Chapter 11––see, in particular, Problems 11.19 and 11.21. (That Chapter 11 dealt with level series of before-tax cash flows is obviously inessential.)

# Appendix A

## Compound Interest Factors—Annual Compounding

COMPOUND INTEREST FACTORS – ANNUAL COMPOUNDING

INTEREST RATE = 0.25 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0025 | 1.0000 | 1.00250 | .0000 |
| 2 | 1.0050 | 2.0025 | .50188 | .4994 |
| 3 | 1.0075 | 3.0075 | .33500 | .9983 |
| 4 | 1.0100 | 4.0150 | .25156 | 1.4969 |
| 5 | 1.0126 | 5.0251 | .20150 | 1.9950 |
| 6 | 1.0151 | 6.0376 | .16813 | 2.4927 |
| 7 | 1.0176 | 7.0527 | .14429 | 2.9900 |
| 8 | 1.0202 | 8.0704 | .12641 | 3.4869 |
| 9 | 1.0227 | 9.0905 | .11250 | 3.9834 |
| 10 | 1.0253 | 10.1133 | .10138 | 4.4794 |
| 11 | 1.0278 | 11.1385 | .09228 | 4.9750 |
| 12 | 1.0304 | 12.1664 | .08469 | 5.4702 |
| 13 | 1.0330 | 13.1968 | .07828 | 5.9650 |
| 14 | 1.0356 | 14.2298 | .07278 | 6.4594 |
| 15 | 1.0382 | 15.2654 | .06801 | 6.9534 |
| 16 | 1.0408 | 16.3035 | .06384 | 7.4469 |
| 17 | 1.0434 | 17.3443 | .06016 | 7.9401 |
| 18 | 1.0460 | 18.3876 | .05688 | 8.4328 |
| 19 | 1.0486 | 19.4336 | .05396 | 8.9251 |
| 20 | 1.0512 | 20.4822 | .05132 | 9.4170 |
| 21 | 1.0538 | 21.5334 | .04894 | 9.9085 |
| 22 | 1.0565 | 22.5872 | .04677 | 10.3995 |
| 23 | 1.0591 | 23.6437 | .04479 | 10.8901 |
| 24 | 1.0618 | 24.7028 | .04298 | 11.3804 |
| 25 | 1.0644 | 25.7646 | .04131 | 11.8702 |
| 26 | 1.0671 | 26.8290 | .03977 | 12.3596 |
| 27 | 1.0697 | 27.8961 | .03835 | 12.8485 |
| 28 | 1.0724 | 28.9658 | .03702 | 13.3371 |
| 29 | 1.0751 | 30.0382 | .03579 | 13.8252 |
| 30 | 1.0778 | 31.1133 | .03464 | 14.3130 |
| 31 | 1.0805 | 32.1911 | .03356 | 14.8003 |
| 32 | 1.0832 | 33.2716 | .03256 | 15.2872 |
| 33 | 1.0859 | 34.3547 | .03161 | 15.7736 |
| 34 | 1.0886 | 35.4406 | .03072 | 16.2597 |
| 35 | 1.0913 | 36.5292 | .02988 | 16.7454 |
| 40 | 1.1050 | 42.0132 | .02630 | 19.1673 |
| 45 | 1.1189 | 47.5661 | .02352 | 21.5789 |
| 50 | 1.1330 | 53.1887 | .02130 | 23.9802 |
| 55 | 1.1472 | 58.8819 | .01948 | 26.3710 |
| 60 | 1.1616 | 64.6467 | .01797 | 28.7514 |
| 65 | 1.1762 | 70.4839 | .01669 | 31.1215 |
| 70 | 1.1910 | 76.3944 | .01559 | 33.4812 |
| 75 | 1.2059 | 82.3792 | .01464 | 35.8305 |
| 80 | 1.2211 | 88.4392 | .01381 | 38.1694 |
| 85 | 1.2364 | 94.5753 | .01307 | 40.4980 |
| 90 | 1.2520 | 100.7885 | .01242 | 42.8162 |
| 95 | 1.2677 | 107.0797 | .01184 | 45.1241 |
| 100 | 1.2836 | 113.4500 | .01131 | 47.4216 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 0.50 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0050 | 1.0000 | 1.00500 | .0000 |
| 2 | 1.0100 | 2.0050 | .50375 | .4988 |
| 3 | 1.0151 | 3.0150 | .33667 | .9967 |
| 4 | 1.0202 | 4.0301 | .25313 | 1.4938 |
| 5 | 1.0253 | 5.0503 | .20301 | 1.9900 |
| 6 | 1.0304 | 6.0755 | .16960 | 2.4855 |
| 7 | 1.0355 | 7.1059 | .14573 | 2.9801 |
| 8 | 1.0407 | 8.1414 | .12783 | 3.4738 |
| 9 | 1.0459 | 9.1821 | .11391 | 3.9668 |
| 10 | 1.0511 | 10.2280 | .10277 | 4.4589 |
| 11 | 1.0564 | 11.2792 | .09366 | 4.9501 |
| 12 | 1.0617 | 12.3356 | .08607 | 5.4406 |
| 13 | 1.0670 | 13.3972 | .07964 | 5.9302 |
| 14 | 1.0723 | 14.4642 | .07414 | 6.4190 |
| 15 | 1.0777 | 15.5365 | .06936 | 6.9069 |
| 16 | 1.0831 | 16.6142 | .06519 | 7.3940 |
| 17 | 1.0885 | 17.6973 | .06151 | 7.8803 |
| 18 | 1.0939 | 18.7858 | .05823 | 8.3658 |
| 19 | 1.0994 | 19.8797 | .05530 | 8.8504 |
| 20 | 1.1049 | 20.9791 | .05267 | 9.3342 |
| 21 | 1.1104 | 22.0840 | .05028 | 9.8172 |
| 22 | 1.1160 | 23.1944 | .04811 | 10.2993 |
| 23 | 1.1216 | 24.3104 | .04613 | 10.7806 |
| 24 | 1.1272 | 25.4320 | .04432 | 11.2611 |
| 25 | 1.1328 | 26.5591 | .04265 | 11.7407 |
| 26 | 1.1385 | 27.6919 | .04111 | 12.2195 |
| 27 | 1.1442 | 28.8304 | .03969 | 12.6975 |
| 28 | 1.1499 | 29.9745 | .03836 | 13.1747 |
| 29 | 1.1556 | 31.1244 | .03713 | 13.6510 |
| 30 | 1.1614 | 32.2800 | .03598 | 14.1265 |
| 31 | 1.1672 | 33.4414 | .03490 | 14.6012 |
| 32 | 1.1730 | 34.6086 | .03389 | 15.0750 |
| 33 | 1.1789 | 35.7817 | .03295 | 15.5480 |
| 34 | 1.1848 | 36.9606 | .03206 | 16.0202 |
| 35 | 1.1907 | 38.1454 | .03122 | 16.4915 |
| 40 | 1.2208 | 44.1588 | .02765 | 18.8359 |
| 45 | 1.2516 | 50.3242 | .02487 | 21.1595 |
| 50 | 1.2832 | 56.6452 | .02265 | 23.4624 |
| 55 | 1.3156 | 63.1258 | .02084 | 25.7447 |
| 60 | 1.3489 | 69.7700 | .01933 | 28.0064 |
| 65 | 1.3829 | 76.5821 | .01806 | 30.2475 |
| 70 | 1.4178 | 83.5661 | .01697 | 32.4680 |
| 75 | 1.4536 | 90.7265 | .01602 | 34.6679 |
| 80 | 1.4903 | 98.0677 | .01520 | 36.8474 |
| 85 | 1.5280 | 105.5943 | .01447 | 39.0065 |
| 90 | 1.5666 | 113.3109 | .01383 | 41.1451 |
| 95 | 1.6061 | 121.2224 | .01325 | 43.2633 |
| 100 | 1.6467 | 129.3337 | .01273 | 45.3613 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 0.75 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0075 | 1.0000 | 1.00750 | .0000 |
| 2 | 1.0151 | 2.0075 | .50563 | .4981 |
| 3 | 1.0227 | 3.0226 | .33835 | .9950 |
| 4 | 1.0303 | 4.0452 | .25471 | 1.4907 |
| 5 | 1.0381 | 5.0756 | .20452 | 1.9851 |
| 6 | 1.0459 | 6.1136 | .17107 | 2.4782 |
| 7 | 1.0537 | 7.1595 | .14717 | 2.9701 |
| 8 | 1.0616 | 8.2132 | .12926 | 3.4608 |
| 9 | 1.0696 | 9.2748 | .11532 | 3.9502 |
| 10 | 1.0776 | 10.3443 | .10417 | 4.4384 |
| 11 | 1.0857 | 11.4219 | .09505 | 4.9253 |
| 12 | 1.0938 | 12.5076 | .08745 | 5.4110 |
| 13 | 1.1020 | 13.6014 | .08102 | 5.8954 |
| 14 | 1.1103 | 14.7034 | .07551 | 6.3786 |
| 15 | 1.1186 | 15.8137 | .07074 | 6.8606 |
| 16 | 1.1270 | 16.9323 | .06656 | 7.3413 |
| 17 | 1.1354 | 18.0593 | .06287 | 7.8207 |
| 18 | 1.1440 | 19.1947 | .05960 | 8.2989 |
| 19 | 1.1525 | 20.3387 | .05667 | 8.7759 |
| 20 | 1.1612 | 21.4912 | .05403 | 9.2516 |
| 21 | 1.1699 | 22.6524 | .05165 | 9.7261 |
| 22 | 1.1787 | 23.8223 | .04948 | 10.1994 |
| 23 | 1.1875 | 25.0010 | .04750 | 10.6714 |
| 24 | 1.1964 | 26.1885 | .04568 | 11.1422 |
| 25 | 1.2054 | 27.3849 | .04402 | 11.6117 |
| 26 | 1.2144 | 28.5903 | .04248 | 12.0800 |
| 27 | 1.2235 | 29.8047 | .04105 | 12.5470 |
| 28 | 1.2327 | 31.0282 | .03973 | 13.0128 |
| 29 | 1.2420 | 32.2609 | .03850 | 13.4774 |
| 30 | 1.2513 | 33.5029 | .03735 | 13.9407 |
| 31 | 1.2607 | 34.7542 | .03627 | 14.4028 |
| 32 | 1.2701 | 36.0148 | .03527 | 14.8636 |
| 33 | 1.2796 | 37.2849 | .03432 | 15.3232 |
| 34 | 1.2892 | 38.5646 | .03343 | 15.7816 |
| 35 | 1.2989 | 39.8538 | .03259 | 16.2387 |
| 40 | 1.3483 | 46.4465 | .02903 | 18.5058 |
| 45 | 1.3997 | 53.2901 | .02627 | 20.7421 |
| 50 | 1.4530 | 60.3943 | .02406 | 22.9476 |
| 55 | 1.5083 | 67.7688 | .02226 | 25.1223 |
| 60 | 1.5657 | 75.4241 | .02076 | 27.2665 |
| 65 | 1.6253 | 83.3709 | .01949 | 29.3801 |
| 70 | 1.6872 | 91.6201 | .01841 | 31.4634 |
| 75 | 1.7514 | 100.1833 | .01748 | 33.5163 |
| 80 | 1.8180 | 109.0725 | .01667 | 35.5391 |
| 85 | 1.8873 | 118.3001 | .01595 | 37.5318 |
| 90 | 1.9591 | 127.8790 | .01532 | 39.4946 |
| 95 | 2.0337 | 137.8225 | .01476 | 41.4277 |
| 100 | 2.1111 | 148.1445 | .01425 | 43.3311 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 1.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0100 | 1.0000 | 1.01000 | .0000 |
| 2 | 1.0201 | 2.0100 | .50751 | .4975 |
| 3 | 1.0303 | 3.0301 | .34002 | .9934 |
| 4 | 1.0406 | 4.0604 | .25628 | 1.4876 |
| 5 | 1.0510 | 5.1010 | .20604 | 1.9801 |
| 6 | 1.0615 | 6.1520 | .17255 | 2.4710 |
| 7 | 1.0721 | 7.2135 | .14863 | 2.9602 |
| 8 | 1.0829 | 8.2857 | .13069 | 3.4478 |
| 9 | 1.0937 | 9.3685 | .11674 | 3.9337 |
| 10 | 1.1046 | 10.4622 | .10558 | 4.4179 |
| 11 | 1.1157 | 11.5668 | .09645 | 4.9005 |
| 12 | 1.1268 | 12.6825 | .08885 | 5.3815 |
| 13 | 1.1381 | 13.8093 | .08241 | 5.8607 |
| 14 | 1.1495 | 14.9474 | .07690 | 6.3384 |
| 15 | 1.1610 | 16.0969 | .07212 | 6.8143 |
| 16 | 1.1726 | 17.2579 | .06794 | 7.2886 |
| 17 | 1.1843 | 18.4304 | .06426 | 7.7613 |
| 18 | 1.1961 | 19.6147 | .06098 | 8.2323 |
| 19 | 1.2081 | 20.8109 | .05805 | 8.7017 |
| 20 | 1.2202 | 22.0190 | .05542 | 9.1694 |
| 21 | 1.2324 | 23.2392 | .05303 | 9.6354 |
| 22 | 1.2447 | 24.4716 | .05086 | 10.0998 |
| 23 | 1.2572 | 25.7163 | .04889 | 10.5626 |
| 24 | 1.2697 | 26.9735 | .04707 | 11.0237 |
| 25 | 1.2824 | 28.2432 | .04541 | 11.4831 |
| 26 | 1.2953 | 29.5256 | .04387 | 11.9409 |
| 27 | 1.3082 | 30.8209 | .04245 | 12.3971 |
| 28 | 1.3213 | 32.1291 | .04112 | 12.8516 |
| 29 | 1.3345 | 33.4504 | .03990 | 13.3044 |
| 30 | 1.3478 | 34.7849 | .03875 | 13.7557 |
| 31 | 1.3613 | 36.1327 | .03768 | 14.2052 |
| 32 | 1.3749 | 37.4941 | .03667 | 14.6532 |
| 33 | 1.3887 | 38.8690 | .03573 | 15.0995 |
| 34 | 1.4026 | 40.2577 | .03484 | 15.5441 |
| 35 | 1.4166 | 41.6603 | .03400 | 15.9871 |
| 40 | 1.4889 | 48.8864 | .03046 | 18.1776 |
| 45 | 1.5648 | 56.4811 | .02771 | 20.3273 |
| 50 | 1.6446 | 64.4632 | .02551 | 22.4363 |
| 55 | 1.7285 | 72.8525 | .02373 | 24.5049 |
| 60 | 1.8167 | 81.6697 | .02224 | 26.5333 |
| 65 | 1.9094 | 90.9366 | .02100 | 28.5217 |
| 70 | 2.0068 | 100.6763 | .01993 | 30.4703 |
| 75 | 2.1091 | 110.9128 | .01902 | 32.3793 |
| 80 | 2.2167 | 121.6715 | .01822 | 34.2492 |
| 85 | 2.3298 | 132.9790 | .01752 | 36.0801 |
| 90 | 2.4486 | 144.8633 | .01690 | 37.8724 |
| 95 | 2.5735 | 157.3538 | .01636 | 39.6265 |
| 100 | 2.7048 | 170.4814 | .01587 | 41.3426 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = **1.25** PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0125 | 1.0000 | 1.01250 | .0000 |
| 2 | 1.0252 | 2.0125 | .50939 | .4969 |
| 3 | 1.0380 | 3.0377 | .34170 | .9917 |
| 4 | 1.0509 | 4.0756 | .25786 | 1.4845 |
| 5 | 1.0641 | 5.1266 | .20756 | 1.9752 |
| 6 | 1.0774 | 6.1907 | .17403 | 2.4638 |
| 7 | 1.0909 | 7.2680 | .15009 | 2.9503 |
| 8 | 1.1045 | 8.3589 | .13213 | 3.4348 |
| 9 | 1.1183 | 9.4634 | .11817 | 3.9172 |
| 10 | 1.1323 | 10.5817 | .10700 | 4.3975 |
| 11 | 1.1464 | 11.7139 | .09787 | 4.8758 |
| 12 | 1.1608 | 12.8604 | .09026 | 5.3520 |
| 13 | 1.1753 | 14.0211 | .08382 | 5.8262 |
| 14 | 1.1900 | 15.1964 | .07831 | 6.2982 |
| 15 | 1.2048 | 16.3863 | .07353 | 6.7682 |
| 16 | 1.2199 | 17.5912 | .06935 | 7.2362 |
| 17 | 1.2351 | 18.8111 | .06566 | 7.7021 |
| 18 | 1.2506 | 20.0462 | .06238 | 8.1659 |
| 19 | 1.2662 | 21.2968 | .05946 | 8.6277 |
| 20 | 1.2820 | 22.5630 | .05682 | 9.0874 |
| 21 | 1.2981 | 23.8450 | .05444 | 9.5450 |
| 22 | 1.3143 | 25.1431 | .05227 | 10.0006 |
| 23 | 1.3307 | 26.4574 | .05030 | 10.4542 |
| 24 | 1.3474 | 27.7881 | .04849 | 10.9056 |
| 25 | 1.3642 | 29.1354 | .04682 | 11.3551 |
| 26 | 1.3812 | 30.4996 | .04529 | 11.8024 |
| 27 | 1.3985 | 31.8809 | .04387 | 12.2478 |
| 28 | 1.4160 | 33.2794 | .04255 | 12.6911 |
| 29 | 1.4337 | 34.6954 | .04132 | 13.1323 |
| 30 | 1.4516 | 36.1291 | .04018 | 13.5715 |
| 31 | 1.4698 | 37.5807 | .03911 | 14.0086 |
| 32 | 1.4881 | 39.0504 | .03811 | 14.4438 |
| 33 | 1.5067 | 40.5386 | .03717 | 14.8768 |
| 34 | 1.5256 | 42.0453 | .03628 | 15.3079 |
| 35 | 1.5446 | 43.5709 | .03545 | 15.7369 |
| 40 | 1.6436 | 51.4896 | .03192 | 17.8515 |
| 45 | 1.7489 | 59.9157 | .02919 | 19.9156 |
| 50 | 1.8610 | 68.8818 | .02702 | 21.9295 |
| 55 | 1.9803 | 78.4225 | .02525 | 23.8936 |
| 60 | 2.1072 | 88.5745 | .02379 | 25.8083 |
| 65 | 2.2422 | 99.3771 | .02256 | 27.6741 |
| 70 | 2.3859 | 110.8720 | .02152 | 29.4913 |
| 75 | 2.5388 | 123.1035 | .02062 | 31.2605 |
| 80 | 2.7015 | 136.1188 | .01985 | 32.9822 |
| 85 | 2.8746 | 149.9682 | .01917 | 34.6570 |
| 90 | 3.0588 | 164.7050 | .01857 | 36.2855 |
| 95 | 3.2548 | 180.3862 | .01804 | 37.8682 |
| 100 | 3.4634 | 197.0723 | .01757 | 39.4058 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 1.50 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0150 | 1.0000 | 1.01500 | .0000 |
| 2 | 1.0302 | 2.0150 | .51128 | .4963 |
| 3 | 1.0457 | 3.0452 | .34338 | .9901 |
| 4 | 1.0614 | 4.0909 | .25944 | 1.4814 |
| 5 | 1.0773 | 5.1523 | .20909 | 1.9702 |
| 6 | 1.0934 | 6.2296 | .17553 | 2.4566 |
| 7 | 1.1098 | 7.3230 | .15156 | 2.9405 |
| 8 | 1.1265 | 8.4328 | .13358 | 3.4219 |
| 9 | 1.1434 | 9.5593 | .11961 | 3.9008 |
| 10 | 1.1605 | 10.7027 | .10843 | 4.3772 |
| 11 | 1.1779 | 11.8633 | .09929 | 4.8512 |
| 12 | 1.1956 | 13.0412 | .09168 | 5.3227 |
| 13 | 1.2136 | 14.2368 | .08524 | 5.7917 |
| 14 | 1.2318 | 15.4504 | .07972 | 6.2582 |
| 15 | 1.2502 | 16.6821 | .07494 | 6.7223 |
| 16 | 1.2690 | 17.9324 | .07077 | 7.1839 |
| 17 | 1.2880 | 19.2014 | .06708 | 7.6431 |
| 18 | 1.3073 | 20.4894 | .06381 | 8.0997 |
| 19 | 1.3270 | 21.7967 | .06088 | 8.5539 |
| 20 | 1.3469 | 23.1237 | .05825 | 9.0057 |
| 21 | 1.3671 | 24.4705 | .05587 | 9.4550 |
| 22 | 1.3876 | 25.8376 | .05370 | 9.9018 |
| 23 | 1.4084 | 27.2251 | .05173 | 10.3462 |
| 24 | 1.4295 | 28.6335 | .04992 | 10.7881 |
| 25 | 1.4509 | 30.0630 | .04826 | 11.2276 |
| 26 | 1.4727 | 31.5140 | .04673 | 11.6646 |
| 27 | 1.4948 | 32.9867 | .04532 | 12.0992 |
| 28 | 1.5172 | 34.4815 | .04400 | 12.5313 |
| 29 | 1.5400 | 35.9987 | .04278 | 12.9610 |
| 30 | 1.5631 | 37.5387 | .04164 | 13.3883 |
| 31 | 1.5865 | 39.1018 | .04057 | 13.8131 |
| 32 | 1.6103 | 40.6883 | .03958 | 14.2355 |
| 33 | 1.6345 | 42.2986 | .03864 | 14.6555 |
| 34 | 1.6590 | 43.9331 | .03776 | 15.0731 |
| 35 | 1.6839 | 45.5921 | .03693 | 15.4882 |
| 40 | 1.8140 | 54.2679 | .03343 | 17.5277 |
| 45 | 1.9542 | 63.6142 | .03072 | 19.5074 |
| 50 | 2.1052 | 73.6828 | .02857 | 21.4277 |
| 55 | 2.2679 | 84.5296 | .02683 | 23.2894 |
| 60 | 2.4432 | 96.2147 | .02539 | 25.0930 |
| 65 | 2.6320 | 108.8028 | .02419 | 26.8393 |
| 70 | 2.8355 | 122.3638 | .02317 | 28.5290 |
| 75 | 3.0546 | 136.9728 | .02230 | 30.1631 |
| 80 | 3.2907 | 152.7109 | .02155 | 31.7423 |
| 85 | 3.5450 | 169.6652 | .02089 | 33.2676 |
| 90 | 3.8189 | 187.9299 | .02032 | 34.7399 |
| 95 | 4.1141 | 207.6061 | .01982 | 36.1602 |
| 100 | 4.4320 | 228.8030 | .01937 | 37.5295 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = *2.00* PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0200 | 1.0000 | 1.02000 | .0000 |
| 2 | 1.0404 | 2.0200 | .51505 | .4950 |
| 3 | 1.0612 | 3.0604 | .34675 | .9868 |
| 4 | 1.0824 | 4.1216 | .26262 | 1.4752 |
| 5 | 1.1041 | 5.2040 | .21216 | 1.9604 |
| 6 | 1.1262 | 6.3081 | .17853 | 2.4423 |
| 7 | 1.1487 | 7.4343 | .15451 | 2.9208 |
| 8 | 1.1717 | 8.5830 | .13651 | 3.3961 |
| 9 | 1.1951 | 9.7546 | .12252 | 3.8681 |
| 10 | 1.2190 | 10.9497 | .11133 | 4.3367 |
| 11 | 1.2434 | 12.1687 | .10218 | 4.8021 |
| 12 | 1.2682 | 13.4121 | .09456 | 5.2642 |
| 13 | 1.2936 | 14.6803 | .08812 | 5.7231 |
| 14 | 1.3195 | 15.9739 | .08260 | 6.1786 |
| 15 | 1.3459 | 17.2934 | .07783 | 6.6309 |
| 16 | 1.3728 | 18.6393 | .07365 | 7.0799 |
| 17 | 1.4002 | 20.0121 | .06997 | 7.5256 |
| 18 | 1.4282 | 21.4123 | .06670 | 7.9681 |
| 19 | 1.4568 | 22.8406 | .06378 | 8.4073 |
| 20 | 1.4859 | 24.2974 | .06116 | 8.8433 |
| 21 | 1.5157 | 25.7833 | .05878 | 9.2760 |
| 22 | 1.5460 | 27.2990 | .05663 | 9.7055 |
| 23 | 1.5769 | 28.8450 | .05467 | 10.1317 |
| 24 | 1.6084 | 30.4219 | .05287 | 10.5547 |
| 25 | 1.6406 | 32.0303 | .05122 | 10.9745 |
| 26 | 1.6734 | 33.6709 | .04970 | 11.3910 |
| 27 | 1.7069 | 35.3443 | .04829 | 11.8043 |
| 28 | 1.7410 | 37.0512 | .04699 | 12.2145 |
| 29 | 1.7758 | 38.7922 | .04578 | 12.6214 |
| 30 | 1.8114 | 40.5681 | .04465 | 13.0251 |
| 31 | 1.8476 | 42.3794 | .04360 | 13.4257 |
| 32 | 1.8845 | 44.2270 | .04261 | 13.8230 |
| 33 | 1.9222 | 46.1116 | .04169 | 14.2172 |
| 34 | 1.9607 | 48.0338 | .04082 | 14.6083 |
| 35 | 1.9999 | 49.9945 | .04000 | 14.9961 |
| 40 | 2.2080 | 60.4020 | .03656 | 16.8885 |
| 45 | 2.4379 | 71.8927 | .03391 | 18.7034 |
| 50 | 2.6916 | 84.5794 | .03182 | 20.4420 |
| 55 | 2.9717 | 98.5865 | .03014 | 22.1057 |
| 60 | 3.2810 | 114.0515 | .02877 | 23.6961 |
| 65 | 3.6225 | 131.1262 | .02763 | 25.2147 |
| 70 | 3.9996 | 149.9779 | .02667 | 26.6632 |
| 75 | 4.4158 | 170.7918 | .02586 | 28.0434 |
| 80 | 4.8754 | 193.7720 | .02516 | 29.3572 |
| 85 | 5.3829 | 219.1439 | .02456 | 30.6064 |
| 90 | 5.9431 | 247.1567 | .02405 | 31.7929 |
| 95 | 6.5617 | 278.0850 | .02360 | 32.9189 |
| 100 | 7.2446 | 312.2323 | .02320 | 33.9863 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 3.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0300 | 1.0000 | 1.03000 | .0000 |
| 2 | 1.0609 | 2.0300 | .52261 | .4926 |
| 3 | 1.0927 | 3.0909 | .35353 | .9803 |
| 4 | 1.1255 | 4.1836 | .26903 | 1.4631 |
| 5 | 1.1593 | 5.3091 | .21835 | 1.9409 |
| 6 | 1.1941 | 6.4684 | .18460 | 2.4138 |
| 7 | 1.2299 | 7.6625 | .16051 | 2.8819 |
| 8 | 1.2668 | 8.8923 | .14246 | 3.3450 |
| 9 | 1.3048 | 10.1591 | .12843 | 3.8032 |
| 10 | 1.3439 | 11.4639 | .11723 | 4.2565 |
| 11 | 1.3842 | 12.8078 | .10808 | 4.7049 |
| 12 | 1.4258 | 14.1920 | .10046 | 5.1485 |
| 13 | 1.4685 | 15.6178 | .09403 | 5.5872 |
| 14 | 1.5126 | 17.0863 | .08853 | 6.0210 |
| 15 | 1.5580 | 18.5989 | .08377 | 6.4500 |
| 16 | 1.6047 | 20.1569 | .07961 | 6.8742 |
| 17 | 1.6528 | 21.7616 | .07595 | 7.2936 |
| 18 | 1.7024 | 23.4144 | .07271 | 7.7081 |
| 19 | 1.7535 | 25.1169 | .06981 | 8.1179 |
| 20 | 1.8061 | 26.8704 | .06722 | 8.5229 |
| 21 | 1.8603 | 28.6765 | .06487 | 8.9231 |
| 22 | 1.9161 | 30.5368 | .06275 | 9.3186 |
| 23 | 1.9736 | 32.4529 | .06081 | 9.7093 |
| 24 | 2.0328 | 34.4265 | .05905 | 10.0954 |
| 25 | 2.0938 | 36.4593 | .05743 | 10.4768 |
| 26 | 2.1566 | 38.5530 | .05594 | 10.8535 |
| 27 | 2.2213 | 40.7096 | .05456 | 11.2255 |
| 28 | 2.2879 | 42.9309 | .05329 | 11.5930 |
| 29 | 2.3566 | 45.2189 | .05211 | 11.9558 |
| 30 | 2.4273 | 47.5754 | .05102 | 12.3141 |
| 31 | 2.5001 | 50.0027 | .05000 | 12.6678 |
| 32 | 2.5751 | 52.5028 | .04905 | 13.0169 |
| 33 | 2.6523 | 55.0778 | .04816 | 13.3616 |
| 34 | 2.7319 | 57.7302 | .04732 | 13.7018 |
| 35 | 2.8139 | 60.4621 | .04654 | 14.0375 |
| 40 | 3.2620 | 75.4013 | .04326 | 15.6502 |
| 45 | 3.7816 | 92.7199 | .04079 | 17.1556 |
| 50 | 4.3839 | 112.7969 | .03887 | 18.5575 |
| 55 | 5.0821 | 136.0716 | .03735 | 19.8600 |
| 60 | 5.8916 | 163.0534 | .03613 | 21.0674 |
| 65 | 6.8300 | 194.3328 | .03515 | 22.1841 |
| 70 | 7.9178 | 230.5941 | .03434 | 23.2145 |
| 75 | 9.1789 | 272.6309 | .03367 | 24.1634 |
| 80 | 10.6409 | 321.3630 | .03311 | 25.0353 |
| 85 | 12.3357 | 377.8570 | .03265 | 25.8349 |
| 90 | 14.3005 | 443.3489 | .03226 | 26.5667 |
| 95 | 16.5782 | 519.2720 | .03193 | 27.2351 |
| 100 | 19.2186 | 607.2877 | .03165 | 27.8444 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 4.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0400 | 1.0000 | 1.04000 | .0000 |
| 2 | 1.0816 | 2.0400 | .53020 | .4902 |
| 3 | 1.1249 | 3.1216 | .36035 | .9739 |
| 4 | 1.1699 | 4.2465 | .27549 | 1.4510 |
| 5 | 1.2167 | 5.4163 | .22463 | 1.9216 |
| 6 | 1.2653 | 6.6330 | .19076 | 2.3857 |
| 7 | 1.3159 | 7.8983 | .16661 | 2.8433 |
| 8 | 1.3686 | 9.2142 | .14853 | 3.2944 |
| 9 | 1.4233 | 10.5828 | .13449 | 3.7391 |
| 10 | 1.4802 | 12.0061 | .12329 | 4.1773 |
| 11 | 1.5395 | 13.4864 | .11415 | 4.6090 |
| 12 | 1.6010 | 15.0258 | .10655 | 5.0343 |
| 13 | 1.6651 | 16.6268 | .10014 | 5.4533 |
| 14 | 1.7317 | 18.2919 | .09467 | 5.8659 |
| 15 | 1.8009 | 20.0236 | .08994 | 6.2721 |
| 16 | 1.8730 | 21.8245 | .08582 | 6.6720 |
| 17 | 1.9479 | 23.6975 | .08220 | 7.0656 |
| 18 | 2.0258 | 25.6454 | .07899 | 7.4530 |
| 19 | 2.1068 | 27.6712 | .07614 | 7.8342 |
| 20 | 2.1911 | 29.7781 | .07358 | 8.2091 |
| 21 | 2.2788 | 31.9692 | .07128 | 8.5779 |
| 22 | 2.3699 | 34.2480 | .06920 | 8.9407 |
| 23 | 2.4647 | 36.6179 | .06731 | 9.2973 |
| 24 | 2.5633 | 39.0826 | .06559 | 9.6479 |
| 25 | 2.6658 | 41.6459 | .06401 | 9.9925 |
| 26 | 2.7725 | 44.3117 | .06257 | 10.3312 |
| 27 | 2.8834 | 47.0842 | .06124 | 10.6640 |
| 28 | 2.9987 | 49.9676 | .06001 | 10.9909 |
| 29 | 3.1187 | 52.9663 | .05888 | 11.3120 |
| 30 | 3.2434 | 56.0849 | .05783 | 11.6274 |
| 31 | 3.3731 | 59.3283 | .05686 | 11.9371 |
| 32 | 3.5081 | 62.7015 | .05595 | 12.2411 |
| 33 | 3.6484 | 66.2095 | .05510 | 12.5396 |
| 34 | 3.7943 | 69.8579 | .05431 | 12.8324 |
| 35 | 3.9461 | 73.6522 | .05358 | 13.1198 |
| 40 | 4.8010 | 95.0255 | .05052 | 14.4765 |
| 45 | 5.8412 | 121.0294 | .04826 | 15.7047 |
| 50 | 7.1067 | 152.6671 | .04655 | 16.8122 |
| 55 | 8.6464 | 191.1592 | .04523 | 17.8070 |
| 60 | 10.5196 | 237.9907 | .04420 | 18.6972 |
| 65 | 12.7987 | 294.9684 | .04339 | 19.4909 |
| 70 | 15.5716 | 364.2905 | .04275 | 20.1961 |
| 75 | 18.9453 | 448.6314 | .04223 | 20.8206 |
| 80 | 23.0498 | 551.2450 | .04181 | 21.3718 |
| 85 | 28.0436 | 676.0901 | .04148 | 21.8569 |
| 90 | 34.1193 | 827.9833 | .04121 | 22.2826 |
| 95 | 41.5114 | 1012.7846 | .04099 | 22.6550 |
| 100 | 50.5049 | 1237.6237 | .04081 | 22.9800 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 5.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0500 | 1.0000 | 1.05000 | .0000 |
| 2 | 1.1025 | 2.0500 | .53780 | .4878 |
| 3 | 1.1576 | 3.1525 | .36721 | .9675 |
| 4 | 1.2155 | 4.3101 | .28201 | 1.4391 |
| 5 | 1.2763 | 5.5256 | .23097 | 1.9025 |
| 6 | 1.3401 | 6.8019 | .19702 | 2.3579 |
| 7 | 1.4071 | 8.1420 | .17282 | 2.8052 |
| 8 | 1.4775 | 9.5491 | .15472 | 3.2445 |
| 9 | 1.5513 | 11.0266 | .14069 | 3.6758 |
| 10 | 1.6289 | 12.5779 | .12950 | 4.0991 |
| 11 | 1.7103 | 14.2068 | .12039 | 4.5144 |
| 12 | 1.7959 | 15.9171 | .11283 | 4.9219 |
| 13 | 1.8856 | 17.7130 | .10646 | 5.3215 |
| 14 | 1.9799 | 19.5986 | .10102 | 5.7133 |
| 15 | 2.0789 | 21.5786 | .09634 | 6.0973 |
| 16 | 2.1829 | 23.6575 | .09227 | 6.4736 |
| 17 | 2.2920 | 25.8404 | .08870 | 6.8423 |
| 18 | 2.4066 | 28.1324 | .08555 | 7.2034 |
| 19 | 2.5270 | 30.5390 | .08275 | 7.5569 |
| 20 | 2.6533 | 33.0660 | .08024 | 7.9030 |
| 21 | 2.7860 | 35.7193 | .07800 | 8.2416 |
| 22 | 2.9253 | 38.5052 | .07597 | 8.5730 |
| 23 | 3.0715 | 41.4305 | .07414 | 8.8971 |
| 24 | 3.2251 | 44.5020 | .07247 | 9.2140 |
| 25 | 3.3864 | 47.7271 | .07095 | 9.5238 |
| 26 | 3.5557 | 51.1135 | .06956 | 9.8266 |
| 27 | 3.7335 | 54.6691 | .06829 | 10.1224 |
| 28 | 3.9201 | 58.4026 | .06712 | 10.4114 |
| 29 | 4.1161 | 62.3227 | .06605 | 10.6936 |
| 30 | 4.3219 | 66.4388 | .06505 | 10.9691 |
| 31 | 4.5380 | 70.7608 | .06413 | 11.2381 |
| 32 | 4.7649 | 75.2988 | .06328 | 11.5005 |
| 33 | 5.0032 | 80.0638 | .06249 | 11.7566 |
| 34 | 5.2533 | 85.0670 | .06176 | 12.0063 |
| 35 | 5.5160 | 90.3203 | .06107 | 12.2498 |
| 40 | 7.0400 | 120.7998 | .05828 | 13.3775 |
| 45 | 8.9850 | 159.7002 | .05626 | 14.3644 |
| 50 | 11.4674 | 209.3480 | .05478 | 15.2233 |
| 55 | 14.6356 | 272.7126 | .05367 | 15.9664 |
| 60 | 18.6792 | 353.5837 | .05283 | 16.6062 |
| 65 | 23.8399 | 456.7980 | .05219 | 17.1541 |
| 70 | 30.4264 | 588.5285 | .05170 | 17.6212 |
| 75 | 38.8327 | 756.6537 | .05132 | 18.0176 |
| 80 | 49.5614 | 971.2288 | .05103 | 18.3526 |
| 85 | 63.2544 | 1245.0871 | .05080 | 18.6346 |
| 90 | 80.7304 | 1594.6073 | .05063 | 18.8712 |
| 95 | 103.0347 | 2040.6935 | .05049 | 19.0689 |
| 100 | 131.5013 | 2610.0252 | .05038 | 19.2337 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 6.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0600 | 1.0000 | 1.06000 | .0000 |
| 2 | 1.1236 | 2.0600 | .54544 | .4854 |
| 3 | 1.1910 | 3.1836 | .37411 | .9612 |
| 4 | 1.2625 | 4.3746 | .28859 | 1.4272 |
| 5 | 1.3382 | 5.6371 | .23740 | 1.8836 |
| 6 | 1.4185 | 6.9753 | .20336 | 2.3304 |
| 7 | 1.5036 | 8.3938 | .17914 | 2.7676 |
| 8 | 1.5938 | 9.8975 | .16104 | 3.1952 |
| 9 | 1.6895 | 11.4913 | .14702 | 3.6133 |
| 10 | 1.7908 | 13.1808 | .13587 | 4.0220 |
| 11 | 1.8983 | 14.9716 | .12679 | 4.4213 |
| 12 | 2.0122 | 16.8699 | .11928 | 4.8113 |
| 13 | 2.1329 | 18.8821 | .11296 | 5.1920 |
| 14 | 2.2609 | 21.0151 | .10758 | 5.5635 |
| 15 | 2.3966 | 23.2760 | .10296 | 5.9260 |
| 16 | 2.5404 | 25.6725 | .09895 | 6.2794 |
| 17 | 2.6928 | 28.2129 | .09544 | 6.6240 |
| 18 | 2.8543 | 30.9057 | .09236 | 6.9597 |
| 19 | 3.0256 | 33.7600 | .08962 | 7.2867 |
| 20 | 3.2071 | 36.7856 | .08718 | 7.6051 |
| 21 | 3.3996 | 39.9927 | .08500 | 7.9151 |
| 22 | 3.6035 | 43.3923 | .08305 | 8.2166 |
| 23 | 3.8197 | 46.9958 | .08128 | 8.5099 |
| 24 | 4.0489 | 50.8156 | .07968 | 8.7951 |
| 25 | 4.2919 | 54.8645 | .07823 | 9.0722 |
| 26 | 4.5494 | 59.1564 | .07690 | 9.3414 |
| 27 | 4.8223 | 63.7058 | .07570 | 9.6029 |
| 28 | 5.1117 | 68.5281 | .07459 | 9.8568 |
| 29 | 5.4184 | 73.6398 | .07358 | 10.1032 |
| 30 | 5.7435 | 79.0582 | .07265 | 10.3422 |
| 31 | 6.0881 | 84.8017 | .07179 | 10.5740 |
| 32 | 6.4534 | 90.8898 | .07100 | 10.7988 |
| 33 | 6.8406 | 97.3432 | .07027 | 11.0166 |
| 34 | 7.2510 | 104.1838 | .06960 | 11.2276 |
| 35 | 7.6861 | 111.4348 | .06897 | 11.4319 |
| 40 | 10.2857 | 154.7620 | .06646 | 12.3590 |
| 45 | 13.7646 | 212.7435 | .06470 | 13.1413 |
| 50 | 18.4202 | 290.3359 | .06344 | 13.7964 |
| 55 | 24.6503 | 394.1720 | .06254 | 14.3411 |
| 60 | 32.9877 | 533.1282 | .06188 | 14.7909 |
| 65 | 44.1450 | 719.0829 | .06139 | 15.1601 |
| 70 | 59.0759 | 967.9322 | .06103 | 15.4613 |
| 75 | 79.0569 | 1300.9487 | .06077 | 15.7058 |
| 80 | 105.7960 | 1746.5999 | .06057 | 15.9033 |
| 85 | 141.5789 | 2342.9817 | .06043 | 16.0620 |
| 90 | 189.4645 | 3141.0752 | .06032 | 16.1891 |
| 95 | 253.5463 | 4209.1042 | .06024 | 16.2905 |
| 100 | 339.3021 | 5638.3681 | .06018 | 16.3711 |

COMPOUND INTEREST FACTORS – ANNUAL COMPOUNDING

INTEREST RATE = 7.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0700 | 1.0000 | 1.07000 | .0000 |
| 2 | 1.1449 | 2.0700 | .55309 | .4831 |
| 3 | 1.2250 | 3.2149 | .38105 | .9549 |
| 4 | 1.3108 | 4.4399 | .29523 | 1.4155 |
| 5 | 1.4026 | 5.7507 | .24389 | 1.8650 |
| 6 | 1.5007 | 7.1533 | .20980 | 2.3032 |
| 7 | 1.6058 | 8.6540 | .18555 | 2.7304 |
| 8 | 1.7182 | 10.2598 | .16747 | 3.1465 |
| 9 | 1.8385 | 11.9780 | .15349 | 3.5517 |
| 10 | 1.9672 | 13.8164 | .14238 | 3.9461 |
| 11 | 2.1049 | 15.7836 | .13336 | 4.3296 |
| 12 | 2.2522 | 17.8885 | .12590 | 4.7025 |
| 13 | 2.4098 | 20.1406 | .11965 | 5.0648 |
| 14 | 2.5785 | 22.5505 | .11434 | 5.4167 |
| 15 | 2.7590 | 25.1290 | .10979 | 5.7583 |
| 16 | 2.9522 | 27.8881 | .10586 | 6.0897 |
| 17 | 3.1588 | 30.8402 | .10243 | 6.4110 |
| 18 | 3.3799 | 33.9990 | .09941 | 6.7225 |
| 19 | 3.6165 | 37.3790 | .09675 | 7.0242 |
| 20 | 3.8697 | 40.9955 | .09439 | 7.3163 |
| 21 | 4.1406 | 44.8652 | .09229 | 7.5990 |
| 22 | 4.4304 | 49.0057 | .09041 | 7.8725 |
| 23 | 4.7405 | 53.4361 | .08871 | 8.1369 |
| 24 | 5.0724 | 58.1767 | .08719 | 8.3923 |
| 25 | 5.4274 | 63.2490 | .08581 | 8.6391 |
| 26 | 5.8074 | 68.6765 | .08456 | 8.8773 |
| 27 | 6.2139 | 74.4838 | .08343 | 9.1072 |
| 28 | 6.6488 | 80.6977 | .08239 | 9.3289 |
| 29 | 7.1143 | 87.3465 | .08145 | 9.5427 |
| 30 | 7.6123 | 94.4608 | .08059 | 9.7487 |
| 31 | 8.1451 | 102.0730 | .07980 | 9.9471 |
| 32 | 8.7153 | 110.2182 | .07907 | 10.1381 |
| 33 | 9.3253 | 118.9334 | .07841 | 10.3219 |
| 34 | 9.9781 | 128.2588 | .07780 | 10.4987 |
| 35 | 10.6766 | 138.2369 | .07723 | 10.6687 |
| 40 | 14.9745 | 199.6351 | .07501 | 11.4233 |
| 45 | 21.0025 | 285.7493 | .07350 | 12.0360 |
| 50 | 29.4570 | 406.5289 | .07246 | 12.5287 |
| 55 | 41.3150 | 575.9286 | .07174 | 12.9215 |
| 60 | 57.9464 | 813.5204 | .07123 | 13.2321 |
| 65 | 81.2729 | 1146.7552 | .07087 | 13.4760 |
| 70 | 113.9894 | 1614.1342 | .07062 | 13.6662 |
| 75 | 159.8760 | 2269.6574 | .07044 | 13.8136 |
| 80 | 224.2344 | 3189.0627 | .07031 | 13.9273 |
| 85 | 314.5003 | 4478.5761 | .07022 | 14.0146 |
| 90 | 441.1030 | 6287.1854 | .07016 | 14.0812 |
| 95 | 618.6697 | 8823.8535 | .07011 | 14.1319 |
| 100 | 867.7163 | 12381.6618 | .07008 | 14.1703 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 8.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0800 | 1.0000 | 1.08000 | .0000 |
| 2 | 1.1664 | 2.0800 | .56077 | .4808 |
| 3 | 1.2597 | 3.2464 | .38803 | .9487 |
| 4 | 1.3605 | 4.5061 | .30192 | 1.4040 |
| 5 | 1.4693 | 5.8666 | .25046 | 1.8465 |
| 6 | 1.5869 | 7.3359 | .21632 | 2.2763 |
| 7 | 1.7138 | 8.9228 | .19207 | 2.6937 |
| 8 | 1.8509 | 10.6366 | .17401 | 3.0985 |
| 9 | 1.9990 | 12.4876 | .16008 | 3.4910 |
| 10 | 2.1589 | 14.4866 | .14903 | 3.8713 |
| 11 | 2.3316 | 16.6455 | .14008 | 4.2395 |
| 12 | 2.5182 | 18.9771 | .13270 | 4.5957 |
| 13 | 2.7196 | 21.4953 | .12652 | 4.9402 |
| 14 | 2.9372 | 24.2149 | .12130 | 5.2731 |
| 15 | 3.1722 | 27.1521 | .11683 | 5.5945 |
| 16 | 3.4259 | 30.3243 | .11298 | 5.9046 |
| 17 | 3.7000 | 33.7502 | .10963 | 6.2037 |
| 18 | 3.9960 | 37.4502 | .10670 | 6.4920 |
| 19 | 4.3157 | 41.4463 | .10413 | 6.7697 |
| 20 | 4.6610 | 45.7620 | .10185 | 7.0369 |
| 21 | 5.0338 | 50.4229 | .09983 | 7.2940 |
| 22 | 5.4365 | 55.4568 | .09803 | 7.5412 |
| 23 | 5.8715 | 60.8933 | .09642 | 7.7786 |
| 24 | 6.3412 | 66.7648 | .09498 | 8.0066 |
| 25 | 6.8485 | 73.1059 | .09368 | 8.2254 |
| 26 | 7.3964 | 79.9544 | .09251 | 8.4352 |
| 27 | 7.9881 | 87.3508 | .09145 | 8.6363 |
| 28 | 8.6271 | 95.3388 | .09049 | 8.8289 |
| 29 | 9.3173 | 103.9659 | .08962 | 9.0133 |
| 30 | 10.0627 | 113.2832 | .08883 | 9.1897 |
| 31 | 10.8677 | 123.3459 | .08811 | 9.3584 |
| 32 | 11.7371 | 134.2135 | .08745 | 9.5197 |
| 33 | 12.6760 | 145.9506 | .08685 | 9.6737 |
| 34 | 13.6901 | 158.6267 | .08630 | 9.8208 |
| 35 | 14.7853 | 172.3168 | .08580 | 9.9611 |
| 40 | 21.7245 | 259.0565 | .08386 | 10.5699 |
| 45 | 31.9204 | 386.5056 | .08259 | 11.0447 |
| 50 | 46.9016 | 573.7702 | .08174 | 11.4107 |
| 55 | 68.9139 | 848.9232 | .08118 | 11.6902 |
| 60 | 101.2571 | 1253.2133 | .08080 | 11.9015 |
| 65 | 148.7798 | 1847.2481 | .08054 | 12.0602 |
| 70 | 218.6064 | 2720.0801 | .08037 | 12.1783 |
| 75 | 321.2045 | 4002.5566 | .08025 | 12.2658 |
| 80 | 471.9548 | 5886.9354 | .08017 | 12.3301 |
| 85 | 693.4565 | 8655.7061 | .08012 | 12.3772 |
| 90 | 1018.9151 | 12723.9386 | .08008 | 12.4116 |
| 95 | 1497.1205 | 18701.5069 | .08005 | 12.4365 |
| 100 | 2199.7613 | 27484.5157 | .08004 | 12.4545 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 9.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0900 | 1.0000 | 1.09000 | .0000 |
| 2 | 1.1881 | 2.0900 | .56847 | .4785 |
| 3 | 1.2950 | 3.2781 | .39505 | .9426 |
| 4 | 1.4116 | 4.5731 | .30867 | 1.3925 |
| 5 | 1.5386 | 5.9847 | .25709 | 1.8282 |
| 6 | 1.6771 | 7.5233 | .22292 | 2.2498 |
| 7 | 1.8280 | 9.2004 | .19869 | 2.6574 |
| 8 | 1.9926 | 11.0285 | .18067 | 3.0512 |
| 9 | 2.1719 | 13.0210 | .16680 | 3.4312 |
| 10 | 2.3674 | 15.1929 | .15582 | 3.7978 |
| 11 | 2.5804 | 17.5603 | .14695 | 4.1510 |
| 12 | 2.8127 | 20.1407 | .13965 | 4.4910 |
| 13 | 3.0658 | 22.9534 | .13357 | 4.8182 |
| 14 | 3.3417 | 26.0192 | .12843 | 5.1326 |
| 15 | 3.6425 | 29.3609 | .12406 | 5.4346 |
| 16 | 3.9703 | 33.0034 | .12030 | 5.7245 |
| 17 | 4.3276 | 36.9737 | .11705 | 6.0024 |
| 18 | 4.7171 | 41.3013 | .11421 | 6.2687 |
| 19 | 5.1417 | 46.0185 | .11173 | 6.5236 |
| 20 | 5.6044 | 51.1601 | .10955 | 6.7674 |
| 21 | 6.1088 | 56.7645 | .10762 | 7.0006 |
| 22 | 6.6586 | 62.8733 | .10590 | 7.2232 |
| 23 | 7.2579 | 69.5319 | .10438 | 7.4357 |
| 24 | 7.9111 | 76.7898 | .10302 | 7.6384 |
| 25 | 8.6231 | 84.7009 | .10181 | 7.8316 |
| 26 | 9.3992 | 93.3240 | .10072 | 8.0156 |
| 27 | 10.2451 | 102.7231 | .09973 | 8.1906 |
| 28 | 11.1671 | 112.9682 | .09885 | 8.3571 |
| 29 | 12.1722 | 124.1354 | .09806 | 8.5154 |
| 30 | 13.2677 | 136.3075 | .09734 | 8.6657 |
| 31 | 14.4618 | 149.5752 | .09669 | 8.8083 |
| 32 | 15.7633 | 164.0370 | .09610 | 8.9436 |
| 33 | 17.1820 | 179.8003 | .09556 | 9.0718 |
| 34 | 18.7284 | 196.9823 | .09508 | 9.1933 |
| 35 | 20.4140 | 215.7108 | .09464 | 9.3083 |
| 40 | 31.4094 | 337.8824 | .09296 | 9.7957 |
| 45 | 48.3273 | 525.8587 | .09190 | 10.1603 |
| 50 | 74.3575 | 815.0836 | .09123 | 10.4295 |
| 55 | 114.4083 | 1260.0918 | .09079 | 10.6261 |
| 60 | 176.0313 | 1944.7921 | .09051 | 10.7683 |
| 65 | 270.8460 | 2998.2885 | .09033 | 10.8702 |
| 70 | 416.7301 | 4619.2232 | .09022 | 10.9427 |
| 75 | 641.1909 | 7113.2321 | .09014 | 10.9940 |
| 80 | 986.5517 | 10950.5741 | .09009 | 11.0299 |
| 85 | 1517.9320 | 16854.8003 | .09006 | 11.0551 |
| 90 | 2335.5266 | 25939.1842 | .09004 | 11.0726 |
| 95 | 3593.4971 | 39916.6350 | .09003 | 11.0847 |
| 100 | 5529.0408 | 61422.6755 | .09002 | 11.0930 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 10.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.1000 | 1.0000 | 1.10000 | .0000 |
| 2 | 1.2100 | 2.1000 | .57619 | .4762 |
| 3 | 1.3310 | 3.3100 | .40211 | .9366 |
| 4 | 1.4641 | 4.6410 | .31547 | 1.3812 |
| 5 | 1.6105 | 6.1051 | .26380 | 1.8101 |
| 6 | 1.7716 | 7.7156 | .22961 | 2.2236 |
| 7 | 1.9487 | 9.4872 | .20541 | 2.6216 |
| 8 | 2.1436 | 11.4359 | .18744 | 3.0045 |
| 9 | 2.3579 | 13.5795 | .17364 | 3.3724 |
| 10 | 2.5937 | 15.9374 | .16275 | 3.7255 |
| 11 | 2.8531 | 18.5312 | .15396 | 4.0641 |
| 12 | 3.1384 | 21.3843 | .14676 | 4.3884 |
| 13 | 3.4523 | 24.5227 | .14078 | 4.6988 |
| 14 | 3.7975 | 27.9750 | .13575 | 4.9955 |
| 15 | 4.1772 | 31.7725 | .13147 | 5.2789 |
| 16 | 4.5950 | 35.9497 | .12782 | 5.5493 |
| 17 | 5.0545 | 40.5447 | .12466 | 5.8071 |
| 18 | 5.5599 | 45.5992 | .12193 | 6.0526 |
| 19 | 6.1159 | 51.1591 | .11955 | 6.2861 |
| 20 | 6.7275 | 57.2750 | .11746 | 6.5081 |
| 21 | 7.4002 | 64.0025 | .11562 | 6.7189 |
| 22 | 8.1403 | 71.4027 | .11401 | 6.9189 |
| 23 | 8.9543 | 79.5430 | .11257 | 7.1085 |
| 24 | 9.8497 | 88.4973 | .11130 | 7.2881 |
| 25 | 10.8347 | 98.3471 | .11017 | 7.4580 |
| 26 | 11.9182 | 109.1818 | .10916 | 7.6186 |
| 27 | 13.1100 | 121.0999 | .10826 | 7.7704 |
| 28 | 14.4210 | 134.2099 | .10745 | 7.9137 |
| 29 | 15.8631 | 148.6309 | .10673 | 8.0489 |
| 30 | 17.4494 | 164.4940 | .10608 | 8.1762 |
| 31 | 19.1943 | 181.9434 | .10550 | 8.2962 |
| 32 | 21.1138 | 201.1378 | .10497 | 8.4091 |
| 33 | 23.2252 | 222.2515 | .10450 | 8.5152 |
| 34 | 25.5477 | 245.4767 | .10407 | 8.6149 |
| 35 | 28.1024 | 271.0244 | .10369 | 8.7086 |
| 40 | 45.2593 | 442.5926 | .10226 | 9.0962 |
| 45 | 72.8905 | 718.9048 | .10139 | 9.3740 |
| 50 | 117.3909 | 1163.9085 | .10086 | 9.5704 |
| 55 | 189.0591 | 1880.5914 | .10053 | 9.7075 |
| 60 | 304.4816 | 3034.8164 | .10033 | 9.8023 |
| 65 | 490.3707 | 4893.7073 | .10020 | 9.8672 |
| 70 | 789.7470 | 7887.4696 | .10013 | 9.9113 |
| 75 | 1271.8954 | 12708.9537 | .10008 | 9.9410 |
| 80 | 2048.4002 | 20474.0021 | .10005 | 9.9609 |
| 85 | 3298.9690 | 32979.6903 | .10003 | 9.9742 |
| 90 | 5313.0226 | 53120.2261 | .10002 | 9.9831 |
| 95 | 8556.6760 | 85556.7605 | .10001 | 9.9889 |
| 100 | 13780.6123 | 137796.1234 | .10001 | 9.9927 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 12.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.1200 | 1.0000 | 1.12000 | .0000 |
| 2 | 1.2544 | 2.1200 | .59170 | .4717 |
| 3 | 1.4049 | 3.3744 | .41635 | .9246 |
| 4 | 1.5735 | 4.7793 | .32923 | 1.3589 |
| 5 | 1.7623 | 6.3528 | .27741 | 1.7746 |
| 6 | 1.9738 | 8.1152 | .24323 | 2.1720 |
| 7 | 2.2107 | 10.0890 | .21912 | 2.5515 |
| 8 | 2.4760 | 12.2997 | .20130 | 2.9131 |
| 9 | 2.7731 | 14.7757 | .18768 | 3.2574 |
| 10 | 3.1058 | 17.5487 | .17698 | 3.5847 |
| 11 | 3.4785 | 20.6546 | .16842 | 3.8953 |
| 12 | 3.8960 | 24.1331 | .16144 | 4.1897 |
| 13 | 4.3635 | 28.0291 | .15568 | 4.4683 |
| 14 | 4.8871 | 32.3926 | .15087 | 4.7317 |
| 15 | 5.4736 | 37.2797 | .14682 | 4.9803 |
| 16 | 6.1304 | 42.7533 | .14339 | 5.2147 |
| 17 | 6.8660 | 48.8837 | .14046 | 5.4353 |
| 18 | 7.6900 | 55.7497 | .13794 | 5.6427 |
| 19 | 8.6128 | 63.4397 | .13576 | 5.8375 |
| 20 | 9.6463 | 72.0524 | .13388 | 6.0202 |
| 21 | 10.8038 | 81.6987 | .13224 | 6.1913 |
| 22 | 12.1003 | 92.5026 | .13081 | 6.3514 |
| 23 | 13.5523 | 104.6029 | .12956 | 6.5010 |
| 24 | 15.1786 | 118.1552 | .12846 | 6.6406 |
| 25 | 17.0001 | 133.3339 | .12750 | 6.7708 |
| 26 | 19.0401 | 150.3339 | .12665 | 6.8921 |
| 27 | 21.3249 | 169.3740 | .12590 | 7.0049 |
| 28 | 23.8839 | 190.6989 | .12524 | 7.1098 |
| 29 | 26.7499 | 214.5828 | .12466 | 7.2071 |
| 30 | 29.9599 | 241.3327 | .12414 | 7.2974 |
| 31 | 33.5551 | 271.2926 | .12369 | 7.3811 |
| 32 | 37.5817 | 304.8477 | .12328 | 7.4586 |
| 33 | 42.0915 | 342.4294 | .12292 | 7.5302 |
| 34 | 47.1425 | 384.5210 | .12260 | 7.5965 |
| 35 | 52.7996 | 431.6635 | .12232 | 7.6577 |
| 40 | 93.0510 | 767.0914 | .12130 | 7.8988 |
| 45 | 163.9876 | 1358.2300 | .12074 | 8.0572 |
| 50 | 289.0022 | 2400.0182 | .12042 | 8.1597 |

COMPOUND INTEREST FACTORS – ANNUAL COMPOUNDING

INTEREST RATE = 15.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.1500 | 1.0000 | 1.15000 | .0000 |
| 2 | 1.3225 | 2.1500 | .61512 | .4651 |
| 3 | 1.5209 | 3.4725 | .43798 | .9071 |
| 4 | 1.7490 | 4.9934 | .35027 | 1.3263 |
| 5 | 2.0114 | 6.7424 | .29832 | 1.7228 |
| 6 | 2.3131 | 8.7537 | .26424 | 2.0972 |
| 7 | 2.6600 | 11.0668 | .24036 | 2.4498 |
| 8 | 3.0590 | 13.7268 | .22285 | 2.7813 |
| 9 | 3.5179 | 16.7858 | .20957 | 3.0922 |
| 10 | 4.0456 | 20.3037 | .19925 | 3.3832 |
| 11 | 4.6524 | 24.3493 | .19107 | 3.6549 |
| 12 | 5.3503 | 29.0017 | .18448 | 3.9082 |
| 13 | 6.1528 | 34.3519 | .17911 | 4.1438 |
| 14 | 7.0757 | 40.5047 | .17469 | 4.3624 |
| 15 | 8.1371 | 47.5804 | .17102 | 4.5650 |
| 16 | 9.3576 | 55.7175 | .16795 | 4.7522 |
| 17 | 10.7613 | 65.0751 | .16537 | 4.9251 |
| 18 | 12.3755 | 75.8364 | .16319 | 5.0843 |
| 19 | 14.2318 | 88.2118 | .16134 | 5.2307 |
| 20 | 16.3665 | 102.4436 | .15976 | 5.3651 |
| 21 | 18.8215 | 118.8101 | .15842 | 5.4883 |
| 22 | 21.6447 | 137.6316 | .15727 | 5.6010 |
| 23 | 24.8915 | 159.2764 | .15628 | 5.7040 |
| 24 | 28.6252 | 184.1678 | .15543 | 5.7979 |
| 25 | 32.9190 | 212.7930 | .15470 | 5.8834 |
| 26 | 37.8568 | 245.7120 | .15407 | 5.9612 |
| 27 | 43.5353 | 283.5688 | .15353 | 6.0319 |
| 28 | 50.0656 | 327.1041 | .15306 | 6.0960 |
| 29 | 57.5755 | 377.1697 | .15265 | 6.1541 |
| 30 | 66.2118 | 434.7451 | .15230 | 6.2066 |
| 31 | 76.1435 | 500.9569 | .15200 | 6.2541 |
| 32 | 87.5651 | 577.1005 | .15173 | 6.2970 |
| 33 | 100.6998 | 664.6655 | .15150 | 6.3357 |
| 34 | 115.8048 | 765.3654 | .15131 | 6.3705 |
| 35 | 133.1755 | 881.1702 | .15113 | 6.4019 |
| 40 | 267.8635 | 1779.0903 | .15056 | 6.5168 |
| 45 | 538.7693 | 3585.1285 | .15028 | 6.5830 |
| 50 | 1083.6574 | 7217.7163 | .15014 | 6.6205 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 20.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.2000 | 1.0000 | 1.20000 | .0000 |
| 2 | 1.4400 | 2.2000 | .65455 | .4545 |
| 3 | 1.7280 | 3.6400 | .47473 | .8791 |
| 4 | 2.0736 | 5.3680 | .38629 | 1.2742 |
| 5 | 2.4883 | 7.4416 | .33438 | 1.6405 |
| 6 | 2.9860 | 9.9299 | .30071 | 1.9788 |
| 7 | 3.5832 | 12.9159 | .27742 | 2.2902 |
| 8 | 4.2998 | 16.4991 | .26061 | 2.5756 |
| 9 | 5.1598 | 20.7989 | .24808 | 2.8364 |
| 10 | 6.1917 | 25.9587 | .23852 | 3.0739 |
| 11 | 7.4301 | 32.1504 | .23110 | 3.2893 |
| 12 | 8.9161 | 39.5805 | .22526 | 3.4841 |
| 13 | 10.6993 | 48.4966 | .22062 | 3.6597 |
| 14 | 12.8392 | 59.1959 | .21689 | 3.8175 |
| 15 | 15.4070 | 72.0351 | .21388 | 3.9588 |
| 16 | 18.4884 | 87.4421 | .21144 | 4.0851 |
| 17 | 22.1861 | 105.9306 | .20944 | 4.1976 |
| 18 | 26.6233 | 128.1167 | .20781 | 4.2975 |
| 19 | 31.9480 | 154.7400 | .20646 | 4.3861 |
| 20 | 38.3376 | 186.6880 | .20536 | 4.4643 |
| 21 | 46.0051 | 225.0256 | .20444 | 4.5334 |
| 22 | 55.2061 | 271.0307 | .20369 | 4.5941 |
| 23 | 66.2474 | 326.2369 | .20307 | 4.6475 |
| 24 | 79.4968 | 392.4842 | .20255 | 4.6943 |
| 25 | 95.3962 | 471.9811 | .20212 | 4.7352 |
| 26 | 114.4755 | 567.3773 | .20176 | 4.7709 |
| 27 | 137.3706 | 681.8528 | .20147 | 4.8020 |
| 28 | 164.8447 | 819.2233 | .20122 | 4.8291 |
| 29 | 197.8136 | 984.0680 | .20102 | 4.8527 |
| 30 | 237.3763 | 1181.8816 | .20085 | 4.8731 |
| 31 | 284.8516 | 1419.2579 | .20070 | 4.8908 |
| 32 | 341.8219 | 1704.1095 | .20059 | 4.9061 |
| 33 | 410.1863 | 2045.9314 | .20049 | 4.9194 |
| 34 | 492.2235 | 2456.1176 | .20041 | 4.9308 |
| 35 | 590.6682 | 2948.3411 | .20034 | 4.9406 |
| 40 | 1469.7716 | 7343.8578 | .20014 | 4.9728 |
| 45 | 3657.2620 | 18281.3099 | .20005 | 4.9877 |
| 50 | 9100.4382 | 45497.1908 | .20002 | 4.9945 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 25.00 PERCENT,

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.2500 | 1.0000 | 1.25000 | .0000 |
| 2 | 1.5625 | 2.2500 | .69444 | .4444 |
| 3 | 1.9531 | 3.8125 | .51230 | .8525 |
| 4 | 2.4414 | 5.7656 | .42344 | 1.2249 |
| 5 | 3.0518 | 8.2070 | .37185 | 1.5631 |
| 6 | 3.8147 | 11.2588 | .33882 | 1.8683 |
| 7 | 4.7684 | 15.0735 | .31634 | 2.1424 |
| 8 | 5.9605 | 19.8419 | .30040 | 2.3872 |
| 9 | 7.4506 | 25.8023 | .28876 | 2.6048 |
| 10 | 9.3132 | 33.2529 | .28007 | 2.7971 |
| 11 | 11.6415 | 42.5661 | .27349 | 2.9663 |
| 12 | 14.5519 | 54.2077 | .26845 | 3.1145 |
| 13 | 18.1899 | 68.7596 | .26454 | 3.2437 |
| 14 | 22.7374 | 86.9495 | .26150 | 3.3559 |
| 15 | 28.4217 | 109.6868 | .25912 | 3.4530 |
| 16 | 35.5271 | 138.1085 | .25724 | 3.5366 |
| 17 | 44.4089 | 173.6357 | .25576 | 3.6084 |
| 18 | 55.5112 | 218.0446 | .25459 | 3.6698 |
| 19 | 69.3889 | 273.5558 | .25366 | 3.7222 |
| 20 | 86.7362 | 342.9447 | .25292 | 3.7667 |
| 21 | 108.4202 | 429.6809 | .25233 | 3.8045 |
| 22 | 135.5253 | 538.1011 | .25186 | 3.8365 |
| 23 | 169.4066 | 673.6264 | .25148 | 3.8634 |
| 24 | 211.7582 | 843.0329 | .25119 | 3.8861 |
| 25 | 264.6978 | 1054.7912 | .25095 | 3.9052 |
| 26 | 330.8722 | 1319.4890 | .25076 | 3.9212 |
| 27 | 413.5903 | 1650.3612 | .25061 | 3.9346 |
| 28 | 516.9879 | 2063.9515 | .25048 | 3.9457 |
| 29 | 646.2349 | 2580.9394 | .25039 | 3.9551 |
| 30 | 807.7936 | 3227.1743 | .25031 | 3.9628 |
| 31 | 1009.7420 | 4034.9678 | .25025 | 3.9693 |
| 32 | 1262.1774 | 5044.7098 | .25020 | 3.9746 |
| 33 | 1577.7218 | 6306.8872 | .25016 | 3.9791 |
| 34 | 1972.1523 | 7884.6091 | .25013 | 3.9828 |
| 35 | 2465.1903 | 9856.7613 | .25010 | 3.9858 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 30.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.3000 | 1.0000 | 1.30000 | .0000 |
| 2 | 1.6900 | 2.3000 | .73478 | .4348 |
| 3 | 2.1970 | 3.9900 | .55063 | .8271 |
| 4 | 2.8561 | 6.1870 | .46163 | 1.1783 |
| 5 | 3.7129 | 9.0431 | .41058 | 1.4903 |
| 6 | 4.8268 | 12.7560 | .37839 | 1.7654 |
| 7 | 6.2749 | 17.5828 | .35687 | 2.0063 |
| 8 | 8.1573 | 23.8577 | .34192 | 2.2156 |
| 9 | 10.6045 | 32.0150 | .33124 | 2.3963 |
| 10 | 13.7858 | 42.6195 | .32346 | 2.5512 |
| 11 | 17.9216 | 56.4053 | .31773 | 2.6833 |
| 12 | 23.2981 | 74.3270 | .31345 | 2.7952 |
| 13 | 30.2875 | 97.6250 | .31024 | 2.8895 |
| 14 | 39.3738 | 127.9125 | .30782 | 2.9685 |
| 15 | 51.1859 | 167.2863 | .30598 | 3.0344 |
| 16 | 66.5417 | 218.4722 | .30458 | 3.0892 |
| 17 | 86.5042 | 285.0139 | .30351 | 3.1345 |
| 18 | 112.4554 | 371.5180 | .30269 | 3.1718 |
| 19 | 146.1920 | 483.9734 | .30207 | 3.2025 |
| 20 | 190.0496 | 630.1655 | .30159 | 3.2275 |
| 21 | 247.0645 | 820.2151 | .30122 | 3.2480 |
| 22 | 321.1839 | 1067.2796 | .30094 | 3.2646 |
| 23 | 417.5391 | 1388.4635 | .30072 | 3.2781 |
| 24 | 542.8008 | 1806.0026 | .30055 | 3.2890 |
| 25 | 705.6410 | 2348.8033 | .30043 | 3.2979 |
| 26 | 917.3333 | 3054.4443 | .30033 | 3.3050 |
| 27 | 1192.5333 | 3971.7776 | .30025 | 3.3107 |
| 28 | 1550.2933 | 5164.3109 | .30019 | 3.3153 |
| 29 | 2015.3813 | 6714.6042 | .30015 | 3.3189 |
| 30 | 2619.9956 | 8729.9855 | .30011 | 3.3219 |
| 31 | 3405.9943 | 11349.9811 | .30009 | 3.3242 |
| 32 | 4427.7926 | 14755.9755 | .30007 | 3.3261 |
| 33 | 5756.1304 | 19183.7681 | .30005 | 3.3276 |
| 34 | 7482.9696 | 24939.8985 | .30004 | 3.3288 |
| 35 | 9727.8604 | 32422.8681 | .30003 | 3.3297 |

COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING

INTEREST RATE = 40.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.4000 | 1.0000 | 1.40000 | .0000 |
| 2 | 1.9600 | 2.4000 | .81667 | .4167 |
| 3 | 2.7440 | 4.3600 | .62936 | .7798 |
| 4 | 3.8416 | 7.1040 | .54077 | 1.0923 |
| 5 | 5.3782 | 10.9456 | .49136 | 1.3580 |
| 6 | 7.5295 | 16.3238 | .46126 | 1.5811 |
| 7 | 10.5414 | 23.8534 | .44192 | 1.7664 |
| 8 | 14.7579 | 34.3947 | .42907 | 1.9185 |
| 9 | 20.6610 | 49.1526 | .42034 | 2.0422 |
| 10 | 28.9255 | 69.8137 | .41432 | 2.1419 |
| 11 | 40.4957 | 98.7391 | .41013 | 2.2215 |
| 12 | 56.6939 | 139.2348 | .40718 | 2.2845 |
| 13 | 79.3715 | 195.9287 | .40510 | 2.3341 |
| 14 | 111.1201 | 275.3002 | .40363 | 2.3729 |
| 15 | 155.5681 | 386.4202 | .40259 | 2.4030 |
| 16 | 217.7953 | 541.9883 | .40185 | 2.4262 |
| 17 | 304.9135 | 759.7837 | .40132 | 2.4441 |
| 18 | 426.8789 | 1064.6971 | .40094 | 2.4577 |
| 19 | 597.6304 | 1491.5760 | .40067 | 2.4682 |
| 20 | 836.6826 | 2089.2064 | .40048 | 2.4761 |
| 21 | 1171.3556 | 2925.8889 | .40034 | 2.4821 |
| 22 | 1639.8978 | 4097.2445 | .40024 | 2.4866 |
| 23 | 2295.8569 | 5737.1423 | .40017 | 2.4900 |
| 24 | 3214.1997 | 8032.9993 | .40012 | 2.4925 |
| 25 | 4499.8796 | 11247.1990 | .40009 | 2.4944 |
| 26 | 6299.8314 | 15747.0785 | .40006 | 2.4959 |
| 27 | 8819.7640 | 22046.9099 | .40005 | 2.4969 |
| 28 | 12347.6696 | 30866.6739 | .40003 | 2.4977 |
| 29 | 17286.7374 | 43214.3435 | .40002 | 2.4983 |
| 30 | 24201.4324 | 60501.0809 | .40002 | 2.4988 |

**COMPOUND INTEREST FACTORS - ANNUAL COMPOUNDING**

**INTEREST RATE = 50.00 PERCENT**

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.5000 | 1.0000 | 1.50000 | .0000 |
| 2 | 2.2500 | 2.5000 | .90000 | .4000 |
| 3 | 3.3750 | 4.7500 | .71053 | .7368 |
| 4 | 5.0625 | 8.1250 | .62308 | 1.0154 |
| 5 | 7.5938 | 13.1875 | .57583 | 1.2417 |
| 6 | 11.3906 | 20.7813 | .54812 | 1.4226 |
| 7 | 17.0859 | 32.1719 | .53108 | 1.5648 |
| 8 | 25.6289 | 49.2578 | .52030 | 1.6752 |
| 9 | 38.4434 | 74.8867 | .51335 | 1.7596 |
| 10 | 57.6650 | 113.3301 | .50882 | 1.8235 |
| 11 | 86.4976 | 170.9951 | .50585 | 1.8713 |
| 12 | 129.7463 | 257.4927 | .50388 | 1.9068 |
| 13 | 194.6195 | 387.2390 | .50258 | 1.9329 |
| 14 | 291.9293 | 581.8585 | .50172 | 1.9519 |
| 15 | 437.8939 | 873.7878 | .50114 | 1.9657 |
| 16 | 656.8408 | 1311.6817 | .50076 | 1.9756 |
| 17 | 985.2613 | 1968.5225 | .50051 | 1.9827 |
| 18 | 1477.8919 | 2953.7838 | .50034 | 1.9878 |
| 19 | 2216.8378 | 4431.6756 | .50023 | 1.9914 |
| 20 | 3325.2567 | 6648.5135 | .50015 | 1.9940 |
| 21 | 4987.8851 | 9973.7702 | .50010 | 1.9958 |
| 22 | 7481.8276 | 14961.6553 | .50007 | 1.9971 |
| 23 | 11222.7415 | 22443.4829 | .50004 | 1.9980 |
| 24 | 16834.1122 | 33666.2244 | .50003 | 1.9986 |
| 25 | 25251.1683 | 50500.3366 | .50002 | 1.9990 |

# Appendix B

## Nominal versus Effective Interest Rates

| | EFFECTIVE INTEREST RATES | | | | | |
|---|---|---|---|---|---|---|
| NOMINAL INTEREST RATE | FREQUENCY OF COMPOUNDING | | | | | |
| | SEMIANNUALLY | QUARTERLY | MONTHLY | WEEKLY | DAILY | CONTINUOUSLY |
| .01 | .010025 | .010038 | .010046 | .010049 | .010050 | .010050 |
| .02 | .020100 | .020151 | .020184 | .020197 | .020201 | .020201 |
| .03 | .030225 | .030339 | .030416 | .030446 | .030453 | .030455 |
| .04 | .040400 | .040604 | .040742 | .040795 | .040808 | .040811 |
| .05 | .050625 | .050945 | .051162 | .051246 | .051267 | .051271 |
| .06 | .060900 | .061364 | .061678 | .061800 | .061831 | .061837 |
| .07 | .071225 | .071859 | .072290 | .072458 | .072501 | .072508 |
| .08 | .081600 | .082432 | .083000 | .083220 | .083278 | .083287 |
| .09 | .092025 | .093083 | .093807 | .094089 | .094162 | .094174 |
| .10 | .102500 | .103813 | .104713 | .105065 | .105156 | .105171 |
| .11 | .113025 | .114621 | .115719 | .116148 | .116260 | .116278 |
| .12 | .123600 | .125509 | .126825 | .127341 | .127475 | .127497 |
| .13 | .134225 | .136476 | .138032 | .138644 | .138802 | .138828 |
| .14 | .144900 | .147523 | .149342 | .150057 | .150243 | .150274 |
| .15 | .155625 | .158650 | .160755 | .161583 | .161798 | .161834 |
| .16 | .166400 | .169859 | .172271 | .173223 | .173470 | .173511 |
| .17 | .177225 | .181148 | .183892 | .184976 | .185258 | .185305 |
| .18 | .188100 | .192519 | .195618 | .196845 | .197164 | .197217 |
| .19 | .199025 | .203971 | .207451 | .208831 | .209190 | .209250 |
| .20 | .210000 | .215506 | .219391 | .220934 | .221336 | .221403 |
| .21 | .221025 | .227124 | .231439 | .233156 | .233604 | .233678 |
| .22 | .232100 | .238825 | .243597 | .245499 | .245994 | .246077 |
| .23 | .243225 | .250609 | .255864 | .257962 | .258509 | .258600 |
| .24 | .254400 | .262477 | .268242 | .270547 | .271149 | .271249 |
| .25 | .265625 | .274429 | .280732 | .283256 | .283916 | .284025 |
| .26 | .276900 | .286466 | .293334 | .296090 | .296810 | .296930 |
| .27 | .288225 | .298588 | .306050 | .309050 | .309834 | .309964 |
| .28 | .299600 | .310796 | .318881 | .322136 | .322988 | .323130 |
| .29 | .311025 | .323089 | .331826 | .335351 | .336274 | .336427 |
| .30 | .322500 | .335469 | .344889 | .348696 | .349692 | .349859 |
| .31 | .334025 | .347936 | .358069 | .362171 | .363246 | .363425 |
| .32 | .345600 | .360489 | .371367 | .375778 | .376935 | .377128 |
| .33 | .357225 | .373130 | .384784 | .389519 | .390761 | .390968 |
| .34 | .368900 | .385859 | .398321 | .403394 | .404725 | .404948 |
| .35 | .380625 | .398676 | .411980 | .417404 | .418830 | .419068 |
| .36 | .392400 | .411582 | .425761 | .431553 | .433075 | .433329 |
| .37 | .404225 | .424577 | .439665 | .445839 | .447463 | .447735 |
| .38 | .416100 | .437661 | .453693 | .460265 | .461996 | .462285 |
| .39 | .428025 | .450835 | .467847 | .474833 | .476673 | .476981 |
| .40 | .440000 | .464100 | .482126 | .489543 | .491498 | .491825 |
| .41 | .452025 | .477455 | .496533 | .504397 | .506471 | .506818 |
| .42 | .464100 | .490902 | .511069 | .519396 | .521594 | .521962 |
| .43 | .476225 | .504440 | .525733 | .534542 | .536869 | .537258 |
| .44 | .488400 | .518070 | .540528 | .549836 | .552296 | .552707 |
| .45 | .500625 | .531793 | .555454 | .565279 | .567878 | .568312 |
| .46 | .512900 | .545608 | .570513 | .580873 | .583615 | .584074 |
| .47 | .525225 | .559517 | .585705 | .596620 | .599511 | .599994 |
| .48 | .537600 | .573519 | .601032 | .612520 | .615565 | .616074 |
| .49 | .550025 | .587616 | .616495 | .628576 | .631780 | .632316 |
| .50 | .562500 | .601807 | .632094 | .644788 | .648157 | .648721 |

# Appendix C

## Compound Interest Factors— Continuous Compounding

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 0.25 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0025 | 1.0000 | 1.00250 | .0000 |
| 2 | 1.0050 | 2.0025 | .50188 | .4994 |
| 3 | 1.0075 | 3.0075 | .33500 | .9983 |
| 4 | 1.0101 | 4.0150 | .25157 | 1.4969 |
| 5 | 1.0126 | 5.0251 | .20150 | 1.9950 |
| 6 | 1.0151 | 6.0377 | .16813 | 2.4927 |
| 7 | 1.0177 | 7.0528 | .14429 | 2.9900 |
| 8 | 1.0202 | 8.0704 | .12641 | 3.4869 |
| 9 | 1.0228 | 9.0906 | .11251 | 3.9833 |
| 10 | 1.0253 | 10.1134 | .10138 | 4.4794 |
| 11 | 1.0279 | 11.1387 | .09228 | 4.9750 |
| 12 | 1.0305 | 12.1666 | .08470 | 5.4702 |
| 13 | 1.0330 | 13.1970 | .07828 | 5.9650 |
| 14 | 1.0356 | 14.2301 | .07278 | 6.4594 |
| 15 | 1.0382 | 15.2657 | .06801 | 6.9533 |
| 16 | 1.0408 | 16.3039 | .06384 | 7.4469 |
| 17 | 1.0434 | 17.3447 | .06016 | 7.9400 |
| 18 | 1.0460 | 18.3881 | .05689 | 8.4327 |
| 19 | 1.0486 | 19.4342 | .05396 | 8.9250 |
| 20 | 1.0513 | 20.4828 | .05132 | 9.4169 |
| 21 | 1.0539 | 21.5341 | .04894 | 9.9083 |
| 22 | 1.0565 | 22.5880 | .04677 | 10.3994 |
| 23 | 1.0592 | 23.6445 | .04480 | 10.8900 |
| 24 | 1.0618 | 24.7037 | .04298 | 11.3802 |
| 25 | 1.0645 | 25.7655 | .04131 | 11.8700 |
| 26 | 1.0672 | 26.8300 | .03977 | 12.3594 |
| 27 | 1.0698 | 27.8972 | .03835 | 12.8483 |
| 28 | 1.0725 | 28.9670 | .03703 | 13.3369 |
| 29 | 1.0752 | 30.0395 | .03579 | 13.8250 |
| 30 | 1.0779 | 31.1147 | .03464 | 14.3127 |
| 31 | 1.0806 | 32.1926 | .03357 | 14.8000 |
| 32 | 1.0833 | 33.2732 | .03256 | 15.2869 |
| 33 | 1.0860 | 34.3565 | .03161 | 15.7734 |
| 34 | 1.0887 | 35.4425 | .03072 | 16.2594 |
| 35 | 1.0914 | 36.5312 | .02988 | 16.7450 |
| 40 | 1.1052 | 42.0158 | .02630 | 19.1669 |
| 45 | 1.1191 | 47.5694 | .02353 | 21.5784 |
| 50 | 1.1331 | 53.1928 | .02130 | 23.9795 |
| 55 | 1.1474 | 58.8870 | .01948 | 26.3702 |
| 60 | 1.1618 | 64.6528 | .01797 | 28.7505 |
| 65 | 1.1764 | 70.4911 | .01669 | 31.1204 |
| 70 | 1.1912 | 76.4029 | .01559 | 33.4799 |
| 75 | 1.2062 | 82.3890 | .01464 | 35.8290 |
| 80 | 1.2214 | 88.4504 | .01381 | 38.1678 |
| 85 | 1.2368 | 94.5881 | .01308 | 40.4961 |
| 90 | 1.2523 | 100.8030 | .01242 | 42.8141 |
| 95 | 1.2681 | 107.0960 | .01184 | 45.1218 |
| 100 | 1.2840 | 113.4682 | .01132 | 47.4190 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 0.50 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0050 | 1.0000 | 1.00501 | .0000 |
| 2 | 1.0101 | 2.0050 | .50376 | .4988 |
| 3 | 1.0151 | 3.0151 | .33668 | .9967 |
| 4 | 1.0202 | 4.0302 | .25314 | 1.4938 |
| 5 | 1.0253 | 5.0504 | .20302 | 1.9900 |
| 6 | 1.0305 | 6.0757 | .16960 | 2.4854 |
| 7 | 1.0356 | 7.1061 | .14574 | 2.9800 |
| 8 | 1.0408 | 8.1418 | .12784 | 3.4738 |
| 9 | 1.0460 | 9.1826 | .11391 | 3.9667 |
| 10 | 1.0513 | 10.2286 | .10278 | 4.4588 |
| 11 | 1.0565 | 11.2799 | .09367 | 4.9500 |
| 12 | 1.0618 | 12.3364 | .08607 | 5.4404 |
| 13 | 1.0672 | 13.3983 | .07965 | 5.9300 |
| 14 | 1.0725 | 14.4654 | .07414 | 6.4188 |
| 15 | 1.0779 | 15.5379 | .06937 | 6.9067 |
| 16 | 1.0833 | 16.6158 | .06520 | 7.3938 |
| 17 | 1.0887 | 17.6991 | .06151 | 7.8800 |
| 18 | 1.0942 | 18.7878 | .05824 | 8.3654 |
| 19 | 1.0997 | 19.8820 | .05531 | 8.8500 |
| 20 | 1.1052 | 20.9816 | .05267 | 9.3338 |
| 21 | 1.1107 | 22.0868 | .05029 | 9.8167 |
| 22 | 1.1163 | 23.1975 | .04812 | 10.2988 |
| 23 | 1.1219 | 24.3138 | .04614 | 10.7800 |
| 24 | 1.1275 | 25.4357 | .04433 | 11.2605 |
| 25 | 1.1331 | 26.5632 | .04266 | 11.7401 |
| 26 | 1.1388 | 27.6963 | .04112 | 12.2188 |
| 27 | 1.1445 | 28.8351 | .03969 | 12.6968 |
| 28 | 1.1503 | 29.9797 | .03837 | 13.1739 |
| 29 | 1.1560 | 31.1300 | .03714 | 13.6501 |
| 30 | 1.1618 | 32.2860 | .03599 | 14.1256 |
| 31 | 1.1677 | 33.4478 | .03491 | 14.6002 |
| 32 | 1.1735 | 34.6155 | .03390 | 15.0739 |
| 33 | 1.1794 | 35.7890 | .03295 | 15.5469 |
| 34 | 1.1853 | 36.9684 | .03206 | 16.0190 |
| 35 | 1.1912 | 38.1537 | .03122 | 16.4903 |
| 40 | 1.2214 | 44.1699 | .02765 | 18.8342 |
| 45 | 1.2523 | 50.3385 | .02488 | 21.1574 |
| 50 | 1.2840 | 56.6632 | .02266 | 23.4598 |
| 55 | 1.3165 | 63.1480 | .02085 | 25.7416 |
| 60 | 1.3499 | 69.7970 | .01934 | 28.0027 |
| 65 | 1.3840 | 76.6143 | .01806 | 30.2431 |
| 70 | 1.4191 | 83.6042 | .01697 | 32.4629 |
| 75 | 1.4550 | 90.7710 | .01603 | 34.6621 |
| 80 | 1.4918 | 98.1192 | .01520 | 36.8408 |
| 85 | 1.5296 | 105.6535 | .01448 | 38.9990 |
| 90 | 1.5683 | 113.3785 | .01383 | 41.1368 |
| 95 | 1.6080 | 121.2991 | .01326 | 43.2541 |
| 100 | 1.6487 | 129.4202 | .01274 | 45.3510 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 0.75 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0075 | 1.0000 | 1.00753 | .0000 |
| 2 | 1.0151 | 2.0075 | .50565 | .4981 |
| 3 | 1.0228 | 3.0226 | .33836 | .9950 |
| 4 | 1.0305 | 4.0454 | .25472 | 1.4906 |
| 5 | 1.0382 | 5.0759 | .20454 | 1.9850 |
| 6 | 1.0460 | 6.1141 | .17109 | 2.4781 |
| 7 | 1.0539 | 7.1601 | .14719 | 2.9700 |
| 8 | 1.0618 | 8.2140 | .12927 | 3.4606 |
| 9 | 1.0698 | 9.2758 | .11534 | 3.9500 |
| 10 | 1.0779 | 10.3457 | .10419 | 4.4381 |
| 11 | 1.0860 | 11.4235 | .09507 | 4.9250 |
| 12 | 1.0942 | 12.5095 | .08747 | 5.4106 |
| 13 | 1.1024 | 13.6037 | .08104 | 5.8950 |
| 14 | 1.1107 | 14.7061 | .07553 | 6.3781 |
| 15 | 1.1191 | 15.8168 | .07075 | 6.8600 |
| 16 | 1.1275 | 16.9359 | .06657 | 7.3407 |
| 17 | 1.1360 | 18.0634 | .06289 | 7.8200 |
| 18 | 1.1445 | 19.1994 | .05961 | 8.2982 |
| 19 | 1.1532 | 20.3439 | .05668 | 8.7751 |
| 20 | 1.1618 | 21.4971 | .05405 | 9.2507 |
| 21 | 1.1706 | 22.6589 | .05166 | 9.7251 |
| 22 | 1.1794 | 23.8295 | .04949 | 10.1983 |
| 23 | 1.1883 | 25.0089 | .04751 | 10.6702 |
| 24 | 1.1972 | 26.1972 | .04570 | 11.1408 |
| 25 | 1.2062 | 27.3944 | .04403 | 11.6102 |
| 26 | 1.2153 | 28.6006 | .04249 | 12.0784 |
| 27 | 1.2245 | 29.8159 | .04107 | 12.5453 |
| 28 | 1.2337 | 31.0404 | .03974 | 13.0110 |
| 29 | 1.2430 | 32.2741 | .03851 | 13.4754 |
| 30 | 1.2523 | 33.5170 | .03736 | 13.9386 |
| 31 | 1.2618 | 34.7693 | .03629 | 14.4005 |
| 32 | 1.2712 | 36.0311 | .03528 | 14.8612 |
| 33 | 1.2808 | 37.3023 | .03434 | 15.3207 |
| 34 | 1.2905 | 38.5832 | .03345 | 15.7789 |
| 35 | 1.3002 | 39.8736 | .03261 | 16.2359 |
| 40 | 1.3499 | 46.4731 | .02905 | 18.5021 |
| 45 | 1.4014 | 53.3248 | .02628 | 20.7374 |
| 50 | 1.4550 | 60.4383 | .02407 | 22.9418 |
| 55 | 1.5106 | 67.8236 | .02227 | 25.1153 |
| 60 | 1.5683 | 75.4912 | .02077 | 27.2582 |
| 65 | 1.6282 | 83.4517 | .01951 | 29.3704 |
| 70 | 1.6905 | 91.7164 | .01843 | 31.4521 |
| 75 | 1.7551 | 100.2969 | .01750 | 33.5034 |
| 80 | 1.8221 | 109.2053 | .01669 | 35.5244 |
| 85 | 1.8917 | 118.4541 | .01597 | 37.5153 |
| 90 | 1.9640 | 128.0563 | .01534 | 39.4762 |
| 95 | 2.0391 | 138.0255 | .01477 | 41.4072 |
| 100 | 2.1170 | 148.3755 | .01427 | 43.3084 |

COMPOUND INTEREST FACTORS – CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 1.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0101 | 1.0000 | 1.01005 | .0000 |
| 2 | 1.0202 | 2.0101 | .50755 | .4975 |
| 3 | 1.0305 | 3.0303 | .34006 | .9933 |
| 4 | 1.0408 | 4.0607 | .25631 | 1.4875 |
| 5 | 1.0513 | 5.1015 | .20607 | 1.9800 |
| 6 | 1.0618 | 6.1528 | .17258 | 2.4708 |
| 7 | 1.0725 | 7.2146 | .14866 | 2.9600 |
| 8 | 1.0833 | 8.2871 | .13072 | 3.4475 |
| 9 | 1.0942 | 9.3704 | .11677 | 3.9333 |
| 10 | 1.1052 | 10.4646 | .10561 | 4.4175 |
| 11 | 1.1163 | 11.5698 | .09648 | 4.9000 |
| 12 | 1.1275 | 12.6860 | .08888 | 5.3809 |
| 13 | 1.1388 | 13.8135 | .08244 | 5.8600 |
| 14 | 1.1503 | 14.9524 | .07693 | 6.3376 |
| 15 | 1.1618 | 16.1026 | .07215 | 6.8134 |
| 16 | 1.1735 | 17.2645 | .06797 | 7.2876 |
| 17 | 1.1853 | 18.4380 | .06429 | 7.7601 |
| 18 | 1.1972 | 19.6233 | .06101 | 8.2310 |
| 19 | 1.2092 | 20.8205 | .05808 | 8.7002 |
| 20 | 1.2214 | 22.0298 | .05544 | 9.1677 |
| 21 | 1.2337 | 23.2512 | .05306 | 9.6336 |
| 22 | 1.2461 | 24.4848 | .05089 | 10.0978 |
| 23 | 1.2586 | 25.7309 | .04891 | 10.5604 |
| 24 | 1.2712 | 26.9895 | .04710 | 11.0213 |
| 25 | 1.2840 | 28.2608 | .04543 | 11.4805 |
| 26 | 1.2969 | 29.5448 | .04390 | 11.9381 |
| 27 | 1.3100 | 30.8417 | .04247 | 12.3941 |
| 28 | 1.3231 | 32.1517 | .04115 | 12.8484 |
| 29 | 1.3364 | 33.4748 | .03992 | 13.3010 |
| 30 | 1.3499 | 34.8112 | .03878 | 13.7520 |
| 31 | 1.3634 | 36.1611 | .03770 | 14.2013 |
| 32 | 1.3771 | 37.5245 | .03670 | 14.6490 |
| 33 | 1.3910 | 38.9017 | .03576 | 15.0950 |
| 34 | 1.4049 | 40.2926 | .03487 | 15.5394 |
| 35 | 1.4191 | 41.6976 | .03403 | 15.9821 |
| 40 | 1.4918 | 48.9370 | .03048 | 18.1710 |
| 45 | 1.5683 | 56.5475 | .02773 | 20.3190 |
| 50 | 1.6487 | 64.5483 | .02554 | 22.4261 |
| 55 | 1.7333 | 72.9593 | .02376 | 24.4926 |
| 60 | 1.8221 | 81.8015 | .02227 | 26.5187 |
| 65 | 1.9155 | 91.0971 | .02103 | 28.5045 |
| 70 | 2.0138 | 100.8692 | .01996 | 30.4505 |
| 75 | 2.1170 | 111.1424 | .01905 | 32.3567 |
| 80 | 2.2255 | 121.9423 | .01825 | 34.2235 |
| 85 | 2.3396 | 133.2960 | .01755 | 36.0513 |
| 90 | 2.4596 | 145.2317 | .01694 | 37.8402 |
| 95 | 2.5857 | 157.7794 | .01639 | 39.5907 |
| 100 | 2.7183 | 170.9705 | .01590 | 41.3032 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 1.25 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERV FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0126 | 1.0000 | 1.01258 | .0000 |
| 2 | 1.0253 | 2.0126 | .50945 | .4969 |
| 3 | 1.0382 | 3.0379 | .34175 | .9917 |
| 4 | 1.0513 | 4.0761 | .25791 | 1.4844 |
| 5 | 1.0645 | 5.1274 | .20761 | 1.9750 |
| 6 | 1.0779 | 6.1919 | .17408 | 2.4635 |
| 7 | 1.0914 | 7.2698 | .15013 | 2.9500 |
| 8 | 1.1052 | 8.3612 | .13218 | 3.4344 |
| 9 | 1.1191 | 9.4664 | .11822 | 3.9167 |
| 10 | 1.1331 | 10.5854 | .10705 | 4.3969 |
| 11 | 1.1474 | 11.7186 | .09791 | 4.8750 |
| 12 | 1.1618 | 12.8660 | .09030 | 5.3511 |
| 13 | 1.1764 | 14.0278 | .08387 | 5.8251 |
| 14 | 1.1912 | 15.2043 | .07835 | 6.2970 |
| 15 | 1.2062 | 16.3955 | .07357 | 6.7668 |
| 16 | 1.2214 | 17.6017 | .06939 | 7.2346 |
| 17 | 1.2368 | 18.8232 | .06570 | 7.7002 |
| 18 | 1.2523 | 20.0599 | .06243 | 8.1638 |
| 19 | 1.2681 | 21.3122 | .05950 | 8.6254 |
| 20 | 1.2840 | 22.5803 | .05686 | 9.0848 |
| 21 | 1.3002 | 23.8643 | .05448 | 9.5422 |
| 22 | 1.3165 | 25.1645 | .05232 | 9.9975 |
| 23 | 1.3331 | 26.4810 | .05034 | 10.4508 |
| 24 | 1.3499 | 27.8141 | .04853 | 10.9019 |
| 25 | 1.3668 | 29.1640 | .04687 | 11.3511 |
| 26 | 1.3840 | 30.5308 | .04533 | 11.7981 |
| 27 | 1.4014 | 31.9149 | .04391 | 12.2431 |
| 28 | 1.4191 | 33.3163 | .04259 | 12.6860 |
| 29 | 1.4369 | 34.7354 | .04137 | 13.1269 |
| 30 | 1.4550 | 36.1723 | .04022 | 13.5657 |
| 31 | 1.4733 | 37.6273 | .03915 | 14.0025 |
| 32 | 1.4918 | 39.1006 | .03815 | 14.4372 |
| 33 | 1.5106 | 40.5924 | .03721 | 14.8699 |
| 34 | 1.5296 | 42.1030 | .03633 | 15.3005 |
| 35 | 1.5488 | 43.6326 | .03550 | 15.7291 |
| 40 | 1.6487 | 51.5740 | .03197 | 17.8413 |
| 45 | 1.7551 | 60.0276 | .02924 | 19.9027 |
| 50 | 1.8682 | 69.0265 | .02707 | 21.9137 |
| 55 | 1.9887 | 78.6057 | .02530 | 23.8745 |
| 60 | 2.1170 | 88.8027 | .02384 | 25.7857 |
| 65 | 2.2535 | 99.6573 | .02261 | 27.6477 |
| 70 | 2.3989 | 111.2120 | .02157 | 29.4608 |
| 75 | 2.5536 | 123.5120 | .02067 | 31.2257 |
| 80 | 2.7183 | 136.6052 | .01990 | 32.9429 |
| 85 | 2.8936 | 150.5429 | .01922 | 34.6129 |
| 90 | 3.0802 | 165.3794 | .01863 | 36.2363 |
| 95 | 3.2789 | 181.1728 | .01810 | 37.8138 |
| 100 | 3.4903 | 197.9849 | .01763 | 39.3459 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 1.50 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0151 | 1.0000 | 1.01511 | .0000 |
| 2 | 1.0305 | 2.0151 | .51136 | .4963 |
| 3 | 1.0460 | 3.0456 | .34346 | .9900 |
| 4 | 1.0618 | 4.0916 | .25952 | 1.4813 |
| 5 | 1.0779 | 5.1534 | .20916 | 1.9700 |
| 6 | 1.0942 | 6.2313 | .17559 | 2.4563 |
| 7 | 1.1107 | 7.3255 | .15162 | 2.9400 |
| 8 | 1.1275 | 8.4362 | .13365 | 3.4213 |
| 9 | 1.1445 | 9.5637 | .11968 | 3.9000 |
| 10 | 1.1618 | 10.7082 | .10850 | 4.3763 |
| 11 | 1.1794 | 11.8701 | .09936 | 4.8501 |
| 12 | 1.1972 | 13.0495 | .09174 | 5.3213 |
| 13 | 1.2153 | 14.2467 | .08530 | 5.7901 |
| 14 | 1.2337 | 15.4620 | .07979 | 6.2564 |
| 15 | 1.2523 | 16.6957 | .07501 | 6.7202 |
| 16 | 1.2712 | 17.9480 | .07083 | 7.1816 |
| 17 | 1.2905 | 19.2192 | .06714 | 7.6404 |
| 18 | 1.3100 | 20.5097 | .06387 | 8.0967 |
| 19 | 1.3298 | 21.8197 | .06094 | 8.5506 |
| 20 | 1.3499 | 23.1494 | .05831 | 9.0020 |
| 21 | 1.3703 | 24.4993 | .05593 | 9.4509 |
| 22 | 1.3910 | 25.8695 | .05377 | 9.8973 |
| 23 | 1.4120 | 27.2605 | .05180 | 10.3413 |
| 24 | 1.4333 | 28.6725 | .04999 | 10.7828 |
| 25 | 1.4550 | 30.1058 | .04833 | 11.2218 |
| 26 | 1.4770 | 31.5608 | .04680 | 11.6584 |
| 27 | 1.4993 | 33.0378 | .04538 | 12.0925 |
| 28 | 1.5220 | 34.5371 | .04407 | 12.5241 |
| 29 | 1.5450 | 36.0591 | .04285 | 12.9533 |
| 30 | 1.5683 | 37.6040 | .04171 | 13.3800 |
| 31 | 1.5920 | 39.1723 | .04064 | 13.8043 |
| 32 | 1.6161 | 40.7644 | .03964 | 14.2261 |
| 33 | 1.6405 | 42.3804 | .03871 | 14.6455 |
| 34 | 1.6653 | 44.0209 | .03783 | 15.0625 |
| 35 | 1.6905 | 45.6862 | .03700 | 15.4770 |
| 40 | 1.8221 | 54.3979 | .03350 | 17.5131 |
| 45 | 1.9640 | 63.7881 | .03079 | 19.4890 |
| 50 | 2.1170 | 73.9096 | .02864 | 21.4052 |
| 55 | 2.2819 | 84.8194 | .02690 | 23.2622 |
| 60 | 2.4596 | 96.5789 | .02547 | 25.0609 |
| 65 | 2.6512 | 109.2543 | .02427 | 26.8018 |
| 70 | 2.8577 | 122.9169 | .02325 | 28.4859 |
| 75 | 3.0802 | 137.6436 | .02238 | 30.1140 |
| 80 | 3.3201 | 153.5173 | .02163 | 31.6869 |
| 85 | 3.5787 | 170.6273 | .02097 | 33.2056 |
| 90 | 3.8574 | 189.0699 | .02040 | 34.6710 |
| 95 | 4.1579 | 208.9489 | .01990 | 36.0842 |
| 100 | 4.4817 | 230.3761 | .01945 | 37.4462 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 2.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0202 | 1.0000 | 1.02020 | .0000 |
| 2 | 1.0408 | 2.0202 | .51520 | .4950 |
| 3 | 1.0618 | 3.0610 | .34689 | .9867 |
| 4 | 1.0833 | 4.1228 | .26275 | 1.4750 |
| 5 | 1.1052 | 5.2061 | .21228 | 1.9600 |
| 6 | 1.1275 | 6.3113 | .17865 | 2.4417 |
| 7 | 1.1503 | 7.4388 | .15463 | 2.9200 |
| 8 | 1.1735 | 8.5891 | .13663 | 3.3950 |
| 9 | 1.1972 | 9.7626 | .12263 | 3.8667 |
| 10 | 1.2214 | 10.9598 | .11144 | 4.3351 |
| 11 | 1.2461 | 12.1812 | .10230 | 4.8002 |
| 12 | 1.2712 | 13.4273 | .09468 | 5.2619 |
| 13 | 1.2969 | 14.6985 | .08824 | 5.7203 |
| 14 | 1.3231 | 15.9955 | .08272 | 6.1754 |
| 15 | 1.3499 | 17.3186 | .07794 | 6.6272 |
| 16 | 1.3771 | 18.6685 | .07377 | 7.0757 |
| 17 | 1.4049 | 20.0456 | .07009 | 7.5209 |
| 18 | 1.4333 | 21.4505 | .06682 | 7.9628 |
| 19 | 1.4623 | 22.8839 | .06390 | 8.4014 |
| 20 | 1.4918 | 24.3461 | .06128 | 8.8368 |
| 21 | 1.5220 | 25.8380 | .05890 | 9.2688 |
| 22 | 1.5527 | 27.3599 | .05675 | 9.6976 |
| 23 | 1.5841 | 28.9126 | .05479 | 10.1231 |
| 24 | 1.6161 | 30.4967 | .05299 | 10.5453 |
| 25 | 1.6487 | 32.1128 | .05134 | 10.9643 |
| 26 | 1.6820 | 33.7615 | .04982 | 11.3800 |
| 27 | 1.7160 | 35.4435 | .04842 | 11.7925 |
| 28 | 1.7507 | 37.1595 | .04711 | 12.2018 |
| 29 | 1.7860 | 38.9102 | .04590 | 12.6078 |
| 30 | 1.8221 | 40.6963 | .04477 | 13.0106 |
| 31 | 1.8589 | 42.5184 | .04372 | 13.4102 |
| 32 | 1.8965 | 44.3773 | .04274 | 13.8065 |
| 33 | 1.9348 | 46.2738 | .04181 | 14.1997 |
| 34 | 1.9739 | 48.2086 | .04094 | 14.5897 |
| 35 | 2.0138 | 50.1824 | .04013 | 14.9765 |
| 40 | 2.2255 | 60.6663 | .03668 | 16.8630 |
| 45 | 2.4596 | 72.2528 | .03404 | 18.6714 |
| 50 | 2.7183 | 85.0578 | .03196 | 20.4028 |
| 55 | 3.0042 | 99.2096 | .03028 | 22.0588 |
| 60 | 3.3201 | 114.8497 | .02891 | 23.6409 |
| 65 | 3.6693 | 132.1346 | .02777 | 25.1507 |
| 70 | 4.0552 | 151.2375 | .02681 | 26.5899 |
| 75 | 4.4817 | 172.3494 | .02600 | 27.9604 |
| 80 | 4.9530 | 195.6817 | .02531 | 29.2640 |
| 85 | 5.4739 | 221.4679 | .02472 | 30.5028 |
| 90 | 6.0496 | 249.9660 | .02420 | 31.6786 |
| 95 | 6.6859 | 281.4613 | .02375 | 32.7937 |
| 100 | 7.3891 | 316.2689 | .02336 | 33.8499 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 3.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0305 | 1.0000 | 1.03045 | .0000 |
| 2 | 1.0618 | 2.0305 | .52296 | .4925 |
| 3 | 1.0942 | 3.0923 | .35384 | .9800 |
| 4 | 1.1275 | 4.1865 | .26932 | 1.4625 |
| 5 | 1.1618 | 5.3140 | .21864 | 1.9400 |
| 6 | 1.1972 | 6.4758 | .18488 | 2.4125 |
| 7 | 1.2337 | 7.6730 | .16078 | 2.8801 |
| 8 | 1.2712 | 8.9067 | .14273 | 3.3427 |
| 9 | 1.3100 | 10.1779 | .12871 | 3.8002 |
| 10 | 1.3499 | 11.4879 | .11750 | 4.2529 |
| 11 | 1.3910 | 12.8378 | .10835 | 4.7005 |
| 12 | 1.4333 | 14.2287 | .10073 | 5.1433 |
| 13 | 1.4770 | 15.6621 | .09430 | 5.5811 |
| 14 | 1.5220 | 17.1390 | .08880 | 6.0139 |
| 15 | 1.5683 | 18.6610 | .08404 | 6.4419 |
| 16 | 1.6161 | 20.2293 | .07989 | 6.8649 |
| 17 | 1.6653 | 21.8454 | .07623 | 7.2831 |
| 18 | 1.7160 | 23.5107 | .07299 | 7.6964 |
| 19 | 1.7683 | 25.2267 | .07010 | 8.1048 |
| 20 | 1.8221 | 26.9950 | .06750 | 8.5084 |
| 21 | 1.8776 | 28.8171 | .06516 | 8.9072 |
| 22 | 1.9348 | 30.6947 | .06303 | 9.3012 |
| 23 | 1.9937 | 32.6295 | .06110 | 9.6904 |
| 24 | 2.0544 | 34.6232 | .05934 | 10.0748 |
| 25 | 2.1170 | 36.6776 | .05772 | 10.4545 |
| 26 | 2.1815 | 38.7946 | .05623 | 10.8294 |
| 27 | 2.2479 | 40.9761 | .05486 | 11.1996 |
| 28 | 2.3164 | 43.2240 | .05359 | 11.5652 |
| 29 | 2.3869 | 45.5404 | .05241 | 11.9261 |
| 30 | 2.4596 | 47.9273 | .05132 | 12.2823 |
| 31 | 2.5345 | 50.3869 | .05030 | 12.6339 |
| 32 | 2.6117 | 52.9214 | .04935 | 12.9810 |
| 33 | 2.6912 | 55.5331 | .04846 | 13.3235 |
| 34 | 2.7732 | 58.2243 | .04763 | 13.6614 |
| 35 | 2.8577 | 60.9975 | .04685 | 13.9948 |
| 40 | 3.3201 | 76.1830 | .04358 | 15.5953 |
| 45 | 3.8574 | 93.8259 | .04111 | 17.0874 |
| 50 | 4.4817 | 114.3242 | .03920 | 18.4750 |
| 55 | 5.2070 | 138.1397 | .03769 | 19.7623 |
| 60 | 6.0496 | 165.8094 | .03649 | 20.9538 |
| 65 | 7.0287 | 197.9570 | .03551 | 22.0541 |
| 70 | 8.1662 | 235.3072 | .03470 | 23.0677 |
| 75 | 9.4877 | 278.7019 | .03404 | 23.9996 |
| 80 | 11.0232 | 329.1193 | .03349 | 24.8543 |
| 85 | 12.8071 | 387.6961 | .03303 | 25.6368 |
| 90 | 14.8797 | 455.7526 | .03265 | 26.3516 |
| 95 | 17.2878 | 534.8229 | .03232 | 27.0032 |
| 100 | 20.0855 | 626.6895 | .03205 | 27.5963 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 4.00 PERCENT

| N | SINGLE-PAVMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0408 | 1.0000 | 1.04081 | .0000 |
| 2 | 1.0833 | 2.0408 | .53081 | .4900 |
| 3 | 1.1275 | 3.1241 | .36090 | .9733 |
| 4 | 1.1735 | 4.2516 | .27602 | 1.4500 |
| 5 | 1.2214 | 5.4251 | .22514 | 1.9201 |
| 6 | 1.2712 | 6.6465 | .19127 | 2.3834 |
| 7 | 1.3231 | 7.9178 | .16711 | 2.8402 |
| 8 | 1.3771 | 9.2409 | .14903 | 3.2904 |
| 9 | 1.4333 | 10.6180 | .13499 | 3.7339 |
| 10 | 1.4918 | 12.0513 | .12379 | 4.1709 |
| 11 | 1.5527 | 13.5432 | .11465 | 4.6013 |
| 12 | 1.6161 | 15.0959 | .10705 | 5.0252 |
| 13 | 1.6820 | 16.7120 | .10065 | 5.4425 |
| 14 | 1.7507 | 18.3940 | .09518 | 5.8534 |
| 15 | 1.8221 | 20.1447 | .09045 | 6.2578 |
| 16 | 1.8965 | 21.9668 | .08633 | 6.6558 |
| 17 | 1.9739 | 23.8633 | .08272 | 7.0473 |
| 18 | 2.0544 | 25.8371 | .07951 | 7.4326 |
| 19 | 2.1383 | 27.8916 | .07666 | 7.8114 |
| 20 | 2.2255 | 30.0298 | .07411 | 8.1840 |
| 21 | 2.3164 | 32.2554 | .07181 | 8.5503 |
| 22 | 2.4109 | 34.5717 | .06974 | 8.9104 |
| 23 | 2.5093 | 36.9826 | .06785 | 9.2644 |
| 24 | 2.6117 | 39.4919 | .06613 | 9.6122 |
| 25 | 2.7183 | 42.1036 | .06456 | 9.9539 |
| 26 | 2.8292 | 44.8219 | .06312 | 10.2896 |
| 27 | 2.9447 | 47.6511 | .06180 | 10.6193 |
| 28 | 3.0649 | 50.5958 | .06058 | 10.9431 |
| 29 | 3.1899 | 53.6607 | .05945 | 11.2609 |
| 30 | 3.3201 | 56.8506 | .05840 | 11.5730 |
| 31 | 3.4556 | 60.1707 | .05743 | 11.8792 |
| 32 | 3.5966 | 63.6263 | .05653 | 12.1797 |
| 33 | 3.7434 | 67.2230 | .05569 | 12.4746 |
| 34 | 3.8962 | 70.9664 | .05490 | 12.7638 |
| 35 | 4.0552 | 74.8626 | .05417 | 13.0475 |
| 40 | 4.9530 | 96.8625 | .05113 | 14.3845 |
| 45 | 6.0496 | 123.7332 | .04889 | 15.5918 |
| 50 | 7.3891 | 156.5532 | .04720 | 16.6775 |
| 55 | 9.0250 | 196.6396 | .04590 | 17.6498 |
| 60 | 11.0232 | 245.6012 | .04488 | 18.5172 |
| 65 | 13.4637 | 305.4031 | .04409 | 19.2882 |
| 70 | 16.4446 | 378.4453 | .04345 | 19.9710 |
| 75 | 20.0855 | 467.6593 | .04295 | 20.5737 |
| 80 | 24.5325 | 576.6254 | .04255 | 21.1038 |
| 85 | 29.9641 | 709.7170 | .04222 | 21.5687 |
| 90 | 36.5982 | 872.2754 | .04196 | 21.9751 |
| 95 | 44.7012 | 1070.8247 | .04174 | 22.3295 |
| 100 | 54.5982 | 1313.3333 | .04157 | 22.6376 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 5.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0513 | 1.0000 | 1.05127 | .0000 |
| 2 | 1.1052 | 2.0513 | .53877 | .4875 |
| 3 | 1.1618 | 3.1564 | .36808 | .9667 |
| 4 | 1.2214 | 4.3183 | .28284 | 1.4375 |
| 5 | 1.2840 | 5.5397 | .23179 | 1.9001 |
| 6 | 1.3499 | 6.8237 | .19782 | 2.3544 |
| 7 | 1.4191 | 8.1736 | .17362 | 2.8004 |
| 8 | 1.4918 | 9.5926 | .15552 | 3.2382 |
| 9 | 1.5683 | 11.0845 | .14149 | 3.6678 |
| 10 | 1.6487 | 12.6528 | .13031 | 4.0892 |
| 11 | 1.7333 | 14.3015 | .12119 | 4.5025 |
| 12 | 1.8221 | 16.0347 | .11364 | 4.9077 |
| 13 | 1.9155 | 17.8569 | .10727 | 5.3049 |
| 14 | 2.0138 | 19.7724 | .10185 | 5.6941 |
| 15 | 2.1170 | 21.7862 | .09717 | 6.0753 |
| 16 | 2.2255 | 23.9032 | .09311 | 6.4487 |
| 17 | 2.3396 | 26.1287 | .08954 | 6.8143 |
| 18 | 2.4596 | 28.4683 | .08640 | 7.1720 |
| 19 | 2.5857 | 30.9279 | .08360 | 7.5221 |
| 20 | 2.7183 | 33.5137 | .08111 | 7.8646 |
| 21 | 2.8577 | 36.2319 | .07887 | 8.1996 |
| 22 | 3.0042 | 39.0896 | .07685 | 8.5270 |
| 23 | 3.1582 | 42.0938 | .07503 | 8.8471 |
| 24 | 3.3201 | 45.2519 | .07337 | 9.1599 |
| 25 | 3.4903 | 48.5721 | .07186 | 9.4654 |
| 26 | 3.6693 | 52.0624 | .07048 | 9.7638 |
| 27 | 3.8574 | 55.7317 | .06921 | 10.0551 |
| 28 | 4.0552 | 59.5891 | .06805 | 10.3395 |
| 29 | 4.2631 | 63.6443 | .06698 | 10.6170 |
| 30 | 4.4817 | 67.9074 | .06600 | 10.8877 |
| 31 | 4.7115 | 72.3891 | .06509 | 11.1517 |
| 32 | 4.9530 | 77.1006 | .06424 | 11.4091 |
| 33 | 5.2070 | 82.0536 | .06346 | 11.6601 |
| 34 | 5.4739 | 87.2606 | .06273 | 11.9046 |
| 35 | 5.7546 | 92.7346 | .06205 | 12.1429 |
| 40 | 7.3891 | 124.6132 | .05930 | 13.2435 |
| 45 | 9.4877 | 165.5462 | .05731 | 14.2024 |
| 50 | 12.1825 | 218.1052 | .05586 | 15.0329 |
| 55 | 15.6426 | 285.5923 | .05477 | 15.7480 |
| 60 | 20.0855 | 372.2475 | .05396 | 16.3604 |
| 65 | 25.7903 | 483.5149 | .05334 | 16.8822 |
| 70 | 33.1155 | 626.3851 | .05287 | 17.3245 |
| 75 | 42.5211 | 809.8341 | .05251 | 17.6979 |
| 80 | 54.5982 | 1045.3872 | .05223 | 18.0116 |
| 85 | 70.1054 | 1347.8435 | .05201 | 18.2742 |
| 90 | 90.0171 | 1736.2049 | .05185 | 18.4931 |
| 95 | 115.5843 | 2234.8710 | .05172 | 18.6751 |
| 100 | 148.4132 | 2875.1708 | .05162 | 18.8258 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 6.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0618 | 1.0000 | 1.06184 | .0000 |
| 2 | 1.1275 | 2.0618 | .54684 | .4850 |
| 3 | 1.1972 | 3.1893 | .37538 | .9600 |
| 4 | 1.2712 | 4.3866 | .28981 | 1.4251 |
| 5 | 1.3499 | 5.6578 | .23858 | 1.8802 |
| 6 | 1.4333 | 7.0077 | .20454 | 2.3254 |
| 7 | 1.5220 | 8.4410 | .18031 | 2.7607 |
| 8 | 1.6161 | 9.9629 | .16221 | 3.1862 |
| 9 | 1.7160 | 11.5790 | .14820 | 3.6020 |
| 10 | 1.8221 | 13.2950 | .13705 | 4.0080 |
| 11 | 1.9348 | 15.1171 | .12799 | 4.4043 |
| 12 | 2.0544 | 17.0519 | .12048 | 4.7911 |
| 13 | 2.1815 | 19.1064 | .11418 | 5.1684 |
| 14 | 2.3164 | 21.2878 | .10881 | 5.5363 |
| 15 | 2.4596 | 23.6042 | .10420 | 5.8949 |
| 16 | 2.6117 | 26.0638 | .10020 | 6.2442 |
| 17 | 2.7732 | 28.6755 | .09671 | 6.5845 |
| 18 | 2.9447 | 31.4487 | .09363 | 6.9156 |
| 19 | 3.1268 | 34.3934 | .09091 | 7.2379 |
| 20 | 3.3201 | 37.5202 | .08849 | 7.5514 |
| 21 | 3.5254 | 40.8403 | .08632 | 7.8562 |
| 22 | 3.7434 | 44.3657 | .08438 | 8.1525 |
| 23 | 3.9749 | 48.1091 | .08262 | 8.4403 |
| 24 | 4.2207 | 52.0840 | .08104 | 8.7199 |
| 25 | 4.4817 | 56.3047 | .07960 | 8.9912 |
| 26 | 4.7588 | 60.7864 | .07829 | 9.2546 |
| 27 | 5.0531 | 65.5452 | .07709 | 9.5101 |
| 28 | 5.3656 | 70.5983 | .07600 | 9.7578 |
| 29 | 5.6973 | 75.9639 | .07500 | 9.9980 |
| 30 | 6.0496 | 81.6612 | .07408 | 10.2307 |
| 31 | 6.4237 | 87.7109 | .07324 | 10.4560 |
| 32 | 6.8210 | 94.1346 | .07246 | 10.6743 |
| 33 | 7.2427 | 100.9556 | .07174 | 10.8855 |
| 34 | 7.6906 | 108.1983 | .07108 | 11.0899 |
| 35 | 8.1662 | 115.8889 | .07047 | 11.2876 |
| 40 | 11.0232 | 162.0915 | .06801 | 12.1809 |
| 45 | 14.8797 | 224.4584 | .06629 | 12.9295 |
| 50 | 20.0855 | 308.6449 | .06508 | 13.5519 |
| 55 | 27.1126 | 422.2849 | .06420 | 14.0654 |
| 60 | 36.5982 | 575.6828 | .06357 | 14.4862 |
| 65 | 49.4024 | 782.7483 | .06311 | 14.8288 |
| 70 | 66.6863 | 1062.2574 | .06278 | 15.1060 |
| 75 | 90.0171 | 1439.5553 | .06253 | 15.3291 |
| 80 | 121.5104 | 1948.8543 | .06235 | 15.5078 |
| 85 | 164.0219 | 2636.3359 | .06222 | 15.6503 |
| 90 | 221.4064 | 3564.3390 | .06212 | 15.7633 |
| 95 | 298.8674 | 4817.0122 | .06204 | 15.8527 |
| 100 | 403.4288 | 6507.9442 | .06199 | 15.9232 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 7.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0725 | 1.0000 | 1.07251 | .0000 |
| 2 | 1.1503 | 2.0725 | .55502 | .4825 |
| 3 | 1.2337 | 3.2228 | .38280 | .9534 |
| 4 | 1.3231 | 4.4565 | .29690 | 1.4126 |
| 5 | 1.4191 | 5.7796 | .24553 | 1.8603 |
| 6 | 1.5220 | 7.1987 | .21142 | 2.2964 |
| 7 | 1.6323 | 8.7206 | .18718 | 2.7211 |
| 8 | 1.7507 | 10.3529 | .16910 | 3.1344 |
| 9 | 1.8776 | 12.1036 | .15513 | 3.5364 |
| 10 | 2.0138 | 13.9812 | .14403 | 3.9272 |
| 11 | 2.1598 | 15.9950 | .13503 | 4.3069 |
| 12 | 2.3164 | 18.1547 | .12759 | 4.6755 |
| 13 | 2.4843 | 20.4711 | .12136 | 5.0333 |
| 14 | 2.6645 | 22.9554 | .11607 | 5.3804 |
| 15 | 2.8577 | 25.6199 | .11154 | 5.7168 |
| 16 | 3.0649 | 28.4775 | .10762 | 6.0428 |
| 17 | 3.2871 | 31.5424 | .10421 | 6.3585 |
| 18 | 3.5254 | 34.8295 | .10122 | 6.6640 |
| 19 | 3.7810 | 38.3549 | .09858 | 6.9596 |
| 20 | 4.0552 | 42.1359 | .09624 | 7.2453 |
| 21 | 4.3492 | 46.1911 | .09416 | 7.5215 |
| 22 | 4.6646 | 50.5404 | .09229 | 7.7881 |
| 23 | 5.0028 | 55.2050 | .09062 | 8.0456 |
| 24 | 5.3656 | 60.2078 | .08912 | 8.2940 |
| 25 | 5.7546 | 65.5733 | .08776 | 8.5335 |
| 26 | 6.1719 | 71.3279 | .08653 | 8.7643 |
| 27 | 6.6194 | 77.4998 | .08541 | 8.9867 |
| 28 | 7.0993 | 84.1192 | .08440 | 9.2009 |
| 29 | 7.6141 | 91.2185 | .08347 | 9.4070 |
| 30 | 8.1662 | 98.8326 | .08263 | 9.6052 |
| 31 | 8.7583 | 106.9987 | .08185 | 9.7958 |
| 32 | 9.3933 | 115.7570 | .08115 | 9.9790 |
| 33 | 10.0744 | 125.1504 | .08050 | 10.1550 |
| 34 | 10.8049 | 135.2248 | .07990 | 10.3239 |
| 35 | 11.5883 | 146.0297 | .07936 | 10.4860 |
| 40 | 16.4446 | 213.0056 | .07720 | 11.2017 |
| 45 | 23.3361 | 308.0489 | .07575 | 11.7769 |
| 50 | 33.1155 | 442.9218 | .07477 | 12.2347 |
| 55 | 46.9931 | 634.3155 | .07408 | 12.5957 |
| 60 | 66.6863 | 905.9161 | .07361 | 12.8781 |
| 65 | 94.6324 | 1291.3358 | .07328 | 13.0973 |
| 70 | 134.2898 | 1838.2723 | .07305 | 13.2664 |
| 75 | 190.5663 | 2614.4121 | .07289 | 13.3959 |
| 80 | 270.4264 | 3715.8070 | .07278 | 13.4946 |
| 85 | 383.7533 | 5278.7607 | .07270 | 13.5695 |
| 90 | 544.5719 | 7496.6976 | .07264 | 13.6260 |
| 95 | 772.7843 | 10644.0999 | .07260 | 13.6685 |
| 100 | 1096.6332 | 15110.4764 | .07257 | 13.7003 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 8.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0833 | 1.0000 | 1.08329 | .0000 |
| 2 | 1.1735 | 2.0833 | .56330 | .4800 |
| 3 | 1.2712 | 3.2568 | .39034 | .9467 |
| 4 | 1.3771 | 4.5280 | .30413 | 1.4002 |
| 5 | 1.4918 | 5.9052 | .25263 | 1.8404 |
| 6 | 1.6161 | 7.3970 | .21848 | 2.2676 |
| 7 | 1.7507 | 9.0131 | .19424 | 2.6817 |
| 8 | 1.8965 | 10.7637 | .17619 | 3.0829 |
| 9 | 2.0544 | 12.6602 | .16227 | 3.4713 |
| 10 | 2.2255 | 14.7147 | .15125 | 3.8470 |
| 11 | 2.4109 | 16.9402 | .14232 | 4.2102 |
| 12 | 2.6117 | 19.3511 | .13496 | 4.5611 |
| 13 | 2.8292 | 21.9628 | .12882 | 4.8998 |
| 14 | 3.0649 | 24.7920 | .12362 | 5.2265 |
| 15 | 3.3201 | 27.8569 | .11918 | 5.5415 |
| 16 | 3.5966 | 31.1770 | .11536 | 5.8449 |
| 17 | 3.8962 | 34.7736 | .11204 | 6.1369 |
| 18 | 4.2207 | 38.6698 | .10915 | 6.4178 |
| 19 | 4.5722 | 42.8905 | .10660 | 6.6879 |
| 20 | 4.9530 | 47.4627 | .10436 | 6.9473 |
| 21 | 5.3656 | 52.4158 | .10237 | 7.1963 |
| 22 | 5.8124 | 57.7813 | .10059 | 7.4352 |
| 23 | 6.2965 | 63.5938 | .09901 | 7.6642 |
| 24 | 6.8210 | 69.8903 | .09760 | 7.8836 |
| 25 | 7.3891 | 76.7113 | .09632 | 8.0937 |
| 26 | 8.0045 | 84.1003 | .09518 | 8.2947 |
| 27 | 8.6711 | 92.1048 | .09414 | 8.4870 |
| 28 | 9.3933 | 100.7759 | .09321 | 8.6707 |
| 29 | 10.1757 | 110.1693 | .09236 | 8.8461 |
| 30 | 11.0232 | 120.3449 | .09160 | 9.0136 |
| 31 | 11.9413 | 131.3681 | .09090 | 9.1734 |
| 32 | 12.9358 | 143.3094 | .09026 | 9.3257 |
| 33 | 14.0132 | 156.2452 | .08969 | 9.4708 |
| 34 | 15.1803 | 170.2584 | .08916 | 9.6090 |
| 35 | 16.4446 | 185.4387 | .08868 | 9.7405 |
| 40 | 24.5325 | 282.5472 | .08683 | 10.3069 |
| 45 | 36.5982 | 427.4161 | .08563 | 10.7426 |
| 50 | 54.5982 | 643.5351 | .08484 | 11.0738 |
| 55 | 81.4509 | 965.9467 | .08432 | 11.3230 |
| 60 | 121.5104 | 1446.9283 | .08398 | 11.5088 |
| 65 | 181.2722 | 2164.4686 | .08375 | 11.6461 |
| 70 | 270.4264 | 3234.9129 | .08360 | 11.7469 |
| 75 | 403.4288 | 4831.8281 | .08349 | 11.8203 |
| 80 | 601.8450 | 7214.1457 | .08343 | 11.8735 |
| 85 | 897.8473 | 10768.1458 | .08338 | 11.9119 |
| 90 | 1339.4308 | 16070.0911 | .08335 | 11.9394 |
| 95 | 1998.1959 | 23979.6640 | .08333 | 11.9591 |
| 100 | 2980.9580 | 35779.3601 | .08332 | 11.9731 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 9.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.0942 | 1.0000 | 1.09417 | .0000 |
| 2 | 1.1972 | 2.0942 | .57169 | .4775 |
| 3 | 1.3100 | 3.2914 | .39800 | .9401 |
| 4 | 1.4333 | 4.6014 | .31150 | 1.3878 |
| 5 | 1.5683 | 6.0347 | .25988 | 1.8206 |
| 6 | 1.7160 | 7.6030 | .22570 | 2.2388 |
| 7 | 1.8776 | 9.3190 | .20148 | 2.6424 |
| 8 | 2.0544 | 11.1966 | .18349 | 3.0316 |
| 9 | 2.2479 | 13.2510 | .16964 | 3.4065 |
| 10 | 2.4596 | 15.4990 | .15869 | 3.7674 |
| 11 | 2.6912 | 17.9586 | .14986 | 4.1145 |
| 12 | 2.9447 | 20.6498 | .14260 | 4.4479 |
| 13 | 3.2220 | 23.5945 | .13656 | 4.7680 |
| 14 | 3.5254 | 26.8165 | .13146 | 5.0750 |
| 15 | 3.8574 | 30.3419 | .12713 | 5.3691 |
| 16 | 4.2207 | 34.1993 | .12341 | 5.6507 |
| 17 | 4.6182 | 38.4200 | .12020 | 5.9201 |
| 18 | 5.0531 | 43.0382 | .11741 | 6.1776 |
| 19 | 5.5290 | 48.0913 | .11497 | 6.4234 |
| 20 | 6.0496 | 53.6202 | .11282 | 6.6579 |
| 21 | 6.6194 | 59.6699 | .11093 | 6.8815 |
| 22 | 7.2427 | 66.2893 | .10926 | 7.0945 |
| 23 | 7.9248 | 73.5320 | .10777 | 7.2972 |
| 24 | 8.6711 | 81.4568 | .10645 | 7.4900 |
| 25 | 9.4877 | 90.1280 | .10527 | 7.6732 |
| 26 | 10.3812 | 99.6157 | .10421 | 7.8471 |
| 27 | 11.3589 | 109.9969 | .10327 | 8.0122 |
| 28 | 12.4286 | 121.3558 | .10241 | 8.1686 |
| 29 | 13.5991 | 133.7844 | .10165 | 8.3168 |
| 30 | 14.8797 | 147.3835 | .10096 | 8.4572 |
| 31 | 16.2810 | 162.2632 | .10034 | 8.5899 |
| 32 | 17.8143 | 178.5442 | .09978 | 8.7155 |
| 33 | 19.4919 | 196.3585 | .09927 | 8.8340 |
| 34 | 21.3276 | 215.8504 | .09881 | 8.9460 |
| 35 | 23.3361 | 237.1780 | .09839 | 9.0516 |
| 40 | 36.5982 | 378.0038 | .09682 | 9.4950 |
| 45 | 57.3975 | 598.8626 | .09584 | 9.8207 |
| 50 | 90.0171 | 945.2382 | .09523 | 10.0569 |
| 55 | 141.1750 | 1488.4633 | .09485 | 10.2262 |
| 60 | 221.4064 | 2340.4098 | .09460 | 10.3464 |
| 65 | 347.2344 | 3676.5279 | .09445 | 10.4309 |
| 70 | 544.5719 | 5771.9782 | .09435 | 10.4898 |
| 75 | 854.0588 | 9058.2984 | .09428 | 10.5307 |
| 80 | 1339.4308 | 14212.2744 | .09424 | 10.5588 |
| 85 | 2100.6456 | 22295.3179 | .09422 | 10.5781 |
| 90 | 3294.4681 | 34972.0534 | .09420 | 10.5913 |
| 95 | 5166.7544 | 54853.1321 | .09419 | 10.6002 |
| 100 | 8103.0839 | 86032.8702 | .09419 | 10.6063 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 10.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.1052 | 1.0000 | 1.10517 | .0000 |
| 2 | 1.2214 | 2.1052 | .58019 | .4750 |
| 3 | 1.3499 | 3.3266 | .40578 | .9334 |
| 4 | 1.4918 | 4.6764 | .31901 | 1.3754 |
| 5 | 1.6487 | 6.1683 | .26729 | 1.8009 |
| 6 | 1.8221 | 7.8170 | .23310 | 2.2101 |
| 7 | 2.0138 | 9.6391 | .20892 | 2.6033 |
| 8 | 2.2255 | 11.6528 | .19099 | 2.9806 |
| 9 | 2.4596 | 13.8784 | .17723 | 3.3423 |
| 10 | 2.7183 | 16.3380 | .16638 | 3.6886 |
| 11 | 3.0042 | 19.0563 | .15765 | 4.0198 |
| 12 | 3.3201 | 22.0604 | .15050 | 4.3362 |
| 13 | 3.6693 | 25.3806 | .14457 | 4.6381 |
| 14 | 4.0552 | 29.0499 | .13959 | 4.9260 |
| 15 | 4.4817 | 33.1051 | .13538 | 5.2001 |
| 16 | 4.9530 | 37.5867 | .13178 | 5.4608 |
| 17 | 5.4739 | 42.5398 | .12868 | 5.7086 |
| 18 | 6.0496 | 48.0137 | .12600 | 5.9437 |
| 19 | 6.6859 | 54.0634 | .12367 | 6.1667 |
| 20 | 7.3891 | 60.7493 | .12163 | 6.3780 |
| 21 | 8.1662 | 68.1383 | .11985 | 6.5779 |
| 22 | 9.0250 | 76.3045 | .11828 | 6.7669 |
| 23 | 9.9742 | 85.3295 | .11689 | 6.9454 |
| 24 | 11.0232 | 95.3037 | .11566 | 7.1139 |
| 25 | 12.1825 | 106.3269 | .11458 | 7.2727 |
| 26 | 13.4637 | 118.5094 | .11361 | 7.4223 |
| 27 | 14.8797 | 131.9731 | .11275 | 7.5630 |
| 28 | 16.4446 | 146.8528 | .11198 | 7.6954 |
| 29 | 18.1741 | 163.2975 | .11129 | 7.8197 |
| 30 | 20.0855 | 181.4716 | .11068 | 7.9365 |
| 31 | 22.1980 | 201.5572 | .11013 | 8.0459 |
| 32 | 24.5325 | 223.7551 | .10964 | 8.1485 |
| 33 | 27.1126 | 248.2876 | .10920 | 8.2446 |
| 34 | 29.9641 | 275.4003 | .10880 | 8.3345 |
| 35 | 33.1155 | 305.3644 | .10845 | 8.4185 |
| 40 | 54.5982 | 509.6290 | .10713 | 8.7620 |
| 45 | 90.0171 | 846.4044 | .10635 | 9.0028 |
| 50 | 148.4132 | 1401.6532 | .10588 | 9.1691 |
| 55 | 244.6919 | 2317.1038 | .10560 | 9.2826 |
| 60 | 403.4288 | 3826.4266 | .10543 | 9.3592 |
| 65 | 665.1416 | 6314.8791 | .10533 | 9.4105 |
| 70 | 1096.6332 | 10417.6438 | .10527 | 9.4444 |
| 75 | 1808.0424 | 17181.9591 | .10523 | 9.4668 |
| 80 | 2980.9580 | 28334.4297 | .10521 | 9.4815 |
| 85 | 4914.7688 | 46721.7452 | .10519 | 9.4910 |
| 90 | 8103.0839 | 77037.3034 | .10518 | 9.4972 |
| 95 | 13359.7268 | 127019.2091 | .10518 | 9.5012 |
| 100 | 22026.4658 | 209425.4400 | .10518 | 9.5038 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 12.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUNO-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.1275 | 1.0000 | 1.12750 | .0000 |
| 2 | 1.2712 | 2.1275 | .59753 | .4700 |
| 3 | 1.4333 | 3.3987 | .42172 | .9202 |
| 4 | 1.6161 | 4.8321 | .33445 | 1.3506 |
| 5 | 1.8221 | 6.4481 | .28258 | 1.7615 |
| 6 | 2.0544 | 8.2703 | .24841 | 2.1531 |
| 7 | 2.3164 | 10.3247 | .22435 | 2.5257 |
| 8 | 2.6117 | 12.6411 | .20660 | 2.8796 |
| 9 | 2.9447 | 15.2528 | .19306 | 3.2153 |
| 10 | 3.3201 | 18.1974 | .18245 | 3.5332 |
| 11 | 3.7434 | 21.5176 | .17397 | 3.8337 |
| 12 | 4.2207 | 25.2610 | .16708 | 4.1174 |
| 13 | 4.7588 | 29.4817 | .16142 | 4.3848 |
| 14 | 5.3656 | 34.2405 | .15670 | 4.6364 |
| 15 | 6.0496 | 39.6061 | .15275 | 4.8728 |
| 16 | 6.8210 | 45.6557 | .14940 | 5.0946 |
| 17 | 7.6906 | 52.4767 | .14655 | 5.3025 |
| 18 | 8.6711 | 60.1673 | .14412 | 5.4969 |
| 19 | 9.7767 | 68.8384 | .14202 | 5.6785 |
| 20 | 11.0232 | 78.6151 | .14022 | 5.8480 |
| 21 | 12.4286 | 89.6383 | .13865 | 6.0058 |
| 22 | 14.0132 | 102.0669 | .13729 | 6.1527 |
| 23 | 15.7998 | 116.0801 | .13611 | 6.2893 |
| 24 | 17.8143 | 131.8799 | .13508 | 6.4160 |
| 25 | 20.0855 | 149.6942 | .13418 | 6.5334 |
| 26 | 22.6464 | 169.7797 | .13339 | 6.6422 |
| 27 | 25.5337 | 192.4261 | .13269 | 6.7428 |
| 28 | 28.7892 | 217.9598 | .13208 | 6.8357 |
| 29 | 32.4597 | 246.7490 | .13155 | 6.9215 |
| 30 | 36.5982 | 279.2087 | .13108 | 7.0006 |
| 31 | 41.2644 | 315.8070 | .13066 | 7.0734 |
| 32 | 46.5255 | 357.0714 | .13030 | 7.1404 |
| 33 | 52.4573 | 403.5968 | .12997 | 7.2020 |
| 34 | 59.1455 | 456.0542 | .12969 | 7.2586 |
| 35 | 66.6863 | 515.1996 | .12944 | 7.3105 |
| 40 | 121.5104 | 945.2031 | .12855 | 7.5114 |
| 45 | 221.4064 | 1728.7205 | .12808 | 7.6392 |
| 50 | 403.4288 | 3156.3822 | .12781 | 7.7191 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 15.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.1618 | 1.0000 | 1.16183 | .0000 |
| 2 | 1.3499 | 2.1618 | .62440 | .4626 |
| 3 | 1.5683 | 3.5117 | .44660 | .9004 |
| 4 | 1.8221 | 5.0800 | .35868 | 1.3137 |
| 5 | 2.1170 | 6.9021 | .30672 | 1.7029 |
| 6 | 2.4596 | 9.0191 | .27271 | 2.0685 |
| 7 | 2.8577 | 11.4787 | .24895 | 2.4110 |
| 8 | 3.3201 | 14.3364 | .23159 | 2.7311 |
| 9 | 3.8574 | 17.6565 | .21847 | 3.0295 |
| 10 | 4.4817 | 21.5139 | .20832 | 3.3070 |
| 11 | 5.2070 | 25.9956 | .20030 | 3.5645 |
| 12 | 6.0496 | 31.2026 | .19388 | 3.8028 |
| 13 | 7.0287 | 37.2522 | .18868 | 4.0228 |
| 14 | 8.1662 | 44.2809 | .18442 | 4.2255 |
| 15 | 9.4877 | 52.4471 | .18090 | 4.4119 |
| 16 | 11.0232 | 61.9348 | .17798 | 4.5829 |
| 17 | 12.8071 | 72.9580 | .17554 | 4.7394 |
| 18 | 14.8797 | 85.7651 | .17349 | 4.8823 |
| 19 | 17.2878 | 100.6448 | .17177 | 5.0126 |
| 20 | 20.0855 | 117.9326 | .17031 | 5.1312 |
| 21 | 23.3361 | 138.0182 | .16908 | 5.2390 |
| 22 | 27.1126 | 161.3542 | .16803 | 5.3367 |
| 23 | 31.5004 | 188.4669 | .16714 | 5.4251 |
| 24 | 36.5982 | 219.9673 | .16638 | 5.5050 |
| 25 | 42.5211 | 256.5655 | .16573 | 5.5771 |
| 26 | 49.4024 | 299.0866 | .16518 | 5.6420 |
| 27 | 57.3975 | 348.4890 | .16470 | 5.7004 |
| 28 | 66.6863 | 405.8865 | .16430 | 5.7529 |
| 29 | 77.4785 | 472.5728 | .16395 | 5.8000 |
| 30 | 90.0171 | 550.0513 | .16365 | 5.8421 |
| 31 | 104.5850 | 640.0684 | .16340 | 5.8799 |
| 32 | 121.5104 | 744.6534 | .16318 | 5.9136 |
| 33 | 141.1750 | 866.1638 | .16299 | 5.9437 |
| 34 | 164.0219 | 1007.3388 | .16283 | 5.9706 |
| 35 | 190.5663 | 1171.3607 | .16269 | 5.9945 |
| 40 | 403.4288 | 2486.6727 | .16224 | 6.0798 |
| 45 | 854.0588 | 5271.1883 | .16202 | 6.1264 |
| 50 | 1808.0424 | 11166.0078 | .16192 | 6.1515 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 20.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERV FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.2214 | 1.0000 | 1.22140 | .0000 |
| 2 | 1.4918 | 2.2214 | .67157 | .4502 |
| 3 | 1.8221 | 3.7132 | .49071 | .8675 |
| 4 | 2.2255 | 5.5353 | .40206 | 1.2528 |
| 5 | 2.7183 | 7.7609 | .35025 | 1.6068 |
| 6 | 3.3201 | 10.4792 | .31683 | 1.9306 |
| 7 | 4.0552 | 13.7993 | .29387 | 2.2255 |
| 8 | 4.9530 | 17.8545 | .27741 | 2.4929 |
| 9 | 6.0496 | 22.8075 | .26525 | 2.7344 |
| 10 | 7.3891 | 28.8572 | .25606 | 2.9515 |
| 11 | 9.0250 | 36.2462 | .24899 | 3.1459 |
| 12 | 11.0232 | 45.2712 | .24349 | 3.3194 |
| 13 | 13.4637 | 56.2944 | .23917 | 3.4736 |
| 14 | 16.4446 | 69.7581 | .23574 | 3.6102 |
| 15 | 20.0855 | 86.2028 | .23300 | 3.7307 |
| 16 | 24.5325 | 106.2883 | .23081 | 3.8367 |
| 17 | 29.9641 | 130.8209 | .22905 | 3.9297 |
| 18 | 36.5982 | 160.7850 | .22762 | 4.0110 |
| 19 | 44.7012 | 197.3832 | .22647 | 4.0819 |
| 20 | 54.5982 | 242.0844 | .22553 | 4.1435 |
| 21 | 66.6863 | 296.6825 | .22477 | 4.1970 |
| 22 | 81.4509 | 363.3689 | .22415 | 4.2432 |
| 23 | 99.4843 | 444.8197 | .22365 | 4.2831 |
| 24 | 121.5104 | 544.3040 | .22324 | 4.3175 |
| 25 | 148.4132 | 665.8145 | .22290 | 4.3471 |
| 26 | 181.2722 | 814.2276 | .22263 | 4.3724 |
| 27 | 221.4064 | 995.4999 | .22241 | 4.3942 |
| 28 | 270.4264 | 1216.9063 | .22222 | 4.4127 |
| 29 | 330.2996 | 1487.3327 | .22208 | 4.4286 |
| 30 | 403.4288 | 1817.6323 | .22195 | 4.4421 |
| 31 | 492.7490 | 2221.0610 | .22185 | 4.4536 |
| 32 | 601.8450 | 2713.8101 | .22177 | 4.4634 |
| 33 | 735.0952 | 3315.6551 | .22170 | 4.4717 |
| 34 | 897.8473 | 4050.7503 | .22165 | 4.4787 |
| 35 | 1096.6332 | 4948.5976 | .22160 | 4.4847 |
| 40 | 2980.9580 | 13459.4438 | .22148 | 4.5032 |
| 45 | 8103.0839 | 36594.3225 | .22143 | 4.5111 |
| 50 | 22026.4658 | 99481.4427 | .22141 | 4.5144 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 25.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.2840 | 1.0000 | 1.28403 | .0000 |
| 2 | 1.6487 | 2.2840 | .72185 | .4378 |
| 3 | 2.1170 | 3.9327 | .53830 | .8350 |
| 4 | 2.7183 | 6.0497 | .44932 | 1.1929 |
| 5 | 3.4903 | 8.7680 | .39808 | 1.5131 |
| 6 | 4.4817 | 12.2584 | .36560 | 1.7975 |
| 7 | 5.7546 | 16.7401 | .34376 | 2.0486 |
| 8 | 7.3891 | 22.4947 | .32848 | 2.2687 |
| 9 | 9.4877 | 29.8837 | .31749 | 2.4605 |
| 10 | 12.1825 | 39.3715 | .30942 | 2.6266 |
| 11 | 15.6426 | 51.5539 | .30342 | 2.7696 |
| 12 | 20.0855 | 67.1966 | .29891 | 2.8921 |
| 13 | 25.7903 | 87.2821 | .29548 | 2.9964 |
| 14 | 33.1155 | 113.0725 | .29287 | 3.0849 |
| 15 | 42.5211 | 146.1879 | .29087 | 3.1595 |
| 16 | 54.5982 | 188.7090 | .28932 | 3.2223 |
| 17 | 70.1054 | 243.3071 | .28814 | 3.2748 |
| 18 | 90.0171 | 313.4126 | .28722 | 3.3186 |
| 19 | 115.5843 | 403.4297 | .28650 | 3.3550 |
| 20 | 148.4132 | 519.0140 | .28595 | 3.3851 |
| 21 | 190.5663 | 667.4271 | .28552 | 3.4100 |
| 22 | 244.6919 | 857.9934 | .28519 | 3.4305 |
| 23 | 314.1907 | 1102.6853 | .28493 | 3.4474 |
| 24 | 403.4288 | 1416.8760 | .28473 | 3.4612 |
| 25 | 518.0128 | 1820.3048 | .28457 | 3.4725 |
| 26 | 665.1416 | 2338.3176 | .28445 | 3.4817 |
| 27 | 854.0588 | 3003.4592 | .28436 | 3.4892 |
| 28 | 1096.6332 | 3857.5180 | .28428 | 3.4953 |
| 29 | 1408.1048 | 4954.1512 | .28423 | 3.5002 |
| 30 | 1808.0424 | 6362.2560 | .28418 | 3.5042 |
| 31 | 2321.5724 | 8170.2984 | .28415 | 3.5075 |
| 32 | 2980.9580 | 10491.8708 | .28412 | 3.5101 |
| 33 | 3827.6258 | 13472.8288 | .28410 | 3.5122 |
| 34 | 4914.7688 | 17300.4546 | .28408 | 3.5139 |
| 35 | 6310.6881 | 22215.2235 | .28407 | 3.5153 |

COMPOUND INTEREST FACTORS – CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 30.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.3499 | 1.0000 | 1.34986 | .0000 |
| 2 | 1.8221 | 2.3499 | .77542 | .4256 |
| 3 | 2.4596 | 4.1720 | .58955 | .8029 |
| 4 | 3.3201 | 6.6316 | .50065 | 1.1342 |
| 5 | 4.4817 | 9.9517 | .45034 | 1.4222 |
| 6 | 6.0496 | 14.4334 | .41914 | 1.6701 |
| 7 | 8.1662 | 20.4830 | .39868 | 1.8815 |
| 8 | 11.0232 | 28.6492 | .38476 | 2.0601 |
| 9 | 14.8797 | 39.6724 | .37507 | 2.2099 |
| 10 | 20.0855 | 54.5521 | .36819 | 2.3343 |
| 11 | 27.1126 | 74.6376 | .36326 | 2.4370 |
| 12 | 36.5982 | 101.7503 | .35969 | 2.5212 |
| 13 | 49.4024 | 138.3485 | .35709 | 2.5897 |
| 14 | 66.6863 | 187.7510 | .35519 | 2.6452 |
| 15 | 90.0171 | 254.4373 | .35379 | 2.6898 |
| 16 | 121.5104 | 344.4544 | .35276 | 2.7255 |
| 17 | 164.0219 | 465.9649 | .35200 | 2.7540 |
| 18 | 221.4064 | 629.9868 | .35145 | 2.7766 |
| 19 | 298.8674 | 851.3932 | .35103 | 2.7945 |
| 20 | 403.4288 | 1150.2606 | .35073 | 2.8086 |
| 21 | 544.5719 | 1553.6894 | .35050 | 2.8197 |
| 22 | 735.0952 | 2098.2613 | .35034 | 2.8283 |
| 23 | 992.2747 | 2833.3565 | .35021 | 2.8351 |
| 24 | 1339.4308 | 3825.6312 | .35012 | 2.8404 |
| 25 | 1808.0424 | 5165.0619 | .35005 | 2.8445 |
| 26 | 2440.6020 | 6973.1044 | .35000 | 2.8476 |
| 27 | 3294.4681 | 9413.7063 | .34997 | 2.8501 |
| 28 | 4447.0667 | 12708.1744 | .34994 | 2.8520 |
| 29 | 6002.9122 | 17155.2412 | .34992 | 2.8535 |
| 30 | 8103.0839 | 23158.1534 | .34990 | 2.8546 |
| 31 | 10938.0192 | 31261.2373 | .34989 | 2.8555 |
| 32 | 14764.7816 | 42199.2565 | .34988 | 2.8561 |
| 33 | 19930.3704 | 56964.0381 | .34988 | 2.8566 |
| 34 | 26903.1861 | 76894.4085 | .34987 | 2.8570 |
| 35 | 36315.5027 | 103797.5946 | .34987 | 2.8573 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 40.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.4918 | 1.0000 | 1.49182 | .0000 |
| 2 | 2.2255 | 2.4918 | .89314 | .4013 |
| 3 | 3.3201 | 4.7174 | .70381 | .7402 |
| 4 | 4.9530 | 8.0375 | .61624 | 1.0214 |
| 5 | 7.3891 | 12.9905 | .56880 | 1.2507 |
| 6 | 11.0232 | 20.3796 | .54089 | 1.4346 |
| 7 | 16.4446 | 31.4027 | .52367 | 1.5800 |
| 8 | 24.5325 | 47.8474 | .51272 | 1.6933 |
| 9 | 36.5982 | 72.3799 | .50564 | 1.7804 |
| 10 | 54.5982 | 108.9782 | .50100 | 1.8467 |
| 11 | 81.4509 | 163.5763 | .49794 | 1.8965 |
| 12 | 121.5104 | 245.0272 | .49591 | 1.9337 |
| 13 | 181.2722 | 366.5376 | .49455 | 1.9611 |
| 14 | 270.4264 | 547.8098 | .49365 | 1.9813 |
| 15 | 403.4288 | 818.2362 | .49305 | 1.9960 |
| 16 | 601.8450 | 1221.6650 | .49264 | 2.0066 |
| 17 | 897.8473 | 1823.5101 | .49237 | 2.0143 |
| 18 | 1339.4308 | 2721.3574 | .49219 | 2.0198 |
| 19 | 1998.1959 | 4060.7881 | .49207 | 2.0237 |
| 20 | 2980.9580 | 6058.9840 | .49199 | 2.0265 |
| 21 | 4447.0667 | 9039.9420 | .49194 | 2.0285 |
| 22 | 6634.2440 | 13487.0088 | .49190 | 2.0299 |
| 23 | 9897.1291 | 20121.2528 | .49187 | 2.0309 |
| 24 | 14764.7816 | 30018.3818 | .49186 | 2.0316 |
| 25 | 22026.4658 | 44783.1634 | .49185 | 2.0321 |
| 26 | 32859.6257 | 66809.6292 | .49184 | 2.0325 |
| 27 | 49020.8011 | 99669.2549 | .49183 | 2.0327 |
| 28 | 73130.4418 | 148690.0560 | .49183 | 2.0329 |
| 29 | ********** | 221820.4978 | .49183 | 2.0330 |
| 30 | ********** | 330918.2971 | .49183 | 2.0331 |

COMPOUND INTEREST FACTORS - CONTINUOUS COMPOUNDING

NOMINAL INTEREST RATE = 50.00 PERCENT

| N | SINGLE-PAYMENT COMPOUND-AMOUNT FACTOR (F/P) | UNIFORM-SERIES COMPOUND-AMOUNT FACTOR (F/A) | UNIFORM-SERIES CAPITAL-RECOVERY FACTOR (A/P) | GRADIENT SERIES FACTOR (A/G) |
|---|---|---|---|---|
| 1 | 1.6487 | 1.0000 | 1.64872 | .0000 |
| 2 | 2.7183 | 2.6487 | 1.02626 | .3775 |
| 3 | 4.4817 | 5.3670 | .83504 | .6798 |
| 4 | 7.3891 | 9.8487 | .75026 | .9154 |
| 5 | 12.1825 | 17.2377 | .70673 | 1.0944 |
| 6 | 20.0855 | 29.4202 | .68271 | 1.2271 |
| 7 | 33.1155 | 49.5058 | .66892 | 1.3235 |
| 8 | 54.5982 | 82.6212 | .66082 | 1.3922 |
| 9 | 90.0171 | 137.2194 | .65601 | 1.4404 |
| 10 | 148.4132 | 227.2365 | .65312 | 1.4737 |
| 11 | 244.6919 | 375.6497 | .65138 | 1.4964 |
| 12 | 403.4288 | 620.3416 | .65033 | 1.5117 |
| 13 | 665.1416 | 1023.7704 | .64970 | 1.5219 |
| 14 | 1096.6332 | 1688.9120 | .64931 | 1.5287 |
| 15 | 1808.0424 | 2785.5452 | .64908 | 1.5332 |
| 16 | 2980.9580 | 4593.5876 | .64894 | 1.5361 |
| 17 | 4914.7688 | 7574.5456 | .64885 | 1.5380 |
| 18 | 8103.0839 | 12489.3144 | .64880 | 1.5393 |
| 19 | 13359.7268 | 20592.3984 | .64877 | 1.5401 |
| 20 | 22026.4658 | 33952.1252 | .64875 | 1.5406 |
| 21 | 36315.5027 | 55978.5910 | .64874 | 1.5409 |
| 22 | 59874.1417 | 92294.0937 | .64873 | 1.5411 |
| 23 | 98715.7710 | 152168.2354 | .64873 | 1.5413 |
| 24 | ********** | 250884.0064 | .64873 | 1.5413 |
| 25 | ********** | 413638.7978 | .64872 | 1.5414 |

# Appendix D

## Annual versus Continuous Uniform Payment Factors

| NOMINAL INTEREST RATE, % | RATIO OF ANNUAL TO CONTINUOUS UNIFORM PAYMENTS $(A/\bar{A})$ |
|---|---|
| 1 | 1.005017 |
| 2 | 1.010067 |
| 3 | 1.015151 |
| 4 | 1.020269 |
| 5 | 1.025422 |
| 6 | 1.030609 |
| 7 | 1.035831 |
| 8 | 1.041088 |
| 9 | 1.046381 |
| 10 | 1.051709 |
| 11 | 1.057073 |
| 12 | 1.062474 |
| 13 | 1.067911 |
| 14 | 1.073384 |
| 15 | 1.078895 |
| 16 | 1.084443 |
| 17 | 1.090029 |
| 18 | 1.095652 |
| 19 | 1.101314 |
| 20 | 1.107014 |
| 21 | 1.112753 |
| 22 | 1.118531 |
| 23 | 1.124348 |
| 24 | 1.130205 |
| 25 | 1.136102 |
| 26 | 1.142039 |
| 27 | 1.148016 |
| 28 | 1.154035 |
| 29 | 1.160095 |
| 30 | 1.166196 |
| 31 | 1.172339 |
| 32 | 1.178524 |
| 33 | 1.184752 |
| 34 | 1.191022 |
| 35 | 1.197336 |
| 36 | 1.203693 |
| 37 | 1.210094 |
| 38 | 1.216538 |
| 39 | 1.223028 |
| 40 | 1.229562 |
| 41 | 1.236141 |
| 42 | 1.242766 |
| 43 | 1.249436 |
| 44 | 1.256153 |
| 45 | 1.262916 |
| 46 | 1.269726 |
| 47 | 1.276583 |
| 48 | 1.283488 |
| 49 | 1.290441 |
| 50 | 1.297443 |

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