Student's Manual to Accompany

Introduction to Probability Models

Tenth Edition

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Academic Press is an imprint of Elsevier

Academic Press is an imprint of Elsevier
30 Corporate Drive, Suite 400, Burlington, MA 01803, USA
525 B Street, Suite 1900, San Diego, California 92101-4495, USA
Elsevier, The Boulevard, Langford Lane, Kidlington, Oxford, OX5 1GB, UK

ISBN: 978-0-12-381446-3

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Typeset by: diacriTech, India

09 10 9 8 7 6 5 4 3 2 1

Contents

Chapter 1

1. $S = \{(R,R),(R,G),(R,B),(G,R),(G,G),(G,B),$ $(B,R),(B,G),(B,B)\}$

   The probability of each point in S is 1/9.

3. $S = \{(e_1, e_2, \ldots, e_n),\ n \geq 2\}$ where $e_i \in$ (heads, tails}. In addition, $e_n = e_{n-1} =$ heads and for $i = 1, \ldots, n-2$ if $e_i =$ heads, then $e_{i+1} =$ tails.

   $$P\{4 \text{ tosses}\} = P\{(t,t,h,h)\} + P\{(h,t,h,h)\}$$
   $$= 2\left[\frac{1}{2}\right]^4 = \frac{1}{8}$$

5. $\frac{3}{4}$. If he wins, he only wins \$1, while if he loses, he loses \$3.

7. If $(E \cup F)^c$ occurs, then $E \cup F$ does not occur, and so E does not occur (and so E^c does); F does not occur (and so F^c does) and thus E^c and F^c both occur. Hence,

   $$(E \cup F)^c \subset E^c F^c$$

   If $E^c F^c$ occurs, then E^c occurs (and so E does not), and F^c occurs (and so F does not). Hence, neither E or F occurs and thus $(E \cup F)^c$ does. Thus,

   $$E^c F^c \subset (E \cup F)^c$$

   and the result follows.

9. $F = E \cup FE^c$, implying since E and FE^c are disjoint that $P(F) = P(E) + P(FE)^c$.

11. $$P\{\text{sum is } i\} = \begin{cases} \dfrac{i-1}{36}, & i = 2, \ldots, 7 \\ \dfrac{13-i}{36}, & i = 8, \ldots, 12 \end{cases}$$

13. Condition an initial toss

    $$P\{\text{win}\} = \sum_{i=2}^{12} P\{\text{win} \mid \text{throw } i\} P\{\text{throw } i\}$$

    Now,

    $$P\{\text{win} \mid \text{throw } i\} = P\{i \text{ before } 7\}$$
    $$= \begin{cases} 0 & i = 2, 12 \\ \dfrac{i-1}{5+1} & i = 3, \ldots, 6 \\ 1 & i = 7, 11 \\ \dfrac{13-i}{19-1} & i = 8, \ldots, 10 \end{cases}$$

    where above is obtained by using Problems 11 and 12.

    $P\{\text{win}\} \approx .49$.

17. $$\text{Prob}\{\text{end}\} = 1 - \text{Prob}\{\text{continue}\}$$
    $$= 1 - P(\{H,H,H\} \cup \{T,T,T\})$$
    $$= 1 - [\text{Prob}(H,H,H) + \text{Prob}(T,T,T)].$$

    $$\text{Fair coin: Prob}\{\text{end}\} = 1 - \left[\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right]$$
    $$= \frac{3}{4}$$

    $$\text{Biased coin: } P\{\text{end}\} = 1 - \left[\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}\right]$$
    $$= \frac{9}{16}$$

19. $E =$ event at least 1 six $P(E)$

    $$= \frac{\text{number of ways to get } E}{\text{number of sample pts}} = \frac{11}{36}$$

    $D =$ event two faces are different $P(D)$

    $$= 1 - \text{Prob(two faces the same)}$$
    $$= 1 - \frac{6}{36} = \frac{5}{6} P(E|D) = \frac{P(ED)}{P(D)} = \frac{10/36}{5/6} = \frac{1}{3}$$

21. Let C = event person is color blind.

$P(\text{Male}|C)$

$$= \frac{P(C|\text{Male})\,P(\text{Male})}{P(C|\text{Male}\,P(\text{Male}) + P(C|\text{Female})\,P(\text{Female})}$$

$$= \frac{.05 \times .5}{.05 \times .5 + .0025 \times .5}$$

$$= \frac{2500}{2625} = \frac{20}{21}$$

23. $P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$

$$= P(E_1)\frac{P(E_1E_2)}{P(E_1)}\frac{P(E_1E_2E_3)}{P(E_1E_2)}\cdots\frac{P(E_1\cdots E_n)}{P(E_1\cdots E_{n-1})}$$

$$= P(E_1\cdots E_n)$$

25. (a) $P\{pair\} = P\{\text{second card is same denomination as first}\}$

$= 3/51$

(b) $P\{\text{pair}|\text{different suits}\}$

$$= \frac{P\{\text{pair, different suits}\}}{P\{\text{different suits}\}}$$

$= P\{\text{pair}\}/P\{\text{different suits}\}$

$$= \frac{3/51}{39/51} = 1/13$$

27. $P(E_1) = 1$
$P(E_2|E_1) = 39/51$, since 12 cards are in the ace of spades pile and 39 are not.

$P(E_3|E_1E_2) = 26/50$, since 24 cards are in the piles of the two aces and 26 are in the other two piles.

$P(E_4|E_1E_2E_3) = 13/49$

So

$P\{\text{each pile has an ace}\} = (39/51)(26/50)(13/49)$

29. (a) $P(E|F) = 0$

(b) $P(E|F) = P(EF)/P(F) = P(E)/P(F) \geq P(E) = .6$

(c) $P(E|F) = P(EF)/P(F) = P(F)/P(F) = 1$

31. Let S = event sum of dice is 7; F = event first die is 6.

$$P(S) = \frac{1}{6}P(FS) = \frac{1}{36}P(F|S) = \frac{P(F|S)}{P(S)}$$

$$= \frac{1/36}{1/6} = \frac{1}{6}$$

33. Let S = event student is sophomore; F = event student is freshman; B = event student is boy; G = event student is girl. Let x = number of sophomore girls; total number of students = $16 + x$.

$$P(F) = \frac{10}{16+x}P(B) = \frac{10}{16+x}P(FB) = \frac{4}{16+x}$$

$$\frac{4}{16+x} = P(FB) = P(F)P(B) = \frac{10}{16+x}$$

$$\frac{10}{16+x} \Rightarrow x = 9$$

35. (a) 1/16

(b) 1/16

(c) 15/16, since the only way in which the pattern H, H, H, H can appear before the pattern T, H, H, H is if the first four flips all land heads.

37. Let W = event marble is white.

$$P(B_1|W) = \frac{P(W|B_1)P(B_1)}{P(W|B_1)P(B_1) + P(W|B_2)P(B_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5}$$

39. Let W = event woman resigns; A, B, C are events the person resigning works in store A, B, C, respectively.

$P(C|W)$

$$= \frac{P(W|C)P(C)}{P(W|C)P(C) + P(W|B)P(B) + P(W|A)P(A)}$$

$$= \frac{.70 \times \frac{100}{225}}{.70 \times \frac{100}{225} + .60 \times \frac{75}{225} + .50\,\frac{50}{225}}$$

$$= \frac{70}{225}\Big/\frac{140}{225} = \frac{1}{2}$$

41. Note first that since the rat has black parents and a brown sibling, we know that both its parents are hybrids with one black and one brown gene (for if either were a pure black then all their offspring would be black). Hence, both of their offspring's genes are equally likely to be either black or brown.

(a) $P(\text{2 black genes} \mid \text{at least one black gene})$

$$= \frac{P(\text{2 black genes})}{P(\text{at least one black gene})}$$

$$= \frac{1/4}{3/4} = 1/3$$

(b) Using the result from part (a) yields the following:

P(2 black genes | 5 black offspring)

$$= \frac{P(\text{2 black genes})}{P(\text{5 black offspring})}$$

$$= \frac{1/3}{1(1/3) + (1/2)^5(2/3)}$$

$$= 16/17$$

where P(5 black offspring) was computed by conditioning on whether the rat had 2 black genes.

43. Let $i =$ event coin was selected; $P(H|i) = \dfrac{i}{10}$.

$$P(5|H) = \frac{P(H|5)P(5)}{\sum_{i=1}^{10} P(H|i)P(i)} = \frac{\frac{5}{10} \cdot \frac{1}{10}}{\sum_{i=1}^{10} \frac{1}{10} \cdot \frac{1}{10}}$$

$$= \frac{5}{\sum_{i=1}^{10} i} = \frac{1}{11}$$

45. Let $B_i =$ event i^{th} ball is black; $R_i =$ event i^{th} ball is red.

$$P(B_1|R_2) = \frac{P(R_2|B_1)P(B_1)}{P(R_2|B_1)P(B_1) + P(R_2|R_1)P(R_1)}$$

$$= \frac{\frac{r}{b+r+c} \cdot \frac{b}{b+r}}{\frac{r}{b+r+c} \cdot \frac{b}{b+r} + \frac{r+c}{b+r+c} \cdot \frac{r}{b+r}}$$

$$= \frac{rb}{rb + (r+c)r}$$

$$= \frac{b}{b+r+c}$$

47. 1. $0 \le P(A|B) \le 1$

2. $P(S|B) = \dfrac{P(SB)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

3. For disjoint events A and D

$$P(A \cup D|B) = \frac{P((A \cup D)B)}{P(B)}$$

$$= \frac{P(AB \cup DB)}{P(B)}$$

$$= \frac{P(AB) + P(DB)}{P(B)}$$

$$= P(A|B) + P(D|B)$$

Direct verification is as follows:

$P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B)$

$$= \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(B)} + \frac{P(ABC^c)}{P(BC^c)}\frac{P(BC^c)}{P(B)}$$

$$= \frac{P(ABC)}{P(B)} + \frac{P(ABC^c)}{P(B)}$$

$$= \frac{P(AB)}{P(B)}$$

$$= P(A|B)$$

Chapter 2

1. $P\{X=0\} = \begin{bmatrix}7\\2\end{bmatrix} \Big/ \begin{bmatrix}10\\2\end{bmatrix} = \dfrac{14}{30}$

3. $P\{X=-2\} = \dfrac{1}{4} = P\{X=2\}$

   $P\{X=0\} = \dfrac{1}{2}$

5. $P\{\max = 6\} = \dfrac{11}{36} = P\{\min = 1\}$

   $P\{\max = 5\} = \dfrac{1}{4} = P\{\min = 2\}$

   $P\{\max = 4\} = \dfrac{7}{36} = P\{\min = 3\}$

   $P\{\max = 3\} = \dfrac{5}{36} = P\{\min = 4\}$

   $P\{\max = 2\} = \dfrac{1}{12} = P\{\min = 5\}$

   $P\{\max = 1\} = \dfrac{1}{36} = P\{\min = 6\}$

7. $p(0) = (.3)^3 = .027$

   $p(1) = 3(.3)^2(.7) = .189$

   $p(2) = 3(.3)(.7)^2 = .441$

   $p(3) = (.7)^3 = .343$

9. $p(0) = \dfrac{1}{2}, \quad p(1) = \dfrac{1}{10}, \quad p(2) = \dfrac{1}{5},$

   $p(3) = \dfrac{1}{10}, \quad p(3.5) = \dfrac{1}{10}$

11. $\dfrac{3}{8}$

13. $\displaystyle\sum_{i=7}^{10} \binom{10}{i} \left[\frac{1}{2}\right]^{10}$

15. $\dfrac{P\{X=k\}}{P\{X=k-1\}}$

$$= \frac{\dfrac{n!}{(n-k)!\,k!}p^k(1-p)^{n-k}}{\dfrac{n!}{(n-k+1)!(k-1)!}p^{k-1}(1-p)^{n-k+1}}$$

$$= \frac{n-k+1}{k}\,\frac{p}{1-p}$$

   Hence,

$$\frac{P\{X=k\}}{P\{X=k-1\}} \geq 1 \leftrightarrow (n-k+1)p > k(1-p)$$
$$\leftrightarrow (n+1)p \geq k$$

   The result follows.

17. Follows since there are $\dfrac{n!}{x_1!\cdots x_r!}$ permutations of n objects of which x_1 are alike, x_2 are alike, …, x_r are alike.

19. $P\{X_1 + \cdots + X_k = m\}$

$$= \begin{bmatrix}n\\m\end{bmatrix}(p_1 + \cdots + p_k)^m(p_{k+1} + \cdots + p_r)^{n-m}$$

21. $1 - \left[\dfrac{3}{10}\right]^5 - 5\left[\dfrac{3}{10}\right]^4\left[\dfrac{7}{10}\right] - \begin{bmatrix}5\\2\end{bmatrix}\left[\dfrac{3}{10}\right]^3\left[\dfrac{7}{10}\right]^2$

23. In order for X to equal n, the first $n-1$ flips must have $r-1$ heads, and then the n^{th} flip must land heads. By independence the desired probability is thus

$$\begin{bmatrix}n-1\\r-1\end{bmatrix}p^{r-1}(1-p)^{n-r}xp$$

25. A total of 7 games will be played if the first 6 result in 3 wins and 3 losses. Thus,

$$P\{7 \text{ games}\} = \binom{6}{3}p^3(1-p)^3$$

   Differentiation yields

$$\frac{d}{dp}P\{7\} = 20\left[3p^2(1-p)^3 - p^3 3(1-p)^2\right]$$
$$= 60p^2(1-p)^2\left[1 - 2p\right]$$

Thus, the derivative is zero when $p = 1/2$. Taking the second derivative shows that the maximum is attained at this value.

27. $P\{\text{same number of heads}\} = \sum_i P\{A = i, B = i\}$

$$= \sum_i \binom{k}{i}(1/2)^k \binom{n-k}{i}(1/2)^{n-k}$$
$$= \sum_i \binom{k}{i}\binom{n-k}{i}(1/2)^n$$
$$= \sum_i \binom{k}{k-i}\binom{n-k}{i}(1/2)^n$$
$$= \binom{n}{k}(1/2)^n$$

Another argument is as follows:

$P\{\#\text{ heads of } A = \#\text{ heads of } B\}$
$= P\{\#\text{ tails of } A = \#\text{ heads of } B\}$

since coin is fair

$= P\{k - \#\text{ heads of } A = \#\text{ heads of } B\}$
$= P\{k = \text{total \# heads}\}$

29. Each flip after the first will, independently, result in a changeover with probability 1/2. Therefore,

$$P\{k \text{ changeovers}\} = \binom{n-1}{k}(1/2)^{n-1}$$

33. $c\int_{-1}^{1}\left(1-x^2\right)dx = 1$

$$c\left[x - \frac{x^3}{3}\right]\Bigg|_{-1}^{1} = 1$$
$$c = \frac{3}{4}$$

$$F(y) = \frac{3}{4}\int_{-1}^{1}(1-x^2)dx$$
$$= \frac{3}{4}\left[y - \frac{y^3}{3} + \frac{2}{3}\right], \qquad -1 < y < 1$$

35. $P\{X > 20\} = \int_{20}^{\infty}\frac{10}{x^2}dx = \frac{1}{2}$

37. $P\{M \le x\} = P\{\max(X_1, \ldots, X_n) \le x\}$

$$= P\{X_1 \le x, \ldots, X_n \le x\}$$
$$= \prod_{i=1}^{n} P\{X_i \le x\}$$
$$= x^n$$

$$f_M(x) = \frac{d}{dx}P\{M \le x\} = nx^{n-1}$$

39. $E[X] = \frac{31}{6}$

41. Let X_i equal 1 if a changeover results from the i^{th} flip and let it be 0 otherwise. Then

$$\text{number of changeovers} = \sum_{i=2}^{n} X_i$$

As,

$$E[X_i] = P\{X_i = 1\} = P\{\text{flip } i-1 \ne \text{flip } i\}$$
$$= 2p(1-p)$$

we see that

$$E[\text{number of changeovers}] = \sum_{i=2}^{n} E[X_i]$$
$$= 2(n-1)p(1-p)$$

43. (a) $X = \sum_{i=1}^{n} X_i$

(b) $E[X_i] = P\{X_i = 1\}$
$= P\{\text{red ball } i \text{ is chosen before all } n \text{ black balls}\}$
$= 1/(n+1)$ since each of these $n+1$ balls is equally likely to be the one chosen earliest

Therefore,

$$E[X] = \sum_{i=1}^{n} E[X_i] = n/(n+1)$$

45. Let N_i denote the number of keys in box i, $i = 1, \ldots, k$. Then, with X equal to the number of collisions we have that $X = \sum_{i=1}^{k}(N_i - 1)^+ = \sum_{i=1}^{k}(N_i - 1 + I\{N_i = 0\})$ where $I\{N_i = 0\}$ is equal to 1 if $N_i = 0$ and is equal to 0 otherwise. Hence,

$$E[X] = \sum_{i=1}^{k} (rp_i - 1 + (1-p_i)^r) = r - k + \sum_{i=1}^{k}(1-p_i)^r$$

Another way to solve this problem is to let Y denote the number of boxes having at least one key, and then use the identity $X = r - Y$, which is true since only the first key put in each box does not result in a collision. Writing $Y = \sum_{i=1}^{k} I\{N_i > 0\}$ and taking expectations yields

$$E[X] = r - E[Y] = r - \sum_{i=1}^{k}[1-(1-p_i)^r] = r - k + \sum_{i=1}^{k}(1-p_i)^r$$

47. Let X_i be 1 if trial i is a success and 0 otherwise.

(a) The largest value is .6. If $X_1 = X_2 = X_3$, then

$$1.8 = E[X] = 3E[X_1] = 3P\{X_1 = 1\}$$

and so

$$P\{X = 3\} = P\{X_1 = 1\} = .6$$

That this is the largest value is seen by Markov's inequality, which yields

$$P\{X \geq 3\} \leq E[X]/3 = .6$$

(b) The smallest value is 0. To construct a probability scenario for which $P\{X = 3\} = 0$ let U be a uniform random variable on (0, 1), and define

$$X_1 = \begin{matrix} 1 & \text{if } U \leq .6 \\ 0 & \text{otherwise} \end{matrix}$$

$$X_2 = \begin{matrix} 1 & \text{if } U \geq .4 \\ 0 & \text{otherwise} \end{matrix}$$

$$X_3 = \begin{matrix} 1 & \text{if either } U \leq .3 \quad \text{or} \quad U \geq .7 \\ 0 & \text{otherwise} \end{matrix}$$

It is easy to see that

$$P\{X_1 = X_2 = X_3 = 1\} = 0$$

49. $E[X^2] - (E[X])^2 = Var(X) = E(X - E[X])^2 \geq 0$. Equality when $Var(X) = 0$, that is, when X is constant.

51. $N = \sum_{i=1}^{r} X_j$ where X_i is the number of flips between the $(i-1)^{st}$ and i^{th} head. Hence, X_i is geometric with mean $1/p$. Thus,

$$E[N] = \sum_{i=1}^{r} E[X_i] = \frac{r}{p}$$

53. $\frac{1}{n+1}, \quad \frac{1}{2n+1} - \left[\frac{1}{n+1}\right]^2.$

55. (a) $P(Y = j) = \sum_{i=0}^{j} \binom{j}{i} e^{-2\lambda} \lambda^j / j!$

$$= e^{-2\lambda} \frac{\lambda^j}{j!} \sum_{i=0}^{j} \binom{j}{i} 1^i 1^{j-i}$$

$$= e^{-2\lambda} \frac{(2\lambda)^j}{j!}$$

(b) $P(X = i) = \sum_{j=i}^{\infty} \binom{j}{i} e^{-2\lambda} \lambda^j / j!$

$$= \frac{1}{i!} e^{-2\lambda} \sum_{j=i}^{\infty} \frac{1}{(j-i)!} \lambda^j$$

$$= \frac{\lambda^i}{i!} e^{-2\lambda} \sum_{k=0}^{\infty} \lambda^k / k!$$

$$= e^{-\lambda} \frac{\lambda^i}{i!}$$

(c) $P(X = i,\ Y - X = k) = P(X = i, Y = k + i)$

$$= \binom{k+i}{i} e^{-2\lambda} \frac{\lambda^{k+i}}{(k+i)!}$$

$$= e^{-\lambda} \frac{\lambda^i}{i!} e^{-\lambda} \frac{\lambda^k}{k!}$$

showing that X and $Y - X$ are independent Poisson random variables with mean λ. Hence,

$$P(Y - X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

57. It is the number of successes in $n + m$ independent p-trials.

59. (a) Use the fact that $F(X_i)$ is a uniform $(0, 1)$ random variable to obtain

$$p = P\{F(X_1) < F(X_2) > F(X_3) < F(X_4)\} = P\{U_1 < U_2 > U_3 < U_4\}$$

where the $U_i, i = 1, 2, 3, 4$, are independent uniform $(0, 1)$ random variables.

(b) $p = \int_0^1 \int_{x_1}^1 \int_0^{x_2} \int_{x_3}^1 dx_4 dx_3 dx_2 dx_1$

$= \int_0^1 \int_{x_1}^1 \int_0^{x_2} (1 - x_3) dx_3 dx_2 dx_1$

$= \int_0^1 \int_{x_1}^1 (x_2 - x_2^2/2) dx_2 dx_1$

$= \int_0^1 (1/3 - x_1^2/2 + x_1^3/6) dx_1$

$= 1/3 - 1/6 + 1/24 = 5/24$

(c) There are 5 (of the 24 possible) orderings such that $X_1 < X_2 > X_3 < X_4$. They are as follows:

$X_2 > X_4 > X_3 > X_1$

$X_2 > X_4 > X_1 > X_3$

$X_2 > X_1 > X_4 > X_3$

$X_4 > X_2 > X_3 > X_1$

$X_4 > X_2 > X_1 > X_3$

61. (a) $f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y} dy$

$= \lambda e^{-\lambda x}$

(b) $f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y} dx$

$= \lambda^2 y e^{-\lambda y}$

(c) Because the Jacobian of the transformation $x = x, w = y - x$ is 1, we have

$$f_{X,W}(x, w) = f_{X,Y}(x, x + w) = \lambda^2 e^{-\lambda(x+w)} = \lambda e^{-\lambda x} \lambda e^{-\lambda w}$$

(d) It follows from the preceding that X and W are independent exponential random variables with rate λ.

63. $\phi(t) = \sum_{n=1}^{\infty} e^{tn}(1 - p)^{n-1} p$

$= pe^t \sum_{n=1}^{\infty} ((1 - p)e^t)^{n-1}$

$= \dfrac{pe^t}{1 - (1 - p)e^t}$

65. $Cov(X_i, X_j) = Cov(\mu_i + \sum_{k=1}^n a_{ik} Z_k, \mu_j + \sum_{t=1}^n a_{jt} Z_t)$

$= \sum_{t=1}^n \sum_{k=1}^n Cov(a_{jk} Z_k, a_{jt} Z_t)$

$= \sum_{t=1}^n \sum_{k=1}^n a_{ik} a_{jt} Cov(Z_k, Z_t)$

$= \sum_{k=1}^n a_{ik} a_{jk}$

where the last equality follows since

$$Cov(Z_k, Z_t) = \begin{matrix} 1 & \text{if } k = t \\ 0 & \text{if } k \neq t \end{matrix}$$

67. $P\{5 < X < 15\} \geq \dfrac{2}{5}$

69. $\Phi(1) - \Phi\left[\dfrac{1}{2}\right] = .1498$

71. (a) $P\{X = i\} = \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ k - i \end{bmatrix} \Big/ \begin{bmatrix} n + m \\ k \end{bmatrix}$

$i = 0, 1, \ldots, min(k, n)$

(b) $X = \sum_{i=1}^k X_i$

$E[X] = \sum_{i=1}^K E[X_i] = \dfrac{kn}{n + m}$

since the i^{th} ball is equally likely to be either of the $n + m$ balls, and so $E[X_i] = P\{X_i = 1\} = \dfrac{n}{n + m}$

$X = \sum_{i=1}^n Y_i$

$E[X] = \sum_{i=1}^n E[Y_i]$

$= \sum_{i=1}^n P\{i^{th} \text{ white ball is selected}\}$

$= \sum_{i=1}^n \dfrac{k}{n + m} = \dfrac{nk}{n + m}$

73. As N_i is a binomial random variable with parameters (n, P_i), we have (a) $E[N_i] = nP_{ji}$ (b) $Var(X_i) = nP_i = (1 - P_i)$; (c) for $i \neq j$, the covariance of N_i and N_j can be computed as

$$Cov(N_i, N_j) = Cov\left[\sum_k X_k, \sum_k Y_k\right]$$

where $X_k(Y_k)$ is 1 or 0, depending upon whether or not outcome k is type $i(j)$. Hence,

$$Cov(N_i, N_j) = \sum_k \sum_\ell Cov(X_k, Y_\ell)$$

Now for $k \neq \ell, Cov(X_k, Y_\ell) = 0$ by independence of trials and so

$$\begin{aligned} Cov\,(N_i, N_j) &= \sum_k Cov(X_k, Y_k) \\ &= \sum_k (E[X_k Y_k] - E[X_k]E[Y_k]) \\ &= -\sum_k E[X_k]E[Y_k] \text{ (since } X_k Y_k = 0) \\ &= -\sum_k P_i P_j \\ &= -nP_i P_j \end{aligned}$$

(d) Letting

$$Y_i = \begin{cases} 1, & \text{if no type } i\text{'s occur} \\ 0, & \text{otherwise} \end{cases}$$

we have that the number of outcomes that never occur is equal to $\sum_1^r Y_i$ and thus,

$$\begin{aligned} E\left[\sum_1^r Y_i\right] &= \sum_1^r E[Y_i] \\ &= \sum_1^r P\{\text{outcomes } i \text{ does not occur}\} \\ &= \sum_1^r (1 - P_i)^n \end{aligned}$$

75. (a) Knowing the values of $N_1, \ldots, N_j$ is equivalent to knowing the relative ordering of the elements $a_1, \ldots, a_j$. For instance, if $N_1 = 0, N_2 = 1, N_3 = 1$ then in the random permutation a_2 is before a_3, which is before a_1. The independence result follows for clearly the number of $a_1, \ldots, a_i$ that follow a_{i+1} does not probabilistically depend on the relative ordering of $a_1, \ldots, a_i$.

(b) $P\{N_i = k\} = \dfrac{1}{i}, \quad k = 0, 1, \ldots, i-1$

which follows since of the elements $a_1, \ldots, a_{i+1}$ the element a_{i+1} is equally likely to be first or second or … or $(i+1)^{st}$.

(c) $E[N_i] = \dfrac{1}{i}\sum_{k=0}^{i-1} k = \dfrac{i-1}{2}$

$$E[N_i^2] = \frac{1}{i}\sum_{k=0}^{i-1} k^2 = \frac{(i-1)(2i-1)}{6}$$

and so

$$\begin{aligned} Var(N_i) &= \frac{(i-1)(2i-1)}{6} - \frac{(i-1)^2}{4} \\ &= \frac{i^2 - 1}{12} \end{aligned}$$

77. If $g_1(x, y) = x + y,\ g_2(x, y) = x - y$, then

$$J = \begin{vmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} \end{vmatrix} = 2$$

Hence, if $U = X + Y,\ V = X - Y$, then

$$\begin{aligned} f_{U,V}(u, v) &= \frac{1}{2} f_{X,\,Y}\left[\frac{u+v}{2}, \frac{u-v}{2}\right] \\ &= \frac{2}{4\tau\sigma^2}\, exp\left[-\frac{1}{2\sigma^2}\left[\left[\frac{u+v}{2} - \mu\right]^2 \right.\right. \\ &\quad \left.\left. + \left[\frac{u-v}{2} - \mu\right]^2\right]\right] \\ &= \frac{e-\mu^2/\sigma^2}{4\tau\sigma^2}\, exp\left[\frac{u\mu}{\sigma^2} - \frac{u^2}{4\sigma^2}\right] \\ &\quad exp\left\{-\frac{v^2}{4\sigma^2}\right\} \end{aligned}$$

79. $K'(t) = \dfrac{E\left[Xe^{tX}\right]}{E\left[e^{tX}\right]}$

$$K''(t) = \frac{E\left[e^{tX}\right]E\left[X^2 e^{tX}\right] - E^2\left[Xe^{tX}\right]}{E^2\left[e^{tX}\right]}$$

Hence,

$$K'(0) = E[X]$$

$$K''(0) = E[X^2] - E^2[X] = \text{Var}(X)$$

Chapter 3

1. $\sum_x p_{X|Y^{(x|y)}} = \frac{\sum_x p(x,y)}{p_{Y(y)}} = \frac{p_{Y(y)}}{p_{Y(y)}} = 1$

3. $E[X|Y=1]=2$

$E[X|Y=2]=\frac{5}{3}$

$E[X|Y=3]=\frac{12}{5}$

5. (a) $P\{X=i|Y=3\} = P\{i$ white balls selected when choosing 3 balls from 3 white and 6 red$\}$

$$= \frac{\begin{bmatrix}3\\ i\end{bmatrix}\begin{bmatrix}6\\ 3-i\end{bmatrix}}{\begin{bmatrix}9\\ 3\end{bmatrix}}, \quad i = 0,1,2,3$$

(b) By same reasoning as in (a), if $Y=1$, then X has the same distribution as the number of white balls chosen when 5 balls are chosen from 3 white and 6 red. Hence,

$$E[X|Y=1] = 5\frac{3}{9} = \frac{5}{3}$$

7. Given $Y=2$, the conditional distribution of X and Z is

$P\{(X,Z)=(1,1)|Y=2\} = \frac{1}{5}$

$P\{(1,2)|Y=2\}=0$

$P\{(2,1)|Y=2\}=0$

$P\{(2,2)|Y=2\} = \frac{4}{5}$

So,

$E[X|Y=2] = \frac{1}{5} + \frac{8}{5} = \frac{9}{5}$

$E[X|Y=2, Z=1] = 1$

9.
$$\begin{aligned} E[X|Y=y] &= \sum_x xP\{X=x|Y=y\} \\ &= \sum_x xP\{X=x\} \quad \text{by independence} \\ &= E[X] \end{aligned}$$

11. $E[X|Y=y] = C\int_{-y}^{y} x(y^2-x^2)dx = 0$

13. The conditional density of X given that $X>1$ is

$$f_{X|X>1}(x) = \frac{f(x)}{P\{X>1\}} = \frac{\lambda \exp^{-\lambda x}}{\exp^{-\lambda}} \text{ when } x>1$$

$$E[X|X>1] = \exp^{\lambda}\int_1^{\infty} x\lambda \exp^{-\lambda x}\, dx = 1 + 1/\lambda$$

by integration by parts.

15.
$$\begin{aligned} f_{X|Y=y}(x|y) &= \frac{\frac{1}{y}\exp^{-y}}{f_y(y)} = \frac{\frac{1}{y}\exp^{-y}}{\int_0^y \frac{1}{y}\exp^{-y}\, dx} \\ &= \frac{1}{y}, \quad 0<x<y \end{aligned}$$

$$E[X^2|Y=y] = \frac{1}{y}\int_0^y x^2 dx = \frac{y^2}{3}$$

17. With $K = 1/P\{X=i\}$, we have that

$$\begin{aligned} f_{Y|X}(y|i) &= KP\{X=i|Y=y\}f_Y(y) \\ &= K_1 e^{-y} y^i e^{-\alpha y} y^{a-1} \\ &= K_1 e^{-(1+\alpha)y} y^{a+i-1} \end{aligned}$$

where K_1 does not depend on y. But as the preceding is the density function of a gamma random variable with parameters $(s+i, 1+\alpha)$ the result follows.

19. $\int E[X|Y=y]f_Y(y)dy$

$$= \int\int x f_{X|Y}(x|y)dx f_Y(Y)dy$$
$$= \int\int x \frac{f(x,y)}{f_Y(y)} dx f_Y(y)dy$$
$$= \int x \int f(x \cdot y)dydx$$
$$= \int x f_X(x)dx$$
$$= E[X]$$

21. (a) $X = \sum_{i=1}^{N} T_i$

(b) Clearly N is geometric with parameter 1/3; thus, $E[N] = 3$.

(c) Since T_N is the travel time corresponding to the choice leading to freedom it follows that $T_N = 2$, and so $E[T_N] = 2$.

(d) Given that $N = n$, the travel times $T_i i = 1, \ldots, n-1$ are each equally likely to be either 3 or 5 (since we know that a door leading back to the nine is selected), whereas T_n is equal to 2 (since that choice led to safety). Hence,

$$E\left[\sum_{i=1}^{N} T_i | N = n\right] = E\left[\sum_{i=1}^{n-1} T_i | N = n\right] + E[T_n|N=n] = 4(n-1)+2$$

(e) Since part (d) is equivalent to the equation

$$E\left[\sum_{i=1}^{N} T_i | N\right] = 4N - 2$$

we see from parts (a) and (b) that

$$E[X] = 4E[N] - 2 = 10$$

23. Let X denote the first time a head appears. Let us obtain an equation for $E[N|X]$ by conditioning on the next two flips after X. This gives

$$E[N|X] = E[N|X,h,h]p^2 + E[N|X,h,t]pq + E[N|X,t,h]pq + E[N|X,t,t]q^2$$

where $q = 1 - p$. Now

$$E[N|X,h,h] = X+1, E[N|X,h,t] = X+1$$
$$E[N|X,t,h] = X+2, E[N|X,t,t] = X+2+E[N]$$

Substituting back gives

$$E[N|X] = (X+1)(p^2+pq) + (X+2)pq + (X+2+E[N])q^2$$

Taking expectations, and using the fact that X is geometric with mean $1/p$, we obtain

$$E[N] = 1 + p + q + 2pq + q^2/p + 2q^2 + q^2 E[N]$$

Solving for $E[N]$ yields

$$E[N] = \frac{2 + 2q + q^2/p}{1 - q^2}$$

25. (a) Let F be the initial outcome.

$$E[N] = \sum_{i=1}^{3} E[N|F=i]p_i = \sum_{i=1}^{3}\left(1 + \frac{2}{p_i}\right)p_i = 1 + 6 = 7$$

(b) Let $N_{1,2}$ be the number of trials until both outcome 1 and outcome 2 have occurred. Then

$$E[N_{1,2}] = E[N_{1,2}|F=1]p_1 + E[N_{1,2}|F=2]p_2 + E[N_{1,2}|F=3]p_3$$
$$= \left(1 + \frac{1}{p_2}\right)p_1 + \left(1 + \frac{1}{p_1}\right)p_2 + (1 + E[N_{1,2}])p_3$$
$$= 1 + \frac{p_1}{p_2} + \frac{p_2}{p_1} + p_3 E[N_{1,2}]$$

Hence,

$$E[N_{1,2}] = \frac{1 + \frac{p_1}{p_2} + \frac{p_2}{p_1}}{p_1 + p_2}$$

27. Condition on the outcome of the first flip to obtain

$$E[X] = E[X|H]p + E[X|T](1-p) = (1 + E[X])p + E[X|T](1-p)$$

Conditioning on the next flip gives

$$E[X|T] = E[X|TH]p + E[X|TT](1-p) = (2 + E[X])p + (2 + 1/p)(1-p)$$

where the final equality follows since given that the first two flips are tails the number of additional flips is just the number of flips needed to obtain a head. Putting the preceding together yields

$$E[X] = (1 + E[X])p + (2 + E[X])p(1-p) + (2 + 1/p)(1-p)^2$$

or

$$E[X] = \frac{1}{p(1-p)^2}$$

29. Let $q_i = 1 - p_i,\ i = 1.2$. Also, let h stand for hit and m for miss.

(a) $$\mu_1 = E[N|h]p_1 + E[N|m]q_1$$
$$= p_1(E[N|h,h]p_2 + E[N|h,m]q_2) + (1 + \mu_2)q_1$$
$$= 2p_1p_2 + (2 + \mu_1)p_1q_2 + (1 + \mu_2)q_1$$

The preceding equation simplifies to

$$\mu_1(1 - p_1q_2) = 1 + p_1 + \mu_2q_1$$

Similarly, we have that

$$\mu_2(1 - p_2q_1) = 1 + p_2 + \mu_1q_2$$

Solving these equations gives the solution.

$$h_1 = E[H|h]p_1 + E[H|m]q_1$$
$$= p_1(E[H|h,h]p_2 + E[H|h,m]q_2) + h_2q_1$$
$$= 2p_1p_2 + (1 + h_1)\,p_1q_2 + h_2q_1$$

Similarly, we have that

$$h_2 = 2p_1p_2 + (1 + h_2)p_2q_1 + h_1q_2$$

and we solve these equations to find h_1 and h_2.

31. Let L_i denote the length of run i. Conditioning on X, the initial value gives

$$E[L_1] = E[L_1|X = 1]p + E[L_1|X = 0](1 - p)$$
$$= \frac{1}{1-p}p + \frac{1}{p}(1 - p)$$
$$= \frac{p}{1-p} + \frac{1-p}{p}$$

and

$$E[L_2] = E[L_2|X = 1]p + E[L_2|X = 0](1 - p)$$
$$= \frac{1}{p}p + \frac{1}{1-p}(1 - p)$$
$$= 2$$

33. Let $I(A)$ equal 1 if the event A occurs and let it equal 0 otherwise.

$$E\left[\sum_{i=1}^{T} R_i\right] = E\left[\sum_{i=1}^{\infty} I(T \ge i)R_i\right]$$
$$= \sum_{i=1}^{\infty} E[I\,(T \ge i)\,R_i]$$
$$= \sum_{i=1}^{\infty} E[I(T \ge i)]E[R_i]$$
$$= \sum_{i=1}^{\infty} P\{T \ge i\}E[R_i]$$
$$= \sum_{i=1}^{\infty} \beta^{i-1}E[R_i]$$
$$= E\left[\sum_{i=1}^{\infty} \beta^{i-1}R_i\right]$$

35. $$np_1 = E[X_1]$$
$$= E[X_1|X_2 = 0](1 - p_2)^n + E[X_1|X_2 > 0][1 - (1 - p_2)^n]$$
$$= n\frac{p_1}{1-p_2}(1 - p_2)^n + E[X_1|X_2 > 0][1 - (1 - p_2)^n]$$

yielding the result

$$E[X_1|X_2 > 0] = \frac{np_1(1 - (1 - p_2)^{n-1})}{1 - (1 - p_2)^n}$$

37. (a) $E[X] = (2.6 + 3 + 3.4)/3 = 3$

(b) $E[X^2] = [2.6 + 2.6^2 + 3 + 9 + 3.4 + 3.4^2]/3 = 12.1067$, and $Var(X) = 3.1067$

39. Let N denote the number of cycles, and let X be the position of card 1.

(a) $$m_n = \frac{1}{n}\sum_{i=1}^{n} E[N|X = i] = \frac{1}{n}\sum_{i=1}^{n}(1 + m_{n-1})$$
$$= 1 + \frac{1}{n}\sum_{j=1}^{n-1} m_j$$

(b) $m_1 = 1$

$$m_2 = 1 + \frac{1}{2} = 3/2$$
$$m_3 = 1 + \frac{1}{3}(1 + 3/2) = 1 + 1/2 + 1/3 = 11/6$$
$$m_4 = 1 + \frac{1}{4}(1 + 3/2 + 11/6) = 25/12$$

(c) $m_n = 1 + 1/2 + 1/3 + \cdots + 1/n$

(d) Using recursion and the induction hypothesis gives

$$m_n = 1 + \frac{1}{n}\sum_{j=1}^{n-1}(1 + \cdots + 1/j)$$
$$= 1 + \frac{1}{n}(n - 1 + (n-2)/2 + (n-3)/3 + \cdots + 1/(n-1))$$
$$= 1 + \frac{1}{n}[n + n/2 + \cdots + n/(n-1) - (n-1)]$$
$$= 1 + 1/2 + \cdots + 1/n$$

(e) $N = \sum_{i=1}^{n} X_i$

(f) $m_n = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{i \text{ is last of } 1,\ldots,i\}$
$$= \sum_{i=1}^{n} 1/i$$

(g) Yes, knowing for instance that $i+1$ is the last of all the cards $1,\ldots,i+1$ to be seen tells us nothing about whether i is the last of $1,\ldots,i$.

(h) $Var(N) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n}(1/i)(1-1/i)$

41. Let N denote the number of minutes in the maze. If L is the event the rat chooses its left, and R the event it chooses its right, we have by conditioning on the first direction chosen:

$$E(N) = \frac{1}{2}E(N|L) + \frac{1}{2}E(N|R)$$
$$= \frac{1}{2}\left[\frac{1}{3}(2) + \frac{2}{3}(5 + E(N))\right] + \frac{1}{2}[3 + E(N)]$$
$$= \frac{5}{6}E(N) + \frac{21}{6}$$
$$= 21$$

43. $E[T|\chi_n^2] = \dfrac{1}{\sqrt{\chi_n^2/n}}E[Z|\chi_n^2] = \dfrac{1}{\sqrt{\chi_n^2/n}}E[Z] = 0$

$$E[T^2|\chi_n^2] = \frac{n}{\chi_n^2}E[Z^2|\chi_n^2] = \frac{n}{\chi_n^2}E[Z^2] = \frac{n}{\chi_n^2}$$

Hence, $E[T] = 0$, and

$$\text{Var}(T) = E[T^2] = E\left[\frac{n}{\chi_n^2}\right]$$
$$= n\int_0^\infty \frac{1}{x}\,\frac{\frac{1}{2}e^{-x/2}(x/2)^{\frac{n}{2}-1}}{\Gamma(n/2)}\,dx$$
$$= \frac{n}{2\Gamma(n/2)}\int_0^\infty \frac{1}{2}e^{-x/2}(x/2)^{\frac{n-2}{2}-1}\,dx$$
$$= \frac{n\Gamma(n/2-1)}{2\Gamma(n/2)}$$
$$= \frac{n}{2(n/2-1)}$$
$$= \frac{n}{n-2}$$

45. Now

$$E[X_n|X_{n-1}] = 0, \quad Var(X_n|X_{n-1}) = \beta X_{n-1}^2$$

(a) From the above we see that

$$E[X_n] = 0$$

(b) From (a) we have that $Var(x_n) = E[X_n^2]$. Now

$$E[X_n^2] = E\{E[X_n^2|X_{n-1}]\}$$
$$= E[\beta X_{n-1}^2]$$
$$= \beta E[X_{n-1}^2]$$
$$= \beta^2 E[X_{n-2}^2]$$
$$\cdot$$
$$= \beta^n X_0^2$$

47. $E[X^2Y^2|X] = X^2E[Y^2|X]$
$$\geq X^2(E[Y|X])^2 = X^2$$

The inequality following since for any random variable U, $E[U^2] \geq (E[U])^2$ and this remains true when conditioning on some other random variable X. Taking expectations of the above shows that

$$E[(XY)^2] \geq E[X^2]$$

As

$$E[XY] = E[E[XY|X]] = E[XE[Y|X]] = E[X]$$

the result follows.

49. Let A be the event that A is the overall winner, and let X be the number of games played. Let Y equal the number of wins for A in the first two games.

$$P(A) = P(A|Y=0)P(Y=0) + P(A|Y=1)P(Y=1) + P(A|Y=2)P(Y=2)$$
$$= 0 + P(A)2p(1-p) + p^2$$

Thus,

$$P(A) = \frac{p^2}{1-2p(1-p)}$$

$$E[X] = E[X|Y=0]P(Y=0) + E[X|Y=1]P(Y=1) + E[X|Y=2]P(Y=2)$$
$$= 2(1-p)^2 + (2+E[X])2p(1-p) + 2p^2$$
$$= 2 + E[X]2p(1-p)$$

Thus,

$$E[X] = \frac{2}{1-2p(1-p)}$$

51. Let α be the probability that X is even. Conditioning on the first trial gives

$$\alpha = P(\text{even}|X=1)p + P(\text{even}|X>1)(1-p) = (1-\alpha)(1-p)$$

Thus,

$$\alpha = \frac{1-p}{2-p}$$

More computationally

$$\alpha = \sum_{n=1}^{\infty} P(X=2n) = \frac{p}{1-p}\sum_{n=1}^{\infty}(1-p)^{2n} = \frac{p}{1-p}\,\frac{(1-p)^2}{1-(1-p)^2} = \frac{1-p}{2-p}$$

53. $P\{X=n\} = \int_0^\infty P\{X=n|\lambda\}e^{-\lambda}d\lambda$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!}e^{-\lambda}d\lambda$$
$$= \int_0^\infty e^{-2\lambda}\lambda^n\frac{d\lambda}{n!}$$
$$= \int_0^\infty e^{-t}t^n\frac{dt}{n!}\left[\frac{1}{2}\right]^{n+1}$$

The result follows since

$$\int_0^\infty e^{-t}t^n dt = \Gamma(n+1) = n!$$

57. Let X be the number of storms.

$$P\{X\geq 3\} = 1 - P\{X\leq 2\}$$
$$= 1 - \int_0^5 P\{X\leq 2|\Lambda = x\}\frac{1}{5}dx$$
$$= 1 - \int_0^5 [e^{-x} + xe^{-x} + e^{-x}x^2/2]\frac{1}{5}dx$$

59. (a) $P(A_iA_j) = \sum_{k=0}^{n} P(A_iA_j|N_i=k)\binom{n}{k}p_i^k(1-p_i)^{n-k}$

$$= \sum_{k=1}^{n} P(A_j|N_i=k)\binom{n}{k}p_i^k(1-p_i)^{n-k}$$
$$= \sum_{k=1}^{n-1}\left[1-\left(1-\frac{p_j}{1-p_i}\right)^{n-k}\right]\binom{n}{k} \times p_i^k(1-p_i)^{n-k}$$
$$= \sum_{k=1}^{n-1}\binom{n}{k}p_i^k(1-p_i)^{n-k} - \sum_{k=1}^{n-1} \times\left(1-\frac{p_j}{1-p_i}\right)^{n-k}\binom{n}{k} \times p_i^k(1-p_i)^{n-k}$$
$$= 1-(1-p_i)^n - p_i^n - \sum_{k=1}^{n-1}\binom{n}{k} \times p_i^k(1-p_i-p_j)^{n-k}$$
$$= 1-(1-p_i)^n - p_i^n - [(1-p_j)^n - (1-p_i-p_j)^n - p_i^n]$$
$$= 1 + (1-p_i-p_j)^n - (1-p_i)^n - (1-p_j)^n$$

where the preceding used that conditional on $N_i = k$, each of the other $n-k$ trials independently results in outcome j with probability $\frac{p_j}{1-p_i}$.

(b) $P(A_iA_j) = \sum_{k=1}^{n} P(A_iA_j|F_i=k)\,p_i(1-p_i)^{k-1} + P(A_iA_j|F_i>n)\,(1-p_i)^n$

$$= \sum_{k=1}^{n} P(A_j|F_i=k)\,p_i(1-p_i)^{k-1}$$
$$= \sum_{k=1}^{n}\left[1-\left(1-\frac{p_j}{1-p_i}\right)^{k-1}(1-p_j)^{n-k}\right] \times p_i(1-p_i)^{k-1}$$

(c) $P(A_iA_j) = P(A_i) + P(A_j) - P(A_i\cup A_j)$

$$= 1-(1-p_i)^n + 1-(1-p_j)^n - [1-(1-p_i-p_j)^n]$$
$$= 1 + (1-p_i-p_j)^n - (1-p_i)^n - (1-p_j)^n$$

61. (a) $m_1 = E[X|h]p_1 + E[H|m]q_1 = p_1 + (1+m_2)q_1 = 1 + m_2q_1.$

Similarly, $m_2 = 1 + m_1 q_2$. Solving these equations gives

$$m_1 = \frac{1 + q_1}{1 - q_1 q_2}, \quad m_2 = \frac{1 + q_2}{1 - q_1 q_2}$$

(b) $P_1 = p_1 + q_1 P_2$

$P_2 = q_2 P_1$

implying that

$$P_1 = \frac{p_1}{1 - q_1 q_2}, \quad P_2 = \frac{p_1 q_2}{1 - q_1 q_2}$$

(c) Let f_i denote the probability that the final hit was by 1 when i shoots first. Conditioning on the outcome of the first shot gives

$$f_1 = p_1 P_2 + q_1 f_2 \quad \text{and} \quad f_2 = p_2 P_1 + q_2 f_1$$

Solving these equations gives

$$f_1 = \frac{p_1 P_2 + q_1 p_2 P_1}{1 - q_1 q_2}$$

(d) and (e) Let B_i denote the event that both hits were by i. Condition on the outcome of the first two shots to obtain

$$P(B_1) = p_1 q_2 P_1 + q_1 q_2 P(B_1) \to P(B_1) = \frac{p_1 q_2 P_1}{1 - q_1 q_2}$$

Also,

$$P(B_2) = q_1 p_2 (1 - P_1) + q_1 q_2 P(B_2) \to P(B_2) = \frac{q_1 p_2 (1 - P_1)}{1 - q_1 q_2}$$

(f) $E[N] = 2p_1 p_2 + p_1 q_2 (2 + m_1) + q_1 p_2 (2 + m_1) + q_1 q_2 (2 + E[N])$

implying that

$$E[N] = \frac{2 + m_1 p_1 q_2 + m_1 q_1 p_2}{1 - q_1 q_2}$$

63. Let S_i be the event there is only one type i in the final set.

$$P\{S_i = 1\} = \sum_{j=0}^{n-1} P\{S_i = 1 | T = j\} P\{T = j\}$$
$$= \frac{1}{n} \sum_{j=0}^{n-1} P\{S_i = 1 | T = j\}$$
$$= \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{n - j}$$

The final equality follows because given that there are still $n - j - 1$ uncollected types when the first type i is obtained, the probability starting at that point that it will be the last of the set of $n - j$ types consisting of type i along with the $n - j - 1$ yet uncollected types to be obtained is, by symmetry, $1/(n - j)$. Hence,

$$E\left[\sum_{i=1}^{n} S_i\right] = nE[S_i] = \sum_{k=1}^{n} \frac{1}{k}$$

65. (a) $P\{Y_n = j\} = 1/(n + 1), \quad j = 0, \ldots, n$

(b) For $j = 0, \ldots, n - 1$

$$P\{Y_{n-1} = j\} = \sum_{i=0}^{n} \frac{1}{n+1} P\{Y_{n-1} = j | Y_n = i\}$$
$$= \frac{1}{n+1} (P\{Y_{n-1} = j | Y_n = j\} + P\{Y_{n-1} = j | Y_n = j + 1\})$$
$$= \frac{1}{n+1} (P(\text{last is nonred} | j \text{ red}) + P(\text{last is red} | j + 1 \text{ red})$$
$$= \frac{1}{n+1} \left(\frac{n - j}{n} + \frac{j + 1}{n}\right) = 1/n$$

(c) $P\{Y_k = j\} = 1/(k + 1), \quad j = 0, \ldots, k$

(d) For $j = 0, \ldots, k - 1$

$$P\{Y_{k-1} = j\} = \sum_{i=0}^{k} P\{Y_{k-1} = j | Y_k = i\} P\{Y_k = i\}$$
$$= \frac{1}{k+1} (P\{Y_{k-1} = j | Y_k = j\} + P\{Y_{k-1} = j | Y_k = j + 1\})$$
$$= \frac{1}{k+1} \left(\frac{k - j}{k} + \frac{j + 1}{k}\right) = 1/k$$

where the second equality follows from the induction hypothesis.

67. A run of j successive heads can occur in the following mutually exclusive ways: (i) either there is a run of j in the first $n - 1$ flips, or (ii) there is no j-run in the first $n - j - 1$ flips, flip $n - j$ is a tail, and the next j flips are all heads. Consequently, (a) follows. Condition on the time of the first tail:

$$P_j(n) = \sum_{k=1}^{j} P_j(n - k) p^{k-1} (.1 - p) + p^j, \quad j \le n$$

69. (a) Let $I(i,j)$ equal 1 if i and j are a pair and 0 otherwise. Then

$$E\left[\sum_{i<j} I(i,j)\right] = \binom{n}{2}\frac{1}{n}\frac{1}{n-1} = 1/2$$

Let X be the size of the cycle containing person 1. Then

$$Q_n = \sum_{i=1}^{n} P\{\text{no pairs}|X = i\}1/n = \frac{1}{n}\sum_{i\neq 2} Q_{n-i}$$

73. Condition on the value of the sum prior to going over 100. In all cases the most likely value is 101. (For instance, if this sum is 98 then the final sum is equally likely to be either 101, 102, 103, or 104. If the sum prior to going over is 95 then the final sum is 101 with certainty.)

75. (a) Since A receives more votes than B (since $a > a$) it follows that if A is not always leading then they will be tied at some point.

(b) Consider any outcome in which A receives the first vote and they are eventually tied, say $a, a, b, a, b, a, b, b\ldots$. We can correspond this sequence to one that takes the part of the sequence until they are tied in the reverse order. That is, we correspond the above to the sequence $b, b, a, b, a, b, a, a\ldots$ where the remainder of the sequence is exactly as in the original. Note that this latter sequence is one in which B is initially ahead and then they are tied. As it is easy to see that this correspondence is one to one, part (b) follows.

(c) Now,
$P\{B$ receives first vote and they are eventually tied$\}$
$= P\{B$ receives first vote$\}= n/(n+m)$
Therefore, by part (b) we see that
$P\{$eventually tied$\}= 2n/(n+m)$
and the result follows from part (a).

77. We will prove it when X and Y are discrete.

(a) This part follows from (b) by taking $g(x,y) = xy$.

(b)
$$E[g(X,Y)|Y = \bar{y}] = \sum_y \sum_x g(x,y) P\{X = x, Y = y|Y = \bar{y}\}$$

Now,

$$P\{X = x, Y = y|Y = \bar{y}\} = \begin{cases} 0, & \text{if } y \neq \bar{y} \\ P\{X = x,\ Y = \bar{y}\}, & \text{if } y = \bar{y} \end{cases}$$

So,

$$\begin{aligned} E\left[g(X,Y)|Y = \bar{y}\right] &= \sum_k g(x,\bar{y})P\{X = x|Y = \bar{y}\} \\ &= E[g(x,\ \bar{y})|Y = \bar{y} \end{aligned}$$

(c)
$$\begin{aligned} E[XY] &= E[E[XY|Y]] \\ &= E[YE[X|Y]] \qquad \text{by (a)} \end{aligned}$$

79. Let us suppose we take a picture of the urn before each removal of a ball. If at the end of the experiment we look at these pictures in reverse order (i.e., look at the last taken picture first), we will see a set of balls increasing at each picture. The set of balls seen in this fashion always will have more white balls than black balls if and only if in the original experiment there were always more white than black balls left in the urn. Therefore, these two events must have same probability, i.e., $n - m/n + m$ by the ballot problem.

81. (a) $f(x) = E[N] = \int_0^1 E[N|X_1 = y]dy$

$$E[N|X_1 = y] = \begin{cases} 1 & \text{if } y < x \\ 1 + f(y) & \text{if } y > x \end{cases}$$

Hence,

$$f(x) = 1 + \int_x^1 f(y)dy$$

(b) $f'(x) = -f(x)$

(c) $f(x) = ce^{-x}$. Since $f(1) = 1$, we obtain that $c = e$, and so $f(x) = e^{1-x}$.

(d) $P\{N > n\} = P\{x < X_1 < X_2 < \cdots < X_n\} = (1-x)^n/n!$ since in order for the above event to occur all of the n random variables must exceed x (and the probability of this is $(1-x)^n$), and then among all of the $n!$ equally likely orderings of this variables the one in which they are increasing must occur.

(e)
$$\begin{aligned} E[N] &= \sum_{n=0}^{\infty} P\{N > n\} \\ &= \sum_n (1-x)^n/n! = e^{1-x} \end{aligned}$$

83. Let I_j equal 1 if ball j is drawn before ball i and let it equal 0 otherwise. Then the random variable of interest is $\sum_{j\neq i} I_j$. Now, by considering the first

time that either i or j is withdrawn we see that $P\{j \text{ before } i\} = w_j/(w_i + w_j)$. Hence,

$$E\left[\sum_{j\neq i} I_j\right] = \sum_{j\neq i} \frac{w_j}{w_i + w_j}$$

85. Consider the following ordering:

$e_1, e_2, \ldots, e_{l-1}, i, j, e_{l+1}, \ldots, e_n$ where $P_i < P_j$

We will show that we can do better by interchanging the order of i and j, i.e., by taking $e_1, e_2, \ldots, e_{l-1}, j, i, e_{l+2}, \ldots, e_n$. For the first ordering, the expected position of the element requested is

$$E_{i,j} = P_{e_1} + 2P_{e_2} + \cdots + (l-1)P_{e_{l-1}} + lp_i + (l+1)P_j + (l+2)P_{e_{l+2}} + \cdots$$

Therefore,

$$E_{i,j} - E_{j,i} = l(P_i - P_j) + (l+1)(P_j - P_i) = P_j - P_i > 0$$

and so the second ordering is better. This shows that every ordering for which the probabilities are not in decreasing order is not optimal in the sense that we can do better. Since there are only a finite number of possible orderings, the ordering for which $p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_n$ is optimum.

87. (a) This can be proved by induction on m. It is obvious when $m = 1$ and then by fixing the value of x_1 and using the induction hypothesis, we see that there are $\sum_{i=0}^{n} \begin{bmatrix} n-i+m-2 \\ m-2 \end{bmatrix}$ such solutions. As $\begin{bmatrix} n-i+m-2 \\ m-2 \end{bmatrix}$ equals the number of ways of choosing $m-1$ items from a set of size $n+m-1$ under the constraint that the lowest numbered item selected is number $i+1$ (that is, none of $1, \ldots, i$ are selected where $i+1$ is), we see that

$$\sum_{i=0}^{n} \begin{bmatrix} n-i+m-2 \\ m-2 \end{bmatrix} = \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}$$

It also can be proven by noting that each solution corresponds in a one-to-one fashion with a permutation of n ones and $(m-1)$ zeros. The correspondence being that x_1 equals the number of ones to the left of the first zero, x_2 the number of ones between the first and second zeros, and so on. As there are $(n+m-1)!/n!(m-1)!$ such permutations, the result follows.

(b) The number of positive solutions of $x_1 + \cdots + x_m = n$ is equal to the number of nonnegative solutions of $y_1 + \cdots + y_m = n - m$, and thus there are $\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$ such solutions.

(c) If we fix a set of k of the x_i and require them to be the only zeros, then there are by (b) (with m replaced by $m-k$) $\begin{bmatrix} n-1 \\ m-k-1 \end{bmatrix}$ such solutions. Hence, there are $\begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n-1 \\ m-k-1 \end{bmatrix}$ outcomes such that exactly k of the X_i are equal to zero, and so the desired probability is $\begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n-1 \\ m-k-1 \end{bmatrix} \Big/ \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}$.

89. Condition on the value of I_n. This gives

$$\begin{aligned} P_n(K) &= P\left\{\sum_{j=1}^{n} jI_j \leq K | I_n = 1\right\} 1/2 \\ &\quad + P\left\{\sum_{j=1}^{n} jI_j \leq K | I_n = 0\right\} 1/2 \\ &= P\left\{\sum_{j=1}^{n-1} jI_j + n \leq K\right\} 1/2 \\ &\quad + P\left\{\sum_{j=1}^{n-1} jI_j \leq K\right\} 1/2 \\ &= [P_{n-1}(k-n) + P_{n-1}(K)]/2 \end{aligned}$$

91. $\dfrac{1}{p^5(1-p)^3} + \dfrac{1}{p^2(1-p)} + \dfrac{1}{p}$

95. With $\alpha = P(S_n < 0 \text{ for all } n > 0)$, we have

$$-E[X] = \alpha = p_{-1}\beta$$

Chapter 4

1. $P_{01} = 1, \quad P_{10} = \frac{1}{9}, \quad P_{21} = \frac{4}{9}, \quad P_{32} = 1$

$$P_{11} = \frac{4}{9}, \quad P_{22} = \frac{4}{9}$$

$$P_{12} = \frac{4}{9}, \quad P_{23} = \frac{1}{9}$$

3.

$P =$

	(RRR)	(RRD)	(RDR)	(RDD)	(DRR)	(DRD)	(DDR)	(DDD)
(RRR)	.8	.2	0	0	0	0	0	0
(RRD)			.4	.6				
(RDR)					.6	.4		
(RDD)							.4	.6
(DRR)	.6	.4						
(DRD)			.4	.6				
(DDR)					.6	.4		
(DDD)							.2	.8

where $D =$ dry and $R =$ rain. For instance, (DDR) means that it is raining today, was dry yesterday, and was dry the day before yesterday.

5. Cubing the transition probability matrix, we obtain P^3:

$$\begin{bmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{bmatrix}$$

Thus,

$$\begin{aligned} E[X_3] &= P(X_3 = 1) + 2P(X_3 = 2) \\ &= \frac{1}{4}P^3_{01} + \frac{1}{4}P^3_{11} + \frac{1}{2}P^3_{21} \\ &\quad + 2\left[\frac{1}{4}P^3_{02} + \frac{1}{4}P^3_{12} + \frac{1}{2}P^3_{22}\right] \end{aligned}$$

7. $$\begin{aligned} P^2_{30} + P^2_{31} &= P_{31}P_{10} + P_{33}P_{11} + P_{33}P_{31} \\ &= (.2)(.5) + (.8)(0) + (.2)(0) + (.8)(.2) \\ &= .26 \end{aligned}$$

9. It is not a Markov chain because information about previous color selections would affect probabilities about the current makeup of the urn, which would affect the probability that the next selection is red.

11. The answer is $\dfrac{P^4_{2,2}}{1 - P^4_{2,0}}$ for the Markov chain with transition probability matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{bmatrix}$$

13. $P^n_{ij} = \sum_k P^{n-r}_{ik} P^r_{kj} > 0$

15. Consider any path of states $i_0 = i, i_1, i_2, \ldots, i_n = j$ such that $P_{i_k i_{k+1}} > 0$. Call this a path from i to j. If j can be reached from i, then there must be a path from i to j. Let $i_0, \ldots, i_n$ be such a path. If all of the values $i_0, \ldots, i_n$ are not distinct, then there is a subpath from i to j having fewer elements (for instance, if $i, 1, 2, 4, 1, 3, j$ is a path, then so is $i, 1, 3, j$). Hence, if a path exists, there must be one with all distinct states.

17. $\sum_{i=1}^{n} Y_j/n \to E[Y]$ by the strong law of large numbers. Now $E[Y] = 2p - 1$. Hence, if $p > 1/2$, then $E[Y] > 0$, and so the average of the Y_is converges in this case to a positive number, which implies that $\sum_1^n Y_i \to \infty$ as $n \to \infty$. Hence, state 0 can be visited only a finite number of times and so must be transient. Similarly, if $p < 1/2$, then $E[Y] < 0$, and so $\lim \sum_1^n Y_i = -\infty$, and the argument is similar.

19. The limiting probabilities are obtained from

$$r_0 = .7r_0 + .5r_1$$
$$r_1 = .4r_2 + .2r_3$$
$$r_2 = .3r_0 + .5r_1$$
$$r_0 + r_1 + r_2 + r_3 = 1$$

and the solution is

$$r_0 = \frac{1}{4}, \quad r_1 = \frac{3}{20}, \quad r_2 = \frac{3}{20}, \quad r_3 = \frac{9}{20}$$

The desired result is thus

$$r_0 + r_1 = \frac{2}{5}$$

21. The transition probabilities are

$$P_{i,j} = \begin{cases} 1 - 3\alpha, & \text{if } j = i \\ \alpha, & \text{if } j \neq i \end{cases}$$

By symmetry,

$$P_{ij}^n = \frac{1}{3}(1 - P_{ii}^n), \quad j \neq i$$

So, let us prove by induction that

$$P_{i,j}^n = \begin{cases} \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n, & \text{if } j = i \\ \frac{1}{4} - \frac{1}{4}(1 - 4\alpha)^n, & \text{if } j \neq i \end{cases}$$

As the preceding is true for $n = 1$, assume it for n. To complete the induction proof, we need to show that

$$P_{i,j}^{n+1} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^{n+1}, & \text{if } j = i \\ \frac{1}{4} - \frac{1}{4}(1 - 4\alpha)^{n+1}, & \text{if } j \neq i \end{cases}$$

Now,

$$\begin{aligned} P_{i,i}^{n+1} &= P_{i,i}^n P_{i,i} + \sum_{j \neq i} P_{i,j}^n P_{j,i} \\ &= \left(\frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n\right)(1 - 3\alpha) \\ &\quad + 3\left(\frac{1}{4} - \frac{1}{4}(1 - 4\alpha)^n\right)\alpha \\ &= \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n(1 - 3\alpha - \alpha) \\ &= \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^{n+1} \end{aligned}$$

By symmetry, for $j \neq i$

$$P_{ij}^{n+1} = \frac{1}{3}\left(1 - P_{ii}^{n+1}\right) = \frac{1}{4} - \frac{1}{4}(1 - 4\alpha)^{n+1}$$

and the induction is complete.

By letting $n \to \infty$ in the preceding, or by using that the transition probability matrix is doubly stochastic, or by just using a symmetry argument, we obtain that $\pi_i = 1/4$.

23. (a) Letting 0 stand for a good year and 1 for a bad year, the successive states follow a Markov chain with transition probability matrix P:

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

Squaring this matrix gives P^2:

$$\begin{pmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{pmatrix}$$

Hence, if S_i is the number of storms in year i then

$$\begin{aligned} E[S_1] &= E[S_1 | X_1 = 0]P_{00} + E[S_1 | X_1 = 1]P_{01} \\ &= 1/2 + 3/2 = 2 \end{aligned}$$

$$\begin{aligned} E[S_2] &= E[S_2 | X_2 = 0]P_{00}^2 + E[S_2 | X_2 = 1]P_{01}^2 \\ &= 5/12 + 21/12 = 26/12 \end{aligned}$$

Hence, $E[S_1 + S_2] = 25/6$.

(b) Multiplying the first row of P by the first column of P^2 gives

$$P_{00}^3 = 5/24 + 7/36 = 29/72$$

Hence, conditioning on the state at time 3 yields

$$\begin{aligned} P(S_3 = 0) &= P(S_3 = 0 | X_3 = 0)\frac{29}{72} + P(S_3 = 0 | X_3 = 1) \\ &\quad \times \frac{43}{72} = \frac{29}{72}e^{-1} + \frac{43}{72}e^{-3} \end{aligned}$$

(c) The stationary probabilities are the solution of

$$\pi_0 = \pi_0\frac{1}{2} + \pi_1\frac{1}{3}$$

$$\pi_0 + \pi_1 = 1$$

giving

$$\pi_0 = 2/5, \quad \pi_1 = 3/5.$$

Hence, the long-run average number of storms is $2/5 + 3(3/5) = 11/5$.

25. Letting X_n denote the number of pairs of shoes at the door the runner departs from at the beginning of day n, then $\{X_n\}$ is a Markov chain with transition probabilities

$$\begin{aligned} P_{i,i} &= 1/4, \quad 0 < i < k \\ P_{i,i-1} &= 1/4, \quad 0 < i < k \\ P_{i,k-i} &= 1/4, \quad 0 < i < k \\ P_{i,k-i+1} &= 1/4, \quad 0 < i < k \end{aligned}$$

The first equation refers to the situation where the runner returns to the same door she left from and then chooses that door the next day; the second to the situation where the runner returns to the opposite door from which she left from and then chooses the original door the next day; and so on. (When some of the four cases above refer to the same transition probability, they should be added together. For instance, if $i = 4$, $k = 8$, then the preceding states that $P_{i,i} = 1/4 = P_{i,k-i}$. Thus, in this case, $P_{4,4} = 1/2$.) Also,

$$P_{0,0} = 1/2$$
$$P_{0,k} = 1/2$$
$$P_{k,k} = 1/4$$
$$P_{k,0} = 1/4$$
$$P_{k,1} = 1/4$$
$$P_{k,k-1} = 1/4$$

It is now easy to check that this Markov chain is doubly stochastic—that is, the column sums of the transition probability matrix are all 1—and so the long-run proportions are equal. Hence, the proportion of time the runner runs barefooted is $1/(k+1)$.

27. The limiting probabilities are obtained from

$$r_0 = \frac{1}{9} r_1$$
$$r_1 = r_0 + \frac{4}{9} r_1 + \frac{4}{9} r_2$$
$$r_2 = \frac{4}{9} r_1 + \frac{4}{9} r_2 + r_3$$
$$r_0 + r_1 + r_2 + r_3 = 1$$

and the solution is $r_0 = r_3 = \frac{1}{20}$, $r_1 = r_2 = \frac{9}{20}$.

29. Each employee moves according to a Markov chain whose limiting probabilities are the solution of

$$\prod_1 = .7\prod_1 + .2\prod_2 + .1\prod_3$$
$$\prod_2 = .2\prod_1 + .6\prod_2 + .4\prod_3$$
$$\prod_1 + \prod_2 + \prod_3 = 1$$

Solving yields $\prod_1 = 6/17, \prod_2 = 7/17, \prod_3 = 4/17$. Hence, if N is large, it follows from the law of large numbers that approximately 6, 7, and 4 of each 17 employees are in categories 1, 2, and 3.

31. Let the state on day n be 0 if sunny, 1 if cloudy, and 2 if rainy. This gives a three-state Markov chain with transition probability matrix

	0	1	2
0	0	1/2	1/2
$P =$ 1	1/4	1/2	1/4
2	1/4	1/4	1/2

The equations for the long-run proportions are

$$r_0 = \frac{1}{4} r_1 + \frac{1}{4} r_2$$
$$r_1 = \frac{1}{2} r_0 + \frac{1}{2} r_1 + \frac{1}{4} r_2$$
$$r_2 = \frac{1}{2} r_0 + \frac{1}{4} r_1 + \frac{1}{2} r_2$$
$$r_0 + r_1 + r_2 = 1$$

By symmetry it is easy to see that $r_1 = r_2$. This makes it easy to solve and we obtain the result

$$r_0 = \frac{1}{5}, \quad r_1 = \frac{2}{5}, \quad r_2 = \frac{2}{5}$$

33. Consider the Markov chain whose state at time n is the type of exam number n. The transition probabilities of this Markov chain are obtained by conditioning on the performance of the class. This gives the following:

$$P_{11} = .3(1/3) + .7(1) = .8$$
$$P_{12} = P_{13} = .3(1/3) = .1$$
$$P_{21} = .6(1/3) + .4(1) = .6$$
$$P_{22} = P_{23} = .6(1/3) = .2$$
$$P_{31} = .9(1/3) + .1(1) = .4$$
$$P_{32} = P_{33} = .9(1/3) = .3$$

Let r_i denote the proportion of exams that are type $i, i = 1, 2, 3$. The r_i are the solutions of the following set of linear equations:

$$r_1 = .8\, r_1 + .6\, r_2 + .4\, r_3$$
$$r_2 = .1\, r_1 + .2\, r_2 + .3\, r_3$$
$$r_1 + r_2 + r_3 = 1$$

Since $P_{i2} = P_{i3}$ for all states i, it follows that $r_2 = r_3$. Solving the equations gives the solution

$$r_1 = 5/7, \quad r_2 = r_3 = 1/7$$

35. The equations are

$$r_0 = r_1 + \frac{1}{2} r_2 + \frac{1}{3} r_3 + \frac{1}{4} r_4$$
$$r_1 = \frac{1}{2} r_2 + \frac{1}{3} r_3 + \frac{1}{4} r_4$$
$$r_2 = \frac{1}{3} r_3 + \frac{1}{4} r_4$$
$$r_3 = \frac{1}{4} r_4$$
$$r_4 = r_0$$
$$r_0 + r_1 + r_2 + r_3 + r_4 = 1$$

The solution is

$$r_0 = r_4 = 12/37, \quad r_1 = 6/37, \quad r_2 = 4/37,$$
$$r_3 = 3/37$$

37. Must show that

$$\pi_j = \sum_i \pi_i P_{i,j}^k$$

The preceding follows because the right-hand side is equal to the probability that the Markov chain with transition probabilities $P_{i,j}$ will be in state j at time k when its initial state is chosen according to its stationary probabilities, which is equal to its stationary probability of being in state j.

39. Because recurrence is a class property it follows that state j, which communicates with the recurrent state i, is recurrent. But if j were positive recurrent, then by the previous exercise i would be as well. Because i is not, we can conclude that j is null recurrent.

41. (a) The number of transitions into state i by time n, the number of transitions originating from state i by time n, and the number of time periods the chain is in state i by time n all differ by at most 1. Thus, their long-run proportions must be equal.

(b) $r_i P_{ij}$ is the long-run proportion of transitions that go from state i to state j.

(c) $\sum_j r_i P_{ij}$ is the long-run proportion of transitions that are into state j.

(d) Since r_j is also the long-run proportion of transitions that are into state j, it follows that

$$r_j = \sum_j r_i P_{ij}$$

43. Consider a typical state—say, 1 2 3. We must show

$$\prod_{123} = \prod_{123} P_{123,123} + \prod_{213} P_{213,123} + \prod_{231} P_{231,123}$$

Now $P_{123,123} = P_{213,123} = P_{231,123} = P_1$ and thus,

$$\prod_{123} = P_1\left[\prod_{123} + \prod_{213} + \prod_{231}\right]$$

We must show that

$$\prod_{123} = \frac{P_1 P_2}{1 - P_1}, \prod_{213} = \frac{P_2 P_1}{1 - P_2}, \prod_{231} = \frac{P_2 P_3}{1 - P_2}$$

satisfies the above, which is equivalent to

$$P_1 P_2 = P_1\left[\frac{P_2 P_1}{1 - P_2} + \frac{P_2 P_3}{1 - P_2}\right]$$
$$= \frac{P_1}{1 - P_2} P_2 (P_1 + P_3)$$
$$= P_1 P_2 \quad \text{since } P_1 + P_3 = 1 - P_2$$

By symmetry all of the other stationary equations also follow.

45. (a) 1, since all states communicate and thus all are recurrent since state space is finite.

(b) Condition on the first state visited from i.

$$x_i = \sum_{j=1}^{N-1} P_{ij} x_j + P_{iN}, \quad i = 1, \ldots, N-1$$
$$x_0 = 0, \quad x_N = 1$$

(c) Must show

$$\frac{i}{N} = \sum_{j=1}^{N-1} \frac{j}{N} P_{ij} + P_{iN}$$
$$= \sum_{j=0}^{N} \frac{j}{N} P_{ij}$$

and follows by hypothesis.

47. $\{Y_n, n \geq 1\}$ is a Markov chain with states (i, j).

$$P_{(i,j),(k,\ell)} = \begin{cases} 0, & \text{if } j \neq k \\ P_{j\ell}, & \text{if } j = k \end{cases}$$

where $P_{j\ell}$ is the transition probability for $\{X_n\}$.

$$\lim_{n \to \infty} P\{Y_n = (i,j)\} = \lim_n P\{X_n = i, X_{n+1} = j\}$$
$$= \lim_n [P\{X_n = i\} P_{ij}]$$
$$= r_i P_{ij}$$

49. (a) No.

$$\lim P\{X_n = i\} = pr^1(i) + (1-p)r^2(i)$$

(b) Yes.

$$P_{ij} = pP_{ij}^{(1)} + (1-p)P_{ij}^{(2)}$$

53. With $\pi_i(1/4)$ equal to the proportion of time a policyholder whose yearly number of accidents is Poisson distributed with mean 1/4 is in Bonus-Malus state i, we have that the average premium is

$$\frac{2}{3}(326.375) + \frac{1}{3}[200\pi_1(1/4) + 250\pi_2(1/4) + 400\pi_3(1/4) + 600\pi_4(1/4)]$$

55. $S_{11} = P\{\text{offspring is aa} \mid \text{both parents dominant}\}$

$$= \frac{P\{\text{aa, both dominant}\}}{P\{\text{both dominant}\}}$$

$$= \frac{r^2 \frac{1}{4}}{(1-q)^2} = \frac{r^2}{4(1-q)^2}$$

$$S_{10} = \frac{P\{\text{aa, 1 dominant and 1 recessive parent}\}}{P\{\text{1 dominant and 1 recessive parent}\}}$$

$$= \frac{P\{\text{aa, 1 parent aA and 1 parent aa}\}}{2q(1-q)}$$

$$= \frac{2qr\frac{1}{2}}{2q(1-q)}$$

$$= \frac{r}{2(1-q)}$$

57. Let A be the event that all states have been visited by time T. Then, conditioning on the direction of the first step gives

$$P(A) = P(A|\text{clockwise})p + P(A|\text{counterclockwise})q$$

$$= p\frac{1-q/p}{1-(q/p)^n} + q\frac{1-p/q}{1-(p/q)^n}$$

The conditional probabilities in the preceding follow by noting that they are equal to the probability in the gambler's ruin problem that a gambler that starts with 1 will reach n before going broke when the gambler's win probabilities are p and q.

59. Condition on the outcome of the initial play.

61. With $P_0 = 0,\ P_N = 1$

$$P_i = \alpha_i P_{i+1} + (1-\alpha_i)P_{i-1}, \quad i = 1, \ldots, N-1$$

These latter equations can be rewritten as

$$P_{i+1} - P_i = \beta_i(P_i - P_{i-1})$$

where $\beta_i = (1-\alpha_i)/\alpha_i$. These equations can now be solved exactly as in the original gambler's ruin problem. They give the solution

$$P_i = \frac{1 + \sum_{j=1}^{i-1} C_j}{1 + \sum_{j=1}^{N-1} C_j}, \quad i = 1, \ldots, N-1$$

where

$$C_j = \prod_{i=1}^{j} \beta_i$$

(c) P_{N-i}, where $\alpha_i = (N-i)/N$

65. $r \geq 0 = P\{X_0 = 0\}$. Assume that $r \geq P\{X_{n-1} = 0\}$

$$P\{X_n = 0 = \sum_j P\{X_n = 0|X_1 = j\}P_j$$

$$= \sum_j [P\{X_{n-1} = \}]^j P_j$$

$$\leq \sum_j r^j P_j$$

$$= r$$

67. (a) Yes, the next state depends only on the present and not on the past.

(b) One class, period is 1, recurrent.

(c) $P_{i,i+1} = P\frac{N-i}{N}, \quad i = 0, 1, \ldots, N-1$

$P_{i,i-1} = (1-P)\frac{i}{N}, \quad i = 1, 2, \ldots, N$

$P_{i,i} = P\frac{i}{N} + (1-p)\frac{(N-i)}{N}, \quad i = 0, 1, \ldots, N$

(d) See (e).

(e) $r_i = \begin{bmatrix} N \\ i \end{bmatrix} p^i(1-p)^{N-i}, \quad i = 0, 1, \ldots, N$

(f) Direct substitution or use Example 7a.

(g) Time $= \sum_{j=i}^{N-1} T_j$, where T_j is the number of flips to go from j to $j+1$ heads. T_j is geometric with $E[T_j] = N/j$. Thus, $E[\text{time}] = \sum_{j=i}^{N-1} N/j$.

69. $r(n_1, \ldots, n_m) = \frac{M!}{n_1, \ldots, n_m!}\left[\frac{1}{m}\right]^M$

We must now show that

$$r(n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots)\frac{n_j + 1}{M}\frac{1}{M-1}$$

$$= r(n_1, \ldots, n_i, \ldots, n_j, \ldots)\frac{i}{M}\frac{1}{M-1}$$

or $\frac{n_j + 1}{(n_i - 1)!(n_j + 1)!} = \frac{n_i}{n_i! n_j!}$, which follows.

71. If $r_j = c\frac{P_{ij}}{P_{ji}}$, then

$$r_j P_{jk} = c\frac{P_{ij}P_{jk}}{P_{ji}}$$

$$r_k P_{kj} = c\frac{P_{jk}P_{kj}}{P_{ki}}$$

and are thus equal by hypothesis.

73. It is straightforward to check that $r_i P_{ij} = r_j P_{ji}$. For instance, consider states 0 and 1. Then

$r_0 p_{01} = (1/5)(1/2) = 1/10$

whereas

$r_1 p_{10} = (2/5)(1/4) = 1/10$

75. The number of transitions from i to j in any interval must equal (to within 1) the number from j to i since each time the process goes from i to j in order to get back to i, it must enter from j.

77. (a) $$\sum_a y_{ja} = \sum_a E_\beta\left[\sum_n a^n I_{\{X_n = j, a_n = a\}}\right]$$
$$= E_\beta\left[\sum_n a^n \sum_a I_{\{X_n = j, a_n = a\}}\right]$$
$$= E_\beta\left[\sum_n a^n I_{\{X_n = j\}}\right]$$

(b) $$\sum_j \sum_a y_{ja} = E_\beta\left[\sum_n a^n \sum_j I_{\{X_n = j\}}\right]$$
$$= E_\beta\left[\sum a^n\right] = \frac{1}{1-\alpha}$$

$$\sum_a y_{ja}$$
$$= b_j + E_\beta\left[\sum_{n=1}^{\infty} = a^n I_{\{X_n = j\}}\right]$$
$$= b_j + E_\beta\left[\sum_{n=0}^{\infty} a^{n+1} I_{\{X_{n+1} = j\}}\right]$$
$$= b_j + E_\beta\left[\sum_{n=0}^{\infty} = a^{n+1} \sum_{i,a} I_{\{X_n = i, a_n = a\}} I_{(X_{n+1} = j\}}\right]$$
$$= b_j + \sum_{n=0}^{\infty} a^{n+1} \sum_{i,a} E_\beta\left[I_{\{X_n = i, a_n = a\}}\right] P_{ij}(a)$$
$$= b_j + a \sum_{i,a} \sum_n a^n E_\beta\left[I_{(X_n = i, a_n = a\}}\right] P_{ij}(a)$$
$$= b_j + a \sum_{i,a} y_{ia} P_{ij}(a)$$

(c) Let $d_{j,a}$ denote the expected discounted time the process is in j, and a is chosen when policy β is employed. Then by the same argument as in (b):

$$\sum_a d_{ja}$$
$$= b_j + a \sum_{i,a} \sum_n a^n E_\beta[I\{X_n = i,\ a_n = a\}]\, P_{ij}(a)$$
$$= b_j + a \sum_{i,a} \sum_n a^n E_\beta\left[I_{\{X_n = i\}}\right] \frac{y_{ia}}{\sum_a y_{ia}} P_{ij}(a)$$
$$= b_j + a \sum_{i,a} \sum_a d_{ia,} \frac{y_{ia}}{\sum_a y_{ia}} P_{ij}(a)$$

and we see from Equation (9.1) that the above is satisfied upon substitution of $d_{ia} = y_{ia}$. As it is easy to see that $\sum_{i,a} d_{ia} = \dfrac{1}{1-a}$, the result follows since it can be shown that these linear equations have a unique solution.

(d) Follows immediately from previous parts. It is a well-know result in analysis (and easily proven) that if $\lim_{n\to\infty} a_n/n = a$ then $\lim_{n\to\infty} \sum_i^n a_i/n$ also equals a. The result follows from this since

$$E[R(X_n)] = \sum_j R(j) P\{X_n = j\}$$
$$= \sum_i R(j) r_j$$

Chapter 5

1. (a) e^{-1} (b) e^{-1}

3. The conditional distribution of X, given that $X > 1$, is the same as the unconditional distribution of $1 + X$. Hence, (a) is correct.

5. e^{-1} by lack of memory.

7. $P\{X_1 < X_2 | \min(X_1, X_2) = t\}$

$$= \frac{P\{X_1 < X_2, \min(X_1, X_2) = t\}}{P\{\min(X_1, X_2) = t\}}$$

$$= \frac{P\{X_1 = t, X_2 > t\}}{P\{X_1 = t, X_2 > t\} + P\{X_2 = t, X_1 > t\}}$$

$$= \frac{f_1(t)\bar{F}_2(t)}{f_1(t)\bar{F}_2(t) + f_2(t)\bar{F}_1(t)}$$

Dividing though by $\bar{F}_1(t)\bar{F}_2(t)$ yields the result. (For a more rigorous argument, replace $'' = t''$ by $'' \in (t, t + \epsilon)''$ throughout, and then let $\epsilon \to 0$.)

9. Condition on whether machine 1 is still working at time t, to obtain the answer,

$$1 - e^{-\lambda_1 t} + e^{-\lambda_1 t}\frac{\lambda_1}{\lambda_1 + \lambda_2}$$

11. (a) Using Equation (5.5), the lack of memory property of the exponential, as well as the fact that the minimum of independent exponentials is exponential with a rate equal to the sum of their individual rates, it follows that

$$P(A_1) = \frac{n\mu}{\lambda + n\mu}$$

and, for $j > 1$,

$$P(A_j | A_1 \cdots A_{j-1}) = \frac{(n - j + 1)\mu}{\lambda + (n - j + 1)\mu}$$

Hence,

$$p = \prod_{j=1}^{n} \frac{(n - j + 1)\mu}{\lambda + (n - j + 1)\mu}$$

(b) When $n = 2$,
$P\{\max Y_i < X\}$

$$= \int_0^\infty P\{\max Y_i < X | X = x\}\lambda e^{-\lambda x} dx$$
$$= \int_0^\infty P\{\max Y_i < x\}\lambda e^{-\lambda x} dx$$
$$= \int_0^\infty (1 - e^{-\mu x})^2 \lambda e^{-\lambda x} dx$$
$$= \int_0^\infty (1 - 2e^{-\mu x} + e^{-2\mu x})^2 \lambda e^{-\lambda x} dx$$
$$= 1 - \frac{2\lambda}{\lambda + \mu} + \frac{\lambda}{2\mu + \lambda}$$
$$= \frac{2\mu^2}{(\lambda + \mu)(\lambda + 2\mu)}$$

13. Let T_n denote the time until the n^{th} person in line departs the line. Also, let D be the time until the first departure from the line, and let X be the additional time after D until T_n. Then,

$$E[T_n] = E[D] + E[X]$$
$$= \frac{1}{n\theta + \mu} + \frac{(n - 1)\theta + \mu}{n\theta + \mu} E[T_{n-1}]$$

where $E[X]$ was computed by conditioning on whether the first departure was the person in line. Hence,

$$E[T_n] = A_n + B_n E[T_{n-1}]$$

where

$$A_n = \frac{1}{n\theta + \mu}, \qquad B_n = \frac{(n - 1)\theta + \mu}{n\theta + \mu}$$

Solving gives the solution

$$E[T_n] = A_n + \sum_{i=1}^{n-1} A_{n-i} \prod_{j=n-i+1}^{n} B_j$$
$$= A_n + \sum_{i=1}^{n-1} 1/(n\theta + \mu)$$
$$= \frac{n}{n\theta + \mu}$$

Another way to solve the preceding is to let I_j equal 1 if customer n is still in line at the time of the $(j-1)^{st}$ departure from the line, and let X_j denote the time between the $(j-1)^{st}$ and j^{th} departure from line. (Of course, these departures only refer to the first n people in line.) Then

$$T_n = \sum_{j=1}^{n} I_j X_j$$

The independence of I_j and X_j gives

$$E[T_n] = \sum_{j=1}^{n} E[I_j]E[X_j]$$

But,

$$E[I_j] = \frac{(n-1)\theta + \mu}{n\theta + \mu} \cdots \frac{(n-j+1)\theta + \mu}{(n-j+2)\theta + \mu} = \frac{(n-j+1)\theta + \mu}{n\theta + \mu}$$

and

$$E[X_j] = \frac{1}{(n-j+1)\theta + \mu}$$

which gives the result.

15. Let T_i denote the time between the $(i-1)^{th}$ and the i^{th} failure. Then the T_i are independent with T_i being exponential with rate $(101-i)/200$. Thus,

$$E[T] = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{101-i}$$

$$Var(T) = \sum_{i=1}^{5} Var(T_i) = \sum_{i=1}^{5} \frac{(200)^2}{(101-i)^2}$$

17. Let C_i denote the cost of the i^{th} link to be constructed, $i = 1, \ldots, n-1$. Note that the first link can be any of the $\binom{n}{2}$ possible links. Given the first one, the second link must connect one of the 2 cities joined by the first link with one of the $n-2$ cities without any links. Thus, given the first constructed link, the next link constructed will be one of $2(n-2)$ possible links. Similarly, given the first two links that are constructed, the next one to be constructed will be one of $3(n-3)$ possible links, and so on. Since the cost of the first link to be built is the minimum of $\binom{n}{2}$ exponentials with rate 1, it follows that

$$E[C_1] = 1 \Big/ \binom{n}{2}$$

By the lack of memory property of the exponential it follows that the amounts by which the costs of the other links exceed C_1 are independent exponentials with rate 1. Therefore, C_2 is equal to C_1 plus the minimum of $2(n-2)$ independent exponentials with rate 1, and so

$$E[C_2] = E[C_1] + \frac{1}{2(n-2)}$$

Similar reasoning then gives

$$E[C_3] = E[C_2] + \frac{1}{3(n-3)}$$

and so on.

19. (c) Letting $A = X_{(2)} - X_{(1)}$ we have

$$E[X_{(2)}] = E[X_{(1)}] + E[A] = \frac{1}{\mu_1 + \mu_2} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2} + \frac{1}{\mu_1}\frac{\mu_2}{\mu_1 + \mu_2}$$

The formula for $E[A]$ being obtained by conditioning on which X_i is largest.

(d) Let I equal 1 if $X_1 < X_2$ and let it be 2 otherwise. Since the conditional distribution of A (either exponential with rate μ_1 *or* μ_2) is determined by I, which is independent of $X_{(1)}$, it follows that A is independent of $X_{(1)}$.

Therefore,

$$Var(X_{(2)}) = Var(X_{(1)}) + Var(A)$$

With $p = \mu_1/(\mu_1 + \mu_2)$ we obtain, upon conditioning on I,

$$E[A] = p/\mu_2 + (1-p)/\mu_1,$$

$$E[A^2] = 2p/\mu_2^2 + 2(1-p)/\mu_1^2$$

Therefore,

$$Var(A) = 2p/\mu_2^2 + 2(1-p)/\mu_1^2 - (p/\mu_2 + (1-p)/\mu_1)^2$$

Thus,

$$Var(X_{(2)}) = 1/(\mu_1 + \mu_2)^2 + 2[p/\mu_2^2 + (1-p)/\mu_1^2] - (p/\mu_2 + (1-p)/\mu_1)^2$$

21. $E[\text{time}] = E[\text{time waiting at 1}] + 1/\mu_1 + E[\text{time waiting at 2}] + 1/\mu_2$

Now,

$$E[\text{time waiting at 1}] = 1/\mu_1,$$

$$E[\text{time waiting at 2}] = (1/\mu_2)\frac{\mu_1}{\mu_1 + \mu_2}$$

The last equation follows by conditioning on whether or not the customer waits for server 2. Therefore,

$E[\text{time}] = 2/\mu_1 + (1/\mu_2)[1 + \mu_1/(\mu_1 + \mu_2)]$

23. (a) 1/2.
(b) $(1/2)^{n-1}$: whenever battery 1 is in use and a failure occurs the probability is 1/2 that it is not battery 1 that has failed.
(c) $(1/2)^{n-i+1}$, $i > 1$.
(d) T is the sum of $n-1$ independent exponentials with rate 2μ (since each time a failure occurs the time until the next failure is exponential with rate 2μ).
(e) Gamma with parameters $n-1$ and 2μ.

25. Parts (a) and (b) follow upon integration. For part (c), condition on which of X or Y is larger and use the lack of memory property to conclude that the amount by which it is larger is exponential rate λ. For instance, for $x < 0$,

$$fx - y(x)dx$$
$$= P\{X < Y\}P\{-x < Y - X < -x + dx | Y > X\}$$
$$= \frac{1}{2}\lambda e^{\lambda x}dx$$

For (d) and (e), condition on I.

27. (a) $\dfrac{\mu_1}{\mu_1 + \mu_3}$
(b) $\dfrac{\mu_1}{\mu_1 + \mu_3}\,\dfrac{\mu_2}{\mu_2 + \mu_3}$
(c) $\displaystyle\sum_i \frac{1}{\mu_i} + \frac{\mu_1}{\mu_1 + \mu_3}\,\frac{\mu_2}{\mu_2 + \mu_3}\,\frac{1}{\mu_3}$
(d) $\displaystyle\sum_i \frac{1}{\mu_i} + \frac{\mu_1}{\mu_1 + \mu_2}\left[\frac{1}{\mu_2} + \frac{\mu_2}{\mu_2 + \mu_3}\,\frac{1}{\mu_3}\right] + \frac{\mu_2}{\mu_1 + \mu_2}\,\frac{\mu_1}{\mu_1 + \mu_3}\,\frac{\mu_2}{\mu_2 + \mu_3}\,\frac{1}{\mu_3}$

29. (a)
$$f_{X|X+Y(x|c)} = Cf_{X.\,X+Y(x,c)}$$
$$= C_1 f_{X_Y(x,c-x)}$$
$$= f_X(x)f_Y(c-x)$$
$$= C_2 e^{-\lambda x}e^{-\mu(c-x)}, \quad 0 < x < c$$
$$= C_3 e^{-(\lambda-\mu)x}, \quad 0 < x < c$$

where none of the C_i depend on x. Hence, we can conclude that the conditional distribution is that of an exponential random variable conditioned to be less than c.

(b) $E[X|X + Y = c] = \dfrac{1 - e^{-(\lambda-\mu)c}(1 + (\lambda - \mu)c)}{\lambda(1 - e^{-(\lambda-\mu)c})}$

(c) $c = E[X + Y|X + Y = c] = E[X|X + Y = c] + E[Y|X + Y = c]$

implying that

$$E[Y|X + Y = c] = c - \frac{1 - e^{-(\lambda-\mu)c}(1 + (\lambda - \mu)c)}{\lambda(1 - e^{-(\lambda-\mu)c})}$$

31. Condition on whether the 1 PM appointment is still with the doctor at 1:30, and use the fact that if she or he is then the remaining time spent is exponential with mean 30. This gives

$$E[\text{time spent in office}]$$
$$= 30(1 - e^{-30/30}) + (30 + 30)e^{-30/30}$$
$$= 30 + 30e^{-1}$$

33. (a) By the lack of memory property, no matter when Y fails the remaining life of X is exponential with rate λ.
(b)
$$E[\min(X, Y)\,|X > Y + c]$$
$$= E[\min(X, Y)\,|X > Y, X - Y > c]$$
$$= E[\min(X, Y)\,|X > Y]$$

where the final equality follows from (a).

37. $\dfrac{1}{\mu} + \dfrac{1}{\lambda}$

39. (a) $196/2.5 = 78.4$
(b) $196/(2.5)^2 = 31.36$

We use the central limit theorem to justify approximating the life distribution by a normal distribution with mean 78.4 and standard deviation $\sqrt{31.36} = 5.6$. In the following, Z is a standard normal random variable.

(c) $P\{L < 67.2\} \approx P\left\{Z < \dfrac{67.2 - 78.4}{5.6}\right\} = P\{Z < -2\} = .0227$

(d) $P\{L > 90\} \approx P\left\{Z > \dfrac{90 - 78.4}{5.6}\right\} = P\{Z > 2.07\} = .0192$

(e) $P\{L > 100\} \approx P\left\{Z > \dfrac{100 - 78.4}{5.6}\right\} = P\{Z > 3.857\} = .00006$

41. $\lambda_1/(\lambda_1 + \lambda_2)$

43. Let S_i denote the service time at server $i, i = 1, 2$ and let X denote the time until the next arrival. Then, with p denoting the proportion of customers that are served by both servers, we have

$$p = P\{X > S_1 + S_2\}$$
$$= P\{X > S_1\}PX > S_1 + S_2|X > S_1\}$$
$$= \frac{\mu_1}{\mu_1 + \lambda}\frac{\mu_2}{\mu_2 + \lambda}$$

45. $E[N(T)] = E[E[N(T)|T]] = E[\lambda T] = \lambda E[T]$

$$E[TN(T)] = E[E[TN(T)|T]] = E[T\lambda T] = \lambda E[T^2]$$
$$E[N^2(T)] = E\left[E[N^2(T)|T]\right] = E[\lambda T + (\lambda T)^2]$$
$$= \lambda E[T] + \lambda^2 E[T^2]$$

Hence,

$$Cov(T, N(T)) = \lambda E[T^2] - E[T]\lambda E[T] = \lambda\sigma^2$$

and

$$Var(N(T)) = \lambda E[T] + \lambda^2 E[T^2] - (\lambda E[T])^2$$
$$= \lambda\mu + \lambda^2\sigma^2$$

47. (a) $1/(2\mu) + 1/\lambda$

(b) Let T_i denote the time until both servers are busy when you start with i busy servers $i = 0, 1$. Then,

$$E[T_0] = 1/\lambda + E[T_1]$$

Now, starting with 1 server busy, let T be the time until the first event (arrival or departure); let $X = 1$ if the first event is an arrival and let it be 0 if it is a departure; let Y be the additional time after the first event until both servers are busy.

$$E[T_1] = E[T] + E[Y]$$
$$= \frac{1}{\lambda + \mu} + E[Y|X = 1]\frac{\lambda}{\lambda + \mu}$$
$$+ E[Y|X = 0]\frac{\mu}{\lambda + \mu}$$
$$= \frac{1}{\lambda + \mu} + E[T_0]\frac{\mu}{\lambda + \mu}$$

Thus,

$$E[T_0] - \frac{1}{\lambda} = \frac{1}{\lambda + \mu} + E[T_0]\,\frac{\mu}{\lambda + \mu}$$

or

$$E[T_0] = \frac{2\lambda + \mu}{\lambda^2}$$

Also,

$$E[T_1] = \frac{\lambda + \mu}{\lambda^2}$$

(c) Let L_i denote the time until a customer is lost when you start with i busy servers. Then, reasoning as in part (b) gives that

$$E[L_2] = \frac{1}{\lambda + \mu} + E[L_1]\frac{\mu}{\lambda + \mu}$$
$$= \frac{1}{\lambda + \mu} + (E[T_1] + E[L_2])\frac{\mu}{\lambda + \mu}$$
$$= \frac{1}{\lambda + \mu} + \frac{\mu}{\lambda^2} + E[L_2]\frac{\mu}{\lambda + \mu}$$

Thus,

$$E[L_2] = \frac{1}{\lambda} + \frac{\mu(\lambda + \mu)}{\lambda^3}$$

49. (a) $P\{N(T) - N(s) = 1\} = \lambda(T - s)e^{-\lambda(T-s)}$

(b) Differentiating the expression in part (a) and then setting it equal to 0 gives

$$e^{-\lambda(T-s)} = \lambda(T - s)e^{-\lambda(T-s)}$$

implying that the maximizing value is

$$s = T - 1/\lambda$$

(c) For $s = T - 1/\lambda$, we have that $\lambda(T - s) = 1$ and thus,

$$P\{N(T) - N(s) = 1\} = e^{-1}$$

51. Condition on X, the time of the first accident, to obtain

$$E[N(t] = \int_0^\infty E[N(t)|X = s]\beta e^{-\beta s}ds$$
$$= \int_0^t (1 + \alpha(t - s))\beta e^{-\beta s}ds$$

53. (a) e^{-1}

(b) $e^{-1} + e^{-1}(.8)e^{-1}$

55. As long as customers are present to be served, every event (arrival or departure) will, independently of other events, be a departure with probability $p = \mu/(\lambda + \mu)$. Thus $P\{X = m\}$ is the probability that there have been a total of m tails at the moment that the n^{th} head occurs, when independent flips of a coin having probability p of coming up heads are made: that is, it is the probability that the n^{th} head occurs on trial number $n + m$. Hence,

$$p\{X = m\} = \binom{n + m - 1}{n - 1} p^n(1 - p)^m$$

■

57. (a) e^{-2}

(b) 2 p.m.

59. The unconditional probability that the claim is type 1 is 10/11. Therefore,

$$P(1|4000) = \frac{P(4000|1)P(1)}{P(4000|1)P(1) + P(4000|2)P(2)}$$
$$= \frac{e^{-4}10/11}{e^{-4}10/11 + .2e^{-.8}1/11}$$

61. (a) Poisson with mean $cG(t)$.

(b) Poisson with mean $c[1 - G(t)]$.

(c) Independent.

63. Let X and Y be respectively the number of customers in the system at time $t + s$ that were present at time s, and the number in the system at $t + s$ that were not in the system at time s. Since there are an infinite number of servers, it follows that X and Y are independent (even if given the number is the system at time s). Since the service distribution is exponential with rate μ, it follows that given that $X(s) = n$, X will be binomial with parameters n and $p = e^{-\mu t}$. Also Y, which is independent of $X(s)$, will have the same distribution as $X(t)$.

Therefore, Y is Poisson with mean $\lambda \int_0^t e^{-\mu y} dy$

$= \lambda(1 - e^{-\mu t})/\mu$

(a) $E[X(t + s)|X(s) = n]$

$= E[X|X(s) = n] + E[Y|X(s) = n]$.

$= ne^{-\mu t} + \lambda(1 - e^{-\mu t})/\mu$

(b) $Var(X(t + s)|X(s) = n)$

$= Var(X + Y|X(s) = n)$

$= Var(X|X(s) = n) + Var(Y)$

$= ne^{-\mu t}(1 - e^{-\mu t}) + \lambda(1 - e^{-\mu t})/\mu$

The above equation uses the formulas for the variances of a binomial and a Poisson random variable.

(c) Consider an infinite server queuing system in which customers arrive according to a Poisson process with rate λ, and where the service times are all exponential random variables with rate μ. If there is currently a single customer in the system, find the probability that the system becomes empty when that customer departs.

Condition on R, the remaining service time:

$P\{\text{empty}\}$

$$= \int_0^\infty P\{\text{empty}|R = t\}\mu e^{-\mu t} dt$$
$$= \int_0^\infty \exp\left\{-\lambda \int_0^t e^{-\mu y} dy\right\} \mu e^{-\mu t} dt$$
$$= \int_0^\infty \exp\left\{-\frac{\lambda}{\mu}(1 - e^{-\mu t})\right\} \mu e^{-\mu t} dt$$
$$= \int_0^1 e^{-\lambda x/\mu} dx$$
$$= \frac{\mu}{\lambda}(1 - e^{-\lambda/\mu})$$

where the preceding used that $P\{\text{empty}|R = t\}$ is equal to the probability that an $M/M/\infty$ queue is empty at time t.

65. This is an application of the infinite server Poisson queue model. An arrival corresponds to a new lawyer passing the bar exam, the service time is the time the lawyer practices law. The number in the system at time t is, for large t, approximately a Poisson random variable with mean $\lambda\mu$ where λ is the arrival rate and μ the mean service time. This latter statement follows from

$$\int_0^n [1 - G(y)]dy = \mu$$

where μ is the mean of the distribution G. Thus, we would expect $500 \cdot 30 = 15,000$ lawyers.

67. If we count a satellite if it is launched before time s but remains in operation at time t, then the number of items counted is Poisson with mean $m(t) = \int_0^s \bar{G}(t - y)dy$. The answer is $e^{-m(t)}$.

69. (a) $1 - e^{-\lambda(t-s)}$

(b) $e^{-\lambda s}e^{-\lambda(t-s)}[\lambda(t - s)]^3/3!$

(c) $4 + \lambda(t - s)$

(d) $4s/t$

71. Let $U_1, \ldots$ be independent uniform $(0, t)$ random variables that are independent of $N(t)$, and let $U_{(i,n)}$ be the i^{th} smallest of the first n of them.

$$P\left\{\sum_{i=1}^{N(t)} g(S_i) < x\right\}$$
$$= \sum_n P\left\{\sum_{i=1}^{N(t)} g(S_i) < x|N(t)=n\right\} P\{N(t)=n\}$$
$$= \sum_n P\left\{\sum_{i=1}^{n} g(S_i) < x|N(t)=n\right\} P\{N(t)=n\}$$
$$= \sum_n P\left\{\sum_{i=1}^{n} g(U_{(i,n)}) < x\right\} P\{N(t)=n\}$$

(Theorem 5.2)

$$= \sum_n P\left\{\sum_{i=1}^{n} g(U_i) < x\right\} P\{N(t)=n\}$$
$$\left(\sum_{i=1}^{n} g(U_{(i,n)}) = \sum_{i=1}^{n} g(U_i)\right)$$
$$= \sum_n P\left\{\sum_{i=1}^{n} g(U_i) < x|N(t)=n\right\} P\{N(t)=n\}$$
$$= \sum_n P\left\{\sum_{i=1}^{N(t)} g(U_i) < x|N(t)=n\right\} P\{N(t)=n\}$$
$$= P\left\{\sum_{i=1}^{N(t)} g(U_i) < x\right\}$$

73. (a) It is the gamma distribution with parameters n and λ.

(b) For $n \geq 1$,
$P\{N = n|T = t\}$
$$= \frac{P\{T = t|N = n\}p(1-p)^{n-1}}{f_T(t)}$$
$$= C\frac{(\lambda t)^{n-1}}{(n-1)!}(1-p)^{n-1}$$
$$= C\frac{(\lambda(1-p)t)^{n-1}}{(n-1)!}$$
$$= e^{-\lambda(1-p)t}\frac{(\lambda(1-p)t)^{n-1}}{(n-1)!}$$

where the last equality follows since the probabilities must sum to 1.

(c) The Poisson events are broken into two classes, those that cause failure and those that do not. By Proposition 5.2, this results in two independent Poisson processes with respective rates λp and $\lambda(1-p)$. By independence it follows that given that the first event of the first process occurred at time t the number of events of the second process by this time is Poisson with mean $\lambda(1-p)t$.

75. (a) $\{Y_n\}$ is a Markov chain with transition probabilities given by

$$P_{0j} = a_j, \quad P_{i,i-1+j} = a_j, \quad j \geq 0$$

where

$$a_j = \int \frac{e^{-\lambda t}(\lambda t)^j}{j!} dG(t)$$

(b) $\{X_n\}$ is a Markov chain with transition probabilities

$$P_{i,i+1-j} = \beta_j,\ j = 0, 1, \ldots, i, P_{i,0} = \sum_{k=i+1}^{\infty} \beta_j$$

where

$$\beta_j = \int \frac{e^{-\mu t}(\mu t)^j}{j!} dF(t)$$

77. (a) $\dfrac{\mu}{\lambda+\mu}$

(b) $\dfrac{\lambda}{\lambda+\mu}\dfrac{2\mu}{\lambda+2\mu}$

(c) $\displaystyle\prod_{i=1}^{j-1}\frac{\lambda}{\lambda+i\mu}\frac{j\mu}{\lambda+j\mu},\ j > 1$

(d) Conditioning on N yields the solution; namely $\displaystyle\sum_{j=1}^{\infty}\frac{1}{j}P(N=j)$

(e) $\displaystyle\sum_{j=1}^{\infty}P(N=j)\sum_{i=0}^{j}\frac{1}{\lambda+i\mu}$

79. Consider a Poisson process with rate λ in which an event at time t is counted with probability $\lambda(t)/\lambda$ independently of the past. Clearly such a process will have independent increments. In addition,

$$P\{2 \text{ or more counted events in}(t, t+h)\}$$
$$\leq P\{2 \text{ or more events in}(t, t+h)\}$$
$$= o(h)$$

and

$$P\{1 \text{ counted event in } (t, t+h)\}$$
$$= P\{1 \text{ counted} \mid 1 \text{ event}\}P(1 \text{ event})$$

$+ \; P\{1 \text{ counted} \mid \geq 2 \text{ events}\}P\{\geq 2\}$

$$= \int_t^{t+h} \frac{\lambda(s)}{\lambda}\frac{ds}{h}(\lambda h + o(h)) + o(h)$$

$$= \frac{\lambda(t)}{\lambda}\lambda h + o(h)$$

$$= \lambda(t)h + o(h)$$

81. (a) Let S_i denote the time of the ith event, $i \geq 1$. Let $t_i + h_i < t_{i+1}$, $t_n + h_n \leq t$.

$P\{t_i < S_i < t_i + h_i,\ i = 1, \ldots, n | N(t) = n\}$

$$= \frac{P\{1 \text{ event in } (t_i, t_i + h_i),\quad i = 1, \ldots, n, \text{ no events elsewhere in } (0, t)\}}{P\{N(t) = n\}}$$

$$= \frac{\left[\prod_{i=1}^{n} e^{-(m(t_i+h_i)-m(t_i))}[m(t_i + h_i) - m(t_i)]\right] e^{-[m(t)-\sum_i m(t_i+h_i)-m(t_i)]}}{e^{-m(t)}[m(t)]^n/n!}$$

$$= \frac{n\prod_{i}^{n}[m(t_i + h_i) - m(t_i)]}{[m(t)]^n}$$

Dividing both sides by $h_1 \cdots h_n$ and using the fact that $m(t_i + h_i) - m(t_i) = \int_{t_i}^{t_i+h} \lambda(s)\ ds = \lambda(t_i)h + o(h)$ yields upon letting the $h_i \to 0$:

$f_{S_1 \cdots S_2}(t_1, \ldots, t_n | N(t) = n)$

$$= n! \prod_{i=1}^{n}[\lambda(t_i)/m(t)]$$

and the right-hand side is seen to be the joint density function of the order statistics from a set of n independent random variables from the distribution with density function $f(x) = m(x)/m(t), x \leq t$.

(b) Let $N(t)$ denote the number of injuries by time t. Now given $N(t) = n$, it follows from part (b) that the n injury instances are independent and identically distributed. The probability (density) that an arbitrary one of those injuries was at s is $\lambda(s)/m(t)$, and so the probability that the injured party will still be out of work at time t is

$$p = \int_0^t P\{\text{out of work at } t | \text{injured at } s\}\frac{\lambda(s)}{m(t)}d\zeta$$

$$= \int_0^t [1 - F(t - s)]\frac{\lambda(s)}{m(t)}d\zeta$$

Hence, as each of the $N(t)$ injured parties have the same probability p of being out of work at t, we see that

$$E[X(t)]|N(t)] = N(t)p$$

and thus,

$$E[X(t)] = pE[N(t)]$$

$$= pm(t)$$

$$= \int_0^t [1 - F(t - s)]\lambda(s)\ ds$$

83. Since $m(t)$ is increasing it follows that nonoverlapping time intervals of the $\{N(t)\}$ process will correspond to nonoverlapping intervals of the $\{N_o(t)\}$ process. As a result, the independent increment property will also hold for the $\{N(t)\}$ process. For the remainder we will use the identity

$$m(t + h) = m(t) + \lambda(t)h + o(h)$$

$P\{N(t + h) - N(t) \geq 2\}$

$$= P\{N_o[m(t + h)] - N_o[m(t)] \geq 2\}$$

$$= P\{N_o[m(t) + \lambda(t)h + o(h)] - N_o[m(t)] \geq 2\}$$

$$= o[\lambda(t)h + o(h)] = o(h)$$

$P\{N(t + h) - N(t) = 1\}$

$$= P\{N_o[m(t) + \lambda(t)h + o(h)] - N_o[m(t)] = 1\}$$

$= P\{1$ event of Poisson process in interval of length $\lambda(t)h + o(h)]\}$

$$= \lambda(t)h + o(h)$$

85. \$ 40,000 and \$$1.6 \times 10^8$.

87. $Cov[X(t), X(t + s)]$

$$= Cov[X(t), X(t) + X(t + s) - X(t)]$$

$$= Cov[X(t), X(t)] + Cov[X(t), X(t + s) - X(t)]$$

$= Cov[X(t), X(t)]$ by independent increments

$$= Var[X(t)] = \lambda t E[Y^2]$$

89. Let T_i denote the arrival time of the first type i shock, $i = 1, 2, 3$.

$P\{X_1 > s, X_2 > t\}$

$$= P\{T_1 > s, T_3 > s, T_2 > t, T_3 > t\}$$

$$= P\{T_1 > s, T_2 > t, T_3 > \max(s, t)\}$$

$$= e^{-\lambda_1 s}e^{-\lambda_2 t}e^{-\lambda_3 \max(s,t)}$$

91. To begin, note that

$$P\left[X_1 > \sum_2^n X_i\right]$$

$$= P\{X_1 > X_2\}P\{X_1 - X_2 > X_3|X_1 > X_2\}$$

$$= P\{X_1 - X_2 - X_3 > X_4|X_1 > X_2 + X_3\}\ldots$$

$$= P\{X_1 - X_2 \cdots - X_{n-1} > X_n|X_1 > X_2 + \cdots + X_{n-1}\}$$

$$= (1/2)^{n-1}$$

Hence,

$$P\left\{M > \sum_{i=1}^n X_i - M\right\} = \sum_{i-1}^n P\left\{X_1 > \sum_{j\neq i}^n X_i\right\} = n/2^{n-1}$$

93. (a) $\max(X_1, X_2) + \min(X_1, X_2) = X_1 + X_2$.

(b) This can be done by induction:

$$\max\{(X_1, \ldots, X_n)$$

$$= \max(X_1, \max(X_2, \ldots, X_n))$$

$$= X_1 + \max(X_2, \ldots, X_n) - \min(X_1, \max(X_2, \ldots, X_n))$$

$$= X_1 + \max(X_2, \ldots, X_n) - \max(\min(X_1, X_2), \ldots, \min(X_1, X_n)).$$

Now use the induction hypothesis.

A second method is as follows:

Suppose $X_1 \leq X_2 \leq \cdots \leq X_n$. Then the coefficient of X_i on the right side is

$$1 - \begin{bmatrix} n-i \\ 1 \end{bmatrix} + \begin{bmatrix} n-i \\ 2 \end{bmatrix} - \begin{bmatrix} n-i \\ 3 \end{bmatrix} + \cdots$$

$$= (1-1)^{n-i}$$

$$= \begin{cases} 0, & i \neq n \\ 1, & i = n \end{cases}$$

and so both sides equal X_n. By symmetry the result follows for all other possible orderings of the $X's$.

(c) Taking expectations of (b) where X_i is the time of the first event of the i^{th} process yields

$$\sum_i \lambda_i^{-1} - \sum_i \sum_{<j} (\lambda_i + \lambda_j)^{-1} + \sum_i \sum_{<j} \sum_{<k} (\lambda_i + \lambda_j + \lambda_k)^{-1} - \cdots + (-1)^{n+1} \left[\sum_1^n \lambda_i\right]^{-1}$$

95. $$E[L|N(t) = n] = \frac{\int xg(x)e^{-xt}(xt)^n dx}{\int g(x)e^{-xt}(xt)^n dx}$$

Conditioning on L yields

$$E[N(s)|N(t) = n]$$

$$= E[E[N(s)|N(t) = n, L]|N(t) = n]$$

$$= E[n + L(s - t)|N(t) = n]$$

$$= n + (s - t)E[L|N(t) = n]$$

For (c), use that for any value of L, given that there have been n events by time t, the set of n event times are distributed as the set of n independent uniform $(0, t)$ random variables. Thus, for $s < t$

$$E[N(s)|N(t) = n] = ns/t$$

97. With $C = 1/P(N(t) = n)$, we have

$$f_{L|N(t)}(\lambda|n) = Ce^{-\lambda t}\frac{(\lambda t)^n}{n!}pe^{-p\lambda}\frac{(p\lambda)^{m-1}}{(m-1)!}$$

$$= Ke^{-(p+t)\lambda}\lambda^{n+m-1}$$

where K does not depend on λ. But we recognize the preceding as the gamma density with parameters $n + m, p + t$, which is thus the conditional density.

Chapter 6

1. Let us assume that the state is (n, m). Male i mates at a rate λ with female j, and therefore it mates at a rate λm. Since there are n males, matings occur at a rate λnm. Therefore,

   $v_{(n,m)} = \lambda nm$

   Since any mating is equally likely to result in a female as in a male, we have

   $$P_{(n,m);(n+1,m)} = P_{(n,m)(n,m+1)} = \frac{1}{2}$$

3. This is not a birth and death process since we need more information than just the number working. We also must know which machine is working. We can analyze it by letting the states be

   b : both machines are working
   1 : 1 is working, 2 is down
   2 : 2 is working, 1 is down
   0_1: both are down, 1 is being serviced
   0_2: both are down, 2 is being serviced

   $$v_b = \mu_1 + \mu_2,\ v_1 = \mu_1 + \mu,\ v_2 = \mu_2 + \mu,$$

   $$v_{0_1} = v_{0_2} = \mu$$

   $$P_{b,1} = \frac{\mu_2}{\mu_2 + \mu_1} = 1 - P_{b,2}, \quad P_{1,b} = \frac{\mu}{\mu + \mu_1}$$
   $$= 1 - P_{1,0_2}$$
   $$P_{2,b} = \frac{\mu}{\mu + \mu_2} = 1 - P_{2,0_1}, \quad P_{0_1,1} = P_{0_2,2} = 1$$

5. (a) Yes.
   (b) It is a pure birth process.
   (c) If there are i infected individuals then since a contact will involve an infected and an uninfected individual with probability $i\,(n-i)\,/\binom{n}{2}$, it follows that the birth rates are $\lambda_i = \lambda i(n-i)/\binom{n}{2}$, $i = 1, \ldots, n$. Hence,

   $$E[\text{time all infected}] = \frac{n(n-1)}{2\lambda}\sum_{i=1}^{n} 1/[i(n-i)]$$

7. (a) Yes!
   (b) For $n = (n_1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1})$ let

   $$S_i(n) = (n_1, \ldots, n_{i-1}, n_{i+1} + 1, \ldots, n_{k-1}),$$
   $$i = 1, \ldots, k-2$$
   $$S_{k-1}(n) = (n_1, \ldots, n_i, n_{i+1}, \ldots n_{k-1} - 1),$$
   $$S_0(n) = (n_1 + 1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1})$$

   Then

   $$q_n, S_1(n) = n_i\mu, \qquad i = 1, \ldots, k-1$$
   $$q_n, S_0(n) = \lambda$$

9. Since the death rate is constant, it follows that as long as the system is nonempty, the number of deaths in any interval of length t will be a Poisson random variable with mean μt. Hence,

   $$P_{ij}(t) = e^{-\mu t}(\mu t)^{i-j}/(i-j)!, \quad 0 < j \le i$$
   $$P_{i,0}(t) = \sum_{k=i}^{\infty} e^{-\mu t}(\mu t)^k/k!$$

11. (b) Follows from the hint upon using the lack of memory property and the fact that ϵ_i, the minimum of $j-(i-1)$ independent exponentials with rate λ, is exponential with rate $(j-i+1)\lambda$.
   (c) From (a) and (b)

   $$P\{T_1 + \cdots + T_j \le t\} = P\left\{\max_{1\le i\le j} X_i \le t\right\}$$
   $$= (1 - e^{-\lambda t})^j$$

   (d) With all probabilities conditional on $X(0) = 1$

   $$P_{1j}(t) = P\{X(t) = j\}$$
   $$= P\{X(t) \ge j\} - P\{X(t) \ge j+1\}$$
   $$= P\{T_1 + \cdots + T_j \le t\}$$
   $$- P\{T_1 + \cdots + T_{j+1} \le t\}$$

   (e) The sum of independent geometrics, each having parameter $p = e^{-\lambda t}$, is negative binomial with parameters i, p. The result follows since starting with an initial population of i is equivalent to having i independent Yule processes, each starting with a single individual.

13. With the number of customers in the shop as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = 3, \quad \mu_1 = \mu_2 = 4$$

Therefore

$$P_1 = \frac{3}{4}P_0, \quad P_2 = \frac{3}{4}, \quad P_1 = \left[\frac{3}{4}\right]^2 P_0$$

And since $\sum_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{16}{37}$$

(a) The average number of customers in the shop is

$$P_1 + 2P_2 = \left[\frac{3}{4} + 2\left[\frac{3}{4}\right]^2\right] P_0$$
$$= \frac{30}{16}\left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{30}{37}$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1 - P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{16} \cdot \frac{16}{37} = \frac{28}{37}$$

(c) Now $\mu = 8$, and so

$$P_0 = \left[1 + \frac{3}{8} + \left[\frac{3}{8}\right]^2\right]^{-1} = \frac{64}{97}$$

So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left[\frac{3}{8}\right]^2 \frac{264}{97} = 1 - \frac{9}{97} = \frac{88}{97}$$

The rate of added customers is therefore

$$\lambda\left[\frac{88}{97}\right] - \lambda\left[\frac{28}{37}\right] = 3\left[\frac{88}{97} - \frac{28}{37}\right] = 0.45$$

The business he does would improve by 0.45 customers per hour.

15. With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \ \lambda_i = 0, \quad i \geq 4$$
$$\mu_1 = 2, \ \mu_2 = \mu_3 = 4$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0, \ P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0, \ P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0$$

And therefore,

$$P_0 = \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right]^{-1} = \frac{32}{143}$$

(a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1 - P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \times \frac{32}{143} = \frac{116}{143}$$

(b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0 \ P_2 = \frac{3}{4}P_1 = \left[\frac{3}{4}\right]^2 P_0 \ P_3 = \left[\frac{3}{4}\right]^3 P_0$$

and $P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2 + \left[\frac{3}{4}\right]^3\right]^{-1} = \frac{64}{175}$

So that now

$$1 - P_3 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = \frac{148}{175}$$

17. Say the state is 0 if the machine is up, say it is i when it is down due to a type i failure, $i = 1, 2$. The balance equations for the limiting probabilities are as follows.

$$\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$\mu_1 P_1 = \lambda p P_0$$
$$\mu_2 P_2 = \lambda(1 - p)P_0$$
$$P_0 + P_1 + P_2 = 1$$

These equations are easily solved to give the results

$$P_0 = (1 + \lambda p/\mu_1 + \lambda(1 - p)/\mu_2)^{-1}$$
$$P_1 = \lambda p P_0/\mu_1, \qquad P_2 = \lambda(1 - p)P_0/\mu_2$$

19. There are 4 states. Let state 0 mean that no machines are down, state 1 that machine 1 is down and 2 is up, state 2 that machine 1 is up and 2 is down, and 3 that both machines are down. The balance equations are as follows:

$$(\lambda_1 + \lambda_2)P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$(\mu_1 + \lambda_2)P_1 = \lambda_1 P_0 + \mu_1 P_3$$
$$(\lambda_1 + \mu_2)P_2 = \lambda_2 P_0$$
$$\mu_1 P_3 = \mu_2 P_1 + \mu_1 P_2$$
$$P_0 + P_1 + P_2 + P_3 = 1$$

These equations are easily solved and the proportion of time machine 2 is down is $P_2 + P_3$.

21. How we have a birth and death process with parameters

$$\lambda_i = \lambda, \quad i = 1, 2$$
$$\mu_i = i\mu, \quad i = 1, 2$$

Therefore,

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2/2}$$

and so the probability that at least one machine is up is higher in this case.

23. Let the state denote the number of machines that are down. This yields a birth and death process with

$$\lambda_0 = \frac{3}{10}, \ \lambda_1 = \frac{2}{10}, \ \lambda_2 = \frac{1}{10}, \ \lambda_i = 0, \quad i \geq 3$$
$$\mu_1 = \frac{1}{8}, \ \mu_2 = \frac{2}{8}, \ \mu_3 = \frac{2}{8}$$

The balance equations reduce to

$$P_1 = \frac{3/10}{1/8} P_0 = \frac{12}{5} P_0$$
$$P_2 = \frac{2/10}{2/8} P_1 = \frac{4}{5} P_1 = \frac{48}{25} P_0$$
$$P_3 = \frac{1/10}{2/8} P_2 = \frac{4}{10} P_3 = \frac{192}{250} P_0$$

Hence, using $\sum_0^3 P_i = 1$ yields

$$P_0 = \left[1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250}\right]^{-1} = \frac{250}{1522}$$

(a) Average number not in use

$$= P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}$$

(b) Proportion of time both repairmen are busy

$$= P_2 + P_3 = \frac{672}{1522} = \frac{336}{761}$$

25. If $N_i(t)$ is the number of customers in the ith system $(i = 1, 2)$, then let us take $\{N_1(t), N_2(t)\}$ as the state. The balance equation are with $n \geq 1, m \geq 1$.

(a) $\lambda P_{0,0} = \mu_2 P_{0,1}$

(b) $P_{n,0}(\lambda + \mu_1) = \lambda P_{n-1,0} + \mu_2 P_{n,1}$

(c) $P_{0,m}(\lambda + \mu_2) = \mu_1 P_{1,m-1} + \mu_2 P_{0,m+1}$

(d) $P_{n,m}(\lambda + \mu_1 + \mu_2) = \lambda P_{n-1,m} + \mu_1 P_{n+1,m-1} + \mu_2 P_{n,m+1}$

We will try a solution of the form $C\alpha^n \beta^m = P_{n,m}$. From (a), we get

$$\lambda C = \mu_2 C\beta = \beta = \frac{\lambda}{\mu_2}$$

From (b),

$$(\lambda + \mu_1) C\alpha^n = \lambda C\alpha^{n-1} + \mu_2 C\alpha^n \beta$$

or

$$(\lambda + \mu_1)\alpha = \lambda + \mu_2\alpha\beta = \lambda + \mu_2\alpha\frac{\lambda}{\mu} = \lambda + \lambda\alpha$$

and $\quad \mu_1\alpha = \lambda \Rightarrow \alpha = \dfrac{\lambda}{\mu_1}$

To get C, we observe that $\sum_{n,m} P_{n,m} = 1$

but

$$\sum_{n,m} P_{n,m} = C\sum_n \alpha^n \sum_m \beta^m = C\left[\frac{1}{1-\alpha}\right]\left[\frac{1}{1-\beta}\right]$$

and $C = \left[1 - \dfrac{\lambda}{\mu_1}\right]\left[1 - \dfrac{\lambda}{\mu_2}\right]$

Therefore a solution of the form $C\alpha^n\beta^n$ must be given by

$$P_{n,m} = \left[1 - \frac{\lambda}{\mu_1}\right]\left[\frac{\lambda}{\mu_1}\right]^n \left[1 - \frac{\lambda}{\mu_2}\right]\left[\frac{\lambda}{\mu_2}\right]^m$$

It is easy to verify that this also satisfies (c) and (d) and is therefore the solution of the balance equations.

27. It is a Poisson process by time reversibility. If $\lambda > \delta\mu$, the departure process will (in the limit) be a Poisson process with rate $\delta\mu$ since the servers will always be busy and thus the time between departures will be independent random variables each with rate $\delta\mu$.

29. (a) Let the state be S, the set of failed machines.

(b) For $i \in S, j \in S^c$,

$$q_{S,\, S-i} = \mu_i/|S|, q_{S,\, S+j} = \lambda_j$$

where $S - i$ is the set S with i deleted and $S + j$ is similarly S with j added. In addition, $|S|$ denotes the number of elements in S.

(c) $P_S q_{S,\, S-i} = P_{S-i} q_{S-i,\, S}$

(d) The equation in (c) is equivalent to

$$P_S\mu_i/|S| = P_{S-i}\lambda_i$$

or

$$P_S = P_{S-i}|S|\lambda_i/\mu_i$$

Iterating this recursion gives

$$P_S = P_0(|S|)!\prod_{i\in S}(\lambda_i/\mu_i)$$

where 0 is the empty set. Summing over all S gives

$$1 = P_0\sum_S(|S|)!\prod_{i\in S}(\lambda_i/\mu_i)$$

and so

$$P_S = \frac{(|S|)!\prod_{i\in S}(\lambda_i/\mu_i)}{\sum_S(|S|)!\prod_{i\in S}(\lambda_i/\mu_i)}$$

As this solution satisfies the time reversibility equations, it follows that, in the steady state, the chain is time reversible with these limiting probabilities.

31. (a) This follows because of the fact that all of the service times are exponentially distributed and thus memoryless.

(b) Let $n = (n_1, \ldots, n_i, \ldots, n_j, \ldots, n_r)$, where $n_i > 0$ and let $n' = (n_1, \ldots, n_i - 1, \ldots, n_j - 1, \ldots, n_r)$. Then $q_{n,\,n'} = \mu_i/(r-1)$.

(c) The process is time reversible if we can find probabilities $P(n)$ that satisfy the equations

$$P(n)\mu_i/(r-1) = P(n')\mu_j/(r-1)$$

where n and n' are as given in part (b). The above equations are equivalent to

$$\mu_i P(n) = \mu_j/P(n')$$

Since $n_i = n'_i + 1$ and $n'_j = n_j + 1$ (where n_k refers to the k^{th} component of the vector n), the above equation suggests the solution

$$P(n) = C\prod_{k=1}^{r}(1/\mu_k)^n k$$

where C is chosen to make the probabilities sum to 1. As $P(n)$ satisfies all the time reversibility equations it follows that the chain is time reversible and the $P(n)$ given above are the limiting probabilities.

33. Suppose first that the waiting room is of infinite size. Let $X_i(t)$ denote the number of customers at server $i, i = 1, 2$. Then since each of the $M/M/1$ processes $\{X_i(t)\}$ is time-reversible, it follows by Problem 28 that the vector process $\{(X_1(t), X_2(t)), t \geq 0\}$ is a time-reversible Markov chain. Now the process of interest is just the truncation of this vector process to the set of states A where

$$A = \{(0,\, m) : m \leq 4\} \cup \{(n,\, 0) : n\ \leq 4\}$$
$$\cup\, \{(n,\, m) : nm > 0, n + m \leq 5\}$$

Hence, the probability that there are n with server 1 and n with server 2 is

$$P_{n,\,m} = k(\lambda_1/\mu_1)^n(1 - \lambda_1/\mu_1)(\lambda_2/\mu_2)^m(1 - \lambda_2/\mu_2),$$
$$= C(\lambda_1/\mu_1)^n(\lambda_2/\mu_2)^m, \quad (n, m) \in A$$

The constant C is determined from

$$\sum P_{n,\,n} = 1$$

where the sum is over all $(n,\ m)$ in A.

35. We must find probabilities P_i^n such that

$$P_i^n q_{ij}^n = P_j^n q_{ji}^n$$

or

$$\begin{aligned} cP_i^n q_{ij} &= P_j^n q_{ji}, && \text{if } i \in A, j \notin A \\ P_i q_{ij} &= cP_j^n q_{ji}, && \text{if } i \notin A, j \in A \\ P_i q_{ij} &= P_j q_{ji}, && \text{otherwise} \end{aligned}$$

Now, $P_i q_{ij} = P_j q_{ji}$ and so if we let

$$P_i^n = \begin{matrix} kP_i/c & \text{if } i \in A \\ kP_i & \text{if } i \notin A \end{matrix}$$

then we have a solution to the above equations. By choosing k to make the sum of the P_j^n equal to 1, we have the desired result. That is,

$$k = \left(\sum_{i\in A} P_i/c - \sum_{i\notin A} P_i\right)^{-1}$$

37. The state of any time is the set of down components at that time. For $S \subset \{1, 2, \ldots, n\}$, $i \notin S, j \in S$

$$q(S, S + i) = \lambda_i$$

$$q(S, S - j) = \mu_j\alpha^{|S|}$$

where $S + i = S \cup \{i\}$, $S - j = S \cap \{j\}^c$, $|S| =$ number of elements in S.

The time reversible equations are

$$P(S)\mu_i\alpha^{|S|} = P(S-i)\lambda_i, \quad i \in S$$

The above is satisfied when, for $S = \{i_1, i_2, \ldots, i_k\}$

$$P(S) = \frac{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}}{\mu_{i_1}\mu_{i_2}\cdots\mu_{i_k}\alpha^{k(k+1)/2}}P(\phi)$$

where $P(\phi)$ is determined so that

$$\sum P(S) = 1$$

where the sum is over all the 2^n subsets of $\{1, 2, \ldots, n\}$.

39. $E[0(t)|x(0) = 1] = t - E[\text{time in } 1|X(0) = 1]$

$$= t - \frac{\lambda t}{\lambda + \mu} - \frac{\mu}{(\lambda + \mu)^2}[1 - e^{-(\lambda+\mu)t}]$$

The final equality is obtained from Example 7b (or Problem 38) by interchanging λ and μ.

41. (a) Letting T_i denote the time until a transition out of i occurs, we have

$$P_{ij} = P\{X(Y) = j\} = P\{X(Y) = j \mid T_i < Y\}$$

$$\times \frac{v_i}{v_i + \lambda} + P\{X(Y) = j|Y \le T_i\}\frac{\lambda}{\lambda + v_i}$$

$$= \sum_k P_{ik}P_{kj}\frac{v_i}{v_i + \lambda} + \frac{\delta_{ij}\lambda}{\lambda + v_i}$$

The first term on the right follows upon conditioning on the state visited from i (which is k with probability P_{ik}) and then using the lack of memory property of the exponential to assert that given a transition into k occurs before time Y then the state at Y is probabilistically the same as if the process had started in state k and we were interested in the state after an exponential time with rate λ. As $q_{ik} = v_iP_{ik}$, the result follows.

(b) From (a)

$$(\lambda + v_i)\bar{P}_{ij} = \sum_k q_{ik}\bar{P}_{kj} + \lambda\delta_{ij}$$

or

$$-\lambda\delta_{ij} = \sum_k r_{ik}\bar{P}_{kj} - \lambda\bar{P}_{ij}$$

or, in matrix terminology,

$$-\lambda I = R\bar{P} - \lambda I\bar{P}$$
$$= (R - \lambda I)\bar{P}$$

implying that

$$\bar{P} = -\lambda I(R - \lambda I)^{-1} = -(R/\lambda - I)^{-1}$$
$$= (I - R/\lambda)^{-1}$$

(c) Consider, for instance,

$$P\{X(Y_1 + Y_2) = j|X(0) = i\}$$

$$= \sum_k P\{X(Y_1 + Y_2) = j|X(Y_1) = k, X(0) = i)$$
$$P\{X(Y_1) = k|X(0) = i\}$$

$$= \sum_k P\{X(Y_1 + Y_2) = j|X(Y_1) = k\}\bar{P}_{ik}$$

$$= \sum_k P\{X(Y_2) = j|X(0) = k\}\bar{P}_{ik}$$

$$= \sum_k \bar{P}_{kj}\bar{P}_{ik}$$

and thus the state at time $Y_1 + Y_2$ is just the 2-stage transition probabilities of $\bar{P}_{ij}$. The general case can be established by induction.

(d) The above results in exactly the same approximation as Approximation 2 in Section 6.8.

Chapter 7

1. (a) Yes, (b) no, (c) no.

3. By the one-to-one correspondence of $m(t)$ and F, it follows that $\{N(t), t \geq 0\}$ is a Poisson process with rate 1/2. Hence,

$$P\{N(5) = 0\} = e^{-5/2}$$

5. The random variable N is equal to $N(I) + 1$ where $\{N(t)\}$ is the renewal process whose interarrival distribution is uniform on (0, 1). By the results of Example 2c,

$$E[N] = a\,(1) + 1 = e$$

7. Once every five months.

9. A job completion constitutes a reneval. Let T denote the time between renewals. To compute $E[T]$ start by conditioning on W, the time it takes to finish the next job:

$$E[T] = E[E[T|W]]$$

Now, to determine $E[T|W = w]$ condition on S, the time of the next shock. This gives

$$E[T|W = w] = \int_0^\infty E[T|W = w, S = x]\lambda e^{-\lambda x} dx$$

Now, if the time to finish is less than the time of the shock then the job is completed at the finish time; otherwise everything starts over when the shock occurs. This gives

$$E[T|W = w, S = x] = \begin{cases} x + E[T], & \text{if} \quad x < w \\ w, & \text{if} \quad x \geq w \end{cases}$$

Hence,

$$E[T|W = w]$$

$$= \int_0^w (x + E[T])\lambda e^{-\lambda x} dx + w \int_w^\infty \lambda e^{-\lambda x} dx$$

$$= E[T][1 - e^{-\lambda w}] + 1/\lambda - we^{-\lambda w} - \frac{1}{\lambda} e^{-\lambda w} - we^{-\lambda w}$$

Thus,

$$E[T|W] = (E[T] + 1/\lambda)(1 - e^{-\lambda W})$$

Taking expectations gives

$$E[T] = (E[T] + 1/\lambda)(1 - E[e^{-\lambda W}])$$

and so

$$E[T] = \frac{1 - E[e^{-\lambda W}]}{\lambda E[e^{-\lambda W}]}$$

In the above, W is a random variable having distribution F and so

$$E[e^{-\lambda W}] = \int_0^\infty e^{-\lambda w} f(w) dw$$

11. $$\frac{N(t)}{t} = \frac{1}{t} + \frac{\text{number of renewals in } (X_1, t)}{t}$$

Since $X_1 < \infty$, Proposition 3.1 implies that

$$\frac{\text{number of renewals in } (X_1, t)}{t} - \frac{1}{\mu} \text{ as } t - \infty.$$

13. (a) N_1 and N_2 are stopping times. N_3 is not.

(b) Follows immediately from the definition of I_i.

(c) The value of I_i is completely determined from $X_1, \ldots, X_{i-1}$ (e.g., $I_i = 0$ or 1 depending upon whether or not we have stopped after observing $X_1, \ldots, X_{i-1}$). Hence, I_i is independent of X_i.

(d) $\sum_{i=1}^\infty E[I_i] = \sum_{i=1}^\infty P\{N \geq i\} = E[N]$

(e) $E\left[X_1 + \cdots + X_{N_1}\right] = E[N_1]E[X]$

But $X_1 + \cdots + X_{N_1} = 5, E[X] = p$ and so

$E[N_1] = 5/p$

$E\left[X_1 + \cdots + X_{N_2}\right] = E[N_2]E[X]$

$E[X] = p, E[N_2] = 5p + 3(1 - p) = 3 + 2p$

$E\left[X_1 + \cdots + X_{N_2}\right] = (3 + 2p)p$

15. (a) X_i = amount of time he has to travel after his *ith* choice (we will assume that he keeps on making choices even after becoming free). N is the number of choices he makes until becoming free.

(b) $E[T] = E\left[\sum_1^N X_i\right] = E[N]E[X]$

N is a geometric random variable with $P = 1/3$, so

$$E[N] = 3, E[X] = \frac{1}{3}(2+4+6) = 4$$

Hence, $E[T] = 12$.

(c) $E\left[\sum_1^N X_i | N = n\right] = (n-1)\frac{1}{2}(4+6) + 2 = 5n - 3$, since given $N = n, X_1, \ldots, X_{n-1}$ are equally likely to be either 4 or 6, $X_n = 2$, $E\left(\sum_1^n X_i\right) = 4n$.

(d) From (c),

$$E\left[\sum_1^N X_i\right] = E\left[5N - 3\right] = 15 - 3 = 12$$

17. (i) Yes. (ii) No—Yes, if F exponential.

19. Since, from Example 2c, $m(t) = e^t - 1, 0 < t \le 1$, we obtain upon using the identity $t + E[Y(t)] = \mu[m(t) + 1]$ that $E[Y(1)] = e/2 - 1$.

21. $\dfrac{\mu_G}{\mu + 1/\lambda}$, where μ_G is the mean of G.

23. Using that $E[X] = 2p - 1$, we obtain from Wald's equation when $p \neq 1/2$ that

$$\begin{aligned} E[T](2p-1) &= E\left[\sum_{j=1}^T X_j\right] \\ &= (N-i)\frac{1-(q/p)^i}{1-(q/p)^N} - i\left[1 - \frac{1-(q/p)^i}{1-(q/p)^N}\right] \\ &= N\frac{1-(q/p)^i}{1-(q/p)^N} - i \end{aligned}$$

yielding the result:

$$E[T] = \frac{N\dfrac{1-(q/p)^i}{1-(q/p)^N} - i}{2p-1}, \quad p \neq 1/2$$

When $p = 1/2$, we can easily show by a conditioning argument that $E[T] = i(N-i)$

25. Say that a new cycle begins each time a train is dispatched. Then, with C being the cost of a cycle, we obtain, upon conditioning on $N(t)$, the number of arrivals during a cycle, that

$$\begin{aligned} E[C] &= E[E|C|N(t)]] = E[K + N(t)ct/2] \\ &= k + \lambda ct^2/2 \end{aligned}$$

Hence,

$$\text{average cost per unit time} = \frac{E[C]}{t} = \frac{K}{t} + \lambda ct/2$$

Calculus shows that the preceding is minimized when $t = \sqrt{2K/(\lambda c)}$, with the average cost equal to $\sqrt{2\lambda Kc}$.

On the other hand, the average cost for the N policy of Example 7.12 is $c(N-1)/2 + \lambda K/N$. Treating N as a continuous variable yields that its minimum occurs at $N = \sqrt{2\lambda K/c}$, with a resulting minimal average cost of $\sqrt{2\lambda Kc} - c/2$.

27. Say that a new cycle begins when a machine fails; let C be the cost per cycle; let T be the time of a cycle.

$$E[C] = K + \frac{c_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{c_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{c_1}{\lambda_1}$$

$$E[T] = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{1}{\lambda_1}$$

T the long-run average cost per unit time is $E[C]/E[T]$.

29. (a) Imagine that you are paid a reward equal to W_i on day i. Since everything starts over when a busy period ends, it follows that the reward process constitutes a renewal reward process with cycle time equal to N and with the reward during a cycle equal to $W_1 + \cdots + W_N$. Thus $E[W]$, the average reward per unit time, is $E[W_1 + \cdots + W_N]/E[N]$.

(b) The sum of the times in the system of all customers and the total amount of work that has been processed both start equal to 0 and both increase at the same rate. Hence, they are always equal.

(c) This follows from (b) by looking at the value of the two totals at the end of the first busy period.

(d) It is easy to see that N is a stopping time for the $L_i, i \ge 1$, and so, by Wald's Equation, $E\left[\sum_{i=1}^N L_i\right] = E[L]E[N]$. Thus, from (a) and (c), we obtain that $E[W] = E[L]$.

31. $P\{E(t) > x|A(t) = s\}$

$$= P\{0 \text{ renewals in } (t, t+x]|A(t) = s\}$$
$$= P\{\text{interarrival} > x + s|A(t) = s\}$$
$$= P\{\text{interarrival} > x + s|\text{interarrival} > s\}$$
$$= \frac{1 - F(x+s)}{1 - F(s)}$$

33. Let B be the amount of time the server is busy in a cycle; let X be the remaining service time of the person in service at the beginning of a cycle.

$$E[B] = E[B|X < t](1 - e^{-\lambda t}) + E[B|X > t]e^{-\lambda t}$$
$$= E[X|X < t](1 - e^{-\lambda t}) + \left(t + \frac{1}{\lambda + \mu}\right)e^{-\lambda t}$$
$$= E[X] - E[X|X > t]e^{-\lambda t} + \left(t + \frac{1}{\lambda + \mu}\right)e^{-\lambda t}$$
$$= \tfrac{1}{\mu} - \left(t + \frac{1}{\mu}\right)e^{-\lambda t} + \left(t + \frac{1}{\lambda + \mu}\right)e^{-\lambda t}$$
$$= \frac{1}{\mu}\left[1 - \frac{\lambda}{\lambda + \mu}e^{-\lambda t}\right]$$

More intuitively, writing $X = B + (X - B)$, and noting that $X - B$ is the additional amount of service time remaining when the cycle ends, gives

$$E[B] = E[X] - E[X - B]$$
$$= \frac{1}{\mu} - \frac{1}{\mu}P(X > B)$$
$$= \frac{1}{\mu} - \frac{1}{\mu}e^{-\lambda t}\frac{\lambda}{\lambda + \mu}$$

The long-run proportion of time that the server is busy is $\frac{E[B]}{t + 1/\lambda}$.

35. (a) We can view this as an $M/G/\infty$ system where a satellite launching corresponds to an arrival and F is the service distribution. Hence,

$$P\{X(t) = k\} = e^{-\lambda(t)}[\lambda(t)]^k/k!$$

where $\lambda(t) = \lambda\int_0^t (1 - F(s))ds$.

(b) By viewing the system as an alternating renewal process that is on when there is at least one satellite orbiting, we obtain

$$\lim P\{X(t) = 0\} = \frac{1/\lambda}{1/\lambda + E[T]}$$

where T, the on time in a cycle, is the quantity of interest. From part (a)

$$\lim P\{X(t) = 0\} = e^{-\lambda\mu}$$

where $\mu = \int_0^\infty (1 - F(s))ds$ is the mean time that a satellite orbits. Hence,

$$e^{-\lambda\mu} = \frac{1/\lambda}{1/\lambda + E[T]}$$

and so

$$E[T] = \frac{1 - e^{-\lambda\mu}}{\lambda e^{-\lambda\mu}}$$

37. (a) This is an alternating renewal process, with the mean off time obtained by conditioning on which machine fails to cause the off period.

$$E[\text{off}] = \sum_{i=1}^{3} E[\text{off}|i \text{ fails}]P\{i \text{ fails}\}$$
$$= (1/5)\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + (2)\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + (3/2)\frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$$

As the on time in a cycle is exponential with rate equal to $\lambda_1 + \lambda_2 + \lambda_3$, we obtain that p, the proportion of time that the system is working is

$$p = \frac{1/(\lambda_1 + \lambda_2 + \lambda_3)}{E[C]}$$

where

$$E[C] = E[\text{cycle time}]$$
$$= 1/(\lambda_1 + \lambda_2 + \lambda_3) + E[\text{off}]$$

(b) Think of the system as a renewal reward process by supposing that we earn 1 per unit time that machine 1 is being repaired. Then, r_1, the proportion of time that machine 1 is being repaired is

$$r_1 = \frac{(1/5)\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}}{E[C]}$$

(c) By assuming that we earn 1 per unit time when machine 2 is in a state of suspended animation, shows that, with s_2 being the proportion of time that 2 is in a state of suspended animation,

$$s_2 = \frac{(1/5)\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + (3/2)\frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}}{E[C]}$$

39. Let B be the length of a busy period. With S equal to the service time of the machine whose failure

initiated the busy period, and T equal to the remaining life of the other machine at that moment, we obtain

$$E[B] = \int E[B|S=s]g(s)ds$$

Now,

$$E[B|S=s] = E[B|S=s,\ T \le s](1-e^{-\lambda s})$$
$$+ E[B|S=s, T>s]e^{-\lambda s}$$
$$= (s+E[B])(1-e^{-\lambda s}) + se^{-\lambda s}$$
$$= s + E[B](1-e^{-\lambda s})$$

Substituting back gives

$$E[B] = E[S] + E[B]E[1-e^{-\lambda s}]$$

or

$$E[B] = \frac{E[S]}{E[e^{-\lambda s}]}$$

Hence,

$$E[\text{idle}] = \frac{1/(2\lambda)}{1/(2\lambda) + E[B]}$$

41. $$\int_0^1 \frac{(1-F(x)dx}{\mu}$$

$$= \begin{cases} \displaystyle\int_0^1 \frac{2-x}{2} \quad dx = \frac{3}{4} \text{ in part (i)} \\ \displaystyle\int_0^1 e^{-x} \quad dx = 1 - e^{-1} \text{ in part (ii)} \end{cases}$$

43. Since half the interarrival times will be exponential with mean 1 and half will be exponential with mean 2, it would seem that because the exponentials with mean 2 will last, on average, twice as long, that

$$\bar{F}_e(x) = \frac{2}{3}e^{-x/2} + \frac{1}{3}e^{-x}$$

With $\mu = (1)1/2 + (2)1/2 = 3/2$ equal to the mean interarrival time

$$\bar{F}_e(x) = \int_x^\infty \frac{\bar{F}(y)}{\mu} dy$$

and the earlier formula is seen to be valid.

45. The limiting probabilities for the Markov chain are given as the solution of

$$r_1 = r_2\frac{1}{2} + r_3$$
$$r_2 = r_1$$
$$r_1 + r_2 + r_3 = 1$$

or

$$r_1 = r_2 = \frac{2}{5}, \quad r_3 = \frac{1}{5}$$

(a) $r_1 = \dfrac{2}{5}$

(b) $P_i = \dfrac{r_i\mu_i}{\sum_i r_i\mu_i}$ and so,

$$P_1 = \frac{2}{9},\ P_2 = \frac{4}{9},\ P_3 = \frac{3}{9}.$$

47. (a) By conditioning on the next state, we obtain the following:

$$\mu_j = E[\text{time in } i]$$
$$= \sum E[\text{time in } i|\text{next state is } j]P_{ij}$$
$$= \sum_i t_{ij}P_{ij}$$

(b) Use the hint. Then,

$$E[\text{reward per cycle}]$$
$$= E[\text{reward per cycle}|\text{next state is } j]P_{ij}$$
$$= t_{ij}P_{ij}$$

Also,

$$E[\text{time of cycle}] = E[\text{time between visits to } i]$$

Now, if we had supposed a reward of 1 per unit time whenever the process was in state i and 0 otherwise then using the same cycle times as above we have that

$$P_i = \frac{E[\text{reward is cycle}]}{E[\text{time of cycle}]} = \frac{\mu_i}{E[\text{time of cycle}]}$$

Hence,

$$E[\text{time of cycle}] = \mu_i/P_i$$

and so

$$\text{average reward per unit time} = t_{ij}P_{ij}P_i/\mu_i$$

The above establishes the result since the average reward per unit time is equal to the proportion of time the process is in i and will next enter j.

49. Think of each interarrival time as consisting of n independent phases—each of which is exponentially distributed with rate λ—and consider the semi–Markov process whose state at any time is the phase of the present interarrival time. Hence, this semi-Markov process goes from state 1 to 2 to 3 … to n to 1, and so on. Also the time spent in each state has the same distribution. Thus, clearly the

limiting probabilities of this semi-Markov chain are $P_i = 1/n, i = 1, \ldots, n$. To compute $\lim P\{Y(t) < x\}$, we condition on the phase at time t and note that if it is $n-i+1$, which will be the case with probability $1/n$, then the time until a renewal occurs will be the sum of i exponential phases, which will thus have a gamma distribution with parameters i and λ.

51. It is an example of the inspection paradox. Because every tourist spends the same time in departing the country, those questioned at departure constitute a random sample of all visiting tourists. On the other hand, if the questioning is of randomly chosen hotel guests then, because longer staying guests are more likely to be selected, it follows that the average time of the ones selected will be larger than the average of all tourists. The data that the average of those selected from hotels was approximately twice as large as from those selected at departure are consistent with the possibility that the time spent in the country by a tourist is exponential with a mean approximately equal to 9.

55. $E[T(1)] = (.24)^{-2} + (.4)^{-1} = 19.8611$,
$E[T(2)] = 24.375$, $E[T_{12}] = 21.875$,
$E[T_{2,1}] = 17.3611$. The solution of the equations

$$19.861 = E[M] + 17.361P(2)$$
$$24.375 = E[M] + 21.875P(1)$$
$$1 = P(1) + P(2)$$

gives the results

$$P(2) \approx .4425, \; E[M] \approx 12.18$$

57. $$P\{\sum_{i=1}^{T} X_i > x\} = P\{\sum_{i=1}^{T} X_i > x | T = 0\}(1-\rho) + P\{\sum_{i=1}^{T} X_i > x | T > 0\}\rho$$

$$= P\{\sum_{i=1}^{T} X_i > x | T > 0\}\rho$$

$$= \rho \int_0^\infty P\{\sum_{i=1}^{T} X_i > x | T > 0, X_1 = y\} \frac{\bar{F}(y)}{\mu} dy$$

$$= \frac{\rho}{\mu} \int_0^x P\{\sum_{i=1}^{T} X_i > x | T > 0, X_1 = y\} \bar{F}(y) dy + \frac{\rho}{\mu} \int_x^\infty \bar{F}(y) dy$$

$$= \frac{\rho}{\mu} \int_0^x h(x-y)\bar{F}(y)dy + \frac{\rho}{\mu} \int_x^\infty \bar{F}(y)dy$$

$$= h(0) + \frac{\rho}{\mu} \int_0^x h(x-y)\bar{F}(y)dy - \frac{\rho}{\mu} \int_0^x \bar{F}(y)dy$$

where the final equality used that

$$h(0) = \rho = \frac{\rho}{\mu} \int_0^\infty \bar{F}(y)dy$$

Chapter 8

1. (a) E[number of arrivals]

$$= E[E\{\text{number of arrivals}|\text{service period is } S\}]$$
$$= E[\lambda S]$$
$$= \lambda/\mu$$

(b) $P\{0 \text{ arrivals}\}$

$$= E[P\{0 \text{ arrivals}|\text{service period is } S\}]$$
$$= E[P\{N(S) = 0\}]$$
$$= E[e^{-\lambda S}]$$
$$= \int_0^x e^{-\lambda s} \mu e^{-\mu s} ds$$
$$= \frac{\mu}{\lambda + \mu}$$

3. Let $C_M =$ Mary's average cost/hour and $C_A =$ Alice's average cost/hour.

Then, $C_M = \$3 + \$1\times$ (Average number of customers in queue when Mary works),

and $C_A = \$C + \$1 \times$ (Average number of customers in queue when Alice works).

The arrival stream has parameter $\lambda = 10$, and there are two service parameters—one for Mary and one for Alice:

$\mu_M = 20, \quad \mu_A = 30.$

Set $L_M =$ average number of customers in queue when Mary works and
$L_A =$ average number of customers in queue when Alice works.

Then using Equation (3.2), $L_M = \dfrac{10}{(20-10)} = 1$

$$L_A = \frac{10}{(20-10)} = \frac{1}{2}$$

So $C_M = \$3 + \$1/\text{customer} \times L_M$ customers
$= \$3 + \$1$
$= \$4/\text{hour}$

Also, $C_A = \$C + \$1/\text{customer} \times L_A$ customers
$$= \$C + \$1 \times \frac{1}{2}$$
$$= \$C + \frac{1}{2} \text{ / hour}$$

(b) We can restate the problem this way: If $C_A = C_M$, solve for C.

$$4 = C + \frac{1}{2} \Rightarrow C = \$3.50/\text{hour}$$

i.e., \$3.50/hour is the most the employer should be willing to pay Alice to work. At a higher wage his average cost is lower with Mary working.

5. Let I equal 0 if $W_Q^* = 0$ and let it equal 1 otherwise. Then,

$$E[W_Q^*|I = 0] = 0$$
$$E[W_Q^*|I = 1] = (\mu - \lambda)^{-1}$$
$$Var(W_Q^*|I = 0) = 0$$
$$Var(W_Q^*|I = 1) = (\mu - \lambda)^{-2}$$

Hence,

$$E[Var(W_Q^*|I] = (\mu - \lambda)^{-2}\lambda/\mu$$
$$Var(E[W_Q^*|I]) = (\mu - \lambda)^{-2}\lambda/\mu(1 - \lambda/\mu)$$

Consequently, by the conditional variance formula,

$$Var(W_Q^*) = \frac{\lambda}{\mu(\mu-\lambda)^2} + \frac{\lambda}{\mu^2(\mu - \lambda)}$$

7. To compute W for the $M/M/2$, set up balance equations as

$$\lambda p_0 = \mu p_1 \qquad \text{(each server has rate } \mu)$$
$$(\lambda + \mu)p_1 = \lambda p_0 + 2\mu p_2$$
$$(\lambda + 2\mu)p_n = \lambda p_{n-1} + 2\mu p_{n+1}, \qquad n \geq 2$$

These have solutions $P_n = \rho^n/2^{n-1}p_0$ where $\rho = \lambda/\mu$.

The boundary condition $\sum_{n=0}^{\infty} P_n = 1$ implies

$$P_0 = \frac{1-\rho/2}{1+\rho/2} = \frac{(2-\rho)}{(2+\rho)}$$

Now we have P_n, so we can compute L, and hence W from $L = \lambda W$:

$$\begin{aligned} L = \sum_{n=0}^{\infty} np_n &= \rho p_0 \sum_{n=0}^{\infty} n \left[\frac{\rho}{2}\right]^{n-1} \\ &= 2p_0 \sum_{n=0}^{\infty} n \left[\frac{\rho}{2}\right]^{n} \\ &= 2\frac{(2-\rho)}{(2+\rho)} \frac{(\rho/2)}{(1-\rho/2)^2} \\ &= \frac{4\rho}{(2+\rho)(2-\rho)} \\ &= \frac{4\mu\lambda}{(2\mu+\lambda)(2\mu-\lambda)} \end{aligned}$$

From $L = \lambda W$ we have

$$W = W_{m/m/2} = \frac{4\mu}{(2\mu+\lambda)(2\mu-\lambda)}$$

The $M/M/1$ queue with service rate 2μ has

$$Wm/m/1 = \frac{1}{2\mu - \lambda}$$

from Equation (3.3). We assume that in the $M/M/1$ queue, $2\mu > \lambda$ so that the queue is stable. But then $4\mu > 2\mu + \lambda$, or $\frac{4\mu}{2\mu+\lambda} > 1$, which implies $Wm/m/2 > Wm/m/1$.

The intuitive explanation is that if one finds the queue empty in the $M/M/2$ case, it would do no good to have two servers. One would be better off with one faster server.

Now let $W_Q^1 = W_Q(M/M/1)$

$W_Q^2 = W_Q(M/M/2)$

Then,

$$W_Q^1 = Wm/m/1 - 1/2\mu$$

$$W_Q^2 = Wm/m/2 - 1/\mu$$

So,

$$W_Q^1 = \frac{\lambda}{2\mu(2\mu-\lambda)} \qquad (3.3)$$

and

$$W_Q^2 = \frac{\lambda^2}{\mu(2\mu-\lambda)(2\mu+\lambda)}$$

Then,

$$W_Q^1 > W_Q^2 \Leftrightarrow \frac{1}{2} > \frac{\lambda}{(2\mu+\lambda)}$$
$$\lambda < 2\mu$$

Since we assume $\lambda < 2\mu$ for stability in the $M/M/1$, $W_Q^2 < W_Q^1$ whenever this comparison is possible, i.e., whenever $\lambda < 2\mu$.

9. Take the state to be the number of customers at server 1. The balance equations are

$$\mu P_0 = \mu P_1$$
$$2\mu P_j = \mu P_{j+1} + \mu P_{j-1}, \quad 1 \le j < n$$
$$\mu P_n = \mu P_{n-1}$$
$$1 = \sum_{j=0}^{n} P_j$$

It is easy to check that the solution to these equations is that all the P_js are equal, so $P_j = 1/(n+1)$, $j = 0, \ldots, n$.

11. (a) $\lambda P_0 = \alpha\mu P_1$

$$(\lambda + \alpha\mu)P_n = \lambda P_{n-1} + \alpha\mu P_{n+1}, \quad n \ge 1$$

These are exactly the same equations as in the $M/M/1$ with $\alpha\mu$ replacing μ. Hence,

$$P_n = \left[\frac{\lambda}{\alpha\mu}\right]^n \left[1 - \frac{\lambda}{\alpha\mu}\right], \quad n \ge 0$$

and we need the condition $\lambda < \alpha\mu$.

(b) If T is the waiting time until the customer first enters service, then conditioning on the number present when he arrives yields

$$\begin{aligned} E[T] &= \sum_n E[T|n \text{ present}]P_n \\ &= \sum_n \frac{n}{\mu} P_n \\ &= \frac{L}{\mu} \end{aligned}$$

Since $L = \sum nP_n$, and the P_n are the same as in the $M/M/1$ with λ and $\alpha\mu$, we have that $L = \lambda/(\alpha\mu - \lambda)$ and so

$$E[T] = \frac{\lambda}{\mu(\alpha\mu - \lambda)}$$

(c) $P\{\text{enters service exactly } n \text{ times}\}$

$$= (1-\alpha)^{n-1}\alpha$$

(d) This is expected number of services $\times$ mean services time $= 1/\alpha\mu$

(e) The distribution is easily seen to be memoryless. Hence, it is exponential with rate $\alpha\mu$.

13. Let the state be the idle server. The balance equations are

$$\text{Rate Leave} = \text{Rate Enter},$$

$$(\mu_2 + \mu_3)P_1 = \frac{\mu_1}{\mu_1 + \mu_2}P_3 + \frac{\mu_1}{\mu_1 + \mu_3}P_2,$$

$$(\mu_1 + \mu_3)P_2 = \frac{\mu_2}{\mu_2 + \mu_3}P_1 + \frac{\mu_2}{\mu_2 + \mu_1}P_3,$$

$$\mu_1 + \mu_2 + \mu_3 = 1.$$

These are to be solved and the quantity P_i represents the proportion of time that server i is idle.

15. There are four states $= 0, 1_A, 1_B, 2$. Balance equations are

$$2P_0 = 2P_{1_B}$$

$$4P_{1_A} = 2P_0 + 2P_2$$

$$4P_{1_B} = 4P_{1_A} + 4P_2$$

$$6P_2 = 2P_{1_B}$$

$$P_0 + P_{1_A} + P_{1_B} + P_2 = 1 \Rightarrow P_0 = \frac{3}{9}$$

$$P_{1_A} = \frac{2}{9}, P_{1_B} = \frac{3}{9}, P_2 = \frac{1}{9}$$

(a) $P_0 + P_{1_B} = \dfrac{2}{3}$

(b) By conditioning upon whether the state was 0 or 1_B when he entered we get that the desired probability is given by

$$\frac{1}{2} + \frac{1}{2}\frac{2}{6} = \frac{4}{6}$$

(c) $P_{1_A} + P_{1_B} + 2P_2 = \dfrac{7}{9}$

(d) Again, condition on the state when he enters to obtain

$$\frac{1}{2}\left[\frac{1}{4} + \frac{1}{2}\right] + \frac{1}{2}\left[\frac{1}{4} + \frac{2}{6}\frac{1}{2}\right] = \frac{7}{12}$$

This could also have been obtained from (a) and (c) by the formula $W = \dfrac{L}{\lambda a}$.

That is, $W = \dfrac{\frac{7}{9}}{2\left[\frac{2}{3}\right]} = \dfrac{7}{12}$.

17. The state space can be taken to consist of states $(0,0), (0,1), (1,0), (1,1)$, where the i^{th} component of the state refers to the number of customers at server i, $i = 1, 2$. The balance equations are

$$2P_{0,0} = 6P_{0,1}$$

$$8P_{0,1} = 4P_{1,0} + 4P_{1,1}$$

$$6P_{1,0} = 2P_{0,0} + 6P_{1,1}$$

$$10P_{1,1} = 2P_{0,1} + 2P_{1,0}$$

$$1 = P_{0,0} + P_{0,1} + P_{1,0} + P_{1,1}$$

Solving these equations gives $P_{0,0} = 1/2$, $P_{0,1} = 1/6, P_{1,0} = 1/4, P_{1,1} = 1/12$.

(a) $P_{1,1} = 1/12$

(b) $W = \dfrac{L}{\lambda_a} = \dfrac{P_{0,1} + P_{1,0} + 2P_{1,1}}{2(1 - P_{1,1})} = \dfrac{7}{22}$

(c) $\dfrac{P_{0,0} + P_{0,1}}{1 - P_{1,1}} = \dfrac{8}{11}$

19. (a) Say that the state is $(n, 1)$ whenever it is a good period and there are n in the system, and say that it is $(n, 2)$ whenever it is a bad period and there are n in the system, $n = 0, 1$.

(b) $(\lambda_1 + \alpha_1)P_{0,1} = \mu P_{1,1} + \alpha_2 P_{0,2}$

$$(\lambda_2 + \alpha_2)P_{0,2} = \mu P_{1,2} + \alpha_1 P_{0,1}$$

$$(\mu + \alpha_1)P_{1,1} = \lambda_1 P_{0,1} + \alpha_2 P_{1,2}$$

$$(\mu + \alpha_2)P_{1,2} = \lambda_2 P_{0,2} + \alpha_1 P_{1,1}$$

$$P_{0,1} + P_{0,2} + P_{1,1} + P_{1,2} = 1$$

(c) $P_{0,1} + P_{0,2}$

(d) $\lambda_1 P_{0,1} + \lambda_2 P_{0,2}$

21. (a) $\lambda_1 P_{10}$

(b) $\lambda_2(P_0 + P_{10})$

(c) $\lambda_1 P_{10}/[\lambda_1 P_{10} + \lambda_2(P_0 + P_{10})]$

(d) This is equal to the fraction of server 2's customers that are type 1 multiplied by the proportion of time server 2 is busy. (This is true since the amount of time server 2 spends with a customer does not depend on which type of customer it is.) By (c) the answer is thus

$$(P_{01} + P_{11})\lambda_1 P_{10}/[\lambda_1 P_{10} + \lambda_2(P_0 + P_{10})]$$

23. (a) The states are $n, n \geq 0$, and b. State n means there are n in the system and state b means that a breakdown is in progress.

(b) $\beta P_b = a(1 - P_0)$

$\lambda P_0 = \mu P_1 + \beta P_b$

$(\lambda + \mu + a)P_n = \lambda P_{n-1} + \mu P_{n+1}, \quad n \geq 1$

(c) $W = L/\lambda_n = \sum_{n=1}^{\infty} nP_a/[\lambda(1 - P_b)]$

(d) Since rate at which services are completed = $\mu(1 - P_0 - P_b)$ it follows that the proportion of customers that complete service is

$\mu(1 - P_0 - P_b)/\lambda_a$
$= \mu(1 - P_0 - P_b)/[\lambda(1 - P_b)]$

An equivalent answer is obtained by conditioning on the state as seen by an arrival. This gives the solution

$$\sum_{n=0}^{\infty} P_n[\mu/(\mu + a)]^{n+1}$$

where the above uses that the probability that $n + 1$ services of present customers occur before a breakdown is $[\mu/(\mu + a)]^{n+1}$.

(e) P_b

25. (a)

$$\lambda P_0 = \mu_A P_A + \mu_B P_B$$

$$(\lambda + \mu_A)P_A = a\lambda P_0 + \mu_B P_2$$

$$(\lambda + \mu_B)P_B = (1 - a)\lambda P_0 + \mu_A P_2$$

$$(\lambda + \mu_A + \mu_B)P_n = \lambda P_{n-1} + (\mu_A + \mu_B)P_{n+1},$$

$n \geq 2$ where $P_1 = P_A + P_B$.

(b) $L = P_A + P_B + \sum_{n=2}^{\infty} nP_n$

Average number of idle servers = $2P_0 + P_A + P_B$.

(c) $P_0 + P_B + \dfrac{\mu_A}{\mu_A + \mu_B} \sum_{n=2}^{\infty} P_n$

27. (a) The special customer's arrival rate is act θ because we must take into account his service time. In fact, the mean time between his arrivals will be $1/\theta + 1/\mu_1$. Hence, the arrival rate is $(1/\theta + 1/\mu_1)^{-1}$.

(b) Clearly we need to keep track of whether the special customer is in service. For $n \geq 1$, set

$P_n = Pr\{n$ customers in system regular customer in service$\}$,

$P_n^S = Pr\{n$ customers in system, special customer in service$\}$, and

$P_0 = Pr\{0$ customers in system$\}$.

$$(\lambda + \theta)P_0 = \mu P_1 + \mu_1 P_1^S$$

$$(\lambda + \theta + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1} + \mu_1 P_{n+1}^S$$

$$(\lambda + \mu)P_n^S = \theta P_{n-1} + \lambda P_{n-1}^S,$$

$$n \geq 1 \; [P_0^S = P_0]$$

(c) Since service is memoryless, once a customer resumes service it is as if his service has started anew. Once he begins a particular service, he will complete it if and only if the next arrival of the special customer is after his service. The probability of this is Pr {Service < Arrival of special customer} = $\mu/(\mu + \theta)$, since service and special arrivals are independent exponential random variables. So,

$Pr\{$bumped exactly n times$\}$

$= (1 - \mu/(\mu + \theta))^n(\mu/(\mu + \theta))$

$= (\theta/(\mu + \theta))^n(\mu/(\mu + \theta))$

In essence, the number of times a customer is bumped in service is a geometric random variable with parameter $\mu/(\mu + \theta)$.

29. (a) Let state 0 mean that the server is free; let state 1 mean that a type 1 customer is having a wash; let state 2 mean that the server is cutting hair; and let state 3 mean that a type 3 is getting a wash.

(b) $\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$

$\mu_1 P_1 = \lambda p_1 P_0$

$\mu_2 P_2 = \lambda p_2 P_0 + \mu_1 P_3$

$\mu_1 P_3 = \lambda p_3 P_0$

$P_0 + P_1 + P_2 + P_3 = 1$

(c) P_2

(d) λP_0

Direct substitution now verifies the equation.

31. The total arrival rates satisfy

$$\lambda_1 = 5$$

$$\lambda_2 = 10 + \frac{1}{3}5 + \frac{1}{2}\lambda_3$$

$$\lambda_3 = 15 + \frac{1}{3}5 + \lambda_2$$

Solving yields that $\lambda_1 = 5$, $\lambda_2 = 40$, $\lambda_3 = 170/3$. Hence,

$$L = \sum_{i=1}^{3} \frac{\lambda_i}{\mu_i - \lambda_i} = \frac{82}{13}$$

$$W = \frac{L}{r_1 + r_2 + r_3} = \frac{41}{195}$$

33. (a) Use the Gibbs sampler to simulate a Markov chain whose stationary distribution is that of the queuing network system with $m-1$ customers. Use this simulated chain to estimate $P_{i,\,m-1}$, the steady state probability that there are i customers at server j for this system. Since, by the arrival theorem, the distribution function of the time spent at server j in the m customer system is $\sum_{i=0}^{m-1} P_{i,\,m-1} G_{i+1}(x)$, where $G_k(x)$ is the probability that a gamma (k, μ) random variable is less than or equal to x, this enables us to estimate the distribution function.

(b) This quantity is equal to the average number of customers at server j divided by m.

35. Let S and U denote, respectively, the service time and value of a customer. Then U is uniform on (0, 1) and

$$E[S|U] = 3 + 4U, \quad Var(S|U) = 5$$

Hence,

$$E[S] = E\{E[S|U]\} = 3 + 4E[U] = 5$$

$$\begin{aligned} Var(S) &= E[Var(S|U)] + Var(E[S|U]) \\ &= 5 + 16Var(U) = 19/3 \end{aligned}$$

Therefore,

$$E[S^2] = 19/3 + 25 = 94/3$$

(a) $W = W_Q + E[S] = \dfrac{94\lambda/3}{1 - \delta\lambda} + 5$

(b) $W_Q + E[S|U = x] = \dfrac{94\lambda/3}{1 - \delta\lambda} + 3 + 4x$

37. (a) The proportion of departures leaving behind 0 work

= proportion of departures leaving an empty system

= proportion of arrivals finding an empty system

= proportion of time the system is empty (by Poisson arrivals)

$= P_0$

(b) The average amount of work as seen by a departure is equal to the average number it sees multiplied by the mean service time (since no customers seen by a departure have yet started service). Hence,

Average work as seen by a departure

= average number it sees $\times\ E[S]$

= average number an arrival sees $\times\ E[S]$

$= LE[S]$ by Poisson arrivals

$$= \lambda(W_Q + E[S])E[S]$$

$$= \frac{\lambda^2 E[S]E[S^2]}{\lambda - \lambda E[S]} + \lambda(E[S])^2$$

39. (a) $a_0 = P_0$ due to Poisson arrivals. Assuming that each customer pays 1 per unit time while in service the cost identity (2.1) states that

Average number in service $= \lambda E[S]$

or

$$1 - P_0 = \lambda E[S]$$

(b) Since a_0 is the proportion of arrivals that have service distribution G_1 and $1 - a_0$ the proportion having service distribution G_2, the result follows.

(c) We have

$$P_0 = \frac{E[I]}{E[I] + E[B]}$$

and $E[I] = 1/\lambda$ and thus,

$$\begin{aligned} E[B] &= \frac{1 - P_0}{\lambda P_0} \\ &= \frac{E[S]}{1 - \lambda E[S]} \end{aligned}$$

Now from (a) and (b) we have

$$E[S] = (1 - \lambda E[S])E[S_1] + \lambda E[S]E[S_2]$$

or

$$E[S] = \frac{E[S_1]}{1 + \lambda E[S_1] + \lambda E[S_2]}$$

Substitution into $E[B] = E[S]/(1 - \lambda E[S])$ now yields the result.

41. $E[N] = 2, E[N^2] = 9/2, E[S^2] = 2E^2[S] = 1/200$

$$W = \frac{\frac{1}{20}\frac{5}{2}/4 + 4 \cdot 2/400}{1 - 8/20} = \frac{41}{480}$$

$$W_Q = \frac{41}{480} - \frac{1}{20} = \frac{17}{480}$$

43. Problem 42 shows that if $\mu_1 > \mu_2$, then serving 1's first minimizes average wait. But the same argument works if $c_1\mu_1 > c_2\mu_2$, i.e.,

$$\frac{E(S_1)}{c_1} < \frac{E(S_2)}{\mu_1}$$

45. By regarding any breakdowns that occur during a service as being part of that service, we see that this is an $M/G/1$ model. We need to calculate the first two moments of a service time. Now the time of a service is the time T until something happens (either a service completion or a breakdown) plus any additional time A. Thus,

$$E[S] = E[T + A]$$
$$= E[T] + E[A]$$

To compute $E[A]$ we condition upon whether the happening is a service or a breakdown. This gives

$$E[A] = E[A|\text{service}]\frac{\mu}{\mu + \alpha} + E[A|\text{breakdown}]\frac{\alpha}{\mu + \alpha}$$
$$= E[A|\text{breakdown}]\frac{\alpha}{\mu + \alpha}$$
$$= (1/\beta + E[S])\frac{\alpha}{\mu + \alpha}$$

Since, $E[T] = 1/(\alpha + \mu)$ we obtain

$$E[S] = \frac{1}{\alpha + \mu} + (1/\beta + E[S])\frac{\alpha}{\mu + \alpha}$$

or

$$E[S] = 1/\mu + \alpha/(\mu\beta)$$

We also need $E[S^2]$, which is obtained as follows.

$$E[S^2] = E[(T + A)^2]$$
$$= E[T^2] + 2E[AT] + E[A^2]$$
$$= E[T^2] + 2E[A]E[T] + E[A^2]$$

The independence of A and T follows because the time of the first happening is independent of whether the happening was a service or a breakdown. Now,

$$E[A^2] = E[A^2|\text{breakdown}]\frac{\alpha}{\mu + \alpha}$$
$$= \frac{\alpha}{\mu + \alpha}E[(\text{down time} + S^{\alpha})^2]$$
$$= \frac{\alpha}{\mu + \alpha}\left\{E[\text{down}^2] + 2E[\text{down}]E[S] + E[S^2]\right\}$$
$$= \frac{\alpha}{\mu + \alpha}\left\{\frac{2}{\beta^2} + \frac{2}{\beta}\left[\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right] + E[S^2]\right\}$$

Hence,

$$E[S^2] = \frac{2}{(\mu + \beta)^2} + 2\left[\frac{\alpha}{\beta(\mu + \alpha)} + \frac{\alpha}{\mu + \alpha}\left(\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right)\right] + \frac{\alpha}{\mu + \alpha}\left\{\frac{2}{\beta^2} + \frac{2}{\beta}\left[\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right] + E[S^2]\right\}$$

Now solve for $E[S^2]$. The desired answer is

$$W_Q = \frac{\lambda E[S^2]}{2(1 - \lambda E[S])}$$

In the above, S^{α} is the additional service needed after the breakdown is over. S^{α} has the same distribution as S. The above also uses the fact that the expected square of an exponential is twice the square of its mean.

Another way of calculating the moments of S is to use the representation

$$S = \sum_{i=1}^{N}(T_i + B_i) + T_{N+1}$$

where N is the number of breakdowns while a customer is in service, T_i is the time starting when service commences for the i^{th} time until a happening occurs, and B_i is the length of the i^{th} breakdown. We now use the fact that, given N, all of the random variables in the representation are independent exponentials with the T_i having rate $\mu + \alpha$ and the B_i having rate β. This yields

$$E[S|N] = (N + 1)/(\mu + \alpha) + N/\beta$$
$$Var(S|N) = (N + 1)/(\mu + \alpha)^2 + N/\beta^2$$

Therefore, since $1 + N$ is geometric with mean $(\mu + \alpha)/\mu$ (and variance $\alpha(\alpha + \mu)/\mu^2$) we obtain

$$E[S] = 1/\mu + \alpha/(\mu\beta)$$

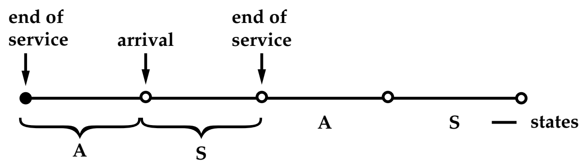
and, using the conditional variance formula,

$$Var(S) = [1/(\mu + \alpha) + 1/\beta]^2\alpha(\alpha + \mu)/\mu^2 + 1/[\mu(\mu + \alpha)] + \alpha/\mu\beta^2)$$

47. For $k = 1$, Equation (8.1) gives

$$P_0 = \frac{1}{1 + \lambda E(S)} = \frac{(\lambda)}{(\lambda) + E(S)} \qquad P_1 = \frac{\lambda(ES)}{1 + \lambda E(S)}$$
$$= \frac{E(S)}{\lambda + E(S)}$$

One can think of the process as an *alteracting renewal process*. Since arrivals are Poisson, the time until the next arrival is still exponential with parameter λ.



The basic result of alternating renewal processes is that the limiting probabilities are given by

$$P\{\text{being in "state } S\text{"}\} = \frac{E(S)}{E(A) + E(S)} \text{ and}$$

$$P\{\text{being in "state } A\text{"}\} = \frac{E(A)}{E(A) + E(S)}$$

These are exactly the Erlang probabilities given above since $E[A] = 1/\lambda$. Note this uses Poisson arrivals in an essential way, viz., to know the distribution of time until the next arrival after a service is still exponential with parameter λ.

49. $$P_3 = \frac{\frac{(\lambda E[S])^3}{3!}}{\sum_{j=0}^{3} \frac{(\lambda E[S])^j}{j!}}, \quad \lambda = 2, E[S] = 1$$
$$= \frac{8}{38}$$

51. Note that when all servers are busy, the departures are exponential with rate $k\mu$. Now see Problem 26.

53. $1/\mu_F < k/\mu_G$, where μ_F and μ_G are the respective means of F and G.

Chapter 9

1. If $x_i = 0$, $\phi(x) = \phi(0_i, x)$.
   If $x_i = 1$, $\phi(x) = \phi(1_i, x)$.

3. (a) If ϕ is series, then $\phi(x) = \min_i x_i$ and so $\phi^D(\underline{x}) = 1 - \min_i (1 - x_i) = \max x_i$, and vice versa.
   (b) $\phi^{D,D}(x) = 1 - \phi^D(1 - x)$

   $$= 1 - [1 - \phi(1 - (1 - x))]$$

   $$= \phi(x)$$

   (c) An $n - k + 1$ of n.
   (d) Say $\{1, 2, \ldots, r\}$ is a minimal path set. Then $\phi(\underbrace{1, 1, \ldots,}_{r} 1, 0, 0, \ldots 0) = 1$, and so

   $\phi^D(\underbrace{0,\ 0, \ldots,}_{r} 0, 1, 1, \ldots, 1) = 1 - \phi(1, 1, \ldots, 1, 0, 0, \ldots, 0) = 0$, implying that $\{1, 2, \ldots, r\}$ is a cut set. We can easily show it to be minimal. For instance,

   $$\phi^D(\underbrace{0, 0, \ldots,}_{r-1} 0, 1, 1, \ldots, 1)$$

   $$= 1 - \phi(\underbrace{1, 1, \ldots,}_{r-1} 1, 0, 0, \ldots, 0) = 1,$$

   since $\phi(\underbrace{1, 1, \ldots,}_{r-1} 1, 0, 0, \ldots, 0) = 0$ since $\{1, 2, \ldots, r - 1\}$ is not a path set.

5. (a) Minimal path sets are

   $\{1, 8\}$, $\{1, 7, 9\}$, $\{1, 3, 4, 7, 8\}$, $\{1, 3, 4, 9\}$,
   $\{1, 3, 5, 6, 9\}$, $\{1, 3, 5, 6, 7, 8\}$, $\{2, 5, 6, 9\}$,
   $\{2, 5, 6, 7, 8\}$, $\{2, 4, 9\}$, $\{2, 4, 7, 8\}$,
   $\{2, 3, 7, 9\}$, $\{2, 3, 8\}$.

   Minimal cut sets are

   $\{1, 2\}$, $\{2, 3, 7, 8\}$, $\{1, 3, 4, 5\}$, $\{1, 3, 4, 6\}$,
   $\{1, 3, 7, 9\}$, $\{4, 5, 7, 8\}$, $\{4, 6, 7, 8\}$, $\{8, 9\}$.

7. $\{1, 4, 5\}$, $\{3\}$, $\{2, 5\}$.

9. (a) A component is irrelevant if its functioning or not functioning can never make a difference as to whether or not the system functions.
   (b) Use the representation (2.1.1).
   (c) Use the representation (2.1.2).

11. $r(p) = P\{\text{either } x_1x_3 = 1 \text{ or } x_2x_4 = 1\}$

    $P\{\text{either of 5 or 6 work}\}$

    $$= (p_1p_3 + p_2p_4 - p_1p_3p_2p_4)$$

    $$(p_5 + p_6 - p_5p_5)$$

13. Taking expectations of the identity

    $$\phi(X) = X_i\phi(1_i, X) + (1 - X_i)\phi(0_i, X)$$

    noting the independence of X_i and $\phi(1_i, X)$ and of $\phi(0_i, X)$.

15. (a) $\frac{7}{32} \le \text{r}\left[\frac{1}{2}\right] \le 1 - \left[\frac{7}{8}\right]^3 = \frac{169}{512}$
    The exact value is $r(1/2) = 7/32$, which agrees with the minimal cut lower bound since the minimal cut sets $\{1\}$, $\{5\}$, $\{2, 3, 4\}$ do not overlap.

17. $E[N^2] = E[N^2|N > 0]P\{N > 0\}$

    $$\ge (E[N|N > 0])^2 P\{N > 0\}$$

    since $E[X^2] \ge (E[X])^2$.

    Thus,

    $$E[N^2]P\{N > 0\} \ge (E[N|N > 0]P\{N > 0\})^2$$

    $$= (E[N])^2$$

    Let N denote the number of minimal path sets having all of its components functioning. Then $r(p) = P\{N > 0\}$.

    Similarly, if we define N as the number of minimal cut sets having all of its components failed, then $1 - r(p) = P\{N > 0\}$.

In both cases we can compute expressions for $E[N]$ and $E[N^2]$ by writing N as the sum of indicator (i.e., Bernoulli) random variables. Then we can use the inequality to derive bounds on $r(p)$.

19. $X_{(i)}$ is the system life of an $n-i+1$ of n system each having the life distribution F. Hence, the result follows from Example 5e.

21. (a) (i), (ii), (iv) − (iv) because it is two-of-three.
 (b) (i) because it is series, (ii) because it can be thought of as being a series arrangement of 1 and the parallel system of 2 and 3, which as $F_2 = F_3$ is IFR.
 (c) (i) because it is series.

23. (a) $\bar{F}(t) = \prod_{i=1}^{n} F_i(t)$

$$\lambda_F(t) = \frac{\frac{d}{dt}\bar{F}(t)}{\bar{F}(t)} = \frac{\sum_{j=1}^{n} F_j'(t) \prod_{i\neq j} F_j(t)}{\prod_{i=1}^{n} F_i(t)}$$

$$= \frac{\sum_{j=1}^{n} F_j'(t)}{F_j(t)}$$

$$= \sum_{j=1}^{n} \lambda_j(t)$$

(b) $F_t(a) = P\{\text{additional life of } t\text{-year-old} > a\}$

$$= \frac{\prod_{1}^{n} F_i(t+a)}{F_i(t)}$$

where F_i is the life distribution for component i. The point being that as the system is series, it follows that knowing that it is alive at time t is equivalent to knowing that all components are alive at t.

25. For $x \geq \xi$,

$$1-p = 1-F(\xi) = 1-F(x(\xi/x)) \geq [1-F(x)]^{\xi/x}$$

since IFRA.

Hence,

$$1-F(x) \leq (1-p)^{x/\xi} = e^{-\theta x}$$

For $x \leq \xi$,

$$1-F(x) = 1-F(\xi(x/\xi)) \geq [1-F(\xi)]^{x/\xi}$$

since IFRA.

Hence,

$$1-F(x) \geq (1-p)^{x/\xi} = e^{-\theta x}$$

27. If $p > p_0$, then $p = {p_0}^{\alpha}$ for some $a \in (0,1)$. Hence,

$$r(p) = r({p_0}^{\alpha}) \geq [r(p_0)]^{\alpha} = {p_0}^{\alpha} = p$$

If $p < p_0$, then $p_0 = p^{\alpha}$ for some $a \in (0,1)$. Hence,

$$p^{\alpha} = p_0 = r(p_0) = r(p^{\alpha}) \geq [r(p)]^{\alpha}$$

29. Let X denote the time until the first failure and let Y denote the time between the first and second failure. Hence, the desired result is

$$EX + EY = \frac{1}{\mu_1 + \mu_2} + EY$$

Now,

$$E[Y] = E[Y|\mu_1 \text{ component fails first}]\frac{\mu_1}{\mu_1+\mu_2} + E[Y|\mu_2 \text{ component fails first}]\frac{\mu_2}{\mu_1+\mu_2}$$

$$= \frac{1}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_1}\frac{\mu_2}{\mu_1+\mu_2}$$

31. Use the remark following Equation (6.3).

33. The exact value can be obtained by conditioning on the ordering of the random variables. Let M denote the maximum, then with $A_{i,j,k}$ being the even that $X_i < X_j < X_k$, we have that

$$E[M] = \sum E[M|A_{i,j,k}]P(A_{i,j,k})$$

where the preceding sum is over all 6 possible permutations of $1, 2, 3$. This can now be evaluated by using

$$P(A_{i,j,k}) = \frac{\lambda_i}{\lambda_i + \lambda_j + \lambda_k}\frac{\lambda_j}{\lambda_j + \lambda_k}$$

$$E[M|A_{i,j,k}] = \frac{1}{\lambda_i + \lambda_j + \lambda_k} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k}$$

35. (a) It follows when $i = 1$ since $0 = (1-1)^n = 1 - \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} \cdots \pm \begin{bmatrix} n \\ n \end{bmatrix}$. So assume it true for i and consider $i+1$. We must show that

$$\begin{bmatrix} n-1 \\ i \end{bmatrix} = \begin{bmatrix} n \\ i+1 \end{bmatrix} - \begin{bmatrix} n \\ i+2 \end{bmatrix} + \cdots \pm \begin{bmatrix} n \\ n \end{bmatrix}$$

which, using the induction hypothesis, is equivalent to

$$\begin{bmatrix} n-1 \\ i \end{bmatrix} = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}$$

which is easily seen to be true.

(b) It is clearly true when $i = n$, so assume it for i. We must show that

$$\begin{bmatrix} n-1 \\ i-2 \end{bmatrix} = \begin{bmatrix} n \\ i-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + \cdots \pm \begin{bmatrix} n \\ n \end{bmatrix}$$

which, using the induction hypothesis, reduces to

$$\begin{bmatrix} n-1 \\ i-2 \end{bmatrix} = \begin{bmatrix} n \\ i-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}$$

which is true.

Chapter 10

1. $X(s) + X(t) = 2X(s) + X(t) - X(s)$.

   Now $2X(s)$ is normal with mean 0 and variance $4s$ and $X(t) - X(s)$ is normal with mean 0 and variance $t - s$. As $X(s)$ and $X(t) - X(s)$ are independent, it follows that $X(s) + X(t)$ is normal with mean 0 and variance $4s + t - s = 3s + t$.

3. $E[X(t_1)X(t_2)X(t_3)]$

$$\begin{aligned}
&= E[E[X(t_1)X(t_2)X(t_3) \mid X(t_1), X(t_2)]]\\
&= E[X(t_1)X(t_2)E[X(t_3) \mid X(t_1), X(t_2)]]\\
&= E[X(t_1)X(t_2)X(t_2)]\\
&= E[E[X(t_1)E[X^2(t_2) \mid X(t_1)]]\\
&= E[X(t_1)E[X^2(t_2) \mid X(t_1)]] \qquad (*)\\
&= E[X(t_1)\{(t_2 - t_1) + X^2(t_1)\}]\\
&= E[X^3(t_1)] + (t_2 - t_1)E[X(t_1)]\\
&= 0
\end{aligned}$$

   where the equality $(*)$ follows since given $X(t_1)$, $X(t_2)$ is normal with mean $X(t_1)$ and variance $t_2 - t_1$. Also, $E[X^3(t)] = 0$ since $X(t)$ is normal with mean 0.

5. $P\{T_1 < T_{-1} < T_2\} = P\{\text{hit 1 before} - 1 \text{ before } 2\}$

$$\begin{aligned}
&= P\{\text{hit 1 before } -1\}\\
&\quad \times P\{\text{hit } -1 \text{ before 2} \mid \text{hit 1 before } -1\}\\
&= \frac{1}{2}P\{\text{down 2 before up 1}\}\\
&= \frac{1}{2}\frac{1}{3} = \frac{1}{6}
\end{aligned}$$

   The next to last equality follows by looking at the Brownian motion when it first hits 1.

7. Let $M = \{\max_{t_1 \le s \le t_2} X(s) > x\}$. Condition on $X(t_1)$ to obtain

$$P(M) = \int_{-\infty}^{\infty} P(M|X(t_1) = y)\frac{1}{\sqrt{2\pi t_1}}\, e^{-y^2/2t_1} dy$$

   Now, use that

$$P(M|X(t_1) = y) = 1, \quad y \ge x$$

   and, for $y < x$

$$\begin{aligned}
P\big(M|X(t_1) = y\big) &= P\{\max_{0<s<t_2-t_1} X(s) > x - y\}\\
&= 2P\{X(t_2 - t_1) > x - y\}
\end{aligned}$$

11. Let $X(t)$ denote the value of the process at time $t = nh$. Let $X_i = 1$ if the i^{th} change results in the state value becoming larger, and let $X_i = 0$ otherwise. Then, with $u = e^{\sigma\sqrt{h}}$, $d = e^{-\sigma\sqrt{h}}$

$$\begin{aligned}
X(t) &= X(0)u^{\sum_{i=1}^n X_i} d^{n - \sum_{i=1}^n X_i}\\
&= X(0)d^n \left(\frac{u}{d}\right)^{\sum_{i=1}^n X_i}
\end{aligned}$$

   Therefore,

$$\begin{aligned}
\log\left(\frac{X(t)}{X(0)}\right) &= n\log(d) + \sum_{i=1}^{n} X_i \log(u/d)\\
&= -\frac{t}{h}\sigma\sqrt{h} + 2\sigma\sqrt{h}\sum_{i=1}^{t/h} X_i
\end{aligned}$$

   By the central limit theorem, the preceding becomes a normal random variable as $h \to 0$. Moreover, because the X_i are independent, it is easy to see that the process has independent increments. Also,

$$\begin{aligned}
&E\left[\log\left(\frac{X(t)}{X(0)}\right)\right]\\
&= -\frac{t}{h}\,\sigma\sqrt{h} + 2\sigma\sqrt{h}\,\frac{t}{h}\,\frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{h})\\
&= \mu t
\end{aligned}$$

   and

$$\begin{aligned}
Var\left[\log\left(\frac{X(t)}{X(0)}\right)\right] &= 4\sigma^2 h\frac{t}{h}p(1-p)\\
&\to \sigma^2 t
\end{aligned}$$

   where the preceding used that $p \to 1/2$ as $h \to 0$.

13. If the outcome is i then our total winnings are

$$x_i o_i - \sum_{j \neq i} x_j = \frac{o_i(1+o_i)^{-1} - \sum_{j \neq i}(1+o_j)^{-1}}{1 - \sum_k (1+o_k)^{-1}}$$

$$= \frac{(1+o_i)(1+o_i)^{-1} - \sum_j (1+o_j)^{-1}}{1 - \sum_k (1+o_k)^{-1}}$$

$$= 1$$

15. The parameters of this problem are

$\sigma = .05, \quad \sigma = 1, \quad x_o = 100, \quad t = 10.$

(a) If $K = 100$ then from Equation (4.4)

$$b = [.5 - 5 - \log(100/100)]/\sqrt{10}$$

$$= -4.5\sqrt{10} = -1.423$$

and

$$c = 100\phi(\sqrt{10} - 1.423) - 100e^{-.5}\phi(-1.423)$$

$$= 100\phi(1.739) - 100e^{-.5}[1 - \phi(1.423)]$$

$$= 91.2$$

The other parts follow similarly.

17. $E\,[B(t)|B(u),\ 0 \le u \le s]$

$$= E[B(s) + B(t) - B(s)|B(u),\ 0 \le u \le s]$$

$$= E[B(s)|B(u),\ 0 \le u \le s]$$

$$+ \ E[B(t) - B(s)|B(u), 0 \le u \le s]$$

$= B(s) + E[B(t) - B(s)]$ by independent increments

$$= B(s)$$

19. Since knowing the value of $Y(t)$ is equivalent to knowing $B(t)$ we have

$$E[Y(t)|Y(u),\ 0 \le u \le s]$$

$$= e^{-c^2t/2}E[e^{cB(t)}|B(u),\ 0 \le u \le s]$$

$$= e^{-c^2t/2}E[e^{cB(t)}|B(s)]$$

Now, given $B(s)$, the conditional distribution of $B(t)$ is normal with mean $B(s)$ and variance $t - s$. Using the formula for the moment generating function of a normal random variable we see that

$$e^{-c^2t/2}E[e^{cB(t)}|B(s)]$$

$$= e^{-c^2t/2}e^{cB(s)+(t-s)c^2/2}$$

$$= e^{-c^2s/2}e^{cB(s)}$$

$$= Y(s)$$

Thus, $\{Y(t)\}$ is a Martingale.

$$E[Y(t)] = E[Y(0)] = 1$$

21. By the Martingale stopping theorem

$$E[B(T)] = E[B(0)] = 0$$

But, $B(T) = (x - \mu T)/\sigma$ and so

$$E[(x - \mu T)/\sigma] = 0$$

or

$$E[T] = x/\mu$$

23. By the Martingale stopping theorem we have

$$E[B(T)] = E[B(0)] = 0$$

Since $B(T) = [X(T) - \mu T]/\sigma$ this gives the equality

$$E[X(T) - \mu T] = 0$$

or

$$E[X(T)] = \mu E[T]$$

Now

$$E[X(T)] = pA - (1-p)B$$

where, from part (c) of Problem 22,

$$p = \frac{1 - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}$$

Hence,

$$E[T] = \frac{A(1 - e^{2\mu B/\sigma^2}) - B(e^{-2\mu A/\sigma^2} - 1)}{\mu(e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2})}$$

25. The means equal 0.

$$Var\left[\int_0^1 t dX(t)\right] = \int_0^1 t^2 dt = \frac{1}{3}$$

$$Var\left[\int_0^1 t^2 dX(t)\right] = \int_0^1 t^4 dt = \frac{1}{5}$$

27. $E[X(a^2t)/a] = \frac{1}{a}E[X(a^2t)] = 0$

For $s < t$,

$$Cov(Y(s), Y(t)) = \frac{1}{a^2}\,Cov(X(a^2s), X(a^2t))$$
$$= \frac{1}{a^2}\,a^2 s = s$$

As $\{Y(t)\}$ is clearly Gaussian, the result follows.

29. $\{Y(t)\}$ is Gaussian with

$E[Y(t)] = (t+1)E(Z[t/(t+1)]) = 0$

and for $s \le t$

$Cov(Y(s), Y(t))$

$$= (s+1)(t+1)\,Cov\left[Z\left[\frac{s}{s+1}\right],\quad Z\left[\frac{t}{t+1}\right]\right]$$

$$= (s+1)(t+1)\frac{s}{s+1}\left[1 - \frac{t}{t+1}\right] \qquad (*)$$

$$= s$$

where $(*)$ follows since $Cov(Z(s), Z(t)) = s(1-t)$. Hence, $\{Y(t)\}$ is Brownian motion since it is also Gaussian and has the same mean and covariance function (which uniquely determines the distribution of a Gaussian process).

31. (a) Starting at any time t the continuation of the Poisson process remains a Poisson process with rate λ.

(b) $E[Y(t)Y(t+s)]$

$$= \int_0^\infty E[Y(t)Y(t+s) \mid Y(t) = y]\lambda e^{-\lambda y}dy$$
$$= \int_0^\infty yE[Y(t+s) \mid Y(t) = y]\lambda e^{-\lambda y}dy$$
$$+ \int_s^\infty y(y-s)\lambda e^{-\lambda y}dy$$
$$= \int_0^s y\frac{1}{\lambda}\lambda e^{-\lambda y}dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y}dy$$

where the above used that

$E[Y(t)Y(t+s)|Y(t) = y]$

$$= \begin{cases} yE(Y(t+s)) = \frac{y}{\lambda}, & \text{if } y < s \\ y(y-s), & \text{if } y > s \end{cases}$$

Hence,
$Cov(Y(t), Y(t+s))$

$$= \int_0^s ye^{-y\lambda}dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y}dy - \frac{1}{\lambda^2}$$

33. $Cov(X(t), X(t+s))$

$$= Cov(Y_1 \cos wt + Y_2 \sin wt,$$
$$Y_1 \cos w(t+s) + Y_2 \sin w(t+s))$$
$$= \cos wt \cos w(t+s) + \sin wt \sin w(t+s)$$
$$= \cos(w(t+s) - wt)$$
$$= \cos ws$$

Chapter 11

1. (a) Let u be a random number. If $\sum_{j=1}^{i-1} P_j < u \le \sum_{j=1}^{i} P_j$ then simulate from F_i.

$\left(\text{In the above } \sum_{j=1}^{i-1} P_j \equiv 0 \text{ when } i = 1.\right)$

(b) Note that

$$F(x) = \frac{1}{3}F_1(X) + \frac{2}{3}F_2(x)$$

where

$$F_1(x) = 1 - e^{-2x}, \quad 0 < x < \infty$$

$$F_2(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x \end{cases}$$

Hence, using (a), let U_1, U_2, U_3 be random numbers and set

$$X = \begin{cases} \dfrac{-\log\, U_2}{2}, & \text{if } U_1 < 1/3 \\ U_3, & \text{if } U_1 > 1/3 \end{cases}$$

The above uses the fact that $\dfrac{-\log\, U_2}{2}$ is exponential with rate 2.

3. If a random sample of size n is chosen from a set of $N + M$ items of which N are acceptable then X, the number of acceptable items in the sample, is such that

$$P\{X = k\} = \begin{bmatrix} N \\ k \end{bmatrix} \begin{bmatrix} M \\ n-k \end{bmatrix} \Big/ \begin{bmatrix} N+M \\ k \end{bmatrix}$$

To simulate X note that if

$$I_j = \begin{cases} 1, & \text{if the } j^{th} \text{ section is acceptable} \\ 0, & \text{otherwise} \end{cases}$$

then

$P\{I_j = 1 | I_1, \ldots, I_{j-1}\} = \dfrac{N - \sum_1^{j-1} I_i}{N + M - (j-1)}$. Hence, we can simulate $I_1, \ldots, I_n$ by generating random numbers $U_1, \ldots, U_n$ and then setting

$$I_j = \begin{cases} 1, & \text{if } U_j < \dfrac{N - \sum_1^{j-1} I_i}{N + M - (j-1)} \\ 0, & \text{otherwise} \end{cases}$$

$X = \sum_{j=1}^{n} I_j$ has the desired distribution.

Another way is to let

$$X_j = \begin{cases} 1, & \text{the } j^{th} \text{ acceptable item is in the sample} \\ 0, & \text{otherwise} \end{cases}$$

and then simulate $X_1, \ldots, X_N$ by generating random numbers $U_1, \ldots, U_N$ and then setting

$$X_j = \begin{cases} 1, & \text{if } U_j < \dfrac{N - \sum_{i=1}^{j-1} I_i}{N + M - (j-1)} \\ 0, & \text{otherwise} \end{cases}$$

$X = \sum_{j=1}^{N} X_j$ then has the desired distribution.

The former method is preferable when $n \le N$ and the latter when $N \le n$.

7. Use the rejection method with $g(x) = 1$. Differentiating $f(x)/g(x)$ and equating to 0 gives the two roots 1/2 and 1. As $f(.5) = 30/16 > f(1) = 0$, we see that $c = 30/16$, and so the algorithm is

Step 1: Generate random numbers U_1 and U_2.

Step 2: If $U_2 \le 16(U_1^2 - 2U_1^3 + U_1^4)$, set $X = U_1$. Otherwise return to step 1.

13. $P\{X = i\} = P\{Y = i | U \leq P_Y/CQ_Y\}$

$$= \frac{P\{Y = i, U \leq P_Y/CQ_Y\}}{K}$$

$$= \frac{Q_i P\{U \leq P_Y/CQ_Y | Y = i\}}{K}$$

$$= \frac{Q_i P_i/CQ_i}{K}$$

$$= \frac{P_i}{CK}$$

where $K = P\{U \leq P_Y/CQ_Y\}$. Since the above is a probability mass function it follows that $KC = 1$.

15. Use $2\mu = X$.

17. (a) Generate the $X_{(i)}$ sequentially using that given $X_{(1)}, \ldots, X_{(i-1)}$ the conditional distribution of $X_{(i)}$ will have failure rate function $\lambda_i(t)$ given by

$$\lambda_i(t) = \begin{cases} 0, & t < X_{(i-1)}, \\ & X_{(0)} \equiv 0. \\ (n-i+1)\lambda(t), & t > X_{(i-1)} \end{cases}$$

(b) This follows since as F is an increasing function the density of $U_{(i)}$ is

$$f_{(i)}(t) = \frac{n!}{(i-1)!(n-i)}(F(t))^{i-1}$$

$$\times (F(t))^{n-i} f(t)$$

$$= \frac{n!}{(i-1)!(n-i)} t^{i-1}(1-t)^{n-i},$$

$$0 < t < 1$$

which shows that $U_{(i)}$ is beta.

(c) Interpret Y_i as the i^{th} interarrival time of a Poisson process. Now given $Y_1 + \cdots + Y_{n+1} = t$, the time of the $(n+1)^{st}$ event, it follows that the first n event times are distributed as the ordered values of n uniform $(0, t)$ random variables. Hence,

$$\frac{Y_1 + \cdots + Y_i}{Y_1 + \cdots + Y_{n+i}}, \quad i = 1, \ldots, n$$

will have the same distribution as $U_{(1)}, \ldots, U_{(n)}$.

(d) $f_{U_{(1)},\ldots|U_{(n)}}(y_1, \ldots, y_{n-1}|y_n)$

$$= \frac{f(y_1, \ldots, y_n)}{f_{U_{(n)}}(y_n)}$$

$$= \frac{n!}{ny^{n-1}}$$

$$= \frac{(n-1)!}{y^{n-1}}, 0 < y_1 < \cdots < y_{n-1} < y$$

where the above used that

$$F_{U_{(n)}}(y) = P\{\max U_i \leq y\} = y^n$$

and so

$$F_{U_{(n)}}(y) = ny^{n-1}$$

(e) Follows from (d) and the fact that if $F(y) = y^n$ then $F^{-1}(U) = U^{1/n}$.

21. $P_{m+1}\{i_1, \ldots, i_{k-1}, m+1\}$

$$= \sum_{\substack{j \leq m \\ j \neq i_1, \ldots, i_{k-1}}} P_m\{i_1, \ldots, i_{k-1}, j\} \frac{k}{m+1} \frac{1}{k}$$

$$= (m-(k-1)) \frac{1}{\begin{bmatrix} m \\ k \end{bmatrix}} \frac{1}{m+1} \frac{1}{\begin{bmatrix} m+1 \\ k \end{bmatrix}}$$

25. See Problem 4.

27. First suppose $n = 2$.

$$Var(\lambda X_1 + (1-\lambda)X_2) = \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2.$$

The derivative of the above is $2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2$ and equating to 0 yields

$$\lambda = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} = \frac{1/\sigma_1^2}{1/\sigma_1^2 + 1/\sigma_2^2}$$

Now suppose the result is true for $n-1$. Then

$$Var\left[\sum_{i=1}^{n} \lambda_i X_i\right] = Var\left[\sum_{i=1}^{n-1} \lambda_i X_i\right] + Var(\lambda_n X_n)$$

$$= (1-\lambda_n)^2\, Var\left[\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} X_i\right]$$

$$+ \lambda_n^2\, Var\, X_n$$

Now by the inductive hypothesis for fixed λ_n the above is minimized when

$$(*)\ \frac{\lambda_i}{1-\lambda_n} = \frac{1/\sigma_i^2}{\sum_{j=1}^{n-1} 1/\sigma_j^2}, \quad i = 1, \ldots, n-1$$

Hence, we now need choose λ_n so as to minimize

$$(1-\lambda_n)^2 \frac{1}{\sum_{j=1}^{n-1} 1/\sigma_j^2} + \lambda_n^2\, \sigma_n^2$$

Calculus yields that this occurs when

$$\lambda_n = \frac{1}{1+\sigma_n^2 \sum_{j=1}^{n-1} 1/\sigma_j^2} = \frac{1/\sigma_n^2}{\sum_{j=1}^{n} 1/\sigma_j^2}$$

Substitution into (*) now gives the result.

29. Use Hint.

31. Since $E[W_n|D_n] = D_n + \mu$, it follows that to estimate $E[W_n]$ we should use $D_n + \mu$. Since $E[D_n|W_n] \neq W_n - \mu$, the reverse is not true and so we should use the simulated data to determine D_n and then use this as an estimate of $E[D_n]$.

33. (a) $E[X^2] \leq E[aX] = aE[X]$

    (b) $Var(X) = E[X^2] - E^2[X] \leq aE[X] - E^2[X]$

    (c) From (b) we have that

$$Var(X) \leq a^2 \left(\frac{E[X]}{a}\right)$$

$$\left(1 - \frac{E[X]}{a}\right) \leq a^2 \max_{0<p<1} p(1-p) = a^2/4$$

35. Use the estimator $\sum_{i=1}^{k} N_i/k^2$ where N_i = number of $j = 1, \ldots, k : X_i < Y_j$.