

# Lecture Notes in Mathematics 1327

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Determinantal Rings

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# Preface

Determinantal rings and varieties have been a central topic of commutative algebra and algebraic geometry. Their study has attracted many prominent researchers and has motivated the creation of theories which may now be considered part of general commutative ring theory. A coherent treatment of determinantal rings is lacking however.

We are algebraists, and therefore the subject will be treated from an algebraic point of view. Our main approach is via the theory of algebras with straightening law. Its axioms constitute a convenient systematic framework, and the standard monomial theory on which it is based yields computationally effective results. This approach suggests (and is simplified by) the simultaneous treatment of the coordinate rings of the Schubert subvarieties of Grassmannians, a program carried out very strictly.

Other methods have not been neglected. Principal radical systems are discussed in detail, and one section each is devoted to invariant and representation theory. However, free resolutions are (almost) only covered for the “classical” case of maximal minors.

Our personal view of the subject is most visibly expressed by the inclusion of Sections 13–15 in which we discuss linear algebra over determinantal rings. In particular the technical details of Section 15 (and perhaps only these) are somewhat demanding.

The bibliography contains several titles which have not been cited in the text. They mainly cover topics not discussed: geometric methods and ideals generated by minors of symmetric matrices and Pfaffians of alternating ones.

We have tried hard to keep the text as self-contained as possible. The basics of commutative algebra supplied by Part I of Matsumura’s book [Mt] (and some additions given in Section 16) suffice as a foundation for Sections 3–7, 9, 10, and 12. Whenever necessary to draw upon notions and results not covered by [Mt], for example divisor class groups and canonical modules in Section 8, precise references have been provided. It is no surprise that multilinear algebra plays a role in a book on determinantal rings, and in Sections 2 and 13–15 we expect the reader not to be frightened by exterior and symmetric powers. Even Section 11 which connects our subject and the representation theory of the general linear groups, does not need an extensive preparation; the linear reductivity of these groups is the only essential fact to be imported. The rudiments on Ext and Tor contained in every introduction to homological algebra will be used freely, though rarely, and some familiarity with affine and projective varieties, as developed in Chapter I of Hartshorne’s book [Ha.2], is helpful.

We hope this text will serve as a reference. It may be useful for seminars following a course in commutative ring theory. A vast number of notions, results, and techniques can be illustrated significantly by applying them to determinantal rings, and it may even be possible to reverse the usual sequence of “theory” and “application”: to learn abstract commutative algebra through the exploration of the special class which is the subject of this book.

Each section contains a subsection “Comments and References” where we have collected the information on our sources. The references given should not be considered

assignments of priority too seriously; they rather reflect the authors' history in learning the subject and give credit to the colleagues in whose works we have participated. While it is impossible to mention all of them here, it may be fair to say that we could not have written this text without the fundamental contributions of Buchsbaum, de Concini, Eagon, Eisenbud, Hochster, Northcott, and Procesi.

The first author gave a series of lectures on determinantal rings at the Universidade federal de Pernambuco, Recife, Brazil, in March and April 1985. We are indebted to Aron Simis who suggested to write an extended version for the IMPA subseries of the Lecture Notes in Mathematics. (By now it has become a very extended version).

Finally we thank Petra Düvel, Werner Lohmann and Matthias Varelmann for their help in the production of this book. We are grateful to the staff of the Computing Center of our university, in particular Thomas Haarmann, for generous cooperation and providing excellent printing facilities.

Vechta, January 1988

WINFRIED BRUNS

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# Contents

<b>1. Preliminaries</b>	<b>1</b>
A. Notations and Conventions	1
B. Minors and Determinantal Ideals	3
C. Determinantal Rings and Varieties	4
D. Schubert Varieties and Schubert Cycles	6
E. Comments and References	9
<b>2. Ideals of Maximal Minors</b>	<b>10</b>
A. Classical Results on Height and Grade	10
B. The Perfection of $I_m(X)$ and Some Consequences	13
C. The Eagon-Northcott Complex	16
D. The Complex of Gulliksen and Negård	22
E. Comments and References	25
<b>3. Generically Perfect Ideals</b>	<b>27</b>
A. The Transfer of Perfection	27
B. The Substitution of Indeterminates by a Regular Sequence	30
C. The Transfer of Integrity and Related Properties	34
D. The Bound for the Height of Specializations	36
E. Comments and References	36
<b>4. Algebras with Straightening Law on Posets of Minors</b>	<b>38</b>
A. Algebras with Straightening Law	38
B. $G(X)$ as an ASL	40
C. The Linear Independence of the Standard Monomials in $G(X)$	43
D. $B[X]$ as an ASL	45
E. Comments and References	48
<b>5. The Structure of an ASL</b>	<b>50</b>
A. ASL Structures on Residue Class Rings	50
B. Syzygies and the Straightening Law	53
C. Nilpotents, Regular Elements and Dimension	54
D. Wonderful Posets and the Cohen-Macaulay Property	58
E. The Arithmetical Rank of Certain Ideals	61
F. Comments and References	63
<b>6. Integrity and Normality. The Singular Locus</b>	<b>64</b>
A. Integrity and Normality	64
B. The Singular Locus	67
C. Comments and References	72

<b>7. Generic Points and Invariant Theory</b>	<b>73</b>
A. A Generic Point for $R_{r+1}(X)$	73
B. Invariants and Absolute Invariants	74
C. The Main Theorem of Invariant Theory for GL and SL	76
D. Remarks on Invariant Theory	80
E. The Classical Generic Point for $G(X; \gamma)$	82
F. $G(X; \gamma)$ and $R(X; \delta)$ as Rings of Invariants	84
G. The Depth of Modules of Invariants	88
H. Comments and References	91
<b>8. The Divisor Class Group and the Canonical Class</b>	<b>93</b>
A. The Divisor Class Group	93
B. The Canonical Class of $R_{r+1}(X)$	97
C. The General Case	100
D. Comments and References	104
<b>9. Powers of Ideals of Maximal Minors</b>	<b>105</b>
A. Ideals and Subalgebras of Maximal Minors	105
B. ASL Structures on Graded Algebras Derived from an Ideal	108
C. Graded Algebras with Respect to Ideals of Maximal Minors	112
D. The Depth of Powers of Ideals of Maximal Minors	117
E. Comments and References	120
<b>10. Primary Decomposition</b>	<b>122</b>
A. Symbolic Powers of Determinantal Ideals	122
B. The Symbolic Graded Ring	124
C. Primary Decomposition of Products of Determinantal Ideals	126
D. Comments and References	133
<b>11. Representation Theory</b>	<b>135</b>
A. The Filtration of $K[X]$ by the Intersections of Symbolic Powers	135
B. Bitableaux and the Straightening Law Revisited	137
C. The Decomposition of $K[X]$ into Irreducible $G$ -Submodules	141
D. $G$ -Invariant Ideals	145
E. $U$ -Invariants and Algebras Generated by Minors	149
F. Comments and References	152
<b>12. Principal Radical Systems</b>	<b>153</b>
A. A Propedeutic Example. Principal Radical Systems	153
B. A Principal Radical System for the Determinantal Ideals	155
C. The Perfection of Determinantal Ideals	158
D. Comments and References	160
<b>13. Generic Modules</b>	<b>162</b>
A. The Perfection of the Image of a Generic Map	162
B. The Perfection of a Generic Module	165
C. Homological Properties of Generic Modules	171
D. Comments and References	173

<b>14. The Module of Kähler Differentials</b>	<b>174</b>
A. Perfection and Syzygies of Some Determinantal Ideals	175
B. The Lower Bound for the Depth of the Differential Module	177
C. The Syzygetic Behaviour of the Differential Module	181
D. Comments and References	183
<b>15. Derivations and Rigidity</b>	<b>184</b>
A. The Lower Bound for the Depth of the Module of Derivations	184
B. The Perfection of the Module of Derivations	189
C. Syzygetic Behaviour and Rigidity	197
D. Comments and References	201
<b>16. Appendix</b>	<b>202</b>
A. Determinants and Modules. Rank	202
B. Grade and Acyclicity	206
C. Perfection and the Cohen-Macaulay Property	209
D. Dehomogenization	211
E. How to Compare “Torsionfree”	213
F. The Theorem of Hilbert-Burch	217
G. Comments and References	218
<b>Bibliography</b>	<b>219</b>
<b>Index of Notations</b>	<b>229</b>
<b>Subject Index</b>	<b>232</b>





# 1. Preliminaries

This section serves two purposes. Its Subsections A and B list the ubiquitous basic notations. In C and D we introduce the principal objects of our investigation and relate them to their geometric counterparts.

## A. Notations and Conventions

Generally we will use the terminology of [Mt] which seems to be rather standard now. In some inessential details our notations differ from those of [Mt]; for example we try to save parentheses whenever they seem dispensable. A main difference is the use of the attributes “local” and “normal”: for us they always include the property of being noetherian. In the following we explain some notations and list the few conventions the reader is asked to keep in mind throughout.

All rings and algebras are commutative and have an element 1. Nevertheless we will sometimes list “commutative” among the hypotheses of a proposition or theorem in order to signalize that the ring under consideration is only supposed to be an *arbitrary* commutative ring. A *reduced* ring has no nilpotent elements. The *spectrum of a ring*  $A$ ,  $\text{Spec } A$  for short, is the set of its prime ideals endowed with the Zariski topology. The *radical* of an ideal  $I$  is denoted  $\text{Rad } I$ . The *dimension* of  $A$  is denoted  $\dim A$ , and the *height* of  $I$  is abbreviated  $\text{ht } I$ .

All the modules  $M$  considered will be unitary, i.e.  $1x = x$  for all  $x \in M$ .  $\text{Ann } M$  is the *annihilator* of  $M$ , and the *support* of  $M$  is given by

$$\text{Supp } M = \{P \in \text{Spec } A : M_P \neq 0\}.$$

We use the notion of *associated prime ideals* only for finitely generated modules over noetherian rings:

$$\text{Ass } M = \{P \in \text{Spec } A : \text{depth } M_P = 0\}.$$

The *depth* of a module  $M$  over a local ring is the length of a maximal  $M$ -sequence in the maximal ideal. The *projective dimension* of a module is denoted  $\text{pd } M$ . We remind the reader of the equation of Auslander and Buchsbaum for finitely generated modules over local rings  $A$ :

$$\text{pd } M + \text{depth } M = \text{depth } A \quad \text{if} \quad \text{pd } M < \infty$$

(cf. [Mt], p. 114, Exercise 4). If a module can be considered a module over different rings (in a natural way), an index will indicate the ring with respect to which an invariant is formed: For example,  $\text{Ann}_A M$  is the annihilator of  $M$  as an  $A$ -module. Instead of Matsumura’s  $\text{depth}_I(M)$  we use  $\text{grade}(I, M)$  and call it, needless to say, the *grade of  $I$  with respect to  $M$* ; cf. 16.B for a discussion of grade. The *rank*  $\text{rk } F$  of a free module  $F$  is the number of elements of one of its bases. We discuss a more general concept of rank in 16.A:  $M$  has *rank*  $r$  if  $M \otimes Q$  is a free  $Q$ -module of rank  $r$ ,  $Q$  denoting the total ring of

fractions of  $A$ . The rank of a linear map is the rank of its image. The *length* of a module  $M$  is indicated by  $\lambda(M)$ .

The notations of homological algebra concerning  $\text{Hom}$ ,  $\otimes$ , and their derived functors seem to be completely standardized; for them we refer to [Rt]. Let  $A$  be a ring,  $M$  and  $N$   $A$ -modules, and  $f: M \rightarrow N$  a homomorphism. We put

$$M^* = \text{Hom}_A(M, A)$$

and

$$f^* = \text{Hom}_A(f, A): N^* \rightarrow M^*.$$

$M^*$  and  $f^*$  are called the *duals* of  $M$  and  $f$ .

For the *symmetric* and *exterior powers* of  $M$  (cf. [Bo.1] for multilinear algebra) we use the symbols

$$\bigwedge^i M \quad \text{and} \quad S_j(M)$$

resp. Sometimes we shall have to refer to bases of  $F^*$ ,  $\bigwedge^i F$  and  $\bigwedge^i F^*$ , given a basis  $e_1, \dots, e_n$  of the free module  $F$ . The *basis* of  $F^*$  dual to  $e_1, \dots, e_n$  is denoted by  $e_1^*, \dots, e_n^*$ . For  $I = (i_1, \dots, i_k)$  the notation  $e_I$  is used as an abbreviation of  $e_{i_1} \wedge \dots \wedge e_{i_k}$ , whereas  $e_I^*$  expands into  $e_{i_1}^* \wedge \dots \wedge e_{i_k}^*$ . (The notation  $e_I$  will be naturally extended to arbitrary families of elements of a module.)

We need some combinatorial notations. A subset  $I \subset \mathbf{Z}$  also represents the sequence of its elements in ascending order. For subsets  $I_1, \dots, I_n \subset \mathbf{Z}$  we let

$$\sigma(I_1, \dots, I_n)$$

denote the signum of the permutation  $I_1 \dots I_n$  (given by juxtaposition) of  $I_1 \cup \dots \cup I_n$  relative to its natural order, provided the  $I_i$  are pairwise disjoint; otherwise  $\sigma(I_1, \dots, I_n) = 0$ . A useful formula:

$$\sigma(I_1, \dots, I_n) = \sigma(I_1, \dots, I_{n-1})\sigma(I_1 \cup \dots \cup I_{n-1}, I_n).$$

For elements  $i_1, \dots, i_n \in \mathbf{Z}$  we define

$$\sigma(i_1, \dots, i_n) = \sigma(\{i_1\}, \dots, \{i_n\}).$$

The cardinality of a set  $I$  is denoted  $|I|$ . For a set  $I$  we let

$$S(m, I) = \{J: J \subset I, |J| = m\}.$$

Last, not least, by

$$1, \dots, \widehat{i}, \dots, n$$

we indicate that  $i$  is to be omitted from the sequence  $1, \dots, n$ .

### B. Minors and Determinantal Ideals

Let  $U = (u_{ij})$  be an  $m \times n$  matrix over a ring  $A$ . For indices  $a_1, \dots, a_t, b_1, \dots, b_t$  such that  $1 \leq a_i \leq m, 1 \leq b_i \leq n, i = 1, \dots, t$ , we put

$$[a_1, \dots, a_t | b_1, \dots, b_t] = \det \begin{pmatrix} u_{a_1 b_1} & \cdots & u_{a_1 b_t} \\ \vdots & & \vdots \\ u_{a_t b_1} & \cdots & u_{a_t b_t} \end{pmatrix}.$$

We do not require that  $a_1, \dots, a_t$  and  $b_1, \dots, b_t$  are given in ascending order. The symbol  $[a_1, \dots, a_t | b_1, \dots, b_t]$  has a twofold meaning:  $[a_1, \dots, a_t | b_1, \dots, b_t] \in A$  as just defined, and

$$[a_1, \dots, a_t | b_1, \dots, b_t] \in \mathbf{N}^t \times \mathbf{N}^t$$

as an ordered pair of  $t$ -tuples of non-negative integers. Clearly  $[a_1, \dots, a_t | b_1, \dots, b_t] = 0$  if  $t > \min(m, n)$ . For systematic reasons it is convenient to let

$$[\emptyset | \emptyset] = 1.$$

If  $a_1 \leq \dots \leq a_t$  and  $b_1 \leq \dots \leq b_t$  we say that  $[a_1, \dots, a_t | b_1, \dots, b_t]$  is a  $t$ -minor of  $U$ . Of course, as an element of  $A$  every  $[a_1, \dots, a_t | b_1, \dots, b_t]$  is a  $t$ -minor up to sign. We call  $t$  the *size* of  $[a_1, \dots, a_t | b_1, \dots, b_t]$ .

Very often we shall have to deal with the case  $t = \min(m, n)$ . Our standard assumption will be  $m \leq n$  then, and we use the simplified notation

$$[a_1, \dots, a_m] = [1, \dots, m | a_1, \dots, a_m].$$

The  $m$ -minors are called the *maximal minors*, those of size  $m - 1$  the *submaximal minors*. (In section 9 the notion “maximal minor” will be used in a slightly more general sense.)

The ideal generated by the  $t$ -minors of  $U$  is denoted

$$I_t(U).$$

The reader may check that  $I_t(U)$  is invariant under invertible linear transformations:

$$I_t(U) = I_t(VUW)$$

for invertible matrices  $V, W$  of formats  $m \times m$  and  $n \times n$  resp.

Sometimes we will need the *matrix of cofactors* of an  $m \times m$  matrix:

$$\begin{aligned} \text{Cof } U &= (c_{ij}), \\ c_{ij} &= (-1)^{i+j} [1, \dots, \widehat{j}, \dots, m | 1, \dots, \widehat{i}, \dots, m]. \end{aligned}$$

### C. Determinantal Rings and Varieties

Let  $B$  be a commutative ring, and consider an  $m \times n$  matrix

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{m1} & \cdots & X_{mn} \end{pmatrix}$$

whose entries are independent indeterminates over  $B$ . The principal objects of our study are the residue class rings

$$R_t(X) = B[X]/I_t(X),$$

$B[X]$  of course denoting the polynomial ring  $B[X_{ij} : i = 1, \dots, m, j = 1, \dots, n]$ . The ideal  $I_t(X)$  is generated by the  $t$ -minors of  $X$ , cf. B. Whenever we shall discuss properties of  $R_t(X)$  which are usually defined for noetherian rings only (for example the dimension or the Cohen-Macaulay property), it will be assumed that  $B$  is noetherian.

Over an algebraically closed field  $B = K$  of coefficients one can immediately associate a geometric object with the ring  $R_t(X)$ . Having chosen bases in an  $m$ -dimensional vector space  $V$  and an  $n$ -dimensional vector space  $W$  one identifies  $\text{Hom}_K(V, W)$  with the  $mn$ -dimensional affine space of  $m \times n$  matrices, of which  $K[X]$  is the coordinate ring. Under this identification the subvariety defined by  $I_t(X)$  corresponds to

$$L_{t-1}(V, W) = \{ f \in \text{Hom}_K(V, W) : \text{rk } f \leq t - 1 \}.$$

We want to associate the letter  $r$  with “rank”, and so we replace  $t$  by  $r + 1$ . Furthermore we put  $L(V, W) = \text{Hom}_K(V, W)$ .

It is not surprising that the geometry of  $L_r(V, W)$  reflects certain properties of the linear maps  $f \in L_r(V, W)$ . Let us consider the following two elementary statements which will lead us quickly to some nontrivial information on  $L_r(V, W)$ : (a) The map  $f$  can be factored through  $K^r$ . (b) Let  $U \subset V$  be a vector subspace of dimension  $r$  and  $\tilde{U}$  a supplement of  $V$ , i.e.  $V = U \oplus \tilde{U}$ ; if  $f|U$  is injective, then there exist unique linear maps  $g: \tilde{U} \rightarrow U$ ,  $h: U \rightarrow W$  such that  $f(u \oplus \tilde{u}) = h(u) + h(g(\tilde{u}))$  for all  $u \in U$ ,  $\tilde{u} \in \tilde{U}$  (in fact,  $h = f|U$ ).

Statement (a) shows that the morphism

$$L(V, K^r) \times L(K^r, W) \longrightarrow L_r(V, W),$$

given by the composition of maps, is surjective. Being an epimorphic image of an irreducible variety,  $L_r(V, W)$  is irreducible itself. An application of (b): It is easy to see that the subset

$$M = \{ f \in L_r(V, W) : f|U \text{ injective} \}$$

is a nonempty open subvariety of  $L_r(V, W)$ : One chooses a basis of  $V$  containing a basis of  $U$ ; then  $M$  is the union of subsets of  $L_r(V, W)$  each of which is defined by the non-vanishing of a determinantal function. By property (b) we have an isomorphism

$$L(\tilde{U}, U) \times (L(U, W) \setminus L_{r-1}(U, W)) \longrightarrow M.$$

Since the variety on the left is an open subvariety of  $L(\tilde{U}, U) \times L(U, W)$ , we conclude at once that

$$\begin{aligned} \dim L_r(V, W) &= \dim M = \dim (L(\tilde{U}, U) \times L(U, W)) = (m - r)r + rn \\ &= mr + nr - r^2. \end{aligned}$$

Furthermore  $M$  is non-singular. Varying  $U$  one observes that all the points  $f \in L_r(V, W) \setminus L_{r-1}(V, W)$  are non-singular:

- (1.1) Proposition.** (a)  $L_r(V, W)$  is an irreducible subvariety of  $L(V, W)$ .  
 (b) It has dimension  $mr + nr - r^2$ .  
 (c) It is non-singular outside  $L_{r-1}(V, W)$ .

The only completely satisfactory information on  $R_{r+1}(X)$  we can draw from (1.1), is its dimension:

$$\dim R_{r+1}(X) = mr + nr - r^2$$

Part (a) only shows that the radical of  $I_{r+1}(X)$  is prime, and unfortunately there seems to be no easy way to prove that  $I_{r+1}(X)$  is a radical ideal itself (over every reduced ring  $B$  of coefficients). Once this is known one can of course directly reverse (c): The generators of the ideal of  $L_r(V, W)$  have all their partial derivatives in  $I_r(X)$ , and the Jacobi criterion (or the definition of non-singularity, depending on ones point of view) implies in conjunction with (c) that  $L_{r-1}(V, W)$  is the singular locus of  $L_r(V, W)$ .

Proposition (1.1) and its proof have been included not only in order to enrich these introductory considerations by some substantial results. We shall encounter algebraic versions of the ideas underlying its proof several times again.

It would be very difficult (for us, at least) to investigate the rings  $R_t(X)$  without viewing them as the most prominent members of a larger class of rings of type  $B[X]/I$  which we call *determinantal rings*. Their defining ideals  $I$  can be described as follows: Given integers

$$1 \leq u_1 < \dots < u_p \leq m, \quad 0 \leq r_1 < \dots < r_p < m,$$

and

$$1 \leq v_1 < \dots < v_q \leq n, \quad 0 \leq s_1 < \dots < s_q < n,$$

the ideal  $I$  is generated by the

$(r_i + 1)$ -minors of the first  $u_i$  rows

and the

$(s_j + 1)$ -minors of the first  $v_j$  columns,

$i = 1, \dots, p, j = 1, \dots, q$ . Later on we shall introduce a systematic notion for determinantal rings which is hard to motivate at this stage.

In order to relate the general class of determinantal rings just introduced to the geometric description of  $R_{r+1}(X)$  given above, one chooses bases  $d_1, \dots, d_m$  and  $e_1, \dots, e_n$  of  $V$  and  $W$  resp.,  $K$  being an algebraically closed field,  $V$  and  $W$  vector spaces of dimensions  $m$  and  $n$ . Let

$$V_k = \sum_{i=1}^k Kd_i \quad \text{and} \quad W_k^* = \sum_{i=1}^k Ke_i^*$$

( $e_1^*, \dots, e_n^*$  is the basis dual to  $e_1, \dots, e_n$ , cf. A above).

Then the ideal  $I$  above defines the *determinantal variety*

$$\{f \in \text{Hom}_K(V, W) : \text{rk } f|_{V_{u_i}} \leq r_i, \text{rk } f^*|_{W_{v_j}^*} \leq s_j, \quad i = 1, \dots, p, j = 1, \dots, q\}.$$

The reader may try to find and to prove the analogue of (1.1) for the variety just defined. It will of course be included in the main results of the Sections 5 and 6.

## D. Schubert Varieties and Schubert Cycles

In the sections 4–9 we shall treat a second class of rings simultaneously with the determinantal rings: the homogeneous coordinate rings of the Schubert varieties (generalized to an arbitrary ring of coefficients) which we call *Schubert cycles* for short. There are two reasons for our treatment of Schubert cycles: (i) They are important objects of algebraic geometry. (ii) Their combinatorial structure is simpler than that of determinantal rings, and most often it is easier to prove a result first for Schubert cycles and to descend to determinantal rings afterwards. Algebraically one can consider every determinantal ring as a dehomogenization of a Schubert cycle (cf. 16.D and (5.5)). In geometric terms one passes from a (projective) Schubert variety to an (affine) determinantal variety by removing a hyperplane “at infinity”.

The first step in the construction of the Schubert varieties is the description of the Grassmann varieties in which they are embedded as subvarieties. While a projective space gives a geometric structure to the set of one-dimensional subspaces of a vector space, a Grassmann variety does this for the set of  $m$ -dimensional subspaces,  $m$  fixed. Let  $K$  be an algebraically closed field,  $V$  an  $n$ -dimensional vector space over  $K$ , and  $e_1, \dots, e_n$  a basis of  $V$ . In a first attempt to assign “coordinates” to a vector subspace  $W$ ,  $\dim W = m$ , one chooses a basis  $w_1, \dots, w_m$  of  $W$  and represents  $w_1, \dots, w_m$  as linear combinations of  $e_1, \dots, e_n$ :

$$w_i = \sum_{j=1}^n x_{ij} e_j, \quad i = 1, \dots, m.$$

Unfortunately the assignment  $W \rightarrow (x_{ij})$  is not well-defined, since  $(x_{ij})$  depends on the basis  $w_1, \dots, w_m$  of  $W$ . Exactly the matrices

$$T \cdot (x_{ij}), \quad T \in \text{GL}(m, K),$$

represent  $W$ . However, the *Plücker coordinates*

$$p = ([a_1, \dots, a_m] : 1 \leq a_1 < \dots < a_m \leq n)$$

formed by the  $m$ -minors of  $(x_{ij})$  remains almost invariant if  $(x_{ij})$  is replaced by  $T \cdot (x_{ij})$ ; it is just replaced by a scalar multiple: The point of projective space with homogeneous coordinates  $p$  depends only on  $W$ ! Thus one has found a well-defined map

$$\mathcal{P} : \{W \subset V : \dim W = m\} \longrightarrow \mathbf{P}^N(K), \quad N = \binom{n}{m} - 1.$$

It is called the *Plücker map*.

This construction can of course be given in more abstract terms. With each subspace  $W$ ,  $\dim W = m$ , one associates the embedding

$$i_W: W \longrightarrow V.$$

Then the  $m$ -th exterior power

$$\bigwedge^m i_W: \bigwedge^m W \longrightarrow \bigwedge^m V$$

maps  $\bigwedge^m W$  onto a one-dimensional subspace of  $\bigwedge^m V$  which in turn corresponds to a point in  $\mathbf{P}(\bigwedge^m V) \cong \mathbf{P}^N(K)$ .

It is easy to see that the Plücker map is injective. Let  $p = \mathcal{P}(W) = \mathcal{P}(\widetilde{W})$ . For reasons of symmetry we may assume that the first coordinate of  $p$  is nonzero. Then we can find bases  $w_1, \dots, w_m$  and  $\widetilde{w}_1, \dots, \widetilde{w}_m$  of  $W$  and  $\widetilde{W}$  resp. such that

$$w_i = e_i + \sum_{j=m+1}^n x_{ij} e_j, \quad \widetilde{w}_i = e_i + \sum_{j=m+1}^n \widetilde{x}_{ij} e_j, \quad i = 1, \dots, m.$$

Looking at the  $m$ -minors  $[1, \dots, \widehat{i}, \dots, m, k]$  of the  $m \times n$  matrices of coefficients appearing in the preceding equations one sees immediately that  $w_i = \widetilde{w}_i$  for  $i = 1, \dots, m$ , hence  $W = \widetilde{W}$ .

It takes considerably more effort to describe the image of  $\mathcal{P}$ . The map  $\mathcal{P}$  is induced by a morphism  $\widetilde{\mathcal{P}}$  of affine spaces;  $\widetilde{\mathcal{P}}$  assigns to each  $m \times n$  matrix the tuple of its  $m$ -minors. Let  $X$  be an  $m \times n$  matrix of indeterminates, and let  $Y_{[a_1, \dots, a_m]}$ ,  $1 \leq a_1 < \dots < a_m \leq n$ , denote the coordinate functions of  $\mathbf{A}^{N+1}(K)$ . Then the homomorphism of coordinate rings associated with  $\widetilde{\mathcal{P}}$  is given as

$$\begin{aligned} \varphi: K[Y_{[a_1, \dots, a_m]}: 1 \leq a_1 < \dots < a_m \leq n] &\longrightarrow K[X], \\ Y_{[a_1, \dots, a_m]} &\longrightarrow [a_1, \dots, a_m], \end{aligned}$$

$[a_1, \dots, a_m]$  specifying an  $m$ -minor of  $X$  now. We denote the image of  $\varphi$  by

$$\mathbf{G}(X);$$

it is the  $K$ -subalgebra of  $K[X]$  generated by the  $m$ -minors of  $X$ . By construction it is clear that the affine variety defined by the ideal  $\text{Ker } \varphi$  is the Zariski closure of  $\text{Im } \widetilde{\mathcal{P}}$ , whereas the corresponding projective variety is the closure of  $\text{Im } \mathcal{P}$ . Much more is true:

- (1.2) Theorem.** (a)  $\mathcal{P}$  maps the set of  $m$ -dimensional subspaces of  $V$  bijectively onto the projective variety with homogeneous coordinate ring  $\mathbf{G}(X)$ .  
 (b)  $\widetilde{\mathcal{P}}$  maps the  $mn$ -dimensional affine space of  $m \times n$  matrices over  $K$  surjectively onto the affine variety with coordinate ring  $\mathbf{G}(X)$ .

Part (a) obviously follows from (b). In order to prove (b) one first has to describe the variety belonging to  $\mathbf{G}(X)$  as a subvariety of  $\mathbf{A}^{N+1}(K)$ . This problem will be solved

in (4.7). Secondly one has to show the surjectivity of  $\tilde{\mathcal{P}}$ , a question which will naturally come across us in Section 7, cf. (7.14).

The projective variety appearing in (1.2),(a) is usually denoted by  $G_m(V)$  and called the *Grassmann variety* of  $m$ -dimensional subspaces of  $V$ . (A different choice of a basis for  $V$  only yields a different embedding into  $\mathbf{P}^N(K)$ ; all these embeddings are projectively equivalent.)

The argument which showed the injectivity of  $\mathcal{P}$  helps us to determine the dimension of  $G_m(V)$ : the open affine subvariety of  $G_m(V)$  complementary to the hyperplane given by the vanishing of  $Y_{[1,\dots,m]}$ , is isomorphic to the affine space of dimension  $m(\dim V - m)$ , hence

$$\dim G_m(V) = m(\dim V - m).$$

(Note that we are using (1.2) here!) Varying the hyperplane one furthermore sees that  $G_m(V)$  is non-singular. The non-singularity of  $G_m(V)$  can also be deduced from another basic fact. The group  $\mathrm{GL}(V)$  of automorphisms of  $V$  acts transitively on  $G_m(V)$ , since two  $m$ -dimensional subspaces of  $V$  differ by an automorphism of  $V$  only. On the other hand this action is induced by the natural action of  $\mathrm{GL}(V)$  on  $\mathbf{P}(\bigwedge^m V)$  (via  $\bigwedge^m V$ ); so  $\mathrm{GL}(V)$  operates transitively as a group of automorphisms on the Grassmann variety  $G_m(V)$ .

**(1.3) Theorem.**  $G_m(V)$  is a non-singular variety of dimension  $m(\dim V - m)$ .

To define the Schubert subvarieties one considers the flag of subspaces associated with the given basis  $e_1, \dots, e_n$  of  $V$  taken in reverse order:

$$V_j = \sum_{i=n-j+1}^n K e_i, \quad 0 = V_0 \subset \dots \subset V_n = V.$$

Let  $1 \leq a_1 < \dots < a_m \leq n$  be a sequence of integers. Then the *Schubert subvariety*  $\Omega(a_1, \dots, a_m)$  of  $G_m(V)$  is defined by

$$\Omega(a_1, \dots, a_m) = \{ W \in G_m(V) : \dim W \cap V_{a_i} \geq i \text{ for } i = 1, \dots, m \}.$$

The varieties thus defined of course depend on the flag of subspaces chosen. But the automorphism group of  $V$  acts transitively on the set of flags, and its action induced on  $G_m(V)$  makes corresponding Schubert subvarieties differ by an automorphism of  $G_m(V)$  only. Hence  $\Omega(a_1, \dots, a_m)$  is essentially determined by  $(a_1, \dots, a_m)$ . It is indeed justified to call  $\Omega(a_1, \dots, a_m)$  a variety:

**(1.4) Theorem.**  $\Omega(a_1, \dots, a_m)$  is the closed subvariety of  $G_m(V)$  defined by the vanishing of all the coordinate functions

$$Y_{[b_1, \dots, b_m]}, \quad b_i < n - a_{m-i+1} + 1 \text{ for some } i, 1 \leq i \leq m.$$

PROOF: The proof is simpler if we dualize our notations first. Let  $c_i = n - a_i$  and  $W_j = \sum_{k=1}^j K e_k$ . Then  $V = V_{n-j} \oplus W_j$  and there is a projection  $\pi_j: V \rightarrow W_j$ ,  $\mathrm{Ker} \pi_j = V_{n-j}$ . By definition

$$\Omega(a_1, \dots, a_m) = \{ W \in G_m(V) : \dim \pi_{c_i}(W) \leq m - i \text{ for } i = 1, \dots, m \}.$$



After the choice of a basis  $w_1, \dots, w_m$ , the subspace  $W$  is represented by the matrix  $(x_{uv})$ ,  $w_u = \sum_{v=1}^n x_{uv} e_v$ . One obviously has

$$\dim \pi_{c_i}(W) \leq m - i \quad \iff \quad \mathbf{I}_{m-i+1}(x_{uv} : 1 \leq v \leq c_i) = 0,$$

and in case this condition holds, every  $m$ -minor which has at least  $m - i + 1$  of its columns among the first  $c_i$  columns of  $(x_{uv})$ , vanishes. Thus all the coordinate functions named in the theorem vanish on  $\Omega(a_1, \dots, a_m)$ . Conversely, if  $\mathbf{I}_{m-i+1}(x_{uv} : 1 \leq v \leq c_i) \neq 0$ , then there is an  $m$ -minor of  $(x_{uv})$  different from zero and having at least  $m - i + 1$  of its columns among the first  $c_i$  ones of  $(x_{uv})$ . —

For arbitrary rings  $B$  of coefficients the *Schubert cycle associated with*  $\Omega(a_1, \dots, a_m)$  is the residue class ring of  $G(X)$  with respect to the ideal generated by all the minors  $[b_1, \dots, b_m]$  such that  $b_i < n - a_{m-i+1} + 1$  for some  $i$ .

## E. Comments and References

The references given below have been included to manifest the geometric significance of determinantal and Schubert varieties. We have restricted ourselves to books (with one exception) since any selection of research articles would inevitably turn out superficial and random. (After all, the AMS classification scheme contains the keys “Determinantal varieties” and “Schubert varieties”.)

The classical source for “the geometry of determinantal loci” is Room’s book [Rm]. It gives plenty of information on the early history of our subject. The decisive treatment of Schubert varieties has been given by Hodge and Pedoe in their monograph [HP]. Among the recent books on algebraic geometry those of Arabello, Cornalba, Griffiths, and Harris [ACGH], Fulton [Fu], and Griffiths and Harris [GH] contain sections on determinantal and/or Schubert varieties. Kleiman and Laksov’s article [KmL] may serve as a pleasant introduction.

## 2. Ideals of Maximal Minors

Though many of the results of this section are covered by the subsequent investigations, cf. Sections 4, 5, and 6, it seems worth to look for those properties of determinantal rings which have been well known for a long time just as the methods they are proved by. In particular one has a rather direct approach to the results concerning the residue class ring  $B[X]/I$  where  $I$  is the ideal generated by the maximal minors of  $X$ .

The second part of the section deals with free resolutions of  $I_t(X)$  in two comparatively simple cases. The first one will be that of maximal minors and after it we shall treat the case in which  $m = n$ ,  $t = n - 1$ , digressing slightly from the title of this section.

### A. Classical Results on Height and Grade

Let  $A$  be an arbitrary commutative ring and  $U = (u_{ij})$  an  $m \times n$  matrix,  $m \leq n$ , of elements in  $A$ . As in Section 1 we denote by  $I_t(U)$  the ideal in  $A$  generated by the  $t$ -minors of  $U$ . There are two observations, simple but often used:

(i)  $I_t(U)$  is invariant under elementary row or column transformations.

(ii) If the element  $u_{mn}$  is a unit in  $A$ , then  $I_t(U) = I_{t-1}(\tilde{U})$  where  $\tilde{U} = (\tilde{u}_{ij})$  is an  $(m-1) \times (n-1)$  matrix,  $\tilde{u}_{ij} = u_{ij} - u_{mj}u_{in}u_{mn}^{-1}$ .

Our investigations concerning properties of  $I_t(X)$  begin with a height formula. There is an upper bound which only depends on  $t$  and the size of the matrix.

**(2.1) Theorem.** *Let  $A$  be a noetherian ring and  $U = (u_{ij})$  an  $m \times n$  matrix of elements in  $A$ . If  $I_t(U) \neq A$  then*

$$\text{ht } I_t(U) \leq (m - t + 1)(n - t + 1).$$

PROOF: By induction on  $t$ . If  $t = 1$ , the inequality is Krull's principal ideal theorem. Let  $t > 1$  and take a minimal prime ideal  $P$  of  $I_t(U)$ . We must show that  $\text{ht } P \leq (m - t + 1)(n - t + 1)$ . Localizing at  $P$  we may assume that  $A$  is a local ring with maximal ideal  $P$ ,  $I_t(U)$  being  $P$ -primary.

If an element of  $U$  is a unit in  $A$ , the theorem follows from the inductive hypothesis and the observation (ii) made above. We may therefore suppose that  $u_{ij} \in P$  for all  $i, j$ .

Let  $T$  be an indeterminate over  $A$ . We consider the  $m \times n$  matrix

$$U' = \begin{pmatrix} u_{11} + T & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{pmatrix}$$

Then  $I_t(U') \subset PA[T]$  and  $I_t(U') + TA[T] = I_t(U)A[T] + TA[T]$ . From the lemma below it follows that  $P' = PA[T]$  is a minimal prime ideal of  $I_t(U')$ . Because of  $\text{ht } P' = \text{ht } P$  we may replace  $P$  by  $P'$ . After localizing the ring  $A[T]$  at  $P'$ , the element  $u_{11} + T$  becomes a unit. As noticed above, the inequality then follows from the inductive hypothesis. —

**(2.2) Remark.** Theorem (2.5) will show that the bound in (2.1) cannot be improved in general. However, under special circumstances one has much better estimates: If  $I_{t+1}(U) = 0$  and  $I_t(U) \neq A$ , then

$$\text{ht } I_t(U) \leq m + n - 2t + 1.$$

The condition  $I_{t+1}(U) = 0$  holds if  $U$  is a matrix of rank  $t$ . More generally, if  $\tilde{U}$  is a  $p \times q$  submatrix of  $U$  and  $u \geq t$ , then

$$\text{ht } I_t(U)/I_u(\tilde{U}) \leq (m - t + 1)(n - t + 1) - (p - u + 1)(q - u + 1).$$

In a ring satisfying the saturated chain condition ([Ka], p. 99) the last inequality is equivalent to

$$\text{ht } I_t(U) - \text{ht } I_u(\tilde{U}) \leq (m - t + 1)(n - t + 1) - (p - u + 1)(q - u + 1).$$

Thus all the ideals  $I_u(\tilde{U})$  have their maximal height along with  $I_t(U)$ , in particular no  $u$ -minor,  $u \geq t$ , can be zero. We refer the reader to [Br.5] for these results. —

**(2.3) Lemma.** *Let  $A$  be a local ring with maximal ideal  $P$  and let  $I$  be a  $P$ -primary ideal. In the polynomial ring  $A[T]$ , let  $I' \subset PA[T]$  be an ideal which has  $I$  as residue modulo  $TA[T]$ . Then  $PA[T]$  is a minimal prime ideal of  $I'$ .*

PROOF: The hypothesis on  $I'$  yields an isomorphism  $A[T]/(I' + TA[T]) \cong A/I$ . Therefore  $PA[T] + TA[T]$  is a minimal prime ideal of  $I' + TA[T]$ . Now let  $Q' \subset PA[T]$  be a minimal prime ideal of  $I'$ . In the ring  $A[T]/Q'$  the ideal  $(Q' + TA[T])/Q'$  is a principal ideal with  $(PA[T] + TA[T])/Q'$  as one of its minimal prime ideals. From  $\text{ht}(PA[T] + TA[T])/Q' \leq 1$  and the chain of prime ideals  $Q' \subset PA[T] \subset PA[T] + TA[T]$  we get  $Q' = PA[T]$ . —

If  $U = X$  the inequality in (2.1) actually becomes an equality. This will be proved by a localization argument frequently used in the sequel.

**(2.4) Proposition.** *Let  $X = (X_{ij})$  and  $Y = (Y_{ij})$  be matrices of indeterminates over the ring  $B$  of sizes  $m \times n$  and  $(m - 1) \times (n - 1)$ , resp. Then the substitution*

$$\begin{aligned} X_{ij} &\longrightarrow Y_{ij} + X_{mj}X_{in}X_{mn}^{-1}, & 1 \leq i \leq m - 1, & \quad 1 \leq j \leq n - 1, \\ X_{mj} &\longrightarrow X_{mj}, & X_{in} &\longrightarrow X_{in} \end{aligned}$$

induces an isomorphism

$$B[X][X_{mn}^{-1}] \cong B[Y][X_{m1}, \dots, X_{mn}, X_{1n}, \dots, X_{m-1,n}][X_{mn}^{-1}]$$

which maps the extension of  $I_t(X)$ ,  $t \geq 1$ , to the extension of  $I_{t-1}(Y)$ . In particular this isomorphism induces an isomorphism

$$\text{R}_t(X)[x_{mn}^{-1}] \cong \text{R}_{t-1}(Y)[X_{m1}, \dots, X_{mn}, X_{1n}, \dots, X_{m-1,n}][X_{mn}^{-1}]$$

where  $x_{mn}$  denotes the residue class of  $X_{mn}$  in  $\text{R}_t(X)$ .

PROOF: The substitution given in the proposition of course induces a homomorphism

$$\varphi: B[X][X_{mn}^{-1}] \longrightarrow B[Y][X_{m1}, \dots, X_{mn}, X_{1n}, \dots, X_{m-1,n}][X_{mn}^{-1}].$$

Analogously we get a homomorphism  $\psi: B[Y][X_{m1}, \dots, X_{m-1,n}][X_{mn}^{-1}] \longrightarrow B[X][X_{mn}^{-1}]$  by substituting

$$Y_{ij} \longrightarrow X_{ij} - X_{mj}X_{in}X_{mn}^{-1}, \quad X_{mj} \longrightarrow X_{mj}, \quad X_{in} \longrightarrow X_{in}$$

Evidently  $\varphi$  and  $\psi$  are inverse to each other. From the remark (ii) made above, it follows that  $I_t(X)B[X][X_{mn}^{-1}] = I_{t-1}(\tilde{X})$  where  $\tilde{X} = (X_{ij} - X_{mj}X_{in}X_{mn}^{-1})$ . Clearly  $\varphi$  maps  $I_{t-1}(\tilde{X})$  to the ideal generated by  $I_{t-1}(Y)$ . —

**(2.5) Theorem.** *Let  $X = (X_{ij})$  be an  $m \times n$  matrix of indeterminates over the noetherian ring  $B$ . Then*

$$\text{grade } I_t(X) = (m - t + 1)(n - t + 1)$$

if  $1 \leq t \leq \min(m, n) + 1$ .

PROOF: In view of (2.1) we must only prove that  $(m - t + 1)(n - t + 1)$  is a lower bound for  $\text{grade } I_t(X)$ . The cases  $t = 1$  and  $t = \min(m, n) + 1$  are trivial. Let  $1 < t \leq \min(m, n)$  and  $P$  be a prime ideal in  $B[X]$  containing  $I_t(X)$ . We will show that  $\text{depth } B[X]_P \geq (m - t + 1)(n - t + 1)$ .

Certainly  $\text{depth } B[X]_P \geq mn > (m - t + 1)(n - t + 1)$  if  $P$  contains all the indeterminates  $X_{ij}$ . Otherwise we may assume that  $X_{mn} \notin P$ . Consider the isomorphism  $B[X][X_{mn}^{-1}] \cong B[Y][X_{m1}, \dots, X_{m-1,n}][X_{mn}^{-1}]$  from (2.4). Using well-known grade formulas and the inductive hypothesis, we get

$$\begin{aligned} \text{depth } B[X]_P &\geq \text{grade } I_t[X]B[X][X_{mn}^{-1}] \\ &= \text{grade } I_t[Y]B[Y][X_{m1}, \dots, X_{m-1,n}][X_{mn}^{-1}] \\ &\geq \text{grade } I_t[Y] \\ &= (m - t + 1)(n - t + 1). \quad \text{—} \end{aligned}$$

Though the following result is not covered by the title of this subsection, it is included here since its proof is another effective application of (2.4).

**(2.6) Theorem.** *Let  $R = R_t(X)$ ,  $P$  be a prime ideal of  $R$  and  $Q = P \cap B$ . Then  $R_P$  is regular if and only if  $B_Q$  is regular and  $P \not\supseteq I_{t-1}(X)/I_t(X)$ .*

PROOF: The statement is obvious if  $t = 1$ . Suppose that  $t > 1$ .

Abbreviating  $I_s = I_s(X)$ , we claim that  $R_P$  cannot be regular if  $P \supset I_1/I_t$ . Otherwise we may assume that  $P$  is a minimal prime of  $I_1/I_t$  and  $B = B_Q$ . Since  $B$  is an integral domain,  $I_1/I_t = P$ . Therefore  $Q \subset I_1/I_t \cap B = 0$ , and  $B$  is a field, say  $K$ . Now it suffices to note that  $(K[X]/I_t)_{I_1/I_t} = K[X]_{I_1/I_t}/I_t K[X]_{I_1}$  is not regular since  $I_t \subset I_1^2$ .

For the rest of the proof we may therefore assume that the residue class  $x_{mn}$  of  $X_{mn}$  is not contained in  $P$ . According to (2.4) we have an isomorphism

$$R[x_{mn}^{-1}] \cong R_{t-1}(Y)[X_{m1}, \dots, X_{m-1,n}][X_{mn}^{-1}].$$

Let  $\tilde{P}$  be the contraction to  $R_{t-1}(Y)$  of the image of  $PR[x_{mn}^{-1}]$  under this isomorphism. Then  $P$  contains  $I_{t-1}/I_t$  if and only if  $\tilde{P}$  contains  $I_{t-2}(Y)/I_{t-1}(Y)$ . And  $R_P$  is regular if and only if  $R_{t-1}(Y)_{\tilde{P}}$  is regular. The inductive hypothesis now immediately yields the required result. —

## B. The Perfection of $I_m(X)$ and Some Consequences

From now on we shall restrict our attention to the ideal  $I_m(X)$  in  $B[X]$ , generated by the maximal minors of  $X$ . It will be shown that  $I_m(X)$  is a perfect ideal. Because of (2.5) this means that  $\text{pd}_{B[X]} R_m(X) = n - m + 1$ . In Subsection C we shall prove this equation by constructing a free resolution of  $R_m(X)$  over  $B[X]$ . On the other hand there is a simple proof, which does not use a concrete free resolution. We formulate the following theorem for arbitrary matrices  $U$ . Apart from giving a more general result, this formulation is better adapted to the method of proof being used below.

**(2.7) Theorem.** *Let  $A$  be a noetherian ring and  $U$  an  $m \times n$  matrix,  $m \leq n$ , with entries in  $A$ . Suppose  $\text{grade } I_m(U) = n - m + 1$ . Then  $I_m(U)$  is a perfect ideal.*

Using (2.5) and (16.19) we obtain:

**(2.8) Corollary.** *The ideal  $I_m(X)$  is perfect. In particular  $R_m(X)$  is a Cohen-Macaulay ring if this holds for  $B$ .*

As we shall see in Section 3, it would be equally justified to call (2.7) a corollary of (2.8). (2.7) will follow from Proposition (2.9). In the proof of (2.9) we will frequently use arguments from 16.A, and we assume that the reader is familiar with the material of that subsection.

**(2.9) Proposition.** *Let  $A$  be a noetherian ring,  $F$  and  $G$  free  $A$ -modules of ranks  $m$  and  $n$ , resp. Further, let  $f: F \rightarrow G$  be a homomorphism such that the ideal  $I_m(f)$  has grade at least  $p \geq 1$ . Then  $f$  is injective, and,  $M$  denoting the cokernel of  $f$ ,  $\bigwedge^{p-1} M$  is torsionfree and  $\text{pd } \bigwedge^p M \leq p$ .*

First we will derive (2.7) from (2.9). Let  $f: A^m \rightarrow A^n$  given by  $U$ , and  $r = n - m$ . Denote by  $u_1, \dots, u_m$  the rows of  $U$ , and consider the map

$$\nu: \bigwedge^r A^n \longrightarrow \bigwedge^n A^n, \quad \nu(x) = x \wedge u_1 \wedge \cdots \wedge u_m,$$

Obviously  $\text{Im } \nu = I_m(U)$  by an identification  $\bigwedge^n A^n \cong A$ . Put  $M = \text{Coker } f$ . Then we have a presentation

$$\begin{array}{ccccccc} A^m \otimes \bigwedge^{r-1} A^n & \longrightarrow & \bigwedge^r A^n & \longrightarrow & \bigwedge^r M & \longrightarrow & 0, \\ x \otimes y & & \longrightarrow & f(x) \wedge y, & & & \end{array}$$

so  $\nu$  factors through  $\bigwedge^r M$ . Since  $\text{rk } \bigwedge^r M = \text{rk } \text{Im } \nu = 1$ , and  $\bigwedge^r M$  is torsionfree by (2.9), we conclude  $\text{pd } \text{Im } \nu = \text{pd } \bigwedge^r M = r$ .

**PROOF OF (2.9):** By induction on  $m$ . The proposition is trivial for  $m = 0$  ( $I_m(f) = A$  in this case). Let  $m > 0$ . Since  $p \geq 1$ ,  $\text{Im } f$  has rank  $m$ , and  $f$  is injective for trivial reasons. Furthermore there is nothing to prove if  $p = 1$ , and we can proceed by induction

on  $p$ . By the inductive hypothesis with respect to  $p$ ,  $\text{pd } \bigwedge^{p-1} M \leq p-1$ . Since  $M_P$  is free for all prime ideals  $P$  in  $A$  with  $\text{depth } A_P < p$ , we get

$$\text{depth } \bigwedge^{p-1} M \otimes A_P \geq \min(1, \text{depth } A_P)$$

for all prime ideals  $P$ . Consequently  $\bigwedge^{p-1} M$  is torsionfree.

Write  $F = F' \oplus A$ ,  $F'$  free of rank  $m-1$ , let  $f': F' \rightarrow G$  be the restriction of  $f$ , and put  $M' = \text{Coker } f'$ . Since  $I_{m-1}(f') \supset I_m(f)$ , the inductive hypothesis on  $m$  can be applied to  $M'$ . We claim that there exists an exact  $A$ -sequence

$$0 \longrightarrow \bigwedge^{p-1} M \longrightarrow \bigwedge^p M' \longrightarrow \bigwedge^p M \longrightarrow 0.$$

This immediately yields  $\text{pd } \bigwedge^p M \leq p$ .

Let  $\pi: M' \rightarrow M$  be the natural projection, and  $y$  a generator of  $\text{Ker } \pi$ . Then we have canonical presentations

$$\begin{aligned} \bigwedge^{p-1} M' &\xrightarrow{\sigma} \bigwedge^p M' \xrightarrow{\bigwedge^p \pi} \bigwedge^p M \longrightarrow 0, \\ \bigwedge^{p-2} M' &\longrightarrow \bigwedge^{p-1} M' \longrightarrow \bigwedge^{p-1} M \longrightarrow 0, \\ &x \longrightarrow x \wedge y. \end{aligned}$$

The second of these presentations shows that the map  $\sigma$  introduced in the first one factors through  $\bigwedge^{p-1} M$ . Since  $\bigwedge^{p-1} M$  is torsionfree and  $\text{rk } \bigwedge^{p-1} M = \text{rk } \text{Ker } \bigwedge^p \pi$ , we obtain  $\text{Ker } \bigwedge^p \pi \cong \bigwedge^{p-1} M$ , as desired. —

As a consequence of (2.8) one can answer questions about the ideals  $I_m(X)$  which, from a naive point of view, concern their prime [sic] properties.

**(2.10) Theorem.** *If  $B$  is an integral domain, then  $I_m(X)$  is a prime ideal.*

PROOF: One may assume  $B$  to be noetherian, for the general statement is easily reduced to this case. Then we use induction on  $m$ . If  $m = 1$ , the theorem is obvious. We assume that  $m > 1$ . Since  $I_m(X)$  is perfect of grade  $n - m + 1$  and  $\text{grade } I_1(X) = mn > n - m + 1$ , the ideal  $I_1(X)$  is not contained in any associated prime ideal of  $I_m(X)$ , cf. (16.17).

Denote by  $x_{ij}$  the residue class of  $X_{ij}$  in  $R = R_m(X)$ . Since  $R[x_{mn}^{-1}]$  is a domain by (2.4) and the inductive hypothesis, there is exactly one associated prime ideal  $P$  of  $R$  such that  $x_{mn} \notin P$ . If  $P$  is the single associated prime ideal, then  $x_{mn}$  is not a zero-divisor in  $R$ , and  $R$  is a domain, too. Suppose there is a second associated prime ideal  $Q \neq P$ . By what we have stated above and since  $x_{mn} \in Q$ , there is some  $x_{ij} \notin Q$ . Arguing inductively again, we get  $x_{ij} \in P$ . Now  $PR[x_{mn}^{-1}] = 0$ , but the image of  $x_{ij}$  in  $PR[x_{mn}^{-1}]$  is different from 0, cf. (2.4). Contradiction! —

**(2.11) Theorem.** *If  $B$  is reduced (a normal domain), then  $R_m(X)$  is reduced (a normal domain), too.*

PROOF: Suppose that  $B$  is a domain. Then  $R = R_m(X)$  is a domain by (2.10). In order to show that  $B$  is reduced or normal resp. we apply criteria based on Serre's conditions.

The statements are obvious if  $m = 1$ . Let  $m > 1$  and suppose that  $B$  is reduced (a normal domain). Consider a prime ideal  $P$  in  $R$  such that  $\text{depth } R_P = 0$  ( $\leq 1$ ). Then  $\text{grade } P = 0$  ( $\leq 1$ ). Because of  $\text{grade } I_1(X) = mn > n - m + 2$  there is an indeterminate  $X_{ij}$  which has residue class  $x_{ij}$  not contained in  $P$ . Clearly we may assume  $x_{ij} = x_{mn}$ . Then by (2.4) and the inductive hypothesis  $R[x_{mn}^{-1}]$  is reduced (a normal domain). Consequently  $R_P$  is reduced (a normal domain), too. —

**(2.12) Remark.** In Section 5 we shall prove that  $I_t(X)$  is a perfect ideal for every  $t$ ,  $1 \leq t \leq m$ . Since  $\text{grade } I_1(X) > \text{grade } I_t(X) + 1$  if  $t > 1$ , the arguments in the proofs of (2.10) and (2.11) demonstrate that (2.10) and (2.11) hold for arbitrary  $t$  (the case  $t = 1$  being trivial).

**(2.13) Remark.** As to converse statements of (2.8) and (2.12), applying (2.4) one easily deduces that  $B$  is reduced (a normal domain, a Cohen-Macaulay ring) if  $B[X]/I_t(X)$  is reduced (a normal domain, a Cohen-Macaulay ring). —

So far we have used Corollary (2.8) only, and it seems adequate to discuss an application of (2.7) which is independent of (2.8). Let  $y_1, \dots, y_k$  be elements of a commutative ring  $A$ ,  $J$  the ideal generated by them, and  $Y$  the  $m \times (m + k - 1)$ -matrix

$$\begin{pmatrix} y_1 & y_2 & y_3 & \cdots & \cdots & y_k & 0 & \cdots & 0 \\ 0 & y_1 & y_2 & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \cdots & 0 & y_1 & y_2 & y_3 & \cdots & \cdots & y_k \end{pmatrix}.$$

For trivial reasons  $I_m(Y) \subset J^m$ . We claim  $I_m(Y) = J^m$ . It is of course enough to prove this for the case in which  $A = \mathbf{Z}[y_1, \dots, y_k]$ , the  $y_i$  being indeterminates. Arguing inductively we conclude  $y_1 J^{m-1} \subset I_m(Y)$  and  $Ay_1 + I_m(Y) = Ay_1 + J^m$ . Next it follows that  $Ay_1 \cap I_m(Y) = y_1 J^{m-1} = Ay_1 \cap J^m$ , and altogether this yields the desired equality. Letting  $n = m + k - 1$  we have

$$n - m + 1 = k,$$

and (2.7) implies that  $J^m$  is perfect (of grade  $k$ ) if  $\text{grade } J = k$ :

**(2.14) Proposition.** *Let  $A$  be a noetherian ring, and  $y_1, \dots, y_k$  an  $A$ -sequence. Then all the ideals  $J^m$ ,  $m \geq 1$ , are perfect (of grade  $k$ ).*

The matrix  $Y$  above helps us to get more information on the rings  $R_m(X)$ . Given an  $m \times n$  matrix  $X$  of indeterminates, we put  $k = n - m + 1$  and choose  $Y_1, \dots, Y_k$  as indeterminates over  $B$ . Let  $S = B[Y_1, \dots, Y_k]/I_m(Y)$ . Then the substitution which assigns each entry of  $X$  the corresponding entry of  $Y$  (formed from  $Y_1, \dots, Y_k$ ), induces surjections

$$\psi: B[X] \longrightarrow S \quad \text{and} \quad \varphi: R_m(X) \longrightarrow S.$$

The kernel of  $\psi$  is generated by the linear polynomials

$$\begin{aligned} X_{ij}, & \quad j - i < 0 \quad \text{or} \quad j - i > k - 1, \\ X_{ij} - X_{i-1, j-1}, & \quad i = 2, \dots, m, \quad 0 \leq j - i \leq k - 1, \end{aligned}$$

and the ideal  $I_m(X)$ . The residue classes of the polynomials listed generate the kernel of  $\varphi$ . Their number is exactly

$$\begin{aligned} nm - (n - m + 1) &= \text{grade Ker } \psi - \text{grade } I_m(X) \\ &= \text{grade Ker } \varphi \end{aligned}$$

by virtue of (16.18): both  $\text{Ker } \psi$  and  $I_m(X)$  are perfect. Here we assume  $B$  to be noetherian, of course. Since the generators of  $\text{Ker } \varphi$  are homogeneous (of degree 1), one concludes easily that they form an  $R_m(X)$ -sequence (in any order). This fact makes it possible to transfer information from  $R_m(X)$  to  $S$  and vice versa. After all,  $S$  can be considered a well-understood  $B$ -algebra.

We use the connection between  $R_m(X)$  and  $S$  to compute the multiplicity of  $R_m(X)$  in case  $B = K$  is a field. The graded  $K$ -algebra  $R_m(X)$  then has a well-defined multiplicity (given by the multiplicity of its localization with respect to the irrelevant maximal ideal). We refer the reader to [Na] for multiplicity theory.

**(2.15) Proposition.** *Let  $B = K$  be a field,  $X$  an  $m \times n$  matrix of indeterminates and  $\underline{y}$  the  $R_m(X)$ -sequence generating  $\text{Ker } \varphi$ , as specified above. Then the multiplicity of  $R_m(X)$  is given by*

$$e(R_m(X)) = \lambda(R_m(X)/\underline{y}R_m(X)) = \binom{n}{m-1}.$$

PROOF: Since the sequence  $\underline{y}$  is a “superficial sequence” (defined to be a sequence of superficial elements in the same way as an  $A$ -sequence is a sequence of elements not dividing zero), the multiplicities of  $R_m(X)$  and  $R_m(X)/\underline{y}R_m(X)$  coincide. The multiplicity of the latter ring is just its length. —

One could further exploit the relationship between  $R_m(X)$  and  $S$  in order to determine the Gorenstein rings among the rings  $R_m(X)$ . We shall do this in (2.21), based on a different argument.

### C. The Eagon-Northcott Complex

In the preceding subsection we have investigated the ideal  $I_m(X)$  by considering  $X$  as the matrix of a linear map  $f: F \rightarrow G$ . In this subsection it is better to start from the dual map  $f^*: G^* \rightarrow F^*$ . To avoid notational complications we replace  $G^*$  and  $F^*$  by  $G$  and  $F$  and  $f^*$  by a map  $g: G \rightarrow F$ . Instead the map  $f$  will be treated as the dual of  $g$ , and the ideal  $I_m(f)$  of Subsection B is  $I_m(g)$  below. While the perfection of  $I_m(g)$  has been proved already, cf. (2.8), we will construct a free resolution of the corresponding residue class ring and some related modules. The approach taken in the following may be rather abstract, but it is certainly very effective.



Let  $A$  be an arbitrary ring, and suppose that  $F$  and  $G$  are finitely generated free  $A$ -modules of rank  $m$  and  $n$ , resp. Since the natural homomorphism

$$G^* \otimes_A F \rightarrow \text{Hom}_A(G, F)$$

is an isomorphism in this situation, one may view every  $A$ -homomorphism  $g: G \rightarrow F$  an element of  $G^* \otimes F$ . The free module  $F$  is the degree 1 homogeneous part of the symmetric algebra  $S(F)$ , so we can consider  $g$  even an element of

$$G^* \otimes S(F) \cong \text{Hom}_{S(F)}(G \otimes S(F), S(F)).$$

Viewed as an  $S(F)$ -linear form on

$$\widehat{G} = G \otimes S(F),$$

$g$  gives rise to a Koszul complex (cf. [Bo.4], §9)

$$\mathcal{C}(g): 0 \longrightarrow \bigwedge^n \widehat{G} \xrightarrow{\partial} \bigwedge^{n-1} \widehat{G} \xrightarrow{\partial} \dots \xrightarrow{\partial} \widehat{G} \xrightarrow{\partial} S(F) \longrightarrow 0,$$

the map  $\partial: \bigwedge^{i+1} \widehat{G} \longrightarrow \bigwedge^i \widehat{G}$  being defined by

$$\partial(x_1 \wedge \dots \wedge x_{i+1}) = \sum_{j=1}^{i+1} (-1)^{j+1} g(x_j) x_1 \wedge \dots \wedge \widehat{x}_j \wedge \dots \wedge x_{i+1}.$$

As a complex of  $A$ -modules  $\mathcal{C}(g)$  splits into direct summands

$$\begin{aligned} \mathcal{C}_i(g): \dots \longrightarrow 0 \longrightarrow \bigwedge^i G \otimes S_0(F) \longrightarrow \bigwedge^{i-1} G \otimes S_1(F) \longrightarrow \dots \\ \longrightarrow \bigwedge^1 G \otimes S_{i-1}(F) \longrightarrow \bigwedge^0 G \otimes S_i(F) \longrightarrow 0. \end{aligned}$$

We fix orientations  $\gamma$  on  $F^*$  and  $\delta$  on  $G^*$ , i.e. isomorphisms  $\gamma: \bigwedge^m F^* \longrightarrow A$  and  $\delta: \bigwedge^n G^* \longrightarrow A$ . Let

$$r = n - m.$$

Then for  $i = 0, \dots, r$  we can splice the  $A$ -dual  $\mathcal{C}_{r-i}^*(g)$  of  $\mathcal{C}_{r-i}(g)$  and  $\mathcal{C}_i(g)$  to a sequence

$$\begin{aligned} \mathcal{D}_i(g): 0 \longrightarrow \left(\bigwedge^0 G \otimes S_{r-i}(F)\right)^* \xrightarrow{\partial^*} \dots \xrightarrow{\partial^*} \left(\bigwedge^{r-i} G \otimes S_0(F)\right)^* \xrightarrow{\nu_i} \bigwedge^i G \otimes S_0(F) \xrightarrow{\partial} \dots \\ \xrightarrow{\partial} \bigwedge^0 G \otimes S_i(F) \longrightarrow 0, \end{aligned}$$

where  $\nu_i$  is described as follows: First one defines  $\nu_i$  as a map

$$\bigwedge^{r-i} G^* \longrightarrow \left(\bigwedge^i G^*\right)^*$$

by

$$(\nu_i(x))(y) = \delta(x \wedge y \wedge \bigwedge^m g^*(z)), \quad x \in \bigwedge^{r-i} G^*, \quad y \in \bigwedge^i G^*, \quad z = \gamma^{-1}(1),$$

and then one regards  $\nu_i$  as a map  $(\bigwedge^{r-i} G \otimes S_0(F))^* \longrightarrow \bigwedge^i G \otimes S_0(F)$  via the natural isomorphisms  $\bigwedge^{r-i} G^* \cong (\bigwedge^{r-i} G)^* \cong (\bigwedge^{r-i} G \otimes S_0(F))^*$  and  $(\bigwedge^i G^*)^* \cong \bigwedge^i G \cong \bigwedge^i G \otimes S_0(F)$ .

An easy calculation shows that

$$\nu_i \circ \partial^* = 0, \quad \partial \circ \nu_i = 0.$$

Furthermore  $\gamma$ ,  $\delta$ , and, hence,  $\nu_i$  are unique up to a unit factor. So  $\mathcal{D}_i(g)$  is a complex whose homology depends only on  $g$ . In order to specify homology modules we consider  $\bigwedge^0 G \otimes S_i(F)$  to be in position 0 and  $(\bigwedge^0 G \otimes S_{r-i}(F))^*$  in position  $r+1$ . Then

$$\begin{aligned} H_0(\mathcal{D}_0(g)) &= A/I_m(g), \\ H_0(\mathcal{D}_i(g)) &= S_i(\text{Coker } g), \quad i > 0. \end{aligned}$$

The second of these equations is quite obvious whereas one has to analyze  $\nu_0$  to observe that  $\text{Im } \nu_0 = I_m(g)$ .

Our purpose will be achieved when the following theorem has been proved:

**(2.16) Theorem.** *Let  $A$  be a noetherian ring,  $g: G \rightarrow F$  a homomorphism of finitely generated free  $A$ -modules. Put  $n = \text{rk } G$ ,  $m = \text{rk } F$  and choose orientations  $\gamma$ ,  $\delta$  of  $F^*$  and  $G^*$ , resp. Suppose  $m \leq n$  and  $\text{grade } I_m(g) = n - m + 1$ . Then the following holds:*

- (a) *The complexes  $\mathcal{D}_i(g)$ ,  $0 \leq i \leq n - m$ , are acyclic.*
- (b)  *$\mathcal{D}_0(g)$  resolves  $A/I_m(g)$ ,  $\mathcal{D}_i(g)$ ,  $i = 1, \dots, r$ , resolves  $S_i(\text{Coker } g)$ .*
- (c)  *$A/I_m(g)$  and  $S_i(\text{Coker } g)$ ,  $i = 1, \dots, r$ , are perfect  $A$ -modules.*

If we look at the next to the last homomorphism of  $\mathcal{D}_0(g)$ , we see that in the situation of (2.16) the first syzygy module of  $I_m(g)$  is generated by the “expected” relations:  $U$  being a matrix representing  $g$ , they are obtained by Laplace column expansion of the  $(m+1)$ -minors of all matrices which result from  $U$  by doubling a row.

Of course (2.16) can be applied to the case in which  $g$  is given by an  $n \times m$  matrix  $X$  of indeterminates over a noetherian ring  $B$ , and part of it has already been proved (cf. (2.8)). In Section 13 we shall again take up the problem concerning the perfection of  $\text{Coker } g$ . More generally the map  $x: R^n \rightarrow R^m$  will be investigated where  $R = R_t(X)$  and  $x$  is given by the residue classes of the entries of  $X$ .  $\text{Coker } x$  will turn out to be a perfect  $B[X]$ -module if and only if  $n \geq m$ .

Only part (a) of (2.16) needs a proof; (b) and (c) then follow easily from what has been said above. The complexes  $\mathcal{D}_i(g)$  are complexes of free  $A$ -modules of length  $r+1 = n - m + 1$ . By virtue of the exactness criterion (16.16) it is enough to show that their localizations  $\mathcal{D}_i(g)_P$ ,  $P \not\supseteq I_m(g)$ , are split-exact. For these prime ideals  $P$  the localization  $g_P$  is surjective, so we have reduced (2.16) to the following proposition.

**(2.17) Proposition.** *Let  $A$  be a noetherian ring,  $g: G \rightarrow F$  a homomorphism of finitely generated free  $A$ -modules. If  $g$  is surjective, then the complexes  $\mathcal{D}_i(g)$  are split-exact.*

PROOF: By the definition of  $\mathcal{D}_i(g)$  one has

$$H_j(\mathcal{D}_i(g)) = \begin{cases} H_0(\mathcal{C}_i(g)) & \text{if } j = 0 \text{ and } i > 0, \\ H_j(\mathcal{C}_i(g)), & j = 1, \dots, i - 1, \\ \text{Ker } \partial / \text{Im } \nu_i, & j = i, \\ \text{Ker } \nu_i / \text{Im } \partial^*, & j = i + 1, \\ H^{r+1-j}(\mathcal{C}_{r-i}^*(g)), & j = i + 2, \dots, r, \\ H^0(\mathcal{C}_{r-i}^*(g)) & \text{if } j = r + 1 \text{ and } i < r. \end{cases}$$

We may assume that  $A$  has exactly one maximal ideal. Then  $\text{Ker } g$  is a free direct summand of  $G$ . As stated above,  $\text{Ker } \partial / \text{Im } \nu_i$  and  $\text{Ker } \nu_i / \text{Im } \partial^*$  do not depend on the orientations  $\gamma$  and  $\delta$ . Therefore one may take a basis  $x_1, \dots, x_m$  of  $F$  and a basis  $y_1, \dots, y_n$  of  $G$  such that  $g(y_k) = x_k$ ,  $k = 1, \dots, m$ ,  $g(y_k) = 0$ ,  $k = m + 1, \dots, n$ , to define  $\gamma$  and  $\delta$  by

$$\gamma(x_1^* \wedge \dots \wedge x_m^*) = 1 \quad \text{and} \quad \delta(y_1^* \wedge \dots \wedge y_n^*) = 1$$

( $x_1^*, \dots, x_m^*$  being the basis dual to  $x_1, \dots, x_m$  etc.). With these data it is very easy to calculate that  $\text{Im } \partial^* = \text{Ker } \nu_i$ ,  $\text{Ker } \partial = \text{Im } \nu_i$ . The rest essentially follows from:

**(2.18) Proposition.** *Let  $A$  be a commutative ring,  $g: G \rightarrow F$  a surjective homomorphism of finitely generated free  $A$ -modules. If  $g$  is surjective, then*

$$H_j(\mathcal{C}(g)) = 0 \quad \text{for } j > \text{rk } G - \text{rk } F$$

and

$$H_j(\mathcal{C}_i(g)) = 0 \quad \text{for } j \neq i.$$

Let us first finish the proof of (2.17). Proposition (2.18) shows that

$$0 \longrightarrow H_i(\mathcal{C}_i(g)) \longrightarrow \bigwedge^i G \otimes S_0(F) \xrightarrow{\partial} \dots \xrightarrow{\partial} \bigwedge^0 G \otimes S_i(F) \longrightarrow 0$$

is a split-exact sequence of  $A$ -modules. Therefore its dual is split-exact, too. Taking into account that this holds for  $i = 0, \dots, r$ , (2.17) follows immediately. —

As just seen, the important part of (2.18) is the second equation. The first one can be viewed a special case of the general theorem concerning the vanishing of Koszul homology ([No.6], Theorem 4, p. 262): the image of the linear form  $g: \widehat{G} \rightarrow S(F)$  is the ideal  $\bigoplus_{i \geq 1} S_i(F)$ . After the choice of a basis for  $F$  one can identify  $S(F)$  with a polynomial ring over  $A$  within which  $\text{Im } g$  is just the ideal generated by the indeterminates, an ideal of grade  $\text{rk } F$  (with the suitable definition of grade if  $A$  is non-noetherian).

PROOF OF (2.18): As in the proof of (2.17) it is useful (and harmless) to assume that  $A$  has exactly one maximal ideal.

One proceeds by induction on  $\text{rk } G - \text{rk } F$ . In case  $\text{rk } G = \text{rk } F$ , the argument just explained shows that  $H_j = 0$  for  $j > 0$  (without any reference to the notion “grade”): the Koszul complex associated with (the linear form given by) a sequence of indeterminates is acyclic in positive degrees (cf. [Bo.4], § 9, no. 6, Prop. 5). Furthermore  $H_j(\mathcal{C}_0(g)) = 0$  for  $j > 0$  by definition of  $\mathcal{C}_0(g)$ .

Let now  $\text{rk } G > \text{rk } F$ . Then one splits  $G$  into a direct sum  $G = H \oplus Ae$ ,  $H$  free,  $\text{rk } H = \text{rk } G - 1$ ,  $e \in \text{Ker } g$ . Let  $h = g|_H$ . The decomposition induces split-exact sequences

$$(1) \quad 0 \longrightarrow \bigwedge^{i+1} H \longrightarrow \bigwedge^{i+1} G \longrightarrow \bigwedge^i H \longrightarrow 0,$$

the map on the left being the natural embedding, the map on the right sending  $x \wedge e$ ,  $x \in \bigwedge^i H$ , to  $x$  and vanishing on  $\bigwedge^{i+1} H$ .

Passing to  $S(F)$  one obtains a diagram

$$\begin{array}{cccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{C}(h): & 0 & \longrightarrow & 0 & \longrightarrow & \bigwedge^{n-1} \widehat{H} & \longrightarrow & \dots & \longrightarrow & \bigwedge^1 \widehat{H} & \longrightarrow & \bigwedge^0 \widehat{H} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \mathcal{C}(g): & 0 & \longrightarrow & \bigwedge^n \widehat{G} & \longrightarrow & \bigwedge^{n-1} \widehat{G} & \longrightarrow & \dots & \longrightarrow & \bigwedge^1 \widehat{G} & \longrightarrow & \bigwedge^0 \widehat{G} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \mathcal{C}(h)[-1]: & 0 & \longrightarrow & \bigwedge^{n-1} \widehat{H} & \longrightarrow & \bigwedge^{n-2} \widehat{H} & \longrightarrow & \dots & \longrightarrow & \bigwedge^0 \widehat{H} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 
 \end{array}$$

whose split-exact columns are induced by (1). It is easy to check that this diagram is commutative, whence we have an exact sequence

$$\dots \longrightarrow H_j(\mathcal{C}(h)) \longrightarrow H_j(\mathcal{C}(g)) \longrightarrow H_{j-1}(\mathcal{C}(h)) \longrightarrow \dots$$

of homology modules. The first equation follows immediately.

For the demonstration of the second we regard the diagram above as a diagram of graded  $S(F)$ -modules. For convenience one chooses the graduation of  $\bigwedge^0 \widehat{G} = S(F)$  as the natural one, and then shifts all the other graduations such that every homomorphism is of degree zero. The  $i$ -th homogeneous part of  $H_j(\mathcal{C}(g))$  is then given by

$$H_j(\mathcal{C}(g))_i = \frac{\text{Ker} \left[ \bigwedge^j G \otimes S_{i-j}(F) \longrightarrow \bigwedge^{j-1} G \otimes S_{i-j+1}(F) \right]}{\text{Im} \left[ \bigwedge^{j+1} G \otimes S_{i-j-1}(F) \longrightarrow \bigwedge^j G \otimes S_{i-j}(F) \right]} = H_j(\mathcal{C}_i(g)).$$

Analogously

$$H_j(\mathcal{C}(h))_i = H_j(\mathcal{C}_i(h)),$$

whereas

$$H_j(\mathcal{C}(h)[-1])_i = H_{j-1}(\mathcal{C}_{i-1}(h)).$$

The decomposition of the exact homology sequence above makes the second equation evident now. —

**(2.19) Remarks.** (a) It is not difficult to identify the homology modules  $H_i(\mathcal{C}_i(g))$ ,  $i = 0, \dots, r$ , in the situation of (2.16) or (2.18). The reader may check that

$$H_i(\mathcal{C}_i(g)) = \left(\bigwedge^i M\right)^*, \quad M = \text{Coker } g^*.$$

Furthermore

$$H^{r-i}(\mathcal{C}_{r-i}^*(g)) = \bigwedge^{r-i} M,$$

so  $\mathcal{C}_{r-i}^*(g)$  resolves  $\bigwedge^{r-i} M$ ,  $i = 0, \dots, r$  (and  $\mathcal{C}_{r+1}^*(g)$  resolves  $\bigwedge^{r+1} M$ ). The map  $\nu_i$  can be interpreted (or constructed) as an isomorphism

$$\bigwedge^{r-i} M \cong \left(\bigwedge^i M\right)^*$$

derived from the linear form  $\nu_0: \bigwedge^r M \rightarrow A$ .

(b) The complexes discussed so far do not exhaust the class of resolutions which can be extracted from the complexes  $\mathcal{C}(g)$ . We prove the following results only for the case  $R = B[X]$ ,  $B$  noetherian,  $g: G \rightarrow F$  given by the  $n \times m$  matrix  $X$  of indeterminates with respect to bases  $e_1, \dots, e_n$  and  $d_1, \dots, d_m$  of  $G$  and  $F$ . To indicate this clearly we use  $X$  in place of  $g$ .

(i) *If  $n \leq m$ , then the complex  $\mathcal{C}(X)$  is acyclic. It resolves  $S(\text{Coker } X)$  over  $S(F)$ ; its homogeneous component  $\mathcal{C}_i(X)$  resolves  $S_i(\text{Coker } X)$  for all  $i \geq 0$ . In particular  $\text{pd } S_i(\text{Coker } X) = \min(i, n)$ .*

It is easy to see (and will be proved in (12.4)) that  $X(e_1), \dots, X(e_n)$  is an  $S(F)$ -regular sequence ( $S(F)$  is the polynomial ring  $R[d_1, \dots, d_m]$ ). Therefore the Koszul complex  $\mathcal{C}(X)$  is acyclic.

It has been shown in [Av.2] that in general  $g(e_1), \dots, g(e_n)$  is an  $S(F)$ -regular sequence if and only if  $\text{grade } I_{n-i}(g) \geq m - n + i + 1$  for  $i = 0, \dots, n - 1$ . Thus (i) holds under this more general condition.

(ii) *If  $n \geq m$ , then the conclusion of (2.18) holds for  $\mathcal{C}(X)$ . In particular  $\mathcal{C}_i(X)$  resolves  $S_i(\text{Coker } X)$  and  $\text{pd } S_i(\text{Coker } X) = \min(i, n)$  for all  $i \geq n - m + 1$  (whereas, by (2.16),  $\text{pd } S_i(\text{Coker } X) = n - m + 1$  for  $i = 1, \dots, n - m$ ).*

This is proved by the same induction as (2.18) starting with the case  $n = m$  covered by (i). In fact, the exactness of the sequence of complexes used in the proof of (2.18) does not depend on the special hypotheses there nor on the choice of  $e$  such that  $G = H \oplus Ae$ ,  $H$  free. (The reader may investigate whether (ii) can be generalized in the same way as (i).) —

The resolution of  $R_m(X)$  obtained from (2.16) carries much more information about  $R_m(X)$  than just its perfection. For example, one can compute its canonical module (cf. 16.C) and decide whether it is a Gorenstein ring.

**(2.20) Theorem.** *Let  $B$  be a Cohen-Macaulay ring having a canonical module  $\omega_B$  and  $X$  an  $m \times n$  matrix of indeterminates over  $B$ ,  $m \leq n$ . Furthermore let  $C$  denote the cokernel of the map  $B[X]^n \rightarrow B[X]^m$  given by the transpose  $X^*$  of  $X$ . Then  $S_{n-m}(C) \otimes_B \omega_B$  is a canonical module of  $R_m(X)$ .*

PROOF: Let  $g$  denote the map given by  $X^*$ ,  $r = n - m$ , and  $R = R_m(X)$ . Then

$$\begin{aligned} \omega_R &= \text{Ext}_{B[X]}^{r+1}(R, \omega_{B[X]}) = H^{r+1}(\text{Hom}_{B[X]}(\mathcal{C}_0(g), B[X])) \\ &= H^{r+1}(\mathcal{C}_0^*(g) \otimes_{B[X]} \omega_B) = H^{r+1}(\mathcal{C}_0^*(g) \otimes_{B[X]} (B[X] \otimes \omega_B)) \\ &= H^{r+1}(\mathcal{C}_0^*(g) \otimes_{B[X]} (B[X] \otimes_B \omega_B)) \\ &= S_r(C) \otimes \omega_B. \quad \text{—} \end{aligned}$$

**(2.21) Corollary.** *Let  $X$  be an  $m \times n$  matrix of indeterminates over the noetherian ring  $B$ . Then  $R_m(X)$  is a Gorenstein ring if and only if (i)  $B$  is a Gorenstein ring and (ii)  $m = 1$  or  $m = n$ .*

PROOF: The “if”-part is obvious (without (2.20)). Assume that  $R = R_m(X)$  is a Gorenstein ring. As in the case of the Cohen-Macaulay property (cf. (2.13)) we deduce that  $B$  is a Gorenstein ring (using a suitable argument stated in [Wt]). Let  $P$  be a prime ideal in  $A$  containing the entries of  $X$  and  $r, g, C$  as in (2.20).  $S_r(C)_P$  is the canonical module of  $R_P$ . By the definition of  $\mathcal{C}_0(g)$  the minimal number of generators of  $S_r(C)_P$  is  $\text{rk } S_r(B[X]^m)$ . It has to be 1 if  $R_P$  is a Gorenstein ring. —

Later we will determine the canonical module for each of the rings  $R_t(X)$ , cf. Sections 8 and 9. Then the canonical module will be described as an ideal of  $R_t(X)$  (provided  $B$  is Gorenstein). The reader may try to derive such a description from (2.20).

## D. The Complex of Gulliksen and Negård

We shall now construct a finite free resolution of  $I_t(X)$  for the case in which  $m = n$ ,  $t = n - 1$ ,  $n \geq 2$ . Let  $A$  be an arbitrary commutative ring. By  $\mathcal{M}_n(A)$  we denote the ring of  $n \times n$  matrices with entries in  $A$ . We also use the structure of  $\mathcal{M}_n(A)$  as a free  $A$ -module of rank  $n^2$ . Let  $U \in \mathcal{M}_n(A)$ . Then the complex of  $A$ -modules

$$\mathcal{G}(U): 0 \longrightarrow \mathcal{G}(U)_4 \xrightarrow{d_4} \mathcal{G}(U)_3 \xrightarrow{d_3} \mathcal{G}(U)_2 \xrightarrow{d_2} \mathcal{G}(U)_1 \xrightarrow{d_1} \mathcal{G}(U)_0 \longrightarrow 0$$

is given as follows: Put  $\mathcal{G}(U)_0 = \mathcal{G}(U)_4 = A$ ,  $\mathcal{G}(U)_1 = \mathcal{G}(U)_3 = \mathcal{M}_n(A)$ . To get  $\mathcal{G}(U)_2$  we consider the zero-sequence

$$(2) \quad A \xrightarrow{\iota} \mathcal{M}_n(A) \oplus \mathcal{M}_n(A) \xrightarrow{\pi} A$$

where  $\iota(a) = (aE, aE)$ ,  $E$  being the unit matrix of  $\mathcal{M}_n(A)$ , and  $\pi(V, W) = \text{trace}(V - W)$ . Let  $E_{ij}$ ,  $1 \leq i, j \leq n$ , be the canonical basis of  $\mathcal{M}_n(A)$ . Then  $\text{Ker } \pi$  is generated by the elements  $(E_{ij}, 0)$ ,  $i \neq j$ ,  $(0, E_{uv})$ ,  $u \neq v$ ,  $(E_{ii}, E_{11})$ ,  $1 \leq i \leq n$ , and  $(0, E_{uu} - E_{11})$ ,  $2 \leq u \leq n$ . Since  $\text{Im } \iota$  is generated by  $\sum_{i=1}^n (E_{ii}, E_{ii}) = \sum_{i=1}^n (E_{ii}, E_{11}) + \sum_{u=2}^n (0, E_{uu} - E_{11})$ ,

$\mathcal{G}(U)_2 = \text{Ker } \pi / \text{Im } \iota$  is a free  $A$ -module. Now let  $\tilde{U}$  be the matrix of cofactors of  $U$ . Then we put

$$d_1(V) = \text{trace}(\tilde{U}V), \quad d_4(a) = a\tilde{U}.$$

To define  $d_2, d_3$  we consider the zero-sequence

$$(3) \quad \mathcal{M}_n(A) \xrightarrow{\psi} \mathcal{M}_n(A) \oplus \mathcal{M}_n(A) \xrightarrow{\varphi} \mathcal{M}_n(A)$$

where  $\psi(V) = (UV, VU)$ ,  $\varphi(V, W) = VU - UW$ . Clearly  $\text{Im } \iota \subset \text{Ker } \varphi$  and  $\text{Im } \psi \subset \text{Ker } \pi$  so that we may define  $d_2, d_3$  as the maps induced by  $\varphi$  and  $\psi$ , resp.

A trivial calculation shows that  $d_i \circ d_{i+1} = 0$ ,  $i = 1, 2, 3$ , whence  $\mathcal{G}(U)$  is in fact a complex. Furthermore  $\text{Im } d_1 = \text{I}_{n-1}(U)$ . We make another trivial observation: If  $h: A \rightarrow A'$  is a homomorphism of commutative rings and if  $h(U)$  denotes the matrix obtained from  $U$  by applying  $h$  to the entries of  $U$ , then one has a natural isomorphism of  $A'$ -complexes

$$\mathcal{C}(U) \otimes_A A' \cong \mathcal{C}(h(U)).$$

**(2.22) Proposition.** *The complex  $\mathcal{G}(U)$  is self-dual.*

PROOF: We have to define isomorphisms  $\nu_i: \mathcal{G}(U)_i \rightarrow [\mathcal{G}(U)_{4-i}]^*$ ,  $0 \leq i \leq 4$ , such that  $\nu_{i-1} \circ d_i = d_{4-i+1}^* \circ \nu_i$  for  $i = 1, \dots, 4$ . Let  $\nu_0 = \nu_4$  be the canonical isomorphism  $A \rightarrow A^*$ . Next we take the canonical basis  $E_{ij}$  of  $\mathcal{M}_n(A)$ ,  $1 \leq i, j \leq n$ , and its dual  $E_{ij}^*$  to define  $\nu: \mathcal{M}_n(A) \rightarrow \mathcal{M}_n(A)^*$  by  $\nu(E_{ij}) = E_{ij}^*$ . Put  $\nu_1 = \nu_3 = \nu$ . Let  $\iota, \pi$  be the maps from the sequence (2) above and denote by  $\chi: \text{Ker } \pi \rightarrow \mathcal{M}_n(A) \oplus \mathcal{M}_n(A)$  the canonical injection. Then  $\chi^* \circ (\nu, -\nu)$  as well as the elements of  $\text{Im}(\chi^* \circ (\nu, -\nu))$  vanish on  $\text{Im } \iota$ . Consequently  $\chi^* \circ (\nu, -\nu)$  induces a homomorphism  $\nu_2: \mathcal{G}(U)_2 \rightarrow \mathcal{G}(U)_2^*$  which is easily seen to be bijective. The equations  $\nu_{i-1} \circ d_i = d_{4-i+1}^* \circ \nu_i$  may be verified directly. —

**(2.23) Proposition.** *If  $U$  is invertible, then  $\mathcal{G}(U)$  is split-exact.*

PROOF: It is no problem to see by direct computation that  $\text{H}(\mathcal{G}(U)) = 0$  in the case under consideration. On the other hand the proposition will follow from the next one once we have shown that  $\text{H}_2(\mathcal{G}(U)) = 0$ . For this purpose let  $V, W \in \mathcal{M}_n(A)$  and suppose  $VU - UW = 0$ . Let  $\tilde{U}$  be the matrix of cofactors of  $U$  and put  $Z = (\det U)^{-1}\tilde{U}V$ . Then  $UZ = V$  and  $ZU = (\det U)^{-1}\tilde{U}VU = (\det U)^{-1}\tilde{U}UW = W$ . —

**(2.24) Proposition.** *Let  $N$  be any  $A$ -module. Then the ideal  $\text{I}_{n-1}(U)$  annihilates  $\text{H}_i(\mathcal{G}(U) \otimes_A N)$  for  $i \neq 2$ .*

PROOF: Let  $E_{ij}$ ,  $1 \leq i, j \leq n$ , be the canonical basis of  $M = \mathcal{M}_n(A)$ . We consider the Koszul complex

$$\mathcal{K}: \dots \bigwedge^2 M \xrightarrow{\partial_2} M \xrightarrow{\partial_1} A \longrightarrow 0$$

derived from the linear form  $\partial_1 = d_1: M \rightarrow A$ . We claim that  $\text{Im } \partial_2 \subset \text{Im } d_2$ . In

connection with (2.22) this yields a commutative diagram

$$\begin{array}{ccccccc}
 & & & \bigwedge^2 M & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & A \\
 & & & \downarrow f & & \parallel & & \parallel \\
 \mathcal{G}(U)_4 & \xrightarrow{d_4} & \mathcal{G}(U)_3 & \xrightarrow{d_3} & \mathcal{G}(U)_2 & \xrightarrow{d_2} & \mathcal{G}(U)_1 & \xrightarrow{d_1} & \mathcal{G}(U)_0 \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & & & \\
 \mathcal{G}(U)_0^* & \xrightarrow{d_1^*} & \mathcal{G}(U)_1^* & \xrightarrow{d_2^*} & \mathcal{G}(U)_2^* & & & & \\
 \parallel & & \parallel & & \downarrow f^* & & & & \\
 A^* & \xrightarrow{\partial_1^*} & M^* & \xrightarrow{\partial_2^*} & \bigwedge^2 M^* & & & & 
 \end{array}$$

Since  $\text{Im } \partial_1 = I_{n-1}(U)$  annihilates the homology of  $\mathcal{K} \otimes_A N$  as well as that of  $\mathcal{K}^* \otimes N$  ([Bo.4], § 9, no. 1, Cor. 2, p. 148), tensoring of the diagram by  $N$  then proves the statement of the proposition.

As to the proof of  $\text{Im } \partial_2 \subset \text{Im } d_2$ , let  $\pi, \varphi$  the maps from (2) and (3) above and  $I = \{1, \dots, n\}$ . An easy computation shows that

$$\partial_2(E_{iu} \wedge E_{iv}) = \pm \sum_{j \neq i} \sigma(j, I \setminus i) [1, \dots, \hat{i}, \dots, \hat{j}, \dots, n | 1, \dots, \hat{u}, \dots, \hat{v}, \dots, n] \varphi(E_{ij}, 0)$$

if  $u \neq v$ , so that  $\partial_2(E_{iu} \wedge E_{iv}) \in \text{Im } d_2$ . In the same way one obtains  $\partial_2(E_{iu} \wedge E_{ju}) \in \text{Im } d_2$ .

Finally let  $i \neq j$ ,  $u \neq v$ . Then

$$\begin{aligned}
 & \partial_2(E_{iu} \wedge E_{jv}) = \\
 & \pm \left( \sum_{k \neq i, j} \sigma(u, I \setminus v) \sigma(k, I \setminus i) [1, \dots, \hat{i}, \dots, \hat{k}, \dots, n | 1, \dots, \hat{u}, \dots, \hat{v}, \dots, n] \varphi(E_{jk}, 0) \right. \\
 & + \sigma(u, I \setminus v) \sigma(j, I \setminus i) [1, \dots, \hat{i}, \dots, \hat{j}, \dots, n | 1, \dots, \hat{u}, \dots, \hat{v}, \dots, n] \varphi(E_{jj}, E_{uu}) \\
 & \left. + \sum_{w \neq u, v} \sigma(j, I \setminus i) \sigma(w, I \setminus v) [1, \dots, \hat{i}, \dots, \hat{j}, \dots, n | 1, \dots, \hat{v}, \dots, \hat{w}, \dots, n] \varphi(0, E_{wu}) \right)
 \end{aligned}$$

so that  $\partial_2(E_{iu} \wedge E_{jv}) \in \text{Im } d_2$ . —

**(2.25) Proposition.** *Let  $N$  be an  $A$ -module. Then the ideal  $(I_{n-1}(U))^2$  annihilates  $H_2(\mathcal{G}(U) \otimes_A N)$ .*

PROOF: Consider  $A$  as an algebra over the ring  $A' = A[X_{ij} : 1 \leq i, j \leq n]$  via the substitution  $X_{ij} \rightarrow u_{ij}$  where  $U = (u_{ij})$ . Let

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$$

be an exact sequence of  $A'$ -modules,  $F$  being free. Then one obtains an exact sequence

$$(4) \quad H_2(\mathcal{G}(X) \otimes_{A'} F) \longrightarrow H_2(\mathcal{G}(U) \otimes_A N) \longrightarrow H_1(\mathcal{G}(X) \otimes_{A'} K)$$



where  $X = (X_{ij})$ , as usual. Put  $d = \det X$  and let  $L$  denote the cokernel of the canonical embedding  $A' \rightarrow A'[d^{-1}]$ . By (2.23) the homology of  $\mathcal{G}(X) \otimes_{A'} A'[d^{-1}]$  vanishes. Therefore  $H_2(\mathcal{G}(X)) \simeq H_3(\mathcal{G}(X) \otimes_{A'} L)$ . Now  $I_{n-1}(X)$  annihilates  $H_3(\mathcal{G}(X) \otimes_{A'} L)$  and  $H_1(\mathcal{G}(X) \otimes_{A'} K)$  by (2.24). Since (4) is an exact sequence  $H_2(\mathcal{G}(U) \otimes_A N)$  is annihilated by  $(I_{n-1}(U))^2$ . —

From (2.24), (2.25) and the acyclicity criterion (16.16) we get

**(2.26) Theorem.** *Let  $A$  be a noetherian ring,  $U$  an  $n \times n$  matrix with entries in  $A$ . Assume that  $\text{grade } I_{n-1}(U) \geq 4$ . Then  $\mathcal{G}(U)$  is acyclic.*

In view of (2.5) we obtain, in particular, that  $\mathcal{G}(X)$  yields a free resolution of  $I_{n-1}(X)$  and  $R_{n-1}(X)$  is a perfect  $B[X]$ -module, so  $R_{n-1}(X)$  is a Cohen-Macaulay ring if this holds for  $B$ . Because of (2.22) the Gorenstein property is also preserved in passing over from  $B$  to  $R_{n-1}(X)$ . This (and (2.21), of course) is a special case of Corollary (8.9) below which says that  $R_t(X)$  is a Gorenstein ring if and only if (i)  $B$  is a Gorenstein ring and (ii)  $m = 1$  or  $m = n$ .

## E. Comments and References

The history of determinantal ideals in case  $t > 1$  seems to begin with Macaulay [Ma]. He stated (2.1) when  $A$  is a polynomial ring over a field and  $t = m$  ([Ma], Section 53). See also [Gb], pp. 199–204, for a simple proof). After a slight generalization of this result, due to Northcott ([No.1], Theorem 9), the general case has been treated by Eagon ([Ea.1], Corollary 4.1). Our proof together with (2.3) is drawn from [EN.1] (Theorem 3).

The localization argument of (2.4) was used, perhaps not for the first time, by Northcott in proving (2.10) ([No.2], Proposition 2). Our version can be found in [Ea.2] (Proof of Theorem 2). (2.5) goes back to Northcott in case  $t = m$  ([No.2], Proposition 1), to Mount ([Mo]) in case  $B$  is a field (of characteristic zero), and to Eagon in the general case ([Ea.2], Theorem 2).

(2.7) is exactly Corollary 5.2 in [Ea.1]; our proof (i.e. (2.9)) is taken from [Ve.2]. The Cohen-Macaulay property of  $R_m(X)$  stated in (2.8), was already proved by Northcott ([No.1], Theorems 10 and 11). More precisely he showed that for a matrix  $U$  with entries in a Cohen-Macaulay ring  $A$  the residue class ring  $A/I_m(U)$  is Cohen-Macaulay, too, if  $I_m(U)$  has the maximally possible grade. This assertion as well as the idea of the proof, which goes by an inductive argument using the knowledge of the first syzygy module of  $I_m(U)$ , is a generalization of corresponding considerations in [Ma], Section 53. The generalizations of (2.7), (2.8) and (2.10) to  $I_t(X)$  for arbitrary  $t$  were proved by Hochster and Eagon ([HE.2], Theorem 1). That of (2.10) has a precursor due to Mount ([Mo]) in case  $B$  is a field (of characteristic zero). For  $t = 2$  the results corresponding to (2.8) and (2.10) had already been proved by Sharpe ([Sh.1], Theorem 3 and Corollary to Theorem 1, resp.) who had also shown the Cohen-Macaulay property of  $A/I_2(U)$  in case the entries of  $U$  belong to a Cohen-Macaulay ring  $A$  and  $I_2(U)$  has maximal grade ([Sh.2], Theorem). In proving these statements Sharpe followed the idea of proof Macaulay and Northcott had applied already: He concurrently computed the first syzygy module of  $I_2(U)$ . (2.11) and (2.12) together with their proofs are drawn from [HE.2] (Corollary 3).

A rather remarkable proof of the perfection of the ideals  $I_m(U)$  has been given by Huneke in [Hu.2]. Huneke concludes the perfection of  $I_m(U)$ ,  $m \leq n$ , from the fact that the ideals  $I_m(U)$ ,  $n = m + 1$ , are even “strongly Cohen-Macaulay”.

The representation of a power of an ideal as a determinantal ideal may be an old idea (though it appears in [BR.1], p. 215 without further reference), and in [Ka], p. 107 it is said that (2.14) goes back to Macaulay. The multiplicity of  $R_m(X)$  has been calculated in [EN.3] as part of an investigation of the Hilbert functions of rings of type  $A/I_m(U)$  based on the Eagon-Northcott complex.

The Eagon-Northcott complex has a long and extensive history. It begins with Hilbert who computed explicitly what we call the Koszul complex derived from a finite sequence of indeterminates ([Hi], p. 229). Gaeta then seems to be the first who indicated a free resolution of  $I_m(U)$  in a comparatively general case ([Ga]); he considered an  $m \times n$  matrix  $U$ ,  $m \leq n$ , the entries of which are homogeneous polynomials over a field such that  $\text{grade} I(U) = n - m + 1$ . The first general construction was given by Eagon and Northcott in [EN.1]: The differentiation  $d_i$  of their complex

$$0 \longrightarrow \bigwedge^n G \otimes S_r(F) \xrightarrow{d_r} \dots \bigwedge^m G \xrightarrow{d_0} A \longrightarrow 0$$

defined by means of bases of  $G$  and  $F$  in an obvious manner, depends on the special choice of the basis for  $F$  (in case  $i > 0$ ) and  $G$  ( $i = 0$ ), resp. Our presentation, i.e.  $\mathcal{D}_0(g)$ , is independent of the bases chosen for  $F$  and  $G$ , up to the definition of  $\nu_0$ . It goes back to [BE.1].

There are numerous generalizations of the Eagon-Northcott complex in a similar direction as we took in Subsection C, and we do not claim the following list to be complete. From the complexes considered in [Bu.1], [BR.1], [BR.2], [Gv], [BE.4], one gets (minimal) free resolutions of  $\text{Coker} \bigwedge^p g$  and  $\text{Coker} S_p(g)$ ,  $1 \leq p \leq m$ , if  $\text{grade} I(g) = n - m + 1$ . With the same assumption the complexes in [Le.1], [Le.2] yield (minimal) free resolutions of  $\bigwedge^p \text{Coker} g^*$ , cf. (2.19), those in [Wm] corresponding resolutions of  $\bigwedge^p \text{Coker} g^*$  and  $S_p(\text{Coker} g^*)$ ,  $1 \leq p \leq n - m + 1$ . The last three papers and [BE.4] make use of divided powers which are also applied to the construction in [BV] giving (minimal) free resolutions of  $S_p(\text{Coker} g)$ ,  $1 \leq p \leq n - m$  as in (2.16). It seems that the complexes  $\mathcal{D}_i(g)$  have first been constructed by Kirby ([Ki]) in terms of bases.

References for (2.20) and (2.21) will be given in Section 9.

[BE.4] also covers the Gulliksen-Negård complex for which we followed the original treatment in [GN]. With [Po] containing a (minimal) free resolution of  $I_{n-1}(U)$  in case  $U$  is an  $n \times (n + 1)$ -matrix (and  $\text{grade} I_{n-1}(U) = 6$ ), we finish our list of “classical” contributions to the problem of constructing free resolutions for determinantal rings.

After some attempts which were more or less effective, Lascoux [Ls] was the first who found a minimal free resolution of  $R_t = B[X]/I_t(X)$ ,  $1 \leq t \leq m$ , over  $B[X]$  in case  $B$  contains the field of rational numbers. This resolution has also been constructed in different ways by Nielsen [Ni.1] and Roberts [Rb.1]. In [PW.1] Pragacz and Weyman give “another approach to Lascoux’s resolution”. It is at present not known whether a minimal free resolution of  $\mathbf{Z}[X]/I_t(X)$  exists which remains minimal after tensoring over  $\mathbf{Z}$  with any ring  $B$ . Of course such a resolution exists in the maximal minor case (see above). Akin, Buchsbaum and Weyman [ABW.1] gave a positive answer in the submaximal minor case by an explicit construction, following an idea first applied in [Bu.3]. For further discussion of the subject we refer the reader to [Ni.2], [Rb.2], [Rb.3].

### 3. Generically Perfect Ideals

The determinantal ring  $B[X]/I_t(X)$ , where  $X$  is a matrix of indeterminates, may be written as

$$B[X]/I_t(X) \cong (\mathbf{Z}[X]/I_t(X)) \otimes_{\mathbf{Z}} B.$$

It arises from the corresponding object over the “generic” ring  $\mathbf{Z}$  by extension of coefficients. In this section we want to study how the arithmetic properties of  $\mathbf{Z}[X]/I$ ,  $I$  an ideal in  $\mathbf{Z}[X]$ , carry over to  $(\mathbf{Z}[X]/I) \otimes_{\mathbf{Z}} B$  under the (inevitable) assumption that  $\mathbf{Z}[X]/I$  is  $\mathbf{Z}$ -flat.

Another type of extension to be investigated below is the substitution of a sequence of indeterminates  $X_1, \dots, X_n$  by an  $A$ -sequence  $x_1, \dots, x_n$ ,  $A$  a  $\mathbf{Z}$ -algebra.

#### A. The Transfer of Perfection

**(3.1) Proposition.** *Let  $M$  be a (not necessarily finitely generated)  $\mathbf{Z}$ -module. Then  $M$  is flat if and only if it is torsionfree.*

PROOF: A flat module is always torsionfree. For the converse one uses that a finitely generated torsionfree  $\mathbf{Z}$ -module is free. Thus  $M$ , being the direct limit of its finitely generated submodules, is the direct limit of flat  $\mathbf{Z}$ -modules and therefore flat itself. —

Throughout this section  $X$  will merely denote a finite collection  $X_1, \dots, X_n$  of indeterminates. The most important property which descends from a  $\mathbf{Z}$ -flat  $\mathbf{Z}[X]$ -module  $M$  to  $M \otimes_{\mathbf{Z}} B$ , is perfection:  $\text{grade } M \otimes_{\mathbf{Z}} B = \text{grade } M$ , and for a free  $\mathbf{Z}[X]$ -resolution  $\mathcal{F}$  of length  $\text{grade } M$  the complex  $\mathcal{F} \otimes_{\mathbf{Z}} B$  is a free resolution over  $B[X]$ . This will be shown in Theorem (3.3) below. (Note that every finitely generated  $\mathbf{Z}[X]$ -module has finite projective dimension for obvious reasons and that every projective  $\mathbf{Z}[X]$ -module is free, cf. [Qu].)

DEFINITION. A finitely generated  $\mathbf{Z}[X]$ -module  $M$  is called *generically perfect* (of grade  $g$ ) if it is perfect (of grade  $g$ ) and faithfully flat as a  $\mathbf{Z}$ -module. An ideal  $I$  is called generically perfect, if  $\mathbf{Z}[X]/I$  is generically perfect.

Before we state the main theorem on generically perfect modules, we want to indicate how this definition could be modified:

**(3.2) Proposition.** *A finitely generated  $\mathbf{Z}[X]$ -module  $M$  is generically perfect of grade  $g$  if and only if  $M$  is a perfect  $\mathbf{Z}[X]$ -module of grade  $g$ , and for every prime number  $p$  the  $(\mathbf{Z}/\mathbf{Z}p)[X]$ -module  $M \otimes_{\mathbf{Z}} (\mathbf{Z}/p\mathbf{Z})$  is perfect of grade  $g$ .*

PROOF: The implication “only if” is covered by Theorem (3.3) below, whereas for the “if” part we only need to prove that  $M$  is torsionfree as a  $\mathbf{Z}$ -module:  $M \otimes (\mathbf{Z}/\mathbf{Z}p) \neq 0$  for all prime numbers  $p$  by hypothesis. Assume that an associated prime  $P \subset \mathbf{Z}[X]$  of  $M$  contains a prime number  $p$ . Since a perfect module is unmixed,  $P$  is a minimal prime of  $M$ , so  $\text{grade } P = g$ , and  $P/\mathbf{Z}[X]p$  is a minimal prime of  $M \otimes (\mathbf{Z}/\mathbf{Z}p)$ , thus  $\text{grade } P/\mathbf{Z}[X]p = g$ , too. This is a contradiction. —

**(3.3) Theorem.** *Let  $M$  be a finitely generated  $\mathbf{Z}[X]$ -module which is faithfully flat over  $\mathbf{Z}$ . Then the following properties are equivalent (all tensor products taken over  $\mathbf{Z}$ ):*

(a)  $M$  is (generically) perfect of grade  $g$ .

(b) For every noetherian ring  $B$ ,  $M \otimes B$  is a perfect  $B[X]$ -module of grade  $g$ .

(c) For every prime number  $p$ ,  $M \otimes (\mathbf{Z}/\mathbf{Z}p)$  is perfect of grade  $g$ .

If  $M$  is generically perfect of grade  $g$ , then for a  $\mathbf{Z}[X]$ -free resolution  $\mathcal{F}$  of  $M$  of length  $g$  the complex  $\mathcal{F} \otimes B$  is a  $B[X]$ -free resolution of  $M \otimes B$ .

PROOF: The implications (b)  $\Rightarrow$  (c) and (b)  $\Rightarrow$  (a) are trivial. Let us discuss the implication (c)  $\Rightarrow$  (a). We have to show that  $M_Q$  is perfect of grade  $g$  over  $\mathbf{Z}[X]_Q$  for every maximal ideal  $Q$  of  $\mathbf{Z}[X]$  for which  $M_Q \neq 0$ . There would be no chance to utilize (c) if  $Q$  would not contain a prime number  $p$ . However such a prime number  $p \in Q$  exists, cf. [Bo.2], § 3, no. 4, Théorème 3, Corollaire 1 for example. To have a compact notation let  $A = \mathbf{Z}[X]$ ,  $S = A/AP$ ,  $P = Q/AP$ . Since  $M_Q \neq 0$ ,  $(M/pM)_P = (M/pM)_Q \neq 0$ , so

$$g = \text{grade}_{S_P}(M/pM)_P = \text{pd}_{S_P}(M/pM)_P.$$

This implies  $\text{pd}_{A_Q} M_Q = g$ , since  $p$  is not a zero-divisor of  $M_Q$  (by flatness over  $\mathbf{Z}$ !). On the other hand one has

$$\text{grade } M = \text{grade}_A M/pM - 1 = (\text{grade}_S M/pM + 1) - 1 = g,$$

using again that  $p$  is not a zero-divisor of  $M$  (and  $A$ ) and  $M \otimes (\mathbf{Z}/\mathbf{Z}p) \neq 0$ .

For the proof of the implication (a)  $\Rightarrow$  (b) we need a lemma.

**(3.4) Lemma.** *Let  $A$  be a noetherian ring, and  $M$  a perfect  $A$ -module of grade  $g$ . Then the sets of zero-divisors of  $M$  and  $\text{Ext}_A^g(M, A)$  coincide.*

PROOF: Since  $\text{Supp } \text{Ext}_A^g(M, A) \subset \text{Supp } M$  in general, and

$$M = \text{Ext}_A^g(\text{Ext}_A^g(M, A), A)$$

here,  $M$  and  $\text{Ext}_A^g(M, A)$  have the same support, and a prime ideal  $P$  is associated to  $M$  if and only if  $M_P \neq 0$  and  $\text{depth } A_P = g$ . Then it is associated to  $\text{Ext}_A^g(M, A)$ , too, and vice versa. —

Let  $M$  be generically perfect of grade  $g$  now,  $A = \mathbf{Z}[X]$ ,  $S = B[X]$ , and

$$\mathcal{F}: 0 \longrightarrow G_g \longrightarrow G_{g-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0$$

a free resolution of  $M$ . Over  $\mathbf{Z}$  the modules  $G_j$  are flat, so

$$\text{H}_i(\mathcal{F} \otimes B) = \text{Tor}_i^{\mathbf{Z}}(M, B) = 0$$

for all  $i \geq 1$ . Hence  $\mathcal{F} \otimes B$  is a free resolution of the  $S$ -module  $M \otimes B$ , and  $\text{pd } M \otimes B \leq g$ .

The crucial point is to show that  $\text{grade } M \otimes B = g$ . By  $*$  we denote the functor  $\text{Hom}_A(\dots, A)$ , by  ${}^\vee$  the functor  $\text{Hom}_S(\dots, S)$ .  $\mathcal{F}^*$  is a free resolution of  $\text{Ext}_A^g(M, A)$ , a

flat  $\mathbf{Z}$ -module by (3.4). As above we conclude that

$$\mathcal{F}^* \otimes B = (\mathcal{F} \otimes B)^\vee$$

is a free resolution (of the  $S$ -module  $\text{Ext}_S^g(M \otimes B, S)$ ), and  $\text{Ext}_S^i(M \otimes B, S) = 0$  for  $i = 1, \dots, g - 1$ . Since  $M \otimes B \neq 0$  is granted, we have

$$\text{grade } M \otimes B \geq g \geq \text{pd } M \otimes B,$$

as desired.

The last contention of the theorem has been proved already. —

We shall apply (3.3) mainly to finitely generated graded  $\mathbf{Z}[X]$ -modules  $M$ , in particular cyclic ones. Then  $M$  is flat over  $\mathbf{Z}$  if and only if it is free (and therefore faithfully flat), for  $M$  is a direct sum of finitely generated  $\mathbf{Z}$ -modules. Our main approach to the investigation of determinantal rings starts with the construction of an explicit  $\mathbf{Z}$ -basis of the determinantal rings (over  $\mathbf{Z}$ ), and therefore (3.3) is ideally suited to reduce the problem of proving perfection for the determinantal ideals to the case where the ring  $B$  of coefficients is a field.

Theorem (3.3) also explains why the resolutions constructed in Section 2 look the same regardless of  $B$ : A resolution over  $\mathbf{Z}[X]$  turns into a resolution over  $B[X]$  upon tensoring with  $B$ .

Often one encounters determinantal ideals of matrices whose entries cannot be regarded as a family of algebraically independent elements generating the ambient ring over a ring of coefficients, for example when the ambient ring is local. In general these ideals are anything but perfect. On the other hand we have seen that an ideal of maximal minors is perfect as soon as its grade is sufficiently large: the generic resolution specializes to an acyclic complex then, and gives a free resolution of the desired length. This fact admits a far-reaching generalization:

**(3.5) Theorem.** *Let  $A$  be a noetherian ring, and  $M$  a perfect  $A$ -module of grade  $g$ . Let  $S$  be a noetherian  $A$ -algebra such that  $\text{grade } M \otimes S \geq g$  and  $M \otimes S \neq 0$ . Then  $M \otimes S$  is perfect of grade  $g$  (and  $\text{grade}(\text{Ann } M)S = g$ ). Furthermore  $\mathcal{F} \otimes S$  is a free resolution of  $M \otimes S$  for every free resolution  $\mathcal{F}$  of  $M$  of length  $g$ .*

PROOF: Note that  $\text{Ann}(M \otimes S)$  and  $(\text{Ann } M)S$  have the same radical (by Nakayama's lemma). Let  $P$  be a prime ideal of  $S$  such that  $\text{grade } P < g$ . Then  $Q = A \cap P \not\subseteq \text{Ann } M$ , and

$$\mathcal{F} \otimes S_P = (\mathcal{F} \otimes A_Q) \otimes S_P$$

is split-exact. Now the claim follows from the acyclicity lemma (16.16). —

For a typical application of (3.5) we consider an  $S$ -sequence  $x_1, \dots, x_n$  and the powers  $I^k$  of the ideal  $I$  generated by it. Let  $A = \mathbf{Z}[X_1, \dots, X_n]$ ,  $J = \sum AX_i$ , and  $M = A/J^k$ . Then  $M$  is generically perfect (by (16.19), say), and one concludes immediately that  $I^k$  is a perfect ideal (of grade  $n$ ): Proposition (2.14) has been proved again, and perhaps in a simpler fashion now.

In the situations of (3.3) and (3.5) not only perfection, but also a free resolution is preserved under the extension considered. As a particular consequence, the canonical module of the extension is obtained as the extension of the canonical module:

**(3.6) Theorem.** *Let  $I$  be a generically perfect ideal in  $\mathbf{Z}[X]$  and  $R = \mathbf{Z}[X]/I$ . Then:*

- (a) *The canonical module  $\omega_R$  is generically perfect, too.*  
 (b) *For a Cohen-Macaulay ring  $B$  with canonical module  $\omega_B$  one has*

$$\omega_{R \otimes_{\mathbf{Z}} B} \cong \omega_R \otimes_{\mathbf{Z}} \omega_B.$$

- (c) *For every Cohen-Macaulay  $\mathbf{Z}[X]$ -algebra  $S$  such that  $\text{grade } IS \geq \text{grade } I$  one has*

$$\omega_{R \otimes_{\mathbf{Z}[X]} S} \cong \omega_R \otimes_{\mathbf{Z}[X]} \omega_S,$$

*provided  $S$  has a canonical module  $\omega_S$ .*

PROOF: Let  $g = \text{grade } I$ . Then  $\omega_R = \text{Ext}_{\mathbf{Z}[X]}^g(R, \mathbf{Z}[X])$ , hence part (a) is a by-product of the proof of (3.3), and parts (b) and (c) are proved essentially in the same way as Theorem (2.20). —

## B. The Substitution of Indeterminates by a Regular Sequence

In the following we shall have to work with associated graded rings and modules. Let  $A$  be a ring,  $I \subset A$  an ideal. The associated graded ring with respect to  $I$  is

$$\text{Gr}_I A = \bigoplus_{i \geq 0} I^i / I^{i+1},$$

and for an  $A$ -module  $M$  the associated graded module is given by

$$\text{Gr}_I M = \bigoplus_{i \geq 0} I^i M / I^{i+1} M;$$

it carries the structure of a  $\text{Gr}_I A$ -module in a natural way. To each element  $x \in M$  we associate its leading form  $x^* \in \text{Gr}_I M$  by

$$\begin{aligned} x^* &= x \bmod I^{d+1} M && \text{if } x \in I^d M \setminus I^{d+1} M, \quad \text{and} \\ x^* &= 0 && \text{if } x \in \bigcap_{i \geq 0} I^i M. \end{aligned}$$

For a submodule  $U \subset M$  the form module  $U^* \subset \text{Gr}_I M$  is generated by the elements  $x^*, x \in U$ . Then obviously

$$(1) \quad \text{Gr}_I(M/U) \cong (\text{Gr}_I M) / U^*.$$

If  $M = A$  and  $J \subset A$  is an ideal, the isomorphism (1) implies that

$$(1') \quad \text{Gr}_{\overline{I}} \overline{A} \cong (\text{Gr}_I A) / J^*$$

where we let  $\overline{A} = A/J$ ,  $\overline{I} = (I + J)/J$ .

Let  $A$  be a noetherian ring and  $x_1, \dots, x_n$  an  $A$ -sequence. Then the associated graded ring  $\text{Gr}_I A$  with respect to the ideal  $I$  generated by  $x_1, \dots, x_n$  is a polynomial ring over  $A/I$ , cf. [Re], the indeterminates being represented by the residue classes  $x_1^*, \dots, x_n^*$  modulo  $I^2$  of  $x_1, \dots, x_n$ . Suppose that  $M$  is a generically perfect module over  $\mathbf{Z}[X]$ ,  $X = (X_1, \dots, X_n)$ . Then, by (3.3) the module

$$\overline{M} = M \otimes_{\mathbf{Z}} A/I$$

is a perfect  $(\text{Gr}_I A)$ -module. We would like to conclude that

$$\widetilde{M} = M \otimes_{\mathbf{Z}[X]} A$$

is a perfect  $A$ -module, where  $A$  is made a  $\mathbf{Z}[X]$ -algebra via the substitution  $X_i \rightarrow x_i$ . From (3.5) it is clear that we only need to know that  $\text{grade}(\text{Ann } M)A \geq \text{grade } M$ . Since  $\text{grade}(\text{Ann } M)(\text{Gr}_I A) \geq \text{grade } M$ , this should hold if  $(\text{Ann } M)A$  and  $(\text{Ann } M)(\text{Gr}_I A)$  can be related in a reasonable fashion.

The following example shows that  $\widetilde{M}$  may not be perfect if one replaces indeterminates by an  $A$ -sequence without further precautions. Let  $A = \mathbf{Z}[U, V, W]$ . Then  $x_1 = U$ ,  $x_2 = V(1-U)$ ,  $x_3 = W(1-U)$  is an  $A$ -sequence. However  $(\mathbf{Z}[X_1, X_2, X_3]/(X_2, X_3)) \otimes A$  (via the substitution  $X_i \rightarrow x_i$ ) is not a perfect  $A$ -module. Though  $x_2^*, x_3^*$  is a  $\text{Gr}_I A$ -sequence,  $\text{grade}(x_2, x_3) = 1$  only, and  $(x_2, x_3)$  is not even unmixed. The difficulty arises from the fact that

$$\bigcap_{k \geq 0} x_3^k R = 0 \quad \text{in } R = A/Ax_2.$$

The usual way out is the assumption that  $I$  be contained in the Jacobson radical.

**(3.7) Lemma.** *Let  $A$  be a noetherian ring,  $I, J$  ideals in  $A$ ,  $x \in A$ .*

- (a) *If  $x^*$  is not a zero-divisor modulo  $J^*$ , then  $(J + Ax)^* = J^* + (\text{Gr}_I A)x^*$ .*
- (b) *If furthermore  $I$  is contained in the Jacobson radical of  $A$ , then  $x$  is not a zero-divisor modulo  $J$ .*

PROOF: The isomorphism (1') above readily reduces the problem to the case in which  $J = 0$ . Then the first statement follows from the equation  $a^*x^* = (ax)^*$  which always holds if  $a^*x^* \neq 0$ , and the second statement is trivial:  $a^* \neq 0$  for all  $a \in A$ . —

**(3.8) Lemma.** *Let  $A$  be a noetherian ring,  $I, J$  ideals in  $A$  such that  $I$  is contained in the Jacobson radical of  $A$ . Suppose that  $\text{grade } J^* \geq g$ . Then  $J^*$  contains a  $(\text{Gr}_I A)$ -sequence  $x_1^*, \dots, x_g^*$ ,  $x_i \in J$ . Therefore  $x_1, \dots, x_g$  is an  $A$ -sequence, and, in particular,  $\text{grade } J \geq g$ .*

PROOF: A homogeneous ideal which is not composed of zero-divisors must contain a form which is not a zero-divisor. The rest is induction on  $g$ , the inductive step relying on the preceding lemma. —

We return to the situation discussed above ( $\widetilde{M} = M \otimes_{\mathbf{Z}[X]} A$ ,  $\overline{M} = M \otimes_{\mathbf{Z}} A/I$ ). If  $I$  is contained in the Jacobson radical, then by (3.8)

$$\text{grade}(\text{Ann } \widetilde{M}) \geq \text{grade}(\text{Ann } \overline{M})^*.$$

In general (cf. (3.10),(c) below)  $\text{grade}(\text{Ann } \widetilde{M})^* < \text{grade}(\text{Ann } \overline{M})$ , and the argument breaks down. However, if  $M$  is a graded  $\mathbf{Z}[X]$ -module, then  $\text{Ann } M$  is generated by

forms  $f_1, \dots, f_s \in \mathbf{Z}[X]$  of positive degree (otherwise  $M$  would not be  $\mathbf{Z}$ -flat!). The ideal  $(\text{Ann } M)A$  is generated by the elements  $f_i(x) \in I$  whereas  $(\text{Ann } M)(\text{Gr}_I A)$  is generated by the  $f_i(x^*)$ . Since for a homogeneous polynomial  $f \in A[X]$

$$(f(x))^* = f(x^*) \quad \text{or} \quad f(x^*) = 0,$$

we conclude that  $((\text{Ann } M)A)^*$  already contains  $(\text{Ann } M)(\text{Gr}_I A)$ . So

$$\begin{aligned} \text{grade } M &= \text{grade}(\text{Ann } M)(\text{Gr}_I A) \leq \text{grade}((\text{Ann } M)A)^* \\ &\leq \text{grade}(\text{Ann } M)A \leq \text{grade}(\text{Ann } \widetilde{M}), \end{aligned}$$

and quoting Theorem (3.5) we complete the proof of:

**(3.9) Theorem.** *Let  $M$  be a generically perfect graded  $\mathbf{Z}[X]$ -module of grade  $g$ . Let  $A$  be a noetherian ring,  $x_1, \dots, x_n$  an  $A$ -sequence such that  $I = \sum_{i=1}^n Ax_i$  is contained in the Jacobson radical of  $A$ . Then, via the substitution  $X_i \rightarrow x_i$ , the  $A$ -module  $M \otimes A$  is perfect of grade  $g$ .*

**(3.10) Remarks.** (a) The hypothesis on  $I$  can be slightly weakened. Whether  $M \otimes A$  is perfect of grade  $g$ , can be decided from the localizations  $M \otimes A_Q$ , where  $Q$  runs through the maximal ideals containing  $(\text{Ann } M)A$ . Therefore one may localize first, and it suffices that  $I \subset Q$  for these maximal ideals  $Q$ .

(b) The assumption that  $I$  be contained in the Jacobson radical can be replaced by the hypothesis that  $A$  is graded and the  $x_i$  are forms of positive degree. We leave the necessary modifications to the reader. (Any hypothesis covering both cases has a rather artificial flavour.)

(c) Let  $p \in \mathbf{Z}$ ,  $p \neq 0, \pm 1$ ,  $A = \mathbf{Q}[[U, V]]/(UV - pV)$ , and  $x_1$  the residue class of  $U$  in  $A$ . The module  $M = \mathbf{Z}[X_1]/(X_1 - p)$  is generically perfect and  $x_1$  is contained in the Jacobson radical. Nevertheless  $M \otimes A$  is not perfect, an example demonstrating that the assumption on  $M$  being graded is essential.

(d) Theorem (3.9) has obvious consequences for determinantal ideals. Whenever  $C$  is a matrix whose entries form a regular sequence inside the Jacobson radical, then the ideals  $I_t(C)$  are perfect. (It will be proved in (5.18) that the ideals  $I_t(X)$  are generically perfect.) Guided by this example we want to indicate a second approach to the proof of (3.9). Let  $X$  be an  $m \times n$  matrix of indeterminates over  $\mathbf{Z}$ . As we shall see in (5.9) there is a sequence  $y_1, \dots, y_s$ ,  $s = mn - (m - t + 1)(n - t + 1)$ , of elements in  $I_1(X)$  such that

$$\text{Rad}(I_t(X) + \sum_{i=1}^s \mathbf{Z}[X]y_i) = I_1(X).$$

If  $C$  is as above and  $\varphi: \mathbf{Z}[X] \rightarrow A$  the substitution  $X \rightarrow C$ , then

$$\text{Rad}(I_t(C) + \sum_{i=1}^s A\varphi(y_i)) = \text{Rad } I_1(C)$$

and it follows that

$$\text{grade } I_t(C) \geq \text{grade } I_1(C) - s = (m - t + 1)(n - t + 1)$$

since  $I_1(C)$  is contained in the Jacobson radical (cf. [Ka], Theorem 127). —



Besides  $M \otimes_{\mathbf{Z}[X]} A$  and  $M \otimes_{\mathbf{Z}} (A/I) = M \otimes_{\mathbf{Z}[X]} (\text{Gr}_I A)$  there is a third module of interest, namely  $\text{Gr}_I(M \otimes A)$ . This is a graded module over  $\text{Gr}_I A$  generated by its forms of degree zero, and unless the same, possibly after a shift of the graduation, holds for  $M$  itself, we cannot expect that  $\text{Gr}_I(M \otimes A) \cong M \otimes (\text{Gr}_I A)$ .

**(3.11) Proposition.** *Let  $M$  be a graded  $\mathbf{Z}$ -flat (thus  $\mathbf{Z}$ -free)  $\mathbf{Z}[X]$ -module generated by its forms of lowest degree. Then, with  $A, x_1, \dots, x_n$ , and  $I$  as in (3.9) one has*

$$\text{Gr}_I(M \otimes A) \cong \widetilde{M} \otimes (\text{Gr}_I A),$$

the tensor products taken over  $\mathbf{Z}[X]$ .

PROOF: After a shift we may assume that  $M$  is generated by its forms of degree zero. Consider a homogeneous representation

$$R^p \xrightarrow{g} R^n \xrightarrow{f} R^m \longrightarrow M \longrightarrow 0$$

over  $R = \mathbf{Z}[X]$  in which the elements of the canonical basis of  $R^m$  are assigned the degree 0, and those of the canonical basis  $e_1, \dots, e_n$  of  $R^n$  are assigned degrees  $\varepsilon_1, \dots, \varepsilon_n$  such that  $\varepsilon_i = \deg f(e_i), i = 1, \dots, n$ . Let  $d_1, \dots, d_p$  the canonical basis of  $R^p$ , and  $\delta_j = \deg g(d_j), j = 1, \dots, p$ . We may assume that none of  $f(e_1), \dots, f(e_n)$  or  $g(d_1), \dots, g(d_p)$  is divisible by any  $q \in \mathbf{Z}, q \neq \pm 1$ , since  $M$  and  $\text{Im } f$  are  $\mathbf{Z}$ -flat. Therefore  $\overline{M} = M \otimes_{\mathbf{Z}} (A/I)$  has the homogeneous representation

$$(2) \quad G^p \xrightarrow{\overline{g}} G^n \xrightarrow{\overline{f}} G^m \longrightarrow \overline{M} \longrightarrow 0, \quad G = \text{Gr}_I A,$$

in which none of  $\overline{f}(e_i \otimes 1)$  or  $\overline{g}(d_j \otimes 1)$  is zero,  $i = 1, \dots, n, j = 1, \dots, p$ . After tensoring the representation of  $M$  with  $A$  (over  $R$ ) we obtain a zero-sequence

$$A^p \xrightarrow{\tilde{g}} A^n \xrightarrow{\tilde{f}} A^m \longrightarrow \widetilde{M} \longrightarrow 0, \quad \widetilde{M} = M \otimes A,$$

which is exact at  $A^m$ . Since  $\overline{f}(e_i \otimes 1) \neq 0$ , we have  $\tilde{f}(e_i \otimes 1) \in I^{\varepsilon_i} A^m, \tilde{f}(e_i \otimes 1) \notin I^{\varepsilon_i+1} A^m$ . Hence  $(\tilde{f}(e_i \otimes 1))^* = \overline{f}(e_i \otimes 1)$ , and  $\text{Im } \tilde{f} \subset (\text{Im } \tilde{f})^*$ , where  $*$  now denotes leading forms in  $A^m$ , of course. We have to prove that  $(\text{Im } \tilde{f})^* \subset \text{Im } \overline{f}$ , too (cf. the isomorphism (1) above). The crucial argument will be that one can “lift” any relation of the  $\overline{f}(e_i \otimes 1)$  because of the exactness of (2). Let  $\sum a_j \tilde{f}(e_j \otimes 1) \in I^u A^m$ . It suffices to show that there are  $\tilde{a}_j \in I^{u-\varepsilon_j}$  such that

$$\sum a_j \tilde{f}(e_j \otimes 1) = \sum \tilde{a}_j \tilde{f}(e_j \otimes 1).$$

Suppose that  $\deg a_j^* = \alpha_j$  and that there is a  $j$  with  $\alpha_j < u - \varepsilon_j$ . Let

$$\kappa = \max_j \{ u - \varepsilon_j - \alpha_j \}$$

and

$$b_j = \begin{cases} a_j & \text{if } u - \varepsilon_j - \alpha_j = \kappa, \\ 0 & \text{else.} \end{cases}$$

Then  $\sum b_j^* \bar{f}(e_j \otimes 1) = 0$  and  $\deg(b_1^*, \dots, b_n^*) = u - \kappa$ . Therefore there are homogeneous elements  $r_1, \dots, r_p$ ,  $\deg r_k = u - \kappa - \delta_k$  or  $r_k = 0$ , such that

$$(b_1^*, \dots, b_n^*) = \sum r_k \bar{g}(d_k \otimes 1).$$

Let  $s_1, \dots, s_p \in A$  such that  $s_k^* = r_k$ ,  $k = 1, \dots, p$ , and define

$$(\tilde{a}_1, \dots, \tilde{a}_n) = (a_1, \dots, a_n) - \sum s_k \tilde{g}(d_k \otimes 1).$$

For the  $j$ -th component  $d_{k_j}$  of  $g(d_k)$  one has

$$d_{k_j} = 0 \quad \text{or} \quad \deg d_{k_j} = \delta_k - \varepsilon_j.$$

The image of  $d_{k_j}$  under the extension  $\mathbf{Z}[X] \rightarrow A$  is the  $j$ -th component  $\tilde{d}_{k_j}$  of  $\tilde{g}(d_k \otimes 1)$ . We claim that

$$(3) \quad \sum s_k \tilde{d}_{k_j} \equiv b_j \equiv a_j \pmod{I^{u-\varepsilon_j-\kappa+1}}.$$

Since obviously

$$\sum a_j \tilde{f}(e_j \otimes 1) = \sum \tilde{a}_j \tilde{f}(e_j \otimes 1),$$

we are done, for an iterated application of the procedure leading from  $a_j$  to  $\tilde{a}_j$  will produce a representation  $\sum \tilde{a}_j \tilde{f}(e_j \otimes 1)$  as desired. The congruence (3) is likewise obvious, since  $b_j \equiv a_j$  by definition, and the residue class of  $s_k \tilde{d}_{k_j}$  modulo  $I^{u-\varepsilon_j-\kappa+1}$  is  $r_k \bar{d}_{k_j}$ ,  $\bar{d}_{k_j}$  being the  $j$ -th component of  $\bar{g}(d_k \otimes 1)$ , as seen by arguing with degrees. —

The hypothesis that  $M$  be generated by its forms of least degree is satisfied for cyclic graded  $\mathbf{Z}[X]$ -modules, so in particular for our standard example, the residue class rings modulo determinantal ideals. A typical application of (3.11) will be given in (3.13).

### C. The Transfer of Integrity and Related Properties

As pointed out already, one of the consequences of Theorem (3.3) for determinantal ideals, say, is that their perfection can be proved by considering fields only as the rings of coefficients. A similar reduction works for the properties of being a radical ideal, a prime ideal or even a prime ideal with a normal residue class ring.

**(3.12) Proposition.** *Let  $J$  be an ideal of  $\mathbf{Z}[X]$  such that  $\mathbf{Z}[X]/J$  is faithfully flat over  $\mathbf{Z}$ . Suppose that  $(\mathbf{Z}[X]/J) \otimes_{\mathbf{Z}} K$  is reduced (a (normal) integral domain) whenever  $K$  is a field. Then  $(\mathbf{Z}[X]/J) \otimes_{\mathbf{Z}} B$  is reduced (a (normal) domain) if  $B$  is noetherian and reduced (a (normal) domain).*

PROOF: Let  $F$  denote the total ring of fractions of  $B$ . Then the embedding  $B \rightarrow F$  extends to an embedding

$$(\mathbf{Z}[X]/J) \otimes B \rightarrow (\mathbf{Z}[X]/J) \otimes F.$$

If  $B$  is reduced (a domain) then  $F$  is a direct product of finitely many fields (a field), so  $(\mathbf{Z}[X]/J) \otimes F$  is a direct product of reduced rings (a domain). It remains to consider normality, for which we can use the properties of flat extensions, since  $A = (\mathbf{Z}[X]/J) \otimes B$  is  $B$ -flat. The normality of  $A$  follows from the normality of  $B$  and the normality of the fibers of the extension  $B \rightarrow A$  which are given as  $(\mathbf{Z}[X]/J) \otimes (B_Q/QB_Q)$ ,  $Q$  running through the prime ideals of  $B$ . —

This proposition has a variant concerning extensions  $\mathbf{Z}[X] \rightarrow A$  as discussed in (3.9) and (3.11).

**(3.13) Proposition.** *Let  $J$  be a homogeneous ideal in  $\mathbf{Z}[X]$  such that  $\mathbf{Z}[X]/J$  is  $\mathbf{Z}$ -flat and  $(\mathbf{Z}[X]/J) \otimes_{\mathbf{Z}} K$  is reduced (a (normal) domain) for all fields  $K$ . Let  $A$  be a noetherian ring with an  $A$ -sequence  $x_1, \dots, x_n$  inside its Jacobson radical such that  $A/I$ ,  $I = \sum_{i=1}^n Ax_i$ , is reduced (a (normal) domain). Then, via the substitution  $X_i \rightarrow x_i$ ,  $(\mathbf{Z}[X]/J) \otimes A$  is reduced (a (normal) domain).*

PROOF: Let  $G = \text{Gr}_I A$ . Then, by virtue of (3.11), one has  $(JA)^* = JG$ . Proposition (3.12) implies that  $G/JG$  is reduced (a (normal) domain) and this in turn forces  $A/JA$  to have the property claimed (cf. [ZS], p. 250). —

Again the hypotheses of (3.13) could be slightly weakened as indicated in (3.10),(a).

The following proposition will sometimes help to compute the grade or height of an ideal.

**(3.14) Proposition.** *Let  $J$  be an ideal in  $\mathbf{Z}[X]$  such that  $R = \mathbf{Z}[X]/J$  is faithfully flat. Suppose that  $I$  is an ideal in  $R$  such that  $\text{grade } I(R \otimes_{\mathbf{Z}} K) \geq k$  for all fields  $K$ . Then  $\text{grade } I(R \otimes_{\mathbf{Z}} B) \geq k$  for all noetherian rings  $B$ , and if  $\text{grade } I(R \otimes_{\mathbf{Z}} K) = k$  throughout, then always  $\text{grade } I(R \otimes_{\mathbf{Z}} B) = k$ . Analogous statements hold for height.*

PROOF: Let  $P$  be a prime ideal of  $A = R \otimes_{\mathbf{Z}} B$ ,  $P \supset IA$ ,  $Q = B \cap P$ , and  $K = B_Q/QB_Q$ . Then

$$\text{depth } A_P = \text{depth } B_Q + \text{depth } A_P \otimes_A K$$

since  $A$  is a flat  $B$ -algebra.  $A_P \otimes_A K$  is a localization of  $R \otimes_{\mathbf{Z}} K$  with respect to a prime ideal containing  $I(R \otimes_{\mathbf{Z}} K)$ , hence

$$\text{depth } A_P \geq \text{depth } A_P \otimes_A K \geq k$$

as desired.

Suppose now that  $\text{grade } I(R \otimes_{\mathbf{Z}} K) = k$  for all fields  $K$ . Then we choose  $P$  as follows. First we pick a minimal prime  $Q$  of  $B$ . Next we take a minimal prime ideal  $\tilde{P}$  of  $I(R \otimes_{\mathbf{Z}} K)$ ,  $K = B_Q/QB_Q$ , such that  $\text{depth}(R \otimes_{\mathbf{Z}} K)_{\tilde{P}} = k$ . Then the preimage  $P$  of  $\tilde{P}$  in  $A$  satisfies  $P \cap B = Q$ ,  $(R \otimes_{\mathbf{Z}} K)_{\tilde{P}} = A_P \otimes_A K$ , so  $\text{depth } A_P = k$  by the equation above.

In order to get the statements for height, one replaces depth by dimension. —

We note a consequence which will be used several times below:

**(3.15) Corollary.** *Suppose that (the image of)  $x \in R$  is not a zero-divisor in  $R \otimes_{\mathbf{Z}} K$  for all fields  $K$ . Then  $x$  is not a zero-divisor in  $R \otimes_{\mathbf{Z}} B$  for every ring  $B$ .*

In fact, if  $x$  is a zero-divisor in  $R \otimes_{\mathbf{Z}} B$  for some commutative ring  $B$ , then there is a finitely generated  $\mathbf{Z}$ -algebra  $\tilde{B} \subset B$  such that  $x$  is a zero-divisor in  $R \otimes_{\mathbf{Z}} \tilde{B}$ , and one obtains a contradiction from (3.14).

### D. The Bound for the Height of Specializations

Let  $M$  be a perfect  $\mathbf{Z}[X]$ -module of grade  $g$ ,  $A$  a noetherian  $\mathbf{Z}[X]$ -algebra,  $P$  a minimal prime ideal of  $\text{Ann}(M \otimes A)$ . Then, applying (3.5) to  $M \otimes A_P$ , we see that

$$\text{grade } P \leq \text{grade } PA_P \leq g.$$

This inequality can be sharpened:

**(3.16) Theorem.** *Let  $M$  be a perfect  $\mathbf{Z}[X]$ -module of grade  $g$ ,  $A$  a noetherian  $\mathbf{Z}[X]$ -algebra. Then  $\text{ht } P \leq g$  for every minimal prime ideal  $P$  of  $\text{Ann}(M \otimes A)$ .*

PROOF: We may obviously assume that  $A$  is local and  $P$  its maximal ideal, and, completing if necessary, that  $A$  is a complete local ring.  $P$  is a minimal prime of  $(\text{Ann } M)A$ , too. By the Cohen structure theorem  $A$  may be written  $S/I$ , where  $S$  is a regular local ring. The extension  $\mathbf{Z}[X] \rightarrow A$  can be factored through  $S$ , and

$$\text{ht}(\text{Ann } M)S = \text{grade}(\text{Ann } M)S \leq g.$$

On the other hand  $((\text{Ann } M)S + I)/I = (\text{Ann } M)A$ ; so  $(\text{Ann } M)S + I$  is primary with respect to the maximal ideal of  $S$ . By Serre's intersection theorem ([Se], Théorème 3, p. V-18)

$$\text{ht}(\text{Ann } M)S + \text{ht } I \geq \text{ht}((\text{Ann } M)S + I),$$

and

$$\text{ht}((\text{Ann } M)S + I) = \dim S = \text{ht}(\text{Ann } M)A + \text{ht } I,$$

so

$$g \geq \text{ht}(\text{Ann } M)S \geq \text{ht}(\text{Ann } M)A.—$$

For the determinantal ideals (3.16) has been known to us already; it was proved in this special case by more direct methods in (2.1). Taking  $M = \mathbf{Z}[X]/(\sum X_i \mathbf{Z}[X])$  we see that (3.16) is a generalization of Krull's principal ideal theorem.

### E. Comments and References

The notion “generically perfect” was introduced by Eagon and Northcott in [EN.2] using the description in (3.2) as a definition. As their main results one may consider (3.5) ([EN.2], Corollary 1, p. 158) and (3.16) ([EN.2], Theorem 3). The non-obvious implication (a)  $\Rightarrow$  (b) of (3.3) was proved by Hochster [Ho.1]. In Hochster's terminology (3.3) states that every generically perfect module is strongly generically perfect. Our proof of (3.3) is a substantial simplification for the case considered by us, namely  $\mathbf{Z}$  as the base ring. The theory of generic perfection can be extended in different directions: (i) One can work relative to a base ring  $\Lambda$  and consider  $\Lambda$ -algebras throughout. (ii) The “noetherian” hypothesis can be dropped after the introduction of an adequate definition of grade for general commutative rings. (iii) As a minor modification, one can weaken the hypothesis “faithfully flat” into “flat” allowing in (3.3), say, that “ $M \otimes B = 0$  or  $M \otimes B$  is perfect ...”. We refer to [Ba], [Ho.5], [No.3], [No.4], [No.5] and [No.6] for more information.

There is however one generalization the reader can perform without substantial changes in the proofs:  $\mathbf{Z}$  as the base can be replaced by any Dedekind domain or field  $D$ , since the properties of  $\mathbf{Z}$  are used only for the equivalence of “torsionfree” and “flat”. Then of course only  $D$ -algebras may be considered and (3.3),(c) has to be modified in an obvious way.

With the same generalization, Theorem (3.9) was proved by Eagon and Hochster in [EH], where the replacement of indeterminates by elements in a regular sequence was investigated in a more general situation. The method of proof employed in [EH] is indicated in (3.10),(d). Our proof of (3.9) and those of (3.11) and (3.13) are patterned after [No.2], where Northcott considered ideals of maximal minors. Part of (3.13) can also be found in [Ng.2]. Proposition (3.12) is taken from [Ho.3] where  $B$  is not supposed to be noetherian, an assumption which simplifies the proof for “normal”. The example (3.10),(c) was given in [EH].

Since (3.5) indicates how one could derive results for “non-generic” determinantal ideals from those on the ideals  $I_t(X)$ , this may be an appropriate place to list some articles in which such determinantal ideals have been investigated. Definitive results have been obtained by Eisenbud in [Ei.2] on ideals  $I_t(\overline{X})$  where  $\overline{X}$  denotes the matrix of residue classes of the entries of  $X$  in a ring  $B[X]/J$ ,  $J$  being generated by linear forms in the indeterminates. The case  $t = \min(m, n)$  had previously been treated by Giusti and Merle ([GM]).

The ultimate generalization of (3.16) would be the “homological height conjecture”: Let  $R \rightarrow S$  be a homomorphism of noetherian rings, and  $M$  an  $R$ -module; then  $\text{ht } P \leq \text{pd } M$  for every minimal prime ideal  $P \supset (\text{Ann } M)S$ . It is known to hold if  $S$  contains a field, cf. [Ho.9].

## 4. Algebras with Straightening Law on Posets of Minors

Among the residue class rings  $B[X]/I$  the most easily accessible ones are those for which the ideal  $I$  is generated by a set of monomials, since one can use the structure of  $B[X]$  as a free  $B$ -module very favourably:  $I$  itself is generated as a  $B$ -module by a subset of the monomial  $B$ -basis of  $B[X]$ . The multiplication table with respect to this basis is very simple, a property inherited by  $B[X]/I$ .

With respect to the monomial basis of  $B[X]$ , a minor of  $X$  is a very complicated expression. Therefore it is desirable to find a new basis of  $B[X]$  which contains the minors and as many of their products as possible. The construction of such a basis is the main object of this section. This basis will consist of monomials whose factors are minors of  $X$ , and whether such a monomial is an element of the basis can be decided by a simple combinatorial criterion.

The set of maximal minors of a matrix has a combinatorially simpler structure than the set of all minors: one needs only one set of indices to specify a maximal minor, and all the maximal minors have the same size. Therefore it is simpler to treat the rings  $G(X)$  first and to derive the structure sought for  $B[X]$  afterwards (from  $G(\tilde{X})$  for an extended matrix  $\tilde{X}$ ).

### A. Algebras with Straightening Law

When all the minors of  $X$  appear in a  $B$ -basis of  $B[X]$ , then, apart from trivial cases, it is impossible that a product of two elements of the basis is in the basis always; nevertheless one has sufficient control over the multiplication table. This situation is met often enough to justify the introduction of a special class of algebras:

**DEFINITION.** Let  $A$  be a  $B$ -algebra and  $\Pi \subset A$  a finite subset with a partial order  $\leq$ , called a *poset* for short.  $A$  is a *graded algebra with straightening law* (on  $\Pi$ , over  $B$ ) if the following conditions hold:

(H<sub>0</sub>)  $A = \bigoplus_{i \geq 0} A_i$  is a graded  $B$ -algebra such that  $A_0 = B$ ,  $\Pi$  consists of homogeneous elements of positive degree and generates  $A$  as a  $B$ -algebra.

(H<sub>1</sub>) The products  $\xi_1 \dots \xi_m$ ,  $m \in \mathbf{N}$ ,  $\xi_i \in \Pi$ , such that  $\xi_1 \leq \dots \leq \xi_m$  are linearly independent. They are called *standard monomials*.

(H<sub>2</sub>) (Straightening law) For all incomparable  $\xi, v \in \Pi$  the product  $\xi v$  has a representation

$$\xi v = \sum a_\mu \mu, \quad a_\mu \in B, \quad a_\mu \neq 0, \quad \mu \text{ standard monomial,}$$

satisfying the following condition: every  $\mu$  contains a factor  $\zeta \in \Pi$  such that  $\zeta \leq \xi$ ,  $\zeta \leq v$  (it is of course allowed that  $\xi v = 0$ , the sum  $\sum a_\mu \mu$  being empty).

The rather long notation “algebra with straightening law” will be abbreviated by *ASL*.

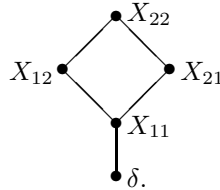
We shall see in Proposition (4.1) that the standard monomials form in fact a basis of  $A$  as a  $B$ -module, the *standard basis* of  $A$ . The representation of an element  $x \in A$  as

a linear combination of standard monomials is called its *standard representation*. The relations in  $(H_2)$  will be referred to as the *straightening relations*.

To be formally precise one would better consider a partially ordered set  $\Pi$  outside  $A$  and an injection  $\Pi \rightarrow A$ . We have preferred to avoid this notational complication and warn the reader that  $\Pi$  (or a subset of it) may be treated as a subset of different rings, in particular when  $A$  and a residue class ring of  $A$  occur simultaneously. Similarly we do not distinguish between a formal monomial in  $\Pi$  and the corresponding ring element. Condition  $(H_1)$  of course says that the family of ring elements parametrized by the formal standard monomials is linearly independent. Whenever a function is defined on (a subset of) the set of monomials by reference to the factors of the monomials, then such a definition properly applies to the formal monomials.

Before we discuss an example, one simple observation: If  $A$  is a graded ASL over  $B$  on  $\Pi$  and  $C$  a  $B$ -algebra, then  $A \otimes C$  is a graded ASL over  $C$  on  $\Pi$  in a natural way.

The polynomial ring  $B[T_1, \dots, T_u]$  is a graded ASL in a trivial fashion: one orders  $T_1, \dots, T_u$  linearly. For a less trivial example we let  $X$  be a  $2 \times 2$ -matrix,  $\delta$  its determinant. We order the set  $\Pi$  of minors of  $X$  according to the diagram



The conditions  $(H_0)$  and  $(H_2)$  are obviously satisfied: Only  $X_{12}$  and  $X_{21}$  are incomparable, and the straightening law consists of the single relation  $X_{12}X_{21} = X_{11}X_{22} - \delta$ . Replacing every occurrence of the product  $X_{12}X_{21}$  in a monomial by  $X_{11}X_{22} - \delta$  one obtains a representation as a linear combination of standard monomials. Furthermore one has a bijective degree-preserving correspondence between the ordinary monomials and the standard monomials:

$$X_{11}^i X_{12}^j X_{21}^k X_{22}^l \longleftrightarrow \begin{cases} X_{11}^i X_{12}^{j-k} \delta^k X_{22}^l & \text{if } j \geq k, \\ X_{11}^i X_{21}^{k-j} \delta^j X_{22}^l & \text{if } k > j. \end{cases}$$

Therefore the standard monomials must be linearly independent, and  $B[X]$  is an ASL on  $\Pi$ . It is much more difficult to establish the analogous result for bigger matrices.

**(4.1) Proposition.** *Let  $A$  be a graded ASL over  $B$  on  $\Pi$ . Then:*

- (a) *The standard monomials generate  $A$  as a  $B$ -module, thus forming a  $B$ -basis of  $A$ .*
- (b) *Furthermore every monomial  $\mu = \xi_1 \dots \xi_m$ ,  $\xi_i \in \Pi$ , has a standard representation in which every standard monomial contains a factor  $\xi \leq \xi_1, \dots, \xi_m$ .*

PROOF: For  $\xi \in \Pi$  let  $u(\xi) = |\{\delta \in \Pi : \xi \leq \delta\}|$  and  $w(\xi) = 3^{u(\xi)}$ ; for  $\mu = \xi_1 \dots \xi_m$ ,  $\xi_i \in \Pi$ , we put  $w(\mu) = \sum_{i=1}^m w(\xi_i)$ . (This is an example of a definition properly applying to the formal monomials.) Obviously  $w(\xi v) < w(\mu)$  for all the monomials  $\mu$  appearing on the right side of the standard representation  $\xi v = \sum a_\mu \mu$ .

Because of  $(H_1)$  it is enough for part (a) to show that every monomial is a linear combination of standard monomials. If all the factors  $\xi_1, \dots, \xi_m$  of  $\mu$  are comparable,  $\mu$  is a standard monomial. Otherwise two of the factors are incomparable. Replacing their

product by the right side of the corresponding straightening relation produces a linear combination of monomials which, if different from 0, have a greater value than  $\mu$  under the function  $w$ . On the other hand their values are bounded above since they have the same degree as homogeneous elements of  $A$ :  $w(\mu) \leq d \cdot 3^{|\Pi|}$  for monomials  $\mu$  of degree  $d$ . Thus we are through by descending induction. The easy proof of the second assertion, a similar induction, is left to the reader. —

The preceding proof shows that the standard representation of an element of  $A$  can be obtained by successive applications of the straightening relations, regardless of the order in which the steps of “straightening” are performed. As a consequence the straightening relations generate the defining ideal of  $A$ :

**(4.2) Proposition.** *Let  $A$  be a graded ASL over  $B$  on  $\Pi$ , and  $T_\xi$ ,  $\xi \in \Pi$  a family of indeterminates over  $B$ . For each monomial  $\mu = \xi_1 \dots \xi_m$ ,  $\xi_i \in \Pi$ , let  $T_\mu = T_{\xi_1} \dots T_{\xi_m}$ . Then the kernel of the epimorphism*

$$\varphi : B[T_\xi : \xi \in \Pi] \longrightarrow A, \quad T_\xi \longrightarrow \xi,$$

is generated by the elements  $T_\xi T_\nu - \sum a_\mu T_\mu$  representing the straightening relations.

PROOF: Let  $f \in \text{Ker } \varphi$ ,  $f = \sum b_\mu T_\mu$ ,  $b_\mu \in B$ . If all the monomials  $\mu$  are standard monomials,  $b_\mu = 0$  for all  $\mu$ . Otherwise we apply the straightening procedure indicated above: we subtract successively multiples of the elements representing the straightening relations. Thus we create a sequence  $f = f_1, f_2, \dots, f_n$  of polynomials in  $\text{Ker } \varphi$ , whose successive terms differ by a multiple of such an element and for which  $f_n$ , representing a linear combination of standard monomials, is zero. —

### B. $G(X)$ as an ASL

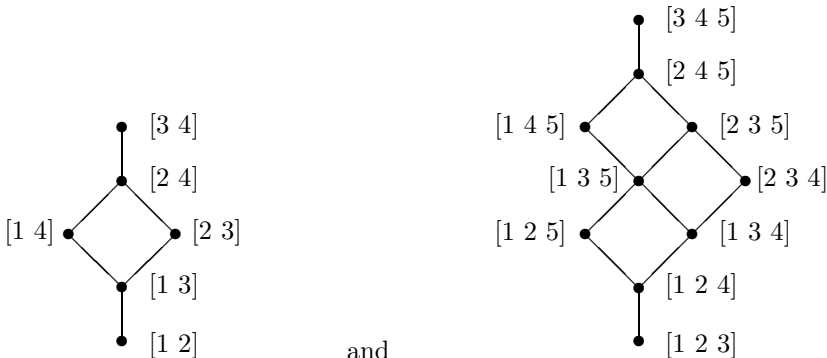
Let  $B$  be a commutative ring and  $X$  an  $m \times n$ -matrix of indeterminates over  $B$ ,  $m \leq n$ . As a  $B$ -algebra  $G(X)$  is generated by the set

$$\Gamma(X)$$

of maximal minors of  $X$ , cf. 1.D.  $\Gamma(X)$  is ordered partially in the following way:

$$[i_1, \dots, i_m] \leq [j_1, \dots, j_m] \iff i_1 \leq j_1, \dots, i_m \leq j_m.$$

Only in the cases  $n = m$  and  $n = m + 1$  the set  $\Gamma(X)$  is linearly ordered. For  $m = 2, n = 4$  and  $m = 3, n = 5$  the partial orders have the diagrams





$\Gamma(X)$  can be considered as a subset of the poset  $\mathbf{N}^m$  in a natural way, and it inherits from  $\mathbf{N}^m$  the structure of a distributive lattice, the lattice operations  $\sqcap$  and  $\sqcup$  given by

$$\begin{aligned} [i_1, \dots, i_m] \sqcap [j_1, \dots, j_m] &= [\min(i_1, j_1), \dots, \min(i_m, j_m)], \\ [i_1, \dots, i_m] \sqcup [j_1, \dots, j_m] &= [\max(i_1, j_1), \dots, \max(i_m, j_m)]; \end{aligned}$$

one has  $\delta_1 \leq \delta_2$  if and only if  $\delta_1 \sqcap \delta_2 = \delta_1$ .

**(4.3) Theorem.**  $G(X)$  is a graded ASL on  $\Gamma(X)$ .

Condition  $(H_0)$  is obviously satisfied. The linear independence of the standard monomials will be proved in Subsection C below. In the first part of the PROOF we want to show that condition  $(H_2)$  holds, assuming linear independence of the standard monomials. By virtue of Proposition (4.1) this implies that the standard monomials generate  $G(X)$  as a  $B$ -module. We shall not describe the straightening relations themselves explicitly; they will result from the *Plücker relations*.

**(4.4) Lemma.** (*Plücker relations*) For every  $m \times n$ -matrix,  $m \leq n$ , with elements in a commutative ring and all indices  $a_1, \dots, a_k, b_1, \dots, b_m, c_1, \dots, c_s \in \{1, \dots, n\}$  such that  $s = m - k + l - 1 > m$ ,  $t = m - k > 0$  one has

$$\sum_{\substack{i_1 < \dots < i_t \\ i_{t+1} < \dots < i_s \\ \{1, \dots, s\} = \{i_1, \dots, i_s\}}} \sigma(i_1, \dots, i_s) [a_1, \dots, a_k, c_{i_1}, \dots, c_{i_t}] [c_{i_{t+1}}, \dots, c_{i_s}, b_1, \dots, b_m] = 0.$$

PROOF: It suffices to prove this for a matrix  $X$  of indeterminates over  $\mathbf{Z}$ . We consider the  $\mathbf{Z}[X]$ -module  $C$  generated by the columns of  $X$ . As a  $\mathbf{Z}[X]$ -module it has rank  $m$ . Let  $\alpha: C^s \rightarrow \mathbf{Z}[X]$  be given by

$$\begin{aligned} \alpha(y_1, \dots, y_s) = \sum_{\pi \in \text{Sym}(1, \dots, s)} \sigma(\pi) \det(X_{a_1}, \dots, X_{a_k}, y_{\pi(1)}, \dots, y_{\pi(t)}) \\ \cdot \det(y_{\pi(t+1)}, \dots, y_{\pi(s)}, X_{b_1}, \dots, X_{b_m}), \end{aligned}$$

$X_j$  denoting the  $j$ -th column of  $X$ ,  $\text{Sym}(1, \dots, s)$  the group of permutations of  $\{1, \dots, s\}$ . It is straightforward to check that  $\alpha$  is a multilinear form on  $C^s$ . When two of the vectors  $y_i$  coincide, every term in the expansion of  $\alpha$ , which does not vanish anyway, is cancelled by a term of the opposite sign:  $\alpha$  is alternating. Since  $s > \text{rk } C$ ,  $\alpha = 0$ .

We fix a subset  $\{i_1, \dots, i_t\}$ ,  $i_1 < \dots < i_t$ , of  $\{1, \dots, s\}$ . Then, for all  $\pi$  such that  $\pi(\{1, \dots, t\}) = \{i_1, \dots, i_t\}$  the summand corresponding to  $\pi$  in the expansion of  $\alpha$  equals

$$\sigma(i_1, \dots, i_s) \det(X_{a_1}, \dots, X_{a_k}, y_{i_1}, \dots, y_{i_t}) \det(y_{i_{t+1}}, \dots, y_{i_s}, X_{b_1}, \dots, X_{b_m}),$$

$i_{t+1}, \dots, i_s$  chosen as above. Therefore each of these terms occurs  $t!(s-t)!$  times in the expansion of  $\alpha$ . In  $\mathbf{Z}[X]$  the factor  $t!(s-t)!$  may be cancelled. —

The first Plücker relation occurs for a  $2 \times 4$ -matrix:

$$[1 \ 2][3 \ 4] - [1 \ 3][2 \ 4] + [1 \ 4][2 \ 3] = 0$$

(corresponding to  $k = 1$ ,  $a_1 = 1$ ,  $l = 3$ ,  $(c_1, c_2, c_3) = (2, 3, 4)$ ). It is, solved for  $[1\ 4][2\ 3]$ , the single straightening relation for this case. The Plücker relation

$$[1\ 4\ 6][2\ 3\ 5] + [1\ 2\ 4][3\ 5\ 6] - [1\ 3\ 4][2\ 5\ 6] + [1\ 2\ 6][3\ 4\ 5] \\ - [1\ 3\ 6][2\ 4\ 5] - [1\ 2\ 3][4\ 5\ 6] = 0$$

corresponds, after reordering the columns, to  $k = 1$ ,  $a_1 = 1$ ,  $l = 3$ ,  $b_3 = 5$ ,  $(c_1, \dots, c_4) = (4, 6, 2, 3)$ . It is not a straightening relation, the first product is the worst “twisted” one, however: for it incomparability results from the second position already, whereas, for the fourth and fifth term, the first two positions are comparable. They are straightened by the Plücker relations

$$[1\ 2\ 6][3\ 4\ 5] - [1\ 2\ 3][4\ 5\ 6] + [1\ 2\ 4][3\ 5\ 6] - [1\ 2\ 5][3\ 4\ 6] = 0, \\ [1\ 3\ 6][2\ 4\ 5] + [1\ 2\ 3][4\ 5\ 6] + [1\ 3\ 4][2\ 5\ 6] - [1\ 3\ 5][2\ 4\ 6] = 0.$$

After substitution we finally obtain

$$[1\ 4\ 6][2\ 3\ 5] = -[1\ 2\ 3][4\ 5\ 6] - [1\ 2\ 5][3\ 4\ 6] + [1\ 3\ 5][2\ 4\ 6].$$

This stepwise straightening where at each step the number of comparable positions is shifted up by one, works in general:

**(4.5) Lemma.** *Let  $[a_1, \dots, a_m]$ ,  $[b_1, \dots, b_m] \in \Gamma(X)$ ,  $a_i \leq b_i$  for  $i = 1, \dots, k$ ,  $a_{k+1} > b_{k+1}$  ( $k$  may be 0). We put*

$$l = k + 2, \quad s = m + 1, \quad (c_1, \dots, c_s) = (a_{k+1}, \dots, a_m, b_1, \dots, b_{k+1}).$$

*Then, in the Plücker relation corresponding to these data, all the terms*

$$[d_1, \dots, d_m][e_1, \dots, e_m] \neq 0 \quad \text{and different from} \quad [a_1, \dots, a_m][b_1, \dots, b_m]$$

*have the following properties (after arranging the indices in ascending order):*

$$(i) [d_1, \dots, d_m] \leq [a_1, \dots, a_m] \quad \text{and} \quad (ii) d_1 \leq e_1, \dots, d_{k+1} \leq e_{k+1}.$$

**PROOF:** Since  $b_1 < \dots < b_{k+1} < a_{k+1} < \dots < a_m$ ,  $[d_1, \dots, d_m]$  arises from  $[a_1, \dots, a_m]$  by a replacement of some of the  $a_i$  by smaller indices. This implies (i) and  $d_i \leq e_i$  for  $i = 1, \dots, k$ . Furthermore  $d_{k+1} \in \{a_1, \dots, a_k, b_1, \dots, b_{k+1}\}$ , so  $d_{k+1} \leq b_{k+1}$ , and  $e_{k+1} \in \{a_{k+1}, \dots, a_m, b_{k+1}, \dots, b_m\}$ , so  $b_{k+1} \leq e_{k+1}$ . —

After these preparations we return to the proof of Theorem (4.3). It follows immediately from (4.5) by induction on  $k$  that every product  $\alpha\beta$  of minors  $\alpha, \beta \in \Gamma(X)$  can be expressed by a linear combination of standard monomials  $\delta\varepsilon$ ,  $\delta, \varepsilon \in \Gamma(X)$  such that  $\delta \leq \alpha$ ,  $\delta \leq \varepsilon$ . In order to show that this representation satisfies condition (H<sub>2</sub>), we assume that the standard monomials are linearly independent. When a product  $\alpha\beta$  of incomparable minors is given, we first straighten it in the order  $\alpha\beta$  obtaining a representation in which  $\delta \leq \alpha$  for all standard monomials occurring. Then we straighten it in the order  $\beta\alpha$  obtaining a representation in which  $\delta \leq \beta$  always. By linear independence both representations coincide, and (H<sub>2</sub>) follows.

The reader may wonder whether one needs linear independence of standard monomials in proving  $(H_2)$ . The following example indicates the main difficulty in deriving  $(H_2)$  directly from Lemma (4.5): Applying (4.5) once in order to “straighten” the product  $[1\ 5\ 6][2\ 3\ 4]$  (with  $(c_1, \dots, c_4) = (5, 6, 2, 3)$ ) one gets an intermediate result containing the product  $[1\ 3\ 5][2\ 4\ 6]$ , a standard monomial violating the condition in  $(H_2)$ !

When one reverses the partial order on  $\Gamma(X)$ , the set of standard monomials remains unchanged. Reversing the partial order corresponds to reversing the sequence of columns of  $X$  which may be viewed as an automorphism of  $B[X]$  and  $G(X)$ . This automorphism maps the elements of  $\Gamma(X)$  to the minors of the new matrix (up to sign). Therefore  $G(X)$  is an ASL with respect to the reverse order on  $\Gamma(X)$ , too, and the straightening relations must satisfy  $(H_2)$  and the dual condition simultaneously:

$$\alpha\beta = \sum a_i\gamma_i\delta_i, \quad a_i \in \mathbf{Z}, \quad \alpha, \beta, \gamma_i, \delta_i \in \Gamma(X), \quad \gamma_i \leq \delta_i, \quad \gamma_i \leq \alpha, \beta, \quad \delta_i \geq \alpha, \beta.$$

We call an ASL on  $\Pi$  *symmetric* if it is an ASL with respect to the reverse order on  $\Pi$ , too. Thus we may state:

**(4.6) Corollary.**  $G(X)$  is a symmetric ASL on  $\Gamma(X)$ .

$G(X)$  was defined as a subalgebra of  $B[X]$ . As a consequence of Theorem (4.3) and Proposition (4.2) one gets a representation:

**(4.7) Corollary.** Let  $B$  be a commutative ring,  $X$  an  $m \times n$ -matrix of indeterminates over  $B$ ,  $m \leq n$ , and  $\Gamma(X)$  the set of  $m$ -minors of  $X$ . Then  $G(X)$ , the  $B$ -subalgebra of  $B[X]$  generated by  $\Gamma(X)$ , is the residue class ring of  $B[T_\gamma : \gamma \in \Gamma(X)]$  modulo the ideal generated by the elements corresponding to the Plücker relations with  $s = m + 1$  and  $a_1 \leq \dots \leq a_m, b_1 \leq \dots \leq b_m$ .

In fact, by (4.2),  $G(X)$  is defined by the straightening relations, and these were obtained by iterated applications of the Plücker relations mentioned. It follows from (4.7) that the Plücker relations generate the defining ideal of the Grassmann variety  $G_m(K^n)$ ,  $K$  an algebraically closed field, and that  $G(X)$  is isomorphic to its homogeneous coordinate ring. A particular consequence of (4.7) (actually an abstract consequence of (4.2) and (4.3)):  $G(X)$  arises from the corresponding object over  $\mathbf{Z}$  by extension of coefficients. This will be needed soon.

### C. The Linear Independence of the Standard Monomials in $G(X)$

It remains to prove the linear independence of the standard monomials in  $G(X)$ . For simplicity we write

$$i \in [a_1, \dots, a_m] \quad \iff \quad i = a_j \quad \text{for some } j.$$

We say that  $(i, j)$ ,  $i < j$ , is a *special pair* for  $[a_1, \dots, a_m]$  if  $i \in [a_1, \dots, a_m]$ ,  $j \notin [a_1, \dots, a_m]$ , and that  $(i, j)$  is *extraspecial* for  $[a_1, \dots, a_m]$  if  $(i, j)$  is the lexicographically smallest special pair for  $[a_1, \dots, a_m]$ . Let a finite subset  $S \neq \emptyset$  of the set of standard monomials be given,  $(i_0, j_0)$  being the lexicographically smallest pair which is extraspecial for some factor of some  $\mu \in S$ . We prove the linear independence of the standard monomials by descending induction on  $(i_0, j_0)$ .

The greatest possible extraspecial pair is  $(n - m + 1, n + 1)$ . If  $(i_0, j_0) = (n - m + 1, n + 1)$ , then  $S$  consists only of powers of  $[n - m + 1, \dots, n]$  which certainly are linearly independent. Suppose that  $(i_0, j_0)$  is smaller than  $(n - m + 1, n + 1)$ . Then  $i_0 \leq n - m$ ,  $j_0 \leq n$ .

For  $\delta \in \Gamma(X)$ ,  $\mu \in S$  let

$$\begin{aligned}\Phi(\delta) &= \begin{cases} \delta & \text{if } (i_0, j_0) \text{ is not special for } \delta, \\ \delta & \text{with } i_0 \text{ replaced by } j_0 \text{ (and ordered again) otherwise,} \end{cases} \\ \Phi(\mu) &= \Phi(\delta_1) \dots \Phi(\delta_u) \quad (\mu = \delta_1 \dots \delta_u, \delta_i \in \Gamma(X)), \\ v(\mu) &= |\{k : (i_0, j_0) \text{ is special for } \delta_k\}|.\end{aligned}$$

Note: If  $\delta \in \Gamma(X)$  is a factor of  $\mu \in S$  and  $(i_0, j_0)$  is special for  $\delta$ , then  $(i_0, j_0)$  is extraspecial. The purpose of  $\Phi$  is to push up  $(i_0, j_0)$ : For such a minor  $\delta$ , the extraspecial pair of  $\Phi(\delta)$  is greater than  $(i_0, j_0)$ . In the following lemma the elements of  $S$  should be considered formal monomials.

- (4.8) Lemma.** (a) Let  $\gamma, \delta \in \Gamma(X)$  be factors of  $\mu \in S$ . If  $\gamma \leq \delta$ , then  $\Phi(\gamma) \leq \Phi(\delta)$ .  
 (b) For  $\mu \in S$  the monomial  $\Phi(\mu)$  is again standard.  
 (c) Let  $\mu, \nu \in S$  such that  $v(\mu) = v(\nu)$ . If  $\mu \neq \nu$ , then  $\Phi(\mu) \neq \Phi(\nu)$ .

PROOF: (a) If  $(i_0, j_0)$  is not special for  $\gamma$ , then  $\Phi(\gamma) = \gamma \leq \delta \leq \Phi(\delta)$ . Let  $(i_0, j_0)$  be special for  $\gamma$ . Then, by choice of  $(i_0, j_0)$ ,

$$\begin{aligned}\gamma &= [i_0, i_0 + 1, \dots, j_0 - 1, g_k, \dots, g_m], & g_k > j_0, \\ \Phi(\gamma) &= [i_0 + 1, \dots, j_0, g_k, \dots, g_m].\end{aligned}$$

Since  $\delta \geq \gamma$ ,  $\delta$  starts with an element  $\geq i_0$ . If it starts with  $i_0$ , then

$$\delta = [i_0, \dots, j_0 - 1, d_k, \dots, d_m], \quad d_k \geq g_k > j_0,$$

and  $\Phi(\delta) = [i_0 + 1, \dots, j_0, d_k, \dots, d_m] \geq \Phi(\gamma)$ ; otherwise  $\Phi(\delta) = \delta \geq \Phi(\gamma)$ , since  $\delta$  starts with an element  $\geq i_0 + 1$ , its elements increase by at least one and from position  $k$  upward nothing has changed.

(b) follows directly from (a). For (c) one may assume that  $\mu = \gamma_1 \dots \gamma_t$ ,  $\nu = \delta_1 \dots \delta_t$ ,  $\gamma_1 \leq \dots \leq \gamma_t$ ,  $\delta_1 \leq \dots \leq \delta_t$ . If  $\Phi(\mu) = \Phi(\nu)$ , then, by virtue of (a),  $\Phi(\gamma_i) = \Phi(\delta_i)$  for  $i = 1, \dots, t$ .

Suppose first that  $(i_0, j_0)$  is special for  $\gamma_i$  if and only if it is special for  $\delta_i$ ,  $i = 1, \dots, t$ . Then  $\Phi(\gamma_i) \neq \Phi(\delta_i)$  if  $\gamma_i \neq \delta_i$ , so  $\Phi(\mu) \neq \Phi(\nu)$ . Otherwise there are  $r, s$  such that  $(i_0, j_0)$  is special for  $\gamma_r$  and  $\delta_s$ , and not special for  $\gamma_s$  and  $\delta_r$ . One may assume  $s < r$ , hence  $\gamma_s < \gamma_r$ . Then

$$\begin{aligned}\gamma_r &= [i_0, \dots, j_0 - 1, g_k, \dots, g_m], \\ \delta_s &= [i_0, \dots, j_0 - 1, d_k, \dots, d_m].\end{aligned}$$

If  $\Phi(\gamma_s) = \Phi(\delta_s)$  then  $\gamma_s = \Phi(\gamma_s) = \Phi(\delta_s) = [i_0 + 1, \dots, j_0, \dots]$ , contradicting  $\gamma_s < \gamma_r$ . —

Suppose that  $\sum_{\mu \in S} a_\mu \mu = 0$ . We extend the ring  $B[X]$  by adjoining a new indeterminate  $W$  and consider an automorphism  $\alpha$  of  $B[X][W]$ :

$$\alpha|B = \text{id}, \quad \alpha(W) = W, \quad \alpha(X_{st}) = X_{st} \quad \text{if } t \neq i_0, \quad \alpha(X_{ui_0}) = X_{si_0} + WX_{sj_0}.$$

On the matrix  $X$  this automorphism acts as an elementary transformation adding the  $W$ -fold of column  $j_0$  to column  $i_0$ . For a minor  $\delta \in \Gamma(X)$  one has

$$\alpha(\delta) = \begin{cases} \delta & \text{if } (i_0, j_0) \text{ is not special for } \delta, \\ \delta \pm W\Phi(\delta) & \text{if } (i_0, j_0) \text{ is special for } \delta, \end{cases}$$

and for a monomial  $\mu \in S$

$$\alpha(\mu) = \pm W^{v(\mu)}\Phi(\mu) + \text{ terms of lower degree in } W.$$

Let  $v_0 = \max\{v(\mu) : \mu \in S\}$  and  $S_0 = \{\mu \in S : v(\mu) = v_0\}$ . Then  $v_0 \geq 1$ ,  $S_0 \neq \emptyset$ , and

$$0 = \sum_{\mu \in S} a_\mu \alpha(\mu) = \pm W^{v_0} \sum_{\mu \in S_0} a_\mu \Phi(\mu) + y_{v_0-1} W^{v_0-1} + \cdots + y_0, \quad y_i \in \mathbb{G}(X).$$

Therefore  $\sum_{\mu \in S_0} a_\mu \Phi(\mu) = 0$ . As observed above, the lexicographically smallest special pair for the monomials  $\Phi(\mu)$  is greater than  $(i_0, j_0)$ . By virtue of Lemma (4.8) the monomials  $\Phi(\mu)$ ,  $\mu \in S_0$ , are pairwise distinct standard monomials. The inductive hypothesis on  $(i_0, j_0)$  now implies

$$a_\mu = 0 \quad \text{for } \mu \in S_0,$$

and by induction on  $v_0$  (or  $|S|$ ) we conclude that the standard monomials  $\mu \in S$  are linearly independent. The proof of Theorem (4.3) is complete. —

We want to illustrate the last part of the proof by means of an example. Let  $m = 2$  and suppose that

$$a_1[1 \ 2][3 \ 4] + a_2[1 \ 2][1 \ 4] + a_3[1 \ 3]^2 + a_4[1 \ 3][1 \ 4] = 0.$$

Then  $(i_0, j_0) = (1, 2)$ ,  $v_0 = 2$ ,  $S_0 = \{[1 \ 3]^2, [1 \ 3][1 \ 4]\}$ ,  $\alpha([1 \ 3]) = [1 \ 3] + W[2 \ 3]$ ,  $\alpha([1 \ 4]) = [1 \ 4] + W[2 \ 4]$ . The highest degree in  $W$  is 2, and

$$a_3[2 \ 3]^2 + a_4[2 \ 3][2 \ 4] = 0,$$

the smallest special pair now being  $(2, 3)$ .

#### D. $B[X]$ as an ASL

Now the polynomial ring  $B[X]$  itself will be considered. We build the matrix  $\tilde{X}$  from  $X$  and  $m$  new columns attached to the right side of  $X$ :

$$\tilde{X} = \begin{pmatrix} X_{11} & \cdots & X_{1n} & X_{1n+1} & \cdots & X_{1n+m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X_{m1} & \cdots & X_{mn} & X_{mn+1} & \cdots & X_{mn+m} \end{pmatrix}.$$

Then we map  $B[\tilde{X}]$  onto  $B[X]$  by sending every element of  $\tilde{X}$  to the corresponding element of the following matrix:

$$\begin{pmatrix} X_{11} & \cdots & X_{1n} & 0 & \cdots & \cdots & 0 & 1 \\ & & & \vdots & & \ddots & \ddots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & 0 & \ddots & \ddots & & \vdots \\ X_{m1} & \cdots & X_{mn} & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Let  $\varphi : G(\tilde{X}) \rightarrow B[X]$  be the induced homomorphism,  $\delta = [b_1, \dots, b_m] \in \Gamma(\tilde{X})$ ,  $\Gamma(\tilde{X})$  denoting the set of  $m$ -minors of  $\tilde{X}$ , of course. Then, for  $\delta \neq [n+1, \dots, n+m]$ ,

$$(*) \quad \varphi(\delta) = \pm [a_1, \dots, a_t | b_1, \dots, b_t]$$

where  $t = \max\{i : b_i \leq n\}$  and  $a_1, \dots, a_t$  have been chosen such that

$$\{a_1, \dots, a_t, n+m+1-b_m, \dots, n+m+1-b_{t+1}\} = \{1, \dots, m\}.$$

For combinatorial purpose we write  $\varphi(\delta) = [a_1, \dots, a_t | b_1, \dots, b_t]$  whenever one of the equations  $(*)$  is satisfied. The minor  $[n+1, \dots, n+m]$  is mapped to

$$\varepsilon = (-1)^{m(m-1)/2},$$

and  $\varphi$  maps  $\Gamma(\tilde{X}) \setminus \{[n+1, \dots, n+m]\}$  bijectively onto

$$\Delta(X),$$

the set of all minors of  $X$ . In particular the  $B$ -algebra homomorphism  $\varphi$  is surjective, and, as we shall see in Lemma (4.10),  $[n+1, \dots, n+m] - \varepsilon$  generates  $\text{Ker } \varphi$ . This fact almost immediately implies that  $B[X]$  is an ASL on  $\Delta(X)$ ,  $\Delta(X)$  inheriting its order from  $\Gamma(\tilde{X})$ . The map  $\varphi$  is chosen such that the inherited order is just the ‘‘natural’’ order on  $\Delta(X)$ : Let

$$\begin{aligned} [a_1, \dots, a_u | b_1, \dots, b_u] \leq [c_1, \dots, c_v | d_1, \dots, d_v] \\ \iff u \geq v, \quad a_1 \leq c_1, \dots, a_v \leq c_v, \quad b_1 \leq d_1, \dots, b_v \leq d_v. \end{aligned}$$

**(4.9) Lemma.** *Let  $\gamma, \delta \in \Gamma(\tilde{X}) \setminus \{[n+1, \dots, n+m]\}$ . Then (disregarding signs)  $\gamma \leq \delta$  if and only if  $\varphi(\gamma) \leq \varphi(\delta)$ .*

PROOF: Let

$$\begin{aligned} \gamma &= [b_1, \dots, b_m], & \delta &= [d_1, \dots, d_m], \\ \varphi(\gamma) &= [a_1, \dots, a_u | b_1, \dots, b_u], & \varphi(\delta) &= [c_1, \dots, c_v | d_1, \dots, d_v]. \end{aligned}$$

Suppose first that  $\gamma \leq \delta$ . Then obviously  $v \leq u$  and  $b_1 \leq d_1, \dots, b_v \leq d_v$ . We assume  $a_1 \leq c_1, \dots, a_w \leq c_w$  and  $a_{w+1} > c_{w+1}$  in order to derive a contradiction. Since  $c_{w+1} \notin \{a_1, \dots, a_u\}$ , there is a  $t$  such that  $c_{w+1} = (n+m+1) - b_t$ . Then

$$(n+m+1) - d_t \leq (n+m+1) - b_t = c_{w+1} < a_{w+1},$$

even

$$(n + m + 1) - d_t < c_{w+1},$$

equality being excluded. The indices which are smaller than  $(n + m + 1) - b_t$ , are

$$a_1, \dots, a_w, (n + m + 1) - b_m, \dots, (n + m + 1) - b_{t+1},$$

hence  $(n + m + 1) - b_t = w + m - t + 1$ . On the other hand the following indices are smaller than  $c_{w+1} = (n + m + 1) - b_t$ :

$$c_1, \dots, c_w, (n + m + 1) - d_m, \dots, (n + m + 1) - d_t.$$

So  $(n + m + 1) - b_t \geq w + m - t + 2$ , a contradiction.

Let now  $\varphi(\gamma) \leq \varphi(\delta)$ . This implies  $v \leq u$  and  $b_1 \leq d_1, \dots, b_v \leq d_v$ . Again we want to reach a contradiction and suppose that  $b_1 \leq d_1, \dots, b_w \leq d_w, b_{w+1} > d_{w+1}$ . Then  $b_{w+1} > d_{w+1} \geq d_{v+1} > n$ . Consequently there exists a  $t$  such that

$$a_t = (m + n + 1) - d_{w+1}.$$

There are at least  $m - w + t - 1$  indices smaller than  $(m + n + 1) - d_{w+1}$ :

$$a_1, \dots, a_{t-1}, (m + n + 1) - b_m, \dots, (m + n + 1) - b_{w+1},$$

in particular  $(m + n + 1) - d_{w+1} \geq m - w + t$ . On the other hand

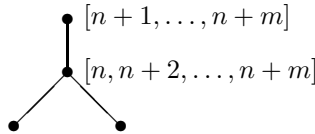
$$(m + n + 1) - d_{w+1} \leq (m + n + 1) - d_{w+1}, \dots, (m + n + 1) - d_{v+1}.$$

Hence  $m - w + t - 1 + (w + 1) - (v + 1) + 1 \leq m$ , so  $t \leq v$ . Since  $a_1 \leq c_1, \dots, a_v \leq c_v$ , all the indices smaller than  $(m + n + 1) - d_{w+1}$  occur among

$$c_1, \dots, c_{t-1}, (m + n + 1) - d_m, \dots, (m + n + 1) - d_{w+2},$$

again a contradiction. —

Lemma (4.9) shows that as a poset  $\Delta(X)$  is isomorphic to  $\Gamma(\tilde{X}) \setminus \{[n+1, \dots, n+m]\}$ . Since the top of  $\Gamma(\tilde{X})$  looks like



$\Gamma(\tilde{X}) \setminus \{[n+1, \dots, n+m]\}$  and  $\Delta(X)$  are distributive lattices, too.

**(4.10) Lemma.** (a)  $G(\tilde{X})([n+1, \dots, n+m] - \varepsilon)$  is a prime ideal if  $B$  is an integral domain.

(b)  $\text{Ker } \varphi = G(\tilde{X})([n+1, \dots, n+m] - \varepsilon)$ .

PROOF: (a) Consider the commutative diagram

$$\begin{array}{ccc} B[T_\gamma : \gamma \in \Gamma(\tilde{X})] & \xrightarrow{\psi} & B[T_\gamma : \gamma \in \Gamma(\tilde{X}) \setminus \{[n+1, \dots, n+m]\}] \\ \pi \downarrow & & \downarrow \chi \\ G(\tilde{X}) & \longrightarrow & G(\tilde{X})/G(\tilde{X})([n+1, \dots, n+m] - \varepsilon) \end{array}$$

of epimorphisms, where  $\psi(T_{[n+1, \dots, n+m]}) = \varepsilon$ . Since  $\text{Ker } \chi = \psi(\text{Ker } \pi)$  it is enough to know that  $\psi$  maps homogeneous prime ideals  $P$  not containing  $T_{[n+1, \dots, n+m]}$  onto prime ideals. The map  $\psi$  is just the “dehomogenization” with respect to  $\tilde{T} = \varepsilon T_{[n+1, \dots, n+m]}$ , and therefore has the desired property, cf. (16.26).

(b) Since both  $G(X)$  and  $B[X]$  as well as the map  $\varphi$  arise from the corresponding objects over  $\mathbf{Z}$  by tensoring with  $B$ , it is sufficient to prove (b) in the case  $B = \mathbf{Z}$ . Since

$$\dim G(\tilde{X}) = mn + 1 + \dim B,$$

as will be shown in Section 5,

$$\dim G(\tilde{X})/G(\tilde{X})([n+1, \dots, n+m] - \varepsilon) = \dim B[X].$$

By virtue of (a) both of them are integral domains, and the epimorphism induced by  $\varphi$  is an isomorphism. —

Now all the arguments for the proof of the main result have been collected:

**(4.11) Theorem.**  *$B[X]$  is a graded ASL on  $\Delta(X)$ .*

PROOF: It follows directly from (4.9) that the standard monomials in  $\Gamma(\tilde{X}) \setminus [m+1, \dots, m+n]$  are mapped to standard monomials in  $\Delta(X)$  (up to sign). Property (H<sub>2</sub>) cannot be destroyed, since the maximal element of  $\Gamma(\tilde{X})$  is replaced by  $\varepsilon$ : any monomial appearing on the right side of a straightening relation in  $G(\tilde{X})$  contains a factor different from  $[n+1, \dots, n+m]$ . The only critical point is whether the standard monomials in  $\Delta(X)$  are linearly independent. Suppose we have a relation  $\sum a_\mu \varphi(\mu) = 0$ ,  $\mu$  representing a standard monomial in  $\Gamma(\tilde{X})$  not containing  $[n+1, \dots, n+m]$ . Then, by virtue of (4.10)

$$\sum a_\mu \mu = (\varepsilon - [n+1, \dots, n+m]) \sum b_\nu \nu,$$

$\nu$  representing a standard monomial, too. It is obvious that such an equation can only hold if all the coefficients  $a_\mu, b_\nu$  are zero. —

For a generalization in the next section we record:

**(4.12) Proposition.**  *$B[X]$  is the dehomogenization of  $G(\tilde{X})$  with respect to  $\varepsilon[n+1, \dots, n+m]$ .*

The geometric analogue of (4.12) has been observed above Theorem (1.3): The affine  $mn$ -space is the open subvariety of the projective variety  $G_m(K^{n+m})$  complementary to the hyperplane defined by  $[n+1, \dots, n+m]$  (or any of the coordinate hyperplanes).

## E. Comments and References

The first standard monomial theory was established by Hodge [Hd] for the homogeneous coordinate rings of the Grassmannians and their Schubert subvarieties. Having found an explicite basis, he could derive the “postulation formula” for the Schubert subvarieties (previously conjectured by him and proved by Littlewood) in an elementary manner. (In algebraic language the “postulation formula” is an explicit formula for the dimension of the  $i$ -th homogeneous component of the homogeneous coordinate ring of a



projective variety.) A complete treatment was given by Hodge and Pedoe in their classical monograph [HP]; the tacit assumption that the ring of coefficients contains the rational numbers is only used there in proving that the relations in (4.4) are linear combinations of the relations

$$\sum_{k=1}^{m+1} (-1)^k [a_1, \dots, a_{j-1}, b_k, a_{j+1}, \dots, a_m] [b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_{m+1}]$$

instead of establishing them directly. (In positive characteristic the just-mentioned relations are not sufficient in general to generate the ideal of Plücker relations, cf. [Ab.2].)

More recent accounts of this standard monomial theory were given by Laksov [La.1] and Musili [Mu]. Musili's article is fairly selfcontained; his proof for the linear independence of the standard monomials will be indicated in Section 6. It is actually simpler than the one given whose merits will however become apparent in Section 11.

Like all the other authors we essentially follow Hodge's "canonical" way in proving that the standard monomials generate the  $B$ -module  $G(X)$ . The proof of the linear independence is borrowed from DeConcini's, Procesi's and Eisenbud's article [DEP.1]. The only place however, where we could find a proof for the validity of  $(H_2)$ , is Lemma 2.1 of Hochster's paper [Ho.3]; Hochster also observed that  $G(X)$  has the property dual to  $(H_2)$ .

Our derivation of (4.11) from (4.3) is taken from [DEP.1] again, where priority for Theorem (4.11) is attributed to Doubilet, Rota and Stein [DRS]. The geometric relationship between the Grassmann variety and the affine space is classical, however; an algebraic argument involving standard products of arbitrary minors can be found in [Mo] already; cf. also [HE.2], p. 1045.

The notion "algebra with straightening law" is drawn from Eisenbud's introductory survey [Ei.1] of the more voluminous monograph [DEP.2], in which the name "Hodge algebra" is used for the members of a more general class and ASLs figure as "ordinal Hodge algebras".

## 5. The Structure of an ASL

In this section we want to derive the properties of determinantal rings and Schubert cycles which follow from the general theory of ASLs and the particular nature of the partially ordered sets  $\Gamma(X)$  and  $\Delta(X)$  introduced in the preceding section. Determinantal rings and Schubert cycles inherit their structure as an ASL from  $B[X]$  and  $G(X)$ , simply because their defining ideals are generated by an ideal in  $\Delta(X)$  and  $\Gamma(X)$  resp.

We shall see that ASLs are reduced over reduced rings  $B$  and that ASLs on posets of a certain class (containing the distributive lattices) are Cohen-Macaulay rings over Cohen-Macaulay rings  $B$ . Furthermore there is a simple combinatorial formula for the dimension of an ASL, for the proof of which one needs “natural” regular elements of an ASL. One of the lemmas on which the formula for dimension is based, is general enough to supply an upper bound for the number of elements needed to generate certain ideals up to radical. This has consequences for the number of equations defining a determinantal or Schubert variety.

### A. ASL Structures on Residue Class Rings

In order to apply ASL theory to determinantal rings and Schubert cycles one first has to show that these rings are ASLs. This will follow readily from the fact that their defining ideals have a system of generators which is distinguished in regard of the underlying poset.

**(5.1) Proposition.** *Suppose  $A$  is a graded ASL on  $\Pi$  over  $B$ .*

- (a) *Let  $\Psi \subset \Pi$ ,  $I = A\Psi$ . If  $I$  is generated as a  $B$ -module by all the standard monomials containing a factor  $\xi \in \Psi$ , then  $A/I$  is again a graded ASL on  $\Pi \setminus \Psi$  (in a natural way).*
- (b) *In particular  $A/A\Omega$  is a graded ASL on  $\Pi \setminus \Omega$  if  $\Omega$  is an ideal in  $\Pi$  (i.e.  $\xi \in \Omega$  and  $v \leq \xi$  implies  $v \in \Omega$ ).*

PROOF: Part (a) is obvious. In (b) the ideal  $A\Omega$  is generated by all the monomials containing a factor  $\xi \in \Omega$ . Thus (b) follows directly from (a) and Proposition (4.1), (b). —

If  $\Pi$  has a single maximal element  $\pi$ , then  $\Psi = \{\pi\}$  satisfies the hypothesis of (5.1),(a) (though it is not an ideal, provided  $\Pi \neq \Psi$ ). This is a trivial but useful example.

En passant we note:

**(5.2) Proposition.** *Let  $\Omega$  and  $\Psi$  be ideals in  $\Pi$ . Then  $A\Omega \cap A\Psi$  is generated by the ideal  $\Omega \cap \Psi$  in  $\Pi$ .*

PROOF: Every standard monomial in the standard representation of an element of  $A\Omega \cap A\Psi$  has to contain a factor  $\omega \in \Omega$  and a factor  $\psi \in \Psi$ . At least one of them lies in  $\Omega \cap \Psi$ . —

Together with the trivial statement that  $A\Omega + A\Psi$  is generated by  $\Omega \cup \Psi$ , Proposition (5.2) shows that the ideals  $A\Omega$ ,  $\Omega$  an ideal in  $\Pi$ , form a distributive lattice with respect to intersections and sums (which is isomorphic to the lattice of ideals  $\Omega \subset \Pi$ ).

In order to have a compact description of our examples and for systematic reasons we introduce one more piece of notation:

DEFINITION. Let  $\Sigma \subset \Pi$ . The *ideal generated by  $\Sigma$*  in  $\Pi$  is the smallest ideal in  $\Pi$  containing  $\Sigma$ :

$$\{\xi \in \Pi: \xi \leq \sigma \text{ for a } \sigma \in \Sigma\},$$

whereas the *ideal cogenerated by  $\Sigma$*  in  $\Pi$  is the greatest ideal disjoint from  $\Sigma$ :

$$\{\xi \in \Pi: \xi \not\geq \sigma \text{ for every } \sigma \in \Sigma\}.$$

As usual let  $X$  be an  $m \times n$  matrix of indeterminates over  $B$ ,  $\Delta(X)$  its set of minors, partially ordered as introduced in 4.D. The ideal  $I_t(X)$  is generated by the  $t$ -minors and contains every  $u$ -minor such that  $u \geq t$ : it contains all the minors  $\gamma \leq \delta$  for a  $t$ -minor  $\delta$ . One has

$$I_t(X) = B[X]\Sigma,$$

$\Sigma$  being the ideal in  $\Delta(X)$  generated by  $[m-t+1, \dots, m|n-t+1, \dots, n]$ , equivalently: the ideal cogenerated by  $[1, \dots, t-1|1, \dots, t-1]$ . The last description is the most convenient one, and as will be seen shortly, the defining ideals of all the determinantal ideals can be described in this way. For  $\delta \in \Delta(X)$  we let

$$\begin{aligned} I(X; \delta) &= B[X]\{\pi \in \Delta(X): \pi \not\geq \delta\}, \\ R(X; \delta) &= B[X]/I(X; \delta), \quad \text{and} \\ \Delta(X; \delta) &= \{\pi \in \Delta(X): \pi \geq \delta\}. \end{aligned}$$

In exploring  $R(X; \delta)$ ,  $\delta$  fixed, we shall have to consider ideals of the form  $I(X; \varepsilon)/I(X; \delta)$ . It is therefore convenient to write

$$I(x; \varepsilon) = I(X; \varepsilon)/I(X; \delta)$$

then. (There is of course no need for the notations  $R(x; \varepsilon)$  and  $\Delta(x; \varepsilon)$ .) Let  $\delta = [a_1, \dots, a_r|b_1, \dots, b_r]$ . Then  $I(X; \delta)$  is generated by the

$$\left. \begin{array}{l} s\text{-minors of the rows } 1, \dots, a_s - 1 \\ s\text{-minors of the columns } 1, \dots, b_s - 1 \end{array} \right\} s = 1, \dots, r, \quad \text{and the} \\ (r+1)\text{-minors of } X.$$

So the rings  $R(X; \delta)$  are determinantal rings in the sense of 1.C, and conversely the determinantal rings  $B[X]/I$  are of type  $R(X; \delta)$ . Let

$$1 \leq u_1 < \dots < u_p \leq m, \quad 0 \leq r_1 < \dots < r_p < m$$

and

$$1 \leq v_1 < \dots < v_q \leq n, \quad 0 \leq s_1 < \dots < s_q < n,$$

such that  $I$  is generated by the

$$(r_i + 1)\text{-minors of the first } u_i \text{ rows}$$

and the

$$(s_j + 1)\text{-minors of the first } v_j \text{ columns,}$$

$i = 1, \dots, p, j = 1, \dots, q$ . In general this system of generators is far from being minimal: If  $u_{i+1} \leq u_i + r_{i+1} - r_i$ , then all the  $(r_{i+1} + 1)$ -minors of the rows  $1, \dots, u_{i+1}$  are linear combinations of the  $(r_i + 1)$ -minors of the rows  $1, \dots, u_i$ . Furthermore all the  $(r + 1)$ -minors are in  $I$  if  $r + 1$  is given as

$$r + 1 = \min(r_p + 1 + m - u_p, s_q + 1 + n - v_q)$$

In case  $r_p + 1 \geq r + 1$  we can discard the  $(r_p + 1)$ -minors of the rows  $1, \dots, u_p$ , since they are contained in the ideal generated by the  $(s_q + 1)$ -minors of the columns  $1, \dots, v_q$ . Similar observations apply to the ‘‘column-defined’’ generators, and therefore it is no restriction to assume that

$$\begin{aligned}
 & u_{i+1} > u_i + r_{i+1} - r_i, & v_{j+1} > v_j + s_{j+1} - s_j, \\
 (*) & & i = 1, \dots, p - 1, \quad j = 1, \dots, q - 1, \\
 & r_p + 1 < s_q + 1 + n - v_q, & s_q + 1 < r_p + 1 + m - u_p.
 \end{aligned}$$

Now we can describe  $\delta$  such that  $I = I(X; \delta)$ :

$$\begin{aligned}
 \delta = & [(1, \dots, r_1), (u_1 + 1, \dots, u_1 + (r_2 - r_1)), \dots, (u_p + 1, \dots, u_p + (r_{p+1} - r_p))] \\
 & [(1, \dots, s_1), (v_1 + 1, \dots, v_1 + (s_2 - s_1)), \dots, (v_q + 1, \dots, v_q + (s_{q+1} - s_q))],
 \end{aligned}$$

where of course  $r_{p+1} = s_{q+1} = r + 1$  and the blocks of consecutive integers in the row and column parts of  $\delta$  have been enclosed in parentheses.

**(5.3) Theorem.** (a) *The determinantal rings  $B[X]/I$  are given exactly by the rings  $R(X; \delta)$ ,  $\delta \in \Delta(X)$ .*

(b)  *$R(X; \delta)$  is a graded ASL on  $\Delta(X; \delta)$ .*

(c)  *$\Delta(X; \delta)$  is a distributive lattice.*

The analogues of  $I(X; \delta)$ ,  $R(X; \delta)$ ,  $\Delta(X; \delta)$  with respect to  $G(X)$  are

$$\begin{aligned}
 J(X; \gamma) &= G(X)\{\delta \in \Gamma: \delta \not\geq \gamma\}, \\
 G(X; \gamma) &= G(X)/J(X; \gamma), \quad \text{and} \\
 \Gamma(X; \gamma) &= \{\delta \in \Gamma: \delta \geq \gamma\}.
 \end{aligned}$$

Analogous to the notation  $I(x; \varepsilon)$  introduced above we may write

$$J(x; \delta) = J(X; \delta)/J(X; \gamma)$$

when a ring  $G(X; \gamma)$  is investigated. It follows directly from (1.4) that for  $\gamma = [a_1, \dots, a_m]$  the ring  $G(X; \gamma)$  is the Schubert cycle associated with

$$\Omega(n - a_m + 1, n - a_{m-1} + 1, \dots, n - a_1 + 1).$$

- (5.4) Theorem.** (a) *The rings  $G(X; \gamma)$  are exactly the Schubert cycles.*  
 (b)  $G(X; \gamma)$  *is a graded ASL on  $\Gamma(X; \gamma)$ .*  
 (c)  $\Gamma(X; \gamma)$  *is a distributive lattice.*

All this is evident now. In 4.D we have extended the matrix  $X$  in order to get the representation

$$B[X] = G(\tilde{X})/G(\tilde{X})([n+1, \dots, n+m] - \varepsilon), \quad \varepsilon = (-1)^{m(m-1)/2}.$$

For  $\delta \in \Delta(X)$ ,  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r]$ , we choose

$$\tilde{\delta} = [b_1, \dots, b_r, n+m+1 - \tilde{a}_{m-r}, \dots, n+m+1 - \tilde{a}_1],$$

$\{\tilde{a}_1, \dots, \tilde{a}_{m-r}\}$  being complementary to  $\{a_1, \dots, a_r\}$  in  $\{1, \dots, m\}$ . Then, by virtue of (4.9), the epimorphism  $G(\tilde{X}) \rightarrow B[X]$  maps a generating set of  $J(\tilde{X}; \tilde{\delta})$  onto a set of generators of  $I(X; \delta)$ . Therefore one obtains immediately:

**(5.5) Theorem.** *With the notations just introduced,  $R(X; \delta)$  is the dehomogenization of  $G(\tilde{X}; \tilde{\delta})$  with respect to  $\varepsilon[n+1, \dots, n+m]$ .*

The geometric significance of (5.5) has been indicated briefly in 1.D. We will use (5.5) mainly to transfer information from  $G(\tilde{X}; \tilde{\delta})$  to  $R(X; \delta)$ .

## B. Syzygies and the Straightening Law

As we have seen in (4.2), an ASL  $A$  is defined in terms of generators and relations by its underlying poset  $\Pi$  and the straightening relations. In a very similar way the module of syzygies of an ideal  $A\Psi$ ,  $\Psi$  an ideal in  $\Pi$ , is determined by the straightening relations involving elements of  $\Psi$  (and the ‘‘Koszul’’ relations corresponding to pairs of comparable elements of  $\Psi$ ).

To have a compact notation in the following proposition we denote the smallest factor of a standard monomial  $\mu$  by  $\alpha(\mu)$  and the product of the remaining factors by  $\omega(\mu)$ . Note that for every standard monomial  $\mu$  in a straightening relation  $\xi\psi = \sum a_\mu\mu$  one has  $\alpha(\mu) \in \Psi$  if  $\Psi$  is a poset ideal and  $\psi \in \Psi$ .

**(5.6) Proposition.** *Let  $A$  be a graded ASL on  $\Pi$  over  $B$ ,  $\Psi \subset \Pi$  an ideal, and  $e_\psi, \psi \in \Psi$ , denote the canonical basis of the free  $A$ -module  $A^\Psi$ .*

(a) *Then the kernel  $U$  of the natural epimorphism*

$$A^\Psi \longrightarrow A\Psi, \quad e_\psi \longrightarrow \psi,$$

*is generated by the elements*

$$\varphi e_\psi - \psi e_\varphi, \quad \varphi, \psi \in \Psi, \varphi < \psi,$$

*and the elements*

$$\xi e_\psi - \sum a_\mu \omega(\mu) e_{\alpha(\mu)}, \quad \xi \in \Pi, \psi \in \Psi, \xi \text{ and } \psi \text{ incomparable},$$

*corresponding to the straightening relations  $\xi\psi = \sum a_\mu\mu$ .*

(b) Suppose that the submodule  $\tilde{U} \subset U$  contains elements

$$\xi e_\psi - \sum_{\varphi < \psi} c_{\xi\psi\varphi} e_\varphi, \quad c_{\xi\psi\varphi} \in A,$$

for all elements  $\xi \in \Pi$ ,  $\psi \in \Psi$  such that  $\xi \not\leq \psi$ . Then  $\tilde{U} = U$ .

PROOF: Since, with the notations of part (a),  $\alpha(\mu) < \psi$ , (b) is a generalization of (a). Let

$$u = \sum_{\psi \in \Psi} d_\psi e_\psi, \quad d_\psi \in A,$$

be an element of  $U$ . Each  $d_\psi$  has a standard representation

$$d_\psi = \sum_{\mu \in M_\psi} a_{\mu\psi} \mu, \quad a_{\mu\psi} \in B, \quad a_{\mu\psi} \neq 0.$$

Suppose that  $\psi \leq \alpha(\mu)$  for all the standard monomials  $\mu \in M_\psi$ . Then, by the linear independence of the standard monomials, one has  $u = 0$ . This observation is the base of an inductive proof: Modulo  $\tilde{U}$  every term  $a_{\mu\psi} \mu e_\psi$  with  $\alpha(\mu) \not\leq \psi$  can be replaced by a linear combination of the elements  $e_\varphi$ ,  $\varphi < \psi$ , and after finitely many iterations one obtains the zero element of  $U$ , as has just been seen. In other words, one creates a sequence

$$u = u_n, u_{n-1}, \dots, u_0 = 0, \quad u_{i+1} \equiv u_i \pmod{\tilde{U}}, \quad i = 0, \dots, n-1. \quad -$$

In order to prove that a given set of relations of the elements  $\psi \in \Psi$  generates  $U$  one will of course show that all the relations required in (b) can be obtained as linear combinations of the given ones.

### C. Nilpotents, Regular Elements and Dimension

A general and extremely important property of ASLs is that they have no nilpotent elements, provided  $B$  has no nilpotents:

**(5.7) Proposition.** *Let  $A$  be a graded ASL on  $\Pi$  over  $B$ . Then  $A$  is reduced if (and only if)  $B$  is reduced.*

PROOF: The proof of the nontrivial statement is by induction on  $|\Pi|$ . In case  $|\Pi| = 1$ ,  $A$  is the polynomial ring in one indeterminate over  $B$ . Let  $|\Pi| > 1$ . Let  $x \in A$  be nilpotent and suppose  $x \neq 0$ . We choose a minimal element  $\xi \in \Pi$ . By induction  $x \in A\xi$ , so  $x = \xi^d y$  such that  $y \notin A\xi$  and  $d \geq 1$ , simply from consideration of the standard representation. If  $\xi$  is the single minimal element of  $\Pi$ , then it is not a zero-divisor obviously, and  $y$  is nilpotent, contradicting the inductive hypothesis. Otherwise there is a second minimal element  $v$ , and by inductive reasoning again  $x = \xi^d v^e z$ ,  $e \geq 1$ . However,  $\xi v = 0$  from (H<sub>2</sub>). —

It follows easily from (5.7) that for general  $B$  the nilradical of  $A$  is the extension of the nilradical of  $B$ .

The argument that  $\xi v = 0$  for a minimal element  $\xi \in \Pi$  and an element  $v \in \Pi$  not comparable to it, will be used several times below.

**(5.8) Corollary.** *If the ring  $B$  of coefficients is reduced, then all the rings  $R(X; \delta)$  and  $G(X; \gamma)$  are reduced.*

In particular, they are the coordinate rings of the varieties associated with them (under a suitable hypothesis on  $B$ ). Our next goal is the computation of their dimensions. For this purpose we exhibit a natural candidate for a system of parameters in a certain localization (which under special circumstances can also serve as a maximal regular sequence). For an element  $\xi \in \Pi$  we define its *rank* by:

$$\text{rk } \xi = k \quad \iff \quad \text{there is a chain } \xi = \xi_k > \xi_{k-1} > \cdots > \xi_1, \quad \xi_i \in \Pi, \\ \text{and no such chain of greater length exists.}$$

For a subset  $\Omega \subset \Pi$  let

$$\text{rk } \Omega = \max\{\text{rk } \xi : \xi \in \Omega\}.$$

(The preceding definition of rank differs from the usual one in combinatorics which gives a result smaller by 1. In order to reconcile the two definitions the reader should imagine an element  $-\infty$  added to  $\Pi$ , vaguely representing  $0 \in A$ .)

**(5.9) Lemma.** *Let  $\Omega \subset \Pi$  be an ideal,  $k = \text{rk } \Omega$ , and  $x_i = \sum_{\xi \in \Omega}^{\text{rk } \xi = i} \xi$ . Then*

$$\text{Rad } A\Omega = \text{Rad } \sum_{i=1}^k Ax_i.$$

PROOF: Let  $\zeta_1, \dots, \zeta_m$  be the minimal elements of  $\Omega$ , and  $J = \text{Rad } \sum_{i=1}^k Ax_i$ . Then, since  $\zeta_u \zeta_v = 0$  for  $u \neq v$  and  $\zeta_1 + \cdots + \zeta_m \in J$ ,  $\zeta_u^2 \in J$ , hence  $\zeta_u \in J$ ,  $u = 1, \dots, m$ . The rest is induction. —

A particularly simple example is  $\Omega = \Pi = \Delta(X; [1|1])$ , the poset underlying  $R_2(X)$ :  $\text{rk } \Omega = m + n - 1$  and

$$x_i = \sum_{u+v=i+1} [u|v], \quad i = 1, \dots, m + n - 1;$$

the  $x_i$  are the sums over the diagonals of the matrix.

Especially for  $\Omega = \Pi$  one has  $\text{Rad } \sum_{i=1}^{\text{rk } \Pi} Ax_i = \text{Rad } A\Pi$ . When  $B = K$  is a field,  $A\Pi$  is the irrelevant maximal ideal, so  $\dim A \leq \text{rk } \Pi$ . This bound turns out to be precise, and the general case regarding  $B$  can be treated via the dimension formula for flat extensions.

**(5.10) Proposition.** *Let  $A$  be a graded ASL on  $\Pi$  over the noetherian ring  $B$ . Then*

$$\dim A = \dim B + \text{rk } \Pi \quad \text{and} \quad \text{ht } A\Pi = \text{rk } \Pi.$$

PROOF: Let  $P$  be a prime ideal of  $A$ , and  $Q = B \cap P$ . Since  $A_P$  is a localization of  $A \otimes B_Q$ , it is a flat local extension of  $B_Q$ , so

$$\dim A_P = \dim B_Q + \dim A_P \otimes (B_Q/QB_Q).$$

The ring  $A_P \otimes (B_Q/QB_Q)$  is just  $(A/QA)_{P/QA}$ , hence a localization of  $A \otimes (B_Q/QB_Q)$  which is a graded ASL over the field  $B_Q/QB_Q$  on  $\Pi$ . From what has been said above

$$\dim A_P \leq \dim B + \text{rk } \Pi,$$

so

$$\dim A \leq \dim B + \text{rk } \Pi.$$

Now it is enough to show that  $\text{ht } A\Pi \geq \text{rk } \Pi$ . Let  $\Omega$  be the set of minimal elements of  $\Pi$ . As we shall see in Lemma (5.11), the element  $\sum_{\xi \in \Omega} \xi$  is not a zero-divisor of  $A$ , thus

$$\text{ht}(\Pi \setminus \Omega)(A/A\Omega) \leq \text{ht } A\Pi - 1.$$

On the other hand, arguing by induction,

$$\text{ht}(\Pi \setminus \Omega)(A/A\Omega) = \text{rank}(\Pi \setminus \Omega) = \text{rank} \Pi - 1. \quad \text{---}$$

**(5.11) Lemma.** *Let  $\Omega \subset \Pi$  consist of pairwise incomparable elements, and suppose that every maximal chain (linearly ordered subset) of  $\Pi$  intersects  $\Omega$ . Then  $x = \sum_{\xi \in \Omega} \xi$  is not a zero-divisor of  $A$ .*

PROOF: Suppose that  $yx = 0$ ,  $y \neq 0$ . Let  $\text{Min } \Pi$  be the set of minimal elements of  $\Pi$ . By induction on  $|\Pi|$  one immediately obtains the following auxiliary claim: (\*) Let  $\pi \in \text{Min } \Pi$ ,  $\pi \notin \Omega$ ; then  $y \in A\pi$ ,  $y = \pi^d y'$ ,  $d \geq 1$ ,  $y' \notin A\pi$  and  $y'x \neq 0$ .

Case 1:  $\Omega = \text{Min } \Pi$ . We pick a standard monomial  $\mu_0$  in the standard representation  $y = \sum a_\mu \mu$ . There is an  $\omega_0 \in \Omega$  such that  $\omega_0 \mu_0$  is a standard monomial, and  $\omega_0 \mu_0$  can not appear in the standard representations of any of the products  $\omega \mu$ ,  $\omega \in \Omega$ ,  $\omega \neq \omega_0$  or  $\mu \neq \mu_0$ ! Therefore  $\omega_0 \mu_0$  occurs in the standard representation of  $yx$  with a nonzero coefficient. Contradiction.

Case 2:  $|\text{Min } \Pi| = 1$ ,  $\text{Min } \Pi = \{\pi\}$ . Then either  $\Omega = \{\pi\}$ , a case covered already, or  $\pi \notin \Omega$ . In the latter case the contradiction results from (\*), since  $\pi$  is not a zero-divisor.

Case 3:  $|\text{Min } \Pi \setminus \Omega| \geq 2$ ,  $\pi_1, \pi_2 \in \text{Min } \Pi \setminus \Omega$ ,  $\pi_1 \neq \pi_2$ . Then, by (\*),  $y \in A\pi_1 \cap A\pi_2 = 0$ .

Case 4:  $|\text{Min } \Pi \setminus \Omega| = 1$ . Let  $\pi \in \text{Min } \Pi$ ,  $\pi \notin \Omega$ . Excluding case 2, we may assume that there is a  $\sigma \in (\text{Min } \Pi) \cap \Omega$ . Write

$$x = x' + x'', \quad x' = \sum_{\xi \in (\text{Min } \Pi) \cap \Omega} \xi.$$

We want to construct a subset  $\Omega'$  of  $\Pi$  which satisfies the hypothesis of the lemma modulo  $A\sigma$ . Let

$$\Omega' = (\Omega \setminus \{\sigma\}) \cup \Omega'', \quad \Omega'' = \{\tau : \tau \text{ an upper neighbour of } \sigma \\ \text{not comparable to any } \omega \in \Omega, \omega \neq \sigma\}.$$

Then  $\Omega'$  consists of incomparable elements. A maximal chain  $\Gamma$  in  $\Pi \setminus \{\sigma\}$  which does not intersect  $\Omega \setminus \{\sigma\}$ , passes through an upper neighbour  $\rho$  of  $\sigma$ ,  $\rho \in \text{Min}(\Pi \setminus \{\sigma\})$ . If  $\rho \in \Omega''$ ,  $\Gamma \cap \Omega' \neq \emptyset$ . Otherwise  $\rho$  is comparable to an  $\omega \in \Omega \setminus \{\sigma\}$ . Since  $\rho$  is minimal in  $\Pi \setminus \{\sigma\}$ ,  $\rho < \omega$ , a fortiori  $\sigma < \omega$ , in contradiction to the hypothesis on  $\Omega$ .

Let  $\tau \in \Omega''$  and suppose  $\tau \geq \pi$ . Then there is a maximal chain in  $\Pi \setminus \{\sigma\}$  starting from  $\pi$  and passing through  $\tau$ . This chain has to intersect  $\Omega \setminus \{\sigma\}$  which is impossible by definition of  $\Omega''$ . So  $\pi$  and  $\tau \in \Omega''$  are incomparable. Let  $\tilde{x} = \sum_{\xi \in \Omega'} \xi$ . Then

$$\tilde{x} = x' - \sigma + x'' + \sum_{\tau \in \Omega''} \tau$$



and

$$y\tilde{x} = yx - y'\pi^d\sigma + y'\pi^d \sum_{\tau \in \Omega''} \tau = 0.$$

$\Omega'$  satisfies the hypothesis of the lemma modulo  $A\sigma$ . Therefore  $y \in A\sigma \cap A\pi = 0$ , a contradiction settling case 4.

Since  $\text{Min } \Pi \subset \Omega$  implies  $\text{Min } \Pi = \Omega$ , all the possible cases have been covered. —

A subset  $\Omega$  of  $\Pi$  which satisfies the hypothesis of (5.11) is of course maximal with respect to having pairwise incomparable elements. This weaker property is not sufficient for  $x$  to be not a zero-divisor. For

$$\Pi = \begin{array}{ccc} \xi & & \rho \\ \bullet & \diagdown & \bullet \\ \pi & & \sigma \\ \bullet & & \bullet \end{array} \quad \text{and} \quad \rho\xi = \rho\pi = \sigma\pi = 0$$

one has  $\sigma\xi(\rho + \pi) = 0$ .

After Proposition (5.10) the computation of  $\dim A$  is a purely combinatorial problem. Let again  $X$  be an  $m \times n$  matrix of indeterminates over  $B$ ,  $\gamma = [a_1, \dots, a_m] \in \Gamma(X)$ . Any maximal chain in  $\Gamma(X; \gamma)$  starts at  $\gamma$ , and one moves to an upper neighbour raising exactly one index by 1. Therefore

$$\text{rk } \Gamma(X; \gamma) = \sum_{i=1}^m (n - m + i - a_i) = m(n - m) + \frac{m(m + 1)}{2} - \sum_{i=1}^m a_i + 1.$$

The rank of  $\Delta(X; \delta)$  can most conveniently be computed by relating it to  $\Gamma(\tilde{X}; \tilde{\delta})$  as in Theorem (5.5): For  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r]$  one has

$$\tilde{\delta} = [b_1, \dots, b_r, (n + m + 1) - \tilde{a}_{m-r}, \dots, (n + m + 1) - \tilde{a}_1],$$

$\{\tilde{a}_1, \dots, \tilde{a}_{m-r}\}$  being complementary to  $\{a_1, \dots, a_r\}$  in  $\{1, \dots, m\}$ . An easy computation gives

$$\text{rk } \Delta(X; \delta) = \text{rk } \Gamma(\tilde{X}; \tilde{\delta}) - 1 = (m + n)r - \sum_{i=1}^r (a_i + b_i) + r.$$

**(5.12) Corollary.** *Let  $B$  be a noetherian ring,  $X$  an  $m \times n$  matrix of indeterminates over  $B$ .*

(a) *Let  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r] \in \Delta(X)$ . Then*

$$\dim R(X; \delta) = \dim B + (m + n)r - \sum_{i=1}^r (a_i + b_i) + r,$$

*in particular, for  $\delta = [1, \dots, t - 1 | 1, \dots, t - 1]$ :*

$$\dim R_t(X) = \dim B + (m + n - t + 1)(t - 1).$$

(b) *Let  $\gamma = [a_1, \dots, a_m] \in \Gamma(X)$ . Then*

$$\dim G(X; \gamma) = \dim B + m(n - m) + \frac{m(m + 1)}{2} - \sum_{i=1}^m a_i + 1,$$

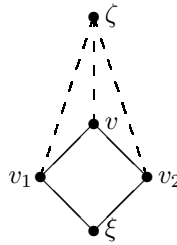
*in particular, for  $\gamma = [1, \dots, m]$ :*

$$\dim G(X) = \dim B + m(n - m) + 1.$$

### D. Wonderful Posets and the Cohen-Macaulay Property

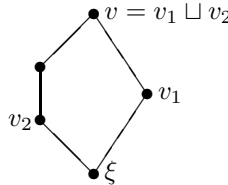
As noticed in (5.3) and (5.4) the posets  $\Delta(X; \delta)$  and  $\Gamma(X; \gamma)$  are distributive lattices. We shall see below that this implies the Cohen-Macaulay property for the corresponding rings (provided  $B$  is Cohen-Macaulay). However, a weaker condition will turn out to be sufficient already, a condition which can be controlled rather easily in an inductive proof:

DEFINITION. A partially ordered set  $\Pi$  is called *wonderful* (in systematic combinatorial language: locally upper semi-modular) if the following holds after a smallest and a greatest element  $-\infty$  and  $\infty$  resp. have been added to  $\Pi$ : If  $v_1$  and  $v_2$  are upper neighbours of  $\xi \in \Pi \cup \{-\infty, \infty\}$  and  $v_1, v_2 < \zeta \in \Pi \cup \{-\infty, \infty\}$  then there is an upper neighbour  $v$  of  $v_1, v_2$  such that  $v \leq \zeta$ , pictorially



the existence of  $v$  being required.

In a lattice  $\Pi$  there is of course only one choice for  $v$ :  $v = v_1 \sqcup v_2$ . In general  $v_1 \sqcup v_2$  need not to be an upper neighbour of  $v_1$  and  $v_2$ , as the following example indicates:



A distributive lattice is always wonderful: Suppose there is an  $\eta \in \Pi$ ,  $v_2 < \eta < v$ . We put  $\omega = \eta \sqcap v_1$ . Then

$$\xi = v_1 \sqcap v_2 \leq v_1 \sqcap \eta = \omega \leq v_1,$$

leaving the cases  $\omega = \xi$  or  $\omega = v_1$ . In the last case  $v_1 \leq \eta$ , so  $v = v_1 \sqcup v_2 \leq \eta < v$ , a contradiction, whereas in the first (and critical, cf. the example above) case

$$\begin{aligned} (\eta \sqcap v_1) \sqcup v_2 &= \xi \sqcup v_2 = v_2, & \text{but also} \\ (\eta \sqcap v_1) \sqcup v_2 &= (\eta \sqcup v_2) \sqcap (v_1 \sqcup v_2) = \eta \sqcap v = \eta, \end{aligned}$$

again a contradiction.

For a lattice  $\Pi$  one could obviously weaken the condition for being wonderful: A lattice is already wonderful if it is upper semi-modular, i.e. if elements  $v_1$  and  $v_2$  with a common lower neighbour  $\xi$  also have a common upper neighbour. For posets in general this weaker property does not imply that an ASL is Cohen-Macaulay; a counterexample will be discussed below.

The next lemma collects some combinatorial properties of wonderful posets.

**(5.13) Lemma.** *Let  $\Pi$  be a wonderful poset.*

- (a) *If  $\Omega \subset \Pi$  is an ideal and if for all minimal elements  $v_1, v_2$  of  $\Pi \setminus \Omega$  and all  $\zeta \in \Pi \setminus \Omega$  such that  $v_1, v_2 < \zeta$  there is a common upper neighbour  $v \leq \zeta$  of  $v_1, v_2$ , then  $\Pi \setminus \Omega$  is wonderful.*
- (b) *If  $\Pi$  has a single minimal element  $\xi$ , then  $\Pi \setminus \{\xi\}$  is wonderful.*
- (c) *Let  $\Omega$  be the ideal cogenerated by a subset of  $\text{Min } \Pi$ . Then  $\Pi \setminus \Omega$  is wonderful.*
- (d) *Every maximal chain in  $\Pi$  has length  $\text{rk } \Pi$ .*
- (e) *Suppose that  $\Pi$  has minimal elements  $\xi_1, \dots, \xi_k$ ,  $k \geq 2$ , let  $\Omega$  be the ideal cogenerated by  $\xi_1$ ,  $\Psi$  the ideal cogenerated by  $\{\xi_2, \dots, \xi_k\}$ . Then:*
  - (i)  *$\Pi \setminus \Omega$ ,  $\Pi \setminus \Psi$  and  $\Pi \setminus (\Psi \cup \Omega)$  are wonderful.*
  - (ii)  *$\text{rk}(\Pi \setminus \Omega) = \text{rk}(\Pi \setminus \Psi) = \text{rk } \Pi$ , whereas  $\text{rk}(\Pi \setminus (\Psi \cup \Omega)) = \text{rk } \Pi - 1$ .*
  - (iii)  *$\Omega \cap \Psi = \emptyset$ .*

PROOF: Part (a) is rather trivial, and parts (b) and (c) follow immediately from (a), whereas (d) is proved by induction on  $|\Pi|$ : Let  $\xi_1 < \dots < \xi_k$  and  $v_1 < \dots < v_l$  be maximal chains in  $\Pi$ . If  $\xi_1 = v_1$ , one passes to

$$(\Pi \setminus \{\text{ideal cogenerated by } \xi_1\}) \setminus \{\xi_1\}$$

which is wonderful by virtue of (c) and (b). Otherwise  $\xi_1$  and  $v_1$  have a common upper neighbour  $\zeta_2$  (in  $\Pi \cup \{-\infty, \infty\}$  they both are upper neighbours of  $-\infty$ ). There is a maximal chain  $\xi_1 < \zeta_2 < \dots < \zeta_m$ . Applying the argument of the case  $\xi_1 = v_1$  twice, we see that the chains  $\xi_1 < \dots < \xi_k$ ,  $\xi_1 < \zeta_2 < \dots < \zeta_m$ ,  $v_1 < \zeta_2 < \dots < \zeta_m$ ,  $v_1 < \dots < v_l$  all have the same length.

In (e) the assertions concerning  $\Pi \setminus \Omega$ ,  $\Pi \setminus \Psi$  follow directly from (c) and (d). Furthermore  $\Pi \setminus (\Psi \cup \Omega)$  does not contain any minimal element of  $\Pi$ , but it contains a common upper neighbour of  $\xi_1$  and  $\xi_2$ , unless it is empty; therefore  $\text{rk}(\Pi \setminus (\Omega \cup \Psi)) = \text{rk } \Pi - 1$ .  $\Omega \cap \Psi = \emptyset$  is trivial. It only remains to prove that  $\Pi \setminus (\Omega \cup \Psi)$  is wonderful, and here we need the full strength of the property “wonderful”! We want to apply (a) and consider minimal elements  $v_1, v_2$  of  $\Pi \setminus (\Omega \cup \Psi)$ . Then  $v_1, v_2 \geq \xi_1$ . The crucial point is to show that  $v_1$  and  $v_2$  are both upper neighbours of  $\xi_1$ ; then (a) can be applied. Suppose  $v_1$  is not an upper neighbour of  $\xi_1$ . Since  $v_1 > \xi_1$  and  $v_1 > \xi_i$  for some  $i \in \{2, \dots, k\}$ ,  $\xi_1$  and  $\xi_i$  have an upper neighbour  $\zeta < v_1$ . This is a contradiction:  $\zeta \in \Pi \setminus (\Omega \cup \Psi)$ , too. —

**(5.14) Theorem.** *Let  $B$  be a Cohen-Macaulay ring,  $\Pi$  a wonderful poset, and  $A$  a graded ASL over  $B$  on  $\Pi$ . Then  $A$  is a Cohen-Macaulay ring, too.*

The proof of the theorem is by induction on  $|\Pi|$ , and Lemma (5.13) contains the combinatorial arguments. The algebraic arguments will be the Cohen-Macaulay criterion for flat extensions and the following lemma which is also crucial in the proof of Hochster-Eagon for the perfection of determinantal ideals (cf. Section 12).

**(5.15) Lemma.** *Let  $K$  be a field,  $A = \bigoplus_{i \geq 0} A_i$  a graded  $K$ -algebra with  $A_0 = K$ .*

- (a) *Let  $x \in A$  be homogeneous of positive degree such that  $x$  is not a zero-divisor. Then  $A$  is Cohen-Macaulay if and only if  $A/Ax$  is Cohen-Macaulay.*
- (b) *Let  $I, J$  be homogeneous ideals such that*

$$\dim A/I = \dim A/J = \dim A, \quad \dim A/(I + J) = \dim A - 1, \quad \text{and} \quad I \cap J = 0.$$

*Suppose that  $A/I$  and  $A/J$  are Cohen-Macaulay. Then  $A$  is Cohen-Macaulay if and only if  $A/(I + J)$  is Cohen-Macaulay.*

PROOF: By virtue of (16.20) we may first localize with respect to the irrelevant maximal ideal. The local analogues of (a) and (b) are easy to prove. For (a) one observes  $\dim A/Ax = \dim A - 1$  and  $\text{depth } A/Ax = \text{depth } A - 1$ , whereas for (b) it is crucial that in the exact sequence

$$0 \longrightarrow A/(I \cap J) \longrightarrow A/I \oplus A/J \longrightarrow A/(I + J) \longrightarrow 0$$

$A/(I \cap J)$  can be replaced by  $A$ :

$$0 \longrightarrow A \longrightarrow A/I \oplus A/J \longrightarrow A/(I + J) \longrightarrow 0$$

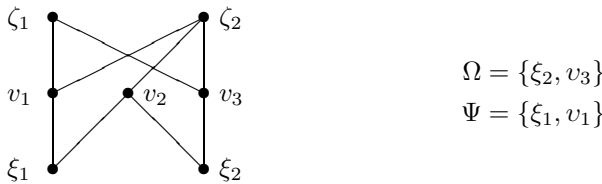
is exact. The middle term has depth equal to  $\dim A$ . If  $\text{depth } A = \dim A$ , then  $\text{depth } A/(I + J) \geq \dim A - 1 = \dim A/(I + J)$ . Conversely, if  $A/(I + J)$  is Cohen-Macaulay, then  $\text{depth } A/(I + J) = \dim A - 1$ , and  $\text{depth } A \geq \dim A$ . —

Let us prove (5.14) now. For a prime ideal  $P$  of  $A$  the localization  $A_P$  is Cohen-Macaulay if and only if for  $Q = P \cap B$  the rings  $B_Q$  and  $(B_Q/QB_Q) \otimes A_P$  are Cohen-Macaulay. The last ring is a localization of  $(B_Q/QB_Q) \otimes A$ , a graded ASL over a field. Hence we may assume that  $B = K$  is a field. Now one applies induction on  $|\Pi|$ . If  $A$  has a single minimal element  $\xi$ , it follows from (5.13),(b) and (5.15),(a) that  $A$  is Cohen-Macaulay. Otherwise there are minimal elements  $\xi_1, \dots, \xi_k$ ,  $k \geq 2$ . Let  $\Omega$  and  $\Psi$  be chosen as in (5.13),(e) and  $I = A\Omega$ ,  $J = A\Psi$ . Then, by virtue of (5.13),(e) and induction, the hypothesis of (5.15),(b) is satisfied and  $A/(I + J)$  is Cohen-Macaulay. We conclude that  $A$  is Cohen-Macaulay itself. —

In the same manner as Theorem (5.14) one proves the following generalization:

**(5.16) Proposition.** *Let  $B$  be a noetherian ring, and  $A$  a graded ASL on a wonderful poset over  $B$ . Then  $A$  satisfies Serre’s condition  $(S_n)$  if (and only if)  $B$  satisfies  $(S_n)$ .*

The following example for  $\Pi$  may show that the condition “wonderful” cannot be weakened in an obvious way:



Though every pair of elements of the same rank has a common upper neighbour, an ASL over  $\Pi$  cannot be Cohen-Macaulay. Since  $\Pi \setminus \Omega$  and  $\Pi \setminus \Psi$  are wonderful,  $A/A\Omega$  and  $A/A\Psi$  are Cohen-Macaulay. Moreover  $A/(A\Omega + A\Psi)$  has dimension one less than  $A$  and is not Cohen-Macaulay, hence  $A$  cannot be Cohen-Macaulay by virtue of (5.15),(b).

It remains to specialize (5.14) for determinantal rings and Schubert cycles. Their underlying posets are distributive lattices, hence wonderful, as remarked above.

**(5.17) Corollary.** *Let  $B$  be a Cohen-Macaulay ring. Then all the rings  $R(X; \delta)$  and  $G(X; \gamma)$  are Cohen-Macaulay rings, too.*

Using the theory of generic perfection one can strengthen and generalize (5.17):

**(5.18) Corollary.** *Let  $X$  be an  $m \times n$  matrix of indeterminates.*

(a) *For  $B = \mathbf{Z}$  the ideal  $I(X; \delta)$  is generically perfect. Hence  $I(X; \delta)$  is perfect over an arbitrary noetherian ring, and, with  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r]$ ,*

$$\text{grade} I(X; \delta) = mn - (m + n + 1)r + \sum_{i=1}^r (a_i + b_i).$$

(b) *Let  $\widehat{J}(X; \gamma)$  denote the kernel of the epimorphism  $B[Y_\rho: \rho \in \Gamma(X)] \rightarrow G(X; \gamma)$  induced by the substitution  $Y_\rho \rightarrow \rho$ . For  $B = \mathbf{Z}$  the ideal  $\widehat{J}(X; \gamma)$  is generically perfect. Hence  $\widehat{J}(X; \gamma)$  is perfect over an arbitrary noetherian ring, and, with  $\gamma = [a_1, \dots, a_m]$ ,*

$$\text{grade} \widehat{J}(X; \gamma) = \binom{n}{m} - m(n - m) - \frac{m(m + 1)}{2} + \sum_{i=1}^m a_i - 1.$$

PROOF: By virtue of (3.3) it is enough to prove the corollary as far as it applies to fields  $B$ , for which perfection follows from (5.17) via (16.20). The formulas for grade result immediately from those for dimension in (5.12) when  $B$  is a field. —

As we observed in Section 2, the Cohen-Macaulay property of the rings  $R_t(X)$  implies that they are (normal) domains whenever the ring  $B$  of coefficients is a (normal) domain. We shall see in the following section that in this case all the rings  $R(X; \delta)$  and  $G(X; \gamma)$  are (normal) domains. For later application we note a generalization of (5.17) and (5.18):

**(5.19) Proposition.** *Let  $\Omega$  be an ideal in  $\Delta(X)$  or  $\Gamma(X)$  such that the minimal elements of its complement have a common lower neighbour if there are at least two minimal elements. Then (5.17) and (5.18) hold mutatis mutandis for the ideal generated by  $\Omega$  in  $B[X]$  or  $\Gamma(X)$  resp. and the residue class ring defined by it.*

PROOF: It follows readily from (5.13),(a) that the complement of  $\Omega$  is a wonderful poset. —

Needless to say, the theory of generic perfection applies to all the ideals in (5.19), in particular to the ideals  $I(X; \delta)$  and  $J(X; \gamma)$ ; the specific consequences are left to the reader.

### E. The Arithmetical Rank of Certain Ideals

One of the main problems of algebraic geometry is the determination of the minimal number of equations defining a given variety. As a by-product of the theory of ASLs we can obtain an upper bound for this number in the case of the determinantal rings. The corresponding algebraic problem is to find the minimal number of elements which generate a given ideal up to radical. For an ideal  $I$  in a commutative noetherian ring  $S$  let therefore the *arithmetical rank* of  $I$  be given by

$$\text{ar} I = \min \left\{ k : \text{there exist } x_1, \dots, x_k \in I \text{ such that } \text{Rad } I = \text{Rad} \sum_{i=1}^k Sx_i \right\}.$$

The following proposition is a direct consequence of Lemma (5.9).

**(5.20) Proposition.** *Let  $A$  be a graded ASL over  $B$  on  $\Pi$ , and  $\Omega \subset \Pi$  an ideal. Then*

$$\text{ara } A\Omega \leq \text{rk } \Omega.$$

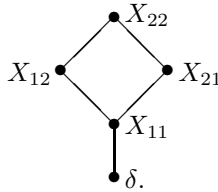
Of course Lemma (5.9) does not only supply this bound; it also shows how to find a sequence  $x_1, \dots, x_k$ ,  $k = \text{rk } \Omega$ , such that  $\text{rad } A\Omega = \text{rad } \sum Ax_i$ . The case in which  $A = B[X]$ ,  $X$  an  $m \times n$  matrix,  $I = I_t(X)$  is particularly simple, since the corresponding ideal has a single maximal element, namely  $[m - t + 1, \dots, m | n - t + 1, \dots, n]$ .

**(5.21) Corollary.** *Let  $X$  be an  $m \times n$  matrix. Then*

$$\text{ara } I_t(X) \leq mn - t^2 + 1.$$

A generalization to arbitrary ideals  $I(X; \delta)$ ,  $J(X; \gamma)$  is left to the reader.

Unfortunately we do not know how to derive a lower bound from ASL theory in general. The problem one is faced can already be illustrated by means of the example  $A = B[X]$ ,  $X$  a  $2 \times 2$  matrix, with the poset



and  $\Omega = \{\delta, X_{11}, X_{12}\}$ :  $\text{rk } \Omega$  even exceeds the minimal number of generators of  $A\Omega$ . However, one can reverse (5.20) if  $A$  is a symmetric ASL. As stated in (4.6),  $G(X)$  is a symmetric ASL, and therefore all the  $G(X; \gamma)$  are symmetric, too. Another class of symmetric ASLs is given by the *discrete* ASLs, in which every straightening relation has the form  $\xi v = 0$ . (In the general theory of ASLs the discrete ones play a central role, cf. [DEP.2].) Discrete ASLs are graded in a natural way: assign the degree 1 to every element of  $\Pi$ .

**(5.22) Proposition.** *Let  $A$  be a symmetric graded ASL on  $\Pi$ . Then for every ideal  $\Omega \subset \Pi$  one has*

$$\text{ara } A\Omega = \text{rk } \Omega.$$

**PROOF:** The complement of  $\Omega$  is an ideal in  $\Pi$  equipped with the reverse order. Since  $A$  is symmetric,  $S = A/(\Pi \setminus \Omega)A$  is again an ASL, the underlying poset being  $\Omega$  with its order reversed. Now obviously  $\text{ara } A\Omega \geq \text{ara } S\Omega$ , and by Krull's Principal Ideal Theorem  $\text{ara } S\Omega \geq \text{ht } S\Omega = \text{rk } \Omega$  (cf. (5.10) for the last equation). —

In particular the Schubert variety with homogeneous coordinate ring  $G(X; \gamma)$  can be defined as a subvariety of the ambient Grassmann variety by  $\text{rk}(\Gamma(X) \setminus \Gamma(X; \gamma))$  equations, but not by a smaller number of equations.

In general the arithmetical rank may go down when one passes from  $G(X)$  to  $B[X]$ , as the example above shows. Without further or completely different arguments one can therefore not conclude that the bound in (5.21) is sharp. Hochster has given an invariant-theoretic argument for the case of maximal minors,  $B$  containing a field of

characteristic 0. We shall discuss Hochster's argument in Section 7. Newstead uses topological arguments in order to show that the bound in (5.21) is an equality for  $t = 2$ ,  $B$  again containing a field of characteristic zero, cf. [Ne], p. 180, Example (i), (a). As Cowsik told us, Newstead's argument goes through for every  $t$  and can be transferred to characteristic  $p > 0$  via the use of étale cohomology. (There is of course no restriction in assuming that  $B$  is a field; otherwise one factors by a maximal ideal of  $B$  first.)

## F. Comments and References

Our representation of ASL theory follows Eisenbud [Ei.1]. However we avoid the passage to the discrete ASL in proving (5.7), (5.9) and (5.10), and in the proof of (5.14) we have replaced an argument of Musili ([Mu], Proposition 1.3) by the closely related (5.15), drawn from [HE.2], section 4. (5.20), (5.21), (5.22) seem to be new, at least in regard to the method of proof.

Since all our examples are graded, we have made "graded" a standard assumption. This allows us to weaken the ASL axioms slightly (relative to [Ei.1]) as indicated in [DEP.2], Proposition 1.1. Proposition (4.2) is the only result for which the assumption "graded" seems to be unavoidable after one has made the conclusion of (4.1) an axiom. The reader may check that the assumption "graded" is not essential for (5.7) and (5.10)(cf. also [DEP.2], Prop 6.1). Without the assumption "graded" Theorem (5.14) is to be replaced by the statement that the sequence  $x_1, \dots, x_k$ ,  $k = \text{rk } \Pi$ , constructed for (5.9) is an  $A$ -regular sequence, cf. [Ei.1].

The Cohen-Macaulay property of the determinantal rings was first proved by Hochster and Eagon in [HE.2] without a standard monomial theory, cf. Section 12. Shortly later Laksov and Hochster proved that the homogeneous coordinate rings of the Schubert subvarieties of the Grassmannians are Cohen-Macaulay, cf. [La.1] and [Ho.3]. Their rather similar proofs were then followed by a proof of Musili [Mu], which differs in the technicalities of the induction step only, and the proof of Theorem (5.14) may be considered an abstract version of it. The proof of Hochster and Laksov has also been reproduced in [ACGH].

The theory of ASLs is a connection between combinatorics and commutative algebra. For a development of ASL theory from a more combinatorial view-point we refer the reader to [Bc].

## 6. Integrity and Normality. The Singular Locus

As we have noticed in (2.12) already, it follows from a localization argument and the Cohen-Macaulay property that the rings  $R_t(X)$  are (normal) domains whenever the ring  $B$  of coefficients is a (normal) domain. In this section we want to extend this result to all the Schubert cycles and determinantal rings. Furthermore their singular locus will be computed.

### A. Integrity and Normality

As a normality criterion we shall use Lemma (16.24): *Let  $S$  be a noetherian ring, and  $x \in S$  such that  $x$  is not a zero-divisor,  $S/Sx$  is reduced, and  $S[x^{-1}]$  is normal. Then  $S$  is normal.*

In a graded ASL  $A$ , whose underlying poset has a single minimal element, this element is a natural candidate for  $x$ :  $A/Ax$  is a graded ASL again and therefore reduced if  $B$  is reduced (cf. (5.1), (5.7)). Then it has “only” to be checked, whether  $A[x^{-1}]$  is normal. In the cases of interest to us the ring  $A[x^{-1}]$  has a particularly simple structure:

**(6.1) Lemma.** *Let  $X$  be an  $m \times n$  matrix of indeterminates over  $B$ ,  $m \leq n$ ,  $\gamma = [a_1, \dots, a_m] \in \Gamma(X)$  and*

$$\Psi = \{ [d_1, \dots, d_m] \in \Gamma(X; \gamma) : a_i \notin [d_1, \dots, d_m] \text{ for at most one index } i \}.$$

Then

$$G(X; \gamma)[\gamma^{-1}] = B[\Psi][\gamma^{-1}],$$

the extensions being formed within the total ring of quotients of  $G(X; \gamma)$ , and notably, the set  $\Psi$  is algebraically independent over  $B$ . Therefore  $G(X; \gamma)[\gamma^{-1}]$  is isomorphic to  $B[T_1, \dots, T_d][T_1^{-1}]$ ,  $d = \dim G(X; \gamma) - \dim B$ ,  $T_1, \dots, T_d$  indeterminates.

PROOF: The inclusion “ $\supset$ ” is clear. We show that  $[b_1, \dots, b_m] \in B[\Psi][\gamma^{-1}]$  for all  $[b_1, \dots, b_m] \in \Gamma(X; \gamma)$  by induction on the number  $k$  of indices  $i$  such that  $b_i \notin [a_1, \dots, a_m]$ . For  $k = 0$  and  $k = 1$ ,  $[b_1, \dots, b_m] \in \Psi$  by definition. Let  $k > 1$  and choose an index  $j$  such that  $b_j \notin [a_1, \dots, a_m]$ . We use the Plücker relation (4.4), the data “...” of (4.4) corresponding to the present ones in the following manner:

$$\begin{aligned} \text{“}k\text{”} &= 0, & \text{“}(b_2, \dots, b_m)\text{”} &= (b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_m), \\ \text{“}l\text{”} &= 2, & \text{“}(c_1, \dots, c_s)\text{”} &= (a_1, \dots, a_m, b_j), \\ \text{“}s\text{”} &= m + 1. \end{aligned}$$

In this relation all the terms different from

$$[a_1, \dots, a_m][b_j, b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_m] = (-1)^{j-1}[a_1, \dots, a_m][b_1, \dots, b_m]$$



and  $\neq 0$  in  $G(X; \gamma)$  have the form  $\delta \varepsilon$  such that  $\delta \in \Psi$  and  $\varepsilon$  has only  $k - 1$  indices not occurring in  $\gamma$ . Solving for  $[a_1, \dots, a_m][b_1, \dots, b_m]$  and dividing by  $\gamma$ , one gets  $[b_1, \dots, b_m] \in B[\Psi][\gamma^{-1}]$ .

In proving the algebraic independence of  $\Psi$  we first consider a field  $B$  of coefficients. If  $x$  is not a zero-divisor in a finitely generated algebra  $A$  over a field, one has  $\dim A = \dim A[x^{-1}]$ . An easy count yields  $|\Psi| = \text{rk } \Gamma(X; \gamma)$ : there are  $n - a_i - (m - i)$  elements in  $\Psi$  which do not contain  $a_i$ . (The rank of  $\Gamma(X; \gamma)$  has been computed above (5.12)). So

$$\begin{aligned} |\Psi| &= \text{rk } \Gamma(X; \gamma) = \dim G(X; \gamma) \\ &= \dim G(X; \gamma)[\gamma^{-1}] = \dim B[\Psi][\gamma^{-1}] = \dim B[\Psi], \end{aligned}$$

and  $\Psi$  is algebraically independent.

Let now  $B = \mathbf{Z}$ . Since  $\Psi$  is algebraically independent over  $\mathbf{Q}$ , it is algebraically independent over  $\mathbf{Z}$ . In order to derive the general case one needs that  $G(X; \gamma)/\mathbf{Z}[\Psi]$  is  $\mathbf{Z}$ -flat. This is equivalent to

$$\text{Tor}_1^{\mathbf{Z}}(G(X; \gamma)/\mathbf{Z}[\Psi], \mathbf{Z}/p\mathbf{Z}) = 0$$

for all prime numbers  $p$ , and this again follows from the case of a field of coefficients considered already. —

The following lemma will be needed in Section 7, in particular for the proof of Theorem (1.2) given there:

**(6.2) Lemma.** *Let  $S$  be a  $B$ -algebra, and suppose that  $\varphi, \psi: G(X; \gamma) \rightarrow S$  are  $B$ -algebra homomorphisms. If  $\varphi(\gamma)$  is not a zero-divisor and  $\varphi(\delta) = \psi(\delta)$  for all  $\delta \in \Psi$ ,  $\Psi$  as in (6.1), then  $\varphi = \psi$ .*

PROOF: Consider the commutative diagram in which the vertical arrows are injections:

$$\begin{array}{ccc} G(X; \gamma) & \xrightarrow{\varphi \quad (\psi)} & S \\ \downarrow & & \downarrow \\ G(X; \gamma)[\gamma^{-1}] & \xrightarrow{\varphi[\gamma^{-1}] \quad (\psi[\gamma^{-1]})} & S[\varphi(\gamma)^{-1}] \end{array}$$

By virtue of (6.1) and hypothesis:  $\varphi[\gamma^{-1}] = \psi[\gamma^{-1}]$ . —

If  $B$  is an integral domain,  $G(X; \gamma)$ , a subring of the domain  $B[\Psi][\gamma^{-1}]$ , is a domain, too, and for normal  $B$  the ring  $B[\Psi][\gamma^{-1}]$  is even normal, so normality of  $G(X; \gamma)$  then follows from the criterion cited above. The ring  $R(X; \delta)$  arises from  $G(\tilde{X}; \tilde{\delta})$  by dehomogenization with respect to  $\pm[n + 1, \dots, n + m]$  as stated in (5.5). So  $R(X; \delta)$  is a (normal) domain, too, by virtue of (16.23).

**(6.3) Theorem.** *Let  $B$  be a (normal) domain,  $X$  an  $m \times n$  matrix,  $m \leq n$ , of indeterminates, and  $\gamma \in \Gamma(X)$ ,  $\delta \in \Delta(X)$ . Then  $G(X; \gamma)$  and  $R(X; \delta)$  are (normal) domains.*

Though a determinantal analogue of (6.1) has not been needed for the proof of (6.3), it will be useful later.

**(6.4) Lemma.** *Let  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r] \in \Delta(X)$ , and*

$$\Psi = \{ [a_i | b_j]: i, j = 1, \dots, t \} \cup \{ \tilde{\delta} \in \Delta(X; \delta): \tilde{\delta} \text{ differs from } \delta \text{ in exactly one index} \}.$$

*Then  $R(X; \delta)[\delta^{-1}] = B[\Psi][\delta^{-1}]$ , and  $\Psi$  is algebraically independent over  $B$ . Thus  $R(X; \delta)[\delta^{-1}]$  is isomorphic to*

$$B[T_1, \dots, T_d][\zeta^{-1}], \quad \zeta \in B[T_1, \dots, T_d], \quad d = \dim R(X; \delta) - \dim B,$$

*$T_1, \dots, T_d$  indeterminates. If  $B$  is an integral domain,  $\zeta$  is a prime element.*

PROOF: For  $R(X; \delta) \subset B[\Psi][\delta^{-1}]$  it is enough that  $[u|v] \in B[\Psi][\delta^{-1}]$  for all  $[u|v] \in \Delta(X; \delta)$ . Suppose first, that  $u = a_i$ . Then (in  $B[X]$  already)

$$[u, a_1, \dots, a_r | v, b_1, \dots, b_r] = 0.$$

Expansion of this minor along row  $u$  shows that  $[u|v]$  can be expressed (over  $\mathbf{Z}$ ) by the  $[a_i | b_j] \in \Psi$ ,  $[a_1, \dots, a_r | v, b_1, \dots, \hat{b}_i, \dots, b_r] \in \Psi$  and  $\delta^{-1}$ . Let  $u$  be arbitrary now. In  $R(X; \delta)$  one has

$$[u, a_1, \dots, a_r | v, b_1, \dots, b_r] = 0,$$

and now one expands along column  $v$ , expressing  $[u|v]$  by the  $[a_i | v] \in B[\Psi][\delta^{-1}]$  (“ $\in$ ” has been shown already),  $[u, a_1, \dots, \hat{a}_i, \dots, a_r | b_1, \dots, b_r] \in \Psi$  and  $\delta^{-1}$ . In proving the algebraic independence of  $\Psi$  one proceeds as in the proof of (6.1). At this point one can derive the contention of (6.3) with respect to  $R(X; \delta)$  or use (6.3) directly in order to conclude that  $\zeta$ , being the determinant of a matrix of indeterminates, is a prime element over a domain  $B$ . —

The representation of  $R(X; \delta)$  as a dehomogenization of  $G(\tilde{X}; \tilde{\delta})$  renders  $R(X; \delta)[\delta^{-1}]$  a residue class ring of  $B[\tilde{\Psi}][\tilde{\delta}^{-1}]$ ,  $\tilde{\Psi}$  constructed for  $\tilde{\delta}$  according to (6.1). The reader may find the resulting representation of  $R(X; \delta)[\delta^{-1}]$ .

Theorem (6.3) has consequences for a more general class of rings.

**(6.5) Corollary.** *Let  $B$  be an integral domain,  $\Omega \subset \Gamma(X)$  an ideal. Then the minimal prime ideals of  $\Omega G(X)$  are the ideals  $J(X; \gamma)$ ,  $\gamma$  a minimal element of  $\Gamma(X) \setminus \Omega$ , and  $\Omega G(X)$  is their intersection. The analogous statement holds for ideals  $\Omega \subset \Delta(X)$ .*

In fact, the ideals  $J(X; \gamma)$  are prime, and  $\Omega G(X) = \bigcap J(X; \gamma)$  follows from  $\Omega = \bigcap (\Gamma(X) \setminus \Gamma(X; \gamma))$  by virtue of (5.2). We leave it to the reader to find the most general version (in regard to  $B$ ) of (6.5) and to prove the following corollary (as an application of (3.15), say):

**(6.6) Corollary.** *Let  $B$  be an arbitrary ring,  $\Omega \subset \Gamma(X)$  an ideal. An element  $\gamma \in \Gamma(X) \setminus \Omega$  is not a zero-divisor modulo  $\Omega G(X)$  if and only if it is comparable to every minimal element of  $\Gamma(X) \setminus \Omega$ . The analogous statement holds for ideals  $\Omega \subset \Delta(X)$ .*

### B. The Singular Locus

Let  $B$  be a field momentarily. Then every localization of  $G(X; \gamma)$  with respect to a prime ideal not containing  $\gamma$  is a localization of a polynomial ring over  $B$ , and therefore regular. The element  $\gamma$  is distinguished only in the combinatorial structure of  $\Gamma(X; \gamma)$ . For the purpose of (6.1) every element of  $G(X; \gamma)$  which can be mapped to  $\gamma$  by an automorphism of  $G(X; \gamma)$ , is as good as  $\gamma$  itself. In the extreme case in which  $\gamma = [1, \dots, m]$ ,  $G(X; \gamma) = G(X)$ , every element of  $\Gamma(X)$  can be mapped (up to sign) to  $\gamma$  by a suitable permutation of the columns of  $X$  which of course induces an automorphism of  $G(X)$ . In general we can only use the permutations which leave  $J(X; \gamma)$  invariant. Every permutation  $\pi$  of  $\{1, \dots, n\}$  induces a permutation of  $\Gamma(X)$  (which up to sign has the same effect as the corresponding automorphism). Let  $\gamma = [a_1, \dots, a_m]$ . If

$$\pi(\{a_i, \dots, n\}) = \{a_i, \dots, n\} \quad \text{for } i = 1, \dots, m$$

then certainly  $\pi(\delta) \in J(X; \gamma)$  for all  $\delta \in \Gamma(X; \gamma)$ , this being equivalent to the invariance of  $J(X; \gamma)$  under (the automorphism induced by)  $\pi$ . The example  $\gamma = [1, \dots, m]$  however shows that the condition above is too coarse: an appropriate condition must take care of how  $[a_1, \dots, a_m]$  breaks into *blocks* of consecutive integers

$$\beta_0 = (a_1, \dots, a_{k_1}), \beta_1 = (a_{k_1+1}, \dots, a_{k_2}), \dots, \beta_s = (a_{k_s+1}, \dots, a_m).$$

For systematic reasons we let  $k_0 = 0$ ,  $k_{s+1} = m$ ,  $a_{m+1} = n + 1$ . Similarly we decompose the complement of  $\gamma$  with respect to the interval  $\{a_1, \dots, n\}$  to obtain the *gaps* of  $\gamma$ :

$$\chi_0 = (a_{k_1} + 1, \dots, a_{k_1+1} - 1), \dots, \chi_s = (a_m + 1, \dots, n).$$

Here  $\chi_s$  is empty if  $a_m = n$ . If a permutation  $\pi$  satisfies the condition

$$(1) \quad \pi(\beta_i \cup \chi_i) = \beta_i \cup \chi_i, \quad i = 0, \dots, s,$$

then  $\pi$  certainly leaves  $\Gamma(X; \gamma)$  invariant as a set, thus maps  $\Gamma(X) \setminus \Gamma(X; \gamma)$  onto itself, and induces an automorphism of  $G(X; \gamma)$ . An element  $\delta = [b_1, \dots, b_m] \in \Gamma(X; \gamma)$  can be mapped to  $\gamma$  by such a permutation if and only if

$$(2) \quad b_{k_i} \in \beta_{i-1} \cup \chi_{i-1}, \quad i = 1, \dots, s + 1.$$

Let  $\Sigma(X; \gamma)$  be the set of elements  $\delta \in \Gamma(X; \gamma)$  which satisfy (2). It is an ideal in the partially ordered set  $\Gamma(X; \gamma)$ !

To give an example: Let  $m = 4$ ,  $n = 7$ ,  $\gamma = [1 \ 3 \ 4 \ 6]$ . Then the blocks and gaps of  $\gamma$  are

$$\beta_0 = (1), \beta_1 = (3 \ 4), \beta_2 = (6) \quad \text{and} \quad \chi_0 = (2), \chi_1 = (5), \chi_2 = (7).$$

$\Sigma(X; \gamma)$  consists of all  $\delta = [b_1, \dots, b_4] \in \Gamma(X; \gamma)$  such that  $b_1 \leq 2$ ,  $b_2 \leq 5$ .

**(6.7) Theorem.** *Let  $B$  be a noetherian ring,  $X$  an  $m \times n$  matrix of indeterminates,  $\gamma \in \Gamma(X)$ ,  $P$  a prime ideal of  $G(X; \gamma)$ , and  $Q = B \cap P$ . Suppose that  $\Gamma(X; \gamma)$  is not a chain. Then  $G(X; \gamma)_P$  is regular, if and only if  $B_Q$  is regular and  $P$  does not contain the ideal  $\Sigma(X; \gamma)G(X; \gamma)$ .*

The case in which  $\Gamma(X; \gamma)$  is a chain, is trivial:  $G(X; \gamma)$  is a polynomial ring over  $B$  then. It is easy to check that  $\Gamma(X; \gamma)$  is a chain if and only if  $\gamma = [a_1, n - m + 2, \dots, n]$  or  $\gamma = [n - m, \dots, a_j, a_j + 2, \dots, n]$ . Before we prove (6.7) we supplement it by a description of the minimal elements in the complement of  $\Sigma(X; \gamma)$  and a slight strengthening.

**(6.8) Supplement to (6.7).** *One has  $\Sigma(X; \gamma) = \Gamma(X; \gamma)$  if and only if (i)  $s = 0$  or (ii)  $s = 1$  and  $a_m = n$  (the latter implying  $\chi_1 = \emptyset$ ). If (iii)  $s \geq 1$  and  $\chi_1 \neq \emptyset$ , then the minimal elements in the complement of  $\Sigma(X; \gamma)$  with respect to  $\Gamma(X; \gamma)$  are*

$$\begin{aligned} \sigma_1 &= [(a_1, \dots, a_{k_1-1}), (a_{k_1+1}, \dots, a_{k_2}, a_{k_2} + 1), \beta_2, \dots, \beta_s], \\ &\vdots \\ \sigma_t &= [\beta_0, \dots, \beta_{t-2}, (a_{k_{t-1}+1}, \dots, a_{k_t-1}), (a_{k_t+1}, \dots, a_{k_{t+1}}, a_{k_{t+1}} + 1), \beta_{t+1}], \end{aligned}$$

where  $t = s$  if  $\chi_s \neq \emptyset$ , and  $t = s - 1$  if  $\chi_s = \emptyset$  (in the first of these cases we let  $\beta_{t+1} = \emptyset$ ). In case (iii) the localizations of  $G(X; \gamma)$  with respect to a prime ideal  $P \supset \Sigma(X; \gamma)G(X; \gamma)$  are not even factorial domains.

In our example  $t = 2$ ,  $\sigma_1 = [3 \ 4 \ 5 \ 6]$ ,  $\sigma_2 = [1 \ 3 \ 6 \ 7]$ .

If the singular locus of  $\text{Spec } B$  is closed, then the singular locus of  $G(X; \gamma)$  is closed, too, and in case (iii) of (6.8) its minimal elements are the prime ideals  $(Q + J(X; \sigma_i)G(X; \gamma))$ , where  $Q$  runs through the minimal singular prime ideals of  $B$  and  $i = 1, \dots, t$ . (Similar statements hold in the remaining cases.)

PROOF OF (6.7) AND (6.8): The “if” part of (6.7) has been indicated already: Let  $S = G(X; \gamma)$ . The ring  $S_P$  is a localization of  $(S \otimes B_Q)[\delta^{-1}]$  for some  $\delta \in \Sigma = \Sigma(X; \gamma)$ . Since  $\delta$  can be mapped to  $\gamma$  by an automorphism of  $S \otimes B_Q$ ,  $S_P$  is regular by (6.1).

For the converse we first note that regularity of  $S_P$  implies regularity of  $B_Q$  through flatness, and factoriality implies that  $B_Q$  is a domain (at least). Next we may assume that  $P$  is a minimal prime ideal of  $S\Sigma$ , in order to derive a contradiction. Then, after having replaced  $B$  by  $B_Q$ , we conclude  $Q = 0$  from (6.6), and  $B = K$  is a field.

In case (i) of (6.8) condition (2) holds for every  $\delta \geq \gamma$ , so  $\Sigma = \Gamma(X; \gamma)$  and  $P = S\Sigma$  is the irrelevant maximal ideal of a graded  $K$ -algebra generated by its 1-forms. If  $\Gamma(X; \gamma)$  is not a chain, the dimension of the  $K$ -vector space of 1-forms differs from the Krull dimension of  $S$ :  $S_P$  is not regular. In case (ii) one has  $a_j = n - (m - j)$ , and therefore, letting  $\delta = [b_1, \dots, b_m]$ ,  $b_j = n - (m - j)$  for  $j \geq k_1 + 1$ . This implies  $b_{k_1} < a_{k_1+1}$ , and we are through by the same argument.

In case (iii) one certainly has  $\Sigma \neq \Gamma(X; \gamma)$ , since  $\sigma_1, \dots, \sigma_t \notin \Sigma$  and  $t \geq 1$ . It is easy to see that  $\sigma_1, \dots, \sigma_t$  are the minimal elements of the complement of  $\Sigma \subset \Gamma(X; \gamma)$ . By our assumption on  $P$  being minimal over  $S\Sigma$ ,  $P = J(x; \sigma_j) (= J(X; \sigma_j)/J(X; \gamma)$ , cf. 5.A) for an index  $j$  (cf. (6.5)). Since  $S/S\Sigma$  is reduced,  $\Sigma$  generates  $PS_P$ , in particular contains a minimal system of generators of  $PS_P$ ; its elements are irreducible. The permutations  $\pi$  which satisfy (1), have the property corresponding to (1) for  $\sigma_j$ , too: The sets  $\beta_i \cup \chi_i$  for  $\gamma$  coincide with the corresponding sets for  $\sigma_j$ ! Therefore these permutations leave

$\Gamma(X; \sigma_j)$  and  $\Gamma(X; \gamma) \setminus \Gamma(X; \sigma_j)$  invariant: they induce automorphisms of  $S_P$ . Since  $\gamma$  can be moved to every element of  $\Sigma$  by such a permutation,  $\gamma$  is an irreducible element of  $S_P$ . On the other hand it cannot be prime: Let

$$\tau_1 = [\beta_0, \dots, \beta_{j-2}, (a_{k_{j-1}+1}, \dots, a_{k_j-1}), a_{k_j} + 1, \beta_j, \dots, \beta_s]$$

and

$$\tau_2 = [\beta_0, \dots, \beta_{j-1}, (a_{k_j+1}, \dots, a_{k_{j+1}-1}), a_{k_{j+1}} + 1, \beta_{j+1}, \dots, \beta_s].$$

Then  $\tau_1, \tau_2$  are upper neighbours of  $\gamma$ , and  $\tau_1, \tau_2 \leq \sigma_j$ , so  $P = J(x; \sigma_j)$  contains  $J(x; \tau_1)$  and  $J(x; \tau_2)$ , two different minimal primes of  $S_\gamma$ , excluding that  $\gamma$  is prime in  $S_P$ . —

**(6.9) Remarks.** (a) As stated already,  $G(X; \gamma)$  is a polynomial ring over  $B$  if  $\Gamma(X; \gamma)$  is a chain. (6.7) shows that the converse is likewise true: Suppose that  $G(X; \gamma)$  is a polynomial ring over  $B$ . Then  $G(X; \gamma) \otimes (B/Q)$  is a polynomial ring over the field  $B/Q$ ,  $Q$  a maximal ideal of  $B$ . So all the localizations of  $G(X; \gamma) \otimes (B/Q)$  are regular, and  $\Gamma(X; \gamma)$  must be a chain by (6.7).

(b) Since the cue “factorial” has been given already, we should point out that in the exceptional cases (i) and (ii) of (6.8) the ring  $G(X; \gamma)$  is indeed factorial, provided  $B$  is factorial:  $\gamma$  has only a single upper neighbour then, so is prime by (6.5), and the factoriality of  $G(X; \gamma)[\gamma^{-1}]$  implies the factoriality of  $G(X; \gamma)$  itself; cf. Section 8 for a detailed discussion.

(c) We have started the proof of (6.8) by trying to find as many elements of  $\Gamma(X; \gamma)$  which are conjugate to  $\gamma$  under an automorphism of  $G(X; \gamma)$ , and have found the set  $\Sigma(X; \gamma)$  of such elements. After (6.8) it is clear that elements  $\sigma$  outside  $\Sigma(X; \gamma)$  are not conjugate to  $\gamma$  under a  $B$ -automorphism: the  $B$ -algebras  $G(X; \gamma)[\gamma^{-1}]$  and  $G(X; \gamma)[\sigma^{-1}]$  are not isomorphic. The structure of  $G(X; \gamma)[\sigma_i^{-1}]$  will be revealed in (8.11).

(d) Without restriction one can exclude the case  $a_m = n$  first, and thus reduce the number of cases to be considered in (6.8). In fact, if  $\gamma = [a_1, \dots, a_p, n - (m - p) + 1, \dots, n]$ , then

$$G(X; \gamma) \cong G(X'; \gamma')$$

where  $X'$  is a  $p \times (n - (m - p))$  matrix of indeterminates and  $\gamma' = [a_1, \dots, a_p]$ . We leave it to the reader to check that the map which sends  $[b_1, \dots, b_p, n - (m - p) + 1, \dots, n] \in \Gamma(X; \gamma)$  to  $[b_1, \dots, b_p] \in \Gamma(X'; \gamma')$ , is well-defined and an isomorphism. —

The most convenient way to find the singular locus of  $R(X; \delta)$  is again the method of dehomogenization. Though very suggestive, the automorphism argument (now in conjunction with (6.4)) does not produce the correct result in all cases, as will be demonstrated below.

We write  $R(X; \delta)$  as the dehomogenization of  $G(\tilde{X}; \tilde{\delta})$  again. It is immediate from (16.26) that the ideal  $I$  of  $R(X; \delta)$  generated by an ideal  $\Omega \subset \Delta(X; \delta)$  is the dehomogenization of the ideal  $J$  of  $G(\tilde{X}; \tilde{\delta})$  generated by the corresponding ideal  $\tilde{\Omega}$  in  $\Gamma(\tilde{X}; \tilde{\delta})$ :  $J$  is homogeneous,  $\pm[n + 1, \dots, n + m]$  is not a zero-divisor modulo  $J$  (since it is the maximal element of the poset underlying the ASL  $G(\tilde{X}; \tilde{\delta})/J$ ), and the generating set  $\tilde{\Omega}$  is mapped (up to sign) onto  $\Omega$ . Let  $\Xi(X; \delta)$  be the subset of  $\Delta(X; \delta)$  corresponding to  $\Sigma(\tilde{X}; \tilde{\delta}) \subset \Gamma(\tilde{X}; \tilde{\delta})$ . Then, by virtue of (16.28) and (6.7), a localization  $R(X; \delta)_P$  is regular if and only if  $B_{P \cap B}$  is regular and  $P \not\supset \Xi(X; \delta)$ . It only remains to give a description

of  $\Xi(X; \delta)$  in terms of  $\delta$ . We state the result, leaving the translation back and forth to the reader.

Let  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r]$ . We decompose the row part  $[a_1, \dots, a_r]$  into its blocks:

$$[a_1, \dots, a_r] = [\beta_0, \dots, \beta_u], \quad \beta_i = (a_{k_i+1}, \dots, a_{k_{i+1}}).$$

Then we let

$$\xi_i = [\beta_0, \dots, \beta_{i-2}, (a_{k_{i-1}+1}, \dots, a_{k_{i-1}}), (a_{k_i+1}, \dots, a_{k_{i+1}}, a_{k_{i+1}+1}), \\ \beta_{i+1}, \dots, \beta_u | b_1, \dots, b_r],$$

$i = 1, \dots, u-1$ , and  $i = u$  if  $a_r < m$  and  $u \geq 1$ . Analogously one constructs elements  $v_j$ ,  $j = 1, \dots, w-1$ , and  $j = w$  if  $b_r < n$  and  $w \geq 1$ , for the column part. In the exceptional case  $a_r = m$  the element  $\xi_u$  is given by

$$\xi_u = [\beta_0, \dots, \beta_{u-2}, (a_{k_{u-1}+1}, \dots, a_{k_{u-1}}), (a_{k_u+1}, \dots, a_r) | b_1, \dots, b_{r-1}],$$

and if  $b_r = n$  the element  $v_w$  is chosen analogously. Finally,

$$\zeta = [a_1, \dots, a_{r-1} | b_1, \dots, b_{r-1}].$$

**(6.10) Theorem.** *Let  $B$  be a noetherian ring,  $X$  an  $m \times n$  matrix of indeterminates, and  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r] \in \Delta(X)$ . Assume that  $\delta \neq [m-r+1, \dots, m | b_1, \dots, b_1+r-1]$  and  $\delta \neq [a_1, \dots, a_1+r-1 | n-r+1, \dots, n]$ . Let  $P$  be a prime ideal of  $R(X; \delta)$  and  $Q = B \cap P$ . Then the localization  $R_P$  is regular if and only if  $B_Q$  is regular and  $P \not\supseteq \Xi(X; \delta)$ , where  $\Xi(X; \delta)$  is given as follows:*

(i) *If  $r = 1$ , then  $\Xi(X; \delta) = \Delta(X; \delta)$ .*

(ii) *If  $r > 1$ ,  $a_r < m$  and  $b_r < n$ ,  $\Xi(X; \delta)$  is the ideal in  $\Delta(X; \delta)$  cogenerated by*

$$\xi_1, \dots, \xi_u, v_1, \dots, v_w, \zeta.$$

(iii) *If  $r > 1$ ,  $a_r = m$  or  $b_r = n$ ,  $\Xi(X; \delta)$  is the ideal in  $\Delta(X; \delta)$  cogenerated by*

$$\xi_1, \dots, \xi_u, v_1, \dots, v_w.$$

The singular locus of  $R_{r+1}(X) = R(X; [1, \dots, r | 1, \dots, r])$  has been computed in (2.6) already. This case is recovered in (ii):  $u = w = 0$  then, and the singular locus is determined by  $\zeta = [1, \dots, r-1 | 1, \dots, r-1]$ . Again one of the exceptional cases  $\delta = [m-r+1, \dots, m | b_1, \dots, b_1+r-1]$  and  $\delta = [a_1, \dots, a_1+r-1 | n-r+1, \dots, n]$  occurs if and only if  $R(X; \delta)$  is a polynomial ring over  $B$ . (The “if” part is obvious, and for the “only if” part one argues as in (6.9),(a).)

The reader may check that only in the cases (i) and (ii)  $\Xi(X; \delta)$  is the set of elements of  $\Delta(X; \delta)$  which are conjugates of  $\delta$  (up to sign) under row and column permutations of  $X$ . That the set of conjugates fails to give the singular locus in general can also be seen from the following example:  $B = K$  a field,  $m = 2$ ,  $n = 3$ ,  $\delta = [1 \ 2 | 1 \ 3]$ . The prime ideal  $P = I(x; [1 | 1])$  has height 1, since  $[1 | 1]$  is an upper neighbour of  $\delta$ . By (6.3) the local ring  $R(X; \delta)_P$  is regular, though  $P$  contains all the conjugates of  $\delta$ . The exceptional nature of case (iii) is easily explained: Let  $\tilde{\delta} = [\tilde{a}_1, \dots, \tilde{a}_m]$ . Then  $a_r < m$  and  $b_r < n$  if and

only if  $\tilde{a}_{r+1} = n + 1 > b_r + 1 = \tilde{a}_r + 1$ . Therefore in cases (i) and (ii) every permutation  $\pi$  satisfying condition (1) above induces an automorphism of  $R(X; \delta)$ .

We can combine the different cases of (6.10) to a single statement if we choose to describe determinantal ideals by their generators. It has been noted in 5.A already that the ideal  $I = I(X; \delta)$  has a system of generators consisting of the

$$(r_i + 1)\text{-minors of the rows } 1, \dots, u_i, \quad i = 1, \dots, p,$$

and the

$$(s_j + 1)\text{-minors of the columns } 1, \dots, v_j, \quad j = 1, \dots, q,$$

where the  $r_i, u_i, s_j, q_j$  are suitably chosen integers satisfying the conditions

$$\begin{aligned} 0 \leq r_1 < \dots < r_p < m, \quad 0 \leq s_1 < \dots < s_q < n, \\ u_{i+1} > u_i + (r_{i+1} - r_i), \quad v_{j+1} > v_j + (s_{j+1} - s_j), \quad i = 1, \dots, p-1, \quad j = 1, \dots, q-1, \end{aligned}$$

and

$$r_p + 1 < s_q + 1 + n - v_q, \quad s_q + 1 < r_p + 1 + m - u_p.$$

**(6.11) Theorem.** *Let  $B$  be a noetherian ring,  $X$  an  $m \times n$  matrix of indeterminates. Suppose that the ideal  $I$  is generated as just specified. Then for a prime ideal  $P$  of  $R = B[X]/I$  the localization  $R_P$  is regular if and only if  $B_Q$  is regular for  $Q = B \cap P$  and  $P$  does not contain the ideal*

$$P_1 \cap \dots \cap P_p \cap Q_1 \cap \dots \cap Q_q,$$

where  $P_i$  is generated by the  $r_i$ -minors of rows  $1, \dots, u_i$ , and the  $Q_j$  are defined analogously for the columns.

The derivation of (6.11) from (6.10) can be left to the reader.

After one has explicitly described the singular locus of the rings  $G(X; \gamma)$  and  $R(X; \delta)$  one can compute its codimension. The best possible general estimate is given in the following proposition:

**(6.12) Proposition.** *Let  $B$  be a noetherian ring which satisfies Serre's condition  $(R_2)$ . Then all the rings  $G(X; \gamma)$  and  $R(X; \delta)$  satisfy  $(R_2)$ , too.*

PROOF: Because of (16.28) it is enough to consider the rings  $R = G(X; \gamma)$ . Let  $P$  be a singular prime ideal of  $R$  and  $Q = B \cap P$ . If  $B_Q$  is singular, then  $\dim R_P \geq \dim B_Q \geq 3$ . Thus we may assume that  $B = B_Q$  is a regular local ring and  $P$  is minimal among the singular prime ideals of  $R$ . In the cases (i) and (ii) of (6.10) one has  $P = \Gamma(X; \gamma)R$ , hence  $\text{ht } P = \text{rk } \Gamma(X; \gamma) \geq 3$  (if  $\text{rk } \Gamma(X; \gamma) \leq 2$ ,  $R$  is a polynomial ring over  $B$ ). In case (iii) of (6.10)  $P = J(X; \sigma_i)/J(X; \gamma)$  for a suitable  $i$ , and there are at least two elements  $\pi < \rho$  of  $\Gamma(X; \gamma)$  strictly between  $\gamma$  and  $\sigma_i$ , and therefore  $\text{ht } P \geq 3$  because of (6.3). —

It is easy to see that  $(R_2)$  is the best we can expect in general; take  $\gamma = [1 \ 3 \ 5]$  for example or  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r]$  such that  $a_r = m - 1$ ,  $b_r = n - 1$ .

### C. Comments and References

The key lemma (6.1) is essentially Lemma 3.11 from Hochster's article [Ho.3], whereas a variant of (6.4) seems to appear first in [Br.3] (for  $R_{r+1}(X)$ ). "Classically" the integrity of the Schubert cycles  $G(X; \gamma)$  is proved by the construction of generic points, cf. Section 7. Hochster shows the "if" part of (6.7) using the automorphism argument and concludes the normality of  $G(X; \gamma)$  from the Cohen-Macaulay property and the Serre condition  $(R_1)$ ; as we have seen, even  $(R_2)$  follows from (6.7). The singular locus of  $G(X; \gamma)$  is given (in the language of Schubert varieties) by Svanes in [Sv.1], p. 451, (5.5.2).

References for the integrity of the rings  $R_{r+1}(X)$  were given in Section 2. Their normality was first proved by Hochster and Eagon [HE.2] as a consequence of the Cohen-Macaulay property and  $(R_1)$ , the latter resulting from a demonstration of the "if" part of (6.11) (as far as the rings  $R(X; \delta)$  are treated in [HE.2]). (6.10) and (6.11) may be considered a natural generalization of their results.



## 7. Generic Points and Invariant Theory

The main objective of this section is to describe the rings  $R_{r+1}(X)$  and  $G(X)$ , more generally  $R(X; \delta)$  and  $G(X; \gamma)$ , as the rings of invariants of actions of linear groups on polynomial rings, thereby solving classical problems of invariant theory. This requires the construction of suitable embeddings into polynomial rings, and the embeddings constructed below are generic points. Furthermore we illustrate the connection between invariant theory and the ideal theory of  $R_{r+1}(X)$ .

### A. A Generic Point for $R_{r+1}(X)$

DEFINITION. Let  $B$  be a commutative ring,  $A$  a  $B$ -algebra. A homomorphism  $\varphi$  from  $A$  into a polynomial ring  $B[W]$  is called a *generic point* if every homomorphism from  $A$  to a field  $L$  factors through  $\varphi$ :

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B[W] \\ & \searrow & \swarrow \\ & & L \end{array}$$

Let us consider  $A = R_{r+1}(X)$  as a simple example. The image  $U$  of the matrix  $X$  with respect to a homomorphism from  $A$  into a field  $L$  satisfies the condition  $\text{rk } U \leq r$ . The homomorphism  $L^m \rightarrow L^n$  given by  $U$  can therefore be factored through  $L^r$ , and the matrix  $U$  may be written

$$U = V_1 V_2$$

where  $V_1$  is an  $m \times r$  matrix,  $V_2$  an  $r \times n$  matrix. So we take an  $m \times r$  matrix  $Y$  and an  $r \times n$  matrix  $Z$  of (independent) indeterminates over  $B$  and factor the homomorphism  $A \rightarrow L$  through

$$\varphi : A \rightarrow B[Y, Z], \quad X \rightarrow YZ,$$

by substituting  $V_1$  for  $Y$ ,  $V_2$  for  $Z$ . Thus  $\varphi$  is a generic point. The existence of a generic point has consequences which are known to us for the rings under consideration. When we shall discuss a different approach to the theory of determinantal rings in Section 12 starting from scratch, part (c) of the following proposition will be extremely useful though. The reader should note that the construction of generic points for the rings  $R(X; \delta)$  below only relies on elementary matrix algebra!

**(7.1) Proposition.** *Let  $\varphi : A \rightarrow B[W]$  be a generic point for the  $B$ -algebra  $A$ .*

- (a) *The kernel of  $\varphi$  is contained in the nilradical of  $A$ .*
- (b) *If  $B$  is reduced, then  $\text{Ker } \varphi$  is the nilradical of  $A$ .*
- (c) *If  $B$  is an integral domain, then the nilradical of  $A$  is prime.*
- (d) *If  $B$  is a domain and  $A$  is reduced, then  $\varphi$  is injective and  $A$  a domain itself.*

All this is evident. If  $B$  is a domain, then it follows from (d) and (5.7) that the generic point constructed for  $R_{r+1}(X)$  is an embedding. But, all we need to prove this in general, is the fact that  $[1, \dots, r | 1, \dots, r]$  is not a zero-divisor in  $R_{r+1}(X)$ :

**(7.2) Theorem.** *For every ring  $B$  the homomorphism  $\varphi: R_{r+1}(X) \rightarrow B[Y, Z]$ ,  $X \rightarrow YZ$ , is a generic point and an embedding.*

PROOF: Over an arbitrary commutative ring  $S$  a matrix  $U$  which has  $I_{r+1}(U) = 0$  and an  $r$ -minor which is a unit in  $S$ , can be factored  $U = V_1 V_2$  as above. So we only need an embedding  $R_{r+1}(X) \rightarrow S$  such that this condition is true for the image  $U$  of  $X$ ; then the embedding factors through  $\varphi$ . A suitable  $S$  is supplied by  $R_{r+1}(X)[\delta^{-1}]$ ,  $\delta = [1, \dots, r | 1, \dots, r]$ . —

The argument just given is typical for many proofs below: After the inversion of a suitable minor the matrix under consideration can be manipulated like a matrix over a field.

The ring  $G(X)$  is defined as a subring of  $B[X]$ . Let  $\psi: G(X) \rightarrow L$  be a homomorphism into a field. Then the “vector”  $(\psi(\gamma): \gamma \in \Gamma(X))$  satisfies the Plücker relations, and one can factor  $\psi$  through  $B[X]$  if and only if it is possible to construct a matrix  $U$  over  $L$  such that its set of Plücker coordinates is  $(\psi(\gamma): \gamma \in \Gamma(X))$ . This is guaranteed by Theorem (1.2) which, however, still waits for the completion of its proof. We shall complete its proof within the proof of (7.14) below where it will also be stated that the embedding  $G(X) \rightarrow B[X]$  is a generic point.

## B. Invariants and Absolute Invariants

In the situation of (7.2) let  $T$  be an element of  $GL(r, B)$ , i.e. an invertible  $r \times r$  matrix over  $B$ . Then

$$YZ = YT^{-1}TZ,$$

so the entries of  $YZ$  are invariant under the substitution  $Y \rightarrow YT^{-1}$ ,  $Z \rightarrow TZ$  considered as an automorphism of  $B[Y, Z]$ . As  $T$  runs through  $G = GL(r, B)$ , this defines an action of  $G$  on  $B[Y, Z]$  as a group of  $B$ -automorphisms. For  $T \in G$  and a polynomial  $f(Y, Z) \in B[Y, Z]$  one puts

$$T(f) = f(YT^{-1}, TZ).$$

The hope that  $B[YZ] \cong R_{r+1}(X)$  is (always) the ring  $B[Y, Z]^G$  of invariants under the action of  $G$  is immediately disappointed: Consider  $B = \mathbf{Z}$ ,  $m = n = r = 1$ . This failure is however caused by a notion of invariant too naive to work for commutative rings in general; a ring like  $\mathbf{Z}$  or a finite field simply has not enough units.

DEFINITION. An element  $f \in B[Y, Z]$  is called an *absolute GL-invariant* if for every ring homomorphism  $\varphi: B \rightarrow S$  the element  $f$  is mapped to an invariant of  $GL(r, S)$  under the natural extension  $B[Y, Z] \rightarrow S[Y, Z]$ .

We shall also consider the action of the special linear groups

$$SL(r, B) = \{T \in GL(r, B) : \det T = 1\}$$

on  $B[Y, Z]$  as a subgroup of  $GL(r, B)$ , and absolute SL-invariants are defined analogously. The absolute invariants are just the invariants of the “general element” of  $GL(r, B)$  and  $SL(r, B)$  resp.:

**(7.3) Proposition.** *Let  $U$  be an  $r \times r$  matrix of indeterminates over  $B$ ,  $\Delta$  its determinant,  $S_1 = B[U][\Delta^{-1}]$ ,  $S_2 = B[U]/B[U](\Delta - 1)$ , and denote the matrix of residue classes in  $S_1$  by  $U$  again. Then  $f \in B[Y, Z]$  is an absolute GL-invariant if and only if it is (as an element of  $S_1[Y, Z]$ ) invariant under the action of  $U$  on  $S_1[Y, Z]$ . The analogous statement holds with GL replaced by SL and  $S_1$  by  $S_2$ .*

PROOF: Let  $S$  be a  $B$ -algebra,  $u \in \text{GL}(r, S)$ . Then one has a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & S_1 \\ & \searrow & \swarrow \psi \\ & & S \end{array}$$

such that  $\psi$  sends  $U$  to  $u$ . The action of  $u$  on  $S[Y, Z]$  restricts to an action on  $\psi(S_1)[Y, Z]$ , on which it is induced by the action of  $U$  on  $S_1[Y, Z]$ . Therefore invariants of  $U$  are mapped to invariants of  $u$ . The same argument works for SL. —

If the ring  $B$  has enough elements (units) then every invariant is already absolutely invariant.

**(7.4) Proposition.** *If  $B$  is a domain with infinitely many elements (units), then every invariant of  $\text{SL}(r, B)$  ( $\text{GL}(r, B)$ ) in  $B[Y, Z]$  is absolutely invariant.*

PROOF: We take  $S_2$  as in the preceding proposition. Let  $L$  be its field of fractions. (The verification that  $S_2$  is a domain is left to the reader.) For the contention regarding SL, it suffices now to show that every invariant in  $B[Y, Z]$  is invariant under the action of  $\text{SL}(r, L)$  on  $L[Y, Z]$ . The group  $\text{SL}(r, L)$  is generated by the elementary transformations  $E_{ij}(t)$ ,  $t \in L$ ,  $i \neq j$ , where  $E_{ij}(t)$  is the identity matrix except that its entry at position  $(i, j)$  is  $t$ . For  $t \in B$  we have  $E_{ij}(t) \in \text{SL}(r, B) (\subset \text{SL}(r, L)$  in a natural way). It is more than required if we show that every element of  $L[Y, Z]$  which is invariant under the actions of the  $E_{ij}(t)$ ,  $t \in B$ , is an invariant of  $\text{SL}(r, L)$ .

Let  $g \in L[Y, Z]$ ,  $g = \sum a_\mu \mu$ ,  $\mu$  running through the monomials in the indeterminates of  $Y$  and  $Z$ ,  $a_\mu \in L$ . Then

$$E_{ij}(t)(g) = \sum p_{ij\mu g}(t) \mu$$

with polynomials  $p_{ij\mu g}$  in one variable over  $L$ , as is easily checked. The invariance of  $g = \sum b_\mu \mu$  under  $E_{ij}(t)$ ,  $t \in B$ , is expressed by the equations

$$p_{ij\mu g}(t) = b_\mu$$

for all  $t \in B$ , all  $i, j, \mu$ . Since the polynomial  $p_{ij\mu g}$  takes the value  $b_\mu$  infinitely often, it has to be constant on  $L$ , so  $g$  is invariant under  $E_{ij}(t)$ ,  $t \in L$ .

In order to prove the statement about GL, we consider the field of fractions  $L$  of  $S_1$ ,  $S_1$  as in (7.3). The group  $\text{GL}(r, L)$  is generated by  $\text{SL}(r, L)$  and the matrices  $E_1(t)$ ,  $t \in L \setminus \{0\}$ , where  $E_1(t)$  is the identity matrix except having  $t$  in its position  $(1, 1)$ . As above every polynomial  $g$  defines functions  $q_{\mu g}(t)$ , sending  $t$  to the coefficient of  $E_1(t)(g)$  with respect to the monomial  $\mu$ . These functions are now rational functions defined on  $L \setminus \{0\}$ , each of them taking a constant value at the points  $t$  which are units in  $B$ , if  $g$  is an invariant of  $\text{GL}(r, B)$ . Therefore  $q_{\mu g}(t)$  is constant then. (Expressed very briefly, we have used that  $\text{SL}(r, L)$  and  $\text{GL}(r, L)$  are generated by one-dimensional subgroups in which the additive group and multiplicative group resp. of  $B$  are Zariski dense.) —

For the computation of the absolute SL-invariants of  $B[Y, Z]$  we need to know how they behave under the action of  $\text{GL}(r, B)$ .

**(7.5) Proposition.** *With the notations introduced, let  $f \in B[Y, Z]$  be an absolute SL-invariant which is bihomogeneous with respect to the indeterminates in  $Y$  and  $Z$  of partial degrees  $d_1$  and  $d_2$  resp. Then  $d_1 - d_2$  is a multiple of  $r$  (in  $\mathbf{Z}$ ),  $d_2 - d_1 = tr$ , and*

$$T(f) = (\det T)^t f$$

for every  $B$ -algebra  $S$  and every  $r \times r$  matrix  $T$  over  $S$ .

In invariant theory this is briefly expressed as:  $f$  is an absolute *semi-invariant* of weight  $t$  (or  $\det^t$ ).

PROOF: We consider the extension  $B \rightarrow S_1$  as in (7.3). It is enough to prove the contention for  $T = U$ . We further extend  $S_1$  to

$$S = S_1[W]/S_1[W](\Delta - W^r),$$

$W$  a new indeterminate. Over  $S$  the matrix  $U$  factors as

$$U = w(w^{-1}U),$$

$w$  denoting the residue class of  $W$ . Note that  $\det w^{-1}U = 1$ . Therefore

$$U(f) = (w(w^{-1}U))(f) = w(f) = f(Yw^{-1}, wZ) = w^{d_2-d_1}f.$$

$S$  is a free  $S_1$ -module with the basis  $1, \dots, w^{r-1}$ . Since  $U(f) \in S_1 = S_1 \cdot 1 \subset S$ , we conclude  $d_2 - d_1 \equiv 0 \pmod{r}$ . —

### C. The Main Theorem of Invariant Theory for GL and SL

Now we are well-prepared to state and to prove the theorem which describes the rings of the absolute GL- and SL-invariants of  $B[Y, Z]$ .

**(7.6) Theorem.** *Let  $B$  be a commutative ring,  $Y$  an  $m \times r$  matrix and  $Z$  an  $r \times n$  matrix of indeterminates,  $r, m, n \geq 1$ .*

(a) *The ring of absolute GL-invariants of  $B[Y, Z]$  is  $B[YZ] \cong \mathbf{R}_{r+1}(X)$ ,  $X$  being an  $m \times n$  matrix of indeterminates over  $B$ .*

(b) *The  $B$ -subalgebra  $A$  of absolute SL-invariants of  $B[Y, Z]$  is generated by the entries of  $YZ$ , the  $r$ -minors of  $Y$ , and the  $r$ -minors of  $Z$ .*

Conditions under which the attribute “absolute” can be omitted, are given in (7.4). For the determinantal rings mainly the case  $r < \min(m, n)$  is of interest. Under invariant-theoretic aspects this restriction should be avoided, and so we allow arbitrary values of  $m, n, r$  in (7.6). The  $B$ -algebra  $A$  in part (b) will be analyzed to some extent in (9.21).

As an immediate corollary we obtain  $G(X)$  as a ring of invariants:

**(7.7) Corollary.** *Let  $B$  be a commutative ring, and  $X$  an  $m \times n$  matrix of indeterminates over  $B$ . Then  $G(X)$  is the ring of absolute invariants under the action  $X \rightarrow TX$  of  $\mathrm{SL}(m, B)$  on  $B[X]$ .*

In fact, it is easy to see that  $A \cap B[Z] = G(Z)$ , and so (7.7) follows from (7.6),(b). Nevertheless we want to give a separate proof which, relative to our preparations, is very short. Its basic idea will be applied again in the proof of (7.6),(a).

PROOF OF (7.7): Certainly the elements of  $G(X)$  are absolutely invariant. One first observes that it is harmless to enlarge the matrix  $X$  by adding columns: If  $\widehat{X}$  is the “bigger” matrix, then the action of SL on  $B[X]$  is induced by that on  $B[\widehat{X}]$ , and obviously  $G(X) = B[X] \cap G(\widehat{X})$ . The action of SL leaves the homogeneous components of  $B[X]$  invariant. Therefore we may first assume  $n \geq m$  and secondly that a given invariant element  $f$  is homogeneous of degree  $d$ , say.

Let  $\widetilde{X}$  consist of the first  $m$  columns of  $X$ , and put  $U = \text{Cof } \widetilde{X}$ . Then by virtue of (7.5) (with  $d_1 = 0$ )  $d = tm$ ,  $t \geq 0$ , and

$$U(f) = (\det U)^t f.$$

On the other hand the entries of  $UX$  are elements of  $G(X)$ ! Furthermore  $\det U = (\det \widetilde{X})^{m-1} = [1, \dots, m]^{m-1}$ . Thus

$$[1, \dots, m]^{t(m-1)} f = U(f) = f(UX) \in G(X).$$

The rest is very easy for us (though it is certainly the difficult part of the proof from a neutral point of view):

$$B[X] = G(X) \oplus C$$

where  $C$  is the  $B$ -submodule generated by all standard monomials containing a factor outside  $\Gamma(X)$ . Since  $[1, \dots, m]$  is the minimal element of  $\Delta(X)$ , multiplication by it maps  $C$  injectively into itself, whence  $f \in G(X)$ . —

In the PROOF OF (7.6),(a) we use similar arguments. Enlarging  $m$  and  $n$  if necessary, we may assume that  $m > r$ ,  $n > r$ . In order to prove the nontrivial inclusion, it is enough to consider invariants  $f$  which are homogeneous with respect to the variables in  $Y$ , of degree  $d$ , say. Let  $\widetilde{Y}$  denote the submatrix of  $Y$  consisting of the first  $r$  rows,  $\widetilde{Z}$  the submatrix of  $Z$  formed from the first  $r$  columns. Over  $B[Y, Z][(\det \widetilde{Y}\widetilde{Z})^{-1}]$  the absolute invariance of  $f$  implies

$$f = f(Y\widetilde{Y}^{-1}, \widetilde{Y}Z),$$

so by elementary matrix algebra

$$\begin{aligned} f &= f(Y\widetilde{Z}(\widetilde{Y}\widetilde{Z})^{-1}, \widetilde{Y}Z) \\ &= f(Y\widetilde{Z}(\det \widetilde{Y}\widetilde{Z})^{-1} \text{Cof}(\widetilde{Y}\widetilde{Z}), \widetilde{Y}Z) \\ &= (\det \widetilde{Y}\widetilde{Z})^{-d} f(Y\widetilde{Z} \text{Cof}(\widetilde{Y}\widetilde{Z}), \widetilde{Y}Z). \end{aligned}$$

The entries of  $Y\widetilde{Z}$ ,  $\text{Cof}(\widetilde{Y}\widetilde{Z})$ ,  $\widetilde{Y}Z$  all are in  $B[YZ]$ . Thus one has

$$(\det \widetilde{Y}\widetilde{Z})^d f \in B[YZ],$$

and it suffices to prove

$$(1) \quad (\det \widetilde{Y}\widetilde{Z})B[YZ] = (\det \widetilde{Y}\widetilde{Z})B[Y, Z] \cap B[YZ].$$

This is equivalent to the injectivity of the homomorphism

$$\varphi: R/R\delta \longrightarrow B[Y, Z]/\det(\widetilde{Y}\widetilde{Z})B[Y, Z],$$

$R = R_{r+1}(X)$ ,  $\delta = [1, \dots, r|1, \dots, r]$ ,  $\varphi$  induced by the embedding  $R \rightarrow B[Y, Z]$  as above. By virtue of (6.6) the element  $\delta' = [1, \dots, r-1, r+1|1, \dots, r-1, r+1]$  is not a zero-divisor modulo  $\delta$ , since it is greater than the upper neighbours  $[1, \dots, r-1, r+1|1, \dots, r]$  and  $[1, \dots, r|1, \dots, r-1, r+1]$  of  $\delta$ . Therefore the natural map  $R/R\delta \rightarrow (R/R\delta)[\delta'^{-1}]$  is an injection. It can be factored through  $\varphi$  since the image of the matrix  $X$  in  $(R/R\delta)[\delta'^{-1}]$  can be factored into a product of an  $m \times r$  matrix and an  $r \times n$  matrix. This finishes the proof of (7.6),(a).

Before embarking on the proof of (7.6),(b), we want to point out that (7.6),(b) is equivalent to ideal-theoretic properties of  $R_{r+1}(X)$ . This is already true for (7.6),(a): we have used such a property in order to prove (7.6),(a); cf. also the remark following the proof of (7.8). Some notations have to be introduced. Let

$$\tilde{P} = (\det \tilde{Y})B[Y, Z] \quad \text{and} \quad \tilde{Q} = (\det \tilde{Z})B[Y, Z],$$

$P$  be the ideal generated by the  $r$ -minors of the first  $r$  rows of  $YZ$ ,  $Q$  the corresponding ideal for the first  $r$  columns.

**(7.8) Lemma.** *Let  $m > r$  and  $n > r$ . Then the following are equivalent:*

- (a) (7.6), (b).
- (b)  $P^j = \tilde{P}^j \cap B[YZ]$  and  $Q^j = \tilde{Q}^j \cap B[YZ]$  for all  $j \geq 1$ .
- (c)  $P^j$  and  $Q^j$  are primary with radicals  $P$  and  $Q$  resp. for all  $j \geq 1$ , provided  $B$  is an integral domain.
- (d)  $[1, \dots, r-1, r+1|1, \dots, r]$  is not a zero-divisor modulo  $P^j$ ,  $[1, \dots, r, |1, \dots, r-1, r+1]$  is not a zero-divisor modulo  $Q^j$  for all  $j \geq 1$ .

PROOF OF (7.8): (a)  $\Rightarrow$  (b): All the  $B$ -submodules appearing in (b) are bihomogeneous in the bigraded  $B$ -module  $B[Y, Z]$ , the first graduation taken with respect to  $Y$ , and the second one with respect to  $Z$ . Let  $x \in \tilde{P}^j \cap B[YZ]$  be homogeneous (thus bihomogeneous of partial degrees  $d_1 = d_2$ ),  $x = p^j y$ ,  $p = \det \tilde{Y}$ ,  $y \in B[Y, Z]$  bihomogeneous. Then  $y$  is an absolute SL-invariant, and

$$T(y) = (\det T)^j y$$

for all matrices  $T$ . Since the product of an  $r$ -minor of  $Y$  and an  $r$ -minor of  $Z$  is in  $B[YZ]$ , (7.6),(b) implies that  $y$  can be written as a linear combination of (standard) monomials of length  $j$  in the  $r$ -minors of  $Z$  with coefficients in  $B[YZ]$ . Multiplied by  $p^j$ , such a monomial is sent into  $P^j$ . The statement on the powers of  $Q$  is proved similarly.

(b)  $\Rightarrow$  (c): Obvious, since the powers of principal primes are primary.

(c)  $\Rightarrow$  (d): (c) implies that  $\mathbf{Z}[YZ]/P^j$  and  $\mathbf{Z}[YZ]/Q^j$  are  $\mathbf{Z}$ -flat, and (3.15) reduces (d) to the case of a field  $B = K$ , in which (d) is a trivial consequence of (c).

(d)  $\Rightarrow$  (b): This is proved in a similar fashion as equation (1) above.

(b)  $\Rightarrow$  (a): Without restriction one may assume that a given absolute SL-invariant  $f$  is bihomogeneous of partial degrees  $d_1$  and  $d_2$  resp. We discuss the case  $d_2 \geq d_1$ , the case  $d_1 \geq d_2$  being analogous. By virtue of (7.5):  $d_2 - d_1 = tr$ ,  $t \in \mathbf{Z}$ ,  $t \geq 0$ . Let  $p = \det \tilde{Y}$  as above. Obviously  $p^t f$  is an absolute GL-invariant, so  $p^t f \in \tilde{P}^t \cap B[YZ] = P^t$ . Write  $p^t f$  as a linear combination of (standard) monomials of length  $t$  in the  $r$ -minors of the first  $r$  rows of  $YZ$  with coefficients in  $B[YZ]$ , and note that such an  $r$ -minor divided by  $p$  gives an  $r$ -minor of  $Z$ . —

It is not difficult to see that (7.6),(a) is equivalent to (b), (c), (d) of (7.8) with  $j = 1$ , and via (6.6) we have derived (7.6),(a) from the fact that  $P$  and  $Q$  are prime ideals over a domain.

We shall prove independently in Section 9 that  $P$  and  $Q$  have primary powers over a domain, and a reference to (9.18) would be the shortest proof of (7.6),(b). A more direct argument is in order, however.

**(7.9) Lemma.** (a) *Let  $f \in B[YZ]$  be homogeneous of degree 1 with respect to the indeterminates in the  $j$ -th row of  $Y$ . Then the row  $j$  appears exactly once in every standard monomial in the standard representation  $f = \sum a_\mu \mu$  of  $f$  as an element of  $R_{r+1}(X)$ .*

(b) *Let  $t \in \mathbf{Z}$ ,  $t \geq 1$ , and suppose  $m \geq tr$ . Let  $f \in B[YZ]$  satisfy the hypothesis of (a) for each  $j$ ,  $1 \leq j \leq tr$ , and assume that  $f$  vanishes after the substitution of linearly dependent vectors (over a  $B$ -algebra) for the rows  $(k-1)r+1, \dots, kr$ ,  $1 \leq k \leq t$  arbitrary. Then in the standard representation of  $f$  the first  $r$  factors of each standard monomial have row parts*

$$[1, \dots, r], [r+1, \dots, 2r], \dots, [(t-1)r+1, \dots, tr]$$

*and none of the remaining factors contains a row  $j$ ,  $1 \leq j \leq tr$ .*

PROOF OF (7.9): Part (a) is almost trivial: multiply the  $j$ -th row of  $Y$  by a new indeterminate  $W$ , and use the linear independence of the standard monomials over  $B[W]$ . Under the hypothesis of (b)  $f$  vanishes modulo  $P$  (as in (7.8)), which is generated by a poset ideal of the poset underlying  $R_{r+1}(X)$ . Therefore every standard monomial in the standard representation of  $f$  has a minor  $[1, \dots, r | \dots]$  as its first factor. Splitting it off, one can argue inductively because of (a). —

PROOF OF (7.6),(b): Without restriction let  $m > r$ ,  $n > r$ , and  $f \in B[Y, Z]$  be a bihomogeneous absolute SL-invariant of partial degrees  $d_1$  and  $d_2$  resp. Suppose that  $d_2 \geq d_1$  and let  $t$  be given by (7.5).

So far we have only repeated the first lines of the proof of (7.8),(b)  $\Rightarrow$  (a). The essential trick now is the introduction of a new  $tr \times r$  matrix of indeterminates which we pile on top of  $Y$  such that the resulting matrix  $\widehat{Y}$  has  $Y$  in its rows  $tr+1, \dots, tr+m$ . Let  $y_k$  be the determinant of the matrix consisting of the rows  $(k-1)r+1, \dots, kr$  of  $\widehat{Y}$ . The element

$$g = f y_1 \dots y_t$$

is an absolute GL-invariant because of (7.5), and we can apply (7.9),(b) to it. Since

$$[(k-1)r, \dots, kr | b_1, \dots, b_r] / y_k$$

is the  $r$ -minor of the columns  $b_1, \dots, b_r$  of  $Z$ , the result follows after division of  $g$  by  $y_1 \dots y_t$ . —

In the proof of (7.8) the hypothesis “ $m > r$  and  $n > r$ ” is only needed for (d) and the implications (c)  $\Rightarrow$  (d) and (d)  $\Rightarrow$  (b). Therefore (7.6),(b) also implies the first part of the following corollary whose second part follows directly from (7.8):

**(7.10) Corollary.** *Let  $B$  be an integral domain,  $X$  an  $m \times n$  matrix of indeterminates,  $m \leq n$ .*

(a) *The prime ideal  $I_m(X)$  has primary powers.*

(b) *Let  $r < \min(m, n)$ . Then the prime ideals  $P$  and  $Q$  generated by the  $r$ -minors of any  $r$  rows and any  $r$  columns resp. of the matrix of residue classes in  $R_{r+1}(X)$  have primary powers.*

### D. Remarks on Invariant Theory

In “classical” invariant theory one considers a group  $G$  of linear transformations on the vector space  $K^p$ ,  $K$  a field, preferably  $K = \mathbf{C}$ , and wants to compute explicitly the polynomial functions  $f$  in  $p$  variables which satisfy the equation

$$f(x) = f(g(x)) \quad \text{for all } g \in G, x \in K^p,$$

and are therefore called *invariants*. The *first main problem* is to determine a finite set  $f_1, \dots, f_q$  of “basic” invariants, i.e. invariants  $f_1, \dots, f_q$  such that every invariant is a polynomial in  $f_1, \dots, f_q$ . (A paradigm for the solution of the first main problem is Newton’s theorem on symmetric functions.) The *second main problem* is solved if one has found all the relations of  $f_1, \dots, f_q$ , a relation being a polynomial  $h$  in  $q$  variables such that  $h(f_1, \dots, f_q) = 0$ .

In modern language  $G$  is a linear algebraic group over a (algebraically closed) field  $K$ , and  $G$  operates on a finite dimensional  $K$ -vector space  $V$  via a morphism or an antimorphism  $G \rightarrow \text{GL}(V)$  of linear algebraic groups (cf. [Hm], [Fo], [Kr], [MF]; the survey [Ho.8] suffices for our purpose). Such a morphism is called a *rational representation* of  $G$ . It makes  $V$  a  $G$ -module; more generally an arbitrary vector space  $W$  is a  $G$ -module if it is the union of an ascending chain of finite dimensional  $G$ -modules. The ring of polynomial functions on  $V$  is the symmetric algebra  $S(V^*)$ .  $G$  acts on  $V^*$  via the composition of the representation  $G \rightarrow \text{GL}(V)$  and the natural anti-isomorphism  $\text{GL}(V) \rightarrow \text{GL}(V^*)$ , sending each automorphism of  $V$  to its dual. Then  $S(V^*)$  becomes a  $G$ -module after the natural extension of the action on  $V^*$  to an action of  $S(V^*)$ : every automorphism of  $V^*$  induces an algebra automorphism of  $S(V^*)$ . In (7.6) and (7.7) we have let  $\text{SL}(r, B)$  and  $\text{GL}(r, B)$  operate directly on the space of 1-forms of a symmetric algebra of a free module over  $B$ . These theorems comprise the solutions of the first main problem for the actions under consideration. In the situation of (7.6),(a) and (7.7) the solution of the second main problem is also well-known to us.

From a geometric view-point  $V$  is the affine  $n$ -space over  $K$ ,  $A = S(V^*)$  is its coordinate ring,  $G$  acts on the affine variety  $V$ . The ring  $A^G$  of invariants is the subalgebra of functions constant on the orbits of the action of  $G$ . The first main problem has a solution if and only if  $A^G$  can be considered the coordinate ring of an affine variety  $\tilde{V}$ . Then the surjection  $V \rightarrow \tilde{V}$  has a universal property: every morphism defined on  $V$  which is constant on the orbits, factors through  $\tilde{V}$ . Thus  $\tilde{V}$  comes as close as possible to being the quotient of  $V$  modulo  $G$ . It is therefore called the *algebraic quotient* of  $V$  with respect to  $G$ , whereas the geometric quotient may not exist: there may be nonclosed orbits.

All this explains the significance of (7.6) and (7.7) for invariant theory. Conversely we can use the results of (algebraic and geometric) invariant theory to gain further knowledge about our objects. This is mainly possible in characteristic zero because the groups  $\text{GL}(n, K)$  and  $\text{SL}(n, K)$  (and direct products of them) are linearly reductive then, and very strong theorems hold for invariants of linearly reductive groups. Linear reductivity can be characterized by each of the following properties:

(i) *Every (finite dimensional)  $G$ -module is completely reducible, i.e. the direct sum of simple  $G$ -modules (motivating the name “reductive”).*



(ii) *In every (finite dimensional)  $G$ -module  $V$  the  $G$ -submodule  $V^G = \{x \in V : g(x) = x \text{ for all } g \in G\}$  of invariants has a (for  $V^G$  necessarily unique)  $G$ -complement. (The  $G$ -homomorphism  $\rho : V \rightarrow V^G$ ,  $\rho|_{V^G} = \text{id}$ , is called the Reynolds operator.)*

(iii) *For every surjective  $G$ -homomorphism  $V \rightarrow W$  the induced map  $V^G \rightarrow W^G$  is surjective, too.*

We now assume that  $A$  is a finitely generated  $K$ -algebra and a  $G$ -module such that the elements of  $G$  act as  $K$ -algebra automorphisms. Then (a) is quite evident:

(a) *Let  $A$  be a domain. Then  $A^G$  is the intersection of its own field of fractions with  $A$ . In particular  $A^G$  is normal if  $A$  is normal (and  $A^G$  noetherian).*

Suppose furthermore that  $G$  is linearly reductive. Then the first main problem always has a solution:

(b) *If  $A$  is noetherian, then  $A^G$  is noetherian; if  $A$  is a finitely generated  $K$ -algebra, then  $A^G$  is finitely generated.*

We should point out that (b) already holds under the weaker assumption that  $G$  is reductive; cf. [Ho.8] for this notion. In characteristic 0 reductivity and linear reductivity are equivalent, whereas in positive characteristic the groups  $\text{GL}(r, K)$  and  $\text{SL}(r, K)$  are not linearly reductive if  $r \geq 2$ . Property (ii) of linearly reductive groups implies:

(c) *As an  $A^G$ -module  $A$  splits as  $A = A^G \oplus C$ ,  $C$  being the  $G$ -complement of  $A^G$ ; the Reynolds operator is an  $A^G$ -homomorphism. (Cf. [Fo], p. 156, Lemma 5.4 or (7.22) below).*

The deep properties (d) and (e) of linearly reductive groups are given by the theorem of Hochster-Roberts [HR], [Ke.5] and the even stronger and more general theorem of Boutot [Bt] resp.:

(d) *If  $A$  is regular, then  $A^G$  is Cohen-Macaulay.*

(e) *If  $\text{char } K = 0$  and  $A$  has rational singularities, then  $A^G$  has rational singularities.*

We cannot discuss the notion of rational singularity here and refer the reader to [KKMS] and [BS]. If  $A$  has rational singularities, then it is Cohen-Macaulay. We shall see below that  $G(X; \gamma)$  and  $R(X; \delta)$  are invariants of groups acting on polynomial rings, the groups being reductive in characteristic zero. Thus we conclude:

**(7.11) Theorem.** *Let  $B = K$  be an algebraically closed field of characteristic zero. Then the affine and projective varieties corresponding to  $G(X; \gamma)$  and  $R(X; \delta)$  have rational singularities for all  $\gamma \in \Gamma(X)$ ,  $\delta \in \Delta(X)$ .*

Even (a) above contains some new information about  $G(X)$ , say. An application of (c) is discussed in the following remark:

**(7.12) Remark.** In (5.21) we have given an upper bound for the arithmetical rank of the ideals  $I_t(X)$ :  $\text{ara } I_t(X) \leq mn - t^2 + 1$ . Here we want to demonstrate that this bound is sharp in case  $t = m \leq n$  if  $B$  admits a homomorphism  $B \rightarrow K$  to a field of characteristic zero. Evidently we may then assume that  $B = K$ . A lower bound of  $\text{ara } I_m(X)$  is supplied by the cohomological dimension

$$\min\{i : H_I^i(K[X]) \neq 0\}, \quad I = I_m(X),$$

cf. [Ha.1], p. 414, Example 2. Here  $H_I^i(\dots)$  is the cohomology with support in  $I$ . Let  $J = I \cap G(X)$ . Then, for the  $G(X)$ -algebra  $K[X]$

$$H_I^i(K[X]) = H_J^i(K[X]) = H_J^i(G(X)) \oplus H_J^i(\tilde{C}),$$

$\tilde{C}$  being the  $\mathrm{SL}(m, K)$ -complement of  $G(X)$  in  $K[X]$  (observe that  $I = JK[X]$ ). For  $i = mn - m^2 + 1$  we have  $H_J^i(G(X)) \neq 0$ , since  $i = \dim G(X)$  (cf. [HK], p. 37–39). Warning: Apart from trivial cases,  $\tilde{C}$  is not the  $K$ -vector space complement  $C$  appearing in the proof of (7.7). It seems hopeless to compute  $\tilde{C}$  explicitly.

The preceding argument can neither be generalized to the case  $t < \min(m, n)$  (cf. (10.16)), nor be applied in characteristic  $p > 0$  if  $t = m < n$ : By virtue of [PS], Proposition (4.1), p. 110 one has  $H_J^i(K[X]) = 0$  for all  $i \geq \mathrm{ht} I$ , in particular for  $i = mn - m^2 + 1$ , and the argument based on cohomological dimension breaks down. Another consequence: A Reynolds operator does not exist! —

**(7.13) Remark.** Let  $K$  be an algebraically closed field and consider the action of  $\mathrm{SL}(m, K)$  on the  $mn$ -dimensional affine space  $V$  of matrices as in (7.7). It follows directly from Theorem (1.2) that the points  $\neq 0$  in the affine variety  $G$  with coordinate ring  $G(X)$  (embedded into  $\mathbf{A}^N$ ,  $N = \binom{n}{m}$ ) correspond bijectively to the orbits of  $\mathrm{SL}(m, K)$  containing a matrix of rank  $m$ . This fact indicates that  $G$  comes close to being the algebraic quotient of  $V$  with respect to the action of  $\mathrm{SL}(m, K)$ , and one is justified to ask whether Theorem (1.2) does already prove (7.7). It does so, provided one has shown the normality of  $G(X)$ , because of the following criterion (cf. [Kr], 3.4, p. 105 for the statement in characteristic 0): *Let  $V$  be an irreducible affine algebraic variety,  $G$  a reductive group acting on  $V$ , and  $\pi: V \rightarrow W$  a surjective morphism from  $V$  to a normal affine variety  $W$ , which is constant on the orbits. Suppose that  $W$  contains a dense subset  $U$  such that  $\pi^{-1}(v)$  contains exactly one closed orbit for every  $v \in U$ . Then  $W$  is the algebraic quotient of  $V$  with respect to  $G$ .* (The reductivity of  $G$  guarantees the a priori existence of a quotient, and the normality of  $W$  then allows one to conclude that it is isomorphic to  $W$ .) It is not difficult to prove (7.6) by means of this criterion (cf. [Kr], 4.1 for  $\mathrm{GL}(r, K)$ ); the normality of the algebra  $A$  in (7.6),(b) will be proved in (9.21) independently. —

We now proceed to give invariant-theoretic descriptions of the rings  $G(X; \gamma)$  and  $R(X; \delta)$  in general. The arguments needed consist of iterative applications of the ideas underlying the proofs of (7.6) and (7.7). We start by giving the “classical” generic point for  $G(X; \gamma)$ .

### E. The Classical Generic Point for $G(X; \gamma)$

Let  $\varphi: B[X] \rightarrow S$  be a  $B$ -algebra homomorphism,  $U$  the image of the matrix  $X$  in  $S$ . Then the induced homomorphism  $G(X) \rightarrow S$  factors through  $G(X; \gamma)$  if and only if  $\varphi(\delta) = 0$  for all  $\delta \not\geq \gamma = [a_1, \dots, a_m]$ . So we can hope to find a generic point for  $G(X; \gamma)$  if we choose for  $U$  a “generic” matrix for which the minors  $\delta \not\geq \gamma$  vanish. This is certainly true, if  $I_k$  (first  $a_k - 1$  columns of  $U$ ) = 0 for  $k = 1, \dots, m$ . Thus let  $U_\gamma$  be the following matrix whose entries  $U_{ij}$  are indeterminates over  $B$ :

$$\begin{pmatrix} 0 & \cdots & 0 & U_{1a_1} & \cdots & U_{1a_2-1} & U_{1a_2} & \cdots & U_{1a_3-1} & \cdots & U_{1a_m} & \cdots & U_{1n} \\ & & & 0 & \cdots & 0 & U_{2a_2} & \cdots & U_{2a_3-1} & & & & \\ & & & & & & 0 & \cdots & 0 & \cdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & U_{ma_m} & \cdots & U_{mn} \end{pmatrix}$$

**(7.14) Theorem.** (a) *The  $B$ -algebra homomorphism  $B[X] \rightarrow B[U_\gamma]$ ,  $X \rightarrow U_\gamma$ , induces an embedding  $\varphi: G(X; \gamma) \rightarrow B[U_\gamma]$ , thus an isomorphism  $G(X; \gamma) \cong G(U_\gamma)$ .*  
 (b) *The embedding  $G(X) \rightarrow B[X]$  is a generic point, as is  $\varphi$  for every  $\gamma \in \Gamma(X)$ .*

We give two PROOFS OF PART (a), the second one being contained in Remark (7.16). The first proof is more “advanced”: we use that  $\gamma$  is not a zero-divisor of  $G(X; \gamma)$ . It runs like that of (7.2): we only need to factor the embedding  $G(X; \gamma) \rightarrow G(X; \gamma)[\gamma^{-1}]$  through  $\varphi$ . In order to find such a factorization we have to construct a matrix  $V$  over  $G(X; \gamma)[\gamma^{-1}]$  which has the same shape as  $U_\gamma$  and whose  $m$ -minors are the elements  $\delta \in \Gamma(X; \gamma)$ . Such a problem we have faced already: the construction of a subspace (or matrix) with given Plücker coordinates is the last step in the proof of Theorem (1.2)!

**(7.15) Lemma.** *Let  $S$  be a  $B$ -algebra, and  $\psi: G(X; \gamma) \rightarrow S$  a  $B$ -algebra homomorphism. Suppose that  $\psi(\gamma)$  is a unit in  $S$ . Then there is a matrix  $V$  of the same shape as  $U_\gamma$  such that  $\psi(\delta)$  is the minor of  $V$  with the same columns as  $\delta$  for all  $\delta \in \Gamma(X; \gamma)$ .*

PROOF: The key role plays the set  $\Psi$  defined in (6.1):

$$\Psi = \{ \delta = [d_1, \dots, d_m] \in \Gamma(X; \gamma) : a_i \notin [d_1, \dots, d_m] \text{ for at most one } i \}.$$

First we let all those entries of  $V$  be zero which correspond to zero entries of  $U_\gamma$ . The remaining entries at positions  $(k, l)$  are defined as follows: Remove  $a_k$  from  $\{a_1, \dots, a_m\}$  and replace it by  $l$ . If  $l = a_j$  for some  $j \neq k$ , the entry is zero. Otherwise

$$\{a_1, \dots, a_{k-1}, l, a_{k+1}, \dots, a_m\}$$

defines, after arrangement in ascending order, an element  $\delta$  of  $\Psi$ . Then we take

$$\begin{aligned} & \sigma(l, a_2, \dots, a_m)\psi(\delta) && \text{if } k = 1, \\ & \sigma(a_1, \dots, a_{k-1}, l, a_{k+1}, \dots, a_m)\psi(\delta)\psi(\gamma)^{-1} && \text{if } k \neq 1, \end{aligned}$$

as the entry of  $V$ . One checks that the minor with the same columns as  $\delta$  equals  $\psi(\delta)$  for all  $\delta \in \Psi$ .

We now have two homomorphisms  $G(X; \gamma) \rightarrow S$ : first  $\psi$ , and secondly the composition of  $\varphi: G(X; \gamma) \rightarrow B[U_\gamma]$  with the homomorphism  $B[U_\gamma] \rightarrow S$  arising from the substitution  $U_\gamma \rightarrow V$ . Since they coincide on  $\Psi$ , they are equal, cf. (6.2), and the second homomorphism sends  $\delta \in \Gamma(X; \gamma)$  to the minor of  $V$  with the same columns as  $\delta$ . —

For the PROOF OF (7.14),(b) we first show that  $G(X) \rightarrow B[X]$  is a generic point. Let  $\psi: G(X) \rightarrow L$  be a homomorphism to a field  $L$ . If  $\psi(\delta) = 0$  for all  $\delta \in \Gamma(X)$ , then  $\psi$  factors through  $B[X]$  for trivial reasons. Otherwise we may assume on the grounds of symmetry that  $\psi([1, \dots, m]) \neq 0$ , and then (7.15) settles the problem.

Let now  $\gamma \in \Gamma(X)$  be arbitrary, and  $\psi: G(X; \gamma) \rightarrow L$  again a homomorphism to a field. By what has just been shown, there is a matrix  $V$  such that the minor of  $V$  with the same columns as  $\delta$  is  $\psi(\delta)$  for all  $\delta \geq \gamma$ , and zero otherwise. Over a field such a matrix can be transformed into one of shape  $U_\gamma$  by an application of elementary row operations. —

**(7.16) Remark.** The second proof of (7.14),(a) is given mainly because it provides a new (and perhaps simpler) demonstration of the linear independence of the standard monomials in  $G(X)$ . We choose new notations: Let  $\tilde{G}(X)$  be the residue class ring of the polynomial ring  $B[T_\gamma: \gamma \in \Gamma(X)]$  modulo the ideal generated by the Plücker relations, and  $\tilde{G}(X; \gamma)$  the residue class ring of  $\tilde{G}(X)$  with respect to the ideal generated by the residue classes of the  $T_\delta, \delta \not\geq \gamma$ . Then we have a homomorphism  $\tilde{G}(X; \gamma) \rightarrow G(U_\gamma)$  since the maximal minors of  $U_\gamma$  satisfy the defining relations of  $\tilde{G}(X; \gamma)$ . Furthermore it follows as in the proof of (4.1) that  $\tilde{G}(X; \gamma)$  is generated as a  $B$ -module by the standard monomials in the residue classes of  $T_\delta, \delta \geq \gamma$ . In order to show that the homomorphism  $\tilde{G}(X; \gamma) \rightarrow G(U_\gamma)$  is an isomorphism it is enough to prove that the standard monomials in the maximal minors  $\delta$  of  $U_\gamma, \delta \geq \gamma$ , are linearly independent! This is done by descending induction in the partially ordered set  $\Gamma(X)$ . Suppose  $0 = \sum b_\mu \mu$  where  $\mu$  runs through these standard monomials,  $b_\mu \in B, b_\mu = 0$  for all but a finite number of standard monomials. Let  $\delta > \gamma$ . The matrix  $U_\delta$  has nonzero entries only where  $U_\gamma$  has indeterminate entries. So we have a well defined substitution  $U_\gamma \rightarrow U_\delta$  inducing a commutative diagram

$$\begin{array}{ccc} \tilde{G}(X; \gamma) & \longrightarrow & \tilde{G}(X; \delta) \\ \downarrow & & \downarrow \\ G(U_\gamma) & \longrightarrow & G(U_\delta). \end{array}$$

By induction hypothesis we conclude  $b_\mu = 0$  for all  $\mu$  not containing  $\gamma$  as a factor. But  $\gamma \in \Gamma(U_\gamma)$  is a product of indeterminates, so certainly not a zero-divisor, and this implies at once that  $b_\mu = 0$  for all  $\mu$  after a second application of the inductive hypothesis: (7.14),(a) is proved again. —

**F.  $G(X; \gamma)$  and  $R(X; \delta)$  as Rings of Invariants**

Multiplication of  $U_\gamma$  by an element of the special linear group does not define an automorphism of  $B[U_\gamma]$  in general. In order to represent  $G(X; \gamma)$  as a ring of invariants we must “symmetrize” the matrix  $U_\gamma$  first. Let  $a_0 = 1, a_{m+1} = n + 1$ , and

$$Z_k = \begin{pmatrix} Z_{1a_k} & \cdots & Z_{1a_{k+1}-1} \\ \vdots & & \vdots \\ Z_{ka_k} & \cdots & Z_{ka_{k+1}-1} \end{pmatrix},$$

$k = 0, \dots, m$ , matrices of indeterminates (as they appear as submatrices of  $U_\gamma$ ). For  $k = 0, \dots, m - 1$  we choose  $(k + 1) \times k$ -matrices  $\tilde{Z}_k$  of indeterminates such that the entries of all the matrices  $Z_k, \tilde{Z}_k$  are algebraically independent over  $B$ . Then we let

$$Z_\gamma = \left( \tilde{Z}_{m-1} \cdots \tilde{Z}_0 Z_0 \mid \tilde{Z}_{m-1} \cdots \tilde{Z}_1 Z_1 \mid \cdots \mid \tilde{Z}_{m-1} Z_{m-1} \mid Z_m \right)$$

by iuxtaposing the products  $\tilde{Z}_{m-1} \cdots \tilde{Z}_k Z_k$  as indicated to form the  $m \times n$  matrix  $Z_\gamma$ . It is clear that

$$(*) \quad I_k(\text{first } a_k - 1 \text{ columns of } Z_\gamma) = 0$$

for  $k = 1, \dots, m$ . Therefore the substitution  $X \rightarrow Z_\gamma$  induces a homomorphism

$$\chi: G(X; \gamma) \longrightarrow G(Z_\gamma) \subset B[\widehat{Z}_\gamma],$$

$\widehat{Z}_\gamma$  denoting the collection of all the entries of the  $Z_k, \widetilde{Z}_k$ . It also induces a homomorphism

$$\omega: R(X; [1, \dots, m|a_1, \dots, a_m]) \longrightarrow B[Z_\gamma] \subset B[\widehat{Z}_\gamma].$$

**(7.17) Proposition.** *The homomorphisms*

$$\chi: G(X; \gamma) \longrightarrow B[\widehat{Z}_\gamma] \quad \text{and} \quad \omega: R(X; [1, \dots, m|a_1, \dots, a_m]) \longrightarrow B[\widehat{Z}_\gamma]$$

are embeddings and generic points.

PROOF: Substituting the corresponding submatrix of  $U_\gamma$  for  $Z_k$  and the  $(k+1) \times k$  matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

for  $\widetilde{Z}_k$ , one factors the embedding  $G(X; \gamma) \rightarrow B[U_\gamma]$  through  $\chi$  to get the claim for  $\chi$ . As soon as  $\gamma$  is invertible, or over a field, a matrix to which  $X$  (considered over  $R(X; \gamma)$ ) specializes, can be “decomposed” in the same way as  $Z_\gamma$ . (It is of course only a problem of elementary linear algebra to find such a decomposition; for special reasons we shall however have to outline the construction of a decomposition in Section 12, proof of (12.3).) —

We introduce group actions on  $B[\widehat{Z}_\gamma]$ . Let

$$H = \prod_{k=0}^{m-1} \text{GL}(k, B).$$

The group  $\text{GL}(k, B)$  acts on  $B[\widehat{Z}_\gamma]$  by the substitution

$$\begin{aligned} Z_k &\longrightarrow TZ_k, \\ \widetilde{Z}_k &\longrightarrow \widetilde{Z}_k T^{-1}, \\ \widetilde{Z}_{k-1} &\longrightarrow T \widetilde{Z}_{k-1} \quad (k > 0), \end{aligned}$$

$T \in \text{GL}(k, B)$ . These actions for various  $k$  commute with each other; so they define an action of  $H$  on  $B[\widehat{Z}_\gamma]$ . Finally we let the group  $\text{SL}(m, B)$  act by

$$\begin{aligned} \widetilde{Z}_{m-1} &\longrightarrow T \widetilde{Z}_{m-1}, \\ Z_m &\longrightarrow TZ_m, \end{aligned}$$

giving an action of  $\overline{H} = \text{SL}(m, B) \times H$  on  $B[\widehat{Z}_\gamma]$ .

**(7.18) Theorem.** (a)  $B[Z_\gamma] \cong \mathbf{R}(X; [1, \dots, m|a_1, \dots, a_m])$  is the ring of absolute  $H$ -invariants of  $B[\widehat{Z}_\gamma]$ .

(b)  $\mathbf{G}(Z_\gamma) \cong \mathbf{G}(X; \gamma)$  is the ring of absolute  $\overline{H}$ -invariants of  $B[\widehat{Z}_\gamma]$ .

For this theorem one of course extends the definition of absolute invariants given above. It is obvious that the Propositions (7.3), (7.4), and (7.5) hold again after the necessary modifications. For a quick proof of (7.18) in characteristic 0 see (7.21) below.

PROOF: Part (a) is proved by induction. It is evident for  $m = 1$ . Let  $m > 1$ . The indeterminates in  $Z_m$  are not affected by the action of  $H$ . Therefore it is enough to show that the entries of

$$(\widetilde{Z}_{m-1} \dots \widetilde{Z}_0 Z_0 | \dots | \widetilde{Z}_{m-1} Z_{m-1}) = \widetilde{Z}_{m-1} Z_{\gamma'},$$

where  $\gamma' = [a_1, \dots, a_{m-1}]$ , generate the ring of absolute invariants after restricting the action of  $H$  to the polynomial ring in the entries of  $\widetilde{Z}_k, Z_k, k = 0, \dots, m-1$ . Let  $H' = \prod_{k=0}^{m-2} \mathbf{GL}(k, B)$ . By induction the ring of absolute invariants of  $H'$  is  $B[Z_{\gamma'}]$  and the action of  $H$  can be restricted to  $B[\widetilde{Z}_{m-1}, Z_{\gamma'}]$ . Therefore it is now sufficient to show that the ring of absolute invariants of  $B[\widetilde{Z}_{m-1}, Z_{\gamma'}]$  under the action of  $\mathbf{GL}(m-1, B)$  is  $B[\widetilde{Z}_{m-1} Z_{\gamma'}]$ .

The rest of the proof is mainly a repetition of the arguments given for (7.6),(a). First we may enlarge  $Z_{m-1}$  by adding a further column of indeterminates at the right to reach a situation in which the number of columns of  $Z_{\gamma'}$  exceeds  $a_{m-1}$ . One now inverts the minor  $\delta' = [1, \dots, m-1|a_1, \dots, a_{m-1}]$  of  $\widetilde{Z}_{m-1} Z_{\gamma'}$  and applies the substitution trick with  $\widetilde{Y} \widetilde{Z}$  replaced by the product of the submatrix consisting of the first  $m-1$  rows of  $\widetilde{Z}_{m-1}$  with the submatrix consisting of columns  $a_1, \dots, a_{m-1}$  of  $Z_{\gamma'}$ . Then one is left to prove that

$$\delta' B[\widetilde{Z}_{m-1} Z_{\gamma'}] = \delta' B[\widetilde{Z}_{m-1}, Z_{\gamma'}] \cap B[\widetilde{Z}_{m-1} Z_{\gamma'}],$$

and this can also be done in analogy with (7.6), this time  $[1, \dots, m-1|a_1+1, \dots, a_{m-1}+1]$  being inverted instead of  $[1, \dots, r-1, r+1|1, \dots, r-1, r+1]$ . The details can be left to the reader.

For part (b) we write  $B[\widehat{Z}_\gamma]$  in the form  $\widetilde{B}[\widetilde{Z}_{m-1}, Z_m]$ ,  $\widetilde{B} = B[\text{remaining variables}]$ . Every absolute  $\mathbf{SL}$ -invariant of  $\widetilde{B}[\widetilde{Z}_{m-1}, Z_m]$  has absolutely invariant homogeneous components  $f$  which satisfy the equation

$$T(f) = (\det T)^j f$$

for every  $T \in \mathbf{GL}(m, S)$ ,  $S$  a  $B$ -algebra,  $j = (\deg f)/m$ . This implies that an invariant  $f \in B[\widehat{Z}_\gamma]$  is in  $\mathbf{G}(Z_\gamma)[\gamma^{-1}]$  ( $\gamma$  taken as a minor of  $Z_\gamma$ ), and the equation

$$\gamma \mathbf{G}(Z_\gamma) = \gamma B[Z_\gamma] \cap \mathbf{G}(Z_\gamma),$$

which finishes the proof, is demonstrated as in the proof of (7.7):  $B[Z_\gamma]$  has a standard basis inherited from  $\mathbf{R}(X; [1, \dots, m|a_1, \dots, a_m])$ . —

It remains to consider the general case of  $\mathbf{R}(X; \delta)$ ,  $\delta = [b_1, \dots, b_r|c_1, \dots, c_r]$ . Let  $\gamma_1 = [b_1, \dots, b_r]$ ,  $\gamma_2 = [c_1, \dots, c_r]$ , and construct matrices  $Z_{\gamma_1}, Z_{\gamma_2}$  as above,  $Y_{\gamma_1}$  as the

transpose of  $Z_{\gamma_1}$ . The collection of indeterminates needed for  $Y_{\gamma_1}$  is denoted by  $\widehat{Y}_{\gamma_1}$ , that for  $Z_{\gamma_2}$  by  $\widehat{Z}_{\gamma_2}$ . Let  $H = \prod_{k=0}^{r-1} \text{GL}(k, B)$ . Then  $H \times H$  acts on  $B[\widehat{Y}_{\gamma_1}, \widehat{Z}_{\gamma_2}]$ , extending the action of the first component on  $B[\widehat{Y}_{\gamma_1}]$  and the action of the second one on  $B[\widehat{Z}_{\gamma_2}]$ . Furthermore we let  $\text{GL}(r, B)$  operate by the substitution

$$\begin{aligned} \widetilde{Y}_{r-1} &\longrightarrow \widetilde{Y}_{r-1}T^{-1}, & Y_r &\longrightarrow Y_rT^{-1}, \\ \widetilde{Z}_{r-1} &\longrightarrow T\widetilde{Z}_{r-1}, & Z_r &\longrightarrow TZ_r. \end{aligned}$$

This action commutes with that of  $H \times H$ , resulting in an action of  $G = H \times \text{GL}(r, B) \times H$ .

**(7.19) Theorem.** *The substitution  $X \longrightarrow Y_{\gamma_1}Z_{\gamma_2}$  induces an embedding*

$$R(X; \delta) \longrightarrow B[\widehat{Y}_{\gamma_1}, \widehat{Z}_{\gamma_2}]$$

*which is a generic point. The image  $B[Y_{\gamma_1}Z_{\gamma_2}]$  is the ring of absolute  $G$ -invariants of  $B[\widehat{Y}_{\gamma_1}, \widehat{Z}_{\gamma_2}]$ .*

The proof may be left to the reader. Again one should note that the attribute “absolute” is superfluous if  $B$  is a domain containing infinitely many units.

**(7.20) Remark.** The groups in (7.6) and (7.19) for  $\delta = [1, \dots, r|1, \dots, r]$  are different, as are those appearing in (7.7) and (7.18) for  $\gamma = [1, \dots, m]$ . In fact one can “minimize” the construction for  $\gamma$  by first applying (6.9),(d) and decomposing  $\gamma$  into its blocks as in subsection 6.B:

$$\gamma = [\beta_0, \dots, \beta_s], \quad \beta_i = (a_{k_i+1}, \dots, a_{k_{i+1}}).$$

Again we simultaneously consider the gaps

$$\chi_0 = (a_{k_1} + 1, \dots, a_{k_1+1} - 1), \dots, \chi_s = (a_m + 1, \dots, n).$$

Then one chooses matrices  $Z_i, i = 1, \dots, s + 1$ , and  $\widetilde{Z}_i, i = 1, \dots, s$ , of sizes

$$k_i \times (|\beta_{i-1}| + |\chi_{i-1}|) \quad \text{and} \quad k_{i+1} \times k_i \quad \text{resp.},$$

and obtains an analogue of (7.17) for the substitution

$$X \longrightarrow \left( 0 \mid \widetilde{Z}_s \dots \widetilde{Z}_1 Z_1 \mid \dots \mid \widetilde{Z}_s Z_s \mid Z_{s+1} \right),$$

an analogue of (7.18),(a) for the operation of  $H' = \prod_{i=1}^s \text{GL}(k_i, B)$ , and an analogue of (7.18),(b) for the operation of  $\text{SL}(m, B) \times H'$ . Similarly one can “minimize” (7.19). —

**(7.21) Remark.** The proofs of (7.18) and (7.19) can be simplified if  $B = K$  is a field of characteristic zero: In the inductive step of the proof of (7.18),(a) and the proof of (b) one can directly appeal to Theorem (7.6),(a) and Corollary (7.7) resp.: Take matrices  $W$  and  $Y$  with indeterminate entries and of the formats  $\widetilde{Z}_{m-1}$  and  $Z_{\gamma'}$  resp. Then the action of  $\text{GL}(m-1, B)$  on  $B[\widetilde{Z}_{m-1}, Z_{\gamma'}]$  is induced by that on  $B[W, Y]$  via the substitution  $W \rightarrow \widetilde{Z}_{m-1}, Y \rightarrow Z_{\gamma'}$ , and the claim follows immediately from (7.6),(a) by the reductivity of  $\text{GL}(m-1, B)$ , cf. property (iii) of linearly reductive groups. Similarly one concludes (7.18),(b) directly from (7.7). —

## G. The Depth of Modules of Invariants

Certain modules over rings of invariants arise as modules of invariants, and this fact can be used to study some of their properties. For simplicity we assume in this subsection (except for (7.25)) that  $B = K$  is a field.

Let  $G$  be a linear algebraic group over  $K$  which acts on a  $K$ -algebra  $S$  such that  $S$  is a  $G$ -module. Furthermore we consider an  $S$ - $G$ -module  $M$ , i.e. an  $S$ -module  $M$  which is simultaneously a  $G$ -module such that

$$g(ax) = g(a)g(x) \quad \text{for all } g \in G, a \in S, x \in M.$$

In particular  $S$  itself is an  $S$ - $G$ -module. Obviously the module

$$M^G = \{x \in M : g(x) = x \text{ for all } g \in G\}$$

of invariants is an  $S^G$ -module. If  $G$  is linearly reductive (cf. D), then there is hope that  $M^G$  may be accessible for a more detailed analysis:

**(7.22) Proposition.** *With the notations introduced so far, suppose that  $S$  is noetherian,  $M$  is finitely generated, and  $G$  is linearly reductive. Let  $\rho_S$  and  $\rho_M$  denote the Reynolds operators of  $S$  and  $M$ .*

(a)  *$\text{Ker } \rho_M$  is an  $S^G$ -module, so  $M = M^G \oplus \text{Ker } \rho_M$  is a decomposition of  $S^G$ -modules, and  $\rho_M : M \rightarrow M^G$  is an  $S^G$ -homomorphism:*

$$\rho_M(bx) = b\rho_M(x) \quad \text{for all } b \in S^G, x \in M.$$

Furthermore

$$\rho_M(ay) = \rho_S(a)y \quad \text{for all } a \in S, y \in M^G.$$

(b)  *$M^G$  is a finitely generated module over the noetherian ring  $S^G$ .*

PROOF:  $M$  splits as a  $G$ -module:  $M = M^G \oplus C$ ,  $C = \text{Ker } \rho_M$ . For the first statement in (a) one has to prove that  $C$  is an  $S^G$ -module.  $G$  being linearly reductive,  $C$  is the sum of its irreducible  $G$ -submodules  $N$ . It is enough to show that  $bN \subset C$  for all  $b \in S^G$ . Since  $b \in S^G$ , the map  $N \rightarrow bN$  is a  $G$ -homomorphism, hence 0 or an isomorphism. In the first case certainly  $bN \subset C$ , in the second  $bN$  is an irreducible  $G$ -submodule of  $M$  on which  $G$  cannot operate trivially, for otherwise it would operate trivially on  $N$  itself, and  $N \subset M^G$ . By construction,  $C$  is the sum of all irreducible  $G$ -modules of  $M$  with nontrivial  $G$ -action, so  $bN \subset C$ .

Let now  $a \in S$ ,  $y \in M^G$ . Write  $a = b + c$ ,  $b = \rho_S(a)$ . Then

$$\rho_M(ay) = \rho_M((b + c)y) = by + \rho_M(cy) = \rho_S(a)y + \rho_M(cy).$$

So we have to verify that  $\rho_M(cy) = 0$  for  $c \in \text{Ker } \rho_S$ ,  $y \in M^G$ . The argument is similar to the one above: one takes an irreducible  $G$ -submodule  $T \subset \text{Ker } \rho_S$ , and studies the  $G$ -homomorphism  $T \rightarrow Ty$ .

For (b) it is enough to show that  $M^G$  is a noetherian  $S^G$ -module. Let  $L \subset M^G$  be an  $S^G$ -submodule. Then  $SL = L \oplus (\text{Ker } \rho_S)L$ ,  $(\text{Ker } \rho_S)L \subset \text{Ker } \rho_M$  by virtue of (a), and every strictly ascending chain of  $S^G$ -submodules of  $M^G$  gives rise to a strictly ascending chain of  $S$ -submodules of  $M$ . —

We are interested in the grades of ideals  $I \subset S^G$  with respect to  $M^G$ . In the most important case for us, in which the objects under consideration are graded and  $I$  is the irrelevant maximal ideal of  $S^G$ , this grade coincides with the depth of  $(M^G)_I$ , whence the title of this subsection.



**(7.23) Proposition.** *Under the hypotheses of the preceding proposition let  $I \subset S^G$  be an ideal. Then*

$$\text{grade}(I, M^G) \geq \text{grade}(SI, M).$$

PROOF:  $M^G$  is a direct summand of  $M$ , thus  $\text{grade}(I, M^G) \geq \text{grade}(I, M)$ . The proof of the equation  $\text{grade}(I, M) = \text{grade}(SI, M)$  is left to reader. —

In general the estimate in (7.23) is not optimal as is demonstrated drastically by the  $S$ - $G$ -module  $S$  itself: then  $S^G$  is a Cohen-Macaulay ring by the theorem of Hochster-Roberts ([HR]) if  $S$  is Cohen-Macaulay, but  $\text{grade} SI < \text{grade} I$  in general. On the other hand it is sharp sometimes, cf. the subsequent discussion of the example  $S = B[Y, Z]$ ,  $S^G = B[YZ] \cong R_{r+1}(X)$ .

Examples of  $S$ - $G$ -modules can be constructed as follows: One chooses a finite-dimensional  $G$ -module  $V$ ; then the  $S$ -module  $M = V \otimes_K S$  becomes an  $S$ - $G$ -module under the  $G$ -action

$$g(v \otimes a) = g(v) \otimes g(a) \quad \text{for all } v \in V, a \in S.$$

Since  $M$  is free as an  $S$ -module, the inequality in (7.23) reduces to

$$\text{grade}(I, M^G) \geq \text{grade} SI.$$

$K$  itself becomes a  $G$ -module via the characters  $\chi: G \rightarrow \text{GL}(1, K)$ , and one can study the  $G$ -action

$$g_\chi(a) = \chi(g)g(a) \quad \text{for all } g \in G, a \in S$$

of  $G$  on  $S$ . The invariants under this action are precisely the semi-invariants of weight  $\chi^{-1}$ :

$$g_\chi(a) = a \quad \iff \quad g(a) = \chi^{-1}(a)a.$$

In the case of interest to us, namely  $S = B[Y, Z]$ ,  $G = \text{GL}(r, K)$ , all the characters are given by the powers of  $\det$ , and furthermore we have already computed the module  $D_j$  of semi-invariants of weight  $\det^j$ :

$$\begin{aligned} D_j &= B[YZ]\{\delta_1 \dots \delta_j : \delta_i \in \Gamma(Z)\} & \text{if } j \geq 0, \\ D_j &= B[YZ]\{\gamma_1 \dots \gamma_j : \gamma_i \in \Gamma(Y)\} & \text{if } j \leq 0, \end{aligned}$$

as follows immediately from (7.6),(b). Let  $\gamma \in \Gamma(Y)$ , the rows of  $\gamma$  being  $a_1, \dots, a_r$ . Then for all  $j \geq 0$

$$\gamma^j D_j = P^j, \quad \text{so} \quad D_j \cong P^j,$$

$P$  being the ideal generated by the  $r$ -minors of the rows  $a_1, \dots, a_r$  of  $YZ$  in  $B[YZ] \cong R_{r+1}(X)$ . Similarly one has  $D_{-j} \cong Q^j$ ,  $Q$  being the ideal generated by the  $r$ -minors of any  $r$  columns. We formulate the final result in terms of  $R_{r+1}(X)$ .

**(7.24) Proposition.** *Let  $B = K$  be a field of characteristic 0,  $X$  an  $m \times n$  matrix of indeterminates over  $K$ ,  $m \leq n$ . Let  $J$  and  $r$  be given as follows: (i)  $r = m$ ,  $J = I_m(X)$ , or (ii)  $r < m$ ,  $J$  the ideal generated by the  $r$ -minors of any  $r$  rows or any  $r$  columns resp. of the matrix  $x$  of residue classes in  $R = R_{r+1}(X)$ . Let furthermore  $I = I_1(X)R$ . Then*

$$\begin{aligned} \text{grade}(I, R/J^j) &\geq mr - 1 && \text{if } n \geq m + r, \\ \text{grade}(I, R/J^j) &\geq mr - \frac{(n - m - r)^2}{4} - 1 && \text{else.} \end{aligned}$$

This result will be improved in 9.D, cf. the examples (9.27):  $B$  may be an arbitrary noetherian ring, the first inequality is correct regardless of  $n \geq m + r$ , and if  $J$  is the “column ideal”, then  $\text{grade}(I, R/J^j) \geq nr - 1$ . These estimates, on the other hand, are optimal: one has equality for  $j$  large.

It remains to compute  $\text{grade}_{I_1}(YZ)B[Y, Z]$ . We restrict ourselves to the case of interest to us.

**(7.25) Proposition.** *Let  $B$  be a noetherian ring,  $Y$  and  $Z$  matrices of indeterminates of sizes  $m \times r$  and  $r \times n$  resp.,  $r \leq m \leq n$ . Then*

$$\begin{aligned} \text{grade}_{I_1}(YZ)B[Y, Z] &= mr && \text{if } n \geq m + r, \\ \text{grade}_{I_1}(YZ)B[Y, Z] &= mr - \left\lceil \frac{(n - m - r)^2}{4} \right\rceil && \text{else.} \end{aligned}$$

[...] denoting the integral part. The same equations hold for “height”.

We sketch two PROOFS. The first one uses the theory of varieties of complexes ([DS]). Let  $I = I_1(YZ)$ . Then  $R = B[Y, Z]/I$  is a Hodge algebra in the sense of [DEP.2], in particular it is a free  $B$ -module. It is enough to consider fields  $B$  (cf. (3.14)).  $R$  is reduced now (actually, over any reduced  $B$ ), and the minimal prime ideals of  $I$  in  $B[Y, Z]$  are given by

$$P_i = I + I_{r-i+1}(Y) + I_{i+1}(Z), \quad i = 0, \dots, r.$$

In fact, let  $P$  be any minimal prime ideal of  $R$ . The residue classes of the matrices  $Y$  and  $Z$  over  $R/P$  define a complex

$$(R/P)^m \xrightarrow{f} (R/P)^r \xrightarrow{g} (R/P)^n.$$

Since  $\text{rk } f + \text{rk } g \leq r$ , the preimage of  $P$  in  $B[Y, Z]$  has to contain one of the ideals  $P_i$ , namely  $P_{\text{rk } g}$ . On the other hand the  $P_i$  are prime ideals ([DS], Theorem 2.11) and

$$\text{grade } P_i = rn + i^2 - (n - m + r)i$$

by virtue of [DS], Lemma 2.3. Now one takes the minimum of these grades.

The second proof is elementary. It goes by induction on  $r$ . If  $r = 1$ , then  $I = I_1(Y)I_1(Z)$ , and the formula for  $\text{grade } I$  is obviously correct. We invert  $Y_{mr}$  and perform elementary row transformations on  $Y$  to obtain

$$\begin{pmatrix} \tilde{Y}_{11} & \cdots & \tilde{Y}_{1r-1} & 0 \\ \vdots & & \vdots & \vdots \\ \tilde{Y}_{m-1,1} & \cdots & \tilde{Y}_{m-1,r-1} & 0 \\ Y_{m1} & \cdots & Y_{m,r-1} & Y_{mr} \end{pmatrix}.$$

Let  $\tilde{R} = B[Y, Z][Y_{mr}^{-1}]$ ,  $\tilde{Y}$  the  $(m - 1) \times (r - 1)$  matrix in the left upper corner of the matrix above, and  $\tilde{Z}$  the  $(r - 1) \times n$  matrix of the first  $r - 1$  rows of  $Z$ . The entries of  $\tilde{Y}$  and  $\tilde{Z}$  are algebraically independent over

$$\tilde{B} = B[Y_{m1}, \dots, Y_{mr}, Y_{1r}, \dots, Y_{m-1,r}][Y_{mr}^{-1}],$$

as are the elements of the product  $Y_m Z$  of the last row  $Y_m$  of  $Y$  and  $Z$  over  $\tilde{B}[\tilde{Y}, \tilde{Z}]$ . Furthermore

$$\tilde{R} = (\tilde{B}[\tilde{Y}, \tilde{Z}])[Y_m Z] \quad \text{and} \quad I\tilde{R} = I_1(\tilde{Y}, \tilde{Z})\tilde{R} + I_1(Y_m Z)\tilde{R},$$

hence

$$\text{grade } I\tilde{R} = \text{grade } I_1(\tilde{Y}, \tilde{Z}) + n.$$

Letting  $\hat{R} = B[Y, Z][Z_{rn}^{-1}]$ , one concludes similarly that

$$\text{grade } I\hat{R} = \text{grade } I_1(\hat{Y}, \hat{Z}) + m,$$

$\hat{Y}, \hat{Z}$  being constructed analogously. Since obviously

$$\text{grade } I = \min(\text{grade } I\tilde{R}, \text{grade } I\hat{R}),$$

the claim follows by the inductive hypothesis. In order to obtain the equations for height, one replaces “grade” by “height” throughout. —

## H. Comments and References

Theorem (7.6) and Corollary (7.7) are classical for fields  $B = K$  of characteristic zero, cf. [We]. The characteristic free version of (7.7) is essentially due to Igusa [Ig], and in their final form presented here, (7.6) and (7.7) were given by de Concini and Procesi [DP]; cf. [BB] and [Ri] for possibly simpler or more elementary proofs. The proofs of (7.7) and (7.6),(a) result from an attempt to understand Igusa’s arguments. Our treatment is certainly close to [DP], from which we copied the proofs of (7.7),(b) and (7.5). At least for (7.6),(a), however, the standard monomial theory is not essential; it can be derived from the result of Hochster and Eagon ([HE.2]) already, cf. Section 12. In order to avoid the intricacies of the notion of algebraic group over general commutative rings we have restricted the definition of “absolute invariant” to concrete situations.

Our notion of “generic point” is inspired by Hochster and Eagon’s article [HE.2] in which the construction of generic points plays a central role, cf. Section 12. The generic points for  $G(X; \gamma)$  in (7.14) were given by Hodge [Hd], and the proof of the linear independence of the standard monomials in (7.16) is taken from Musili [Mu]. The construction of the generic points in (7.17) and the invariant theoretic description (7.18) are borrowed from [HE.2], Sections 7 and 8, and [Ho.3], Section 5. Hochster proves (7.18) in characteristic zero by the reductivity argument indicated in Remark (7.21). We have freed his constructions from the assumption of characteristic zero and generalized to all the rings  $R(X; \delta)$ . The determination of the semi-invariants of the group  $H$  in (7.19) and a generalization of (7.10) are left to the reader. The ideal-theoretic consequences to be expected will be proved in Section 9 by methods perhaps more convenient.

The first (unpublished) proof of Hochster and Eagon for the perfection of determinantal ideals was based on invariant theory, in particular the existence of a Reynolds operator  $K[Y, Z] \rightarrow K[YZ]$  when  $K$  is a field of characteristic zero, cf. [HE.2], Introduction. On the other hand we quote from [HR], p. 118: "... determinantal loci have ... ultimately motivated the conjecture of ... the Main Theorem" of [HR] mentioned above. Further examples for which the theorem of Hochster-Roberts implies the Cohen-Macaulay property are listed in [HR]. Cf. [Ke.5] for a generalization of the theorem of Hochster-Roberts and a simplification of its proof.

The rationality of the singularities of the Schubert varieties was first proved by Kempf [Ke.4]. Their homogeneous coordinate rings are the  $G(X; \gamma)$ , and the varieties corresponding to  $R(X; \delta)$  are open subvarieties of the Schubert varieties, so they have rational singularities, too.

Remark (7.12) was communicated to us by M. Hochster, and Subsection G owes its existence to discussions with J. Herzog.

References for (7.10) and (7.24) will be given in Section 9 where results of the same kind will be derived in a more general context. We do not know of an invariant-theoretic approach in the literature, however.

## 8. The Divisor Class Group and the Canonical Class

This section is devoted to the study of the divisor class groups of the Schubert cycles  $G(X; \gamma)$  and the determinantal rings  $R(X; \delta)$  (over a normal ring  $B$  of coefficients). Their computation has been prepared in Section 6, and will turn out rather easy. If  $B$  is a Cohen-Macaulay ring with a canonical module  $\omega_B$ ,  $G(X; \gamma)$  and  $R(X; \delta)$  have canonical modules, too, which, under the assumption of normality, are completely determined by their divisor class, called the *canonical class*. The crucial case in the computation of the canonical class is  $R_2(X)$ , to which the general case can be reduced by surprisingly simple localization arguments. As an application we determine the Gorenstein rings among the rings under consideration. In Section 9 we shall give a complete description of the canonical module in terms of the standard monomial basis.

### A. The Divisor Class Group

For the theory of divisorial ideals and the (divisor) class group  $\text{Cl}(S)$  of a normal domain  $S$  we refer the reader to [Fs] (or [Bo.3]). The main tool for the computation of the class groups of the rings  $G(X; \gamma)$  and  $R(X; \delta)$  is Nagata's theorem which relates  $\text{Cl}(S)$  and the class groups of its rings of quotients, cf. [Fs], § 7 (or [Bo.3], § 1, no. 10, Prop. 17).

It has been proved in (6.3) that  $G(X; \gamma)$  and  $R(X; \delta)$  are normal domains when the ring  $B$  of coefficients is a normal domain. Therefore  $G(X; \gamma)$  and  $R(X; \delta)$  have well-defined class groups then. The normality of  $G(X; \gamma)$  and  $R(X; \delta)$  has been demonstrated by showing that the rings  $G(X; \gamma)[\gamma^{-1}]$  and  $R(X; \delta)[\delta^{-1}]$  arise from a polynomial ring over  $B$  after the inversion of a prime element, rendering their class groups naturally isomorphic with  $\text{Cl}(B)$ , cf. (6.1) and (6.4). Let us write  $R$  for  $G(X; \gamma)$  and  $R(X; \delta)$  and  $\varepsilon$  for  $\gamma$  and  $\delta$  resp. Since  $R$  is a flat extension of  $B$ , the embedding  $B \rightarrow R$  induces a homomorphism  $\text{Cl}(B) \rightarrow \text{Cl}(R)$ , and the composition

$$\text{Cl}(B) \rightarrow \text{Cl}(R) \rightarrow \text{Cl}(R[\varepsilon^{-1}]) \cong \text{Cl}(B)$$

is just the natural isomorphism from  $\text{Cl}(B)$  to  $\text{Cl}(R[\varepsilon^{-1}])$ .

Naturality here means: These maps are induced by homomorphisms of the groups of divisors which send a divisorial ideal to its extension. It follows at once that

$$\text{Cl}(R) = \text{Cl}(B) \oplus U,$$

the subgroup  $U$  being generated by the classes of the minimal prime ideals of  $\varepsilon$  by virtue of Nagata's theorem. Corollary (6.5) names these prime ideals, and we will specify them below. Let them be denoted by  $P_0, \dots, P_u$  here. Since  $R/R\varepsilon$  is reduced, we have

$$R\varepsilon = \bigcap_{i=0}^u P_i$$

and thus the relation  $\sum_{i=0}^u \text{cl}(P_i) = 0$ . We claim: This is the only relation between the classes  $\text{cl}(P_i)$ , and every subset of  $u$  of them is linearly independent. Suppose that

$$\sum_{i=0}^u t_i \text{cl}(P_i) = 0.$$

Then  $\sum_{i=0}^u t_i \text{div}(P_i)$  is a principal divisor  $\text{div}(Rf)$ ,  $f$  in the field of fractions of  $R$ . The divisor  $\text{div}(P_i)$  is contained in the kernel of the homomorphism  $\text{Div}(R) \rightarrow \text{Div}(R[\varepsilon^{-1}])$  of groups of divisors, whence the element  $f$  is a unit in  $R[\varepsilon^{-1}]$ . Since  $R[\varepsilon^{-1}]$  arises from a polynomial ring over  $B$  by inversion of a prime element, namely  $\varepsilon$ , we have

$$f = g\varepsilon^m,$$

$g$  a unit in  $B$ ,  $m \in \mathbf{Z}$ . So

$$\sum_{i=0}^u t_i \text{div}(P_i) = \text{div}(Rf) = m \text{div}(R\varepsilon) = \sum_{i=0}^u m \text{div}(P_i).$$

Since the divisors  $\text{div}(P_i)$ ,  $i = 0, \dots, u$ , are linearly independent, we conclude that  $t_i = m$  for  $i = 0, \dots, u$  as desired. Therefore

$$\text{Cl}(R) = \text{Cl}(B) \oplus \mathbf{Z}^u,$$

and every set of  $u$  of the classes of  $P_0, \dots, P_u$  generates the direct summand  $\mathbf{Z}^u$ .

Let  $\Pi = \Gamma(X)$  or  $\Pi = \Delta(X)$  resp. Then the ideal defining  $R$  as a residue class ring of  $G(X)$  or  $B[X]$  is generated by an ideal  $\Omega$  of  $\Pi$ ,  $\Omega$  itself being cogenerated by  $\varepsilon$ . The ideal defining  $R/R\varepsilon$  is generated by  $\Omega \cup \{\varepsilon\}$ , and its minimal prime ideals are generated by the ideals of  $\Pi$  which are cogenerated by the upper neighbours of  $\varepsilon$  in  $\Pi$ . Within  $R$  this means that the minimal prime ideals of  $R\varepsilon$  have the form

$$J(x; \zeta) = J(X; \zeta)/J(X; \gamma) \quad \text{or} \quad I(x; \zeta) = I(X; \zeta)/I(X; \delta),$$

$\zeta$  running through the upper neighbours of  $\gamma$  or  $\delta$ .

We deal with  $G(X; \gamma)$  first. In Section 6 we have broken  $\gamma = [a_1, \dots, a_m]$  into its blocks  $\beta_0, \dots, \beta_s$  of consecutive integers:

$$\gamma = [\beta_0, \dots, \beta_s], \quad \beta_i = (a_{k_i+1}, \dots, a_{k_{i+1}}).$$

Each  $\beta_i$  is followed by the gap

$$\chi_i = (a_{k_{i+1}} + 1, \dots, a_{k_{i+1}+1} - 1),$$

the sequence of integers properly between the last element of  $\beta_i$  and the first element of  $\beta_{i+1}$ , the last gap  $\chi_s$  being possibly empty. Obviously  $\gamma$  has as many upper neighbours as there are nonempty gaps  $\chi_i$ , and the upper neighbours are

$$\zeta_i = [\beta_0, \dots, \beta_{i-1}, (a_{k_i+1}, \dots, a_{k_{i+1}} - 1), a_{k_{i+1}} + 1, \beta_{i+1}, \dots, \beta_s],$$

$i = 0, \dots, s$  if  $a_m < n$ ,  $i = 0, \dots, s - 1$  if  $a_m = n$ .

**(8.1) Theorem.** *Let  $B$  be a noetherian normal domain,  $X$  an  $m \times n$  matrix of indeterminates,  $\gamma$  an element of the poset  $\Gamma(X)$  of its  $m$ -minors,  $\gamma = [a_1, \dots, a_m]$ . Then the class group of  $G(X; \gamma)$  is given by*

$$\text{Cl}(G(X; \gamma)) = \begin{cases} \text{Cl}(B) \oplus \mathbf{Z}^s & \text{if } a_m < n \text{ or } s = 0, \\ \text{Cl}(B) \oplus \mathbf{Z}^{s-1} & \text{otherwise.} \end{cases}$$

*The summand  $\text{Cl}(B)$  arises naturally from the embedding  $B \rightarrow G(X; \gamma)$ , and the summand  $\mathbf{Z}^s$  or  $\mathbf{Z}^{s-1}$  is generated by the classes of any set of  $s$  or  $s - 1$  resp. of the prime ideals  $J(x; \zeta_i)$ .*

Note that one can simplify the formulation of (8.1) if one first applies the reduction to the case  $a_m < n$  as indicated in (6.9),(d).

**(8.2) Corollary.**  *$G(X; \gamma)$  is factorial if and only if  $B$  is factorial and there is at most one nonempty gap in  $\gamma$ .*

In particular  $G(X)$  itself is factorial. The condition for  $\gamma$  in (8.2) is satisfied exactly in the cases in which  $\Sigma(X; \gamma) = \Gamma(X; \gamma)$ , cf. (6.8).

In order to determine the upper neighbours of  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r] \in \Delta(X)$  we have to decompose the row part  $[a_1, \dots, a_r]$  and the column part  $[b_1, \dots, b_r]$  similarly, obtaining  $u + 1$  blocks for  $[a_1, \dots, a_r]$  and  $v + 1$  blocks for  $[b_1, \dots, b_r]$ . There arise upper neighbours  $\zeta_i$  from raising a row index,  $i = 0, \dots, u$ , unless  $a_r = m$ , in which case  $i = 0, \dots, u - 1$ . Similary one obtains the upper neighbours  $\eta_j$  determined by the column part. In case  $a_r = m$  and  $b_r = n$  there is the further upper neighbour  $\vartheta = [a_1, \dots, a_{r-1} | b_1, \dots, b_{r-1}]$ , apart from the (trivial) case  $r = 1$ .

**(8.3) Theorem.** *Let  $B$  be a noetherian normal domain,  $X$  an  $m \times n$  matrix of indeterminates,  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r]$  an element of the poset of its minors. Then the class group of  $R(X; \delta)$  is given by*

$$\text{Cl}(R(X; \delta)) = \begin{cases} \text{Cl}(B) \oplus \mathbf{Z}^{u+v} & \text{if } a_r = m \text{ or } b_r = n, \\ \text{Cl}(B) \oplus \mathbf{Z}^{u+v+1} & \text{otherwise.} \end{cases}$$

*The summand  $\text{Cl}(B)$  arises naturally from the embedding  $B \rightarrow R(X; \delta)$ , and the direct summand  $\mathbf{Z}^{u+v}$  or  $\mathbf{Z}^{u+v+1}$  is generated by the classes of any  $u + v$  or  $u + v + 1$  resp. prime ideals corresponding to the upper neighbours of  $\delta$ .*

Evidently  $R(X; \delta)$  is factorial if and only if  $B$  is factorial and

$$\delta = [m - r + 1, \dots, m | b_1, \dots, b_1 + r - 1] \quad \text{or} \quad \delta = [a_1, \dots, a_1 + r - 1 | n - r + 1, \dots, n],$$

equivalently, if it is a polynomial ring over  $B$  (cf. the discussion below (6.10)).  $R(X; \delta)$  can be viewed as arising from a suitable ring  $G(\tilde{X}; \tilde{\delta})$  by dehomogenization with respect to  $\pm[n + 1, \dots, n + m]$ . As in Subsection 16.D one has a natural commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S \cong A[T, T^{-1}] \\ & \searrow \pi & \nearrow \\ & & A \end{array}$$

$R = G(\tilde{X}; \tilde{\delta})$ ,  $A = R(X; \delta)$ ,  $T$  an indeterminate,  $S = R[[n + 1, \dots, n + m]^{-1}]$ . There

is always a natural epimorphism  $\text{Cl}(R) \rightarrow \text{Cl}(S)$ , and a natural isomorphism  $\text{Cl}(A) \rightarrow \text{Cl}(S)$ . Here the resulting epimorphism  $\text{Cl}(R) \rightarrow \text{Cl}(A)$  is an isomorphism. This follows from (8.1) and (8.3) since the upper neighbours of  $\delta$  and  $\tilde{\delta}$  are in one-one correspondence, and the ideals in  $A$  and  $R$  resp. “cogenerated” by them correspond to each other under  $\pi$  as in (16.26).

A by-product: The maximal element  $\mu$  of  $\Gamma(X; \gamma)$  is always a prime element (over an integral domain  $B$ ). It is enough to show this for a field  $B$  since  $G(X; \gamma)/\mu G(X; \gamma)$  is a graded ASL (cf. (3.12)). Then we write  $G(X; \gamma)$  as  $G(\tilde{X}; \tilde{\delta})$ . Since  $\text{Cl}(R) \rightarrow \text{Cl}(S)$  is an isomorphism, the minimal prime ideal of  $\mu$  must be principal. Being irreducible,  $\mu$  is prime itself.

We single out the most important case of (8.3):

**(8.4) Corollary.** *The class group of  $R_{r+1}(X)$ ,  $0 < r < \min(m, n)$ , is*

$$\text{Cl}(R_{r+1}(X)) = \text{Cl}(B) \oplus \mathbf{Z},$$

*the summand  $\mathbf{Z}$  generated by the class of the prime ideal  $P$  generated by the  $r$ -minors of  $r$  arbitrary rows or its negative, the class of the prime ideal  $Q$  generated by the  $r$ -minors of  $r$  arbitrary columns.*

In fact, the generators specified in (8.3) correspond to the first  $r$  rows or first  $r$  columns. Since an automorphism exchanging the rows leaves each of the prime ideals  $Q$  in (8.4) invariant, the induced automorphism of  $\text{Cl}(R_{r+1}(X))$  is the identity, and the same holds for a permutation of the columns. (The reader may describe the isomorphisms between the prime ideals  $P$  or  $Q$  resp. directly; cf. also the discussion above (7.24).)

**(8.5) Remarks.** (a) Let  $B = K$  be a field. The completion  $\widehat{R}$  of  $R = G(X; \gamma)$  or  $R = R(X; \delta)$  with respect to its irrelevant maximal ideal is again normal (since the associated graded ring of  $\widehat{R}$  with respect to its maximal ideal, namely  $R$ , is normal; cf. also (3.13)). In general one has only an injection  $\text{Cl}(R) \rightarrow \text{Cl}(\widehat{R})$ . The ring under consideration satisfies the Serre condition  $(R_2)$  (cf. (6.12)) and is Cohen-Macaulay; therefore  $\text{Cl}(R) \rightarrow \text{Cl}(\widehat{R})$  is even an isomorphism here ([Fl], (1.5)).

(b) In the preceding section we have described the rings  $G(X; \gamma)$  and  $R(X; \delta)$  as rings of (absolute) invariants of linear algebraic groups acting on polynomial rings  $A$  over  $B$ . Assume that  $B = K$  is an algebraically closed field. Then it is well-known that (under hypotheses satisfied for our objects)  $R = A^G$  is a factorial domain if  $G$  is connected and has a trivial character group  $G^*$  (cf. [Kr], p. 100, Satz 2 and p. 101, Bemerkungen). It follows that  $G(X)$  and the  $K$ -algebra of  $\text{SL}(r, K)$ -invariants of  $K[Y, Z]$  are factorial domains (cf. (7.6) and (7.7)).

The main result of [Mg] connects  $G^*$  and  $\text{Cl}(A^G)$  under much more general circumstances:  $G$  is supposed to be a connected algebraic group acting rationally on a normal affine  $K$ -algebra  $A$ . Suppose for simplicity that  $A$  is factorial. Then, by [Mg], Theorem 6,  $\text{Cl}(A^G)$  is a homomorphic image of  $G^*$ . The reader may investigate  $G(X; \gamma)$  and  $R(X; \delta)$  from this point of view. It is clear that one can only expect an isomorphism  $\text{Cl}(A^G) \cong G^*$  if  $G$  is taken as “small” as possible; in this regard (7.20) may be useful. —

With the hypotheses and notations of the preceding corollary, the ideal generated by an  $r$ -minor (of the matrix of residue classes) is the intersection of the prime ideals



$P$  and  $Q$  generated by the  $r$ -minors of its rows and columns resp. On the contrary the lower order minors are prime elements.

**(8.6) Proposition.** *Let  $B$  be an (arbitrary) integral domain. Then for  $s < r$  an  $s$ -minor of the matrix of residue classes is a prime element of  $R_{r+1}(X)$ .*

We outline the PROOF, leaving the details to the reader's scrutiny. It is enough to consider the cases in which  $B = \mathbf{Z}$  or  $B$  is a field: The case  $B = \mathbf{Z}$  provides the flatness argument needed for (3.12). For an inductive reasoning let  $s = 1$  first,  $R = R_{r+1}(X)$ . Because of (2.4) and (8.4) the natural epimorphism  $\text{Cl}(R) \rightarrow \text{Cl}(R[x_{mn}^{-1}])$  is an isomorphism, whence  $x_{mn}$ , being irreducible, is prime. Let  $s > 1$  now. Since  $x_{mn}$  is a prime element and  $R$  is a domain,

$$\delta = [m - s + 1, \dots, m | n - s + 1, \dots, n], x_{mn}$$

is an  $R$ -sequence. In order to prove that  $\overline{R} = R/\delta R$  is a domain, it is now enough to show that  $\overline{R}[(\overline{x_{mn}})^{-1}]$  is a domain, and this follows from (2.4) in conjunction with the inductive hypothesis.

### B. The Canonical Class of $R_{r+1}(X)$

A canonical module (cf. 16.C) of a normal Cohen-Macaulay domain  $S$  is a reflexive  $S$ -module of rank 1, therefore (isomorphic to) a divisorial ideal and completely determined by its class which is called a *canonical class* (cf. [Fs], § 12 or [HK], 7. Vortrag). We want to compute the canonical classes of  $G(X; \gamma)$  and  $R(X; \delta)$ , and to decide which of these rings are Gorenstein rings:  $S$  is Gorenstein if and only if  $S$  is a canonical module of itself.

Let  $B$  be a Cohen-Macaulay ring possessing a canonical module  $\omega_B$ ,  $R$  a generically perfect residue class ring of a polynomial ring  $\mathbf{Z}[X]$ , and  $S = R \otimes_{\mathbf{Z}} B$ . From (3.6) we know that

$$\omega_S = \omega_R \otimes_{\mathbf{Z}} \omega_B$$

is a canonical module of  $S$ . For the rings under consideration this formula can be refined.

**(8.7) Proposition.** *Let  $R$  be one of the rings  $G(X; \gamma)$  or  $R(X; \delta)$  defined over  $\mathbf{Z}$ . Let  $B$  be a normal Cohen-Macaulay domain having a canonical module  $\omega_B$ , and  $S = R \otimes_{\mathbf{Z}} B$ .*

- (a) *The modules  $\omega_B \otimes S$  and  $\omega_R \otimes S$  are divisorial ideals. The class of  $\omega_R \otimes S$  is in the free direct summand  $F$  of  $\text{Cl}(S)$  appearing in (8.1) and (8.3) resp. Under the isomorphism  $F \rightarrow \text{Cl}(S \otimes L)$ ,  $L$  the field of fractions of  $B$ , it is mapped to  $\text{cl}(\omega_R \otimes L)$ .*
- (b) *An element of  $\text{Cl}(S)$  represents a canonical module of  $S$  if and only if it has the form  $\text{cl}(\omega_B \otimes S) + \text{cl}(\omega_R \otimes S)$  for a canonical module  $\omega_B$  of  $B$ , so is unique up to the choice of  $\omega_B$ .*

PROOF: (a)  $\omega_B \otimes S$  is a divisorial ideal, since the extension  $B \rightarrow S$  is flat. Let  $A$  be a polynomial ring over  $\mathbf{Z}$  of which  $R$  is a residue class ring. By virtue of (3.6)  $\omega_R$  is generically perfect of the same grade as  $R$ . So  $\omega_R \otimes_{\mathbf{Z}} S = \omega_R \otimes_{\mathbf{Z}} B$  is a perfect  $A \otimes_{\mathbf{Z}} B$ -module, and one has  $\text{depth } \omega_R \otimes S_P = \text{depth } S_P$  for every prime ideal  $P$  of  $S$ . It has rank 1, as can be seen by passing from  $R$  to  $S[\varepsilon^{-1}]$  through  $R[\varepsilon^{-1}]$ , where  $\varepsilon$  is the

minimal element of the poset defining  $R$ :

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R[\varepsilon^{-1}] & \longrightarrow & S[\varepsilon^{-1}]. \end{array}$$

Being locally a maximal Cohen-Macaulay module, it is torsionfree, so isomorphic with an ideal  $I$  of  $S$ , and  $S/I$ , if  $\neq 0$ , is a Cohen-Macaulay ring of dimension  $\dim S - 1$ . Being equal to  $S$  or unmixed of height 1,  $I$  is divisorial. Since  $\omega_R \otimes S[\varepsilon^{-1}]$  is free of rank 1, the class of  $\omega_R \otimes S$  is in the kernel of  $\text{Cl}(S) \rightarrow \text{Cl}(S[\varepsilon^{-1}])$ , thus in  $F$ . The last statement is obvious.

(b) We have learnt that  $\omega_R \otimes_{\mathbf{Z}} \omega_B = (\omega_R \otimes S) \otimes_S (\omega_B \otimes S)$  is a canonical module of  $S$ , and the class of the tensor product is the sum of the classes. Thus a class  $\text{cl}(\omega_B \otimes S) + \text{cl}(\omega_R \otimes S)$  represents a canonical module of  $S$ . Conversely let a class  $c = c_1 + c_2$ ,  $c_1 \in \text{Cl}(B)$ ,  $c_2$  in the free direct summand, represent a canonical module. The class  $c_1$  contains the extension of a divisorial ideal  $I$  of  $B$ , whose extension to  $S[\varepsilon^{-1}]$  becomes a canonical module of  $S[\varepsilon^{-1}]$ . Using the characterization in [HK], Satz 6.1,d) (for example) and the properties of the extension  $B \rightarrow S[\varepsilon^{-1}]$ , it is easy to show that  $I$  must be a canonical module of  $B$ . In order to isolate  $c_2$ , we consider the extension  $S \rightarrow S \otimes_B L = R \otimes_{\mathbf{Z}} L$ . An extension of a divisorial ideal in the class  $c_2$  then is a canonical module of  $R \otimes_{\mathbf{Z}} L$ , and so is  $\omega_R \otimes_{\mathbf{Z}} L$ . The passage from  $R \otimes_{\mathbf{Z}} L$  to its localization with respect to the irrelevant maximal ideal induces an isomorphism of class groups ([Fs], Corollary 10.3). Since the canonical module of a local ring is uniquely determined, we finally conclude  $c_2 = \text{cl}(\omega_R \otimes S)$ . —

As we shall see, the general case  $R = G(X; \gamma)$  or  $R = R(X; \delta)$  can be reduced in a strikingly simple manner to the case  $R = R_2(X)$ . We start by noting the result for the case  $R = R_{r+1}(X)$ :

**(8.8) Theorem.** *Let  $B$  be a normal Cohen-Macaulay domain with a canonical module  $\omega_B$ ,  $X$  an  $m \times n$  matrix of indeterminates,  $0 < r < \min(m, n)$ ,  $R = R_{r+1}(X)$ . Then a divisorial ideal  $\omega$  with class*

$$\text{cl}(\omega) = \text{cl}(\omega_B R) + m \text{cl}(P) + n \text{cl}(Q)$$

*is a canonical module of  $R$ . (As in (8.4),  $P$  is the prime ideal generated by the  $r$ -minors of any  $r$  rows,  $Q$  the prime ideal generated by the  $r$ -minors of any  $r$  columns.) Every canonical module of  $R$  has this representation, and up to the choice of  $\omega_B$  it is unique.*

Since  $\text{cl}(P) = -\text{cl}(Q)$ , the difference of  $m$  and  $n$  determines the class of  $\omega$  already.

**(8.9) Corollary.** *Let  $B$  be an (arbitrary) noetherian ring. Then  $R_{r+1}(X)$ ,  $0 < r < \min(m, n)$ , is Gorenstein if and only if  $B$  is a Gorenstein ring and  $m = n$ .*

PROOF: Along flat local extensions the Gorenstein property behaves like the Cohen-Macaulay property: For a prime ideal  $I$  of  $R = R_{r+1}(X)$  and  $J = B \cap I$  the localization  $R_I$  is Gorenstein if and only if both  $B_J$  and  $(B_J/JB_J) \otimes R_I$  have this property (cf. [Wt]). As usual, this argument reduces the general case to the one in which  $B$  is a field, and for which the corollary is a direct consequence of the theorem. —

The PROOF OF (8.8) in the crucial case  $r = 1$  is an induction on  $m + n$ . Because of (8.7) it is enough to treat the case in which  $B = K$  is a field. For the minimal choice  $m = n = 2$  of  $m$  and  $n$ , (8.8) is true:  $R$  is the residue class ring of  $K[X]$  modulo a principal ideal, so Gorenstein. In the inductive step we want to descend from  $R$  to the residue class ring modulo the elements in the last row or column of the matrix, thereby passing to a “smaller” ring.

**(8.10) Lemma.** *Let  $A$  be a normal Cohen-Macaulay domain, and  $I$  a prime ideal of height 1 in  $A$  such that  $A/I$  is again a normal Cohen-Macaulay domain. Let  $P_1, \dots, P_u$  be prime ideals of height 1 in  $A$  and suppose that the class of  $I$  and the class of a canonical module  $\omega$  of  $A$  have representations*

$$\text{cl}(I) = \sum_{i=1}^u s_i \text{cl}(P_i) \quad \text{and} \quad \text{cl}(\omega) = \sum_{i=1}^u r_i \text{cl}(P_i).$$

Assume further that:

- (i)  $r_i - s_i \geq 0$  for  $i = 1, \dots, u$ .
- (ii)  $\text{Ann}(P_i^{(r_i - s_i)} / P_i^{r_i - s_i}) \not\subseteq P_i + I$  for  $i = 1, \dots, u$ .
- (iii) The ideals  $(P_i + I)/I$  are distinct prime ideals of height 1 in  $A/I$ . Then  $A/I$  has a canonical module  $\omega_{A/I}$  with

$$\text{cl}(\omega_{A/I}) = \sum_{i=1}^u (r_i - s_i) \text{cl}((P_i + I)/I).$$

We first finish the proof of (8.8). Without restriction we may assume that  $m \geq n$ , so  $m \geq 3$ . Let  $P$  be the prime ideal generated by the elements in the first row,  $Q$  the prime ideal corresponding to the first column, and  $I$  being generated by the elements of the last row. Whatever  $\omega$  is, its class can be written  $\text{cl}(\omega) = p \text{cl}(P) + q \text{cl}(Q)$  with  $p, q > 0$ , since  $\text{cl}(P)$  and  $\text{cl}(Q)$  generate  $\text{Cl}(R)$  and  $\text{cl}(P) + \text{cl}(Q) = 0$ . Now  $\text{cl}(I) = \text{cl}(P)$ . The lemma gives

$$\begin{aligned} \text{cl}(\omega_{A/I}) &= (p - 1) \text{cl}((P + I)/I) + q \text{cl}((Q + I)/I) \\ &= (m - 1) \text{cl}((P + I)/I) + n \text{cl}((Q + I)/I) \end{aligned}$$

by induction, and  $p - q = m - n$  as desired. The hypotheses of the lemma are indeed satisfied: Except for (ii), everything is trivial (for (iii) note that  $m \geq 3$ ). Condition (ii) holds, since  $P$  and  $Q$  become principal when a matrix element not occurring in  $P$ ,  $Q$ , or  $I$  is inverted (one may take  $[2|n]$ ).

Let now  $r > 1$ . The reader may argue inductively, using the isomorphism in (2.4) which allows one to pass from the data  $(m, n, r)$  to the data  $(m - 1, n - 1, r - 1)$  after the inversion of  $[m|n]$ . (8.8) is again covered by (8.14). —

PROOF OF (8.10): A canonical module  $\omega_{A/I}$  is given by  $\text{Ext}_A^1(A/I, \omega)$ . So we have an exact sequence

$$0 \longrightarrow \text{Hom}_A(A, \omega) \longrightarrow \text{Hom}_A(I, \omega) \longrightarrow \omega_{A/I} \longrightarrow 0.$$

$\text{Hom}_A(I, \omega)$  is the quotient  $\omega : I$  within the field of fractions of  $A$ , and

$$\text{cl}(\omega : I) = \text{cl}(\omega) - \text{cl}(I) = \sum_{i=1}^u (r_i - s_i) \text{cl}(P_i).$$

Let  $t_i = r_i - s_i$ . Then  $\text{Hom}_A(I, \omega)$  is isomorphic to  $J = P_1^{(t_1)} \cap \dots \cap P_u^{(t_u)}$ . Since all the exponents are non-negative,  $J \subset A$ , and we have an exact sequence

$$0 \longrightarrow \tilde{J} \longrightarrow J \longrightarrow \omega_{A/I} \longrightarrow 0.$$

The ideal  $\tilde{J}$  is isomorphic to  $\omega$  and contains  $IJ$ . Since  $I$  is a prime ideal different from  $P_1, \dots, P_u$ , the smallest divisorial ideal containing  $IJ$  is  $I \cap J \subset \tilde{J}$ . On the other hand no proper quotient of  $J/I \cap J$  can be (isomorphic to) a nonzero ideal in  $A/I$ . We have  $\tilde{J} = I \cap J$ , and must prove that the equality in

$$J/(I \cap J) \cong (J + I)/I = \overline{P}_1^{(t_1)} \cap \dots \cap \overline{P}_u^{(t_u)}$$

holds, where  $\overline{P}_i = (P_i + I)/I$ . Hypothesis (ii) implies that

$$(P^{(t_i)} + I)/I \subset \overline{P}^{(t_i)}.$$

Therefore one has a chain of inclusions

$$\overline{P}_1^{t_1} \dots \overline{P}_u^{t_u} \subset (J + I)/I \subset \overline{P}_1^{(t_1)} \cap \dots \cap \overline{P}_u^{(t_u)},$$

the last ideal being the smallest divisorial ideal containing  $\overline{P}_1^{t_1} \dots \overline{P}_u^{t_u}$ , and the desired equality holds since  $(J + I)/I \cong \omega_{A/I}$  is likewise divisorial. —

### C. The General Case

Next we treat the case  $R = G(X; \gamma)$  for which we may again assume that  $B$  is a field (cf. (8.7)). The class of the canonical module has a representation

$$\text{cl}(\omega) = \sum_{i=0}^t \kappa_i \text{cl}(\mathcal{J}(x; \zeta_i))$$

where  $t = s$  if  $a_m < n$ ,  $t = s - 1$  if  $a_m = n$ ,  $\gamma$  having  $s + 1$  blocks, the  $\zeta_i$  being the upper neighbours of  $\gamma$ . Since  $\sum \text{cl}(\mathcal{J}(x; \zeta_i)) = 0$ , the differences

$$\kappa_{i-1} - \kappa_i, \quad i = 1, \dots, t,$$

determine  $\text{cl}(\omega)$  uniquely. In (6.8) we introduced the elements

$$\sigma_i = [\beta_0, \dots, \beta_{i-2}, (a_{k_{i-1}+1}, \dots, a_{k_i-1}), (a_{k_i+1}, \dots, a_{k_{i+1}}, a_{k_{i+1}} + 1), \beta_{i+1}, \dots, \beta_s]$$

and we noted that the localizations of  $R$  with respect to the prime ideals  $J(x; \sigma_i)$  are not factorial. All the more,  $R[\sigma_i^{-1}]$  is not factorial, and from the minimal primes of  $\gamma$  only

$$J(x; \zeta_{i-1}) \quad \text{and} \quad J(x; \zeta_i)$$

survive in  $R[\sigma_i^{-1}]$ , since  $\sigma_i$  is contained in all the other ones ( $\sigma_i \not\geq \zeta_j$  for  $j \neq i-1, i$ ). Let  $S = R[\sigma_i^{-1}]$ . Then

$$\text{cl}(\omega_S) = \kappa_{i-1} \text{cl}(J(x; \zeta_{i-1})) + \kappa_i \text{cl}(J(x; \zeta_i)).$$

At this point we have to determine the structure of  $S$ . Analogously with (6.1) we let

$$\Psi = \{ \delta \in \Gamma(X; \gamma) : \delta \text{ differs from } \sigma_i \text{ in at most one index} \}$$

and subdivide  $\Psi$  in

$$\Psi_1 = \{ \delta \in \Psi : \delta \geq \sigma_i \} \quad \text{and} \quad \Psi_2 = \{ \delta \in \Psi : \delta \not\geq \sigma_i \}.$$

Evidently,  $\Psi_2$  contains those  $\delta \in \Psi$  which arise from  $\sigma_i$  by replacement of an element of the block

$$\tilde{\beta}_i = (a_{k_i+1}, \dots, a_{k_i+1} + 1)$$

by an element of the gap

$$\tilde{\chi}_{i-1} = (a_{k_{i-1}} + 1, \dots, a_{k_{i-1}} - 1).$$

Let  $\tilde{\beta}_{i-1} = (a_{k_{i-1}+1}, \dots, a_{k_{i-1}})$ ,  $p = |\tilde{\chi}_{i-1}|$ ,  $q = |\tilde{\beta}_i|$ . We choose a  $p \times q$  matrix  $T$  and an independent family  $\{T_\psi : \psi \in \Psi_1\}$  of indeterminates over  $B$ .

**(8.11) Lemma.** *The substitution  $T_\psi \longrightarrow \psi$ ,  $\psi \in \Psi_1$ , and*

$$T_{jk} \longrightarrow [\beta_0, \dots, \beta_{i-2}, \tilde{\beta}_{i-1}, a_{k_{i-1}} + j, \tilde{\beta}_i \setminus \{a_{k_{i+1}} + 2 - k\}, \beta_{i+1}, \dots, \beta_s]$$

*induces an isomorphism*

$$(B[T]/I_2(T))[T_\psi : \psi \in \Psi_1][T_\sigma^{-1}] \rightarrow R[\sigma_i^{-1}].$$

*Furthermore the prime ideal  $P$  generated in  $B[T]/I_2(T)$  by the elements of the first row of  $T$  extends to  $J(x; \zeta_{i-1})R[\sigma_i^{-1}]$ , the prime ideal  $Q$  generated by the elements of the first column extends to  $J(x; \zeta_i)R[\sigma_i^{-1}]$ .*

This lemma finishes the computation of  $\text{cl}(\omega)$ . It follows immediately from (8.8) that

$$\begin{aligned} \kappa_{i-1} - \kappa_i &= |\tilde{\chi}_{i-1}| - |\tilde{\beta}_i| \\ &= (|\chi_{i-1}| + 1) - (|\beta_i| + 1) \\ &= |\chi_{i-1}| - |\beta_i|. \end{aligned}$$

Before we state the main result, (8.11) should be proved.

PROOF OF 8.11: The substitution induces a surjective map

$$B[T][T_\psi: \psi \in \Psi_1][T^{-1}] \longrightarrow R[\sigma_i^{-1}].$$

This is proved as in (6.1). To see that  $I_2(T)$  is sent to zero, we look at the Plücker relation, for which, with the notations of (4.4),

$$\begin{aligned} "[a_1, \dots, a_k]" &= [\beta_0, \dots, \beta_{i-2}, \tilde{\beta}_{i-1}, \tilde{\beta}_i \setminus \{a_{k_{i+1}} + 2 - k\}, \beta_{i+1}, \dots, \beta_s], \\ "s" &= m + 1, \end{aligned}$$

$$"[c_1, \dots, c_s]" = [a_{k_{i-1}+j}, \beta_0, \dots, \beta_{i-2}, \tilde{\beta}_{i-1}, a_{k_{i-1}+u}, \tilde{\beta}_i \setminus \{a_{k_{i+1}}+2-v\}, \beta_{i+1}, \dots, \beta_s].$$

At most three indices in  $"[c_1, \dots, c_s]"$  do not occur in  $"[a_1, \dots, a_k]"$ , so at most three products can appear in this relation, one of which drops out in  $G(X; \gamma)$ : the "second" factor which contains both  $a_{k_{i-1}} + j$  and  $a_{k_{i-1}} + u$  is  $\not\subseteq \gamma$ . This leaves the desired relation. In order to show injectivity it is enough to prove that the ring on the left side has the same dimension as  $R[\sigma_i^{-1}]$ . This is easily checked if one remembers that  $|\Psi_1| = \dim G(X; \sigma_i) - \dim B$  (cf. (6.1)).

It is obvious that the extension of  $P$  is contained in  $J(x; \zeta_{i-1})R[\sigma_i^{-1}]$ ; since the latter is divisorial, inclusion implies equality. For  $Q$  one argues similarly. (It is of course also possible to prove equality directly.) —

**(8.12) Theorem.** *Let  $B$  be a normal Cohen-Macaulay domain having a canonical module  $\omega_B$ ,  $X$  an  $m \times n$  matrix of indeterminates, and  $\gamma \in G(X)$ . Then a canonical module of  $R = G(X; \gamma)$  is given by a divisorial ideal with class*

$$\text{cl}(\omega_B R) + \sum_{i=0}^t \kappa_i \text{cl}(J(x; \zeta_i))$$

such that

$$\kappa_{i-1} - \kappa_i = |\chi_{i-1}| - |\beta_i|, \quad i = 1, \dots, t,$$

where  $t = s$  if  $a_m < n$ ,  $t = s - 1$  if  $a_m = n$ ,  $\beta_0, \dots, \beta_s$  are the blocks of  $\gamma$ , and  $\chi_{i-1}$  is the gap between  $\beta_{i-1}$  and  $\beta_i$ . Every canonical module of  $R$  has this representation, and up to the choice of  $\omega_B$  it is unique.

**(8.13) Corollary.** *Let  $B$  be an (arbitrary) noetherian ring. Then  $G(X; \gamma)$  is a Gorenstein ring if and only if  $B$  is Gorenstein and  $|\chi_{i-1}| = |\beta_i|$  for  $i = 1, \dots, t$ .*

(8.12) has been completely proved already, and (8.13) follows from it as (8.9) followed from (8.8). In particular we have  $|\chi_{i-1}| = |\beta_i|$  for  $i = 1, \dots, t$  if  $t = 0$ , in which case  $G(X; \gamma)$  is factorial (over a factorial  $B$ ).

The easiest way to deal with  $R = R(X; \delta)$  is to relate it to  $\tilde{R} = G(\tilde{X}; \tilde{\delta})$  as usual (cf. (5.5)):

$$R = \tilde{R}/\tilde{R}y, \quad y = [n + 1, \dots, n + m] \pm 1.$$

Assuming that  $B$  is a field one writes the canonical module  $\omega_{\tilde{R}}$  of  $\tilde{R}$  as  $\bigcap \tilde{P}_i^{(\kappa_i)}$ ,  $\tilde{P}_i$  running through the prime ideals corresponding to the upper neighbours of  $\tilde{\delta}$ , and  $\kappa_i \geq 0$ . Then

$$(*) \quad \text{cl}(\omega_R) = \sum \kappa_i \text{cl}(P_i)$$

where the sum is now extended over the prime ideals corresponding to the upper neighbours of  $\delta$ , and  $P_i$  is the image of  $\tilde{P}_i$  in  $R$ . The equation (\*) can be derived from (8.10), but it is easier to use the properties of dehomogenization. Since  $y$  is not a zero-divisor,

$$\omega_R = \omega_{\tilde{R}}/y\omega_{\tilde{R}},$$

and since  $y$  is not a zero-divisor modulo  $\tilde{\omega}$  (as an ideal) one has  $y\omega_{\tilde{R}} = \omega_{\tilde{R}} \cap \tilde{R}y$ , hence

$$\omega_R = (\omega_{\tilde{R}} + R\tilde{y})/R\tilde{y}.$$

Now (\*) follows from (16.27): dehomogenization preserves primary decomposition. We remind the reader that the upper neighbours of  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r]$  have been named above (8.3):  $\zeta_i$  and  $\eta_i$  arising from raising a row and a column index resp., and, in case  $a_r = m, b_r = n, \vartheta = [a_1, \dots, a_{r-1} | b_1, \dots, b_{r-1}]$ . The blocks of  $[a_1, \dots, a_r]$  are

$$\beta_0, \dots, \beta_u \quad \text{with gaps} \quad \chi_0, \dots, \chi_u,$$

those of  $[b_1, \dots, b_r]$  are denoted

$$\beta_0^*, \dots, \beta_v^* \quad \text{with gaps} \quad \chi_0^*, \dots, \chi_v^*.$$

Furthermore let  $w = u$  if  $a_r < m, w = u - 1$  if  $a_r = m, z = v$  if  $b_r < n, z = v - 1$  if  $b_r = n$ .

Relating the blocks and gaps of  $[a_1, \dots, a_r]$  and  $[b_1, \dots, b_r]$  to those of  $\tilde{\delta}$ , the reader will easily derive the following theorem:

**(8.14) Theorem.** *Let  $B$  be a normal Cohen-Macaulay domain having a canonical module  $\omega_B, X$  an  $m \times n$  matrix of indeterminates, and  $\delta \in \Delta(X)$ . Then a canonical module of  $R = R(X; \delta)$  is given by a divisorial ideal with class*

$$\text{cl}(\omega_B R) + \begin{cases} \sum_{i=0}^w \kappa_i \text{cl}(\mathbf{I}(x; \zeta_i)) + \sum_{i=0}^z \lambda_i \text{cl}(\mathbf{I}(x; \eta_i)) & \text{if } a_r < m \text{ or } b_r < n, \\ \sum_{i=0}^w \kappa_i \text{cl}(\mathbf{I}(x; \zeta_i)) + \sum_{i=0}^z \lambda_i \text{cl}(\mathbf{I}(x; \eta_i)) + \mu \text{cl}(\mathbf{I}(x; \vartheta)) & \text{if } a_r = m \text{ and } b_r = n, \end{cases}$$

where

$$\begin{aligned} \kappa_{i-1} - \kappa_i &= |\chi_{i-1}| - |\beta_i|, & i &= 1, \dots, w, \\ \lambda_{i-1} - \lambda_i &= |\chi_{i-1}^*| - |\beta_i^*|, & i &= 1, \dots, z, \\ \lambda_z - \kappa_w &= \begin{cases} |\chi_z^*| - |\chi_w| (= n - b_r - (m - a_r)) & \text{if } a_r < m, b_r < n, \\ (|\chi_z^*| + |\beta_{w+1}|) - |\chi_w| & \text{if } a_r = m, b_r < n, \\ |\chi_z^*| - (|\beta_{z+1}^*| + |\chi_w|) & \text{if } a_r < m, b_r = n, \end{cases} \\ \mu - \kappa_w &= |\beta_{w+1}| - |\chi_w| & \text{if } a_r = m, b_r = n, \\ \mu - \lambda_z &= |\beta_{z+1}^*| - |\chi_z^*| & \text{if } a_r = m, b_r = n. \end{aligned}$$

Every canonical module of  $R$  has this representation, and up to the choice of  $\omega_B$  it is unique.

Again,  $R$  is Gorenstein if and only if  $B$  is Gorenstein, and all the differences listed above vanish (as far as they apply to a specific  $R$ ).

Within divisor theory the preceding theorems are completely satisfactory. Nevertheless they suffer from an ideal-theoretic deficiency: We don't have a concrete description of the symbolic powers of the prime ideals generating the class group. As we shall see in the next section, they coincide with the ordinary powers. Only for  $R_{r+1}(X)$  this has been proved already, cf. (7.10).

### D. Comments and References

For the simplest case  $R_2(X)$ ,  $X$  a  $2 \times 2$  matrix, the class group is computed in Fossum's book ([Fs], § 14), and Theorems (8.1) and (8.3) may be viewed as natural generalizations, the intermediate case  $R_{r+1}(X)$  being covered by [Br.3]. The factoriality of  $G(X)$  was proved by Samuel ([Sa], p. 38), cf. also [Ho.3], Corollary 3.15.

The computation of the canonical class was initiated in [Br.6] for  $R_{r+1}(X)$ . According to [Hu.1], p. 500 this case was also solved by Hochster. Yoshino [Yo.1] computed the canonical module of  $R_m(X)$  directly from the Eagon-Northcott resolution (cf. Section 2). Svanes determined the Gorenstein rings among the homogeneous coordinate rings of the Schubert varieties and derived (8.9), cf. [Sv.1], pp. 451,452. The first attempt towards (8.9) was made by Eagon [Ea.2] who obtained the result for ideals of maximal minors. Goto [Go.1] proved the necessity of  $m = n$  in (8.9) in general and the sufficiency for the case  $r = 1$ .

Stanley showed that the Gorenstein property is reflected in the Hilbert function of a graded Cohen-Macaulay domain. This fact can also be used to determine the Gorenstein rings among the rings  $R(X; \delta)$  and  $G(X; \gamma)$ , cf. [St].



## 9. Powers of Ideals of Maximal Minors

In Section 7 we have derived results on the powers of certain ideals in the rings  $R_{r+1}(X)$  by invariant-theoretic methods (cf. (7.10) and (7.24)). The ideals considered there are  $I_m(X)$ , the ideal generated by the  $m$ -minors of an  $m \times n$  matrix of indeterminates, and the ideals  $P$  and  $Q$  appearing in the description of the class group and the canonical class of  $R_{r+1}(X)$  in Section 8. In this section we want to investigate more generally the powers of ideals in  $G(X; \gamma)$  and  $R(X; \delta)$  which can justifiably be called *ideals of maximal minors*. They share a remarkable property: their generators in the poset underlying  $R(X; \delta)$  or  $G(X; \gamma)$  generate a sub-ASL in a natural way. The graded algebras related to the powers of such ideals, the ordinary and extended Rees algebra, and the associated graded ring, are again ASLs over wonderful posets and (normal) domains if the ring  $B$  of coefficients is a (normal) domain. In particular, the ideals have primary powers (over an integral  $B$ ), and one obtains a lower bound of their depths (in suitable localizations).

### A. Ideals and Subalgebras of Maximal Minors

Let  $U$  be a matrix over a commutative ring. If  $I_{r+1}(U) = 0$  and  $I_r(U) \neq 0$ , then an  $r$ -minor  $\neq 0$  is called a *maximal minor* of  $U$ . In  $R(X; \delta)$  (considered over an arbitrary commutative ring  $B$ ) an ideal  $I$  is said to be an *ideal of maximal minors* if it is generated by the maximal minors of a submatrix  $U$  of the matrix of residue classes of  $X$  which consists of the first  $u$  rows or first  $v$  columns,  $1 \leq u \leq m$ ,  $1 \leq v \leq n$ . More formally, the ideals of maximal minors are the ideals

$$I(X; \varepsilon)/I(X; \delta),$$

$\delta = [a_1, \dots, a_r | b_1, \dots, b_r]$ ,  $\varepsilon = [a_1, \dots, a_{k-1}, \tilde{a}_k, \tilde{a}_{k+1}, \dots, \tilde{a}_{\tilde{r}} | b_1, \dots, b_{\tilde{r}}]$ , where  $\tilde{a}_k$  is a given integer such that

$$a_k < \tilde{a}_k \leq a_{k+1}$$

and  $\varepsilon$  is the smallest element in  $\Delta(X; \delta)$  whose row part starts as  $a_1, \dots, a_{k-1}, \tilde{a}_k$ , or  $\varepsilon = [a_1, \dots, a_{\tilde{r}} | b_1, \dots, b_{l-1}, \tilde{b}_l, \dots, \tilde{b}_{\tilde{r}}]$  with a similar condition. We allow the extreme cases  $k = r$  and  $\tilde{a}_k = m + 1$ , so  $\varepsilon = [a_1, \dots, a_{r-1} | b_1, \dots, b_{r-1}]$  and  $I$  being generated by all the  $r$ -minors then. The ideal indicated is generated by the  $k$ -minors of the rows  $1, \dots, \tilde{a}_k - 1$ , and the condition  $\tilde{a}_k \leq a_{k+1}$  guarantees that the  $(k + 1)$ -minors of these rows are zero. For simplicity we call the corresponding ideals

$$J(X; \varepsilon)/J(X; \gamma),$$

$\gamma = [a_1, \dots, a_m]$ ,  $\varepsilon = [a_1, \dots, a_{k-1}, \tilde{a}_k, \dots, \tilde{a}_m]$ ,  $\tilde{a}_k \leq a_{k+1}$ , and  $\varepsilon$  as small as possible, ideals of maximal minors, too, and say that  $\varepsilon$  defines an ideal of maximal minors. Note that all the elements in  $\Delta(X; \delta)$  or  $\Gamma(X; \gamma)$  which have been important for the structure of  $R(X; \delta)$  or  $G(X; \gamma)$ , define ideals of maximal minors: the upper neighbours of  $\gamma$  or  $\delta$  as well as the elements describing the singular locus.

The crucial property of ideals of maximal minors is given by the following lemma:

**(9.1) Lemma.** *Let  $\varepsilon$  define an ideal of maximal minors in  $G(X; \gamma)$  or  $R(X; \delta)$  and  $\Omega = \Gamma(X; \gamma) \setminus \Gamma(X; \varepsilon)$  or  $\Omega = \Delta(X; \delta) \setminus \Delta(X; \varepsilon)$ . Let  $\xi, v \in \Omega$  be incomparable. Then every standard monomial appearing in the standard representation*

$$\xi v = \sum a_\mu \mu, \quad a_\mu \neq 0,$$

*is the product of two factors in  $\Omega$ .*

PROOF: Consider the case  $G(X; \gamma)$  first,  $\gamma = [a_1, \dots, a_m]$ . Then every standard monomial appearing on the right side of the straightening relation has exactly two factors, and the union of their indices coincides with the union of the indices of  $\xi, v$ . Now  $\zeta \in \Gamma(X; \gamma)$  is in  $\Omega$  if and only if it has  $k$  indices  $< a_k$ . On the other hand  $\zeta$  cannot have  $k+1$  indices  $< a_k$ , for  $\zeta \not\leq \gamma$  then. Since  $\xi$  and  $v$  together contain  $2k$  indices  $< a_k$  (counted with multiplicities), and both factors of  $\mu$  can have at most  $k$  such indices, both of them have exactly  $k$  of them, and so are in  $\Omega$ .

Again it is useful to consider  $R(X; \delta)$  arising from  $G(\tilde{X}; \tilde{\delta})$  in the usual way. For every element  $\zeta \in \Delta(X; \delta)$  let  $\tilde{\zeta}$  denote the corresponding element in  $\Gamma(\tilde{X}; \tilde{\delta})$ . Then

$$\tilde{\Omega} = \Gamma(\tilde{X}; \tilde{\delta}) \setminus \Gamma(\tilde{X}; \tilde{\varepsilon}) = \{ \tilde{\zeta} : \zeta \in \Omega \},$$

and one checks immediately that  $\tilde{\varepsilon}$  defines an ideal of maximal minors in  $G(\tilde{X}; \tilde{\delta})$ . Let  $\tilde{\xi}\tilde{v} = \sum a_{\tilde{\mu}} \tilde{\mu}$  be the standard representation of  $\tilde{\xi}\tilde{v}$ . From the first part of the proof we know that each of the  $\tilde{\mu}$  has both its factors in  $\tilde{\Omega}$ , so does not contain  $[n+1, \dots, n+m]$ . In passing from  $G(\tilde{X}; \tilde{\delta})$  to  $R(X; \delta)$  one only replaces  $[n+1, \dots, n+m]$  by  $(-1)^{m(m-1)/2}$ . (It is of course as easy to argue directly for  $R(X; \delta)$ .) —

**(9.2) Corollary.** *Let  $S$  be the  $B$ -submodule generated by the standard monomials which have all their factors in  $\Omega$ . Then  $S$  is a subalgebra of  $G(X; \gamma)$  or  $R(X; \delta)$  resp., and therefore automatically a graded ASL on  $\Omega$ .*

In fact, the argument that proved (4.1), shows that  $S$  is a subalgebra, and the rest is obvious. The properties of  $S$  will be noted below: they are as good as one could reasonably hope for.

**(9.3) Corollary.** *Let  $I$  be the ideal of maximal minors generated by the ideal  $\Omega$  in  $\Gamma(X; \gamma)$  or  $\Delta(X; \delta)$ . Then  $I^j$  is the submodule of  $G(X; \gamma)$  or  $R(X; \delta)$  resp. generated by the standard monomials containing at least  $j$  factors in  $\Omega$ , so as an ideal it is generated by the standard monomials of length  $j$  in  $\Omega$ .*

PROOF:  $I^j$  obviously contains the  $B$ -submodule and the ideal mentioned, and the first statement implies the second one. Furthermore  $I^j$  is generated as a  $B$ -module by all the monomials containing at least  $j$  factors from  $\Omega$ . So it is sufficient that every standard monomial appearing in the standard representation of such a monomial contains at least  $j$  factors from  $\Omega$ , too. This standard representation is produced by repeated straightening of pairs of incomparable elements. However, in a straightening relation  $\xi v = \sum a_\mu \mu$  with one factor in  $\Omega$  every  $\mu$  has to contain such a factor, too, since  $\Omega$  is an ideal in the poset underlying  $G(X; \gamma)$  or  $R(X; \delta)$ , and if both  $\xi \in \Omega$  and  $v \in \Omega$ , then  $\mu$  contains (exactly) two factors from  $\Omega$  by virtue of (9.1). —

**(9.4) Proposition.** *Let  $\varepsilon, \Omega$ , and  $S$  be as above. Then:*

- (a)  $\Omega$  is a sublattice of  $\Gamma(X; \gamma)$  or  $\Delta(X; \delta)$  resp.
- (b)  $S$  is a Cohen-Macaulay ring if (and only if)  $B$  is Cohen-Macaulay.
- (c) Let  $B$  be noetherian.
  - (i) If  $R = G(X; \gamma)$ , then  $\dim R - \dim S = k(n - m + k - \tilde{a}_k + 1)$ .
  - (ii) If  $R = R(X; \delta)$  and  $\varepsilon = [a_1, \dots, a_{k-1}, \tilde{a}_k, \dots, \tilde{a}_{\tilde{r}} | b_1, \dots, b_{\tilde{r}}]$ , then

$$\dim R - \dim S = k(m + k - \tilde{a}_k + 1) - 1.$$

*An analogous formula holds for  $\varepsilon = [a_1, \dots, a_{\tilde{r}} | b_1, \dots, b_{l-1}, \tilde{b}_l, \dots, \tilde{b}_{\tilde{r}}]$ .*

- (d)  $S$  is a (normal) integral domain, if (and only if)  $B$  is a (normal) integral domain.

We outline the PROOF; the reader may supply the details (should there be any). Part (a) is quite evident, and (b) follows from (5.14). For (c) one counts the number of steps one needs to climb from the (single) maximal element of  $\Omega$  to that of  $\Gamma(X; \gamma)$  or  $\Delta(X; \delta)$ , (All the maximal chains in a distributive lattice have the same length.) For (d) one proceeds as in Section 6. Let first  $R = G(X; \gamma)$ , and  $\Psi$  as in (6.1). Then  $\Psi \cap \Omega$  has  $\dim S$  elements, and one checks that

$$S[\gamma^{-1}] \cong B[T_\psi : \psi \in \Psi \cap \Omega][T_\gamma^{-1}].$$

For  $R = R(X; \delta)$  one constructs  $\tilde{R} = G(\tilde{X}; \tilde{\delta})$ ,  $\tilde{\varepsilon}$ ,  $\tilde{\Omega}$ ,  $\tilde{S}$  and observes that  $\tilde{S}$  is mapped isomorphically onto  $S$  here; cf. the last part of the proof of (9.1). —

The reader may try to find the class group of  $S$  and the canonical class.

**(9.5) Remark.** *The Segre product*

$$\bigoplus_{k=0}^{\infty} A_k \otimes \tilde{A}_k$$

of graded ASLs  $A = \bigoplus A_k$  on  $\Pi$  and  $\tilde{A} = \bigoplus \tilde{A}_k$  on  $\tilde{\Pi}$  is a graded ASL on the poset

$$\bigcup_{k=0}^{\infty} \{ \xi \otimes \tilde{\xi} : \xi \in \Pi \cap A_k, \tilde{\xi} \in \tilde{\Pi} \cap \tilde{A}_k \}$$

ordered by the decree

$$\xi \otimes \tilde{\xi} \leq v \otimes \tilde{v} \iff \xi \leq v, \tilde{\xi} \leq \tilde{v}.$$

The straightforward verification of this fact can be left to the reader, and we mention it only because some of the ASLs  $S$  considered in (9.4) can be viewed as such Segre products. Let  $X$  be an  $m \times n$  matrix of indeterminates,  $\delta = [1, \dots, r | 1, \dots, r]$ , and  $\varepsilon = [1, \dots, r-1 | r, \dots, r-1]$ . Then  $S$  is the Segre product of  $G(Y)$  and  $G(Z)$  where  $Y$  is an  $r \times m$  matrix and  $Z$  is an  $r \times n$  matrix. Note that this includes the case  $S = R_2(X)$  in which  $S$  is the Segre product of two polynomial rings in  $m$  and  $n$  variables resp. —

### B. ASL Structures on Graded Algebras Derived from an Ideal

The algebras derived from an ideal  $I$  in a ring  $A$  we want to consider, are

the (ordinary) *Rees algebra*  $\mathcal{R}_I(A) = \bigoplus_{j=0}^{\infty} I^j T^j \subset A[T]$ ,  $T$  an indeterminate,

the *extended Rees algebra*  $\widehat{\mathcal{R}}_I(A) = \mathcal{R}_I(A) \oplus \bigoplus_{j=1}^{\infty} AT^{-j} \subset A[T, T^{-1}]$ , and

the *associated graded ring*  $\text{Gr}_I A = \bigoplus_{j=0}^{\infty} I^j / I^{j+1}$ .

One has the representations

$$\begin{aligned} \text{Gr}_I A &= \mathcal{R}_I(A) / I \mathcal{R}_I(A), \\ \text{Gr}_I A &= \widehat{\mathcal{R}}_I(A) / T^{-1} \widehat{\mathcal{R}}_I(A). \end{aligned}$$

We suppose that  $\bigcap_{j=0}^{\infty} I^j = 0$ . Then every element  $x \in A$  has a well-defined degree with respect to the filtration of  $A$  by the powers of  $I$ :

$$\text{grad } x = j \quad \text{if } x \in I^j \setminus I^{j+1}.$$

The element  $xT^{\text{grad } x}$  in  $\mathcal{R}_I(A) \subset \widehat{\mathcal{R}}_I(A)$  and its residue class in  $\text{Gr}_I A$  will both be denoted by  $x^*$  and called the *leading form* of  $x$  with respect to  $I$ .

Let there be given a graded ASL  $A$  on  $\Pi$  over a ring  $B$  of coefficients,  $\Omega$  an ideal in the poset  $\Pi$ , and  $I = A\Omega$ . We say that  $I$  (or  $\Omega$ ) is *straightening-closed* if every standard monomial  $\mu$  appearing in the standard representation  $\xi v = \sum a_{\mu} \mu$  of incomparable elements  $\xi, v \in \Omega$  contains at least two factors in  $\Omega$ . Note that automatically  $\bigcap_{j=0}^{\infty} I^j = 0$ :

**(9.6) Proposition.** *Let  $A$  be a graded ASL on  $\Pi$  over  $B$ , and  $\Omega \subset \Pi$  an ideal such that  $I = A\Omega$  is straightening-closed. Then  $I^j$  is the  $B$ -submodule of  $A$  generated by all standard monomials with at least  $j$  factors in  $\Omega$ .*

This proposition is proved as Corollary (9.3).

**(9.7) Theorem.** *Let  $A$  be a graded ASL on  $\Pi$  over  $B$ , and  $\Omega \subset \Pi$  an ideal such that  $I = A\Omega$  is straightening-closed. Then the extended Rees algebra  $\widehat{\mathcal{R}}_I(A)$  is a graded ASL over  $B[T^{-1}]$  on the poset*

$$\Pi^* = \{ \xi^* : \xi \in \Pi \}$$

ordered by:  $\xi^* \leq v^* \iff \xi \leq v$ .

PROOF: One has  $\widehat{\mathcal{R}}_I(A) = \bigoplus_{j=-\infty}^{\infty} R_j T^j$ . Each  $R_j$  is a graded  $B$ -module since  $I$  is a homogeneous ideal:  $R_j = \bigoplus_{k=0}^{\infty} R_{jk}$ , and  $\widehat{\mathcal{R}}_I(A)$  then is a graded  $B[T^{-1}]$ -algebra with homogeneous components

$$\widehat{R}_k = \bigoplus_{j=-\infty}^{\infty} R_{jk} T^j.$$

Evidently  $\Pi^*$  generates  $\widehat{\mathcal{R}}_I(A)$  as a  $B[T^{-1}]$ -algebra.

The ring  $A[T, T^{-1}] = A \otimes B[T, T^{-1}]$  is obviously a graded ASL on  $\Pi$  over  $B[T, T^{-1}]$ . Since  $\Pi^*$  arises from  $\Pi$  over  $B[T, T^{-1}]$  by multiplication of its elements with units of  $B[T, T^{-1}]$ ,  $A[T, T^{-1}]$  is also an ASL on  $\Pi^*$ , implying the linear independence of the standard monomials in  $\Pi^*$  over the smaller ring  $B[T^{-1}]$ .

It follows from the preceding proposition that

$$\xi^* = \xi \quad \text{for } \xi \in \Pi \setminus \Omega \quad \text{and} \quad \xi^* = \xi T \quad \text{for } \xi \in \Omega.$$

Let  $\xi, v \in \Pi$  be incomparable with standard representation  $\xi v = \sum a_\mu \mu$ . If  $\mu = \pi_1, \dots, \pi_k, \pi_j \in \Pi$ , then  $\mu^* = \pi_1^*, \dots, \pi_k^*$ , and

$$\xi^* v^* = \sum a_\mu T^{j_\mu} \mu^*$$

is the standard representation of  $\xi^* v^*$  over  $B[T, T^{-1}]$ . By the hypotheses on  $I$ ,  $j_\mu = \text{grad } \xi v - \text{grad } \mu \leq 0$  for all  $\mu$ , and we have a standard representation over  $B[T^{-1}]$ . —

**(9.8) Corollary.** *The associated graded ring  $\text{Gr}_I A$  is a graded ASL on (the image of)  $\Pi^*$  over  $B$ .*

PROOF: In passing from  $\widehat{\mathcal{R}}_I(A)$  to

$$\text{Gr}_I A = \widehat{\mathcal{R}}_I(A) \otimes_{B[T^{-1}]} (B[T^{-1}]/T^{-1}B[T^{-1}]) = \widehat{\mathcal{R}}_I(A) \otimes_{B[T^{-1}]} B$$

we have only “extended” the ring of coefficients. —

**(9.9) Corollary.** *If moreover  $\Pi$  is wonderful and  $B$  is a Cohen-Macaulay ring, then  $\widehat{\mathcal{R}}_I(A)$  and  $\text{Gr}_I A$  are Cohen-Macaulay rings, too.*

Let  $\xi v = \sum a_\mu \mu$  be a straightening relation in  $A$ . Then

$$\xi^* v^* = \sum a_\mu T^{j_\mu} \mu^*, \quad j_\mu = \text{grad } \xi v - \text{grad } \mu,$$

is the corresponding straightening relation in  $\widehat{\mathcal{R}}_I(A)$ , and in  $\text{Gr}_I A$  it transforms into

$$\xi^* v^* = \sum_{j_\mu=0} a_\mu \mu^*,$$

so one obtains this relation from  $\sum a_\mu \mu$  by dropping all the terms on the right side which have higher degree with respect to  $I$  than  $\xi v$ .

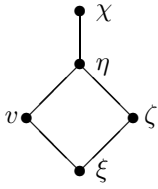
We want to make the ordinary Rees algebra  $\mathcal{R}_I(A)$  an ASL over  $B$ . Obviously  $\Pi^*$  does not generate  $\mathcal{R}_I(A)$  as a  $B$ -algebra: the elements  $\xi \in \Omega \subset A \subset \mathcal{R}_I(A)$  are not representable by polynomials in  $\Pi^*$  with coefficients in  $B$ , and we have to “double”  $\Omega$ : Let

$$\Pi \uplus \Omega: = \Pi \cup \Omega^* \subset \mathcal{R}_I(A)$$

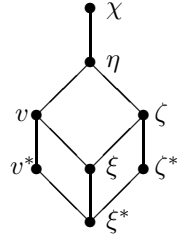
where the subsets  $\Pi$  and  $\Omega^*$  are ordered naturally and every other relation is given by

$$\xi^* < v \quad \text{for } \xi \in \Omega, v \in \Pi \quad \text{such that } \xi \leq v.$$

$\Pi \uplus \Omega$  is evidently a partially ordered subset of  $\mathcal{R}_I(A)$ . For example, if  $\Pi$  is given by



and  $\Omega = \{\xi, v, \zeta\}$ , then  $\Pi \uplus \Omega$  is

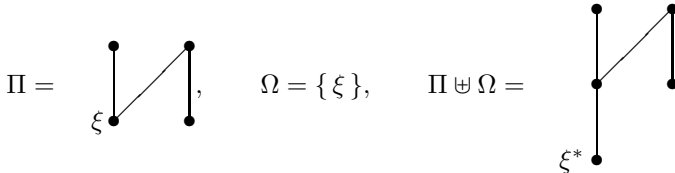


**(9.10) Theorem.** *Let  $A$  be a graded ASL on  $\Pi$  over  $B$ ,  $\Omega \subset \Pi$  an ideal such that  $I = A\Omega$  is straightening-closed. Then  $\mathcal{R}_I(A)$  is a graded ASL on  $\Pi \uplus \Omega$  over  $B$ .*

PROOF:  $\mathcal{R}_I(A)$  is obviously a graded  $B$ -algebra and generated by  $\Pi \uplus \Omega$ . The full polynomial ring  $A[T]$  is a graded ASL on  $\Pi \cup \{T\}$  if we declare  $T$  to be the maximal (or minimal) element of  $\Pi \cup \{T\}$ . Since for a standard monomial  $\mu = \pi_1 \dots \pi_k$ ,  $\pi_j \in \Pi \uplus \Omega$ , the factors from  $\Omega^*$  have to proceed the factors from  $\Pi$ , the standard monomials in  $\Pi \uplus \Omega$  correspond bijectively to those standard monomials in  $\Pi \cup \{T\}$  whose degree with respect to  $T$  does not exceed the number of factors from  $\Omega$ . Therefore the standard monomials in  $\Pi \uplus \Omega$  are linearly independent. In order to write down the straightening relations we represent every standard monomial  $\mu$  as  $\mu = \alpha_\mu \beta_\mu \omega_\mu$ ,  $\alpha_\mu$  being the smallest factor,  $\beta_\mu$  the second (if present), and  $\omega_\mu$  the “tail”. There are three types of straightening relations, always derived from the straightening relation  $\xi v = \sum a_{\mu} \mu$  in  $A$  (and, in case  $\xi, v \in \Omega$ ,  $\xi > v$ , from the relation  $\xi v = v\xi$ ):

- (i)  $\xi, v \in \Pi$ :  $\xi v = \sum a_{\mu} \mu$ ,
- (ii)  $\xi \in \Omega, v \in \Pi$ :  $\xi^* v = \sum a_{\mu} \alpha_{\mu}^* \beta_{\mu} \omega_{\mu}$ ,
- (iii)  $\xi, v \in \Omega$ :  $\xi^* v^* = \sum a_{\mu} \alpha_{\mu}^* \beta_{\mu}^* \omega_{\mu}$ . —

For the extended Rees algebra and the associated graded algebra the poset  $\Pi$  has only been replaced by an isomorphic copy. As the example above shows,  $\Pi \uplus \Omega$  need not be wonderful without further hypothesis:  $v^*$  and  $z^*$  are upper neighbours of  $\xi^*$ , but don't have a common upper neighbour. Such an obstruction does not occur, if  $\Omega$  is *self-covering*: every upper neighbour of elements  $v, z \in \Omega$  which have a common lower neighbour  $\xi \in \Omega \cup \{-\infty\}$ , is in  $\Omega$ . For the examples of interest to us,  $\Omega$ , being a sublattice of a lattice  $\Pi$  then, is always self-covering. As the following example shows, even this is not completely sufficient:



**(9.11) Lemma.** *Suppose that  $\Pi$  is wonderful and  $\Omega$  a self-covering ideal in  $\Pi$  containing all the minimal elements of  $\Pi$ . Then  $\Pi \uplus \Omega$  is wonderful.*

PROOF: The definition of the partial order on  $\Pi \uplus \Omega$  implies: (a)  $\xi \in \Omega^*$  has a single upper neighbour  $\eta \in \Pi$ , and  $\xi = \eta^*$ . (b) If  $v \in \Pi$  and  $z \geq v$  then  $z \in \Pi$ .

Let  $\widehat{\Pi} = (\Pi \uplus \Omega) \cup \{\infty, -\infty\}$ , and suppose that  $v_1, v_2 \in \Pi \uplus \Omega$ ,  $v_1 \neq v_2$ , have a common lower neighbour  $\xi \in \widehat{\Pi}$  and  $v_1, v_2 \leq \zeta \in \widehat{\Pi}$ . Because of (a) and (b) we have to consider the cases:

- (i)  $v_1, v_2 \in \Pi$ ,  $\zeta \notin \Omega^*$ ;      (ii)  $v_1 \in \Pi$ ,  $v_2 \in \Omega^*$ ,  $\zeta \notin \Omega^*$ ;
- (iii)  $v_1, v_2 \in \Omega^*$ ,  $\zeta \in \Omega^*$ ;      (iv)  $v_1, v_2 \in \Omega^*$ ,  $\zeta \notin \Omega^*$ .

Case (i) is trivial, and in case (iii) one only needs that  $\Omega$  is an ideal in the wonderful poset  $\Pi$ . In case (iv) we write  $v_i = \omega_i^*$ . Since  $\xi \in \Omega^* \cup \{-\infty\}$ ,  $\omega_1$  and  $\omega_2$  have a common lower neighbour in  $\Pi$  or are minimal elements of  $\Pi$ . Furthermore  $\omega_1, \omega_2 \leq \zeta$ , so they have a common upper neighbour  $\tau \leq \zeta$ , and necessarily  $\tau \in \Omega$ . Consequently  $\tau^* \leq \zeta$ , and  $\tau^*$  is an upper neighbour of  $v_1$  and  $v_2$ . In case (ii) it is impossible that  $\xi = -\infty$  since  $v_1$  is not minimal in  $\Pi \uplus \Omega$ : If  $v_1 \notin \Omega$ , then  $v_1$  is not even minimal in  $\Pi$ , and otherwise  $v_1^* < v_1$ . The case  $\xi = -\infty$  being excluded, necessarily  $\xi \in \Omega^*$ , and it follows immediately that  $\xi = v_1^*$ . Now  $v_2 = v^*$  for a  $v \in \Omega$ , and  $v$  is a suitable common upper neighbour of  $v_1$  and  $v_2$ . —

**(9.12) Corollary.** *Let  $A$  be a graded ASL on  $\Pi$  over  $B$ . Suppose that  $B$  is a Cohen-Macaulay ring, and  $\Pi$  a wonderful poset. Let  $\Omega$  be an ideal in  $\Pi$  such that  $\Omega$  is self-covering, contains all the minimal elements of  $\Pi$ , and  $I = A\Omega$  is straightening-closed. Then  $\mathcal{R}_I(A)$  is a Cohen-Macaulay ring, too.*

**(9.13) Remark.** Besides the Rees algebra(s) and the associated graded ring there is another commutative algebra defined by “powers” of  $I$ : the symmetric algebra

$$S(I) = \bigoplus_{j=0}^{\infty} S_j(I).$$

The natural epimorphisms  $S_j(I) \rightarrow I^j$ , sending a product of  $j$  elements of  $I$  in  $S_j(I)$  to their product in  $I^j$ , defines a natural epimorphism

$$S(I) \longrightarrow \mathcal{R}_I(A).$$

It would be unreasonable to expect that this epimorphism is an isomorphism for our rings and ideals, except under very rare circumstances. One has a commutative diagram

$$\begin{array}{ccc} A[T_\omega : \omega \in \Omega] & \xrightarrow{\varphi} & S(I) \\ \psi \searrow & & \swarrow \\ & & \mathcal{R}_I(A), \end{array}$$

the indeterminate  $T_\omega$  being sent to  $\omega \in S_1(I)$  and  $\omega^*$  resp. Being a graded ASL,  $\mathcal{R}_I(A)$  is represented over  $B$  by its generators  $\Pi \uplus \Omega$  and the straightening relations. Therefore the kernel of  $\psi$  is generated by the elements representing the straightening relations of types (ii) and (iii) in the proof of (9.10). The elements of type (ii) are in the kernel of  $\varphi$ , too. So  $S(I)$  and  $\mathcal{R}_I(A)$  are isomorphic if  $\Omega$  is linearly ordered and no relations of type (iii) are present. —

### C. Graded Algebras with Respect to Ideals of Maximal Minors

Without further ado we draw the consequences of the results in Subsections A and B:

**(9.14) Theorem.** *Let  $R = G(X; \gamma)$  or  $R = R(X; \delta)$  over a ring  $B$  of coefficients,  $I$  an ideal of maximal minors in  $R$ ,  $\Pi = \Gamma(X; \gamma)$  or  $\Pi = \Delta(X; \delta)$  resp., and  $\Omega$  the ideal in  $\Pi$  generating  $I$ .*

- (a) *Then  $\text{Gr}_I R$  and  $\mathcal{R}_I(R)$  are graded ASLs over  $B$  on  $\Pi^* \subset \text{Gr}_I R$  and  $\Pi \uplus \Omega$  resp.*
- (b) *If  $B$  is a Cohen-Macaulay ring, then  $\text{Gr}_I R$ ,  $\widehat{\mathcal{R}}_I(R)$ , and  $\mathcal{R}_I(R)$  are Cohen-Macaulay rings, too.*

Next we want to prove that  $\text{Gr}_I R$ ,  $\widehat{\mathcal{R}}_I(R)$ , and  $\mathcal{R}_I(R)$  are (normal) domains over a (normal) domain  $B$ . First we observe that the sub-ASL generated by  $\Omega$  is present in  $\text{Gr}_I R$  (and  $\mathcal{R}_I(R)$ ), too:

**(9.15) Lemma.** *Under the hypothesis of the preceding theorem  $\Omega^*$  generates a sub-ASL of  $\text{Gr}_I R$ . It is isomorphic to the sub-ASL generated by  $\Omega$  in  $R$ .*

This is obvious: the straightening relation for incomparable  $\xi^*$  and  $v^*$  in  $\Omega^*$  is produced from that for  $\xi$  and  $v$  in  $\Omega$  by “starring” all the factors  $\zeta \in \Pi$  occurring. (Remember that a graded ASL is completely determined by its straightening relations!)

Different from our usual procedure we start with the case  $R = R(X; \delta)$  for the investigation of integrity and normality, mainly because we regard expansions of determinants more “visible” than Plücker relations. Let  $A = \text{Gr}_I R$ , and assume that  $B$  is normal. The element  $\delta^*$  is minimal in the poset underlying  $A$ , and one would like to show that  $A[(\delta^*)^{-1}]$  is normal in order to apply (16.24) then. Let

$$\delta = [a_1, \dots, a_r | b_1, \dots, b_r],$$

and the element  $\varepsilon$  defining  $I$  be given by

$$\varepsilon = [a_1, \dots, a_{k-1}, \tilde{a}_k, \dots, \tilde{a}_r | b_1, \dots, b_r].$$

In  $R$  we expand the minor  $\delta$  along its first  $k$  rows:

$$(1) \quad \delta = \sum_C \pm [a_1, \dots, a_k | C] [a_{k+1}, \dots, a_r | \{b_1, \dots, b_r\} \setminus C],$$

$C$  running through the subsets of cardinality  $k$  of  $\{b_1, \dots, b_r\}$ . Every term in this equation has exactly one factor in  $\Omega$ :  $\delta$  on the left and  $[a_1, \dots, a_k | C]$  on the right side. Because of (9.3) none of these factors is in  $I^2$ , and  $[a_{k+1}, \dots, a_r | \{b_1, \dots, b_r\} \setminus C] \notin I$ . Therefore

$$(2) \quad \delta^* = \sum \pm [a_1, \dots, a_k | C]^* [a_{k+1}, \dots, a_r | \{b_1, \dots, b_r\} \setminus C]^*$$

in  $A = \text{Gr}_I R$ . Let  $\tilde{A} = A[(\delta^*)^{-1}]$ . It suffices to show that  $\tilde{A}_P$  is normal for the prime ideals  $P$  of  $\tilde{A}$ . Since  $\delta^*$  is a unit in  $\tilde{A}$ , one of the elements  $[a_1, \dots, a_k | C]^*$  has to be a unit in  $\tilde{A}_P$ , too. Eventually it is enough to prove normality for the extensions

$$A[(a_1, \dots, a_k | C^*)^{-1}].$$



**(9.16) Lemma.** *With the hypotheses of (9.14) and the notations just introduced, let  $\zeta = [a_1, \dots, a_k|C]^*$ ,  $C = [c_1, \dots, c_k]$ . Furthermore Let  $S$  be the sub-ASL of  $A$  generated by  $\Omega^*$ ,  $U$  denote a  $k \times k$  matrix, and  $V$  an  $(m - \tilde{a}_k + 1) \times k$  matrix of indeterminates over  $B$ . Then the homomorphism*

$$S[\zeta^{-1}, U, V]/I_k(U) \xrightarrow{\varphi} A[\zeta^{-1}]$$

*which is the identity on  $S \subset A$ , sends  $U_{ij}$  to the residue class of  $[a_i|c_j]$  in  $R/I$  and  $V_{uv}$  to the residue class of  $[\tilde{a}_k - 1 + u|c_v]$ , is an isomorphism.*

PROOF: Let  $[u|v]^\wedge$  be the residue class of  $[u|v]$  in  $R/I$ . Since  $[a_1, \dots, a_k|b_1, \dots, b_k] \in I$ , the determinant of the matrix formed by the elements  $[a_i|c_j]^\wedge$  (which is different from  $[a_1, \dots, a_k|c_1, \dots, c_k]^*$ !) is zero, and the homomorphism is well-defined.

The rings and the homomorphism  $\varphi$  under consideration are constructed from the corresponding objects over  $\mathbf{Z}$  by tensoring with  $B$ , since both rings are, roughly spoken, defined by their straightening relations. So we may assume that  $B$  is a noetherian integral domain. A glance at (9.4),(c) shows that the dimensions are equal (note that  $\dim A = \dim R$ ), and it is enough that the homomorphism  $\varphi$  is surjective. As an  $S$ -algebra,  $A$  is generated by the elements  $[i|j]^\wedge$ . Let first  $i < \tilde{a}_k$  and  $j = c_v$ . If  $i \in \{a_1, \dots, a_k\}$ ,  $[i|j]^\wedge \in \text{Im } \varphi$  by definition. Otherwise we look at the equation

$$[i|c_v][a_1, \dots, a_k|C] = \sum_u \pm [a_u|c_v][\{a_1, \dots, a_k, i\} \setminus \{a_u\}|C]$$

in  $R$  which simply results from the Laplace expansion of a minor with two equal columns. If  $k = 1$ , then  $[i, c_v]^\wedge = 0$ , and  $[i, c_v]^\wedge \in \text{Im } \varphi$  trivially. If  $k > 1$ , the  $k$ -minors  $\neq 0$  in this equation all lie in  $I \setminus I^2$ , and the 1-minors are in  $R \setminus I$ . Therefore in  $A$  one has

$$\begin{aligned} [a_u|c_v]^\wedge &= [a_u|c_v]^*, \\ [i|c_v]^\wedge &= [i|c_v]^* = \zeta^{-1} \sum_u \pm [a_u|c_v]^* [\{a_1, \dots, a_k, i\} \setminus \{a_u\}|C]^*, \end{aligned}$$

and  $[i|c_v]^\wedge \in \text{Im } \varphi$ . Combined with the definition of  $\varphi$ , we conclude  $[i|c_v]^\wedge \in \text{Im } \varphi$  for all  $i = 1, \dots, m$  and all  $v = 1, \dots, k$ .

In order to “cover”  $[i|j]^\wedge$  with  $i \in \{a_1, \dots, a_k\}$ ,  $j \notin \{c_1, \dots, c_k\}$  one works with the relation

$$[a_u|j][a_1, \dots, a_k|C] = \sum_v \pm [a_u|c_v][a_1, \dots, a_k|\{c_1, \dots, c_k, j\} \setminus \{c_v\}],$$

and finally for  $[i|j]^\wedge$  with  $i \notin \{a_1, \dots, a_k\}$ ,  $j \notin \{c_1, \dots, c_k\}$  the equation

$$\begin{aligned} [i|j][a_1, \dots, a_k|C] &= [a_1, \dots, a_k, i|c_1, \dots, c_k, j] \\ &+ \sum_v \pm [i|c_v][a_1, \dots, a_k|\{c_1, \dots, c_k, j\} \setminus \{c_v\}] \end{aligned}$$

implies a suitable equation in  $A$ : the  $k$ -minors and the  $(k + 1)$ -minor appearing all are in

$I \setminus I^2$ , unless they are zero or  $k = 1, i < \tilde{a}_k$ , in which case  $[i, j]^\wedge = 0 \in \text{Im } \varphi$  anyway. —

**(9.17) Theorem.** *Let  $R = G(X; \gamma)$  or  $R = R(X; \delta)$  over a ring  $B$  of coefficients, and  $I$  be an ideal of maximal minors in  $R$ . If  $B$  is a (normal) domain, then  $\text{Gr}_I R, \widehat{\mathcal{R}}_I(R)$ , and  $\mathcal{R}_I(R)$  are (normal) domains, too.*

PROOF: Let  $R = R(X; \delta)$ ,  $A = \text{Gr}_I R$ . Since  $\delta^*$  is not a zero-divisor of  $A$ ,  $A$  is a subring of  $A[(\delta^*)^{-1}]$ . Every localization of  $A[(\delta^*)^{-1}]$  is a localization of one of the rings  $A[\zeta^{-1}]$  as in the preceding lemma.  $A[\zeta^{-1}]$  is a (normal) domain by virtue of the lemma, (9.4), and (6.3). A little exercise shows that  $A[(\delta^*)^{-1}]$  cannot contain a nontrivial idempotent (if  $B$  has none), and therefore  $A[(\delta^*)^{-1}]$  is a (normal) domain together with all its localizations. So  $A$  itself is a domain, and normal, when  $A[(\delta^*)^{-1}]$  is normal, cf. (16.24).

In case  $R = G(X; \gamma)$ ,  $A = \text{Gr}_I R$ , we view  $R$  as the homogenization of  $\overline{R} = R(\overline{X}; \overline{\delta})$ ,  $\overline{R} = R/R(y \pm 1)$ ,  $y = [n - m + 1, \dots, n]$ . Let  $\overline{I}$  be the dehomogenization of  $I$ . Since  $y^*$  is the maximal element in the poset underlying the ASL  $A$ ,  $y^* \pm 1 = (y \pm 1)^*$  is not a zero-divisor, consequently

$$\text{Gr}_{\overline{I}} \overline{R} = A/A(y^* \pm 1)$$

(cf. (3.7)). Obviously  $A$  can be viewed as a graded  $B$ -algebra in which  $y^*$  is an element of degree 1. Since  $A/Ay^*$  is an ASL and therefore reduced, we can apply (16.24) and conclude integrity and normality of  $A$  from the corresponding properties of  $\text{Gr}_{\overline{I}} \overline{R}$ .

For the Rees algebras integrity is not an issue. Since

$$\mathcal{R}_I(R)[(\eta^*)^{-1}] = \widehat{\mathcal{R}}_I(R)[\eta^{-1}],$$

$\eta = \delta$  or  $\eta = \gamma$  resp., it is enough to prove normality for  $\widehat{\mathcal{R}}_I(R)$ , one more application of (16.24). Now

$$\widehat{\mathcal{R}}_I(R)[(T^{-1})^{-1}] = R[T, T^{-1}]$$

is normal and

$$\widehat{\mathcal{R}}_I(R)/T^{-1}\widehat{\mathcal{R}}_I(R) = \text{Gr}_I R$$

is certainly reduced. (So the normality of  $\mathcal{R}_I(R)$  and  $\widehat{\mathcal{R}}_I(R)$  results from (9.8) already.) —

Generalizing (7.10) we obtain:

**(9.18) Corollary.** *If  $B$  is an integral domain, then an ideal  $I$  of maximal minors in  $R$  (is prime and) has primary powers:  $I^j = I^{(j)}$  for all  $j \geq 0$ .*

PROOF: Suppose that the contention is false, and let  $k$  be the smallest exponent for which  $I^k \neq I^{(k)}$ . For  $x \in I^{(k)} \setminus I^k$  there exists an  $y \in R \setminus I$  such that  $yx \in I^k$ . By assumption on  $k$ ,  $x \in I^{k-1}$ , and  $y^*x^* = 0$  in  $\text{Gr}_I R$ , contradicting the integrity of  $\text{Gr}_I R$ . —

This corollary allows us to complete the description of the canonical module whose class was computed in the preceding section:

**(9.19) Corollary.** *Let  $B$  be an Cohen-Macaulay ring having a canonical module  $\omega_B$ . If the integers  $\kappa_i \geq 0$  satisfy the condition in (8.12) or the analogous condition in (8.14) resp., then a canonical module of  $R$  is given (as a  $B$ -module) by the direct sum*

$$\omega_R = \bigoplus \omega_B \mu,$$

$\mu$  ranging over the standard monomials which have at least  $\kappa_i$  factors in  $\Gamma(X; \gamma) \setminus \Gamma(X; \zeta_i)$  or  $\Delta(X; \delta) \setminus \Delta(X; \zeta_i)$  resp., the elements  $\zeta_i$  being the upper neighbours of  $\gamma$  or  $\delta$  resp. (The assumption  $\kappa_i \geq 0$  can always be satisfied.)

PROOF: Let  $R_0$  be the ring  $G(X; \gamma)$  or  $R(X; \delta)$  over the integers  $\mathbf{Z}$ . Then

$$\omega_R = \omega_B \otimes_{\mathbf{Z}} \omega_{R_0}$$

reducing everything to  $R_0$ . We have  $\omega_{R_0} = \bigcap P_i^{\kappa_i}$  now,  $P_i = J(x; \zeta_i)$  or  $P_i = I(x; \zeta_i)$ , and  $P_i^{\kappa_i}$  has a basis consisting of standard monomials as given by (9.6). —

**(9.20) Remark.** In principle (9.19) allows the computation of the Cohen-Macaulay type of  $G(X; \gamma)$  and  $R(X; \delta)$  over a field, say. (The Cohen-Macaulay type is the minimal number of generators of the canonical module.) A relatively simple case is  $R = R_{r+1}(X)$ . Assume that  $m \leq n$ ,  $k = n - m$ . Then  $\omega_R \cong Q^k$ ,  $Q$  generated by the  $r$ -minors of the first  $r$  columns of the matrix of residue classes, and a minimal system of generators of  $Q^k$  (in  $R$  as well as in the localization with respect to the irrelevant maximal ideal) is given by the standard monomials of length  $k$  in the  $r$ -minors of the first  $r$  columns. Therefore it coincides with the minimal number of generators of the  $k$ -th power of the irrelevant maximal ideal of  $G(Y)$ ,  $Y$  an  $r \times m$  matrix, and the type of  $R_{r+1}(X)$  can be read off from the Hilbert series of  $G(Y)$ . The latter has been computed explicitly in [HP], p. 387, Theorem III. J. Brennan communicated the following expression for the type of  $R_{r+1}(X)$ :

$$\frac{\binom{n-m+r}{r} \cdots \binom{n-1}{r}}{\binom{r}{r} \cdots \binom{m-1}{r}}.$$

In the cases in which the generators of  $Q$  are linearly ordered (i.e.  $r + 1 = m$  or  $r = 1$ ) this simplifies to  $\binom{n-1}{n-m}$ , a result which also follows directly from (9.19). —

**(9.21) Remark.** As in 7.C let  $R = R_{r+1}(X) \cong B[YZ]$ ,  $Y$  be an  $m \times r$  matrix and  $Z$  an  $r \times n$  matrix of indeterminates over  $B$ . In the following we want to analyze the algebra  $A$  generated by the entries of the product matrix  $YZ$ , the  $r$ -minors of  $Y$  and the  $r$ -minors of  $Z$ . It has been demonstrated (cf. (7.6),(b)) that  $A$  is the ring of absolute  $SL(r, B)$ -invariants of  $B[Y, Z]$ . In view of Remark (7.13) it is desirable to prove the normality of  $A$  independently from invariant theory. For the rest of this remark we assume that  $B$  is a normal domain. We sketch the arguments, leaving some details to the reader. As usual, let  $P$  ( $Q$ ) be the ideal in  $R \cong B[YZ]$  generated by the  $r$ -minors of the first  $r$  rows (columns), and  $T$  an independent indeterminate over  $R$ . Furthermore  $[a_1, \dots, a_r]_Y$  is the  $r$ -minor of the rows  $a_1, \dots, a_r$  of  $Y$ , whereas  $[b_1, \dots, b_r]_Z$  denotes the  $r$ -minor of the columns  $b_1, \dots, b_r$  of  $Z$ . Finally,  $\delta = [1, \dots, r | 1, \dots, r]$  (as a minor of  $X = YZ$  in  $R$ ).

(a) *The assignment*

$$\begin{aligned} [a_1, \dots, a_r]_Y &\longrightarrow [a_1, \dots, a_r | 1, \dots, r] \delta^{-1} T^{-1}, \\ [b_1, \dots, b_r]_Z &\longrightarrow [1, \dots, r | b_1, \dots, b_r] T, \end{aligned}$$

induces an isomorphism of  $R$ -algebras

$$A \cong \bigoplus_{j=1}^{\infty} Q^j \delta^{-j} T^{-j} \oplus R \oplus \bigoplus_{j=1}^{\infty} P^j T^j = \bigoplus_{j=-\infty}^{\infty} P^j T^j.$$

For the object on the right side, the powers  $P^j$ ,  $j < 0$ , are of course to be considered fractionary ideals of the domain  $R$ . The equality in the preceding formula is easily checked if one applies the valuations associated with the divisorial prime ideals of  $R$ :

$$v_P(\delta) = v_Q(\delta) = 1, \quad v_I(\delta) = 0 \quad \text{for } I \neq P, Q.$$

Furthermore one needs of course that the powers of  $P$  and  $Q$  are divisorial ideals (by virtue of 9.18). In order to prove the isomorphism on the left, one first observes that  $A$  is a graded subalgebra  $\bigoplus_{j=-\infty}^{\infty} A_j$  of  $B[Y, Z]$  where  $A_j$  contains the bihomogeneous elements of partial degrees  $d_1$  with respect to  $Y$  and  $d_2$  with respect to  $Z$  such that  $d_2 - d_1 = jr$ . The equations

$$[a_1, \dots, a_r]_Y [b_1, \dots, b_r]_Z = [a_1, \dots, a_r | b_1, \dots, b_r]$$

then suffice to show that the assignment given induces  $B$ -isomorphisms

$$A_j \cong \begin{cases} P^j & \text{for } j \geq 0, \\ Q^{-j} & \text{for } j < 0, \end{cases}$$

whose direct sum is an  $R$ -algebra isomorphism. In the following we identify  $A$  and its isomorphic copy.

(b) *Let  $S = \mathcal{R}_P(R) \subset A$ . Then*

$$A \subset \bigcap_{I \neq (PT)S} S_I \subset S[(\delta T)^{-1}],$$

*the intersection being extended over the divisorial prime ideals  $I \neq (PT)S$  of  $S$ .*

In fact,  $S[(\delta T)^{-1}]$  is the intersection of all the localizations  $S_I$ ,  $I$  a divisorial prime,  $\delta T \notin I$ . This explains the inclusion on the right side; for the one on the left we note that

$$A \subset \bigcup_{j=0}^{\infty} (S : ((PT)S)^j) = \bigcap_{I \neq (PT)S} S_I,$$

the operation  $:$  being performed in the field of fractions of  $S$ . However, (b) is only a preparation for (c):

(c) *One has*

$$A = \bigcap_{I \neq (PT)S} S_I = \bigoplus_{j=-\infty}^{\infty} P^j T^j.$$

*A is a normal domain. If B is factorial then A is factorial, too.*

In order to prove the equality claimed, one uses the second inclusion in (b), and shows that every element  $s$  of  $S[(\delta T)^{-1}]$  such that  $s((PT)S)^j \subset S$  for some  $j$ , is an element of  $A$ . Being an intersection of discrete valuation rings (and noetherian)  $A$  must be normal. For a quick proof of the last statement one applies [HV], Theorem,(a), p. 183: The extension  $R \rightarrow S$  induces an isomorphism of divisor class groups. Therefore the class of  $PS$  generates  $\text{Cl}(S)$ . Now  $PS$  and  $(PT)S$  are isomorphic ideals of  $S$ , so  $\text{cl}(PS) = \text{cl}((PT)S)$ , and  $\text{cl}((PT)S)$  is in the kernel of the natural epimorphism  $\text{Cl}(S) \rightarrow \text{Cl}(A)$ , cf. [Fs], § 7. (The last statement in (c) can be generalized:  $\text{Cl}(A) \cong \text{Cl}(B)$ .) —

### D. The Depth of Powers of Ideals of Maximal Minors

For a local ring  $R$  with maximal ideal  $P$  and an ideal  $I \subset R$  the analytic spread  $l(I)$  is defined by

$$l(I) = \dim \text{Gr}_I R / P \text{Gr}_I R,$$

cf. [NR], [Bh.2], and [Bd]. For a graded ASL  $A$  on  $\Pi$  over a noetherian ring  $B$ , and an ideal  $I \subset A_{\Pi}$  the corresponding quantity is

$$\dim \text{Gr}_I A / \Pi \text{Gr}_I A.$$

If  $B$  is a field, then  $\dim \text{Gr}_I A / \Pi \text{Gr}_I A = l(I_{A_{\Pi}})$ , since  $(\text{Gr}_I A) \otimes A_{A_{\Pi}} = \text{Gr}_{I_{A_{\Pi}}} A_{A_{\Pi}}$  and  $(\text{Gr}_I A / \Pi \text{Gr}_I A) \otimes A_{A_{\Pi}} = \text{Gr}_I A / \Pi \text{Gr}_I A$ . It is easy to determine  $\text{Gr}_I A / \Pi \text{Gr}_I A$  and its dimension for our objects.

**(9.22) Proposition.** *Let B be a noetherian ring, and  $R = G(X; \gamma)$  or  $R = R(X; \delta)$ , I an ideal of maximal minors,  $\Omega$  the ideal in  $\Pi = \Gamma(X; \gamma)$  or  $\Pi = \Delta(X; \delta)$  generating I and S the sub-ASL generated by  $\Omega$ .*

(a) *Then  $\text{Gr}_I R / \Pi \text{Gr}_I R$  is a homomorphic image of S and*

$$\dim \text{Gr}_I R / \Pi \text{Gr}_I R \leq \dim B + \text{rk } \Omega.$$

(b) *If  $R = G(X; \gamma)$  or  $R = R(X; \delta)$ ,  $\delta = [a_1, \dots, a_r | b_1, \dots, b_r]$  and  $\Omega$  consists of r-minors only, then  $\text{Gr}_I R / \Pi \text{Gr}_I R \cong S$  and*

$$\dim \text{Gr}_I R / \Pi \text{Gr}_I R = \dim B + \text{rk } \Omega.$$

PROOF: (a) It has been noticed in (9.15) that  $S$  can be regarded the sub-ASL of  $\text{Gr}_I R$  generated by  $\Omega^*$ . Since the generators of the  $B$ -algebra  $\text{Gr}_I R$  outside  $\Omega^*$  are killed in passing to  $\text{Gr}_I R / \Pi \text{Gr}_I R$ , the latter ring is a homomorphic image of  $S$  which by (5.10) has dimension  $\dim B + \text{rk } \Omega$ .

(b) We have to show that  $(\Pi \text{Gr}_I R) \cap S = 0$ . For this it is sufficient that every standard monomial in the standard representation of an element in  $\Pi \text{Gr}_I R$  contains a factor from  $\Pi^* \setminus \Omega^*$ . In view of the straightening procedure outlined in (4.1) this

is equivalent to the appearance of at least one factor from  $\Pi^* \setminus \Omega^*$  in every standard monomial on the right hand side of a straightening relation

$$\xi^* v^* = \sum a_\mu \mu \quad (\text{in } \text{Gr}_I R!)$$

with  $\xi^* \in \Pi^* \setminus \Omega^*$ . Since this equation is homogeneous in the graded ring  $\text{Gr}_I R$ , a standard monomial  $\mu$  can have at most one factor in  $\Omega^*$ . In case  $R = G(X; \gamma)$  every such  $\mu$  has automatically two factors. In the other case of (b) one argues as follows: If  $v^* \in \Pi^* \setminus \Omega^*$  too, then every  $\mu$  entirely consists of factors from  $\Pi^* \setminus \Omega^*$ , and if  $v \in \Omega^*$ , then every  $\mu$  must have two factors for reasons of degree in  $R$ . —

The inequality in (a) can indeed be strict, as is demonstrated by the example  $R = B[X]$ ,  $X$  an  $m \times n$  matrix with  $m \geq 2$ ,  $I$  the ideal generated by the elements in the first row.

As an  $A$ -module the associated graded ring represents the properties common to all of the quotients  $I^j/I^{j+1}$ . A quantity which can be rather comfortably computed by means of the associated graded ring, is the minimum of their depths if  $A$  is local. The global analogue for our objects is  $\text{grade}(A\Pi, I^j/I^{j+1})$ , the length of a maximal  $(I^j/I^{j+1})$ -sequence in  $A\Pi$ . In view of a later application the following proposition is kept more general than needed presently.

**(9.23) Proposition.** *Let  $A$  be a noetherian ring and let  $\mathcal{F} = (I_j)_{j \geq 0}$ ,  $I_0 = A$ , be a multiplicative filtration of  $A$  by ideals such that the associated graded ring  $\text{Gr}_{\mathcal{F}} A$  is noetherian. Consider  $\text{Gr}_{\mathcal{F}} A$  as an  $A$ -algebra via the natural epimorphism  $A \rightarrow A/I_1$ , and let  $J \subset A$  be an ideal. Then*

$$\min \text{grade}(J, A/I_j) = \min \text{grade}(J, I_j/I_{j+1}) = \text{grade } J\text{Gr}_{\mathcal{F}} A.$$

PROOF: The left equation follows from the behaviour of grade along the exact sequences

$$0 \longrightarrow I_j/I_{j+1} \longrightarrow A/I_{j+1} \longrightarrow A/I_j \longrightarrow 0.$$

If  $J\text{Gr}_{\mathcal{F}} A$  contains an element which is not a zero-divisor of  $\text{Gr}_{\mathcal{F}} A$ , then  $J$  is not contained in the preimage of any of the (finitely many) associated prime ideals of  $\text{Gr}_{\mathcal{F}} A$ , so contains an element which is not a zero-divisor on any of  $I_j/I_{j+1}$ . Conversely, if  $\text{grade } J\text{Gr}_{\mathcal{F}} A = 0$ , then  $J$  must annihilate a homogeneous element  $\neq 0$  in  $\text{Gr}_{\mathcal{F}} A$  of degree  $d$ , say, and so  $\text{grade}(J, I_d/I_{d+1}) = 0$ . The rest is induction based on the equation  $(Ax)^* = (\text{Gr}_{\mathcal{F}} A)x^*$  for an element  $x \in A$  which is not a zero-divisor of  $\text{Gr}_{\mathcal{F}} A$ ,  $*$  again denoting “leading form”. —

**(9.24) Corollary.** *Let  $A$  be a local ring,  $P$  its maximal ideal, and  $I \subset A$  an ideal. Then*

$$\min \text{depth } A/I^j \leq \text{ht } P\text{Gr}_I A \leq \dim \text{Gr}_I A - l(I).$$

*If  $\text{Gr}_I A$  is a Cohen-Macaulay ring, one has equality throughout.*

**(9.25) Corollary.** *Let  $R = G(X; \gamma)$  or  $R = R(X; \delta)$  over a noetherian ring  $B$  of coefficients,  $I$  an ideal of maximal minors in  $R$ , and  $\Omega$  the ideal in  $\Pi = G(X; \gamma)$  or  $\Pi = R(X; \delta)$  resp. generating  $I$ . Then*

$$\min \text{grade}(R\Pi, R/I^j) = \min \text{grade}(R\Pi, I^j/I^{j+1}) \geq \text{rk } \Pi - \text{rk } \Omega,$$

and if the hypothesis of part (b) of (9.22) is fulfilled, one has equality.

PROOF: Let  $B = \mathbf{Z}$  first. Then the defining ideal of  $\text{Gr}_I R$  as a residue class ring of the polynomial ring  $\mathbf{Z}[T_\pi : \pi \in \Pi^*]$  is generically perfect as a consequence of (9.14). Because of (9.23) and (3.14) it is enough that  $\text{ht } \Pi \text{Gr}_I R \geq \text{rk } \Pi - \text{rk } \Omega$  (with equality under the hypothesis of part (b) of (9.22)) whenever  $B$  is a field, and this is guaranteed by (9.22). —

The best information we can give on the behaviour of  $\text{grade}(J, R/I^j)$  as a function of  $j$ , is the following proposition.

**(9.26) Proposition.** *Let  $A$  be a noetherian ring,  $I, J$  ideals of  $A$  such that  $\text{ht } I \geq 1$  and  $\text{Gr}_I A$  is a Cohen-Macaulay ring. If  $\text{grade}(J, A/I^k) = \min \text{grade}(J, A/I^j)$ , then  $\text{grade}(J, A/I^k) = \text{grade}(J, A/I^{k+1})$ .*

PROOF: Suppose that  $\min \text{grade}(J, A/I^j) \geq 1$ . Then there exists an  $x \in J$  such that  $x^*$  is not a zero-divisor of  $\text{Gr}_I A$ . This fact triggers a proof by induction (observe that  $\text{ht}(I + Ax)/Ax \geq 1$ ), and one need only deal with the case  $\text{grade}(J, A/I^k) = 0$ . Since  $\text{ht } I \geq 1$ ,  $\dim \text{Gr}_I A/I^* \text{Gr}_I A = \dim A/I < \dim A = \dim \text{Gr}_I A$ . So  $I \setminus I^2$  contains an element  $y$  for which  $y^*$  is not a zero-divisor of  $\text{Gr}_I A$ . Multiplication by  $y$  then induces an embedding

$$A/I^k \longrightarrow A/I^{k+1},$$

whence  $\text{grade}(J, A/I^{k+1}) = 0$ , too. —

**(9.27) Examples.** In the following we assume that  $B = K$  is a field. Because of (3.14) the grade formulas generalize to arbitrary noetherian rings. They improve Proposition (7.24).

(a)  $R = B[X]$ ,  $X$  an  $m \times n$  matrix,  $m \leq n$ ,  $I = I_m(X)$ . Then

$$\min \text{grade}(I_1(X), R/I^j) = m^2 - 1.$$

It will be shown later (cf. (14.12)) that  $\text{grade}(I_1(X), R/I^2) = 3$  if  $m = 2$ , and the preceding proposition then implies  $\text{grade}(I_1(X), R/I^j) = 3$  for all  $j \geq 2$ . Another completely known case is  $n = m + 1$ . Since  $I \cong \text{Coker } X$  (the linear map  $X: R^m \rightarrow R^{m+1}$  given by the matrix  $X$ , cf. (16.36) for the isomorphism) and  $I^j \cong S_j(I)$  by (9.13), we conclude from (2.19),(b)(i) that  $\text{pd } R/I^j = \min(j, m) + 1$ , hence

$$\begin{aligned} \text{grade}(I_1(X), R/I^j) &= m(m+1) - j - 1 && \text{for } j = 1, \dots, m, \\ \text{grade}(I_1(X), R/I^j) &= m^2 - 1 && \text{for } j \geq m, \end{aligned}$$

because of the equation of Auslander-Buchsbaum and the equality  $\text{grade}(I_1(X), M) = \text{depth } M_{I_1(X)}$  for graded  $K[X]$ -modules  $M$ . (We believe that  $\text{grade}(I_1(X), R/I^j)$  always behaves in a regular manner; cf. the discussion below (10.8).)

(b) More generally let  $R = R_{r+1}(X)$ ,  $X$  as in (a),  $I = I_r(X)/I_{r+1}(X)$ . Then

$$\min \text{grade}(I_1(X), R/I^j) = r^2 - 1.$$

(c) If  $R$  is as in (b) and  $Q$  the ideal generated by the  $r$ -minors of any  $r$  columns, then

$$\min \text{grade}(I_1(X), R/Q^j) = nr - 1.$$

For the ideal  $P$  generated by the  $r$ -minors of any  $r$  rows one has

$$\min \text{grade}(I_1(X), R/P^j) = mr - 1.$$

Since  $Q^{n-m}$  is a canonical module and therefore a maximal Cohen-Macaulay module,

$$\text{grade}(I_1(X), R/Q^{n-m}) = \dim R - 1 = (m + n - r)r - 1,$$

and the minimum can only be attained for exponents  $> n - m$ .

(d) The analysis of the example (c) can certainly be carried further. We content ourselves with the case in which  $r + 1 = m \leq n$ :

$$\begin{aligned} \text{grade}(I_1(X), R/Q^j) &= nm - (n - m + 1) - 1 && \text{for } j = 1, \dots, n - m + 1, \\ \text{grade}(I_1(X), R/Q^j) &= nm - j - 1 && \text{for } j = n - m + 1, \dots, n, \\ \text{grade}(I_1(X), R/Q^j) &= nm - n - 1 && \text{for } j \geq n. \end{aligned}$$

and  $R, Q, \dots, Q^{n-m+1}, P$  are the only Cohen-Macaulay modules of rank 1 (up to isomorphism). Note that the canonical module  $Q^{n-m}$  is not the last one in the sequence of powers of  $Q$  to be a Cohen-Macaulay module.

Every Cohen-Macaulay module of rank 1 is a divisorial ideal and therefore isomorphic to a power of  $Q$ , cf. (8.4). In order to compute the multiplicity of  $R$  we have considered the  $R$ -sequence

$$\underline{y} = \{ [i|j] : j - i < 0 \text{ or } j - i > n - m \} \cup \{ [i|j] - [i-1|j-1] : 0 \leq j - i \leq n - m \}$$

It generates an  $I_1(X)R$ -primary ideal, and one has

$$e(R) = \lambda(R/R\underline{y}) = \binom{n}{m-1},$$

cf. (2.15). Let  $M$  be a Cohen-Macaulay module of rank 1. Then  $\dim M = \dim R$ , and  $M$  is a maximal Cohen-Macaulay module. Since this property localizes,

$$(*) \quad \lambda(M/\underline{y}M) = \binom{n}{m-1}$$

by virtue of [He.2], Proposition 1.1. (For a graded  $R$ -module the property of being a maximal Cohen-Macaulay module also globalizes, cf. (16.20): it is equivalent to being perfect over  $K[X]$  of grade equal to  $\text{grade } I_m(X)$ ). Therefore the validity of (\*) is sufficient for the modules under consideration to be Cohen-Macaulay.)

The minimal number of generators of all divisorial ideals, but the listed ones, already exceeds  $\binom{n}{m-1}$ , excluding them from being Cohen-Macaulay modules. The ideal  $P$  certainly is a Cohen-Macaulay module. For the powers of  $Q$  we use the free resolution over  $K[X]$  constructed in (2.16) and (2.19),(b)(ii). Let  $f: K[X]^n \rightarrow K[X]^m$  be given by the matrix  $X^*$ , and  $\bar{f} = f \otimes R$ .  $\text{Coker } f$  is annihilated by  $I_m(X)$  (cf. (16.2)), so  $\text{Coker } \bar{f} \cong \text{Coker } f$ . Since  $\text{rk } \bar{f} = m - 1$ ,  $\text{Coker } f$  has rank 1 as an  $R$ -module. Being a perfect  $K[X]$ -module, it is (isomorphic to) a (divisorial) ideal. Sending its  $i$ -th ‘‘canonical’’



generator to  $(-1)^{i+1}[1, \dots, \widehat{i}, \dots, m | 1, \dots, m-1]$ , one maps it onto  $Q$ , so  $Q \cong \text{Coker } f$ . For the formation of the symmetric powers  $S_j(Q)$ ,  $j \geq 1$ , it makes no difference whether we consider  $Q$  as an  $R$ -module or a  $K[X]$ -module. In conjunction with (9.13), (2.16) and (2.19),(b)(ii) therefore provide the projective dimension of all the powers  $Q^j$ . The three equations above now follow as those in (a).

For the general case in regard to  $m, n, r$  the preceding discussion at least implies that the number of (isomorphism classes of) Cohen-Macaulay modules of rank 1 over  $R_{r+1}(X)$  is always finite. —

## E. Comments and References

The investigation of powers of determinantal ideals was initiated by Hochster ([Ho.6]) who showed that  $I_m(X)$  has primary powers if  $X$  is an  $m \times (m+1)$  matrix, cf. also [ASV], p. 67, Beispiel 6.2. His result was generalized by Ngo ([Ng.1]) to  $m \times n$  matrices; Ngo investigated the associated graded ring by the method of principal radical systems ([HE.2]). Huneke showed in [Hu.1] that straightening-closed ideals are generated by “weak d-sequences” ([Hu.1], Proposition 1.3). This allowed him to prove the equality of ordinary and symbolic powers for the ideals  $I$  discussed in (9.27) and to compute  $\text{min grade}(I_1(X), I^j)$ . Example (9.27),(c) was treated in [Br.6] by an ad hoc method. A special case of (9.27),(a) appeared in [Ro]; the special result for  $n = m+1$  is taken from [AH]. The divisor class groups of the Rees algebras with respect to ideals of maximal minors can be computed by the results of [HV].

In (9.5) it has been pointed out that certain rings appearing in (9.4) can be viewed as Segre products. Conditions under which Segre products of Cohen-Macaulay rings are Cohen-Macaulay again are investigated by Chow ([Ch]). Chow’s results in particular imply that  $R_2(X)$  is a Cohen-Macaulay ring.

The material of Subsection B is taken from [Ei.1], [DEP.2], Section 2, and [EiH]. The extended Rees algebra and the associated graded ring can be treated in greater generality; one only needs that the filtrations on which they are based satisfy a certain condition, cf. [DEP.2].

There are more ideals satisfying the hypotheses of (9.12), say, than just the ideals of maximal minors considered above. It is quite obvious that some of their subideals share their characteristic properties; cf. [AS.1], [AS.2], [BrS], [BNS] for a detailed analysis of certain ideals of this type.

In [Hu.3] Huneke has determined all the values of  $m, n, t$  such that  $\mathcal{R}_{I_t(X)}(B[X]) \cong S(I_t(X))$ ,  $X$  an  $m \times n$  matrix of indeterminates.

Proposition (9.23) and its corollary generalize Burch’s inequality ([Bh.2]), cf. also [Bd].

## 10. Primary Decomposition

Let  $X$  be an  $m \times n$  matrix of indeterminates over a domain  $B$ ,  $m \leq n$ . Contrary to the ideal  $I_m(X)$  (and  $I_1(X)$ , of course) the ideals  $I_t(X)$ ,  $2 \leq t \leq m-1$ , have non-primary powers. In this section we shall determine the symbolic powers of the  $I_t(X)$ , discuss the “symbolic graded ring” as the proper analogue for  $I_t(X)$  of the ordinary associated graded ring for  $I_m(X)$ , and finally compute a primary decomposition for products  $I_{t_1}(X) \dots I_{t_s}(X)$ , essentially under the condition that  $B$  contains a field of characteristic zero. It is a remarkable fact that the primary decomposition depends on characteristic.

### A. Symbolic Powers of Determinantal Ideals

In (2.4) we have established an isomorphism which will prove very useful here: Let  $Y$  be an  $(m-1) \times (n-1)$  matrix of indeterminates; then the substitution

$$\begin{aligned} X_{ij} &\longrightarrow Y_{ij} + X_{mj}X_{in}X_{mn}^{-1}, & 1 \leq i \leq m-1, & \quad 1 \leq j \leq n-1, \\ X_{mj} &\longrightarrow X_{mj}, & X_{in} &\longrightarrow X_{in} \end{aligned}$$

induces an isomorphism

$$\varphi: B[X][X_{mn}^{-1}] \longrightarrow B[Y][X_{m1}, \dots, X_{mn}, X_{1n}, \dots, X_{m-1,n}][X_{mn}^{-1}]$$

whose inverse is given by  $Y_{ij} \longrightarrow X_{ij} - X_{mj}X_{in}X_{mn}^{-1}$ ,  $X_{mj} \longrightarrow X_{mj}$ ,  $X_{in} \longrightarrow X_{in}$ . For simplicity we identify the two rings by putting

$$Y_{ij} = X_{ij} - X_{mj}X_{in}X_{mn}^{-1},$$

remembering of course that the  $Y_{ij}$  are algebraically independent over  $B$ . In order to distinguish minors of  $X$  and  $Y$  we write  $[\dots|\dots]_X$  and  $[\dots|\dots]_Y$ .

**(10.1) Lemma.** (a) *For all minors  $[a_1, \dots, a_s|b_1, \dots, b_s]_Y$  one has*

$$[a_1, \dots, a_s|b_1, \dots, b_s]_Y = X_{mn}^{-1}[a_1, \dots, a_s, m|b_1, \dots, b_s, n]_X.$$

(b) *Let  $B$  be an integral domain,  $R = B[X]$ ,  $S = B[X][X_{mn}^{-1}]$ . Then*

$$\begin{aligned} I_t(X)^{(k)} &= (I_t(X)^{(k)}S) \cap R, \\ I_t(X)^{(k)}S &= (I_{t-1}(Y)^{(k)}S) \end{aligned}$$

for all  $t$ ,  $2 \leq t \leq m$ , and all  $k \in \mathbf{N}$ .

The equation in (a) is proved using the invariance of determinants under elementary transformations. The first equation in (b) follows from  $R \subset S \subset R_{I_t(X)}$ , and for the second it is important that the extensions  $R \rightarrow S$  and  $B[Y] \rightarrow S$  commute with the formation of symbolic powers.

The symbolic powers of  $I_1(X)$  coincide with the ordinary powers for trivial reasons; one has  $I_k(X) \subset I_1(X)^{(k)}$  for all  $k$  and  $\delta \notin I_1(X)^{(k+1)}$  for a  $k$ -minor  $\delta$ ,  $k \geq 1$ . Starting with  $t = 1$  and applying (10.1) inductively one gets:

**(10.2) Proposition.** *Let  $B$  be an integral domain. Then*

$$I_{t+k-1}(X) \subset I_t(X)^{(k)}$$

for all  $k$ , and  $\delta \notin I_t(X)^{(k+1)}$  for a  $(t+k-1)$ -minor  $\delta$  if  $1 \leq k \leq m-t+1$ .

The symbolic powers of a prime ideal  $P$  form a multiplicative filtration,

$$P^{(k)}P^{(l)} \subset P^{(k+l)},$$

and this fact together with (10.2) determines the symbolic powers  $I_t(X)^{(k)}$  completely as will be seen below. Because of (10.2) the degree of a  $(t+k-1)$ -minor with respect to this filtration is  $k$ . Therefore we define the function  $\gamma_t$  (for arbitrary  $t$ ) by

$$\gamma_t(\delta) = \begin{cases} 0 & \text{if } \delta \text{ is an } s\text{-minor, } s < t, \\ s-t+1 & \text{if } \delta \text{ is an } s\text{-minor, } s \geq t, \end{cases}$$

and extend this definition to the set of all (formal) monomials of minors by

$$\gamma_t(\delta_1 \dots \delta_p) = \sum_{i=1}^p \gamma_t(\delta_i).$$

Let  $J(t, k)$  be the ideal generated by all the monomials  $\pi$  such that  $\gamma_t(\pi) \geq k$ . Then  $J(t, k) \subset I_t(X)^{(k)}$ , and since we have obviously equality for  $t = 1$ , we could prove equality for all  $t$  by induction via (10.1) if we knew that  $J(t, k)S \cap R = J(t, k)$ , equivalently, that  $X_{mn}$  is not a zero-divisor modulo  $J(t, k)$ .

**(10.3) Lemma.**  *$J(t, k)$  is generated as a  $B$ -module by the standard monomials  $\mu$  such that  $\gamma_t(\mu) \geq k$ . In particular,  $X_{mn}$  is not a zero-divisor modulo  $J(t, k)$  if  $t \geq 2$ .*

PROOF: The proof of Proposition (4.1) details the “straightening procedure” by which repeated applications of the straightening relations transform an arbitrary monomial into its standard representation. It therefore suffices that in a straightening relation  $\xi v = \sum a_\mu \mu$  one has  $\gamma_t(\mu) \geq \gamma_t(\xi) + \gamma_t(v)$  for all  $\mu$ . This is easily seen to be true if one takes into account that  $\mu$  and  $\xi v$  have the same degree as polynomials in the entries of  $X$  and that  $\mu$  has at most two factors.

The second statement is obvious now: For every standard monomial  $\mu$  the product  $\mu X_{mn}$  is a standard monomial again, and  $\gamma_t(\mu X_{mn}) = \gamma_t(\mu)$  for  $t \geq 2$ . —

**(10.4) Theorem.** *Let  $B$  be an integral domain. Then for all  $t$ ,  $1 \leq t \leq m$ , and all  $k$  the  $k$ -th symbolic power of  $I_t(X)$  is generated by the (standard) monomials  $\mu$  such that  $\gamma_t(\mu) \geq k$ . Equivalently,*

$$I_t(X)^{(k)} = \sum I_{t+\kappa_1-1}(X) \dots I_{t+\kappa_s-1}(X),$$

the sum being extended over all  $\kappa_1, \dots, \kappa_s \geq 1$ ,  $s \leq k$ , such that  $\kappa_1 + \dots + \kappa_s \geq k$ . Furthermore  $\mu \in I_t(X)^{(k)}$  if and only if  $\gamma_t(\mu) \geq k$ .

PROOF: Only the last statement for non-standard monomials still needs a proof. In the next subsection we will introduce the associated graded ring with respect to the

filtration given by the symbolic powers of  $I_t(X)$ . This ring is a domain, cf. (10.7). So the leading forms of the minors of  $X$  are not zero-divisors, and for  $\mu = \delta_1 \dots \delta_p$ ,  $\delta_i \in \Delta(X)$ , one therefore has

$$\mu^* = \delta_1^* \dots \delta_p^*,$$

\* denoting leading form. Thus the degree of  $\mu^*$  is the sum of the degrees of its factors  $\delta_i^*$ , whence it coincides with  $\gamma_t(\mu)$ . —

**(10.5) Remark.** Without essential changes the ideals  $I_t(X)/I_u(X) \subset R_u(X)$ ,  $1 \leq t \leq u$  can be considered. (10.4) remains true modulo  $I_u(X)$ , in other words:

$$(I_t(X)/I_u(X))^{(k)} = (I_t(X)^{(k)} + I_u(X))/I_u(X).$$

The generalization to  $R(X; \delta)$  is not immediate, because the induction argument breaks down. Nevertheless we expect that  $(I_t(X)R(X; \delta))^{(k)} = I_t(X)^{(k)}R(X; \delta)$  throughout. —

### B. The Symbolic Graded Ring

For a prime ideal  $P$  in a ring  $A$  the ring

$$\text{Gr}_P^{(0)}A = \bigoplus_{j \geq 0} P^{(j)} / P^{(j+1)}$$

should properly be called the graded ring associated with the filtration by symbolic powers. In general, one cannot say much about it; it may even be non-noetherian though  $A$  is noetherian (cf. [Rb.4]). As in the case of ordinary powers we denote the leading form of  $x \in A$  by  $x^*$ , in  $\text{Gr}_P^{(0)}A$  as well as in the “extended symbolic Rees ring”

$$\widehat{\mathcal{R}}_P^{(0)}(A) = \bigoplus_{j=0}^{\infty} P^{(j)}T^j \oplus \bigoplus_{j=1}^{\infty} AT^{-j} \subset A[T, T^{-1}].$$

In order to make  $\text{Gr}_P^{(0)}A$  and  $\widehat{\mathcal{R}}_P^{(0)}(A)$  well-defined objects over every ring  $B$ , we consider  $I_t(X)^{(k)}$  to be given by the description in (10.4) if  $B$  is not a domain.

**(10.6) Theorem.** *Let  $B$  be a commutative ring,  $X$  an  $m \times n$  matrix of indeterminates,  $m \leq n$ , and  $\Delta$  its poset of minors. Let  $1 \leq t \leq m$  and  $P = I_t(X)$ . Then  $\widehat{\mathcal{R}}_P^{(0)}(B[X])$  is a graded ASL on  $\Delta^*$  over  $B[T^{-1}]$ , and  $\text{Gr}_P^{(0)}B[X]$  is a graded ASL on  $\Delta^*$  over  $B$ ,  $\Delta^*$  inheriting its partial order from  $\Delta$  as in (9.7).*

PROOF: This theorem is proved in the same fashion as (9.7) and (9.8). One needs of course that  $\Delta^*$  generates the extended symbolic Rees ring as follows from (10.4), and that in a straightening relation  $\xi v = \sum a_\mu \mu$  the inequality  $\gamma_t(\xi v) \leq \gamma_t(\mu)$  holds for all  $\mu$ . —

**(10.7) Corollary.** (a) *If  $B$  is a Cohen-Macaulay ring, then the rings considered in (10.6) are Cohen-Macaulay rings, too.*

(b) *If  $B$  is reduced (a (normal) domain), then the rings considered in (10.6) are reduced ((normal) domains).*

Part (a) and the assertion on being reduced are immediate. For (b) one applies (16.24) after the inversion of the maximal element  $X_{mn}^*$  of  $\Delta^*$  (modulo which the rings under consideration are again ASLs) together with induction on  $t$ .

In the following proposition we consider  $\text{Gr}_P^{(0)}B[X]$  a  $B[X]$ -algebra via the natural epimorphism  $B[X] \rightarrow B[X]/P \subset \text{Gr}_P^{(0)}B[X]$ .

**(10.8) Proposition.** *Let  $B$  be a noetherian domain. Then with the notations of (10.6), one has*

$$\min \text{grade}(\mathbf{I}_1(X), P^{(j)}/P^{(j+1)}) = \text{grade} \mathbf{I}_1(X) \text{Gr}_P^{\circ} B[X] = t^2 - 1.$$

PROOF: The first equality follows from (9.23). Let now  $R = B[X]$ ,  $S = \text{Gr}_P^{\circ} R$ , and  $J = \mathbf{I}_1(X)S$ . The ideal  $J$  is generated by the subset

$$\tilde{\Delta} = \{ \delta^* : \delta \text{ an } s\text{-minor, } 1 \leq s < t \}$$

of  $\Delta^*$ . We want to show that  $S/J$  is a graded ASL on  $\Omega = \Delta^* \setminus \tilde{\Delta}$ . By Proposition (5.1),(a) it is enough to show that as a  $B$ -module  $J$  is generated by the standard monomials containing a factor from  $\tilde{\Delta}$ , and, by reference to the straightening procedure, one only has to show that in a straightening relation

$$\xi^* v^* = \sum a_{\mu} \mu \quad (\text{in } S!)$$

every standard monomial  $\mu$  contains a factor from  $\tilde{\Delta}$  if  $\xi^* \in \tilde{\Delta}$ . If additionally  $v^* \in \tilde{\Delta}$ , then this is the straightening relation in  $R/P$ , and every  $\mu$  consists entirely of factors from  $\tilde{\Delta}$ . Let  $v^* \notin \tilde{\Delta}$ . If  $\mu = \zeta^* \eta^*$ ,  $\zeta, \eta \in \Delta$ ,  $\zeta \leq \eta$ , then  $\eta^* \geq \xi^*$  and  $\eta^* \in \tilde{\Delta}$ . (Remember that, after all, the straightening relations are inherited from  $\Gamma(\tilde{X})$ .) If  $\mu = \nu^*$ ,  $\nu \in \Delta$ , then, as polynomials in  $B[X]$ ,

$$\deg \nu = \deg \xi + \deg v > \deg v$$

and

$$\gamma_t(\xi v) = \gamma_t(v) < \gamma_t(\nu).$$

This is impossible, since the straightening relations are homogeneous equations in the graded ring  $S$ .

So  $S/J$  is a graded ASL over the wonderful poset  $\Omega$ . As in (9.25) one first reduces to the case in which  $B$  is a field (via (3.14)). Since  $S$  is Cohen-Macaulay then, one has

$$\begin{aligned} \text{grade} \mathbf{I}_1(X)S &= \text{rk } \Delta - \text{rk } \Omega \\ &= mn - (mn - t^2 + 1) = t^2 - 1. \quad - \end{aligned}$$

It will be shown in (14.12) that for all  $t \geq 2$  one has

$$\begin{aligned} \text{grade}(\mathbf{I}_1(X), \mathbf{I}_t(X)/\mathbf{I}_t(X)^{(2)}) &= \text{grade}(\mathbf{I}_1(X), \mathbf{R}_{t-1}(X)) + 3 \\ &= (m + n - t + 2)(t - 2) + 3. \end{aligned}$$

Therefore

$$\begin{aligned} d &= \text{grade}(\mathbf{I}_1(X), \mathbf{R}_t(X)) - \text{grade}(\mathbf{I}_1(X), \mathbf{I}_t(X)/\mathbf{I}_t(X)^{(2)}) \\ &= m + n - 2t \end{aligned}$$

divides

$$\begin{aligned} \text{grade}(\mathbf{I}_1(X), \mathbf{R}_t(X)) - \min \text{grade}(\mathbf{I}_1(X), \mathbf{I}_t(X)^{(j)}/\mathbf{I}_t(X)^{(j+1)}) \\ = (m + n - 2t)(t - 1). \end{aligned}$$

We believe that  $\text{grade}(\mathbf{I}_1(X), \mathbf{I}_t(X)^{(j)}/\mathbf{I}_t(X)^{(j+1)})$  goes down by  $d$  if  $j$  is increased by 1 until it reaches its minimal value (and stays constant then). Admittedly there is not much support for this claim, cf. (9.27),(a).

### C. Primary Decomposition of Products of Determinantal Ideals

None of the results proved so far depends on the characteristic of the ring  $B$  of coefficients. Quite surprisingly, the primary decomposition of products  $I_{t_1}(X) \dots I_{t_s}(X)$ , in particular of powers  $I_t(X)^k$ , cannot be given without reference to the characteristic of  $B$ , and we shall succeed in complete generality only for the rings containing the rational numbers.

Let  $B$  be an integral domain,  $X$  an  $m \times n$  matrix of indeterminates. The smallest symbolic power of  $I_j(X)$  containing  $I_t(X)$  is  $I_j(X)^{e(j,t)}$ , where

$$e(j, t) = \begin{cases} t - j + 1 & \text{if } 1 \leq j \leq t, \\ 0 & \text{if } t < j, \end{cases}$$

$t$  arbitrary. This implies immediately the inclusion “ $\subset$ ” in:

**(10.9) Theorem.** *Let  $B$  be an integral domain,  $X$  an  $m \times n$  matrix of indeterminates, and  $t_1, \dots, t_s$  integers,  $1 \leq t_i \leq \min(m, n)$ . Let  $w = \max t_i$ , and suppose that  $(\min(t_i, m - t_i, n - t_i))!$  is invertible in  $B$  for  $i = 1, \dots, s$ . Then*

$$I_{t_1}(X) \dots I_{t_s}(X) = \bigcap_{j=1}^w I_j(X)^{e_j}, \quad e_j = \sum_{i=1}^s e(j, t_i),$$

is a (possibly redundant) primary decomposition.

It will be indicated in (10.12) how to refine this decomposition to an irredundant one.

As a specific example we take  $n \geq m \geq 3$ ,  $s = 2$ ,  $t_1 = t_2 = 2$ . Then (10.9) in conjunction with (10.4) says

$$I_2(X)^2 = I_1(X)^4 \cap (I_3(X) + I_2(X)^2).$$

In particular the product of a 1-minor and a 3-minor must be in  $I_2(X)^2$ , the first nontrivial case of the following lemma which is the crucial argument in the proof of (10.9).

**(10.10) Lemma.** *Let  $B = \mathbf{Z}$ , and  $F(i, j)$  be the  $\mathbf{Z}$ -submodule of  $\mathbf{Z}[X]$  generated by the products  $\delta_1 \delta_2$  of the  $i$ -minors  $\delta_1$  and the  $j$ -minors  $\delta_2$ . Then for  $\pi = [a_1, \dots, a_u | b_1, \dots, b_u]$ ,  $\rho = [c_1, \dots, c_v | d_1, \dots, d_v]$ ,  $u \leq v - 2$ , and*

$$\tilde{u} = \max ( |\{a_1, \dots, a_u\} \cap \{c_1, \dots, c_v\}|, |\{b_1, \dots, b_u\} \cap \{d_1, \dots, d_v\}| )$$

one has

$$(u + 1 - \tilde{u})! \pi \rho \in F(u+1, v-1).$$

(We include the case  $u = 0$  in which  $\pi = 1$ .)

**PROOF OF (10.9):** The inclusion “ $\subset$ ” has been noticed already. The converse is proved by induction on  $s$ , the case  $s = 1$  being trivial. Consider a (standard) monomial  $\mu = \delta_1 \dots \delta_p$  of minors  $\delta_i$  such that  $\gamma_j(\mu) \geq e_j$  for  $j = 1, \dots, w$ . If one of the minors  $\delta_i$

has size  $w$ , one is through by induction. Otherwise we arrange the factors  $\delta_1, \dots, \delta_p$  in ascending order relative to their sizes, and split  $\mu$  into the product

$$\mu_1 = \delta_1 \dots \delta_q$$

of minors of size  $< w$ , and

$$\mu_2 = \delta_{q+1} \dots \delta_p$$

of minors of size  $> w$ . Let  $u$  and  $v$  be the sizes of  $\delta_q$  and  $\delta_{q+1}$  resp. Applying (10.10) to  $\delta_q \delta_{q+1}$  we get a representation

$$\mu = \sum a_i \nu_i, \quad a_i \in B, \quad \nu_i = \delta_1 \dots \delta_{q-1} \zeta_i \eta_i \delta_{q+2} \dots \delta_p,$$

in which  $\zeta_i$  has size  $u + 1$  and  $\eta_i$  has size  $v - 1$ . Evidently

$$\begin{aligned} \gamma_j(\nu_i) &= \gamma_j(\mu), & j &= 1, \dots, u + 1, \\ \gamma_j(\nu_i) &= \gamma_j(\mu) - 1, & j &= u + 2, \dots, w \end{aligned}$$

By induction on  $v - u$  or by reference to the case in which a  $w$ -minor is present, we are through if  $\gamma_j(\mu) > e_j$  for  $j = u + 2, \dots, w$ . It remains the case in which  $\gamma_r(\mu) = e_r$  for some  $r$ ,  $u + 2 \leq r \leq w$ .

One may assume that  $t_1, \dots, t_k < r - 1$  and  $t_{k+1}, \dots, t_s \geq r - 1$ . Let

$$\begin{aligned} J_1 &= \prod_{i=1}^k I_{t_i}(X), & J_2 &= \prod_{i=k+1}^s I_{t_i}(X), \\ e_j^1 &= \sum_{i=1}^k e(j, t_i), & e_j^2 &= \sum_{i=k+1}^s e(j, t_i). \end{aligned}$$

Then  $\gamma_j(\mu) = \gamma_j(\mu_2)$  for  $j \geq r - 1$ . Furthermore  $\gamma_{r-1}(\mu) - \gamma_r(\mu) = p - q$  and  $e_{r-1} - e_r = s - k$ . Since  $\gamma_{r-1}(\mu) \geq e_{r-1}$  it follows that  $p - q \geq s - k$ , whence  $\mu_2 \in J_2$  for trivial reasons. We claim:  $\gamma_j(\mu_1) \geq e_j^1$  for  $j \leq r - 2$  and finish the proof by induction on  $s$ . Relating  $\gamma_{r+1}$  and  $\gamma_r$ ,  $e_{r+1}$  and  $e_r$  one gets

$$\begin{aligned} \gamma_r(\mu) - \gamma_{r+1}(\mu) &= p - q, \\ e_r - e_{r+1} &\leq s - k. \end{aligned}$$

Therefore  $p - q \leq s - k$ , too, and  $p - q = s - k$ . Since

$$\begin{aligned} \gamma_j(\mu_2) &= \gamma_r(\mu_2) + (p - q)(r - j) = e_r + (s - k)(r - j) \\ &= e_j^2 \end{aligned}$$

for all  $j \leq r$ ,  $\gamma_j(\mu_1) \geq e_j^1$  for all  $j \leq r - 2$  as claimed. —

PROOF OF (10.10): In case  $u = 0$  the contention is a trivial consequence of Laplace expansion. Let  $u > 0$  and suppose that

$$\tilde{u} = |\{a_1, \dots, a_u\} \cap \{c_1, \dots, c_v\}|,$$

transposing if necessary. We use descending induction on  $\tilde{u}$ , starting with the maximal value  $\tilde{u} = u$ . The fundamental relation which is crucial for this case as well as for the inductive step, is supplied by the following lemma. Its very easy proof is left to the reader:

**(10.11) Lemma.** *One has*

$$\sum_{i=1}^{u+1} (-1)^{i-1} [a_1, \dots, \widehat{a}_i, \dots, a_{u+1} | b_1, \dots, b_u] [a_i, c_2, \dots, c_v | d_1, \dots, d_v] \in F(u+1, v-1).$$

In proving (10.10) for  $\tilde{u} = u$  we may assume that  $c_1 \notin \{a_1, \dots, a_u\}$ . Then, with  $a_{u+1} = c_1$ , all the terms of the sum in (10.11) except

$$(-1)^u [a_1, \dots, a_u | b_1, \dots, b_u] [c_1, \dots, c_v | d_1, \dots, d_v]$$

are zero, and  $\pi\rho \in F(u+1, v-1)$  as desired.

Let  $\tilde{u} < u$  now and again  $c_1 \notin \{a_1, \dots, a_u\}$ . We put  $\beta = (b_1, \dots, b_u)$ ,  $\delta = (d_1, \dots, d_v)$ ,  $a_{u+1} = c_1$ . The terms  $[a_1, \dots, \widehat{a}_i, \dots, a_{u+1} | \beta] [a_i, c_2, \dots, c_v | \delta]$  with  $a_i \in \{c_2, \dots, c_v\}$  drop out, and

$$(1) \quad \sum_{\substack{i \\ a_i \notin \{c_2, \dots, c_v\}}} (-1)^{i-1} [a_1, \dots, \widehat{a}_i, \dots, a_{u+1} | \beta] [a_i, c_2, \dots, c_v | \delta] \in F(u+1, v-1).$$

We claim: If  $a_i \notin \{c_2, \dots, c_v\}$ , then

$$(2) \quad (u - \tilde{u})! (-1)^{i-1} [a_1, \dots, \widehat{a}_i, \dots, a_{u+1} | \beta] [a_i, c_2, \dots, c_v | \delta] \\ \equiv (u - \tilde{u})! (-1)^u [a_1, \dots, a_u | \beta] [c_1, \dots, c_v | \delta] \pmod{F(u+1, v-1)}.$$

Multiplying the sum in (1) by  $(u - \tilde{u})!$  and applying the preceding congruence we get

$$(u + 1 - \tilde{u})! [a_1, \dots, a_u | \beta] [c_1, \dots, c_v | \delta] \in F(u+1, v-1)$$

as desired.

In order to prove (2) one replaces the rows  $a_i$  and  $c_1$  of  $X$  both by the sum of these rows, creating a matrix  $\tilde{X}$ . Let  $\tilde{\pi}$  and  $\tilde{\rho}$  be the minors of  $\tilde{X}$  arising from  $\pi$  and  $\rho$  under the substitution  $X \rightarrow \tilde{X}$ . The minors  $\tilde{\pi}$  and  $\tilde{\rho}$  both can be interpreted as minors of a matrix with  $m - 1$  rows, and then have  $\tilde{u} + 1$  rows in common. The  $\mathbf{Z}$ -module  $F(u+1, v-1)$  relative to the new matrix is contained in  $F(u+1, v-1)$  relative to  $X$ , and by induction,

$$(u + 1 - (\tilde{u} + 1))! \tilde{\pi} \tilde{\rho} \in F(u+1, v-1).$$

On the other hand

$$\tilde{\pi} \tilde{\rho} = [a_1, \dots, a_i, \dots, a_u | \beta] [a_i, c_2, \dots, c_v | \delta] + [a_1, \dots, c_1, \dots, a_u | \beta] [a_i, c_2, \dots, c_v | \delta] \\ + [a_1, \dots, a_i, \dots, a_u | \beta] [c_1, c_2, \dots, c_v | \delta] + [a_1, \dots, c_1, \dots, a_u | \beta] [c_1, c_2, \dots, c_v | \delta].$$

The inductive hypothesis applies to the first and the fourth term on the right side of this equation, whence

$$(u - \tilde{u})! (\pi\rho + [a_1, \dots, c_1, \dots, a_u | \beta] [a_i, c_2, \dots, c_v | \delta]) \in F(u+1, v-1). \quad -$$

The intersection in (10.9) is obviously redundant if  $s = 1$ ,  $t_1 > 1$  or  $t_1 = \dots = t_s = \min(m, n) > 1$ . In the latter case  $I_{t_1}(X) \dots I_{t_s}(X)$  is primary itself, cf. (9.18). The following proposition shows how to single out the essential primary ideals.



**(10.12) Proposition.** *Let  $B$  be an integral domain,  $X$  an  $m \times n$  matrix of indeterminates such that  $m \leq n$  (for notational simplicity). Furthermore let  $e_1, \dots, e_w$  be given as in (10.9). Then*

$$I_k(X)^{(e_k)} \not\supseteq \bigcap_{\substack{j=1 \\ j \neq k}}^w I_j(X)^{(e_j)} \iff e_{k+1} \leq (m-k)(g_k-1),$$

$g_k$  denoting the number of indices  $i$ ,  $1 \leq i \leq s$ , such that  $t_i \geq k$ .

PROOF: Passing to a ring of quotients does not affect the question whether a primary ideal  $Q$  is irredundant in a given decomposition, provided  $Q$  stays a proper ideal under extension. Therefore we may invert a 1-minor if  $k > 1$ , and eventually reduce the proposition to the case in which  $k = 1$ . So one has to show:

$$I_1(X)^{(e_1)} \not\supseteq \bigcap_{j=2}^w I_j(X)^{(e_j)} \iff e_2 \leq (m-1)(s-1),$$

$s$  as in (10.9) denoting the number of factors of the product to be decomposed. Observe that  $e_1 = e_2 + s$ , so

$$(3) \quad e_2 \leq (m-1)(s-1) \iff e_1 - 1 \leq m(s-1).$$

We write  $e_1 - 1 = qm + r$ ,  $q, r \in \mathbf{Z}$ ,  $0 \leq r < m$  and choose an  $m$ -minor  $\delta$  and an  $r$ -minor  $\varepsilon$  ( $\varepsilon = 1$  if  $r = 0$ ). Then it is easy to see that

$$\gamma_j(\mu) \leq \gamma_j(\delta^q \varepsilon), \quad j = 1, \dots, m$$

for all (standard) monomials  $\mu$  such that  $\gamma_1(\mu) < e_1$ . Therefore

$$(4) \quad I_1(X)^{(e_1)} \not\supseteq \bigcap_{j=2}^w I_j(X)^{(e_j)} \iff \gamma_j(\delta^q \varepsilon) \geq e_j \quad \text{for } j = 2, \dots, w.$$

We now show that the right sides of (3) and (4) are equivalent. Suppose first that  $e_1 - 1 > m(s-1)$ . Since, on the other hand,  $e_1 \leq ms$ , one has  $q = s-1$  and  $r > 0$ , so  $\delta^q \varepsilon$  has exactly  $s$  factors ( $\neq 1$ ), and

$$\gamma_2(\delta^q \varepsilon) = e_1 - 1 - s < e_1 - s = e_2.$$

Suppose now that  $e_1 - 1 \leq m(s-1)$ . Then

$$\gamma_2(\delta^q \varepsilon) \geq e_1 - 1 - (s-1) = e_2.$$

Observe in the following that the differences  $e_i - e_{i+1}$  form a non-increasing sequence (i.e.  $e_{i+2} - e_{i+1} \leq e_{i+1} - e_i$  for all  $i$ ) and that  $\gamma_j(\delta^q \varepsilon) - \gamma_{j+1}(\delta^q \varepsilon)$  can only take the values  $q$  and  $q+1$ . In order to obtain a contradiction we assume that there exists a  $v$  such that  $\gamma_v(\delta^q \varepsilon) \geq e_v$ , but  $\gamma_{v+1}(\delta^q \varepsilon) < e_{v+1}$ . Then for all  $j \geq v$  one has

$$\begin{aligned} \gamma_j(\delta^q \varepsilon) - \gamma_{j+1}(\delta^q \varepsilon) &\geq \gamma_v(\delta^q \varepsilon) - \gamma_{v+1}(\delta^q \varepsilon) - 1 \\ &\geq e_v - e_{v+1} \\ &\geq e_j - e_{j+1}. \end{aligned}$$

Summing up these differences for  $j = v+1$  to  $j = m+1$  one obtains the desired contradiction since  $\gamma_{m+1}(\delta^q \varepsilon) = e_{m+1} = 0$ . —

An immediate consequence of (10.9) and (10.12) is the following irredundant primary decomposition of the powers of the ideals  $I_t(X)$ :

**(10.13) Corollary.** *Let  $B$  be an integral domain,  $X$  an  $m \times n$  matrix,  $m \leq n$ . Suppose that  $(\min(t, m - t))!$  is invertible in  $B$ . Then*

$$I_t(X)^s = \bigcap_{j=r}^t I_j(X)^{((t-j+1)s)}, \quad r = \max(1, m - s(m - t)),$$

*is an irredundant primary decomposition.*

**(10.14) Remarks.** (a) If one defines  $I_t(X)^{(k)}$  by means of the description given in (10.4), i.e.  $I_t(X)^{(k)}$  is the  $B$ -submodule generated by the standard monomials  $\mu$  such that  $\gamma_t(\mu) \geq k$ , then the intersection formulas of (10.9) and (10.13) hold for every ring in which the elements  $(\min(t_i, m - t_i, n - t_i))!$ ,  $i = 1, \dots, s$ , are units.

(b) The proof of (10.9) shows that for  $B = \mathbf{Z}$  the ideal  $\bigcap_{j=1}^w I_j(X)^{(e_j)}$  is the  $\mathbf{Z}$ -torsion of  $\mathbf{Z}[X]$  modulo  $I_{t_1}(X) \dots I_{t_s}(X)$ .

(c) If in addition to the hypotheses of (10.13)  $B$  is noetherian, then the associated prime ideals of  $I_t(X)^s$  are precisely the ideals  $I_j(X)$ ,  $j = r, \dots, t$ . If  $m$  is large compared to  $t$ , then  $I_1(X)$  is associated with  $I_t(X)^k$  for all  $k \geq 2$ , and if  $t \leq m - 1$ , then  $I_1(X)$  is associated with  $I_t(X)^k$  for  $k \geq t$ , as is easily seen. We will show below that the latter fact holds over every noetherian domain.

(d) The example given below shows that it is not possible to remove the assumption on the characteristic of  $B$  in (10.9) or (10.13). It should be noted however that in (10.13) it becomes void not only in the cases  $t = 1$  or  $t = m$ , but also when  $t = m - 1$ , the case of submaximal minors. On the other hand these are the only exceptional cases for (10.13), cf. the end of (g) below.

(e) Without any change in their statements or proofs, the results (10.9) – (10.13) carry over from the polynomial ring  $B[X]$  to the residue class ring  $R_{r+1}(X)$ , provided of course that one considers products and powers of the ideals  $I_t(X)$ ,  $t \leq r$ , only. Cf. (10.5) for the corresponding remark in regard to the symbolic powers.

(f) With the notations of (10.10) the order of  $\pi\rho$  modulo  $F(u+1, v-1)$  may be smaller than  $(u + 1 - \tilde{u})!$ . For example let  $u = 1$  and  $v > 2$  be an even number. Then  $\pi\rho \in F(2, v-1)$ . The general case follows from the case  $\pi = [1|1]$ ,  $\rho = [2, \dots, v+1|2, \dots, v+1]$  by specialization. One has

$$(5) \sum_{i=1}^{v+1} (-1)^{i-1} [i|1][1, \dots, \widehat{i}, \dots, v+1|2, \dots, v+1] = [1, \dots, v+1|1, \dots, v+1] \in F(2, v-1)$$

by Laplace expansion, and

$$\begin{aligned} & (-1)^{i-1} [i|1][1, \dots, \widehat{i}, \dots, v+1|2, \dots, v+1] \\ & \quad + (-1)^{j-1} [j|1][1, \dots, \widehat{j}, \dots, v+1|2, \dots, v+1] \in F(2, v-1) \end{aligned}$$

by virtue of (10.11), and the sum in (5) has an odd number of terms.

(g) On the other hand the order of  $\pi\rho$  modulo  $F(u-1, v+1)$  is greater than 1 in general. We claim:  $[1|1][2\ 3\ 4|2\ 3\ 4] \notin F(2, 2)$  and  $(F(1, 3) + F(2, 2))/F(2, 2) \cong \mathbf{Z}/2\mathbf{Z}$  if  $m = n = 4$ .

A simple observation helps to reduce the amount of computation needed to prove this. Let  $\mu = \delta_1 \dots \delta_k$ ,  $\delta_i \in \Delta(X)$ . Then the *support* of  $\mu$  is the smallest submatrix of  $X$  from which all the minors  $\delta_i$  can be taken. It is fairly obvious that  $B[X]$  and all the modules  $F(u, v)$  (and  $F(u_1, \dots, u_t)$  as a self-suggesting generalization) decompose into the direct sum of their submodules generated by monomials with a fixed support. In order to show  $[1|1][2\ 3\ 4|2\ 3\ 4] \notin F(2, 2)$  it is therefore enough to consider the submodule  $N$  of  $F(2, 2)$  which is generated by the products  $\delta_1 \delta_2$ ,  $\delta_i$  a 2-minor, whose support is the entire  $4 \times 4$  matrix  $X$ .

In order to write  $2[1|1][2\ 3\ 4|2\ 3\ 4]$  as an element of  $F(2, 2)$  we follow the proof of (10.10). The reader who has proved (10.11) knows that

$$[1|1][1\ 2\ 3|2\ 3\ 4] = -[1\ 2|1\ 2][1\ 3|3\ 4] + [1\ 2|1\ 3][1\ 3|2\ 4] - [1\ 2|1\ 4][1\ 3|2\ 3]$$

and

$$(6) \quad [1|1][2\ 3\ 4|2\ 3\ 4] - [2|1][1\ 3\ 4|2\ 3\ 4] \\ = [1\ 2|1\ 2][3\ 4|3\ 4] - [1\ 2|1\ 3][3\ 4|2\ 4] + [1\ 2|1\ 4][3\ 4|2\ 3].$$

Disregarding all terms of support smaller than  $X$ , one gets from the first of these equations:

$$(7) \quad [1|1][2\ 3\ 4|2\ 3\ 4] + [2|1][1\ 3\ 4|2\ 3\ 4] \\ = -[1\ 3|1\ 2][2\ 4|3\ 4] + [1\ 3|1\ 3][2\ 4|2\ 4] - [1\ 3|1\ 4][2\ 4|2\ 3] \\ - [2\ 3|1\ 2][1\ 4|3\ 4] + [2\ 3|1\ 3][1\ 4|2\ 4] - [2\ 3|1\ 4][1\ 4|2\ 3].$$

Addition of (6) and (7) yields the desired representation of  $2[1|1][2\ 3\ 4|2\ 3\ 4]$ , and it is enough to prove that the nine products appearing in it are part of a  $\mathbf{Z}$ -basis of  $N$ .  $N$  has the same rank as the submodule generated by the standard monomials with support  $X$  in  $F(0, 4)$ ,  $F(1, 3)$ , and  $F(2, 2)$ . An easy count yields  $\text{rk } N = 14$  whereas 18 products  $\delta_1 \delta_2$  of 2-minors  $\delta_i$  have support  $X$ . Relations of these products are produced by equating two expansions of  $[1\ 2\ 3\ 4|1\ 2\ 3\ 4]$  along two rows or two columns. Let  $R_{ij}$  be the expansion along rows  $i, j$  and  $C_{ij}$  the expansion along columns  $i, j$ . It is not difficult to see that the relations

$$C_{12} - C_{13} = 0, \quad C_{12} - C_{14} = 0, \quad C_{12} - R_{12} = 0, \quad C_{12} - R_{13} = 0$$

can be solved for four products none of which appears in the representation of  $2[1|1][2\ 3\ 4|2\ 3\ 4]$  derived above.

The second claim follows very easily now: The generators of  $F(1, 3)$  with support smaller than  $X$  are in  $F(2, 2)$ , and those with support  $X$  all have order 2 modulo  $F(2, 2)$ . In conjunction with (10.11) this shows that they are congruent to each other modulo  $F(2, 2)$ .

The usual inductive technique (cf. (10.1)) allows one to conclude that

$$[1, \dots, t-1|1, \dots, t-1][1, \dots, t-2, t, t+1, t+2|1, \dots, t-2, t, t+1, t+2] \notin F(t, t)$$

if  $t \geq 2$ , and  $X$  is at least a  $(t+2) \times (t+2)$  matrix. Therefore the list of exceptional cases in which (10.13) holds without an assumption on the characteristic of  $B$ , is complete as given in (d) above.

In (10.17) we shall see that the preceding computations shed some light on the structure of the subalgebra of  $\mathbf{Z}[X]$  generated by the  $t$ -minors of  $X$ . —

(h) Under the hypotheses of (10.13) the ideal  $I_t(X)^s$  is generated as a  $B$ -module by standard monomials. As the preceding example shows, this is not true in general over  $\mathbf{Z}$ , and not even over a field:  $[1\ 2\ 3|1\ 2\ 3][4|4]$  appears with coefficient 1 in the standard representation of  $[2\ 3|1\ 2][1\ 4|3\ 4]$ , but  $[1\ 2\ 3|1\ 2\ 3][4|4] \notin I_2(X)^2$  if  $B$  is a field of characteristic 2 and  $X$  (at least) a  $4 \times 4$  matrix. —

Some of the consequences of (10.13) hold without an assumption on the characteristic of  $B$ .

**(10.15) Proposition.** *Let  $B$  be a noetherian domain,  $X$  an  $m \times n$  matrix of indeterminates.*

- (a) *Let  $t < \min(m, n)$ . Then the ideals  $I_j(X)$ ,  $1 \leq j \leq t$ , are associated prime ideals of  $I_t(X)^s$  for  $s \geq t$ .*
- (b) *If  $B$  contains a field, then the associated prime ideals of  $I_{t_1}(X) \dots I_{t_s}(X)$  are among the ideals  $I_j(X)$ ,  $j = 1, \dots, \max t_i$ .*

PROOF: If  $B$  contains a field  $K$ ,  $B[X]/I_{t_1}(X) \dots I_{t_s}(X)$  is a flat  $B$ -algebra, since  $B[X]/I_{t_1}(X) \dots I_{t_s}(X) = (K[X]/I_{t_1}(X) \dots I_{t_s}(X)) \otimes_K B$ . The usual technique (involving the fibers of  $B \rightarrow B[X]/I_{t_1}(X) \dots I_{t_s}(X)$ ) reduces part (b) to the case in which  $B$  is a field itself. One now observes that  $I_1(X)$  is a maximal ideal and applies the usual inductive trick of inverting an element of  $X$ .

Part (a) is a statement about the localizations of  $B[X]$  with respect to the prime ideals  $I_j(X)$ . Inverting  $B \setminus \{0\}$  first we may assume that  $B$  is a field again and use part (b). Since an element of  $X$  is not contained in any of the ideals  $I_j(X)$ ,  $2 \leq j \leq t$ , it is now enough to show that  $X_{11}$ , say, is a zero-divisor modulo  $I_t(X)^s$  for  $s \geq t$ . Let  $\tilde{X}$  be the  $(t + 1) \times (t + 1)$  submatrix corresponding to the first  $t + 1$  rows and columns. One has a natural inclusion  $B[\tilde{X}] \rightarrow B[X]$  and a natural epimorphism  $B[X] \rightarrow B[\tilde{X}]$  whose composition is the identity on  $B[\tilde{X}]$ . As remarked above, the conclusion of (10.9) holds for  $B[\tilde{X}]$  without an assumption on the characteristic of  $B$ . So  $X_{11}$  is a zero-divisor mod  $I_t(\tilde{X})^s$  whence it is a zero-divisor modulo  $I_t(X)^s$ . —

As a consequence of (10.15) one has  $\text{grade}(I_1(X), B[X]/I_t(X)^s) = 0$  for  $1 \leq t < \min(m, n)$  and  $s \geq t$ . By virtue of (9.23) this implies

$$\text{grade}_{I_1(X)} \text{Gr}_{I_t(X)} B[X] = 0.$$

One can say more:

**(10.16) Proposition.** *Let  $B$  be a noetherian domain,  $X$  an  $m \times n$  matrix of indeterminates,  $t$  an integer,  $1 \leq t \leq \min(m, n)$ ,  $I = I_t(X)$ , and  $A = B[X]$ .*

- (a)  *$\text{Gr}_I A / I_1(X) \text{Gr}_I A$  is isomorphic to the  $B$ -subalgebra  $S$  generated by the  $t$ -minors of  $X$ .*
- (b) *If  $t < \min(m, n)$ , the field of fractions of  $B[X]$  is algebraic over the field of fractions of  $S$ . Minors of size  $\geq t$  are even integral over  $S$ .*
- (c) *If  $t < \min(m, n)$ ,  $I_1(X) \text{Gr}_I A$  is a minimal prime ideal.*

PROOF: Part (a) holds more generally for ideals  $J$  of  $B[X]$  which are generated by homogeneous polynomials of constant degree. Then  $J^k / I_1(X) J^k$  is isomorphic to the  $B$ -submodule of  $B[X]$  generated by the products of length  $k$  in the generators of  $J$ , and

these isomorphisms are compatible with the multiplications  $J^k/I_1(X) \cdot J^k \times J^p/I_1(X) \cdot J^p \rightarrow J^{k+p}/I_1(X) \cdot J^{k+p}$ .

It is enough to show (b) for  $(t + 1) \times (t + 1)$  matrices  $X$ . The matrix  $\text{Cof } X$  of its cofactors has entries in  $S$  and

$$(\det X)^t = \det(\text{Cof } X) \in S,$$

proving the second statement and implying that the entries of  $X^{-1} = (\det X)^{-1}(\text{Cof } X)$  are algebraic over the field of fractions of  $S$ . The extension field generated by them contains the entries of  $X = (X^{-1})^{-1}$ , too, as one sees by taking cofactors again.

Part (c) concerns the localization of  $\text{Gr}_I A$  with respect to  $I_1(X)\text{Gr}_I A$ . Since furthermore the inversion of  $B \setminus \{0\}$  commutes with the formation of the associated graded ring, there is no harm in assuming that  $B$  is a field. Dimension can now be measured by transcendence degree. Hence

$$\dim \text{Gr}_I A = mn = \dim \text{Gr}_I A/I_1(X)\text{Gr}_I A. \text{ —}$$

For  $2 \leq t < \min(m, n)$  the ring  $\text{Gr}_{I_t(X)} B[X]$  does not seem to have an attractive structure. It is not even reduced:  $\delta \in I_t(X) \setminus I_t(X)^2$  for a  $(t+1)$ -minor  $\delta$ , but  $\delta^t \in I_t(X)^{t+1}$  (in arbitrary characteristic!), so  $(\delta^*)^t = 0$ .

**(10.17) Remark.** We know the structure of  $S$  very precisely in the trivial case  $t = 1$ ,  $S = B[X]$ , and the case  $t = \min(m, n)$ ,  $S = G(X)$ . Another case is completely explained by (10.16),(b):  $m = n = t + 1$ . In this case the  $t$ -minors of  $X$  are algebraically independent. As in the proof of (6.1) it is enough to consider a field of coefficients, for which the algebraic independence follows from (10.16),(b). By representation-theoretic methods we shall show in 11.E that  $S$  is a normal Cohen-Macaulay ring over a field of characteristic zero. In fact, there seems to be no characteristic-free access to the rings  $S$ , except in the special cases in which the primary decomposition of  $I_t(X)^s$  is independent of characteristic. Let  $S_0$  be the  $\mathbf{Z}$ -algebra of  $\mathbf{Z}[X]$  generated by the  $t$ -minors of  $X$ . The example in (10.14),(g) demonstrates that  $\mathbf{Z}[X]/S_0$  is not  $\mathbf{Z}$ -flat in general. For  $B = \mathbf{Z}/2\mathbf{Z}$ ,  $X$  at least a  $4 \times 4$  matrix, and  $t = 2$  the natural epimorphism  $S_0/2S_0 \rightarrow S$  is not injective, for its kernel contains the residue class of  $2[1|1][2\ 3\ 4|2\ 3\ 4]$ . This element is even nilpotent in  $S_0/2S_0$ ! Since

$$\varepsilon = [1|1][2\ 3\ 4|2\ 3\ 4] - [2|1][1\ 3\ 4|2\ 3\ 4] \in F(2, 2),$$

one has

$$\begin{aligned} 2[1|1]^2[2\ 3\ 4|2\ 3\ 4]^2 &= 2([1|1][1\ 3\ 4|2\ 3\ 4])([2|1][2\ 3\ 4|2\ 3\ 4]) \\ &\quad + 2[1|1][2\ 3\ 4|2\ 3\ 4]\varepsilon \in S_0 \end{aligned}$$

by (10.11), and therefore  $4[1|1]^2[2\ 3\ 4|2\ 3\ 4]^2 \in 2S_0$ .

## D. Comments and References

The symbolic powers of the ideals  $I_t(X)$  have first been computed by de Concini, Eisenbud, and Procesi ([DEP.1], Section 7). We have reproduced their proof in Subsection A. In [DEP.2], Section 10 it has been indicated how to consider the symbolic graded ring and the symbolic extended Rees algebra as an ASL.

The article [DEP.1] is the source for the primary decomposition of products of determinantal ideals, too. Our proof of (10.9) seems to be new, however. Since it does not depend on representation theory (different from the one in [DEP.1]), it allows us to refine the hypotheses on the characteristic of the ring of coefficients. We have followed [DEP.1] essentially in the determination of the irredundant primary components.

Proposition (10.16) has been observed in [CN].

## 11. Representation Theory

Though some of the results of this section hold over quite general rings  $B$  of coefficients, we will assume throughout that  $B = K$  is a field which, in this introduction, has characteristic 0. Let  $X$  be an  $m \times n$  matrix of indeterminates,  $T_1 \in \mathrm{GL}(m, K)$ ,  $T_2 \in \mathrm{GL}(n, K)$ . Then the substitution

$$X \longrightarrow T_1 X T_2^{-1}$$

induces a  $K$ -algebra automorphism of  $K[X]$ , and  $K[X]$  becomes a  $G$ -module,  $G = \mathrm{GL}(m, K) \times \mathrm{GL}(n, K)$ . The group  $G$  is linearly reductive, and  $K[X]$  has a decomposition into irreducible  $G$ -submodules. This decomposition is our main objective. Furthermore the  $G$ -stable ideals of  $K[X]$  will be determined in conjunction with the characterization of the prime and primary ones among them. In the last subsection we will indicate that important properties of the rings  $R_{r+1}(X)$  and their subalgebras generated by minors of a fixed size can be derived by the method of  $U$ -invariants,  $U$  being the unipotent radical of the maximal torus in a Borel subgroup of  $G$ .

### A. The Filtration of $K[X]$ by the Intersections of Symbolic Powers

The determinantal ideals  $I_t(X)$ , their products, and their symbolic powers are obviously  $G$ -stable ideals. In this subsection we study a filtration of  $K[X]$  by certain intersections of the symbolic powers. This filtration is an important tool in the investigation of the  $G$ -structure of  $K[X]$ . In characteristic zero it coincides with the filtration by the products of the ideals  $I_t(X)$ , cf. (11.2).

Whether a monomial  $\mu = \delta_1 \dots \delta_p$ ,  $\delta_i \in \Delta(X)$ , belongs to the symbolic power  $I_t(X)^{(k)}$  only depends on the size of its factors  $\delta_i$ : By virtue of (10.4)

$$\mu \in I_t(X)^{(k)} \quad \iff \quad \gamma_t(\mu) = \sum_{i=1}^p \gamma_t(\delta_i) \geq k$$

where

$$\gamma_t(\delta_i) = \begin{cases} 0 & \text{if } \delta_i \text{ is an } s\text{-minor, } s < t, \\ s - t + 1 & \text{if } \delta_i \text{ is an } s\text{-minor, } s \geq t. \end{cases}$$

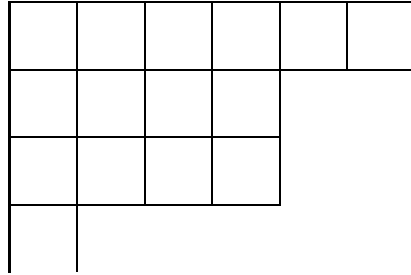
It will be very convenient to extend the notion of size from minors to monomials, for which it is called *shape*. We arrange the factors  $\delta_i$  such that their sizes form a non-increasing sequence:  $\delta_i$  is an  $s_i$ -minor and  $s_i \geq s_j$  if  $i \leq j$ . The shape of  $\mu$  is the sequence

$$|\mu| = (s_1, \dots, s_p).$$

More pictorially, the shape of a monomial can be described by a (*Young*) *diagram*: The diagram corresponding to a non-increasing (!) sequence  $\sigma = (s_1, \dots, s_p)$ , simply denoted by  $(s_1, \dots, s_p)$ , is the subset

$$\{ (i, j) \in \mathbf{N}_+ \times \mathbf{N}_+ : j \leq s_i \}$$

of  $\mathbf{N}_+ \times \mathbf{N}_+$ . One can depict such a diagram as a sequence of rows of boxes. For example  $(6, 4, 4, 1)$  is represented by:



If  $\sigma = (s_1, \dots, s_p)$  with  $s_p \neq 0$  we call  $s_1$  the number of the columns and  $p$  the number of the rows of  $\sigma$ . It is tacitly understood that the diagrams considered in connection with  $K[X]$  have at most  $\min(m, n)$  columns.

Let  $\sigma = (s_1, \dots, s_p)$  and  $\mu$  a monomial of shape  $\sigma$ . Without ambiguity we then define  $\gamma_t(\sigma)$  by

$$\gamma_t(\sigma) = \gamma_t(\mu).$$

Obviously  $\gamma_t(\sigma)$  is the number of “boxes” of  $\sigma$  in its  $t$ -th column or further right.

The filtration we want to study is formed by the ideals

$$I^{(\sigma)} = \bigcap_t I_t(X)^{\gamma_t(\sigma)},$$

$\sigma$  running through the diagrams with at most  $\min(m, n)$  columns. As noted above, for a monomial  $\mu$  one has

$$\mu \in I^{(\sigma)} \iff \gamma_t(\sigma) \leq \gamma_t(\mu) \quad \text{for all } t.$$

This motivates the introduction of a partial order on diagrams:

$$\sigma_1 \leq \sigma_2 \iff \gamma_t(\sigma_1) \leq \gamma_t(\sigma_2) \quad \text{for all } t.$$

As subsets of  $\mathbf{N}_+ \times \mathbf{N}_+$ , the diagrams are also partially ordered by the inclusion  $\subset$ . It is clear that  $\leq$  refines  $\subset$ .

Using the new notations, we recapitulate the main properties of  $I^{(\sigma)}$ :

- (11.1) Proposition.** (a)  $I^{(\sigma)}$  is the  $K$ -subspace of  $K[X]$  generated by the monomials  $\mu$  of shape  $\geq \sigma$ .  
 (b)  $I^{(\sigma)}$  has a basis given by the standard monomials of shape  $\geq \sigma$ .  
 (c)  $I^{(\sigma)}$  is a  $G$ -submodule of  $K[X]$ .



For fields of characteristic 0 the ideal  $I^{(\sigma)}$  can also be described as a product of determinantal ideals. Let  $I^\sigma$  be the ideal generated by all monomials of shape  $\sigma = (s_1, \dots, s_p)$ :

$$I^\sigma = \prod_{i=1}^p I_{s_i}(X).$$

**(11.2) Proposition.** *Let  $\sigma$  be a diagram.*

- (a)  $I^\sigma$  is the  $K$ -subspace generated by all monomials  $\mu$  such that  $|\mu| \supset \sigma$ . It is a  $G$ -submodule of  $K[X]$ .
- (b) If  $\text{char } K = 0$ , then  $I^\sigma = I^{(\sigma)}$ .

PROOF: (a) is trivial and (b) follows at once from (10.9). —

### B. Bitableaux and the Straightening Law Revisited

Let  $\nu$  be a monomial of shape  $\sigma$ . Since  $I^{(\sigma)}$  has a basis of standard monomials, the standard monomials  $\mu$  appearing in the standard representation

$$\nu = \sum a_\mu \mu, \quad a_\mu \in K, \quad a_\mu \neq 0,$$

all have shape  $\geq \sigma$ . This representation can be split into two parts:

$$\nu = \sum_{|\mu|=\sigma} a_\mu \mu + \sum_{|\mu|<\sigma} a_\mu \mu.$$

In order to analyze the  $G$ -structure of  $K[X]$ , we need some information about the first of these summands. In some sense it can be computed by a separate consideration of the “row part” and the “column part” of  $\nu$  (cf. (11.4),(c)). For this purpose we need a more flexible notation.

Let  $\sigma$  be a diagram. A *tableau*  $\Sigma$  of shape  $\sigma$  on  $\{1, \dots, m\}$  is a map

$$\Sigma: \sigma \longrightarrow \{1, \dots, m\}.$$

Pictorially,  $\Sigma$  “writes” a number between 1 and  $m$  into each of the “boxes” of  $\sigma$ . An example:

3	2	6	7	7
1	4	5	6	
2	3			
1				

is a tableau of shape  $(5, 4, 2, 1)$  on  $\{1, \dots, 7\}$ . A *bitableau*  $(\Sigma|\mathbf{T})$  of shape  $\sigma$  is a pair of tableaux  $\Sigma, \mathbf{T}$  of shape  $\sigma$ . In the following it is always understood that  $\Sigma$  is a tableau on  $\{1, \dots, m\}$ , while  $\mathbf{T}$  has values in  $\{1, \dots, n\}$ . It is clear what is meant by a row of a tableau.

The *content*  $c(\Sigma)$  of a tableau  $\Sigma$  is the function that counts the number of occurrences of a number in a tableau:

$$c(\Sigma)(u) = |\{(i, j) : \Sigma(i, j) = u\}|.$$

The content  $c(\Sigma|\mathbf{T})$  of a bitableau is the pair  $(c(\Sigma)|c(\mathbf{T}))$ .

To each bitableau we associate an element of  $K[X]$  in the following manner:

$$(\Sigma|\mathbf{T}) = \delta_1 \dots \delta_p$$

where  $p$  is the number of rows of  $\Sigma$  (or  $\mathbf{T}$ ) and  $\delta_i$  the determinant of the matrix

$$([\Sigma(i, j)|\mathbf{T}(i, k)] : j = 1, \dots, s_p, k = 1, \dots, s_p).$$

Up to sign,  $(\Sigma|\mathbf{T})$  (as an element of  $K[X]$ ) is a monomial which has the same shape as  $(\Sigma|\mathbf{T})$ .

A tableau is called *standard* if its rows are increasing (i.e.  $\Sigma(i, j) < \Sigma(i, k), \mathbf{T}(i, j) < \mathbf{T}(i, k)$  for  $j < k$ ) and its columns are non-decreasing. A *standard bitableau* is a pair of standard tableaux. Obviously the standard bitableaux are in bijective correspondence with the standard monomials. Therefore we can reformulate part of the ASL axioms for  $K[X]$  in the language of tableaux. On this occasion we note an additional property of the standard representation:

**(11.3) Theorem.** *Each bitableau  $(\Sigma|\mathbf{T})$  (satisfying the restrictions due to the size of  $X$ ) has a representation*

$$(\Sigma|\mathbf{T}) = \sum_i a_i (\Sigma_i|\mathbf{T}_i), \quad a_i \in K, a_i \neq 0, \quad (\Sigma_i|\mathbf{T}_i) \text{ standard}.$$

Furthermore  $c(\Sigma|\mathbf{T}) = c(\Sigma_i|\mathbf{T}_i)$  for all  $i$ .

The last statement can be derived from the straightening procedure: an application of a Plücker relation only renders an exchange of indices, none gets lost and none is created (neither in  $G(\tilde{X})$  nor in  $K[X]$ ). A direct proof: In order to test the equality  $(c(\Sigma))(u) = (c(\Sigma_i))(u)$  one multiplies the row  $u$  of  $X$  by a new indeterminate  $W$  and exploits the linear independence of the standard bitableaux (monomials) over  $K[W]$  (One can further refine (11.3) by extending the partial order  $\leq$  from diagrams to tableaux, cf. [DEP.1].)

The tableaux  $\Sigma$  of shape  $\sigma$  on  $\{1, \dots, m\}$  are partially ordered in a natural way by component-wise comparison. The smallest tableau  $K_\sigma$  with respect to this partial order

is called the *initial tableau* of shape  $\sigma$ , while the maximal one  $\overline{K}_\sigma$  is called *final*:

$K_\sigma =$	<table border="1" style="display: inline-table;"><tr><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>...</td></tr><tr><td>1</td><td>2</td><td>3</td><td>...</td><td></td><td></td></tr><tr><td>1</td><td>2</td><td>...</td><td></td><td></td><td></td></tr><tr><td>1</td><td>...</td><td></td><td></td><td></td><td></td></tr><tr><td>⋮</td><td></td><td></td><td></td><td></td><td></td></tr></table>	1	2	3	4	5	...	1	2	3	...			1	2	...				1	...					⋮					
1	2	3	4	5	...																										
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$\overline{K}_\sigma =$	<table border="1" style="display: inline-table;"><tr><td>...</td><td><math>m-4</math></td><td><math>m-3</math></td><td><math>m-2</math></td><td><math>m-1</math></td><td><math>m</math></td></tr><tr><td>...</td><td><math>m-3</math></td><td><math>m-2</math></td><td><math>m-1</math></td><td><math>m</math></td><td></td></tr><tr><td>...</td><td><math>m-3</math></td><td><math>m-2</math></td><td><math>m-1</math></td><td><math>m</math></td><td></td></tr><tr><td>...</td><td><math>m-1</math></td><td><math>m</math></td><td></td><td></td><td></td></tr><tr><td>...</td><td></td><td></td><td></td><td></td><td></td></tr></table>	...	$m-4$	$m-3$	$m-2$	$m-1$	$m$	...	$m-3$	$m-2$	$m-1$	$m$		...	$m-3$	$m-2$	$m-1$	$m$		...	$m-1$	$m$				...					
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A bitableau is *left (right) initial* if it is of the form  $(K_\sigma|T)$  ( $(\Sigma|K_\sigma)$ ). The *left and right final* bitableaux are defined correspondingly. We put

$$\Lambda_\sigma = (K_\sigma|K_\sigma) \quad \text{and} \quad \overline{\Lambda}_\sigma = (\overline{K}_\sigma|\overline{K}_\sigma)$$

where  $\overline{K}_\sigma$  is of course a final tableau on  $\{1, \dots, n\}$  if it appears on the right side of a bitableau. A last piece of notation:

$$I_{>\sigma}^{(\sigma)}$$

is the  $K$ -subspace generated by all (standard) monomials of shape  $> \sigma$  (and thus an ideal).

**(11.4) Lemma.** (a) *A left initial bitableau  $(K_\sigma|T)$  has a standard representation*

$$(K_\sigma|T) = \sum_i a_i (K_\sigma|T_i).$$

Analogous statements hold for “right” in place of “left” and final bitableaux.

(b) *With the notations of (a) one has for every tableau  $\Sigma$  of shape  $\sigma$ :*

$$(\Sigma|T) \equiv \sum_i a_i (\Sigma|T_i) \pmod{I_{>\sigma}^{(\sigma)}}.$$

(c) *If furthermore  $(\Sigma|K_\sigma) = \sum_j b_j (\Sigma_j|K_\sigma)$  is the standard representation of  $(\Sigma|K_\sigma)$ , then*

$$(\Sigma|T) \equiv \sum_{i,j} a_i b_j (\Sigma_j|T_i) \pmod{I_{>\sigma}^{(\sigma)}}.$$

PROOF: (a)  $K_\sigma$  is the only standard tableau which has content  $c(K_\sigma)$ . So part (a) is a trivial consequence of (11.3).

(b) holds trivially if  $T$  is a standard tableau. If not, we may certainly assume that the rows of  $\Sigma$  and  $T$  are increasing, and we can switch to the language of monomials. Let

$$(K_\sigma|T) = \delta_1 \dots \delta_p, \quad \delta_i \in \Delta(X), \quad |\delta_i| \geq |\delta_{i+1}|.$$

In the proof of (4.1) every monomial has been assigned a “weight”. This is increased by an application of the straightening law: if one replaces a product  $\delta_i \delta_j$  of incomparable minors by its standard representation and expands the resulting expression, the monomials on the right hand side have a higher weight. This leads to a proof by induction as we shall see below.

First we deal with the crucial case  $p = 2$ . Let

$$\mu = \delta_1 \delta_2 = [a_1, \dots, a_u | b_1, \dots, b_u] [c_1, \dots, c_v | d_1, \dots, d_v], \quad u \geq v.$$

The corresponding left initial monomial is

$$\varepsilon_1 \varepsilon_2 = [1, \dots, u | b_1, \dots, b_u] [1, \dots, v | d_1, \dots, d_v].$$

As in Section 4 we relate  $K[X]$  to  $G(\tilde{X})$ . Then

$$\tilde{\varepsilon}_1 \tilde{\varepsilon}_2 = [b_1, \dots, b_u, n + 1, \dots, n + m - u] [d_1, \dots, d_v, n + 1, \dots, n + m - v].$$

Assume that  $b_i \leq d_i$  for  $i = 1, \dots, k$ , but  $b_{k+1} > d_{k+1}$ . In order to straighten the product  $\tilde{\varepsilon}_1 \tilde{\varepsilon}_2$  one applies a Plücker relation from (4.4) as in (4.5). The “same” Plücker relation is applied to  $\tilde{\delta}_1 \tilde{\delta}_2$ , and the crucial point is to show that any formal term on the right hand side of the relation for  $\tilde{\varepsilon}_1 \tilde{\varepsilon}_2$  which drops out for this choice of minors, also drops out for  $\tilde{\delta}_1 \tilde{\delta}_2$  or gives a term of shape  $> |\delta_1 \delta_2|$  back in  $K[X]$ . Observe in the following that the indices  $n + 1, \dots, n + m - v$  of the second factor are not involved in the exchange of indices within the Plücker relation. If a formal term vanishes for  $\tilde{\varepsilon}_1 \tilde{\varepsilon}_2$  because of a coincidence among the indices  $b_1, \dots, b_u, d_1, \dots, d_v$ , it also vanishes for  $\tilde{\delta}_1 \tilde{\delta}_2$ . If it is zero because of a coincidence among the indices  $n + 1, \dots, n + m - u, n + 1, \dots, n + m - v$  then one of the indices  $n + 1, \dots, n + m - u$  must have travelled from the “left” factor to the “right” factor. Of course this term may drop out for  $\tilde{\delta}_1 \tilde{\delta}_2$ , too; if not, it forces the “right” factor to be a minor of smaller size in  $K[X]$ , as desired.

The preceding arguments have proved the following assertion: There are elements  $f_i \in K$  such that

$$\begin{aligned} \varepsilon_1 \varepsilon_2 &= \sum_i f_i \xi_{i1} \xi_{i2}, \\ \delta_1 \delta_2 &\equiv \sum_i f_i v_{i1} v_{i2} \pmod{I_{>}^{(|\delta_1 \delta_2|)}}, \end{aligned}$$

$\xi_{ij}$  has the same row part as  $\varepsilon_j$ ,  $v_{ij}$  has the same row part as  $\delta_j$ , and the column parts of  $\xi_{ij}$  and  $v_{ij}$  coincide,  $j = 1, 2$ ; furthermore the column parts of  $\xi_{i1}$  and  $\xi_{i2}$  are comparable in the first  $k + 1$  positions. Therefore induction on  $k$  finishes the case  $p = 2$ .

Now we deal with the general case for  $p$ . Suppose that the column parts of  $\delta_k$  and  $\delta_{k+1}$  are incomparable. Let  $(K_\sigma | T) = \varepsilon_1 \dots \varepsilon_p$ . Then the (column parts of)  $\varepsilon_k$  and  $\varepsilon_{k+1}$  are incomparable, too, and we substitute the standard representation of  $\varepsilon_k \varepsilon_{k+1}$  into the product  $\varepsilon_1 \dots \varepsilon_p$ :

$$\varepsilon_1 \dots \varepsilon_p = \sum_i f_i \varepsilon_1 \dots \varepsilon_{k-1} \xi_{i1} \xi_{i2} \varepsilon_{k+2} \dots \varepsilon_p.$$

From the case  $p = 2$ :

$$\delta_1 \dots \delta_p \equiv \sum_i f_i \delta_1 \dots \delta_{k-1} v_{i1} v_{i2} \delta_{k+2} \dots \delta_p \pmod{\delta_1 \dots \delta_{k-1} \delta_{k+2} \dots \delta_p I_{>}^{(|\delta_k \delta_{k+1}|)}}$$

where the row and the column parts of  $\varepsilon_k, \varepsilon_{k+1}, \delta_k, \delta_{k+1}, \xi_{ij}, v_{ij}$  are related as above. Since

$$\delta_1 \dots \delta_{k-1} \delta_{k+2} \dots \delta_p I_{>}^{(|\delta_k \delta_{k+1}|)} \subset I_{>}^{(\sigma)},$$

the result follows by induction on the “weight” as indicated already.

Part (c) results immediately from (b) and its “right” analogue. —

### C. The Decomposition of $K[X]$ into Irreducible $G$ -Submodules

Let now  $L_\sigma$  be the  $K$ -subspace of  $K[X]$  generated by all the right initial bitableaux of shape  $\sigma$ , and  ${}_\sigma L$  the corresponding object for “left”.  $L_\sigma$  is certainly a  $\text{GL}(m, K)$ -submodule of  $K[X]$ , and  ${}_\sigma L$  is a  $\text{GL}(n, K)$ -submodule. Letting  $G$  act by

$$(g, h)(x_1 \otimes x_2) = g(x_1) \otimes h(x_2), \quad x_1 \in L_\sigma, \quad x_2 \in {}_\sigma L, \quad g \in \text{GL}(m, K), \quad h \in \text{GL}(n, K),$$

one makes  $L_\sigma \otimes {}_\sigma L$  a  $G$ -module.

**(11.5) Theorem.** (a)  ${}_\sigma L$  has a basis given by the standard bitableaux  $(K_\sigma | T)$ . A corresponding statement holds for  $L_\sigma$ .

(b) There is a  $G$ -isomorphism  $\varphi: L_\sigma \otimes {}_\sigma L \longrightarrow I^{(\sigma)} / I_{>}^{(\sigma)}$  such that

$$\varphi((\Sigma | K_\sigma) \otimes (K_\sigma | T)) = (\Sigma | T) + I_{>}^{(\sigma)}$$

for all tableaux  $\Sigma, T$  of shape  $\sigma$ .

PROOF: Part (a) is proved by part (a) of Lemma (11.4). Restricting the formula in (b) to the standard tableaux one therefore defines an isomorphism of  $K$ -vector spaces, whereupon the formula is valid for all tableaux because of part (c) of (11.4). Evidently  $\varphi$  is compatible with the actions of  $G$  on  $L_\sigma \otimes {}_\sigma L$  and  $I^{(\sigma)} / I_{>}^{(\sigma)}$  resp. —

Next we analyze the structure of  ${}_\sigma L$  and  $L_\sigma$ . On the grounds of symmetry it is enough to consider  ${}_\sigma L$ . For reasons which will become apparent in Subsection E below, it is useful to investigate the action of the subgroup  $U^-(n, K)$  which consists of the lower triangular matrices with the entry 1 on all diagonal positions. The subgroup  $U^+(n, K)$  is defined analogously.

Again the crucial argument is given as a lemma.

**(11.6) Lemma.** Let  $K$  be an infinite field. Then every nontrivial  $U^-(n, K)$ -submodule of  $K[X]$  contains a nonzero element of the  $K$ -subspace generated by all (standard) right final bitableaux.

PROOF: For the proof of the linear independence of the standard monomials in  $G(X)$  we have studied the effect of the elementary transformation  $\alpha$ ,

$$X_{st} \xrightarrow{\alpha} X_{st} \quad \text{for } t \neq i_0, \quad X_{s i_0} \xrightarrow{\alpha} X_{s i_0} + W X_{s j_0}, \quad W \xrightarrow{\alpha} W,$$

on a linear combination  $\sum_{\mu \in S} a_\mu \mu$  of standard monomials, cf. 4.C;  $W$  is a new indeter-

minate, and  $(i_0, j_0)$  is the lexicographically smallest special pair occurring in the factors of the monomials  $\mu \in S$ . Let now  $\delta = [a_1, \dots, a_t | b_1, \dots, b_t] \in \Delta(X)$ . Then we say (in this proof) that  $(i, j)$ ,  $i < j$ , is *column-special* for  $\delta$  if  $i \in \{b_1, \dots, b_t\}$ ,  $j \notin \{b_1, \dots, b_t\}$ . Consider an element  $\sum_{\mu \in S} a_\mu \mu$  of  $K[X]$  given in its standard representation. If  $j > n$  for every column-special pair appearing in the factors of the monomials  $\mu \in S$ , then each  $\mu \in S$  corresponds to a right final bitableau. Otherwise we take  $(i_0, j_0)$  to be the lexicographically smallest of all the column-special pairs  $(i, j)$  with  $j \leq n$  (as far as they occur “in  $S$ ”). As in Section 4 we define

$$\begin{aligned} \Phi(\delta) &= \begin{cases} \delta & \text{if } (i_0, j_0) \text{ is not column-special for } \delta, \\ \delta & \text{with } i_0 \text{ replaced by } j_0 \text{ in the column part (and ordered again) otherwise,} \end{cases} \\ \Phi(\mu) &= \Phi(\delta_1) \dots \Phi(\delta_u) \quad (\mu = \delta_1 \dots \delta_u, \delta_i \in \Delta(X)), \\ v(\mu) &= |\{k : (i_0, j_0) \text{ is column-special for } \delta_k\}|. \end{aligned}$$

We leave it to the reader to check that the analogue of Lemma (4.8) holds:

- (a) Let  $\gamma, \delta \in \Delta(X)$  be factors of  $\mu \in S$ . If  $\gamma \leq \delta$ , then  $\Phi(\gamma) \leq \Phi(\delta)$ .
- (b) For  $\mu \in S$  the monomial  $\Phi(\mu)$  is again standard.
- (c) Let  $\mu, \nu \in S$  such that  $v(\mu) = v(\nu)$ . If  $\mu \neq \nu$ , then  $\Phi(\mu) \neq \Phi(\nu)$ .

Put  $v_0 = \max\{v(\mu) : \mu \in S\}$ . Then for the elementary transformation  $\alpha$  above one has as in Section 4

$$\alpha\left(\sum_{\mu \in S} a_\mu \mu\right) = \pm W^{v_0} \sum_{\mu \in S_0} a_\mu \Phi(\mu) + \sum_{i=0}^{v_0-1} W^i y_i, \quad y_i \in K[X],$$

where  $S_0 = \{\mu \in S : v(\mu) = v_0\}$  is nonempty and the standard monomials  $\Phi(\mu)$ ,  $\mu \in S_0$ , are pairwise distinct. Furthermore the lexicographically smallest among all the column-special pairs  $(i, j)$ ,  $j \leq n$ , occurring “in  $S_0$ ” is greater than  $(i_0, j_0)$ , provided there is left such a pair.

Now we replace the indeterminate  $W$  by an element  $w \in K$ , obtaining an element  $\alpha_w \in U^-(n, K)$ , and

$$\alpha_w\left(\sum_{\mu \in S} a_\mu \mu\right) = \pm w^{v_0} \sum_{\mu \in S_0} a_\mu \Phi(\mu) + \sum_{i=0}^{v_0-1} w^i y_i, \quad y_i \in K[X].$$

Let  $V$  be the  $U^-(n, K)$ -submodule generated by  $\sum a_\mu \mu$ . It is enough to show that  $\sum_{\mu \in S_0} a_\mu \Phi(\mu) \in V$ , and now the hypothesis that  $K$  is infinite plays an essential role: If  $\sum_{i=0}^v w^i x_i = 0$  for vectors  $x_0, \dots, x_v$  in a  $K$ -vector space and all  $w \in K$ , then  $x_0 = \dots = x_v = 0$ . Apply this to  $K[X]/V$ . —

**(11.7) Proposition.** *Let  $K$  be an infinite field.*

- (a) *Then every nontrivial  $U^-(n, K)$ -submodule of  ${}_\sigma L$  contains  $(K_\sigma | \overline{K}_\sigma)$ .*
- (b)  *${}_\sigma L$  is  $U^-(n, K)$ -indecomposable: it does not contain a nontrivial direct  $U^-(n, K)$ -summand. All the more, it is  $GL(n, K)$ -indecomposable.*
- (c) *If  $\text{char } K = 0$ , then  ${}_\sigma L$  is  $GL(n, K)$ -irreducible: it does not contain a nontrivial proper  $GL(n, K)$ -submodule.*

*Analogous statements hold for  $U^+(n, K)$ ,  $U^-(m, K)$ , and  $U^+(m, K)$ .*

PROOF: Part (a) follows directly from (11.4),(a) and the preceding lemma, (b) is an immediate consequence of (a), and (c) results from (b) since  $\text{GL}(n, K)$  is linearly reductive if  $\text{char } K = 0$ . —

It is easy to see that (11.7),(c) is wrong in positive characteristic  $p$ . The case  $m = 2$ ,  $n = 1$  provides a “universal” counterexample: Consider the subspace generated by the  $p$ -th powers of the indeterminates; it is a  $\text{GL}(2, K)$ -subspace of  $L_\sigma$ ,  $\sigma = (1, \dots, 1)$ , because of the mathematics-made-easy binomial formula for  $p$ -th powers in characteristic  $p$ .

**(11.8) Corollary.** *Let  $K$  be an infinite field, and  $\sigma, \tau$  diagrams. Then  ${}_\sigma L$  and  ${}_\tau L$  are isomorphic as  $\text{GL}(n, K)$ -modules if and only if  $\sigma = \tau$ . An analogous statement holds for  $L_\sigma$ ,  $L_\tau$ , and  $\text{GL}(m, K)$ .*

PROOF: One has to show that  $\sigma$  can be reconstructed from the  $\text{GL}(n, K)$ -action on  ${}_\sigma L$ . The  $K$ -module of  $U^-(n, K)$ -invariants of  ${}_\sigma L$  is one-dimensional, generated by  $(K_\sigma | \overline{K}_\sigma)$  because of (11.7),(a) and the definition of  ${}_\sigma L$ . Consider the subgroup  $D(n, K)$  of diagonal matrices in  $\text{GL}(n, K)$ , and let  $d_1, \dots, d_n$  be the elements in the diagonal of  $d \in D(n, K)$ . Then

$$(*) \quad d(K_\sigma | \overline{K}_\sigma) = \left( \prod_{i=1}^n d_i^{-e_i} \right) (K_\sigma | \overline{K}_\sigma)$$

where  $e_i$  is the multiplicity with which the column index  $i$  appears in  $\overline{K}_\sigma$ . Conversely, the exponents  $e_i$  are uniquely determined by the equation (\*) if  $d$  runs through  $D(n, K)$ . They in turn characterize  $\sigma$ . —

**(11.9) Remark.** It is a fundamental theorem of representation theory that the representations  $\text{GL}(m, K) \rightarrow \text{GL}(L_\sigma)$ ,  $\sigma$  running through the diagrams with at most  $m$  columns, are the only irreducible polynomial representations of  $\text{GL}(m, K)$  over a field of characteristic zero. (A representation  $\text{GL}(m, K) \rightarrow \text{GL}(V)$  is polynomial if it is given by a polynomial map in the entries of the matrices in  $\text{GL}(m, K)$ .) For the representation theory of  $\text{GL}(m, K)$  the reader may consult [Gn]. —

As a preliminary stage to the  $G$ -decomposition of  $K[X]$  we shall study its decomposition over  $\text{GL}(m, K)$  and  $\text{GL}(n, K)$ . Let  $H$  be a linearly reductive group and  $V$  an  $H$ -module. Then  $V$  decomposes into the direct sum  $\bigoplus V_\omega$ , where  $V_\omega$  is the submodule formed by the sum of all irreducible  $H$ -submodules of  $V$  which have a given isomorphism type  $\omega$ .  $V_\omega$  is called the *isotypic component of type  $\omega$* . (Of course  $V_\omega = 0$  if none such submodule occurs in  $V$ .) As a consequence every  $H$ -submodule  $U \subset V$  decomposes into the direct sum  $\bigoplus U_\omega$ ,  $U_\omega = U \cap V_\omega$ .

**(11.10) Proposition.** *Let  $K$  be a field of characteristic 0, and  $M_\sigma$  denote a  $G$ -complement of  $I_{>}^{(\sigma)}$  in  $I^{(\sigma)}$ .*

(a) *Then  $K[X] = \bigoplus M_\sigma$ , the sum being extended over the diagrams  $\sigma$  with at most  $\min(m, n)$  columns.*

(b)  *$M_\sigma$  is the isotypic  $\text{GL}(n, K)$ -component of  $K[X]$  of type  $L_\sigma$ , as well as the isotypic  $\text{GL}(n, K)$ -component of  $K[X]$  of type  ${}_\sigma L$ .*

(c) *Therefore  $M_\sigma$  is the unique  $G$ -complement of  $I_{>}^{(\sigma)}$  in  $I^{(\sigma)}$ .*

PROOF: By (11.5) the  $GL(m, K)$ -module  $M_\sigma$  is a direct sum of  $GL(m, K)$ -modules of type  $L_\sigma$ , and  $L_\sigma$  is irreducible, cf. (11.7),(c). Since  $L_\sigma$  and  $L_\tau$  are non-isomorphic according to (11.8), one has

$$M_\sigma \cap \sum_{\tau \neq \sigma} M_\tau = 0,$$

as discussed above. The rest of (a) is a dimension argument: Let  $d$  be the degree of a monomial of shape  $\sigma$ ; since  $L_\sigma \subset M_\sigma$ , the elements of  $M_\sigma$  have degree  $d$ , and  $M_\sigma$  has the same dimension as the  $K$ -subspace  $V_\sigma$  generated by the standard monomials of shape  $\sigma$ , (cf. (11.5) again). Furthermore the  $d$ -th homogeneous component of  $K[X]$  is the direct sum of the subspaces  $V_\sigma$ , the sum being extended over all the diagrams  $\sigma$  with exactly  $d$  boxes (and at most  $\min(m, n)$  columns).

Parts (a) and (b) have been proved now, and the uniqueness of  $M_\sigma$  follows directly from (b). —

The main objective of this section is of course the following theorem which reveals the  $G$ -structure of  $K[X]$  in characteristic 0.

**(11.11) Theorem.** *Let  $K$  be a field of characteristic 0.*

(a) *Every nontrivial  $G$ -submodule  $V$  of  $M_\sigma$  contains  $\Lambda_\sigma = (K_\sigma | K_\sigma)$  (and  $\bar{\Lambda}_\sigma = (\bar{K}_\sigma | \bar{K}_\sigma)$ ). Therefore  $M_\sigma$  is irreducible as a  $G$ -module.*

(b) *The direct sum  $K[X] = \bigoplus M_\sigma$  is a decomposition into pairwise non-isomorphic irreducible  $G$ -modules.*

PROOF: Only (a) needs a proof, since (b) follows directly from (a) and the preceding proposition. Let  $U = \bigoplus L_\tau$ . By virtue of the  $U^+(n, K)$ -variant of (11.6) we have  $V \cap \bigoplus L_\tau \neq 0$ . The only isotypic component  $L_\tau$  of  $U$  which can intersect  $M_\sigma$  nontrivially, is  $L_\sigma$ . Thus  $L_\sigma \subset V$ , and, a fortiori,  $\Lambda_\sigma \in V$ . (Here we use of course (11.7),(c) and (11.8).) —

**(11.12) Remark.** Let  $K$  be a field of characteristic 0. One should note that  $M_\sigma$  is generated as a  $G$ -module by any left (or right) initial (or final) bitableau  $(\Sigma|T)$  of shape  $\sigma$  since, as above,  $L_\sigma$  and  ${}_\sigma L$  and their “final” analogues are contained in  $M_\sigma$ . On the other hand the  $G$ -module generated by an arbitrary bitableau of shape  $\sigma$ , even a standard one, always contains  $M_\sigma$ , but may be bigger. It contains  $M_\sigma$  since it is part of  $I^{(\sigma)}$  and not contained in  $I_>^{(\sigma)}$ . As an example, consider  $m = 2, n = 2$ , and let  $V$  be the subspace of homogeneous polynomials of degree 2 in  $K[X]$ . Then

$$V = M_{(2)} \oplus M_{(1,1)},$$

$M_{(2)}$  has the basis  $[1\ 2|1\ 2]$ , and  $M_{(1,1)}$  has the basis

$$\begin{aligned} & [i|j][i|k], [r|t][s|t], \quad i, j, k, r, s, t \in \{1, 2\}, \\ & [1|1][2|2] + [2|1][1|2]. \end{aligned}$$

As a  $G$ -module  $V$  is generated by  $[1|1][2|2]$ , for example. —

In an application below it will be useful to have at least an upper approximation to the  $G$ -module generated by a bitableau.



**(11.13) Proposition.** *Let  $(\Sigma|T)$  be a bitableau of shape  $\sigma$ , and  $S$  be the set of diagrams  $\tau$  (with at most  $\min(m, n)$  columns) such that (i)  $\tau \geq \sigma$ , and (ii) there is a standard bitableau  $(\Sigma'|T')$  of shape  $\tau$  with the same content as  $(\Sigma|T)$ . Then (the  $G$ -submodule generated by)  $(\Sigma|T)$  is contained in  $\bigoplus_{\tau \in S} M_\tau$ .*

PROOF: Let  $S'$  be the set of all diagrams  $\tau$  such that  $\tau \geq \sigma$  and there is a standard bitableau  $(\Sigma'|T')$  of shape  $\tau$  with  $c(\Sigma') = c(\Sigma)$ . By symmetry it is enough to show that

$$(\Sigma|T) \in \bigoplus_{\tau \in S'} M_\tau.$$

Descending inductively with respect to  $\leq$ , we may further suppose that all the bitableaux  $(\Sigma'|T')$  of shape  $> \sigma$  and with  $c(\Sigma') = c(\Sigma)$  are contained in

$$N'_\sigma = \bigoplus_{\tau \in S', \tau > \sigma} M_\tau.$$

Let now  $V$  be the  $K$ -subspace generated by all bitableaux  $(\Sigma|T')$ ,  $T'$  a standard tableau of shape  $\sigma$ .  $V$  is a  $GL(n, K)$ -submodule, as well as  $V + N'_\sigma$ . Arguing inductively via (11.4),(b) and (11.3) one has  $(\Sigma|T) \in V + N'_\sigma$ . Because of (11.4),(b) again, the assignment

$$(K_\sigma|\Xi) \longrightarrow (\Sigma|\Xi), \quad \Xi \text{ a standard tableau of shape } \sigma,$$

defines an isomorphism  ${}_\sigma L \longrightarrow (V + N'_\sigma)/N'_\sigma$ . Therefore

$$V + N'_\sigma \cong {}_\sigma L \oplus N'_\sigma$$

as  $GL(n, K)$ -modules. Every  $GL(n, K)$ -submodule of  $K[X]$  of type  ${}_\sigma L$  is contained in  $M_\sigma$ , so  $(\Sigma|T) \in M_\sigma \oplus N'_\sigma$  as claimed. —

At this point the reader should note that the attributes “initial” or “final” could always have been replaced by “nested”: A tableau  $\Sigma$  of shape  $\sigma = \{s_1, \dots, s_p\}$  is *nested* if

$$\{ \Sigma(i, j) : 1 \leq j \leq s_i \} \supset \{ \Sigma(k, j) : 1 \leq j \leq s_k \}$$

for all  $i, k = 1, \dots, p, i < k$ .

### D. $G$ -Invariant Ideals

In this section  $K$  has characteristic 0 throughout. In the decomposition  $K[X] = \bigoplus M_\sigma$  the irreducible  $G$ -submodules are pairwise non-isomorphic. Therefore every  $G$ -submodule of  $K[X]$  has the form

$$\bigoplus_{\sigma \in S} M_\sigma$$

for some subset  $S$  of the set of diagrams (with at most  $\min(m, n)$  columns). As a remarkable fact, the ideals among the  $G$ -submodules correspond to the ideals in the set

of diagrams partially ordered by  $\supset$ :  $S$  is called a  $D$ -ideal if it satisfies the following condition:

$$\sigma \in S, \tau \supset \sigma \implies \tau \in S.$$

**(11.14) Theorem.** *Let  $K$  be a field of characteristic 0. Then a  $G$ -submodule  $\bigoplus_{\sigma \in S} M_\sigma$  is an ideal if and only if  $S$  is a  $D$ -ideal.*

The theorem will follow at once from the description of the  $G$ -submodules  $I_\sigma$  corresponding to the “principal”  $D$ -ideals given in (11.15): For a diagram  $\sigma$  we put

$$I_\sigma = \bigoplus_{\tau \supset \sigma} M_\tau.$$

Obviously

$$I_\sigma \subset I^{(\sigma)} = \bigoplus_{\tau \geq \sigma} M_\tau.$$

The determinantal ideals  $I_t(X)$  are given as

$$I_t(X) = I_{(t)},$$

where  $(t)$  is the diagram with a single row of  $t$  boxes. (One applies (11.14) or observes that  $\tau \supset (t)$  if and only if  $\tau \geq (t)$ .)

**(11.15) Proposition.**  *$I_\sigma$  is the ideal generated by  $M_\sigma$ . It is the smallest  $G$ -stable ideal containing  $\Lambda_\sigma$ .*

PROOF: Since  $\Lambda_\sigma$  generates the  $G$ -module  $M_\sigma$ , it is enough to prove the first statement. Let  $J = M_\sigma K[X]$ . Then  $J$  is a  $G$ -stable ideal. If we can show that  $\Lambda_\tau \in J$  for the upper neighbours  $\tau$  of  $\sigma$  with respect to  $\subset$ , then  $I_\sigma \subset J$  follows by induction. An upper neighbour of  $\tau$  differs from  $\sigma$  in exactly one row in which it has one more box (including the case in which  $\sigma$  has no box in the pertaining row). Let  $\sigma = (s_1, \dots, s_p)$ , allowing  $s_p = 0$ . Then  $\tau = (t_1, \dots, t_p)$  with  $t_k = s_k + 1$  for exactly one  $k$ , and  $t_i = s_i$  otherwise. We switch to monomials:

$$\Lambda_\sigma = \delta_1 \dots \delta_p, \quad \delta_i = [1, \dots, s_i | 1, \dots, s_i].$$

Then

$$\begin{aligned} \Lambda_\tau &= \left( \prod_{i \neq k} \delta_i \right) [1, \dots, s_k + 1 | 1, \dots, s_k + 1] \\ &= \left( \prod_{i \neq k} \delta_i \right) \sum_{j=1}^{s_k+1} \pm [j | s_k + 1] [1, \dots, \widehat{j}, \dots, s_k + 1 | 1, \dots, s_k] \\ &= \sum_{j=1}^{s_k+1} \pm [j | s_k + 1] \left( \left( \prod_{i \neq k} \delta_i \right) [1, \dots, \widehat{j}, \dots, s_k + 1 | 1, \dots, s_k] \right). \end{aligned}$$

The bitableau corresponding to  $\left( \left( \prod_{i \neq k} \delta_i \right) [1, \dots, \widehat{j}, \dots, s_k + 1 | 1, \dots, s_k] \right)$  is in  $L_\sigma \subset M_\sigma$ .

The converse inclusion  $J \subset I_\sigma$  is proved once we can show that

$$(1) \quad [i|j]\Lambda_\sigma \in I_\sigma$$

for all entries  $[i|j]$  of the matrix  $X$ , since first every element of  $M_\sigma$  is a  $K$ -linear combination of  $G$ -conjugates of  $\Lambda_\sigma$ , and secondly  $I_\tau \subset I_\sigma$  for  $\tau \supset \sigma$ . Write  $[i|j]\Lambda_\sigma = (\Sigma|T)$  as a standard bitableau, and let  $\tilde{\sigma}$  be its shape. It is an easy exercise to show that  $\tau \supset \sigma$  for every diagram  $\tau$  such that (i)  $\tau \geq \tilde{\sigma}$  and (ii) there is a standard bitableau  $(\Sigma'|T')$  of shape  $\tau$  such that  $c(\Sigma'|T') = c(\Sigma|T)$ . Now the desired inclusion results from (11.13). —

The preceding theorem sets up a bijective correspondence

$$S \longleftrightarrow I(S) = \bigoplus_{\sigma \in S} M_\sigma$$

between  $D$ -ideals  $S$  and the  $G$ -stable ideals of  $K[X]$ . This correspondence preserves set-theoretic inclusions, and makes the set of  $G$ -stable ideals a distributive lattice, transferring  $\cap$  and  $\cup$  into the intersection and sum resp. of ideals. In order to carry the correspondence even further we define a multiplication of diagrams: For  $\sigma = (s_1, \dots, s_p)$  and  $\tau = (t_1, \dots, t_q)$

$$\sigma\tau$$

is the diagram with row lengths  $s_1, \dots, s_p, t_1, \dots, t_q$  arranged in non-increasing order. Obviously one has

$$(*) \quad \Lambda_\sigma \Lambda_\tau = \Lambda_{\sigma\tau},$$

and this equation makes it plausible that there is a correspondence between the multiplicative properties of  $D$ -ideals and their counterparts in  $K[X]$ .

A  $D$ -ideal  $S$  is called

- radical* if  $(\sigma^k \in S \implies \sigma \in S)$ ,
- prime* if  $(\sigma\tau \in S \implies \sigma \in S \text{ or } \tau \in S)$ ,
- primary* if  $(\sigma\tau \in S, \sigma \notin S \implies \tau^k \in S \text{ for some } k)$ .

Furthermore one puts  $\text{Rad } S = \{ \sigma : \sigma^k \in S \}$ . Obviously  $\text{Rad } S$  is a  $D$ -ideal.

**(11.16) Theorem.** *Let  $K$  be a field of characteristic 0. A  $D$ -ideal  $S$  is radical, prime or primary if and only if  $I(S)$  has the corresponding property. Furthermore  $I(\text{Rad } S) = \text{Rad } I(S)$ , and the only  $G$ -stable radical ideals are the prime ideals  $I_t(X)$ .*

PROOF: The equation  $(*)$  above immediately guarantees the implication “ $\Leftarrow$ ” in the first statement, as well as the inclusion  $I(\text{Rad } S) \subset \text{Rad } I(S)$ .

Let now  $S$  be an arbitrary  $D$ -ideal  $\neq \emptyset$ , and choose  $\sigma \in S$  such that  $\sigma$  has its first row as short as possible,  $\sigma = (t, \dots)$  say. Then  $(t)^k \in S$  for  $k$  large, and obviously  $\text{Rad } S = \{ \tau : \tau \supset (t) \}$ . Thus every radical  $D$ -ideal is prime and of the form  $\{ \tau : \tau \supset (t) \}$  for some  $t$ . Furthermore, since  $I_t(X)$  is a prime ideal,

$$I_t(X) = I(\text{Rad } S) \subset \text{Rad } I(S) \subset \text{Rad } I(\text{Rad } S) = I_t(X),$$

proving  $I(\text{Rad } S) = \text{Rad } I(S)$  and the third claim.

It only remains to show the implication “ $\implies$ ” in the first statement for the property “primary”. We shall however prove this implication completely without using that the ideals  $I_t(X)$  are prime ideals, obtaining a new proof of the latter fact for fields of characteristic 0.

**(11.17) Lemma.** *The associated prime ideals of a  $G$ -stable ideal are themselves  $G$ -stable.*

Before proving (11.17) we conclude the proof of (11.16). Suppose that  $S$  is primary. It follows from the lemma and the arguments above that the associated prime ideals of  $I(S)$  are among the ideals  $I_t(X)$  (regardless whether these ideals are known to be prime). In order to obtain a contradiction we assume that  $I(S)$  is not primary. Then  $\text{Rad}(I(S)) = I_t(X)$ , say, and  $I(S)$  has another associated prime ideal  $I_u(X)$ . The latter annihilates the ideal  $J = I(S) : I_u(X)$  modulo  $I(S)$ ,  $J \subset I_t(X)$  and  $J$  containing  $I(S)$  strictly. Since  $I_u(X)$  and  $I(S)$  are  $G$ -stable,  $J$  is  $G$ -stable, too, and thus contains an irreducible  $G$ -submodule  $M_\tau$ ,  $\tau \notin S$ . So  $\Lambda_{(u)}\Lambda_\tau \in I(S)$ ,  $(u)\tau \in S$ ,  $(u) \notin \text{Rad } S$ ,  $\tau \notin S$ , contradicting the hypothesis that  $S$  be primary. Let now  $S$  be even prime. Then  $I(S) = I_t(X)$  for some  $t$ , and  $I_t(X)$  is at least primary by what has just been shown. If it is not prime, then  $\text{Rad } I_t(X) = I_u(X)$  for some  $u < t$ , implying  $(u)^k \in S$  for some  $k$ ; however  $(u) \notin S$ . —

PROOF OF (11.17): Let  $P_1, \dots, P_q$  be the associated prime ideals. The action of an element  $g$  permutes the set  $\{P_1, \dots, P_q\}$ . Let

$$A_i = \{g \in G : g(P_1) = P_i\}.$$

Then  $A_i \cap A_j = \emptyset$  for  $i \neq j$ , and  $A_1 \cup \dots \cup A_q = G$ . As an affine algebraic set  $G$  is connected, and the  $A_i$  are Zariski-closed subsets of  $G$ . Now  $A_1 \neq \emptyset$ , and so  $g(P_1) = P_1$  for all  $g \in G$ . (Cf. [HR], Lemma 10.3 for an explicit statement of the arguments just used.) —

Let  $I = \bigoplus_{\sigma \in S} M_\sigma$  be a  $G$ -stable ideal. Let  $T$  be the set of minimal elements of  $S$  with respect to  $\subset$ . Then by (11.14) one has

$$I = \sum_{\sigma \in T} I_\sigma,$$

and this description is obviously irredundant. Furthermore  $T$  must be finite. The (radical and, therefore,) prime ideals among the  $G$ -stable ideals have been determined already, and it remains to characterize the primary ones.

**(11.18) Proposition.** *Let  $K$  be a field of characteristic 0, and  $I \subset K[X]$  a  $G$ -stable ideal written irredundantly as  $I = \sum_{\sigma \in T} I_\sigma$  as above. Let  $t$  be the shortest length of a row of any of the diagrams  $\sigma \in T$ .*

- (a) *Then  $I$  is primary if and only if some  $\sigma \in T$  is rectangular of width  $t$ , i.e.  $\sigma = (t, \dots, t)$ . In this case  $\text{Rad } I = I_t(X)$ .*
- (b)  *$I_\sigma$  is primary if and only if  $\sigma$  is rectangular.*

PROOF: In view of (11.16) one has to show that the given condition is equivalent to  $S = \{\sigma : \Lambda_\sigma \in I\}$  being a primary  $D$ -ideal. Suppose first that  $S$  is primary, and let  $\sigma \in T$ ,  $\sigma = (s_1, \dots, s_p)$ ,  $s_p = t$ . Then  $(s_1, \dots, s_{p-1}) \notin S$ , but  $(s_1, \dots, s_{p-1})(t) \in S$ , and so  $(t)^k \in S$  for some  $k$ . Now there is a  $\tau \in T$  such that  $(t)^k \supset \tau$ , and  $\tau$  must be rectangular of width  $t$ . Conversely assume that there is a  $\sigma \in T$ ,  $\sigma$  rectangular of width  $t$ . Then  $\text{Rad } S = \{\tau : \tau \supset (t)\}$ . If  $\tau_1\tau_2 \in S$ , then at least one of them has a row of length  $\geq t$ , and so is in  $\text{Rad } S$ . (b) is a special case of (a). —

An immediate consequence:

**(11.19) Corollary.** *Let  $\sigma$  be a diagram and  $t_1, \dots, t_k, t_1 > \dots > t_k$ , the numbers which occur as row lengths of  $\sigma$ . Let  $\sigma_i$  be the largest rectangular diagram of width  $t_i$  contained in  $\sigma$ . Then  $I_\sigma = I_{\sigma_1} \cap \dots \cap I_{\sigma_k}$  is an irredundant primary decomposition, and  $I_{t_1}(X), \dots, I_{t_k}(X)$  are the associated prime ideals of  $I_\sigma$ .*

The preceding discussion may suggest that the  $G$ -invariant ideals are very manageable objects. To some extent, however, this impression is deceptive: we do not have a description in terms of generators, say. Vice versa, it can be quite difficult to describe the  $D$ -ideal corresponding to a very “concrete” ideal, for example a power of  $I_t(X)$ , without having a primary decomposition. In principle this can be done by purely representation-theoretic methods, cf. the following remarks.

**(11.20) Remarks.** (a) The first problem remaining open in the consideration above is how to determine the  $D$ -ideal  $S$  such that  $I_\sigma I_\tau = \bigoplus_{\rho \in S} M_\rho$ . It has been solved in [Wh], at the expense of more representation theory.

(b) Another problem is the primary decomposition of an arbitrary  $G$ -invariant ideal. There is at least an algorithm by which it can be computed, cf. [DEP.1], p. 153.

(c) The combination of the methods mentioned in (a) and (b) should lead to a primary decomposition of the powers of the ideals  $I_t(X)$ , including the determination of the symbolic powers (in characteristic 0). In [DEP.1] one finds a “mixed” approach: the symbolic powers are determined as in Section 10, whereas the proof of (11.2) is based on representation theory. The reader consulting [DEP.1] should note that Lemma 6.2 of [DEP.1] is covered by the case  $\tilde{u} = u$  of (10.10).

(d) A remarkable fact:  $I^{(\sigma)}$  is the integral closure of  $I_\sigma$ , cf. [DEP.1], Section 8. This is the key to the computation of the integral closure of an arbitrary  $G$ -stable ideal also given in [DEP.1]. —

### E. $U$ -Invariants and Algebras Generated by Minors

In Subsection 7.D we have outlined that certain properties of a  $K$ -algebra  $R$  are inherited by the ring  $R^G$  of invariants, the linear algebraic group  $G$  acting rationally on  $R$  as a group of  $K$ -algebra automorphisms. Here we want to study a situation in which the direction of inheritance can be reversed. Let  $B \subset G$  a Borel subgroup and  $U$  the radical of the maximal torus in  $B$  (cf. [Hm] and [Kr] for the notions of the theory of algebraic groups). Then  $R$  shares important properties with the ring  $R^U$  of  $U$ -invariants:

**(11.21) Theorem.** *Let  $K$  be an algebraically closed field of characteristic 0,  $G$  a (linearly) reductive linear algebraic group acting rationally on a finitely generated  $K$ -algebra  $R$  as a group of  $K$ -algebra automorphisms, and  $U$  as above.*

- (a) *Then  $R^U$  is a finitely generated  $K$ -algebra.*
- (b)  *$R$  is normal if and only if  $R^U$  is normal.*
- (c)  *$R$  has rational singularities if and only if  $R^U$  has rational singularities.*

Cf. [Hd] and [Gh] for (a), [LV] for (b), and [Bn] for (c); (a) and (b) are also proved in [Kr].

In our case  $G = \text{GL}(m, K) \times \text{GL}(n, K)$  acts on  $K[X]$ , and a suitable subgroup  $U$  is given by  $U^-(m, K) \times U^+(n, K)$ , the Borel subgroup being the direct product of the lower

triangular matrices in  $GL(m, K)$  and the upper triangular matrices in  $GL(n, K)$ , and the maximal torus being formed by the direct product of the subgroup of diagonal matrices. The results of Subsection C below make the computation of  $K[X]^U$  a very easy problem. There is of course nothing to learn about  $K[X]$  from (11.21), but we can simultaneously study the induced action of  $G$  on  $R_{r+1}(X)$ , and even further that on the subalgebras of  $R_{r+1}(X)$  generated by the minors of fixed size  $t$  (of the matrix of residue classes).

**(11.22) Proposition.** *Let  $K$  be a field of characteristic 0. Then the ring of  $U$ -invariants of  $R_{r+1}(X)$ ,  $0 \leq r \leq \min(m, n)$ , is generated by the “initial” minors  $\delta_k = [1, \dots, k|1, \dots, k]$ ,  $1 \leq k \leq r$ , and therefore a polynomial ring in  $r$  indeterminates.*

PROOF: Let  $D_r$  denote the set of diagrams with at most  $r$  columns. Since, as a  $G$ -module,

$$R_{r+1}(X) \cong \bigoplus_{\sigma \in D_r} M_\sigma,$$

the first statement is equivalent to: The subspace  $V$  of  $U$ -invariant elements in  $M_\sigma$  is one-dimensional and generated by  $\Lambda_\sigma = (K_\sigma|K_\sigma)$ . The  $U^+(n, K)$ -variant of (11.7) implies that  $V \subset L_\sigma$ , and its  $U^-(m, K)$ -variant forces  $V \subset {}_\sigma L$ , hence  $V \subset L_\sigma \cap {}_\sigma L = K\Lambda_\sigma$ . On the other hand  $\Lambda_\sigma$  is a  $U$ -invariant. The second statement is obvious: Every monomial in the “initial” minors is standard. —

In conjunction with (11.21) the preceding proposition yields a representation-theoretic proof of the normality of the rings  $R_{r+1}(X)$ , and a new proof for the rationality of their singularities (cf. also (7.11)) including the Cohen-Macaulay property. (Normality and the Cohen-Macaulay property descend if one restricts the field of coefficients.)

Let  $S \subset R_{r+1}(X)$  be the  $K$ -subalgebra generated by the  $t$ -minors,  $t$  fixed,  $1 \leq t \leq r$ . If  $t = 1$ , then  $S = R_{r+1}(X)$ , and if  $t = r$ , then  $S$  is a subalgebra of maximal minors, cf. Subsection 9.A. These rings can be considered well-understood over every ring of coefficients, as well as the case  $m = u = r$ ,  $t = m - 1$ . Under the latter hypothesis  $S$  is a polynomial ring over  $K$ , cf. (10.17), where it has also been pointed out that there seems to be no characteristic-free approach to the rings  $S$  in general. Using the theory of  $U$ -variants we can prove that all these rings  $S$  behave well in characteristic 0. However, we have to draw heavily upon the theory of rings generated by monomials, as developed in [Ho.2], and the results of [Ke.5] and [Bt].

Let  $Y_1, \dots, Y_u$  generate the free commutative semigroup  $N$  in  $u$  variables, the composition written multiplicatively. We consider the elements of  $N$  as monomials in the variables  $Y_1, \dots, Y_u$ . Let  $M$  be a subsemigroup.

(a)  $M$  is called *normal* ([Ho.2]) if it is finitely generated and if the equation  $\pi\nu^k = \mu^k$  for elements  $\pi, \nu, \mu \in M$  implies that  $\pi = \rho^k$  for some  $\rho \in M$ . It is called a *full* subsemigroup ([Ho.2]), if  $\pi\nu = \mu$  for  $\nu, \mu \in M$ ,  $\pi \in N$  implies that  $\pi \in M$ . If  $M$  is a finitely generated *radical* subsemigroup ( $\pi^k \in M \implies \pi \in M$ ) then  $M$  is certainly normal.

(b) A normal subsemigroup  $M$  of monomials can be embedded into a (possibly different) free commutative semigroup  $N'$  generated by variables  $Z_1, \dots, Z_v$  such that it is a full subsemigroup of  $N'$ . Cf. [Ho.2], Proposition 1.

(c) Let  $B$  be an arbitrary commutative ring, and  $M$  a full semigroup of  $N$ . Then the  $B$ -submodule generated by all the monomials  $\pi \in N \setminus M$  is obviously a  $B[M]$ -submodule of  $B[Y_1, \dots, Y_u]$ , and one has a Reynolds operator  $B[Y_1, \dots, Y_u] \longrightarrow B[M]$ .

(d) Let  $K$  be a field, and  $M$  a normal semigroup of monomials. Then  $K[M]$  is normal ([Ho.2], Proposition 1) and a Cohen-Macaulay ring. This is the main result of [Ho.2], but follows (now) directly from (b), (c), and [Ke.5], Theorem 0.2: Because of (b) and (c)  $K[M]$  is a finitely generated pure subalgebra of  $K[Z_1, \dots, Z_v]$ .

(e) If  $K$  is a field of characteristic 0, then the last-mentioned fact implies that  $K[M]$  has rational singularities by the main result of [Bt].

**(11.23) Theorem.** *Let  $K$  be a field of characteristic 0,  $X$  an  $m \times n$  matrix of indeterminates over  $K$ ,  $R = R_{r+1}(X)$ ,  $0 \leq r \leq \min(m, n)$ , and  $t$  an integer,  $1 \leq t \leq r$ . Furthermore let  $S$  be the subalgebra of  $R$  generated by the  $t$ -minors of the matrix of residue classes. Then  $S$  is a normal Cohen-Macaulay domain. It has rational singularities if  $K$  is algebraically closed.*

PROOF: In view of the preceding discussion it is enough to show that the ring of  $U$ -invariants of  $S$  is of the form  $K[M]$  for a normal semigroup  $M$  of monomials. Let  $A$  be the ring of  $U$ -invariants of  $R$ . It is a polynomial ring in the “initial” minors  $\delta_1, \dots, \delta_r \in R$  of  $X$  by (11.22).

Let  $J = I_t(X)/I_{r+1}(X)$ . Then, with  $R_j$  denoting the  $j$ -th homogeneous component of  $R$ , one has

$$S = \bigoplus_j (R_{jt} \cap J^j)$$

Since  $R_{jt}$  and  $J^j$  have a basis consisting of the standard monomials they contain, the same is true for  $S$ , and consequently for  $A \cap S$ , the ring of  $U$ -invariants of  $S$ . So  $A \cap S$  is of the form  $K[M]$ ,  $M$  being a subsemigroup of monomials in  $\delta_1, \dots, \delta_r$ . (11.21),(a) implies that  $M$  is finitely generated. (This can also be proved directly.) Now

$$\delta_1^{k_1}, \dots, \delta_r^{k_r} \in S$$

if and only if

$$(1) \quad \sum_{i=1}^r ik_i \equiv 0 \pmod t$$

and, with the notations introduced below (10.2) and above (10.9),

$$(2) \quad \sum_{i=1}^r k_i \gamma_j(\delta_i) \geq \frac{1}{t} \left( \sum_{i=1}^r ik_i \right) e(j, t), \quad j = 1, \dots, r,$$

because of (10.4) and (10.13). The monomials satisfying (1) certainly form a full subsemigroup, and those satisfying (2) a radical subsemigroup. Being finitely generated and the intersection of a full subsemigroup and a radical subsemigroup,  $M$  is clearly normal. —

We don't see how to avoid the detour via the  $U$ -invariants in the proof of (11.23). None of the extensions  $S \rightarrow R_{r+1}(X)$ ,  $S \cap A \rightarrow A$  has a Reynolds operator in general; they are not even pure extensions: there are ideals  $I$  in  $S$  such that  $IR_{r+1}(X) \cap S \neq I$ . As an example take  $m, n \geq 3$ ,  $t = 2$ . Then  $[1|1]^2 \notin S$ ,  $[1 \ 2 \ 3|1 \ 2 \ 3]^2 \in S$  and  $[1|1]^2[1 \ 2 \ 3|1 \ 2 \ 3]^2 \in S$ . In particular the application of Hochster's results on normal semigroups seems to be essential.

## F. Comments and References

We cannot comment adequately on the representation-theoretic context of Theorem (11.11) here, instead we refer the reader to [ABW.2] (where the decomposition of  $K[X]$  is derived in a different way), the introduction of [DEP.1], and [Gn]. Apart from some details of the proofs, our treatment follows [DEP.1] closely. We have added (11.13) which is only implicit in [DEP.1]. One of the main applications of (11.11) in [DEP.1] is the primary decomposition of products of determinantal ideals for which representation theory is dispensable however, as seen in Section 10. We have already pointed to Whitehead's solution ([Wh]) of a problem left open in [DEP.1], cf. (11.20),(a).

References to the literature on  $U$ -invariants have been given in Subsection E. The inclusion of the method of  $U$ -invariants has been suggested by Kraft's book ([Kr]). Theorem (11.23) seems to be new.



## 12. Principal Radical Systems

All the results in the Sections 4 – 11 depend on standard monomial theory, and therefore have a combinatorial flavour. The first (published) proof of the perfection of determinantal ideals, given by Hochster and Eagon in [HE.2], avoids the use of standard monomials. It is “pure” commutative algebra, and may to some extent be rated simpler than the ASL approach. It has been employed in the investigation of other classes of ideals, too, and is of principal importance. Therefore we develop it in detail, although we cannot derive essential new results about determinantal ideals.

The proof of perfection uses the same inductive reasoning as the proof in Section 5. It is based on two auxiliary arguments: (i) A certain element  $x$  is not a zero-divisor modulo an ideal  $I$ ; (ii) an ideal  $I$  is represented as  $I = I_1 \cap I_2$  (with additional information on  $I_1 + I_2$ ). Whereas the validity of these auxiliary arguments is quite obvious in the ASL approach (the hard part being the verification of the ASL axioms), their demonstration is the central problem now. It is only natural to consider (i) and (ii) as problems on (primary or) prime decomposition. As pointed out in Section 7, generic points are readily constructed. So the crucial problem is to show that the ideals under consideration are radical ideals, and this is done by means of an inductive scheme called a *principal radical system*.

### A. A Propedeutic Example. Principal Radical Systems

In order to separate the pattern of the proof from its combinatorial details we discuss an example first, the ideal  $I_2(X)$ ,  $X$  an  $m \times n$  matrix of indeterminates. The main goal is to prove its perfection (from which further properties can be derived by localization arguments, cf. (2.10) – (2.12)):

(1) The ideal  $J_1 = I_2(X)$  is perfect of grade  $(m - 1)(n - 1)$ .

It follows from (3.2) and (3.3) that it is enough to consider noetherian domains  $B$  as rings of coefficients. Let  $A = B[X]$ . Auxiliary ideals are

$$\begin{aligned} J_2 &= I_2(X) + AX_{11}, \\ J_3 &= I_2(X) + \sum_{i=1}^m AX_{i1}, \\ J_4 &= I_2(X) + \sum_{j=1}^n AX_{1j}. \end{aligned}$$

We now make the crucial assumption:

(2) The ideals  $J_1, \dots, J_4$  are radical ideals.

As pointed out in 7.A, elementary linear algebra provides us with a generic point for  $A/J_1$  (and analogously for  $A/J_3$  and  $A/J_4$ ). Furthermore  $J_3J_4 \subset J_2$ , and one concludes immediately:

- (3)  $J_1, J_3, J_4$  are prime ideals. In addition (a)  $X_{11}$  is not a zero-divisor mod  $J_1$ , and (b)  $J_2 = J_3 \cap J_4$ .

At this point it is clear that  $\mathbf{Z}[X]/J_1$  is  $\mathbf{Z}$ -free. In proving perfection one can therefore restrict the ring of coefficients to be a field.

The grade of  $J_1$  has been computed in (2.5). Writing  $J_3 = I_2(\tilde{X}) + \sum_{i=1}^m AX_{i1}$ ,  $\tilde{X}$  consisting of the last  $n - 1$  columns of  $X$ , and representing  $J_4$  and  $J_3 + J_4$  in a similar way, one has:

- (4)  $\text{grade } J_2 = \text{grade } J_3 = \text{grade } J_4 = \text{grade } J_1 + 1$ , and  $\text{grade}(J_3 + J_4) = \text{grade } J_1 + 2$ .

The auxiliary arguments are complete now. Inductively one may suppose that  $J_3$ ,  $J_4$ , and  $J_3 + J_4$  are perfect ideals. Then it follows from (3),(b) and (4) that  $J_2$  is perfect by virtue of Lemma (5.15),(b), and part (a) of this lemma, in conjunction with (3),(a), implies the desired perfection of  $J_1$ .

Statement (2) above which has only been an assumption so far, is proved by induction, too. At least we know from the existence of generic points:

- (5)  $\text{Rad } J_1$  is prime. In particular  $X_{11}$  is not a zero-divisor modulo  $\text{Rad } J_1$ .

Assume for the first part of the inductive proof of (2) that  $J_2 = J_1 + AX_{11}$  is a radical ideal. Let  $y \in A$  be nilpotent modulo  $J_1$ . Then

$$y = y_1 + z_1X_{11}, \quad y_1 \in J_1, \quad z_1 \in A.$$

Since  $X_{11}$  is not a zero-divisor modulo  $\text{Rad } J_1$ , one has iteratively

$$y = y_u + z_uX_{11}^u, \quad y_u \in J_1, \quad z_u \in A$$

for all  $u \geq 1$ . Arguing in the graded ring  $A/J_1$ , we immediately conclude that  $y \in J_1$ , as desired.

Unfortunately there seems to be no way to derive the radical property of  $J_2$  from that of  $J_3$ ,  $J_4$ , and  $J_3 + J_4$  which we could safely assume to be radical. We are forced to enlarge the class of ideals:

$$G_v = J_1 + \sum_{j=1}^v AX_{1j},$$

$$H_v = J_1 + I_1(X_{ij} : 1 \leq i \leq m, 1 \leq j \leq v).$$

Observe that  $G_1 = J_2$ ,  $G_n = J_4$ ,  $H_1 = J_3$ . By descending induction on  $v$  one now sees that all the ideals  $G_v$  are radical.  $G_n$  and  $H_1, \dots, H_n$  may be assumed to be prime. Let  $1 \leq v \leq n$ . Then:

- (6)  $X_{1v}H_{v-1} \subset G_{v-1} \subset H_{v-1}$  and  $X_{1v}$  is not a zero-divisor mod  $\text{Rad } H_{v-1}$ .

Let  $y \in A$  be nilpotent modulo  $G_{v-1}$ . Then  $y \in G_v$  (by induction!),

$$y = y_1 + z_1 X_{1v}, \quad y_1 \in G_{v-1}, \quad z_1 \in A.$$

Next  $z_1 X_{1v} \in \text{Rad } H_{v-1}$ , so  $z_1 \in \text{Rad } H_{v-1} = H_{v-1}$ , and  $y \in G_{v-1}$ .

This scheme of reasoning can be cast in abstract form:

**(12.1) Theorem.** *Let  $A$  be a noetherian ring, and  $\mathcal{F}$  a family of ideals in  $A$ , partially ordered by inclusion. Suppose that for every member  $I \in \mathcal{F}$  one of the following assumptions is fulfilled:*

- (a)  *$I$  is a radical ideal.*
- (b) *There exists an element  $x \in A$  such that  $I + Ax \in \mathcal{F}$  and*
  - (i)  *$x$  is not a zero-divisor modulo  $\text{Rad } I$  and  $\bigcap_{i=0}^{\infty} (I + Ax^i)/I = 0$ , or*
  - (ii) *there exists an ideal  $J \in \mathcal{F}$ ,  $J \supset I$ ,  $J \neq I$ , such that  $xJ \subset I$  and  $x$  is not a zero-divisor modulo  $\text{Rad } J$ .*

*Then all the ideals  $I \in \mathcal{F}$  are radical ideals.*

A family of ideals satisfying the hypothesis of (12.1) is called a *principal radical system*. The attribute “principal” refers to the fact that one ascends in the system by adding a principal ideal to a given ideal.

In our example above the family  $\mathcal{F}$  consists of the ideals  $J_1 = I_2(X)$ ,  $G_1, \dots, G_n$ ,  $H_1, \dots, H_n$ . For  $H_1, \dots, H_n$  and  $G_n$  the assumption (a) is fulfilled by induction on the size of the matrix, (b),(i) holds for  $J_1$ , and (b),(ii) is valid for  $G_1, \dots, G_{n-1}$ .

The theorem is proved by noetherian induction with the same arguments as in the example above.

**(12.2) Remarks.** (a) We may replace the hypothesis that  $A$  be noetherian by the weaker assumption that every subfamily of  $\mathcal{F}$  has a maximal element. In most applications  $\mathcal{F}$  will even be finite of course.

(b) The family of ideals can be replaced by a family of submodules of a (finitely generated)  $A$ -module, the role of the radical then played by a certain “envelope”  $E(\dots)$  such that  $M \subset E(M)$ , and  $M \subset N$  implies  $E(M) \subset E(N)$ . The conclusion is that  $M = E(M)$  for all  $M \in \mathcal{F}$ . —

## B. A Principal Radical System for the Determinantal Ideals

In the following it will be inconvenient to stick too much to the notations used for the exploration of determinantal rings from the ASL point of view. We introduce a new description. Let  $X$  be an  $m \times n$  matrix of indeterminates, and  $s_0, \dots, s_r$  integers such that  $0 \leq s_0 < \dots < s_r = n$ . Then

$$I(s_0, \dots, s_r)$$

denotes the ideal generated by the collection of  $t$ -minors,  $1 \leq t \leq r+1$ , of the submatrix formed by the columns  $1, \dots, s_{t-1}$  of  $X$ . Obviously

$$I(s_0, \dots, s_r) = I(X; [1, \dots, r | s_0 + 1, \dots, s_{r-1} + 1])$$

and

$$I_t(X) = I(0, \dots, t-2, n).$$

The ideals corresponding to  $G_1, \dots, G_n$  in the example above are given by

$$I(s_0, \dots, s_r; v) = I(s_0, \dots, s_r) + \sum_{j=1}^v X_{1j} B[X],$$

$B$  as usual denoting the ring of coefficients.

**(12.3) Lemma.** *Let  $B$  be an integral domain and  $v = s_w$  for some  $w, 0 \leq w \leq r$ . Then the radical of  $I(s_0, \dots, s_r; v)$  is a prime ideal.*

PROOF: Let first  $w = 0$ , that is  $I(s_0, \dots, s_r; v) = I(s_0, \dots, s_r)$ . Denote this ideal by  $I$ . Based on completely elementary linear algebra we have already constructed a generic point for  $R = B[X]/I$  in (7.19), and thus proved the lemma for these ideals, cf. (7.1). Since we have to refer to the details of this construction in order to obtain the claim for  $w > 0$ , we repeat it here. Let  $s_{-1} = 0$  and choose a matrix

$$Z_k = \begin{pmatrix} Z_{1s_{k-1}+1} & \cdots & Z_{1s_k} \\ \vdots & & \vdots \\ Z_{ks_{k-1}+1} & \cdots & Z_{ks_k} \end{pmatrix}$$

of indeterminates,  $k = 0, \dots, r$ . Then one puts

$$Z = (\tilde{Z}_{r-1} \cdots \tilde{Z}_0 Z_0 | \cdots | \tilde{Z}_{r-1} Z_{r-1} | Z_r)$$

where  $\tilde{Z}_j$  is a  $(j + 1) \times j$  matrix of (new) indeterminates. In the (relative to (7.19)) special case considered here, one simply takes an  $m \times r$  matrix  $Y$  of indeterminates, and then the substitution

$$X \longrightarrow YZ$$

induces the generic point  $R \rightarrow B[\hat{Y}, \hat{Z}]$ : If  $L$  is a field and  $R \rightarrow L$  a  $B$ -homomorphism, then the matrix to which the matrix  $X$  (modulo  $I$ ) specializes can be decomposed in the same way as  $YZ$ ; this gives rise to a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & B[\hat{Y}, \hat{Z}] \\ & \searrow & \swarrow \\ & & L \end{array}$$

as desired. There are of course various ways to construct such a decomposition, and below we shall outline a specific one in order to guarantee an extra condition.

If  $w = r$ , then  $I(s_0, \dots, s_r; v)$  is of the form  $I(\tilde{s}_0, \dots, \tilde{s}_r)B[X] + \sum_{j=1}^v X_{1j} B[X]$ ,  $I(\tilde{s}_0, \dots, \tilde{s}_r)$  taken with respect to the rows  $2, \dots, m$  of  $X$ . So one may assume  $0 < w < r$ . We write

$$I(s_0, \dots, s_r; v) = I + J,$$

$I = I(s_0, \dots, s_r)$ ,  $J = \sum_{j=1}^v X_{1j} B[X]$ . Let  $R = B[X]/I$  be as above, and  $\bar{J} = JR$ . Then for every  $B$ -homomorphism  $R/\bar{J} \rightarrow L$ ,  $L$  a field, the composition  $R \rightarrow R/\bar{J} \rightarrow L$  can be

factored through  $B[\widehat{Y}, \widehat{Z}]$ . It is enough that at least one such a factorization gives rise to a commutative diagram

$$\begin{array}{ccc} R/\overline{J} & \longrightarrow & B[\widehat{Y}, \widehat{Z}]/P, \\ & \searrow & \swarrow \\ & & L \end{array}$$

$P$  a fixed prime ideal in  $B[\widehat{Y}, \widehat{Z}]$ . Here is the only point in this section where we have to work a little. We choose  $P$  as the ideal generated by the coefficients in the first row of

$$Y \widetilde{Z}_{r-1} \cdots \widetilde{Z}_w.$$

Let  $x$  be the image of the matrix  $X$  under  $B[X] \rightarrow R \rightarrow L$ . In order to reach the desired factorization  $R/\overline{J} \rightarrow B[\widehat{Y}, \widehat{Z}]/P \rightarrow L$  we now specify how to decompose  $x$ . The matrices appearing in the decomposition of  $x$  are denoted by small letters. For systematic reasons it is convenient to write  $\widetilde{Z}_r = Y$ .

First we represent  $x$  as

$$x = (x_0 | \dots | x_r)$$

where the separators  $|$  are placed after the columns  $s_0, \dots, s_{r-1}$  as in  $Z$  above. Since  $\text{rk } x \leq r$  there is an  $m \times r$  matrix  $\widetilde{z}_r$  such that its column space equals the column space of  $x = (x_0 | \dots | x_r)$ . Next we find an  $r \times (r-1)$  matrix  $\widetilde{z}_{r-1}$  for which the column space of  $\widetilde{z}_r \widetilde{z}_{r-1}$  coincides with the column space of  $(x_0 | \dots | x_{r-1})$ . Continuing this procedure we have eventually chosen matrices  $\widetilde{z}_r, \dots, \widetilde{z}_0$  of formats  $m \times r, r \times (r-1), \dots, 1 \times 0$  such that  $\widetilde{z}_r \dots \widetilde{z}_j$  has the same column space as  $(x_0 | \dots | x_j)$ . The choice of  $z_0, \dots, z_r$  is the last step (and no problem, of course). The matrix  $(x_0 | \dots | x_w)$  has only zeros in its first row, and since its column space equals that of  $\widetilde{z}_r \dots \widetilde{z}_w$ , the latter matrix has zeros in its first row, too. This is exactly the condition to be satisfied in order to factor  $R/\overline{J} \rightarrow L$  through  $B[\widehat{Y}, \widehat{Z}]/P$ .

It remains to show that  $P$  is a prime ideal. The generators of  $P$  are the entries of the  $1 \times s_w$  matrix

$$Y_1 \widetilde{Z}_{r-1} \cdots \widetilde{Z}_w,$$

$Y_1$  denoting the first row of  $Y$ . The number of columns of  $Y_1, \widetilde{Z}_{r-1}, \dots, \widetilde{Z}_w$  is decreasing from left to right, and the claim therefore follows inductively from the lemma below:

**(12.4) Lemma.** *Let  $A$  be a noetherian ring, and  $f_1, \dots, f_u$  elements of  $A$  generating an ideal  $I$  of grade  $g$ . Let  $U$  be an  $u \times v$  matrix of indeterminates  $X_{ij}$  over  $A$ .*

(a) *If  $v \leq g$ , then the elements  $\sum_{i=1}^u f_i X_{ij}, j = 1, \dots, v$ , form an  $A[U]$ -sequence.*

(b) *If  $A$  is a domain and  $v < g$ , then the ideal  $J$  generated by them is a prime ideal.*

PROOF: Inductive reasoning reduces both (a) and (b) immediately to the case  $v = 1$ . Since every zero-divisor in  $A[U]$  is annihilated by an element of  $A$ ,  $\sum_{i=1}^u f_i X_{i1}$  cannot be a zero-divisor. This proves (a) already.

For (b) we have  $g \geq 2$ , and grade  $IA[U]/J \geq 1$  because of (a). There is no harm in assuming that  $f_1, \dots, f_u \neq 0$ . We first show that  $A[U]/J$  is reduced. To be reduced is a local property, and it certainly suffices that the rings  $(A[U]/J)[f_i^{-1}]$  are domains. This is easy to see:

$$(A[U]/J)[f_i^{-1}] \cong ((A[f_i^{-1}][U]) / (\text{extension of } J)),$$

and over  $A[f_i^{-1}]$  the generator of  $J$  becomes an indeterminate.

Since  $(A[U]/J)[f_i^{-1}]$  is a domain,  $f_i$  must be contained in all the minimal primes of  $J$  but one. Since  $f_i f_j \notin J$  for all  $i, j$ , the “excluded” minimal prime must be the same for all  $i$ . Since, on the other hand,  $\text{grade } IA[U]/J \geq 1$ , there cannot be a second minimal prime: it would contain  $f_1, \dots, f_u$ . —

Now it is easy to show:

**(12.5) Proposition.** *Let  $B$  be a noetherian domain,  $X$  an  $m \times n$  matrix of indeterminates over  $B$ . Then the ideals  $I(s_0, \dots, s_r; v)$ ,  $0 \leq r \leq \min(m, n)$ ,  $0 \leq v \leq n$ , form a principal radical system. Hence all these ideals are radical.*

PROOF: First we invoke induction on the size of the matrix to conclude that all the ideals  $I(s_0, \dots, s_r; n)$  are radical ideals. For the other ideals  $I = I(s_0, \dots, s_r; v)$  one may suppose that  $v \geq s_0$ . In order to show that assumption (b) of (12.1) is fulfilled for all of them we take  $x = X_{1v+1}$ . Then  $I + xB[X] = I(s_0, \dots, s_r; v + 1)$ . Case (i):  $v = s_w$  for some  $w$ . Then  $x$  is not a zero-divisor mod  $\text{Rad } I$ . Otherwise it would be nilpotent modulo  $I$  by (12.3); this is impossible because the  $B$ -homomorphism  $B[X] \rightarrow B$ ,  $X_{1v+1} \rightarrow 1$ ,  $X_{ij} \rightarrow 0$  for all other indeterminates, factors through  $B[X]/I$ . Since  $B[X]/I$  is graded and the residue class of  $x$  has positive degree,  $\bigcap_{i=0}^{\infty} (I + x^i B[X])/I = 0$ . Case (ii):  $s_w < v < s_{w+1}$  for some  $w$ . Let

$$J = I(s_0, \dots, s_{w-1}, v, s_{w+1}, \dots, s_r; v).$$

and  $y = [a_1, \dots, a_{w+1} | b_1, \dots, b_{w+1}]$  be a generator of  $J$  not already in  $I$ . Then Laplace expansion of  $[1, a_1, \dots, a_{w+1} | b_1, \dots, b_{w+1}, v + 1] \in I$  along its first row shows  $xy \in I$ . It is seen as in case (i) that  $x$  is not a zero-divisor modulo  $\text{Rad } J$ . —

**(12.6) Corollary.** *Let  $B$  be a noetherian domain,  $X$  an  $m \times n$  matrix of indeterminates over  $B$ .*

(a) *If  $v = s_w$ , then  $I(s_0, \dots, s_r; v)$  is a prime ideal, and  $X_{1v+1}$  is not a zero-divisor modulo  $I(s_0, \dots, s_r; v)$ .*

(b) *Let  $s_w < v < s_{w+1}$ . Then*

$$I(s_0, \dots, s_r; v) = I(s_0, \dots, s_r; s_{w+1}) \cap I(s_0, \dots, s_{w-1}, v, s_{w+1}, \dots, s_r; v)$$

*is the prime decomposition of  $I(s_0, \dots, s_r; v)$ .*

**(12.7) Remark.** As a consequence of (12.6) the  $\mathbf{Z}$ -algebras  $\mathbf{Z}[X]/I(s_0, \dots, s_r; v)$  are  $\mathbf{Z}$ -free. Therefore one may use (3.12) in order to relax the hypotheses of (12.5) and (12.6). As far as the property “radical” is concerned, it is enough to assume that  $B$  is reduced; for “prime” one only needs that  $B$  is an integral domain.

### C. The Perfection of Determinantal Ideals

Looking back to the example in Subsection A one notes that the only missing step in the proof of perfection is an analogue of (4), a formula for the grade of the ideals  $I(s_0, \dots, s_r; v)$ .

**(12.8) Proposition.** *Let  $B$  be a noetherian ring,  $X$  an  $m \times n$  matrix of indeterminates. Then*

$$\text{grade } I(s_0, \dots, s_r) = mn - (m+n)r + \frac{r(r+1)}{2} + \sum_{i=0}^{r-1} s_i,$$

and if  $s_{w-1} < v \leq s_w$ ,  $1 \leq w \leq r$ ,

$$\text{grade } I(s_0, \dots, s_r; v) = \text{grade } I(s_0, \dots, s_r) + w.$$

PROOF: By virtue of (12.7) and (3.14) we may assume that  $B = K$  is a field. The chain

$$I(s_0, \dots, s_r; s_{w-1}) \subsetneq I(s_0, \dots, s_r; v) \subset I(s_0, \dots, s_r; s_w)$$

of inclusions reduces the second equation to the special case in which  $v = s_w$ . Since  $I(s_0, \dots, s_r; s_w)$  is a minimal prime ideal of

$$I(s_0, \dots, s_r; s_{w-1}) + X_{1s_{w-1}+1}K[X],$$

the second equation can be derived inductively from the first one.

If  $s_0 > 0$ , one can write

$$I(s_0, \dots, s_r) = \tilde{I}(0, s_1 - s_0, \dots, s_r - s_0) + \sum_{i=1}^m \sum_{j=1}^{s_0} X_{ij}K[X],$$

the ideal  $\tilde{I}(\dots)$  being taken from a smaller matrix of indeterminates in an obvious way. Thus we are left with the case  $s_0 = 0$ , for which we remind the reader of the inductive argument (2.4). Here it is of course convenient to invert  $X_{11}$  and to perform elementary transformations with respect to the first row and column. A glance at the generating set of  $I(s_0, \dots, s_r)$  shows that the extension of  $I(s_0, \dots, s_r)$  in  $K[X][X_{11}^{-1}]$  can be identified with the extension of an ideal

$$I(t_0, \dots, t_{r-1}), \quad t_i = s_{i+1} - 1,$$

taken from an  $(m-1) \times (n-1)$  matrix of indeterminates. —

**(12.9) Theorem.** *Let  $B$  be a noetherian ring,  $X$  an  $m \times n$  matrix of indeterminates. Then the ideals  $I(s_0, \dots, s_r; s_w)$ ,  $0 \leq w \leq r$ , and  $I(s_0, \dots, s_r; s_w + 1)$ ,  $0 \leq w < r$ , are perfect. In particular the ideals  $I_i(X)$  are perfect.*

PROOF: Again it is harmless to work with a field  $B$ , cf. (3.3), thus rendering Lemma (5.15) applicable. By part (a) of (5.15) (and (16.20)) the perfection of  $I(s_0, \dots, s_r; s_w)$  follows from that of  $I(s_0, \dots, s_r; s_w + 1)$ , a larger ideal, unless  $s_w = n$ , for which case we invoke induction on  $m$ . If  $s_w + 1 < s_{w+1}$ , one writes

$$I(s_0, \dots, s_r; s_w + 1) = I(s_0, \dots, s_{w-1}, s_w + 1, s_{w+1}, \dots, s_r; s_w + 1) \cap I(s_0, \dots, s_r; s_{w+1}).$$

Since the sum of the ideals on the right hand side is

$$I(s_0, \dots, s_{w-1}, s_w + 1, s_{w+1}, \dots, s_r; s_{w+1}),$$

and all ideals involved have the “correct” grade according to (12.8), a reference to (5.15),(b) finishes the proof. —

**(12.10) Remarks.** (a) It seems worthwhile to look back to the sections 5 – 11 and to check which of the results in these sections, as far as they apply to the ideals  $I(s_0, \dots, s_r)$ , can be derived from (12.5) – (12.9). The properties of being a radical or a prime ideal and the grade formula are covered explicitly, as well as perfection, of course.

Section 6: Lemma (6.4) only builds on the dimension (or grade) formula, and one concludes normality of the residue class ring. The computation of the singular locus is as easy as in (2.6) if one uses the inductive device sketched in the proof of (12.8).

Section 7: The construction of generic points (at least over domains  $B$ ) is a main argument in that section. The proof of (7.6),(a), relies on the fact that the minor  $[1, \dots, r-1, r+1 | 1, \dots, r-1, r+1]$  is not a zero-divisor modulo  $[1, \dots, r | 1, \dots, r]$  in  $R_{r+1}(X)$ . This can be derived from (12.9) and (12.7), since the minimal prime ideals of  $I_{r+1}(X) + [1, \dots, r | 1, \dots, r]B[X]$  (over a domain) belong to the class of ideals considered. This is no longer true for  $I(s_0, \dots, s_r) + [1, \dots, r | s_0 + 1, \dots, s_{r-1} + 1]B[X]$ , and it is doubtful whether one can derive the result of (7.19) for  $I(s_0, \dots, s_r)$ . At least in characteristic zero there is a loop-hole, however, cf. (7.21).

Section 8: For the reason just mentioned the computation of the divisor class group of  $B[X]/I(s_0, \dots, s_r)$  is certainly not immediate in the general case, though it is in the case  $R_{r+1}(X)$ , for which one can then compute the canonical class.

Section 9 – 11: Here the full strength of the ASL structure on  $B[X]$  is used, and there seems to be no chance to obtain the main results without considerable effort. This does not exclude the possibility of constructing principal radical systems containing the ideals of interest, and at least in one case this has been successful, cf. [Ng.1].

(b) The choice of a principal radical system embracing the ideals  $I_t(X)$  is by no means unique! In fact, the ideals  $J(X; \gamma) \subset G(X)$  have been investigated in [Ho.3] by a blend of methods based on a principal radical system and standard monomial theory. It doesn't seem a very bold speculation to believe that a principal radical system containing the ideals  $I(X; \delta)$  can be constructed even without standard monomial theory (though one will certainly need the partial order on  $\Delta(X)$  as a systemizing tool).

(c) It should be possible to explore the rings  $B[X]/I(s_0, \dots, s_r; s_w)$  in regard to their divisor class group, canonical class etc. At least they are normal over a normal domain  $B$ , cf. [HE.2], p. 1024, Corollary 3.

(d) A modification of the scheme of proof developed in this section has been suggested in [KIL.1]. It avoids generic points and exploits dimension-theoretic arguments in order to prove that “ $x$  is not a zero-divisor modulo  $\text{Rad } I$ ” or “ $\dots \text{Rad } J$ ” resp. In fact, if all the minimal prime ideals  $P$  of  $I$  have height  $\leq h$ , but  $\text{ht } I + Ax \geq h + 1$ , then  $x$  is not a zero-divisor modulo  $\text{Rad } I$ . —

## D. Comments and References

The main source for this section is Hochster and Eagon's fundamental article [HE.2] whose line of reasoning is followed very closely. The construction of a generic point for  $I(s_0, \dots, s_r)$  is their Proposition 25, (12.3) corresponds to Proposition 29, part 1), (12.4) combines the Propositions 21 and 22. (12.8) reproduces Proposition 32 of [HE.2], with a different proof however, and the derivation of (12.9) is exactly as given in [HE.2], section 11.

The notion of principal radical system as defined in (12.1) has been suggested by Ngo, cf. [Ng.2], Proposition 3. It certainly simplifies the definition of [HE.2]. Ngo discusses a



generalization called a principal system of ideals. A module-theoretic version as indicated in (12.2),(b) has been used in [Br.7].

In [KIL.1] Kleppe and Laksov give a detailed account of their modification to the proof of Hochster and Eagon, as pointed out in (12.10),(d). Originally they had developed it for their investigation of ideals generated by pfaffians ([KIL.2]), which have been treated by Marinov ([Mr.1],[Mr.2],[Mr.3]) in complete analogy to [HE.2]. Other applications of the method of principal radical systems are to be found in [Ho.3] (cf. (12.10),(b)), Kutz's work [Ku] on ideals generated by minors of symmetric matrices, and [Ng.2] (cf. 9.E). It is interesting to note that the classes of ideals studied in the papers mentioned have later been explored by standard monomial methods, too, cf. [DEP.2] for a survey. Nonetheless there seem to be situations in which the method of principal radical systems solves a problem which cannot be tackled by standard monomial methods, cf. [HL].

## 13. Generic Modules

Once more let  $X = (X_{ij})$  be an  $m \times n$  matrix of indeterminates over a noetherian ring  $B$  and  $r$  a nonnegative integer. Put  $R = R_{r+1}(X)$ . In this section we shall investigate the image and the cokernel of the map  $x: R^m \rightarrow R^n$  given by the matrix of the residue classes of the indeterminates  $X_{ij}$ . The map  $x$  and its cokernel  $C$  have the following universal properties: Let  $S$  be a (noetherian)  $B$ -algebra. If  $f: S^m \rightarrow S^n$  is a homomorphism of rank  $r$  represented by a matrix  $(u_{ij})$ , then  $f = x \otimes S$ ,  $S$  made an  $R$ -algebra via the substitution  $X_{ij} \rightarrow u_{ij}$ . If  $M$  is an  $S$ -module given by  $n$  generators and  $m$  relations and of rank  $\geq n - r$ , then  $M = C \otimes S$  (since  $M$  is represented by a map  $S^m \rightarrow S^n$  of rank  $\leq r$ ). The universal properties of  $x$  and  $C$  justify the notions *generic map* and *generic module*.

The main results of the section will be that  $\text{Im } x$  is a perfect  $B[X]$ -module (with one trivial exception) and that  $\text{Coker } x$  is perfect (provided  $r \geq 1$ ) if and only if  $m \geq n$ . Some special cases have been treated already: For  $r \geq \min(m, n)$  we refer to (2.16). This result also implies that  $\text{Coker } x$  is perfect in case  $r + 1 = n \leq m$  since  $I_{r+1}(X)$  annihilates  $\text{Coker } X$ , then.

### A. The Perfection of the Image of a Generic Map

We start with a simple lemma which will be used several times. Its proof consists in a repeated application of (16.14),(b).

**(13.1) Lemma.** *Let  $S$  be a noetherian ring,  $I$  an ideal in  $S$ , and  $M$  a finitely generated  $S$ -module,  $M = M_0 \supset \dots \supset M_s = 0$  a filtration of  $M$ . Then*

$$\text{grade}(I, M) \geq \min\{\text{grade}(I, M_i/M_{i+1}) : 0 \leq i \leq s-1\}.$$

The crucial step in proving the perfection of  $\text{Im } x$  is contained in:

**(13.2) Proposition.** *Let  $r < \min(m, n)$ ,  $C = \text{Coker } x$ .*

(a)  *$C$  is a reflexive  $R$ -module.*

(b) *There exists a free submodule  $F$  of  $C$  such that  $C/F$  is annihilated by  $I(X; \delta)$ ,  $\delta = [1, \dots, r|1, \dots, r-1, r+1]$ , and as an  $R(X; \delta)$ -module is isomorphic with the ideal in  $R(X; \delta)$  generated by the residue classes of the  $r$ -minors  $[1, \dots, r|1, \dots, \widehat{k}, \dots, r+1]$ ,  $1 \leq k \leq r$ .*

PROOF: Let  $z_1, \dots, z_n$  denote the canonical basis of  $R^n$ . We put

$$F_i = \sum_{j=i+1}^n Rz_j \quad \text{mod } \text{Im } x, \quad 0 \leq i \leq r,$$

$F_{r+1} = 0$ , and

$$\delta_i = [1, \dots, r|1, \dots, \widehat{i}, \dots, r+1], \quad 1 \leq i \leq r+1.$$

$I(X; \delta_i)$  is the ideal generated by the  $(r + 1)$ -minors of  $X$  and the  $i$ -minors of its first  $i$  columns. Then for  $i = 1, \dots, r + 1$ :

- (i)  $C/F_i$  is annihilated by  $I(X; \delta_i)$ .
- (ii)  $F_{i-1}/F_i$  is an  $R(X; \delta_i)$ -free submodule of  $C/F_i$ .
- (iii) The map  $x$  induces an exact sequence

$$R(X; \delta_i)^m \xrightarrow{x^{(i-1)}} R(X; \delta_i)^{i-1} \longrightarrow C/F_{i-1} \longrightarrow 0,$$

the  $m \times j$  matrix  $x^{(j)}$  consisting of (the residues of) the first  $j$  columns of  $x$ .

We shall finish the proof of (13.2) first before demonstrating the assertions (i)–(iii). Obviously  $C_P$  is  $R_P$ -free for all prime ideals  $P \subset R$  which do not contain  $I_r(x)$ . From (ii), (5.18), and (16.18) we obtain

$$\begin{aligned} \text{grade}(I_r(x), F_{i-1}/F_i) &= \text{grade}(I_r(x), R(X; \delta_i)) \\ &= \text{grade} I(X; [1, \dots, r - 1 | 1, \dots, \hat{i}, \dots, r]) / I(X; \delta_i) \\ &= \text{grade} I(X; [1, \dots, r - 1 | 1, \dots, \hat{i}, \dots, r]) - \text{grade} I(X; \delta_i) \\ &= m + n - 2r \end{aligned}$$

for  $i = 1, \dots, r$ , and  $\text{grade}(I_r(x), F_r) = m + n - 2r + 1$ . Now (a) is an immediate consequence of (13.1) (cf. (16.33)). As to (b) we put  $F = F_r$  and let  $J$  be the ideal in  $R(X; \delta)$  generated by the  $r$ -minors  $\delta_k$ ,  $1 \leq k \leq r$ . From (iii) one gets a presentation

$$R(X; \delta)^m \xrightarrow{x^{(r)}} R(X; \delta)^r \longrightarrow C/F \longrightarrow 0$$

by tensoring with  $R(X; \delta)$ . On the other hand there is a zero-sequence

$$(1) \quad R(X; \delta)^m \xrightarrow{x^{(r)}} R(X; \delta)^r \xrightarrow{h} J \longrightarrow 0$$

where the surjective map  $h$  is defined by

$$h(\bar{z}_k) = (-1)^{k+1} \delta_k, \quad 1 \leq k \leq r,$$

$\bar{z}_1, \dots, \bar{z}_r$  being the canonical basis of  $R(X; \delta)^r$ . So we obtain a surjection  $C/F \longrightarrow J$ . Furthermore  $C/F$  has rank 1 and is torsionfree (as an  $R(X; \delta)$ -module): It is free of rank 1 at all prime ideals  $P \subset R(X; \delta)$  which do not contain the ideal  $J$ . Using (13.1) and (ii) once more, we get

$$\text{grade}(J, C/F) \geq \min\{\text{grade}(J, F_i/F_{i+1}): 0 \leq i \leq r - 1\} \geq 1.$$

Consequently  $C/F \cong J$  (and (1) is exact).

(i)–(iii) will be proved by descending induction on  $i$ . Since the first  $r$  columns of  $x$  are linearly independent over  $R$ ,  $F_r$  is a free submodule of  $C$ . Obviously  $x$  induces an exact sequence

$$R^m \xrightarrow{x^{(r)}} R^r \longrightarrow C/F_r \longrightarrow 0.$$

So the assertions hold for  $i = r + 1$ . Let  $1 \leq i \leq r$ . By the inductive hypothesis we have an exact sequence

$$R(X; \delta_{i+1})^m \xrightarrow{x^{(i)}} R(X; \delta_{i+1})^i \longrightarrow C/F_i \longrightarrow 0,$$

so  $C/F_i$  is annihilated by  $I(X; \delta_i)$  (cf. (16.2)). Tensoring with  $R(X; \delta_i)$  yields an exact sequence

$$R(X; \delta_i)^m \xrightarrow{x^{(i)}} R(X; \delta_i)^i \longrightarrow C/F_i \longrightarrow 0.$$

Since the first  $i - 1$  columns of  $x^{(i)}$  are linearly independent over  $R(X; \delta_i)$ ,  $F_{i-1}/F_i$  is an  $R(X; \delta_i)$ -free submodule of  $C/F_i$  of rank 1, and (iii) is an immediate consequence. —

**(13.3) Remark.** By the way, the proof of (13.2) shows that the first syzygy module of the ideal in (13.2),(b) is as one expects at first sight: cf. the exact sequence (1). The exactness of (1) can also be derived from (5.6) or even checked directly. —

Taking into account the special structure of the ideal described in (13.2),(b) we are now able to prove the main result of this subsection.

**(13.4) Theorem.** *Choose notations as at the beginning of the section. Then  $\text{Im } x$  is a perfect  $B[X]$ -module except for the case in which  $r \geq n$  and  $m > n$ , and  $\text{Coker } x$  is an almost perfect  $B[X]$ -module, i.e.*

$$\text{grade Coker } x \geq \text{pd Coker } x - 1.$$

PROOF: Assume first that  $r \geq \min(m, n)$ , the case in which  $I_{r+1}(X) = 0$ . If  $m \leq n$ , then  $\text{Im } x$  is free and  $\text{pd Coker } x = 1$ . In case  $m > n$  we obtain the (almost) perfection of  $\text{Coker } x$  from (2.16).

Suppose now that  $r < \min(m, n)$ . By Proposition (13.2)  $\text{Coker } x$  is a torsionfree  $R$ -module, so  $\text{grade Coker } x = \text{grade } R$  (over  $B[X]$ ). Therefore the second assertion follows from the first one via the exact sequence  $0 \rightarrow \text{Im } x \rightarrow R^n \rightarrow \text{Coker } x \rightarrow 0$ .

Because of (3.3) we only have to prove that  $\text{Im } x$  is generically perfect and that  $\text{Coker } x$  is  $\mathbf{Z}$ -flat in case  $B = \mathbf{Z}$ . Since  $\text{Im } x$  is a graded torsionfree  $R$ -module, it is a free  $\mathbf{Z}$ -module. The same is true for  $\text{Coker } x$  because of (13.2). It remains to show that  $\text{Im } x$  is perfect if  $B = \mathbf{Z}$ . This is equivalent with the fact that  $(\text{Im } x)_P$  is a Cohen-Macaulay module over  $R_P$  for all prime ideals  $P \subset R$  (cf. 16.19), or that

$$\text{depth } C_P \geq \dim R_P - 1,$$

$C = \text{Coker } x$ . In view of (13.2),(b) it will be enough that

$$\text{depth}(R(X; \delta)/J)_P \geq \dim R_P - 2$$

for all prime ideals  $P \subset R$ ,  $P \in \text{Supp}(R(X; \delta)/J)$ , where  $J$  is the ideal in  $R(X; \delta)$  generated by the residue classes of the elements  $[1, \dots, r|1, \dots, \widehat{k}, \dots, r + 1]$ ,  $1 \leq k \leq r$ . It is easy to check that  $R(X; \delta)/J = B[X]/\Omega B[X]$ ,  $\Omega$  being the ideal in  $\Delta(X)$  cogenerated by the elements

$$\delta_1 = [1, \dots, r - 1, r + 1|1, \dots, r - 1, r + 1], \quad \delta_2 = [1, \dots, r|1, \dots, r - 1, r + 2].$$

By (5.19)  $R(X; \delta)/J$  is a Cohen-Macaulay ring, and obviously  $\dim R(X; \delta)/J = \dim R - 2$ . The proof of (13.4) is complete now. —

**(13.5) Remarks.** (a) It is a simple but noteworthy fact that under the assumptions of (13.4) all  $R$ -syzygies of Coker  $x$  are perfect  $B[X]$ -modules along with  $\text{Im } x$ .

(b) In [Br.7] the following generalized version of Theorem (13.4) has been stated: *Let  $A$  be a noetherian ring and  $u = (u_{ij})$  be an  $m \times n$  matrix of elements in  $A$ . Suppose that  $0 \leq r \leq \min(m, n - 1)$  and that  $I_{r+1}(u)$  has (the maximally possible) grade  $(m - r)(n - r)$ . Denote by  $R$  the residue class ring  $A/I_{r+1}(u)$ . Let  $\bar{u}: R^m \rightarrow R^n$  be the map given by the matrix of the residue classes of the elements  $u_{ij}$ . Assume further that  $I_r(\bar{u})$  contains an element which is not a zero-divisor of  $R$ . Then  $\text{Im } \bar{u}$  and hence all higher syzygies of  $\text{Coker } \bar{u}$  are perfect  $A$ -modules.*

To prove this, one has only to change the arguments which reduce the general case to the generic one: Let  $X = (X_{ij})$  be an  $m \times n$  matrix of indeterminates over  $\mathbf{Z}$ . Then  $A$  is a  $\mathbf{Z}[X]$ -algebra via the substitution  $X_{ij} \rightarrow u_{ij}$ . Put  $S = \mathbf{Z}[X]/I_{r+1}(X)$  and denote by  $x: S^m \rightarrow S^n$  the map given by the matrix of the residue classes of the elements  $X_{ij}$ . Consider the natural surjection  $h: \text{Im } x \otimes R \rightarrow \text{Im}(x \otimes R)$ . By the assumption on  $I_r(\bar{u})$  one obtains that  $h \otimes R_P$  is an isomorphism of free  $R_P$ -modules of positive rank whenever  $P$  is an associated prime ideal of  $R$ . By (3.5) we can derive that  $\text{Im } x \otimes R = \text{Im } x \otimes A$  is a perfect  $A$ -module. But then  $\text{Im } x \otimes R$  is necessarily a torsionfree  $R$ -module and consequently  $h$  is an isomorphism. —

### B. The Perfection of a Generic Module

To get further information on the generic module, we consider the following (more or less well known) homomorphism which will also play an essential role in the next two sections.

Let  $F, G$  be modules over an arbitrary ring  $A$ ,  $f: F \rightarrow G$  an  $A$ -homomorphism and  $r, s, t$  integers such that  $0 \leq r \leq \min(s, t)$ . Then

$$\varphi_{f,r}: \bigwedge^s F \otimes \bigwedge^t G^* \longrightarrow \bigwedge^{s-r} F \otimes \bigwedge^{t-r} G^*$$

is defined by

$$\varphi_{f,r}(y_I \otimes z_J^*) = \sum_{\substack{U \in \mathcal{S}(r, I) \\ V \in \mathcal{S}(r, J)}} \sigma(U, I \setminus U) \sigma(V, J \setminus V) z_V^* (\bigwedge^r f(y_U)) y_{I \setminus U} \otimes z_{J \setminus V}^*,$$

$y_I = y_{i_1} \wedge \cdots \wedge y_{i_s}$ ,  $y_{i_\sigma} \in F$ , and  $z_J^* = z_{j_1}^* \wedge \cdots \wedge z_{j_t}^*$ ,  $z_{j_r}^* \in G^*$ . Clearly  $\varphi_{f,0}$  is the identity map.

Here we are interested in the case in which  $s = r + 1$ ,  $t = r$ . Then obviously  $f \circ \varphi_{f,r} = 0$  if  $\text{rk } f \leq r$ . If moreover  $\text{Im } f$  is a free direct summand of  $G$  and  $\text{rk } f = r$ , then

$$\bigwedge^{r+1} F \otimes \bigwedge^r G^* \xrightarrow{\varphi_{f,r}} F \xrightarrow{f} G$$

is split-exact. Adopting the notations of the introduction, we can show:

**(13.6) Proposition.** *Suppose  $r < \min(m, n)$ . Then the sequence*

$$(2) \quad \bigwedge^{r+1} R^m \otimes \bigwedge^r (R^n)^* \xrightarrow{\varphi_{x,r}} R^m \xrightarrow{x} R^n$$

and its dual are exact.

PROOF: For  $r = 0$  there is nothing to prove. Let  $r \geq 1$  and  $\varphi = \varphi_{x,r}$ . It is clear that (2) is split exact if localized at a prime ideal of  $R$  which does not contain the ideal  $I_r(x)$ . In particular (2) is exact in depth 0. So it suffices to show that  $\text{Coker } \varphi$  is a torsionfree  $R$ -module. By what we have just mentioned this will follow from the inequality  $\text{grade}(I_r(x), \text{Coker } \varphi) \geq 1$ .

We argue as in the proof of (13.2): Let  $y_1, \dots, y_m$  be the canonical basis of  $R^m$ ,

$$F_i = \sum_{j=1}^{m-i} R y_j \pmod{\text{Im } \varphi}, \quad 0 \leq i \leq m - r,$$

$F_{m-r+1} = 0$ , and

$$\gamma_i = [1, \dots, r - 1, m - i | 1, \dots, r], \quad 0 \leq i \leq m - r.$$

Then for  $i = 1, \dots, m - r + 1$ :

- (i)  $(\text{Coker } \varphi)/F_i$  is annihilated by  $I(X; \gamma_i)$ .
- (ii)  $F_{i-1}/F_i$  is an  $R(X; \gamma_i)$ -free submodule of  $(\text{Coker } \varphi)/F_i$ .

From these assertions and (13.1) we obtain

$$\begin{aligned} \text{grade}(I_r(x), \text{Coker } \varphi) &\geq \text{grade}(I_r(x), R(X; \gamma_0)) \\ &= \text{grade } I_r(x) - \text{grade } I(X; [1, \dots, r - 1, m | 1, \dots, r]) \\ &= 1. \end{aligned}$$

To prove (i) we observe that  $\text{Im } \varphi$  is generated by the elements

$$\sum_{i \in I} \sigma(i, I \setminus i) [I \setminus i | J] y_i, \quad I \in S(r+1, m), \quad J \in S(r, n).$$

As to (ii) we assume that there is an equation

$$a y_{m-i+1} = \sum_{j=1}^{m-i} a_j y_j + y,$$

$a, a_j \in R, y \in \text{Im } \varphi$ . Then

$$a x_{m-i+1} = \sum_{j=1}^{m-i} a_j x_j$$

where  $x_k$  denotes the  $k$ -th row of  $x$ . Elementary determinantal calculation yields

$$a \gamma_{i-1} \in I(X; \gamma_{i-1})/I_{r+1}(X)$$

and consequently  $a \in I(X; \gamma_{i-1})/I_{r+1}(X)$  since  $\gamma_{i-1}$  is not a zero-divisor mod  $I(X; \gamma_{i-1})$  (cf. (5.11)). This proves (ii).

To demonstrate the exactness of the dual to (2) we replace  $x$  by  $x^*$ . Then the dual to (2) becomes the sequence

$$(3) \quad R^m \xrightarrow{x} R^n \xrightarrow{\psi} \bigwedge^{r+1} R^n \otimes \bigwedge^r (R^m)^*$$

(up to canonical isomorphisms) where  $\psi = (\varphi_{x^*, r})^*$ . Since (3) is exact in depth 0 and Coker  $x$  is torsionfree (cf. (13.2),(a)), the exactness of (3) follows immediately. —

**(13.7) Remark.** A routine argument shows that the cokernel of the map  $\psi$  in the sequence (3) is torsionfree: Coker  $\psi$  is free in depth less than  $m+n-2r+1$  and Coker  $x$  is reflexive (cf. (13.2),(a)). This fact will be used in the proof of the following theorem. —

Of course it cannot be expected that Coker  $x$  is a perfect  $B[X]$ -module in general: If  $r \geq m$  and  $m < n$  then  $R = B[X]$  and Coker  $x$  fails to be perfect since it has projective dimension 1 and rank  $\geq 1$ . On the other side (2.16) says that Coker  $x$  is perfect in case  $r \geq n$  and  $m \geq n$  as we have indicated already in the introduction. The following theorem describes completely how the perfection of Coker  $x$  depends on the size of  $X$ .

**(13.8) Theorem.** *With the notations of the introduction Coker  $x$  is a perfect  $B[X]$ -module if and only if (i)  $r = 0$  or (ii)  $r \geq 1$  and  $m \geq n$ .*

PROOF: Obviously Coker  $x$  is perfect if  $r = 0$ . Taking into account what has just been said we may further assume that  $1 \leq r < \min(m, n)$ .

We first consider the case in which  $B = \mathbf{Z}$ . Since  $C = \text{Coker } x$  is almost perfect, we have  $\text{depth } C_P \geq \dim R_P - 1$  for all  $P \in \text{Spec } R$ . Perfection of  $C$  means  $\text{depth } C_P = \dim R_P$  for all  $P \in \text{Spec } R$  and hence is equivalent to  $\text{Ext}_R^1(C, \omega_R) = 0$  by the local duality theorem,  $\omega_R$  being a canonical module of  $R$  (cf. [HK], 4.10 and 5.2). One has  $C^* = \text{Ker } x^*$ , so this module is a third (actually a fourth) syzygy by (13.2),(a). Furthermore  $C_P$  and  $C_P^*$  are free  $R_P$ -modules for all  $P \in \text{Spec } R$  such that  $P \not\supseteq I_r(x)$ . Since  $\text{grade } I_r(x) = m+n-2r+1 \geq 3$ ,  $C^*$  is 3-torsionless (cf. (16.33)) and hence  $\text{Ext}_R^1(C^{**}, R) = \text{Ext}_R^1(C, R) = 0$ . In case  $m = n$  this already shows that  $C$  is perfect,  $R$  being a Gorenstein ring then (cf. (8.9)).

In Section 9 we gave a representation of  $\omega_R$  as an ideal of  $R$ . From the exact sequence  $0 \rightarrow \omega_R \rightarrow R \rightarrow R/\omega_R \rightarrow 0$  one derives an exact sequence

$$0 \longrightarrow \text{Hom}_R(C, \omega_R) \longrightarrow C^* \xrightarrow{h} \text{Hom}_R(C, R/\omega_R) \longrightarrow \text{Ext}_R^1(C, \omega_R) \longrightarrow 0.$$

Thus the perfection of  $C$  is equivalent with the fact that  $h$  is surjective. Denote by  $x^1, \dots, x^n$  the columns of  $x$ . Then  $C^*$  can be identified with the submodule of all  $(a_1, \dots, a_n) \in (R^n)^*$  such that  $a_1x^1 + \dots + a_nx^n = 0$ , and a homomorphism  $\beta: C \rightarrow R/\omega_R$  can be lifted to an element  $(b_1, \dots, b_n) \in (R^n)^*$  with the property  $b_1x^1 + \dots + b_nx^n \in \omega_R(R^m)^*$ .

In case  $m > n$  the canonical module  $\omega_R$  is given by  $P^{m-n}$  where  $P$  is the ideal in  $R$  generated by the  $r$ -minors of the first  $r$  rows of  $x$  (cf. (9.20)). Lemma (13.9) below says that there is an element  $\alpha \in C^*$  which is congruent to  $(b_1, \dots, b_n)$  modulo  $\omega_R(R^n)^*$ . This means  $h(\alpha) = \beta$ . Consequently  $h$  is surjective and  $C$  is perfect in this case.

Let  $m < n$  and denote by  $M$  the prime ideal  $I_r(x)$ . Clearly  $C$  is not perfect if  $\text{depth } C_M < \text{depth } R_M$ . According to (2.4) we have an isomorphism

$$(4) \quad R[x_{mn}^{-1}] \cong R_r(Y)[X_{m1}, \dots, X_{m-1,n}][X_{mn}^{-1}].$$

Put  $\tilde{R} = R_r(Y)$ ,  $\tilde{M} = I_{r-1}(Y)$  and let  $\tilde{C}$  be the corresponding generic module. Denote by  $y$  the matrix of residue classes modulo  $I_r(Y)$  of the indeterminates  $Y_{ij}$ . Then the map  $x \otimes R[x_{mn}^{-1}]$  is represented by the matrix

$$(5) \quad \begin{pmatrix} & & & 0 \\ & & & \vdots \\ & y & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

so  $C \otimes R[x_{mn}^{-1}] \cong \tilde{C} \otimes R[x_{mn}^{-1}]$ . Since  $\tilde{R}_{\tilde{M}} \rightarrow R_M$  is a local and flat extension, the depth inequality above is equivalent to  $\text{depth } \tilde{C}_{\tilde{M}} < \text{depth } \tilde{R}_{\tilde{M}}$ . We may therefore assume that  $r = 1$ . Furthermore we can replace the base ring  $\mathbf{Z}$  by the field  $\mathbf{Q}$  of rational numbers because  $M \cap \mathbf{Z} = \{0\}$ . But then it suffices to show that  $C$  is not a perfect  $\mathbf{Q}[X]$ -module (cf. (16.20)). In the case under consideration the canonical module  $\omega_R$  of  $R$  is given by  $Q^s$ ,  $s = n - m$ , where  $Q$  is the ideal in  $R$  generated by the entries of the first column of  $x$ . Take  $b_1 = x_{11}^{s-1}, b_2 = \dots = b_n = 0$ . Then  $(b_1, \dots, b_n)$  induces an element of  $\text{Hom}_R(C, R/Q^s)$ . If there were  $a_1, \dots, a_n \in R$  such that  $a_1x^1 + \dots + a_nx^n = 0$  and  $b_j - a_j \in Q^s$ , one in particular would get  $x_{11}^{s-1} \in Q^s + J$ ,  $J$  being the ideal in  $R$  generated by the components of  $x^2, \dots, x^n$ . This is obviously impossible. Therefore the homomorphism  $h$  above is not surjective.

Now we treat the general case for  $B$ . Let  $m \geq n$ . Since the cokernel of the map  $(\varphi_{x^*,r})^*$  in (13.6) is  $\mathbf{Z}$ -flat (cf. (13.7)), the perfection of  $\text{Coker } x$  follows from the fact that it is  $B$ -free and perfect in case  $B = \mathbf{Z}$  (cf. (3.3)). It remains to prove that  $C_B = C \otimes_{\mathbf{Z}} B$  is not perfect if  $1 \leq r < m < n$ . From (13.4) and the considerations above we obtain that  $\text{pd}_{\mathbf{Z}[X]} C = \text{grade}_{I_{r+1}(X)} C + 1$ . Let

$$\mathcal{F}: 0 \longrightarrow F_t \longrightarrow \dots \longrightarrow F_0$$

be a  $\mathbf{Z}[X]$ -free resolution of  $C$  of minimal length. Since  $C$  and the modules  $F_j$  are  $\mathbf{Z}$ -flat,  $\mathcal{F} \otimes_{\mathbf{Z}} B$  is a  $B[X]$ -free resolution of  $C_B$ . Let  $J$  be a prime ideal in  $B[X]$  containing  $I_r(X)$ , and  $I$  the preimage of  $J$  in  $\mathbf{Z}[X]$ . Then  $(\mathcal{F} \otimes_{\mathbf{Z}} B) \otimes_{B[X]} B[X]_J$  is a  $B[X]_J$ -free resolution of  $(C_B)_J$ . To see that it has minimal length, we consider the canonical isomorphism

$$(\mathcal{F} \otimes_{\mathbf{Z}} B) \otimes_{B[X]} B[X]_J \cong (\mathcal{F} \otimes_{\mathbf{Z}[X]} \mathbf{Z}[X]_I) \otimes_{\mathbf{Z}[X]_I} B[X]_J.$$

$\mathcal{F} \otimes_{\mathbf{Z}[X]} \mathbf{Z}[X]_I$  has minimal length in view of the inequality  $\text{depth } C_M < \text{depth } R_M$  above. Furthermore the extension  $\mathbf{Z}[X]_I \rightarrow B[X]_J$  is local. Consequently  $\text{pd}(C_B)_J = t = \text{grade}(C_B)_J + 1$ . —



**(13.9) Lemma.** *Denote by  $P$  the ideal in  $R$  generated by the  $r$ -minors of the first  $r$  rows of  $x$ . Let further  $s$  be a positive integer and  $b_1, \dots, b_k \in R$  such that  $b_1x^1 + \dots + b_kx^k \in P^sR^m$ ,  $x^j$  being the  $j$ -th column of  $x$ . Then there are elements  $a_1, \dots, a_k \in R$  such that  $a_1x^1 + \dots + a_kx^k = 0$  and  $b_j - a_j \in P^s$  for  $j = 1, \dots, k$ .*

PROOF: Of course we may assume that  $r \geq 1$ . Let  $Q$  be the ideal in  $R$  generated by the  $r$ -minors of the first  $k - 1$  columns of  $x$ . Taking linear combinations of  $b_1x_{i1} + \dots + b_kx_{ik}$ ,  $1 \leq i \leq m$ , with suitable minors of  $x$  as coefficients we get

$$b_k\delta \in P^s + Q$$

where

$$\delta = \begin{cases} [1, \dots, k - 1, r + 1 | 1, \dots, k] & \text{if } k \leq r, \\ [1, \dots, r - 1, k | 1, \dots, r - 1, k] & \text{if } k > r. \end{cases}$$

In case  $B$  is a field,  $\delta$  is not a zero-divisor modulo  $P^s + Q$  since  $P^s + Q$  is  $(P + Q)$ -primary and  $\delta \notin P + Q$  (cf. (9.18)). Furthermore  $R/(P^s + Q)$  is  $\mathbf{Z}$ -free in case  $B = \mathbf{Z}$ . From (3.14) we then obtain that  $\delta$  is not a zero-divisor modulo  $P^s + Q$  for arbitrary  $B$ .

Consequently  $b_k \in P^s + Q$ . Obviously this implies the assertion in case  $k \leq r$ . Assume that  $k > r$ . Let  $J \in S(r, k - 1)$  and put  $\tilde{J} = J \cup \{k\}$ . Then

$$\sum_{j \in \tilde{J}} \sigma(j, \tilde{J} \setminus j) [I | \tilde{J} \setminus j] x^j = 0 \quad \text{for all } I \in S(r, m).$$

A suitable linear combination of these determinantal relations of  $x^1, \dots, x^k$  yields a relation  $(a_1, \dots, a_k)$  such that  $b_k - a_k \in P^s$ . Induction on  $k$  now completes the proof of (13.9). —

Since  $\text{Coker } x$  fails to be perfect in case  $1 \leq r \leq m < n$  the cokernel of the map  $\psi = (\varphi_{x^*, r})^*$  certainly cannot be perfect in this case. It requires a little more effort to see that  $\text{Coker } \psi$  is not perfect except for  $m = n$ .

**(13.10) Proposition.** *The cokernel of the map  $\psi = (\varphi_{x^*, r})^*$  is perfect if and only if  $m = n$ .*

PROOF: There is to prove something only in case  $m \geq n$ . Assume first that  $B = \mathbf{Z}$ . If  $m = n$  then  $D = \text{Coker } \psi$  is almost perfect, so perfection of  $D$  is equivalent to  $\text{Ext}_R^1(D, R) = 0$  since  $R$  is a Gorenstein ring in that case. The vanishing of  $\text{Ext}_R^1(D, R)$  in any case follows from (13.6).

Suppose now that  $m > n$ . Imitating part of the proof of (13.8) we write  $M = I_r(x)$  and prove that  $\text{depth } D_M < \text{depth } R_M$ . Again we use the isomorphism (4), put  $\tilde{R} = R_r(Y)$ ,  $\tilde{M} = I_{r-1}(Y)$  and  $\tilde{D}$  the cokernel of the map  $(\varphi_{x^*, r-1})^*$ ,  $y$  denoting the matrix of residue classes modulo  $I_r(Y)$  of the indeterminates  $Y_{ij}$ . Since  $x \otimes R[x_{mn}^{-1}]$  can be represented by the matrix (5), we obtain that  $D \otimes R[x_{mn}^{-1}] \cong \tilde{D} \otimes R[x_{mn}^{-1}] \oplus F$ ,  $F$  being a free  $R[x_{mn}^{-1}]$ -module. Consequently  $\text{depth } D_M < \text{depth } R_M$  is equivalent to  $\text{depth } \tilde{D}_{\tilde{M}} < \text{depth } \tilde{R}_{\tilde{M}}$ , so we may assume that  $r = 1$ . As in the proof of (13.8) we can replace the base ring  $\mathbf{Z}$  by  $\mathbf{Q}$  and have only to show that  $D$  is not a perfect  $\mathbf{Q}[X]$ -module. Since  $\text{Ext}_R^1(D, R) = 0$  this is equivalent to the fact that the natural homomorphism

$$\text{Hom}_R(D, R) \xrightarrow{h} \text{Hom}_R(D, R/P^{m-n})$$

is not surjective,  $P$  denoting the ideal in  $R$  generated by the elements of the first row of  $x$ . (Remember that  $P^{m-n}$  is a canonical module of  $R$  in the case under consideration.) Put  $s = m - n$  and denote by  $y_1, \dots, y_m$  and  $z_1, \dots, z_n$  the canonical bases of  $R^m$  and  $R^n$  resp. Looking at the sequence (3) in (13.6) we see that an element of  $\text{Hom}_R(D, R/P^s)$  comes from an element

$$\sum_{i=1}^m \sum_{J \in S(2,n)} b_{i,J} z_J^* \otimes y_i, \quad b_{i,J} \in R,$$

such that for  $1 \leq j \leq n$

$$\sum_{i=1}^m \sum_{\substack{J \in S(2,n) \\ j \in J}} \sigma(j, J \setminus j) b_{i,J} [i | J \setminus j] \in P^s,$$

and every such element induces an element of  $\text{Hom}_R(D, R/P^s)$ . Now we put

$$b_{i,J} = \begin{cases} x_{11}^{s-1} & \text{if } i = 1, J = \{1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\sum_{i,J} b_{i,J} z_J^* \otimes y_i$  obviously induces an element  $\beta$  of  $\text{Hom}_R(D, R/P^s)$ .  $\beta$  lies in the image of  $h$  if and only if there is an element  $\sum_{i,J} a_{i,J} z_J^* \otimes y_i$ ,  $a_{i,J} \in R$ , such that for  $1 \leq j \leq n$

$$(6) \quad \sum_{i=1}^m \sum_{\substack{J \in S(2,n) \\ j \in J}} \sigma(j, J \setminus j) a_{i,J} [i | J \setminus j] = 0,$$

and  $b_{i,J} - a_{i,J} \in P^s$ ,  $1 \leq i \leq m$ ,  $J \in S(2, n)$ . But there is no such element: (6) in particular implies that  $a_{1,\{1,2\}}$  is contained in the ideal  $I$  of  $R$  generated by the elements of the last  $m - 1$  rows of  $x$  and  $x_{13}, \dots, x_{1n}$ . Obviously  $x_{11}^{s-1} \notin P^s + I$ .

Since  $C$  is perfect if  $m \geq n$ ,  $D$  must be a second syzygy in this case. Now the rest is mutatis mutandis a copy of the last part of the proof given for (13.8). —

**(13.11) Remarks.** (a) Actually we proved in (13.8) and (13.10) resp. that for any prime ideal  $P$  of  $R$  which contains  $I_r(x)$  (i) in case  $m < n$  the module  $(\text{Coker } x)_P$  and (ii) in case  $m > n$  the module  $(\text{Coker } \psi)_P$  is not perfect over the corresponding localization of  $B[X]$ .

(b) Theorem (13.8) allows a generalization analogous with that of (13.4) (cf. [Br.7]): Let  $A; u, r, R, \bar{u}$  be as in (13.5), (b) and assume that  $\text{grade } I_{r+1}(u) = (m - r)(n - r)$ . If  $m \geq n$  then  $\text{Coker } \bar{u}$  and hence all syzygies of  $\text{Coker } \bar{u}$  in an  $R$ -free resolution of  $\text{Coker } \bar{u}$  are perfect  $A$ -modules. If  $m < n$ ,  $I_r(\bar{u}) \neq R$  and  $I_r(\bar{u})$  contains an element which is not a zero-divisor of  $R$ , then  $\text{Coker } \bar{u}$  is not a perfect  $A$ -module.

Let  $x$  be as in the proof of (13.5), (b). Since  $\text{Coker } \bar{u} = \text{Coker}(x \otimes A) = \text{Coker } x \otimes A$ , the first assertion follows immediately from (13.8). To prove the second, one shows that for any  $P \in \text{Spec } R$ ,  $P \supset I_r(\bar{u})$ ,  $\text{Coker } \bar{u} \otimes R_P$  is not a perfect module over the corresponding localization of  $A$ . In doing so one may directly assume that  $R$  and  $A$  are

local,  $P$  being the maximal ideal of  $R$ . The preimage  $Q$  of  $P$  in  $\mathbf{Z}[X]$  contains  $I_r(X)$ . Put  $C = \text{Coker } x$ . From (13.4) we obtain  $\text{pd } C_Q = \text{grade } C_Q + 1$  over  $\mathbf{Z}[X]_Q$ . Let

$$\mathcal{F}: 0 \longrightarrow F_t \xrightarrow{f_t} F_{t-1} \longrightarrow \cdots \longrightarrow F_0$$

be a minimal  $\mathbf{Z}[X]_Q$ -free resolution of  $C_Q$ . Since  $\text{Coker } \bar{u} = C \otimes A$  has positive rank over  $R$ , we have  $\text{grade } \text{Coker } \bar{u} = \text{grade } I_{r+1}(\bar{u}) = t - 1$ . So it suffices to show that  $\mathcal{F} \otimes A$  is a minimal  $A$ -free resolution of  $C \otimes A$ . For every prime ideal  $I$  of  $\mathbf{Z}[X]$  such that  $I \not\supseteq I_{r+1}(X)$  the complex  $\mathcal{F} \otimes \mathbf{Z}[X]_I$  is split-acyclic. Hence  $\mathcal{F} \otimes A$  is split-acyclic at all prime ideals having grade smaller than  $t - 1$ . The map  $f_t$  finally splits at all primes  $I \not\supseteq I_r(X)$  since  $C_I$  is free and thus a perfect module over  $\mathbf{Z}[X]_I$ . Consequently  $f_t \otimes A$  splits at all prime ideals of  $A$  whose grade is smaller than  $t$ . From (16.16) it follows that  $\mathcal{F} \otimes A$  is acyclic. The extension  $\mathbf{Z}[X]_Q \rightarrow A$  being local,  $\mathcal{F} \otimes A$  is a minimal  $A$ -free resolution of  $C \otimes A$ . —

### C. Homological Properties of Generic Modules

In this section we investigate some homological properties of  $\text{Coker } x$  where  $x$  is as in the introduction. We start with a simple observation concerning projective dimension.

**(13.12) Proposition.** *Coker  $x$  has finite projective dimension as an  $R$ -module if and only if  $r = 0$  or  $r \geq \min(m, n)$ . In case  $r \geq \min(m, n)$  one has  $\text{pd}_R \text{Coker } x = 1$  if  $m < n$  and  $\text{pd}_R \text{Coker } x = \text{grade } I_n(X) = m - n + 1$  otherwise. If  $1 \leq r < \min(m, n)$  and  $P$  is a prime ideal in  $R$  then the following properties are equivalent:*

- (i)  $\text{pd}_{R_P}(\text{Coker } x)_P < \infty$ .
- (ii)  $(\text{Coker } x)_P$  is a free  $R_P$ -module.
- (iii)  $P \not\supseteq I_r(x)$ .

PROOF: The essential part of the second statement has been proved in Section 2. Thus the “if”-part of the first one is clear. The “only if”-part is an immediate consequence of the third assertion. In case  $1 \leq r < \min(m, n)$ ,  $P \not\supseteq I_r(x)$  if and only if  $(\text{Im } x)_P$  is a free direct summand of  $R_P^n$  (cf. (16.3)), so (ii) is equivalent to (iii). Furthermore  $\text{pd}_{R_P}(\text{Coker } x)_P < \infty$  implies  $\text{pd}_{R_P}(\text{Coker } x)_P \leq 1$  because of (13.4). Thus  $(\text{Ker } x)_P = (\text{Coker } x^*)_P^*$  is free. The same is true for  $(\text{Coker } x^*)_P$  since  $\text{Coker } x^*$  is reflexive. Consequently  $x_P^*$  splits as well as  $x_P$ , and  $(\text{Coker } x)_P$  is free. —

There is a sharp trichotomy concerning the homological properties of  $\text{Coker } x$  between the cases  $m = n$ ,  $m < n$  and  $m > n$  which is not immediately apparent from the former considerations. There is nothing to say in case  $r = 0$ . If  $r \geq m$  and  $m < n$  then  $\text{Coker } x$  is an  $(n - m)$ -th syzygy but not an  $(n - m + 1)$ -th one, and  $\text{Ext}_R^1(\text{Coker } x, R) \neq 0$ . In case  $r \geq n$  and  $m \geq n$ ,  $\text{Coker } x$  is an  $R$ -torsion module which is perfect as an  $R$ -module and therefore  $\text{Ext}_R^i(\text{Coker } x, R) = 0$  for  $i = 1, \dots, m - n$ ,  $\text{Ext}_R^{m-n+1}(\text{Coker } x, R) \neq 0$ . The remaining cases are more interesting. We start with  $m = n$ :

**(13.13) Theorem.** *Suppose that  $1 \leq r < m = n$ . Then  $\text{Ext}_R^i(\text{Coker } x, R) = \text{Ext}_R^i((\text{Coker } x)^*, R) = 0$  for all  $i \geq 1$ . In particular  $\text{Coker } x$  is an infinite syzygy module, i.e. there is an infinite exact sequence*

$$0 \longrightarrow \text{Coker } x \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_t \longrightarrow F_{t+1} \longrightarrow \cdots$$

with free  $R$ -modules  $F_i$ .

PROOF: In case  $B = \mathbf{Z}$  the ring  $R$  is Gorenstein. Since  $C = \text{Coker } x$  is a maximal Cohen-Macaulay module,  $\text{Ext}_R^i(C, R) = \text{Ext}_R^i(C^*, R) = 0$  for all  $i \geq 1$ . Let

$$\mathcal{F}: \longrightarrow F_t \xrightarrow{f_t} \dots \longrightarrow F_1 \xrightarrow{f_1} F_0$$

be an  $R$ -free resolution of  $C^*$ . Then  $\mathcal{F}^*$  is acyclic. By the usual argumentation based on  $\mathbf{Z}$ -flatness we obtain that  $\mathcal{F} \otimes_{\mathbf{Z}} B$  is an  $R \otimes_{\mathbf{Z}} B$ -free resolution of  $C^* \otimes_{\mathbf{Z}} B$  and that  $\mathcal{F}^* \otimes_{\mathbf{Z}} B$  is exact for every (noetherian) ring  $B$ . This implies  $\text{Ext}_R^i(C^*, R) = 0$  for all  $i \geq 1$  in the general case. One equally gets  $\text{Ext}_R^i(C, R) = 0$  for all  $i \geq 1$ . —

In case  $m \neq n$  the homological invariants of  $\text{Coker } x$  turn out to be grade-sensitive with respect to the ideal  $I_r(x)$ .

**(13.14) Theorem.** *Suppose that  $1 \leq r < \min(m, n)$  and put  $s = \text{grade } I_r(x)$ . Then in case*

(a)  $m > n$ :

- (i)  $\text{Coker } x$  is an  $s$ -th syzygy but not an  $(s + 1)$ -th syzygy.
- (ii)  $\text{Ext}_R^i(\text{Coker } x, R) = 0$  for  $i = 1, \dots, s - 1$ ,  $\text{Ext}_R^s(\text{Coker } x, R) \neq 0$ .

(b)  $m < n$ :

- (i)  $\text{Coker } x$  is an  $(s - 1)$ -th syzygy but not an  $s$ -th syzygy.
- (ii)  $\text{Ext}_R^i(\text{Coker } x, R) = 0$  for  $i = 1, \dots, s$ ,  $\text{Ext}_R^{s+1}(\text{Coker } x, R) \neq 0$ .

PROOF: First we use (16.32) to get that (a),(i) is equivalent to (b),(ii) and (b),(i) equivalent to (a),(ii): This holds since  $D(\text{Coker } x) = \text{Coker } x^*$ .

Part (i) of (b) is an easy consequence of (13.4) and (13.11),(a): Since  $\text{Coker } x$  is almost perfect,  $\text{depth}(\text{Coker } x)_P \geq s - 1$  for all  $P \in \text{Spec } R$ ,  $P \supset I_r(x)$ , and if  $\text{depth } R_P = s$  for such a prime ideal then  $\text{depth}(\text{Coker } x)_P = s - 1$  because  $(\text{Coker } x)_P$  is not perfect. Finally we repeat that in any case  $(\text{Coker } x)_P$  is free whenever  $P \in \text{Spec } R$ ,  $P \not\supset I_r(x)$ .

Similarly part (i) of (a) follows from (13.8) and (13.11),(a):  $\text{Coker } x$  is perfect, so

$$\text{depth}(\text{Coker } x)_P \geq \min(s, \text{depth } R_P)$$

and consequently  $\text{Coker } x$  is an  $s$ -th syzygy. If it would be an  $(s + 1)$ -th one, then  $\text{Ext}_R^i(D(\text{Coker } x), R) = 0$  for  $i = 1, \dots, s + 1$  (cf. (16.34)). But  $D(\text{Coker } x) = \text{Coker } x^*$  has a free resolution

$$\dots \longrightarrow \bigwedge^{r+1} (R^n)^* \otimes \bigwedge^r R^m \xrightarrow{\varphi_{x^*, r}} (R^n)^* \xrightarrow{x^*} (R^m)^*$$

by (13.6), so  $(\text{Coker } x^*)^* = \text{Ker } x$  would be an  $(s + 3)$ -th and  $\text{Coker } \psi$  (cf. (3)) an  $s$ -th syzygy. This is impossible since  $\text{depth}(\text{Coker } \psi)_P = s - 1$  for all  $P \in \text{Spec } R$  such that  $P \supset I_r(x)$  and  $\text{depth } R_P = s$ . —

**(13.15) Remark.** (13.12), (13.13) and (13.14) have obvious generalizations to the case considered in (13.5),(b) and (13.11),(b). Details may be found in [Br.7] or left to the reader. —

**(13.16) Remark.** We conclude the section with a few observations concerning  $B[X]$ -free resolutions of  $\text{Im } x$  and  $\text{Coker } x$ . The complexity to construct such a resolution (which should be minimal), is certainly comparable to the complexity of the corresponding problem for determinantal ideals. Besides the case in which  $r \geq \min(m, n)$

the maximal minor case seems to be the only one for which results are available. Let  $r + 1 = \min(m, n)$  in the following.

(i) If  $m \geq n$  then a minimal  $B[X]$ -free resolution of  $\text{Coker } x$  is given by the complex  $\mathcal{D}_1(x)$  constructed in Section 2 (cf. (2.16)), since  $\text{Coker } x = \text{Coker } X$  in this case. To get such a resolution for  $\text{Im } x$ , we use the following observation: Let  $A$  be a commutative ring,  $f: F \rightarrow G$  a homomorphism of free  $A$ -modules,  $r$  a non-negative integer, and  $\varphi_{f,r}: \bigwedge^{r+1} F \otimes \bigwedge^r G^* \rightarrow F$  the homomorphism defined in Subsection B. Then  $I_{r+1}(f)F \subset \text{Im } \varphi_{f,r}$ . (The easy proof is left to the reader). Applied to the special situation just considered it yields an isomorphism

$$\text{Coker } \varphi_{X,r} \cong \text{Im } x.$$

A candidate for a minimal  $B[X]$ -free resolution of  $\text{Coker } \varphi_{X,r}$  can be found in [BE.4], p. 270: It is not hard to check that the complex  $L_1^{1,r+1}X$  defined there is acyclic and its map  $d_1$  is nothing but  $\varphi_{X,r}$ .

(ii) In case  $m \leq n$  we observe that the kernel of the map  $g: B[X]^m \rightarrow R^n$  induced by  $X$  is generated by the elements

$$\sum_{i=1}^m (-1)^{i+1} [I \setminus i | J] y_i, \quad I = \{1, \dots, m\}, J \in S(r, n),$$

$y_1, \dots, y_m$  being the canonical basis of  $B[X]^m$  (cf. the observation made in (i).) Next we consider the isomorphism

$$\bigwedge^{m-1} (B[X]^m)^* \xrightarrow{h} B[X]^m,$$

where  $h(y^*)(z^*)$  is the coefficient of  $y^* \wedge z^*$  with respect to  $y_1^* \wedge \dots \wedge y_m^*$ ,  $y^* \in \bigwedge^{m-1} (B[X]^m)^*$ ,  $z^* \in (B[X]^m)^*$ . One readily checks that  $\text{Ker } g = h(\text{Im } \bigwedge^{m-1} X^*)$ , so  $\text{Im } x$  is isomorphic with  $\text{Coker } \bigwedge^{m-1} X^*$ . Corollary 3.2 in [BE.4] provides a minimal  $B[X]$ -free resolution of  $\text{Coker } \bigwedge^{m-1} X^*$ . One may use the resolution of  $\text{Im } x$  just mentioned and the resolution of  $R$  given in Section 2 (cf. (2.16)) to get a resolution of  $\text{Coker } x$  by constructing the mapping cylinder of a chain map induced by  $X$ . The resolution of  $\text{Coker } x$  thus obtained is not minimal, not even if  $m < n$ , the case in which it has minimal length; in fact, already the system of generators of the first syzygy module turns out to be non-minimal. The resolution of  $\text{Coker } x$  constructed in [Av.2], Proposition 7, is for the same reason not minimal. —

### D. Comments and References

References to the results of this section in case  $r \geq \min(m, n)$  have been given in Section 2. The main content is taken from [Br.7]. For the perfection of the generic module (cf. (13.8)) and its homological properties (cf. Subsection C) this applies also to the method of proof. There is a difference in demonstrating Theorem (13.4): While we use a simple filtration argument (cf. (13.2)) and the results of Section 5 concerning wonderful posets, in [Br.7] the inductive methods of Hochster and Eagon (cf. [HE.2] and Section 12) have been exploited to obtain the perfection of  $\text{Im } x$ . Additional literature to the subject treated in (13.16) may be found in 2.E.

## 14. The Module of Kähler Differentials

Throughout this section  $X = (X_{ij})$  is an  $m \times n$  matrix of indeterminates over a noetherian ring  $B$ ,  $r$  an integer such that  $1 \leq r < \min(m, n)$  and  $R = R_{r+1}(X) = B[X]/I_{r+1}(X)$ . We shall investigate the module  $\Omega_{R/B}^1$  of (Kähler) differentials of  $R/B$ . (The reader who wants detailed information about this concept and its importance in local algebra is referred to the books of Kunz [Kn] or Scheja [Sch].) We are mainly interested in computing  $\text{grade}(I_1(X), \Omega_{R/B}^1)$ . For this purpose the special structure of the poset  $\Delta(X)$  together with the general results on ASLs of Section 5 will be found very useful, once more.

The module  $\Omega_{R/B}^1$  is closely related to  $I_{r+1}(X)/I_{r+1}(X)^2$  via the exact sequence

$$I_{r+1}(X)/I_{r+1}(X)^2 \longrightarrow \Omega_{B[X]/B}^1/I_{r+1}(X)\Omega_{B[X]/B}^1 \longrightarrow \Omega_{R/B}^1 \longrightarrow 0$$

which can be improved to

$$0 \longrightarrow I_{r+1}(X)/I_{r+1}(X)^{(2)} \longrightarrow \Omega_{B[X]/B}^1/I_{r+1}(X)\Omega_{B[X]/B}^1 \longrightarrow \Omega_{R/B}^1 \longrightarrow 0$$

if  $B$  is a domain, so computing  $\text{grade}(I_1(X), \Omega_{R/B}^1)$  means to compute the grade of  $I_1(X)$  with respect to  $I_{r+1}(X)/I_{r+1}(X)^{(2)}$  in this case. Thus our investigations are connected with Proposition (10.8) which gives a lower bound for  $\text{grade}(I_1(X), I_{r+1}(X)/I_{r+1}(X)^{(2)})$ . We shall see that this bound is not sharp except for the extreme cases in which  $m = n = r + 1$  or  $r = 1$ .

Of course the computation of  $\text{grade}(I_1(X), \Omega_{R/B}^1)$  is equivalent to the computation of  $\text{grade}(I_1(X), M)$ , in general,  $M$  denoting the kernel of the projection

$$\Omega_{B[X]/B}^1/I_{r+1}(X)\Omega_{B[X]/B}^1 \longrightarrow \Omega_{R/B}^1.$$

For technical reasons we shall primarily deal with  $M$ .

To get a lower bound for  $\text{grade}(I_1(X), M)$  we shall construct a (finite) filtration of  $M$ , the quotients of which are isomorphic to direct sums of certain good-natured ideals in the rings  $R(X; \delta)$ ,  $\delta = [1, \dots, r-1, s|1, \dots, r-1, t]$ . These ideals are investigated in the first subsection. The second deals with the filtration. It is not hard to see that the lower bound obtained for  $\text{grade}(I_1(X), \Omega_{R/B}^1)$  is an upper bound, too. Finally we shall discuss the syzygetic behaviour of  $\Omega_{R/B}^1$ .

### A. Perfection and Syzygies of Some Determinantal Ideals

Let  $s, t$  denote integers such that  $r \leq s \leq m, r \leq t \leq n$ . We put

$$\delta_{st} = [1, \dots, r - 1, s | 1, \dots, r - 1, t].$$

The ideal in  $R(X; \delta_{st})$ , we are interested in, is generated by the residue classes of all  $r$ -minors  $[a_1, \dots, a_r | b_1, \dots, b_r]$  of  $X$  such that  $a_r = s, b_r = t$ . More formally one can give the following description: Consider the ideal

$$\{ \alpha \in \Delta(X) : \alpha \not\geq \delta_{s+1,t} \text{ and } \alpha \not\geq \delta_{s,t+1} \}$$

in  $\Delta(X)$  cogenerated by  $\delta_{s+1,t}, \delta_{s,t+1}$  (cf. Section 5; of course we allow the extreme cases  $\delta_{m+1,t} = \delta_{s,m+1} = [1, \dots, r - 1 | 1, \dots, r - 1]$ .) This generates an ideal  $I(X; \delta_{s+1,t}, \delta_{s,t+1})$  in  $B[X]$ . The quotient

$$I(x; \delta_{s+1,t}, \delta_{s,t+1}) = I(X; \delta_{s+1,t}, \delta_{s,t+1}) / I(X; \delta_{st})$$

is the ideal in  $R(X; \delta_{st})$  we have in mind. The special case in which  $s = r, t = r + 1$  has already been treated in Section 13. In accordance with the notation just introduced we put

$$\Delta(X; \delta_{s+1,t}, \delta_{s,t+1}) = \{ \gamma \in \Delta(X) : \gamma \geq \delta_{s+1,t} \text{ or } \gamma \geq \delta_{s,t+1} \}.$$

**(14.1) Proposition.** *Choose notations as above. Then  $I(X; \delta_{s+1,t}, \delta_{s,t+1})$  is a perfect ideal of  $B[X]$ . Furthermore*

$$\text{rk } \Delta(X; \delta_{s+1,t}, \delta_{s,t+1}) = \text{rk } \Delta(X; \delta_{st}) - 1.$$

PROOF: The minimal elements in  $\Delta(X; \delta_{s+1,t}, \delta_{s,t+1})$  are exactly

$$\begin{array}{ll} \delta_{s+1,t}, \delta_{s,t+1} & \text{if } s < m, t < n, \\ \delta_{s+1,t} & \text{if } s \leq m, t = n, \\ \delta_{s,t+1} & \text{if } s = m, t \leq n. \end{array}$$

In any case these elements are upper neighbours of  $\delta_{st}$ , which is the only minimal element of  $\Delta(X; \delta_{st})$ . Hence  $I(X; \delta_{s+1,t}, \delta_{s,t+1})$  is perfect in view of (5.19), and the rank formula is obvious. —

For an application in the next subsection we need a description of the first syzygy of  $I(x; \delta_{s+1,t}, \delta_{s,t+1})$  as an  $R(X; \delta_{st})$ -module. The following proposition generalizes (13.3):

**(14.2) Proposition.** *Let  $s, t$  be integers such that  $r \leq s \leq m, r \leq t \leq n$ . We put  $\delta_{st} = [1, \dots, r - 1, s | 1, \dots, r - 1, t]$  as above and  $\bar{R} = R(X; \delta_{st})$ . Let  $y_1, \dots, y_s$  and  $z_1, \dots, z_t$  be the canonical bases of  $\bar{R}^s, \bar{R}^t$ , and consider  $\bar{R}^{s-1}, (\bar{R}^{t-1})^*$  as submodules of  $\bar{R}^s, (\bar{R}^t)^*$  generated by  $y_1, \dots, y_{s-1}$  and  $z_1^*, \dots, z_{t-1}^*$  resp. Denote by  $\bar{x}: \bar{R}^s \rightarrow \bar{R}^t$  the map given by the  $s \times t$  matrix which arises from  $X$  by cancelling the last  $m - s$  rows and the last  $n - t$  columns. Let*

$$\bar{\varphi}: \bigwedge^{r-1} \bar{R}^{s-1} \otimes \bigwedge^{r-1} (\bar{R}^{t-1})^* \longrightarrow \bar{R},$$

be the composition of the map

$$\bigwedge^{r-1} \overline{R}^{s-1} \otimes \bigwedge^{r-1} (\overline{R}^{t-1})^* \longrightarrow \bigwedge^r \overline{R}^s \otimes \bigwedge^r (\overline{R}^t)^*, \quad y_I \otimes z_J^* \longrightarrow y_{I \cup \{s\}} \otimes z_{J \cup \{t\}}^*$$

and  $\varphi_{\overline{x}, r}$ . Then the kernel of  $\overline{\varphi}$  is generated by the elements

$$\sum_{i \in I} \sigma(i, I \setminus i)[i|j] y_{I \setminus i} \otimes z_J^*, \quad I \in S(r, s-1), \quad j \in S(1, n), \quad J \in S(r-1, t-1),$$

and

$$\sum_{j \in J} \sigma(j, J \setminus j)[i|j] y_I \otimes z_{J \setminus j}^*, \quad J \in S(r, t-1), \quad i \in S(1, m), \quad I \in S(r-1, s-1).$$

PROOF: Let  $\tilde{N}$  be the submodule generated by these elements. Obviously

$$\overline{\varphi}(y_I \otimes z_J^*) = [I, s|J, t],$$

$I \in S(r-1, s-1)$ ,  $J \in S(r-1, t-1)$ , so  $\tilde{N} \subset \text{Ker } \overline{\varphi}$ . Proposition (5.6),(b) provides the proof of the opposite inclusion: Put

$$\Psi = \{ [I, s|J, t] : I \in S(r-1, s-1), \quad J \in S(r-1, t-1) \}.$$

$\Psi$  is an ideal in  $\Delta(X; \delta_{st})$ . Let  $[I, s|J, t] \in \Psi$ ,  $[K|L] \in \Delta(X; \delta_{st})$  such that  $[K|L] \not\prec [I, s|J, t]$ . We claim that

$$[K|L] y_I \otimes z_J^* \in \tilde{N} + \sum_{[U|V] < [I|J]} \overline{R} y_U \otimes z_V^*.$$

To show this we write  $[K|L] = [k_1, \dots, k_u | l_1, \dots, l_u]$ ,  $[I|J] = [i_1, \dots, i_{r-1} | j_1, \dots, j_{r-1}]$ . By assumption there is a  $\rho \leq r-1$  such that  $k_\rho < i_\rho$  or  $l_\rho < j_\rho$ . Suppose that  $k_\rho < i_\rho$  for some  $\rho$ . Denote by  $\sigma$  the smallest such index.

If  $\sigma = u$ , then we put  $\tilde{K} = \{i_1, \dots, i_{u-1}\}$ ,  $\tilde{I} = \{k_1, \dots, k_u, i_u, \dots, i_{r-1}\}$ . From the lemma below (with  $F = \overline{R}^{s-1}$ ,  $G = \overline{R}^{t-1}$ ,  $f$  the matrix  $\overline{x}$  decreased by its last row and its last column,  $v = r-1$ ) we obtain that

$$\varphi_u(y_{\tilde{K}} \otimes y_{\tilde{I}} \otimes z_{\tilde{L}}^*) \otimes z_J^* = \sum_{U \in S(u, \tilde{I})} \sigma(U, \tilde{I} \setminus U)[U|L] y_{\tilde{K}} \wedge y_{\tilde{I} \setminus U} \otimes z_J^* \in \tilde{N}.$$

Since  $[\tilde{K}, \tilde{I} \setminus U | J] < [I|J]$  for all  $U \in S(u, \tilde{I})$  such that  $U \neq K$  and  $[\tilde{K}, \tilde{I} \setminus U | J] \neq 0$ , the claim follows at once.

If  $\sigma < u$ , then the inductive hypothesis on  $u$  yields

$$[K \setminus k_u | L \setminus l_\rho] y_I \otimes z_J^* \in \tilde{N} + \sum_{[U|V] < [I|J]} \overline{R} y_U \otimes z_V^*$$

for  $\rho = 1, \dots, u$ . But  $[K|L] y_I \otimes z_J^*$  is a linear combination of the elements on the left hand side, so we are done in this case, too.

Clearly the proof runs analogously if  $l_\rho < j_\rho$  for some  $\rho$ . —



**(14.3) Lemma.** *Let  $A$  be an arbitrary ring,  $f: F \rightarrow G$  a homomorphism of  $A$ -modules, and  $u, v$  integers such that  $1 \leq u \leq v + 1$ . Consider the map*

$$\varphi_u: \bigwedge^{u-1} F \otimes \bigwedge^{v+1} F \otimes \bigwedge^u G^* \longrightarrow \bigwedge^v F$$

given by

$$\varphi_u(w_K \otimes y_I \otimes z_L^*) = w_K \wedge \varphi_{f,u}(y_I \otimes z_L^*),$$

$w_K = w_{k_1} \wedge \dots \wedge w_{k_{u-1}}$ ,  $y_I = y_{i_1} \wedge \dots \wedge y_{i_{v+1}}$ ,  $z_L^* = z_{l_1}^* \wedge \dots \wedge z_{l_u}^*$ ,  $w_{k_\rho}, y_{i_\sigma} \in F$ ,  $z_{l_\tau}^* \in G^*$ . Then  $\text{Im } \varphi_u \subset \text{Im } \varphi_1$ .

PROOF: We use induction on  $u$ . There is nothing to prove for  $u = 1$ . Let  $u \geq 1$ ,  $w_K = w_{k_1} \wedge \dots \wedge w_{k_u}$ ,  $y_I = y_{i_1} \wedge \dots \wedge y_{i_{v+1}}$ ,  $z_L^* = z_{l_1}^* \wedge \dots \wedge z_{l_{u+1}}^*$ ,  $w_{k_\rho}, y_{i_\sigma} \in F$ ,  $z_{l_\tau}^* \in G^*$ . Then

$$\begin{aligned} & \varphi_{u+1}(w_K \otimes y_I \otimes z_L^*) \\ &= \sum_{U \in S(u+1, I)} \sigma(U, I \setminus U) z_L^* \left( \bigwedge^u f(y_U) \right) w_K \wedge y_{I \setminus U} \\ &= \sum_{U \in S(u+1, I)} \sum_{i \in U} \sigma(U, I \setminus U) \sigma(i, U \setminus i) z_{l_1}^*(f(y_i)) z_{L \setminus l_1}^* \left( \bigwedge^{u-1} f(y_{U \setminus i}) \right) w_K \wedge y_{I \setminus U} \\ &= \sum_{U \in S(u+1, I)} \sum_{i \in U} (-1)^u \sigma(U \setminus i, I \setminus (U \setminus i)) \sigma(i, I \setminus U) z_{l_1}^*(f(y_i)) z_{L \setminus l_1}^* \left( \bigwedge^{u-1} f(y_{U \setminus i}) \right) w_K \wedge y_{I \setminus U} \\ &= \pm \sum_{V \in S(u, I)} \sum_{i \in I \setminus V} \sigma(V, I \setminus V) \sigma(i, (I \setminus V) \setminus i) z_{l_1}^*(f(y_i)) z_{L \setminus l_1}^* \left( \bigwedge^{u-1} f(y_V) \right) w_K \wedge y_{(I \setminus V) \setminus i} \\ &= \pm \sum_{V \in S(u, I)} \sigma(V, I \setminus V) z_{L \setminus l_1}^* \left( \bigwedge^{u-1} f(y_V) \right) \left( \sum_{i \in I \setminus V} \sigma(i, (I \setminus V) \setminus i) z_{l_1}^*(f(y_i)) w_K \wedge y_{(I \setminus V) \setminus i} \right) \\ &\equiv \pm \sum_{V \in S(u, I)} \sigma(V, I \setminus V) z_{L \setminus l_1}^* \left( \bigwedge^{u-1} f(y_V) \right) \left( \sum_{i \in K} \sigma(i, K \setminus i) z_{l_1}^*(f(w_i)) w_{K \setminus i} \wedge y_{I \setminus V} \right) \text{ mod Im } \varphi_1 \\ &\equiv \pm \sum_{i \in K} \sigma(i, K \setminus i) z_{l_1}^*(f(w_i)) \varphi_u(w_{K \setminus i} \otimes y_I \otimes z_{L \setminus l_1}^*) \text{ mod Im } \varphi_1 \\ &\equiv 0 \text{ mod Im } \varphi_1 \quad \text{by the inductive hypothesis. —} \end{aligned}$$

## B. The Lower Bound for the Depth of the Differential Module

The  $R$ -module  $M$  considered in the introduction is generated by the residue classes  $\overline{d\alpha}$  modulo  $I_{r+1}(X)\Omega_{B[X]/B}^1$  of the elements  $d\alpha$  where  $d$  is the universal  $B$ -derivation of  $B[X]$  and  $\alpha$  runs through the  $(r + 1)$ -minors of  $X$ . To have a simpler notation we shall write  $\overline{d\alpha}$  instead of  $\overline{d\alpha}$ . If we identify  $\Omega_{B[X]/B}^1/I_{r+1}(X)\Omega_{B[X]/B}^1$  with  $R^m \otimes (R^n)^*$  via the map

$$\overline{d}X_{ij} \longrightarrow y_i \otimes z_j^*,$$

$y_1, \dots, y_m$  and  $z_1, \dots, z_n$  being the canonical bases of  $R^m, R^n$  resp., then clearly

$$\begin{aligned} \bar{d}([I|J]) &= \sum_{\substack{i \in I \\ j \in J}} \sigma(i, I \setminus i) \sigma(j, J \setminus j) [I \setminus i | J \setminus j] y_i \otimes z_j^* \\ &= \varphi_{x,r}(y_I \otimes z_J^*) \end{aligned}$$

for all  $I \in S(r+1, m), J \in S(r+1, n)$ , the map  $x: R^m \rightarrow R^n$  given by the matrix  $X$  modulo  $I_{r+1}(X)$ .

We start with a simple observation concerning the free locus of  $\Omega$ . It may also be derived from (2.6).

**(14.4) Proposition.** *Let  $P$  be a prime ideal of  $R$ . Then  $M_P$  is a free direct summand of  $R_P^m \otimes (R_P^n)^*$  (of rank  $(m-r)(n-r)$ ) if and only if  $P \not\supseteq I_r(X)/I_{r+1}(X)$ .*

PROOF: The less trivial “if” part is an immediate consequence of the following general fact: Let  $r$  be a nonnegative integer,  $F, G$  modules over an arbitrary ring  $A$  and  $f: F \rightarrow G$  an  $A$ -homomorphism such that  $\text{Im } f$  is a free direct summand of  $G$  and  $\text{rk } f = r$ . Then the image of the map

$$\varphi_{f,r}: \bigwedge^{r+1} F \otimes \bigwedge^{r+1} G^* \longrightarrow F \otimes G^*$$

is the direct summand  $\text{Ker } f \otimes \text{Ker } f^*$  of  $F \otimes G^*$ . —

Let  $r \leq s < m, r \leq t < n$ , and put

$$\begin{aligned} M(s, t) &= \text{submodule of } M \text{ generated by all } \bar{\alpha}, \\ \alpha &= [a_1, \dots, a_{r+1} | b_1, \dots, b_{r+1}], \quad (a_r, b_r) \preceq (s, t) \end{aligned}$$

(“ $\preceq$ ” means “lexicographically  $\leq$ ”). Clearly  $\{M(s, t): r \leq s < m, r \leq t < n\}$  gives an increasing filtration of  $M$  if the pairs  $(s, t)$  are ordered lexicographically. Next we consider the quotients of this filtration:

$$\bar{M}(s, t) = \begin{cases} M(r, r) & \text{if } s = t = r \\ M(s, t)/M(s, t-1) & \text{if } t > r \\ M(s, r)/M(s-1, n-1) & \text{if } s < r, t = r. \end{cases}$$

**(14.5) Proposition.** *Put  $\delta_{st} = [1, \dots, r-1, s | 1, \dots, r-1, t]$  whenever  $r \leq s < m, r \leq t < n$ , and choose notations as above. Then*

- (a)  $\text{Ann}_R \bar{M}(s, t) = I(X; \delta_{st})/I_{r+1}(X)$ .
- (b) *As an  $R(X; \delta_{st})$ -module  $\bar{M}(s, t)$  is isomorphic to the  $(m-s)(n-t)$ -fold direct sum of the ideal  $I(x; \delta_{s+1,t}, \delta_{s,t+1})$ .*

PROOF: Let  $(s, t)$  be such that  $r \leq s < m, r \leq t < n$ . Looking at  $R^{s-1}, R^{t-1}$  as submodules of  $R^m, R^n$  generated by  $y_1, \dots, y_{s-1}$  and  $z_1, \dots, z_{t-1}$  resp., we consider the map

$$\tilde{\varphi} = \tilde{\varphi}(s, i; t, j): \bigwedge^{r-1} R^{s-1} \otimes \bigwedge^{r-1} (R^{t-1})^* \longrightarrow \bar{M}(s, t)$$

which is composed of the inclusion

$$\bigwedge_{r-1} R^{s-1} \otimes \bigwedge_{r-1} (R^{t-1})^* \longrightarrow \bigwedge_{r+1} R^m \otimes \bigwedge_{r+1} (R^n)^*, \quad y_I \otimes z_J^* \longrightarrow y_{I \cup \{s,i\}} \otimes z_{J \cup \{t,j\}},$$

the homomorphism  $\varphi_{x,r}$ , and the residue class map with respect to the corresponding submodule of  $M$ . Obviously

$$\tilde{\varphi}(y_I \otimes z_J^*) = \text{residue class of } d([I, s, i | J, t, j]),$$

$I \in S(r-1, s-1)$ ,  $J \in S(r-1, t-1)$ .

By expansion one obtains

$$(1) \quad \begin{aligned} \sum_{i \in I} \sigma(i, I \setminus i) [i | j] \bar{d}([I \setminus i | J]) &= 0, \\ \sum_{l \in L} \sigma(l, L \setminus l) [k | l] \bar{d}([K | L \setminus l]) &= 0 \end{aligned}$$

for all  $I \in S(r+2, m)$ ,  $j \in S(1, n)$ ,  $J \in S(r+1, n)$ ,  $k \in S(1, m)$ ,  $K \in S(r+1, m)$ ,  $L \in S(r+2, n)$ . We therefore get

$$\begin{aligned} \sum_{i \in I} \sigma(i, I \setminus i) [i | j] y_{I \setminus i} \otimes z_J^* &\in \text{Ker } \tilde{\varphi}, \\ \sum_{l \in L} \sigma(l, L \setminus l) [k | l] y_K \otimes z_{L \setminus l}^* &\in \text{Ker } \tilde{\varphi} \end{aligned}$$

for all  $I \in S(r, s-1)$ ,  $j \in S(1, n)$ ,  $J \in S(r-1, t-1)$ ,  $k \in S(1, m)$ ,  $K \in S(r-1, s-1)$ ,  $L \in S(r, t-1)$ . By elementary determinantal calculations it follows that

$$[K | L] y_I \otimes z_J^*, [\tilde{K} | \tilde{L}] y_I \otimes z_J^* \in \text{Ker } \tilde{\varphi}$$

for all  $K \in S(r, s-1)$ ,  $L \in S(r, n)$ ,  $\tilde{K} \in S(r, m)$ ,  $\tilde{L} \in S(r, t-1)$  and all  $I \in S(r-1, s-1)$ ,  $J \in S(r-1, t-1)$ . This proves the inclusion “ $\supset$ ” of (a).

Put  $\bar{R} = R(X; \delta_{st})$ . Then  $\tilde{\varphi}$  induces an  $\bar{R}$ -homomorphism

$$\bigwedge_{r-1} \bar{R}^{s-1} \otimes \bigwedge_{r-1} (\bar{R}^{t-1})^* \longrightarrow \bar{M}(s, t)$$

whose kernel contains the kernel of the map  $\tilde{\varphi}$  in (14.2). So  $\sum_{i=s+1}^m \sum_{j=t+1}^n \tilde{\varphi}(s, i; t, j)$  induces a surjective map

$$\bigoplus_{s+1}^m \bigoplus_{t+1}^n I(x; \delta_{s+1,t}, \delta_{s,t+1}) \longrightarrow \bar{M}(s, t).$$

The proof of the proposition is complete once we have shown that  $\bar{M}(s, t)$  contains a free  $\bar{R}$ -module of rank  $(m-s)(n-t)$ . This clearly holds if the residue classes of the elements  $d([1, \dots, r-1, s, i | 1, \dots, r-1, t, j])$  are linearly independent over  $\bar{R}$ . Assume that

$$\sum_{i=s+1}^m \sum_{j=t+1}^n a_{ij} \bar{d}([1, \dots, r-1, s, i | 1, \dots, r-1, t, j]) \in \begin{cases} 0 & \text{if } s = t = r, \\ M(s, t-1) & \text{if } t > r, \\ M(s-1, t-1) & \text{if } s > r, t = r \end{cases}$$

with elements  $a_{ij} \in R$ . The left side of this relation has  $\pm a_{ij} \delta_{st}$  as its component belonging to  $\bar{d}X_{ij}$  while the corresponding component of the right side lies in the ideal  $I(X; \delta_{st})/I_{r+1}(X)$ . Since  $\delta_{st}$  is not a zero-divisor of  $R(X; \delta_{st})$  we get

$$a_{ij} \in I(X; \delta_{st})/I_{r+1}(X). \quad \text{—}$$

**(14.6) Corollary.** *For every prime ideal  $P$  in  $B[X]$  containing  $I(X; \delta_{st})$  one has*

$$\text{depth } \overline{M}(s, t)_P = \text{depth } R_P - (s - r) - (t - r).$$

PROOF: Assume first that  $B$  is a field or  $B = \mathbf{Z}$ . From (14.1) we get

$$\text{depth } I(x; \delta_{s+1, t}, \delta_{s, t+1})_P = \text{depth } R(X; \delta_{st})_P,$$

so

$$\text{depth } \overline{M}(s, t)_P = \text{depth } R(X; \delta_{st})_P$$

by (14.5),(b). Using the dimension formula (5.12),(a) we obtain

$$\text{depth } R(X; \delta_{st})_P = \text{depth } R_P - (s - r) - (t - r).$$

It follows in particular that  $\overline{M}(s, t)$  is a torsionfree  $R(X; \delta_{st})$ -module if  $B = \mathbf{Z}$ , so it is  $B$ -flat in case  $B$  is an arbitrary noetherian ring, what we will assume from now on. Put  $Q = P \cap B$  and consider the flat extension  $B_Q \rightarrow R_P$ . Since  $\overline{M}(s, t)_P$  and  $R(X; \delta_{st})_P$  are also flat over  $B_Q$ , the depth formula we have used already in the previous sections (cf. the proof of (3.14) for example) yields

$$\begin{aligned} \text{depth } \overline{M}(s, t)_P &= \text{depth } B_Q + \text{depth}(\overline{M}(s, t)_P \otimes (B_Q/QB_Q)), \\ \text{depth } R(X; \delta_{st})_P &= \text{depth } B_Q + \text{depth}(R(X; \delta_{st})_P \otimes (B_Q/QB_Q)), \\ \text{depth } R_P &= \text{depth } B_Q + \text{depth}(R_P \otimes (B_Q/QB_Q)). \end{aligned}$$

The claim now follows from what we have derived in the field case. —

**(14.7) Proposition.** *For every prime ideal  $P$  in  $B[X]$  containing  $I_r(X)$*

$$\text{depth}(\Omega_{R/B}^1)_P \geq \text{depth } R_P - \text{grade}(I_r(X), R) + 2.$$

Consequently

$$\text{grade}(I, \Omega_{R/B}^1) \geq \text{grade}(I, R) - \text{grade}(I_r(X), R) + 2$$

for all ideals  $I$  in  $B[X]$ ,  $I \supset I_r(X)$ .

PROOF: Once more we consider the first syzygy  $M$  of  $\Omega_{R/B}^1$  and its filtration  $\{M(s, t) : r \leq s < m, r \leq t < n\}$ . From (14.6) and the depth analogue to Lemma (13.1) it follows that

$$\begin{aligned} \text{depth } M_P &\geq \text{depth } R_P - (m - 1 - r) - (n - 1 - r) \\ &= \text{depth } R_P - (m + n - 2r + 1) + 3 \\ &= \text{depth } R_P - \text{grade}(I_r(X), R) + 3. \end{aligned}$$

This proves the proposition. —

**(14.8) Remarks.** (a) The module of relations of  $M$  is generated by the linear relations (1) in the proof of (14.5), in other words: Let  $F = R^m$ ,  $G = R^n$ . Then the sequence

$$\left(\bigwedge^{r+2} F \otimes G^* \otimes \bigwedge^{r+1} G^*\right) \oplus \left(\bigwedge^{r+1} F \otimes F \otimes \bigwedge^{r+2} G^*\right) \xrightarrow{\varphi_1} \bigwedge^{r+1} F \otimes \bigwedge^{r+1} G^* \xrightarrow{\varphi_0} F \otimes G^*,$$

where  $\varphi_0 = \varphi_{x,r}$ ,  $\varphi_1 = (\varphi_{x,1} \otimes 1) \oplus (1 \otimes \varphi_{x,1})$  (cf. 13.B) is exact. To demonstrate this, one has only to look into the proof of (14.5). Another way to obtain exactness is as follows: The sequence is easily seen to be a complex which is (split) exact in depth 0. Furthermore one may treat  $\text{Coker } \varphi_1$  in the same manner as the module  $M$  (cf. (14.5), (14.6) and the proof of (14.7)), to get that

$$\text{depth}(\text{Coker } \varphi_1)_P \geq \min(3, \text{depth } R_P)$$

for all prime ideals  $P \in \text{Spec } R$ . So  $\text{Coker } \varphi_1$  is torsionfree and thus  $\text{Im } \varphi_1 = \text{Ker } \varphi_0$ . —

(b) The module  $M (= I_{r+1}(X)/I_{r+1}(X)^{(2)})$  is a direct  $B$ -summand of the symbolic graded ring

$$\text{Gr}_{I_{r+1}(X)}^{(\cdot)} B[X] = \bigoplus I_{r+1}(X)^{(i)}/I_{r+1}(X)^{(i+1)},$$

and inherits a standard basis from this ASL in a natural way (cf. (10.6) where this has been defined for arbitrary  $B$ ). The filtration considered above is compatible with the standard basis: each submodule  $M(s, t)$  is generated as a  $B$ -module by the elements of the standard basis it contains. The rank argument in the proof of (14.5) can be replaced by a comparison of standard bases.

Of course each of the quotients  $I_{r+1}(X)^{(i)}/I_{r+1}(X)^{(i+1)}$  inherits a standard basis from the symbolic graded ring, and it should be possible to construct similar filtrations for them. These filtrations may yield lower bounds for the depth of  $I_{r+1}(X)^{(i)}/I_{r+1}(X)^{(i+1)}$  as indicated below (10.8). —

### C. The Syzygetic Behaviour of the Differential Module

The inequalities of (14.7) actually are equalities. This, of course, determines the syzygetic behaviour of  $\Omega_{R/B}^1$ . On the other hand we do not use the full truth about  $\text{depth}(\Omega_{R/B}^1)_P$ ,  $P$  a prime ideal in  $B[X]$ , to describe the syzygetic behaviour of  $\Omega_{R/B}^1$ . Besides (14.7) we only need:

**(14.9) Proposition.** *For every minimal prime ideal  $P$  of  $I_r(X)$*

$$\text{depth}(\Omega_{R/B}^1)_P = 2.$$

PROOF: We use induction on  $r$ . Let  $r = 1$ . A simple localization argument shows that we may assume  $P$  to be the only prime ideal in  $B[X]$  containing  $I_1(X)$ . From (14.7) we get

$$\text{depth}(\Omega_{R/B}^1)_P \geq 2,$$

and by (14.4)  $(\Omega_{R/B}^1)_Q$  is free for all prime ideals  $Q$  in  $B[X]$  which are different from  $P$ . If  $\text{depth}(\Omega_{R/B}^1)_P \geq 3$  then every  $R_P$ -sequence consisting of three elements would be an

$(\Omega^1_{R/B})_P$ -sequence. According to (5.11),  $X_{11}, X_{12} + X_{21}, X_{mn}$  form an  $R_P$ -sequence. To show that it is not an  $(\Omega^1_{R/B})_P$ -sequence we put  $\omega = X_{12}\bar{d}X_{11}$ . Then

$$\begin{aligned} X_{mn}\omega &= X_{11}(X_{21}\bar{d}X_{mn} - X_{2n}\bar{d}X_{m1} + X_{m2}\bar{d}X_{1n}) \\ &\quad + (X_{12} + X_{21})(-X_{m1}\bar{d}X_{1n} + X_{mn}\bar{d}X_{11}) \\ &\quad - X_{21}\bar{d}([1 \ m|1 \ n]), \end{aligned}$$

but  $\omega \notin X_{11}(R^m \otimes (R^n)^*) + (X_{12} + X_{21})(R^m \otimes (R^n)^*) + M$ . Thus  $\text{depth}(\Omega^1_{R/B})_P = 2$ .

Assume now that  $r > 1$ . Let  $x_{mn}$  denote the residue class of  $X_{mn}$  in  $R$ . By (2.4) we have an isomorphism

$$R[x_{mn}^{-1}] \cong R_r(Y)[X_{m1}, \dots, X_{mn}, X_{1n}, \dots, X_{m-1,n}][X_{mn}^{-1}],$$

$Y$  being an  $(m-1) \times (n-1)$  matrix of indeterminates over  $B$ , which maps the extension of  $I_r(X)$  to the extension of  $I_{r-1}(Y)$ . Put  $S = R_r(Y)$ ,  $Q = PR[x_{mn}^{-1}] \cap S$ . Since

$$\Omega^1_{R/B} \otimes_R R[x_{mn}^{-1}] \cong \Omega_{R[x_{mn}^{-1}]/B} \cong (\Omega_{S/B} \otimes_S R[x_{mn}^{-1}]) \oplus F,$$

with a free  $R[x_{mn}^{-1}]$ -module  $F$ , and  $S \rightarrow R[x_{mn}^{-1}]$  is a flat extension, we obtain

$$\text{depth}(\Omega^1_{R/B})_P = \text{depth}(\Omega^1_{S/B})_Q + \text{depth } R_P - \text{depth } S_Q.$$

Clearly  $Q$  is a minimal prime ideal of  $I_{r-1}(Y)$ , so

$$\text{depth } S_Q = \text{grade } I_{r-1}(Y) = \text{grade } I_r(X) = \text{depth } R_P.$$

Using the inductive hypothesis we get the required result. —

Now it is easy to prove

**(14.10) Theorem.**  $\Omega^1_{R/B}$  is a second syzygy but not a third one.

PROOF: If  $P$  is a prime ideal in  $B[X]$ ,  $P \supset I_r(X)$ , then  $\text{depth } R_P \geq \text{grade}(I_r(X), R)$ , so  $\text{depth}(\Omega^1_{R/B})_P \geq 2$  in view of (14.7). In all other cases  $(\Omega^1_{R/B})_P$  is  $R_P$ -free. Consequently  $\Omega^1_{R/B}$  is a second syzygy and, if it were a third one, then  $\text{depth}(\Omega^1_{R/B})_P \geq 3$  for all prime ideals  $P \supset I_r(X)$ , which obviously contradicts (14.9). —

**(14.11) Remarks.** (a) To prove the inequality “ $\geq$ ” in (14.9), we only need the lower bound for  $\text{grade}(I_1(X), \Omega^1_{R/B})$  coming from Proposition (10.8) (cf. the introduction). This lower bound also suffices (combined with the usual localization argument) in showing that  $\Omega^1_{R/B}$  is a second syzygy (cf. the proof of (14.10)). —

(b) In the next section we shall give an explicit presentation of  $\Omega^1_{R/B}$  as a second syzygy. —

(14.9) will also be used to prove the main result of this section:

**(14.12) Theorem.** *Let  $P$  be a prime ideal in  $B[X]$  which contains  $I_r(X)$ . Then*

$$\text{depth}(\Omega_{R/B}^1)_P = \text{depth } R_P - \text{grade}(I_r(X), R) + 2.$$

*Consequently*

$$\text{grade}(I, \Omega_{R/B}^1) = \text{grade}(I, R) - \text{grade}(I_r(X), R) + 2$$

*for all ideals  $I$  in  $B[X]$ ,  $I \supset I_r(X)$ .*

PROOF: We only need to prove the first equality. Since  $\Omega_{R/B}^1$  is  $B$ -flat, the usual techniques (used for instance in the proof of (14.7)) allow us to restrict ourselves to the case in which  $B$  is a field. The inequality “ $\geq$ ” has been established in (14.7). Assume that

$$\text{depth}(\Omega_{R/B}^1)_P \geq \text{depth } R_P - \text{grade}(I_r(X), R) + 3$$

and take a  $(\text{grade}(I_r(X), R) - 3)$ -th syzygy  $N$  of  $\Omega_{R/B}^1$ . Then  $\text{depth } N_P = \text{depth } R_P$  and consequently

$$\text{depth}(\Omega_{R/B}^1)_Q \geq \text{depth } R_P - \text{grade}(I_r(X), R) + 3 \geq 3$$

for all prime ideals  $Q$  in  $B[X]$  satisfying  $P \supset Q \supset I_r(X)$ . This contradicts (14.9). —

## D. Comments and References

Special cases of Theorems (14.10), (14.12) have already been treated in [Ve.1] ( $r = 1 = m - 1$ ) and [Br.2] ( $r = 1$ ). As mentioned in 9.E, Theorem (3.5) in [AH] implies the grade formula of (14.12) for  $n = m + 1$  and  $I = I_1(X)$  (cf. (9.27),(a)). Our presentation of the general case follows [Ve.3] where the main results are covered by Theorem (3.4).

## 15. Derivations and Rigidity

With the notations of the previous section we continue the investigation of  $\Omega_{R/B}^1$ . More precisely we shall treat the  $R$ -dual of  $\Omega_{R/B}^1$  which is just the module of  $B$ -derivations from  $R$  to  $R$ . The main result will be that  $(\Omega_{R/B}^1)^*$  is an almost perfect  $B[X]$ -module which is perfect if and only if  $m \neq n$ .

For obvious reasons this result makes it possible to describe the syzygetic behaviour of  $(\Omega_{R/B}^1)^*$  as we did for the generic module in Section 13. As a consequence one obtains some results concerning the rigidity of  $R$  as a  $B$ -algebra in case  $B$  is a (perfect) field (cf. Subsection C).

To have a shorter notation we put  $\Omega = \Omega_{R/B}^1$ . For  $\delta \in \Delta(X)$ ,  $\delta \geq [1, \dots, r | 1, \dots, r]$ , we write

$$I(x; \delta) = I(X; \delta) / I_{r+1}(X),$$

and correspondingly

$$I_r(x) = I_r(X) / I_{r+1}(X)$$

etc. as introduced in Section 5.

### A. The Lower Bound for the Depth of the Module of Derivations

Let  $F, G$  be modules over an arbitrary ring  $A$ ,  $f: F \rightarrow G$  an  $A$ -homomorphism and  $r$  a nonnegative integer. In 13.B we have defined a homomorphism

$$\varphi = \varphi_{f,r}: \bigwedge^{r+1} F \otimes \bigwedge^{r+1} G^* \longrightarrow F \otimes G^*.$$

In case  $A = R$ ,  $F = R^m$ ,  $G = R^n$ ,  $f = x$ , the cokernel of  $\varphi$  represents the module  $\Omega$ . To investigate  $\Omega^*$  we shall define two more homomorphisms  $\chi, \psi$  in the general situation considered above, which are connected with  $\varphi$  in a sequence

$$(1) \quad F_0 \xrightarrow{\varphi} F_1 \xrightarrow{\chi} F_2 \xrightarrow{\psi} F_3$$

where

$$F_0 = \bigwedge^{r+1} F \otimes \bigwedge^{r+1} G^*,$$

$$F_1 = F \otimes G^*,$$

$$F_2 = [F \otimes F^*] \oplus [G \otimes G^*],$$

$$F_3 = A \oplus [G \otimes F^*] \oplus [F \otimes (\bigwedge^{r+1} F \otimes \bigwedge^r G^*)^*] \oplus [(\bigwedge^r F \otimes \bigwedge^{r+1} G^*)^* \otimes G^*].$$



Let

$$\begin{aligned}\chi(y \otimes z^*) &= y \otimes f^*(z^*) + f(y) \otimes z^*, \\ \psi(y \otimes y^* + z \otimes z^*) &= [y^*(y) - z^*(z)] + [f(y) \otimes y^* - z \otimes f^*(z^*)] \\ &\quad + [y \otimes \varphi_{f,r}^*(y^*)] + [\varphi_{f,r}^*(z) \otimes z^*],\end{aligned}$$

for all  $y \in F$ ,  $y^* \in F^*$ ,  $z \in G$ ,  $z^* \in G^*$ ,  $\varphi_{f,r}$  being as in 13.B and  $z$  viewed an element of  $G$  as well as an element of  $G^{**}$  via the canonical map  $G \rightarrow G^{**}$ . (We adopt this convention for analogous situations.)

- (15.1) Proposition.** (a) *If  $\text{rk } f \leq r$ , then (1) is a complex.*  
 (b) *If  $\text{Im } f$  is a free direct summand of  $G$  and  $\text{rk } f = r$ , then (1) is split exact.*

PROOF: (a) Let  $y_I \in \bigwedge^{r+1} F$ ,  $z_J^* \in \bigwedge^{r+1} G^*$ . Then

$$\begin{aligned}\chi \circ \varphi(y_I \otimes z_J^*) &= \sum_{\substack{U \in \mathcal{S}(r,I) \\ V \in \mathcal{S}(r,J)}} \sigma(U, I \setminus U) \sigma(V, J \setminus V) z_V^* \left( \bigwedge^r f(y_U) \right) [y_{I \setminus U} \otimes f^*(z_{J \setminus V}^*) + f(y_{I \setminus U}) \otimes z_{J \setminus V}^*] \\ &= \sum_{U \in \mathcal{S}(r,I)} \sigma(U, I \setminus U) y_{I \setminus U} \otimes \left[ \sum_{V \in \mathcal{S}(r,J)} \sigma(V, J \setminus V) y_U \left( \bigwedge^r f^*(z_V^*) \right) f^*(z_{J \setminus V}^*) \right] \\ &\quad + \sum_{V \in \mathcal{S}(r,J)} \sigma(V, J \setminus V) \left[ \sum_{U \in \mathcal{S}(r,I)} \sigma(U, I \setminus U) z_V^* \left( \bigwedge^r f(y_U) \right) f(y_{I \setminus U}) \right] \otimes z_{J \setminus V}^* \\ &= \sum_{U \in \mathcal{S}(r,I)} \sigma(U, I \setminus U) [y_{I \setminus U} \otimes f^* \circ \varphi_{f^*,r}(z_J^* \otimes y_U)] \\ &\quad + \sum_{V \in \mathcal{S}(r,J)} \sigma(V, J \setminus V) [f \circ \varphi_{f,r}(y_I \otimes z_V^*) \otimes z_{J \setminus V}^*] \\ &= 0\end{aligned}$$

(cf. 13.B). To prove  $\psi \circ \chi = 0$  we take  $y \in F$ ,  $z^* \in G^*$ . Then

$$\begin{aligned}\psi \circ \chi(y \otimes z^*) &= \psi(y \otimes f^*(z^*) + f(y) \otimes z^*) \\ &= [f^*(z^*)(y) - z^*(f(y))] + [f(y) \otimes f^*(z^*) - f(y) \otimes f^*(z^*)] \\ &\quad + [y \otimes \varphi_{f,r}^* \circ f^*(z^*)] + [\varphi_{f,r}^* \circ f(y) \otimes z^*] \\ &= 0.\end{aligned}$$

(b) The assumption on  $f$  guarantees that  $G = \text{Im } f \oplus C$ ,  $C$  being a submodule of  $G$  isomorphic with  $\text{Coker } f$ , and that there are elements  $y_1, \dots, y_r \in F$  whose images under  $f$  form a basis of  $\text{Im } f$ . As one easily checks,

$$\begin{aligned}\text{Ker } \chi &= \text{Ker } f \otimes C^*, \\ \text{Ker } \psi &= (\text{Ker } f \otimes \text{Im } f^*) \oplus (\text{Im } f \otimes C^*) \oplus \sum_{i,j} A(y_i \otimes y_j^* + f(y_i) \otimes f(y_j)^*),\end{aligned}$$

and the modules on the right hand side of these equations are direct summands of  $F_1$  and  $F_2$  resp. Now let  $y \in \text{Ker } f, z^* \in C^*$ . Then

$$\varphi(y_1 \wedge \cdots \wedge y_r \wedge y \otimes f(y_1)^* \wedge \cdots \wedge f(y_r)^* \wedge z^*) = y \otimes z^*,$$

so  $\text{Im } \varphi = \text{Ker } \chi$ , and

$$\begin{aligned} \chi(y \otimes f(y_i)^*) &= y \otimes y_i^*, \\ \chi(y_i \otimes z^*) &= f(y_i) \otimes z^*, \\ \chi(y_i \otimes f(y_j)^*) &= y_i \otimes y_j^* + f(y_i) \otimes f(y_j)^*, \end{aligned}$$

$1 \leq i, j \leq r$ , so  $\text{Im } \chi = \text{Ker } \psi$ . —

**(15.2) Corollary.** *In case  $A = R, F = R^m, G = R^n$  and  $f = x$ , the sequence (1) is exact. Consequently it yields an explicit presentation of  $\Omega = \text{Coker } \varphi$  as a second syzygy.*

PROOF: The assumption of (15.1),(b) is fulfilled if we localize at a prime ideal  $P \not\supseteq I_r(x)$ . In particular  $(\text{Coker } \varphi)_P$  and  $(\text{Coker } \chi)_P$  are  $R_P$ -free, and (1) is split-exact in depth 1. From (14.10) we know that  $\text{Coker } \varphi = \Omega$  is a second syzygy, so  $\text{Coker } \chi$  is torsionfree. But then  $\text{Im } \varphi = \text{Ker } \chi, \text{Im } \chi = \text{Ker } \psi$ . —

As in the corollary let  $A = R, F = R^m, G = R^n$  and  $f = x$ . Then  $\text{Ker } \varphi^* = \Omega^*$ . We will now prove that  $\text{pd}_{B[X]} \text{Coker } \chi^* \leq \text{grade } R + 2$ . This will imply that  $\Omega^*$  is almost perfect. The method of proof is very similar to the one used in Section 14: We shall consider a filtration of  $\text{Coker } \chi^*$ , the quotients of which are direct sums of well-known ideals generated by poset ideals.

Let  $y_1, \dots, y_m$  and  $z_1, \dots, z_n$  be the canonical bases of  $R^m$  and  $R^n$ . Put  $N = \text{Coker } \chi^*$  and abbreviate  $w_{ij} = y_i^* \otimes z_j \pmod{\text{Im } \chi^*}, i \in S(1, m), j \in S(1, n)$ . Let

$$N_{kl} = \sum_{i>k, j>l} R w_{ij} \quad k, l \geq 0.$$

Obviously  $N_{rr} \subset N_{0r} \subset N_{0r} + N_{r0} \subset N$ . Using notations as at the beginning of 14.A we can state the following

- (15.3) Proposition.** (a)  $N_{rr}$  is a free submodule of  $N$ .  
 (b)  $\text{Ann}_R N_{0r}/N_{rr} = I(x; \delta_{r+1, r})$ . As an  $R(X; \delta_{r+1, r})$ -module  $N_{0r}/N_{rr}$  is isomorphic to the  $(n - r)$ -fold direct sum of  $I(x; \delta_{r+2, r}, \delta_{r+1, r+1})$ .  
 (c)  $\text{Ann}_R(N_{0r} + N_{r0})/N_{0r} = I(x; \delta_{r, r+1})$ . As an  $R(X; \delta_{r, r+1})$ -module  $(N_{0r} + N_{r0})/N_{0r}$  is isomorphic to the  $(m - r)$ -fold direct sum of  $I(x; \delta_{r+1, r+1}, \delta_{r, r+2})$ .  
 (d)  $\text{Ann}_R N/(N_{0r} + N_{r0}) = I(x; \delta_{r+1, r+1})$ . As an  $R(X; \delta_{r+1, r+1})$ -module  $N/(N_{0r} + N_{r0})$  is isomorphic to  $I(x; \delta_{r+2, r}, \delta_{r, r+2})$ .

PROOF: (a) Assume that  $\sum_{i>r, j>r} a_{ij} w_{ij} = 0$ . Then in particular

$$\sum_{i>r, j>r} a_{ij} ((y_i^* \otimes z_j) \circ \varphi)(y_{\{1, \dots, r, k\}} \otimes z_{\{1, \dots, r, l\}}^*) = 0, \quad r < k \leq m, r < l \leq n,$$

whence  $a_{kl}[1, \dots, r|1, \dots, r] = 0$ , and consequently  $a_{kl} = 0$ .

(b) Observe that

$$(2) \quad \begin{aligned} \chi^*(y_k^* \otimes y_i) &= \sum_{v=1}^n [i|v] y_k^* \otimes z_v, & i, k \in S(1, m), \\ \chi^*(z_j^* \otimes z_l) &= \sum_{u=1}^m [u|j] y_u^* \otimes z_l, & j, l \in S(1, n). \end{aligned}$$

Therefore  $\sum_{i=1}^r [i|j] w_{il} \in N_{rr}$  for all  $j \in S(1, n)$ ,  $r < l \leq n$ , and consequently

$$[1, \dots, r|J] w_{il} \in N_{rr}, \quad 1 \leq i \leq r < l \leq n, \quad J \in S(r, n).$$

This proves  $I(x; \delta_{r+1, r}) \subset \text{Ann}_R N_{0r}/N_{rr}$ . To show the rest of the assertion we proceed like in the proof of (14.5). Let  $\bar{R} = R(X; \delta_{r+1, r})$  and consider the homomorphisms

$$g_j: (\bar{R}^r)^* \longrightarrow N_{0r}/N_{rr}, \quad g_j(\bar{y}_i^*) = w_{ij} \pmod{N_{rr}}, \quad r < j \leq n, \quad 1 \leq i \leq r$$

( $\bar{y}_1, \dots, \bar{y}_r$  being the canonical basis of  $\bar{R}^r$ ), and

$$\bar{\varphi}: \bigwedge_{i=1}^{r-1} \bar{R}^i \longrightarrow \bar{R}, \quad \bar{\varphi}(\bar{y}_{\{1, \dots, r\} \setminus i}) = [1, \dots, \widehat{i}, \dots, r+1|1, \dots, r], \quad 1 \leq i \leq r.$$

By (14.2)  $\text{Ker } \bar{\varphi}$  is generated by the elements  $\sum_{i=1}^r (-1)^{i+1} [i|k] \bar{y}_{\{1, \dots, r\} \setminus i}$ ,  $k \in S(1, n)$ . Since  $\sum_{i=1}^r [i|k] \bar{y}_i^* \in \text{Ker } g_j$ ,  $r < j \leq n$ ,  $\bar{\varphi}$  induces a surjective map

$$\bigoplus_1^{n-r} I(x; \delta_{r+2, r}, \delta_{r+1, r+1}) \longrightarrow N_{0r}/N_{rr},$$

which is injective, too, since for example the residue classes of  $w_{r, r+1}, \dots, w_{rn}$  modulo  $N_{rr}$  are linearly independent over  $R(X; \delta_{r+1, r})$  as is readily checked: Assume that  $\sum_{j=r+1}^n b_j w_{rj} \in N_{rr}$ ,  $b_j \in R$ , and apply  $\sum_{j=r+1}^n b_j (y_r^* \otimes z_j) \circ \varphi$  to  $y_{\{1, \dots, r+1\}} \otimes z_{\{1, \dots, r, k\}}^*$ ,  $r < k \leq n$ .

(c) Analogously with (b) we obtain that  $\text{Ann}_R N_{r0}/N_{rr} = I(x; \delta_{r, r+1})$  and that  $N_{r0}/N_{rr}$  is isomorphic to the  $(m-r)$ -fold direct sum of  $I(x; \delta_{r+1, r+1}, \delta_{r, r+2})$  as an  $R(X; \delta_{r, r+1})$ -module. Since  $(N_{0r} + N_{r0})/N_{0r} \cong N_{r0}/(N_{0r} \cap N_{r0})$  and  $N_{rr} \subset N_{0r} \cap N_{r0}$ , we get a surjection  $N_{r0}/N_{rr} \longrightarrow (N_{0r} + N_{r0})/N_{0r}$ . The residue classes of  $w_{r+1, r}, \dots, w_{mr}$  in  $(N_{0r} + N_{r0})/N_{0r}$  being linearly independent over  $R(X; \delta_{r, r+1})$ , this map must be injective, too.

(d) The inclusion  $I(x; \delta_{r+1, r+1}) \subset \text{Ann}_R N/(N_{0r} + N_{r0})$  is again an easy consequence of (2). Next we put  $\bar{R} = R(X; \delta_{r+1, r+1})$  and consider the homomorphisms

$$\begin{aligned} g: (\bar{R}^r)^* \otimes \bar{R}^r &\longrightarrow N/(N_{0r} + N_{r0}), \\ g(\bar{y}_i^* \otimes \bar{y}_j) &= w_{ij} \pmod{N_{0r} + N_{r0}}, \quad i, j \in S(1, r) \end{aligned}$$

$(\bar{y}_1, \dots, \bar{y}_r$  being the canonical basis of  $\bar{R}^r$ ), and

$$\bar{\varphi}: \bigwedge^{r-1} \bar{R}^r \otimes \bigwedge^{r-1} (\bar{R}^r)^* \longrightarrow \bar{R},$$

$$\bar{\varphi}(\bar{y}_{\{1, \dots, r\} \setminus i} \otimes \bar{y}_{\{1, \dots, r\} \setminus j}^*) = [1, \dots, \hat{i}, \dots, r+1 | 1, \dots, \hat{j}, \dots, r+1], \quad i, j \in S(1, r).$$

The kernel of  $\bar{\varphi}$  is generated by the elements  $\sum_{i=1}^r (-1)^{i+1} [i|k] \bar{y}_{\{1, \dots, r\} \setminus i} \otimes \bar{y}_{\{1, \dots, r\} \setminus l}^*$  and  $\sum_{j=1}^r (-1)^{j+1} [u|j] \bar{y}_{\{1, \dots, r\} \setminus v} \otimes \bar{y}_{\{1, \dots, r\} \setminus j}^*$ ,  $k \in S(1, n)$ ,  $u \in S(1, m)$ ,  $l, v \in S(1, r)$  (cf. (14.2)). Since  $\sum_{i=1}^r [i|k] \bar{y}_i^* \otimes \bar{y}_l$  and  $\sum_{j=1}^r [u|j] \bar{y}_v^* \otimes \bar{y}_j$  are elements of  $\text{Ker } g$ , we get a surjection

$$I(x; \delta_{r+2, r}, \delta_{r, r+2}) \longrightarrow N/(N_{0r} + N_{r0}).$$

Obviously the residue class of  $w_{rr}$  generates an  $\bar{R}$ -free submodule in  $N/(N_{0r} + N_{r0})$ , whence the map must be bijective. —

**(15.4) Proposition.** *Choose notations as at the beginning of the section. Then  $\Omega^*$  is an almost perfect  $B[X]$ -module.*

PROOF: We consider the filtration

$$N_{rr} \subset N_{0r} \subset N_{0r} + N_{r0} \subset N$$

preceding (15.3). Put  $g = \text{grade } R$ . Then  $\text{pd } N_{rr} = g$  since this module is  $R$ -free and  $R$  is perfect. By (5.18) and (14.1) the rings  $R(X; \delta_{st})$ ,  $B[X]/I(X; \delta_{s+1, t}, \delta_{s, t+1})$ ,  $r \leq s \leq m$ ,  $r \leq t \leq n$ , are perfect, too, of grades  $g + (s-r) + (t-r)$  and  $g + (s-r) + (t-r) + 1$  resp., so

$$\text{pd } N_{0r}/N_{rr} = \text{pd}(N_{0r} + N_{r0})/N_{0r} = g + 1,$$

$$\text{pd } N/(N_{0r} + N_{r0}) = g + 2$$

by (15.3), (b)–(d). Consequently  $\text{pd } N \leq g + 2$  and therefore

$$\text{pd } \text{Im } \chi^* \leq g + 1$$

since  $\text{Im } \chi^*$  is a first  $R$ -syzygy of  $N$ . This shows that  $\text{Im } \chi^*$  is almost perfect.

Consider the inclusion map  $\text{Im } \chi^* \longrightarrow \text{Ker } \varphi^* = \Omega^*$  which is bijective if localized at a prime ideal not containing  $I_r(x)$  (cf. (15.1), (b)). Since  $\text{Im } \chi^*$  is almost perfect, we obtain

$$\text{depth}(\text{Im } \chi^*)_P \geq \text{depth } R_P - 1$$

for all prime ideals  $P$ . Consequently  $\text{Im } \chi^*$  is reflexive because  $(\text{Im } \chi^*)_P$  is free when  $P \not\supseteq I_r(x)$  (cf. (16.33)).  $\text{Ker } \varphi^*$  being torsionfree, we thus get  $\text{Im } \chi^* = \text{Ker } \varphi^*$ . —

**(15.5) Remarks.** (a) It has just been demonstrated that  $\text{Im } \chi^* = \text{Ker } \varphi^*$  in case  $f = x$ . Actually in this case the dual to (1) is exact everywhere as we shall see below.

(b) The system (2) of generators of  $\text{Im } \chi^* = \Omega^*$  is not minimal since the element  $\sum_{u=1}^m y_u^* \otimes y_u - \sum_{v=1}^n z_v^* \otimes z_v$  lies in  $\text{Ker } \chi^*$ . This is nothing but the fact that the Euler-derivation  $\sum_{i,j} [i|j] y_i^* \otimes z_j$  can be written as  $\sum_{u=1}^m \chi^*(y_u^* \otimes y_u)$  and as  $\sum_{v=1}^n \chi^*(z_v^* \otimes z_v)$ . On the other hand arbitrary  $m^2 + n^2 - 1$  elements of (2) form a minimal system of generators for  $\Omega^*$ . —

### B. The Perfection of the Module of Derivations

(15.4) leaves open when  $\Omega^*$  is perfect. To answer this question we need the first syzygy of  $\Omega^*$  and some technical information about intersections of certain determinantal ideals.

**(15.6) Proposition.** *In case  $f = x$ , the dual to (1) is exact.*

PROOF: In view of (15.5),(a) it remains only to show that  $\text{Im } \psi^* = \text{Ker } \chi^*$ . Since the dual to (1) is a zero-sequence which is split-exact at all prime ideals not containing  $I_r(x)$ , it will be enough that  $\text{grade}(I_r(x), \text{Coker } \psi^*) \geq 1$ . For  $\text{Coker } \psi^*$  is torsionfree, then, and consequently the canonical map  $\text{Coker } \psi^* \rightarrow \text{Im } \chi^*$  is bijective.

Put  $C = \text{Coker } \psi^*$  and let  $y_1, \dots, y_m$  and  $z_1, \dots, z_n$  the canonical bases of  $R^m$  and  $R^n$ . We intend to apply (13.1). The filtration of  $C$  needed for (13.1), is obtained as follows: Let

$$(i, j) \prec (k, l) \iff i < k \text{ or } i = k, j > l,$$

and put

$$C_{kl} = \sum_{(i,j) \succeq (k,l)} R y_i^* \otimes y_j + \sum_{u,v} R z_u^* \otimes z_v \pmod{\text{Im } \psi^*}.$$

Let  $(k, l) \in S(1, r) \times S(1, m)$ . We claim:

- (i) The module  $C_{kl}/C_{k,l-1}$  is annihilated by  $I(x; [1, \dots, \widehat{k}, \dots, r, l | 1, \dots, r])$ ,  $m \geq l > r$ , and is free as an  $R(X; [1, \dots, \widehat{k}, \dots, r, l | 1, \dots, r])$ -module;
- (ii) the modules  $C_{kr}/C_{kk}$  and, if  $k > 1$ ,  $C_{k,k-1}/C_{k+1,m}$  are annihilated by the ideal  $I(x; [1, \dots, \widehat{k}, \dots, r+1 | 1, \dots, r])$ , and are free as  $R(X; [1, \dots, \widehat{k}, \dots, r+1 | 1, \dots, r])$ -modules;
- (iii)  $C_{11} = C_{2m}$ , and if  $k > 1$ , then the module  $C_{kk}/C_{k,k-1}$  is annihilated by the ideal  $I(x; [1, \dots, \widehat{k-1}, \dots, r+1 | 1, \dots, r])$  and is a free  $R(X; [1, \dots, \widehat{k-1}, \dots, r+1 | 1, \dots, r])$ -module;
- (iv)  $C_{r+1,m} \cong (\text{Im } x)^{m-r} \bigoplus (\text{Im } x^*)^n$ .

Since  $[m-r+1, \dots, m | 1, \dots, r]$  is not a zero-divisor modulo any of the ideals occurring in (i)–(iii), the claim and (13.1) imply immediately that  $\text{grade}(I_r(x), C) \geq 1$ .

To prove the annihilator assertions of (i)–(iii), we observe that  $\text{Im } \psi^*$  contains the following elements (cf. the definition of  $\psi$  given in Subsection A):

$$(3) \quad \psi^*(1) = \sum_{u=1}^m y_u^* \otimes y_u - \sum_{v=1}^n z_v^* \otimes z_v,$$

$$(4) \quad \psi^*(z_i^* \otimes y_j) = \sum_{u=1}^m [u|i] y_u^* \otimes y_j - \sum_{v=1}^n [j|v] z_i^* \otimes z_v,$$

$$i \in S(1, n), \quad j \in S(1, m),$$

$$(5) \quad \psi^*(y_i^* \otimes y_I \otimes z_J^*) = \sum_{u \in I} \sigma(u, I \setminus u) [I \setminus u | J] y_u^* \otimes y_u,$$

$$i \in S(1, m), \quad I \in S(r+1, m), \quad J \in S(r, n).$$

We abbreviate  $y_{ij} = y_i^* \otimes y_j \pmod{\text{Im } \psi^*}$ . From (4) one gets

$$\sum_{u=1}^k [u|i] y_{ul} \in C_{k+1,m}, \quad l \in S(1, m), \quad i \in S(1, n),$$

so

$$[1, \dots, k|V] y_{kl} \in C_{k+1,m}, \quad l \in S(1, m), \quad V \in S(k, n),$$

(by elementary determinantal calculations), and from (5) we obtain

$$[I|l|J] y_{kl} \in C_{k,l-1}, \quad r < l \in I \in S(r+1, l), \quad J \in S(r, n).$$

This proves the first half of (i) and (ii). Clearly (3) implies  $C_{11} \subset C_{2m}$ . Assume that  $k > 1$ . From (3) and (4) it follows that

$$\begin{aligned} \sum_{u=1}^k y_{uu} &\in C_{k+1,m}, \\ \sum_{u=1}^{k-1} [u|i] y_{u,j-1} &\in C_{k,k-1}, \quad i \in S(1, n), \quad j \in S(1, k-1). \end{aligned}$$

The inclusions of the last line yield

$$[1, \dots, k-1|V] y_{jj} \in C_{k,k-1}, \quad j \in S(1, k-1), \quad V \in S(k-1, n),$$

so  $[1, \dots, k-1|V] y_{kk}$  is an element of  $C_{k,k-1}$  for all  $V \in S(k-1, n)$ .

Now we turn to the second part of (i)–(iii). Since  $\text{Im } \psi^* \subset \text{Ker } \chi^*$ , a relation

$$\sum_{i,j} a_{ij} y_i^* \otimes y_j + \sum_{u,v} b_{uv} z_u^* \otimes z_v \in \text{Im } \psi^*$$

implies

$$(6) \quad \sum_{j=1}^m a_{ij} [j|v] + \sum_{u=1}^n b_{uv} [i|u] = 0, \quad i \in S(1, m), \quad v \in S(1, n)$$

(cf. (2)). To prove the second assertion of (i)–(iii), resp., we therefore shall deduce that a system of equations

$$\begin{aligned} \sum_{u=1}^n b_{uv} [i|u] &= 0, \quad 1 \leq i < k, \quad v \in S(1, n), \\ \sum_{j=1}^l a_{kj} [j|v] + \sum_{u=1}^n b_{uv} [k|u] &= 0, \quad v \in S(1, n), \end{aligned}$$

yields

$$(7) \quad a_{kl} \in I(x; \gamma)$$

where

$$\gamma = \begin{cases} [1, \dots, \widehat{k}, \dots, r, l | 1, \dots, r] & \text{if } l > r, \\ [1, \dots, \widehat{k}, \dots, r+1 | 1, \dots, r] & \text{if } l < k \text{ or } k < l \leq r, \\ [1, \dots, \widehat{k-1}, \dots, r+1 | 1, \dots, r] & \text{if } 1 < l = k. \end{cases}$$

In any case, and provided  $v \in S(1, n)$  has been fixed, multiplying the  $i$ -th equation of the system by  $(-1)^i [1, \dots, \widehat{i}, \dots, k | 1, \dots, k-1]$ ,  $i = 1, \dots, k$ , and summing up leads to

$$[1, \dots, k-1 | 1, \dots, k-1] \sum_{j=1}^l a_{kj} [j | v] \in I(x; [1, \dots, \widehat{k}, \dots, r+1 | 1, \dots, r]),$$

whence

$$\sum_{j=1}^l a_{kj} [j | v] \in I(x; [1, \dots, \widehat{k}, \dots, r+1 | 1, \dots, r]), \quad v \in S(1, n).$$

By the usual determinantal calculations we finally obtain that

$$\delta a_{kl} \in I(x; \gamma),$$

where

$$\delta = \begin{cases} \gamma & \text{if } l > r, \\ [1, \dots, \widehat{k}, \dots, l+1 | 1, \dots, l] & \text{if } k < l \leq r, \\ [1, \dots, l | 1, \dots, l] & \text{if } l < k, \\ [2, \dots, k | 1, \dots, k-1] & \text{if } 1 < l = k. \end{cases}$$

This implies (7).

As to assertion (iv), we consider the surjection

$$[(R^{m-r})^* \otimes R^m] \oplus [(R^n)^* \otimes R^n] \xrightarrow{\pi} C_{r+1, m},$$

which maps  $y_i^* \otimes y_j, z_u^* \otimes z_v$  to their images mod  $\text{Im } \psi^*$ ,  $i, j \in S(1, m)$ ,  $i > r$ ,  $u, v \in S(1, n)$ . So

$$\sum_{\substack{i, j \\ i > r}} a_{ij} y_i^* \otimes y_j + \sum_{u, v} b_{uv} z_u^* \otimes z_v \in \text{Ker } \pi$$

implies

$$\begin{aligned} \sum_{u=1}^n b_{uv} [i | u] &= 0, & i \in S(1, r), v \in S(1, n), \\ \sum_{j=1}^m a_{ij} [j | v] + \sum_{u=1}^n b_{uv} [i | u] &= 0, & r < i \leq m, v \in S(1, n) \end{aligned}$$

(cf. (6)). Since the first  $r$  rows of  $x$  are linearly independent and  $\text{rk } x = r$ , this system of equations is equivalent to

$$\sum_{u=1}^n b_{uv} z_u^* \in \text{Ker } x^*, \quad v \in \text{S}(1, n),$$

$$\sum_{j=1}^m a_{ij} y_j \in \text{Ker } x, \quad r < i \leq m.$$

Thus

$$\sum_{\substack{i,j \\ i>r}} a_{ij} y_i^* \otimes y_j \in \text{Ker}(1 \otimes x), \quad \sum_{u,v} b_{uv} z_u^* \otimes z_v \in \text{Ker}(x^* \otimes 1).$$

Conversely this implies, that

$$\sum_{\substack{i,j \\ i>r}} a_{ij} y_i^* \otimes y_j, \quad \sum_{u,v} b_{uv} z_u^* \otimes z_v \in \text{Im } \psi^*$$

since  $\text{Ker}(1 \otimes x) = \text{Im}(1 \otimes \varphi_{x,r}) \subset \text{Im } \psi^*$ ,  $\text{Ker}(x^* \otimes 1) = \text{Im}(\varphi_{x,r} \otimes 1) \subset \text{Im } \psi^*$  (cf. (13.6) and the definition of  $\psi$  in Subsection A). —

**(15.7) Theorem.** *Choose notations as at the beginning of the section. Then  $\Omega^*$  is a perfect  $B[X]$ -module except for  $m = n$  in which case it is almost perfect.*

PROOF: We know already that  $\Omega^*$  is almost perfect in any case (cf. (15.4)). To prove that it is perfect except for  $m = n$  we reduce to the case  $B = \mathbf{Z}$  as usual: Let  $R_0 = \mathbf{Z}[X]/I_{r+1}(X)$ .  $(\Omega_{R_0/\mathbf{Z}}^1)^*$  is faithfully flat over  $\mathbf{Z}$ , so  $\Omega^* \cong (\Omega_{R_0/\mathbf{Z}}^1)^* \otimes_{\mathbf{Z}} B$  is a perfect  $B[X]$ -module if  $(\Omega_{R_0/\mathbf{Z}}^1)^*$  is a perfect  $\mathbf{Z}[X]$ -module. In case  $(\Omega_{R_0/\mathbf{Z}}^1)^*$  is not perfect, we repeat the argument used in the last paragraph of the proof of (13.8), to obtain that  $\Omega^*$  is not perfect, too.

Let  $B = \mathbf{Z}$ . In case  $m = n$ ,  $R$  is Gorenstein (cf. (8.9)). Put  $P = I_r(x)$ . Then  $\text{depth } \Omega_P = 2$  (cf. (14.9)). Thus  $\Omega_P$  is not a third syzygy. Consequently  $\text{Ext}_{R_P}^1(\Omega_P^*, R_P) \neq 0$ . It follows that  $\text{depth } \Omega_P^* < \text{depth } R_P$ , in particular that  $\Omega^*$  is not perfect in the case just treated. Assume that  $m \neq n$ , say  $m < n$ , and let  $\omega$  be the canonical module of  $R$ . We must show that  $\text{Ext}_R^1(\Omega^*, \omega) = 0$ . Consider the exact sequence

$$F_3^* \xrightarrow{\psi^*} F_2^* \xrightarrow{\chi^*} F_1^*$$

where  $\Omega^* \cong \text{Im } \chi^*$  (cf. (15.6)). Let  $D = \text{Im } \psi^*$ . Obviously  $\text{Ext}_R^1(\Omega^*, \omega) = 0$  is equivalent to the fact that the induced map

$$\text{Hom}_R(F_2^*, \omega) \longrightarrow \text{Hom}_R(D, \omega)$$

is surjective. Put  $s = n - m$ . As we know,  $\omega$  can be represented by  $Q^s$ ,  $Q$  being the ideal in  $R$  generated by the  $r$ -minors of arbitrary  $r$  columns of  $x$ . For technical reasons we assume  $Q$  to be generated by the  $r$ -minors  $[I|2, \dots, r+1]$ ,  $I \in \text{S}(r, m)$ . Let  $h \in \text{Hom}_R(D, Q^s)$ . We have to find a  $g \in \text{Hom}_R(F_2^*, Q^s)$  such that  $g|D = h$ .



Let  $D'$  be the  $\psi^*$ -image of

$$[F^* \otimes (\bigwedge^{r+1} F \otimes \bigwedge^r G^*)] \oplus [(\bigwedge^r F \otimes \bigwedge^{r+1} G^*) \otimes G],$$

the last two components of  $F_3^*$  (cf. Subsection A,  $F = R^m$ ,  $G = R^n$ , of course). Then  $F_2^*/D'$  is a direct sum of  $m$  copies of  $\text{Im } x$  and  $n$  copies of  $\text{Im } x^*$  (cf. (13.6)), so  $\text{Ext}_R^1(F_2^*/D', Q^s) = 0$  by (13.4). It follows that there exists a  $\tilde{g} \in \text{Hom}_R(F_2^*, Q^s)$  such that  $\tilde{g}|D' = h|D'$ . Thus we may assume that  $h|D' = 0$ .

Put  $h_{ij} = h \circ \psi^*(z_j^* \otimes y_i)$ ,  $i \in S(1, m)$ ,  $j \in S(1, n)$  ( $y_1, \dots, y_m$  and  $z_1, \dots, z_n$  being the canonical basis of  $R^m$  and  $R^n$ ). Since  $\psi^*(1)$  generates a free direct summand of  $F_2^*$ , it suffices to find a  $g \in \text{Hom}_R(F_2^*, Q^s)$  satisfying  $g \circ \psi^*(z_j^* \otimes y_i) = h_{ij}$  for all  $i \in S(1, m)$ ,  $j \in S(1, n)$ , and  $g|D' = 0$ . This will be done by the following assertion:

(8) Assume that  $(1, 1) \preceq (i, j) \preceq (m, n)$  ( $\preceq$  means lexicographically less or equal). Then there exists a  $g \in \text{Hom}_R(F_2^*, Q^s)$  such that

$$g \circ \psi^*(z_v^* \otimes y_u) = h_{uv}, \quad (u, v) \preceq (i, j), \\ g|D' = 0.$$

To prove (8) we first observe the relations

$$(9) \quad \sum_{u \in I} \sigma(u, I \setminus u)[I \setminus u | J] h_{uk} = 0, \quad I \in S(r+1, m), \quad J \in S(r, n), \quad k \in S(1, n),$$

$$(10) \quad \sum_{v \in J} \sigma(v, J \setminus v)[I | J \setminus v] h_{lv} = 0, \quad I \in S(r, m), \quad J \in S(r+1, n), \quad l \in S(1, m),$$

which come from  $h|D' = 0$  and from the obvious fact that

$$\sum_{u \in I} \sigma(u, I \setminus u)[I \setminus u | J] z_k^* \otimes y_u - \sum_{l=1}^m [l | k] y_l^* \otimes y_I \otimes z_j^*, \quad I, J, k \text{ as in (7)}, \\ \sum_{v \in J} \sigma(v, J \setminus v)[I | J \setminus v] z_v^* \otimes y_l + \sum_{k=1}^n [l | k] y_I \otimes z_k^* \otimes z_k, \quad I, J, l \text{ as in (8)},$$

are in  $\text{Ker } \psi^*$ . We may further suppose that  $h_{uv} = 0$  for all  $(u, v) \prec (i, j)$ . Let  $i > r$ . Substituting  $\{1, \dots, r, i\}$  for  $I$ ,  $\{1, \dots, r\}$  for  $J$  and  $j$  for  $k$  we obtain from (9) that  $[1, \dots, r | 1, \dots, r] h_{ij} = 0$ , so  $h_{ij} = 0$ . Using (10) we get analogously that  $h_{ij} = 0$  in case  $j > r$ .

One may therefore assume that  $i, j \leq r$ . From (9), with  $I = \{1, \dots, r+1\}$ ,  $J = \{1, \dots, r\}$ ,  $k = j$ , it follows that

$$h_{ij} \in \text{I}(x; [1, \dots, \hat{i}, \dots, r+1 | 1, \dots, r]),$$

and from (10), putting  $I = \{1, \dots, r\}$ ,  $J = \{1, \dots, r+1\}$ ,  $l = i$ , we derive

$$h_{ij} \in \text{I}(x; [1, \dots, r | 1, \dots, \hat{j}, \dots, r+1]),$$

so

$$h_{ij} \in I(x; [1, \dots, \widehat{i}, \dots, r+1 | 1, \dots, r]) \cap I(x; [1, \dots, r | 1, \dots, \widehat{j}, \dots, r+1]) \cap Q^s.$$

By Lemma (15.9),(b) below

$$h_{ij} = \sum_{I \in S(j,m)} a_I [I | 1, \dots, j],$$

where

$$a_I = \sum_{\substack{J \in S(r,m) \\ \{1, \dots, i\} \subset J}} a_{I;J} [J | H],$$

$a_{I;J} \in Q^{s-1}$  and  $H = [2, \dots, r+1]$ . We put

$$g = \sum_{k,l \in S(1,m)} b_{kl} y_k \otimes y_l^*,$$

$$b_{kl} = (-1)^{i+j} \sum_{\substack{I \in S(j,m) \\ k \in I}} \sigma(k, I \setminus k) [I \setminus k | 1, \dots, j-1] \sum_{\substack{J \in S(r,m) \\ \{1, \dots, i\} \subset J}} a_{I;J} [l, J \setminus i | H].$$

Then

$$\begin{aligned} g \circ \psi^*(z_v^* \otimes y_u) &= g \left( \sum_{k=1}^m [k | v] y_k^* \otimes y_u \right) = \sum_{k=1}^m [k | v] b_{ku} \\ &= \sum_{k=1}^m [k | v] (-1)^{j-1} \sum_{\substack{I \in S(j,m) \\ k \in I}} \sigma(k, I \setminus k) [I \setminus k | 1, \dots, j-1] \sum_{\substack{J \in S(r,m) \\ \{1, \dots, i\} \subset J}} a_{I;J} (-1)^{i-1} [u, J \setminus i | H] \\ &= \begin{cases} 0 & \text{for } u < i, \\ \sum_{I \in S(j,m)} \left( \sum_{k \in I} (-1)^{j-1} \sigma(k, I \setminus k) [k | v] [I \setminus k | 1, \dots, j-1] \right) a_I & \text{for } u = i \end{cases} \\ &= \begin{cases} 0 & \text{for } (u, v) \prec (i, j), \\ h_{ij} & \text{for } (u, v) = (i, j). \end{cases} \end{aligned}$$

Trivially  $g \circ \psi^*(y_K \otimes z_L^* \otimes z_k) = 0$ ,  $K \in S(r, m)$ ,  $L \in S(r+1, n)$ ,  $k \in S(1, n)$ . Furthermore

$$\begin{aligned} &g \circ \psi^*(y_i^* \otimes y_I \otimes z_J^*) \\ &= \sum_{u \in I} \sigma(u, I \setminus u) [I \setminus u | J] b_{Iu} \\ &= \sum_{u \in I} \sigma(u, I \setminus u) [I \setminus u | J] (-1)^{i+j} \sum_{\substack{K \in S(j,m) \\ l \in K}} \sigma(l, K \setminus l) [K \setminus l | 1, \dots, j-1] \sum_{\substack{L \in S(r,m) \\ \{1, \dots, i\} \subset L}} a_{K;L} [u, L \setminus i | H] \\ &= \pm \sum_{\substack{K \in S(j,m) \\ l \in K}} \sigma(l, K \setminus l) [K \setminus l | 1, \dots, j-1] \sum_{\substack{L \in S(r,m) \\ \{1, \dots, i\} \subset L}} a_{K;L} \left( \sum_{u \in I} \sigma(u, I \setminus u) [I \setminus u | J] [u, L \setminus i | H] \right) \\ &= 0 \end{aligned}$$

for all  $l \in S(1, m)$ ,  $I \in S(r+1, m)$ ,  $J \in S(r, n)$ , since the sum in parentheses is seen to be zero when  $[u, L \setminus i | H]$  is expanded with respect to the first row. —

**(15.8) Remark.** Let  $m = n$ . The generalization from  $\mathbf{Z}$  to arbitrary  $B$  indicated in the proof of (15.7) actually yields for all prime ideals  $P \supset I_r(x)$  that

$$\text{depth } \Omega_P^* = \text{depth } R_P - 1. —$$

**(15.9) Lemma.** *The ring  $B$  being arbitrary the following holds:*

(a) *For all  $I, J \in S(r, m)$  and  $K, L \in S(r, n)$*

$$I(x; [I|1, \dots, r]) \cdot [I|K] \subset \sum_{U < I} R[U|K],$$

$$I(x; [1, \dots, r|L]) \cdot [J|L] \subset \sum_{V < L} R[J|V].$$

(Of course  $U < I$  and  $V < L$  mean  $[U|K] < [I|K]$  and  $[J|V] < [J|L]$ .)

(b) *Let  $H = \{2, \dots, r+1\}$  and  $Q$  be the ideal in  $R$  generated by the  $r$ -minors  $[I|H]$ ,  $I \in S(r, m)$ . Then for all  $i \in S(1, r)$ ,  $j \in S(1, r+1)$ , and  $s \geq 1$ :*

$$(b_1) \quad Q^s \cap I(x; [1, \dots, \widehat{i}, \dots, r+1|1, \dots, r]) = \sum_{I \in S(r-i, m)} Q^{s-1}[1, \dots, i, I|H]$$

$$(b_2) \quad Q^s \cap I(x; [1, \dots, \widehat{i}, \dots, r+1|1, \dots, r]) \cap I(x; [1, \dots, r|1, \dots, \widehat{j}, \dots, r+1]) \\ = \left( \sum_{I \in S(r-i, m)} Q^{s-1}[1, \dots, i, I|H] \right) \cdot I(x; [1, \dots, r|1, \dots, \widehat{j}, \dots, r+1]).$$

PROOF: (a) Of course it suffices to prove the first inclusion. Moreover we may assume that  $K = \{1, \dots, r\}$  since  $I(x; [I|1, \dots, r])$  is invariant under a permutation of the columns of  $x$ . Let  $I = \{a_1, \dots, a_r\}$ ,  $a_i < a_{i+1}$  for  $i = 1, \dots, r-1$ , and  $\delta$  a  $j$ -minor of the first  $a_j - 1$  rows of  $x$ . If  $\delta$  and  $[I|1, \dots, r]$  are comparable then necessarily  $\delta < [I|1, \dots, r]$ , and  $\delta \cdot [I|1, \dots, r] \in \sum_{U < I} R[U|1, \dots, r]$  for trivial reasons. If they are not, we observe that every standard monomial in the standard representation of  $\delta \cdot [I|1, \dots, r]$  contains a factor  $\mu \leq \delta, [I|1, \dots, r]$ . Then  $\mu = [b_1, \dots, b_r|1, \dots, r]$ ,  $b_i \leq a_i$  for  $i = 1, \dots, r$ . Since  $\mu \neq \delta$  there is an  $i$  such that  $b_i < a_i$ . So  $\delta \cdot [I|1, \dots, r] \in \sum_{U < I} R[U|1, \dots, r]$ .

(b) We abbreviate  $\zeta_j = [1, \dots, r|1, \dots, \widehat{j}, \dots, r+1]$ . (b<sub>1</sub>), (b<sub>2</sub>) are immediate consequences of the following assertion:

(11) Let  $K \in S(r, m)$  and  $\Psi$  be an ideal in the poset  $\{[I|H]: I \in S(r, m)\}$ . Then

$$\sum_{\mu \in \Psi} Q^{s-1} \mu \cap I(x; [K|1, \dots, \widehat{j}, \dots, r+1]) \subset \sum_{I \not\geq K} Q^{s-1} [I|H] + \left( \sum_{\mu \in \Psi} Q^{s-1} \mu \right) \cdot I(x; \zeta_j).$$

(Here  $I \not\geq K$  means  $[I|H] \not\geq [K|H]$ ; we adopt this convention for analogous situations.)

Substituting  $\{1, \dots, \widehat{i}, \dots, r+1\}$  for  $K$ ,  $r+1$  for  $j$  and  $\{[I|H]: I \in S(r, m)\}$  for  $\Psi$ , we obtain (b<sub>1</sub>) from (11). Similarly we get (b<sub>2</sub>): Put  $K = \{1, \dots, r\}$ ,  $\Psi = \{[I|H]: \{1, \dots, i\} \subset I\}$  and observe (b<sub>1</sub>).

To prove (11) we use induction on  $|\Psi|$ . Take

$$a = \sum_{\mu \in \Psi} a_\mu \mu \in I(x; [K|1, \dots, \widehat{j}, \dots, r+1]), \quad a_\mu \in Q^{s-1}.$$

If there is a maximal element  $\tau$  of  $\Psi$  such that  $\tau \not\geq [K|H]$ , then applying the inductive hypothesis to  $a - a_\tau \tau$  yields immediately that  $a$  is in the right hand ideal of (11). So we

may assume that  $\tau \geq [K|H]$  for all maximal elements  $\tau$  of  $\Psi$ . Let  $\tau = [J|H]$  be such an element. Then we get

$$a_{\tau\tau} \in \mathbf{I}(x; [J|1, \dots, \widehat{j}, \dots, r+1])$$

because  $J \geq K$  and  $\mu \not\geq [J|1, \dots, \widehat{j}, \dots, r+1]$  for all  $\mu \in \Psi \setminus \{\tau\}$ . It follows that

$$(12) \quad a_{\tau} \in \mathbf{I}(x; [J|1, \dots, \widehat{j}, \dots, r+1]) \cap Q^{s-1}$$

since  $\tau \geq [J|1, \dots, \widehat{j}, \dots, r+1]$ . Observing

$$\mathbf{I}(x; [J|1, \dots, \widehat{j}, \dots, r+1]) = \mathbf{I}(x; [J|1, \dots, r]) + \mathbf{I}(x; \zeta_j),$$

we obtain

$$\begin{aligned} a_{\tau\tau} &\in \mathbf{I}(x; [J|1, \dots, r])\tau + \tau \cdot \mathbf{I}(x; \zeta_j) \\ &\subset \sum_{I < J} R[I|H] + \sum_{\mu \in \Psi} \mu \cdot \mathbf{I}(x; \zeta_j) \end{aligned}$$

using the first inclusion of (a). So

$$a = \sum_{\mu \in \Psi \setminus \{\tau\}} \tilde{a}_{\mu}\mu + \tilde{a}, \quad \tilde{a}_{\mu} \in R, \tilde{a} \in \sum_{\mu \in \Psi} \mu \cdot \mathbf{I}(x; \zeta_j).$$

Applying the inductive hypothesis we get the result for  $s = 1$ .

Let  $s > 1$ . From (12) and the inductive hypothesis on  $s$  we obtain

$$a_{\tau} \in \sum_{I \not\geq J} Q^{s-2}[I|H] + Q^{s-1}\mathbf{I}(x; \zeta_j)$$

(substituting  $J$  for  $K$  and  $\{[I|H] : I \in \mathbf{S}(r, m)\}$  for  $\Psi$ ). We claim that

$$(13) \quad a_{\tau\tau} \in \sum_{I < J} Q^{s-1}[I|H] + \tau \cdot Q^{s-1}\mathbf{I}(x; \zeta_j).$$

To show this we must only look at terms of the form  $b[L|H]\tau$ ,  $b \in Q^{s-2}$ , in which  $[L|H]$  and  $\tau = [J|H]$  are incomparable. According to (9.1) every standard monomial in the standard representation of  $[L|1, \dots, r][J|1, \dots, r]$  is a product  $[U|1, \dots, r][V|1, \dots, r]$ ,  $U, V \in \mathbf{S}(r, m)$ ,  $U \leq V$ . So  $[L|H]\tau$  has a standard representation whose monomials are of the form  $[U|H][V|H]$ ,  $U, V$  as above. Thus we get

$$b[L|H]\tau \in \sum_{I < J} Q^{s-1}[I|H].$$

So (13) holds. With that we obtain a representation

$$a = \sum_{\mu \in \Psi \setminus \{\tau\}} \tilde{a}_{\mu}\mu + \tilde{a}, \quad \tilde{a}_{\mu} \in Q^{s-1}, \tilde{a} \in \left( \sum_{\mu \in \Psi} \mu Q^{s-1} \right) \mathbf{I}(x; \zeta_j),$$

and the proof of (11) can be finished as for  $s = 1$ . —

### C. Syzygetic Behaviour and Rigidity

We now investigate some homological properties of  $\Omega^*$  as we did for the generic modules in Section 13. Abbreviating

$$s = \text{grade } I_r(x) \quad (= m + n - 2r + 1)$$

we state the following

**(15.10) Theorem.** (a) *Let  $m = n$ . Then  $\Omega^*$  is an  $(s-1)$ -th syzygy or, equivalently,  $\text{Ext}_R^i(\Omega, R) = 0$ ,  $1 \leq i \leq s-3$ .*

(b)  *$\Omega^*$  is an  $s$ -th syzygy in case  $m \neq n$ . Equivalently  $\text{Ext}_R^i(\Omega, R) = 0$ ,  $1 \leq i \leq s-2$ .*

PROOF: Since  $\Omega = \Omega^{**}$  (cf. (14.10)) and  $\Omega_P^*$  is free for all prime ideals  $P \subset R$  such that  $\text{depth } R_P < s$ , the assertions given in (a) and (b) are in fact both equivalent to  $\text{depth } \Omega_P^* \geq \min(s-1, \text{depth } R_P)$  and  $\text{depth } \Omega_P^* \geq \min(s, \text{depth } R_P)$ , resp., for all  $P \in \text{Spec } R$  (cf. (16.33)). From (15.7) we get  $\text{depth } \Omega_P^* \geq \text{depth } R_P - 1$  in case  $m = n$  and  $\text{depth } \Omega_P^* = \text{depth } R_P$  in case  $m \neq n$ . This proves the theorem. —

To derive some supplementary results on the syzygetic behaviour of  $\Omega^*$  we need the map  $\varphi_1$  defined in (14.8),(a), the cokernel of which coincides with the first syzygy  $M$  of  $\Omega$ .

#### (15.11) Supplement to (15.10).

(a) *Let  $m = n$ . Then  $\Omega^*$  is not an  $s$ -th syzygy and consequently  $\text{Ext}_R^{s-2}(\Omega, R) \neq 0$ .*

(b) *Let  $m < n-1$ . Then  $\Omega^*$  is not an  $(s+1)$ -th syzygy and consequently  $\text{Ext}_R^{s-1}(\Omega, R) \neq 0$ .*

(c) *Let  $m = n-1$ . Then:*

(c<sub>1</sub>)  *$\text{Ext}_R^i(\Omega, R) = 0$ ,  $1 \leq i \leq s$ , and  $\text{Ext}_R^{s+1}(\Omega, R) \neq 0$  in case  $r+1 < m$  (and consequently  $\Omega^*$  is at least an  $(s+2)$ -th syzygy).*

(c<sub>2</sub>)  *$\text{Ext}_R^i(\Omega, R) = 0$ ,  $1 \leq i \leq s+1 = 5$ , and  $\text{Ext}_R^6(\Omega, R) \neq 0$  in case  $r+1 = m$  (and therefore  $\Omega^*$  is at least a seventh syzygy).*

PROOF: (a) Because of (15.8)  $\text{depth } \Omega_P^* = s-1$  for all minimal prime ideals  $P$  of  $I_r(x)$ , so  $\Omega^*$  is not an  $s$ -th syzygy.

(b) If  $\Omega^*$  were an  $(s+1)$ -th syzygy, then it would be  $(s+1)$ -torsionless (cf. 16.34)). So  $M^*$  were an  $s$ -th syzygy. We claim however that  $\text{depth } M_P^* = \text{depth } R_P - 1$  for all prime ideals  $P \supset I_r(x)$ . To prove this, we first reduce to the case in which  $B = \mathbf{Z}$ . We may then argue as in the last part of the proof of (13.8).

Let  $B = \mathbf{Z}$ . Clearly  $\text{depth } M_P^* \geq \text{depth } R_P - 1$  for all prime ideals  $P \supset I_r(x)$  since  $\Omega^*$  is perfect. So it is enough to show that  $\text{depth } M_{I_r(x)}^* < \text{depth } R_{I_r(x)}$ . Localizing as in the proof of (14.9) we can reduce to the case in which  $r = 1$ . Furthermore we may replace  $\mathbf{Z}$  by  $\mathbf{Q}$  since  $I_r(x) \cap \mathbf{Z} = 0$ . Then we need only to prove that  $M^*$  is not perfect (cf. (16.20)) or equivalently that  $\text{Ext}_R^1(M^*, Q^{n-m}) \neq 0$ ,  $Q$  being the ideal in  $R$  generated by the entries of the first column of  $x$ . Consider the map

$$h: \text{Hom}_R(M^*, R) \longrightarrow \text{Hom}_R(M^*, R/Q^{n-m})$$

induced by the residue class projection  $R \rightarrow R/Q^{n-m}$ . It is easy to see, that

$$[2|1]^{n-m-1} y_1 \otimes z_1^* - [1|1][2|1]^{n-m-2} y_2 \otimes z_1^*$$

represents an element of  $\text{Hom}_R(M^*, R/Q^{n-m})$  which is not in  $\text{Im } h$  ( $y_1, \dots, y_m$  and  $z_1, \dots, z_n$  denoting the canonical bases of  $R^m$  and  $R^n$ , resp.). So  $h$  is not surjective and consequently  $\text{Ext}_R^1(M^*, R/Q^{n-m}) \neq 0$ .

(c) In case  $r + 1 = m$  the kernel of the map  $\varphi$  (cf. Subsection A) is obviously isomorphic with  $\text{Im } x$ . The assertion of (c<sub>2</sub>) follows therefore from (15.10) and (13.14),(b).

As to (c<sub>1</sub>) we consider the three  $R$ -modules  $\text{Ker } \varphi_1^*(= M^*)$ ,  $\text{Im } \varphi_1^*$  and  $\text{Coker } \varphi_1^*$ . The assertion is an easy consequence of the following claim:

(14)  $\text{Ker } \varphi_1^*$  and  $\text{Im } \varphi_1^*$  are perfect  $B[X]$ -modules whereas

$$\text{depth}(\text{Coker } \varphi_1^*)_P = \text{depth } R_P - 1$$

for all prime ideals  $P \supset I_r(x)$ .

We outline the proof of (14); the computational details are very similar to those used up to now and may be left to the reader: Since  $\text{Ker } \varphi_1^*$  and  $\text{Im } \varphi_1^*$  are  $\mathbf{Z}$ -free in case  $B = \mathbf{Z}$ , the proof of the perfection of  $\text{Ker } \varphi_1^*$  can be reduced to this case (cf. (3.3)). Then it is enough to show that  $\text{Ext}_R^1(M^*, Q) = 0$  where  $Q$  is the ideal in  $R$  generated by the  $r$ -minors of the first  $r$  columns of  $x$ . The same way leads to the perfection of  $\text{Im } \varphi_1^*$ . (Observe that  $\text{depth}(\text{Coker } \varphi_1^*)_P \geq \text{depth } M_P^* - 2$  for all prime ideals  $P$  in  $R$ , so  $\text{Coker } \varphi_1^*$  is  $R$ -torsionfree and therefore  $\mathbf{Z}$ -flat in case  $B = \mathbf{Z}$ .) It follows that  $\text{depth}(\text{Coker } \varphi_1^*)_P \geq \text{depth } R_P - 1$  for all prime ideals  $P$  in  $R$ . To get equality when  $P \supset I_r(x)$ , one reduces to the case in which  $r = 1$  and  $B = \mathbf{Q}$  as one did in the proof of (b). An easy computation yields  $\text{Ext}_R^1(\text{Coker } \varphi_1^*, Q) \neq 0$ . —

**(15.12) Remark.** The proof shows that the assertions of (15.11) remain true if we localize at some prime ideal containing  $I_r(x)$ .—

Finally we shall derive some results concerning the rigidity of determinantal rings, the base ring  $B$  presumed to be a field  $K$  from now. Some concepts and results in a more general situation are needed.

Let  $S$  be a finitely generated  $K$ -algebra,  $S = K[X_1, \dots, X_u]/I$ ,  $X_1, \dots, X_u$  being indeterminates and  $I$  an ideal in  $K[X_1, \dots, X_u]$ . “The” Auslander-Bridger dual of the  $S$ -module  $I/I^2$  (cf. 16.E) will be called an *Auslander-module* of  $S$  and is denoted by  $D_S$ ; up to projective direct summands it does not depend on the special presentation taken for  $S$ .

An  $S$ -algebra  $T$  will be called a *complete intersection over  $S$*  if  $T$  is a factor ring of a polynomial ring  $S[Y_1, \dots, Y_v]$  with respect to an ideal generated by an  $S[Y_1, \dots, Y_v]$ -sequence.

We will not discuss here what it means that  $S$  is rigid. The reader may find detailed information about this concept in the literature (cf. [Ar], [Jä], [KL] or [SI] for instance). The only fact we notice is that in case  $K$  is perfect and  $S$  is reduced,  $S$  is a rigid  $K$ -algebra if and only if  $\text{Ext}_S^1(\Omega_{S/K}^1, S) = 0$ .

**DEFINITION.** Let  $S$  be as above and  $k$  a natural number. Assume  $S$  to be rigid.  $S$  is *k-rigid* if the following condition holds: If  $T$  is a complete intersection over  $S$  whose flat dimension as an  $S$ -module is at most  $k$ , then  $\text{Tor}_i^S(T, D_S) = 0$  for all  $i > 0$ . If  $S$  is  $k$ -rigid for all  $k$ , then  $S$  is *very rigid*.

**(15.13) Proposition.** *Let  $S$  be as above.*

(a) *If  $S$  is  $k$ -rigid and  $a_1, \dots, a_j$  an  $S$ -sequence,  $j \leq k$ , then  $\text{Tor}_i^S(S/(a_1, \dots, a_j)S, D_S) = 0$  for all  $i > 0$ .*

(b) *If  $D_S$  satisfies the condition  $(\tilde{S}_k)$  (cf. 16.E), then  $S$  is  $k$ -rigid.*

PROOF: (a) Assume  $a_1, \dots, a_j$ ,  $1 \leq j \leq k$ , to be an  $S$ -sequence. The  $S$ -module  $\bar{S} = S/(a_1, \dots, a_j)S$  is a complete intersection over  $S$  which has flat dimension at most  $k$ , so  $\text{Tor}_i^S(\bar{S}, D_S) = 0$  for all  $i > 0$ .

(b) Let  $T$  be a complete intersection over  $S$ ,  $\tilde{S} = S[Y_1, \dots, Y_v]$ ,  $T = \tilde{S}/(f_1, \dots, f_l)\tilde{S}$ , where  $f_1, \dots, f_l$  is an  $\tilde{S}$ -sequence. Assume that the flat dimension of  $T$  over  $S$  is at most  $k$ . We claim that

$$(15) \quad \text{depth}(D_S \otimes_S \tilde{S})_{\tilde{Q}} \geq l$$

for all  $\tilde{Q} \in \text{Spec } \tilde{S}$  containing  $f_1, \dots, f_l$ . This implies

$$\text{Tor}_i^S(T, D_S) = \text{Tor}_i^{\tilde{S}}(T, D_S \otimes_S \tilde{S}) = 0$$

for all  $i > 0$ : Let  $l \geq 1$  and

$$\mathcal{F}: 0 \longrightarrow F_l \xrightarrow{\varphi_l} \dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

be the Koszul-complex over  $\tilde{S}$  derived from  $f_1, \dots, f_l$ .  $\mathcal{F}$  is an  $\tilde{S}$ -free resolution of  $T$ . To see that  $\mathcal{F} \otimes_{\tilde{S}} (D_S \otimes_S \tilde{S})$  remains exact, we use Theorem (16.15), the numbers  $r_i$  and the ideals  $J_i$  being defined as there. It is well known that  $\text{Rad } J_i = \text{Rad } \sum_{j=1}^l \tilde{S}f_j$ ,  $i = 1, \dots, l$ . So the exactness we want to prove follows immediately from (15).

To verify (15) let  $Q$  be the image of  $\tilde{Q}$  in  $T$  and  $P$  the preimage of  $\tilde{Q}$  in  $S$  with respect to the canonical maps. From the flatness of the extension  $S_P \rightarrow \tilde{S}_{\tilde{Q}}$  we obtain

$$\text{depth}(D_S \otimes_S \tilde{S})_{\tilde{Q}} = \text{depth}(D_S)_P + \text{depth } \tilde{S}_{\tilde{Q}} - \text{depth } S_P.$$

Since  $\text{depth}(D_S)_P \geq \min(k, \text{depth } S_P)$  by assumption, we are done in case  $k \geq \text{depth } S_P$ . Let  $k < \text{depth } S_P$  and consider elements  $a_1, \dots, a_j \in P$  which form a maximal  $S_P$ -sequence. Denote by  $\mathcal{K} = \mathcal{K}(a_1, \dots, a_j)$  the Koszul-complex (over  $S$ ) derived from  $a_1, \dots, a_j$ . Since the flat dimension of  $T$  over  $S$  is at most  $k$ , we get

$$\begin{aligned} \text{H}_i(\mathcal{K} \otimes_S S_P \otimes_{S_P} T_Q) &= \text{Tor}_i^{S_P}(S_P/(a_1, \dots, a_j)S_P, T_Q) \\ &= \text{Tor}_i^S(S/(a_1, \dots, a_j)S, T_Q) \otimes_S S_P \\ &= \text{Tor}_i^S(S/(a_1, \dots, a_j)S, T) \otimes_T T_Q \\ &= 0 \end{aligned}$$

for  $i > k$ . Thus the ideal  $(a_1, \dots, a_j)T_Q$  has grade at least  $j - k$  (cf. Theorem (16.15) for example). Consequently  $\text{depth } T_Q \geq \text{depth } S_P - k$ . Since  $\text{depth } \tilde{S}_{\tilde{Q}} \geq l + \text{depth } T_Q$ , the proof of (15) is complete now. —

**(15.14) Corollary.** *Let  $S$  be Cohen-Macaulay. Then  $S$  is  $k$ -rigid if and only if  $D_S$  satisfies the condition  $(\tilde{S}_k)$ .*

PROOF: It is easy to see that in the case we consider, a finitely generated  $S$ -module  $M$  satisfies the condition  $(\tilde{S}_k)$  if and only if  $\text{Tor}_i^S(S/(a_1, \dots, a_j)S, M) = 0$  for every  $S$ -sequence  $a_1, \dots, a_j$ ,  $1 \leq j \leq k$ , and all  $i > 0$ . —

Now we specialize to the determinantal ring  $R = R_{r+1}(X)$  with base ring  $B = K$ . As above put  $s = \text{grade } I_r(x)$ . From (15.14) and the syzygetic behaviour of  $\Omega^*$  we derive:

**(15.15) Theorem.** *Assume that  $K$  is a perfect field. Then  $R$  is rigid except for the case in which  $r + 1 = m = n$ . Furthermore:*

- (a) *If  $r + 1 < m = n$  then  $R$  is  $(s - 4)$ -rigid but not  $(s - 3)$ -rigid.*
- (b) *If  $m < n - 1$  then  $R$  is  $(s - 3)$ -rigid but not  $(s - 2)$ -rigid.*
- (c) *Let  $m = n - 1$ .*
  - (c<sub>1</sub>) *If  $r + 1 < m$  then  $R$  is  $(s - 1)$ -rigid but not  $s$ -rigid.*
  - (c<sub>2</sub>) *If  $r + 1 = m$  then  $R$  is very rigid.*

PROOF: First we observe that there is a commutative diagram

$$\begin{array}{ccc}
 \bigwedge^{r+1} R^m \otimes \bigwedge^{r+1} (R^n)^* & \xrightarrow{\varphi} & R^m \otimes (R^n)^* \\
 \tilde{\varphi} \searrow & & \nearrow \\
 & I_{r+1}(X)/I_{r+1}(X)^2 &
 \end{array}$$

where  $\varphi = \varphi_{x,r}$  and

$$\tilde{\varphi}(y_I \otimes z_J^*) = [I|J] \pmod{I_{r+1}(X)^2},$$

$y_1, \dots, y_m$  and  $z_1, \dots, z_n$  being the canonical bases of  $R^m$  and  $R^n$  resp. (cf. Section 14). It is a well known fact that  $I_{r+1}(X)/I_{r+1}(X)^2$  and  $\text{Im } \varphi$  have the same rank as  $R$ -modules. So their  $R$ -duals coincide, and consequently  $\Omega^*$  is a third syzygy of an Auslander-module  $D_R$  of  $R$ . Clearly  $(D_R)_P$  is free for all prime ideals  $P \subset R$  such that  $\text{depth } R_P < s$ .

From (15.7) and (15.8) we therefore obtain that  $D_R$  satisfies the condition  $(\tilde{S}_{s-3})$  in case  $m < n$  and  $(\tilde{S}_{s-4})$  in case  $r + 1 < m = n$ . On the other hand  $D_R$  satisfies  $(\tilde{S}_t)$  for  $t \leq s$  if and only if it is a  $t$ -th syzygy (cf. (16.33)). So  $(\tilde{S}_{s-2})$  and  $(\tilde{S}_{s-3})$  do not hold for  $D_R$  in case  $m < n - 1$  and  $r + 1 < m = n$  resp. (cf. (15.11)). Since  $R$  is Cohen-Macaulay in any case, (a) and (b) follow immediately from (15.14).

Let  $m = n - 1$ . Consider the map  $\varphi_1$  we have defined in (14.8),(a). If  $r + 1 = m$ , we obtain from (9.18) that  $\text{Im } \tilde{\varphi} \cong I_{r+1}(X)/I_{r+1}(X)^{(2)} \cong \text{Im } \varphi$ , so  $\text{Coker } \varphi_1^*$  is an Auslander-module of  $R$ . Moreover  $\text{Coker } \varphi_1^*$  is isomorphic with  $\text{Coker } x^*$  (in the case under consideration) which is a perfect module in view of (13.8). Thus  $\text{Coker } \varphi_1^*$  satisfies  $(\tilde{S}_k)$  for all  $k$ , and (c<sub>2</sub>) holds because of (15.14). In case  $r + 1 < m$  the module  $D_R$  satisfies  $(\tilde{S}_{s-1})$  since  $\Omega^*$  is an  $(s + 2)$ -th syzygy (cf. (15.11),(c<sub>1</sub>)), so  $R$  is  $(s - 1)$ -rigid. If it were  $s$ -rigid,  $D_R$  would be an  $s$ -th syzygy. Then  $\text{Im } \varphi_1^*$ , which is isomorphic with  $\text{Coker } \tilde{\varphi}^*$  in any case, would be an  $(s + 1)$ -th syzygy and thus even  $(s + 1)$ -torsionless (cf. (16.34)). This implies  $\text{Ext}_R^{s-1}(\text{Ker } \varphi, R) = 0$  since  $(\text{Im } \varphi_1^*)^* = \text{Ker } \varphi$ . But  $\text{Ext}_R^{s-1}(\text{Ker } \varphi, R) = \text{Ext}_R^{s+1}(\Omega, R) \neq 0$  (cf. (15.11),(c<sub>1</sub>)). —



## D. Comments and References

The content of the first two subsections is taken from [Ve.4]. The history of the results given in Subsection C is a little more complicated.

(15.10) is due to Svanes ([Sv.3], 6.8.1). The vanishing of  $\text{Ext}_R^1(\Omega, R)$  and, as a consequence, the rigidity of  $R$  in case  $B$  is a perfect field was independently shown by Jähner (cf. [Jä], 7.6)). The methods used by Svanes are far from being elementary (and yield a more general result than (15.10)), in contrast to Jähner's proof which is quite adjusted to the special (determinantal) situation and works by methods very similar to those we developed in the first two subsections (cf. [Jä], (7.1)–(7.5)). We mention that the rings  $R$  have been suspected to be rigid for a long time. The special case in which  $r = 1$ ,  $m = 2 < n$ , has been treated already in [GK]. The statements of (15.10) are sharpened by (15.11) which result can also be found in [Ve.4].

In [Bw], (4.5.4) and (5.1.1) Buchweitz has introduced the concepts of an Auslander-module and  $k$ -rigidity, resp. Part of (15.13) is contained in [Bw], (5.1.3), and (15.15) is a generalization of [Bw], (5.2.1). Theorem (15.15) is also suggested by Buchweitz who has given a somewhat weaker version based on the Theorem of Svanes mentioned above (cf. [Bw], (5.3)).

## 16. Appendix

In the appendix we discuss topics in commutative algebra for which we cannot adequately refer to a standard text book.

### A. Determinants and Modules. Rank

Let  $A$  be an arbitrary commutative ring. With every homomorphism  $f: F \rightarrow G$  of finitely generated free  $A$ -modules  $F$  and  $G$  we associate its determinantal ideals  $I_k(f)$  in the following manner: With respect to bases of  $F$  and  $G$  resp.,  $f$  is given by a matrix  $U$ , and we simply put  $I_k(f) = I_k(U)$ . This definition makes sense since  $I_k(U)$  obviously depends on  $f$  only and not on the bases chosen. It is equally obvious that  $I_k(f)$  is an invariant of the submodule  $\text{Im } f$  of  $G$ . It is a little more surprising that  $I_k(f)$  is completely determined by the isomorphism class of  $\text{Coker } f$ :

**(16.1) Proposition.** *Let  $A$  be a commutative ring,  $M$  an  $A$ -module with finite free presentations*

$$F \xrightarrow{f} G \xrightarrow{g} M \longrightarrow 0 \quad \text{and} \quad \tilde{F} \xrightarrow{\tilde{f}} \tilde{G} \xrightarrow{\tilde{g}} M \longrightarrow 0$$

Let  $n = \text{rk } G$ ,  $\tilde{n} = \text{rk } \tilde{G}$ . Then  $I_{n-k}(f) = I_{\tilde{n}-k}(\tilde{f})$  for all  $k \geq 0$ .

PROOF: Let  $e_1, \dots, e_n$  and  $\tilde{e}_1, \dots, \tilde{e}_{\tilde{n}}$  be bases of  $G$  and  $\tilde{G}$  resp. Then one has equations

$$\tilde{g}(\tilde{e}_i) = \sum_{j=1}^n a_{ij} g(e_j), \quad a_{ij} \in A, \quad i = 1, \dots, \tilde{n}.$$

Therefore  $M$  has a presentation

$$F \oplus A^{\tilde{n}} \xrightarrow{h} G \oplus \tilde{G} \xrightarrow{\tilde{g}} M \longrightarrow 0, \quad \tilde{g}(x, \tilde{x}) = g(x) - \tilde{g}(\tilde{x}),$$

for which  $h$  has the following matrix relative to a matrix  $U = (u_{ij})$  of  $f$ :

$$H = \left( \begin{array}{ccc|cccc} & & & 0 & \cdots & \cdots & 0 \\ & & & \vdots & & & \vdots \\ & & U & 0 & \cdots & \cdots & 0 \\ \hline & & & 1 & 0 & \cdots & 0 \\ & & & 0 & \ddots & \ddots & \vdots \\ & & a_{ij} & \vdots & \ddots & \ddots & 0 \\ & & & 0 & \cdots & 0 & 1 \end{array} \right).$$

Evidently  $I_{n+\tilde{n}-k}(h) = I_{n+\tilde{n}-k}(H) = I_{n-k}(U)$ , and, as stated above, this ideal is determined by  $\text{Ker } \tilde{g}$ . By symmetry  $I_{n+\tilde{n}-k}(h) = I_{\tilde{n}-k}(\tilde{f})$  as well. —

With the notation of the preceding proposition, the ideal  $I_{n-k}(f)$  is also called the  $k$ -th *Fitting invariant* of  $M$ . It is not difficult to see that the zeroth Fitting invariant annihilates  $M$ . Let  $e_1, \dots, e_n$  be a basis of the free  $A$ -module  $G$  as above, and let

$$y_i = \sum_{j=1}^n u_{ij}e_j, \quad u_{ij} \in A,$$

be a system of generators of  $\text{Ker } g$ . We take minors with respect to  $U = (u_{ij})$ . By Laplace expansion

$$\sum_{i=1}^n (-1)^{i+1} [a_1, \dots, \hat{a}_i, \dots, a_n | 1, \dots, \hat{j}, \dots, n] y_{a_i} = [a_1, \dots, a_n | 1, \dots, n] e_j.$$

So all the  $n$ -minors of  $U$  annihilate  $M$ . In general  $\text{Ann } M \neq I_n(f)$ , but the radicals of  $\text{Ann } M$  and  $I_n(f)$  coincide:

**(16.2) Proposition.** *Let  $A, f, M, n$  be as in the preceding proposition. Then*

$$I_n(f) \subset \text{Ann } M \quad \text{and} \quad \text{Rad } I_n(f) = \text{Rad } \text{Ann } M.$$

The inclusion has been proved already. It remains to show that a prime ideal  $P$  does not contain  $I_n(f)$  if  $M_P = 0$ . This may be considered a special case of Proposition (16.3) below.

The ideals  $I_k(f)$  control the minimal number of local generators of  $M$  in the same way as they control the vector space dimension of  $M$  when  $A$  is a field. For a prime ideal  $P$  of  $A$  we denote by  $\mu(M_P)$  the minimal number of generators of the  $A_P$ -module  $M_P$ ; by virtue of Nakayama's lemma:

$$\mu(M_P) = \dim_K M \otimes K, \quad K = A_P/PA_P.$$

**(16.3) Proposition.** *Let  $A$  be a commutative ring and  $M$  an  $A$ -module with finite free presentation  $F \xrightarrow{f} G \rightarrow M \rightarrow 0$ . Let  $n = \text{rk } G$  and  $P$  a prime ideal of  $A$ . Then the following statements are equivalent:*

- (a)  $I_k(f) \not\subset P$ ,
- (b)  $(\text{Im } f)_P$  contains a (free) direct summand of  $G_P$  of rank  $\geq k$ ,
- (c)  $\mu(M_P) \leq n - k$ .

PROOF: We may assume that  $A$  is local with maximal ideal  $P$ . Let  $\overline{\phantom{x}}$  denote residue classes mod  $P$ . The presentation of  $M$  induces a presentation

$$\overline{F} \xrightarrow{\overline{f}} \overline{G} \rightarrow \overline{M} \rightarrow 0.$$

Since  $\overline{I_k(f)} = I_k(\overline{f})$  and because of Nakayama's lemma one can replace  $A, P, M, f$  by  $\overline{A}, \overline{P}, \overline{M}, \overline{f}$  without affecting the validity of (a), (b), or (c). Now we are dealing with finite-dimensional vector spaces over the field  $\overline{A}$ , for which the equivalence of (a), (b), and (c) is trivial. —

For later application we note a consequence of (16.3):

**(16.4) Proposition.** *Let  $A$  be a commutative ring,*

$$\mathcal{F}: 0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0$$

*be a complex of finitely generated free  $A$ -modules, and  $r_k = \sum_{i=k}^n (-1)^{i-k} \text{rk } F_i$ . Let  $P$  be a prime ideal. Then  $\mathcal{F} \otimes_{A_P}$  is split-exact if and only if  $I_{r_k}(f_k) \not\subset P$  for  $k = 1, \dots, n$ .*

PROOF: We may certainly suppose that  $A = A_P$ . Let first  $\mathcal{F}$  be split-exact. Then, by a trivial induction,  $\text{Im } f_k$  is a free direct summand of  $F_{k-1}$ ,  $\text{rk Im } f_k = r_k$ , whence  $I_{r_k}(f_k) \not\subset P$  by virtue of (16.3). For the converse one applies induction, too: One may assume that  $\text{Coker } f_2$  is free of rank  $\leq r_1$ . On the other hand  $\text{Im } f_1$  contains a free direct summand of  $F_0$  whose rank is  $\geq r_1$ . Therefore the natural surjektion  $\text{Coker } f_2 \rightarrow \text{Im } f_1$  is an isomorphism, and, finally,  $\text{Coker } f_1$  is free of rank  $r_0$ . —

Usually the rank of a module  $M$  over an integral domain with field of fractions  $L$  is the dimension of the  $L$ -vector space  $M \otimes L$ . We extend this notion to all rings without attempting to assign a rank to every module.

DEFINITION. Let  $A$  be an arbitrary commutative ring,  $Q$  its total ring of fractions. An  $A$ -module  $M$  has rank  $r$ , abbreviated  $\text{rk } M = r$ , if  $M \otimes Q$  is a free  $Q$ -module of rank  $r$ .

Over a noetherian ring the rank of a module can be computed in several ways:

**(16.5) Proposition.** *Let  $A$  be a noetherian ring, and  $M$  a finitely generated  $A$ -module with a finite free presentation  $F \xrightarrow{f} G \longrightarrow M \longrightarrow 0$ . Let  $n = \text{rk } G$ . Then the following are equivalent:*

- (a)  $M$  has rank  $r$ .
- (b)  $M$  has a free submodule  $N$  of rank  $r$  such that  $M/N$  is a torsion module.
- (c) For all  $P \in \text{Ass } A$  the  $A_P$ -module  $M_P$  is free of rank  $r$ .
- (d)  $I_{n-r}(f)$  contains an element which is not a zero-divisor of  $A$ , and  $I_k(f) = 0$  for all  $k > n - r$ .

PROOF: (a)  $\Rightarrow$  (b): Let  $x_1, \dots, x_r$  be a basis of  $M \otimes Q$ . Multiplying with a suitable element of  $A$  which is not a zero-divisor, we obtain elements  $y_1, \dots, y_r$  (which are images of elements) in  $M$ . Now take  $N = \sum Ay_i$ .

(b)  $\Rightarrow$  (c): This is as trivial as the implication (b)  $\Rightarrow$  (a).

(c)  $\Rightarrow$  (d): Replacing  $A$  by a localization  $A_P$ ,  $P \in \text{Ass } A$ , it is enough to show that  $I_{n-r}(f) = A$  and  $I_k(f) = 0$  for  $k > n - r$  if  $M$  is free of rank  $r$  over the ring  $A$ . One replaces the given presentation by  $0 \longrightarrow M \longrightarrow M \longrightarrow 0$  and applies (16.1) above.

(d)  $\Rightarrow$  (a) Replacing  $A$  by its total ring of fractions  $Q$ , one may assume that  $I_{n-r}(f) = A$  and  $I_k(f) = 0$  for  $k > n - r$ . Let  $P$  be a prime ideal of  $A$ . By virtue of (16.3)  $(\text{Im } f)_P$  contains a free direct summand of  $G_P$ . After a suitable choice of basis for  $G_P$  it is evident that  $M_P$  is free of rank  $r$ . Therefore  $M$  is a projective  $A$ -module of constant local rank  $r$ . The ring  $A$  under consideration has only finitely many maximal ideals, whence  $M$  is free. (The last conclusion can be proved in the following manner: Let  $P_1, \dots, P_u$  be the maximal ideals of  $A$ . There are elements  $a_i \in P_1 \cap \cdots \cap \hat{P}_i \cap \cdots \cap P_u$  such that  $a_i \notin P_i$ . Now one chooses elements  $g_{ij} \in M$ ,  $i = 1, \dots, u$ ,  $j = 1, \dots, r$ , such that  $g_{i1}, \dots, g_{ir}$  are mapped to a basis of  $M/P_i M$ . Obviously  $\sum a_i g_{i1}, \dots, \sum a_i g_{ir}$  then is a basis of  $M$ .) —

The reader may find out which of the implications between (a), (b), (c), and (d) in (16.5) remain valid if one drops the hypothesis “noetherian.”

Over a noetherian ring a projective module has a rank if and only if the rank of its localizations is constant. For finitely generated  $A$ -modules  $M, N$  with ranks the modules  $\text{Hom}_A(M, N), M \otimes N, \bigwedge^k M, S_j(M)$  have ranks, too, and these are computed as for free modules  $M, N$ : after all, their construction commutes with localization.

Rank is an additive function along sequences which are exact in depth 0:

**(16.6) Proposition.** *Let  $A$  be noetherian ring,  $M, M_1, M_2$  finitely generated  $A$ -modules for which there is a sequence of homomorphisms*

$$\mathcal{C}: 0 \longrightarrow M_1 \xrightarrow{f} M \longrightarrow M_2 \longrightarrow 0$$

*such that  $\mathcal{C} \otimes A_P$  is exact for all prime ideals  $P \in \text{Ass } A$ . If two of  $M, M_1, M_2$  have a rank, then the third one has a rank, too, and*

$$\text{rk } M = \text{rk } M_1 + \text{rk } M_2.$$

PROOF: One may directly assume that  $A$  is local of depth 0 and  $\mathcal{C}$  is exact. Only the case in which  $M_1$  and  $M$  are supposed to be free, is nontrivial. Then  $\text{pd } M_2 \leq 1$  and, since  $\text{depth } A = 0$ , even  $\text{pd } M_2 = 0$  by the Auslander-Buchsbaum formula, or one proves this afresh: Let  $P$  be the maximal ideal.  $M_2$  is a free  $A$ -module if and only if

$$\mathcal{C} \otimes A/P: 0 \longrightarrow M_1/PM_1 \xrightarrow{\bar{f}} M/PM \longrightarrow M_2/PM_2 \longrightarrow 0$$

is exact. If it is not exact, there is an element  $x \in M_1 \setminus PM_1$  such that  $f(x) \in PM$ . Because of  $\text{depth } A = 0$  and the injectivity of  $f$ , the element  $x$  is annihilated by a nonzero element of  $A$ . On the other hand it belongs to a basis of  $M_1$ . Contradiction. —

**(16.7) Corollary.** *Let  $A$  be a noetherian ring, and  $M$  an  $A$ -module with a finite free resolution*

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0.$$

*Then  $\text{rk } M = \sum_{i=0}^n (-1)^i \text{rk } F_i$ .*

**(16.8) Corollary.** *Let  $A$  be a noetherian ring, and  $I$  an ideal of  $A$ . Then  $I$  has a rank if and only if  $I = 0$  (and  $\text{rk } I = 0$ ) or  $I$  contains an element not dividing 0 (and  $\text{rk } I = 1$ ).*

PROOF: Consider the exact sequence  $0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$  and apply (16.6). —

The following corollary is absolutely trivial now. We list it because it contains an argument often effective:

**(16.9) Corollary.** *Let  $A$  be a noetherian ring,  $f: M \rightarrow N$  a surjective homomorphism of  $A$ -modules  $M, N$  such that  $\text{rk } M = \text{rk } N$  and  $M$  is torsionfree. Then  $f$  is an isomorphism.*

We conclude this subsection with a description of the free locus of a module with rank.

**(16.10) Proposition.** *Let  $A$  be a noetherian ring,  $M$  a finitely generated  $A$ -module of rank  $r$ . Furthermore let*

$$F \xrightarrow{f} G \longrightarrow M \longrightarrow 0$$

*be a finite free representation of  $M$ , and  $n = \text{rk } G$ . Then  $M_P$  is a free  $A_P$ -module for a prime ideal  $P$  of  $A$  if and only if  $\text{I}_{n-r}(f) \not\subset P$ .*

PROOF: One observes that  $M_P$  is free if and only if  $\mu(M_P) \leq r$  and applies (16.3). —

Extending the presentation to a free resolution the reader may formulate the generalization of (16.10) describing the prime ideals  $P$  such that  $\text{pd } M_P \leq k$ .

### B. Grade and Acyclicity

Let  $A$  be a local ring,  $P$  its maximal ideal, and  $M$  a finitely generated  $A$ -module. The length of a maximal  $M$ -sequence inside  $P$  is usually called  $\text{depth } M$ . In the following definition we replace  $P$  by an arbitrary ideal in a noetherian ring.

DEFINITION. Let  $A$  be a noetherian ring,  $I \subset A$  an ideal, and  $M$  a finitely generated  $A$ -module such that  $IM \neq M$ . Then the *grade of  $I$  with respect to  $M$*  is the length of a maximal  $M$ -sequence in  $I$ . It is denoted by  $\text{grade}(I, M)$ .

The reader may consult [Mt], Ch. 6 for the definition of “ $M$ -sequence.” There the notation  $\text{depth}_I(M)$  is used for  $\text{grade}(I, M)$  (Attention: The first edition of [Mt] differs considerably from the second one in regard to Ch. 6!) It is easy to see that the grade just defined is always finite; in fact, it is bounded by  $\text{ht } I$ .

Very often we shall have  $M = A$ , and therefore we introduce the abbreviation

$$\text{grade } I = \text{grade}(I, A).$$

For systematic reasons it is convenient to cover the case in which  $IM = M$ , too; thus we put  $\text{grade}(I, M) = \infty$  if  $IM = M$ .

It is very important that grade can be computed from homological invariants.

**(16.11) Theorem.** *Let  $A$  be a noetherian ring,  $I \subset A$  an ideal,  $N$  a finitely generated  $A$ -module such that  $\text{Supp } N = \{P \in \text{Spec } A: P \supset I\}$ . Then*

$$\text{grade}(I, M) = \min\{j: \text{Ext}_A^j(N, M) \neq 0\}$$

*for every finitely generated  $A$ -module  $M$ .*

The case in which  $\text{grade}(I, M) < \infty$ , thus  $IM \neq M$ , is an immediate consequence of [Mt], Theorem 28 (and stated on p.102 of [Mt]). If  $\text{grade}(I, M) = \infty$ , one has  $M_P = 0$  for all  $P \in \text{Supp } N$ , thus  $\text{Supp}(\text{Ext}_A^j(N, M)) = \emptyset$  for all  $j$ .

The case  $N = A/I$  of (16.11) suggests that  $\text{grade}(I, M)$  is an invariant of  $A/I$  rather than an invariant of  $I$ . It would even justify to call  $\text{grade}(I, M)$  the grade of  $N$  with respect to  $M$ . We restrict ourselves to the case  $M = A$ .

DEFINITION. Let  $N$  be a finitely generated module over the noetherian ring  $A$ . The *grade of  $N$*  is the grade of  $\text{Ann } N$  with respect to  $A$ , abbreviated  $\text{grade } N$ .

In order to avoid the ambiguity thus introduced we insist on the first meaning of  $\text{grade } I$  whenever  $I$  is considered an ideal. An immediate corollary of (16.11):

**(16.12) Corollary.**  $\text{grade } N \leq \text{pd } N$ .

As a consequence of (16.11) one has  $\text{grade}(I, M) = \text{grade}(\text{Rad } I, M)$ . This follows also from the following local description of grade for which we refer to [Mt], p.105, Proposition:

**(16.13) Proposition.** *With the notations introduced above one has*

$$\text{grade}(I, M) = \inf \{ \text{depth } M_P : P \in \text{Spec } A, P \supset I \}.$$

Another fact implied by (16.11) is the behaviour of grade along exact sequences (which can of course be derived directly from the definition of grade):

**(16.14) Proposition.** *Let  $A$  be a noetherian ring,  $M_1, M_2,$  and  $M_3$  finitely generated  $A$ -modules connected by an exact sequence*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0.$$

*Then one has for every ideal  $I$  of  $A$ :*

- (a)  $\text{grade}(I, M_3) \geq \min(\text{grade}(I, M_1), \text{grade}(I, M_2)) - 1,$
- (b)  $\text{grade}(I, M_2) \geq \min(\text{grade}(I, M_1), \text{grade}(I, M_3)),$
- (c)  $\text{grade}(I, M_1) \geq \min(\text{grade}(I, M_2), \text{grade}(I, M_3) + 1).$

With the notations and hypotheses as in (16.11) let

$$\mathcal{F}: \dots \longrightarrow F_n \longrightarrow \dots \longrightarrow F_1 \xrightarrow{f_1} F_0$$

be a free resolution of  $N = \text{Coker } f_1$ . Put  $m = \text{grade}(I, M)$  and consider the truncation

$$\mathcal{F}_m^*: 0 \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \dots \longrightarrow F_{m-1}^* \longrightarrow F_m^*$$

of the dual  $\text{Hom}_A(\mathcal{F}, A)$ . The inequality “ $\leq$ ” in (16.11) implies that  $\mathcal{F}_m^* \otimes M$  is acyclic. This fact admits a far-reaching generalization:

**(16.15) Theorem.** *Let  $A$  be a noetherian ring, and*

$$\mathcal{F}: 0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{f_1} F_0$$

*a complex of finitely generated free  $A$ -modules. Let  $r_k = \sum_{i=k}^n (-1)^{i-k} \text{rk } F_i$  and  $J_k = I_{r_k}(f_k)$ . Furthermore let  $M$  be a finitely generated  $A$ -module. Then the following statements are equivalent:*

- (a)  $\mathcal{F} \otimes M$  is acyclic.
- (b)  $\text{grade}(J_k, M) \geq k$  for  $k = 1, \dots, n$ .

PROOF: First we prove the implication (b)  $\Rightarrow$  (a) by induction on the length  $n$  of  $\mathcal{F}$ . One may suppose that  $\mathcal{F}' \otimes M$  is acyclic,  $\mathcal{F}'$  given as the truncation

$$\mathcal{F}': 0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_2 \xrightarrow{f_2} F_1$$

of  $\mathcal{F}$ . It “resolves”  $C \otimes M$ ,  $C$  being the cokernel of  $f_2$ . We have to show that the induced homomorphism  $\bar{f}_1 : C \otimes M \rightarrow F_0 \otimes M$  is injective. By virtue of (16.4) the localizations  $\mathcal{F}_P$ ,  $P \in \text{Ass } M$ , are split-exact, and so is  $(\mathcal{F} \otimes M)_P$ . Hence  $(\text{Ker } \bar{f}_1)_P = 0$  for all prime ideals  $P \in \text{Ass } M$ . In order to derive a contradiction let us assume that  $\text{Ker } \bar{f}_1 \neq 0$ . Hence there exists an element  $b \in A$ ,  $b$  neither a unit nor a zero-divisor of  $M$ , such that  $b(\text{Ker } \bar{f}_1) = 0$ . For a prime ideal  $Q$  minimal in  $\text{Supp}(\text{Ker } \bar{f}_1)$  one then has  $\text{depth } M_Q \geq 1$ , but  $\text{depth}(\text{Ker } \bar{f}_1)_Q = 0$ .

Let  $m = \text{depth } M_Q$ . Suppose first  $m \geq n$ . Then an iterated application of (16.14),(a) to the “ $M$ -resolution”  $\mathcal{F}' \otimes M$  of  $C \otimes M$  yields  $\text{depth}(C \otimes M)_Q \geq 1$ . Let next  $0 < m < n$ . Applying (16.4) again, we see that  $\tilde{F} = (\text{Coker } f_{m+1})_Q$  is a free  $A_Q$ -module, and  $(\mathcal{F}' \otimes M)_Q$  decomposes into a split-exact tail

$$0 \rightarrow (F_n \otimes M)_Q \rightarrow \dots \rightarrow (F_m \otimes M)_Q \rightarrow \tilde{F} \otimes M_Q \rightarrow 0$$

and a shorter  $M_Q$ -resolution of  $(C \otimes M)_Q$

$$0 \rightarrow \tilde{F} \otimes M_Q \rightarrow (F_{m-1} \otimes M)_Q \rightarrow \dots \rightarrow (F_1 \otimes M)_Q.$$

By an iterated application of (16.14),(a) again:  $\text{depth}(C \otimes M)_Q \geq 1$ . Since  $\text{Ker } \bar{f}_1 \subset C \otimes M$ ,  $\text{depth}(\text{Ker } \bar{f}_1)_Q \geq 1$  as well, the desired contradiction.

In proving the implication (a)  $\Rightarrow$  (b) we may inductively suppose that  $\text{grade}(J_k, M) \geq k - 1$  for  $k = 1, \dots, n$ . Assume that  $\text{grade}(J_k, M) = k - 1$  for some  $k$ , and let  $P \supset J_k$  be a prime ideal,  $\text{depth } M_P = k - 1$ . In order to derive a contradiction we replace  $A$  by  $A_P$ , and, as above, split off the tail of  $\mathcal{F}$ . Then we substitute the right part of  $\mathcal{F}$  for  $\mathcal{F}$  itself, and  $k$  for  $n$ , and conclude that it is enough to prove that  $\text{depth } M \geq n$ . This follows from the case  $n = 1$ , which we postpone, by induction: Let  $\mathcal{F}'$  be as above. If  $\text{depth } M \geq 1$ , there is a non-unit  $a \in A$  which is not a zero-divisor of  $M$ . Elementary arguments imply the exactness of

$$(\mathcal{F}' \otimes M) \otimes (A/Aa) = \mathcal{F}' \otimes (M/aM).$$

The inductive hypothesis yields  $\text{depth } M/aM \geq n - 1$ , whence  $\text{depth } M \geq n$ .

After all, we have reduced the implication (a)  $\Rightarrow$  (b) to the following statement: Let  $A$  be a local ring,  $f : F_1 \rightarrow F_0$  a homomorphism of finitely generated free  $A$ -modules,  $M$  a finitely generated  $A$ -module for which  $f \otimes M$  is injective; if  $\text{depth } M = 0$ , then  $f$  embeds  $F_1$  as a free direct summand of  $F_0$ . Suppose, not. Then there is an element  $e \in F_1$  which belongs to a basis of  $F_1$  such that  $f(e) \in PF_0$ ,  $P$  being the maximal ideal of  $A$ . On the other hand there is an  $x \in M$ ,  $x \neq 0$ , such that  $Px = 0$ . Now  $e \otimes x \neq 0$ , but  $f(e \otimes x) \in PF_1 \otimes x = F_1 \otimes Px = 0$ , a final contradiction. —

In view of (16.5),(d) and (16.7), the reader may prove that  $I_{r_k+1}(f_k)M = 0$  for  $k = 1, \dots, n$  if  $\mathcal{F} \otimes M$  is exact.

Undoubtedly the most important case of the theorem is the one in which  $M = A$ ; and very often the following weaker version of (b)  $\Rightarrow$  (a) is all one needs:

**(16.16) Corollary.** *If  $\mathcal{F} \otimes A_P$  is split-exact for all prime ideals  $P$  such that  $\text{depth } A_P < n$ , then  $\mathcal{F}$  is exact.*



### C. Perfection and the Cohen-Macaulay Property

In (16.12) it is stated that the projective dimension of a module always bounds its grade:  $\text{grade } M \leq \text{pd } M$ . The modules for which equality is attained, are of particular importance and merit a special attribute:

**DEFINITION.** Let  $A$  be a noetherian ring. A finitely generated  $A$ -module  $M$  such that  $\text{grade } M = \text{pd } M$ , is called *perfect*. By the usual abuse of language, an ideal  $I$  is called perfect if  $A/I$  is a perfect  $A$ -module.

Perfect modules are distinguished by having a perfect resolution: They have a projective resolution  $\mathcal{P}$  of finite length whose dual  $\mathcal{P}^*$  is acyclic, too, cf. (16.11).  $\mathcal{P}^*$  resolves  $\text{Ext}_A^g(M, A)$ ,  $g = \text{grade } M$ , and  $\text{Ext}_A^g(\text{Ext}_A^g(M, A), A) = M$ .

Perfect modules are “grade unmixed”:

**(16.17) Proposition.** *With the notations of the definition, a prime ideal  $P$  of  $A$  is associated to  $M$  if and only if  $M_P \neq 0$  and  $\text{depth } A_P = \text{grade } M$ . Furthermore  $\text{grade } P = \text{grade } M$  for all  $P \in \text{Ass } M$ .*

**PROOF:** Because of  $\text{Ass } M \subset \text{Supp } M$ , we may suppose  $M_P \neq 0$ . If  $\text{depth } A_P = \text{grade } M$ , then  $\text{depth } M_P = 0$  since always

$$\text{grade } M \leq \text{grade } M_P \leq \text{pd } M_P \leq \text{pd } M,$$

and

$$\text{depth } M_P = \text{depth } A_P - \text{pd } M_P$$

by the equation of Auslander and Buchsbaum. Conversely, if  $\text{depth } M_P = 0$ , necessarily  $\text{depth } A_P = \text{grade } M$ . It remains to prove that  $\text{grade } P = \text{depth } A_P$ . Suppose  $\text{grade } P < \text{depth } A_P$ . Then there is a prime ideal  $Q \supset P$  such that  $\text{grade } P = \text{grade } Q = \text{depth } A_Q \geq \text{pd } M_Q \geq \text{pd } M_P$ . Contradiction. —

The preceding proof shows that  $\text{pd } M_P = \text{grade } M_P = \text{grade } M$  for all prime ideals  $P$  in the support of a perfect module  $M$ .

The main objects of our interest will be certain perfect ideals  $I$  in a polynomial ring  $A = B[X_1, \dots, X_n]$ . In the investigation of  $A/I$  it is often important to know that an ideal in  $A/I$  has  $\text{grade} \geq 1$ . In this connection the following proposition will be very useful.

**(16.18) Proposition.** *Let  $A$  be a noetherian ring,  $I$  a perfect ideal in  $A$ , and  $J \supset I$  another ideal. Then*

$$\text{grade } J/I \geq \text{grade } J - \text{grade } I,$$

where  $J/I$  is considered an ideal of  $A/I$ , of course. If  $J$  is perfect, too, one has equality.

**PROOF:** Let  $Q \supset J/I$  be a prime ideal, and  $P$  the preimage of  $Q$  in  $A$ . Then

$$\text{depth}(A/I)_Q = \text{depth}(A/I)_P = \text{depth } A_P - \text{pd}(A/I)_P \geq \text{grade } J - \text{grade } I,$$

and this is enough by (16.13). If  $J$  is perfect, too, we obtain equality by first choosing  $P$  as an associated prime of  $J$ , and  $Q$  as its image in  $A/I$ . —

We say that a noetherian ring  $A$  is a *Cohen-Macaulay* ring if each of its localizations  $A_P$  is Cohen-Macaulay. For modules we adopt the analogous convention. The theory of Cohen-Macaulay rings and modules is developed in [Mt], Sect. 16; cf. also [Ka], Chap. 3. Perfection and the Cohen-Macaulay property are closely related:

**(16.19) Proposition.** *Let  $A$  be a Cohen-Macaulay ring,  $M$  a finitely generated  $A$ -module.*

(a) *If  $M$  is perfect,  $M$  is a Cohen-Macaulay module.*

(b) *If  $M$  is a Cohen-Macaulay module,  $\text{pd } M < \infty$ , and  $\text{Supp } M$  is connected, then  $M$  is perfect.*

PROOF: Suppose  $\text{pd } M < \infty$ ,  $P \in \text{Supp } M$ . Then

$$\text{depth } M_P = \text{depth } A_P - \text{pd } M_P$$

and

$$\dim M_P = \dim A_P - \text{ht } \text{Ann } M_P = \text{depth } A_P - \text{grade } \text{Ann } M_P.$$

Therefore  $M_P$  is perfect if and only if  $M_P$  is a Cohen-Macaulay module.

It only remains to prove that  $M$  is perfect if its localizations  $M_P$ ,  $P \in \text{Supp } M$ , are perfect and  $\text{Supp } M$  is connected. Evidently, the crucial point is that  $\text{pd } M_P$  is constant on  $\text{Supp } M$ . The local perfection of  $M$  implies that

$$\{P: \text{pd } M_P = k\} = \{P: \text{pd } M_P \geq k\} \cap \{P: \text{grade } M_P \leq k\}$$

for all  $k$ ,  $0 \leq k \leq \text{pd } M$ . Both sets on the right side are given as intersections of finitely many closed sets, each of which is the locus of vanishing of a (co)homology module; cf. (16.11) for the rightmost set. Therefore the set on the left side is closed, too. Since  $\text{Supp } M$  is connected, it is nonempty if and only if  $k = \text{pd } M$ . —

In particular, perfection and the Cohen-Macaulay property of  $M$  are equivalent if  $A$  is a polynomial ring over a field or the integers, and  $M$  is a graded  $A$ -module. (Note that  $\text{Supp } M$  is connected if and only if  $A/\text{Ann } M$  has no nontrivial idempotents.) For such modules perfection can even be tested at a single localization.

**(16.20) Proposition.** *Let  $A$  be a polynomial ring over a field,  $P$  its irrelevant maximal ideal, and  $M$  a graded  $A$ -module. Then the following conditions are equivalent:*

(a)  *$M$  is a Cohen-Macaulay module.*

(a')  *$M$  is perfect.*

(b)  *$M_P$  is a Cohen-Macaulay  $A_P$ -module.*

(b')  *$M_P$  is a perfect  $A_P$ -module.*

Only the implication (b)  $\Rightarrow$  (a) needs a proof, and it follows immediately from the fact that a minimal graded resolution of  $M$  over  $A$  becomes a minimal resolution of  $M_P$  over  $A_P$  upon localization.

A very important invariant of a local Cohen-Macaulay ring  $A$  is its canonical (or: dualizing) module  $\omega_A$ , provided such a module exists for  $A$ . We refer to [HK] and [Gr] for its theory. The canonical module is uniquely determined (up to isomorphism).  $A$  is a Gorenstein ring if and only if it is Cohen-Macaulay and  $\omega_A = A$ . A regular local ring is Gorenstein, and a Cohen-Macaulay residue class ring  $A = S/I$  of a local Gorenstein ring  $S$  has a canonical module:

$$\omega_A = \text{Ext}_S^g(A, S), \quad g = \text{grade } I.$$

Let now  $A$  be an arbitrary Cohen-Macaulay ring. An  $A$ -module is called a canonical module of  $A$  if it is a canonical module locally. We denote a canonical module by  $\omega_A$

though it is no longer unique in general: if  $M$  is a projective module of rank 1,  $\omega_A \otimes M$  is a canonical module, too. The characterization of Gorenstein rings remains valid. For the existence theorem quoted we have to require that  $\text{grade } I_P = \text{grade } I$  for all  $P \in \text{Supp } A$  now. Since the ideals  $I$  of interest to us are perfect, this is not an essential restriction. If  $S$  is Gorenstein,  $I \subset S$  a perfect ideal, and  $A = S/I$  resolved by

$$\mathcal{P}: 0 \longrightarrow G_g \longrightarrow \cdots \longrightarrow G_0, \quad G_i \text{ projective,}$$

one has a very direct description of  $\omega_A$ : it is the  $(A)$ -module resolved by  $\mathcal{P}^*$ .

### D. Dehomogenization

The principal objects of our interest are two classes of rings. The rings in the first class are graded, and every ring  $A$  in the second one arises from a ring  $R$  in the first one by dehomogenization:  $A = R/R(x - 1)$  for a suitable element  $x \in R$  of degree 1. The rings  $A$  and  $R$  are much closer related than a ring and its homomorphic images in general, and very often it will be convenient to derive the properties of  $A$  from those of  $R$ . (Geometrically,  $R$  is the homogeneous coordinate ring of a projective variety and  $A$  the coordinate ring of the open affine subvariety complementary to the hyperplane “at infinity” defined by the vanishing of  $x$ .)

Let  $R = \bigoplus_{i \geq 0} R_i$  be a graded ring,  $x \in R_1$  a non-nilpotent element. The natural homomorphism  $\pi: R \rightarrow A$ ,  $A = R/R(x - 1)$ , factors through  $S = R[x^{-1}]$  in a canonical way, so one has a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \pi \searrow & & \swarrow \psi \\ & A & \end{array}$$

$S$  is graded again:  $S = \bigoplus_{i=-\infty}^{\infty} S_i$ ,  $S_i = \{x^j f : j \in \mathbf{Z}, f \in R_{i-j}\}$ . (Though  $R$  may not be a subring of  $S$ , we do not distinguish notationally between elements in  $R$  and their images under the homomorphism  $R \rightarrow S$ . Furthermore we shall write  $I \cap R$  for the preimage of an ideal  $I \subset S$  in  $R$ .)

The structure of the graded ring  $S$  is particularly simple. Evidently:

**(16.21) Proposition.** (a) *For every homogeneous ideal  $I \subset S$ , in particular for  $I = S$ , one has*

$$I = \bigoplus_{i=-\infty}^{\infty} x^i (I \cap S_0).$$

(b) *The natural homomorphism  $S_0[X, X^{-1}] \rightarrow S$ ,  $X \rightarrow x$ , is an isomorphism.*

Furthermore  $A$  is not only a homomorphic image, but a subring of  $S$ , too:

**(16.22) Proposition.** *The homomorphism  $\psi$  maps  $S_0$  isomorphically onto  $A$ .*

PROOF: Obviously the restriction of  $\psi$  to  $S_0$  is surjective. If, on the other hand,  $\psi(fx^{-i}) = 0$  for an element  $f \in R_i$ , then  $\pi(f) = 0$ . So  $f = g(x - 1)$ ,  $g \in R$ . Since  $f$  is homogeneous,  $gx = 0$ ,  $f = -g$ , and  $f = 0$  in  $S$ . —

In the following we shall identify  $A$  with  $S_0$ .

**(16.23) Proposition.** *If  $R$  is reduced (a (normal) domain), then  $A$  is reduced (a (normal) domain), too.*

PROOF:  $S = R[x^{-1}]$  inherits each of the listed properties from  $R$ , and  $A$  is a subring of  $S$ . The only not completely obvious problem is whether normality carries over to  $A$ . It is well known, that  $A$  is normal if and only if  $A[X]$  is normal, and the normality of  $A[X]$  follows from the normality of  $S = A[X, X^{-1}]$  by the following lemma which will be very useful several times:

**(16.24) Lemma.** *Let  $T$  be a noetherian ring, and  $y \in T$  such that  $y$  is not a zero-divisor,  $T/Ty$  is reduced and  $T[y^{-1}]$  is normal. Then  $T$  is normal.*

PROOF: We use Serre's normality criterion:  $T$  is normal if and only if  $T_P$  is regular for every prime ideal  $P$  of  $T$  such that  $\text{depth } T_P \leq 1$ . Let  $P$  be such a prime ideal. If  $y \notin P$ ,  $T_P$  is a localization of the normal ring  $T[y^{-1}]$ , thus regular. Otherwise  $P$  is a minimal prime of  $Ty$ , and  $PT_P = yT_P$ , since  $T/Ty$  is reduced. Having its maximal ideal generated by an element which is not a zero-divisor,  $T_P$  is a regular local ring. —

Since the inversion of  $x$  may destroy pathologies of  $R$ , one cannot reverse (16.23) in complete generality. For the rings of interest to us this is possible however, since the additional hypotheses of the following proposition are satisfied.

**(16.25) Proposition.** *Suppose additionally that  $x$  is not a zero-divisor. Then  $R$  is reduced (a domain) if  $A$  is reduced (a domain). If furthermore  $R$  is noetherian and  $R/Rx$  is reduced, then normality transfers from  $A$  to  $R$ .*

This is immediate now. In the following we want to relate the ideals of  $R$  and  $A$ .

- (16.26) Proposition.** (a) *One has  $\pi(I) = IS \cap A$  for a homogeneous ideal  $I \subset R$ , and  $J = \pi(JS \cap R)$  for every ideal  $J$  of  $A$ .*  
 (b) *By relating the ideals  $I$  and  $\pi(I)$  the homomorphism  $\pi$  sets up a bijective correspondence between the homogeneous ideals of  $R$ , modulo which  $x$  is not a zero-divisor, and all the ideals of  $A$ .*  
 (c) *This correspondence preserves set-theoretic inclusions and intersections. It furthermore preserves the properties of being a prime, primary or radical ideal in both directions.*

PROOF: (a) The ideal  $\pi(I)$  is generated by the images of the homogeneous elements  $f \in I$ . If  $f$  has degree  $d$ , then  $\pi(f) = fx^{-d} \in IS \cap A$ . Conversely, let  $g \in IS \cap A$ . Then  $g = x^k h$ ,  $k \in \mathbf{Z}$ ,  $h \in I$ , and  $g = \psi(g) = \pi(x^k h) = \pi(h) \in \pi(I)$ .

The ideal  $JS$  is homogeneous, so  $JS \cap R$  is homogeneous, and

$$\pi(JS \cap R) = (JS \cap R)S \cap A = JS \cap A = J.$$

(b) A (homogeneous) ideal  $\tilde{I}$  of  $R$  appears as the preimage of a (homogeneous) ideal of  $S$  (namely of its own extension) if and only if  $x$  is not a zero-divisor modulo  $\tilde{I}$ . This establishes a bijective correspondence between the ideals  $I$  under consideration and the homogeneous ideals of  $S$ . The latter are in 1-1-correspondence with the ideals of  $A$  by (16.21),(a), and the first equation in (a) shows that the desired correspondence is induced by  $\pi$ .

(c) The first statement of (c) is completely obvious. The properties of being a prime, primary or radical ideal are preserved in going from  $R$  to its ring of quotients  $S$ , and

also under taking preimages in  $A \subset S$ . Conversely they cannot be destroyed by the extensions  $A \rightarrow A[X]$  and  $A[X] \rightarrow A[X, X^{-1}] = S$ , from where we pass to  $R$  by taking preimages. —

One usually calls  $\pi(I)$  the *dehomogenization* of  $I$ , and  $JS \cap R$  the *homogenization* of  $J$ . As a consequence of (16.26), (c) primary decompositions are preserved:

**(16.27) Proposition.** *Let  $R$  be noetherian,  $I$  a homogeneous ideal modulo which  $x$  is not a zero-divisor, and  $I = \bigcap Q_i$  an irredundant decomposition into homogeneous primary ideals. Then  $\pi(I) = \bigcap \pi(Q_i)$  is an irredundant primary decomposition of  $\pi(I)$ . The analogous statement holds for the process of homogenization.*

Let  $P$  be a homogeneous prime ideal of  $R$ ,  $x \notin P$ , and  $Q$  its dehomogenization. Then

$$R_P = A[X]_{QA[X]}$$

is a localization of  $A_Q[X]$ , and it is clear that  $R_P$  and  $A_Q$  share essentially all ring-theoretic properties:

**(16.28) Proposition.** *Suppose that  $R$  is noetherian. Let  $P$  be a homogeneous prime ideal of  $R$ ,  $x \notin P$ , and  $Q$  its dehomogenization. Then  $R_P$  and  $A_Q$  coincide with respect to the following quantities and properties: dimension, depth, being reduced, integrity, normality, being Cohen-Macaulay, being Gorenstein, regularity.*

In fact, the extension  $A_Q \rightarrow R_P$  is faithfully flat. Its fiber is the field  $(A_Q/QA_Q)(X)$ . Thus (16.28) follows from the properties of flat extensions as given in [Mt], Sect. 21, and, as far as the Gorenstein property is concerned, in [Wt]. (Of course one can give more direct arguments in the special situation of (16.28).)

### E. How to Compare “Torsionfree”

Since the notion “torsionfree” is fairly standard, we have used it without explanation: An  $A$ -module  $M$  is torsionfree if every element of  $A$  which is not a zero-divisor of  $A$ , is not a zero-divisor of  $M$ . In this subsection we introduce several notions which describe higher degrees of being torsionfree, and give conditions under which they are equivalent.

**DEFINITION.** Let  $A$  be a noetherian ring,  $M$  a finitely generated  $A$ -module.  $M$  is called  *$n$ -torsionfree* if every  $A$ -sequence of length at most  $n$  is an  $M$ -sequence, too.

There is a slightly stronger condition of Serre type:

**(16.29) Proposition.** *Let  $A$  be a noetherian ring,  $M$  a finitely generated  $A$ -module. Then  $M$  is  $n$ -torsionfree if it satisfies the condition*

$$(\tilde{S}_n): \text{depth } M_P \geq \min(n, \text{depth } A_P) \quad \text{for all prime ideals } P.$$

It is an exercise on associated prime ideals to prove that  $M$  is  $n$ -torsionfree if and only if

$$\text{depth } M_P \geq \min(n, \text{grade } P) \quad \text{for all prime ideals } P,$$

and this inequality is obviously weaker than  $(\tilde{S}_n)$ . It is furthermore obvious that both properties of (16.29) are equivalent if the localizations  $A_P$  such that  $\text{depth } A_P < n$  are

Cohen-Macaulay rings, since  $\text{grade } P = \text{depth } A_P$  then for all prime ideals  $P$  such that  $\text{depth } A_P \leq n$  (cf. (16.13)).

Let  $A$  be an integral domain momentarily,  $M$  a torsionfree  $A$ -module,  $Q$  the field of fractions of  $A$ . The natural map  $h: M \rightarrow M^{**}$  becomes an isomorphism when tensored with  $Q$ . Since  $M$  is torsionfree, the torsion module  $\text{Ker } h$  must be zero. An epimorphism  $F \rightarrow M^*$ ,  $F$  free, leads to an embedding  $M \hookrightarrow M^{**} \hookrightarrow F^*$ :  $M$  is a submodule of a free  $A$ -module, and therefore a first module of syzygies of an  $A$ -module.

DEFINITION. Let  $A$  be a noetherian ring. An  $A$ -module  $M$  is called an  $n$ -th syzygy if there is an exact sequence

$$0 \longrightarrow M \longrightarrow F_n \longrightarrow \dots \longrightarrow F_1$$

with finitely generated free  $A$ -modules  $F_i$ .

From the behaviour of depth along exact sequences one concludes immediately:

**(16.30) Proposition.** *An  $n$ -th syzygy satisfies  $(\tilde{S}_n)$ .*

An  $A$ -module  $M$  for which the natural map  $h: M \rightarrow M^{**}$  is injective, is called *torsionless*. The argument above shows that a torsionless module is a first syzygy, and conversely a first syzygy is torsionless: An embedding  $M \hookrightarrow F$  extends to a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & F \\ h \downarrow & & \parallel \\ M^{**} & \longrightarrow & F^{**}. \end{array}$$

If  $h$  is an isomorphism,  $M$  is called *reflexive*.

A natural idea how to make  $M$  an  $n$ -th syzygy, is to start with a free resolution of the dual

$$F_n \longrightarrow \dots \longrightarrow F_1 \longrightarrow M^* \longrightarrow 0,$$

to dualize and to replace the embedding  $M^{**} \rightarrow F_1^*$  by its composition with  $M \rightarrow M^{**}$ . This yields a zero-sequence

$$0 \longrightarrow M \longrightarrow F_1^* \longrightarrow \dots \longrightarrow F_n^*.$$

DEFINITION.  $M$  is called  $n$ -torsionless if the preceding sequence is exact.

Since it is irrelevant which resolution of  $M^*$  has been chosen, this definition is justified.

**(16.31) Proposition.** *Let  $A$  be a noetherian ring,  $M$  a finitely generated  $A$ -module.*

- (a) *If  $M$  is  $n$ -torsionless, then it is an  $n$ -th syzygy.*
- (b)  *$M$  is 1-torsionless (2-torsionless) if and only if it is torsionless (reflexive).*
- (c)  *$M$  is  $k$ -torsionless for  $k \geq 3$  if and only if it is reflexive and  $\text{Ext}_A^i(M^*, A) = 0$  for  $i = 1, \dots, k - 2$ .*

The proposition follows readily from the definition of “ $n$ -torsionless”. A somewhat smoother description of “ $n$ -torsionless” can be given by means of the Auslander-Bridger dual of  $M$ : It is the cokernel of  $f^*$  in a finite free presentation

$$F \xrightarrow{f} G \longrightarrow M \longrightarrow 0.$$

Despite its non-uniqueness we denote it by  $D(M)$ . One has  $M = D(D(M))$ . (It is not difficult to prove that  $D(M)$  is unique up to projective direct summands.)

**(16.32) Proposition.** *Let  $A$  be a noetherian ring,  $M$  a finitely generated  $A$ -module,  $h: M \rightarrow M^{**}$  the natural map. Then:*

- (a)  $\text{Ker } h = \text{Ext}_A^1(\text{D}(M), A)$ ,
- (b)  $\text{Coker } h = \text{Ext}_A^2(\text{D}(M), A)$ , and
- (c)  $M$  is  $n$ -torsionless if and only if  $\text{Ext}_A^i(\text{D}(M), A) = 0$  for  $i = 1, \dots, n$ .

PROOF: Because of the preceding proposition and

$$\text{Ext}_A^i(\text{D}(M), A) = \text{Ext}_A^{i-2}(M^*, A)$$

for  $i \geq 3$  it is enough to prove (a) and (b). We choose a finite free presentation of  $M^*$

$$K \longrightarrow H \longrightarrow M^* \longrightarrow 0$$

and splice its dual via the natural homomorphism  $h$  with a presentation

$$F \xrightarrow{f} G \longrightarrow M \longrightarrow 0.$$

Then  $\text{D}(M) = \text{Coker } f^*$ , and one has a commutative diagram

$$\begin{array}{ccccccc} F & \longrightarrow & G & \longrightarrow & H^* & \longrightarrow & K^* \\ & & \downarrow & & \uparrow & & \\ & & M & \longrightarrow & M^{**} & & \end{array}$$

whose upper row has homology  $\text{Ker } h$  at  $G$  and  $\text{Coker } h$  at  $H^*$ . By construction

$$K \longrightarrow H \longrightarrow G^* \longrightarrow F^* \longrightarrow \text{D}(M) \longrightarrow 0$$

is the right end of a free resolution of  $\text{D}(M)$ . —

The most elementary notion among the ones introduced is certainly the property “ $n$ -th syzygy”. On the other hand it is the hardest to control, and the properties  $(\tilde{S}_n)$  and “ $n$ -torsionless” should be regarded as a lower and an upper “homological” approximation. Under certain hypotheses on  $M$  (or  $A$ ) all the properties introduced are equivalent:

**(16.33) Proposition.** *Let  $A$  be a noetherian ring, and  $M$  a finitely generated  $A$ -module such that  $\text{pd } M_P < \infty$  for all prime ideals  $P$  of  $A$  with  $\text{depth } A_P < n$ . Then all the properties*

$$\text{“}n\text{-torsionfree”}, (\tilde{S}_n), \text{ “}n\text{-th syzygy”}, \text{ and “}n\text{-torsionless”}$$

*are equivalent.*

As we shall see below “ $n$ -th syzygy” and “ $n$ -torsionless” are equivalent under a slightly weaker hypothesis on  $M$ .

PROOF: We first show that  $M$  satisfies  $(\tilde{S}_n)$  if it is  $n$ -torsionfree. For a prime ideal  $P$  such that  $\text{grade } P \geq n$  one clearly has  $\text{depth } M_P \geq n$ . Otherwise there is a prime ideal  $Q \supset P$  such that  $\text{depth } A_Q = \text{grade } Q = \text{grade } P < n$ . Then  $\text{depth } M_Q = \text{depth } A_Q$ , and

$M_Q$  has to be a free  $A_Q$ -module because of  $\text{depth } A_Q + \text{pd } M_Q = \text{depth } A_Q$ . Even more  $M_P$  is a free  $A_P$ -module.

Next one proves directly that  $(\tilde{S}_n)$  implies “ $n$ -torsionless”. From the argument just given it follows that a free resolution

$$\cdots \longrightarrow F_n \cdots \longrightarrow F_1 \xrightarrow{f} M^* \longrightarrow 0$$

splits when localized with respect to prime ideals  $P$  such that  $\text{depth } A_P < n$ , and furthermore  $h_P: M_P \rightarrow M_P^{**}$  is an isomorphism. Therefore the cokernel  $N$  of the map  $g$  in

$$(*) \quad 0 \longrightarrow M \xrightarrow{g} F_1^* \longrightarrow \cdots \longrightarrow F_n^*$$

has property  $(\tilde{S}_{n-1})$  and  $\text{pd } N_P < \infty$  for all prime ideals  $P$  such that  $\text{depth } A_P < n$ .  $M$  is certainly torsionfree and  $(\text{Ker } g)_P = \text{Ker } g_P = 0$  for all associated prime ideals  $P$  of  $A$ . Since  $N^* = \text{Ker } f$ , an inductive argument finishes the proof. —

As pointed out above, every first syzygy is 1-torsionless, and this fact signalizes that “ $n$ -th syzygy” and “ $n$ -torsionless” should be equivalent under a weaker hypothesis.

**(16.34) Proposition.** *With the remaining hypotheses of (16.33) suppose furthermore that  $\text{pd } M_P < \infty$  for all prime ideals  $P$  such that  $\text{depth } A_P < n - 1$ . Then every  $n$ -th syzygy is  $n$ -torsionless.*

PROOF: The case  $n = 1$  being settled, we treat  $n = 2$  as a separate case, too. There is an exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow N \longrightarrow 0$$

in which  $N$  is a first syzygy and  $F$  is free. Then we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & N & \longrightarrow & 0. \\ & & h \downarrow & & \parallel & & \downarrow g & & \\ & & M^{**} & \xrightarrow{f} & F^{**} & \longrightarrow & N^{**} & & \end{array}$$

The kernel of  $f$  is  $(\text{Ext}_A^1(N, A))^*$ , hence zero since  $\text{Ext}_A^1(N, A)$  is a torsion module:  $N_P$  is free for all  $P \in \text{Ass } A$  because of  $\text{pd } N_P < \infty$ . Since  $g$  is injective,  $h$  has to be surjective.

Let  $n > 2$  now. We have an exact sequence  $0 \rightarrow M \rightarrow F \rightarrow N \rightarrow 0$  as above, in which  $N$  is  $(n - 1)$ -torsionless, as follows by induction or from the preceding proposition. Dualizing one obtains an exact sequence

$$0 \longrightarrow N^* \longrightarrow F^* \longrightarrow M^* \longrightarrow \text{Ext}_A^1(N, A) \longrightarrow 0.$$

We split this into two exact sequences:

$$\begin{array}{l} 0 \longrightarrow N^* \longrightarrow F^* \longrightarrow K \longrightarrow 0, \\ 0 \longrightarrow K \longrightarrow M^* \longrightarrow \text{Ext}_A^1(N, A) \longrightarrow 0. \end{array}$$

Dualizing again one gets exact sequences

$$\begin{array}{ll} \text{Ext}_A^{i-1}(N^*, A) \longrightarrow \text{Ext}_A^i(K, A) \longrightarrow 0 & \text{for } i \geq 1, \\ \text{Ext}_A^i(\text{Ext}_A^1(N, A), A) \longrightarrow \text{Ext}_A^i(M^*, A) \longrightarrow \text{Ext}_A^i(K, A) & \text{for } i \geq 0. \end{array}$$



Since  $N_P$  is a free  $A_P$ -module for all prime ideals  $P$  such that  $\text{depth } A_P < n - 1$ , one has

$$\text{grade Ext}_A^1(N, A) \geq n - 1,$$

and therefore  $\text{Ext}_A^i(\text{Ext}_A^1(N, A), A) = 0$  for all  $i, 0 \leq i \leq n - 2$  (cf. (16.11)). One readily concludes that

$$\text{Ext}_A^i(M^*, A) = 0 \quad \text{for } i = 2, \dots, n - 2.$$

Since  $n > 2$ ,  $N$  is reflexive, and therefore the linear map  $F^{**} \rightarrow N^{**}$  is the original epimorphism  $F \rightarrow N$ , whence  $\text{Ext}_A^1(K, A) = \text{Ext}_A^1(M^*, A) = 0$ , too. —

**(16.35) Remark.** The hypothesis

$$\text{pd } M_P < \infty \quad \text{for all prime ideals } P \text{ with } \text{depth } A_P < n - 1$$

is only needed to ensure  $\text{grade Ext}_A^1(N, A) \geq n - 1$ . In any case  $\text{depth } N_P = \text{depth } A_P$  for all prime ideals  $P$  with  $\text{depth } A_P < n - 1$ , and arguing with an injective resolution of  $A_P$  one also concludes  $\text{grade Ext}_A^1(N, A) \geq n - 1$  if the localizations  $A_P$  with  $\text{depth } A_P < n - 1$  are Gorenstein rings. Similarly one can replace the condition on  $M$  in (16.33) by the hypothesis:  $A_P$  is Gorenstein for all prime ideals  $P$  such that  $\text{depth } A_P < n$ . (Observe that  $\text{Ext}_A^1(N, A) = 0$  for the module  $N$  constructed in the proof of (16.33).)

### F. The Theorem of Hilbert-Burch

Commutative algebra is not very rich in classification theorems. One of the few examples identifies the ideals  $I$  in a noetherian ring for which  $A/I$  has a free resolution of length 2:

$$(1) \quad 0 \longrightarrow A^m \xrightarrow{f} A^{m+1} \xrightarrow{g} A.$$

Let  $f$  be given by the matrix  $U$  and put  $\delta_i = (-1)^{i+1}[1, \dots, \hat{i}, \dots, m + 1]$ . Choosing the map  $h: A^{m+1} \rightarrow A$  by sending the  $i$ -th element of a basis of  $A^{m+1}$  to  $\delta_i, i = 1, \dots, m + 1$ , one certainly obtains a complex

$$(2) \quad 0 \longrightarrow A^m \xrightarrow{f} A^{m+1} \xrightarrow{h} A.$$

The acyclicity criterion (16.15) applied to the exact sequence (1) yields that  $\text{grade } I \geq 1$ ,  $\text{grade } I_m(f) \geq 2$ . On the other hand,  $I_1(h) = \text{Im } h = I_m(f)$ , so it forces the complex (2) to be exact, too, and we have an isomorphism

$$I \cong \text{Coker } f \cong I_m(f).$$

Since  $\text{grade } I_m(f) \geq 2$  and, thus,  $\text{Ext}_A^1(A/I_m(f), A) = 0$ , the natural homomorphism  $A^* \rightarrow (I_m(f))^*$  is an isomorphism, whence every map  $I_m(f) \rightarrow A$  is a multiplication by an element  $a \in A$ . So  $I = aI_m(f)$ ; because of  $\text{grade } I \geq 1$ ,  $a$  cannot divide zero:

**(16.36) Theorem.** *Let  $A$  be a noetherian ring, and  $I \subset A$  an ideal for which  $A/I$  has a free resolution as (1) above. Then there exists an element  $a \in A$  which is not a zero divisor, such that  $I = aI_m(f)$ .*

This theorem is often called the *Hilbert-Burch theorem* since it has appeared in a special form in [Hi], pp. 239, 240 and has been given its first modern version by Burch [Bh.1]. One should note that its hypotheses are fulfilled if  $A$  is a regular local ring or a polynomial ring over a field,  $\text{grade } I = 2$ , and  $A/I$  is a Cohen-Macaulay ring.

## G. Comments and References

Many of the auxiliary results in this section may be classified as “folklore”, even if some of them should have been documented in the literature.

Proposition (16.1) is a theorem of Fitting [Fi], thus the notion “Fitting invariant”. Our definition of “rank” and its treatment are borrowed from Scheja and Storch ([SS], section 6). It may have appeared elsewhere.

In Subsection B we have given references to Matsumura [Mt] for the basic notion “depth” and its extension “grade”. Another good source for the theory of grade is Kaplansky’s book [Ka], pp. 89 – 103. Our notation  $\text{grade}(I, M)$  is his  $G(I, M)$ ; (16.14), for example, is an exercise on p.103 of [Ka]. The grade of a module as defined below (16.11) has been introduced in Rees’ fundamental paper [Re], the equality in (16.11) serving as the definition.

The utmost important acyclicity criterion (16.15) is (almost) identical with [BE.2], Theorem. It is closely related to the lemme d’acyclicité of Peskine and Szpiro ([PS], (1.8)). Our proof may be new (though perhaps not original).

The notion “perfect” goes back to Macaulay ([Ma], p. 87). Our definition which is copied from Rees [Re] is an abstract and generalized version of Gröbner’s ([Gb], p. 197). The description of the relationship between the properties of being perfect and being Cohen-Macaulay as given above, is just a technical elaboration of Rees’ results ([Re], p. 41).

Our treatment of the process of dehomogenization has been inspired by unpublished lecture notes of Storch, it is certainly the standard one nowadays. A detailed discussion is to be found in [ZS], Ch. VII, §§ 5, 6.

Subsection E is based on Auslander and Bridger’s monograph [ABd]. It is difficult to say something about the notion “ $n$ -torsionfree” and its relatives not being contained in [ABd] already. However, the treatment in [ABd] suffers from a rather heavy technical apparatus, and the inclusion of subsection E should be regarded as an attempt to make the results of [ABd] directly accessible.

The version of the Hilbert-Burch theorem given above has been drawn from [BE.3], Theorem 0. It can be greatly extended, cf. [BE.3], Theorem 3.1.

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## Index of Notations

Notations which seem to be completely standard or have been used in accordance with [Mt], have not been listed.

$\mathbf{N}$	set of non-negative integers
$\mathbf{N}_+$	set of positive integers
$\mathbf{Z}$	ring of integers
$\mathbf{Q}$	field of rational numbers
$\mathbf{C}$	field of complex numbers
$\text{rk } M$	rank of a module, 1, 204
$\text{rk } f$	rank of (the image of) a homomorphism, 2
$\lambda(M)$	length of a module, 2
$M^*$	dual of a module, 2
$f^*$	dual of a homomorphism, 2
$\bigwedge^i M$	$i$ -th exterior power of a module, 2
$S_j(M)$	$j$ -th symmetric power of a module, 2
$e_I, e_I^*$	2
$\sigma(I_1, \dots, I_n), \sigma(i_1, \dots, i_n)$	signum of a permutation, 2
$ I $	cardinality of a set, 2
$S(m, I)$	2
$1, \dots, \widehat{i}, \dots, n$	$i$ is to be omitted from $1, \dots, n$
$[a_1, \dots, a_t   b_1, \dots, b_t]$	$t$ -minor of a matrix, 3
$[a_1, \dots, a_m]$	maximal minor of a matrix, 3
$I_t(U)$	ideal generated by the $t$ -minors of a matrix, 3
$\text{Cof } U$	matrix of cofactors of a matrix, 3
$R_t(X)$	determinantal ring, 4
$L_t(V, W), L(V, W)$	determinantal variety, 4
$\mathbf{P}^N(K)$	projective $N$ -space over the field $K$
$\bigwedge^i f$	$i$ -th exterior power of the homomorphism $f$
$\mathbf{P}(V)$	projective space of the vector space $V$
$\mathbf{A}^N(K)$	affine $N$ -space over the field $K$
$G(X)$	$B$ -subalgebra of $B[X]$ generated by the maximal minors of $X$ , 7
$G_m(V)$	Grassmann variety, 8
$\text{GL}(V)$	group of automorphisms of a vector space, 8
$\Omega(a_1, \dots, a_m)$	Schubert variety, 8
$S(M)$	symmetric algebra of the module $M$

$\mathcal{C}(g)$	17
$\mathcal{C}_i(g), \mathcal{C}_i^*(g)$	17
$\mathcal{D}_i(g)$	17
$\mathcal{C}(X)$	21
$\mathcal{M}_n(A)$	set of $n \times n$ matrices with entries in a ring, 22
$\mathcal{G}(U)$	Gulliksen-Negård complex, 22
grade $M$	grade of a module, 206
$\omega_A$	canonical module of a Cohen-Macaulay ring, 210
$\text{Gr}_I A, \text{Gr}_I M$	associated graded ring, module, 30
$x^*$	leading form, 30
$U^*$	form module, 30
$\Gamma(X)$	set of maximal minors of a matrix, 46
$\Delta(X)$	set of all minors of a matrix, 46
$I(X; \delta), I(x; \delta)$	determinantal ideal, 51
$R(X; \delta)$	determinantal ring, 51
$\Delta(X; \delta)$	51
$I(X; \gamma)$	ideal defining a Schubert cycle, 52
$G(X; \gamma)$	Schubert cycle, 52
$\Gamma(X; \gamma)$	52
$\text{rk } \xi$	rank of an element in a poset, 55
$\text{rk } \Omega$	rank of a subset of a poset, 55
$\text{ara } I$	arithmetical rank of an ideal, 61
$\Sigma(X; \gamma)$	67
$\Xi(X; \delta)$	69
$\text{GL}(r, B)$	group of invertible $r \times r$ matrices over a ring, 74
$A^G$	subring of invariants, 74
$\text{SL}(r, B)$	group of $r \times r$ matrices with determinant 1, 74
$V^G$	subspace of invariants, 81
$H_I^i(A)$	cohomology with support in an ideal, 81
$M^G$	module of invariants, 88
grade( $I, M$ )	grade of an ideal with respect to a module, 206
$\text{Cl}(S)$	divisor class group of a normal domain, 93
$\text{cl}(I)$	divisor class of a fractionary ideal, 94
$\text{div}(I)$	divisor of a fractionary ideal, 94
$\mathcal{R}_I(A)$	Rees algebra, 108
$\widehat{\mathcal{R}}_I(A)$	extended Rees algebra, 108
$\text{grad } x$	108
$\Pi^*$	108
$\Pi \uplus \Omega$	109
$v_P$	valuation associated with a divisorial prime ideal, 116
$l(I)$	analytic spread of an ideal in a local ring, 117

$\text{Gr}_{\mathcal{F}}A$	associated graded ring with respect to a filtration, 118
$\gamma_t(\delta)$	123
$\text{Gr}_P^{(0)}A$	symbolic graded ring, 124
$\widehat{\mathcal{R}}_P^{(0)}A$	extended symbolic Rees ring, 124
$e(j, t), e_j$	126
$F(i, j)$	126
$ \mu $	shape of a monomial, 136
$I^{(\sigma)}$	136
$I^\sigma$	137
$\Sigma$	(Young) tableau, 137
$(\Sigma, T)$	bitableau, 138
$c(\Sigma), c(\Sigma, T)$	content of a tableau, bitableau, 138
$K_\sigma, \overline{K}_\sigma$	139
$\Lambda_\sigma, \overline{\Lambda}_\sigma$	139
$I_{>}^{(\sigma)}$	139
${}_\sigma L, L_\sigma$	141
$U^+(n, K), (U^-(n, K))$	group of upper (lower) triangular $n \times n$ matrices with entry 1 on all diagonal positions, 141
$D(n, K)$	group of invertible diagonal $n \times n$ matrices, 143
$V_\omega$	isotypic component, 143
$M_\sigma$	143
$I_\sigma$	146
$I(S)$	147
$\text{Rad } S$	radical of a $D$ -ideal, 147
$I(s_0, \dots, s_r)$	155
$I(s_0, \dots, s_r; v)$	156
$\varphi_{f,r}$	165
$D(M)$	Auslander-Bridger dual of a module, 214
$\Omega_{R/B}^1, \Omega$	module of Kähler differentials, 174
$\delta_{st}$	175
$I(X; \delta_{s+1,t}, \delta_{s,t+1}), I(x; \delta_{s+1,t}, \delta_{s,t+1})$	175
$\Delta(X; \delta_{s+1,t}, \delta_{s,t+1})$	175
$d\alpha, \overline{d}\alpha$	differential of $\alpha$ , 177
$M(s, t), \overline{M}(s, t)$	178
$\varphi, \chi, \psi$	184
$N_{kl}$	186
$D_S$	Auslander-module, 198
$\mu(M)$	minimal number of generators of a module, 203
$(\widetilde{S}_n)$	Serre type condition for modules, 213

# Subject Index

- absolute invariant, 74, 86
- absolute semi-invariant, 76
- acyclicity criterion, 207, 208
- algebra generated by the  $t$ -minors, 132, 133, 151
- algebra with straightening law (ASL), 38
- algebraic quotient, 80
- analytic spread, 117
- arithmetical rank, 61
  - of a determinantal ideal, 62, 81
  - of an ideal defining a Schubert cycle, 62
  - of an ideal generated by a poset ideal, 62
  - in a symmetric ASL, 62
- ASL, 38
  - discrete, 62
  - symmetric, 43
- ASL-property of
  - a determinantal ring, 52
  - a graded ring derived from a straightening closed ideal, 108, 110
  - a Schubert cycle, 53
  - $B[X]$  on  $\Delta(X)$ , 48
  - $G(X)$  on  $\Gamma(X)$ , 41
- associated graded module, 30
- associated graded ring 30, 108
  - derived from a straightening-closed ideal, 109
  - derived from an ideal of maximal minors, 112, 114
  - derived from the ideal of  $t$ -minors, 132
- associated prime ideals, 1
  - of a  $G$ -stable ideal 148, 149
  - of powers of a determinantal ideal, 132
- Auslander-Bridger dual, 214
- Auslander-Buchsbaum equation, 1
- Auslander-module, 198
  - and rigidity, 199, 200
- bitableau, 138
  - left (right) final, 139
  - left (right) initial, 139
  - standard, 138
- canonical (divisor) class, 93
  - of a determinantal ring, 97, 98, 103
  - of a Schubert cycle, 97, 102
- canonical module, 210
  - of a determinantal ring, 22, 97, 98, 103, 115
  - of a Schubert cycle, 97, 102, 115
- class group, s. divisor class group
- cofactors, matrix of, 3
- Cohen-Macaulay
  - module, 209
  - ring, 209
  - type of a determinantal ring, 115
- Cohen-Macaulay property
  - and perfection, 210
  - of a determinantal ring, 13, 25, 60
  - of a graded ring derived from an ideal of maximal minors, 112
  - of an ASL, 59
  - of a Schubert cycle, 60
- complete intersection, 198
- content of a tableau (bitableau), 138
- decomposition of  $K[X]$  into irreducible  $G$ -modules, 143, 144
- defining ideal
  - of an ASL, 40
  - of  $G(X)$ , 43
- dehomogenization, 211
  - and being reduced, 212
  - determinantal ring as a, 53
  - integrality, 212
  - normality, 212
  - of a homogeneous ideal, 212, 213
  - $B[X]$  as a, 48



- depth, *s.* also grade
  - of a module, 1, 206
- derivations, module of, 184
  - almost perfection, 188
  - perfection, 192
  - system of generators, 188
  - syzygetic behaviour, 197
  - syzygies, 189
- determinantal ideal, 5, 51, 155, 156, 175
  - of a homomorphism, 202
- determinantal ring, 4, 5
  - as a ring of absolute invariants, 76, 86, 87, 160
- determinantal variety, 4, 6
- diagram, 136
- D*-ideal, 145
  - primary, 147
  - prime, 147
  - principal, 146
  - radical, 147
- differentials, module of, 174
  - depth, 180, 181, 183
  - free locus, 178
  - grade, 180, 183
  - presentation as a second syzygy, 186
  - syzygetic behaviour, 182
  - syzygies, 181
- dimension
  - of a determinantal ring, 57
  - of an ASL, 55
  - of a Schubert cycle, 57
- discrete ASL, 62
- divisor class group
  - of a determinantal ring, 95, 96, 160
  - of a Schubert cycle, 95
- dual
  - basis, 2
  - of a homomorphism, 2
  - of a module, 2
- dualizing module, 210
  
- Eagon-Northcott complex, 16
- extended Rees algebra, 108
  - derived from
    - an ideal of maximal minors, 112, 114
    - a straightening-closed ideal, 108, 109
  - extended symbolic Rees ring, 124
    - derived from a determinantal ideal, 124
  - extraspecial pair, 43
- factoriality of a
  - determinantal ring, 95
  - Schubert cycle, 69, 95
- Fitting invariants of a module, 203
- form ideal, form module, 30
- free resolution, 18, 21
  - of an ideal
    - of maximal minors, 18
    - of submaximal minors in a quadratic matrix, 25
  - of a generic module, 172
  - of symmetric and exterior powers, 18, 21
  - of the image of a generic map, 172
- full semigroup, 150
  
- G*-decomposition of  $K[X]$ , 143, 144
- generically perfect
  - ideal, 27
  - module, 27
- generic map, 162
  - free resolution of the image, 172
  - perfection of the image, 164, 165
- generic module, 162
  - almost perfection, 164
  - free resolution, 172
  - perfection, 18, 167, 170
  - projective dimension, 171
  - reflexivity, 162
  - syzygetic behaviour, 171, 172
- generic perfection, 27
  - and substitution of indeterminates, 31, 32
  - of a determinantal ideal, 61
  - of a Schubert cycle, 61
  - of the canonical module, 30
- generic point, 73
  - of a determinantal ring, 74, 85, 87
  - of a Schubert cycle, 83, 85
- G*-module, 80
- Gorenstein property, 210
  - of a determinantal ring, 22, 25, 98, 104
  - of a Schubert cycle, 102

- Gorenstein ring, 210
- grade, 206
  - of a determinantal ideal, 12, 61, 159
  - of a module, 206
  - of an ideal, 206
  - and projective dimension, 207
  - behaviour along exact sequences, 207
  - of an ideal defining a Schubert cycle, 61
  - homological description, 206
  - local description, 207
- grade estimates
  - for modules of invariants, 89
  - for powers of ideals of maximal minors, 90, 118, 119
  - for symbolic powers of a determinantal ideal, 125
- Grassmann variety, 6, 8
- $G$ -stable ideals of  $K[X]$ , 146
- $G$ -submodule generated by a bitableau, 145
- Gulliksen-Negård complex, 22
- height of
  - a determinantal ideal, 12, 61, 159, 175
  - an ideal generated by minors of a matrix, 10, 11
  - an ideal defining a Schubert cycle, 61
  - a specialization of a perfect ideal, 36
- Hilbert-Burch theorem, 218
- homogenization of an ideal, 212, 213
- ideal in a partially ordered set, 50
  - cogenerated by a subset, 51
  - generated by a subset, 51
- ideals of maximal minors, 10, 105
  - free resolution, 18
- indecomposability of the subspace of (right) initial bitableaux, 142
- induction argument for determinantal rings, s. localization argument
- integral closure of a  $G$ -stable ideal, 149
- integrity of a
  - determinantal ring, 14, 65, 158
  - graded ring derived from an ideal of maximal minors, 114
  - Schubert cycle, 65
- invariant, 80
  - absolute, 74, 86
  - theory, main problems, 80
  - versus absolutely invariant, 75
- isotopic component, 142
- Kähler differentials, s. differentials
- Koszul complex, 17
- leading form, 30, 108
- linear independence of standard monomials in  $G(X)$ , 43, 84
- linearly reductive, 80, 81
- localization argument for
  - determinantal rings, 11, 66, 122
  - Schubert cycles, 64
- locally upper semi-modular poset, 58
- maximal minors, 3, 105
  - ideals of, 10, 105
  - powers of ideals of, 106, 114
  - subalgebra of, 106, 107, 112
- minimal primes of an ideal generated by a poset ideal
  - in  $\Delta(X)$ , 66
  - in  $\Gamma(X)$ , 66
- minor of a matrix, 3
- module
  - torsionfree, 213
  - $n$ -torsionfree, 213
  - $n$ -th syzygy, 214
  - $n$ -torsionless, 214
  - of derivations, s. derivations
  - of differentials, s. differentials
  - reflexive, 214
- multiplicity of a (maximal minors) determinantal ring, 16
- normality criterion, 212
- normality of a
  - determinantal ring, 15, 61, 65, 150, 160
  - graded ring derived from an ideal generated by maximal minors, 114
  - Schubert cycle, 65
- normal subsemigroup, 150

- perfect
  - ideal, 209
  - module, 209
- perfection, 209
  - and Cohen-Macaulay property, 210
  - of a determinantal ideal, 13, 18, 25, 61, 159, 175
  - of a Schubert cycle, 61
  - of a specialization, 28, 29
- Plücker
  - coordinates, 6
  - map, 6
  - relations, 41
- poset, 39
- powers of
  - an ideal generated by a regular sequence, 15, 29
  - an ideal of maximal minors, 79, 114
  - determinantal ideals, 79, 114, 130, 149
- primary decomposition of
  - a  $G$ -stable ideal, 148, 149
  - products of determinantal ideals, 126, 130, 149
- prime elements in a determinantal ring, 97
- principal radical system, 155
  - of determinantal ideals, 158
- products of determinantal ideals, 126, 137
- radical
  - of a  $D$ -ideal, 147
  - subsemigroup, 150
- rank of a
  - free module, 1
  - module, 1, 204
  - poset element, 55
  - subset of a poset, 55
- rational representation, 80
- rational singularity, 81, 150
- reduced
  - ASL, 54
  - determinantal ring, 15, 55
  - ring, 1
  - Schubert cycle, 55
- Rees algebra, 108
  - derived from
    - an ideal of maximal minors, 112, 114
    - a straightening-closed ideal, 110, 111
  - extended, 108
- reflexive module, 214
- regular elements in an ASL, 55, 56
- Reynolds operator, 81, 88
- rigidity, 198, 200
- ring of invariants
  - being noetherian, 81
  - normality, 81, 115, 149
  - rational singularities, 81, 149
  - Cohen-Macaulay property, 81
  - existence of a Reynolds operator, 81
- ring of  $U$ -invariants, 149
  - in a determinantal ring, 150
- Schubert cycle, 6, 9
  - as a ring of absolute invariants, 76, 86
- Schubert variety, 6, 8
- Segre product of ASLs, 107
- self-covering poset ideal, 110
- Serre's condition
  - $(R_2)$  for a determinantal ring, 71
  - $(R_2)$  for a Schubert cycle, 71
  - $(S_n)$  for an ASL, 60
- Serre type criterion for torsionfree modules, 213
- shape of a monomial, 135
- singular locus of a
  - determinantal ring, 12, 70, 71, 160
  - Schubert cycle, 68
- special pair, 43
- spectrum of a ring, 1
- standard
  - basis, 38
  - bitableau, 138
  - monomial, 38
  - tableau, 138
- standard monomials as generators of an ASL, 39
- standard representation, 39
  - of a bitableau, 138, 139
  - of a monomial, 39

- straightening
  - law, 38
  - relations, 38
- straightening-closed ideal, 108
  - powers of  $\mathfrak{a}$ , 108
  - extended Rees-algebra derived from  $\mathfrak{a}$ , 108
- submaximal minor, 3
- support of a module, 1
- support of a monomial, 131
- symbolic graded ring, 124
  - derived from a determinantal ideal, 124
- symbolic powers of determinantal ideals, 122, 123, 124, 126, 129, 130, 149
  - intersection of, 136, 137
- symmetric algebra of a straightening-closed ideal, 111
- symmetric ASL, 43
- syzygies of
  - a generic module, 166
  - an ideal generated by a poset ideal, 53
  - determinantal ideals, 164, 175
- syzygy module, 214
- tableau, 137
  - final, 139
  - initial, 139
  - nested, 145
  - standard, 139
- torsionfree module, 213
- torsionless module, 214
- wonderful poset, 58
- Young diagram, 136