Nenad Radakovic
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Borders in Mathematics Pre-Service Teacher Education

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Foreword

The theme of this book is borders in mathematics pre-service teacher education. The chapter authors have identified curricular, programmatic, and societal borders that not only serve to separate and divide but can also inspire and challenge us when they are crossed. The notion of border crossing reminded me of the National Science Foundation's (NSF) emphasis on *convergence research*, which defined it as collaboration around compelling, vexing problems that address societal needs (https://www.nsf.gov/od/oia/convergence/index.jsp). While NSF's focus was on research, the notion applies equally well to teacher education. Certainly, improving mathematics pre-service teacher education is a vexing problem that addresses the great societal need of improving the teaching and learning of mathematics. Many aspects of this vexing problem are related to the borders that are identified in this book (such as borders between research and practice, borders between mathematics and other disciplines, and borders between different demographic and cultural groups). And solutions are likely to be found in convergence—the coming together of people with different histories, perspectives, experiences, knowledge, and theories. This coming together across borders for sustained collaborations can result in new knowledge, frameworks, and ways of thinking that would not be possible without border crossing. The grand challenges of our time are going to be solved through convergence. And the work of educating future mathematics teachers is all about the grand challenges of our time—contributing to a well-educated and equitable society. When we come together across all sorts of boundaries as described in this book, we can create something new through collaboration, which leads to innovation.

The authors of chapters in this book provide a glimpse into many ways that mathematics teacher educators can cross boundaries and the professional growth and innovation that emerge from this border crossing. I offer a few additional thoughts about ways we might challenge ourselves to cross borders to improve mathematics pre-service teacher education. For example, in my college, graduating students in all teacher education programs regularly tell us that they feel unprepared for classroom management. Rather than assume that this topic is the province of other colleagues in teacher preparation or that future teachers will learn to manage a classroom from their field experiences, we might collaborate with peers in cognitive science to expand our knowledge of classroom management and social and emotional learning, among other topics. This border-crossing collaboration could lead to mathematics educators being more aware of what contemporary research tells us about classroom management and current trends in schools, and we could then strategically integrate this information into our mathematics method courses. This collaboration might also help cognitive science colleagues see particular examples from mathematics education that could be used in their educational foundation courses, which would help our mathematics education students see the value of these classes that are often taken before they reach a mathematics education course. A more radical outcome might be a co-taught course that addresses a range of issues covered in a typical educational foundation course in a manner particular to mathematics education. It is worth having the conversation about what topics might best be covered in what venues and knowing what our colleagues are teaching so that we might reference it or build upon it in our courses. Similar border crossing collaborations might involve our colleagues in special education, Teaching English to Speakers of Other Languages, gifted education, speech pathology, school psychology, or counselor education and those who teach foundational courses often labeled as "diversity courses." Or we might look for places of intersection with our colleagues in literacy, science, social studies, art, music, physical education, and other educator preparation fields. For instance, most teacher educators, regardless of discipline, are preparing future teachers to engage students in the type of learning that leads to conceptual understanding. Although called different things in different fields, we have similar objectives and could benefit from understanding how others approach the task of helping future teachers think differently about teaching and learning.

It is also worth considering the "grain size" at which we cross borders. There are many ways we can cross borders as individuals—reaching out to a colleague, attending a conference (or sessions at conference such as the American Educational Research Association) in another field, or doing some independent research. In other cases, we may be part of a small group of faculty (such as a mathematics education program faculty) actively working to build bridges with colleagues in other areas, such as a mathematics department. We might also consider what it means to cross borders as a field of mathematics teacher education. How are we inviting others to cross into our bordered land? Who comes to our conferences? Who reads our publications? Who do we invite to speak at our conferences? Who do we include when we build professional learning teams, research teams, and instructional teams?

In a similar vein, some instances of border crossing occur by happenstance, while others are more deliberate. What would it mean for us as individuals or groups to be intentional about crossing borders? How do we hold ourselves accountable for reading broadly for our own professional knowledge, providing our students with readings from a diverse array of sources, visiting a colleague's classroom, or inviting our P-12 education partners into conversations about our pre-service teacher education programs? How often do we (as individuals and as a collective) glance across a border and decide not to cross without even giving it a try to see what we might learn or contribute? To be sure, we will cross some borders and decide that it is not fruitful and will decide to stay within our own boundaries. But it is more likely that most border crossings will leave us, and ultimately our students, enriched.

This book offers us—as individuals and as a field—multiple examples of border crossing and the challenges and benefits of doing so. I hope these examples inspire readers to examine borders in which they find themselves and to step outside those borders for the purpose of improving mathematics pre-service teacher education.

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Acknowledgments

This book is about mathematics pre-service teacher education—a topic that both of us are passionate about and have thought a lot about but also want to learn more about.

We came to mathematics pre-service teacher education in different ways.

Limin always knew that she wanted to be an educator. As a young girl, she used a red pen to dutifully "evaluate" the quality and completion of each page in her coloring and puzzle books. Her part-time jobs centered around education, for example, as a learn-to-skate instructor and camp counselor. In secondary school, she determined that her destiny was to become a teacher educator. To a teenaged Limin, it was simple. Being a teacher was the most rewarding job in the world—why not be a context where you are surrounded by individuals who also share this same passion and encourage them to thrive? The focus on mathematics entered later. Limin had had a positive school experience generally enjoying all facets of school life and learning. As time went on, she realized her positive experiences were not universally shared and, in particular, mathematics seemed to stand out as a polarizing and emotionally laden part of school (and in fact life) for many people. And so, Limin decided that mathematics teacher education would provide a way to consider how to remove negative sentiments about, and experiences with, mathematics.

Nenad has been excited about mathematics education since he was 14 when he started tutoring his neighbors in mathematics. After getting a degree in mathematics and becoming a teacher, he started to understand the importance of educating teachers to teach powerful mathematics. His first teaching job was being a secondary school teacher in a community in postwar Croatia that was divided among the ethnic lines. Nenad realized that his teaching job involved addressing issues such as hatred and intolerance and that his job was not "simply to teach mathematics." Later, being an elementary and middle school teacher in Canada gave him further insight into the complex role of a teacher in different cultural contexts. There, Nenad thought about concepts such as teacher autonomy, accountability, and pedagogical and content knowledge. Working in a learning center with students with learning disabilities made Nenad realize the incredible power that teachers have. Leading to positive results when advocating for their students, this power also had the ability to marginalize students based on individual differences. For these reasons, Nenad chose to focus on mathematics pre-service teacher education.

Now, in our roles as teacher educators and researchers, we are so fortunate to work with mathematics pre-service teachers from whom we gain so much meaning, knowledge, and fulfillment. We see pre-service teachers as the future of mathematics education. We are inspired by their energy, enthusiasm, dedication, passion, optimism, and willingness to learn. This book is dedicated to them.

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Introduction

Borders are ubiquitous in our world(s). They can be used to define, classify, organize, make sense of, and control. They can also be used to unite or divide. There are many ways that the concept of a border illuminates the field of mathematics education, more specifically mathematics pre-service teacher education. As a consequence, researchers and practitioners also react to the borders in many ways. Within the field of mathematics education, borders are explored in many different contexts, such as exploring mathematics across topics (e.g., geometry, algebra, and probability) and blurring the boundaries between them (e.g., Jagger, 2018; Radakovic & McDougall, 2012); exploring borders between mathematics and other disciplines such as science, the arts, and social studies (e.g., Austin, Thompson, & Beckmann, 2005; Gerofsky, 2013; Lesser, 2014); and challenging gender, cultural, and racial borders (e.g., Esmonde, 2011; Larnell, 2016).

The concept of borders emerges from educational theory and research, and we see it as a powerful way to examine current trends in mathematics pre-service teacher education. Borders are often discussed in relation to wider mathematics education research. Yet, there is a need for additional investigation, particularly in the context of pre-service teacher education, given the impact that teacher education programs have on PSTs' future practice. This edited collection was born out of this need.

As we embarked on this project, we identified some possible borders that we see in our work, have experienced, and recognize from the existing literature. In addition to inviting scholars at various stages of their career, we specifically reached out to individuals whose work delves (previously or currently) into the areas related to borders. In our invitations, we outlined the borders that we had identified; however, we encouraged contributors to reflect on what borders mean in their work and to identify and explore borders beyond our suggestions in a way that was meaningful and authentic to them. As such, although we had some ideas about the organizing principle for the book, the final structure emerged from the contributions.

This book consists of four parts that offer varied perspectives on borders in mathematics pre-service teacher education. In the remainder of this chapter, we outline each part. We use these four parts as an organizational tool to guide the reader through the themes identified by us, as editors, and by the authors. These parts are not mutually exclusive. We acknowledge that by delineating parts, we, as editors, have in fact created borders in our book. Thus, we encourage readers to imagine/challenge our "borders" and consider how the chapters may be extended beyond them.

Part I: Opening: "En La Lucha/In the Struggle for Mathematics Teacher Education Without Borders"

Sandra Crespo's chapter was originally presented as a keynote address at the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA), one of the largest conferences for mathematics education researchers in North America. The chapter serves as an opening to this book because it outlines some borders in mathematics education (including mathematics pre-service teacher education) while optimistically hoping that these borders will cease to exist. Specifically, Crespo reflects on three intellectual divides that drive her research and practice, namely, mathematics/mathematics education, expert/novice, and research/teaching divides. Here in "En La Lucha/In the Struggle for Mathematics Teacher Education Without Borders," Crespo argues that these divides are widespread and far more dangerous than are acknowledged in our field of mathematics education. She suggests a reflexive and collaborative approach to identifying and problematizing the intellectual divides in order to make a difference in communities that mathematics educators serve.

Furthermore, we selected this chapter to open our book as it is an exemplar of the complexity and contradictions of the current historical, political, and social context in which mathematics education takes place. For context, the 38th PME-NA conference took place in Tucson in November 2016, a couple of days before Donald Trump won the presidential election running on a platform that was largely about drawing borders between peoples and nations. The official title of the conference was "Sin Fronteras: Questioning Borders With(in) Mathematics Education," and both the spirit of the conference and organizational principles around choosing the keynote speakers focused on challenging, describing, transforming, and erasing borders (Wood, Turner, & Civil, 2016). Consistent with the conference's theme, presenters were invited to present in Spanish, and Spanish/English interpretation services were available. The intention and hope was that this practice would continue at future conferences.[1]

[1] Although conference papers are accepted in all of the three major languages in North America (English, French, and Spanish), to our knowledge, there have not been presentations or interpretation/translation in French, an official language in Canada, at any of the PME-NA conferences. This includes instances when the conference took place in Canada (e.g., 2004 in Toronto, ON, and 2014 in Vancouver, BC).

Part II: Curricular Borders in Mathematics Pre-service Teacher Education

This part explores curricular borders that exist in mathematics pre-service teacher education. Specifically, we see that these borders can be classified in two ways: (1) within mathematics (e.g., geometry, number sense, and probability) and (2) between mathematics and other subjects (e.g., science, social studies, and the arts).

In schools, mathematics is often presented as a collection of disconnected themes and topics. Mathematics learners often experience mathematics content as a series of disconnected facts (Boaler, 2015). For example, students may have procedural fluency to perform symbolic calculations such as $3^2 + 4^2$. Yet, they are not able to see how this calculation relates to the combined area of two squares. Similarly, Skemp (2006) describes how students may see and be taught about the areas of various figures (e.g., triangles, rectangles, and parallelograms) as disconnected facts without seeing an underlying connection, namely, relating each area to the area of the rectangle. These occurrences signal that the students have compartmentalized information about mathematical procedures without conceptual understanding (Lai, Kinnear, & Fung, 2017). Similarly, research indicates that when students are unable to make connections across mathematical representations, this may also indicate that students lack a depth of mathematical understanding (Pape & Tchoshanov, 2001). It is not solely children (as mathematics learners and doers) who have these problems; pre-service teachers also face similar challenges (Livy, Muir, & Maher, 2012; Stohlmann, Moore, Cramer, & Maiorca, 2015; Tirosh, 2000).

"Continuous Directed Scaling: How Could Dynamic Multiplication and Division Diagrams Be Used to Cross Mathematical Borders?" delves deeper into one approach to disrupt borders within mathematics. Specifically, Justin K. Dimmel and Eric A. Pandiscio challenge typical representations of multiplication and division as presented in US curriculum standards. This is done through bridging analytic geometry concepts with the concepts related to numbers and operations.

Borders between mathematics and other subjects have also existed and, more recently, have been an area of growth and attention. First, the need for interdisciplinary fields has led to the development of science, technology, engineering, and mathematics (STEM) education (Breiner, Harkness, Johnson, & Koehler, 2012; Mohr-Schroeder, Cavalcanti, & Blyman, 2015). Additionally, an increased importance of interdisciplinary education has prompted the development of STEAM (A for the arts) education. From one perspective, arts educators have reacted to the prioritizing of educational initiatives and policies to incorporate STEM into the curriculum by advocating for STEAM (Bequette & Bequette, 2012; Guyotte, Sochacka, Costantino, Kellam, & Walther, 2015; Land, 2013). From another perspective, STEM educators (including mathematics educators) see STEAM as a way to more authentically explore curriculum (Madden et al., 2013; Radziwill, Benton, & Moellers, 2015; Yakman & Lee, 2012). Furthermore, the urgency of dealing with the need for quantitative literacy has influenced some jurisdictions (e.g., Australia) to consider numeracy across the curriculum

(Geiger, Forgasz, & Goos, 2015; Goos, Dole, & Geiger, 2012). Similarly, societal crises, such as climate change, require transdisciplinary solutions that connect and transcend disciplines. Thus, schools need to create curricula and educational opportunities for students to explore connections between disciplines including the connection between mathematics and other subjects (Peterson, 2013). Given the importance of interdisciplinary teaching and learning, teacher education programs have the responsibility of ensuring that pre-service teachers are prepared to incorporate these approaches into their future practice. The remaining three chapters in this part explore relationships between mathematics and other subjects.

In "Crossing Disciplinary Borders in Pre-service Teacher Education: Historical Consciousness as a Tool to Develop Awareness of Mathematical Positionality to Achieve Epistemic Change," Marta Kobiela and Paul Zanazanian present an approach to connect mathematics and history. Their chapter describes how pre-service teachers can engage with historical cases to make sense of mathematics and themselves as mathematics learners and teachers.

Next, Jennifer Hall and Helen Forgasz present a case of when mathematics is introduced across the curriculum. "Secondary Pre-service Teachers' Experiences in a Numeracy Course" describes how numeracy is interwoven throughout the teacher education program at Monash University for *all* pre-service teachers to reinforce the importance of numeracy across grade levels and subject areas. Indeed, there are many ways to integrate mathematics with other disciplines.

Finally, in "Mathematics Crossing Borders: A Comparative Analysis of Models for Integrating Mathematics with Other Disciplines in Pre-service Teacher Education," Merrilyn Goos shares some models of integration that she has used or observed in pre-service teacher education courses. These include mathematical modeling, curriculum integration, and numeracy across the curriculum approaches.

In summary, authors in this part advocate for and/or present ways of addressing curricular borders in mathematics pre-service teacher education. They reinforce an urgent need for mathematics to be meaningfully integrated in teacher education programs in order to ensure that pre-service teachers see relationships and connections within mathematics and a crucial component of our complex/connected reality.

Part III: Programmatic Borders in Mathematics Pre-service Teacher Education

There exists a wide variety of structures and formats of teacher education programs as there is no universal understanding and/or agreement as to the best way to train pre-service teachers. In addition, contextual variables and constraints (e.g., location, resources, and institutional requirements) result in specific programmatic decisions and further contribute to the variance in programs from institution to institution. The chapters in this part explore programmatic borders that exist in mathematics pre-service teacher education.

Decisions made about the structure of teacher education programs create borders. For example, many programs for mathematics pre-service teachers are organized into coursework focused on content knowledge or teaching methods and *practica*. There is also a division created between teachers specializing to teach various populations such as emergent bilinguals, students with disabilities, etc. The first two chapters in this part speak to the borders that are created by the structure of teacher education programs and how these borders may hinder the development of pre-service teachers. The chapters also suggest ways to respond to these borders.

In "Bridging the Gap Between Coursework and Practica: Secondary Mathematics Pre-service Teachers' Perceptions About Their Teacher Education Program," Limin Jao, Nakita Rao, and Alexandra Stewart give voice to secondary mathematics pre-service teachers in order to understand their experiences with various components of their program (i.e., coursework and practica). Their experiences provide insight into the nature of the "bordered" reality of their program and possible ways for teacher education program stakeholders to challenge these borders and create better connections between the components.

Next, Annette Bagger and Helena Roos explore challenges of collaborations occurring between special education teachers in mathematics and mathematics teachers for primary school in the Swedish context. Here in "The Shared Duty of Special Educational Support in Mathematics: Borders and Spaces in Degree Ordinances for Pre-service Teachers," the authors explore the origin of these challenges and suggest how existing programmatic borders could be addressed in order to encourage more fruitful collaboration.

While structural borders in teacher education may have a negative impact on pre-service teachers' development, their prior experiences with mathematics also may be a hindrance. Authentic mathematics education presents mathematics as a powerful, creative, and social activity (Yeh, Ellis, & Koehn Hurtado, 2017), yet many pre-service teachers have not had personal experience with this kind of mathematics. Rather, pre-service teachers typically have only experienced mathematics through mastery of a set of procedures (Bekdemir, 2010; Lewis, 2014). Thus, it is important for teacher education programs to provide opportunities for pre-service teachers to experience learning mathematics in a different way. Yet, merely experiencing mathematics in new ways is not enough to support pre-service teachers' development. Pre-service teachers need to reflect on these experiences and make sense of how their new personal experiences can inform their future practice. For example, pre-service teachers can consider ways in which they found their new experiences to be challenging as mathematics learners and (as future teachers) in which they could mitigate their challenges for their future students.

The next two chapters highlight the value of pre-service teachers experiencing and reflecting upon these experiences as mathematics learners. First, in "Blurring the Border Between Teacher Education and School Classrooms: A Practical

Testing Activity for Both Contexts," Tina Rapke, Marc Husband, and Heather Bourrie present a testing activity. In this activity, pre-service teachers take turns developing test questions and take tests created by their peers. The participating pre-service teachers' reflections of their experiences provide insight into the implications of such an activity in both teacher and K–12 education contexts.

"Teaching the Hungarian Mathematics Pedagogy to American Pre-service Teachers" describes the approach of the Budapest Semesters in Mathematics Education program that has students experiencing common teaching approaches and activities used in the Hungarian mathematics education. Péter Juhász, Anna Kiss, Ryota Matsuura, and Réka Szász share how, in the program, pre-service teachers experience Hungarian mathematics as both learners and educators as they solve and design rich mathematical tasks. The program also challenges cultural and geographic borders, as participants are typically pre-service teachers from the United States.

Teacher education programs provide an opportunity for pre-service teachers to join and/or be enculturated into a community of educators. Typically, pre-service teachers form a community with their peers through face-to-face interactions in classes/coursework and through shared experiences throughout their program. Yet, communities may also be organically formed in other spaces. For example, a Facebook group is created by a group of pre-service teachers to make sense of a class assignment, or social media pages centered around prominent members of the mathematics or mathematics education community (e.g., Francesco Daddi, Dan Meyer, Marian Small, and Eddie Woo) or educational trends (e.g., educational technology, problem-based learning, and STEAM) could inspire the creation of a community. These are spaces where pre-service teachers gather to make sense of their practice and distinct from the teacher education programs. Therefore, there is a border between traditional spaces and informal spaces often led by pre-service teachers.

In order to understand their affordances and challenges that they create, the mathematics teacher education community should make an effort to understand informal online spaces. In "Mathematics Education Communities: Crossing Virtual Boundaries," Josh T. Hertel, Nicole M. Wessman-Enzinger, and Justin K. Dimmel present a framework for understanding virtual mathematics education communities based on the theoretical work of communities of practice and boundary crossings. The authors test their framework on several North American communities in order to understand them and the boundaries associated with them and outline ways that understanding virtual mathematics communities may benefit mathematics pre-service teachers and their practice.

The authors in this part address borders that exist within teacher education programs such as structural borders, borders created by pre-service teachers' traditional mathematics learning experiences, and borders between traditional and emerging spaces for the development of professional communities. Their chapters may serve as inspiration for mathematics pre-service teacher education stakeholders to revisit and/or reimagine the structure of their programs.

Part IV: Societal Borders in Mathematics Pre-service Teacher Education

Like any type of education, mathematics pre-service teacher education has the tendency/capacity to replicate the same inequities and societal divisions/borders that exist in the wider society such as sexism and racism (Dubbs, 2016; Martin, 2009). Mathematics and mathematics education are gendered, heteronormative, and Eurocentric spaces (Greer & Mukhopadhyay, 2012). Yet, mathematics classrooms are typically seen as objective and value-neutral, and mathematics is often thought of as a "universal language" (D'Ambrosio & D'Ambrosio, 1994). Traditionally, bringing in different identities, voices, and perspectives into these spaces has not been allowed. This is why teachers and teacher educators often avoid topics and content that include different identities, voices, and perspectives (Simic-Muller, Fernandes, & Felton-Koestler, 2015).

The chapters in this part suggest ways of challenging borders by bringing forth the identities in different ways. "Queering Mathematics: Disrupting Binary Oppositions in Mathematics Pre-service Teacher Education" challenges heteronormative and gendered space by suggesting how to queer the space. Indeed, queering is a way to erase the border between different identities by including these identities in the conversations. In this chapter, Cathery Yeh and Laurie Rubel draw on existing theories to recognize and challenge borders around gender, sexuality, and other identity categories in mathematics teacher education. The authors suggest ways of working toward the opportunities to blur and queer the borders and to challenge heterosexism and genderism.

In the second chapter in the part, Mahtab Nazemi brings forth student racial identities and suggests implications for this in pre-service teacher education. "Persisting Racialized Discourses Pose New Equity Demands for Teacher Education" explores the ways in which classroom teachers can support students to negotiate and navigate their racial identities while learning mathematics. Here, Nazemi uses a case of an Advanced Placement Statistics classroom to underline the importance of teachers reflecting upon their students' identities in relation to their own identities. Furthermore, she asserts that teacher educators are responsible for encouraging pre-service teachers to develop this awareness/disposition even before having a mathematics classroom of their own. The scope of "Queering Mathematics: Disrupting Binary Oppositions in Mathematics Pre-service Teacher Education" and "Persisting Racialized Discourses Pose New Equity Demands for Teacher Education" goes beyond pre-service teacher education by considering mathematics education more generally; however, both describe the critical role that pre-service teacher education has for shaping the future of mathematics education.

Another voice that has yet to be adequately represented in the field of mathematics pre-service teacher education is that of the immigrant voice. mutindi ndunda and Nenad Radakovic, two immigrant teacher educators, frame the issue of standardization from their own point of view by respecting their own experiences that often get erased in environments that tend to be US-centric and often xenophobic. Their chapter "Standardization and Borders in Mathematics Pre-service Teacher Education: A Duoethnographic Exploration" presents the authors' take on duoethnographic methodology in order to make sense of their teaching practice as immigrant professors in the American South. ndunda and Radakovic use a political border metaphor to understand possible responses to borders created by standardization.

The authors in Part IV make the issues of identity salient, thus revealing the borders that are created in Eurocentric and heteronormative spaces. Their chapters provide a blueprint for including voices in mathematics pre-service teacher education that have historically been silenced.

In conclusion, we hope that this book provides a space for much needed reflection on borders and hope to motivate the community to consider the role that borders play in mathematics pre-service teacher education and their own research and practice. It is important to note that this book serves to provoke questions, and we understand that there are many perspectives, and this is just one. This book is not meant to provide (all of the) answers. Although we have tried to offer diverse perspectives and approaches to challenging, blurring, erasing, and addressing borders, we acknowledge that the book presents only a subsection of the vast and complex concept of borders in mathematics pre-service teacher education. In a way, we see this is a conversation, and we want our community to be part of the conversation. We hope that this book inspires our community/colleagues to continue to engage in the concept/discussion about borders. We are reminded of Argentinian writer Jorge Luis Borges' observation that a book changes every time it is read and that the book gives birth to infinite dialogues between the book and its readers (Borges, 1966). Our hope is that our book will have many lives and that the dialogues and conversations will motivate mathematics education researchers to further understand existing borders and their role in our field.

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Part I
Opening: "En La Lucha/In the Struggle for Mathematics Teacher Education Without Borders"

En La Lucha/in the Struggle for Mathematics Teacher Education Without Borders

Sandra Crespo

1 Introduction

The conference theme of the PME-NA 2016 *Sin Fronteras!—Without Borders* gave me the opportunity to reflect on the main struggles I have faced as a mathematics educator seeking to make a difference within our field of mathematics education. As someone who is constantly crossing geographical, cultural, linguistic, and intellectual borders, I am sensitive to how cultural norms and ideologies are used to justify membership and exclusions of people. This is reflected in my work as a mathematics educator who strives to contribute to improving mathematics education in ways that align with the goals and values of democratic and anti-oppressive education. I am especially interested in learning and teaching practices that redistribute power and challenge stereotypes and hierarchies in the mathematics classroom, and this has pushed me to see social interactions from multiple perspectives and theoretical lenses. I approach my work in collaboration with colleagues, schools, and teachers committed to social change. I do this work by crossing national boundaries across the Dominican Republic, Canada, and the USA by straddling the worlds of elementary/secondary education, of formal/informal mathematics, of theory/practice, and of equity/excellence debates and debacles. More importantly, I have learned to embrace the tension and burden of working within and across these many communities and boundaries.

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While it is true that my work crosses boundaries, this is not unique to my scholarship. I would argue that we are all in some way or another navigating multiple personal and professional communities that require us to negotiate interactions that challenge us and that nurture us. Therefore, I am not claiming that I have something unique to share or to stake claim to a piece of intellectual property that is solely my own. To the contrary, the work I have done over the past 20+ years as a mathematics educator has been possible because it has taken a whole village of collaborators who have helped me to keep front and center my commitment to anti-oppressive education and to remain hopeful that as math educators we can make a difference. My approach here is to reflect on the kinds of boundaries I have had to cross throughout my career to make visible intellectual divides that I consider dangerous and worthy of bridging and eventually take down.

I use "in the struggle/en la lucha" in the title of this chapter to remind myself of Paulo Freire's (1970) pedagogy of hope in which he discusses our struggle as educators to work within the system that oppresses us and that we seek to change. I am also channeling bell hooks' (1994) idea of teaching to transgress, where she calls on educators to find new ways of thinking about teaching and about learning so that our work "does not reinforce systems of domination, imperialism, racism, sexism, elitism" (p. xx). It is in that spirit that I use this opportunity to reflect on my own work and how it has been challenged by pernicious intellectual divides that create adversarial relationship and unwarranted hierarchies in our field and among ourselves.

2 Fronteras Intelectuales and Dangerous Divides

One way to understand intellectual divides is through the ideology of the two cultures, a phenomenon that was discussed by a prominent scholar in the middle of last century. Snow (1959) spoke as a participant in both literary and scientific communities about the deep-rooted divide between two fields—the literary intellectuals and the scientists—and how each exalted its own virtues by vilifying the other's values. He described them as two polar groups: the literary intellectuals at one pole and at the other the scientists—"Between the two a gulf of mutual incomprehension. They have a curious distorted image of each other" (Snow, 1959, p. 4). Snow's characterization highlighted that the literary intellectuals value nuance, subtlety, depth, responsiveness, and imagination, whereas scientists will talk about those qualities as touchy-feely and fuzzy-minded subjectivism. Similarly, the scientists value rationality, objectivity, and functional prose, while literary scholars consider those qualities dull, literal minded, and lacking depth of understanding.

A similar analysis is offered in "Disciplinary Cultures and Tribal Warfare," a chapter in the book *Scandalous Knowledge* by Herrstein Smith (2006), who also explains the dangers of creating intellectual camps and hierarchies. She revisits C. P. Snow's two cultures, adding that the tendency to polarize, compare, and rank ourselves is part of what all social groups do, including academics and intellectuals.

In academic circles, this is known as the ideology of the two cultures and refers to our tendency to identify ourselves with one or more social groups (e.g., religious, ethnic, political, professional), to experience that identity through contrast and comparison to one or more other groups—or, in other words, to experience the world in terms of "us" and "them." This is known as a tendency to self-standardize and other-pathologize, said another way "to see the practices, preferences and beliefs of one's own group as natural, sensible and mature and to see the divergent practices, preferences and beliefs of members of other groups, especially those considered as the 'other,' as absurd, perverse, undeveloped or degenerate" (Herrstein Smith, 2006; p. 113). Another consideration is that this tendency to pathologize the other is self-perpetuating, in that these are invoked and circulated as ideological narratives within and across various communities.

In mathematics education, there are numerous intellectual divides to choose from (see Davis, 2004; Davis, Sumara, & Luce-Kapler, 2014; Stinson & Bullock, 2012). In the 1980s, the quantitative/qualitative debate took center stage as did the constructivism vs. social theories of learning. The 1990s witnessed the cognition vs. communication, and acquisition vs. participation debates (Sfard, 1998), while the 2000s experienced the sociocultural vs. sociopolitical divide (Gutiérrez, 2013). These debates have been played out in the intellectual domain and among academics and eventually have slipped into the everyday conversations of schools and universities as ideological narratives that cast polar opposite characters (reform vs. traditional) battling out intellectual wars. Although these debates have faded, they still frame current conversations and practices in mathematics education. Furthermore, they fall into the dualistic intellectual tradition that Snow (1959) characterized as the ideology of the two cultures and that Herrstein Smith (2006) describes in her writings as the tendency to self-standardize and other-pathologize.

I focus here on three enduring divides that have not had as much play as those named above but are ever present in our everyday practices as mathematics educators and fuel an "us vs. them" mentality as described in the ideology of the two cultures. These are (a) Mathematics/Education, (b) Expert/Novice, and (c) Research/Teaching. I contend that these divides may seem innocuous but are nevertheless more dangerous than they appear to be. As I look back at my work with these three divides in mind, I can see how these have been and still continue to be a challenge in my own scholarship but also to our field more broadly. Looking more specifically at my published work within the PME and PME-NA proceedings, I can see all three divides in each of those articles. When considering which divide was most foregrounded, the following groupings emerged—seven articles foregrounding (a) [the mathematics/education divide], five of them foregrounding (b) [the expert/novice divide], and six articles foregrounding (c) [the research/teaching divide]. Rather than synthesizing the three groupings, I use one representative article to springboard the discussion on each intellectual divide. I purposefully picked articles that are 6–7 years apart, so that they represent broadly the scholarship that I have been engaged in over the past 20 years.

2.1 The Mathematics/Education Divide

Looking back at my very first PME-NA presentation and paper, I can see a clear pushback to the mathematics/education divide by prominently highlighting and questioning the separation between where and how teacher candidates can learn mathematics in their teacher preparation programs. I experienced this divide in my own undergraduate education as I traveled from one side of campus, where I was studying mathematics and physics to the other side of campus where I was taking education classes. This structural divide continues to persist and is very present in my own practice as a mathematics teacher educator. The very structure of teacher preparation programs in general continues to reaffirm the mathematics/education divide by locating the learning of mathematics content in designated math courses and separating it from the learning of teaching methods contained in education courses. Embedded within the structure is the assumption that learning to teach entails learning the content first and the teaching methods second (rather than concurrently).

In *Learning mathematics while learning to teach: Mathematical insights prospective teachers experience when working with students* (Crespo, 2000), I argued that prospective teachers engage in mathematical inquiry within their education courses and in particular when working directly with students. I provided three examples—posing tasks, analyzing students' work, and providing mathematical explanations—where teacher candidates could gain mathematical insights while learning educational methods and theories. This surely is no longer a controversial point, but at the time mathematics educators were just beginning to consider Ma's (1999) and Ball and Bass' (2000) work describing the profound understanding of mathematics entailed in the work of elementary mathematics teaching. The pushback from mathematics educators who dug their feet firmly into the mathematics side of the divide was intense, making anything that they did not recognize as mathematical sound crazy or simply stupid. Therefore, the process of selecting examples that were recognizable as mathematical by those holding dominant perspectives about mathematics was a challenge but key to navigating this divide.

Let me provide a few illustrations. In Crespo (2000), I included several examples to illustrate the ways in which mathematical questions and insights arise when prospective teachers work on teacher preparation course projects that have them exploring mathematics with students. In one example, I shared how, when interviewing a second grader about her strategies for sharing cookies among different number of people, a prospective teacher found her student conjecturing that if the number of cookies was even, it could be shared evenly among people, and that if the number of cookies was odd, it could not. The young student concluded this after having shared several even numbers of cookies, such as sharing 30 cookies among 3 and then 5 people. In this situation, the prospective teacher found herself in a position of exploring this student's conjecture by offering her several more examples to have the student test her conjecture and see whether or not it does or does not work for other cases.

In another example, a prospective teacher had adapted a mathematics problem (Watson, 1988) we had explored in our university class to try it out with fifth graders in her field placement. This problem read:

> Three tired and hungry monsters went to sleep with a bag of cookies. One monster woke up and ate *1/3* of the cookies, then went back to sleep. Later a second monster woke up and ate *1/3* of the remaining cookies, then went back to sleep. Finally, the third monster woke up and ate *1/3* of the remaining cookies. When she was finished there were 8 cookies left. How many cookies were in the bag originally?

The prospective teacher chose to rescale the problem by changing the fractional number in the problem from *1/3* to 1/2. By doing so, she made an interesting discovery, that is, that her students were able to arrive at the correct answer by using a restrictive solution method that in fact does not work for the original version of the problem. Students had approached the problem by multiplying the leftover cookies by 2 (8×2×2×2), basically doubling the leftover cookies three times. Yet, even though this method works for halves, it yields an incorrect answer for thirds, fourths, and any other fractional part. This unexpected outcome launched the prospective teacher into her own mathematical investigation into the reasons for how and why such a minor numerical change could alter the nature of the original problem (Crespo, 2000).

I have made similar and related arguments about mathematics as a practice that occurs and is learned everywhere not solely inside mathematics classrooms and most definitely not solely in coursework offered in mathematics departments. I recognize the history of why and how disciplinary knowledge broke off and was elevated from the everyday knowledge and practices and the privileges that this affords to those of us in the field of mathematics education. However, to me, mathematics is a human practice that belongs to all of us not solely to mathematicians (Bishop, 1990). Hence throughout my career, I have argued that it is especially important for prospective teachers to consider their teaching as a site for mathematical inquiry and for problem posing with their students and to find ways to explore the mathematics that students learn in their communities and in out-of-school contexts. I have continued to address the mathematics/education divide in multiple ways and especially as I have increasingly foregrounded educational equity within the curriculum and pedagogy of the mathematics education courses for future elementary and secondary mathematics teachers. If concerns and pushback about "where is the mathematics?" or "how is this mathematics?" were raised with regard to learning mathematics through learning mathematics pedagogy, the pushback to infusing educational equity in the teaching of mathematics has been even that more forceful.

The divide between mathematics and education continues to be reflected in the intellectual but also in the physical divide found on most university campuses. This divide contributes to the lack of coherence and continuity in the curriculum and pedagogy of teacher preparation (Feiman-Nemser, 2001). Mathematics courses are offered in mathematics departments, taught by instructors who do not address questions that concern educators. Education courses in turn are offered in colleges of education and are typically focused on educational issues without attending to specific content issues. The mathematics methods course is also influenced by this divide. Instructors of these courses often assume that teacher candidates have to "unlearn" oppressive approaches to the teaching and learning of mathematics that they have picked up in the math courses they have taken. The rift between mathematics educators who work in colleges of natural science and mathematics educators who work in colleges of education is very palpable at my current institution and I suspect across many other institutions as well.

As a mathematics educator who has colleagues in the college of natural science and in the college of education, I am constantly challenged by both sides to see their perspective while neither side seems to see their own biases and entrenched ideologies. One side asks and insists on raising the question of "where is the math" whenever the conversation is focused on educational issues that transcend the narrow particulars of the discipline of mathematics as constructed and practiced by research mathematicians. I constantly hear the "where is the math" question raised in faculty meetings, in students' comprehensive exams, in dissertations, and in colloquia. My education colleagues, on the other hand, ask and insist on raising questions about whether mathematics as a discipline can be trusted to embrace democratic ideals when so much of what is wrong and objectionable about today's public schooling can be attributed to the way mathematics is used to exclude and deny access to college to a large majority of non-White students, not to mention the oppressive ways in which mathematics continues to be taught and learned in schools.

To be clear, I consider the mathematics/education divide as dangerous because it shapes interactions among ourselves with colleagues on our campuses and members of various other communities. It instantiates the tendency to self-standardize and other-pathologize discussed earlier. It forcefully comes into play when faculty is engaged in doctoral admissions or discussing prospective colleagues who have or do not have a so-called "strong" mathematical background or do not have a so-called "substantial" classroom teaching experience. With each side digging their heels more deeply into their own camp, they continue to reproduce their perspectives and pathologize the other. The danger lies in how this divide breeds toxic and deficit discourses within our own academic communities which, not surprisingly, is expressed outwardly through our research onto the very communities we are hoping to help (Shields, Bishop, & Mazawi, 2005). This intellectual divide becomes normalized and replicated in our teacher preparation programs and travels to our partner schools. It undermines our goals to make mathematics a subject that many and more diverse groups of students engage with and enjoy, and a subject that supports the democratic values and ideals of public education. Not challenging this divide propagates the ideology that one field of study is more important than the other. It generates categories of students that are liberally applied to elementary prospective teachers and breeds the dominant narrative about elementary teacher candidates' "lack of knowledge" of mathematics. This issue speaks to the next divide—the expert/novice divide—which I discuss next.

2.2 The Expert/Novice Divide

Another divide always present in mathematics education is the categorization of experts and novices. I consider this to be another dangerous divide because the experts become the norm by which everyone else is judged and evaluated. It creates a hierarchy and a social reward system that promotes a rush to mastery, which undermines and shortchanges the process of learning. Additionally, if the category of expert is associated with natural talent as it is often the case for mathematics and for teaching, gaining such expertise becomes unattainable for novices—let those be elementary-age students or teacher candidates in undergraduate mathematics content or methods courses. Worse still, it suggests that only a few can ever be experts in the teaching and learning of mathematics.

The expert/novice divide results in students, teachers, and schools getting cut in the crossfire. In Crespo (2006) and elsewhere, I have argued that prospective teachers are most likely learning mathematics teaching practices that have not yet been documented in the mathematics education literature because the dominant research frames and tools are focused on a very narrow set of desirable teaching practices. If the window for what constitutes an expert performance is narrowly defined, then the bulk of what can and will be observed would be classified as not meeting expert quality, and by default, they become novice performances or worse considered as examples of not very good teaching.

In a 2007 PME-NA research presentation (and at a later PME-NA presentation in Crespo, Oslund, Brakoniecki, Lawrence, & Thorpe, 2009), I discussed how and why we decided to revise our initial assumptions about expert/novice enactments of teaching practice. As a member of another research project, the Teachers for New Era (TNE) project (Battista et al., 2007), I was able to use similar research tools in order to explore the relation between mathematics knowledge for teaching (MKT) and a practice-focused posing-interpreting-responding (PIR) framework (see Table 1). Working on both these projects at the same time allowed me to see quite a few strange results that called into question assumptions about what experts and novices do/do not know and can/cannot do in their teaching of mathematics. Results from the TNE-Math surveys, for example, which were administered concurrently to prospective teachers at different stages in the program (studying math content and study math methods), had us looking at a number of very strange results such as a decline in mathematics knowledge for teaching (MKT), as prospective teachers transitioned from learning about content to learning about teaching practice.

Another curious result was uncovered when the PIR team compared the prospective teachers' MKT and PIR responses to tasks such as those in Table 1. In his 2009 PME-NA presentation, Brakoniecki (2009), then a graduate research assistant to both projects, reported on prospective teachers who had participated in both the TNE and PIR projects. He showcased three prospective teachers who had correctly addressed the MKT question about generalizing a student subtraction algorithm using negative numbers (see Table 2). All three teacher candidates showed that they could apply the alternative algorithm to a new example. However, their instructional

Table 1 TNE and PIR teaching scenarios focused on two-digit subtraction

TNE project—MKT scenario	PIR project—teaching practice scenario
W1. *Imagine that one of your students shows you the following strategy for subtracting whole numbers.* 37 −19 −2 20 18 **W1a**. *Do you think that this strategy will work for any two whole numbers?* *Yes No I don't know* **W1b.** *How do you think the student would use this strategy in the problem below?* 423 −167	**PIR2a**. Imagine you are teaching a lesson about two-digit subtraction and you ask the class to explore different ways to solve the following subtraction. The students look puzzled. What do you imagine *saying* and *doing* next? **37** **−19** **PIR2b.** After giving students some time to work on the task you call on their attention and ask for volunteers to share their strategies. Imagine that one of the students shows the following strategy. What can you imagine *saying* and *doing*? Say a bit about what you would want to accomplish by saying and doing so. **37** **−19** **−2** **20** **18**

Table 2 Three prospective teachers' responses to MKT task and PIR teaching scenario

Responses to MKT task	Dean Response to PIR task	Becky Response to PIR task	Lisa Response to PIR task
Correct response to the TNE task 423 −167 −4 −40 300 256			This strategy can work.
	I would ask the student to rewrite the problem and show each step they took to get to their answer.	I do not like this way—Math for higher on is going to be a lot harder if they learn this now.	The student knows that we start in the ones column. 7–9 = –2. The tens column is also correct, as 30–10 = 20. Now what the student did was combine – with 2 and 20 to get 18. We got the same answer.
	I would want the students to learn the importance of showing their work and how they can use it to retrace their steps in a problem		I would want to let the class know that there is more than one way to solve a problem, and it is important to remember that subtraction of multiple digit numbers involve multiple subtractions, depending on how many places are in the number

responses to the PIR teaching scenario were all very different (see Table 2) and raised all sorts of questions for the PIR team about the relationship between MKT and PIR practices. So here we have three novices, Dean, Becky, and Lisa (all pseudonyms), who demonstrate that they can do the mathematics that is required to assess the validity and generalizability of an alternative computation algorithm that a student may offer in their classroom, but each of them responds quite differently to a hypothetical teaching scenario. Becky disapproves and does not seem to appreciate the value of this algorithm; Dean seems willing to accept students' algorithms as long as they can show and explain their work; and Lisa makes connections between the standard and alternative algorithms as she expresses her view that there is "more than one way to solve a problem."

So what is a mathematics educator to do with these prospective teachers' responses, classify them as high MKT but then low (Becky), medium (Dean), and high (Lisa) with respect to their instructional practice? What are we to do with prospective teachers like Becky in our teacher preparation courses? Fail them and tell them they are not qualified to teach students? We seem to be willing to do so when they do not know the mathematics and not so willing to take such a stance when they do not know teaching practice. These initial insights made it clear to us that without reframing our assumptions about expert and novice performances of PIR practices, we would continue to recreate and reinforce the same type of instruments and make the same kinds of claims about prospective elementary teachers. This would mean and we would continue to propagate the circular and dead-end deficit discourse about students and their teachers (Comber & Kamler, 2004).

In Crespo, Oslund, and Parks (2007), we shared our revised definitions, which then led us to design new kinds of teaching scenario instruments, ones that invited teacher candidates to provide multiple not just one response to the teaching scenarios and ones that invited a more dialogical representation of their practice (see Crespo, Oslund, & Parks, 2011). In the PIR project, we were then able to document more of prospective teachers' strengths (could do and were able to do) than deficits. More importantly, it led us to propose another type of teaching scenario tasks that positioned prospective teachers as creators (not just as reproducers) of teaching practice. In this new type of teaching scenario instrument, prospective teachers represented a whole class mathematical discussion in the form of a classroom dialogue. I argue that these kinds of dialogical scenarios elicit different kinds of representations from prospective teachers that make visible more of the complex and nuanced ways in which they imagine mathematics teaching practice. Unlike much of the research on prospective and practicing teachers of elementary school mathematics, my PIR project documented many ways in which prospective teachers take up the student-centered and equity-oriented pedagogies they are studying during teacher preparation. I argued that by researching dialogical representations of mathematics teaching, researchers and teacher educators can learn more about how prospective teachers transform what they are studying in teacher preparation courses into purposeful and principled teaching actions. This new insight would not have been possible without challenging and questioning the expert/novice divide that is so engrained within mathematics education's research/teaching practices, which is another divide I discuss next.

2.3 The Research/Teaching Divide

The research/teaching divide has been in the education research landscape for a long time as educational research was initially conceived as research *on* teaching and not *with* or *by* teachers. The animosity and distrust between teachers and researchers in the past and still in the present is reminiscent of Snow's (1959) characterization of the two cultures and it can be related to the longstanding divide between the theoretical and the practical. Researchers characterize teaching as resisting change and teachers characterize educational research as irrelevant to their problems of practice. The research/teaching divide became even more heated when some educational researchers proposed the notion of the teacher as researcher, which raised all sorts of debates, pushback, and controversy (Cochran-Smith & Lyttle, 1990, 1999). As someone who studies her own teaching practice and who collaborates with teachers and students in the research process, I have had to negotiate this divide and address questions about whether my scholarship counts as research or whether my research has made any impact in the everyday practice of teachers. These are questions rooted in the process of self-standardizing and other-pathologizing that I alluded to earlier. The tendency to vilify other perspectives rather than embrace the diversity in our field is very much alive and well in our own academic backyards.The research/teaching divide has always puzzled me. As a teacher, I have always considered myself a researcher of the mathematics I was teaching and of my students' learning, simply stated I considered myself a student of my students' mathematical thinking and learning. Therefore, I find the divide between education practitioners and researchers to be unhelpful and unnecessarily elitist. As a doctoral student, I wrote a comprehensive exam paper titled *What does research got to do with teaching?* where I explored the contentious relationship between research and teaching and argued that the two had more things alike than things that were different. To me, learning, teaching, and researching are similar practices rooted in people's desire to inquire and understand what they do not know. Hence, research is no more than another learning practice that has been uprooted from the everyday practices of people and their communities (this is a similar point to the one I made earlier in relation to the mathematics and education divide).

As a researcher interested in educational experiences that are empowering and transformational for students and their teachers, I see the boundary between teaching and researching as an unproductive divide. In my work, teaching involves research and research involves teaching, the two are deeply intertwined. In *Getting smarter together about complex instruction in the mathematics classroom* (Crespo, 2013), I describe an example in my scholarship where research and teaching seamlessly collaborate to advance the goal of promoting equity in the mathematics classroom. Complex instruction (CI) is a collaborative teaching method that addresses inequitable teaching and learning. Applying the theory of status generalization to classroom interactions, Elizabeth Cohen (1994) interpreted students' unequal participation in the classroom as a problem of unequal status. Unequal status breeds competitive behavior, which, in turn, undermines everyone's learning. Status issues are rooted in societal expectations of competence for students who fit and do not fit the dominant culture's views about who is and not intellectually capable. In the mathematics classroom, status issues become visible when students from non-dominant groups seem reluctant to participate in learning activities. Rather than seeing students who under participate in the classroom as either disengaged or unmotivated, Cohen (1994) saw these students as systematically excluded from learning opportunities not only by their teacher but also their peers, but more importantly by the classroom structures which endorsed rather than disrupt competitive forms of interactions among students.

In contrast, complex instruction seeks to not only understand unequal participation in the classroom, it seeks to engineer instructional structures and practices that could disrupt unequal peer interactions in the classroom and to promote a more collaborative learning environment. Rather than setting up the classroom as a competitive space for learning where some students rise to the top and some sink to the bottom, complex instruction sets up the classroom for collaboration and as a place where everyone is expected to succeed and to contribute to a greater understanding than it would be possible by one person alone. In a complex instruction classroom, no one is seen as more or less smart. Instead, everyone's capacities, abilities, and experiences are acknowledged, valued, and nurtured as resources in the classroom.

Consistent with CI's theory about collaborative participatory learning—that no one is as smart as all of us together—my complex instruction colleagues and I have engaged in this work in ways that require and value each other's perspectives. We realize that simply talking about these issues and becoming aware of them is not enough. This work entails inviting practicing and prospective teachers to work with us on these ideas in the context of learning about lesson studies, which is unsurprisingly also a collaborative approach to teachers' professional learning. We design together complex instruction math lessons and investigate together questions about students' access, participation, and learning in collaborative mathematics lessons (see Crespo & Featherstone, 2012 and Featherstone et al., 2011). This has created a collaborative network of researchers and practitioners with a common goal and who share teaching and research insights across institutional settings using all sorts of communication outlets including social media, teacher blogs, research and practitioner journal articles, book chapters and books, workshops, talleres, and community forums.

3 En La Lucha/In the Struggle: Mathematics Educators Sin Fronteras

I return now to the theme of "Sin Fronteras/Without Borders" and how it might be possible to value and embrace diversity of perspectives in light of the issues I have raised here about the intellectual divides we manage to erect in the process of rationalizing and justifying our work as mathematics teacher educators. Here I conclude with two approaches I have taken to counter my own tendency to self-standardize and other-pathologize by pursuing instead a more reflexive and collaborative mathematics education scholarship. A reflexive approach to mathematics education entails holding the mirror back to ourselves to identify ways in which we are complicit in the very things we criticize and seek to change. A reflexive researcher bluntly asks themselves whether their research is making things better or worse

(Kleinsasser, 2010). In this case, consider how it is that we create intellectual divides with our own scholarship and practices. As I consider, for example, the extent to which my research reflects my commitments to anti-oppressive mathematics education, I have to wonder how to best represent these commitments through my research methods and practices and whether my choices and approaches are making things better or worse.

For example, one important commitment I made early on in my career was to write and speak in ways that are accessible, inviting, and free of academic jargon inasmuch as that is possible. This was partially rooted in my own experiences as a speaker of English as an additional language and the challenges of reading academic papers in a non-dominant language. Additionally, as a teacher of mathematics, I worked hard to demystify the aura of super human intellect that is associated with the very compressed shorthand of mathematical symbolism that keeps so many students in the dark and excluded from using and conversing in mathematics. More importantly, I am continually reminded to question my motives and my hopes for the educational research I choose to pursue, by the words of Elliott (1989) one of the authors I read in graduate school.

> Rather than playing the role of theoretical handmaiden of practitioners by helping them clarify, test, develop, and disseminate the ideas which underpin their practices, academics tend to behave like terrorists. We take an idea which underpins teachers' practices, distort it through translation into academic jargon, and thereby "highjack" it from its practical context and the web of interlocking ideas which operate in that context. (Elliott, 1989; p. 7)

Yet as I hold on to this commitment, I also consider the critiques other scholars raise about taking what seems to be a reductionist and simplistic route to explaining complex educational issues. In their view, such an approach to scholarship feeds into rather than challenge the distrust people have of academics and anything that sounds too intellectual or overly complex, whether those ideas come from science or the humanities (e.g., Davis et al., 2014). I also understand that our words are critical and that how we name and talk about people, communities, and students matter and shape our thinking and practices. Therefore, I also participate in discussions that seek to clarify, object, and subvert particular terms and language commonly used in research and in practice, especially language that is offensive and degrading to marginalized students and communities.

The point here is that I have come to accept that there is inherent tension and contradictions within the work we do as researchers in mathematics education and appreciate Elbow's (1983, 2000) notion of embracing contraries as a way to see beyond our tendency to polarize and take sides without fully understanding and considering opposing views. Sfard's (1998) discussion of two metaphors for learning (as acquisition and participation) also takes a similar stance about opposing and contradictory perspectives. I have tried out Elbow's ideas in a recent editorial (Crespo, 2016a) for the *Mathematics Teacher Educator* journal (of which I served as editor from 2014–2018), in order to promote a more educative rather than adversarial approach to reviewing manuscript submissions to the journal. I also explored Elbow's embracing of contraries in a recent publication (Crespo, 2016b) focusing on the challenge to disrupt our tendency to polarize mathematics teaching practice when selecting and using video representations of mathematics teaching. This is an issue that the National Council of Teachers of Mathematics (NCTM) Research Committee (2016) recently discussed and identified as a pernicious storyline that circulate and influence the public perception about mathematics education.

Collaborative research is another way in which I have chosen to pursue research in mathematics education. This is one approach that discourages me from building intellectual divides. I have come to the point of realizing that educational problems are much too big for any one of us to take on and solve by ourselves and that it will take literally a whole village of committed mathematics educators to make the kinds of changes we are all striving to make. All this within a world of higher education and academia that is often driven by competitive policies and reward systems. Although this can create hostile working environments for faculty, it is worth investing in developing collaborative networks with colleagues. Operating under the tenets of complex instruction that together we can learn more than individually, and that each collaborator needs to be willing to learn from each other's perspectives, I continually renew my belief and commitment in collaborative mathematics education research. And as I alluded to earlier, my work is only possible by collaborating with colleagues from all walks of life that are committed to social change.

In addition to the example I offered earlier with my complex instruction colleagues with whom I wrote the book *Smarter Together* (Featherstone et al., 2011), I have also collaborated with another network of educators committed to identifying and challenging oppressive forms of mathematics education research and to making our field more inclusive of diverse perspectives and practices (see Herbel-Eisenmann et al., 2013). Another more recent collaboration is a book of cases for mathematics teacher educators (White, Crespo, & Civil, 2016), which includes a collection of 19 cases from different authors, highlighting dilemmas they experienced while teaching about inequities in mathematics education in the contexts of content and methods courses and professional development contexts. Each case includes commentaries from three different authors. Altogether the perspectives of over 80 mathematics educators are included in this book. The conversations that we have had and that we will continue to have around these cases are very exciting to me and give me hope that together we can and will make a difference in shaping the future of mathematics education research. I am also hopeful that the future generation of mathematics educators will engage with diverse perspectives by embracing contraries and engaging in collaborative research.

To close, I highlight four key commitments that have helped me to not simply navigate borders in mathematics teacher education but to take steps toward challenging them. I invite readers to consider, and add to these commitments, and work toward re-imagining mathematics teacher education without divisive borders. One first step is awareness and a second step is creating a community that will help us reflect on and be accountable to our commitments. Since my plenary in 2016, there have been many more conversations all over the country, at conferences, at our institutions, among educational leaders, and in our professional organizations focusing on the challenges we face as mathematics teacher educators. This makes me increasingly hopeful that our field can move forward and live up to its potential and commitment to preparing future generations of mathematics teachers who will choose: (a) strengths over deficit in a system that rewards other-pathologizing; (b) collaboration in a system that rewards competition; (c) to embrace contraries in a system that rewards divisiveness; and (d) to humanize in a system that rewards dehumanization.

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Part II
Curricular Borders in Mathematics Pre-service Teacher Education

Continuous Directed Scaling: How Could Dynamic Multiplication and Division Diagrams Be Used to Cross Mathematical Borders?

Justin K. Dimmel and Eric A. Pandiscio

1 Introduction

Mathematics, as represented in US schools, is a collection of topics that are arranged in a sequence, through which children are expected to make steady, linear progress. A glance through the table of contents of any US elementary mathematics textbook encapsulates the subject-progression structure. In third grade,[1] for example, one might find lessons on place value for numbers of different orders of magnitude; lessons on adding two- and three-digit numbers; lessons on multiplication and division by specific numbers; and lessons about more general techniques for adding, subtracting, multiplying, and dividing, among other lessons. US elementary students experience mathematics as a sequence of topics from their first days in class through high school and even into college.

Separation is an organizing principle of US school mathematics: arithmetic is a separate subject from geometry, which in turn is separate from subjects like algebra, probability, or trigonometry. Within these broad mathematics subject headings

[1] Such as would be found in *My Math, Grade 3, Vol. 1* (2017, McGraw-Hill).

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students experience over the course of their education are finer distinctions still—for example, whole numbers and their operations are different from integers which are different from fractions which are different from real numbers. To study mathematics in school is to be continually negotiating the borders between its various subject domains.

The organization of mathematics into a sequence of distinct subjects has utility for mathematics education at the K-16 levels. Topics can be arranged so that students develop more basic mathematical skills before they are required to use those skills to solve more challenging problems. Colleges can use the math-subject experiences of its newly admitted students to sort them into various introductory courses. Disciplines that require specific mathematical skills, such as engineering, the physical sciences, and the life sciences, can recommend courses for their students to acquire those skills. And the standard mathematical subject sequence at the K-12 levels provides an organizing structure for student mathematical experiences in schools that facilitates the design, validation, and widespread use of high stakes tests (Au, 2011).

The utility of the topical divisions within school mathematics notwithstanding, sorting mathematics, a broad field of human knowledge, into different subjects presents an overly fractured view of the discipline to students. There has been extensive research documenting how students' experiences of mathematics in schools can reinforce unproductive beliefs about and dispositions toward mathematical activity (e.g., Boaler, 2002; Crawford, Gordon, Nicholas, & Prosser, 1994; Schoenfeld, 1988). The artificial yet tangible borders erected between various subjects, presenting them as distinct when in fact they are inextricably connected, serve as metaphorical borders that impede students' deep learning and dispositions. What can we, as mathematics educators, do to blur, bypass, or otherwise break down the borders between the different subject areas of mathematics?

In this chapter, we explore what is possible by probing the mathematical borders that are typically encountered by US pre-service elementary teachers. We consider first how number systems and their operations are represented as subject areas in elementary mathematics, with a specific focus on the development of multiplicative reasoning. We then consider how the border between arithmetic and geometry could be blurred and softened through the development of dynamic, diagrammatic models of multiplication and division. We conclude with a consideration of how dynamic representations, which are typically considered in the context of geometry, could be used to represent arithmetic ideas in pre-service teacher education.

2 Representations of Multiplication in Elementary Mathematics

Students spend a great deal of their mathematics education learning about number systems and their operations. In the USA, the Common Core State Standards Initiative—a national set of curricular benchmarks that are linked to standardized, end-of-year tests—provides one record of the dominant place of number systems and their operations throughout K-12 mathematics. Mastering the fundamentals of counting lays a foundation for adding and subtracting single-digit whole numbers, then adding and subtracting multi-digit whole numbers. Multiplication and division of whole numbers follow, after which students are ready to extend what they know of these operations to different types of numbers—integers, percents, and the ratio and decimal representations of rational numbers. As they continue their mathematics education into US middle and secondary schools (typically ages 12–18), the picture is filled in for irrational and complex numbers. In school mathematics, operations such as multiplication and division are represented as ever-growing lists of procedural rules that apply in different circumstances.

Concordantly, investigating how students understand number systems and their operations has been a focus of mathematics educationresearch since its inception as a discipline. Brownell (1947) examined the conditions under which it could be said that children know the mathematical meanings of arithmetical procedures. Erlwanger (1973) described the case of Benny, a student enrolled in an individualized program of instruction (IPI). Benny developed his own set of idiosyncratic procedures for solving problems that often resulted in correct responses but that were inconsistent with arithmetic principles. Brown and Burton (1978) developed diagnostic models for identifying and repairing procedural bugs students enact when working through arithmetic algorithms. Recent work has documented elementary students' and pre-service elementary teachers' difficulties with understanding multiplication and division—especially the multiplication and division of fractions (Ma, 2010) and division by zero (Ball, 1990; Cankoy, 2010; Quinn, Lamberg, & Perrin, 2008).

Multiplicative thinking has been a focus of the research on how children develop their early number sense (Larsson, Pettersson, & Andrews, 2017). Researchers have investigated students' intuitive models of multiplication (Fischbein, Deri, Nello, & Marino, 1985), the schemes they use for solving multiplication problems (Steffe, 1988), the conceptual foundations of multiplication in children's thinking, and whether various models of multiplication (e.g., multiplication as repeated addition, multiplication as correspondence) are more or less effective at helping students reason multiplicatively in other settings (Thompson & Saldanha, 2003). In the course of the mathematical preparation of pre-service teachers, it is typical to account for this broad research base by introducing different conceptualizations of multiplication that can be used in different situations. For example, Sowder, Sowder, and Nickerson (2017) identify four views of multiplication in their elementary mathematics textbook, *Reconceptualizing Mathematics for Elementary SchoolTeachers.*

The first is the repeated addition view: "When a whole number of quantities, each with value q, are combined, the resulting quantity has a value of $q + q + q + \ldots$(n addends), or $n \times q$" (Sowder et al., 2017, p. 57). The repeated addition representation of multiplication has a long history in the mathematics educationresearch literature (e.g., Bechtel & Dixon, 1967; Fischbein et al., 1985; Kouba, 1989; Mulligan & Mitchelmore, 1997; Rappaport, 1968; Weaver, 1967). In elementary schools, multiplication as repeated addition is generally the first view of multiplication children encounter, suitable as it is for representing the multiplication of whole numbers. A second view of multiplication is the array or area view: The product of two numbers is represented "as a rectangle n units across and m units down," resulting in a product of $m \times n$ (Sowder et al., 2017, p. 58). This representation is more general than the repeated-addition model and can be used to describe integer, rational, or real number multiplication. It connects an arithmetic idea, multiplication, to a geometric idea, area, and is a clear departure from the conception of multiplication as repeated addition.

A third view of multiplication is the fractional part-of-a-quantity view, also referred to as the *operator view* of multiplication, which "is useful for finding a fractional part of one of the two quantities" (Sowder et al., 2017, p. 59). This is the view of multiplication that links multiplying by a fraction to division—for example, to find the (1/2) part of some whole quantity, multiply the quantity by (1/2), which is the same as dividing the quantity by 2. The fractional part-of-a-quantity view is useful when students are operating with the ratio representations of rational numbers. A fourth view is the fundamental counting principle view of multiplication: "In a case where two acts can be performed, if Act 1 can be performed m ways, and Act 2 can be performed in n ways…then the sequence Act 1-Act 2 can be performed in $m \times n$ ways" (Sowder et al., 2017, p. 60).

This list of views of multiplication in elementary mathematics is not intended to be comprehensive, but rather is meant to be illustrative of the representations of multiplication that are typically encountered in elementary mathematics classrooms. In addition to these theoretical representations of multiplication, physical or virtual manipulatives also play a prominent role in helping students to develop fluency with number systems and their operations in elementary mathematics classrooms. Base 10 blocks, Cuisenaire rods, Fraction tiles, or virtual manipulatives, such as virtual area rectangles, are routinely used in elementary mathematics classrooms to help young learners explore arithmetic concepts. Like the various views of multiplication described above, different manipulatives have utility in different circumstances (Carbonneau, Marley, & Selig, 2013; Laski, Jordan, Daoust, & Murray, 2015; Moyer, Bolyard, & Spikell, 2002; Moyer-Packenham & Westenskow, 2013), though, with the exception of virtual manipulatives, they are generally limited to discrete—rather than continuous—numerical contexts.

While all four views of multiplication have utility in different situations, each is an incomplete representation of multiplication as a binary relationship between real numbers. For example, a key affordance of the repeated addition view is that it grounds multiplication in a process that is familiar to children (Fischbein et al., 1985) and for the natural numbers, multiplication as repeated addition is adequate to facilitate calculation. But the repeated addition view of multiplication has been the subject of criticism (e.g., Steffe, 1988; Thompson & Saldanha, 2003) for being a source of stubborn misconceptions about multiplication that students and even teachers carry throughout their experiences with mathematics. Since the repeated addition view is only sufficient for describing multiplication among natural numbers, some argue that teaching it to students is a disservice that has the potential to do lasting harm (Devlin, 2008). Array or area representations of multiplication avoid the pitfall of defining multiplication as repeated addition and can be used to describe multiplying rational or irrational numbers. However, the area representation has also been critiqued precisely because it blends multiplication with area, which could contribute to the misconception that area cannot be defined independently of multiplication (McLoughlin & Droujkova, 2013).

Another limitation that applies to all four views concerns signed multiplication. In the case of the fundamental counting principle view: What constitutes a negative act? How could two such acts be counted so as to produce a positive product? For the operator view: What is a negative part of a negative whole? Why would a negative part of a negative whole yield a positive product? For the area view: How would one construct a rectangle that has negative side lengths? How would the product of negative lengths produce a positive area? For the repeated addition view: How would one repeatedly add a negative number a negative number of times in such a way that the result would be a positive number? These are examples of questions students could ask of these representations. What would satisfactory answers to these questions look like, and would such answers be clear and convincing to students? Developing a visual representation of multiplication that accounts for the products of signed numbers is a principal motivator for the diagrammatic model of multiplication we describe below.

3 Multiplication as Continuous Directed Scaling

We consider here a continuous directed scaling, model of multiplication that is rooted in geometry. The version of the model we develop below was adapted from McLoughlin and Droujkova (2013). The model we develop is continuous, in the sense that it is defined for all real numbers, as opposed to discrete representations (e.g., repeated addition) that are only defined for natural numbers. The model is directed, in that positive and negative numbers are represented as directions on a set of coordinate axes. The model shows how one directed length can be scaled by a second directed length, using techniques from compass-and-straightedge constructions.

Compass and Straightedge Multiplication Given a unit length and a procedure for constructing parallel lines, it is possible to scale any segment by any other segment, using compass and straightedge (McLoughlin & Droujkova, 2013). One procedure works as follows: Let $\overline{AB}$ and $\overline{CD}$ be the segments to be multiplied, and let $\overline{u}$ be a unit length (Fig. 1).

Construct two perpendicular lines. Copy segments $\overline{AB}$ and $\overline{CD}$ to create segment $\overline{A'B'}$ on one line and segment $\overline{C'D'}$ on the other so that they are at right angles to each other. Copy segment $\overline{u}$ to create $\overline{u'}$ (Fig. 2).

The unit segment provides a reference for scaling one segment by the other. To complete the construction of the product, join the segment from D' to the end of the unit length—call this segment $\overline{m}$. Then, construct the line through B′ parallel

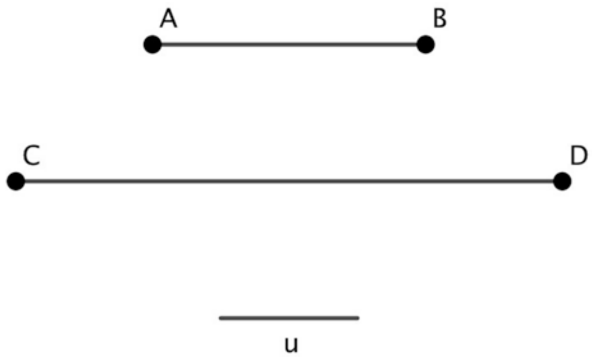

Fig. 1 Segments to be multiplied by compass and straightedge construction

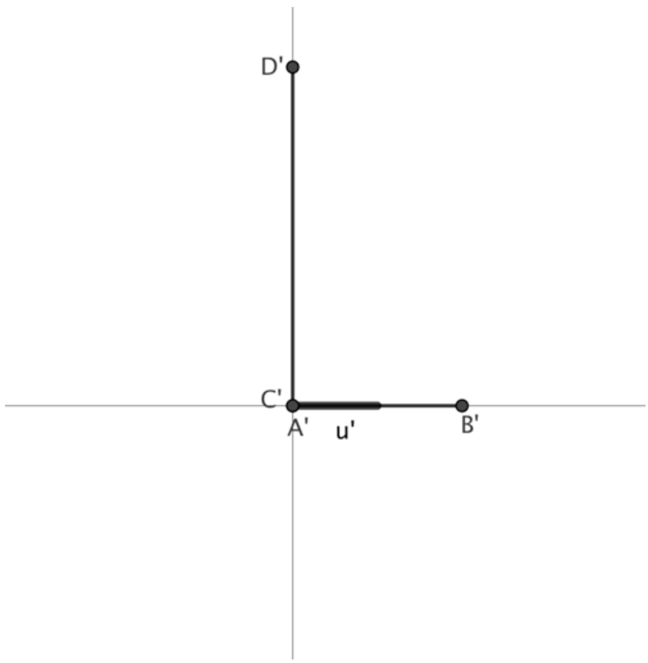

Fig. 2 The segments are copied so that they are perpendicular to each other, and a unit length is marked off at the end of one of the segments. In this figure, the ends of the segments coincide at A' and C'. The unit length, $\overline{u'}$, overlaps $\overline{A'B'}$

to $\overline{m}$. The point where the parallel line intersects the vertical line defines the segment that is the product of $\overline{A'B'}$ and $\overline{C'D'}$. In Fig. 3, the product is the segment $\overline{C'P}$.

To see that the product of the $\overline{A'B'}$ and $\overline{C'D'}$ is $\overline{C'P}$, we can reason from the similarity of the two triangles. Let the triangle with u' as a leg be *A* and the triangle with $\overline{A'B'}$ as a leg be *B*. Then:

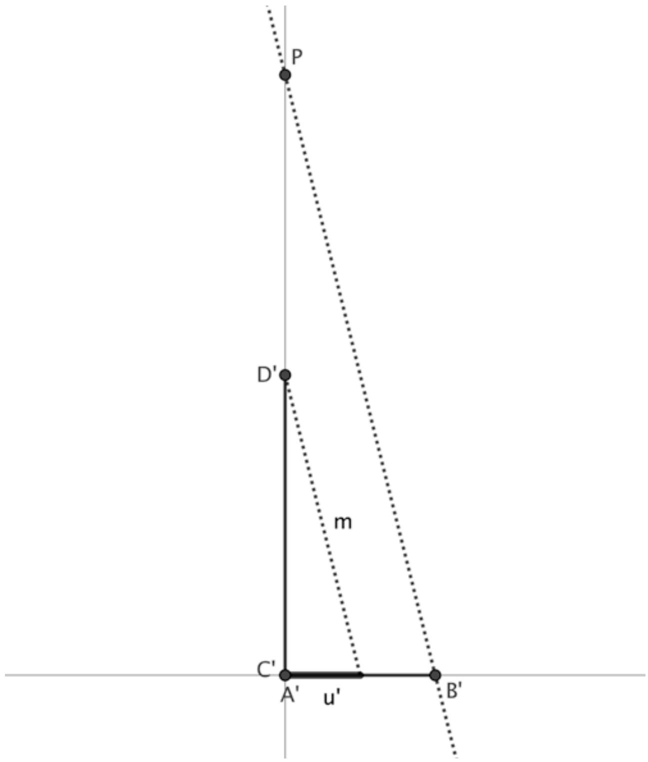

Fig. 3 Where the line parallel to *m* intersects the vertical line (point *P*) defines the segment that is the product of $A'B'$ and $C'D'$

$$A \sim B \rightarrow \overline{\mu'} : \overline{A'}\overline{B'} \,\&\, \overline{C'}\overline{D'} : \overline{C'}\overline{P'}$$

Since $\overline{u}'$ is a unit length, the scale factor that maps $\overline{A'B'}$ to u' is $\overline{A'B'}$. Thus,

$$\overline{C'}\overline{P} = \overline{A'}\overline{B'} * \overline{C'}\overline{D'}$$

Similarly, one could use a compass-and-straightedge procedure to construct the quotient of two segments. In the case of division, the procedure once again begins by copying the dividend and divisor onto perpendicular lines and adjoining a unit length to the end of the divisor. To find the quotient, construct the segment between the endpoints of the dividend and divisor ($\overline{D'B'}$ in Fig. 4). Then, construct the line parallel to that segment through the end of the unit length. The quotient is defined as the point where that parallel line intersects the vertical line (see Fig. 4).

For the division diagram, it is also possible to reason from the similarity of the triangles to see that $\overline{C'P}$ is, in fact, equal to $\overline{C'D'}$ divided by $\overline{A'B'}$. Let $A'B'D'$ be triangle *A* and let the triangle that has $\overline{u'}$ as a side be *B*. Then,

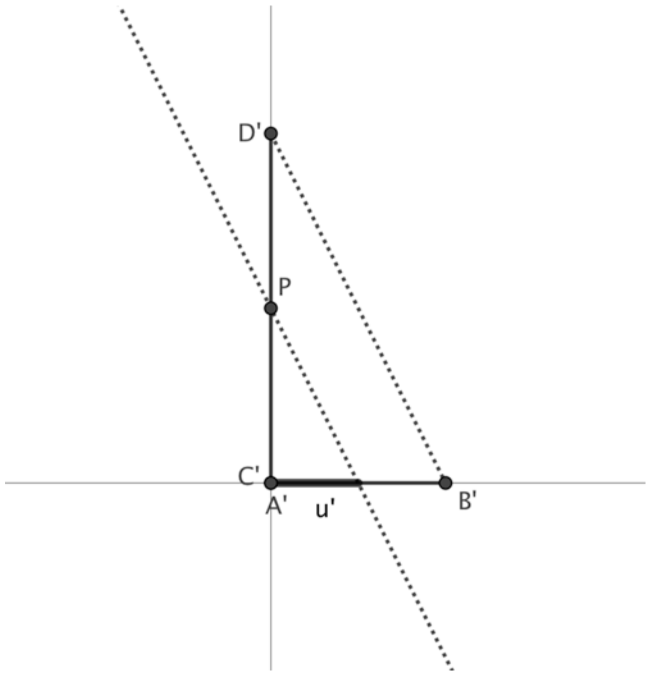

Fig. 4 A diagram that shows $\overline{C'D'}$ divided by $\overline{A'B'}$. The quotient is $\overline{C'P}$

$$A \sim B \rightarrow \overline{A'}\overline{B'} : \overline{\mu'} \;\&\; \overline{A'}\overline{D'} : \overline{A'}\overline{P'}$$

As with the multiplication diagram, since $\overline{u'}$ is a unit length, the scale factor that maps $\overline{A'B'}$ to u' is $(1/\overline{A'B'})$. Thus,

$$\overline{C'}\overline{P} = \frac{\overline{C'}\overline{D'}}{\overline{A'}\overline{B'}}$$

The Multiplication and Division Diagrams as Virtual Manipulatives From the pedagogic viewpoint, it would be impracticable to use a compass and straightedge to physically construct the product or quotient of two numbers—especially for younger students that are still developing their fine motor skills. But in a dynamic geometry environment, the diagrammatic models of multiplication and division can be used to create virtual manipulatives (Moyer et al., 2002), where a continuously variable segment, aligned with the *x*-axis, and a continuously variable segment, aligned with the *y*-axis, yield a segment aligned with the *y*-axis as their product or quotient. Figure 5 shows the diagrammatic model of multiplication as a virtual manipulative realized in GeoGebra. The yellow dot and the blue dot can be slid along the *x* and *y* axes to display different products. The diagram in Fig. 6 shows 2 times 3.

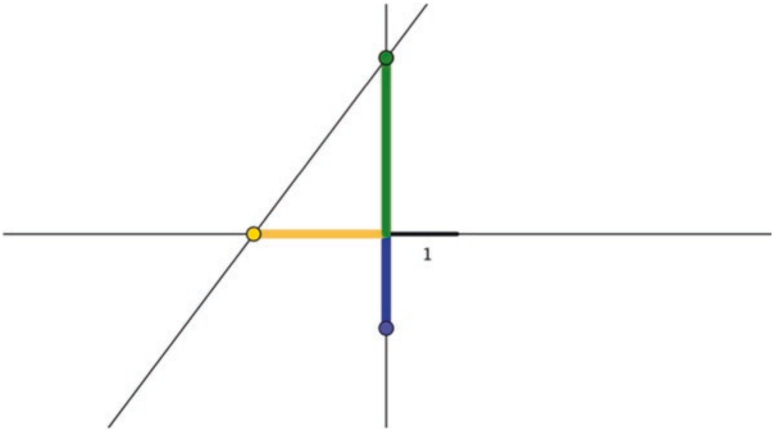

Fig. 5 The diagrammatic model of multiplication as a virtual manipulative in GeoGebra. The blue point and yellow point can be varied to display different products

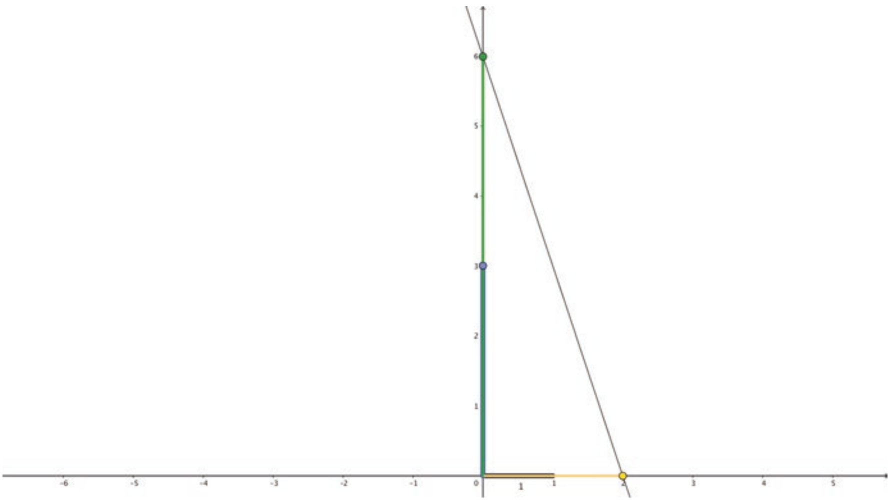

Fig. 6 A diagram showing that 2 times 3 is 6

We used variations in color, stroke weight, and point gauge to visually highlight key features of the dynamic diagram: the heavier stroke weights for the multiplicand, multiplier, and product segments help to make these parts of the display more prominent than the axes, product line, or unit length; variations in color were designed to visually underscore the mathematical operation of the diagram (Dimmel & Herbst, 2015)—that is, yellow and blue combine to produce green; and the larger gauge for the yellow, blue, and green points give them greater emphasis than the other points of their respective segments. Students interact with the diagram by dragging the yellow point or the blue point back and forth across the x- or y-axes (respectively).

The dynamic diagrams represent multiplication and division as *continuous directed scaling* of one quantity by another. The diagrams are thus models of multiplication and division that can visually represent the product of any two real numbers—positive or negative, rational or irrational. Below, we consider how the diagrammatic models differ from the views of multiplication that are typically presented in elementary mathematics classrooms. We suggest activities with the diagrammatic models that could complement the activities students do with other representations of multiplication.

Continuous The diagrammatic models of multiplication and division are continuous in the sense that they are not discrete and can represent the product of any two real numbers. Whole numbers, integers, rational numbers, or irrational numbers are each handled the same way by the diagrammatic models. They present unified models of multiplication and division that do not need to be revised, qualified, or otherwise modified for different number systems. This is significant because in elementary mathematics classrooms, learning how to make sense of different number systems is a border that students continually need to cross as they advance in their education. Children learn one set of concepts for characterizing whole numbers, new concepts for integers, new concepts for rational numbers, and new concepts still for real numbers. It is important for students to learn the properties that distinguish natural numbers from integers or rational numbers from irrational numbers. But the differences among the number sets that are highlighted in the elementary mathematics curriculum can obscure the underlying mathematical unity that these are each real numbers that can be located on a number line. The diagrammatic models of multiplication and division could help to blur the borders between different numbers and highlight that multiplication is defined in the same way over all of them.

That the dynamic diagram defines multiplication for all real numbers is also significant because of *natural number bias*—the tendency to attribute "characteristics and properties of the natural numbers to different kinds of non-natural numbers" (Christou, 2015, p. 748). For example, the idea that *multiplication makes bigger* has been cited as a misconception about multiplication that has its roots in natural number bias (Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). This misconception is not necessarily directly taught to students, but it is nonetheless a property of multiplication that children infer as a result of their years of experience doing arithmetic with natural numbers in elementary mathematics classrooms (Van Hoof et al., 2015). Other properties of the natural numbers—for example, that the set of natural numbers is discrete, that every natural number has a unique predecessor and unique successor—also bias how students conceptualize the arithmetic operations (Christou, 2015). The causes of natural number bias are not yet fully understood, but researchers have argued that children's experiences with arithmetic in schools likely play some role in fostering or sustaining the bias (Ni & Zhou, 2005). We describe below how the multiplication diagram could be used to challenge natural number bias.

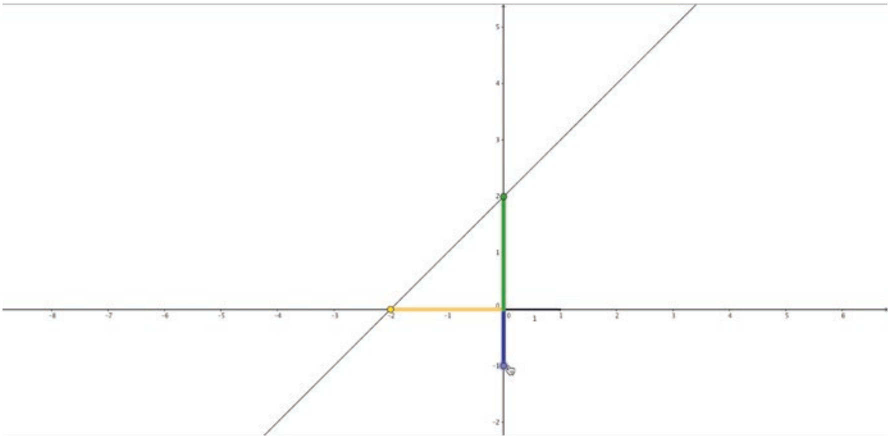

Fig. 7 A multiplication diagram that shows that the product of negative 2 and negative 1 is positive 2

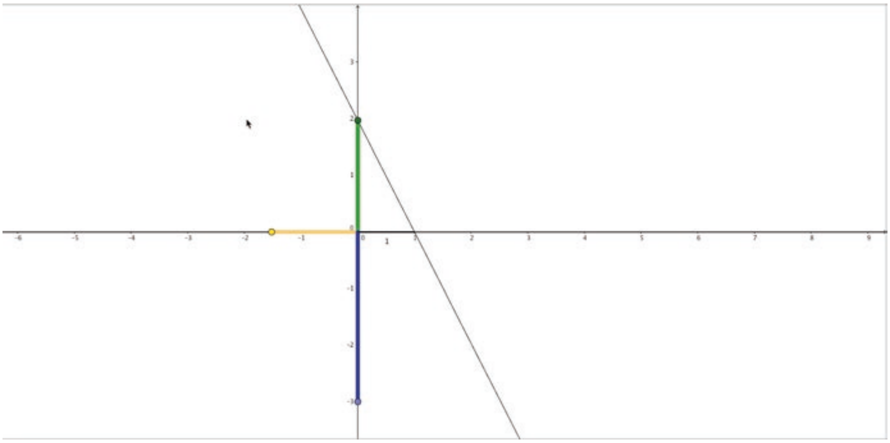

Fig. 8 A division diagram that shows that the negative 3 divided by negative 1.5 is positive 2

Directed The second feature of the diagrammatic models is that they represent multiplication and division as operations on directed lengths. They thus provide natural ways to interpret multiplication or division by positive or negative numbers. For example, the yellow and blue segments shown in Figs. 7 and 8 are aligned with the negative *x* and *y* axes, respectively. They show that the product (Fig. 7) or quotient (Fig. 8) of two negative numbers is a positive number.

In the case where the multiplicand/multiplier or dividend/divisor have mixed signs, the product or quotient would be negative, as shown in Fig. 9. These results are intrinsic features of the diagram. It is true that the behavior of the diagrams do not, on their own, provide an explanation or reason or justification for why this

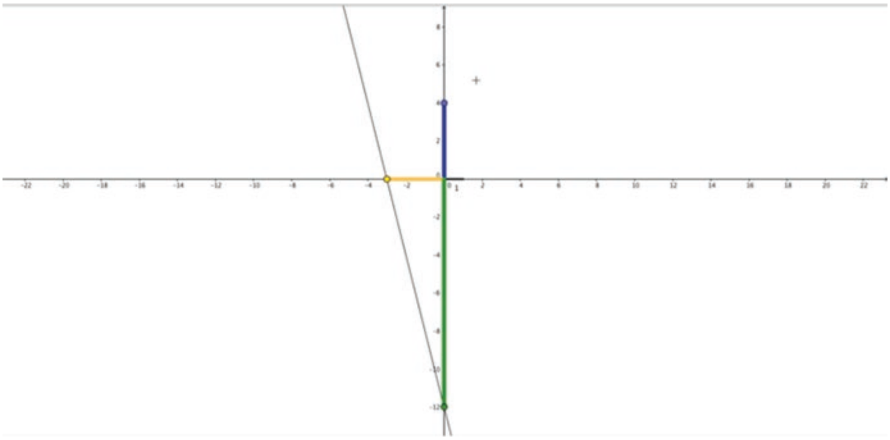

Fig. 9 A multiplication diagram that shows that the product of negative 3 and positive 4 is negative 12

occurs, but they nonetheless provide visual models of multiplication and division that are consistent with the procedures for calculating signed products and quotients.

The sign rules for multiplying and dividing by positive and negative numbers have proven to be challenging for students to grasp in conceptually meaningful ways. The typical representations of multiplication and division that are presented in elementary classrooms have no consistent ways of making sense of why multiplication of negative numbers results in a positive number, or why multiplication of numbers with mixed signs results in a negative number.

For example, the yellow chip/red chip representation of integers uses yellow chips to represent positive quantities and red chips to represent negative quantities (Sowder et al., 2017). But representing signed multiplication with discrete chips involves conceptualizing multiplication as repeated addition. Furthermore, statements such as (2×-4) and (-2×4) require different interpretations under the chip model, even though each represents the same product. In the first case, the repeated addition view states that the quantity (-4) is repeated two times, which results in (-8). But then in the second case, if the repeated addition view were applied in the same way, the result would be that the quantity (4) is repeated (-2) times, which is nonsensical (Sowder et al., 2017).

A way around this is to reinterpret the multiplication-as-repeated addition view with signed numbers to follow one set of rules if the signed number appears second (these are the standard multiplication as repeated addition rules) and a different set of rules—that is, multiplication as repeated subtraction—if the signed number appears first. In that case, the example of (-2×4) could be interpreted as stating, "subtract 4 twice."

It is unclear how the ad hoc stipulations that are necessary to extend the domain of the representations to include signed multiplication would be clear to or convincing for students.

The root issue is that the behavior of the real numbers under signed multiplication is a deep feature of the structure of the real number system. There is no natural-language explanation for this fact, nor is there a compelling physical representation that can illuminate why this is the case. It is the case, in particular, because the field structure of the real numbers requires that it be the case. The multiplication and division diagrams are well behaved under signed products and quotients because they are mathematical models of those operations in ways that collections of physical things cannot be.

Scaling The third feature of the diagrammatic models of multiplication and division is that they represent multiplication and division as scaling of one number by another. From the start, then, the diagrammatic models define multiplication as an operation that is distinct from addition. The diagrams help to visually convey what it means to scale one quantity by another, as the implicit triangles in the diagram provide a visual cue for the concept of proportionality. The scaling that is represented in the diagrammatic models is different from the geometric representation of multiplication that is evident in the array or area representations.

In the area representation, the product of two one-dimensional quantities is a two-dimensional quantity. When viewed from a geometric standpoint, this is a reasonable way to describe area, and, in fact, it helps to illustrate the need to use two-dimensional units when measuring area. But the area that results from the multiplication of two real numbers needs to be re-interpreted as a linear, one-dimensional quantity in order for the area representation of multiplication to be well defined (McLoughlin & Droujkova, 2013). The area representation does not capture the idea of multiplication-as-scaling so much as it introduces a new geometric interpretation for multiplication.

Other Properties of the Models In addition to representing multiplication (and division) as continuous directed scaling, the diagrammatic model has other properties that are mathematically significant. Dynamic geometry software environments, such as GeoGebra, have made possible new modes of mathematical inquiry, new conceptions of geometric relationships, and new activities for the teaching and learning of geometry (Hollebrands, 2007). The capacity to explore, via continuous transformation, a range of diagrammatic realizations of the same underlying figure has been shown to be a powerful affordance of DGS that helps students develop and test geometric conjectures (González & Herbst, 2009).

Algebra solvers, spreadsheets, graphing calculators, or other software applications have enhanced our capacity for operating with numerical representations in comparably transformative ways. For example, arrays of numbers with thousands of entries can be rendered as dynamic graphs and complex surfaces defined analytically by families of curves can be visualized as two- or three-dimensional figures. But such visual representations of numerical information preserve the standard modes of interaction between graphs and the data they are determined by—numbers (or equations) as inputs into a renderer that produces a visual output. The dynamic, diagrammatic model of multiplication we have described here is distinct in that the inputs and outputs are visual representations of numeric quantities. It thus makes it possible to use the modes of interactivity that have proven to be so transformative in dynamic geometry environments to explore arithmetic relationships.

The diagrammatic model of multiplication has different affordances than other physical or virtual manipulatives or calculating tools. The purpose of the dynamic, diagrammatic model is not to facilitate the evaluation of any specific multiplication problem, but rather to facilitate continuous comparisons of multiplication problems by dynamic transformation of the multiplication diagram. Base-10 blocks, Cuisenaire rods, and calculators are more practical resources for children to use to explore discrete sets of specific products and quotients, but they are incapable of modeling the continuous transformation that happens when one segment of variable length is used to scale another. The potential to represent families of quotients and products and to explore how multiplication and division, as binary operations, change as their inputs are continuously varied is a key affordance of the multiplication and division diagrams.

Numbers Between 0 and 1 One activity that is possible with the diagrammatic model of multiplication is investigating how multiplication operates on different sets of numbers, such as numbers that are between 0 and 1. A common misconception held by students and teachers in elementary mathematics courses is that multiplication results in a product that is larger than either factor and that division results in a number that is smaller than the dividend (Graeber, 1993; Graeber, Tirosh, & Glover, 1989). With the dynamic diagrams, students can drag the factors across continuous sets of values between 0 and 1 to explore how such products and quotients challenge these misconceptions. The models also highlight the role that the multiplicative unit plays as a boundary that defines different classes of products.

Investigating Products Between 0 and 1 with Pre-service Teachers We tested this activity with pairs of pre-service elementary teachers during a task-based interview study (Dimmel & Pandiscio, 2017). We recruited elementary pre-service teachers that were enrolled in geometrycontent courses at a public university in New England. Pairs of participants completed a brief orientation to GeoGebra, where they were shown how to drag a point back and forth along a line and how to change the scale (zoom in/out) and span (scroll left/right/up/down) of the display window. Following the orientation, pairs of participants were audio-recorded as they described their explorations of the dynamic diagrams and their dynamic manipulations of the diagram were screen-recorded. The purpose of the study was to document how pre-service teachers explored and made sense of the diagrams. The pairs of pre-service teachers we interviewed showed how the multiplication model could be used to explore the "multiplication makes larger" misconception.

One pair of participants, Sally and Karen,[2] demonstrated how specific states of the diagram could help to visually convey that it is possible for multiplication to

[2] Pseudonyms.

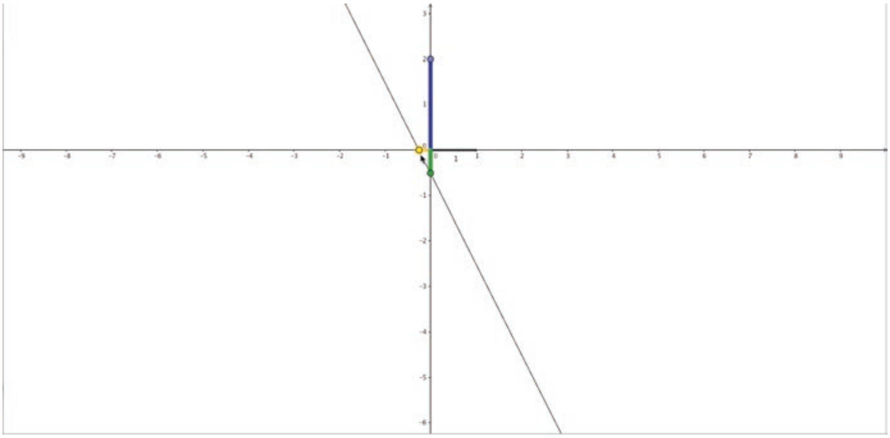

Fig. 10 A multiplication diagram that shows that the product of 2 and −0.25 is −0.5

result in a number that is smaller than at least one of the numbers that is multiplied. As they did so, it seemed that they themselves were using the diagram to discover the conditions under which multiplication could make a number smaller. They began with the diagram shown in Fig. 10, which they indicated was an example of multiplication making a number smaller.

For this part of the discussion, Sally and Karen stated that by “smaller” they were describing the lengths of the segments, not their direction. Next, they moved the yellow point to between −1 and −2 (see Fig. 11), at which time Karen observed, “That’s bigger….so it has to be less than--”; as Karen says “less than,” Sally moves the blue point so that the blue segment is between 0 and 1 (see Fig. 12). Sally stated, “Also, if you do this one,” Sally pauses for a second as she moves the yellow point to between −3 and −4, “the yellow is going to be more than the green” (see Fig. 13).

The researchers then asked, “What happens if both yellow and blue are between 0 and 1? What happens to the green?” The participants used the diagram to answer this question. They moved the yellow point to be between 0 and −1 before giving their answer (see Fig. 14).

Once the blue and yellow points were in position, Sally observed, “It gets smaller than both of them.” As Sally and Karen considered the products of numbers between 0 and 1, they dragged the yellow and blue points through ranges of cases within the [−1,0] and [0, 1] intervals. They did not fixate on specific products but were instead focused on trying to describe a property of multiplication that would apply to all possible products within that range. This type of exploration is an example of an activity that could potentially help pre-service teachers expand their conception of multiplication beyond the natural numbers.

Signed Multiplication We described above how visually representing the structure of signed multiplication is a feature of the model. The diagrammatic model provides a means to explore how the structure of multiplication requires that a negative number times a negative number would produce a positive product. Figure 15 shows a

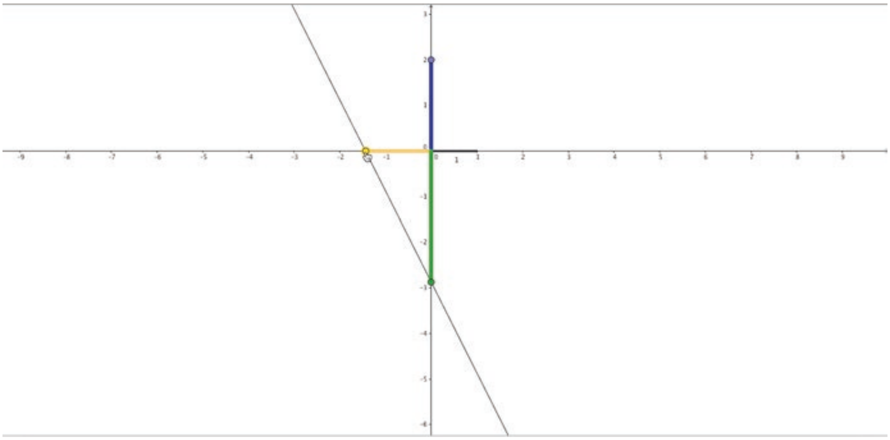

Fig. 11 Karen moves the yellow point and observes that the product is bigger

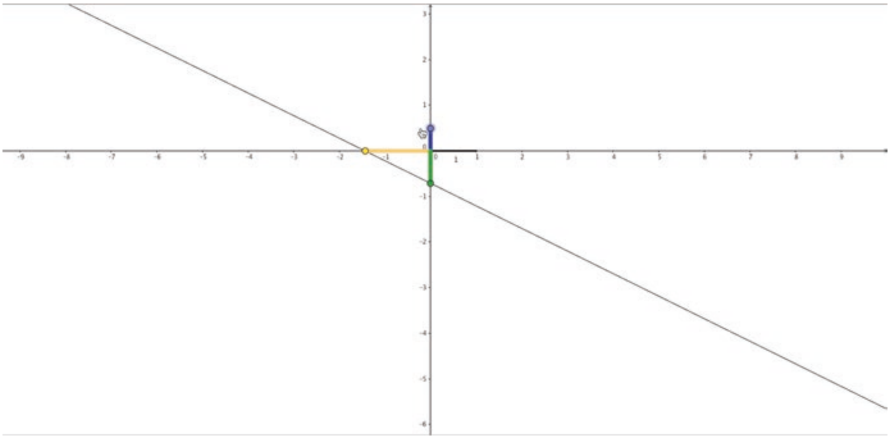

Fig. 12 Sally moves the blue point to between 0 and 1

multiplication diagram for a negative number times a positive number. As the positive factor approaches 0 and then becomes a negative factor (Fig. 16), the diagram shows the product moving from a negative number to a positive number (Fig. 17).

In this example, the diagram shows how small changes in the value of the inputs of the multiplication model—in this case, the blue segment continually decreases—yield correspondingly small changes in the output of the model—the green segment gradually increases as the blue segment gradually decreases. This continuous variation proceeds apace as the blue factor passes through the origin and becomes a negative number, thereby visually demonstrating that for multiplication to be well-behaved (i.e., continuous), the product of two negative numbers is a positive number.

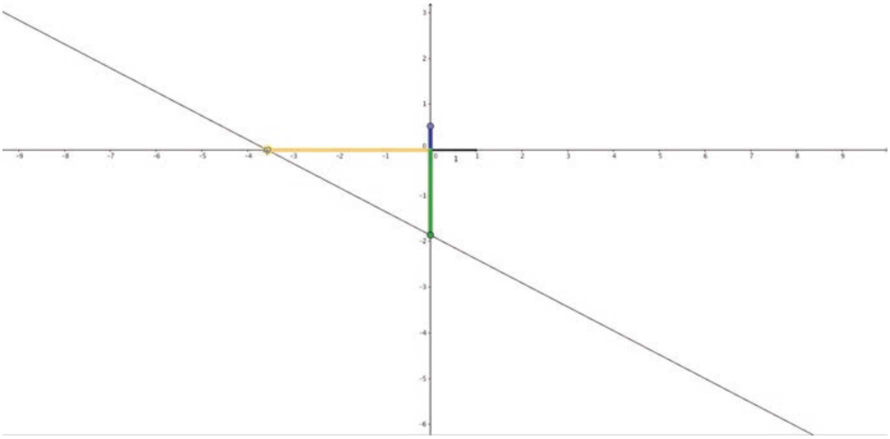

Fig. 13 Sally moves the yellow point to between −3 and −4; then observes, "the [absolute value of] yellow is going to be more than the [absolute value of] green"

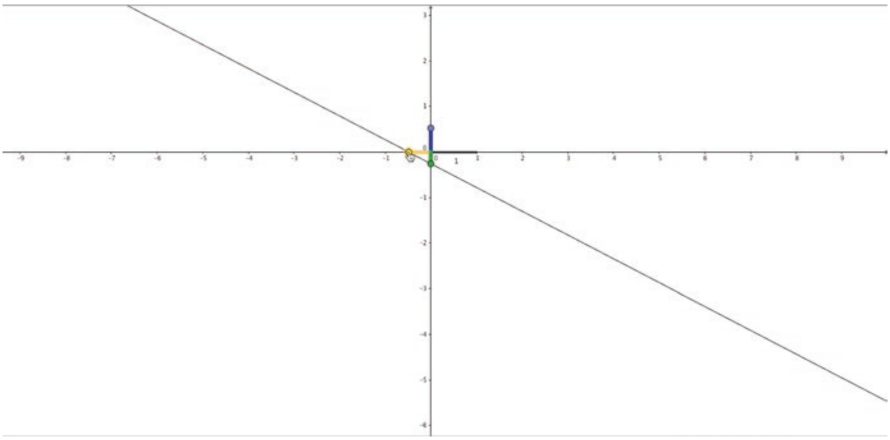

Fig. 14 The participants used the diagram to investigate the products of two numbers that are between 0 and 1

This structural argument is similar to an argument that can be made by comparing the products of different pairs of numbers and looking for a pattern. But the numerical realization of the argument amounts to, at best, a discrete collection of specific instances. The multiplication diagram, by contrast, illustrates the idea by allowing a student to pass through a continuous range of cases. By analogy to dynamic diagrams in the context of geometry—where continuous variation has helped students gain insight about the relationships among geometric figures—the multiplication diagram makes the structure of multiplication accessible to investigation from a visual perspective.

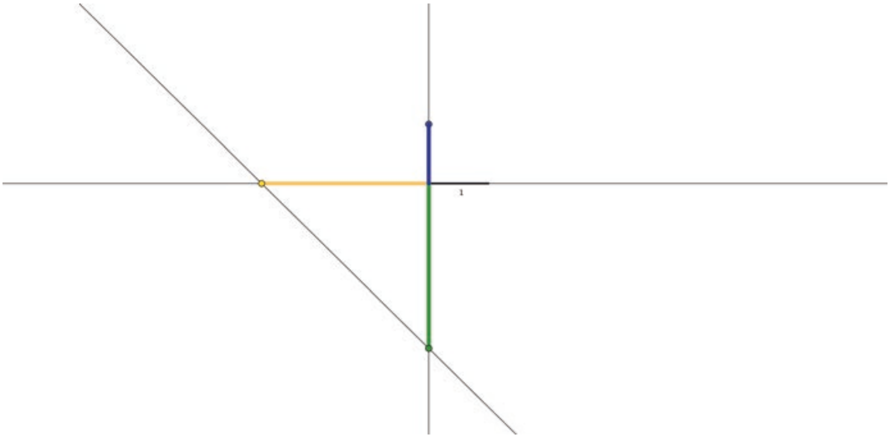

Fig. 15 A positive number (blue segment) times a negative number (yellow segment) is a negative number

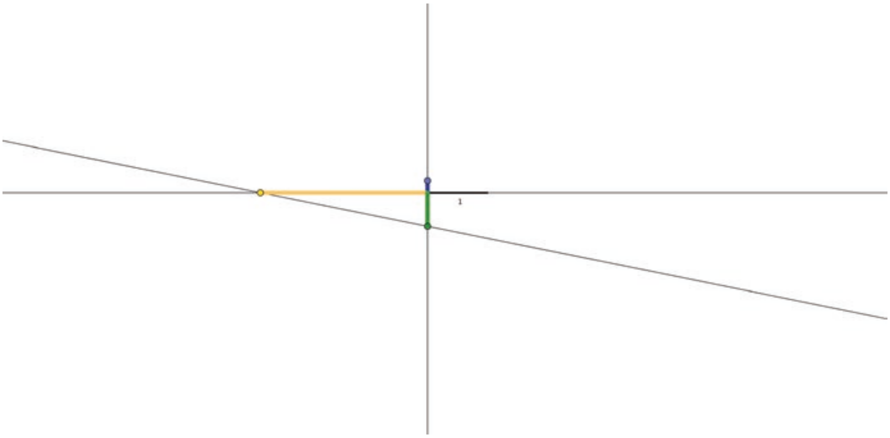

Fig. 16 The positive number approaches 0 and the product increases

One hypothesis underlying our work to develop the diagrammatic models is that more deliberate efforts to foster visual literacy could aid pre-service teachers in mathematical sense-making. We recognize that exploring *that* the diagram represents multiplication does not necessarily mean that pre-service teachers will come to understand, conceptually, *how* the diagram represents multiplication. But the visual structure of the diagram could be conceptually accessible to pre-service elementary teachers. Dake (2007) argues that developing visual literacy can lead to new ways of thinking and communicating; it is akin to learning to think in a new language. The diagrammatic models provide visual resources that could help pre-service teachers understand multiplication as uniform scaling—similar to the notion

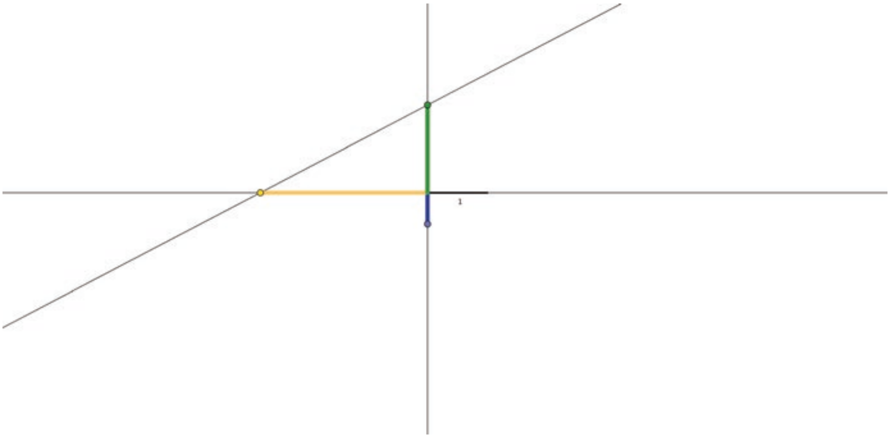

Fig. 17 The positive number becomes a negative number and the product becomes a positive number

of enlarging or shrinking an image by projection, with the parallel line effecting the change in scale. The interview study we conducted with pairs of pre-service teachers did not focus on explaining the model to the participants, but investigating how pre-service teachers would explain/justify how the model works to elementary students is an area for future work.

Division by Zero One final affordance of the diagrammatic models is that they provide a visual representation that division by zero is undefined (Dimmel & Pandiscio, 2020). Students and teachers alike have been shown to have difficulties knowing that or explaining why division is undefined for a zero divisor (Crespo & Nicol, 2006; Tirosh & Graeber, 1989; Tsamir & Sheffer, 2000). With the diagrammatic representation of division, continuous transformations of the diagram can be used to investigate what happens to the quotient as the divisor goes to zero. Figures 18 and 19 show that the quotient grows without bound as the divisor gets closer to zero.

When the divisor is at zero (Fig. 20), the quotient line is parallel to the *y*-axis. Thus, there is no point of intersection, and the quotient is undefined at that point. Learners could keep the divisor at zero and continuously vary the dividend to see that division by zero is undefined for all real numbers.

The diagram provides a visual representation of what it means for division by zero to be undefined. Furthermore, even though division by zero is undefined, it is nonetheless a particular state of the diagram. The visual representation of division by zero could help students appreciate what it means for a quotient to be undefined.

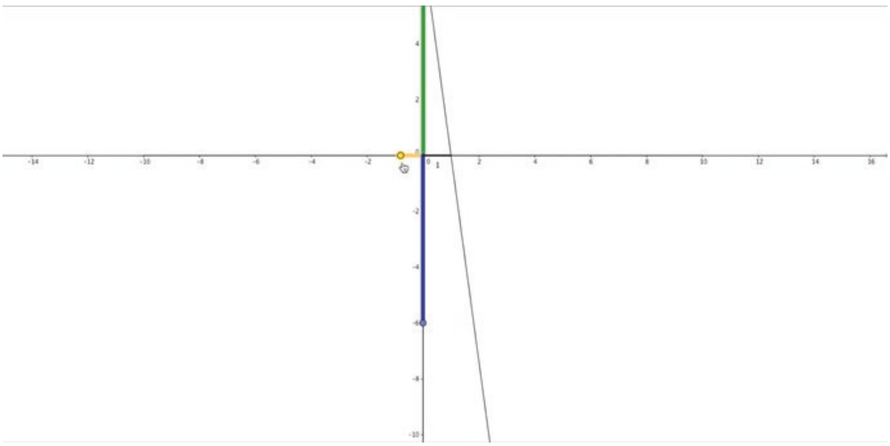

Fig. 18 The quotient increases as the divisor approaches 0

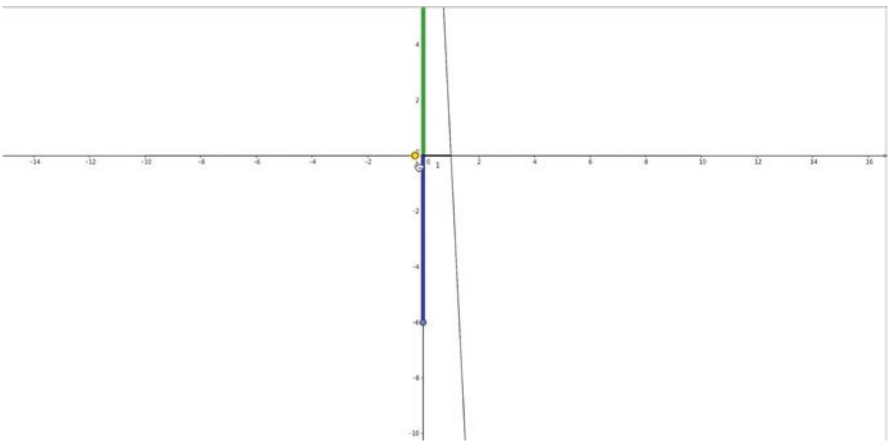

Fig. 19 The quotient continues to increase

4 Diagrammatic Multiplication and Mathematical Borders

We have considered how the elementary mathematics curriculum can seem like crossing a series of borders that separate different sets of numbers, different mathematical topics, and different subjects within the field of mathematics. Arithmetic is a distinct subject area from geometry, and—short of doing calculations with the measurement properties of figures—there are scant opportunities to blur, soften, or otherwise break down the border that keeps them separated. The diagrammatic models of multiplication and division we described here could provide opportunities for students to cross the border between arithmetic and geometry and to use dynamic exploration practices to investigate the structure of the real number system.

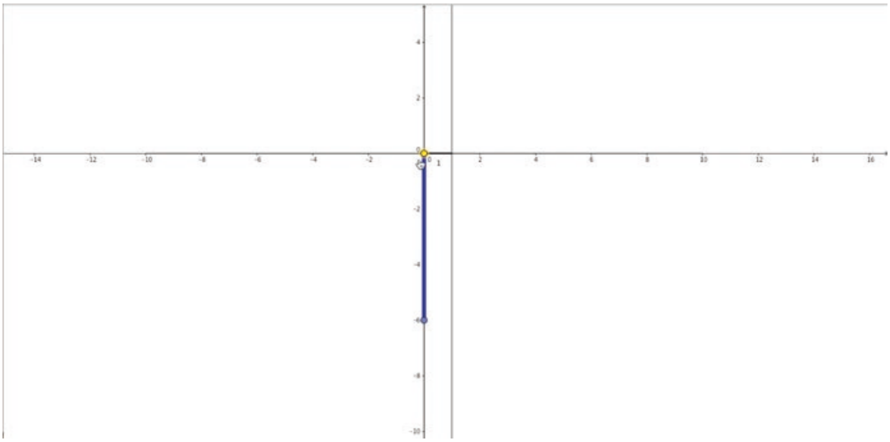

Fig. 20 The divisor is at 0, and the quotient line is parallel to the *y*-axis. The quotient is undefined because there is no point of intersection

Beyond blurring the border between arithmetic and geometry in elementary mathematics, the multiplication and division diagrams could facilitate blurring the border between elementary and disciplinary mathematics. Elementary mathematics, as represented in schools, is enigmatic. The seemingly basic concepts that elementary teachers are expected to develop in their students have deep mathematical roots whose subtleties are challenging for teachers to teach and learners to learn. Elementary students can make observations, ask questions, or share ideas that are vague, unclear, or strange, but that are nonetheless full of mathematical potential.

A well-known example of such a student comment is the case of Sean's Numbers (Ball, 1993; Ball, Lewis, & Thames, 2008). Sean was a third grader in a class taught by Deborah Ball. Ball (1993) described how one day, during a class discussion of even and odd numbers, Sean observed that some numbers could be considered to be both even and odd. Sean offered the number 6 as an example of such a number: It was even because it could be split into two groups without splitting a number in half (this was the definition of even the class had developed); but it could also be considered odd, because it contained three groups of two. After some deliberation, Ball (1993) decided to validate Sean's observation about 6 and stated to the class that Sean had "invented another kind of number that we had not known before" (p. 387).

The case of Sean's numbers is often described as an illustration of the tension between what is expected to be taught—six is not an odd number—and children's ideas—*six is both even and odd*. We see another tension in Sean's numbers. The language that Sean used to describe the number 6 as both "even and odd" is nonstandard, but Sean's recognition that there is something special about 6 and other numbers like it is not trivial. As a third grader, Sean grasped the distinction between *singly-even* and *doubly-even* numbers. A singly-even number is an even number that is congruent to 2 mod 4 (Weisstein, 1996). A doubly-even number is an even number that is divisible by 4 (Weisstein, 1996). The distinction between singly- and doubly-even numbers is significant in several branches of mathematics. For example, in manifold theory, the Euler characteristic (X) of an orientable manifold is even if the dimension of the manifold is singly-even—that is, if the dimension of the manifold is a Sean number (Hoekzema, 2018). The distinction between singly-even and doubly-even integers has its own Wikipedia page that surveys how the distinction is pertinent to topics in pure and applied mathematics.[3] Sean's observation about the number 6 thus captures the tension between disciplinary mathematics and school mathematics; it exemplifies how children can have insights that allow them to peer over the border that separates the math they are expected to learn in schools from the open field of mathematical inquiry.

School may erect borders between elementary and disciplinary mathematics, but Sean's observation about the number 6 shows how students' mathematical imaginations can bypass those borders. As mathematics educators, one of our goals is to establish learning environments where students' mathematical creativity and inquiry can flourish. We believe the diagrammatic models of multiplication and division bring the affordances of dynamic geometry software to bear on students' investigations of the arithmetic structure of the real numbers. Concepts like the rules for signed multiplication or that division by zero is undefined are difficult to motivate or elucidate with real-world things and everyday language. As an additional tool, the visual representations of multiplication and division in dynamic diagrams provide models of these concepts that could help elementary pre-service teachers glimpse their deep mathematical structure.

5 Diagrammatic Models and Pre-service Teacher Education

The diagrammatic models of multiplication and division visually show products and quotients. We see the dynamic diagrams as complementary with the representations of multiplication and division that are typically used in elementary pre-service teacher education. The mathematical structure of the multiplication and division diagrams is conceptually more advanced than other models, in that understanding the diagrammatic models entails facility with compass-and-straightedge constructions and also the capacity for following geometric and proportional reasoning arguments. Still, we do not believe that grasping the mathematical structure of the models is beyond pre-service teachers. On the contrary, we see investigations of *why* the models work as opportunities to develop pre-service teachers' mathematical reasoning, especially as it relates to the conception that multiplication is *scaling*—that is, a distinct mathematical process that cannot be reduced to repeated addition.

We envision using diagrammatic investigations of multiplication and division after pre-service teachers have had the opportunity to investigate these operations

[3] See: https://en.wikipedia.org/wiki/Singly_and_doubly_even

using other, more traditional, models or manipulatives. We see a role for the diagrammatic models in helping pre-service teachers to develop robust concept images of multiplication and division that can help mitigate natural number bias, repair misconceptions about multiplying/dividing numbers between zero and one, and demystify division by zero. Once elementary pre-service teachers have experience using the models to refine their own understanding of multiplication and division, they would be in a position to use the models with elementary students. We have not yet explored using the model with elementary students directly, although we envision that as a natural next step in this line of inquiry.

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Crossing Disciplinary Borders in Pre-service Teacher Education: Historical Consciousness as a Tool to Develop Awareness of Mathematical Positionality to Achieve Epistemic Change

Marta Kobiela and Paul Zanazanian

In this chapter, we use Zanazanian's (2015, 2019) conceptualization of historical consciousness as a framework to support mathematics pre-service teachers in achieving epistemic change about mathematics and mathematics teaching and learning – a change where teachers' beliefs are aligned with approaches to teaching mathematics for meaning. By teaching mathematics for meaning, students develop a range of mathematical competencies, including understanding: Why mathematical ideas are true or valid, how to use a range of procedures and tools, when to use those tools, and how to communicate mathematical reasoning (Kilpatrick, Swafford, & Findell, 2001). Teaching mathematics for meaning requires that teachers shift away from traditional views of mathematics teaching as focused solely on procedures and memorization. This shift necessitates change in individuals' *epistemic beliefs*, that is, their beliefs "about the nature of knowledge and knowing" (Muis, Bendixen, & Haerle, 2006, p. 4). We believe this change can come about through raising awareness of their epistemic cognition, that is, "the processes in which individuals engage in order to consider the criteria, limits, and certainty of knowing" (Maggioni & Parkinson, 2008, p. 446) and the workings of their thinking patterns when engaging with mathematics. Such change is essential to provide students greater autonomy, action, and mathematical understanding – ensuring that they have the skills to cultivate positive relationships with mathematics and have access to engage critically with social and economic issues in our developing world. On this view, epistemic change, for purposes of this chapter, refers to a process where change in epistemic beliefs (the stability, structure, and source of knowledge) results from raising awareness of one's regular (incognizant) thinking patterns when making sense of the world to then make a concerted effort to transform the workings of one's epistemic cognition.

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Our approach is based on Zanazanian's (2015, 2017, 2019) conceptualization of historical consciousness. We consciously present our work in a normative way to provide explicit guidance on how to use historical consciousness as a means or methodology for connecting individuals' mental functioning to their ability to navigate social reality. This is done purposefully in order to reach out to a wider readership beyond academia and into school classrooms and the realm of teaching and learning. In using meanings assigned to the past to navigate social reality, historical consciousness can be the vehicle for epistemic change by allowing pre-service teachers to examine how everyday people use history to think and act in the world, and how they specifically construct social reality to such ends (Zanazanian, 2015, 2017, 2019). We draw upon Zanazanian's (2015, 2019) definition of historical consciousness as rooted in human action – as a way of thinking about one's position in relation to events in time (past, present, future) and as situated in broader social and cultural understandings and contexts. One consequence of this, if steered carefully by a teacher educator, is that pre-service teachers can learn to take critical distance from their knowledge claims when making statements about the world in order to gain awareness of their positionality in relation to their epistemic cognition. Critical distance refers to the extent to which individuals – in our case, pre-service teachers – (Zanazanian, 2019, p. 857). The aim of such an approach is to see the extent to which pre-service teachers seek such fuller understandings of mathematical reality, all the while knowing that it is not possible to do so concretely. Our guiding assumption is that once pre-service teachers gain awareness of the thinking behind their positionality, it will be easier for them to gradually achieve epistemic change – change in their epistemic thinking and positioning – over longer periods of time through sustained reflexive self-analysis.

In this chapter, we ask: How can historical consciousness be used as an approach to support elementary and secondary pre-service teachers to achieve epistemic change regarding mathematics, where epistemic beliefs become aligned with approaches to teaching mathematics for meaning? Our focus in this chapter is situated within the Canadian and US context. We start by providing an overview of the context of educational reform in mathematics education in these countries; how this context has created a need for pre-service teachers' epistemic changes; and what those changes involve. We then describe how history has been used in mathematics teacher education and elaborate on why the historical consciousness framework we propose is a promising approach for supporting epistemic changes in pre-service teachers. Next, we outline a set of design considerations, drawn from the literature on history in mathematics education, to provide guidance for how to implement our historical consciousness approach within a mathematics education context. In the second half of this chapter, we describe what the historical consciousness approach that we propose encompasses and provide a case example – in the context of multi-digit multiplication – of how it might be implemented in a mathematics methods course. We conclude with practical and research considerations in this area. Given that our approach and examples are limited to a euro-centric view, we also discuss limitations and needed areas for future research to expand beyond euro-centrism when integrating mathematics and history.

1 Problem: The Need for Achieving Epistemic Change for Pre-service Teachers

In this section, we justify why pre-service teachers may need support in changing their epistemic beliefs. To do so, we elaborate on why policy changes have moved towards a mathematics-for-meaning approach and away from traditional approaches to teaching mathematics. We then elaborate on our conceptualization of epistemic change, epistemic beliefs, and epistemic cognition and describe what kinds of epistemic beliefs pre-service teachers need to adopt to teach mathematics for meaning.

1.1 The Move Away from Traditional Approaches Towards Mathematics for Meaning

Over the last several decades, numerous reform and policy documents have called for a shift in how mathematics is treated in schools – away from traditional methods of instruction focused solely on procedures and memorization towards a focus on learning mathematics for meaning (Ministère de L'Éducation, du Loisir et du Sport, 2006; National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). Reforms focused on mathematics for meaning encourage students to engage in a range of mathematical practices in which they actively construct mathematical knowledge, including problem solving, posing conjectures and questions, inventing strategies, explaining and justifying claims, identifying patterns, generalizing, and more (Boaler, 2002a; NGA & CCSSO, 2010). The overall aim of these approaches is for students to develop deeper, more robust understandings of mathematics and become aware of their mathematical action – or ability to think and act upon mathematical problems/ideas – with the hopes of becoming autonomous and self-sufficient thinkers and critical contributors to society. Students should ultimately learn skills that will help them engage with the increasing knowledge demands of our society, learning to think critically when solving real, complex problems, when engaging with data from media, and when presented with opposing arguments.

The changes towards mathematics for meaning, at least in the United States, were largely motivated by poor student performance in international exams and by a growing body of research on students' mathematical thinking that provided evidence for the negative effects of traditional instruction (Schoenfeld, 2004). Traditional approaches have been shown to have negative effects on students' mathematical understanding, attitudes towards mathematics, and views of themselves as mathematics learners (Boaler, 1998; Schoenfeld, 2004). For example, through traditional methods, students often misremember definitions or struggle to interpret definitions provided to them (e.g., Dickerson & Pitman, 2016; Roh, 2008). Moreover, mathematical content is cumulative – that is, the content at each grade level builds directly on the previous grade level. Because of this, students' unresolved struggles multiply with each successive grade level. This can have dire consequences: As Moses and colleagues (Moses, Kamii, Swap, & Howard, 1989) describe, mathematical subjects such as algebra can become a "gateway" (p. 424) for higher education and, in turn, economic and social upward mobility. Mathematics success is especially consequential for minority groups, who consistently perform lower on assessments of mathematics proficiency (Schoenfeld, 2004). Traditional mathematics instruction thus jeopardizes efforts to obtain educational and – ultimately – societal and economic equity.

In contrast, mathematics-for-meaning approaches help students overcome the negative consequences of traditional instruction by helping them develop positive relationships with mathematics: Instead of viewing teachers and textbooks as the sole authority, students learn to become authors of mathematics and to see themselves as mathematical actors capable of seeking out, critiquing, and constructing mathematical knowledge (Ball, 1993; Boaler, 2002b; Kobiela & Lehrer, 2015; Lampert, 1990). Teachers can help to cultivate students' action – and, consequently, their positive relationships with mathematics – by inviting and responding to the varied methods and solutions that they bring to their classrooms. Countless studies have shown that through such approaches, students develop deeper and more lasting understandings of content (e.g., Boaler, 1998; Kobiela & Lehrer, 2015; Lehrer & Pritchard, 2002). Moreover, recent research has shown that a mathematics-for-meaning approach carries long-term benefits compared to the traditional approach: Students who were provided opportunities to learn mathematics in these ways in secondary school maintained positive attitudes towards mathematics in adulthood and had higher economic upward mobility, compared to students who had experienced traditional forms of mathematics (Boaler & Selling, 2017).

However, despite the well-documented benefits of teaching mathematics for meaning, traditional methods of instruction still persist today for a number of reasons. To begin, meaningful teaching is challenging – it requires teachers to be attentive, responsive, and welcoming to the range of ideas students bring to class (Ball & Cohen, 1999). Adding to this challenge, these approaches are often unfamiliar to teachers because they differ from their own experiences with learning mathematics (Ball, 1988). Moreover, the reform has often received pushback from parents, teachers, and politicians who feel strongly that students should focus on learning their basic mathematics facts (Schoenfeld, 2004).

1.2 Conceptualizing Epistemic Change Towards Mathematics for Meaning

In order to move away from traditional forms of instruction and instead implement mathematics for meaning, teachers need to achieve epistemic change. As described previously, epistemic change first involves changes in epistemic beliefs. These occur when "individuals actively construct or make meaning of their experiences" (Muis et al., 2006, p. 30), becoming aware of their regular (incognizant) thinking patterns through their social interactions. Throughout their education, teachers experience many classroom environments, each of which provide a particular epistemic climate that, through direct and indirect signals, send messages about the nature of knowledge and knowing (Muis & Duffy, 2013). Teachers' epistemic beliefs are thus constructed over many years of interaction and social enculturation through both their life and work experiences. To shift these deeply held beliefs, teachers must engage in and develop their epistemic cognition (Maggioni & Parkinson, 2008). This includes a process where they become aware of and consciously transform the workings of their epistemic cognition by deliberately reflecting on their thinking, patterns, and, by extension, their epistemic beliefs. In the remainder of this section, we describe the epistemic beliefs that teachers must reflect on to align with a mathematics-for-meaning approach.

According to Muis and colleagues (Muis et al., 2006; Muis & Duffy, 2013), epistemic beliefs can be conceptualized along four dimensions. First, individuals may view knowledge as certain or uncertain – that is, as fixed versus unknown and changing – and as simple versus complex. In line with this dimension, to move towards a mathematics-for-meaning approach, teachers must view mathematics as an evolving discipline rather than a static, unchanging entity with fixed and absolute truths about knowledge (Chrysostomou & Philippou, 2010). At the same time, teachers must come to see mathematics as a humanistic discipline – that is, one in which humans construct and create mathematical knowledge through social exchanges – often a messy, non-linear process. These perspectives reflect how mathematics is treated in the discipline of mathematics. For example, in his historical analysis of the development of the proof for the Euler Characteristic, Lakatos (1976) illustrated how members of the larger mathematical community shifted the direction of discovery through introducing new counterexamples to the initial conjecture. These counterexamples led to re-examination of the proof. The "zig-zag" (p. 42) between proofs and counterexamples yielded suggestions for an improved conjecture.

The second dimension of the epistemic beliefs framework highlights individuals' views of how knowledge is justified – through expert authorities or through one's own opinion and experience. Similarly, the third dimension captures individual's understandings of whether knowledge originates from authority figures or from oneself. Both of these dimensions reflect what individuals believe in terms of who has the authority to engage in and construct mathematics. To move towards a mathematics-for-meaning approach, teachers must shift away from seeing the teacher's role as a disseminator of knowledge – as the sole authority – to one who guides students in developing reasoning (Cross, 2009). In this perspective, mathematical expertise is not innate, but is subject to change and growth (Chrysostomou & Philippou, 2010). Thus, students can become actors and authors of mathematics and have the potential, with the teachers' support, to become mathematical experts. This requires teachers to view mathematics as encompassing forms of reasoning that students can cultivate rather than just formulas and arithmetic operations that are predetermined by external authorities.

Finally, the fourth dimension of epistemic beliefs refers to whether individuals believe that truth is always attainable or not. To align with a mathematic-for-meaning approach, teachers must move away from traditional views of mathematics instruction as always leading to one absolute answer or method and instead believe that there are multiple ways to approach mathematical problems (Chrysostomou & Philippou, 2010). This view is important for understanding the cultural roots of mathematics – as not existing solely as a western enterprise, but as originating in many cultures – each of equal value (Stedall, 2012).

Although shifting teachers' epistemic beliefs in the ways described above is important, this cannot be done without also engaging in and developing their epistemic cognition to become attentive to their thinking processes about the "criteria, limits, and certainty of knowing" (Maggioni & Parkinson, 2008, p. 446). As we describe next, history of mathematics provides a promising epistemic climate for teachers' engagement in epistemic cognition.

2 The Promise of a History of Mathematics Approach to Achieve Pre-service Teachers' Epistemic Change Through Transforming Epistemic Cognition

In this section, we draw upon the literature on history in mathematics education to justify why history of mathematics has the potential for supporting changes in pre-service teachers' epistemic thinking/cognition and to describe how it has been implemented in teacher education programs. History can provoke epistemic cognition by encouraging pre-service teachers to rethink their understandings of and relationships to mathematics – important for achieving change in epistemic thinking towards mathematics for meaning. Integrating history into teacher education programs can support pre-service teachers to understand how mathematical ideas develop (Furinghetti, 2007; Povey, 2014), to gain increased confidence in teaching mathematics (Charalambous, Panaoura, & Philippou, 2009), to acquire deeper and expanded understandings of mathematics (Youchu, 2016), to improve their confidence in their understanding of mathematical topics (Clark, 2012; Fenaroli, Furinghetti, & Somaglia, 2014; Galante, 2014), and to develop an increased openness towards mathematics and empathy towards mathematics learners (Guillemette, 2017).

The history of mathematics has been integrated into both classroom and teacher education contexts for some time now (Jankvist, 2009). In his review, Jankvist (2009) describes two reasons that researchers suggest for integrating history into mathematics learning environments. The first, *history as a goal*, suggests that integration of history can support learners to understand aspects of the history of mathematics, including how mathematics has evolved and developed over time and the role of humans and different cultures in its development. In contrast, the second reason, *history as a tool*, suggests that integration of history can help learners make sense of mathematics in new ways, can motivate and inspire learners to apply mathematics differently, or can help learners (in particular, teachers) understand particular developmental struggles that students might experience.

To achieve these goals within teacher education contexts, history has been integrated using a variety of methods. One common approach has been to ask pre-service teachers to solve historical mathematics problems to help them experience mathematics in a different way (Charalambous et al., 2009; Clark, 2012; Galante, 2014). Similarly, researchers often provide pre-service teachers with original sources to analyze, such as historical textbooks or mathematical papers (Fenaroli et al., 2014; Guillemette, 2017). Other researchers have asked pre-service teachers to complete reflective journals in which they consider connections between a historical topic and mathematics teaching and learning (Galante, 2014) or connections to their own learning (Clark, 2012). In some courses, pre-service teachers conducted extensive research projects to investigate a historical topic in mathematics (Furinghetti, 2007; Galante, 2014). Pre-service teachers also developed lesson plans in which they were required to integrate what they had learned about the history of mathematics for a particular topic (Fenaroli et al., 2014; Furinghetti, 2007; Youchu, 2016). Teacher education programs often leverage more than one of these methods, sometimes complementing these learner-centered approaches with lectures on historical topics (e.g., Charalambous et al., 2009; Fenaroli et al., 2014).

Despite these positive results, studies also highlight cases in which pre-service teachers maintain their traditional epistemic beliefs of mathematics, despite having opportunities to solve mathematical historical problems and to consider connections to the curriculum (e.g., Charalambous et al., 2009). In the following section, we draw upon lessons learned from this body of research to outline considerations for integrating history into mathematics teacher preparation.

3 Design Considerations for Integrating History into Mathematics Teacher Preparation to Achieve Epistemic Change

In order to design effective epistemic climates for achieving epistemic change in pre-service teacher education, we drew upon the research literature on integrating history of mathematics into elementary and secondary teacher education. We identify four considerations for designing activities to help provoke pre-service teachers' engagement in epistemic cognition: (1) pre-service teachers need training to incorporate history into mathematics; (2) teacher educators should consider pre-service teachers' backgrounds when selecting historical excerpts; (3) historical texts need to provide connections for empathy; and (4) historical analysis should incorporate experiencing mathematics. These considerations are not meant to be all encompassing, but highlight some dominant themes in the literature that are relevant to achieving epistemic change towards mathematics for meaning. Moreover, as we elaborate on in the Concluding Thoughts section, one limitation of these considerations is that they are grounded in a euro-centric vision of history, mathematics, and history of mathematics. However, because they are based on existing empirical research on integrating history into mathematics teacher education, we are limited by the foci of that research literature. We elaborate on each of these considerations next. Our proposition below of an anthropocentric and cultural understanding of history's uses for making sense of the world – which once made aware of – can take these considerations further. Based on individuals' everyday thinking of history, they provide input into one's intentions and decision making when faced with mathematical problems of an historical nature.

3.1 Consideration 1: Teachers Need Training to Incorporate History into Mathematics

The first design consideration for integrating history into mathematics teacher training is that it is necessary to train teachers on how to make sense of history. Because mathematics pre-service teachers often have limited experience with thinking like historians (Furinghetti, 2007), they may not be able to engage with the materials in a way that could provoke epistemic cognition. In secondary teacher education programs, although pre-service teachers may take an entire course in the history of mathematics (e.g., Clark, 2012; Youchu, 2016), such courses often use tools from history without providing teachers proper training in how to use those tools (Fauvel, 1991). In elementary teacher education programs, pre-service teachers must take courses from a range of disciplines. However, this may leave little space for mathematics-specific courses, including those focused on history of mathematics. This is concerning given that elementary pre-service teachers may have little experience with the history of mathematics. For example, in a survey of 100 pre-service and in-service elementary teachers on their understanding of key figures and cultures in the history of mathematics, Gazit (2013) found that both groups lacked in their understanding of these topics, scoring an average of only 40.1% correct on the items. Thus, in order to integrate history of mathematics to provoke epistemic cognition, both elementary and secondary mathematics teacher training programs need to provide opportunities for pre-service teachers to learn how to analyze historical texts and apply forms of historical reasoning. Here, however, we focus on everyday individuals' thinking of history, and not historians'.

3.2 Consideration 2: Teacher Educators Should Consider Pre-service Teachers' Backgrounds When Selecting Historical Excerpts

As a second consideration, pre-service teachers need to find the mathematical content in the historical examples accessible. As discussed previously, many efforts to integrate history and mathematics in teacher training include opportunities for pre-service teachers to examine, analyze, or re-create examples from the history of mathematics. If the content is not accessible, pre-service teachers may struggle to connect to and make sense of the historical example – creating a barrier to the reflection needed for engaging in and changing epistemic cognition. For example, Charalambous et al. (2009) found that elementary pre-service teachers felt that the content in the historical examples examined in a course was too challenging, and, as a result, experienced a lack of confidence. In addition, the pre-service teachers in the study reported that they felt that the content in the historical examples was not relevant to what they would be teaching in elementary school. It is possible that these elements had negative effects on how pre-service teachers interpreted and related to the historical examples. The researchers found that by the end of the course, pre-service teachers had stronger beliefs of mathematics as rule-based, lessened beliefs of mathematics as constructivist and dynamic, and felt more negatively towards mathematics – showing trends towards epistemic beliefs that reflected traditional mathematics instruction instead of mathematics for meaning. Thus, in order for pre-service teachers to position themselves and their teaching in relation to historical examples, the mathematical content must feel accessible and relevant to them.

3.3 Consideration 3: Historical Texts Need to Provide Connections for Empathy

Third, in addition to being mathematically accessible, historical texts should be selected to encourage pre-service teachers to develop empathy towards people of the past. Some scholars suggest that original sources (e.g., original textbooks, mathematician's papers) can be one way to attain empathy. For example, Guillemette (2017) found that by making sense of the original texts of mathematicians, pre-service teachers were able to develop empathy for the struggles that mathematicians expressed in their writings. The pre-service teachers also came to see mathematics in new ways – as open and encompassing different ways of thinking, perspectives that they hoped to share with their future students.

Developing empathy for historical actors can impact how pre-service teachers approach their teaching in two ways. First, pre-service teachers may be motivated to foster empathy in similar ways in their future students. One way of doing so can be to have pre-service teachers reflect on the implications of the historical mathematics for students' learning. For example, as part of a history of mathematics course, Clark (2012) asked pre-service teachers to read an historical excerpt of a mathematical explanation. Pre-service teachers were asked to develop the mathematical argument, report to the whole class, consider how using the historical excerpt might impact students' learning of the topic, and reflect on how they might approach their teaching. In their reflective journals, several pre-service teachers noted that they would want to incorporate history into their teaching to impact students' interest in mathematics and to develop their awareness of other cultures in mathematics. Thus, the combination of reading historical texts and considering implications for students' learning allowed these pre-service teachers to consider how they could foster empathy within their future students. Second, by developing empathy for people of the past, pre-service teachers may come to understand and empathize with their future students in new ways. For example, in the study described above by Guillemette (2017), by developing new perspectives on the struggles experienced by mathematicians of the past, pre-service teachers were able to reflect on the experiences that their future students might have when learning mathematics.

3.4 Consideration 4: Historical Analysis Should Incorporate Experiencing Mathematics

Finally, as a fourth consideration, when engaging with historical texts, teachers should have opportunities to experience the mathematics of the past. Doing so provides pre-service teachers opportunities to be able to grapple with mathematics and develop expanded understandings (Povey, 2014) that move beyond epistemic beliefs of mathematics as traditional. For example, in her study, Furinghetti (2007) found that by having pre-service teachers make sense of the mathematics in historical texts, pre-service teachers placed themselves in the perspective of the mathematicians who authored or contributed to the texts. This perspective in turn impacted how they wrote and carried out their lesson plans. In contrast, in the study by Charalambous et al. (2009) described earlier, in which pre-service teachers left the course with more traditional views of mathematics, in their interviews, pre-service teachers acknowledged that they did not have enough opportunities to engage with and experiment with the mathematical content, despite being asked to solve historical mathematics problems. The lack of opportunity to do so may have contributed to their difficulties in accessing mathematics. These results suggest that the ways in which pre-service teachers are asked to engage in mathematics must provide them with opportunities to experience mathematics from the perspective of the historical actors.

The four instructional considerations we outline in this section, which emerge from the mathematics education literature, are intended to guide how teacher educators use history to support pre-service teachers to achieve epistemic change towards mathematics for meaning. As we argue in the next section, historical consciousness can be one approach that surpasses these considerations and supports change in pre-service teachers' epistemic beliefs. We thus illustrate how the four instructional considerations can be integrated with an historical consciousness approach in a mathematics methods course. By engaging in self-analysis through reflection and justification, historical consciousness can allow pre-service teachers to take critical distance from their knowledge claims when making mathematical statements about the world in order to become aware of their differing positionalities when engaging with mathematics (Zanazanian, 2019).

4 Historical Consciousness: An Approach to Encouraging Epistemic Change

In following Zanazanian's (2015, 2017, 2019) theorization, historical consciousness, as an object of inquiry, permits examining how everyday people use history to think and act in the world, and how they specifically construct social reality to such ends. As part of the larger study of human memory, historical consciousness nonetheless stands out from other memory frameworks because of its focus on the individual and how they provide order and meaning to temporal change. As a form of cultural production, history's reach expands here to account for its everyday life uses, with historical consciousness constituting the resulting ideas that emanate when making historical sense of the past (Rüsen, 2017). Connecting everyday people's thinking to the cultural, institutional, and historical contexts of their lives, individuals employ conceptual resources to make meaning and to ultimately attain their historical consciousness' end-point configurations (Wertsch, 1998; Zanazanian, 2015, 2017, 2019). Embedded in the differing epistemologies of their respective thought communities, individuals' uniqueness and autonomy in their thinking and acting arise according to the manner in which they employ the conceptual resources that exist at their disposal in their personal repertoires or cultural toolkits for doing so (Zanazanian, 2015, 2017, 2019).

At a general level, these conceptual resources include both the narrative configurations of the past and the interpretive filters used to make sense of that same past. Located in larger social processes and in the collective consciousness of individuals' groups and broader cultures of belonging, these resources trickle down, mingle, and mix in everyday people's thinking when giving meaning to the past. The more they are prevalent among people, the more they risk being interiorized as the correct way to proceed. Individuals (incognizantly) employ the conceptual resources they deem the most appropriate for the specific contexts they find themselves in (Wertsch, 1998; Zanazanian, 2015, 2017, 2019). Based on Zanazanian's (2015, 2017, 2019) theorization, memory frames underlying the past's narrative configurations help structure understandings of the content of the past that form the "stuff" of collective memory and history – key dates, places, actors, settings as well as sequences of events – whereas history-as-interpretive-filter's memory frames invoke those intellectual functions assigned to history as a mode of thought for reading, organizing, and explicating social reality. If individuals unreflexively instead of reflexively rely on these pre-given memory frames, conventional means for making sense of the past risk arising (Zanazanian, 2015, 2017, 2019). In the latter instance, historical consciousness holds the potential of encouraging open attitudes regarding history's intellectual uses for reading reality and engaging in it. Of importance is the degree to which individuals take critical distance from their resulting knowledge claims when enacting their history-as-interpretive-filter memory frames. The more critical distance they take when doing so, the more the chances that they transcend their own positionalities and self-imposed limits for grasping the world around them (Zanazanian, 2015, 2017, 2019).

Of interest for our chapter, grasping individuals' uses of history-as-interpretive-filter can help seize the role historical consciousness plays in the production of knowledge (Zanazanian, 2019). For analytical purposes, this permits one to see how individuals actually use history for solving life/social (mathematical) problems of a historical nature and the extent to which they consequently develop positive – critical, self-reflexive – mindsets in that regard. Based on this logic, the idea is to develop mindsets that encourage individuals to seek nuances in how they exercise their historical consciousness when constructing understandings of social reality. What this raises is the issue of promoting an understanding for a need for self-reflexivity in how pre-service teachers use history when producing (mathematical) knowledge and when teaching mathematics. This can be done by answering guided reflection questions when having pre-service teachers read about different historical contexts or case studies for using math, as will be elaborated below. On this view, the ultimate end goal is to obtain and develop a clearer understanding of how history is employed and promoted and what this can do for raising pre-service teachers' awareness of their mathematical action and, by extension, their pedagogical abilities when teaching mathematics. By reflecting on their uses of history-as-interpretive-filter, the end result for pre-service teachers is to gain the necessary mechanisms for organizing their understandings of mathematics and its cultural potentials for life purposes in meaningful ways, to then independently channel their ideas to decentralize their thinking for using mathematics more openly and inclusively, mindful of its many possibilities for positive change.

5 Problematizing Pre-service Teachers' Positionality Regarding Multi-digit Multiplication Algorithms: A Case Example of Epistemic Change Through Historical Consciousness

To illustrate how historical consciousness can be used as a self-reflexive tool to promote pre-service teachers' epistemic change, we present three phases of our pedagogical approach. The aim across each phase is to encourage self-analysis through reflection and justification with the end goal of writing self-portraits that

Fig. 1 Example of the US and Canadian standard algorithm for multi-digit multiplication

$$\begin{array}{r} {\scriptstyle 1} \\ 23 \\ \underline{\times\ 41} \\ 23 \\ \underline{+\ 920} \\ 943 \end{array}$$

pre-service teachers can expand on through their teacher education. The three phases are (1) developing one's positionality regarding history and mathematics by having pre-service teachers take ownership of historical ideas from an historical text; (2) engaging autonomously with historical cases to use history to make sense of a mathematical or teaching problem that emerges in social reality; and (3) fostering the habit of taking critical distance when positioning themselves through sustained practice. Phases 1 and 2 involve writing narrative texts that pre-service teachers store until Phase 3, where they analyze and code their thinking inductively. Through this practice, key components that underlie their epistemic cognition can be identified and transformed to promote epistemic change.

We illustrate each step with a hypothetical example grounded in the context of multi-digit multiplication algorithms. A multi-digit algorithm refers to a repeatable method or process for solving arithmetic problems involving numbers with more than one digit (e.g., 23 × 41) (Kilpatrick et al., 2001). Our example is grounded within the US and Canadian context. As we describe in more detail later, in the United States and Canada, the algorithm for solving multi-digit multiplication involves a process of sequentially multiplying each digit of one number with each digit of the other number. These values are added to determine the final product. For example, the procedure for multiplying 23 by 41 using this algorithm is shown in Fig. 1. We begin by justifying why multi-digit multiplication algorithms serve as a rich context for cultivating pre-service teachers' epistemic change towards mathematics for meaning. Because of the centrality of multi-digit algorithms in US and Canadian mathematics curricula and because pre-service teachers have had ample experiences with US and Canadian standard multi-digit algorithms, this topic provides a familiar and thus accessible means for pre-service teachers to make sense of historical cases (Design Consideration 2). By rethinking what it means to learn and teach mathematics within the context of multi-digit multiplication, pre-service teachers can learn to value their students' varied methods and to invite students to develop action as authors of these algorithms – approaches in line with mathematics for meaning.

5.1 Justifying Multi-digit Multiplication as a Context for Cultivating Epistemic Change

Across all operations, the treatment of multi-digit algorithms in Canadian and US schools has been a topic of reoccurring debate. Traditional approaches to mathematics in these countries have focused on asking students to memorize and use the standard algorithm, often using the steps demonstrated by the teacher. In contrast, mathematics reform efforts aligned with mathematics for meaning suggest that students should invent and make sense of a range of algorithms (e.g., NCTM, 2000). As seen more broadly, students who have such opportunities show stronger mathematical understanding than students who learn via traditional approaches (e.g., Hiebert & Wearne, 1993). Moreover, many teachers experienced traditional forms of mathematics instruction during their schooling, in which they were told to follow the standard algorithms, without exposure to other algorithms. This experience can mislead teachers to perceive the standard algorithm as the sole, correct, and/or "best" way to solve multi-digit arithmetic problems – reinforcing their views of mathematics as involving one singular, formulaic approach to solving problems. Yet, across the world, different cultures use different algorithms, many of which have made their way into North American homes with waves of immigrant populations (Philipp, 1996). In order to cultivate students' positive relationships with mathematics, teachers must position their students as mathematical actors by inviting and responding to the varied algorithms that students might bring to their classrooms.

In addition, when students have the opportunity to make sense of a wide range of methods, they develop greater confidence in mathematics (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Being able to come up with multiple strategies to solve mathematical problems and to approach problems flexibly is a key aspect of mathematical competency (Kilpatrick et al., 2001). Research suggests that when teachers use their understanding of children's diverse strategies when teaching, their students' conceptual understanding and problem-solving abilities improve (Fennema et al., 1996). Thus, teachers must be able to recognize and make sense of the validity of a wide range of algorithms – both in order to respond to the diverse methods that students might bring to class and to encourage students to invent and explore different methods. Doing so places students in a position to develop mathematics for meaning – cultivating their autonomy and action as mathematical knowers and thinkers. In what follows, we describe three phases of an instructional sequence that may be used with pre-service teachers. These three phases are illustrated in Table 1.

Table 1 Phases of instructional sequence

Phase	Purpose	Overview of steps for the phase
1. Developing one's positionality regarding history and mathematics	To provide pre-service teachers opportunities to consider the intellectual uses of history for making sense of mathematics. This goes beyond thinking like historians to employing everyday individual's uses of history as a mode of thought.	1. Pre-service teachers read an initial text that outlines issues of the history of mathematics. 2. Pre-service teachers write a reflection piece (Guided Reflection Piece 1) in which they apply the ideas from the text to construct an episode as a hypothetical teaching situation where they connect the importance of history for thinking about mathematics to teaching mathematics for meaning. 3. Pre-service teachers respond to follow-up questions that encourage them to delve deeper into the connections between their epistemic views of mathematics and mathematics teaching and a historical perspective on mathematics (Zanazanian, 2019).
2. Engaging autonomously with historical cases to use history to make sense of problems	To provide pre-service teachers opportunities to apply what they gained from Phase 1 to a different situation in a more autonomous manner by using information in cases exterior to their experiences.	1. Pre-service teachers read historical cases provided to them. 2. Pre-service teachers explain the mathematics in each case and individually write a second guided reflection piece (Guided Reflection Piece 2) using provided prompts.
3. Fostering the habit of taking critical distance when positioning oneself through sustained practice	To provide pre-service teachers opportunities to step back and critically examine their thinking in their two reflection pieces and follow-up ideas to consider their positionality in relation to mathematics, history, and teaching mathematics for meaning.	1. Pre-service teachers review their narratives from Phases 1 and 2 to code what they have written according to questions provided to them. 2. Pre-service teachers synthesize key ideas by completing a set of prompts about the nature of mathematics, history, teaching mathematics for meaning, using history to teach mathematics for meaning, and justifications for each of these points. 3. Pre-service teachers interpret connections across their analyzed "data." 4. Pre-service teachers write a summary of their findings in a brief descriptive report. 5. Pre-service teachers analyze and interpret the reflections and follow-up questions for two other classmates. Within this group, they should then compare their analyses and interpretations for each individual's set of reflections and follow-up questions and come to a final consensus. 6. Pre-service teachers write a reflection in which they use the agreed upon analyses and interpretations to develop a self-portrait of themselves as a mathematics person or as a mathematics teacher.

5.2 Phase 1: Developing One's Positionality Regarding History and Mathematics

In order to support pre-service teachers to use history in thinking about mathematics (Design Consideration 1), they first need opportunities to consider the intellectual uses of history for making sense of mathematics. This goes beyond thinking like historians to employing everyday individual's uses of history as a mode of thought. This is based on Zanazanian's (2015, 2017, 2019) approach developed to measure the impact of individual's historical consciousness on their positioning when making sense of social reality – the extent to which they take critical distance from their knowledge claims when making statements about the world through thinking historically. Thus, the first phase of our instructional approach provides opportunities for pre-service teachers to develop their positionality in relation to history, to mathematics, and to mathematics teaching and learning through self-analysis related to an historical text. Important here is that pre-service teachers have opportunities to reflect on and justify their thinking to help develop awareness of their positionality in relation to their general epistemic beliefs about mathematics and the implications of those beliefs for mathematics teaching and learning.

To help provoke their thinking about their epistemic beliefs, Phase 1's first step thus requires teacher educators to ask pre-service teachers to read an initial text that outlines issues of the history of mathematics, including the nature of mathematics as an evolving and changing discipline, the multiple cultural roots of mathematics, the spread of mathematics, and the role of humans as actors in mathematics. For example, pre-service teachers may read the book, *The History of Mathematics: A Very Short Introduction*, by Jacqueline Stedall (2012). In there, Stedall chronicles the history of mathematics thematically, attending to the stories of a range of people from a variety of places. The chapters consider issues such as the meaning of mathematics, who counts as a mathematician, the spread of mathematical ideas, and the history of mathematics education.

After reading the text, as a second step, the teacher educator should ask preservice teachers to write a reflection piece in which they apply the ideas from the text to construct an episode as a hypothetical teaching situation where they connect the importance of history for thinking about mathematics to teaching mathematics for meaning. An episode, although reflective in nature, should communicate a narrative. Here, it is important that pre-service teachers go beyond simply summarizing ideas from the text and instead apply those ideas through their own cognizant thinking to explain how history can be used to make sense of the nature of mathematics and/or its purpose when teaching. Of key importance here is that pre-service teachers elaborate on the rationale or reasoning behind what they are saying (Zanazanian, 2019). Following Zanazanian (2019), it is through justifying their ideas that we can gain insight into pre-service teachers' thinking processes and what they believe or say about knowledge construction when using history to understand mathematics and to teach it for meaning. In writing this episode, pre-service teachers should develop a narrative that tells a story with details about what individuals were doing and saying in order to position themselves as actors within the story. By "actors," we mean that pre-service teachers should able to imagine themselves as engaging in the roles of the individuals within the story. To accomplish this, pre-service teachers might be asked: *Based on your reading of* Stedall *(*2012*), develop a fictive episode where you would use history to inform how you approach* your *understanding of mathematics and its teaching*. *Please write the episode to communicate the story from beginning to end with as much detail as possible (*e.g., *include what individuals in your story say and do).* To get at their spontaneous thinking, pre-service teachers should write this reflection during class time, with approximately 40 min to do so. This should be done on the spot without access to the readings, to notes or to the Internet. The objective is for them to see how they spontaneously apply knowledge to concrete situations to make meaning without having to search for information on how to do so, but to use what they already know automatically. The reading should help initiate their thinking, but should not be a crutch in their reflective process. It is thus important to make sure that pre-service teachers understand the importance of reading the text beforehand.

Following their reflections, as a third step, the teacher educator should ask pre-service teachers to respond to follow-up questions that encourage them to delve deeper into the connections between their epistemic views of mathematics and a historical perspective on mathematics (Zanazanian, 2019). The key here is to see how they view history as a mode of thought and knowledge construction and to see how they apply this thinking to mathematics. The aim is to get at their epistemic thinking to understand how we can help transform it (towards mathematics for meaning). For example, pre-service teachers may be asked: *(1) Why did you select the episode that you did? What were you trying to say/do? (2) Based on what you have read, what is history and how does it work? (3) How can history help us to better understand mathematics and its key principles and uses? (4) How can history help us teach mathematics to better support sense-making and meaning? (5) Do you think a traditional approach to teaching mathematics (memorization of formulas and patterns) has more value/is more efficient? Why? (6) What do you see as the purpose of mathematics? (7) What is your role as a mathematics teacher?* They should answer these questions in writing and archive them for their upcoming self-portraits in Phase 3.

5.3 Phase 2: Engaging Autonomously with Historical Cases to Use History to Make Sense of Problems

After their initial self-analysis, as the second phase in our pedagogical approach, pre-service teachers should then zoom into a particular problem related to using history for better grasping the nature and function of mathematics and for teaching mathematics for meaning. To do so, pre-service teachers should be presented with one or more historical cases in mathematics. The purpose here is for pre-service teachers to apply what they gained from Phase 1 to a different situation in a more autonomous manner by using information in cases exterior to their experiences. Within the teacher education course, teacher educators may ask pre-service teachers to read the cases and, in approximately 40 min, explain the mathematics in each case (Design Consideration 4) and individually write a second guided reflection piece (Guided Reflection Piece 2) using provided prompts. Prompts for Guided Reflection Piece 2 require pre-service teachers to again write a narrative essay, whereby in writing their episode, they situate their thinking in relation to the past, present, and future (Zanazanian, 2015, 2017, 2019). The prompts must again encourage pre-service teachers to justify and account for their choices, with the intended aim of seeing the extent to which they take critical distance from their knowledge claims when they later analyze their data. For example, for the two cases we describe in the next paragraph, pre-service teachers might be asked to reflect on the following: *Using what you read in the book, look at the two cases provided. Write a fictive episode in which you explain how you would use the historical ideas within the cases to teach students multi-digit multiplication in a way that goes beyond memorizing the standard multiplication algorithm and that shows how mathematical knowledge is constructed. Please write the episode to communicate the story from beginning to end with as much detail as possible (*e.g., *include what individuals in your story say and do).* The purpose of this question is to encourage pre-service teachers to consider the past (i.e., the historical cases) in connection to their future positionality as a teacher and to question their existing (present) assumptions about what mathematics should count as valid in their mathematics classrooms.

To illustrate, we selected two cases to provoke students' positionality in relation to teaching the standard multiplication algorithm: Russian Peasant Multiplication and a method created by Italian abacists that resembles the standard US/Canadian algorithm. The goal here is to see how pre-service teachers apply the historical knowledge gained through their readings and applied in their guided reflection piece and follow-up questions to an episode where they have to use these historical cases to teach mathematics for meaning and to show how mathematical knowledge is constructed. By justifying their thinking within their episodes, pre-service teachers would be able to uncover their positionalities in relation to their epistemic beliefs about multi-digit multiplication algorithms, and ultimately, mathematics more generally. Because most Canadian and US teachers have experiences seeing the standard US/Canadian algorithm as the sole method, the cases can help them see that there can be alternate approaches to solving the mathematical problem developed by humans based on their social needs of the time – a key aspect to reflect on in their written piece.

Each case shows a valid method for solving multi-digit multiplication problems. Both methods served distinct purposes in commerce and satisfied the needs of the individuals who used them at the time. The first case is intended to help pre-service teachers see the validity of an approach that might be less familiar to them, but that is relatively accessible for students and thus can provide students opportunities for developing their autonomy and action as mathematicians. This method relies on recomposing the groups – a core idea in multiplication. Moreover, given that it is a method developed and used by peasants, this method can help to encourage empathy for peoples of the past among pre-service teachers (Design Consideration 3) and disrupt their existing views of mathematics as being an innate skill, reserved for elite mathematicians. The second case will be more familiar to pre-service teachers in North America, but the historical context may not be. This case is intended to help pre-service teachers re-examine their assumptions about what the standard Canadian/US multiplication algorithm is, why it exists, and who it is for. Through their analysis, we hope that pre-service teachers will come to see the standard algorithm not as the sole approach to multiplication, but as one approach that served a particular historical purpose in a given context based in social reality issues, during a time when modern technologies were not accessible for facilitating computations during commerce. Here, the mathematics was connected to human action and their need to adapt to the changing times – a point to help pre-service teachers reflect on the evolving nature of mathematics. By making sense of the human aspects of both cases, pre-service teachers may also see mathematics as more than purely procedural, and instead as developed through reasoning and problem solving.

The first case received the name "Russian Peasant Multiplication" because it was used by Russian peasants for many years and is still used by some in Russia today (Philipp, 1996). To use the method, a person writes down the two numbers to be multiplied. For example, if multiplying 7 × 22 (Fig. 2), one would write 7 next to 22. Next, the person should double the first number (e.g., 7) and halve the second number (e.g., 22) and write these quantities directly underneath the original quantities. In our example, 7 doubled results in 14. This value should be written directly underneath the 7, creating a first column. The 22 halved results in 11. This value should be written directly under the 22, creating a second column. The person should continue by doubling the value in the first column and halving the value in the second column until the value in the second column is 1, as pictured in Fig. 2. To find the product, all the quantities in the left column with a corresponding odd number in the second column are summed. Thus, in our example, we sum 112 + 28 + 14 because the numbers to the right of them are odd numbers (1, 5, and 11, respectively). This results in 154, the final result of 7 × 22. Because the method only requires one to understand how to double and halve, something that can be done with tools – as illustrated by the pebbles in the example, it is accessible to a large number of people.

The second case dates back to the Italian Renaissance and shows a method created by Italian abacists who were tasked to develop methods that could allow merchants to more quickly and efficiently keep track of money in their business exchanges. This method closely resembles the standard North American algorithm often taught in schools. Here, a digit in one number is multiplied by each digit in the second number and then repeated with remaining digits. For example, as illustrated in Fig. 3, to multiply 9876 × 6789, we first multiply the "9" from 6789 by the "6" from 9876 to result in 54. The "4" is written underneath the rightmost column. The "5" is reserved. We then multiply 9 × 7 to result in 63. The "5" from the 54 is added to the "3" from the 63 to result in 8. This is written underneath the second rightmost column. The "6" from the 63 is reserved. We then multiply 9 × 8 to result in 72 and

Case 1:

Dating back before the twentieth century, Russian peasants needed a method for commercial exchange. Their method needed to be accessible and easy to use without great mathematical understanding. They developed an approach that likely involved use of pebbles to keep track of quantities. For example, imagine two peasants who wish to exchange goods. One wants to buy seven animals from the other. Each animal costs 22 coins, leaving them with the challenge to determine how much the buyer owed the seller. The seller of the animals created two columns of holes. In the first row, the seller placed 7 pebbles in 1 hole and 22 in the other. He then proceeded to change the numbers of pebbles in each column in the following way:

7	22
14	11
28	5
56	2
112	1

The seller then combined the 14 pebbles, the 28 pebbles, and the 112 pebbles. He used this to determine that the cost of the animals should be 154 coins. The buyer, following the steps, agreed and was able to proceed with the exchange.

Fig. 2 Case 1: Russian peasant multiplication. The case above was adapted from Ogilvy and Anderson (1988). Although the story about the two peasants is fictive, it is written to reflect the purpose of the use of the algorithm

add the "2" from this quantity to the "6" from the 63. This sum of 8 is written underneath the third rightmost column and the "7" from the 72 is reserved. The 9 is then multiplied by the 9 to result in 81. Because there are no more values to multiply 9 by, we add 81 to the "7" from the 72 to result in 88. The 88 is written underneath the leftmost column. This process continues by multiplying the "8" from the 6789 by each digit in the 9876, creating a row of values (79008) underneath the first computed row of values (88884). As illustrated in Fig. 3, this row is shifted one digit to the left. This process then continues with the remaining digits. Finally, the values in all the rows are summed to result in the final product of 67,048,164. Unlike the Russian Peasant algorithm, to comprehend why this method works, students need to understand what each digit in the numbers represent in terms of place value (i.e., the meaning of each position within a numeral), why that results in shifting the computed rows, and why the values are summed at the end. Although more easily applicable to high values of numbers, the method provides more opportunities for error. Once done, students should keep a record of their second narrative for purposes of self-analysis, which is done in Phase 3 to develop their self-portraits.

Case 2:

During the Renaissance, commercial exchange shifted from one based on barter to one based on money. Due to improvements in shipping technology, merchants began running larger businesses in which they delegated shipping to their employees. They now needed to keep track of the exchange of money. This led to the development of a new type of mathematician, called an abacist, and new methods of computation. Whereas before, barter just required an exchange, merchants now needed to determine the quantitative value of objects. Abacist schools were developed to teach merchants methods of computation. This generated the need to have algorithms that could be communicated and replicated on paper (Radford, 2003). Numerous texts were generated. For example, in *Summa Arithmetica* (1494), Pacioli provided eight methods of calculation for multiplication. One that had the most success he illustrated with the following:

				9	8	7	6
				6	7	8	9
			8	8	8	8	4
		7	9	0	0	8	
	6	9	1	3	2		
5	9	2	5	6			
6	7	0	4	8	1	6	4

Fig. 3 Case 2: Italian abacist multiplication

5.4 *Phase 3: Fostering the Habit of Taking Critical Distance When Positioning Oneself Through Sustained Practice*

In the third and final phase, pre-service teachers step back and critically examine their thinking in their two reflection pieces and follow-up ideas to consider their positionality in relation to mathematics, history, and teaching mathematics for meaning. To assist with this process, pre-service teachers should engage in an in-depth self-analysis of their reflections – much like *inductive coding* in qualitative analysis where categories are developed from the data – in which they identify dominant themes related to their perspectives on mathematics, history, and the particular mathematical topic (Zanazanian, 2019). By interpreting their coded data, pre-service teachers will develop a self-portrait of themselves as a mathematics actor and, by extension, as a mathematics teacher – a reflection of their epistemic beliefs regarding the construction of mathematical knowledge that they can then continue to revisit throughout their program.

For example, to develop core themes for their self-analysis based on their reflection piece and follow-up questions, as a first step for Phase 3, teacher educators should ask pre-service teachers to review their narratives to code what they have written according to these questions: *(1) What perspectives emerge regarding mathematics and its workings? (2) What perspectives emerge regarding history and its workings? (3) What perspectives emerge regarding the teaching of mathematics for meaning? (4) What perspectives emerge about how history can help inform* *teaching mathematics for meaning? (5) What perspectives emerge regarding the justifications behind your reasoning?* Once all these questions are answered, the emerging information for each can be compared and reduced into representative codes (themes) that account for the students' thinking until the point of saturation. As a second step, pre-service teachers can then synthesize key ideas by completing the following prompts related to each of the above questions: *According to the author, (a) Mathematics is… (b) History is… (c) Teaching mathematics for meaning is… (d) Using history to teach mathematics for meaning refers to… and (e) The justifications for their reasoning include….*

To help pre-service teachers further reflect on their claims in their self-analysis, as a third step, teacher educators should ask them to interpret connections across their analyzed "data." For example, they can complete the following prompt: *According to the author, multiplication, math, and history connect in the following ways…*and then answer: *What evidence does the author provide for these connections?* Once they code their data, as a fourth step, teacher educators should have pre-service teachers write a summary of their findings in a brief descriptive report – which would basically comprise the core text or ideas of their portraits (to be developed below).

Following the self-analysis, it is essential to bring in additional perspectives to help pre-service teachers take critical distance from their knowledge claims. To do so, as a fifth step, teacher educators should task them to analyze and interpret the reflections and follow-up questions for two other classmates. Within this group, they should then compare their analyses and interpretations for each individual's set of reflections and follow-up questions and come to a final consensus. By being placed in a position where they have to clarify and defend their claims, this step should provide pre-service teachers with an alternate perspective to their claims and require them to respond to those perspectives.

As a final step, the teacher educator should ask pre-service teachers to write a reflection in which they use the agreed-upon analyses and interpretations to develop a self-portrait of themselves as a mathematics person or as a mathematics teacher. They should use the first person in their writing ("I am…," "my…."). For example, pre-service teachers may be prompted in the following way: *Using the analyses of your reflections and follow-up questions, develop a portrait of yourself as a knower of mathematics and as a mathematics educator. Use the first person when writing your portrait.*

Although the process we describe above may be implemented in a teaching methods course, it should not be treated as a quick solution for achieving epistemic change towards mathematics for meaning. Most pre-service teachers come to their teacher training with a long history of experiencing traditional mathematics (which they should also keep in mind during their self-analysis). These experiences provided distinct epistemic climates that have shaped their epistemic beliefs towards seeing mathematics as certain, simple, originating in external authority, and always attainable. Uprooting these beliefs requires time. The process we describe here is intended to help provoke pre-service teachers to develop awareness of their epistemic beliefs and positionality as a mathematics teacher. This should only be the start of a sustained self-reflexive conversation. Throughout their teacher education program, teacher educators should encourage pre-service teachers to continue to examine their epistemic cognition's workings by revisiting and re-examining their self-portraits through additional reflections and opportunities to examine other historical cases. By doing so, the aim is for pre-service teachers to develop epistemic beliefs about mathematical knowledge aligned with approaches for teaching mathematics for meaning through an evolutionary process where they are self-reflexive, changing and updating their portrait as they go along.

6 Concluding Thoughts

In this chapter, we have attempted to illustrate how historical consciousness, as an emerging concept in the area of history education, can be used as a tool to support shifts in pre-service teachers' epistemic understandings of the nature of mathematics, of learning mathematics, and of teaching mathematics. We have argued that historical consciousness provides pre-service teachers with a framework for examining history's impact on their thinking (Design Consideration 1) and allows pre-service teachers to consider their positionality as mathematics learners and teachers. In our example, we showed how a historical consciousness approach may allow pre-service teachers to question and re-examine their epistemic views of mathematics, which, in turn, may inform their teaching. By changing the historical cases and the reflection questions, this approach can be generalized to help pre-service teachers re-examine their positionality in relation to different aspects of teachers' epistemic beliefs of mathematics, which can impact teaching and learning.

The bulk of research to date has examined the integration of history into secondary mathematics pre-service teacher contexts (e.g., Clark, 2012; Fenaroli et al., 2014; Furinghetti, 2007; Guillemette, 2017; Povey, 2014; Youchu, 2016). Because elementary teachers are trained as generalists and must take courses within both mathematics and social studies, the elementary context provides a unique setting for integrating historical consciousness into pre-service teacher mathematics methods courses. Social studies and mathematics teacher educators can collaborate to introduce historical consciousness within social studies methods courses and then leverage historical consciousness as a tool within mathematics methods courses to help pre-service teachers take critical distance from the claims they make when constructing mathematical knowledge. Such an approach would allow both teacher educators and pre-service teachers to cross-disciplinary borders within the traditionally silo-ed teacher education context. Research could then investigate questions of pre-service teacher learning across these contexts.

In secondary mathematics methods teacher education programs, pre-service teachers generally have more time allocated to studying mathematics and teaching mathematics. For example, several studies highlight contexts where pre-service teachers have entire courses dedicated to the study of the history of mathematics (e.g., Fenaroli et al., 2014; Gazit, 2013; Youchu, 2016). Despite the additional time, teacher educators still need to make room for understanding historical approaches within such courses (Design Consideration 1). We suggest that historical consciousness provides an accessible means for mathematics teacher educators to help pre-service teachers use history to take critical distance from the mathematical knowledge claims they make in order to develop their positionality in relation to mathematics and mathematics teaching.

Although not the focus of this chapter, we anticipate that historical consciousness can also be used in adult education and K-12 education contexts to support mathematics learners to develop positive and productive relationships with mathematics. Through examining historical texts, learners can come to position themselves and their struggles in relation to mathematicians (Fauvel, 1991; Jankvist, 2009). If pre-service teachers learn about the relevance of history of mathematics in their teacher education courses through an historical consciousness approach, this can serve as a model for how they might then incorporate the same approach into their classrooms. Methods courses can then provide opportunities for pre-service teachers to plan lessons using such a framework. Through incorporating historical consciousness within teacher education, teachers will hopefully not only become aware of their own positionality in relation to mathematics, but will provide supportive environments where students can develop positive and productive relationships with mathematics and achieve access to engage critically with social issues.

One limitation of our approach, the examples provided and the underlying design considerations, is that they are grounded in a euro-centric vision of history, mathematics, and history of mathematics. As Zanazanian's (2015, 2019) framework already makes room for non-Western experiences and epistemologies, it can easily attend to different, yet specific, knowledge and learning contexts. With the development of research that highlights how to leverage non-European perspectives in integrating history and mathematics in teacher education, we nonetheless anticipate that our design considerations may be expanded or revised. For example, one consideration might highlight how historical analysis should go beyond euro-centrism to champion historical contributions from underrepresented groups. In addition, design considerations might incorporate other research literatures. At the same time, it is unlikely that simply adding non-European examples to historical analysis will be enough to move beyond a euro-centric vision. More research is also needed to understand how historical consciousness might be expanded or altered to encompass non-euro-centric historical and mathematical ways of thinking, which Zanazanian's (2019) framework already intends to do. We thus hope that our approach is an initial, small step towards larger discussions about how to integrate not only historical and mathematical content, but also historical and mathematical *ways of thinking* across a range of cultures and contexts to better support pre-service teachers in achieving epistemic changes in their thinking and beliefs.

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Secondary Pre-service Teachers' Experiences in a Numeracy Course

Jennifer Hall and Helen Forgasz

1 Introduction

In recent years, numeracy has become an increasing focus of Australian education, from the school level to teacher education programs. Although the focus on numeracy is contemporary, the concept of numeracy has a long history, dating back 60 years, when Crowther (1959) defined numeracy as the mirror image of literacy. Nearly two decades later, similar to Crowther's (1959) idea, Ehrenberg (1977) suggested that numeracy is comprised of "two facets – reading and writing, or *extracting* numerical information and *presenting* it" (p. 277, emphasis in original). Furthermore, Cockcroft (1982) suggested that numeracy teaching should be the responsibility of all teachers, not simply those who teach mathematics. These ideas are echoed in the current Australian Curriculum, where numeracy is one of seven general capabilities, alongside other capabilities such as literacy and information and communication technology (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.-a).

Numeracy has been framed within the national goals for schooling in Australia since the Hobart Declaration on Schooling, released in 1989 (Education Council, 2014b), in which numeracy and literacy were among 10 goals endorsed by all state/territory ministers of education. Ten years later, in the Adelaide Declaration of National Goals for Schooling in the Twenty-first Century (Education Council, 2014a), three main goals and 17 sub-goals were agreed upon. In Goal 2.2, it is stated that "Students should have attained the skills of numeracy and English literacy; such that, every student should be numerate, able to read, write, spell and communicate at an appropriate level" (Education Council, 2014a). In 2008, two major goals were put forward in the Melbourne Declaration on Educational Goals for Young Australians (Ministerial Council on Education, Employment, Training and Youth

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Affairs [MCEETYA], 2008): schooling of excellence and for equity, and promoting successful learners, confident and creative individuals, and active and informed citizens. Numeracy is referred to in the second goal, linked to developing successful learners who "have the essential skills in literacy and numeracy… as a foundation for success in all learning areas" (MCEETYA, 2008, p. 8). A commitment to action was included: "The curriculum will include a strong focus on literacy and numeracy skills" (MCEETYA, 2008, p. 13). ACARA was established in May 2008 to oversee the implementation of the nationwide curriculum. More information on the development of the Australian Curriculum can be found at ACARA (2016).

Within the Australian Curriculum, developing students' general capabilities is the responsibility of teachers at all grade levels and of all subject areas. According to ACARA (n.d.-b), numeracy "encompasses the knowledge, skills, behaviours and dispositions that students need to use mathematics in a wide range of situations" (para. 1). Additionally, being numerate "involves students recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully" (ACARA, n.d.-b, para. 1). Therefore, as conceptualised in the Australian Curriculum, numeracy is not simply about having the mathematical skills to be able to apply mathematics in everyday life; students also need to value mathematics and have the dispositions to want to use mathematics.

In order to be prepared for the numeracy demands of their profession, pre-service teachers (PSTs) in Australia are required to meet two requirements: the Australian Professional Standards for Teachers (APST) and the Literacy and Numeracy Test for Initial Teacher Education (LANTITE). The APST, developed by the Australian Institute of Teaching and School Leadership (AITSL), are a series of standards that graduating and practising teachers must meet. In order to be accredited, universities that provide teacher education programs must demonstrate that the programs offered prepare PSTs to meet the "graduate" level standards (AITSL, 2017c). The seven broad, multi-part standards of the APST are organised into three domains of teaching: professional knowledge, professional practice, and professional engagement. Additionally, the APST are offered at four levels of the teaching profession – graduate, proficient, highly accomplished, and lead – reflecting career stages (AITSL, 2017b). There are two professional standards that are relevant to the numeracy demands of being a teacher:

- Standard 2.5 (Literacy and numeracy strategies): "Know and understand literacy and numeracy teaching strategies and their application in teaching areas"
- Standard 5.4 (Interpret student data): "Demonstrate the capacity to interpret student assessment data to evaluate student learning and modify teaching practice" (AITSL, 2017a)

In 2016, the LANTITE test was introduced in order to assess PSTs' personal literacy and numeracy capabilities. PSTs must pass both components of the LANTITE in order to graduate from a teacher education program in Australia (Australian Council for Educational Research [ACER], 2018c). Passing indicates that PSTs are in the top 30% of the adult population in Australia with regard to personal literacy and numeracy (Australian Government Department of Education and Training, 2017). The numeracy component of the LANTITE is a 2-hr-long computer-based test that contains 65 questions, of which 52 can be completed using a calculator and 13 must be completed without a calculator (ACER, 2018a). The numeracy component addresses the three strands in the Australian Curriculum: Mathematics – Number and Algebra, Measurement and Geometry, and Statistics and Probability – as well as three numeracy contexts that are relevant to being a teacher: personal and community, schools and teaching, and further education and professional learning (ACER 2018b).

As discussed, PSTs in Australia need to be prepared for a profession that has many numeracy requirements, both within and outside the classroom. Before facing the day-to-day numeracy demands of the teaching profession, PSTs must demonstrate that they have sufficient numeracy capabilities, as demonstrated on the LANTITE and by the APST. Unfortunately, according to Klein (2008), while "pre-service teachers are expected to teach their students for numerate participation in a global world… they themselves oftentimes lack the necessary mathematical foundations and strategic and critical skills" (p. 321).

Since numeracy is a general capability in the Australian Curriculum (ACARA, n.d.-a), teachers of all subject areas and of all grade levels must be prepared to meet this expectation. As mentioned, numeracy, by definition, involves taking mathematics beyond mathematics lessons and crossing borders into other subject areas. This border crossing places new demands on teachers. Although Australian elementary school teachers have always been generalists who need to teach mathematics among the many subject areas for which they were responsible, the numeracy requirement of the Australian Curriculum means that these teachers need to develop the skills and confidence to seize opportunities to incorporate mathematics into other contexts and across subject areas. In contrast to elementary school teachers, secondary school teachers have typically been subject area specialists, teaching within their disciplines of expertise. However, the challenges for these teachers are similar to those for the elementary school teachers, that is, developing the capabilities and confidence to incorporate mathematical dimensions into other subject areas, particularly those that may not have had such a focus in the past (e.g., drama, English, foreign languages). The challenges for secondary teachers in STEM (science, technology, engineering, and mathematics) fields, particularly mathematics teachers, are the reverse of the aforementioned groups. While mathematics teachers presumably have very strong mathematical skills, they may not have the understandings of the other aspects of numeracy, particularly the focus on a critical orientation and the application of mathematics in social and community contexts. Hence, in line with Klein's (2008) concerns, we believe that there may be different aspects of numeracy (e.g., mathematical skills, critical thinking) that may pose challenges for a range of teachers as they endeavour to meet the expectations of the Australian Curriculum.

In this chapter, we discuss the numeracy beliefs, confidence, and capabilities of PSTs at Monash University, a prestigious university in Melbourne, Australia. We consider how a required numeracy course, Numeracy for Learners and Teachers (NLT), influenced their views on numeracy and levels of confidence to incorporate numeracy into their teaching. Importantly, we consider whether there are differences between the groups just discussed: elementary versus secondary school PSTs, and STEM versus non-STEM subject area specialists (secondary school PSTs).

1.1 Numeracy for Learners and Teachers

In 2015, the numeracy course, Numeracy for Learners and Teachers (NLT), was introduced at Monash University in the Master of Teaching (MTeach) program. This 2-year teacher preparation program is undertaken by students who have already completed an undergraduate degree. During the MTeach program, students complete one teaching placement and four courses per semester, resulting in four teaching placements (60 days in schools or other educational settings, such as museums or zoos) and 16 courses that must be completed successfully in order for students to graduate from the program. Additionally, students must pass the mandatory, externally set LANTITE test in order to graduate.

The NLT course is mandatory for all PSTs in the MTeach program, save for those preparing to be Early Years teachers (birth to age 8) who complete an alternative numeracy course. Hence, the PSTs who complete NLT are preparing to be Early Years/Primary[1] teachers (birth to Grade 6), Primary teachers (Foundation to Grade 6), Primary/Secondary teachers (Foundation to Grade 10), or Secondary teachers (Grades 7–12). Primary/Secondary PSTs have one subject area specialism (e.g., mathematics, history), whereas Secondary PSTs have two subject area specialisms. For the purposes of this chapter, we will consider Early Years/Primary and Primary PSTs to be "elementary" PSTs, while we will consider Primary/Secondary and Secondary PSTs to be "secondary" PSTs.

NLT focuses on a different numeracy topic each week. In 2017, there were 9 weeks of class, whereas in 2018, there were 10 weeks of class. The first week of the course is an introduction to the concept of numeracy and its relationship to mathematics, as well as its place in Australian education. In the first class, we also introduce students to the 21st Century Numeracy Model (Goos, Geiger, & Dole, 2014), the conceptual framework for numeracy that underpins the course. In this model, contexts (work, citizenship, and personal/social life) are at the heart of numeracy. Furthermore, to be numerate, individuals need to have mathematical knowledge (e.g., estimation skills), particular dispositions (e.g., flexibility), and the ability to use relevant tools (e.g., digital tools like spreadsheets). Importantly, numerate individuals must have a critical orientation, which means that they "evaluate whether the results obtained make sense and are aware of appropriate and inappropriate uses of mathematical thinking to analyze situations and draw conclusions" (Goos et al., 2014, p. 85).

[1] In Australia, Primary refers to the first 7 years of school, Foundation to Grade 6.

The remaining weeks of the course focus on school subjects (e.g., the arts, history), combined with numeracy. Students learn about the links between the subject areas and numeracy, and participate in activities where they explore these connections. Some classes pertain to numeracy demands for students and for teachers, such as the class about financial literacy, where students' skills and understandings of this topic, the links to subjects such as economics and history, and teachers' financial literacy demands (e.g., budgeting, planning field trips, understanding pension plans), are addressed. There is also a class focused on statistical literacy for teaching and assessment, which pertains to teachers' numeracy demands outside the classroom. In 2018, the topic of the additional class was large-scale numeracy assessments (the Australian National Assessment Program – Literacy and Numeracy [NAPLAN] and LANTITE).

Starting in the first year of the NLT course being offered, we decided to undertake a research project to learn about the students' understandings of, and capabilities in, numeracy, as well as their experiences in the NLT course, beyond the information that we would typically receive from university mandated course evaluations. In all 4 years of NLT (2015–2018), students have completed voluntary pre-course and post-course online questionnaires. In 2015 and 2016, individual interviews were also conducted after the course concluded. However, the interviews did not provide any meaningful additional information, so we only continued with the questionnaires.

Here, we report on findings from the pre- and post-course questionnaires from 2017 and 2018. In addition to discussing general trends, we make comparisons between those students who are preparing to be elementary school teachers and those who are preparing to be secondary school teachers. Furthermore, within the secondary PST group, we make comparisons between those with STEM specialisms and those with non-STEM specialisms. In so doing, we address whether there are differences between these groups in terms of their numeracy capabilities, understandings, and confidence levels.

2 Methodology

Survey methods were adopted in the study. There were two questionnaires administered online, one prior to the commencement of NLT and the other at the end of the course. The invitation to complete each questionnaire was announced on the NLT (Moodle) website. Participation was voluntary and anonymous. Hence, there is no way to know if the same students participated in both iterations (i.e., pre-course and post-course) of the questionnaire. The questionnaires have been completed each year since the course began (2015). However, in the first 2 years of NLT, only secondary (2015) or elementary (2016) PSTs were enrolled, so there were no comparison group data to analyse. Hence, we are only using the 2017 and 2018 datasets, when both elementary and secondary PSTs were enrolled simultaneously.

2.1 Data Collection Instruments

Both questionnaires included demographic questions (e.g., gender, age, course information) and featured questions with closed- and open-ended response formats. Some of the closed items had categorical response formats (e.g., yes/unsure/no); others had 5-point Likert-type response formats (e.g., 1 = strongly disagree to 5 = strongly agree). Open-ended items included, for example, requests to explain answers to closed items, as well as answers to mathematical skills items.

The two questionnaires were different, although there were some items that were repeated on both. The pre-course questionnaire contained 57 questions and typically took 15–20 min to complete while the post-course questionnaire contained 29 questions and typically took 10 min to complete. The pre-course questionnaire, but not the post-course questionnaire, included items aimed at gauging students' appreciation of numeracy, as well as items to assess the respondents' mathematical skills. Items aimed to assess the impact of studying NLT were only included on the post-course questionnaire.

2.2 Participants

In Table 1, the characteristics of the pre- and post-course questionnaire respondents are summarised by year of administration. It should be noted that not all students who commenced the questionnaires responded to all of the items; hence, the subgroup (e.g., gender) totals do not always sum to the total number of participants.

From Table 1, it can be seen that in both years, there were far fewer respondents to the post-course questionnaires than to the pre-course questionnaires; this was unsurprising, given that the post-course questionnaires were completed at the end of the semester, when students were busy with assignments. Specifically, in 2017,

Table 1 Pre-course and post-course questionnaire participants

2017				2018			
Pre-course questionnaire							
$n = 81$				$n = 124$			
Elementary		Secondary		Elementary		Secondary	
23		52		32		83	
Men	Women	Men	Women	Men	Women	Men	Women
3	20	15	37	1	30	25	58
Post-course questionnaire							
$n = 30$				$n = 34$			
Elementary		Secondary		Elementary		Secondary	
11		17		8		26	
Men	Women	Men	Women	Men	Women	Men	Women
4	7	4	13	1	7	7	19

when 485 students were enrolled in the course, 16.7% completed the pre-course questionnaire, while only 6.2% completed the post-course questionnaire. Similarly, in 2018, when 578 students were enrolled in the course, 21.5% completed the pre-course questionnaire, while only 5.9% completed the post-course questionnaire. Consistent with the enrolment patterns across the MTeach program, there were more secondary than elementary respondents and far more women than men respondents. Due to the small number of men respondents, subsequent data analyses presented in this chapter are not reported by respondent gender.

Due to the relatively small number of participants in the post-course questionnaires and in the elementary groups in both years, the 2017 and 2018 data were combined in subsequent analyses. This enabled some statistical exploration of differences in beliefs and numeracy skill levels between elementary and secondary respondents, as well as between STEM and non-STEM secondary students.

2.2.1 Secondary Pre-service Teachers' Teaching Specialisms

Elementary PSTs in the MTeach program are generalists; that is, they teach all disciplines covered in the school curriculum and do not have a teaching specialism. In the pre- and post-course questionnaires, the secondary PSTs were asked to indicate which of the 24 teaching specialisations offered through the MTeach program they were studying. If mathematics, physics, chemistry, biology, or information technology was indicated, the student was categorised as a STEM student. Information about the STEM and non-STEM students from the pre- and post-course questionnaires is summarised in Table 2.

As shown in Table 2, there were higher proportions of non-STEM than STEM students who participated in each questionnaire, with just under 30% of respondents to both questionnaires being STEM students.

2.3 Analysis

Due to the different question types, the questionnaire data were analysed in various ways. To begin, descriptive statistics (e.g., means, percentages) were calculated. As appropriate, group comparisons (elementary vs. secondary and STEM vs.

Table 2 Frequencies and percentages of secondary pre-service teachers' specialisms, by questionnaire

Pre-course questionnaire		Post-course questionnaire	
$n = 135$[a]		$n = 43$	
STEM	Non-STEM	STEM	Non-STEM
36 (29%)	88 (71%)	11 (27%)	30 (73%)

[a]Not all participants answered this item

non-STEM) were completed through t-tests and chi-square analyses. The open-ended responses to the mathematical skills questions were first coded as correct or incorrect. Then, codes were applied to indicate the type of response within the "incorrect" category.

3 Findings

In the following sections, we discuss the findings from our analyses of the pre-course and post-course questionnaires. The pre-course questionnaire included mathematical skills questions as well as general questions about the participants' views of numeracy, mathematics, and teaching. The post-course questionnaire included similar general questions, plus questions about the NLT course. As mentioned, the 2017 and 2018 datasets were combined, and comparisons were made between students with STEM and non-STEM specialisms, as well as between those preparing to be elementary school teachers and those preparing to be secondary school teachers. First, we discuss differences in the pre- and post-course questionnaire responses for the combined 2017 and 2018 cohorts to see what influence the course, NLT, had on the PSTs' beliefs and confidence levels. Then, we make comparisons by the aforementioned groups. Specifically, we compare the participants' accuracy and confidence on three mathematical skills questions, as well as their perceptions of their mathematics capabilities more generally.

3.1 Effect of NLT on Pre-service Teachers' Beliefs and Confidence Levels

To determine the effect of studying NLT on the PSTs' beliefs about numeracy, two items appearing on the pre- and post-course questionnaires were of interest:

- Do you believe that there are differences between mathematics and numeracy? (Yes/Unsure/No)
- Are there mathematical demands on teachers in schools apart from what is taught to students? (Yes/Unsure/No)

To explore whether there were differences in the response patterns to these items on the pre-and the post-course questionnaires, chi-square tests were conducted. The results are shown in Table 3.

In Table 3, it can be seen that a higher proportion of students on the post-course questionnaire (82%) than on the pre-course questionnaire (61%) agreed that there were numeracy demands on teachers apart from what is taught to students ($p < 0.05$). Although not statistically significant, there was also an increase in the proportion of participants believing that there were differences between mathematics and

Table 3 Chi-square results for two items, by questionnaire timing

Item	Response	Pre-course	Post-course	χ^2, *df*, *p*-level
Do you believe there are differences between mathematics and numeracy?	Yes	102 (77%)	31 (84%)	*ns*
	Unsure	22 (17%)	4 (11%)	
	No	8 (5%)	2 (5%)	
Are there numeracy demands on teachers in schools apart from what is taught to students?	Yes	75 (61%)	31 (82%)	6.1, 2, 0.047
	Unsure	41 (33%)	5 (13%)	
	No	8 (7%)	2 (5%)	

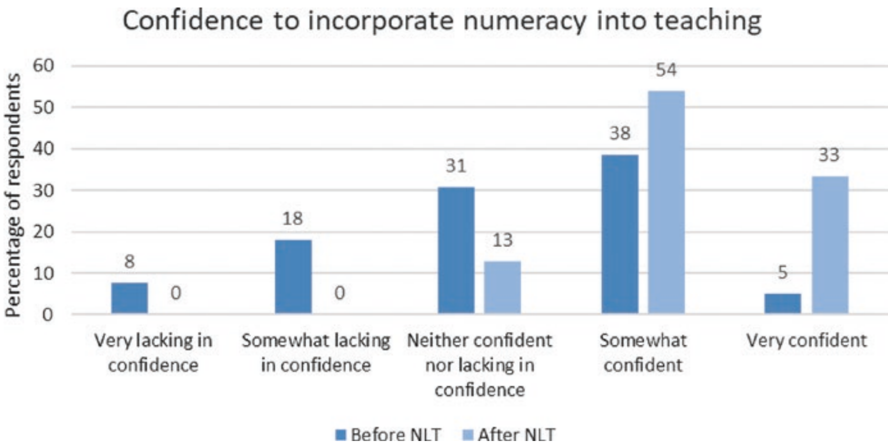

Fig. 1 Levels of confidence to incorporate numeracy into teaching, before and after NLT

numeracy (from 77% to 84%). These differences may suggest that studying NLT positively changed students' views. In the post-course questionnaire, students were specifically asked if NLT had influenced the way that they viewed numeracy. Of the students responding to this item, 31 (80%) indicated that it had.

The post-course questionnaire also included two items aimed at gauging the PSTs' change in confidence about incorporating numeracy into their teaching:

- Before commencing NLT, how confident were you about incorporating numeracy into the teaching of your subject area(s)?
- After completing NLT, how confident are you about incorporating numeracy into the teaching of your subject area(s)?

The reported levels of confidence to incorporate numeracy into teaching before and after studying NLT are illustrated in Fig. 1.

As shown in Fig. 1, before studying NLT, about a quarter of the students (26%) reported that they lacked confidence to incorporate numeracy into their teaching.

After studying NLT, none of the students felt that they lacked confidence and 87% of the students were at least somewhat confident to do so.

3.1.1 Open-Ended Responses About the Impact of NLT

The post-questionnaire included the following open-ended item:

- What is the biggest message you will take away from NLT?

Twenty-six PSTs (11 elementary, 15 secondary [3 STEM/12 non-STEM]) provided responses; only two comments were negative.

Sample responses illustrating the impact of the course on beliefs about numeracy, its place in the curriculum, and on the pre-service students' future responsibilities as teachers are provided below:

> Anyone can teach numeracy in the class – dance teachers, musicians, health teachers – we all have the skills, just need to build our confidence. (Elementary)

> I found that after the tutorials I attended it was somewhat eye-opening for me in a sense that incorporating numeracy into teaching isn't just based upon mathematical formulas and questions but rather there are many ways of introducing numeracy into classrooms activities such as reading articles and identifying the maths within it. (Secondary, non-STEM)

> Don't be intimidated by numeracy, you're actually doing it without realising a lot of the times. (Elementary)

> Be aware of the potential of maths in lessons. (Secondary, non-STEM)

> Numeracy is everywhere. You just need to see it to use it. (Secondary, STEM)

In summary, NLT appears to have had a positive effect on the PSTs' views of numeracy and its role in their teaching and their work as teachers. Furthermore, as reported by the post-course respondents, NLT increased their confidence to incorporate numeracy into their teaching.

3.2 Comparisons by Teaching Qualification Level and Specialism Groups

We were also interested in whether there were any differences in the participants' numeracy capabilities and confidence/self-perceptions by teaching level groups (elementary vs. secondary) and by specialism groups (STEM vs. non-STEM). In the following sections, we report on these groups' outcomes on two types of questions: mathematical skills questions and a question regarding the participants' general perceptions of their mathematics capabilities.

3.2.1 Mathematical Skills Questions

On the pre-course questionnaire only, participants completed six mathematical skills questions, some with multiple parts. Some of these questions had multiple-choice answers, while others had text boxes in which respondents typed their answers. After each question, the participants were asked to rate their confidence in the accuracy of their response (right, wrong, or unsure). We report on findings from three of the questions, selected based on 2015 and 2016 participants' accuracy rates: a question that has generally been done very well (Box Question), one that has been completed with a moderate rate of accuracy (Distance Question), and one that has been done poorly (Code Question).

Box Question In this question, participants had to calculate the mass of the lid for a box shown in a diagram (with and without the lid). The mass of the lid and box together was provided (232 g), as was the mass of the box by itself (186 g). Response options were 46 g (the correct response), 56 g, 144 g, and 54 g. Past participants have completed this question with high levels of accuracy and confidence, and the combined 2017 and 2018 group was no different: 95% of the respondents ($n = 122$) were correct, compared to 90% who thought that they were correct. The proportions of all participants who were correct and of all participants who were confident in their answers are shown in Table 4, separately by teaching qualification levels and by secondary specialism groups.

As shown in Table 4, all groups had similar accuracy rates for this question. However, the non-STEM group was much less confident in their response accuracy, compared to either the STEM group or the elementary group. Note that both the "accuracy" and the "confidence" percentages apply to each group as a whole; hence, some of the participants who were confident that they were correct were not, and vice versa.

Distance Question In this question, participants had to determine which of the following was the longest distance: 0.1203 km, 123 m (the correct response), 1230 cm, or 12,030 mm. The participants were generally 10% less accurate on the Distance Question than they were on the Box Question, which is perhaps unsurprising since the latter simply involved subtraction of whole numbers, whereas the former involved unit conversions. Overall, 82% of the participants ($n = 123$) were correct, compared to 83% who thought that they were correct. The accuracy and confidence levels of each group as a whole are shown in Table 5.

Table 4 Accuracy and confidence rates for the Box Question, by group

Group	n	Provided the correct answer	Confident that answer is correct
Elementary	32	97%	94%
Secondary	89	94%	89%
STEM[a]	26	96%	96%
Non-STEM	59	95%	85%

[a]Not all Secondary participants provided information their subject area specialism(s)

Table 5 Accuracy and confidence levels for the Distance Question, by group

Group	*n*	Provided the correct answer	Confident that answer is correct
Elementary	34	82%	85%
Secondary	88	82%	82%
STEM	25	80%	92%
Non-STEM	59	81%	78%

As with the Box Question, the groups all had similar levels of accuracy on the Distance Question. However, the STEM students were overconfident that their answers were correct: 92% thought that they were correct, but only 80% were actually correct. Hence, several of the STEM students who gave incorrect answers were confident that their answers were correct.

Code Question This question differed from the other two questions discussed, as it was not a multiple-choice question; rather, participants had to type their responses into a text box and therefore could not simply make a guess by selecting one of the provided answers. In the question, an image of a 10-digit (0–9) keypad was shown. The question stated, "Helen's office has a security alarm. To turn it off, Helen has to type her 4-digit code into this keypad. Helen's code is 0051. Including Helen's code, how many four-digit codes are possible?"

Participants' responses were first coded as being correct (10,000, 10^4, or some other equivalent statement) or incorrect (any other numeric response, or a response like "I have no idea" or "Lots"). We also included responses of 9,999 in the "correct" category. Participants providing this response appeared to have understood the concept, but may have interpreted the question to ask for potential combinations other than the sample code; alternatively, they may have omitted the option of 0000, which may not have been considered to be a potential usable code. Within the "incorrect" category, we coded the responses as "10^x, $x \neq 4$" (e.g., "100,000", "1,000"), "other number" (e.g., "3,628,800", "40 million"), and "no numeric answer provided" (e.g., "lots", "Too many"). In total, 109 participants responded to this question, a lower response rate than for the multiple-choice questions discussed earlier. Only 51% of those who responded were correct, and an even smaller proportion (48%) thought that they were correct; of the other (incorrect) answers, the most common response was another number (28% of the question respondents), followed by no numeric answer (12%), and 10^x, $x \neq 4$ (8%). The cross-tabulation comparing the response codes with the participants' reported confidence levels is shown in Table 6, with percentages applying to each row.

Interestingly, all five of the participants who provided a response of 9,999 (i.e., misread the question but appeared to understand the concept) thought that they were correct. Hence, if we only consider those who provided an answer of 10,000, 39 (76%) were confident that they were correct, while 12 (24%) were unsure. The participants who provided the correct response (i.e., either 9,999 or 10,000) were generally quite confident that they were correct. Conversely, participants who provided incorrect responses generally lacked confidence about the accuracy of their responses. It is encouraging to see that so many participants showed awareness of the accuracy of their answers.

When considering the responses by the sub-groups on this question, the response patterns were very similar. Similar proportions of the elementary (48%) and secondary (47%) participants answered this question correctly and thought that they were correct (47% and 45%, respectively). However, the differences between the STEM and non-STEM secondary groups were much more pronounced, with more than a 20% difference in both confidence and accuracy between the groups, favouring the STEM group (40% of non-STEM participants were correct and were confident of their answers, compared to 63% of the STEM group on both measures). Although the percentages were equal for both measures for each group, recall that not all of those who were correct thought that they were, and vice versa.

3.2.2 Perceptions of Mathematics Capabilities

The participants' general perceptions of their mathematical capabilities were queried on the pre-course questionnaire with the following question:

- How good are you at mathematics? (1 = weak, 2 = below average, 3 = average, 4 = good, 5 = excellent)

T-tests by teaching level (elementary/secondary) groups and by STEM/non-STEM groups were conducted. The mean scores and t-test results are shown in Table 7.

Overall, the PSTs considered themselves to have above-average mathematical achievement ($\bar{x} = 3.54$). As shown in Table 7, there was no difference in the mean perceived mathematical achievement levels of elementary and secondary students. However, on average, compared to the non-STEM students ($\bar{x} = 3.45$), the STEM students believed that they had better mathematical capabilities ($\bar{x} = 3.81$); the difference was statistically significant ($p = 0.02$). Although not tested statistically, it

Table 6 Cross-tabulation of Code Question response codes and reported accuracy

Response	Yes	Unsure	No
Correct response: 10,000[a]	44 (79%)	12 (21%)	0 (0%)
$10^\wedge x$, $x \neq 4$	0 (0%)	4 (44%)	5 (56%)
Other number	6 (19%)	21 (68%)	4 (13%)
No numeric answer	2 (15%)	5 (39%)	6 (46%)

[a]This code includes responses of 9,999

Table 7 Pre-service teachers' perceptions of their mathematics capabilities, by group

Mean scores			Mean scores		
Elementary ($n = 33$)	Secondary ($n = 92$)	t, p-level	STEM ($n = 26$)	Non-STEM ($n = 62$)	t, p-level
3.55	3.54	*ns*	3.81	3.45	2.45, 0.02

was noteworthy that the mean score for the non-STEM secondary students (3.45) was lower than for the elementary PSTs (3.55).

4 Concluding Remarks

As demonstrated by the findings presented, we believe that completing Numeracy for Learners and Teachers (NLT) helped the PSTs to gain an understanding of numeracy as a general concept and of the numeracy demands of teaching. Furthermore, as shown by the post-course questionnaire data, the PSTs reported that they gained confidence in their abilities to incorporate numeracy into the subject areas that they will be qualified to teach. In so doing, the PSTs should be able to meet the numeracy general capability expectations of the Australian Curriculum. The knowledge and confidence that the PSTs gained by completing NLT helped them to understand that, by definition, numeracy involves taking mathematics out of the mathematics classroom and crossing borders not only into other subject areas, but also to broader multidisciplinary issues, such as sustainability, and the world outside of school.

The mathematical capabilities and confidence of elementary PSTs is a well-researched issue (e.g., Ball, Hill, & Bass, 2005; Beilock, Gunderson, Ramirez, & Levine, 2010; Bursal & Paznokas, 2006). In this study, we did not find any meaningful differences in either confidence or mathematical capabilities between the elementary and secondary participants overall. It is possible that the elementary PSTs in the course who strongly disliked and/or were very lacking in confidence in their mathematical abilities would not have voluntarily participated in such a study. Hence, the elementary PSTs who took part may have been had stronger mathematics capabilities and confidence than the average elementary pre-service (or practising) teacher. Additionally, it is important to remember that all of the PSTs who participated in this study have already completed an undergraduate degree in another field and have met the high entrance standards of the Master of Teaching program at Monash University, a prestigious university.

In comparison to the well-developed research base about elementary teachers, the mathematical capabilities and confidence of secondary PSTs who are not mathematics specialists is an under-researched area of study. While this may have been for good reason in the past (i.e., such teachers presumably did not have a large focus on mathematics/numeracy in their teaching), with numeracy being a general capability in the Australian Curriculum, it is now the responsibility of all Australian teachers to identify and incorporate pertinent mathematical skills development across all non-mathematics subject areas. Hence, our study contributes to the extant literature as we provide insight into the challenges presented by this border crossing, particularly for non-STEM pre-service secondary teachers' experiences, capabilities, confidence levels, and views.

When comparing the STEM and non-STEM groups of secondary PSTs, we found little difference in their mathematical capability, other than on the Code

Question (which addressed the challenging mathematical topic of combinatorics, even though in a common everyday context). However, there were marked differences in confidence. While the STEM group tended to be overconfident in their answers to some questions, the non-STEM group was not. There are many factors that may have contributed to these findings. For instance, there were proportionally more men than women (relative to the dataset as a whole) in the STEM group than in the non-STEM group. It is well documented (e.g., Bench, Lench, Liew, Miner, & Flores, 2015; Ellis, Fosdick, & Rasmussen, 2016) that boys and men tend to be overconfident about their mathematics capabilities, whereas girls and women are more likely to be realistic or underrate those capabilities. Additionally, societal biases may come into play. Those in the STEM group may be more likely to assume that they are "math people" and thus able to answer the questions, compared to the non-STEM group who may be more likely to presume the opposite.

The border crossings that are expected by the numeracy demands placed on teachers in Australia – per the APST and the Australian Curriculum – require a different type of preparation for PSTs, particularly those in subject areas where numeracy has not typically been a focus. All teachers in Australia need to be confident and capable of incorporating numeracy into all the subject areas that they teach. As we have discussed, the secondary PSTs with non-STEM specialisms may lack confidence in their mathematical capabilities, both in general and as demonstrated on the mathematical skills questions that they completed, relative to the secondary PSTs with STEM specialisms. While this finding is not surprising, it is concerning. If these PSTs do not feel confident in their own mathematical capabilities, they may be hesitant to introduce a numeracy focus in their subject area specialisms. As we have shown, completing NLT helped the participants (in general) to become more confident about incorporating numeracy into their teaching, an encouraging outcome that augurs well for the future. However, the lower levels of confidence in their mathematical capabilities shown by the non-STEM secondary PSTs remain a concern. Further research into this area is needed.

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Mathematics Crossing Borders: A Comparative Analysis of Models for Integrating Mathematics with Other Disciplines in Pre-service Teacher Education

Merrilyn Goos

1 Introduction

Teacher education is replete with borders that define and divide different sites for learning, knowledge categories and people who contribute to the formation of future teachers. Perhaps the most frequently identified border is that which separates theory from practice, symbolised by the separation of the university from the school as sites for learning to teach. The university component of teacher education also involves clearly marked boundaries between the types of knowledge to be gained by future teachers. In Australia, for example, university programs preparing secondary school teachers must include discipline studies providing knowledge of the content to be taught together with professional studies comprising discipline-specific curriculum and pedagogical studies ("methods" courses), general education studies (typically drawing on psychology, sociology, history and philosophy of education) and professional experience in schools (Australian Institute for Teaching and School Leadership [AITSL], 2016).

This chapter looks beyond the well-known divides mentioned above to explore ways in which mathematics crosses disciplinary borders in the context of pre-service teacher education. Disciplinary border crossing in mathematics education is sometimes conceived of in terms of making *connections* between mathematics, other disciplines and the real world. Mathematics curriculum documents in many countries stress the value of such connections. In the USA, the National Council of Teachers of Mathematics (2000) describes Standards that outline the processes of school mathematics, stating that students' understanding of mathematics is enriched by seeing mathematical connections in contexts that relate mathematics to other

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subjects. One of the aims of the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.) is for students to recognise connections between mathematics and other disciplines.

Rather than giving priority to connections, an alternative view defines the border between disciplines and disciplinary communities as a marker of sociocultural *difference*, giving rise to a discontinuity that needs to be negotiated. This view draws on sociocultural theories of learning in communities of practice (Wenger, 1998) and boundary crossing between communities (Akkerman & Bakker, 2011). Boundaries and boundary crossing are thought to carry potential for learning involving dialogical interactions between multiple perspectives and multiple actors. Taking a boundary crossing theoretical perspective permits a deeper and more critical investigation of the types of connections and discontinuities that might exist between mathematics, other disciplines and the real world.

This chapter is organised in the following sections. First, a brief overview of the sociocultural perspective on boundary crossing and four potential learning mechanisms that can operate at the boundaries between disciplines or domains is provided: identification, coordination, reflection and transformation (Akkerman & Bakker, 2011). These mechanisms provide the framework for comparing different models of integrating mathematics with other disciplines in pre-service teacher education. Subsequent sections outline key features of three models of integration: mathematical modelling, curriculum integration and numeracy across the curriculum. For each model, I identify its theoretical and philosophical rationale, positioning of mathematics and assumptions about the relationship of mathematics to other disciplines and barriers to and enablers of integration. The final section summarises the comparison between models in terms of learning mechanisms at the boundary between disciplines and discusses implications for teacher education and classroom practice more generally.

2 A Sociocultural Perspective on Boundary Crossing

There is an emerging body of research on learning mechanisms involved in interdisciplinary work on shared problems. This type of work is becoming increasingly important because of growing specialisation within domains of expertise that requires people to collaborate across boundaries between disciplines and institutions. Akkerman and Bakker's (2011) review of this research literature emphasised that boundaries are markers of "sociocultural difference leading to discontinuity in action or interaction" (p. 133). Boundaries are thus dynamic constructs that can shape new practices through revealing and legitimating difference, translating between different world views and confronting shared problems. Akkerman and Bakker argued that, as a consequence, boundaries carry potential for learning.

Akkerman and Bakker (2011) proposed four potential mechanisms for learning at the boundaries between domains. The first is *identification*, which occurs when the distinctiveness of established practices is challenged or threatened because people find themselves participating in multiple overlapping communities. Identification processes work either by emphasising the differences and tensions between different domains or by legitimising their coexistence. Identification reconstructs the boundaries between practices by delineating more clearly how the practices differ, so that discontinuities are not necessarily overcome. A second learning mechanism involves *coordination* of practices or perspectives via dialogue in order to accomplish the work of translation between two worlds. The aim is to overcome the boundary by facilitating a smooth movement between domains, communities or sites. *Reflection* is the third learning mechanism discussed by Akkerman and Bakker. Boundary crossing can promote reflection on differences between practices that leads to new insights on the nature of one's own and others' practices. Reflection differs from identification, in that it results in an expanded set of perspectives and construction of new identities, whereas identification instead reinforces current practices and identities. The fourth learning mechanism is described as *transformation*, which can lead to a profound change in practice, "potentially even the creation of a new, in-between practice, sometimes called a boundary practice" (p. 146).

Learning at the boundary between domains can be understood by examining the roles of people and objects that cross boundaries. While boundary crossing might evolve spontaneously, it can also be facilitated by *brokers* who build bridges between worlds and connect disciplinary paradigms. Wenger (1998) explained that the job of brokering requires the ability to "cause learning by introducing into a practice elements of another" (p. 109). Brokering is acknowledged to be a complex and an ambiguous role that can leave brokers feeling uncertain about their positioning "in-between" practices and domains.

Objects, as well as people, can facilitate boundary crossing between domains. Star and Griesemer (1989) introduced the idea of *boundary objects* to explain the role of artefacts in simultaneously inhabiting intersecting worlds. Although boundary objects have different meanings within the worlds they connect, they are flexible enough to be accessed by people in separate worlds and to enable these groups to work together. In the context of mathematics crossing disciplinary borders in pre-service teacher education, brokers might be found amongst mathematics teacher educators who collaborate with colleagues in other disciplines to explore models of integration, while the models they create might serve as boundary objects that establish continuity across the disciplines.

3 Mathematical Modelling

3.1 Rationale and Positioning of Mathematics in Modelling

The use of mathematical modelling is often advocated in curriculum documents that encourage teachers to connect mathematics with the real world. While there are many different interpretations of modelling, it is common to refer to a modelling cycle that begins with analysing the real-world situation, then specifying assumptions associated with mathematical concepts or the real-world context, formulating the problem in mathematical terms, solving the mathematical problem, interpreting the solution within the real-world context, validating the solution and using the model to report, explain, predict or design. A simplified version of the modelling cycle is shown in Fig. 1. In practice, modelling is an iterative process that involves interconnections between the phases rather than a sequential progression. Thus, the modelling process involves crossing and re-crossing the boundary between the mathematical world and the real world, and the modelling cycle can be viewed as a boundary object facilitating coordination of perspectives and translation between the two worlds (Akkerman & Bakker, 2011).

Various perspectives on mathematical modelling have been identified in the literature, with each implying a particular role for mathematics and its relationship with the real world. Kaiser and Sririman (2006) distinguished between pragmatic, pedagogical, psychological, subject-related and science-related perspectives: for example, a pragmatic-utilitarian perspective informs realistic modelling where the central aim is to solve practical real-world problems, while educational modelling is aligned with pedagogical and subject-related perspectives to improve students' understanding of the world while they learn mathematical concepts and methods. The contexts for educational modelling can also come from disciplines other than mathematics, such as science, economics, sport or art.

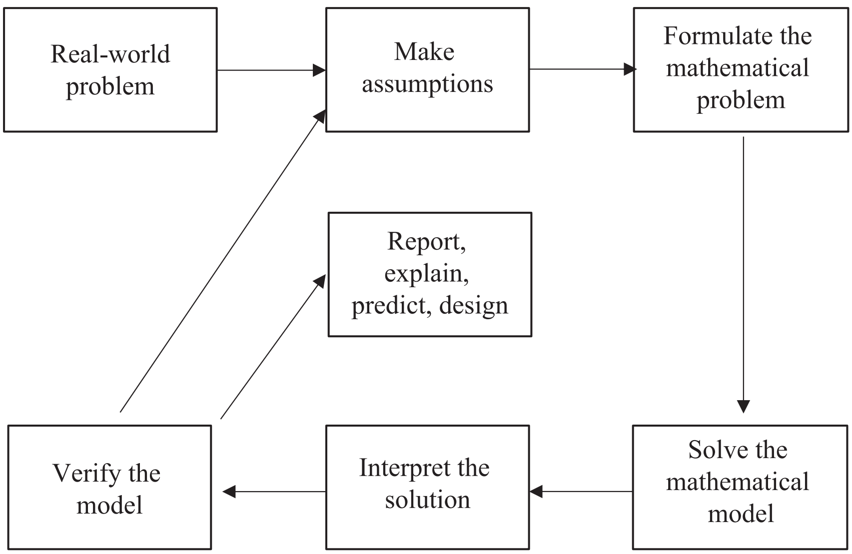

Fig. 1 The modelling cycle

3.2 Impetus for Addressing Mathematical Modelling in Pre-service Teacher Education

Until the recent introduction of a national curriculum for Australian schools (e.g. ACARA, n.d.), each state and territory produced its own curriculum and assessment policies, and several of these jurisdictions had incorporated mathematical modelling into their secondary school mathematics curricula. The state of Queensland introduced modelling into mathematics syllabuses published in 1992, with revisions in 2001 and 2008 that increasingly refined the explicit focus on modelling (Queensland Studies Authority, 2008).

Modelling was progressively embedded into senior secondary mathematics curricula (Grades 11 and 12, students aged 15–17 years) in Queensland via a series of syllabus revisions. However, an analysis of mathematics textbooks used in Queensland and two other jurisdictions found that, although these resources offered many opportunities for students to develop underpinning mathematical competencies for modelling, there was more emphasis on technical skills and application of learned models and procedures than on understanding the modelling process and developing a critical understanding of the social or cultural concerns surrounding the situation being modelled (Stillman, Brown, Faragher, Geiger, & Galbraith, 2013). Consequently, pre-service teachers who rely on textbooks as a teaching resource are unlikely to develop their students' – or their own – capacity to model real-world situations using mathematics. This observation prompts a question of how best to prepare pre-service teachers to teach mathematical modelling.

3.3 Examples of Mathematical Modelling in Pre-service Teacher Education

Two examples from my experience as a teacher educator will illustrate some of the challenges and benefits of preparing future teachers to incorporate mathematical modelling into their classroom practice. The first example takes advantage of the increasing availability of portable digital technologies for data collection and analysis. Graphics calculators can be connected to data logging equipment such as motion detectors and probes that measure temperature, light intensity, pH, dissolved oxygen, heart rate and the frequency of sound waves to investigate physical phenomena and the mathematical models that describe them. A task that engages pre-service teachers in modelling the periodic motion of a pendulum requires them to use a motion detector to collect distance versus time data. The data points can be graphed as a scatter plot, with the aim of finding a function to fit the data (as in Fig. 2).

Pre-service teachers will typically choose the function $y = A\cos[B(x - C)] + D$ and then use their knowledge of the meaning of the parameters A, B, C and D to estimate values based on the data they have recorded. They might also notice that the value of the amplitude A decreases with each successive period, so a refinement to the model is needed in order to represent the amplitude's decay as an exponential function. Inevitably, some pre-service teachers will instead resort to the graphics

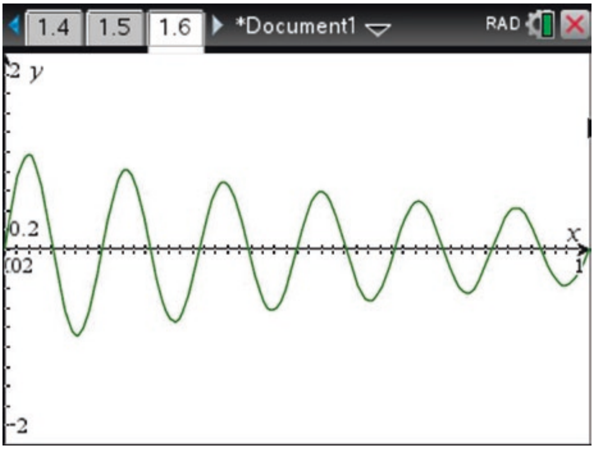

Fig. 2 Modelling the motion of a pendulum with a trigonometric function

calculator's regression function to fit an equation to the data, using the R-squared value to justify the goodness of fit of the resulting model. This data-driven approach can be used to motivate a discussion about the modelling cycle and how it emphasises the need for mathematical reasoning in the formulation of a model instead of deferring to the calculator's "black box" algorithms. One of the challenges for teachers using technology in mathematical modelling, then, is to adopt a critical orientation to the role of technology as a boundary object linking the mathematical and real worlds.

While the pendulum task can help teacher educators introduce pre-service teachers to the modelling process, it does not necessarily develop a disposition to view the world through a mathematical lens. This task would typically be used in a school classroom where students are learning about trigonometric functions, and so, it presupposes that they will draw on this specific knowledge, moving from the mathematical world to the real-world application. Instead, the modelling cycle begins with a real-world situation that provokes mathematisation, and so, teacher educators also need to provide these kinds of experiences to pre-service teachers – as in my second example.

An assessment task for pre-service secondary mathematics teachers involved creating a Maths Trail around the university campus. A Maths Trail is a sequence of outdoor activities and investigations that takes mathematics – and students – out of the classroom into the real world. Maths Trails can promote learning by increasing motivation, inviting students to apply mathematics to real-world situations, creating opportunities for group work and developing students' modelling and problem-solving skills in practical situations (French, 1994). A task from one of the Maths Trails developed by pre-service teachers is shown in Fig. 3, which represents the *real-world problem*.

A solution to this task offered by the pre-service teachers requires investigating the relationship between the sun's position and the amount of sunlight that falls on people seated beneath the pergola. Exposure to the sun will be a maximum when the

There is a pergola positioned adjacent to the university's Social Sciences and Humanities Library. This is a place where staff and students can relax and get some protection from the sun's harmful UV rays. Study the pergola and note that the amount of sunlight that gets through depends on the location of the sun in the sky. How does the amount of sunlight penetrating the pergola vary with time of day? How could you design a pergola, using the same roofing materials as in this one, which would provide full shade at 3 p.m.?

Fig. 3 Pergola task

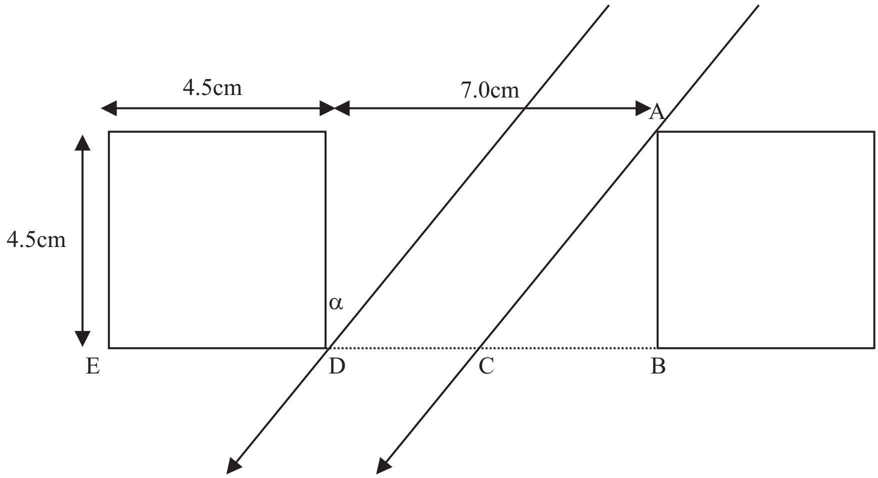

Fig. 4 Sunshine penetrating pergola roof

sun is highest in the sky. The size, spacing and thickness of the slats comprising the pergola roof are of particular importance. It is also helpful to make some simplifying *assumptions*:

1. The slats are positioned lengthwise in a north/south direction.
2. The sun is at its highest point in the sky at noon.
3. There are 12 h between sunrise and sunset.

From these assumptions, it follows that the sun moves through 15° every hour (180° in 12 h). Figure 4 represents the generalised situation when the sun is positioned at an angle of α degrees to the vertical. The pergola is made up of repeating units, and direct measurement reveals that the slats have a width of 4.5 cm with a gap between slats of 7.0 cm. The amount of sunlight that gets through to people beneath the pergola can be expressed as the ratio of distance DC (width of the sunbeam that penetrates for each repeating unit) to distance EB (width of the repeating unit). This represents the proportion of the area under the pergola in sunlight.

Some sample calculations illustrate the changing penetration of sunlight:

At noon, $\alpha = 0$ degrees and DC = DB, so the sunlight proportion = 7/11.5 = 61%.

At 11 a.m. and 1 p.m., $\alpha = 15°$, BC = 4.5 tan α = 1.2 cm, so the sunlight proportion = (7 − 1.2)/11.5 = 50%.

At 8 a.m. and 4 p.m., $\alpha = 60°$, BC = 4.5 tan α = 7.8 cm. Clearly, this is physically impossible in the context of the problem, since BC cannot be less than DB or 7.0 cm. This means that no sunlight is penetrating the pergola.

These calculations can be extended to produce a table of values for the function

$$y = \frac{7 - 4.5 \tan x}{11.5}$$

where y is the sunlight proportion (in decimal form) and x the angle of inclination of the sun from the vertical. This function *formulates the mathematical model* of the real-world situation.

A graphics calculator or spreadsheet can be used to plot this function and see how the sunlight proportion varies with the sun's angle and hence time of day. (It makes sense to restrict the domain to values of x between 0° and 60°, since we know that the sunlight proportion drops to zero before the sun reaches this angle.) Figure 5 thus represents a graphical *solution* to the mathematical model.

By observing where the graph cuts the x-axis, it is possible to estimate the sun's angle (and hence times of day) when the area beneath the pergola is in full shade. Alternatively, the time can be calculated exactly by solving the equation 7 − 4.5 tan $\alpha = 0$ (for values of α between 0° and 90°), which gives $\alpha = 57.3°$. Since we assume the sun moves through 15° every hour, the time frame of interest is 57.3/15 = 3.82 h before and after midday. Thus the *interpretation of the solution* suggests that people can sit beneath the pergola in full shade before 8:11 a.m. and after 3:49 p.m.

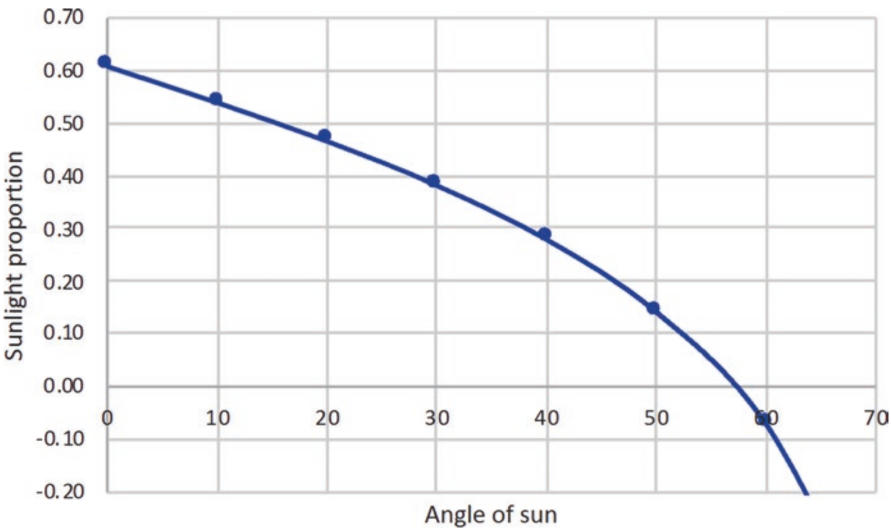

Fig. 5 Graphical solution to the mathematical model

The model can now be used to *design* a pergola, made from the same roofing materials, that gives full shade at 3 p.m. To do this, we can investigate altering the gap between slats, and real-world knowledge suggests the need to place the slats closer together. At 3 p.m., the sun is at an angle of 45°. Let the new distance between slats be d. Then we want to find the value of d such that $d - 4.5 \tan 45° = 0$. Hence $d = 4.5$ cm, and the distance between slats would be reduced to the width of the slats. The model, and assumptions underpinning it, can be *validated* by sitting under the pergola at different times of day and comparing the actual and predicted sunlight proportions. This practical validation might result in adjusting the assumptions and reformulating the model to give a more accurate representation of sunlight conditions.

To assess the pre-service teachers' work, I took the whole class outdoors to try out the Maths Trails they had created and followed this with a peer feedback session when we returned to our classroom. We then created two Maths Trail booklets, one for junior secondary students (Grades 8–10, students aged 12–15 years) and the other for senior secondary students (Grades 11–12, students aged 15–17 years), each comprising seven or eight of the activities the pre-service teachers had developed that could be completed by school students in a day's excursion to the university. The activities were supported by an educational rationale and links to relevant syllabuses and included sample solutions and teaching notes. We subsequently used these Maths Trail booklets as the basis for a whole day professional development event attended by 50 practising teachers, in which the pre-service teachers acted as presenters and Maths Trail guides (Goos et al., 2004).

3.4 Barriers and Enablers for Mathematical Modelling in Pre-service Teacher Education

Australia provides an interesting case study of attempts in different educational jurisdictions to embed mathematical modelling into the secondary school curriculum. In the state of Queensland, introduction of mathematical modelling proceeded gradually through a series of syllabus revisions over an extended period of time. The state had also abolished high-stakes external examinations in the early 1970s and introduced a system of externally moderated school-based assessment, which led to increased teacher professionalism with regard to curriculum and assessment design. Teachers were already accustomed to designing and assessing extended tasks that students undertook in class or at home, an approach that was consistent with mathematical modelling. Although teachers were unfamiliar with mathematical modelling when it was first introduced, the state curriculum authority tolerated a slow evolutionary process in teaching and assessing of modelling that was facilitated by advice from moderation panels of experienced teachers who monitored task design and assessment judgements.

The situation with regard to modelling and its assessment was quite different in the Australian state of Victoria, which had undertaken a major review of the school curriculum at around the time mathematical modelling was introduced in Queensland. Until this time, assessment in Victorian schools had been conducted solely via external examinations, but it was now proposed to introduce a mix of external examinations and school-based assessment in all subjects. At the same time, an innovative mathematics curriculum was introduced that incorporated mathematical modelling activities, all of which had to be formally assessed. According to Stillman's (2007) analysis of this implementation effort, both the extent and pace of change threatened the ability of Victoria's education system to sustain curriculum innovation. Within a few years, school-based assessment of extended modelling tasks was replaced by more traditional assessment of coursework, signalling the end of an assessment-driven educational experiment with a genuine focus on mathematical modelling.

Mathematical modelling can promote learning through crossing boundaries back and forth between the mathematics world and the real world, where the learning mechanism involves *coordination* and translation between these two worlds (Akkerman & Bakker, 2011). The modelling cycle could be regarded as a boundary object facilitating and enabling this coordination. The existence of a syllabus that mandates the teaching and assessment of mathematical modelling in the senior secondary school is another important enabler, since without this systemic support, teachers and teacher educators would find it difficult to justify the inclusion of modelling into their curriculum planning.

4 Curriculum Integration

4.1 Rationale and Positioning of Mathematics in Curriculum Integration

Curriculum integration has long been proposed as a way of helping students to develop richly connected knowledge and discover how this knowledge is used in real-world contexts. Approaches to curriculum organisation differ according to the type of connections made between subject areas. At one extreme is a subject-centred approach and at the other is full curriculum integration, where knowledge from relevant disciplines is brought to bear on problem-solving situations (Woodbury, 1998). In between lie a variety of interdisciplinary approaches that connect subject areas in different ways, for example, by planning separate subjects around a common theme or problem or by unifying some subjects into a single course taught by two or more teachers. Huntley (1998) discusses these variations in terms of three broad categories. She describes an *intradisciplinary* curriculum as one that focuses on a single discipline, such as mathematics. An *interdisciplinary* curriculum still has its focus on one discipline, but it uses other disciplines to support the content of the first domain (e.g. by establishing relevance or context).

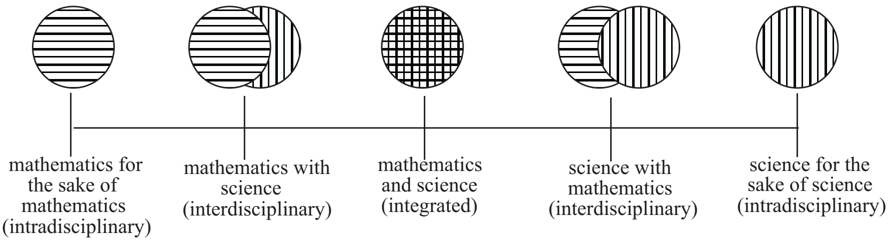

Fig. 6 Mathematics/science integration continuum. (Adapted from Fig. 1 in Huntley (1998, p. 322))

In an *integrated* curriculum, disciplinary boundaries dissolve completely, as concepts and methods of inquiry from one discipline are infused into others.

Huntley (1998) illustrated these different approaches to curriculum integration by proposing a continuum to clarify the degree of overlap between disciplines during instruction. Figure 6 represents the continuum for integrating mathematics and science, with intradisciplinary curricula at either extreme. Huntley defined an interdisciplinary "mathematics *with* science" course as one teaching mathematical topics (represented by the circle filled with horizontal lines) under the cover of a science context (circle filled with vertical lines). On the other hand, in a fully integrated "mathematics *and* science" course, the two disciplines interact and support each other in ways that result in students learning more than just the mathematics and science content (circles overlap completely to form a new pattern). In terms of Akkerman and Bakker's (2011) boundary crossing taxonomy, an interdisciplinary curriculum is likely to promote learning via *identification* with the separate disciplines leading to better understanding of their distinctive features, whereas an integrated curriculum aims for *transformation* and creation of new practices.

4.2 Impetus for Addressing Curriculum Integration in Pre-service Teacher Education

Curriculum integration was one of a number of related reforms to middle schooling implemented in schools in the Australian state of Queensland as part of the Education Department's New Basics Project (Luke et al., 2000). The design of this new curriculum framework began with three questions:

- What are the characteristics of students who are ideally prepared for future economies, cultures and society?
- What are the everyday life worlds that they will have to live in, interact with and transform?
- What are the valuable practices that they will have to "do" in the worlds of work, civic participation, leisure and mass media?

The resulting framework proposed four curriculum organisers to assist teachers, curriculum planners and schools to move towards a critical engagement with new social, technological and economic conditions. These organisers were labelled as life pathways and social futures, multiliteracies and communications media, active citizenship and environments and technologies. These curriculum categories were intended to draw on and combine a range of knowledge categories and disciplines within the traditional school curriculum. The New Basics project was trialled in 20 schools over 4 years, and this initiative provided an innovative context for engaging pre-service teachers in planning curriculum units and assessment tasks that integrated mathematics with other disciplines.

4.3 Examples of Curriculum Integration in Pre-service Teacher Education

In pre-service teacher education, curriculum specialisations are usually taught as discrete "methods" courses – a practice that mirrors the situation in secondary schools, where disciplinary boundaries are preserved. Attempting to breach these boundaries requires changes to organisational structures that not only address timetabling and staffing arrangements, but also encourage professional dialogue between teachers in different subject areas. It could be argued that similar priorities apply to pre-service teacher education programs that aim to prepare graduates for new school environments, such as those anticipated by the New Basics curriculum reform. A history teacher education colleague and I decided to model this kind of professional dialogue by investigating curriculum integration in secondary school mathematics and history (Goos & Mills, 2001). We planned a group assessment task in which mathematics and history pre-service teachers were to work in groups to produce an integrated curriculum unit.

Joint meetings of the mathematics and history curriculum classes took place during a 9-week block at the start of the year, before pre-service teachers began their first practicum placement. Both curriculum groups continued to meet separately throughout this period for subject-specific workshops. The combined classes, which lasted 1 h, were relatively unstructured and designed to provide pre-service teachers with time together to work on their curriculum units; however, we did address topics relevant to this task, in particular, by providing information about the theoretical and policy background to the New Basics reform. An important feature of this project was our desire to model the kind of cross-disciplinary dialogue and intellectual risk taking that we hoped our pre-service students would embrace. We also used questions and feedback from the pre-service teachers to help us design the assessment task, which was comprised of group and individual components. The curriculum unit plan was originally conceived as a group task; however, we soon recognised pre-service teachers' need to evaluate their individual contributions and to comment on the process of working as a cross-curricular

Table 1 Topic and subject matter for the pyramids of Egypt curriculum unit

Topic	History content	Mathematics content
Pyramids of Egypt	When were the pyramids built? Political/social structure of ancient Egypt Geography of Egypt Hieroglyphics Mathematics of ancient Egyptians Religious/burial practices and beliefs Pyramid construction methods	Mass Ratio and proportion Plane and 3D shapes Measurement (length, area, volume, angle, time) Number study and operations Statistics

group. To this end, we asked individuals to write a brief reflective analysis of the benefits and difficulties they experienced, and how problems were dealt with in their group, and to identify implications for collaboration between teachers across different curriculum areas.

The sequence of lessons in the curriculum unit designed by the pre-service teachers was to lead to an assessment task with real-world value and use that would allow junior secondary students (Grades 8–10, students aged 12–15 years) to demonstrate the mathematics and history knowledge and skills they had developed. One such assessment task, for a curriculum unit on the pyramids of Egypt, is reproduced below.

> You have been declared Pharaoh of Egypt! As a monument to your reign, you choose to build a pyramid in your honour. Determine resources required, list environmental impacts, forecast problems that may occur, and construct a scale model of your pyramid. Conduct a feasibility study and report on your findings.

Table 1 summarises the subject matter of this curriculum unit, and Fig. 7 shows sample learning activities that were devised by the pre-service teachers.

4.4 *Barriers and Enablers for Curriculum Integration in Pre-service Teacher Education*

The existence of a large-scale school reform initiative, represented by the New Basics project in the Australian state of Queensland, provided an authentic context for teacher educators to explore curriculum integration with pre-service teachers. Since that time, a new national curriculum has been introduced in Australia (ACARA, n.d.) with a return to strong disciplinary boundaries in the secondary school years. It is difficult to envisage how this curriculum structure could support the type of disciplinary boundary crossing involved in the integrated curriculum approach described above.

Even when curriculum organisation is conducive to integration, barriers remain to be overcome. One of the major obstacles faced by the mathematics pre-service teachers was the realisation that the mathematical aspects of the curriculum unit

Activity #1: Size of the Pyramids of Giza

Pyramid	Side (m)	Height (m)	Base area (m^2)
Khufu	230	146.5	
Khafre	216	140.5	
Menkaure	108	66.5	

How many Olympic-sized swimming pools would fit into the base of Khufu's pyramid?
How many football fields would fit into the base of Khufu's pyramid?
If Khafre's pyramid were as tall as this classroom, how tall would you be?

Activity #2: Construction of the pyramids of Giza
If the density of limestone is 2280 kg/m^3, what is the total weight of Khufu's pyramid?
If the average weight of a limestone block is 2.5 tons, how many blocks comprise Khufu's pyramid? (1 ton = 1016 kg)
Khufu reigned for a minimum of 23 years. How many blocks of limestone needed to be delivered to the pyramid every hour for the pyramid to be completed within Khufu's lifetime:

- if work continued all year round
- if work took place only during the 3 months annual inundation of Egypt.

Fig. 7 Sample learning activities for the pyramids of Egypt curriculum unit

played a secondary role to the history material. One described this as "feeling as if we had to let the history people come up with ideas first, so that we could build from them. It didn't feel as if we were able to suggest maths ideas first …." As a counterpoint to these concerns about collaboration, practising teachers sometimes express fears about curriculum integration requiring generalist teachers, working as individuals, to teach cross-disciplinary units such as those prepared by the pre-service teachers. On the contrary, however, Wallace, Rennie, and Malone (2000) have argued that an integrated curriculum should not be taught by one person since instead it can enable team teaching by disciplinary specialists. Subject-specific expertise therefore becomes more, not less, important, if potentially rich connections are to be made between curriculum areas.

Another barrier to curriculum integration involves organisational constraints within the school, such as timetabling, lesson duration and allocation of teachers to classes. In the project described above, pre-service teachers were allowed to assume that adequate time, resources and personnel would be available to implement their teaching plan. This was a deliberate choice on the part of the teacher educators in order to challenge assumptions about existing school structures and the organisation of secondary teacher education.

Participation in the integrated curriculum project required the mathematics and history pre-service teachers to examine their professional values and disciplinary beliefs. Some groups, defeated by the logistics of collaboration, "atomised the task and worked substantially in isolation". Other individuals reported learning "valuable lessons about diplomacy, compromise and exchange of ideas between teachers". There was some evidence from their written reflections of a growing appreciation of the value of each subject. For example, one mathematics pre-service teacher commented that she "finally began to appreciate that mathematics is instrumental in explaining and extending concepts in other areas and real-world contexts. Rather than making it a lesser subject, this characteristic of mathematics is one of its greatest virtues". The relationship between mathematics and history achieved in this project suggests that the pre-service teachers created an interdisciplinary, rather than integrated, curriculum (Huntley, 1998), with history providing the primary context for teaching selected mathematical concepts. Crossing the boundaries between disciplines led to learning through increased *identification* with one's own discipline, reinforcing disciplinary identities while enabling and legitimating coexistence of mathematics and history within the curriculum unit. The interdisciplinary curriculum units, and the teacher education assessment task that led to their creation, could be considered as boundary objects enabling disciplinary boundary crossing, while the two teacher educators – each with an intellectual commitment to their own discipline as well as to interdisciplinary collaboration – acted as brokers working to connect the disciplines.

5 Numeracy Across the Curriculum

5.1 Rationale and Positioning of Mathematics in Numeracy Across the Curriculum

In many countries, the notion of mathematical literacy as a twenty-first century competency has emerged from international studies such as the OECD's Programme for International Student Assessment (PISA; OECD, 2016). In some English-speaking countries, however, it is more common to speak of *numeracy* rather than mathematical literacy. Being numerate involves more than mastering basic mathematics, because numeracy connects the mathematics learnt at school with out-of-school situations that additionally require problem-solving, critical judgement and making sense of the non-mathematical context.

By the late 1990s, educators and policymakers in Australia had embraced a broad interpretation of numeracy similar to the OECD definition of mathematical literacy: "To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life" (Department of Employment, Education, Training and Youth Affairs, 1997, p. 15). This definition became widely accepted in Australia and formed the basis for much numeracy-related research and curriculum development.

The Quantitative Literacy Design Team (2001) has argued that for numeracy to be useful to students, it must be learnt in multiple contexts as an integral part of all school subjects, not just mathematics. They distinguished between mathematics and numeracy by explaining:

> Mathematics climbs the ladder of abstraction to see, from sufficient height, common patterns in seemingly different things. Abstraction is what gives mathematics its power; it is what enables methods derived from one context to be applied in others. But abstraction is not the focus of numeracy. Instead, numeracy clings to specifics, marshalling all relevant aspects of setting and context to reach conclusions. (pp. 17–18)

The relationship between mathematics, numeracy, other subjects taught in school and real-world contexts is more complex than can be described by the language of "connection" or "integration". My colleagues and I have attempted to capture these multi-faceted relationships by developing the numeracy model shown in Fig. 8 (Goos, Geiger, Dole, Forgasz, & Bennison, 2019).

The model consists of four core elements: attention to real-life and curricular contexts; application of mathematical knowledge; use of physical, representational and digital tools; and promotion of positive dispositions towards the use of mathematics to solve real-world problems. A fifth overarching element – a critical orientation – requires the appropriate selection and application of mathematics to a real-world problem as well as the interpretation and critique of results.

The numeracy model was designed to be readily accessible to teachers as an instrument for planning and reflection. It has been validated through several research and development projects involving Australian teachers in primary and secondary schools and across many subject domains. It has been effective in providing a framework for auditing the numeracy demands of the school curriculum, in helping teachers to recognise the numeracy demands and opportunities in different school subjects and to design numeracy tasks, and in enabling us to trace teachers' own growing understanding of numeracy across the curriculum (Geiger, 2016; Geiger, Goos, & Dole, 2011; Goos, Dole, & Geiger, 2012; Goos, Geiger, & Dole, 2014).

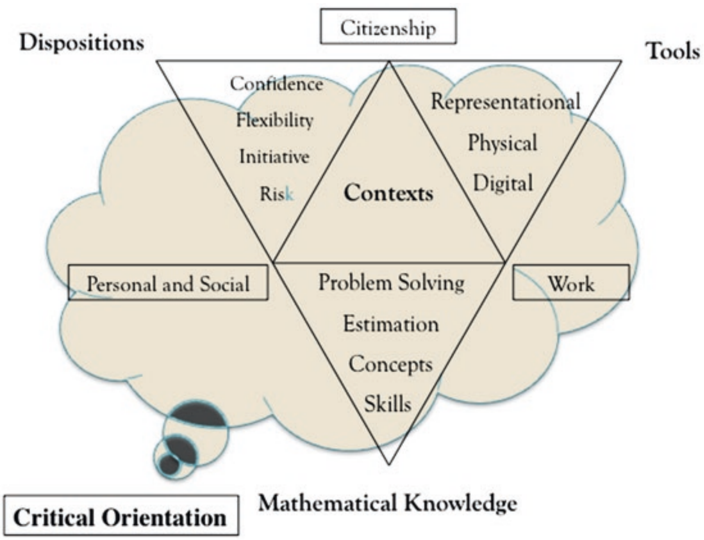

Fig. 8 Numeracy model

The numeracy model functions as a boundary object that connects the intersecting worlds of the school curriculum and its constituent disciplines – including mathematics – with real-world contexts.

5.2 Impetus for Addressing Numeracy Across the Curriculum in Pre-service Teacher Education

In Australia, there has been acknowledgement for many years that numeracy is an across the curriculum commitment. This commitment was first formalised in a national numeracy policy (Department of Education, Training, and Youth Affairs, 2000) and later reinforced by a national numeracy review (Council of Australian Governments, 2008). Inclusion of numeracy as a general capability in the Australian curriculum endorses the expectation that all teachers will be responsible for developing their students' numeracy, no matter what subjects they teach. Yet, the national curriculum for Australian schools is not explicit in setting out how all teachers should achieve this goal. Despite the success of research and development projects that have helped teachers to recognise the numeracy demands and opportunities of the subjects they teach (e.g. Goos et al., 2014; Thornton & Hogan, 2003), numeracy is still widely regarded as the responsibility of the mathematics teacher or department (Carter, Klenowski, & Chalmers, 2015).

In Australia, teacher preparation policies and standards require pre-service teachers to demonstrate competence in understanding and applying numeracy teaching strategies appropriate to their subject area (AITSL, 2017). However, these standards are weakly framed and lack specific detail as to what teachers must know and be able to do in order to achieve curricular goals for numeracy. A variety of approaches can be observed in Australia for developing knowledge of numeracy teaching strategies in pre-service teacher education programs. Such approaches range from semester-long courses to no specific courses at all and instead an effort to incorporate numeracy across the whole teacher education curriculum.

5.3 Example of Numeracy Across the Curriculum in Pre-service Teacher Education

The release of Australia's national numeracy policy in 2000 prompted some early interest in developing numeracy across the curriculum courses for pre-service teacher education students (e.g. Groves, 2001). However, it is likely that such courses will become more common, as universities are now required to meet national accreditation standards that expect graduate teachers will be able to embed numeracy into the subjects they teach (see Forgasz & Hall, 2016, for an evaluation of one such course).

A full description of a current course on numeracy across the curriculum can be found at https://www.courses.uq.edu.au/student_section_loader.php?section=1&profileId=96602 (The University of Queensland, 2014). This course is compulsory for pre-service secondary teachers across all specialist curriculum domains. The topics addressed in each week of the course are presented in Table 2.

A brief case study from Week 7 of the course, showing how a teacher created a numeracy learning opportunity within the subject of Health and Physical Education, is presented in the box below. The case study is intended to be used as a stimulus for secondary pre-service teachers to recognise numeracy learning opportunities within the subjects they will teach.

Case Study of Numeracy Across the Curriculum

An experienced teacher in a small rural school was assigned to teach mathematics, English and health and physical education to a group of Grade 8 students who were in their first year of secondary school (aged 12–13 years). While participating in a research and development project, centred on the numeracy model depicted in Fig. 8, she decided to adapt an activity she had used for several years in order to enhance students' numeracy learning opportunities. The task required students to investigate their level of physical activity over 1 week by using a pedometer. Each student recorded the number of paces taken each day and recorded this information in a shared Excel spreadsheet projected onto an electronic whiteboard at the front of the classroom. At the end of the week, the teacher wanted students to learn how to convert their total number of paces to the distance they had walked over a week. She took the class outside and asked students to estimate 100 m from a common starting point and then walk to this position. She followed the students while laying out a measuring tape to mark off 100 m. Students were asked to walk back to the starting point, 100 m away, and count the number of paces they took. Once back in the classroom, the teacher introduced a template, displayed via the electronic whiteboard, which was designed to assist students to convert paces to kilometres. She modelled the conversion by filling in her own result – 119 paces in 100 m. Students were amazed to find that their teacher had walked 98.8 km in a week, and they eagerly worked on their own personal calculations. They also spontaneously produced Excel graphs to compare distances walked by different students, by boys and girls, and on different days of the week.

Pre-service teachers can be asked to analyse the numeracy opportunities created by this teacher in terms of the numeracy model shown in Fig. 8. Their response might point to measurement (estimation and converting units), number (ratio) and chance and data (collection, organising and representing data) as the *mathematical knowledge* used in this lesson. The students' learning was situated in the real-world

Table 2 Weekly outline of topics in a pre-service teacher education course on numeracy across the curriculum

Week	Topic
1	Understanding numeracy. Exploring definitions of numeracy; differences between mathematics and numeracy; personal conceptions of numeracy; exploring what a numerate person knows and can do.
2	Numeracy across the curriculum. History of the idea of numeracy in Australia and beyond; numeracy as a general capability in the Australian Curriculum.
3	Numeracy in the twenty-first century. A theoretical model and framework for curriculum planning and task design, considering its use to plan and design numeracy activities.
4	The role of ICTs in numeracy. Exploring a range of ICTs, including websites, programs, apps to support the teaching and learning of numeracy.
5	Numeracy as a critical orientation. Re-visiting the numeracy model and exploring ways to promote critical numeracy in students.
6	Numeracy demands of the curriculum. Using the numeracy model to audit the numeracy demands of learning areas in the Australian Curriculum.
7	Numeracy opportunities across the curriculum. Recognising numeracy opportunities in learning areas; case studies of tasks that teachers have developed and used to create numeracy learning opportunities; identifying a numeracy teaching opportunity in a learning area.
8	Principles of numeracy task design. Research examples of numeracy tasks and pedagogical approaches to engaging students in numeracy tasks; designing a numeracy task.
9	Planning for numeracy across the curriculum. Mapping numeracy in curriculum areas, using planning templates, creating rich numeracy tasks in different learning areas.
10	Assessing numeracy learning. Critical analysis of what can be learnt from PISA and national numeracy testing.
11	Challenges and dilemmas. Research on the experiences of Australian teachers in embedding numeracy across the curriculum.
12	Whole school approaches to numeracy. Focus on curriculum leadership approaches to engage colleagues, parents, cross-curricular teams and making school links to enhance numeracy.

context of an outdoor activity that required them to convert personal information – paces walked in a week – into standard measures (kilometres), which in turn were used to compare each student's level of activity with that of others. The teacher used a range of *tools* through the lesson: physical tools such as tape measures to mark out 100 m, representational tools in the form of the template she designed in order to scaffold the conversion of paces into kilometres, and digital tools in the form of electronic calculators and the Excel spreadsheet used for recording students' data. By embedding learning in an outdoor activity that made use of students' personal information, she was attempting to encourage positive *dispositions* towards learning mathematics. This lesson also incorporated aspects of *critical orientation* as students were asked to consider why the distances they had walked differed from each other in relation to both pace length and to students' different levels of activity over a week.

5.4 *Barriers and Enablers for Numeracy Across the Curriculum in Pre-service Teacher Education*

Education policies that require teachers to develop numeracy in all school subjects, and teacher educators to prepare graduates who can meet this challenge, are key enablers of numeracy across the curriculum in pre-service teacher education. But policies alone will not ensure that secondary pre-service teachers experience productive boundary crossing between mathematics and their "home" discipline. Also needed are models of integration that act as boundary objects connecting numeracy, mathematics and other disciplines with the real world. The numeracy model described in this section is a boundary object that serves this purpose.

Tensions can also exist within curricula that maintain strong disciplinary boundaries while simultaneously promoting numeracy as a cross-curricular priority – as is the case in Australia. Teachers of subjects other than mathematics can easily form the view that numeracy is not their responsibility. For these teachers, the challenge is to recognise and attend to the numeracy demands of their subject as a means of enhancing students' disciplinary understanding.

When Australian teachers participated in our research and development projects on numeracy across the curriculum, they were asked to identify the element of the numeracy model, which represented their initial focus at the beginning of the project and also to indicate those elements that assumed greater importance to them as the project progressed. Providing teachers with a copy of the numeracy model they could annotate allowed our research team to gather data on their professional learning trajectory (Geiger et al., 2011). Interviews with teachers also yielded perceptions of learning that resonate with Akkerman and Bakker's (2011) mechanisms for learning at the boundary between domains. One teacher described her learning in the following way:

> During the initial project meeting, where the model was described for what it was to be numerate, exemplar activities were provided that helped me with *knowing* about numeracy. Returning to school and trying out initial ideas was part of me *doing* in relation to numeracy. Eventually, though, the continued interaction of my developing knowing and doing led to my present state where my approach to teaching numeracy had become part of my *being*. I felt that my involvement in the project has changed who I am, both professionally and personally.

This transition from knowing to doing to being is indicative of *reflection* as a learning mechanism that leads to new insights and construction of a new teacher identity resulting from crossing boundaries between disciplines in search of numeracy opportunities.

6 Comparison of Models and Implications for Pre-service Teacher Education

Mathematics can cross disciplinary borders in a variety of ways in pre-service teacher education, and this chapter has explored three such approaches: mathematical modelling, curriculum integration and numeracy across the curriculum. Table 3 summarises key features of each approach, facilitating comparison of their rationales and positionings of mathematics, the impetus for their inclusion in teacher education programs and barriers and enablers for implementation. In addition to these pragmatic considerations, theoretical comparisons can be made between the mechanisms for learning at the boundaries between domains in each approach and the roles of boundary objects and brokers identified.

It appears that the most common rationale for integrating mathematics with other disciplines emphasises *connections* that motivate student learning in real-world contexts, and the impetus for addressing this goal in pre-service teacher education derives from changes to the school curriculum or accreditation standards for teacher education programs. However, a boundary crossing theoretical perspective allows us to see *discontinuities* and differences between domains that should not be ignored. While these discontinuities can offer potential for learning, they also pose challenges for teachers and teacher educators who are discipline-based specialists. The summary of barriers and enablers presented in Table 3 points to some of these challenges. Interestingly, curriculum reform and educational policies are identified as both an impetus and a barrier to boundary crossing, which suggests there is a gap between rhetoric and reality in this space. However, teachers' beliefs, values and perceptions also influence their experience of boundary crossing between domains.

It is possible to conceive of all four learning mechanisms proposed by Akkerman and Bakker (2011) coming into play when teacher educators experiment with integrating mathematics with other disciplines. In my own work with pre-service teachers, I emphasise that mathematical modelling requires *coordination* between mathematics and the real world, while developing numeracy across the curriculum can lead to learning through *reflection* on differences between mathematics and a teacher's primary subject discipline. I would argue that curriculum integration rarely results in *transformation* and creation of new boundary practices, because of the practical difficulties in fully blending the concepts and methods of inquiry of separate disciplines. Instead, it might be more common to experience enhanced *identification* with one's own primary discipline as a result of working with teacher colleagues in another discipline.

In all three integrated approaches in pre-service teacher education, *boundary objects* facilitated crossing between domains, and, in the case of the integrated curriculum and numeracy across the curriculum projects, brought together pre-service and practising teachers from these different domains. Only in the former project was it possible to identify *brokers* who deliberately worked to connect disciplinary

Table 3 Comparison of three models for integrating mathematics with other disciplines in pre-service teacher education

Parameter	Models for integrating mathematics with other disciplines		
	Mathematical modelling	Curriculum integration	Numeracy across the curriculum
Rationale and positioning of mathematics	Motivates connection between mathematics and the real world Enriches learning of mathematical concepts and modelling process	Motivates development of connected knowledge in real-world contexts Connections can create intradisciplinary, interdisciplinary or integrated curricula	Needed for critical citizenship Distinction between mathematics and numeracy Numeracy should be developed in all school subjects
Impetus for addressing in pre-service teacher education	Mathematics syllabus revisions that mandated teaching and assessment of modelling	School reform initiative that trialled an integrated curriculum framework	National numeracy policies and strategies Curriculum endorsing numeracy as cross-curricular responsibility Teacher preparation standards requiring graduate competence in embedding numeracy
Barriers and enablers	Pace and extent of curriculum change Teacher professionalism	Curriculum structure: strong vs. weak disciplinary boundaries Role of mathematics diminished Need for collaboration between subject specialist teachers School organisational constraints Teacher professional values and beliefs (disciplinary vs. interdisciplinary)	Education policies and curriculum structures Teachers' personal conceptions of numeracy
Learning mechanism at the boundary between domains	Coordination	Identification (interdisciplinary) Transformation (integrated)	Reflection
Boundary objects	Modelling cycle	Integrated curriculum unit	Numeracy model
Brokers	Not identified	Teacher educators from two disciplines	Not identified

paradigms, in the form of the two teacher educators from the disciplines of mathematics and history. However, it is possible to imagine how brokers might play a role in the other two approaches. For example, a mathematics and a science teacher educator might decide to work with their pre-service students on modelling tasks in the context of chemistry, physics or biology, or a mathematics teacher educator given responsibility for coordinating a course on numeracy across the curriculum might call on teacher education colleagues for insights into the numeracy demands of other areas of the school curriculum. This observation highlights another significant issue when mathematics crosses borders in pre-service teacher education – mathematics teacher educators themselves need to embrace their roles as brokers in modelling productive collaboration with colleagues in other disciplines.

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Part III
Programmatic Borders in Mathematics Pre-service Teacher Education

Bridging the Gap Between Coursework and Practica: Secondary Mathematics Pre-service Teachers' Perceptions About Their Teacher Education Program

Limin Jao, Nakita Rao, and Alexandra Stewart

Secondary-level mathematics pre-service teacher (PST) education programs typically have various components such as university-based coursework (including mathematics content and mathematics teaching methods courses) and school-based practice teaching blocks (field experiences/practica). Oftentimes, rather than being visibly interlinked, there are logistical and practical borders between these components, making it difficult for PSTs to see the connections between them (Bain & Moje, 2012; Ebby, 2000). As educators and researchers continue to better understand the impact that teacher education programs have on PSTs, it is pertinent to explore these components and their role in PSTs' development.

While some teacher education programs make noteworthy efforts to integrate the programmatic components and endeavor to help PSTs see the relationships between them (Ebby, 2000), most programs are organized by distinctly separate components that rarely overlap. In this chapter, we use the case of a Canadian teacher education program to explore PSTs' experiences in, and perceptions of, this common, "bordered" model of teacher education programs. Gathering first-hand accounts of PSTs' experiences in teacher education programs has proven to be particularly meaningful with regards to improving the programs themselves (Clift & Brady, 2005; Ralph, Walker, & Wimmer, 2009). As such, we asked the following research questions: (1) According to secondary mathematics PSTs, what is the purpose and value of each component of their teacher education program? (2) In what ways did these components influence their development as future secondary mathematics teachers? (3) What (if any) connections did the PSTs make between the components of their program?

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1 Components of a Secondary Mathematics Teacher Education Program

In Canada, there are many programs leading to certification as a secondary mathematics teacher. In some cases, PSTs enter their teacher education program having already completed a degree in their subject of specialization (e.g., Bachelor of Mathematics). These programs[1] assume that the PSTs have the required subject-area content knowledge; thus, the only subject-specific components of their programs are mathematics teaching methods courses and field experiences (in secondary mathematics classes).[2] In other cases, typically for those who have not previously completed a specialist degree, teacher education programs require PSTs take mathematics content courses in addition to the components described earlier. In this chapter, we focus on the latter format of teacher education programs and, in the following sections, describe these subject-specific components in more detail.

1.1 Mathematics Teaching Methods Courses

Mathematics teaching methods courses are usually taken by mathematics PSTs early in their teacher education program and may be coupled with a practicum component (Abell, 2006; Higher Education Commission, 2012). In general, the purpose of teaching methods courses is to improve PSTs' knowledge and provide them with experience of actual classroom teaching through both student and teacher lenses (Kiliç, 2011; Wilkins & Brand, 2004). Through a student lens, mathematics teaching methods courses provide PSTs with the opportunity to explore mathematics subjects (e.g., numbers, geometry, statistics and probability, algebra, and measurement) via manipulatives, problem solving and other investigative approaches (Albayrak & Unal, 2011; Baroody & Coslick, 1998). Through a teacher lens, specific goals of mathematics teaching methods courses include cultivating positive dispositions toward mathematics, developing PSTs' mathematics pedagogy, and establishing why mathematics knowledge is taught (Hodge, 2011).

Although there are a variety of goals for PSTs in mathematics teaching methods courses, a big focus is also on that of introducing different teaching methods for mathematics (Albayrak & Unal, 2011). More specifically, a large part of these courses is supporting PSTs' development of mathematical knowledge for teaching

[1] This format of teacher education program may be at an undergraduate, graduate, or certificate level.

[2] In these programs, PSTs also take general education courses (e.g., assessment and evaluation, educational psychology).

(MKT) (Ball, 1990; Ball, Thames, & Phelps, 2008; Tirosh, 2000). MKT is the "mathematical knowledge that teachers use in classrooms to produce instruction and student growth" (Hill, Ball, & Schilling, 2008, p. 374) and is composed of both subject matter knowledge and pedagogical content knowledge (PCK). While subject matter knowledge is an understanding of how teaching mathematics is similar and yet different from other fields using mathematics (e.g., engineering), PCK is the knowledge held by teachers that allows them to connect what they know about pedagogy with what they know about content and curricula (An, Kulm, & Wu, 2004; Hill et al., 2008).

More than the intended purposes and structure of the courses, mathematics teaching methods courses allow for PSTs to experience a mathematical community that is different from what is experienced elsewhere in the program (Ebby, 2000). With "the assumption that experiencing mathematics differently as learners will cause teachers to reconstruct their beliefs, assumptions, and ultimately their practice" (Ebby, 2000, p. 70), the experiences in the methods courses also aid in the development of new perspectives of PSTs themselves, their peers, and of the subject by giving opportunities for active instruction (Albayrak & Unal, 2011). This is important as secondary mathematics PSTs typically enter their teacher education program, having mostly experienced traditional approaches to teaching and learning mathematics, characterized by teacher-centered and rote learning (Jao, 2018). Yet, research advocates for a reformed approach which involves student-centered and exploratory learning, and strengthening students' mathematical understanding (Hunter, Hunter, Jorgensen, & Choy, 2016; National Council of Teachers of Mathematics, 2014). Mathematics teaching methods courses have been found to support change in PSTs' beliefs toward reform-based ideals (Jao, 2017; Wilkins & Brand, 2004).

In contrast, despite the positive learning outcomes from the mathematics teaching methods courses, some challenges remain. Some instances show PSTs to have trouble taking on the teacher role after these courses, stating a difficulty in transitioning from what they have learned in the courses to their application in the classroom (Stickles, 2015). In other cases, the newly acquired skills and knowledge from the courses get "wiped out" by conflicting situations in PSTs' practica or within the first years of teaching (Wilkins & Brand, 2004). It has been suggested that this is in part because the understandings that PSTs develop from their program's coursework is dependent on their beliefs, dispositions, and experiences (Ebby, 2000). For example, PSTs may not consolidate and solidify their learning if they experience situations in practica, where mentors (e.g., university supervisors, cooperating teachers) encourage teaching approaches that are contradictory to those learned in the teaching methods courses (e.g., teacher-centered approached vs. student-centered approaches). However, it has been found that taking multiple mathematics teaching methods courses in their program (i.e., interspersed throughout their program between other coursework and practica) helps PSTs to reinforce and maintain beliefs and attitudes acquired from these courses (Hart, 2002).

1.2 Mathematics Content Courses

At their core, mathematics content courses provide PSTs with subject-specific content knowledge for teaching. Mathematics content courses are taken mainly by two categories of students: Science, Technology, Engineering, and Mathematics (STEM) majors, and secondary PSTs whose subject of specialization is mathematics (Hodge, Gerberry, Moss, & Staples, 2010). These courses are typically taught by mathematicians, which, when teaching PSTs, inevitably results in the mathematicians adopting roles as teacher educators (Leikin, Meller, & Zazkis, 2017). Secondary mathematics PSTs typically take almost as many mathematics content courses as a mathematics major would: various calculus courses along with several linear algebra, differential equations, real analysis, and abstract algebra courses.

For secondary mathematics PSTs, the purpose of these content courses is to provide PSTs with advanced and robust mathematical content knowledge (MCK), often going beyond the curricula that they are required to teach at the secondary level (Plotz, Froneman, & Nieuwoudt, 2013). Although more advanced than secondary curricula, the mathematical knowledge gained in these content courses is beneficial and facilitates success for both these future teachers and their future students (Leikin et al., 2017). In addition to developing PSTs' MCK, content courses also have many different goals including: supporting PSTs with learning to make connections between various fields of mathematics in order to see the "big picture" (Williams, 2001), and providing opportunities for PSTs to learn about the history of mathematics and what it is like to be a mathematician (Hodge et al., 2010; Leikin et al., 2017). Furthermore, content courses support PSTs' development of mathematical processes (e.g., logical reasoning and problem-solving skills) as well as mathematical communication; provide the ability to solve unique, open-ended, and hypothetical problems, and identify poorly proposed questions; and facilitate the development of confidence and self-assurance in PSTs through the mastery of their content knowledge (Williams, 2001).

Despite the rigorous mathematics education content courses provide, PSTs have difficulty transferring what they learn into the classroom (Dreher, Lindmeier, Heinze, & Niemand, 2018). Specifically, research indicates that there is often a disconnect between content knowledge and relevant teaching methods (i.e., PSTs' MKT) in many teacher education programs (Ball et al., 2008; Wu, 2011). Similar to mathematics teaching methods courses, PCK has been shown to be a major component typically missing from mathematics content courses. As such, researchers have questioned the validity of mixing PSTs with STEM majors in these courses (Hill et al., 2008). Although both groups need to develop abstract and higher mathematics content knowledge, PSTs require the additional component of PCK to fulfill their MKT and better apply their knowledge for teaching at the secondary school level (Hodge et al., 2010).

1.3 Field Experiences

Typically, field experiences (practica) are partnerships between teacher education programs and host schools to allow for PSTs to experience the role of the teacher in contexts similar to those that they will be working in the future (Borman, Mueninghoff, Cotner, & Frederick, 2009). While PSTs have their own personal experiences as students, and therefore have experienced the acculturation effects of attending school, this "apprenticeship of observation" (Lortie, 1975) is incomplete. Being a student does not allow for reflection or rationales in teaching to be shared or discussed. Therefore, practica allow for PSTs to experience authentic apprenticeships prior to starting their own careers (Bullock & Russell, 2009). Generally, practica are aimed to have PSTs experience and practice instructional tasks (e.g., developing and carrying out lessons) and all other facets of school life (e.g., administrative tasks, extra-curricular activities) (Begum & Yarmus, 2013).

Given their in situ and lived experience nature, practica provide opportunities for PSTs to experiment and self-reflect and apply theory learned in coursework to practice (Ralph et al., 2009). Yet, research shows that PSTs alone cannot bridge this "theory-practice gap" (Dillon & O'Connor, 2009). PSTs benefit from explicit links to their practica in different program components (Bain & Moje, 2012; Baumgartner, Koerner, & Rust, 2002). For example, providing PSTs with opportunities for discussions in related concurrent courses has been shown to allow for further professional growth (Begum & Yarmus, 2013).

The placement and frequency of practica within teacher education varies. In some cases (typically programs that focus more on the practical aspects of the profession and assume prior teaching experience), practica occur at the beginning of the program (ECS Educational Policy Site: Teaching Quality, 2007). In contrast, most teacher education programs disperse practica throughout the entire program (Dillon & O'Connor, 2009). In these cases, the practica are often scaffolded such that the practica which take place toward the beginning of the program have PSTs focus on classroom observation and experience working one-on-one with students. Later practica have PSTs take on additional responsibilities culminating in practica during which PSTs take on full teaching responsibilities (Ontario Teachers' Federation, 2010). Other programs may have longer practica, which begin with a short observational period and progressively allow for longer and more involved teaching duration (Begum & Yarmus, 2013; Ontario Teachers' Federation, 2010). Research shows that for practica to yield fruitful reflections and significant growth, they must also be varied in context and setting (Borman et al., 2009).

At host schools, PSTs are typically paired with in-service teachers (cooperating teachers). Cooperating teachers act as mentors and supervise, guide, and facilitate PSTs' induction into the profession (Hobson, Ashby, Malderez, & Tomlinson, 2009). Cooperating teachers play an important role in the socialization of PSTs into the teaching profession by helping PSTs manage their workload and day-today tasks (Bullough & Draper, 2004; Lindgren, 2005; Maldarez, Hobson, Tracey, & Kerr, 2007; Moor et al., 2005; Wang & Odell, 2002). PSTs can also benefit from the feedback provided to them by their cooperating teachers. Whether through informal discussions between classes or formal written assessments of teaching performance, PSTs value specific and timely feedback from their cooperating teachers (Broad & Tessaro, 2009). Research shows that cooperating teachers' mentorship and feedback may result in PSTs feeling less isolated, more confident, and more capable in reflecting on difficult situations and problems (Bullough, 2005; Johnson, Berg, & Donaldson, 2005; Lindgren, 2005; Marable & Raimondi, 2007).

While cooperating teachers can have a positive effect on PSTs, tensions may arise due to the amount of responsibility that is given to the host schools and the cooperating teachers (Breunig, 2005). Views and aims of cooperating teachers may not align with those of the teacher education program, requiring PSTs to navigate conflicting and confusing expectations (Bain & Moje, 2012; Vick, 2006). Similarly, PSTs may find it challenging if their cooperating teachers advocate for and model teaching practices that counter those espoused in their teacher education programs.

2 Research Context and Approach

Our study took place within the context of a 4-year undergraduate teacher education program (Bachelor of Education) at a Canadian university. Graduates of the program are certified to teach at the secondary school level and choose one subject as an area of expertise – in the case of our participants ($n = 6$), mathematics. The teacher education program is comprised of coursework and school-based practica. Courses include subject-specific content courses, subject-specific teaching methods courses, and general education courses taken by all PSTs (e.g., assessment, educational psychology, diverse learners). PSTs have one practicum per academic year each with an increased level of responsibility (i.e., the first practicum is observational and the final practicum has the PSTs taking on close to a full teaching load).

Informed by the exploratory case method (Yin, 2009), a qualitative approach was used to explore secondary mathematics PSTs' experiences in their teacher education program. Data were collected through semi-structured interviews. Specifically, participants were each interviewed by a member of the research team once at the end of their teacher education program and were asked to reflect on their experiences in each of the different components of their teacher education program that had explicit connections to mathematics (i.e., mathematics teaching methods courses, mathematics content courses, and practica). Examples of questions include: How did the practica shape your development as a secondary mathematics PST? Which component was the most meaningful? What did the courses have in common? How were they different? Interviews were audio recorded and transcribed verbatim. To analyze the interviews, we identified quotes in which participants specifically spoke about one of our foci components of their teaching education program: mathematics content courses, mathematics teaching methods courses, and practica. Through an iterative process, we reread the quotes with the intention of seeking and grouping emerging patterns and themes (Saldaña, 2009). We present these themes in the sections that follow. All names are pseudonyms.

3 PSTs' Perspectives About Mathematics Teaching Methods Courses

PSTs noted that mathematics teaching methods courses were valuable to their development as future teachers. PSTs described the teaching methods courses' impact on their teaching views, development as mathematics PSTs, as well as how the teaching methods courses offered them a supportive and safe environment to explore their understandings of mathematics teaching. PSTs recounted how the teaching methods courses reflected what they had seen in the field and what they expected to experience as soon-to-be teachers. As Domino noted,

> [Y]ou know how the university is preparing you for the future and for your job and all that – those [two methods courses] were the most useful [courses] towards preparing me for my job, apart from your field experiences – in terms of class work – [the mathematics teaching methods courses were the] most useful.

PSTs also felt that the teaching methods courses acted as the main precursor to both their practica and future jobs as teachers. As Zorra described, "we learn all this content, and, in the methods course, it's really focused on how you can deliver the content and how can you get students to think about mathematics." Indeed, the PSTs felt that what they learned in the teaching methods course could be applied into their teaching practice during practica. Later, when we share PSTs' perspectives about their field experiences, we further elaborate on how the PSTs made connections between these two components of their teacher education program.

3.1 A Different Hands-On Experience

PSTs expressed that the teaching methods courses offered a hands-on experience that was unique from the rest of their courses. In these courses, PSTs were introduced to and engaged with various mathematics activities. These included rich learning tasks (e.g., The Popcorn Box Task, where students find the maximum volume for a movie theater snack container in the shape of an open-topped box given certain parameters) and mathematics games and puzzles (e.g., an arithmetic maze, where students were challenged to achieve the highest result of a series of calculations based on the route taken in the maze). Magda highlighted this part of the course and explained how her experience of these hands-on activities shaped how she wanted to conduct her future practice:

> My favorite moments…I definitely liked activities. It was a lot of fun, I didn't know there were that many games out there with math. It's true! I never had them in classes…I want to apply them like once every two weeks or…once a month even. It's a lot of fun and really cool to get [students] moving and doing something with math that's just not paper and pencil.

Similarly, James shared how, through hands-on activities in the teaching methods courses, he started becoming conscious of his understandings as both a PST and mathematics learner. "We did [practical hands-on activities] and I became sort of acutely aware of both what's going on in my mind and what might be going on in a student's mind as they're doing something like this." From a student perspective during an activity, James continued, "I remember…we were just like – no idea. I love that, because I'm like, you know, students probably feel like this most of the time." Similarly, Zorra explained that one useful aspect of the teaching methods courses was how assignments from the courses allowed for her to continue "developing [her] content knowledge."

Other PSTs described the methods courses as being the course-based component of their teacher education program that most closely related to their future practice. In particular, Domino described the methods courses as "the most useful class of [my] degree – most useful [courses]." She explained, "[The courses are useful] in terms of what I'm going to be doing. In terms of my actual job." Through the experiences provided within the teaching methods courses, she stressed that "[the courses] provide you with actual experiences… it's sort of a safe space where you can develop as a teacher and refine your teaching skills." Magda explained where she utilized teaching techniques from the teaching methods courses saying, "A lot of the activities we do in [the methods courses] - I try and adapt them to use them in my field experience and my tutoring so it's […] super essential." Ruby reinforced this statement when she spoke about her views on the teaching methods courses: "Everything we talked about was specific to what we were doing." Indeed, while PSTs valued the hands-on learning that took place in their methods courses, the relevance of these experiences to their future careers was particularly appreciated.

3.2 Allowing for Supported Practice

The teaching methods courses offered PSTs a hands-on experience of both teaching and learning mathematics, as well as a safe and open context to do so. The teaching methods courses were useful to PSTs, in that their classmates (fellow PSTs) and course instructors supported their practice of newly acquired techniques. As expressed by Ruby, "Had I not taken this course, I would not be using…I probably would have tried dropping [different teaching practices], or maybe saying like, 'That doesn't work…'." Domino emphasized this by adding that the teaching methods courses "give you practice and allow you to develop as a teacher in a safe environment where you can reflect and grow."

Specifically, the Cycle of Enactment and Investigation (Lampert et al., 2013), during which PSTs engaged in a teaching rehearsal, was particularly meaningful. In the rehearsals, PSTs taught a segment of a mathematics lesson that they had developed to their classmates who played the role of "students". As the "facilitator", the course instructor "paused" the rehearsals in moments where the "teacher" could refine their teaching approach, and led discussions during which all PSTs could collectively considered alternative ways that the "teacher" could have attended to challenging moments in the rehearsal. Ruby described the significance of this experience saying:

> There were things that we discussed [in my rehearsal]…which I wouldn't have caught before I did my Cycle of Enactment. […] When I was stopped, [another PST] said "I feel like you're talking a lot" and I was like, "Oh my God, I'm doing everything I didn't want to be doing". If I hadn't had that moment I probably would have gone into my [practica] and did exactly that.

Indeed, the PSTs emphasized that the teaching methods courses allowed them to test out different mathematics activities and refine their teaching approaches before implementing them in a secondary mathematics classroom during their practica. As Magda said, teaching methods courses are "where you actually get to practice it before applying it in a real live scenario."

3.3 Re-orientating Their Teaching Repertoire

PSTs found the teaching methods courses shifted their beliefs about mathematics teaching. James mentioned that, through these courses, "we're cracking the perception of what's expected of you as a mathematics teacher." PSTs described changes in their perspectives as more student oriented. James continued by identifying the teaching methods courses as having the most impact in terms of the program's coursework:

> [W]hat I thought that those (mathematics teaching) methods courses sort of imparted on me the most is re-orienting yourself to what's going on in the students' minds…it made me sort of think about was, okay, when I'm lecturing up at the board, what are my students doing…I thought mathematics methods courses, more than any other course – so of all the other education courses – tends to try to get us to think about, like, student-centered approach.

Similarly, Ruby said that the teaching methods courses were critical to reframing her beliefs about mathematics teaching. She shared that in "the rest of [her university] career, [the mathematics PSTs] were never addressed. So, any of the ways to approach math, or different ways of presenting student work. Like, different ways of approaching students who have questions [were not important]." Thus, through these courses, not only did PSTs select and add new strategies they had experienced to their repertoire of teaching approaches, but they also started thinking about these strategies from different perspectives.

4 PSTs' Perspectives About Mathematics Content Courses

PSTs shared that mathematics content courses helped them understand what it is like to struggle as a mathematics student and strengthened their mathematical knowledge. Yet, some PSTs found course content to be superfluous and not necessarily applicable or useable in their future careers as secondary mathematics teachers.

4.1 Learning the Struggle

The mathematics content courses were described by PSTs as being the most challenging component of their program. This difficulty, many of them noted, taught them what it is like to be a struggling mathematics student and made them appreciate and understand how their own students might feel. Ruby said that this struggle taught her resilience. Domino, who had always been strong in mathematics, explained that mathematics content courses:

> [H]elped me realize what it is to sort of struggle in math because I think before this I had never really struggled like it was smooth sailing. So that was like a positive sort of thing that I can relate to now […] what it's like to struggle, what it's like to work really, really hard and then you still not understand it and go into a test.

It was valuable for Domino to have this unique experience of struggling with mathematics. Had she not been required to take mathematics courses in the teacher education program, Domino felt that she might not have been able to relate to her future students and their troubles.

Ruby recognized that while her own methods of teaching mathematics might make sense to her, they might be less clear to her students. Taking mathematics content courses and seeing what it is like to have someone else teach content to her was valuable and eye-opening:

> Being in that student role again…and, knowing how my students feel. Because I can sit in my own classroom and watch my own classroom and get everything…because it's me and I know how to explain things so that I understand them. But, to be in a situation where I had a really hard time keeping up, or even understanding the way that the teachers were presenting the material. So, then, again, like, I put myself in my students' shoes.

Recognizing that it could be difficult to learn something due to the approach chosen by the teacher was an important realization for Ruby and made her more conscientious of how her students might struggle with the way she taught mathematics. PSTs obtained unique experiences about learning the struggle of being a mathematics student from mathematics content courses.

4.2 Strengthening Mathematical Knowledge

In spite of the difficulties faced by the PSTs in the mathematics content courses, they valued the opportunity to strengthen their mathematical knowledge prior to teaching and additionally, learn higher-level mathematics. Ruby believed that "Being the master…being very proficient in the subject that you teach. That has its value." Zorra had a predisposition prior to taking the courses that they would not be important – that she already had the content knowledge needed to teach secondary-level mathematics. She explained, however, that:

> Being in this program I realized […] if we don't have [the courses] our knowledge is very basic. And I think you need to at least have an understanding of higher-level mathematics, even to have an understanding of what's going on in [secondary] school. I've had courses in the higher math classes where in [secondary] school you're learning the basic idea but with the higher one you have a better understanding of it. You understand the above and beyond, which I think is very important.

Magda agreed with Zorra that having knowledge mathematical knowledge which exceeds what one is required to teach deepens PSTs' understanding of what it is that they are teaching. Additionally, she recognized that students will likely ask questions that extend beyond the content they are learning, thus requiring the teacher to tap into their higher mathematical knowledge. From her own experiences in practica, Magda recalled:

> Sometimes I have a student that talks about something and they're like "Does this exist?" [T]hat's where you open up…to be like, "You know what? Yeah, this does exist. There's a whole field on this and I can actually show you something." I had a student in my third field experience asking me how to [differentiate] something because his older sibling was doing it…so during lunch he came with some friends and I had them do a bit of derivatives and they were super excited because they're like "Oh my God, I'm doing like really high-level math." So, it's fun to see them do that and it opens up a door for them to explore.

PSTs shared that the higher content knowledge they developed from the mathematics content courses would be useful in cases where students inquire into higher level content and/or seek enrichment. Without this deeper mathematical knowledge, PSTs felt that they may not be able to provide their students with the tools and resources to work ahead and get inspired by higher mathematics.

Zorra appreciated the solidification of content knowledge gained from mathematics content courses and argued that you never know when you are going to have to use some of it, even if it seems irrelevant at the time. Zorra also saw an additional benefit of this enhanced content knowledge. In her words:

> I would say that as much as I hate to admit it, I think it helps a lot more with just solidifying your own knowledge…I think having all this content knowledge helps you. And that can help your students as well. The last thing you want is a teacher who keeps questioning what they're teaching which happened to me a couple times when I had to teach something I'd never done in my life.

Here, Zorra described how a strengthened content knowledge can be reassuring and support a teacher's self-confidence.

James shared that the mathematics content courses were the component in which he could, above and beyond the mathematics content itself, learn what mathematics really is. James said, "math content – for me, it's huge." More specifically, he described the courses as allowing him to overcome misconceptions of mathematics being a tool to achieve something else rather than its own realm saying:

> It's this language – [math] has its own semantics. Like I said, it has its own topic, right? It's not a methodology. It's not a thing. It's not just this thing you build, and then you suddenly turn around, go to the world, and then like use it, right? No, no, no. Like I said, it has its own subject matter.

For James, while mathematics can be used as a tool to accomplish other things, there is also an intrinsic beauty to the essence of mathematics itself that should be seen in all content courses.

Participants recognized that mathematics content courses not only inform and strengthen PSTs' knowledge of the material they will be teaching; the courses also taught them higher mathematics that is nonetheless connected to secondary-level content and made them more knowledgeable of the "bigger picture" of mathematics. PSTs also acknowledged that not every class or interaction with students remain in the confines of the curricula planned for that day; and many students will ask questions that surpass the curricula requiring teachers to help students with this advancement.

4.3 Relevance Toward Their Future Careers

PSTs shared the belief that at their core, the purpose of the mathematics content courses was for PSTs to learn and strengthen mathematical content knowledge. However, PSTs questioned whether or not this knowledge would be used in their future careers. As described by Domino:

> [The content courses were] useless – for the most part. In terms of my degree and like the future work environment, how are differential equations going to serve me when I teach Grade 9 – that kind of thing. It's a big negative that comes to mind. It's sort of something that is a stress that – for something that I [know won't] have a big impact on my career. And it's like, it's just something that I have to get through. And for classes that I wouldn't gain a lot of insight from for my future teaching career.

Many PSTs believed that while learning higher-level mathematics was not necessarily a bad thing, it did not contribute anything to their future careers.

Some PSTs suggested modifications to the coursework in the teacher education program that they felt would be more beneficial for their development. Zorra said:

> Unfortunately, you're not really using much of the content courses in the field or in teaching secondary mathematics unless you're doing a pre-analysis course, it's not directly linked…and I think actually what could help is maybe having one content course where it's solely the topics in high school. Maybe we can add an extra option where you can take a complimentary course where you focus more on the content of high school mathematics.

Here, Zorra suggested a complementary mathematics content course that focuses specifically on secondary content that teachers are actually required to teach to make their content knowledge more applicable to their future careers.

5 PSTs' Perspectives About Field Experiences

PSTs shared that the practica played a positive role in their development as future teachers. PSTs described this component as "extremely useful" and "super valuable." Many PSTs reflected that the practica were the most important component of their teacher education program. As Domino shared, practica are "where you're going to develop and grow the most as a teacher."

5.1 Learning by Observation

PSTs expressed that practica allowed them the opportunity to observe teachers in a secondary school environment. As described by James, the PSTs could "see what [teachers] do, how exactly they do it." He continued by saying, "I used to get to see in, like, the other class which I wasn't teaching…Sometimes seeing things that you wouldn't do, and you know, noticing things, how students react in ways that you don't see when you're up front." Magda echoed this when she said, "I like being able to observe as many people as I can. It gives you an idea of what you can grab and what things you wouldn't want to do."

PSTs had also commented on how the observational part of their practicum allowed them to make connections between their practice and coursework. Domino mentioned how observation allowed her to reflect on what she had learned in the mathematics teaching methods courses and subsequently implement these various teaching strategies into her practice. She said:

> You can sort of remember what the classroom culture is like. You remember, "Okay these were some of the kids, some of the attention problems, then these kids were stronger, this is how these kids learn" so you've observed…you can notice certain things and you sort of bring [in strategies from] your methods course…"How could I apply these activities to that behavior?" You think about it, and then…you actually get to apply it [when you teach].

Zorra provided a different but complementary perspective. She explained,

> I would think about what we learned in [our teaching methods course]…when I was observing, especially for the math teachers. I'd be like okay this is what we learned, let's see if they implement it…subconsciously without all these teachers knowing, because maybe they did not take this class, they are implementing it and I'm like wow.

As shared by the PSTs, classroom observations allowed them to reflect upon teaching approaches that they learned in the teaching methods courses by considering how they might implement these approaches in the context that they were observing, or observing how a classroom teacher implements these approaches.

5.2 Experiencing Life as a Teacher

In addition to observing secondary school teachers in action, PSTs described practica as the context for having a chance to practice teaching in an authentic secondary school context. This seemed to be the greatest benefit of practica for the PSTs. As Domino said, "[Y]ou're teaching…you're doing it. Like you're actually doing the job that you have to do and that you will be doing in the future."

PSTs stressed that having this practical component in their teacher education program was important. PSTs shared that they learned about being a teacher in their coursework, but it was not until their practica (contexts similar to those that they would be teaching in the future) that they could experience being a teacher for themselves. As Magda said,

> [Practica is where] you get to actually see what works for you and what doesn't and what (teaching) methods you like and what you don't because it's good to theorize them and be like, "Oh, that sounds like a cool idea", but then when you practice it you're like, "Oh my God, this is terrible, not for me." So yeah, the real-life experience is (important).

Similarly, in speaking about her learning in the domain of classroom management, Zorra described the importance of practica:

> It's one thing to hear about classroom management (in coursework), it's another thing to see it in practice. So, I think that being in the field, it helps you prepare for that. I learned a lot about classroom management in my [education] courses, but we don't know how it actually works until you see it in practice.

Zorra shared that she could develop her practical knowledge about classroom management, first by observing her cooperating teacher in action, and then was able to further develop her skills once "in front of a classroom." As she said of teaching, "Like anything else, you need practice." Similarly, James spoke of the multiple benefits of practica saying, "I got to observe…and teach…it was like the best of both worlds." This idea of scaffolding their learning (through coursework followed by experiential learning in practica, both through observation and first-hand experience) was shared by all PSTs.

PSTs also shared that practica provided them with opportunities to experience not only the "teaching" aspect of being a teacher, but all facets of life as a teacher. PSTs spoke of a variety of learning experiences including grading, extra-curricular supervision, parent-teacher nights, and field trips. James described these additional experiences and their impact saying:

> [T]he parent-teacher night was awesome. The parents were just there and trying to, like – like, "Okay, what can we do? Like, how do we fix this if they're not doing well?" It was a very helpful situation. And administratively, now, I mean, if I get a job with [this school board], I know what system they use to take attendance, I know all these things…[it's] not just the teaching…It's more like, "Oh, you have to go down there, get that folder, bring it up into all these other things." And so that, of course, that stuff is really valuable.

Here, James shared how experiencing the various non-teaching aspects of school life was equally as valuable to his development as a future teacher.

5.3 *Gaining Confidence and an Identity as a Teacher*

While PSTs shared that the "theoretical" knowledge of teaching that they developed in coursework served as a good foundation for their development as teachers, lived experience in practica was critical to their development of both teaching skills and comfort in school environments. As one PST described his feelings before practica, "you have this, like, fear" (James). Ruby elaborated on this positive impact of practica saying:

> I learnt like, so much from being in [practica]. And, I gained more confidence. Like, the week before (my first practicum), I was like, shaking. I was so anxious. I was like, "How am I ever going to stand in front of 30 people and talk? And, why would they listen to me? Like, why would they listen to me? What do I know? I know nothing." And, I was so nervous. And, it was really even hard for me to visualize myself in a classroom with 30 people. Whereas now, like, that's not even like, a thing anymore. Like, I enjoy doing that. And, had I just been in like, a theory program the whole time, I would still feel that way, now going into the job market, right.

Many PSTs shared that prior to practica, they felt nervous and uncertain about their abilities as a teacher.

5.4 *The Cooperating Teacher as a Key Player*

PSTs felt that their cooperating teachers played a critical role in their practica experiences. As previously described, the experience of practica itself supported PSTs' development; however, PSTs shared that their cooperating teacher could enhance or hinder this development. As described by Magda:

> If your [cooperating teacher] allows you and is open to you trying a whole bunch of things just do it because the more you try the more experience you get and the whole thing is just an experience. If your [cooperating teacher] is open to it and they're willing to let you have that rein to try stuff out go for it. I tried so many things and I'm so happy I did because I got to see what type of style I would like.

In this case, Magda believed that the autonomy provided to her by her cooperating teacher enhanced her ability to develop as a teacher, specifically with regards to her identity and comfort as a teacher.

In practica, not only did the PSTs have a chance to practice teaching, but they also received feedback on it. PSTs found the feedback to be invaluable to their development as teachers. As one PST said, “Every single piece of constructive criticism I got, I really appreciated” (Ruby). Once again, PSTs shared that their cooperating teachers played an important role in this aspect of the practica. Not just feedback providers, cooperating teachers were also a sounding board in subsequent discussions. In speaking of the role of the cooperating teacher, Magda stated, “You have someone to give you feedback and…they’re there to bounce ideas off of.” Similarly, James described his goals when discussing his teaching with his cooperating teachers as, “What’s wrong with what I do? Like, let’s figure it out.”

Finally, PSTs spoke of cooperating teachers as playing a role in the extent to which PSTs got involved in different aspects of school life during practica. While there was an expectation that PSTs would primarily take on teaching responsibilities in the classroom, there seemed to be flexibility regarding the amount of teaching responsibility that cooperating teachers assigned to the PSTs. PSTs shared that their responsibilities ranged between 60% and 100% of their cooperating teachers’ teaching load. PSTs also described additional ways that their workloads varied. Some PSTs spoke of having to develop the lesson plans, activities, and assessments to be used in the classes for which they were responsible, whereas others were solely being responsible for delivering the lessons that were developed by their cooperating teacher. Domino contrasted her experiences in practica to that of some of her peers saying, “I was able to manage [my teaching]…and do extra-curriculars. But I think that some people had to do like lesson plans for every class they taught, in addition to like taking on 100% of the load.” These varied responsibilities, as determined by their cooperating teachers, thus had an impact on the range of experiences that the PSTs could take advantage of.

6 Concluding Thoughts

In this chapter, we share secondary mathematics PSTs’ perceptions and experiences in three different components of their teacher education program: mathematics teaching methods courses, mathematics content courses, and practica. Findings suggest that PSTs see each of these components as having a different purpose and impact on their development as future teachers.

PSTs described the mathematics teaching methods courses as the context in which they could experience exemplary teaching approaches as students and experiment with these approaches in a safe and supportive environment. PSTs spoke of gaining a different hands-on experience, which allowed them to engage with varied strategies through both student and teacher lenses (Akarsu & Kaya, 2012). When speaking of mathematics content courses, PSTs described this component of their teacher education program to be a struggle. Although mostly described as a negative experience, PSTs mentioned how this component helped them understand what their (struggling) students might experience in math. Finally, the practica was the most valued component by the PSTs. PSTs shared that practica scaffolded their learning and allowed them to gain confidence in their teaching practice. As is also described in the existing literature, PSTs found one of the most beneficial elements of practica to be the lived experience of what everyday school life is like (Cohen, Hoz, & Kaplan, 2013).

Whether intentional or not, components of teacher education programs are commonly separate, suggesting a "bordered" reality. Coursework (teaching methods and content courses) are typically university-based and practica are school-based. Teaching methods courses are typically offered by faculties of education, and content courses are offered by departments of their specific subject areas respectively (e.g., departments of mathematics). Research has shown that the purposes and goals of these components may or may not overlap, an issue echoed by the PSTs. This too creates yet another "border" within teacher education programs. Yet, the fact that the PSTs made references across components suggests that while some borders exist to various extents (e.g., program components having disparate goals, as well as taking place in geographically different locations and university departments), the components were not fully distinct. Furthermore, a point of comparison used by all of the PSTs was the relevancy and practicality of what they learned in each component to their future careers. The PSTs regularly described a theory and practice divide – a common challenge described in the literature for teacher education programs (e.g., Cheng, Cheng, & Tang, 2010; Korthagen, 2010; Zeichner, 2010) – between their experiences in their teacher education program and the realities that they would be facing, once in the teaching profession.

The connections that PSTs made between activities, teaching methods, and other theories learned in the mathematics teaching methods courses to what they had experienced in the field during their practica components suggest that PSTs saw relationships between these two components. Although opportunities to understand the role of a teacher through a teacher's lens (e.g., mock-teaching experiences and discussions of case studies of teaching scenarios) may occur in methods courses (Hodge, 2011; Stickles, 2015), these are still within the confines of a university-based context. It is thus reassuring to know that PSTs see similarities in what they perceive to be a relatively theoretical component of their program and the component which is the most practical.

For the mathematics content courses, findings suggest that there was a paradox between PSTs recognizing and appreciating the importance of learning university-level mathematics but also finding it inapplicable to their careers as secondary mathematics teachers, an issue previously reported in the literature (e.g., Ball et al., 2008; Dreher et al., 2018; Hodge et al., 2010; Mewborn, 2003; Wu, 2011). Research suggests that this perceived "pointlessness" stems from the difficulty that PSTs have moving from content theory to their teaching practice (Kari & Lilach, 2005). This struggle may have affected the PSTs' willingness and ease to make further connections between program components. Nevertheless, the fact that PSTs still speak of other components of their program in relation to the content component suggests that PSTs are trying to understand their experiences as a cohesive program.

Finally, it was in the practica component of the teacher education program that PSTs seemed to have the most ease in moving from theory to practice. Prior to practica, the PSTs' knowledge as a teacher was developed through their theoretical learning in coursework. The practica provided the opportunity for PSTs to experience life as a teacher and allowed them to take the critical step of putting theory into practice (Allen & Wright, 2014; Korthagen, 2010). PSTs described their experiences in practica to be characterized by the implementation of different knowledge learned in the other components. PSTs shared that this knowledge furthered their understanding of their practice (Liljedahl et al., 2009).

The PSTs in our study endeavored to make connections, albeit at times, tenuous, between the components of their teacher education program. We wonder on whom the responsibility falls to support PSTs in making these connections. What is the role of teacher education program administrators and teacher educators, those university-based (e.g., teaching methods and content course instructors) or school-based (e.g., cooperating teachers)? Although the PSTs themselves seemed to make links between program components, how best can the aforementioned stakeholders enhance and expedite this process? Moreover, how can the borders between teacher education programs and the realities of the profession be blurred? Yet, research has shown that overloading PSTs with information on their future experiences within their teacher education program might not be fully absorbed as we expect (Fajet, Bello, Leftwich, Mesler, & Shaver, 2005). Perhaps the PSTs' previous personal experiences and readiness to relate to the given information are important to consider (Naylor, Campbell-Evas, & Maloney, 2015). As such, additional questions emerge. Are there benefits in having PSTs develop their own connections as a means to strengthen their professional growth? Can these connections be fully developed when PSTs are left to make them on their own? Perhaps as in most cases, it is a question of balance. Indeed, it is not just the individual components but their cumulative effects that influence PSTs' development, thus requiring a thoughtful (re-)examination of our approaches to secondary mathematics pre-service teacher education (Floden & Meniketti, 2005). Nonetheless, we must remain steadfast in efforts to diminish borders and make connections between components to fully allow future secondary mathematics teachers reach their full potential.

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The Shared Duty of Special Educational Support in Mathematics: Borders and Spaces in Degree Ordinances for Pre-service Teachers

Anette Bagger and Helena Roos

1 Introduction

Special education support in mathematics is a duty shared between different professionals, such as psychologists, special education teachers, mathematics teachers, and teaching assistants (see, e.g. Radford, Bosanquet, Webster, & Blatchford, 2015) each of whom take responsibility for different areas. Therefore, the duties demand collaboration and coordination of different expertise, training, and competencies (Roos & Gadler, 2018; Secher Schmidt, 2016; Sherer, Beswick, DeBlois, Healey, & Opitz, 2016). The method usually used by special education teachers in order to support students is individualised or small-group teaching and supervising the classroom mathematics teachers. Special education teachers often have a relational perspective on difficulties and an equity discourse on education (Göransson, Lindqvist, & Nilholm, 2015).

When it comes to students who face challenges learning mathematics, their achievement can be connected to teachers' skills, and in these cases, research shows that there is a need for teacher professional development (Griffin et al., 2018). Furthermore, students' mathematical development is impacted by teachers' professional understanding and knowledge about how to support such struggling students (Bottge, Rueda, Serlin, Hung, & Kwon, 2007; Gal & Linchevski, 2010; Hinton, Flores, Burton, & Curtis, 2015; Moscardini, 2010; Moscardini, 2015). Professional skills, knowledge, and understanding will play out in the classroom through mathematics and special education teachers' shared assignment to support students. For example, when they prioritise what is important to do to support students, these

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teachers should consider who should be involved in the students' education and why. It is also important for the teachers to analyse and evaluate the design of their teaching approaches and the assignments given to students in order to understand the suitability for different students. We claim that the spaces and borders for the shared assignment begin to form during pre-service teacher training. In the long run, how the priorities are made determines if and how the school provides and develops efficient and appropriate education for every student in mathematics at the local and national levels. Some of these students are considered to be students in need of support in mathematics. There is no consensus on the concept of "students in need" (Heyd-Metzuyanim, 2013). For this chapter, we understand students in need of support in mathematics as not always being about achievement or disabilities, instead it is a need emerging in relation to the context and system of education in mathematics. We thereby agree with Méndez, Lacasa, and Matusov (2008) who state, "Disability is regarded as being located in particular types of activity systems and learning cultures rather than within an individual" (p. 63). We also assert that different education professionals are affected by the structure, routines, and culture of the educational system in the classroom and of the school. These systems and cultures will have an impact on how and when students experience various aspects of teaching as hindering their learning. The needs are thereby closely connected to situated social aspects of teaching and learning: "We have adopted the concept the student <u>in</u> need of special education in mathematics in order to emphasise the social aspect. The word *'in'* is here of great importance. The student is *in* need, not *with* needs" (Bagger & Roos, 2015, p. 34).

At the same time, there seems to be a lack of evaluation of and research on how pre-service teacher training contributes to learning regarding these factors in the educational goals stated in the degree ordinance, at least in the Swedish context. A search was conducted for peer-reviewed journal articles during the last 10 years on the ERIC database using the keywords "teacher education", "special education", "mathematic∗", and "goal∗", and only 11 hits were found. After reading the abstracts, several were found to focus on the role of special education teachers, how to teach mathematical content to students in need of support, or collaboration between types of teachers. While articles focused on in-service teachers and their work, few focused on pre-service teacher training. For the few concerned with pre-service teacher training, none of them focused on the goals of the teacher training programs for mathematics or special education students.

Cultural boundaries influence how teacher education prepares teachers on how to educate students in need of support as well as how to collaborate with other professionals to support these students. For example, the knowledge of mathematics teachers has been shown to reflect current national debates in some countries. More specifically, Blömeke, Suhl, and Döhrmann (2013) describe how this occurred in Taiwan, Singapore, Russia, Poland, the USA, and Norway, and state that "the knowledge profiles of the future teachers matched the respective national debates. This result points to the influences of the cultural context on mathematics teacher knowledge" (p. 795).

National and cultural boundaries need to be acknowledged, explored, and placed in an international context in order to be better understood, as well as to create a high-quality foundation for developing and guiding teacher education (Blömeke, Schmidt, & Hsieh, 2013). This chapter contributes to exploring national and cultural boundaries through the case of Sweden by analysing goals in teacher exams as they are stated in the higher education ordinance. The goals are understood as governing pre-service teachers' opportunities to learn (OTL). OTL as concept has been used in research for approximately half a century and focuses on the goals of education, what is afforded to learn, and how these affordances depend on, among other things, curricula, culture, and programs in education (Wang & Tang, 2013). OTL also goes beyond directions and implementations of curricula and refers to socio-political aspects of who the learner is and for whom education functions, which in turn derives from the context and system at hand and is connected to issues of equity to learn and reach targets (Lester, 2007).

Thus, in this chapter, OTL refers to what is offered through the educational goals in regard to what can be learned and how this affects collaboration between the professions. OTL will affect how pre-service teachers (both mathematics and special education) develop the knowledge needed in their future profession, and for their joint assignment to support students in need of support in the subject of mathematics. The exploration of country-specific borders of the pre-service teachers OTL may serve as a foundation to explore and reflect on other national settings and borders.

1.1 Special Education in Mathematics

Learning in STEM (science, technology, engineering, and mathematics) is often debated and a focus within school systems, and sometimes, mathematics is considered to be the core of these subjects. The development of mathematics has often been stressed as urgent in order to protect democracy and success for both individuals and countries (Erdogan & Stuessy, 2016; Grek, 2009). Professional development in the form of in-service training in specific subjects is often implemented to raise standards and goal achievement among students, as well as students' OTL in these subjects (see, e.g. Piasta, Logan, Pelatti, Capps, & Petrill, 2015). OTL has also been a focus of research in order to "interpret cross-national patterns of math achievement" (Bachman, Votruba-Drzal, El Nokali, & Castle Heatly, 2015, p. 896). In several countries, the task to support student learning is considered as a challenge for teachers and has been identified as the core for further professional development (Schleicher, 2012).

Furthermore, students that do not meet standards or lack the desired competencies or knowledge are talked about in neoliberally governed schools as being threats to future prospects for individuals, education, and society (Bagger, 2016). One example of this is how the Programme for International Student Assessment (PISA)[1] evaluates the skills and competencies of 15-year-olds that are deemed to be needed in adult life (OECD, 2017). This measure involves comparison and ranking between countries and engines educational policy change (Grek, 2009; Kim, 2017). This is an example of an international, goal-oriented, marketing trend of schooling for connecting results with quality and accountability (Rönnberg, 2011). A consequence of international monitoring surveys and national testing is adjustments to education and policy decisions on a national level (Dreher, 2012; Luke, 2011; Martens, Knodel, & Windzio, 2014; Pettersson, 2008; Wrigley, 2010). This reasoning is also applied to different pre-service and in-service teacher education programs and policies in order to improve teachers (Lincove, Osborne, Mills, & Bellows, 2015). To summarise, students that do not reach learning targets in the curricula are consequently labelled as "low achieving" students. Subsequently, these students' results are often attributed to flaws in the education system and accountability issues of the school and ultimately, teacher education.

There are questions regarding who students in needs of support in mathematics really are and what support for these students should or could look like (Lewis & Fisher, 2016), as well as how it differs between schools and across countries. Nevertheless, disadvantaged groups in mathematics are often connected to identity categories, such as gender, race, class, and ethnicity, and there is a challenge for teachers and schools to embrace these categories (Bishop, Tan, & Barkatsas, 2015). Teacher training is meant to prepare future teachers for fruitful collaboration in order to prepare them to support students with special educational needs. Such training aims at supporting inclusive settings:

> Internationally, standards related to teaching and teacher preparation have reflected the expectation that general and special education teachers ought to be prepared to collaborate with each other to meet the needs of students with [special educational needs] in inclusive settings. (Van Ingen, Eskelson, & Allsopp, 2016, p. 74)

Promoting this inclusive stance has been a challenge for teacher training programs. If positive attitudes towards the inclusive approach are not developed, this stance will eventually diminish in teachers' practices in mathematics classrooms (Shade & Stewart, 2001). In addition, how well this collaboration plays out will affect how students in need of support are educated and is ultimately regulated through the goals in the professional exam. For example, preparation on how to collaborate is needed and has to be executed in a way that takes both mathematical knowledge and special educational knowledge into account at the same time.

How educational professions involved in providing special support are organised and work together, varies between countries (Göransson, Lindqvist, Möllås, Almqvist, & Nilholm, 2017). In Sweden, special education teachers and mathematics teachers for primary school (MTPs) are the main professionals involved in giving support to all students (including those in need of support) in mathematics in

[1] PISA is an Organisation of Economic Co-operation and Development (OECD) project that measures 15-year olds' knowledge every third year.

primary school. MTPs are trained as generalists and are responsible for teaching all subject areas, not just mathematics. In Sweden, only teachers who have passed an exam in mathematics are allowed to assess students' mathematical knowledge thus in all schools, students have at least one teacher who has a mathematical background. Special education teachers have different specialisations, including mathematics, whom we will refer to as special education teachers in mathematics (STms). STms have pre-service training in both the subject of mathematics, and to a varied degree, special education and knowledge of special needs. STms do not have their own classes. They sometimes, but not always, have an office or a room in which they collaborate with classroom teachers to work with students in need of support. It is also common for STms to co-teach with mathematics teachers in the classroom.

A division is typically seen between MTPs and STms for two reasons: (1) STms often give support directly to students in order to supplement the teaching provided by MTPs and (2) STms also give consultations to MTPs in order to help them to develop, adjust, and improve their teaching of students in need of support (Sundqvist & Ström, 2015). These are common practices in both Finland and Sweden. In a Finnish context, these kinds of consultations were found to consist of mainly three different types: consultation as counselling, reflective consultation, and cooperative conversations. According to Sundqvist and Ström (2015), the counselling type of consultation is a mainly expert driven approach to consulting and consists of classroom teachers getting transference of special educational knowledge in the shape of information, advice, or suggestion. Reflective consultations are instead participant driven and are more like a stimulating, exploratory conversation in a search for developmental areas and problems to handle through a process of learning that are often relational, social, or system-oriented. Cooperative conversations are conversations in which professional exchange takes place and both parties mutually benefit. Sundqvist and Ström (2015) found all three forms of consultation to be beneficial for various purposes, and that it was essential that each was used in a knowing and reflective way. Consulting counselling are more prominent if the teachers involved are recently educated and lack experience. What limits the teachers is the transferral of knowledge, the student being in the core rather than a system-oriented approach towards challenges, and the risk of the STms' knowledge and interpretations taking over and becoming predominant (Sundqvist & Ström, 2015). Internationally, it has also been shown that teacher education programs do not prepare STms and MTPs for consultations that focus specifically on mathematics, mathematical contexts, and student participation (Van Ingen et al., 2016).

Even in a Swedish context, consultations were found to be important (Roos, 2015). Here, the different spaces where STms work is understood by using Wenger's (1998) notion of communities of practice (CoP). A CoP is a social practice with *mutual engagement between members*. STms have a complex mission, they participate in many CoPs at the same time (such as with MTPs and other STms), and are brokers between CoPs. Hence, special education teachers struggle to find both the time and the space to carry out this brokering and have consultations with MTPs.

Holgersson and Wästerlid (2018) revisited the specialised work of the mathematical development of students in need of support and stated that foundational to STms' duties is to secure organisational prerequisites for all students to learn mathematics. They also described the following as important competencies for STms: pedagogical content knowledge, knowledge of how mathematics is learned and how development in the subject can and should appear, and knowledge about the obstacles encountered by students in need and the ways to overcome these obstacles. Accordingly, STms work directly with students in need by providing special support and facilitating to development of their learning environment. The latter means that approaches, relations, teaching materials, and methods are the main focus for development, and this occurs at three levels: at the individual, group and organisational level. Thus, it is important for STms to hold consultations with teachers or other school professionals, as well as to make sure that the school has systems for monitoring and securing the mathematical development of all students. With this said, STms are typically not the ones assessing or analysing individual students' results. Rather, their responsibilities are often organisational. Then again, it is also quite common for STms to perform some of the trickier or more difficult cases to evaluate or to complete teacher assessments in some way (Holgersson & Wästerlid, 2018).

Holgersson and Wästerlid (2018) have further described that the duties at the group and organisational levels are not prioritised by the school management in terms of developing learning environments and consulting teachers regarding the educational methods, content, and strategies. The phenomenon of STms having difficulties working on organisational issues of development and as consultants has also been identified in a survey of special education teacher training and occupation (Göransson, Lindqvist, Klang, Magnusson, & Nilholm, 2015). The survey points towards this profession as serving a governmental quality function and as a guardian of relational approaches towards learning and inclusion in schools in Sweden, but lacking in authority. In Sweden, teaching is approximately 50% of the duties of special education teachers. Furthermore, the Education Act of Sweden (SFS, 2010:800) does not mention special education teachers to be a necessary part of health care teams, unlike nurses, psychologists, or school social workers. In Sweden, school social workers work for students' well-being on several levels and collaborate with professionals within and outside the school, and are not the same as municipality social workers that investigates families, for example. If students need someone besides their friend, teacher or parent to listen to them, or if students are absent, are involved in bullying or some other disruptive behaviour – the school social worker might become involved (for more reading, see, e.g. Isaksson & Larsson, 2017). When it comes to the special educational competence that should be available in the health care team, it is not made explicit what kind of profession or degree, is required. The jurisdiction of special education teachers thus becomes hard to identify, and there is a discrepancy between what is expected from schools and what is expected from a political stance. Göransson, Lindqvist, Klang, et al. (2015) conclude that the role of special education teachers can rather be seen as an expression of educational policy formulated in terms of a new vocational degree.

1.2 Aim of This Chapter

Professionals, in this case STms and MTPs, are supposed to collaborate and intersect with each other in order to provide support to students in need of support. Aspects that constitute and effect the efficacy of this collaboration include the following:

1. The kinds of knowledge pre-service training affords about the "other" profession (MTPs for STms, and vice versa)
2. Whether the goals in the degree ordinance for the two professions' pre-service training are in harmony with each other
3. How these goals lead to a shared understanding of the assignment, and a joint and solid base of knowledge on the shared assignment to support students in need of support in mathematics

Hence, the aim of this chapter is to contribute knowledge of borders, spaces, and intersections between two pre-service teacher professions that will cooperate in their future teaching: STms and MTPs. This knowledge is constructed by exploring some of the prerequisites for the OTL about the shared duties to provide special education mathematics to students through the goals of teacher exams in Sweden. The research question is, how does the degree ordinance depict the duties of the teacher, the "other" profession, and their future shared task of providing support to students in need of support?

2 Methodology

The case of Sweden is situated in an educational national context that is framed by curricular goals in the Higher Education Ordinance, which create OTL for pre-service teachers. OTL are understood as being constituted partly by exams, and the goals of the exams constitute borders between different kinds of teachers regarding their duties to support students in need of support in mathematics. OTL have been connected to how goals in curricula are reached and measured (Stobart, 2009). It also includes a stance of caring for equal access to fair and meaningful learning for all students, and is then socioculturally constructed as emerging from the interplay between learning environment and the individual (Moss, Pullin, Gee, Haertel, & Young, 2008). OTL within mathematics education have been researched based on the idea that various contexts and systems affect these opportunities (Lester, 2007). Early on, the focus of OTL was often on how goals in curricula were possible to reach, while later research has focused on effects connected to issues of intersectionality and how different learners and groups of learners are affected by the curricula and other system-oriented issues (Lester, 2007). This chapter explores the patterns, intersections, and borders between goals in the exams for pre-service teachers. OTL are provided for pre-service teachers, who are the focus of the goals in their exams. At the same time, students in need of support in mathematics are a special group of students that could be affected by the explored goals.

2.1 The Context of the Study

The Swedish context of higher teacher education is briefly described to facilitate the contextual reading of the results, conclusions, and discussion. STm and MTP exams lead to a Bachelor's degree. Teacher exams – and other university exams for that matter – are regulated by a degree ordinance in which the demands for degrees are set in the Higher Education Ordinance (SFS, 1993:100). The ordinance includes the following headings: extent, goal, prerequisites, and what is required in order to achieve a degree certificate. The requirements for achieving a certificate are specified as the achievement or demonstration of knowledge in three qualitatively different areas: (1) knowledge and understanding, (2) competence and skills, and (3) judgement and approach (SFS, 1993:100).

In Sweden, there are two types of teacher education programs for primary school: one for preschool-class[2] (6-year-old students) to grade 3 (9-year-old students), and one from grade 4 (10-year-olds) to grade 6 (12-year-olds). Both are four years long. There are also two types of training to become a mathematics teacher in secondary school: one for grade 7 (13-year-old students) to grade 9 (16-year-old students), which is 4.5 years long, and one for upper secondary school, which is 5–5.5 years long (a voluntary school in which students are usually 16–19 years old). In Sweden, there are two programs for teachers who wish to specialise in special education. One to become a special pedagogue and another to become a special education teacher. Each program is 1.5 years in length (at an advanced level) and results in a different teaching degree. Both programs require a prior teaching degree, as well as 3 years of working in the profession. The special education teachers program has different specialisations, for example, reading and writing development, and mathematics development. The special pedagogue program works primarily with development of the school, learning environment, teachers' competence – and also with students – but is not specialised within a subject or disability. In this paper, we explore the degree program for STms, and the MTP exams for preschool-class to grade 3, grades 4–6, and grades 7–9 respectively. In other words, these teachers will support and teach students from ages 6 to 16 in mathematics. Sometimes, the STms will be labelled more generally as special education teachers, since all special education teachers share degree ordinance and thereby most of the goals, except for the specialised goals concerning mathematical development.

2.2 Method of Analysis

We have performed a systematic qualitative content analysis (Feucht & Bendixen, 2010) of the degree ordinance attached to the higher education ordinance. The investigation of the qualitative goals in these bachelor exams focused on: (1) knowledge

[2] In Sweden, there are two forms of education before formal schooling: (1) a daycare form of preschool (for 1–5 year olds) and (2) a preschool-class that is part of the school system (as described in this chapter).

and understanding, (2) competence and skills, and (3) judgement and approach. The content was framed in relation to the research questions, and the steps of the analysis procedure were taken from Feucht and Bendixen (2010). Statements connected to the duties of the teacher, the "other" profession, and the future work description regarding the mission to provide support were first selected from the three areas of qualitative goals. The statements were then grouped in regard to similarities in content, and the groups were given explanatory paraphrases as labels signalling what was in common, such as assessment, learning, development, disabilities, and organisation. Finally, groups of statements and statements within groups were compared in regard to how they constituted borders, spaces, and intersections related to their shared duties.

The analysis was performed in close connection to the labelling of goals, or levels of knowledge demands, in the degree ordinance, and the result is presented accordingly. These labels are in common for all degrees in the Higher Education Ordinance, constitute the knowledge deemed to be desirable and needed in order to work independently with the actual profession, and thereby depict OTL described in this steering document. It is important to pay attention to the nuances in knowledge that are displayed in the goals. For example, nuances might be *insights, knowledge, deeper knowledge,* or *understanding,* which could signal a variation in the depth or quality of knowledge. Sometimes, the knowledge is something that is supposed to be shown in assignments during their training, while at other times, it is something that the teacher is supposed to have. It is also important to remember that the STms have always undertaken some basic teacher training, and a teaching degree is a prerequisite for undertaking education to become a special education teacher. This implies that it is possible for STms to be preschool-class teachers with additional mathematics education, subject teachers in mathematics from secondary or upper secondary school, or to have completed one of the two teacher exams investigated in this chapter, namely, to become preschool-class to grade 3 teachers or grades 4–6 teachers.

3 Borders, Spaces, and Intersections

Swedish acts, ordinances, and government agency regulations are published in The Swedish Code of Statues (SFS). The references for education acts, Higher Education Ordinance etc. is therefore SFS followed by the identification number of the actual document.[3] Documents in The Swedish Code of Statues (SFS) are digitally published without page-numbers.

The results and analysis is built as a narrative of the OTL about special support in mathematics within the degree ordinances for STms and MTPs, and between those ordinances. The degree ordinances are part of the Higher Education Ordinance (SFS, 1993:100). This regulates the teacher training at universities, not

[3] For more information see https://svenskforfattningssamling.se/english.html

to be confused with the Education Act (SFS, 2010:800), which regulates the education in schools. In other words, teacher training is regulated in the Higher Education Ordinance and when they thereafter work as teachers in compulsory school, their works and the education they provide, is regulated by the Education Act.

The identified themes were explored under three levels of knowledge demands, or labels of goals, stated in the degree ordinance, which in turn illuminated borders, spaces, and intersections between the exams. Important to remember is that all special education teachers share the same degree ordinance, there are some but very important specific goals within that, that differs in regard to specialisation and that decides the specialisation in the program, and what degree is reached by studying it. Therefore, when the text depicts things that are in common for all directions of special teacher programs, it is marked with the abbreviation "ST" while, if the text concerns only the special direction of mathematics, the abbreviation "STm" is used. An overview of the main findings is displayed in Table 1.

3.1 Knowledge and Understanding

The goals regarding knowledge and understanding are one of the foundations in future professional practice. Three themes emerged: (1) a historical perspective; (2) duties to work for learning; and (3) development, assessment, and grading. Regardless of professional degree (STms or MTPs), teachers are supposed to have insights into relevant research and development work. The criteria for relevance are not stated, but they can be assumed to relate to the other goals and the purpose of the exam.

Table 1 Overview of analytical strategy and main findings

Area of goals → Brief overview of analysis ↓	Knowledge and understanding Foundation	Competence and skills Direction	Values and judgement Focal points
Themes in common	A historical perspective Development or learning Assessment and grading	Foundations Development of practice Development of student knowledge	Focal point of the profession hubs Ethics as a foundation
Interpretation	*ST* Guardian of educational discourses and structures. Carrier of explicit knowledge of disorders. Monitoring the process of assessment. *MTP* Carrier of present knowledge of the individual students' learning (F-3) or development (4–6) in math, and assessment.	*ST* To cooperate, advise, lead, participate, and be critical *MTP* To cooperate, follow, develop, and be developed	*ST* Human rights, collaboration with other professions and schools The hub of the supportive work *MTP* Children's proper and sustainable development The hub of the classroom

3.1.1 A Historical Perspective

STms should have knowledge regarding students in need of support from a historical perspective, as well as deeper knowledge regarding learning in mathematics (SFS, 1993:100). MTPs should also have knowledge regarding historical aspects, but not the explicit history of students in need of support but rather the history of schooling (SFS, 1993:100). This situation is interpreted as STms being responsible for ways (both successful and not) that have been used to support students in the school system and organisation, as well as the ones who foresee future implications in terms of teaching approaches, methods, or materials. This is in contrast to MTPs who are supposed to know about the history of schooling, but without explicit knowledge of the possibilities and pitfalls that might derive from the history and culture of the education system. This makes the special education teacher a guardian of sustainable development of the teaching discourses of students in need of support and creates a shared area of historical awareness. It also creates a limitation between the teaching roles in regard to the focus and possibilities to share understanding in their duties to support students in need of support in mathematics.

3.1.2 Learning and Development

MTPs should have knowledge regarding children's and students' development, needs, and prerequisites required for teaching and learning. However, this is not explicitly related to students in need of support, or the subject of mathematics. Additionally, this could also be interpreted as the knowledge needed being subject-specific. MTPs are also supposed to show knowledge in didactics and methods that are required in the subject of mathematics and in general. MTPs of younger students (preschool-class to grade 3) should also display deeper knowledge in regard to learning mathematics, as well as children's communication and development in general and in language. Teachers of grades 4–6 are not expected to display knowledge of students' learning; instead, the goal is aimed at reaching deeper knowledge regarding mathematics development. Interestingly, MTPs for preschool-class to grade 3 are supposed to know about learning in mathematics, while MTPs for grades 4–6 are supposed to know about development. This indicates that learning comes first, and knowledge in mathematics might be developed thereafter, as if there is no knowledge in mathematics before that. Of course, knowing about students' prerequisites for learning and what is demanded for teaching could very well include teachers' knowledge of their mathematical knowledge. But again, this is not stated.

As for STms, the goals are combined in general terms of learning and development, although not specifically in relation to the subject of mathematics, but rather for students with disabilities: "Demonstrate deeper knowledge of the development and learning of children and pupils and demonstrate knowledge of the development and learning of children and students with disabilities, including neuropsychiatric disorders" (SFS, 1993:100). Additional knowledge about neuropsychiatric disorders was recently added in 2018. This constructs STms as the carrier of knowledge of disabilities, and especially neuropsychological disabilities. Furthermore, there is a border between STms and MTPs, since the latter's goals do not include knowledge of students with disabilities but rather subject-specific learning and development. Therefore, the professions must combine their knowledge so that both the subject-specific knowledge of learning and development and the knowledge of disabilities come into play. Very often, STms work together with teachers when a student advances from grade 3 to 4, and in those cases, STms become the guardian of the progression in learning and development for students with disabilities.

3.1.3 Assessment and Grading

Preschool-class to grade 3 MTPs are supposed to obtain and demonstrate "deeper knowledge of the assessment of student learning and development" (SFS, 1993:100), which is not explicitly in the context of mathematics but can be compared to what these teachers are supposed to know about the students in mathematics, namely, deeper knowledge of student learning. However, these MTPs are also supposed to have knowledge (if not deeper knowledge) in general regarding students' development. Thus, the assessment is framed in relation to development for the preschool-class to grade 3 teachers. For grade 4–6 teachers, however, the assessment is framed in relation to grading and as a display of deeper knowledge: "show deeper knowledge in assessment and grading" (SFS, 1993:100). It is relevant to go back to what kind of focus that grades 4–6 teachers should have in students' mathematical learning: their development. This is not mentioned in relation to assessment and grading, as opposed to the knowledge in assessment of learning and development for preschool-class to grade 3 teachers. Finally, STms are also supposed to "display deeper knowledge of assessment issues and grading" (SFS, 1993:100), implying that STms should understand and be able to promote and support the teachers working with assessment. When looking at issues in assessment, STms appear to exist in the shared space of both assessment and grading and can therefore work together with MTPs by providing support. At the same time, the words "assessment issues" (SFS, 1993:100), imply that STms might be the ones keeping watch over the borders of the duties to assess and grade.

3.2 Competencies and Skills

The competencies and skills that could be most central for providing support for students might be the ones that ensure that this goal is actually a focus in carrying out the teacher's duties. Notably, in regard to the competence and skills of teachers, "mathematics" is mentioned only twice, and that is in the STms' goals and not in the MTPs'. If interpreting this positively, the subject and duties to provide support to students in need of it, is included in "having the subject knowledge needed to teach" and "apply subject didactics and methods needed for teaching and learning" (SFS, 1993:100). The goals for preschool-class to grade 3 and grades 4–6 teachers that can be connected to the shared duties to give support are the same. We understand the STms' ability to support and develop both students and learning environments independently is understood as the core: "show ability to support students and children and to develop learning environments in the school" and to "demonstrate the ability to independently carry out, follow-up and evaluate as well as lead the development of the educational work with the aim of meeting the needs of all children and students" (SFS, 1993:100). In the MTP exam, instead of "independent work", there is a strong emphasis on collaboration in the duties to provide support. The task is expressed as an act of "handling special needs" (SFS, 1993:100). The goals also stress that the teaching duties must derive from and communicate equity in the form of core values: "demonstrate the ability to identify and, in collaboration with others, handle special educational needs", "demonstrate the ability to consider, communicate and anchor equality perspectives in the educational activities", and "demonstrate the ability to consider, communicate and anchor core values including human rights and democratic values in the educational activities" (SFS, 1993:100). There are no differences between the preschool-class to grade 3 and grades 4–6 teachers in these statements.

The intersection between STms and MTPs is understood here as STms promoting, analysing, developing, and challenging the learning environment, and thus challenging MTPs' practices. MTPs are instead obliged to communicate, cooperate, and be receptive in their work in which special needs are handled. Notably, mathematics as a subject is not mentioned explicitly, and the emphasis that MTPs should cooperate and communicate does not have a counterpart in the STms' goals. Overall, providing support is depicted as being carried out in two specific directions of development: (1) development of the learning environment and (2) development of students' knowledge.

3.2.1 Developing the Learning Environment

The STms' duties are twofold, and in the part of developing learning environments, STms are depicted as free, critical, and able catalysts of change and improvement. At the same time, STms are supposed to do this independently, but also while "participating" in teaching. The previous section stated that STms should work independently whilst MTPs should cooperate. Participating is not the same as cooperating. What is actually meant by the learning environment is not stated, but this often means the teachers, teaching methods, materials. As such, STms should "demonstrate the ability to critically and independently initiate, analyse and participate in preventive work and help to eliminate obstacles and difficulties in different learning environments" (SFS, 1993:100). Correspondingly, this development work rests in STms' abilities to do educational investigations, which seem to be slightly different from working on preventive measures and eliminating obstacles and difficulties. This is interpreted as an educational investigation being acted out only after difficulties have already happened, and despite all the literature depicting the power of building on what is strong, functional, and works, preventive work has a kind of dystopian stance. Furthermore, when the difficulties are already apparent, the student is perceived as a passive object and that the student has difficulties, even though they appear, or maybe even "come into existence", in the student's encounter with the environment. STms should "demonstrate [an] in-depth ability to critically and independently carry out educational investigations and analyse difficulties for the individual in the learning environments where the child or the student is taught and staying during preschool-class or school day" (SFS, 1993:100).

These goals of STms stand in contrast with MTPs' goals regarding competence and skills. In this case, the development of the environment rests rather on the ability to reflect on their own and others experiences. In the following expression of "others", STms are to "demonstrate an in-depth ability to critically and independently utilize, systematize and reflect on the experiences of their own and others" (SFS, 1993:100). Furthermore, MTPs should work towards deeper learning and developing their own knowledge and skills in order to teach and support all students, as described by the following: "demonstrate in-depth ability to create the conditions for all students to learn and develop" (SFS, 1993:100). Here, it is taken for granted that students in need of support in mathematics are included among "all students", in the same way that we took for granted that the STms are included in "others". Interestingly, the STms do not have the target of reflecting on their own knowledge and being oriented towards developing it, but rather others' knowledge. This is an aspect of the future shared tasks where the borders between the examined occupations (i.e. STms and MTPs) might appear clearest, namely, in regard to their perspectives of their own and shared experiences and knowledge.

3.2.2 Developing Students' Knowledge

STms' goals are constructed as a mechanism of assessing and reviewing assessments of mathematical development (probably made by teachers). However, mathematical knowledge or learning is not mentioned in the assessment perspective: "the student should also demonstrate the ability to critically review and apply methods for assessing […] children and students' mathematical development" (SFS, 1993:100). Furthermore, cooperation on behalf of STms is finally stressed, as it is needed in establishing and implementing action plans for supporting students in need: "to participate in and in cooperation with stakeholders work to establish and implement action plans for individual students" (SFS, 1993:100). Furthermore, some part of the action plans or teaching of students in need of support can be the duties of STms, as individualised approaches should be a foundation in the knowledge of STms. STms should "demonstrate a profound ability for an individualized approach for children and students in need of special assistance" (SFS, 1993:100).

In comparison, MTPs' duties are to have knowledge to stimulate learning for all. Again, it is not certain that this is perceived as also including students in need of support in mathematics. As described in the degree ordinance, "MTPs should demonstrate the ability to acquire knowledge of and experience with students to stimulate each student's learning and development" (SFS, 1993:100). The MTPs' ability to cooperate is also stressed in the process of planning, evaluating, and developing teaching for all students and students' individual learning. They must "demonstrate the ability to independently, together with others, plan, implement, evaluate, and develop teaching and teaching activities in general to optimally stimulate the learning and development of each student" (SFS, 1993:100). It can be guessed that in the "together with others" part, other school professionals such as STms might be included, but this is not explicitly said. Regarding cooperation, parents and students are mentioned in the context of supporting the individual students' learning, but other professionals within the school are not required to "demonstrate the ability to observe, document, analyse, and assess students' learning and development in relation to the goals of the school and to inform and collaborate with students and their guardians" (SFS, 1993:100).

To sum up, cooperation is stressed in both STms' and MTPs' ordinances, but it is not explicitly said whether the "other" profession is included as a partner in this cooperation. Although, it can be assumed that this is the case. This is more likely in the situation of developing teaching and less likely in the supporting of each student (including students in need). According to the goals stated, STms are supposed to have the ability to stimulate learning for students in need of support, which can thus be understood as a single-handed mission of STms. This contrasts with the consultant conversations that STms are also supposed to be able to hold, as well as the critical and developing approach to teaching. In this circumstance, mathematics is mentioned for a second time: "demonstrate the ability, depending on the specialization chosen, to be a qualified call partner and adviser in matters relating to mathematics difficulties".

3.3 Judgement and Approach

The third area of demands in the exam concerns judgement and approach. Here, the values underlying the Swedish Education Act, and which the teachers are assumed to work to fulfil when they later are employed at teachers, are put to the fore. These values are understood as secured and governed through the decision to educate teacher professionals towards them.

There are both similarities and differences between the teacher exams, and they manifest under two themes: (1) ethics, which is constructed as an underlying fabric in the teacher professions, and (2) hubs, which point towards the connection or core in the duties. Mathematics is not mentioned, nor is there any difference between the two kinds of MTPs, the ones teaching preschool-class to grade 3 and the ones teaching grades 4–6. The MTPs' goal of developing and reflecting on their own knowledge and experiences, as well as that of other professionals, is considered to be a competence and skill. For STms, the ability to identify needs to develop knowledge and competence is stated as a goal within the area of judgement and approach.

3.3.1 Ethics

Both STms and MTPs have to be able to "demonstrate self-knowledge and empathetic ability" (SFS, 1993:100). In order to demonstrate empathy, teachers must know oneself and be able to self-reflect. This also applies to assessment, in which teachers are supposed to exploit, foresee, and detect social and ethical aspects. However, a difference is that STms are encouraged to retrieve these ethical insights from human rights, whilst MTPs are encouraged to retrieve these ethical insights from the students' rights in particular, and the UN Convention on the Rights of the Child (United Nations, 1989)The ethics also relies on the assessment being based on a scientific perspective as teachers should: "demonstrate the ability to make assessments based on relevant scientific, social and ethical aspects with particular regard to […] human rights" or "in particular the child's rights under the Child Convention and sustainable development" (SFS, 1993:100).

3.3.2 Hubs

Connected to judgement and approach is collaboration with other professionals and schools, which is stressed as a requirement for STms. Collaboration was already highlighted as a competence and skill for MTPs. Since the stakeholders are not explicitly defined, it might be assumed that what are intended here are not primarily professionals within the school, but other professionals from the student health team at the school, which includes school nurses, school psychologists, school social workers, principals, and special education teachers. Additional collaborators include other types of schools, such as schools for mentally disabled students,[4] Sámi schools[5] or special schools,[6] and external medical professionals such as pedagogues for visually disabled students, child psychologists or professionals working with rehabilitation. Since this aspect of collaboration is not stated for the MTPs, STms are designated as the hub connecting organisations and professionals surrounding the student, as opposed to the teacher, who is the hub in the classroom at school.

4 Conclusions and Implications

The differences, similarities, and overlaps between the pre-service teacher training for STms and MTPs are understood as constituting important borders and spaces in future work to give students support in mathematics. These borders and spaces

[4] In Sweden, there are specific schools for mentally disabled students with its own curricula.

[5] A school form in parallel with Swedish compulsory school, but with its own curricula, for children who are Sámi (the indigenous people of Sweden).

[6] In Sweden, there are special schools for students with certain exceptionalities (for example, autism). These schools follow the curricula of the Swedish compulsory school.

impact how well-prepared teachers are to collaborate in the shared task of supporting students in need in the subject of mathematics. This study has the potential to contribute to a deeper understanding of pre-service teacher training and its opportunities and threats in preparing for a school that can provide equal and high-quality teaching of students in need of support in mathematics. The key findings show that the borders and intersections are created mainly through placing responsibility, concerns, location of teaching and duties as well as directing ways of relating to their own and the “other” profession. Tensions between aspects of teaching and learning have emerged, regarding for example the order between learning and knowledge, the history and the present, the classroom and schools’ development, assessment as a process or product and learning for all or individualised learning. The resulting friction and synergy will be briefly discussed below.

On July 3, 2018, there was a change in the Higher Education Ordinance (SFS, 1993:100) in Sweden regarding the qualitative goals for pre-service STms. This change implicates a display of specific knowledge of neuropsychological disabilities. What has not changed is the overall goal of the education, which can be interpreted as STms working mainly individually with students. As described in the degree ordinance, “The student should demonstrate the knowledge and ability demanded in order to work independently as a special education teacher for children and students […] in need of support […] in mathematical development” (SFS, 1993:100). The Education Act (SFS, 2010:800), however, remains the same in regard to special education support. In practice, this means that any kind of professional with special educational knowledge with 90 credits of specialised education (not just STms) might be the one assisting the health care team or MTPs to provide support to students in need of support. This is troublesome since earlier research shows that development work at the group and organisational levels is needed in order to secure inclusion and learning for every student, and at the same time, the authority of the special education teacher is weak (Göransson, Lindqvist, Klang, et al., 2015; Holgersson & Wästerlid, 2018). The changes in steering documents are now giving STms even more responsibilities and explicit skills, but without strengthening their authority through school legislation.

The blurred authority and responsibility of special education teaching implies that the borders, intersections, and limits of STms and MTPs must be negotiated between the professionals in every municipality and school. The collaboration in practice is partly decided by school principals in terms of what tasks should be performed by which professionals. The general and sweeping labels in the higher education ordinance regarding the knowledge and cooperation that are needed in providing special support do not help with the division of labour. We believe that the generalised way of stating the goals in teacher education when it comes to collaboration, roles, and shared responsibilities can actually hinder future collaboration.

Looking at the borders, limitations, and intersections of the goals for pre-service teachers, aspects such as the source, direction, and focal points of knowledge appear to be essential. STms are defined as someone who has responsibility for the historical and future perspectives on students in need of support, whilst the MTPs’ responsibilities are placed in the present to a high degree. This creates borders between

STms and MTPs by placing them in different spaces in time. This also puts an expectation on STms to be a broker between spaces, time, and teachers teaching preschool-class to grade 3 and grades 4–6. In addition, it is quite obvious that STms are the carriers and guardians of explicit knowledge about special needs and disabilities and that they are assumed to work with, and promote the development of mathematical knowledge both with students in need of support, and also supporting all students' learning through promoting development in the organisation. The responsibility and ability to carry out these responsibilities are connected to leading, working independently, and having a historical and critical perspective of the school's work with these issues. Furthermore, MTPs are also supposed to be challenged and supported with knowledge about students' mathematical knowledge, obstacles, and ways forward.

Interestingly, STms are not directed to have a critical stance towards their own knowledge or to be reflective in their listening to other teachers' knowledge, which is the case with MTPs. A challenge lies in moving from the role of a teacher to the role of an expert, especially since teachers are often used to working in teams and collaborating, as indicated in the goals of the pre-service teacher training. Something that is not highlighted in the degree ordinances is a goal connected to co-teaching (see, e.g. Weiss, Pellegrino, Regan, & Mann, 2015; Van Ingen et al., 2016). This gives the impression that when collaborating, STms should lead and that perhaps MTPs should follow. Then, according to the goals depicted in the degree ordinance, the collaborative stance with collegial learning spaces seems to disappear when training STms. Hence, the goals create a border of diversity, segregation, and perhaps loneliness or superiority, which might create socioemotional hindrances for future collaborations between these two occupations. Furthermore, the loneliness of the STms implies that they need to seek collaborations outside their own school to promote their development (since there most often is only one STm at a school), in networks in their community, and with universities. Pre-service teacher training could give all pre-service teachers a deeper knowledge of each other's duties and also the possibility of taking shared courses, which might help them begin to make plans and negotiations, collaboration, learning, teaching, leading, and following. This could make teachers' shared duties to support students in need of support in mathematics easier when these pre-service teachers enter into the profession, and the possibility to maybe even engage in co-teaching.

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Blurring the Border Between Teacher Education and School Classrooms: A Practical Testing Activity for Both Contexts

Tina Rapke, Marc Husband, and Heather Bourrie

1 Introduction

In this chapter, we focus on the border between school and university mathematics education courses by describing a testing activity that can be used in both contexts. Many academics (e.g., Ball, 1990; Beswick & Muir, 2013; Gainsburg, 2012; Hart & Swars, 2009) are concerned with the divide between reform-based mathematics recommendations promoted in teacher education courses and what actually happens in school mathematics classrooms (often what actually happens in schools is described using the word traditional). Academics stress that teachers have difficulty in translating general pedagogical recommendations to concrete classroom activities (Grossman, Smagorinski, & Valencia, 1999) and educators "model[ling] high-quality teaching practices becomes paramount" (Schwartz, Walkowiak, Poling, Richardson, & Polly, 2018, p. 62) to the education of pre-service teachers (PSTs). It is essential that in mathematics teacher education courses, we (mathematics teacher educators) practice what we preach and do not default to "do as I say, not as I do" (i.e., fall back ourselves to traditional ways). The assessment activities we implement in mathematics teacher education courses should align with the recommendations that we make about reform-based mathematics. Even more conducive to PSTs' learning, we can implement activities in mathematics education courses that PSTs can use in their future school classrooms. This allows us to offer PSTs extensive experience with concrete classroom activities that can be used in schools.

Basically, we consider traditional mathematics classrooms to emphasize the teacher's preferred solution strategies and reform-based classrooms to focus on students' mathematics solution strategies. Traditional activities, such as *show & tell* (Ball, 2001) place the teacher as the main source of mathematical knowledge in classrooms. If you were to walk into what we conceptualize as a traditional

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mathematics classroom, you would likely see the teacher at the front of the classroom demonstrating steps of their preferred solution strategy while students sit quietly and passively at their desks. You may also see students working individually at their desks completing worksheets that require them to mimic the steps that the teacher has demonstrated. In contrast, reform-based mathematics classrooms emphasize students' ideas as a resource for learning (e.g., National Council of Teachers of Mathematics [NCTM], 2014). If you were to observe what we classify as a reform-based mathematics classroom, you would likely hear a lot of *Math Talk* (Campbell & Bolyard, 2018), i.e., you would likely see students and their teacher sharing, comparing, and analyzing *students'* mathematical ideas.

The divide between what should be happening (i.e., reform-based mathematics) and what is typically happening in mathematics classrooms has been thoroughly discussed in the literature (Gainsburg, 2012). Mathematics education academics have identified a variety of causes of the problem including teachers' knowledge about mathematics content, the difficulty in enacting reform-based mathematics teaching, teachers tending to teach based on their experiences as students, and university academics teaching general conceptual/theoretical ideas and assuming educators can then translate and apply them in their classrooms.

The relationship between being a PST and an in-service teacher within the context of testing becomes extremely relevant when considering the idea of teachers reverting back to and using activities that they experienced as students. Officially, PSTs are enrolled in an education program and as such are students who are adding to their experiences as *students*. In terms of PSTs and in-service teachers falling back to their experiences as students, a traditional mathematics content test that PSTs sit may be perpetuating the divide between what is being promoted in teacher education courses and what is actually happening in school classrooms. In other words, PSTs and in-service teachers reverting to traditional ways may be partially due to teacher education programs. Furthermore, this may be particularly true for testing experiences, as many Canadian universities have PSTs sit Grade 6 and 7 mathematics content tests (Brown, 2016). Having PSTs demonstrate their knowledge of elementary mathematics content through traditional testing, is essentially offering PSTs more experience with traditional activities to fall back on. Thus, it would make more sense to align what happens in education programs with recommendations of reform-based mathematics. Simply put, aligning what PSTs experience in their education programs with the recommendations that are made about reform-based mathematics classrooms serves to bridge the border between what is said and done and ultimately what PSTs will do in their in-service teaching.

For teacher education courses, it is often recommended that PSTs have extensive experience with concrete classroom activities before learning about general pedagogical recommendations. Berliner (1989) suggests that we should offer PSTs extensive experience with practical activities that they can use in their future classrooms to support closing the gap between research and practice. In a review of the literature on the divide between what actually happens in classrooms and what research recommends, Gainsburg (2012) presents the case for first offering PSTs practical tools and activities, then proceeding to explore general/conceptual frameworks. Essentially, the argument is about learning from the concrete and then moving to the general. At one end of the argument, there is the idea that "[o]nly through extensive experience with particular practical tools will teachers derive the general educational concept; teachers must do before they understand" (Gainsburg, 2012, p. 363).

However, researchers also believe that this is difficult because school-based practice teaching blocks often come after course work (Gainsburg, 2012). The National Council for Accreditation of Teacher Education (NCATE, 2010) suggests that teacher education programs have course work "woven around" practicum experiences. This is a way for PSTs to emphasize and gain experience with practical applications of course work. Here, we extend the work on offering PSTs concrete experiences with classroom activities by detailing a testing activity that can be implemented in school classrooms and mathematics teacher education classrooms. We detail an activity of having PSTs develop, sit, and assess tests for one another. We analyze PSTs' written reflections about experiencing such a testing activity, and situate PSTs' perceptions of their learning in related literature and recommendations for reform-based mathematics.

2 Reform-Based Mathematics

The NCTM is a well-established and well-known proponent of reform-based mathematics. Their recommendations to support reforms in mathematics include a focus on multiple students' mathematics ideas, use and posing of questions/tasks that advance students' mathematical ideas, ideas about feedback, and collaboration (NCTM, 2000, 2014).

The NCTM suggests that reform-based mathematics classrooms should include a focus on students' mathematical ideas. The NCTM (2014) clearly states that effective teachers "use evidence of student thinking to assess progress toward mathematical understanding" (p. 3), and that effective teachers encourage discourse among students by "analyzing and comparing student approaches and arguments" (p. 3). Clearly, there is a focus on students' ideas in these suggestions, as the suggestions would not be possible without students' approaches to a mathematics problem. Furthermore, there is emphasis placed on multiple solution strategies because discourse is about analyzing and comparing *students'* approaches and arguments (if it were singular then comparing could not take place). Indeed, literature supports the use of multiple solution strategies in mathematics classrooms. For example, Rittle-Johnson and Star (2007) found that students who compared solution methods to the same problem outperformed students who used only one solution strategy.

In terms of questions, the NCTM (2014) is clear that in reform-based classrooms, teachers are encouraged to use purposeful questions to advance their students' mathematical ideas and draw attention to important mathematics relationships. Specific to posing questions in reform-based classrooms, the NCTM (2000) is apparent that in addition to solving problems, students are also encouraged to become problem posers. Teachers should work with students to "develop a broad range of problem-solving strategies, to pose (formulate) challenging problems, and to learn to monitor and reflect on their own ideas in solving problems" (p. 116). Problem posing has been associated with creativity and active mathematics classrooms that focus on mathematical ideas originating from students (Silver, 1994). Yet, posing questions can be challenging for both the creator and the receiver (Silver, 1997).

The NCTM believes that feedback "helps students in setting goals, assuming responsibility for their own learning, and becoming more independent learners" (NCTM, 2000, p. 2). Moreover, the NCTM (2014) suggests that mathematical success for all can be achieved through "descriptive, accurate, and timely feedback on assessments, including strengths, weaknesses" (p. 5). There is no agreement on the definition of feedback within the literature (Evans, 2013) and there are many different ways to conceptualize feedback. Feedback can be viewed in a range of ways such as: simply as a corrective tool or as a process that provides students with opportunities to dialogue and decide how to improve and refine their work. Evans (2013) states that:

> [T]he emphasis in the literature [in higher education—which would include teacher education] is on feedback as a corrective tool, whereas it should also be seen as a challenge tool, where the learners clearly understand very well and the feedback is an attempt to extend and refine their understandings. (p. 72)

There does seem to be some overlap about effective feedback. For example, there are academics (e.g., Andrew, 2009; Lavey & Shriki, 2014) who emphasize that effective feedback should be concrete. Precisely, feedback should be about improving students' mathematical communication and conceptual understanding rather than simply providing general positive affirmations (e.g., "good job"). Embedding ideals of NCTM, we believe that feedback is crucial to learning. Indeed, many researchers echo the NCTM's suggestions about feedback and assert that feedback is one of the "most critical teaching skills that have been documented as facilitating student achievement" (Scheeler, 2008, p. 146). To think about the vast impact feedback can have on learning, our ideas include literature about feedback being dynamic where the teacher also learns from students through dialogue (e.g., Carless, Salter, Yang, & Lam, 2011), and feedback having the potential to change beliefs and levels of engagement (Nelson & Schunn, 2009).

The NCTM is also explicit in making recommendations about collaboration in reform-based mathematics classrooms. For example, it is asserted that "[w]orking in pairs or small groups enables students to hear different ways of thinking and refine the ways in which they explain their own ideas" (NCTM, 2000, p. 129). This emulates the ideas about analyzing and comparing students' solution strategies but focuses on collaboration. The NCTM extends their recommendations about collaboration to include teachers. NCTM (2000) endorses that mathematics teachers have much to gain from collaborating with other teachers when it says that mathematics teachers can collaborate with colleagues in creating their own learning opportunities. In doing so, they enhance their own mathematical and pedagogical knowledge, to teach their students well. In the end, we consider reform-based mathematics activities (including testing activities) to be ones that emulate ideas about effective feedback, focus on students' ideas, embed a lot of *Math Talk* and collaboration, and involve purposefully selected questions or problem posing.

3 Teacher Learning and Reform-Based Mathematics

The literature about teacher education, feedback, using students' mathematical ideas, and multiple solution strategies is either sparse or point to difficulties. Furthermore, some academics (e.g., Ball, Sleep, Boerst, & Bass, 2009; Hiebert, Morris, Berk, & Jansen, 2007) suggest that PSTs' coursework should follow a practice-based approach—one where PSTs "do instruction, not just hear and talk about it" (Hiebert et al., 2007, p. 459).

It is useful to consider how difficult it can be to enact reform-based methods when considering how to educate PSTs to use students' multiple solutions. Gainsburg (2012) points to the difficulty of using students' ideas because of the uncertainty in what ideas students will share. She explains that "[t]eachers must continuously monitor and respond to students' thinking, design active learning tasks and environments, and cope with uncertainty" (p. 365). Indeed, others have suggested that teachers should anticipate students' mathematical ideas before lessons (Smith & Stein, 2011). Even with anticipating, it must be pointed out that using more than one solution method in mathematics lessons presents a challenge for teachers because it requires teachers and students to go beyond show and tell—it requires connecting students' mathematical ideas (Smith & Stein, 2011). In addition, professional learning opportunities for teachers on using multiple solutions in their classrooms have received poor results. Durkin, Star, and Rittle-Johnson (2017) reported that despite providing teachers with curriculum materials that were specifically designed for teachers to get their students to compare multiple strategies, teachers rarely implemented these materials. Based on these findings, Durkin and her colleagues recommended that additional supports were needed for teachers to learn how to use materials that focused on multiple solution strategies. They go on to recommend that instructional techniques should be researched in this area, so that teachers can effectively do this in their classroom.

Unfortunately, much of the literature about feedback does not focus on educating PSTs to provide feedback themselves as teachers. The research regarding feedback within the context of teacher education programs tends to investigate the type of feedback offered *to* PSTs. For example, Ellis and Loughland (2017) discussed how mentor teachers (in-service teachers of practicum classrooms) offered feedback to PSTs. Similarly, Schwartz et al. (2018) examined effective feedback provided to PSTs on lessons PSTs taught in school-based practice teaching blocks by examining occurrences of feedback that was mathematics specific. Thomas and Sondergeld (2015) is one of the exceptions, as they focused on PSTs providing feedback in their future classrooms through having PSTs practice giving written feedback to school students' work. Based on their findings, Thomas and Sondergeld (2015) articulated that "we cannot assume that [PSTs] will develop skills in best assessment practices without deliberate, scaffolded, and guided instruction" (p. 104). Therefore, it is necessary that education programs consider how descriptive and timely feedback could be approached for PSTs' learning. This is an under-researched area and findings about how PSTs learn to enact best practices in feedback in their future classrooms could not only extend existing research but also be used to inform the design of PSTs' coursework and activities.

In terms of designing courses for PSTs to help them learn about reform-based mathematics, there seems to be a push towards *pedagogies of enactment* (Grossman, Hammerness, & McDonald, 2009). This movement goes against the implicit assumption that adults (including PSTs) learn differently from children—the assumption that PSTs learn from listening and reading, rather than gaining extensive experience with(in) the concrete and then moving to more generalized and overarching theories/ideas. For example, Grossman et al.'s (2009) work proposed that teacher education "move away from a curriculum focused on what teachers need to know" to a "curriculum organized around core practices" (p. 274). Grossman and her colleagues (2009) go on to recommend preparing PSTs by offering opportunities to experience activities "in environments that are less complex than classrooms" (p. 279) such as the environment of teacher education courses. The conclusion is that learners (PSTs and schoolchildren) benefit from extensive experiences with the concrete and then moving to the general. Ideally, PSTs could learn about feedback by providing feedback and learn about using multiple mathematical solution strategies by experiencing activities that provoke multiple solutions strategies. Learning from the concrete and moving to the general could be very efficient for PSTs through activities that could be used within mathematics teacher education courses and within school mathematics classes.

4 Teacher Learning Through Experience

The idea of learning through experience can be found outside of research about educating PSTs and in the learning of in-service teachers. For example, Mason's (2002) work on noticing is a framework that is well established and used extensively by mathematics education researchers (e.g., Chapman, 2015). Mason's framework extends Schön's (1983) work on the reflective practitioner to include the notion of *reflecting-through-action*. *Reflecting-through-action* is about "heighten sensitivity to notice while engaging in practice" (Mason, 2002, p. 15). Mason (2002) speaks about teacher learning in terms of professional noticing. Mason described professional noticing as something that one does when they see someone else doing something (e.g., gesturing, using certain phrases/questions, a task) in a professional setting such as teaching or interacting with a client and they imagine themselves doing something similar when they are acting professionally. Mason and Davis' (2013) perspective on teacher professional learning and noticing is grounded in learning from experience. They go as far to say that:

> [T]he only strategy, the only action that human beings have access to in order to learn from experience is to become consciously aware of recent actions that proved fruitful and then to imagine themselves having this action come to mind in some similar situation in the future. (p. 192)

In the specific case of PSTs, Mason's work can be interpreted to make claims that PSTs learned from their experiences if they can make connections to their future classroom by envisioning themselves enacting a similar activity in their own teaching.

5 The Study

The testing activity can broadly be described as PSTs developing, sitting, and assessing final tests for another. Having students develop tests and solution keys, sit each other's tests, and assess classmates' solutions to the tests is not a new idea and has been implemented in mathematics courses (see Rapke, 2016). Research shows that such testing activities promote students to apply deep approaches to learning while they prepare to sit the "live/actual" test. Here and in line with the NCTM's focus on feedback, we have specifically added a feedback element to the activity (the assessment of classmates' responses to test questions included directions to provide feedback). For ease of description, we say that the feedback portion of the enactment of the activity is the period that begins when PSTs return their test responses to their colleagues who crafted the test questions. We have also changed the context of how PSTs are to craft questions. In mathematics classes, students are told to craft and select questions about the mathematical concepts they learned in the course. PSTs were told to craft questions based on what they think themselves and their classmates should know as future teachers (this may include questions focused on mathematics content or those more focused on pedagogy). For example, a question that focused on mathematics content might include a prompt to share multiple solutions strategies to solve a problem such as "3.5 × 24." Whereas a more pedagogical question might ask about possible prompts a teacher could use to support students to make connections across solution strategies. Allowing students (in this case, PSTs) to choose and craft questions for the test allows their ideas to take center stage (focusing on students' ideas is a crucial component of reform-based mathematics classrooms as outlined in our related literature section). The testing activity also involves *Math Talk*, in line with research (Campbell & Bolyard, 2018). *Math Talk* takes place during the crafting of the tests, assessing of the tests, and the feedback portion of the activity.

This study took place at a large research-intensive Canadian University. The teacher education program requires elementary PSTs to take two mathematics education courses during their 2 years of study. The participants in this study were in their first year of the program in two different sections of the same mathematics education course. The course is characterized as a teaching methods course for elementary PSTs. This course is designed to teach PSTs how to teach mathematics in primary and junior classrooms (with children of ages 5–12). All students enrolled in the course took part in the testing activity. Forty-two PSTs volunteered to participate in this study and have their work be used for research purposes. The study took place during the last two sessions of a course that consisted of 12 sessions. The following provides an overview of the testing activity that took place over 2 days (session 11: Day 1 and session 12: Day 2).

5.1 Day 1—Creating Questions and Solution Keys

In session 11, PSTs were separated into groups consisting of 4–5 members and spent three hours creating four test questions. Criteria for the test questions took place in class and was led by the course instructor who sought agreement on the types of questions that would be posed. The content, for example, had to be based on topics explored in the course like subtraction. The PSTs also wanted the questions to be asked in a way that supported a variety of ways they could demonstrate their understanding of the mathematics that was being tested. PSTs also had to provide a solution key to their questions. They were told that they would use this solution key to assess their classmates' responses and to assign grades (if assigning grades is what their individual group decided they were going to do). Requiring solution keys was purposefully included in the activity because it is something that PSTs seemed to be familiar with, based on their testing experiences in previous mathematics courses. It was also hoped that crafting solution keys would provide PSTs with the opportunity to anticipate multiple solution strategies for each question that they posed. This was a pass/fail activity. The instructor of the course was present during the class and was circulating around the room to offer advice/suggestions. PSTs were told that they passed once the instructor deemed their tests and solution keys acceptable.

5.2 Between Day 1 and Day 2

In between Day 1 and Day 2, each groups' test questions and solution keys were posted on a class website for everyone in the class to view. To prepare for the test, PSTs could practice some or all of the possible questions that could potentially be asked on the final day.

5.3 Day 2—Sitting the Test and Assessing Questions

On the final day of the course, each group (that was formed in Day 1) was randomly paired with another group for the purpose of exchanging/sitting one another's tests. PSTs were given an hour to individually answer each other's test questions in a traditional manner—no resources, no talking, and only access to paper and pencil/pen. Upon completion of their test, PSTs submitted their work to the group who developed the test. This is when the feedback portion of the activity began. Each individual group chose whether or not to assign a grade value or level of achievement between 1 and 4 (with 4 being the highest) and were told that they had to provide feedback that their classmates could use to enhance their work. This facilitated a process whereby PSTs naturally wanted to talk to their classmates who wrote the test. The PSTs were instructed that they would pass the assessment/feedback portion of the test once the PSTs who sat the test were satisfied with the feedback they received.

6 Data Collection

As part of the course requirements, PSTs completed a written reflection about their experiences with the testing activity. The reflection was guided by the question: How has this experience (over two classes) influenced your future teaching? Only PSTs reflections that agreed to participate in the study were collected as data and analyzed for research purposes. Pseudonyms were given to all PSTs to protect their identity. This was also a pass/fail portion of the activity. PSTs were told they would have an opportunity to resubmit their reflections and the instructor would provide feedback, if the instructor deemed the reflection as unacceptable. No PSTs reflections were deemed unacceptable.

7 Data Analysis

The data was analyzed by drawing upon a phenomenographic approach, whereby data are gathered in order to study participants' experiences and perceptions of the phenomenon being investigated (Marton & Booth, 1997; Mason, 2002). Mason (2002) tells us that "[t]he aim [of phenomenography] is to describe and characterise different ways of experiencing" (p. 162). Phenomenography is a qualitative research approach that has a long history in higher education (e.g., Kinnunen & Simon, 2012; Marton & Booth, 1997) and takes a second-order approach. This means that the focus of the research rests upon how people in the situation experience/perceive the phenomenon, rather than how the researcher experiences or understands it. Phenomenography is used to develop qualitatively different categories of ways participants experience/conceptualize a phenomenon. These qualitatively different categories are derived from the collected data by means of iterative procedures (Kinnunen & Simon, 2012). The premise of phenomenography is that knowledge is both subjective and relative (Kinnunen & Simon, 2012). Rich data is derived from a wide range of experiences/conceptions of the same phenomenon. This is accomplished by asking people to explain in their own words about the phenomenon and how they experienced/conceptualized it personally. This research was guided by the question: *In what ways do PSTs describe the learning that took place through creating, sitting, and assessing final tests for one another?*

The second author started the analysis by identifying statements that related to the research question. These comments were re-read by Author 2 and read by Author 1 to familiarize themselves with the developing ideas. Author 1 and 2 started to identify common features within the data to form categories, and then they started to examine what made each category different and similar. Author 1 and Author 2 continued to re-form groups by amalgamating groups and identifying other categories through the iterative process of revisiting data. Author 3 then read through the selected comments and either agreed or disagreed with the categories. Author 3 agreed with the categories and their descriptions.

8 Findings

PSTs' experiences of creating, sitting, and assessing final tests with one another were sorted and classified into four emergent categories that describe PSTs perceptions about:

1. Crafting questions and problem posing
2. Anticipating and making sense of students' mathematical ideas
3. Feedback
4. Collaboration

Each category of description is outlined and direct quotes are provided to illustrate the characteristics of each category and provide insight into the differences between categories. The experiences or perceptions may have come from one or all the participants. The number of comments in each category is not given because phenomenography is concerned with the variety of perceptions and experiences not the most typical.

8.1 Crafting Questions and Problem Posing

In this category, PSTs specifically used the word "question" or "questions" and spoke about what they learned in regard to what is required to craft/pose questions (both test questions and questions PSTs will use in their future classes and on assignments). It was explained that when crafting the text of the question, teachers need to be thoughtful by anticipating how students might interpret the question. Advice was given that you need "be very careful with your wording" (Amy) and that sometimes "all it takes is one word to change for a question to finally make sense to someone" (Amy). Warnings were also given that the language within the question can lead "students down the wrong path" (Amy). To clarify, within the comments was the idea that the wording of the questions would influence the mathematical skills and knowledge that student would demonstrate in their responses. It was clearly stated that words and phrases "provoke the use of the certain [*mathematical*] concepts I would like to see in the student's answers" (Adrian). There was also the idea that the clearer the language (including the wording of instructions within the question) the greater the chances of students demonstrating the mathematical skills and knowledge that the teacher has planned for. Safire said it best when she explained, "words and phrases that are used need to be clear or else the students might not answer the question how we expected them to." It is evident that experiences of developing, sitting and assessing tests can lead to an awareness that teachers need to craft questions using language that minimizes misinterpretations and influences the use of certain skills and knowledge that the teacher meant for students to demonstrate.

PSTs reported that they learned that teachers also "need to be very thoughtful about the numbers" (Nancy) within the questions when they are attempting to have students demonstrate specific mathematical concepts. The numbers within mathematical questions will allow "students to see a relationship between numbers and concepts in math" (Adrian). Moreover, insight was gained that numbers within the question can potentially lead to students experiencing negative feelings and developing negative ideas about mathematics. For instance, Adrian advised, "not choosing numbers that will frustrate students into thinking math is hard." The idea of intentionally choosing numbers within a question becomes apparent, for instance, if the teacher is wanting a Grade 2 student to demonstrate equal sharing but asks a question that involves sharing 27 items among five people.

Comments indicated that "clear"/(not vague) or "good" test questions are questions that "require time and attention to create" (Carly). It was explained that teachers need to be very strategic with their use of language and numbers, so that students demonstrate the mathematical skills and content the teacher envisioned. Furthermore, tests questions should allow the teacher to evidence that students are demonstrating specific expectations from the curriculum. The opposite of clear questions would be "vague questions" and it was clarified that "if the teacher provides a vague question, he or she should expect a diverse set of answers and not restrict the 'correct' answer to a specific answer" (Ed). This means that credit should be given to solutions that did not use the concept that the teacher intended and wanted students to demonstrate. In summary, when PSTs reflected on their experience of the testing activity, they spoke of the following influences: crafting the text of a question, choosing the numbers within a mathematical question, the extensive time required to develop questions, and how questions on tests can cause students to experience negative emotions.

8.2 Anticipating and Making Sense of Students' Mathematical Ideas

The experiences of this category included challenges associated with anticipating a range of students' ideas and/or spoke to the importance of understanding students' mathematical ideas. Comments alluded to struggles PSTs had with anticipating solutions for the answer key that they created to assess their classmates' tests. An idea that appeared within the experiences was about not being able to have a complete answer key because PSTs were unable to anticipate all the responses on the tests. For example, Elsa said: "[E]ven though [*fellow PSTs*] believed they had offered all of the possible solutions to a single question, there still existed many more ways of mathematical thinking they did not expect."

Anticipating test responses was deemed challenging because the test responses contained different strategies from the preferred/anticipated strategy of the PSTs who authored the test. The following statement speaks to why PSTs felt this practice was so difficult. Aleeya reported that no matter how specific the instructions are: "…they often divert to their own ways of thinking and showing their work and completely ignoring ours." Experiencing the difficulties in anticipating how students will respond to test questions led to PSTs reporting more general insights into to future teaching (i.e., what this means for lesson planning, or being a mathematics teacher in general). In terms of preparing for lessons, Crystal acknowledged that: "I guess I have to learn that no matter how much I plan, when I teach, students will find ways to answer my questions that I can't plan for."

Unexpected solutions/answers to test questions prompted PSTs to think about how being a mathematics teacher means that they need grapple with having to anticipate and that they cannot anticipate everything. For example, one PST said: "I knew by the end of the semester to expect the unexpected, but regardless of what you expect, the unexpected will still take you by surprise" (Matilda). The experiences about being comfortable with not being able to anticipate everything included rationales to why teachers should become comfortable in being uncomfortable. Ronny identified the challenge of not being able to anticipate all of his students' ideas but lays important claims to what he feels it means to be a mathematics teacher by saying: "Although I won't be able to anticipate all the answers, I need to be able to anticipate the ways in which my students may answer questions so that I can connect their answers in meaningful ways" (Ronny). Specifically, Ronny identified that he thinks that mathematics teachers need to anticipate multiple solutions because his job as a teacher requires him to connect students' mathematical ideas in "meaningful" ways. Thus, anticipating how students will respond to a mathematics question is related to being able "to understand what it is they [*students*] are doing" (Aleeya).

Descriptions of PSTs experiences implied that teachers can "Tap into your students thinking in order to help them progress in terms of achievement" (Cherry).

Mira echoed Cherry's idea and exemplifies how important she feels it is for teachers to understand students' mathematical ideas by stating: "Once we figure out their thinking we can help them with missing steps or help them develop their thinking further." PSTs were motivated by unexpected solutions to move out of their seats and to ask their colleagues about their ideas. Based on this experience, PSTs recommended that future classroom activities involve "conferencing to better understand student thinking" (Kate).

8.3 Feedback

The statements that fell under this category included the word "feedback" and resulted from PSTs reflecting on the feedback portion of the testing activity. The feedback portion of the testing activity took place at the end of the activity and occurred after PSTs sat the test and submitted their work to the group who made the test they sat.

Comments in this category provided insight into how PSTs conceptualize effective feedback and the complexities of providing effective feedback. It was explained that realization was gained about "how much learning can come from good feedback even if a child got it completely correct" (Cara). One PST described how she could do more than "give check marks or irrelevant comments (ex. good job)" (Mandy) and that understanding and meaning making could be supported by focusing feedback on students' solution methods. Providing feedback to students' solution strategies was also mentioned in terms of benefits to students' learning dispositions. For example, Andrea took note of the importance of providing positive feedback, "as it may contribute to students feeling confident about their own ways of solving a mathematics problem." Statements falling in this category indicated how providing feedback can be difficult because it involves interpretations. Specific comments eluded to an awareness of possible (mis)interpretations students might make based on the nuanced language contained within the feedback. For instance, Carmela said, "I realized that my language may have hidden meanings for a student. Phrases such as 'I like how' can lead a student to think that the way they used to solve is the only one the teacher 'likes' and therefore resort to using only one method to solve." In terms of nuanced language and learning specific mathematical topics, Elsa said that she learned that "when giving feedbacks [sic] it is helpful to go back and look at the specific curriculum expectations" to help choose words that she can use within feedback.

The difficulty of interpreting and making sense of written feedback can be seen in suggestions that the effectiveness of feedback may improve with in-person discussions. This can be observed when Crystal said, "I feel like the feedback would be more valuable in person as well, so we could interactively look at the work together. I feel like my feedback was not as effective just written on the page like that." Ann shared in experiencing difficulty while communicating about how to enhance students' mathematical work as she said she plans to "incorporate talk" in her feedback practices.

In terms of learning to be future teachers, Drake indicated that he learned from the feedback he received from fellow PSTs by saying that he will “definitely use their [fellow PSTs’] feedback to inform future questions I pose to students.” It is interesting to note that a PST said that developing, sitting and assessing tests for one another “was the first time that I provided feedback in my marking which included meaningful and thought-provoking responses, and I am certainly inspired to do the same in my future practice as an educator” (Lisa). PSTs clearly articulated that the testing activity provided them with an opportunity to learn about feedback themselves, by implementing feedback practices and for some it was the first time in their program of study where they had an opportunity to provide meaningful feedback.

8.4 Collaboration

PSTs described the benefits they experienced and envisioned for their future students and teaching based on the collaborative nature they experienced while developing, sitting, and assessing final tests for one another. Comments contained specific benefits of collaboration that they themselves experienced in the testing process and possible benefits of future collaborations with in-service teacher colleagues. The benefits included increasing confidence to sit the test, building friendships, and supporting students to take risks.

It was explained that trepidations about taking the test were eased because developing a test allowed for learning opportunities and confidence building to sit the test. There were experiences that described how PSTs could prepare to sit the test themselves by developing a test. For example, Steph spoke about her own learning of mathematics and pedagogy when she explained that while developing the test “I could observe and learn from my peers.”

There were also reports about how the process increased confidence to sit the test because they felt they were not sitting a test that was out to trick them. For instance, including students in developing test left one PST feeling that the test “isn’t going to include a plan to trick the students into making mistakes” (Andrea). Others reported how collaborations reduced tension and increased confidence because they were able “to witness that although my peers knew more math than I did, I knew how to ask the right math questions” (Juan).

Reflections also included perspectives about how and why their experiences of the testing activity has encouraged them to involve their future students in developing, sitting, and assessing tests to build community or provide support. Building community can be evidenced in terms of “friendships” in Sherry’s statement: “I would try to incorporate this strategy so that new friendships can form in my classroom.” Other perspectives also included the idea of supporting students to succeed. For example, Ainsley said that she would use the testing activity in her future classroom to “support the students’ development and not setting them up to fail.”

While other statements indicated that experiencing the testing activity had them think about collaborating with future colleagues. Elsa explained that her experience helped her "gain insight to the many benefits that can result from collaboration with other teachers in the field." The benefits of future collaboration with other teachers were described in terms of coming up with new and unique ideas. It was explained that new and unique ideas could take the form of lesson plans or teaching strategies. Others described more complex ideas about how students and their mathematics teacher should be in collaboration. Some pointed to how the teacher should be in dialogue with students instead of making quick judgments about student ideas that do not match their own their preferred solution method. The comments gave a sense of ways of being open with students by not just assuming they (PSTs) are right because they are the teacher:

> I don't think I ever had a teacher ask me to explain how I got an answer before marking it … In the future, I hope I remember this in my own teaching and that I'm not quick to judge a student's thinking if it does not match mine (Crystal).

Other comments lined up with these ideas through emphasizing how the responsibilities of teaching should be shared in classrooms. For instance, Steph explained, "the teacher teaching the student sometimes flips around to the student teaching the teacher." Others claimed that experiencing the testing activity made them realize that "it is beneficial to have students leading the class while the teacher provokes conversations though using an inquiry approach" (Sam).

In terms of teachers continuously learning and taking an active role it in the learning environment, it was clearly articulated that "as an educator we need to continuously learn about how students learn in order to develop optimal opportunities" (Mandy). Moreover, it is important that "educators are constantly learning from their students just as much as their students are learning from them" (Malisa). Experiences in this category indicate specific benefits of collaboration that can be realized through enacting the testing activity in mathematics and mathematics education classrooms. They include the benefits of students collaborating during the testing activity and illuminate the collaborative pathways and partnerships between students and teachers that are possible.

9 Discussion

Our findings point to the significant benefits of reform-based mathematics activities that sit in the milieu of school classrooms and teacher education courses—activities that sit on the border because they can be implemented in both contexts and embody general constructs of reform-based mathematics recommendations. Our findings suggest that such activities allow PSTs to learn about general constructs of reform-based mathematics through concrete experience, thus supporting and extending literature about PSTs education (e.g., Gainsburg, 2012). The PSTs perceptions not only indicate that they learned about constructs of reform-based mathematics classrooms but also identified and experienced the worth/outcomes of the activity. As a result, and in terms of teacher learning as Mason (2002) conceptualizes it, PSTs envisioned themselves and their students benefiting from the testing activity in future situations.

In terms of PSTs' learning, our findings point to significant gains that can be realized in terms of bridging the divide between school classrooms and what is recommended in teacher education courses. Not only did PSTs learn about constructs within reform-based mathematics recommendations, but they themselves experienced the value of activities that exhibit constructs from reform-based mathematics recommendations. In turn, PSTs claimed their intentions of implementing the testing activity in their future classrooms. Considering Mason's (2002) conceptualization of learning, many of the PSTs envisioned themselves using the testing process in their own classrooms giving rationale of positive aspects from which they have benefitted. PSTs said they intend on using the testing process in their future classrooms to help their students build friendships and build confidence. There is clear evidence that this testing activity provided PSTs with an experience that is tangible and will be easily applied in their own classrooms. In terms of teacher learning, this should signify that much effort and time should be put towards designing more reform-based mathematics activities that sit in the border between school and teacher education courses (i.e., activities can be implemented in teacher education courses and school classrooms).

As gaining extensive experience with concrete classroom activities is promoted in the literature about mathematics teacher education (e.g., Gainsburg, 2012), our findings point to future directions reform-based mathematics activities can play in teacher education courses. Specifically, PSTs experiencing a reform-based mathematics testing activity that was *implemented within* an education course allowed PSTs to *draw out* and learn about conceptual ideas. Some conceptual ideas they learned about are found in the NCTM's (2000, 2014) recommendations for reform-based mathematics. For example, the PSTs began to think about focusing on and using students' mathematical approaches—a strategy that researchers have deemed difficult (Hiebert, 2013). This is evident in PSTs comments about the challenges in using student thinking as a basis for mathematics teaching. They indicated that using multiple students' solutions in classrooms will require them to take risks and feel comfortable with uncertainty. More importantly, some PSTs learned about why they should be using students' ideas even though it proves to be challenging. PSTs learned that mathematics teaching is about connecting students' approaches in meaningful ways and advancing students' reasoning by helping them develop their thinking further through feedback that focuses on the students' approaches. Our study is significant as researchers (e.g., Durkin et al., 2017; Handal & Herrington, 2003) posit that using multiple solution approaches is challenging and require more research about how to support teachers enact the recommendation (Durkin et al., 2017). We offer a solution to how PSTs (or in-service teachers) can learn about and become comfortable with (1) using multiple solutions in their teaching, and (2) knowing they cannot anticipate everything. Implementing reform-based mathematics activities that sit in the border between school and teacher education programs show notable promise.

Additionally, through experiencing a reform-based mathematics testing activity within an education course, PSTs learned about problem posing. Creating and implementing questions that focus on students' mathematical ideas and draw attention to important mathematics relationships is essential (NCTM, 2014) and problem posing is challenging (Silver, 1997). It is quite significant that PSTs in our study appreciated the challenge of crafting the text within a question and thoughtfully choose numbers within the question to have important concepts and number relationships emerge.

Feedback was one of the constructs our study participants signaled to be an area of significant learning. Our findings about feedback should draw much attention because: (1) they extend existing literature about feedback and PSTs, and (2) the findings indicate that the activity promoted PSTs to learn about effective feedback. The existing literature about feedback mostly involves conceptualizations as feedback as corrective (Evans, 2013) or is about feedback provided to PSTs (e.g., Ellis & Loughland, 2017; Schwartz et al., 2018). Our chapter adds to the small body of literature that investigates implementation of feedback as much more than a corrective tool (e.g., Evans, 2013). Our study participants indicated that through experiencing the testing activity they have come to view feedback differently. Participants explained that they learned that feedback can be useful, even if students get the answer correct. Participants acknowledged that there is much more to be gained through feedback that goes past positive comments (such a "good job"). Participants' comments (such as the ones about going back to the curriculum and focusing on student approaches) clearly show that they have come to view effective feedback as feedback that is specific to the mathematics involved. This finding allows us to conclude that the testing activity promotes PSTs to learn about feedback that align with ideas about effective feedback in the literature (e.g., Andrew, 2009; Lavey & Shriki, 2014; Schwartz et al., 2018).

Participants' experiences about dialogue between the parties involved in feedback extends research literature on PSTs and feedback. There is very little research about PSTs learning about feedback practices they can implement in their future classrooms. Indeed, most of the literature is about feedback provided to PSTs (e.g., Schwartz et al., 2018) and not in the service of PSTs learning about providing feedback themselves. PSTs spoke about possible misinterpretations and how they needed to be thoughtful about the beliefs they could be endorsing with students (i.e., the teacher's methods is better or the only ways to solve a problem). It is important to note, at least one PST believed that the activity was the first time she was able to practice giving meaningful feedback. This is a comment that is substantial but not surprising, as there is very little literature about providing PSTs experiences in the practices of feedback.

Collaboration was another theme that PSTs experienced within the testing activity and is within recommendations for reform-based mathematics. Significantly, PSTs in this study moved beyond experiencing the testing activity as students in a teacher education course, and started to envision how collaborations could be beneficial as future mathematics teachers (e.g., planning lessons with other teachers). Some participants went as far as speaking to collaboration and rethinking their perspectives as a teacher by providing clear and direct comments about blurring the line between them and their future students. They wrote about the “flip flop” between the roles and responsibilities of students and teachers. In other words, PSTs saw themselves encouraging their future students to own their learning and gain independence from the teacher (independence and responsibility are words used by the NCTM (2014)—see our section of reform-based mathematics and feedback). PSTs talked about how teachers should be constantly learning (about how students think mathematically and about new pedagogical strategies). PSTs also talked about the need and want to dialogue with the person who wrote the test. They talked about not judging the students’ work too quickly. In this sense, they were conceptualizing ways of being with students in dialogue where they (as teachers) are learners. The notion of teachers learning through dialogue is a complex idea that can be found within the literature about best practice in feedback (Carless et al., 2011). Given that researchers (e.g., Thomas & Sondergeld, 2015) recommend that we should not assume that PSTs can develop ideas about best assessment practices without intentional and thoughtful scaffolds/activities, the testing activity should be of great interest. PSTs comments clearly suggest that reform-based mathematics activities that can be used in school classrooms and teacher education courses are well worth examining and warrants spending time and effort to develop new activities.

Our findings might be considered to be most substantial in regard to having PSTs take mathematics content tests to receive teaching credentials. Having PSTs write traditional mathematics tests is happening and is a core practice (at least in Ontario, Canada). The comments our participants shared about testing and “tricking” should create huge concern and at the same time, offer hope. It should be concerning that PSTs believed (before experiencing the testing activity) that testing is a punishable act and not an opportunity focused on learning. Indeed, the traditional testing of PSTs on mathematics content could be reinforcing these beliefs. In terms of hope, there is clear indication that PSTs in the study changed their beliefs about testing. PSTs described their plan to implement the activity in their own classrooms because of an increase in their confidence, opportunities teachers and students to learn and providing opportunities for students to build friendships. The changes in these beliefs is not surprising as feedback was a major element in testing activity and research informs us that effective implementations of feedback can change beliefs (Nelson & Schunn, 2009). The testing activity can be put in place of traditional testing and change students’ and teachers’ beliefs about testing from one about “tricking” to thinking about testing as an opportunity to learn and help set up students for success.

It is important to keep in mind that the traditional testing of Canadian PSTs is occurring out of concern that they do not have adequate knowledge of elementary school mathematics (Brown, 2016). The testing activity investigated here was originally developed and studied in a mathematics course (Rapke, 2016). The testing activity exhibits ideas from the literature about learning mathematics (e.g., see previous sections on feedback, collaboration, problem posing, focusing on multiple solution strategies). The testing activity also exhibits the idea that students talking about mathematics is very effective for learning (Campbell & Bolyard, 2018). Based on PSTs' comments, they clearly learned through multiple solution strategies, which research has shown to be valuable for the learning of mathematics (Rittle-Johnson and Star 2007).

Furthermore, in the context of a mathematics course, research was conducted to conclude that the testing activity (without the emphasis of feedback that we added in for this research) encourages students to employ deep approaches to learning mathematics (Rapke, 2016). The PSTs in the current study indicated that the testing activity provided them an opportunity to learn mathematics, due in part to the fact that the activity eased their trepidations and had them feel confident about sitting the test. Again, it is important to keep in mind that the test was based on what PSTs and their colleagues should be able to demonstrate as future mathematics teachers (questions that involved mathematical content). PSTs indicated that they prepared as students to sit the test through observing their classmates, i.e., they learned mathematics. Additionally, PSTs (e.g., Juan and Steph) indicated that they had entry points into learning the mathematics on the test by observing their peers and asking good questions. PSTs inherently learned mathematics content through the testing activity because key ideas from reform-based mathematics (e.g., problem posing, comparing and analyzing students' ideas, feedback, and collaboration) were embedded in the testing activity.

10 Conclusion

The reform-based mathematics testing activity described in this chapter allows for instructors of mathematics teacher education courses to simultaneously get away from "do what I say, not as I do" and provokes PSTs to learn about constructs within recommendation of reform-based mathematics. Specifically, the PSTs in our study learned about using multiple students' solutions strategies, crafting purposeful questions, problem posing, feedback, and collaboration—areas that research indicates teachers and students experience difficulty with. Additionally, experiencing the testing activity supported re-conceptualization of testing to be much more than setting students up to fail.

In line with recommendations about teacher education (Grossman et al., 1999), PSTs demonstrated that they learned by moving from the specific/practical (i.e., testing activity) to general/conceptual (e.g., focusing on multiple students' mathematical ideas, crafting questions that advance mathematical reasoning, feedback, and collaboration). Our study participants were able to reflect upon their experiences within the testing activity and *draw out* constructs from reform-based classrooms. Our findings further reinforce calls for mathematics teacher educators to focus on developing and enacting reform-based mathematics activities that can be used in school and teacher education classrooms. These reform-based mathematics activities have the potential to bridge the divide between what is *actually* happening in school classrooms and what *should* be happening based on research and recommendations made in mathematics teacher education courses. The testing activity supports PSTs to re-conceptualize the purpose/affordances of testing and learn about constructs within recommendation of reform-based mathematics, both of which sets PSTs up for effective reform-based mathematics teaching.

If teacher educators, governing bodies, or policy makers feel the need to test PSTs for teaching credential purposes, we strongly recommend that having PSTs develop, sit and assess tests for one another is a thoughtful pathway forward. The testing activity can be used to help PSTs prepare to sit the exam (learn what they need to know as future teachers), and is aligned with reform-based mathematics recommendations that mathematics educators are promoting.

In the end, we do not see the border between teacher education courses and school classrooms as a being a distinct line between each, but rather we see the border more as an overlap. This overlap contains reform-based mathematics activities that can be implemented within both contexts. The testing activity is an example of something that sits on the border; we visualize many more activities, and hope others will direct time and effort to developing and enacting them with us. These activities are in line with general recommendations about reform-based mathematics, and thus can promote PSTs to learn about the constructs within recommendations. At the same time, these activities provide extensive experience with concrete classroom activities that PSTs can use in their future teaching. The true potential of the activities is that they "crack two nuts with one stroke,"

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Teaching the Hungarian Mathematics Pedagogy to American Pre-service Teachers

Péter Juhász, Anna Kiss, Ryota Matsuura, and Réka Szász

1 Introduction

Budapest Semesters in Mathematics Education (BSME) is a study abroad program for American students to cross cultural and geographic borders between the United States and Hungary as part of their journey to become secondary school mathematics teachers.[1] BSME participants spend a semester in Budapest—the capital and cultural center of Hungary—and learn about the Hungarian mathematics pedagogy that emphasizes problem solving, creativity, and communication. They investigate how to bring this pedagogy into their future American classrooms, thus blending good practices from the two countries.

At BSME, American pre-service teachers immerse themselves in Hungarian mathematics education, as the BSME courses are taught by Hungarian instructors who are practicing secondary school teachers. In addition, the pre-service teachers observe K-12 classrooms in Budapest, and design and teach their own Hungarian-style lessons to Hungarian students (in English). Living in Budapest, BSME participants also immerse themselves in a beautiful historical city with one of the most vibrant cultures in Europe. This chapter focuses on the coursework component of the BSME program. For more information about the other aspects of the BSME experience, please see the program website www.bsmeducation.com.

The BSME academic experience is also based on crossing another border—namely, the border between the roles of the student and the teacher. As students, the American participants learn mathematics through the Hungarian pedagogy; as

[1] The phrase "secondary school" includes both high school (grades 9–12) and middle school (grades 6–8).

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teachers, they reflect on the learning experience and consider how to apply the Hungarian pedagogy to their future classrooms.

This chapter is written by the directors and instructors of BSME. We begin by describing the history and principles of the Hungarian mathematics pedagogy. We then elaborate on the dual roles of the student and the teacher played by BSME participants. Next, we examine two particular aspects of the Hungarian pedagogy—learning through games and learning through problem posing. We conclude by describing the impact that the BSME program has had on the participants.

2 Hungarian Mathematics Pedagogy

The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (NCTM, 2000) begins with the following vision for high-quality and engaging mathematics instruction:

> *Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. Alone or in groups and with access to technology, they work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it.* (p. 3)

This vision aligns closely with how mathematics is taught in many Hungarian classrooms, where a strong and explicit focus is placed on problem solving, creativity, and communication. Hungarian students learn concepts by working on mathematically meaningful problems that emphasize procedural fluency, conceptual understanding, logical thinking, and connections between various topics. For every lesson, an overarching goal is to learn what it means to engage in mathematics and to feel the excitement of discovery (Stockton, 2010).

Mathematical problem solving has had a long tradition in Hungary. The renowned Hungarian mathematician George Pólya wrote several books on the topic (1945, 1954, 1962), including his seminal book *How to Solve It* (1945), which delineated problem-solving heuristics that guide students to discover mathematical concepts through their own work.

While Pólya received worldwide acclaim, it was Tamás Varga who introduced Pólya's vision into Hungarian primary and secondary school classrooms (Szendrei, 2007). During the 1970s, Varga led an educational reform effort that brought new curriculum and teacher training methods to Hungary. Some of Varga's principles included an emphasis on problem solving, giving children freedom to argue and make mistakes, differentiation, and viewing (and teaching) mathematics as a whole discipline rather than as separate sub-fields such as algebra and geometry.

In many Hungarian classrooms today, the teacher carefully selects and sequences problems that bring focus and coherence to each lesson and provide students with opportunities to struggle productively towards understanding. These problems do more than help students learn the mathematical topics—the teacher sees them as vehicles for fostering students' reasoning skills, problem solving, and proof writing (Andrews & Hatch, 2001).

Another hallmark of the Hungarian pedagogy is student presentations. After working on problems individually or in small groups, volunteers come to the front of class to share their solutions. Given the non-trivial nature of these problems, students learn to communicate their thinking with clarity and precision (Andrews & Hatch, 2001). When a student gets stuck, others chime in to offer suggestions. The teacher creates a welcoming environment conducive to sharing students' mathematical experiences.

In such classrooms, the teacher's role is that of a motivator and facilitator (Andrews & Hatch, 2001). The teacher provides encouragement as students engage with the task, offers guidance when a student is stuck, and probes when clarification is needed. After student investigation, the teacher summarizes students' findings by highlighting important ideas embedded in the problems. This summary makes sense and is meaningful, because students already have had the experience of playing around with these ideas on their own.

The Hungarian pedagogy also stresses connections between different areas of mathematics. Even at the high school level, there are no self-contained subjects such as algebra and geometry. These content areas are part of an integrated curriculum focusing on multiple mathematical perspectives. For example, students may solve a task using an approach that combines tools from algebra, geometry, and statistics (Szendrei, 2007). The problems that teachers pose guide students to "develop understanding not only of the topic itself but also its interrelationships with other topics" (Andrews & Hatch, 2001, p. 36). Instead of teaching fragmented pieces of mathematics, teachers develop students' understanding of mathematics as a discipline, with an explicit focus on their problem-solving ability (Szendrei, 2007).

3 BSME Academic Experience

An effective way to develop pre-service teacher knowledge is exposing them to new modes of learning and teaching that challenge their own classroom experiences as students (Watson & Mason, 2007). Another method is analyzing tasks from a school setting from the teacher's perspective (Watson & Sullivan, 2008). At BSME, we connect these by (1) providing productive struggle to participants by posing them challenging secondary school-level mathematical tasks; and (2) having the participants analyze the tasks and reflect on this problem-solving experience from the teacher's point of view. Thus, BSME participants cross the border between the roles of students and teachers.

BSME participants are immersed in a learning environment that closely resembles the Hungarian pedagogy. The impact of such experience on pre-service teachers cannot be overstated, since teachers' "professional choices of actions are the manifestation of what they have learned or are learning" (Watson & Mason, 2007, p. 208). We engage them in rigorous mathematical thinking and challenging mathematical tasks to provide them with firsthand understanding of the values of those experiences for learners. The reflection component aims to develop their pedagogical habit of "adapting mathematical tasks so as to enable them to listen to learners and to develop sensitivity to learners' thinking and obstacles to that thinking" (Watson & Mason, 2007, p. 207). Such awareness will help pre-service teachers make "principled choices of tasks and interaction strategies when working with learners" (Watson & Mason, 2007, p. 207).

To see the Hungarian pedagogy "in action," BSME participants engage in weekly observations of K-12 classrooms in Budapest. Thus, they obtain firsthand experience on how the methods learned in their BSME courses are put into practice. The participants also take part in pre- and post-lesson discussions with Hungarian teachers and students. As a capstone project, BSME participants develop and teach their own lessons to Hungarian secondary school students, to put into practice what they learned and to have further opportunities for reflection.

For the coursework component of the BSME program, participants typically take three to four courses in mathematics education. While each course has a different focus—problem solving, learning through games, technology, classroom implementation, for instance—they have a shared goal of engaging participants in the Hungarian approach to learning and teaching, as described in Sect. 2. These courses are taught by Hungarian instructors who are practicing secondary teachers.

Sections 3.1 and 3.2 describe mathematical tasks and reflection, which form the foundation of the BSME coursework. Sections 4 and 5 delve into two BSME courses and concretely illustrate how each course provides students with opportunities to engage with challenging mathematical tasks (as well as create their own) and to reflect on that experience.

3.1 Mathematical Tasks

BSME participants work on mathematical tasks in which the main challenge is the thinking involved rather than the content knowledge needed. Thus, although these tasks are grounded in secondary school mathematics, the pre-service teachers find them challenging and interesting. Examples of such tasks are given in Sects. 4.1 and 5.2.

For pre-service teachers, benefits of grappling with challenging mathematical tasks include the following:

- Learning how to think like mathematicians—they improve in problem solving, experimenting, problem posing, definition making, and communication.

- Developing and refining their view of teaching—this occurs as they experience a pedagogical approach that likely differs from what they have seen in their own education (Liljedahl et al., 2009).

Cuoco, Goldenberg, and Mark (1996) elaborate on the above notion of thinking like mathematicians. They introduce *mathematical habits of mind*—i.e., the thinking, creation methods, and research techniques that mathematicians employ. Habits such as exploring, conjecturing, making connections, and posing questions form "a repertoire of general heuristics and approaches that can be applied in many different situations" (p. 387) and allow students to "experience what goes on behind the study door before new results are polished and presented" (p. 376). Bass (2011) also discusses the value of providing students with "authentic experience of *doing* mathematics" (p. 3) that helps them develop the habits of mind that are essential in generating new mathematical understanding.

The mathematical tasks we pose to pre-service teachers have the following properties. First, they are "low threshold, high ceiling" in nature, i.e., accessible without much prerequisite knowledge, but offering possibilities for rich exploration; this is important, since we want pre-service teachers to understand that all students can have authentic mathematical experiences. Second, the tasks have multiple entry points, or different ways in which they can be approached. Third, the tasks have complexity and structure that require students to persevere in solving them and to reflect on their strategies. Fourth, in some cases, a task is part of a "thread" of tasks—this notion of task thread will be discussed later in this chapter.

In BSME courses, participants also develop their own tasks that are geared toward secondary school students. Engaging with challenging tasks provide participants with firsthand experience in learning through the Hungarian pedagogy. However, BSME participants are pre-service teachers. Thus, it is essential that they learn to design their own tasks, so they are well equipped to provide similar experiences to their future students. Sample tasks developed by the participants are described in Sects. 4.2 and 5.3.

3.2 Reflection

Reflection is an essential component of any learning experience (Mason & Johnston-Wilder, 2006). It is particularly important for pre-service teachers, who are learning about the learning process itself (Cooney, 1999).

> Through the constant shifting between student and teacher they are given the opportunity … to recast their initial (preconceived) beliefs about what it means to be a teacher, what it means to teach, what it means to learn, and even what it means for something to be mathematics. (Liljedahl et al., 2009, p. 29)

BSME participants play dual roles of doing mathematics as students and reflecting on that experience as teachers. Since they use authentic tasks from the Hungarian secondary curriculum, reflection takes on particularly powerful meaning for these pre-service teachers as:

- They reflect on mathematical content and experience: big underlying ideas, different solution approaches, difficulties faced.
- They analyze the pedagogical context of the task: target student age, prerequisite knowledge, common errors, follow-up activities, curricular connections.
- They reflect on pedagogical approaches used.
- They consider possible adaptations for different groups of students by modifying the task in content, difficulty levels, and instructional methods.

Through this reflection process, BSME participants internalize their own learning experience and begin to develop the mathematical and pedagogical knowledge needed to design their own tasks. We thus envision American pre-service teachers to return from BSME and implement the Hungarian pedagogy with their own future students.

4 *Concept Building Through Games and Manipulatives*—Learning Through Games

Games and manipulatives are effective tools to arouse students' curiosity while engaging them in powerful mathematical exploration (Malone, 1981), and they play an important role in Hungarian mathematics education. Tamás Varga, in his educational reform in the 1970s, placed games and manipulatives at the core of the middle school curriculum (Szendrei, 2007). Zoltán Dienes, the creator of base 10 blocks, used games, manipulatives, stories, and even dance, always starting from the specific students' culture and interests (Dienes, 1973). Dienes stresses free play as an effective means to introduce students to a new mathematical concept (Hirstein, 2007). A popular course at BSME is *Concept Building Through Games and Manipulatives*, which explores how various areas of secondary mathematics are developed as students play with fun mathematical games and hands-on manipulatives that maintain the mathematical integrity and rigor of the underlying ideas (Matsuura & Szász, 2015). As in other BSME courses, participants play the dual roles of the student and teacher. As students, they enjoy playing the various games and learn mathematical concepts from them. As teachers, they reflect on how games can be used to enhance secondary students' mathematics learning. Pre-service teachers acquire a deeper understanding of how mathematical concepts are interwoven into these games. They learn how to guide the students to uncover these concepts through their own playing and exploration. This includes learning about common student misconceptions, and concrete strategies on how to help students work through their errors on their own. As a culminating project, pre-service teachers design their own mathematical game and a corresponding lesson.

4.1 Sample Game: Functions Play Tag

We illustrate the Hungarian approach of learning through games with a sample task used in the BSME course *Concept Building Through Games and Manipulatives*. This task was originally designed by a Hungarian educator Éva Szeredi who taught middle school in Budapest. Modified versions of the task are used through grades 5–9. In this section, we first describe the game, and then the reflection questions posed to the BSME participants.

Mathematical Task
Two students act as "calculators". One receives a card with $2x - 1$, and the other with $x - 3$. Only the two calculators know these expressions. The following table is drawn on the board:

x								
Name 1								
Name 2								

Other students suggest values of x. The calculators compute the corresponding values of their expressions and record them on the table. The winner is the student who suggests a number x at which the two expressions yield the same result. The mathematical aims of the game include the following:

- Finding a number where the two expressions are equal.
- Finding *all* numbers where the two expressions are equal.
- Finding the expressions that the calculators are using.

Students then repeat the process with more difficult pairs of expressions, e.g., $\text{abs}(x - 1)$ and $\text{sgn}(x) + 6$.

Reflection
After playing the game, the instructor poses the following reflection questions:

- *For what age group would each pair of expressions be appropriate?*
- *Besides choosing different pairs of expressions, how could the game and its aims be modified to suit different age groups?*
- *To what mathematical topics is the game connected?*

Discussing these prompts allows BSME participants to share their experience with the game as students, and analyze and reflect on it as teachers. The instructor shares her own experiences with the game as a practicing middle school teacher, and provides insights on teaching the underlying mathematical topics—e.g., functions, function transformations, linear systems, and mental mathematics.

Participants also reflect on the more general process of learning and teaching through games. For example, they discuss features of games that not only make them fun, but also maintain the mathematical rigor and integrity of the concepts that the teacher wishes to convey. Such discussion is particularly useful when the pre-service teachers begin to design their own games.

4.2 Sample Game Designed by a BSME Participant: Slopes, Slopes, Slopes

In addition to experiencing existing games used in Hungarian mathematics classrooms, BSME participants develop their own mathematics games. In this section, we describe a game developed by a pre-service teacher taking the BSME course *Concept Building Through Games and Manipulatives*. Designed for middle and early high school students, its mathematical objective is to help students make and deepen connections between linear equations, their graphs, the slope, and students' prior knowledge about plotting points. We conclude the section with an excerpt of a reflection by the participant who designed this game.

Materials Needed

- Game board (Fig. 1), showing the first quadrant and containing seven lines through the origin with slopes 1/3, 1/2, 2/3, 1, 3/2, 2, and 3.
- Seven slope cards corresponding to the seven lines on the game board. The color on the back of each slope card matches the color of the corresponding line. A slope card is worth one or two points as indicated on the card.
- Red and blue chips, acting as points on the coordinate plane. A chip is worth one point.
- Linear equation card $ay - bx = 0$.

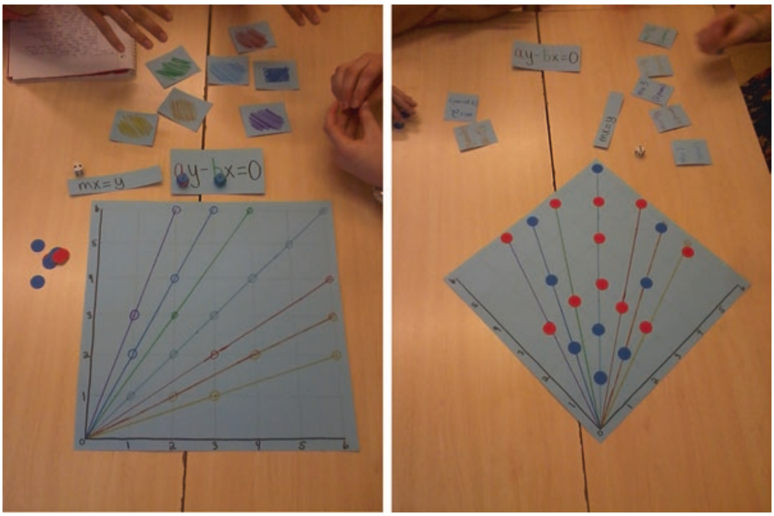

Fig. 1 Game board at the beginning and end of game

- Red and green dice, whose numbers become the values of the coefficients a and b, respectively.
- Regular (white) die.

Game Setup and Overview

- This is a two-person game. Each person chooses his/her own set of chips (red or blue).
- Place the slope cards face down, so the slope values are *not* visible to the players.
- A player can earn points by earning a slope card or placing a chip on the game board.
- The first person to earn 20 points wins the game.

How to Play

1. The first player (indicated by "he") rolls the red and green dice. He substitutes the values of the dice into the linear equation card, and he calculates the slope of that line (e.g., the red and green dice show 4 and 6, respectively; thus the linear equation would be $4y - 6x = 0$ with slope 3/2). On the game board, the player identifies the line that is the graph of his equation. Then he turns over the slope card with the same color as his line. He earns that card if he had correctly calculated the slope and that slope matches the value on the slope card. Otherwise, the player loses his turn.
2. The second player (indicated by "she") proceeds similarly. If after rolling the red and green dice, the second player finds a slope already owned by the first player (e.g., she rolls 2 and 3, which yields the equation $2y - 3x = 0$ with slope 3/2), then she loses her turn. The player also loses her turn if she obtains a line that does not exist on the game board (e.g., $5y - 4x = 0$).
3. Once a player earns one or more slope cards, he may choose to earn points on his line. (He may also choose to earn more slope cards, as described in Step 1.) To do this, the player selects one of his slope cards with slope m. Then he rolls a white die and uses the number as the x value in $y = mx$. If the corresponding point (x, y) is on the game board, then he places a chip on it. Example: Suppose the first player selects his card $m = 3/2$ and rolls 4 on the white die. He obtains the point (4, 6), so he can place a chip there. If such point does not exist or already has a chip on it, the player loses his turn.
4. If a player has all her lines filled with chips, she may choose another person's line and "steal" his points. To do this, the "thief" selects a line owned by her opponent. She then rolls the white die as in Step 3. If the roll yields an empty point, she places her chip there. If the point has the opponent's chip on it, she may replace that chip with her own.

Reflection

After designing the game and implementing it with their BSME classmates, participants reflect on this experience. These reflections (discussed verbally, written, or both) are a required part of all BSME courses. The following excerpt was written by the participant who developed the game "Slopes, slopes, slopes". She discusses possible adaptations to the game and their potential impact on the learning of those playing the game. Such reflection strengthens not only her understanding of this particular game, but also her capacity to design more games in the future.

> *I really enjoyed how my game went. I think there are so many adaptations that could be made, but I think it was successful as is. [Another participant] suggested that I use negative numbers and look into all four quadrants. I think this really complicates the game, and I was doing it more for an introductory level practice, but this would make it more difficult. I like having it at 20 points because it forces students to try for the slope values. I do like the idea of having less points on the m = 1 line to decrease the probability of rolling for a point on that line. Overall, I don't think it took much longer than I was expecting and I think the [other participants] enjoyed [playing] it. If I wanted to focus more on slopes, I could create a larger board so I could include more slopes, which would mean the students would have to calculate more times.* (**Authors' note:** By using dice, the participant included a luck factor in hopes of making the game enjoyable for struggling students who may not be as adept at finding winning strategies.)

4.3 Pre-service Teachers' Development in Designing Mathematical Games

The instructor of the BSME course *Concept Building Through Games and Manipulatives* felt that pre-service teachers initially struggled in two main areas:

- Their conceptual understanding and visual imagery of mathematical ideas were missing, false, or not strong enough despite their theoretical knowledge of the ideas.
- Without plenty of support, they could not design games or manipulatives that would foster students' mathematical understanding.

The instructor found that by having participants grapple with activities from secondary schools that contain manipulation and visualization (e.g., Functions play tag), they acquired vivid mental images of concepts, and their problem solving and posing skills grew considerably. For example, participants improved in their articulation of the underlying mathematical purpose of each game; they also learned to pose questions that were more strongly connected to the intended mathematical ideas.

BSME participants designed their own activities and taught from these activities in micro-teaching lessons with their peers. Through engaging with and reflecting on activities incorporating games and manipulatives, they became more proficient in designing such activities themselves—they became more creative, they could better design activities which illuminate and deepen concepts, and they learned to allow students to discover mathematical ideas on their own.

5 *Discovery Learning: The Pósa Method*—Learning Through Problem Posing

Problem posing is an integral component in the work of teaching, since teachers are responsible for "planning problems that will give students the opportunity to learn important content through their explorations" (NCTM, 2000, p. 341). Furthermore, the Hungarian mathematics pedagogy emphasizes problem posing as being crucial for *students* (Pólya, 1954). We not only want students to be able to solve problems, but also have the curiosity and know-how to pose good questions as mathematicians do. Through problem posing, students become used to thinking about problems without knowing or being afraid of how difficult they are; they often learn about and are fascinated by the existence of unsolved problems in mathematics.

Problem posing enhances students' problem solving (Hashimoto, 1987; Perez, 1986; Silver & Cai, 1996), creativity (Van Harpen & Sriraman, 2013), personal understanding (Brown & Walter, 2005), self-confidence (Mason, 2010), and attitudes toward mathematics (Winograd, 1991). Moreover, the NCTM (1989) recommends that students "have some experience recognizing and formulating their own problems, an activity that is at the heart of doing mathematics" (p. 138). To provide students with *authentic* mathematical experiences—to help them learn to think like mathematicians—problem posing should be an integral part of their learning.

BSME participants learn about problem posing in the course *Discovery Learning: The Pósa Method* (Matsuura & Szász, 2015). This course introduces a method of teaching developed by a Hungarian mathematician and educator Lajos Pósa, well known in Hungary for his mathematics camps for secondary students. Pósa developed a particular style of teaching mathematics in which students discover mathematical concepts by working on tasks that build on each other. The method was originally developed for gifted students in Pósa's mathematics camps, but it also has been successfully implemented in more general school settings (Győri & Juhász, 2017).

In the *Discovery Learning* course, BSME participants again play the dual roles of the student and the teacher. As students, they grapple with tasks from Pósa's mathematics camps—while geared for secondary school students, we have found these tasks to be interesting and challenging for pre-service teachers, too. As teachers, participants reflect on this learning experience, and discuss the ways in which Pósa's principles can be applied to their own teaching.

5.1 Pósa's Method of Teaching

Below, we provide further information on Pósa's method of teaching. Its main goal is the development of students' mathematical knowledge through their own work and discovery. Pósa's teaching relies on *task threads*, well-crafted series of mathematical tasks that build on each other, gradually guiding students toward desired understanding (Győri & Juhász, 2017). The conceptual leap between successive tasks is adjusted to meet students' needs. Pósa compares this set of leaps with a staircase (to the mathematical goal), where steeper steps may be used with stronger students and smaller, more frequent steps may be required for struggling students (Stockton, 2010).

Tasks can take different forms: solving problems, posing questions, making up or agreeing on definitions, playing games, experimenting, just to name a few. When students work on a task thread, they are aware that the tasks build on each other; thus, they need to use knowledge developed in the previous task for the next one. To avoid this kind of explicit help, students are given tasks within the same thread with some time apart—a task thread can span across months or even years—so they often do not realize which tasks build on each other. There are multiple task threads running simultaneously, so students are often working on several tasks belonging to different task threads; this also helps the teacher in navigating student differences.

Constant readaptation is crucial. When planning, the teacher adapts the tasks based on his former experiences with the tasks and on his knowledge of the current students. While teaching, he needs to adjust, as needed, according to students' responses and understanding (Győri & Juhász, 2017).

5.2 Sample Task Thread: Partitioning Points on the Plane

The following task thread comes from the BSME course *Discovery Learning: The Pósa Method*. It consists of five parts and is used with Hungarian gifted students in grades 8–10 over five different lessons, with time between lessons (anywhere from 1 week to 2 months). At BSME, the thread is assigned to the participants over five different class sessions. We describe each part below, along with reflection questions posed to the pre-service teachers. These questions were posed to the participants immediately after completing the part of the mathematical task. They reflected on the questions verbally, facilitated by the instructor. Some questions specifically addressed the task at hand (e.g., "What are the main mathematical ideas in this problem?"). Other questions pertained more generally to the process of teaching and learning through tasks (e.g., "How can teachers use students' incorrect answer to help students learn?"). Again, reflecting on and discussing these questions help them become better equipped to design their own task threads.

In addition to helping BSME participants become more familiar with Pósa's pedagogy, the task thread below also emphasizes problem posing. Ability to pose questions may be developed by reflecting on problem-posing heuristics. The questions in this task thread involve two important techniques. One is examining examples given in a problem and finding common properties (e.g., none of the examples are bounded); then we ask, "Is this property necessary? Why?" This heuristic helps in exploring a topic deeply. Another technique is considering analogues or related concepts (e.g., we start with points of symmetry; then we ask about *axes* of symmetry). This heuristic may require subtle intuition, and sometimes results in the posing of difficult questions.

Mathematical Task: Part 1

BSME participants are given the following problem, on which they work individually or in pairs, while the instructor provides feedback: *100 distinct points are on a plane. Does there always exist a line so that each of the two half-planes defined by the line contain exactly 50 points?*

Reflection Questions

- *What are the main mathematical ideas in this problem?*
- *In what settings could the problem be used with students?*

Mathematical Task: Part 2

The instructor asks the pre-service teachers: *What follow-up questions to the original problem could you pose?* Below are some of the questions which the participants developed:

1. Instead of 100 points, what if there are *n* points (with *n* even)?
2. With 100 points, do there always exist two intersecting lines so that each of the four parts of the plane defined by those two lines contains exactly 25 points?
3. Same as Question 2, but with two *perpendicular* lines.

The instructor then suggests his own follow-up questions:

4. 100 points are on the plane. Does there always exist a circle so that each of the two parts of the plane defined by the circle contains exactly 50 points?
5. Infinitely many points are given on the plane. Does there always exist a line so that each of the two half-planes defined by the line contains infinitely many points?

Participants solve some of these questions in class, each working at their own pace. Some remaining questions are assigned as homework, and some as optional challenges.

Reflection Questions

- *What are some aims of problemposingin a mathematics classroom?*
- *What are some appropriate settings for a problem-posing task?*

Mathematical Task: Part 3

Participants are encouraged the pose more follow-up questions, and they come up with the two below. With Pósa's method, the teacher must *anticipate* the questions that students might pose, and determine in advance whether or not those questions belong in the task thread. Having pre-service teachers pose their own follow-up questions not only deepens their understanding of this task thread, but also fosters their ability to anticipate student questions.

6. Infinitely many points are given on the plane and no three of them are collinear. Does there always exist a line so that each of the two half-planes defined by the line contains infinitely many points?

7. Uncountably many points are given on the plane. Does there always exist a line so that each of the two half-planes defined by the line contains uncountably many points?

Reflection Questions

- *What were some typical incorrect answers that the members of our class gave? What are the misconceptions behind these answers?*
- *How can teachers use students' incorrect answers to help students learn?*

Mathematical Task: Part 4

Participants are again asked to pose a follow-up question. They try to add the restriction that the point-set is bounded. Instead of using the term "bounded," they use imprecise language like *it should be finite* and *it shouldn't go out to infinity*. The instructor facilitates a discussion where the participants develop their own definition of "bounded" and pose this question:

8. A bounded point-set with infinitely many points are given on the plane. Does there always exist a line so that each of the two half-planes defined by the line contains infinitely many points?

Reflection Questions

- *What is the value of having students develop their own mathematical definitions?*
- *Think of other settings in the secondary curriculum that are conducive to having students develop their own definitions.*

Mathematical Task: Part 5

Participants pose a follow-up question once more:

9. A bounded point-set with infinitely many points are given on the plane and no three of the points are collinear. Does there always exist a line so that each of the two half-planes defined by the line contains infinitely many points?

Reflection Questions

In contrast to the previous parts during which participants examined the individual parts separately, in this part, participants consider the set of problems in the task thread as a whole. They first collect and examine all the problems in the thread, including those that they posed themselves. Then they reflect on the following questions.

- *What does it mean for a set of tasks to form a "thread"? What is the value of posing such threads to secondary students?*
- *How do the problems that were posed in this task threadbuild on each other? What happens if students pose different questions, or questions in a different order?*
- *What makes a good follow up question? How can teachers help students take the concept to the "next level" by asking such follow-up questions?*

5.3 *Task Thread Designed by a Participant: Sets and Subsets*

We give an excerpt of a task thread developed by a participant taking the BSME course *Discovery Learning: The Pósa Method*. Designed for mid to late high school students, the mathematical objective of the thread is to further students' understanding of the notion of subsets. We also provide an excerpt of a reflection by this participant.

Overview of the Task Thread
These problems have a common underlying theme of sets and subsets. In each problem, students consider a set of objects (e.g., cards, paths, numbers) and analyze properties of its subsets.

The difficulty of the problems generally increases from one problem to the next. Since many ideas and solution approaches reappear throughout the thread, the difficulty of these problems may depend more on each student's comfort level with the sets of objects in the problems, rather than the actual questions being asked.

Excerpt of Tasks:

Problem 2 A magician has 100 cards numbered 1–100. He puts them into three boxes, a red one, a white one, and a blue one, so that each box contains at least two cards. An audience member chooses a box, takes two cards from that box, and announces the sum of the numbers of the chosen cards. Given this sum, the magician identifies the box from which the cards were chosen. Find a distribution of the cards that makes this "magic" happen.

Solution 2 The following is one of several approaches to this problem: place the first 33 numbers in the first box, the second 33 numbers in the second box, and the last 34 numbers in the third box. If two numbers from the first box are chosen, the sum is at most 65. If two numbers from the second box are chosen, the sum is at least 69 and at most 131. If two numbers from the last box are chosen, the sum is at least 135. Since these sums do not overlap, we can distinguish between the boxes.

Problem 3 Now suppose we have an infinite number of cards, one for each positive integer. The magician puts them into three boxes so that each box contains infinitely many cards.

(a) If an audience member selects two cards from one of the three boxes, is there a distribution that allows the magician to determine the box from which the cards came?
(b) What if three cards are chosen?

Solution 3a Separate the numbers into the three equivalence classes mod 3, and let each box contain an equivalence class. Let the first, second, and third boxes contain the numbers congruent to 1, the numbers congruent to 2, and the numbers congruent to 0, respectively. The sum of two numbers from the first box is congruent to 2, the sum of two numbers from the second box is congruent to 1, and the sum of two numbers from the third box is congruent to 0.

Comment 3a[2] The problem has become much more difficult, as we have transitioned from finitely many objects to infinitely many objects. It is not expected that the students are familiar with modular arithmetic, so a hint will need to be posed, explained using the idea of remainders or considering numbers of the form $3k$, $3k + 1$, and $3k + 2$.

Problem 4 Consider the map below (Fig. 2).

How many routes are there from A to B so that:

(a) the sum of the numbers along the route is even?
(b) the sum of the numbers is divisible by 3?

Solution 4a Observe that the number of routes from A to B is the same as the number of routes from B to A. Consider the leftmost section of paths with 3 and 10. Regardless of the path we take in the other three sections, we can take the 3-path if the sum so far is odd and the 10-path if the sum so far is even—then our entire path will have an even sum. Since each of the paths through the other three sections demands a particular path through the leftmost section, only half of the total paths yield an even sum. Thus, there are $\frac{1}{2} \bullet 2 \bullet 3 \bullet 4 \bullet 5 = 60$ paths.

Solution 4b The second section with 1, 2, and 9 has one path for each equivalence class mod 3. Thus, regardless of our path through the other three sections, there is a unique path through the second section that makes the sum divisible by 3. Thus there are $\frac{1}{3} \bullet 2 \bullet 3 \bullet 4 \bullet 5 = 40$ paths.

Comment 4a And 4b We change gears back to a set of objects slightly more tangible—paths! This will hopefully be a welcome relief to students who benefit more from visualization. When discussing these problems, be sure to note the modular arithmetic connection between solutions to this problem and the previous one.

Problem 5 Let $H = \{1, 2, 3, \ldots, 10\}$.

(a) How many subsets of H are there?
(b) How many subsets contain 1 and 2?
(c) How many subsets contain an even number of elements?
(d) How many subsets have a property that the sum of its elements is a multiple of 4?

Solution 5a To construct a subset of H, we decide whether each element of H is in the subset or not in the subset. Thus, there are 2^{10} subsets.

[2] These are comments written by the participant who designed the task thread.

Solution 5b Since we know 1 and 2 are in the subset, we must decide whether or not each of the remaining eight elements in H is in the subset. Thus, there are 2^8 subsets.

Comment 5a And 5b We return to sets of numbers. Problems 5a and 5b ensure that students have a good tool to count numbers of subsets, since counting subsets may be more challenging than counting paths.

Solution 5c There are 2^9 subsets of $G = \{1, 2, 3, \ldots, 9\}$. If a subset of G has odd cardinality, then include 10 to create a subset of H with an even cardinality. Therefore, there are at least 2^9 subsets of H with even cardinality. However, we can make the same argument to conclude that there are at least 2^9 subsets of H with odd cardinality. Since $2^9 + 2^9 = 2^{10}$, there are exactly 2^9 subsets of H with even cardinality (and the same number with odd cardinality).

Solution 5d There are 2^8 subsets of $\{3, 4, 5, \ldots, 10\}$. We can include either 1, 2, 1 & 2, or neither of the numbers to make sure that the sum of the elements is a multiple of 4. Thus, there are 2^8 subsets of H with this property.

Comment 5c and 5d We again see modular arithmetic. Problem 5d can be challenging, since it is the first instance in which two numbers together are used to build an equivalence class.

Reflection

In the following excerpt, the participant reflects on the various mathematical themes embedded in this task thread. Articulating the purpose of a task thread helps students strengthen their understanding of how the different problems are connected to form a coherent whole.

> *The purpose of this thread is to teach students how to work with subsets. The tasks require students to construct subsets with desired properties, check that these subsets are maximal, and examine why certain types of subsets can or cannot exist. Other key themes appear in this thread. First is modular arithmetic, which shows up in Problems 3,4, and 5. Second is the notion of a proof—e.g., students learn to prove that certain constructions cannot exist or cannot be larger than they already are.* (**Authors' note:** The various counting strategies that were employed, particularly in Problems 4 and 5, also form a theme.)

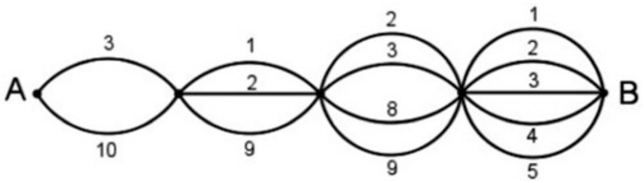

Fig. 2 Map for Problem 4

5.4 Pre-service Teachers' Development in Designing Task Threads

The instructor of the BSME course *Discovery Learning: The Pósa Method*, felt that pre-service teachers initially had difficulties in these areas:

- Identifying the main ideas of problems.
- Understanding how different problems connect to and build on each other.

Through experiencing and reflecting on the process of discovery learning with the Pósa method, they grew in both areas. Based on the formative evaluation of their problem posing and task design, as well as their reflections on those experiences, the instructor noted that the participants became more proficient at developing threads of tasks that build on each other and have similar underlying ideas across different mathematical topics.

As an example of this growth, consider the task thread designed by a participant, described in Sect. 5.3. This thread contains problems about cards, paths, and numbers. While the problems deal with these varying objects and ask different types of questions about them, they all have a common underlying theme of sets and subsets. The problems also increase in difficulty and introduce new solutions approaches as needed—for example, after working with 100 cards in Problem 2, students are asked about infinitely many cards in Problem 3, which requires the use of modular arithmetic. This use of modular arithmetic reappears in later problems, allowing the students to apply the insights gained while working on Problem 3.

6 Summary and Conclusion

In this chapter, we described an approach to pre-service teacher education founded on crossing two types of borders:

- Cultural and geographic borders between the United States and Hungary.
- The border between the roles of the student and the teacher.

At the BSME program, participants (American pre-service teachers) immerse themselves in Hungarian mathematics education. The participants learn about the Hungarian mathematics pedagogy—that emphasizes problem solving, creativity, and communication—from BSME instructors, who are practicing Hungarian secondary school teachers. While the focus of this chapter was on the coursework component of the program, BSME participants make weekly observation visits of K-12 classrooms in Budapest, where they also engage in discussions with Hungarian teachers and students; and they develop and teach their own lessons to Hungarian secondary school students as a capstone project. It is, indeed, an immersion experience in the educational and mathematics culture of Hungary.

In the chapter, we examined two particular aspects of the Hungarian pedagogy—learning through games and learning through problem posing. Moreover, BSME participants play dual roles: as students grappling with the mathematical tasks, and as teachers reflecting on their learning experience. Pre-service teachers engaging in this approach and instructors facilitating their learning experience all found that participants' teacher knowledge developed extensively—not only did their own view of mathematics develop, but through designing and implementing their own lessons, they also successfully put into practice the teaching methods that they learned.

As (informal) evidence for the value of the BSME approach, we present sample feedback from participants. The feedback was collected at the end of the program, anonymously and in written form. The feedback speaks to mathematical tasks and reflection—the foundation of the BSME coursework—and the two courses described in Sects. 4 and 5.

- **On Mathematical Tasks**: *The way in which problems were put together to create a cohesive narrative of a problem thread in order to guide students' (and our) learning was masterful. The opportunity and experience of being able to craft a problem thread on our own was both a fun experience and one where we could really begin to envision how to put this method into action.*
- **On Reflection**: *I feel that the content mastery attained by the [BSME] students was due in a large part to this fluidity of responsibility [of engaging in tasks as students and reflecting on the experience as teachers].*
- **On *Concept Building Through Games and Manipulatives***: *Being able to participate in the activities as a student was one of my favorite parts of this course. Even if I don't remember the particular details of many activities, I know I am now better at creating activities in this style.*
- **On *Discovery Learning: The Pósa Method***: *This class was very hands-on and we were made to do active learning, which is much better, as well as somewhat think about what it means and takes to be a good math teacher by posing interesting problems and having a map of mathematics in one's head.*

Findings from the Third International Mathematics and Science Study (TIMSS) 1999 Video Study suggest that American teachers should place more emphasis on fostering students' conceptual understanding by providing them with opportunities for "solving challenging problems and discussing the relationships that can be constructed among the mathematical facts, procedures, and ideas" (Hiebert & Stigler, 2004, p. 13). However, Hiebert and Stigler (2004) remind us that teaching is very difficult to change, because "most teachers learn to teach by growing up in a culture, watching their own teachers teach, and then adapting these methods for their own practice" (p. 13).

While BSME participants do not grow up in Hungary, the impact of their crossing the cultural and geographical borders cannot be overstated. These American pre-service teachers immerse themselves in Hungarian mathematics education. They learn about mathematics and about teaching from Hungarian educators. The participants reflect on this learning experience and explore how to adapt the

Hungarian pedagogy—which aims to develop students' conceptual understanding through problem solving—for their future classrooms. We believe these are significant steps in changing the way these pre-service teachers think about teaching.

Pre-service teachers come to BSME eager to learn and experience a new culture. The first thing that often happens is the realization that their assumptions about mathematics and teaching may not necessarily be true. They see mathematics they "know" in a new light—adding depth to that particular content but also adding richness to the secondary curriculum as a whole. This business of secondary school mathematics is deeper than they suspected, and the practice of teaching is richer and more complex than they could have imagined before they start the program.

How these experiences translate into their teaching practice when they return to the United States is still an open question. BSME is a fairly new program, and we are only beginning to learn about the impact on participants' practice. For example, a former participant, now a high school teacher in New York, commented that BSME gave him the confidence and know-how to incorporate manipulatives into his teaching and that he has been able to bring some of the Hungarian pedagogy into his classroom (Matsuura & Szász, 2017). We look forward to further investigating BSME's impact on pre-service teachers' practice, and we anticipate a real opportunity for learning in the field of teacher preparation.

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Mathematics Education Communities: Crossing Virtual Boundaries

Joshua T. Hertel, Nicole M. Wessman-Enzinger, and Justin K. Dimmel

The growth of social media has yielded a range of virtual communities focused on issues related to education (Carpenter & Krutka, 2014; Hur & Brush, 2009). These communities, which operate across a range of different platforms, create an evolving landscape for users to navigate. Moreover, interactions within and across virtual communities has become a norm within society at large as well as within mathematics education. The Math Twitter Blog-o-Sphere (MTBoS), Mathematics Stack Exchange, specialized Facebook groups, and myNCTM are just a few examples of communities that are currently popular with mathematics teachers and educators in North America. Similarly, students of mathematics use virtual communities to make records of information that, in earlier times, would have been available through more informal channels. For example, solution clearinghouse sites (e.g., Chegg.com) allow students to request or post answers to problems sets and teacher-rating sites (e.g., RateMyProfessor.com) offer a platform where students can trade information about their instructors.

With the ubiquity of internet-enabled devices, negotiating virtual communities has become a norm within mathematics teaching and learning. Consequently, educators, both new and old, who participate in these communities are encountering issues and ideas that they likely have limited experience with. This raises a number

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of questions for mathematics teacher educators seeking to help themselves, preservice teachers (PSTs), and current teachers understand these virtual communities. For example: How can the differences, similarities, and affordances of communities be highlighted? How can the boundaries that define and separate these communities be made clear? Within this chapter, we seek to address these and related questions by providing a framework for understanding these communities. We then use this framework to examine several communities currently popular within North America.

1 Framing Our Discussion

In establishing a foundation for our discussion of virtual communities, we draw upon two different areas of theory: communities of practice and boundary crossing. As we will discuss, these different theoretical perspectives provide a lens for understanding virtual communities as well as tools for understanding the interaction of users and platforms.

1.1 Virtual Communities of Practice

Virtual communities have flourished as social media platforms have become integral to people's daily lives. We draw a distinction between platforms—the websites that facilitate interactions among groups of users—and the virtual communities[1] that reside on such platforms. Virtual communities, as noted by Ellis et al. (2004), have different features than traditional social networks:

> Virtual communities are both narrow and specialized, in terms of the information posted, but at the same time broadly social and supportive. Consistent evidence suggests that many individuals go to virtual communities because of these social and supportive characteristics: the many weak ties supported by [a] virtual community provide access to a much wider network of people than conventional, social networks. The potential for invisibility regarding normal social cues such as gender, race, class, and age opens up the potential for networking and interaction that may be inhibited elsewhere. (p. 148).

We conceptualize these virtual communities as types of *communities of practice*[2] (Lave & Wenger, 1991; Wenger, 1998; Wenger-Trayner & Wenger-Trayner, 2015).

[1] We recognize the concept of virtual communities is itself an area of some discussion and debate (Ellis, Oldridge, & Vasconcelos, 2004). In particular, virtual communities may be re-creations of existing communities, members of virtual communities may also work within in physical communities, and the interaction between virtual and physical communities is complex.

[2] We acknowledge there are inherent differences between physical and virtual communities of practice. For example, members of virtual communities of practice may never meet face-to-face, share information via artifacts, and may be limited in communication by the context of the platform. Additionally, the situated, co-constructed nature of learning may be different within a virtual community of practice from within a physical community of practice (Ellis et al., 2004).

Within this chapter, we concentrate on virtual communities of practice that are focused on the teaching and learning of mathematics and refer to these groups as virtual mathematics education communities (vMECs), which we contrast with real-world mathematics education communities of practice (rMECs). Although we regard mathematics education communities of practice differently, membership within one community does not preclude membership in another. Moreover, given the interconnected nature of education, we expect that educators are members of multiple mathematics education communities with some being real-world and others virtual. However, one's role or status within a community does not necessarily transfer to other contexts. For example, an influential old-timer within an rMEC may be seen as a relative newcomer within a vMEC. Likewise, a well-known member of a vMEC might be on the periphery within an rMEC.

Our goal in focusing on vMECs is to mold a lens for understanding these communities and provide tools to compare vMECs to rMECs be they practitioner focused, scholarly focused, or otherwise. Drawing on Wenger-Trayner and Wenger-Trayner's (2015) discussion of communities of practice, we consider the three essential characteristics of a vMEC: (1) the community has a shared domain; (2) the community is constituted by engaged participants; and (3) the community is focused on practice.

Shared Domain Broadly, the shared domain of a community of practice is the field, subject matter, and scope of practice around which members cohere. Focusing on vMECs, the breadth of the subject matter is large whereas the scope of the practice is more narrow and centered around actions and routines relevant to the teaching and learning of mathematics. To an outsider, the shared domain of any particular vMEC may be apparent as the discussions, interactions, resources, etc., are related to the teaching and learning of mathematics. However, larger vMECs may be further separated into smaller communities of practice each having a more narrow focus (e.g., mathematics content at the elementary level, the professional development of mathematics teachers). Thus, collectively vMECs make up a larger group that is different from other virtual communities of practice and, at the same time, may form smaller communities that are distinct from one another.

Engaged Participants The people who constitute a vMEC share information and help each other. Members interact, build relationships, collaborate, and discuss ideas from the shared domain. In this way, the shared domain works as an organizational tool and creates a boundary around the content that is discussed. How engagement occurs, however, is influenced by the structure of the platform on which the community resides. In particular, the tools provided by a platform can influence the discourse and interaction that occurs.

For example, the Mathematics Stack Exchange (https://math.stackexchange.com/), a part of the Stack Exchange platform, provides interaction mechanisms that support asking and answering well-posed, specific mathematical questions. Users can increase their reputation (measured in points) in many ways including asking questions that are upvoted or providing answers to questions that are upvoted by other users. An achievement tracking system, in the form of bronze, silver, or gold badges, provides more specific measures of the quality and duration of a user's participation within the community. As users gain more reputation, they are also provided with more tools to structure discourse on the platform. Consequently, the interactions of members are guided by community norms, which are upheld by the aggregate actions of participants in the community.

Other platforms have more relaxed mechanisms for structuring how users interact with each other. For example, spontaneous communities can materialize in the comments sections of online articles. One example of such a phenomenon occurred in 2015 when many news outlets, including The *New York Times,* reported on the Cheryl's Birthday logic puzzle from a Singapore mathematics class that went viral. An online article in the *Times* that outlined a solution to the puzzle generated spirited responses from more than 1000 readers in the comments section (Chang, 2015). However, as the engagement mechanisms in the comments portal are minimal (users can recommend comments, comments accrue points for being recommended), users have limited tools for structuring discourse. In the case of the logic puzzle, the community as mediated through reader comments did not generate clear, mathematically vetted answers. Nor was there a larger focus by the community on issues related to the teaching and learning of mathematics. Instead, the 1247 responses are a haphazard list that includes questions, clarifications, non sequiturs, misunderstandings, witty rejoinders, and novel interpretations.

These examples illustrate some of the different possibilities for engagement in online spaces. Within highly structured platforms such as Stack Exchange, discourse can be shaped as community members work to curate, validate, generate discussion, and maintain norms. In contrast, minimally structured systems like the comment thread of the *Times* provide users with limited tools to engage one another and give shape to the resulting discourse.

Practice Focused We adopt Wenger's (1998) perspective that practice is always social. As such, it involves not just doing something, but performing that activity within a social as well as historical context. Defined in this way, practice includes actions and ideas that are explicit and easily recognized as well as those that are tacit. As Wenger notes,

> [Practice] includes what is said and what is left unsaid; what is represented and what is assumed. It includes the language, tools, documents, images, symbols, well-defined roles, specified criteria, codified procedures, regulations, and contracts that various practices make explicit for a variety of purposes. But it also includes all the implicit relations, tacit conventions, subtle cues, untold rules of thumb, recognizable intuitions, specific perceptions, well-tuned sensitivities, embodied understandings, underlying assumptions, and shared worldviews. Most of these may never be articulated, yet they are unmistakable signs of membership in communities of practice and are crucial to the success of their enterprises. (Wenger, 1998, p. 47).

Our discussion here focuses on practices related to the teaching and learning of mathematics. Some of these practices are specific to mathematics (e.g., proving or disproving relationships within geometric figures, methods of solving systems of linear equations) whereas other practices may be more general to education (e.g., formative assessment techniques). Although the explicit and visible practices of a virtual community may be quickly recognized and adopted by a newcomer, the implicit, untold, underlying assumptions that are part of the community can go unnoticed. The ease of access to virtual communities of practice can also create room for miscommunication since, unlike physical communities that require a real-world connection, participating in virtual communities typically requires no more than visiting a publicly viewable website or registering a free account on a platform.[3] Thus, newcomers can easily participate without understanding the practices of the community and the assumed knowledge base upon which members draw.

1.2 Boundaries

Although passage into and out of different websites appears seamless to the user, we conceptualize this engagement as a type of *boundary crossing* and in doing so take Clarke's (2015) description of a boundary:

> [Boundaries] are constructions, built of language through discourse. However, we respond to boundaries in different ways. Sometimes the boundary appears as a natural feature, like a river, separating one habitat from another; sometimes, as an artefact, like a wall, constructed to enclose or to separate; and, sometimes, as the principles by which the members of a club or society are distinguished from non-members. Given such variation, the nature of boundary crossing itself must take different forms. (p. 170).

Boundary crossing includes describing how professionals who are newcomers in a field overcome barriers by creating a hybrid situation (Akkerman & Bakker, 2011; Suchman, 1994). In this sense, members of a vMEC can be seen as boundary crossers within social media spaces as well as real-world spaces. For example, university professors who are well known within an rMEC may become boundary crossers as they begin to engage in online platforms. In doing so, they may broaden their sphere of influence and cross the boundaries created by their professional context. Likewise, classroom teachers engaged in a vMEC may reshape themselves as experts in professional development for mathematics education thereby crossing the perceived boundary of the classroom and transitioning into a new role. Importantly, unlike boundaries in the real-world, the edges of these virtual boundaries are less apparent, which, as noted, can further conceal the norms and assumptions of community members. For example, a vMEC focused on a particular area of mathematics educationresearch may have established ways for communicating about topics and a shared knowledgebase that is drawn on when interacting. These norms, which help

[3] As a society we are beginning to wrestle with the hidden costs and challenges of such convenient access. For example, one cost is the commercialization of personal user data, which are mined, stored, and distributed by the entities—typically private companies—that own the platforms. Access to this information can provide any individual or group with insight into the characteristics of platform users, which can be used for advertising and promotion.

unify the vMEC, can also serve as barriers to newcomers who have limited background knowledge. In conceptualizing this crossing of boundaries, we draw on the theoretical lens put forward by Clarke (2015) and consider several methods of crossing boundaries.

Methods of Crossing Boundaries Clarke (2015) shares that “one way to cross a boundary is to abolish it” (p. 170). In this view, the act of eliminating or putting an end to practices or systems of exclusion can be seen as a type of crossing. Since social media exists apart from traditional mathematics education communities, it provides opportunities to abolish boundaries that are part of these communities. For example, one might consider the boundaries between higher education and K–12 schools, high school and elementary teachers, pre-service and experienced teachers, or high school and elementary teachers. In each case, the identified groups are part of different systems that have defined boundaries. Social media platforms provide spaces of public communication and allow users to develop communities around common topics (e.g., teaching fifth grade, number talks). This open, public communication allows practitioners to be leveraged as experts. Whereas in the past, communication of pedagogical innovations might have been characterized by dissemination through journals, published books, conference presentations, or professional development sessions conducted by academics, social media allows this one-way, “scholarly”-based dissemination to be abolished. Thus, practitioners at all levels of education can be the leaders in these spaces as they engage in mathematics education conversations that provide evidence from their practice and perspectives.

A second means to cross a boundary is to destroy or demolish it (Clarke, 2015). Whereas abolishing a boundary refers to putting an end to a practice, demolishing a boundary refers to deconstructing a barrier or structure. Although we are concerned with virtual communities, there are a number of structures and barriers that exist within online spaces. For example, membership in a professional organization grants one access to particular resources (e.g., academic journals) and excludes others from these materials. Likewise, holding a position at an institution of higher education provides access to resources through institutional structures as well as through membership in the academic community. Both of these examples highlight barriers that have been created to monetize knowledge and prevent it from being freely shared. Social media sites have created new paths to demolish these obstacles. For example, sites such as researchgate.net and https://www.academia.edu/ allow scholars to network and more easily share their work with others. Similarly, the tools provided by social media allow for members of vMECs to more easily share resources through avenues outside of traditional publishing (e.g., https://nixthetricks.com/, http://www.mathtalks.net/).

Another means of crossing a boundary is by building a bridge. As Clarke (2015) notes “A bridge conveys individuals, groups, ideas or artefacts between domains. It does not interact with the boundary, but passes over it” (p. 172). Here again, the nature of social media spaces make them places that foster this kind of boundary crossing since the sites provide tools for actively building bridges between different online communities of practice. For example, hashtags (e.g., #MTBoS, #mathchat, #iteachfifth, #estimation180) are used ubiquitously on social media platforms and allow members of different physical communities (e.g., novice teachers, expert teachers, math coaches, university professors) to interact in new and novel ways. These interactions may happen by chance, such as when one stumbles upon a relevant hashtag, or may be intentionally structured. For example, the National Council of Teachers of Mathematics supports a monthly Twitter chat around a free preview article. This chat is hosted by the author of the featured article and helps to build bridges between communities (e.g., authors and readers, professors and teachers, math coaches and new teachers).

Finding ideas that are by their nature permeable provides another means to cross a boundary (Clarke, 2015). Drawing on the community of practice lens, we might consider how ideas from the shared domain of the teaching and learning of mathematics can be seen as these kinds of objects. For example, methods of teaching specific mathematical content (e.g., integers, quadratic functions) are permeable objects since they can be easily shared between individuals in different educational communities. Similarly, pedagogical approaches can be shared easily through text, audio, and video. Moreover, social media provides tools to search for and find discussions of mathematical pedagogy in ways that are different from those that existed previously. These permeable ideas present a foundation upon which vMECs can form.

A final means to cross a boundary is to "accept responsibility for its construction (and deconstruction)" (Clarke, 2015, p. 175). For example, the inherent assumption that English is the primary language of mathematics education creates many boundaries within vMECs. For example, many of the hashtags that are used on various social media platforms (e.g., MTBoS, mathchat), only makes sense if one uses English to construct the acronym. Likewise, the practice of establishing a conference hashtag is now common within the field of mathematics education (e.g., AERA19, AMTE2019). Although these hashtags allow for crossing boundaries and fostering interaction, they also create barriers for those outside of the community and, in particular, for individuals who may not speak English. The duality of this boundary crossing and construction is important to recognize. This brings us circuitously back to the purpose of this chapter: What are boundaries of mathematics education communities? How do individuals cross these boundaries? How are we, as mathematics teacher educators, making the boundaries and boundary crossing we engage in—both those constructed and deconstructed—visible to students, colleagues, and the community?

Coordinating Communities of Practice and Boundary Crossing As discussed, we view the virtual communities formed around social media platforms as kinds of communities of practice. In addition, we view engagement in these virtual spaces as a type of boundary crossing. Membership in a community of practice necessarily creates boundaries as discussed by Wenger (1998):

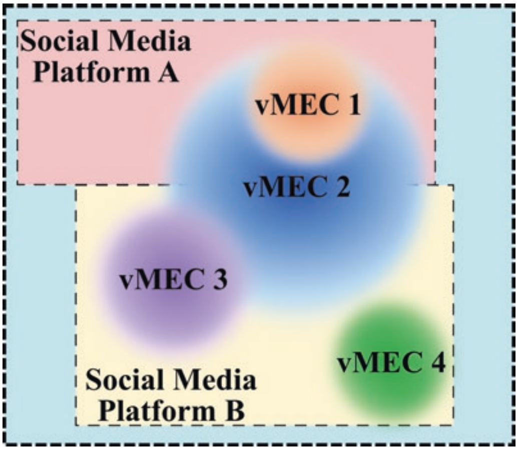

Fig. 1 Relationships between vMECs

> Participation and reification can both contribute to the discontinuity of a boundary. In some cases, the boundary of a community of practice is reified with explicit markers of membership, such as titles, dress, tattoos, degrees, or initiation rites. Of course, the degree to which these markers actually act as boundary depends on their effect on participation. Moreover, the absence of obvious markers does not imply the absence or the looseness of boundaries. The status of outsider can be reified in subtle and not so subtle ways—through barriers to participation—without a reification of the boundary itself. On the school playground, the unmarked but sharp boundary of a clique can be a cruel reality, one for which well-meaning parents and teachers are of little help. The nuances and jargon of a professional group distinguish the inside from the outside as much as do certificates. Not having the style and connections can be as detrimental to an ambitious employee as the lack of a degree from a major business school. A "glassceiling" is sometimes more impenetrable in practice than any official policy or entrance requirement. (p. 104).

Thus, the ideas of boundary crossing and virtual communities of practice are intertwined. As mathematics education communities form within social media spaces, they create boundaries that define membership. These boundaries might reflect real-world characteristics such as degree or status, but may also be based around knowledge of the shared domain. Moreover, the easy access to social media sites on the internet allows for a variety of relationships to form between virtual communities, providing opportunities for boundary crossing or ambiguity of boundaries.

Figure 1 provides a visual representation of the relationships between vMECs. For example, communities might exist wholly within a larger group as vMEC 1 is within vMEC 2. Communities might also intersect with each other as vMEC 2 and vMEC 3 or exist completely apart from other communities, as is the case of vMEC 4. Additionally, the figure shows how mathematics education communities can exist within a particular social media platform as is the case with vMEC 4 or span between platforms, as is the case with vMEC 2.

Viewing these communities from a high-level, the borders between them are most prominent since we visualize them as boundaries. The blurred periphery signifies that the boundary of the vMEC is not well defined and can be easily crossed. However, the communities themselves are not free of borders. Within each community of practice are newcomers, who we describe as *occasional participants*, and old-timers, who we refer to as *routine participants.* Occasional participants reside in the periphery (lighter shading) and participate only sporadically in the community. Routine participants, on the other hand, frequently engage in the vMEC and may gain status through their participation. Individuals may transition from occasional to routine participants through engagement, interaction, and collaboration with other members of the vMEC.

vMECs are dynamic in nature as they continually undergo a process of reformation through changes in membership as well as social media platforms. As discussed, the features of a particular platform may shape the nature and extent of the discourse within a vMEC. Likewise, the rise or decline of a social media platform creates opportunities and challenges for a vMEC. New social media platforms create opportunities for the formation of new vMECs as well as the splintering or wholesale migration of existing vMECs. Similarly, the decline of social media platforms may result in the end of a vMEC.

2 A Framework for Understanding vMECs

Within this section, we draw upon the previously discussed theoretical perspectives and present a framework for examining mathematics education communities. Our goal in creating this framework is to fashion a tool that can be used to compare, contrast, and better understand the mathematics education communities that reside on different social media platforms as well as the relationships between community practices and platform features. We hope that the framework can be used by mathematics teachers, educators, and researchers at all levels. The framework has three components: Membership Expertise, Communication Practices, and Platform Boundaries.

2.1 Component 1: Membership Expertise

The first component that we consider is the expertise level of the membership in a vMEC. By distinguishing between communities in this way, we are seeking to identify characteristics that are shared in general by routine members of the group. These are members who frequently engage and form the core of the vMEC. We acknowledge that characterizing the community in this way may ignore occasional participants and variation among routine members within a group. Within the framework, we consider two degrees of expertise: specialist and generalist.

Specialist communities are built around a narrow focus on a particular idea or topic. Consequently, members of communities may be primarily of individuals within a specific profession. For example, we might consider scholars who work within a specific theoretical framework as a specialized community. Likewise, seventh grade teachers, mathematics coaches at the elementary level, and professional development directors each make up a different specialist community. Members of these communities are distinguished by their expert level knowledge of the shared domain. Moreover, the process of becoming a specialist necessarily involves crossing boundaries that separate the specialist community from outsiders.

The boundaries of specialist communities function to define, preserve, and generate expertise. For example, an academically trained mathematician is in a position to contribute knowledge to the discipline precisely because of the boundaries that were negotiated over the course of their training. Although anyone can have mathematical insight about a particular problem, and amateurs can make mathematical discoveries, the boundary between those who have specialized training in mathematics and those who do not has been useful to advance the discipline of mathematics. Thus, specialist communities maintain, by their very nature, boundaries that can impede newcomers from participation.

Generalist communities, in contrast, are built around broadly understood ideas that, although still specific to a topic or issue, carry far fewer boundaries. The common interest that connects a generalist community forms a shared domain, but this domain is broader in scope than that of specialist communities. Whereas specialist communities have a specific interest that establishes boundaries by its very nature, generalist communities are formed around broader themes and ideas. This loose organization may lead to a more diverse membership in terms of characteristics such as backgrounds and ideologies, since there are fewer borders to participation. The lack of boundaries also creates more opportunities for miscommunication since the norms of the community may not be well established. For example, the practice of teaching mathematics serves as a shared domain for a generalist community. This shared domain is narrow enough to form a community, yet broad enough to allow many different perspectives on teaching mathematics to exist within its borders. The continued discussion around traditional versus reformed teaching approaches within the United States is an example of how members of generalist communities may have differing perspectives about many fundamental ideas within a shared domain.

Interactions between specialist and generalist communities present many opportunities for boundary crossing as individuals from one community can easily interact with members of another. Figure 1 provides a visual of some possible interactions. If we view vMEC 2 as a generalist community and vMEC 1 and vMEC 3 as specialists, we see that members of a specialist community may reside wholly or partially within a larger generalist community or be completely separated. These specialist communities might also intersect with each other to form even more specialized groups. For example, academics working within mathematics education are simultaneously members of the generalist community of mathematics educators as well as members of more specialized groups focused on their research content. Similarly, the generalist group of K–12 teachers of mathematics includes individuals who only teach mathematics as well as others who teach subjects in addition to mathematics.

Table 1 *vMEC communication practices*

Educative practices	Transactional practices
• Asking good questions • The pursuit of knowledge • Curation of content • Rich interactions among community members • Focus on knowledge building • Moderation of community members • Enforcement of community norms • Community member interactions and content generation tend to be closely aligned with the shared domain of the community	• Limited organization or curation of content. • Shallow interactions between community members • Focused around identifying resources to share • Monetization or advertisement of content • Interactions and content generation that are loosely aligned with the shared domain of the community

2.2 Component 2: Communication Practices

As individuals transition from occasional to routine participants, they begin to acclimate to the practices and norms of the community itself. Here we draw a distinction between communities where practices are more educative in nature versus communities where practices are more transactional. As the name implies, educative practices or norms are those that are rooted in the notion that the community exists to educate and enlighten members. Norms within these communities may include critiquing and curating community-generated content, seeking out and welcoming newcomers, and structuring discussion around central topics within the community. Transactional communities, in contrast, have practices and norms based around supplying and consuming content, which does not necessarily foster complex engagement or collaboration. Table 1 provides a summary of some educative and transactional practices.

We see the practices of a vMEC as lying on a spectrum from educative to transactional. The degree to which the practices of a particular vMEC are educative or transactional is based in part on the features of the platform(s) on which the community resides. Consider, for example, the differences between the platforms Stack Exchange and Pinterest. On Stack Exchange, moderation tools are provided for the community to critique and curate content. Members can upvote answers to indicate quality. Pinterest, in contrast, provides tools that allow for easy display and sharing of content but offers limited mechanisms for collaboration or curation. Additionally, monetizing or promoting user-generated content is a feature of the Pinterest platform but is absent from Stack Exchange.

2.3 Component 3: Platform Boundaries

The preceding discussion highlights the interaction and co-constructive nature of virtual communities and the platform on which they reside. When viewed through the lens of boundaries, it becomes clear that the structure of the site or platform may enable the creation of boundaries that affect the engagement of the vMEC. We focus on three aspects of platforms that can create boundaries: access, measurement of popularity, and communication structures.

How members of vMECs engage on a particular platform is ultimately a consequence of their access. Access may happen from outside the vMEC, as part of registration for the social network, or it may be granted by members of the community within a platform. Some platforms have a permeable boundary that allows access to content by search engines or visitors without restrictions. Stack Exchange and Twitter, for example, can each be accessed by anyone outside the community using a search engine (e.g., Google)—the boundaries are permeable with access to viewing community content available to those outside of the immediate vMEC. However, posting and interacting on these platforms does require registering for an account. Other social networks, such as Pinterest, may require that users have an account to fully view content. Platforms may also allow vMECs to create approval processes for members. For example, vMECs active as Facebook groups may have approval processes where those seeking to join the group must answer membership questions. Additionally, the extent to which a user can remain anonymous may create boundaries since members may not wish to provide their real information or, conversely, interact with individuals whom they cannot identify. Some social media platforms, notably Facebook, require that users use their real name and do not allow the use of aliases. Other platforms, such as Twitter, allow users to decide whether or not they want to use their real name for accounts.

Providing tools to gauge the popularity of content is another feature common on social media platforms. The nature of how popularity is measured depends on the platform. For example, Twitter posts can be shared through retweeting. Questions and answers posted on Stack Exchange can be upvoted or downvoted to signify popularity. Pinterest posts can be pinned by users to a board, which is tracked and displayed. On Facebook, popularity is measured with emoticons. This idea of collective shared beliefs, or popularity in a post, tweet, etc., represents a boundary crossing as it presents a shared, collective unity in thinking.

It is important to note that measurements of popularity are not necessarily measurements of content quality. Features of a platform, however, may intertwine quality and popularity. For example, on the Stack Exchange platform the number of up or down votes a particular question or answer has received can be seen as a measure of content quality as judged by community members. In this sense, the Stack Exchange platform breaks a boundary in that it provides some measure of content validity. However, on other platforms may be no clear association between popularity and quality of content (Hertel & Wessman-Enzinger, 2017). Instead, popularity may be a measure of agreement with particular content. For example, the act of sharing another user's content material is an expression that one likely agrees with or advocates the content.

As noted, the structuring of discourse is influenced by features of a platform. In particular, the interface for viewing content establishes boundaries that affect communication and dissemination of information. For example, Facebook and Twitter provide community feeds, where the most current information is filtered to the top of the feed. Individual posts may have comments and discussion within them. Stack Exchange, in contrast, blends the idea of a feed with that of a webpage. Questions and related discussion take place on individual pages. Searching Stack Exchange, however, results in a feed that is comprised of questions.

3 Using the Framework to Understand Current vMECs

We now draw upon the framework and create profiles of several active vMECs. The profiles are intended to illustrate the utility of the framework in comparing, contrasting, and better understanding the mathematics education communities that reside on different social media platforms. These profiles should not be viewed as robust characterizations of each vMEC. Moreover, although we view the framework as useful in comparing and contrasting specific aspects of each vMEC, we caution against reductionism with regard to these communities.

3.1 Profile 1: Math Twitter Blog-o-Sphere (MTBoS)

The Math Twitter Blog-o-Sphere (MTBoS) is a vMEC that interacts across several different platforms. It is a subset of the larger education community that resides on Twitter (Carpenter, 2015; Visser, Evering, & Barrett, 2014). This intentional use of different platforms, including blogs and websites for content creation and Twitter for communication and sharing, is a feature of MTBoS that sets it apart from other vMECs. For example, members create blog posts, which can then be viewed as web pages with organized comments and discussion. Links to these posts can be shared within Twitter feeds and then become referents for the original content. The growth and popularity of the MTBoS has elevated the status of many mathematics educators within the field and attracted researchers interested in examining the potential of the vMEC for teacher professional development and pre-service teacher education (Hsieh, 2017; Larsen, 2016; Larsen & Liljedahl, 2017; Staudt Willet & Reimer, 2018).

Another feature of MTBoS is the high degree of organizational structure that has been generated by the community itself. For example, there are a number of different websites created for the purpose of welcoming new members, compiling resources, and organizing community interaction (e.g., https://mtbos.org/, https://exploremtbos.wordpress.com/, http://mathtwitterblogosphere.weebly.com/). Additionally, although the Twitter and blog platforms allow users to remain anonymous, the members of the MTBoS tend to use real names and affiliations (Table 2).

Table 2 Unpacking MTBoS with the framework

Math Twitter Blog-o-Sphere	
Membership expertise	The community is made up of educators at various levels within the K–12 system, higher education, and in related fields. As such, it is comprised of both specialist and generalists.
Communication practices	The community has a primarily educative focus with members interactive via blog posts, comments, and twitter.
Platform boundaries	*Access*: The boundary between the MTBoS and outside content is permeable. Content can be access without restrictions and located using search engines.
	Popularity measurement: Measures of popularity within the community include retweets, replies to posts, and comments.
	Communication structures: The use of different platforms provides a variety of structures for communication within the vMEC. For example, blog posts can be viewed as pages with organized comments and discussion. Links to blog posts can be shared within twitter posts. These posts can then generate discussion and serve as referents for the original content.

3.2 Profile 2: Mathematics Educators Stack Exchange Community

The Mathematics Educators Stack Exchange Community is a vMEC that is hosted on the Stack Exchange platform. As discussed previously, the community is a venue to ask mathematical questions. These questions are then taken up and answered by members of the vMEC. A feature of the vMEC is that the experience and expertise of users is measured by platform-based badge and reputation systems. For example, a routine participant with thousands of reputation points and multiple gold badges signifies a more experienced, expert user than an occasional participant with only a few points of reputation. Additionally, the reputation and badge system structures the management of the vMEC. Users who have earned sufficient reputation points are granted privileges that allow them to actively manage the discussion that takes place within the community. This include such actions as closing discussion on a question, blocking users who breach the norms of the community, and placing bounties on questions they deem worthy of attention—i.e., a reputation point reward for other users within the community to provide answers to the question.

Another feature of the vMEC is an emphasis on socializing new users into the norms for asking focused, specific questions that are likely to generate clear answers. For example, experienced users that have earned moderator privileges regulate how questions are posed by revising, flagging off-topic posts, noting duplications, or otherwise pointing out how a question could be improved. Thus, the reputation and badge features of the Stack Exchange platform provide incentives for members to engage with one another and foster productive discourse (Table 3).

Table 3 Unpacking Mathematics Educators Stack Exchange community with the framework

Mathematics Educators Stack Exchange Community https://matheducators.stackexchange.com/	
Membership expertise	The community is made up of anyone studying mathematics at any level. The norms for posing and responding to questions are enforced by the expert users in the community. The vMEC is geared more toward specialists because in order to participate in the community one must be able to ask clear, specific, well-posed questions that have verifiable answers.
Communication practices	The community has an educative focus, in that the aim of the community is to assist people that are learning mathematics at any level.
Platform boundaries	*Access*: The boundary is permeable. Content can be accessed without restrictions and located using outside search engines. Participating in the community beyond browsing content requires creating a free account and logging in.
	Popularity measurement: There are three measures for the popularity of a question: Views, votes, and answers. Views record how many users have viewed the full version of a question; upvotes (and downvotes) record the number of users that indicated that a question is a 'good' question; answers indicate the number of answers that have been posted in responses to the question. View, votes, and answers measure different aspects of the community's engagement with a question. The popularity of answers to questions is measured by votes and by acceptance—i.e., whether the original poser of the problem marks the answer as the preferred solution (each question has at most one best answer that is determined by the person that posed the question). In addition to the metrics for measuring the popularity of questions and answers, there are also measures of user's participation, in the form of reputation points awarded and badges earned.
	Communication structures: The stack exchange platform supports user interactions via the mechanisms for asking, answering, and responding to questions. Discourse within the vMEC is managed by members who have earned sufficient reputation points through participation.

3.3 Profile 3: Jo Boaler's How to Learn Math Group

Jo Boaler's How to Learn Math Group is a vMEC that interacts on the Facebook platform. The community is made up Facebook members with an interest in the content of the online course created by Jo Boaler. The group description notes "This is a place where classmates in Jo Boaler's online course can share thoughts and ideas. We share ideas and articles about teaching math." (Jo Boaler's How to Learn Math, 2018) As of this writing, the group has over 31,000 members. Individual posts structure discussion within the group. Post content typically includes questions or thoughts related to teaching or reshares of resources from outside the group (e.g., videos, images, websites).

Two features of the vMEC are the related to the platform of Facebook. First, since Facebook requires that all users use their real name, the identity of group members is known. Members may limit the amount of information outside of their name that is publicly visible and control the degree to which they can be contacted by other platform users. Second, rather than a single popularity measure, the

Table 4 Unpacking Jo Boaler's How to Learn Math Facebook Group with the framework

Jo Boaler's How to Learn Math Facebook Group	
Membership expertise	The community is made up of Facebook members who with interest in Jo Boaler's How to Learn Math online course. This includes educators at various levels within the K–12 system, higher education, and in related fields. As such, the vMEC is comprised of both specialist and generalists.
Communication practices	The community has a primarily educative focus with members engaged in asking questions, discussing ideas, and sharing relevant content.
Platform boundaries	*Access*: The boundary between this group the rest of Facebook is permeable. Anyone on Facebook can find the group, view who is in the group, and see what is posted regardless of whether or not they are members. However, the boundary between this group and the larger world wide web is rigid, as the group cannot be searched from outside of Facebook. As this group is on the Facebook platform, all members must use their real names.
	Popularity measurement: Measures of popularity include emotional reactions to a post or comment (e.g., love, ha-ha, wow, sad, angry), the number of comments and replies made on a post, and the number of times a particular post is shared by members.
	Communication structures: The Facebook platform uses a feed structure. Users can display either the most recent posts or the posts with the most recent activity appearing first. Members can communicate through comments and posts within the group, as well as message each other using the Facebook platform. The group administrator has the privileges to structure discourse by creating announcements, deleting posts, disabling comments, and managing users.

Facebook platform has six emotional reactions that users can choose when reacting to posted content. These reactions are like, love, ha-ha, wow, sad, or angry. Consequently, reactions are able to measure both the popularity of a post, but also members to note agreement, disagreement, excitement, etc. and thereby generate a rough measure of how members of vMEC feel about posted content (Table 4).

4 Concluding Thoughts

In this chapter, we have presented a framework for understanding virtual communities focused on the teaching and learning of mathematics. Rather than waiting for PSTs to stumble upon one of these vMECs, we believe that active discussion of vMECs should be a part of pre-service teacher education. By engaging in discussion of vMEC components such as membership expertise, communication practices, and platform boundaries, mathematics educators can help PSTs recognize the features of a community that may or may not benefit their classroom practice. In doing so, PSTs may also come to better recognize the routine participants who comprise a community of practice and identify the way that boundaries are crossed or formed by the vMEC.

Discussing vMECs also allows for reflection on how society is coming to terms with new streams of information and new avenues for controlling access to information. As recently as the early 1990s, searching for information was mediated primarily through physical card catalogues housed in libraries. Similarly, non-local group discussions of ideas were mediated through letters to the editor, pamphlets, newsletters, or other limited mailings. The affordances of social media to connect large groups of people and provide venues where ideas can be exchanged in real time on a worldwide scale has transformed what it means to generate, archive, and share information.

The fluidity of expertise, information, and access facilitated by social media has also introduced challenges that we are just now coming to terms with. Physically bounded stores of information (e.g., libraries, publications) carried with them implicit boundaries between known/unknown, verified/unverified, or truth/falsehood. For example, the *New York Times'* slogan, "all the news that's fit to print" highlights how the physical act of printing itself creates a boundary between those events that are "newsworthy" and those that are not. However, the old physical boundaries have weakened and, in some instances, are now gone. It is unclear what will emerge from the ephemera of social media to take their place. Against this backdrop, virtual communities of practice can provide guidance and reconfigure boundaries that differentiate between truth and falsehood. Consequently, understanding these communities and the boundaries they cross, create, or demolish should become a focus within our field.

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Part IV
Societal Borders in Mathematics Pre-service Teacher Education

Queering Mathematics: Disrupting Binary Oppositions in Mathematics Pre-service Teacher Education

Cathery Yeh and Laurie Rubel

1 Introduction

Borders – territorial, political, economic, ideological – might seem as if they are predetermined and unchangeable, with a function to separate left from right or in from out. But we can think of borders differently, such as how, in *Theory of the Border*, Thomas Nail (2016) defines borders as processes of social division. As part of this theory, Nail emphasizes that the essence of borders is actually in the "in between," similar to what others, notably Gloria Anzaldúa (1999), have called "borderlands." Borders monitor and exclude and are regulated, patrolled, maintained, and defended by an array of power regimes, but it is the borderlands that are the places of movement, agency, and resistance (Anzaldúa, 1999; Nail, 2016).

In this chapter, we draw on this social definition of borders to elaborate on processes of social division around gender and sexuality[1] in mathematics education.

[1] It is impossible to separate gender and sexuality from race. We are concerned by how this analysis and discussion of gender and sexuality in an analysis seem to ignore race and how such an absence could be interpreted as a claim that race is somehow separate. We chose to focus this chapter on gender and sexuality, most of the time leaving matters of race unstated, but remind readers that gender and sexuality always intersect with race (see Joseph, Hailu, & Boston, 2017; Leyva, 2017).

We deliberately listed authors in reverse alphabetical order, in a nod to our subject of queering, but this chapter represents the shared work of both authors.

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Our goal in this chapter is to support pre-service teachers in recognizing and challenging salient borders around gender, sexuality, and other identity categories in mathematics education. Ultimately, we look toward opportunities for hybridity created by these borders and in the blurring or queering of them. We begin with a review of literature, framed by Anzaldúa's and Nail's ideas about borders, that documents the extent to which sexist and heterosexist ideologies patrol, reinforce, and perpetuate borders in mathematics that marginalize women and queer people. Next, against this backdrop, we present recommendations that will be useful for teachers, teacher educators, and mathematics education researchers alike about how to queer mathematics education.

2 Theoretical Framing and Related Literature

2.1 Queering Borders

Nail (2016) emphasizes that borders are processes of social division, which exist through motions of expansion and expulsion and are "mobile processes designed to redirect, recirculate, and bifurcate social motion" (pp. 220–221). These ideas remind us of Anzaldúa's (1999) scholarship, in which she explains that borders are not simply a topological divide between here and there or us and them. For Anzaldúa and for Nail, borders are spatial divides that separate countries, states, or classrooms but can be located inside the minds of individuals, expressed through values, beliefs, and ways of knowing that people develop through daily interactions. Seen in this way, borders are sites of psychic, social, and cultural marginalization that we inhabit and that inhabits us. Based on her experiences living at the juncture between the USA and Mexico, originally Mexican land, Anzaldúa uses the construct of "borderlands" to highlight the inherent in-betweenness of this space, in which she is neither fully accepted by White feminist academics as Chicana and at the same time nor by the Chicana community as queer. These experiences highlight her personal understanding of the psychological, physical, social, cultural, and even spiritual marginalization that occurs to those who are situated at social identity borders.

And yet, though enmeshed with power and hierarchy, borders are in a "constant place of transition" (Anzaldúa, 1999, p. 36), changing and being changed over time. Most relevant to this chapter, we stress that borders around identity categories like gender and sexuality are not fixed; since they are socially constructed, these borders are in flux, negotiated through practices and discourses across spaces. Yet traditional constructions of borders around sex and gender have been formed around fixed, binary distinctions, namely, that there are two sexes and genders, that people are born with in correspondence, and that normative sexuality is based on a coupling of opposites. The traditional construction goes against overwhelming evidence that there are more than two genders, that gender does not correspond to sex, and there is as wide a variation among people according to their sexuality, preferred gender identity, or choice of gender expression (Hird, 2000; Lorber, 1996). Once sex and gender are no longer understood as binary and no longer understood as equivalent, then a binary understanding of sexuality no longer makes sense.

Queer theory enables us to interrogate and blur these identity borders, by understanding sex, gender, and sexuality as socially and discursively produced, as emerging from social, cultural, and economic contexts, and as processes and actions that are always relational and intersectional. The term "queer," once a marginalized and marginalizing epithet, has been reclaimed to disrupt processes of marginalization[2] and is not meant only as a signifier that represents gay, lesbian, bisexual, and transgendered identities. Rather, "queer" also functions as a verb, as a challenge to identity categories more generally, to guide us to remember that what someone does is not equivalent to what someone is (Butler, 1990). In other words, our identities are not assigned to us at birth, but, instead, we are constantly developing identities through social interaction, informed, of course, by existing social constructions (Butler, 1990). And so, while traditional binary conceptions function as borders in the way that they divide us socially, queer theory highlights how these borders can be interrogated, challenged, and even blurred.

2.2 Gender Borderlands in Our Schools

Schools are discursively gendered spaces with their own particular sets of regulated processes and regimes of truth about sex, gender, and sexuality that tend to marginalize transgender or gender-nonconforming youth (Foucault, 1990; Mellor & Epstein, 2006; Miller, 2011). Consider, as an example, how many elementary schools across the USA sponsor events like "Daddy-Daughter" dances, events that are well meaning yet marginalizing. Beyond the insensitive exclusion of children who do not have fathers, this kind of event casts borders around what it might mean to be someone's child or someone's father. The very social construct of "daughter" is a problematic border for many children, such as those for whom gender and sex do not correspond, or those who might be seen as someone's daughter but for whom that identity is restrictive and inaccurate. Our argument goes beyond identifying the exclusion of genderqueer or transgender children from a school activity. There are many young people who identify as girls but might not be comfortable being "daddy's daughter," a construct that in and of itself presumes heterosexuality. Furthering this same genre of school tradition, most, if not all, high schools in the USA sponsor the culminating social event of a senior prom, a ritual event that involves dressing

[2] David Halperin (2003) credits Teresa de Lauretis with coining "queer theory" during a California conference because of its shock value. Lauretis intentionally builds on its origin as defamatory, intentionally, and "scandalously offensive" (Halperin, 2003, p. 340) to define "queer" as something that means to unsettle, disrupt, and transcend what has been socially and politically accepted as normative identity categories.

up in certain styles and is attended by students in heterosexual couples. Heterosexuality is emphasized through these proms as normative, with many examples of prohibition of same-sex couples or of mandating that tuxedos can only be worn by boys and dresses can only be worn by girls. These high-profile events either exclude the participation of queer youth or put them in the position of having to “come out” as queer in a highly public way.

In classrooms, children are required on a daily basis to publicly identify as “girl” or “boy” in front of their peers. Teachers typically assign genders to their students and often try to balance their groupings of students according to those assigned genders (see Rubel, 2016). A common classroom management strategy is to seat girls next to vocal boys as a strategy to control or limit the boys’ behaviors (Esmonde, Brodie, Dookie, & Takeuchi, 2009; Mellor & Epstein, 2006). Girls and boys are expected to inhabit their bodies differentially and spatially, evidenced by how classroom furniture is spatially arranged in particular ways (e.g., single file, pairs, or small groups) in which children are given a space and refused permission to sit in other spaces, meaning that their bodies are controlled, in gendered and sexualized ways (Gordon, Holland, & Lahelma, 2000; Walkerdine, 1990). Boys, for example, are commonly positioned as mischievous and disruptive in school and active in very bodily ways, while girls are seen as passive, obedient, and still (Gordon et al., 2000; Mellor & Epstein, 2006; Walkerdine, 1990). These positionings in schools contribute toward the common discursive framing of normative heterosexuality as comprising a binary of active masculinity and passive femininity.

Conventional borders around sex, gender, and sexuality maintain these forms of marginalization and oppression and often lead to violence in schools. The current direction in the USA of definitions of gender and corresponding laws about school bathrooms is toward viewing gender only as equivalent to one’s sex “as assigned at birth,” even in the face of troves of counterexamples of transgender people. New anti-trans legislation is being proposed across the USA, despite how these laws effectively deny many young people from feeling comfortable in school and making them vulnerable to harassment and physical violence. For example, nearly all (90%) gender-nonconforming students indicate that they have received negative remarks about their gender expression and more than half reported receiving gender-based physical violence in the past year (Kosciw, Greytak, Palmer, & Boeseen, 2014). Students who are perceived to be gender nonconforming are significantly less likely than their peers to report homophobic and transphobic harassment and assault by students and faculty (Swearer, Turner, Givens, & Pollack, 2008). Although not usually framed as related to borders around gender in schools, school shootings in the USA are almost always perpetrated by boys. Rarely is the fact that this violence is almost always being perpetrated by males raised as an important issue or that school shooting violence is connected to hypermasculinity, homophobia, or violence against transgender people (Kimmel & Mahler, 2003; Kosciw et al., 2014).

Where does mathematics education fit in? Nail’s (2016) classification of borders shows that some borders contain, protect, and maintain flows while others divide and politicize. Indeed, mathematics is employed as a border to support and further processes of social division that contain, protect, and maintain flows of access and opportunity, through what Louie (2017, p. 489) has called its persistent and restrictive "culture of exclusion." Various social systems around mathematics, such as ability tracking or how mathematical ability is narrowly defined, are hallmarks of this restrictiveness and exclusion. Likewise, mathematics is used as a border that divides and politicizes. For example, mathematics is racialized as white (Battey & Leyva, 2016; Gutiérrez, 2017b; Martin, 2012; Stinson, 2006) as evidenced in the historical creation and ongoing use of standardized mathematics assessments as a means to justify separation of students within and between schools by race (Berry, Ellis, & Hughes, 2013; Ellis, 2008; Yeh, 2018).[3] Nail argues (p. 9) that the border is "not the result of a spatial ordering, but precisely the other way around – the spatial ordering of society is what is produced by a series of divisions and circulations of motion made by the border." We make the same theoretical argument here: we do not fault the discipline of mathematics as inherently problematic or claim that mathematics itself produces social orderings, but instead, we highlight that processes of social division produce orderings of society through and using mathematics.

Another way that Western mathematics has been recruited to support processes of social division around gender and sexuality is through its own reliance on binary thinking. There is a plethora of examples whereby binary logic dominates: a quadrilateral, for example, is a rectangle or not, a number is even or odd but never both, a function crosses the x-axis or it does not, and so forth. Since mathematics is positioned as neutral and ascribed power in supposedly representing an abstract truth (Borba & Skovsmose, 1997; Greer & Mukhopadhyay, 2012), its disciplinary emphasis of binary logic perpetuates those ways of thinking as normative and justifies this logic as a kind of natural truth (Rubel, 2016). Any attempts at disrupting or queering binary thinking can, therefore, be seen as incongruous to or even in conflict with mathematics. Instead, mathematics is itself a product of social relations (Greer & Mukhopadhyay, 2012; Mukhopadhyay & Roth, 2012), and so, binary perspectives that dominate social relations like gender and sexuality extend to it. This, of course, suggests that as we blur borders around gender and sexuality in mathematics education, such an expansion will create space for other kinds of logics in mathematics.

2.3 *Gender and Sexuality in Mathematics Education*

Over the past 45 years, the focus in the literature about gender in mathematics education has generally been on describing and understanding sex-based differences in mathematics achievement and participation (Damarin & Erchick, 2010; Leyva, 2017; Rands, 2009, 2016). Notably, these studies tend to ignore distinctions between

[3] We remind readers that systems of oppression around gender and sexuality intersect with and reinforce systems of oppression around race and class. Our choice in this chapter was to "telescope in" (Collins, 1990) on gender and sexuality, despite how these systems of oppression always intersect with race and class.

gender and sex (Leyva, 2017). Many studies explain and justify sex-based achievement differences favoring males using explanations rooted in biology (e.g., Benbow & Stanley, 1982), psychology (e.g., Hyde, Fennema, Ryan, Frost, & Hopp, 1990), or social factors like participation (e.g., Hart, 1989). Other studies argue that any differences in mathematics achievement by sex are too small to be recognized (e.g., Hyde, Fennema, & Lamon, 1990). More recently, Reardon, Fahle, Kalogrides, Podolsky, and Zárate (2018) show that on average, across the entire USA, there is no gender gap in mathematics achievement. However, when achievement data is analyzed to take into account local social contexts and the variability among them, their data showed patterns around gender, mathematics achievement, race, and socioeconomic status. Whereas in school districts serving poor communities of color, girls slightly outperform boys, Reardon and colleagues (2018) show that in affluent, White, suburban school districts boys significantly outperform girls in mathematics. This finding might seem surprising in the face of the typical outward expression of gender equality values in such communities and schools. However, in the context of the predominance of traditional gender roles among parents in those communities, and in the role that affluent White men play at the top of unequal power structures in our society, it becomes less surprising that the system that is being maintained is one that favors their sons.

While the set of sex-based achievement gap studies typically show parity, rates of participation of men and women in mathematics differ dramatically. In most countries, including the USA, female participation drops as soon as mathematics becomes optional and further decreases at higher levels (Gray, 1996). In the USA, for example, in recent years (2014), while 41% of undergraduate mathematics degrees were earned by women, only 29% of mathematics PhDs are women, and women constitute 12% of tenured university mathematics faculty (National Science Foundation, 2017). We can unpack this steep drop-off and reinterpret it not necessarily as a decline in women's participation but instead as increasing exclusion, as they try higher and more powerful credentials in mathematics.

A mechanism of this exclusion is how mathematics is gendered as masculine. Mathematics is constructed as a subject and a field for those who are rational, analytical, emotionally detached, and competitive. More than any other academic subject, faculty in mathematics (most of whom are men) believe that mathematical abilities are innate (Leslie, Cimpian, Meyer, & Freeland, 2015) and the high status of mathematics as a discipline can be attributed in part to its construction as masculine. Mathematical models gain added credibility through the image of mathematics as rational and objective – characteristics associated with masculinity – as opposed to models that are seen as subjective and value-laden (Borba & Skovsmose, 1997). "Pure mathematics," which is limited to concepts and techniques separated from human concern, is elevated to a higher status than applied mathematics (Greer & Mukhopadhyay, 2012).

As a consequence of mathematics being gendered as masculine, doing mathematics then becomes a part of doing masculinity (Mendick, 2005b). The gendering of mathematics as masculine occurs in the context of a constellation of other kinds of related gendered and binary oppositions like competitive/collaborative, fast/slow, or analytical/holistic (Esmonde, 2011; Esmonde & Langer-Osuna, 2013; Mendick, 2006) and explains, for example, the finding that girls in the USA have a greater tendency to believe that mathematics is more suited for boys (Jacobs & Eccles, 1985). It is well documented that schoolteachers' beliefs and biases systematically influence student participation and achievement in mathematics and that teachers have been shown to generally favor boys in the classroom by giving them more extensive answers, attention, reinforcement, and positive feedback (Becker, 1981; Fennema, Carpenter, Jacobs, Franke, & Levi, 1998). Teachers (and, more often, women teachers) have been found to rate boys as more proficient in mathematics, even when compared with girls of similar participation and achievement (Lubienski, Robinson, Crane, & Ganley, 2013). Together, these findings explain why girls have been found to underestimate their mathematical abilities, while boys overestimate theirs (Jones & Smart, 1995).

The masculinization of mathematics can be understood as an exclusionary border for girls and women (as well as anyone negotiating other forms of masculinity) because to choose to participate in mathematics means to be comfortable and to welcome participating and performing in a masculine space (Esmonde, 2011; Mendick, 2003; Walshaw, 2001). Barnes (2000) presented a detailed case study that shows how doing mathematics as doing masculinity can structure a gendered hierarchy of mathematics ability that differentially positions individual girls and boys. In an investigation of relationships between gender and the social construction of mathematical competence in an advanced high school calculus class in Australia organized around collaborative learning, Barnes highlighted how students in the class were making meaning of mathematics and constructing their own mathematical identities, as learners of mathematics, relative to one another, and that gender was salient in these interactions. Barnes found that the classroom organization around collaborative learning was effective and supported students in grappling with significant mathematical ideas. However, at certain points of interaction, the collaborative groups were less than optimal, and this revelation led to Barnes' gender analysis of those episodes.

Barnes (2000) found the discursive production of two subgroup of boys, the Mates, who performed a dominant form of masculinity, and the Technophiles, who were negotiating another nondominant form of masculinity. The Mates were quickest to divert the teacher's attention toward them and were successful by tapping into related social capital around athleticism and a sense of "coolness" as well as intellectual capital in the form of problem-solving behavior and academic praise by teacher. The Technophiles, on the other hand, were seen as outcasts and were often ignored by their peers. The girls in Barnes' study were often treated as "helpers or assistants," whose role became managing the groups and, especially, managing the behavior of the Mates. We see through Barnes' study that when mathematical ability or thinking is framed around masculinity and power, the environment becomes highly competitive and dissuades girls and other boys from participation.

Though often overlooked, the masculinization of mathematics also presents exceptional challenges to queer or gender-nonconforming youth in the ways that negotiating masculinity implies negotiating heterosexuality as well. As an example, consider an approach to inviting the participation of girls in mathematics by assuring them that participation in mathematics will not detract from their heterosexual appeal. For example, in her top-selling popular book series aimed at girls, Danica McKellar (2009, 2011, 2012) pines that "doing mathematics is sexy" and mixes her presentation of school geometry as "girls get curves" with "how do you attract guys?" If there was any doubt about the premise of heterosexuality or how it is leveraged as part of a promise of attractability to men in the face of participation in mathematics, a second book in this series for girls is titled *Hot X: Algebra Exposed* (McKellar, 2011).

One prevalent response to the masculinization of mathematics is to understand femininity as inadequate or irrational and ill-suited to mathematics, like framing the issue as "the girl problem in mathematics" (Campbell, 1995). An example of this kind of "blaming the victim" is the attribution of girls' reticence to participate in mathematics to their characteristics of being less confident or less assertive, which are positioned as innate characteristics of girls rather than as responses coproduced in relation to their learning environments. The implication becomes, then, instead of changing girls to act more like boys or changing the way we parent or educate children, perhaps harder to imagine is in changing mathematics itself as the solution to gender inequities in mathematics (Damarin & Erchick, 2010). A queer twist on this perspective, put forward by Mendick (2005b) citing Butler (1995), premises that gender is not something that you are but rather something that people do, and so we should not assume only two genders. In addition, we should not assume that male bodies are necessarily the only ones who perform or wish to perform masculinity or that people consistently choose one, or even any, gender. Yes, we need to challenge mathematics as a masculine space to make space for broader participation, but we concurrently need to interrogate the very accessibility of masculinity. Who is allowed to perform masculinity and what is the cost of transgressing this border?

3 Queering Mathematics Education

> Knowledge makes me more aware, it makes me more conscious. Knowing is painful because after it happens I can't stay in the same place and be comfortable. I am no longer the same person I was before (Anzaldúa, 1999, p. 70).

There is a growing call for teacher preparation to include a focus on equity, including an expectation that beginning teachers understand "the role of power, privilege, and oppression in the history of mathematics education and [is] equipped to question existing educational systems that produce inequitable learning experiences and outcomes for students" (AMTE, 2017, p. 18). Mathematics methods courses are paramount ways in which beginning teachers begin to recognize and challenge salient borders around gender, sexuality, and other identity categories in mathematics education. We argue that Nail's border theory and Anzaldúa's notion of borderlands provide a framework for the queering of mathematics education during teacher preparation. Even though it is the broader society that establishes and negotiates borders around gender, sexuality, and mathematics ability, what teachers do and say and how they teach, in social settings like mathematics classrooms, can reproduce or interrogate and challenge these borders (Miller, 2011; Yeh & Otis, 2019). Just as border security and immigration officers enforce national borders and exclude those who appear to be threatening the power regime, as teachers, we monitor the borders of what is considered "normal" in our classrooms, what it means to participate, and what it looks like to succeed, and we are tasked with having to rank, order, classify, and effectively exclude students according to perceived differences (Collins, 2013). This means that pre-service teachers hold the promise and ability to support those on the margins while simultaneously critiquing marginality (Souto-Manning & Lanza, 2019). In this section, we present recommendations about how to identify borders in mathematics education during teacher preparation, how to queer those borders, and, finally, how to queer mathematics.

3.1 Identifying Borders

Awareness about how mathematics is gendered is essential for teachers to be able to recognize and challenge salient borders around gender, sexuality, and other identity categories in mathematics education. Dominant discourse around mathematicians in social media depicts them as male, boring, and socially awkward (Mendick, 2005a). As part of their studies of teachers' conceptions about mathematics and who does mathematics, Mewborn and Cross (2007) as well as Lake and Kelly (2014) asked pre-service teachers to draw a mathematician. The produced images of mathematicians yield insights into teachers' beliefs and understandings about mathematicians and, in kind, about mathematics. Both studies found that a large number of pre-service teachers (mostly female) drew images portraying mathematicians as "nerdy, socially inept, middle-aged men working with equations in a lonely room" (Mewborn & Cross, p. 263).

In our own practice, building on the work of Zaskis (2015), we have used this "personification" prompt with pre-service teachers:

> If Math were a person, who would Math be? Write a description (approximately 500 words). This paragraph should address things such as: How long have you known each other? What does Math look like? What does Math act like? How has your relationship with Math changed over time? These questions are intended to help you get started and should not constrain what you choose to write about.

Through this writing activity, teachers use their own voices and experiences to reflect upon power, identity, and the gendering of mathematics. The particular method of eliciting personification challenges pre-service teachers to attribute human qualities to mathematics, thus allowing them to notice their own relationships with mathematics and the way gender, sexuality, and race play roles in those relationships. We have found that through this personification activity, participants often identify how mathematics is masculinized, heteronormative, and omnipotent, such as:

> Mr. Math is your stereotypical, middle school gym teacher. He is a middle-aged, muscular man towering over us as he walked around in his tight, athletic shorts. The very familiar whistle is wrapped around his neck so he can easily blow it furiously at all times. His favorite phase is “Go faster!”

In this sample excerpt, we see the discursive framing of normative heterosexuality and hypermasculinity: physical strength, aggression, and sexuality.

Similar to results found in the draw-a-mathematician studies, these teachers presented mathematics as male and, for most of our teachers (predominantly female), different from themselves. Unlike the results of the “draw-a-mathematician” studies, interestingly, we have not found teachers presenting “Math” as a socially inept “nerd” but instead as men in positions of authority or power, such as the PE coach, king, boyfriend, or lover. Perhaps differences can be attributed to the prompt itself. Elicited personification targets the relationship with mathematics more directly than the draw-a-mathematician task by directly asking about the teacher’s own relationship with mathematics, allowing us and the teachers (many self-identifying as having mathematics anxiety) an opportunity to reflect on the high status of mathematics and its power in shaping school experiences. The personification writing activity allows pre-service teachers to express and share their evolving relationships with mathematics. In our experience, most pre-service teachers typically indicate an initial relationship with mathematics as both emotional and tumultuous, that is, filled with both joy and isolation. In many cases, they describe that relationship turning “sour,” either as mathematics became more challenging, more complex, or less connected to their day-to-day lives.

3.2 Queering Borders

Our assumptions about and understandings of gender, sexuality, and families are reflected in the stories that we tell, in the things that we mathematize, and how we mathematize them. We know that learning is improved when people see relevance in mathematics and can use it as a lens or a window through which to better understand the world (Gutiérrez, 2007). Instead of themes of corporate profit, consumer thriftiness, or middle-class leisure activities often found in US mathematics textbooks, we could be using mathematics to unpack and redress social justice issues around gender and sexuality (e.g., Rands, 2013). Since mathematics includes the identification of trends, forming of projections, and the communication and evaluation of solutions, the process of mathematical modeling could be directed toward addressing problems such as wage earnings gaps between men and women in the workforce and disparities in the rate of hate crimes and police brutality between transgender and cisgender populations.

Learning is improved when school mathematics serves not just as a *window* through which to see the outside world but also as a *mirror* through which people see themselves and their families (Gutiérrez, 2007). Such a commitment necessitates inclusion and representation of gender diversity and of LGBTQ people and their families across our mathematics curricula. This means including and representing diversities of gender identities, sexualities, kinds of families, kinds of couples and how they celebrate, as well as intersections of those diversities. A mathematics curriculum that goes beyond reflecting only the experiences of a narrow range of people will make mathematics more equitably accessible to more people. In addition, such a curriculum will support the dismantling of the existing oppressive and rigid binary categories of gender and corresponding sexuality.

Identity is developed through reclaiming histories of marginalization and violence, articulating oppression's gendered and heteronormative elements, and embracing the fluidity of borders – a concept Anzaldúa (1999) describes as *border consciousness*. We propose that mathematics educators be given and take the space to develop this kind of border consciousness, by interpreting and assessing the ways that gender and sexuality norms are relegated and naturalized by the contextualized realities made available in mathematics curriculum. For example, consider the following prompts used to support pre-service teachers in a process of queering their curriculum:

1. What knowledge (aside from the mathematics) and worldview is assumed by this word problem? What are the problem's assumptions or values?
2. Does this problem reflect your own experiences?
3. Whose experiences are reflected or not included?
4. How could we queer these problems so to reflect a wider number of windows and mirrors for our students?
5. What categories of resistance might you face to these new word problems and how will you respond or get support?

The process of collective analysis increases the visibility of and opportunities to make "unhidden" the tradition of silence – sexism, heterosexism, classism, and consumerism – typically reified through mathematics texts (Bright, 2016). Since any group of people has diverse histories and borderland experiences, this leads to differing interpretations and discoveries that can overlap or contradict. Collective analysis, therefore, enables multivocality, highlights essentialist depictions, and provides alternate discourses that would likely be unavailable if individuals conducted curriculum analysis on their own. As shown in the above set of prompts, after interrogating a problem in terms of assumptions, values, and whose experiences it valorizes, teachers can then undertake a process of "relabeling" (Souto-Manning & Lanza, 2019), toward challenging the written boundaries around gender and sexuality, pushing back against stated stereotypes, and interrogating the problem's implicit values. School students, too, can be engaged in this process of border consciousness and queering word problems, a process whereby they reframe mathematics texts to be better mirrors of their identities, experiences, and values (Yeh & Otis, 2019).

3.3 Queering Mathematics

Representation in mathematics communicates to young people what mathematics is for and whose problems get to be solved with mathematics. And yet, representation and this inclusion approach, as described directly above, or what Rands (2009) has called the "add queers and stir" approach, are crucial but not sufficient. Efforts organized around inclusion and on broader representation of individuals and of gender and sexuality more broadly miss an essential element of queer theory, that identity is not fixed but is constantly fluid and being performed (Rands, 2009, 2016). In other words, we cannot only "mathematize the queer," or work to ensure that mathematics is used as a window that looks out onto social justice issues around gender or sexuality, or that there is enough and appropriate representation in mathematics for it to function as a mirror for queer people (Rands, 2016, p. 188). Instead, we take up Rands' (2016) call that in addition to mathematizing the queer, as we have discussed directly above, we must also queer mathematics itself.

The current emphasis in mathematics education imposes the values of rationalism, objectivism, and abstraction – traits that are often associated with masculinity (Bishop, 1990; Gutiérrez, 2017a). As Fasheh (1982, p. 3) reminds us, although teaching mathematics is often positioned as having the objectives of knowing "certain mathematical facts" and "thinking correctly, logically, and scientifically," we could instead organize mathematics education around teaching young people "to doubt, to inquire, to discover, to see alternatives, and, most important of all, to construct new perspectives and convictions." Similarly, when we take such positions as there is only one right answer (untrue as soon as mathematics is applied to reality) or only one right way to carry out a computation or express proof (absolutely not true), these actions inflict intellectual abuse by discrediting the experiential and mathematical brilliance students possess (Greer, Mukhopadhyay, & Roth, 2012). In addition, "mathematical developments in other cultures, follow different tracks of intellectual inquiry, hold different concepts of truth, different sets of values, different visions of the self, of the Other, of mankind of mature and the planet, and of the cosmos" (D'Ambrosio, 1997, p. 15). Neither Fasheh, Greer et al., nor D'Ambrosio uses the language of "queering" in these stated challenges to mathematics education, but we argue that in interrogating the very foundation of how we think about mathematics and its purposes, these ideas are examples of queering mathematics.

Queering mathematics inquires about and questions boundaries, not only around social categories of gender and sexuality but also around *mathematical* categories (Sheldon, 2019). Sheldon and Rands (2013) and Sheldon (2019) highlight the possibility of queering mathematical concepts like time, infinity, space, measurement, place-value, and more. Mathematics is not universal, "the same for everyone," but a human activity, enriched by the diverse intellectual activity bound in life. Authentic cultural mathematical practices are connected to lived experiences, to bodies, immediate needs, and desires (Greer et al., 2012; Mukhopadhyay & Roth, 2012), implying that mathematical activity can be multisensory, can involve or benefit from our hands, bodies, or eyes, and can involve direct interaction with the physical and social world. Queering mathematics, as well as new kinds of logics and ways of thinking about gender and sexuality, implies creating space for new ways of mathematical thinking. In other words, queering mathematics will support us to interpret existing questions in new ways, ask altogether new questions, challenge premises that seem no longer self-evident, develop new kinds of representations and arguments, see patterns that may have been invisible before, and will ultimately support us in solving both new and heretofore unsolved problems.

4 Conclusion

Social justice issues around gender and sexuality, along with women's legal rights, bodies, and overall role in society, are and remain a political battleground. From differential access to human rights, resources, and professions, inequality in wages for work, who controls women's bodies and their health, to sexual intimidation and assault – all of these pressing issues demonstrate the urgency of our blurring and renegotiating existing borders around sex, gender, and sexuality. On the one hand, pretending that mathematics is somehow separate from these social issues or, on the other hand, that mathematics alone can rectify these injustices is both problematic. Both of these perspectives ignore the role that mathematics plays in creating and justifying borders as social divisions and, as importantly, in creating new or exacerbating existing injustices (Pais & Valero, 2012). A way forward is through queering mathematics education during teacher preparation or, as we have argued and explained here, in identifying borders in mathematics education, queering those borders, and in queering mathematics itself.

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Persisting Racialized Discourses Pose New Equity Demands for Teacher Education

Mahtab Nazemi

> I think it's important [for teachers to] involve current events that you have your students stay on top of what's going on in the world because it's important that they know this stuff. [M]aybe this idea that they heard in class, this real-life topic will encourage something else. Maybe they'll write a book about it or maybe start a movement around it. You never know what you could inspire by just helping kids know what's going on. – Leilani, student interview

Teachers need to connect their classrooms to real-life events and to students' lives, as the effects can be as far-reaching – as Leilani suggests here – such that students might feel encouraged and inspired to use what they have learned to engage with and change the world around them. As a mathematics education community, we have come to know that equitable mathematics teaching needs to be responsive to students and their cultures, as well as affirming to students' identities both academically and socially (e.g., Delpit, 2012; Gay, 2000; Gutiérrez, 2013; Gutstein, 2007; Ladson-Billings, 2014). Complex instruction (CI, Cohen, 1994) is one example of an ambitious and equity-driven form of instruction which has been shown to support students' advancement of conceptual understanding in mathematics and support students to develop productive mathematical identities at the same time (Boaler & Staples, 2008; Jilk, 2010). In brief, CI is organized around rich and complex, group-worthy mathematical tasks. These tasks are conceptual in nature, problem-solving oriented, and open-ended (Cohen & Lotan, 1997). Through these tasks, group work is encouraged in CI. Students are expected to work together to make meaning of concepts and big ideas, rather than one student dominating the conversation or task. Students are held accountable for the learning of every student in the group, for example, through the teacher's insistence that all questions are group questions (rather than individual questions) and through assessments such as group quizzes that require students to support and work with one another (Cohen, Lotan, Scarloss, & Arellano, 1999). Despite its promising context, we know very little in regard to students' experiences (in their own words) with equity-driven initiatives such as CI, making it challenging to understand if/how these forms of instruction can support students' identities, academic and otherwise.

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Recognizing and affirming students' identities and experiences are essential, especially in the context of an equity-driven mathematics classroom. This means putting students and their sense of selves (back) at the center of learning, such that our curriculum and teaching practices are built around them. Knowing about the racialized experiences of students, from a diversity of racialized student of color perspectives, can be advantageous in understanding identity and racial discourse in mathematics learning and across racial lines. In this AP[1] Statistics classroom – the site of this study – the classroom teacher employed CI and was conscious of supporting students and their identities while learning mathematics. While this form of instruction, and this classroom teacher, successfully supported students (who were by and large of color) to participate and succeed in upper level mathematics, these students expressed narratives around learning that were racialized in nature.

This chapter draws upon previous work (Nazemi, 2017) which showed that, in this classroom context that is ambitious and equity-driven with a teacher who is conscious of race and her students' identities, racialized discourses and processes of racialization persisted. In centering students' voices – across various "of color" racial identities – I showed that the complexity of understanding students' racial and academic identities has to do with hearing from *them* about *their* experiences, as embedded in the contexts of learning in which they are situated (Nazemi, 2017). Understanding that race and racialization are salient, even in the context of ambitious and equity-oriented mathematics instruction, urges teachers and teacher educators to consider a new equity demand. Said differently, students' experiences in the mathematics classroom are racialized in nature, a reality that teachers need to be aware of and prepared to respond to. The prevalence of assumptions around who can and cannot know, do, and succeed in mathematics affects how students interact with one another, their classroom teacher, and the mathematics learning they engage with. This chapter starts here, in order to suggest implications for pre-service teacher education, centered around, and responsive to, students' identities and racialized experiences.

1 Theoretical Underpinnings

This chapter draws on sociocultural theories of learning and identity as well as critical race theory (CRT), to center and privilege the racialized narratives of six girls of color who were enrolled in an AP Statistics class, characterized by high-quality implementation of equity-oriented instruction. Specifically, in this section, I describe how CRT builds upon sociocultural theories around learning and identity so that we can understand all contexts of learning as inherently racialized while also shaping our sense of selves.

[1] AP stands for Advanced Placement and is a program in the United States and Canada that offers college level course curriculum and examinations for high school students. Often, upon successful completion of the AP exam, students are able to apply course credit(s) toward their college degree.

Early sociocultural theorists in education demonstrated that thinking, development, and learning depend greatly on the social and cultural contexts in which they take place (e.g., Lave & Wenger, 1991; Rogoff, 1990; Rogoff & Lave, 1984; Vygotsky, 1978). Specifically, through ethnographic studies, Lave and Wenger (1991) showed that learning is "situated" or takes place within social context. This means that the contexts in which students are located have everything to do with what they are learning, who they are becoming, and how they see themselves. In this way, we can think about identity as a never-to-be-complete product of sociocultural histories (Hall as cited in Nazemi, 2017) or "being recognized as a certain 'kind of person' in a given context" (Gee, 2000, p. 99). Wenger (1998) showed us ways in which learning and identity are connected, so that participating in the social context of learning "shapes not only what we do, but also who we are and how we interpret what we do" (p. 4). While sociocultural theory helps to connect learning and identity, this approach does not tend to recognize the imminent role of race and racism in all learning contexts, a shortcoming some scholars have remarked upon (e.g., Esmonde & Booker, 2016; Gutiérrez & Rogoff, 2003; Nasir & Hand, 2006; Nasir & Saxe, 2003).

In order to respond to the shortcomings of sociocultural perspectives, many scholars have focused on the identity-related experiences of students of color so as to recognize the role of race and racism in learning contexts (e.g., Esmonde, Brodie, Dookie, & Takeuchi, 2009; Leyva, 2016; McGee, 2016; Nasir et al., 2013; Oppland-Cordell & Martin, 2015; Shah, 2017; Stinson, 2008; Zavala, 2014). Some scholars – including some of the aforementioned – account for the prevalence of race and the processes of racialization in learning contexts, through employing a CRT lens in education (Ladson-Billings & Tate, 1995; Martin, 2012). CRT helps scholars recognize, center, privilege, and legitimize counter-narratives that speak to the lived experiences of students of color in their own words (Delgado & Stefancic, 1995; Solorzano & Yosso, 2001) while calling out and analyzing the function of race and racism both in students' lives and classrooms, as well as in the educational system as a whole (Tate, 1997). In the context of mathematics learning, CRT has been used by scholars to examine the complexities of the experiences of students of color and to provide a framework for counter-narratives where stories of resilience and success are (re)-centered and shared (e.g., Berry III, 2008; Corey & Bower, 2005; Martin, 2012; Stinson, 2008; Zavala, 2014). Few of these studies – much like the one described in this chapter – employed both CRT and sociocultural theory together (e.g., Esmonde et al., 2009; Leyva, 2016; Zavala, 2014). While a large number of these studies focus squarely on African American boy students (e.g., Berry III, 2008; Corey & Bower, 2005; Martin, 2012; Stinson, 2008), all of these studies, centering the narratives of students of color by way of CRT, allowed scholars to uncover and make sense of students' experiences in mathematics and how these students made sense of and navigated their identities, vis-à-vis learning mathematics. This work has allowed scholars to show ways in which students of color have been successful in mathematics, for example, through resisting, persisting, and carrying positive identities, often in opposition to the dominant narratives about their racial/ethnic identities (e.g., Berry III, 2008; Martin, 2012; Stinson, 2008). In this chapter, it is through the uncovering and centering of students' racialized narratives, along with navigation and negotiation of their identities, that we are able to understand the implications these counter-narratives pose for teacher education.

2 Research Methods and Context

The data presented here is from a 6-month study exploring how six girls of color (Carlin, Gena, Jane, Leilani, Lia, Mya[2]) navigated and negotiated their identities while learning mathematics within an AP Statistics classroom, with a race-conscious White[3] woman teacher who employs equity-driven forms of instruction. The overarching goals of the study were to understand how these girls of color felt their racial identities played a role in their learning mathematics, how they viewed themselves in relation to other students in their classroom, what sort of racialized narratives persisted in spite of the equitable forms of instruction taking place, and how these racialized narratives reflected or ran counter to dominant neoliberal ideologies. I employed qualitative interview study methodology (Miles, Huberman, Huberman, & Huberman, 1994; Yin, 1994) and drew upon standpoint theory (Haraway, 1988) and critical race theory (CRT) along with sociocultural theories of learning and identity, to ensure that I was centering the stories of these girls of color, in order to understand their experiences in their own words. The major data sources for this study included 8 h of interview data, with focal girls of color and their classroom teacher, and field notes based on 23 h of classroom observations.

Students were asked about their racial and ethnic identities and responded with various racial self-identifications. They identified as "South Asian, Cambodian" (Jane) and "African American" (Leilani), or "mixed" (Lia), "mixed race" (Mya), and "multiracial" (Carlin). Gena's self-identification stood out in that, while she identified as Filipino, she first listed all the ways in which she did not identify yet was assumed to by others (Nazemi, 2017). Because of the sensitive nature of asking people about their racial identity, I accepted the language that students provided and respectfully probed for further clarification when given the opportunity. For example, Lia identified as "mixed," and she expanded on this term to say: "My, umm, my mom is White and Black, and then my dad is Black." Yet, when I asked Mya about her racial identity, she said, "Pretty much like mixed race," and throughout our conversation it never felt appropriate or respectful to probe further to know the specifics behind her chosen racial identification.

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[2] All names are pseudonyms to protect participants' anonymity.

[3] To be consistent, I chose to capitalize the word "White" in instances where it marks one's racial identity. For example, "Ms. Williams is a White woman teacher." In other instances, where it is used as a noun or adjective, I do not capitalize it. For example, "Neoliberalism and Institutional racism support white supremacy and white hegemony by leaving whiteness as unmarked."

Overall analyses took place in two phases, where student level data was first examined to explore and center their racialized narratives, and then classroom observation data helped describe classroom instruction as well as how students interacted with one another, their classroom teacher, and the curriculum. The second phase of analysis also situated students and their classroom within the larger social context in which these people and places are located – in order to explore how students' narratives ran in support of, or counter to, greater discourses of institutional racism and neoliberalism.

This study grew out of my work as a mathematics instructional coach in a secondary teacher education program in the Pacific Northwest. I first came to know the classroom teacher, Ms. Williams, through her and her school's partnership with my university. I observed and learned quickly that, unlike any other teacher I had met, Ms. Williams' attention to her students and their identities – both academically and racially – was something unique and worth understanding more closely. This in addition to the AP Statistics classroom being racially diverse in ways unlike any other upper level mathematics classroom[4] I had seen, made for a phenomenon I wished to further explore. This means that in some ways, this classroom was typical of other large urban classrooms in the United States (e.g., Delpit, 1995; Grenfell & James, 1998; Howard, 1999; Nieto, 2004; Stiff & Harvey, 1988; Weinstein, 1985; Zevenbergen, 2003) – given the White and woman identifying teacher and the majority of students being of color. At the same time, this classroom was atypical for an upper level mathematics classroom in terms of the racial composition of students, making it a useful site for exploring the experiences of students of color in mathematics (Bol & Berry, 2005; Viadero, 2002).

My research identity was central to my relationship with the focal students, the classroom teacher, and the school at large. I am a woman of color, who at the time of data collection was completing a doctoral degree in education. Because of space limitations, and for the purpose of this chapter, I have not further discussed my positionality as a researcher here. See Nazemi (2017) for more information about my positionality as well as more about the research design of the study.

3 Key Findings

Overall, findings indicate that even within a classroom context that reflects ambitious and equity-oriented instruction and is organized to support students' academic identities and mathematics learning, this classroom is a site in which racialized discourse persists regarding how students are positioned as doers of mathematics in

[4] The AP Calculus classrooms at this school were racially diverse as well, but in different ways. Briefly speaking, there were a large number of African American or Black and Pacific Islander identifying students in this AP Statistics classroom, unlike the AP Calculus that had a large number of East Asian identifying students in addition to a much larger number of White students.

relation to how they racially identify or are identified by others. This section is organized around three main findings, each one following from the previous one(s). These findings, in order, are called Representation Is Not Enough, Attending to Student Learning and Students' Identities, and Even with Equity-Minded and Ambitious Instruction, Racialized Narratives Persist.

3.1 Representation Is Not Enough

When Ms. Williams first came to Champlain High School, she noticed right away that there was a large number of students of color at this school, and there were varying institutional supports – such as an afterschool one-on-one tutoring session – in support of all students. With her and the AP Calculus teacher's help, both AP Calculus and AP Statistics started being offered at this school, around the same time the school adopted a new attraction as a STEM magnet school.[5] Offering AP mathematics courses at a racially diverse school such as Champlain is exceptional based on usual trends across the United States (Flores, 2007; Oakes, 2005). At the same time, Ms. Williams remarked that her AP Statistics classroom looked different than her colleague's AP Calculus classroom. In speaking to her concern about traditionally underserved students, here Ms. Williams contrasted her AP Statistics classroom to the AP Calculus classroom at Champlain, expressing that these courses were made up of very different populations which could be sending different messages to students:

> [A]nother thing is that socially, the messages that they're getting about where they should be or who should be there, who belongs. You know? Like even the fact that we have AP Stats and we have two sections of it now, and two sections of AP Calculus, it's so exciting to me because when I started here we had neither of those classes, now we have four. So that's really exciting. And then in the last couple years, I mean having kids in any of those classes is powerful. However, if you walk into AP Stats or you walk into AP Calc, again you will see a divide, you will see a much higher proportion of Asian, Asian-descent students in Calc, and a much higher proportion of African American, East African, and still Asian American students, but even among the Asian American, like more of the Pacific Islanders are showing up, or you know Filipino students are showing up in my AP stats. So there's definitely a divide in the, you know, of the color of the students in my class and so they're getting messages about what they can do and can't do.

Above, Ms. Williams noted that students were getting messages about who can and cannot be successful in mathematics. A message that either comes from, or is affirmed through, the contrasting racial composition of AP Statistics versus AP Calculus.

[5] In the United States, "magnet" schools attract students, across regular boundaries or school zones, through providing specialized courses or curricula. In this particular case, the specialized curriculum was in Science, Technology, Engineering, and Mathematics (STEM).

Through informal conversation with Ms. Billings, the AP Calculus teacher at Champlain, I learned her take on the contrast between her and Ms. Williams' AP classes. Ms. Billings, like Ms. Williams, explained that across the mathematics department at Champlain, teachers were concerned about status differences between AP Statistics and AP Calculus. There was the worry that these two AP classes were turning into different streams of mathematics, with AP Statistics being considered the "easier than calc" stream (Field Notes, January 26, 2016).

One major challenge that both Ms. Williams and Ms. Billings described to me was that many students did not see AP Statistics as an Advanced Placement class, a perception confirmed by the fact that compared to AP Calculus, few students took the AP exam for Statistics. Ms. Billings added that students looked at their peers that ended up taking AP Statistics instead of AP Calculus and based on students' racial identities, "low-status students" were seen as connected to AP Statistics while "strong students" were associated with AP Calculus. In response to this concern, teachers were trying to send the message to students that both classes, while different, are challenging. In fact, the year I collected my data, Ms. Billings said that mathematics teachers at Champlain introduced yet another mathematics course which supported students to pursue college mathematics by further assisting them with precalculus curriculum – with the hopes of supporting students to stay in the Calculus/Algebra mathematics stream, rather than assuming AP Statistics was an easy way out (Field Notes, January 26, 2016).

In the case of a classroom like this one, at first glance one could assume that equity had been achieved as there were a large number of students, engaged in an Advanced Placement mathematics course, who identified racially with historically underserved populations. While having a disproportionate number of people of color in a negative circumstance (like low-tracked or remedial mathematics courses) can signal inequity, having an appropriate representation of people of color in a positive circumstance does not necessarily signal equity. In other words, as Gutiérrez (2012a) indicates, access to high-quality mathematics is only one aspect of equity, the other aspects being achievement, power, and identity. Access refers to the resources and learning opportunities that are made available to students, while achievement has to do with access to, participation in, and success with mathematics. Identity has to do with attention to oneself and to others with the intent of supporting one's sense of self, and power is about social transformation (Gutiérrez, 2012a). Even so, in later work Gutiérrez (2012b) pushes these aspects of equity even further by recognizing that access and achievement have to do with *dominant* mathematics – which is the mathematics that is upheld in colonial and Eurocentric contexts of learning. When we attend to identity and power, we begin to push back against dominant forms of mathematics in support of doing *critical* mathematics – finding ways to engage with and change mathematics, in ways that are built around who we are and what we know in our personal and academic lives.

3.2 Attending to Student Learning and Students' Identities

Given Ms. Williams' interest in knowing how students identified racially and ethnically, each year at the beginning of the school year, students were expected to write an anonymous letter, in which they were prompted to talk about their racial/ethnic identities and their experiences with racism. At the same time, Ms. Williams designed instruction to make the relationship between race and various social injustices visible. For example, she described how she designed a group-worthy task about institutional racism and the death penalty and how this was a context, both potentially interesting and relatable for many of her students and useful for showing a power of statistical methods. Here she told me about students' responses about their racial identities and experiences of racism. She relayed that she was surprised at how varied their responses and their levels of ease in disclosing information were – even when it was anonymously done:

> I'm very curious about students' identification, and we try to get some of that out of their letter, but some of them don't open up with that. In stats, we asked a little bit later about experience of racism, because we were going to do a task about racism in the justice system and like statistics that show, like clearly show, that there [are] more incidents of getting the death penalty for African American male, than for White male, and also, based on not just the color of your skin but the killer. Getting the death penalty for having killed someone but also for who you killed. So, like to get into that task we ended up asking them about racism and what do they identify as, and so it was surprising to see what some of them identify as, and it was anonymous so it was surprising to see that some of them were very open and then others gave such vague responses.

For Ms. Williams, it was important at the start of the year to learn about her students' racial identities, especially in preparation for a task she did with students about racism in the US justice system. She found students' writing about their identities "surprising," either because of how forward students were or how vague they were. Through her extensive ethnographic study of Columbus High School, Pollock (2004) classified students' race labels or how they described racial identity, as being either "matter of fact" or "suppressed" all together (p. 9). Students' responses about their racial identity, in the anonymous letter they wrote to Ms. Williams, follow what Pollock (2004) found. Specifically, Ms. Williams' students' responses about their racial identities tended to be straightforward and open (or very matter of fact) or vague (or suppressed) all together.

Throughout our conversations, Ms. Williams expressed how conscious she was about her identity as a White teacher vis-à-vis her students' racial identities. As one example, Ms. Williams described the role she felt her racial identity played in how she related to students:

<u>Ms. Nazemi</u>: Do you feel that your racial identity plays a role in how you relate to your students?

<u>Ms. Williams</u>: Umm I wish that it didn't, but I'm sure that it does. […T]here's definitely like a White female expectation, there just is, I know there is. I'm not someone who tries to take on anything, honestly, that tries to take on anything that's not my own personality but every once in a while when I'll be, I'll say something silly because it is just *my* silly, you know? Some kids will definitely get it, and

> then I'll just see a look on a face that's kind of like…um sorry I'm not being clear…but it comes, like the look on a face kind of says to me that they don't see something genuine. That's so…that's so sad to me that this was my moment of really being genuine, but it's still gonna register with people differently based on their identity.

Ms. Williams described here that her identity as a White teacher had a big impact on how she related to students and how they might or might not relate to her. She expressed the worry that she might be seen as disingenuous when she makes a remark or says something humorous that could be culturally nuanced/specific and that it might be taken up differently depending on the student and how he/she identifies. At the same time in the following excerpt, we can see that Ms. Williams further described white guilt as part of her identity that she tried to be careful about with her students.

> [O]verdoing the sympathy and guilt thing is another part I think of my identity. I think some of my students are tired of that as well. I mean like how much apology do they want? They just, they want action, they want to see things in a way that really fits everybody.

Above, Ms. Williams implied that through apologizing to students for her racial identity, it was like she was asking for sympathy for her guilt around being White, rather than taking the time to adjust to what students' needs are based on their identities. An important aspect of understanding whiteness and White teacher identity,[6] according to Paley (2009), is white guilt, as Ms. Williams has shown us.

Pedagogically, Ms. Williams' AP Statistics classroom, much like the greater Champlain High School context, employed CI, a prominent example of equity-oriented pedagogy (Cohen, 1994), which takes a stance toward instruction that has been shown to support both students' sense of selves and their opportunities to engage in and succeed with ambitious high-quality curriculum (Boaler & Staples, 2008; Cohen, 1994; Cohen & Lotan, 1997; Cohen et al., 1999; Jilk, 2010). In Ms. Williams' AP classroom, like CI classrooms in general, students were familiar with the structures and expectations in place. Since this ambitious and equity-minded approach was typical across the entire mathematics department, students spent years working in groups, supporting one another and/or asking for support from one another, as well as asking the teacher group questions and turning in group quizzes for assessments. Additionally, because this was an AP classroom, students were given the opportunity to write the AP College Board exam in May, prior to which the mathematics content was selected and presented in preparation for the exam. After the AP exam was administered in May, students spent the remainder of the school year working on a group project that involved doing statistical data analysis around a topic of their choice for which they collected data. While this is not within the scope of this chapter, this activity, within a complex instruction framework, was both culturally responsive and relevant to the lives of these students.

[6] A limitation of this study is that I did not engage in a full treatment of the teacher's racial identity. I expect that other work, and future research in mathematics education, takes up issues associated with teachers' racialized identities.

Elsewhere (Nazemi, 2017) I have explored in great detail the principles of CI and how these are implemented at a high-quality level in Ms. Williams' classroom. I described these tenets as Group-Worthy Tasks; Organization of Group Work and Establishing Group Norms; and Student Status: Highlighting Multiple Abilities and Assigning Competence. It is believed that when simultaneously enacted, these principles in CI support equitable participation and increased student learning and success (Cohen & Lotan, 1997). For this purpose of this chapter, I will focus on the third tenet that deals with student status, as this closely relates to recognizing assumptions around students' abilities – often following what Martin (2009) referred to as the "racial hierarchy of mathematical ability" – and disrupting these assumptions.

Drawing on expectation state theory (Berger, Cohen, & Zelditch, 1972) which describes status characteristics in group settings, CI recognizes that students come with varying (high or low) agreed-upon social ranks, which shape students' interactions, participation, and learning. Status is based on perceived intelligence among groups and tends to mostly affect students based on their less normative and privileged identities – around race, class, gender, language, religion, etc. (Cohen & Lotan, 1997). "[S]tatus characteristics become the basis for the group's expectations for competence for its members: low expectations for low-status students, and high expectations for high-status students" (Cohen et al., 1999, p. 7). As an example of a status problem, Cohen (1994) found that popular students and students who are expected to do well academically tended to dominate and influence group discussions. Research has shown that through addressing status problems, CI can "promote equal-status interactions amongst students, creating opportunities for all students to engage with and learn from rigorous math tasks within a cooperative learning environment" (Jilk, 2010, p. 6).

I observed Ms. Williams publicly assigning competence to students seen as having low status in the classroom. For example, in an interview, Jane, who was not routinely seen as a top student, describes here how CI practices supported her to feel successful in Ms. Williams' class:

Ms. Nazemi:	Can you tell me about a specific time this year in Ms. Williams' class where you felt successful in math, and what was it about that time that made you feel successful?
Jane:	So there was a unit we did on correlation, and I was like really understanding the math, and [Ms. Williams] encouraged me to teach others when we were doing these test corrections, because once you get a certain amount of points correct on the test correction, you don't need to do the corrections but just help other people. And I felt like giving me the opportunity to help other people validated what I knew and what I was learning.
Ms. Nazemi:	And that felt good?
Jane:	Yeah.
Ms. Nazemi:	How were your peers with that?
Jane:	They were good. Which is why I like the concept of student-student teaching, because it helps people understand things better.
Ms. Nazemi:	And that was your time to shine!
Jane:	Yeah.

As Jane notes above, Ms. Williams deliberately worked to increase the status of students who were struggling in mathematics compared to their peers. Specifically, Jane described here that Ms. Williams took the time in her classroom to uphold the status of students so that they were seen as a rich source of learning to their peers. Ms. Williams worked to increase her status in her group, positioning her as a great resource for her peers and their learning.

Having a teacher recognize multiple abilities and at the same time publicly assign competence to a student reflects one of the important components to enacting CI (Cohen et al., 1999). Assigning competence, as in this case between Ms. Williams and Jane, is about publicly naming the intellectual contributions of a student within a group, a move that is especially important when a student – like this girl of color – might have lower status in the classroom context. Research has found that through open-ended mathematics curriculum, establishment of norms around accountability and support, and "treating" status problems through "assigning competence" to low-status students, equity can be advanced by supporting students of color to learn, succeed, and demonstrate their learning of mathematics (Boaler & Staples, 2008).

3.3 Even with Equity-Minded and Ambitious Instruction, Racialized Narratives Persist

Even with the affordances of an equity-driven and ambitious form of mathematics instruction, and a thoughtful race-conscious teacher like Ms. Williams, students' experiences showed that racialized discourses still ran deep within and among their peer and racial groups. In this section, I focus on Gena, Leilani, and Carlin who each identified racialized narratives that shaped how they were perceived by others and how they perceived themselves with respect to others, in the context of Ms. Williams' AP Statistics classroom.

When asked, "Do you feel that your racial identity plays a role in learning mathematics [in Ms. Williams' AP Statistics classroom]?", all students except for one (Gena) said no. However, over the course of the interviews, all students discussed ways that were suggestive that their racial identity did indeed impact their sense of selves, their mathematics learning, and their mathematics learning opportunities, in Ms. Williams' AP Statistics classroom. Based on an analysis of interviews with the students, and classroom observations wherever possible, it appeared that the central way in which racial identity mattered for their learning and opportunities to learn was due to how they were racially seen by others and in relation to other racial groups. This was especially true in regard to assumptions about different racial groups' intelligence in mathematics.

Gena, who identified racially as Asian, was the only student who answered "yes" to my question regarding whether she felt her racial identity impacted her learning of mathematics in Ms. Williams' class. In her response, she referenced the well-known stereotype about Asians being good at mathematics (Cvencek, Nasir, O'Connor, Wischnia, & Meltzoff, 2015; Nasir & Shah, 2011). She felt that because of the assumption that all Asians are good at math, during group work she was assumed by her peers to be good at math. Specifically, she said:,"I think it's cuz people are just like 'Asians can do math,' and I'm like no, not all Asians can do math, cuz I know a lot of us don't know what the heck is going on!" In class, I observed Gena working with her group on their final statistics project where they were administering a survey (to their classmates at Champlain) based on a topic they were curious to explore further using statistical methods. Gena's group was made up of one Muslim student (I am basing this on her headscarf), one Black Muslim student (I know this because I interviewed him), and Carlin who is mixed race (focal student in my study). Gena's group decided to explore the relationship between lack of sleep and student's race. Gena was telling me about her girlfriend (who is White identifying) that does not seem to understand why Gena – who has a part-time job on top of keeping up for her studies – cannot seem to get enough sleep. As Gena tells me about her frustrations of keeping up with her job and school and that her girlfriend "doesn't get it" because she does not need to work to help her family,[7] Gena's peers in her group kept asking Gena questions about considerations they need to make for their survey. Gena seemed happy to take on a leading role in her group and use her personal experiences as a motivation behind exploring lack of sleep and race together (Field Notes, May 24, 2016), yet her positioning as a leader in the group and as knowledgeable about the topic – perhaps more experientially rather than mathematically – was noteworthy. Gena's leadership in her group suggested to me that other students likely see her as intelligent and knowing, possibly because, as I observed, she was the only Asian student in her group at that time.

Leilani, who identified herself as African American, is the only other student that brought up the "Asians are good at math" stereotype, again in the context of group work. In her case, however, this assumption was brought up in terms of what this stereotype meant for students who are not Asian and especially students that identified as Black or African American, like her:

> I think sometimes people, you know, like uh the stereotype is that Asians are really good at math, so umm when you're in a group with like all Asians and you're a Black kid, sometimes you might feel like okay they're gonna think I'm not as well as them in this. Then it starts to get to your head, that you know maybe I'm not as well as them, and you know sometimes they say certain things. You'll agree with it even if you know it's not right because you're like, "Oh, they know," but sometimes you're right. So I think it's important that people just be secure with what they know and not try to feed into a stereotype or umm something that's like working against you.

[7] This is another instance where class appeared to be a salient social marker for students. Gena and I had a conversation where I was suggesting that there could be factors and circumstances that, along with race, predict sleeping patterns. She seemed willing to listen to my suggestion but still seemed to see the contrast between her and her girlfriend as strictly racialized.

Leilani's discussion of the stereotypes between Asian and Black students' contrasting mathematics abilities fits with what other scholars have found (Martin, 2009; McGee, 2016; Nasir & Shah, 2011; Shah, 2017). Specifically, it is common for students of color to talk about – and place themselves and their peers within –what Martin (2009) called the "racial hierarchy" of mathematics. In this racial hierarchy, "students who are identified as Asian and White are placed at the top, and students identified as African American, Native American,[8] and Latino are assigned to the bottom" (p. 315). Building on Martin's work which looked closely at the prevalence of racialized discourse in mathematics education (research, practice, etc.) as a whole, Shah (2017) interviewed various students of color in the context of their mathematics classroom. He also found that, according to students, Asian and White students were positioned at the top in terms of ability and performance in mathematics, while students who identified as Pacific Islander or Black were positioned near the bottom.

I asked Leilani for a specific example to illustrate her claim about how poorly Black students see themselves and their mathematics abilities, as compared to Asian students. In response to my query, Leilani further highlighted the negative consequences of the "Asians are good at math" stereotype for other groups of students:

> I think that I've seen people make assumptions about themselves. It's like, sometimes I'll hear, Black kids in class and they'll be like umm, they'll get really good scores on a test and then they, maybe the Asian kids got a lower score, and they'll be like "Wow! I did better than the smart kid," and I'm just like, "Wow you, you think you're not smart…like you should feel that you're smart. You shouldn't think that, that you know that you're less than them in whatever you're doing." I hear people say like, "I gotta get to do the project with the smart kids" or "I gotta sit by the smart kids," you know? So I feel like that's really degrading to, to think that uh you're not smart.

Consistent with Shah's (2017) findings, Leilani, like many students of color, experienced the stereotype that they are not as smart as their Asian peers as "degrading" to their identities as Black learners of mathematics. It is worth noting, however, her counter-narrative is one of resilience, of being aware of assumptions that can have consequences for her and working hard not to let them. Leilani's feelings around coping with the stereotype that Black students are not as good at mathematics relate to what we will next see with Carlin's experience.

Carlin, much like Gena and Leilani, spoke about how her racial identity played a role in learning because of assumptions about intelligence that peers made based on her perceived racial identity. Carlin identified herself as "multiracial," which she later elaborated meant she was Black and White. Different from Gena and

[8] While Martin's (2009) racial hierarchy of mathematics refers to Native American students, an unfortunate shortcoming of much of the important work around race and racialization in mathematics education (and educational research in general) is the omission of Native American and Aboriginal Peoples. I recognize that Native American and Aboriginal Peoples have been the most underserved populations in this county's educational system. While it is not within the scope of this chapter to attend to these populations of students, I want to be explicit that I recognize how ignored and made invisible these groups have been both historically and presently. Future research will tend to this shortcoming more carefully.

Leilani, Carlin described how she could "pass as White," pointing to how context-dependent one's racialized experience can be, especially when one is White passing. Here, Carlin reveals how complicated it is to navigate her multiracial identity, especially given that both racial markers that she identifies are associated with very contrasting assumptions around intelligence (emphasis added in italics below).

Ms. Nazemi:	Do you feel like your racial identity plays a role in your learning?
Carlin:	Not so much, *but since I'm Caucasian, people, people kinda expect that I'm smarter for some reason*. And I'm like well that has nothing to do with my intelligence. It's just… kinda just a statistic. And I was like…they…they like… *I hang around African American, Black people a lot, and they expect me to be smarter for some reason. But I'm just me.* I'm just learning how I learn, and it's…it's weird.
Ms. Nazemi:	How do you know they expect you to be smarter? Like what…
Carlin:	They say it.
Ms. Nazemi:	They say it?
Carlin:	Yeah.
Ms. Nazemi:	They say it, just like straight up.
Carlin:	mm hmm.
Ms. Nazemi:	Like they say, "you're White, you're smarter?"
Carlin:	No, they don't say "you're White, you're smarter." It's like *when I struggle and I ask them for help, they'll be like, "you're White…you should know this." I'm like well that's kinda weird. I'm also Black as well so…should I know… should I not know it because I'm Black?* Like that's weird for them to say like, so I kinda just shrug it off 'cuz I don't let that stuff bother me.

Carlin explained here that passing as White for her meant that she was expected to be smarter. She described above that when she was struggling with a mathematics concept, her peers would tell her that she should know it because she's White, to which she responded that she's also Black. This finding is consistent with Hobbs (2014) who outlined the effects on one's racial and cultural identity, when passing as White. McGee (2016), who drew upon this work, showed that while it might appear that students of color benefit from passing as White, there is also much "lost by partial or full rejection of one's racial and cultural identity" (p. 1654). She further describes that feeling pressured to limit parts of one's racial and cultural identity is an attestation to the enduring and continued manifestation of white privilege through racism and white hegemony.

Assumptions about Carlin and her intelligence shifted when she was seen in comparison to White (boy) classmates.[9] Elsewhere, Carlin describes the way she was viewed in relation to White students in her class (Nazemi, 2017) and that this felt different than when students of color were seeing her as White. In particular, while I did not have the opportunity to observe Carlin interact with her White classmates (as she was not grouped with them when I was observing the classroom), I did observe the two White (boy) students in the classroom and the ways in which they dominated whole class discussions and even their small group discussions as

[9] While gender is not a focus of this chapter, it is important to note that all six focal students are girls of color, and the only two White students in the class are boys, making it difficult at times to separate out racialized and gendered identities.

well. In my observations, both White students were often the first to ask a question and/or offer an answer, often arguing with students' responses in front of the class. More generally, while Carlin relayed that she was often presumed White and presumed smart in the presence of others, when she was in the presence of these White students, she was positioned as less knowing – likely because both her multiracial identity and her gendered identity were seen as inferior to a "full White guy" (to use her language) identity. Attempting to manage how one is seen by others is as degrading as it is futile. Carlin's stories help us to see that while sometimes we cannot verbalize or fully make sense of why or how we are seen by others, the implications for our identities, if not spoken or understood, are deeply felt. At the same time, much like what Gena shared with me about how she felt others assumed her to be smart in mathematics because she is Asian, Carlin shared that she is assumed to be smart when she is seen as White. And, much like Leilani's explanation of Black students being seen as intellectually inferior to Asian students, Carlin's anecdote describes that Black students are also being seen as intellectually inferior to White students. Additionally, in contrast to Carlin's presumed whiteness and smartness, when interacting with a "full White guy" student, she is powerless due to the complexity of her identity, which is not limited to White but includes Black and girl.

More generally, the students I interviewed confirmed that discourses regarding a "racial hierarchy of mathematical ability," where Asian and White students are assumed to be at the top and Black students are assumed to be at the bottom (Martin, 2009), were alive and well in Ms. Williams' class. Regardless of Ms. Williams' stance toward students' racial identities along with her enactment of equity-oriented practices in support of student learning, it is apparent that hierarchical racial narratives circulate and strive having deeply felt consequences for students' sense of selves. Furthermore, while Carlin and Leilani's narratives help us to understand the complicated disconnect between how we see ourselves and how we are seen by others, they also remind of the ways in which our identities are complicated and vary or shift from context to context. The complexity and fluidity of identity are particularly clear in Carlin's stories about how she is seen by others but how that shifts when faced with a White boy student who exerts power over her gendered and racial identities. Consisted with a sociocultural perspective, as taken up in this chapter, we've seen here that identity is an ever-changing complex, and context-dependent, notion that is made up of how we see ourselves, as well as how we are seen by others.

If Ms. Williams' ambitious and equity-driven classroom attends to supporting student learning while being responsive to students' identities and experiences, yet racialized narratives persist, what does this mean for teaching and teaching education?

4 Implications for Pre-service Teacher Education

Various focal students' narratives revealed that racialized assumptions were routinely made by their peers regarding these students and their abilities in mathematics. The narratives regarding students' positioning in terms of intelligence in mathematics and racial group membership reflected what Martin (2009) called the racial hierarchy of mathematics ability. Particularly, students recounted ways in which peers positioned Black students as less capable in mathematics than Asian and White students. However, focal students also exhibited counter-narratives and resisted the racial hierarchy. For example, Leilani, a Black student, explained that while she remained aware of the racial hierarchy, she worked to actively resist dominant racial narratives around Asians being good at mathematics because of what this meant for her and other African American students (Nazemi, 2017). Her counter-narrative allowed her not to let the stereotype work against her. More generally, this study suggests that pedagogical innovations like CI, while equity-oriented, still need to consider the racialized narratives that circulate among the members of a classroom and find ways to disrupt them. After all, even with Ms. Williams' enactment of CI, and in particular the way in which she would assign competence to uplift a student's status, students continued to function within the "racial hierarchy of mathematical ability" allowing it to thrive even in this equity-minded classroom.

Recall that at Champlain, AP Calculus was primarily composed of White and East Asian students, while AP Statistics was primarily African American and Pacific Islander students. The mathematics faculty had been concerned about the perception of AP Calculus and AP Statistics as two racialized mathematics tracks. I recall this because addressing racialized discourse (such as the racial hierarchy of mathematics that students experienced) cannot happen only in the context of one classroom but needs to happen across all classrooms and across entire schools. We need to consider our students as nested within our classrooms and schools and further situated within the larger social contexts and discourses in which these positions are located. Discussions like those among the mathematics teachers at schools like Champlain are necessary, yet from there, immediate action is also necessary. Highlighting student success, persistence, resistance, and resilience, as an increasingly large body of work has done (e.g., Berry III, 2008; McGee & Martin, 2011; Stinson, 2008), is important.[10] At the same time, we cannot stop there. We cannot simply give students access to upper level classrooms yet leave them to fend for themselves in the hopes that they are resilient enough. As Ahmad White suggests, "Resistance is a symptom of the way things are, not the way things necessarily should be" (from the TV show Atlanta). Thus, meeting hardship with resistance or resilience is a symptom of a problem, a problem of inequity. It is not on students of color to take full responsibility for their learning and figure out how to succeed in mathematics, all while remaining true to their sense of selves. After all, while AP Statistics holds a large number of students that are "normally" not present in upper level mathematics classrooms, it was also seen as the lower track of mathematics compared to AP Calculus, which as we know held larger numbers of White and East Asian students – those who are already seen as occupying the top-most tier of the racial hierarchy of mathematics.

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[10] Some scholars have called the resistance of students of color in mathematics as their "oppositional identity." See Gutstein (2002, 2007) and Martin (2000).

Teacher education programs are in a unique position to reach teachers at the earliest point in their careers. With this fact, as well as given that some teachers may eventually consider educational leadership at the school or district level, teacher education programs must take the time and space to engage with and support pre-service teachers to understand the urgent implications of making assumptions about students' abilities based on their identities and tracking students, within and beyond their classrooms. Starting these conversations with pre-service teachers can help educators and school leaders work together to find ways to ensure that their classes fairly represent the demographics of the larger community and are disrupting racial hierarchies and that individual students do not bear, alone, the responsibility for succeeding in mathematics. This is why I say that representation is not enough.

As teachers, whether pre-service or in-service, there is much we can learn from Ms. Williams and her AP Statistics classroom. Ms. Williams thinks deeply about her own identity and takes the time to learn about her students' identities and experiences. One reason she gathers this sort of information on her students – for example, through their anonymous letters at the start of the year – is so that she can build tasks that are relevant and responsive to students' lives, identities, and experiences. Ms. Williams is aware that her position as a White woman teacher has consequences for if/how she relates to her students, a challenge that she continues to recognize and grapple with. She also enacts an equitable and ambitious form of instruction, where she is constantly paying close attention to students' status in the classroom and working to respond to this in ways that uplift students and disrupt assumptions about their abilities in mathematics. Hearing about the racialized narratives or discourses that shape these students' relationship to mathematics and to one another suggests that Ms. Williams tends to her classroom in ways that support students and their sense of selves while learning mathematics. However, it is important to note that while noticing and addressing status issues are important, issues of power and privilege cannot always be fully addressed in the classroom context. Even if students are seen for their multiple abilities with the teacher deliberately noting and addressing unequal participation of students due to varying status in the classroom (Cohen et al., 1999), racialization is a present process, and classroom contexts inevitably affect how students are perceived.

Recognizing the persistence of racialized narratives even in an equity-minded classroom taught by a teacher who works to teach in relation and in response to her students' identities and experiences poses a new equity demand. These findings here suggest that as teachers, we need to continue to do the things that Ms. Williams does – take the time to explore our own identities and recognize how these affect the way we teach and relate to our students, learn about students and their identities and experiences so that we can build our curriculum and pedagogies to be centered around their lives, and respond to assumptions in our classrooms that position students in the "racial hierarchy of mathematical ability" by assigning competence and uplifting students' status. At the same time as doing these things, we must do more. Said simply, the persistence of students' racialized narratives in Ms. Williams classroom suggests that equitable forms of instruction such as CI, which partly support students' learning and identities, are necessary but not sufficient.

In particular, through students' own voices about their own narratives, teachers can help surface what students of color learning mathematics might be experiencing in their mathematics classrooms. Ms. Williams' students' anonymous letter about their racial identities and experiences is one way teachers can begin to do this. Without reaching out to know students, a teacher has limited access to students' identities, usually only how they are identified by school level data and what the student has shared with the teacher. By noting and privileging the voices and racialized experiences of students of color in the mathematics classroom, teachers can better connect with students, learn more about them, and support student learning and students' identities at the same time. This is a tall task. It will require seriously difficult work on the part of teachers – much of whom are White women – who must take the time to examine their own racial identities, for themselves and in relation to their students.

It is important for pre-service teachers to examine their own experiences of learning to help them decide on the kind of teacher they want to be. Or as Alvine (2001) puts it: "When those who plan to teach write about their own early memories of learning, they bring forward their embedded understandings about teaching and learning. In making those understandings explicit, they make them available to themselves" (p. 9). Given the ways in which identities form and shift through learning and teaching experiences, having pre-service teachers explore their identities – academic and personal – in the context of their own experiences of learning can uncover ways in which one's learning and sense of self was simultaneously supported by teachers. In teacher education, we can help pre-service teachers to recall their learning experiences, as well as the ways in which their identities have been shaped and shifted throughout their learning trajectories. In facilitating this work as teacher educators, we can help pre-service teachers begin to understand how their experiences with identities and learning might inform the ways in which they might choose to teach. What kinds of information do teachers wish their own teachers had about them? What things did their teachers do to make them feel included/excluded and capable/ incapable in mathematics? Did they feel they were treated differently because of their gender, race, sexual orientation, class, religion, etc.? What kinds of mathematics learning would have felt relevant to their lives and identities? Considering these things about pre-service teachers' own learning begins the identity work that we want teachers to continue doing well into their teaching careers, so that they do not ever forget these questions and considerations in their own classrooms and with their own students.

In addition to gathering information and making oneself aware of students' racialized experiences and identities, teachers need to find ways to engage with students in conversations around race and racism, within the context of mathematics tasks as well as the broader context of mathematics learning and assumptions around ability. I agree with what Shah (2013) suggested in his dissertation that looked closely at racialized discourse among students of color in mathematics. Specifically, teachers can immediately and consistently address assumptions around race and ability by treating "explicit invocations of racial-mathematical narratives with the same level of gravity as they would blatantly racist statements" (p. 120). One way of doing this, as Ms. Williams shows us, is through assigning competence to uplift the status of a student that might otherwise be seen as occupying a lower tier on the "racial hierarchy of mathematical ability." Student learning and students' sense of selves are what is at stake when racialized narratives circulate in the classroom. It is up to teachers to make note of and disrupt these racialized discourses that have harmful impacts on students' identities, mathematical and otherwise.

The challenging identity work of teachers, along with learning to recognize and disrupt racialized assumptions in the classroom, must start prior to when teachers enter their own classroom and thus in the context of teacher education. Taking the time to explore, articulate, and understand our own identities can contribute to our fuller understanding of the identities of our students and help us to think about how the affordances and constraints of our identities are in relation to the identities and lives of our students. In the context of teacher education, pre-service teachers must be given ample opportunities to explore their own learning experiences and learn to connect these experiences to how their identities are shifting and how they hope to be as teachers. When we come to recognize our students (and model this with pre-service teachers), just like ourselves, as holding shifting identities that are complex and intersectional[11] in nature, we can better nuance all the parts that make up who we are, as well as how these parts are navigated, negotiated, and can be upheld in the contexts in which we are teaching and learning mathematics.

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[11] While intersectionality has not been the framing or focus in this work, I still wish to assert that our identities are complex and intersectional in nature.

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Standardization and Borders in Mathematics Pre-service Teacher Education: A Duoethnographic Exploration

mutindi ndunda and Nenad Radakovic

1 Before We Begin: Opening Fragments[1]

What follows is the result of weekly conversations that took place from January 2018 to May 2019 between two colleagues, both immigrants to the United States, discussing some issues that affect our teaching practice as mathematics teacher educators Two colleagues, mutindi and Nenad, are professors and mathematics teacher educators in a state liberal arts college in Charleston, South Carolina. mutindi is an associate professor with over 20 years of teaching experience in academia. Her research and teaching have spanned fields such as STEM education and women's education in Kenya and in the United States (e.g., ndunda, 2000). She was born and raised in Kenya and has lived in many places including Namibia, Canada, and the Unites States. Nenad grew up in former Yugoslavia and has lived in the United States and Canada. He is an assistant professor in his fifth year of teaching mathematics education and research methods classes. His primary research is on social justice and transdisciplinarity in mathematics education (e.g., Jao & Radakovic, 2018).

We currently live in the American South and are both non-native speakers of English, immigrants who immigrated to the United States via Canada. mutindi immigrated in the 1990s to Canada from Kenya, and Nenad immigrated to Canada in the early 2000s from former Yugoslavia. Students in our classes are mostly white middle class women in their early twenties. As such we are othered both in the place in which we work and the place where we live.

[1] Here we evoke German romantic philosopher Novalis' idea of a fragment according to which a fragment is a genre of writing that is meant for self-reflection and also as a seed for other ideas (Gjesdal, 2014).

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We teach different sections of the same course, mathematics content for K-8 teachers, so we thought that this would be a good context for our discussion. In the beginning, we focused on making sense of a lesson that we co-taught in mutindi's section of the course. We soon realized that we had significantly different interpretations of the lesson due to distinct views on, and experiences with, the curricula and curriculum standards in different contexts (North American, European, and African) and that this should be the focus of our conversations. As our conversations progressed and our views evolved, we had our own personal crisis of representation – how should our views be represented. The conversations then turned to duoethnographic methodology and the ways that we could use it that is consistent with our experiences, needs, and identities.

We use regular script to represent individual voices and for the explanatory sections of the paper. As it will be explained later, there were sections of duoethnography that we could not solely attribute to one voice. This, emergent third voice, is written in italics.

We are weary of the typical linear thesis, what Jagger (2014) calls introductionliteraturereviewmethodologyresultsdiscussion. We find it inauthentic, and it does not depict what we did and how we talked throughout the year. This is why we start our chapter with a dialogue.

2 "Kyuo" or Speaking the "Truth"

mutindi: As a Kamba woman, we have a word, "kyuo," that our elders use to encourage those in the community to speak the "truth" of their experiences. This is because kyuo, if not spoken, cannot be used to make sound judgments. Kyuo spoken within a community of people helps in the making of fair judgment. If a judgment/decision is made without including others' kyuos, the decision/judgment is seen as incomplete by the community. Kyuo, therefore, encourages the inclusion of diverse voices and ideas.

Nenad: This is why I think it is important for us to have this conversation about our practice. Both of us teach the same course, mathematics content for K-8 teachers, and we have the common goal of preparing our students for teaching which involves being able to conceptually understand mathematics. And this common goal comes with common challenges: a large number (if not most of the students) have negative attitudes towards math and are also math anxious, they are not confident in their mathematical knowledge, and do not see math as a creative endeavor. So using our kyuos to make sense of our practice is crucial.

mutindi: The process of listening to you allowed me to question my experiences and thoughts. Our conversation was a process of reflection, and reflecting together is like having two different mirrors in front of you – your own mirror and someone else's mirror – and two mirrors project different images. This way you can see different characteristics of the image.

Nenad: This process of thinking together to solve the problem and making sense of how we think together about our practice brought me to the idea of duoethnography. I was reminded of John Mason's observation that research is making sense of ourselves in relation to others (as mentioned in Zaskis & Koichu, 2015). While we were working on this project, it became clear to us that it was important to make sense of ourselves and our experiences together. We do this not to erase our differences but to highlight and carefully describe them so that we can evolve and evolve together.

For this reason, we picked duoethnography, an emerging methodological approach introduced in the early 2000s (Norris & Sawyer, 2012). Being an emerging methodology, its rules are not completely established (Breault, 2016), and even if the methodology had a strict canon with completely established norms and rules, we would still break them since methodology should be molded to fit the goal – the most authentic description of the phenomena under investigation (Lincoln, Lynham, & Guba, 2011). This is consistent with Norris and Sawyer's (2012) assertion that "methodology should not dictate the form of a conversation of sentient beings in quest of meaning" (as cited in Breault, 2016, p. 778). Therefore, as we started to make sense of borders and what they mean to us, we had to come to terms of what duoethnography is to us. The realization of our vision of duoethnography did not come linearly – it evolved as we were telling stories about ourselves and working through our differences.

3 How We See Duoethnography

mutindi: Duoethnography allowed me to speak in a way that is not limited to a methodology that is sanitized, pure, and benign. In other words, the methodology is not objective and harmless – it has intended and unintended consequences. It allows us to sit together, make meaning, ask each other questions, make statements, and have somebody to deepen the understanding of the experiences. This methodology brought together "two complex beings situated within their own complex social networks" (Breault, 2016, p. 780). This reminds me of a Kamba saying that a calabash shines when it passes through multiple hands. Similarly, the idea gets finer when it passes through multiple minds. Eventually, it becomes difficult to discern where the final product came from. The idea that it is only in your head only speaks when it becomes integrated in the community (even though it is two of us, it is the community). You draw more into your identity, because we are multiple subjectivities.

3.1 The Writing Process: The Emergence of Collaborative Generative Writing

mutindi: The writing process was more than a year long, and it was a thinking process. The ideas that we explored sometimes got refined, sometimes they got rejected, sometimes our favorite ideas got rejected. The process of writing not only helped us reveal our identities but also helped us realize how we perform our identities. As we reflected on our kyuos, we were cognizant of our "colonized" minds that tended to veer towards methodologies that hide our identities.

3.2 Process of Creating Our Duoethnography

As we were communicating our ideas and "speaking out our truths" (our kyuos), we also needed to think about collaborative writingmethodologiesthat would enable us to capture our ideas and represent them. We had private ideas that we made public, but we also had ideas that became shared between the two of us and that we have decided not to share here. We believe that our sharing of ideas is an honest process, but what gets published is a political and subversive decision. In other words, there are aspects of our conversations about our institution, our lives, and our students that are intentionally left unspoken. We tried to preserve the authenticity as much as possible but not at the expenses of our standing in academia.

Writing played an important role in developing our duoethnography. We started with conversations about our practice centered around an activity we co-taught (described later in this chapter). The topics of our conversations included our teaching practice, our students, and our identities. After several meetings, we went through generative writing steps (Boice, 1990) as described by Esmonde (2017). Generative writing was the major strategy that we used to write about and reflect on our experience of teaching in this context. The process involved the following steps:

1. Individually writing for 10 min, not rushing through the process but also not stopping. At this stage, we did not worry about grammar, style, and sentence structure.
2. Individually creating a concept map based on the first step for 10 min.
3. With the concept map from step 2, writing a new version of the document described in Step 1.
4. Presenting our drafts to each other and discussing them.
5. Together, writing our duoethnographic dialogue and the common narrative.

Nenad: I think that you would agree that we started from a need to have our story told and then landed on duoethnography as a method. I also say that we struggled (in a productive, positive way of the word) to work out what this means to us. So, as we worked through the process, we knew that we would not be happy to tell two parallel stories or come to an agreement without extensive discussion. We also recognized that duoethnography is not a transcript and verbatim script of a dialogue. This view is consistent with other researchers' views on duoethnography (Norris & Sawyer, 2012; Zaskis & Koichu, 2015). We are also happy that Zaskis and Koichu (2015) realize that duoethnography is a process of reflection and reconceptualization of researchers' views. This process makes researchers vulnerable, but it also gives them space to vocalize and grow their ideas.

mutindi: The other way to think about this is that this approach allows us to stand close to our experiences as well as to distance ourselves from those experiences. In my practice, I often use the metaphor of standing on the shoulders of people whose views I'm examining. Standing on the shoulders enables me to see better, but it also exerts pressure on the person on whose shoulders I am standing, hence making them vulnerable.

3.3 Context

Nenad: As we were thinking about the context and our place at the college, we started reflecting on our practice as mathematics pre-service teacher educators. We started looking at our specific teaching practices and beliefs about teaching and acknowledged that they do not exist on the vacuum. This reminds me of the depiction of the complex relationship between teachers, students, content, and the environment given by Deborah Lowenberg Ball in her presidential address to the annual conference of the American Educational Research Association (AERA, 2018). She started by discussing an older (and in her mind, not completely authentic) model of teaching that comes from her earlier work, known as "the instructional triangle," showing the complex relationship between teaching, students, content, and the environment (Cohen, Raudenbush, & Ball, 2003). She then revised the model by accentuating the importance that the environment plays in classroom interactions and pointing out that the environment includes sociohistorical factors such as institutional racism and historical injustices. She defines and depicts the environment as the cloud that permeates all aspects of teaching such as teachers, students, content, and their mutual interactions.

3.4 College of Charleston: Historical Context

Nenad: When we look at the "cloud" of environments around the center that Ball proposes, we see that it permeates teaching and student-teacher relationship. The environments also consist of structures, institutions, and histories that shape the place. The environmental cloud that we cannot ignore in our analysis and the one that in our conversations we keep getting back to is the racist and white supremacist legacy and reality of the place where we work and educate educators. The College of Charleston was founded in 1770 and was only open to White men. White women were allowed to enroll starting in 1918 and Black students started to enroll in 1967 even though segregation was abolished in 1954. 250 years since its founding, the legacy of the college as an exclusive space continues. For example, the number of students (undergraduate and graduate) is around 11,000 yet, the percentage of African American students is less than 8% (College of Charleston, 2019) in a state where Black population is 27% (United States Census Bureau, 2019). Although the physical borders and barriers have been removed, the powerful structural barriers are still in place.[2]

mutindi: In this complex environment, where I have committed myself to educating pre-service teachers to be successful math teachers of all children, I often find myself struggling to make that happen. Most of my students don't consider mathematics a subject that they enjoy or the subject that, in their words, "comes easily" to them. Although teachers are critical gatekeepers with power to ether deny or grant access to mathematics, they often don't see themselves as such. The purpose of our collaboration is to reflect on our practice of training mathematically confident and critically literate educators.

[2] For the detailed descriptions of racial disparities in Charleston, please see Stacey Patton's report *The State of Racial Disparities in Charleston County, South Carolina 2005–2015* available at https://rsji.cofc.edu/resources/disparities-report/

3.5 *The Critical Episode*

In the beginning of this project, we taught an hour-long lesson to a group of pre-service teachers, third year teacher education students, based on a counting dots task (Takahashi, 2007). In the environment of mainstream math education in the United States, this is a standard exemplar of a rich mathematical task and has been adopted by many educators including Jo Boaler (Youcubed, n.d.). The task requires students to describe how they see the following pattern in order to help them find the total number of dots (Fig. 1):

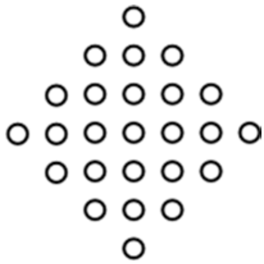
There are many ways for students to interpret this pattern making this a rich task – a task that contains multiple strategies, pathways to the solution, and representations (Boaler, 2015). For example, some students see the increasing sequence of consecutive odd numbers followed by a matching decreasing sequence: $1 + 3 + 5 + 7 + 5 + 3 + 1 = 25$. Others may realize that the pattern consists of a 3×3 and a 4×4 square array: $3^2 + 4^2 = 25$. The task gives students an opportunity to communicate mathematical ideas and to think deeper about mathematical objects (Takahashi, 2007).

This lesson served as the basis for the critical episode in our sense making of our teaching practice. When we were debriefing the lesson, we realized that we interpreted students' ideas differently which created a dissonance in a way that we think about math education. The dissonance caused us to investigate our ideas and to question our role as mathematics teacher educators.

Nenad: As we reflected on our practice and the task, I started to introduce curriculum and professional standards and mapped out what Common Core Mathematical Practice Standards (National Governors Association [NGA], 2010) and Standards for Preparing Teachers of Mathematics (Association of Mathematics Teacher Educators [AMTE], 2017) are aligned with the task.

mutindi: I was surprised by your push towards introducing standards. There was a borderless math, and then you introduced borders with your interjection of standards. We very quickly went to what we learned to be "important." Students to me were problem-solving, and there goes Nenad and starts talking about standards. You brought in the whole idea of structure and of canning the knowledge and the experiences and aligning it to your own standards-based agenda. It surprised me how quickly you went to that. It was so interesting how we landed on standards as if the task only becomes valuable when connected to the standards. Teachers have been conditioned that a task is legitimate only if it can be aligned with the standards.

Fig. 1 Counting dots task



Nenad: Lately I have been in love with standards; as an assistant professor on a tenure track, they give me the sense of belonging, sense of compass, and sense of direction for me and my students. I got enculturated in the practice/problems of math education through my interaction with others who have found their way around standards. It is almost like interpreting a religious text.

mutindi: The standards for pre-service teacher preparation (similar to K-12 standards) have been around in the North American context for decades, and there is no evidence that student and pre-service teacher understanding of mathematics has improved. There is a paradox in the fact that standards list minimum competencies, and yet we are happy when the standards are "met." This standardization means "the erasure of academic freedom resulting in lost opportunities for originality, creativity, dissent, and discovery, the very raisons d'être for educational institutions" (Pinar, 2004, pp. 229–230).

Nenad: I agree with you that standardization leads to lack of flexibility. But I am hoping that if I play by the rules I can bring in the nonmeasurable concepts such as creativity, beauty, and discovery into the standards. For example, if I wanted my students to appreciate the beauty of mathematical objects, I could align the activities with the Common Core practice standard of making sense of structure. The Common Core standard of critiquing the reasoning of others could be transformed into respecting and listening to other people's arguments. As an immigrant, I tend to play by the rules and work within the system. I believe that I don't have the luxury or privilege to break them.

In our conversations, we began to see the idea ofstandardizationas an attempt to create borders, and as immigrants, we immediately thought of political borders and the violence that borders create and perhaps the "protection" that borders bring if you can play by the rules. We started to play with the idea of physical borders as a metaphorfor borders brought bystandardization.

3.6 Toward the "Political Border" Metaphor

Nenad: mutindi, you brought the work of Pinar into our conversations about borders. Particularly the overreliance on standards and attempting an impossible task of making education "teacher proof." Another way that you talked about Pinar is the idea of curriculum being tied to the test. There is also a strong movement in the United States that considers education as a commercial activity. For example, the Council for Accreditation of Educator Preparation (CAEP) refers to teacher education programs as educator preparation providers (CAEP, n.d.). The word "provider" implies that the purpose of a teacher preparation program is to deliver a service rather than being an institution of higher learning dedicated to the education of teachers.

Pinar also talks about social engineering: it works with behaviorist and positivist paradigms (Pinar mentions Edward Thorndike, the behaviorist). The idea is that "evidence-based" research and practice is "evidence based" because education is reduced to "measurable outcomes" rather than seeing education as a complex web of connections in which teachers and students work to co-construct the world.

This standardization brings limitations and constraints on teachers, students, and everybody involved. In other words, the standardization creates borders.

Again, the border is a metaphor. The border is also a strong word that evokes political borders. We are both immigrants from areas where political borders often bring the meaning of turmoil, strife, and suffering. I grew up in Yugoslavia in a town that is now in Croatia, close to the border with Serbia. The border was not enforced before the war because Serbia and Croatia were parts of the same country. There is a town of Sid in Serbia, and I remember travelling with my parents from Sid as a child and entering a village of Tovarnik which is in Croatia. There is a road connecting the town and the village, and before the war, you could travel in between without stopping. I remember being in a car with my mom and dad travelling along the road, and Mom just said, "See this row of trees, along the road, when we pass it, we will be in Croatia." The border between Croatia and Serbia was only bureaucratic, and it was not enforced. Prior to World War II, there was not even a formal border: both places were a part of the same county in various countries such as the Kingdom of Yugoslavia and the Austro-Hungarian Empire. This was before the existence of Serbia and Croatia in the modern sense. During the Yugoslav wars of the 1990s, there was a frontline between the two sides of the conflict (Serbian and Croatian) which could have been seen by the fact that for a long time, the facade of the buildings in Tovarnik were scarred by shrapnel. After the war, the formal border was established, and now instead of trees, there is an official border entry point with customs officials.

mutindi: Creation of political borders comes with power. You may not have an idea of what is in Mexico because you are in Texas: the border separates. The border and what stands on the other side are forbidden places.

Nenad: Borders define identity, and these standards are supposed to shape our identity. By naming the standards, similarly to naming and establishing countries, one also names the border: the activities that align with it and the ones that do not.

mutindi: Who has the power to name the standards?

3.7 Our Response to Borders

Nenad: The question we find ourselves asking is how do we respond to the borders created by standardization? We can choose to react to "the borders" in various ways.

mutindi: One way is to accept the standardization and the borders that are created. If you accept standardization, the "yardstick" that it creates (a standardized way to measure outcomes) gives us a way to compare populations and speculate on their needs which can give a semblance of equity. Sometimes you are afraid of the border because what is on the other side is another country. What do I find there? There are also people who live close to borders but never cross them. I choose to live on the side of the border not covered by standards. I do not cross over to the "land of the standards" because I feel that thinking is forbidden there or at least highly constrained. This reminds me of the banking system described by Paulo Freire in the pedagogy of the oppressed (Freire, 2000). Many oppressed people do not get the time to think. They have no power to move the border. Thinking creates possibilities and opportunities, and many are left out.

Nenad: So, reaction to the border is then also the question of power. You can react to the standards by superficial coverage, or you can embrace standards and try to use them to reach the full potential of the standards. This is a matter of interpretation; you can also recognize other standards and maybe even create other borders. What if you don't recognize the terrain, the standards' terrain?

mutindi: This reminds me of the Saan people of South Africa: they had deep knowledge of the physical terrain before the borders were erected, and they found themselves with no place to call home. These indigenous people, who have the best working environmental conservation skills, are landless and are in danger of dying out. I am thinking about those who understand the terrain and yet are excluded because of the borders. Do we have the courage to transgress and cross the border anyway? Borders have elements of protection, but they also separate; they exclude. They can be sites of oppression.

I look or relook and perhaps reexamine the standards that I have spent a lot of time working with and within; I begin to think about how did I get to this "place" that is so limited? A place that is so controlled, under surveillance, a place of power and powerlessness. With the standards, we have standards police, at every level. Those who really know the standards and those who have to be "forced" to be gatekeepers of the standards. We have gatekeepers, the guards who ensure that what the students are taught/told is what is sanctioned by the standards. Passing standardized tests is the key yardstick. As teachers are made to teach the standards, there's fear of venturing outside the borders because outside the borders is not safe. If you are outside the border, you might not be allowed to come back inside.

I am thinking of the alternatives or maybe what that could be or what it used to be. I am thinking of the places where we can "rupture" these canned knowledges that forbid us from going through the borders. Who really have to stick to the canned knowledge? I am having a lot of questions. Why are we so happy about "coverage" rather than uncoverage? With the limitless amount of knowledge, can we afford to limit ourselves, our students, to this standardized knowledge and justify it as the only knowledge knowable?

4 Discussion and Implications for Mathematics Pre-service Teacher Education

In this chapter, we introduced the idea of speaking our truths to better understand our practice. This resulted in opposing views on standardization. The purpose of duoethnography is not to resolve these issues but rather to help us understand our stance and how to proceed. The synthesis and resolution, as consistent with Norris and Sawyer (2012), are up to the reader. The duoethnographic exploration leads us to look closer into our views on standardization.

Here we evoked border metaphors, because human thinking is metaphoric and metaphors enable us to understand the phenomenon under the investigation (Lakoff & Johnson, 1980). Our goal was to present duoethnography in a way that is truthful to our identities in order to make sense of the issues leant in our context. The metaphor that spoke to us as immigrants was the political border metaphor. Through evoking this metaphor, we identified several ways of reacting to standardization that has implications for mathematics pre-service teacher education. One way to react is to embrace the standards in order to satisfy governing bodies such as CAEP. This is analogous to accepting your position as a citizen of a new formed country and following the established rules and borders. Outside of completely rejecting standardization (analogous to the Saan people), another option is to include the ways to push them through introducing the elements that are not explicitly present. This is the equivalent of maintaining the relationship with communities from both sides of the border within the rules outlined by the authorities. This position of accepting the borders by moving between the two sides of the borders is analogous to Wiseman's and Lunney Borden's (2018) idea of transversing or "cutting across the boundaries" (p. 183). The introduction of the elements that are not explicitly stated in the rules can be an act of transgression (Hooks, 1994).

As mathematics pre-service teacher educators, through speaking out our truths, we were able to identify the tension that existed in our work regarding standardization. This tension is the result of our identities as well as the experiences within the specific political, geographic, and the social context of the American South. True to the spirit of duoethnography, we do not offer a resolution, rather an invitation for readers to conduct their own investigations and meaning makings about borders and standardization.

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Index

N. Radakovic, L. Jao (eds.), *Borders in Mathematics Pre-Service Teacher Education*, https://doi.org/10.1007/978-3-030-44292-7