Engineering Mathematics-1

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Preface

I am deeply gratified by the enthusiastic response shown to all the earlier editions of my book on Engineering Mathematics-1 by the students and teachers throughout the country.

This book has been revised to meet the requirements of the first-year undergraduate course on Engineering Mathematics offered to the students of engineering. The contents have been covered in adequate depth for the semester 1 Syllabi of various universities/ deemed universities across the country.

The book offers a balanced coverage of both theory and problems. Lucid writing style supported by step-by-step solutions to all problems enhances understanding of the concepts. It has an excellent pedagogy with 1212 unsolved problems, 539 solved problems, 669 short answer questions and 914 descriptive questions. For the benefit of the students, the solutions to 2012–13 of Anna University is given in appendix. Additionally, the book is accompanied by a website which provides useful supplements including recent solved question papers.

I hope that the book will be received by both the faculty and the students as enthusiastically as the earlier editions of the book and my other books. Critical evaluation and suggestions for the improvement of the book will be highly appreciated and acknowledged.

T VEERARAJAN

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PART I

Mathematics-I

1. Matrices
2. Sequences and Series
3. Application of Differential Calculus
4. Differential Calculus of Several Variables
5. Multiple Integrals

Chapter 1

Matrices

1.1 INTRODUCTION

In the lower classes, the students have studied a few topics in Elementary Matrix theory. They are assumed to be familiar with the basic definitions and concepts of matrix theory as well as the elementary operations on and properties of matrices. Though the concept of rank of a matrix has been introduced in the lower classes, we briefly recall the definition of rank and working procedure to find the rank of a matrix, as it will be of frequent use in testing the consistency of a system of linear algebraic equations, that will be discussed in the next section.

1.1.1 Rank of a Matrix

Determinant of any square submatrix of a given matrix A is called a minor of A. If the square submatrix is of order r, then the minor also is said to be of order r.

Let A be an $m \times n$ matrix. The rank of A is said to be 'r', if

(i) there is at least one minor of A of order r which does not vanish and
(ii) every minor of A of order $(r + 1)$ and higher order vanishes.

In other words, the rank of a matrix is the largest of the orders of all the non-vanishing minors of that matrix. Rank of a matrix A is denoted by $R(A)$ or $\rho(A)$.

To find the rank of a matrix A, we may use the following procedure:

We first consider the highest order minor (or minors) of A. Let their order be r. If any one of them does not vanish, then $\rho(A) = r$. If all of them vanish, we next consider minors of A of next lower order $(r - 1)$ and so on, until we get a non-zero minor. The order of that non-zero minor is $\rho(A)$.

This method involves a lot of computational work and hence requires more time, as we have to evaluate many determinants. An alternative method to find the rank of a matrix A is given below:

Reduce A to any one of the following forms, (called normal forms) by a series of elementary operations on A and then find the order of the unit matrix contained in the normal form of A:

$$[I_r];\ [I_r \mid O];\ \left[\frac{I_r}{O}\right];\ \left[\frac{I_r \mid O}{O \mid O}\right]$$

Here I_r denotes the unit matrix of order r and O is zero matrix.

By an *elementary operation* on a matrix (denoted as E-operation) we mean any one of the following operations or transformations:

(i) Interchange of any two rows (or columns).
(ii) Multiplication of every element of a row (or column) by any non-zero scalar.
(iii) Addition to the elements of any row (or column), the same scalar multiples of corresponding elements of any other row (or column).

Note ☑ The alternative method for finding the rank of a matrix is based on the property that the rank of a matrix is unaltered by elementary operations.

Finally we observe that we need not necessarily reduce a matrix A to the normal form to find its rank. It is enough we reduce A to an equivalent matrix, whose rank can be easily found, by a sequence of elementary operations on A. The methods are illustrated in the worked examples that follow.

1.2 VECTORS

A set of n numbers $x_1, x_2, \ldots, x_n$ written in a particular order (or an ordered set of n numbers) is called an *n-dimensional vector* or a *vector of order n*. The n numbers are called the components or elements of the vector. A vector is denoted by a single letter X or Y etc. The components of a vector may be written in a row as $X = (x_1, x_2, \ldots, x_n)$

or in a column as $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. These are called respectively row vector and column vector. We note that a row vector of order n is a $1 \times n$ matrix and a column vector of order n is an $n \times 1$ matrix.

1.2.1 Addition of Vectors

The sum of two vectors of the same dimension is obtained by adding the corresponding components.

i.e., if $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$,
then $X + Y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

1.2.2 Scalar Multiplication of a Vector

If k is a scalar and $X = (x_1, x_2, \ldots, x_n)$ is a vector, then the scalar multiple kX is defined as

$$kX = (kx_1, kx_2, \ldots, kx_n).$$

1.2.3 Linear Combination of Vectors

If a vector X can be expressed as $X = k_1X_1 + k_2X_2 + \cdots + k_rX_r$ then X is said to be a linear combination of the vectors $X_1, X_2, \ldots, X_r$.

1.3 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE OF VECTORS

The vectors $X_1, X_2, \ldots, X_r$ are said to be *linearly dependent* if we can find scalars $k_1, k_2, \ldots k_r$, which are not all zero, such that $k_1X_l + k_2X_2 + \cdots + k_rX_r = 0$.

A set of vectors is said to be *linearly independent* if it is not linearly dependent, i.e. the vectors $X_1, X_2, \ldots, X_r$ are linearly independent, if the relation $k_1X_1 + k_2X_2 + \ldots k_rX_r = 0$ is satisfied only when $k_l = k_2 = \cdots = k_r = 0$.

Note ☑ When the vectors $X_1, X_2, \ldots, X_r$ are linearly dependent, then $k_1X_1 + k_2X_2 + \cdots + k_rX_r = 0$, where at least one of the k's is not zero. Let $k_m \neq 0$.

Thus
$$X_m = -\frac{k_1}{k_m} \cdot X_1 - \frac{k_2}{k_m} X_2 - \cdots - \frac{k_r}{k_m} X_r .$$

Thus at least one of the given vectors can be expressed as a linear combination of the others.

1.4 METHODS OF TESTING LINEAR DEPENDENCE OR INDEPENDENCE OF A SET OF VECTORS

Method 1 Using the definition directly.

Method 2 We write the given vectors as row vectors and form a matrix. Using elementary row operations on this matrix, we reduce it to echelon form, i.e. the one in which all the elements in the r^{th} column below the r^{th} element are zero each. If the number of non-zero row vectors in the echelon form equals the number of given vectors, then the vectors are linearly independent. Otherwise they are linearly dependent.

Method 3 If there are n vectors, each of dimension n, then the matrix formed as in method (2) will be a square matrix of order n. If the rank of the matrix equals n, then the vectors are linearly independent. Otherwise they are linearly dependent.

1.5 CONSISTENCY OF A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

Consider the following system of m linear algebraic equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This system can be represented in the matrix form as $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \hdashline a_{m1} & a_{m2} \cdots & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The matrix A is called *the coefficient matrix* of the system, X is the matrix of unknowns and B is the matrix of the constants.

If $B \equiv O$, a zero matrix, the system is called a system of *homogeneous linear equations;* otherwise, the system is called a system of *linear non-homogeneous equations.*

The $m \times (n + 1)$ matrix, obtained by appending the column vector B to the coefficient matrix A as the additional last column, is called the *augmented matrix* of the system and is denoted by $[A, B]$ or $[A \mid B]$.

i.e.
$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \hdashline a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

1.5.1 Definitions

A set of values of $x_1, x_2 \ldots, x_n$. which satisfy all the given m equations simultaneously is called a *solution* of the system.

When the system of equations has a solution, it is said to be *consistent.* Otherwise the system is said to be *inconsistent.*

A consistent system may have either only one or infinitely many solutions. When the system has only one solution, it is called *the unique solution.*

The necessary and sufficient condition for the consistency of a system of linear non-homogeneous equations is provided by a theorem, called Rouches's theorem, which we state below without proof.

1.5.2 Rouche's Theorem

The system of equations $AX = B$ is consistent, if and only if the coefficient matrix A and the augmented matrix $[A, B]$ are of the same rank.

Thus to discuss the consistency of the equations $AX = B$ (m equations in n unknowns), the following procedure is adopted:

We first find $R(A)$ and $R(A, B)$.

(i) If $R(A) \neq R(A, B)$, the equations are inconsistent

(ii) If $R(A) = R(A, B) =$ the number of unknowns n, the equations are consistent and have a unique solution.

In particular, if A is a non-singular (square) matrix, the system $AX = B$ has a unique solution.

(iii) If $R(A) = R(A, B) <$ the number of unknowns n, the equations are consistent and have an infinite number of solutions.

1.5.3 System of Homogeneous Linear Equations

Consider the system of homogeneous linear equations $AX = O$ (m equations in n unknowns)

i.e
$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \hdashline a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = 0 \end{array}$$

This system is always consistent, as $R(A) = R(A, O)$. If the coefficient matrix A is non-singular, the system has a unique solution, namely, $x_1 = x_2 = \cdots = x_n = 0$. This unique solution is called *the trivial solution,* which is not of any importance.

If the coefficient matrix A is singular, i.e. if $|A| = 0$, the system has an infinite number of non-zero or non-trivial solutions.

The method of finding the non-zero solution of a system of homogeneous linear equations is illustrated in the worked examples that follow.

WORKED EXAMPLE 1(a)

Example 1.1 Show that the vectors $X_1 = (1, 1, 2)$, $X_2 = (1, 2, 5)$ and $X_3 = (5, 3, 4)$ are linearly dependent. Also express each vector as a linear combination of the other two.

Method 1

Let $$k_1X_1 + k_2X_2 + k_3X_3 = 0$$

i.e. $$k_1(1, 1, 2) + k_2(1, 2, 5) + k_3(5, 3, 4) = (0, 0, 0)$$

$\therefore$
$$k_1 + k_2 + 5k_3 = 0 \quad (1)$$
$$k_1 + 2k_2 + 3k_3 = 0 \quad (2)$$
$$2k_1 + 5k_2 + 4k_3 = 0 \quad (3)$$

(2) – (1) gives $k_2 - 2k_3 = 0$ or $k_2 = 2k_3$ (4)

Using (4) in (3), $k_1 = -7k_3$ (5)

Taking $k_3 = 1$, we get $k_1 = -7$ and $k_2 = 2$.

Thus $$-7X_1 + 2X_2 + X_3 = 0 \quad (6)$$

$\therefore$ The vectors X_1, X_2, X_3 are linearly dependent.

From (6), we get $$X_1 = \frac{2}{7}X_2 + \frac{1}{7}X_3,$$

$$X_2 = \frac{7}{2}X_1 - \frac{1}{2}X_3 \quad \text{and} \quad X_3 = 7X_1 - 2X_2$$

Method 2

Writing X_1, X_2, X_3 as row vectors, we get

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \left(R_2' = R_2 - R_1,\ R_3' = R_3 - 5R_1\right)$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \left(R_3'' = R_3' + 2R_2'\right)$$

In the echelon form of the matrix, the number of non-zero vectors = 2 (< the number of given vectors).

$\therefore$ X_1, X_2, X_3 are linearly dependent.

Now
$$\begin{aligned} 0 = R_3'' &= R_3' + 2R_2' \\ &= (R_3 - 5R_1) + 2(R_2 - R_1) \\ &= -7R_1 + 2R_2 + R_3 \end{aligned}$$

i.e. $-7X_1 + 2X_2 + X_3 = 0$

As before, $X_1 = \frac{2}{7}X_2 + \frac{1}{7}X_3,\ X_2 = \frac{7}{2}X_1 - \frac{1}{2}X_3$ and $X_3 = 7X_1 - 2X_2$.

Method 3

$$|A| = 0 \quad \therefore R(A) \neq 3; \quad R(A) = 2$$

$\therefore$ The vectors X_1, X_2, X_3 are linearly dependent.

Example 1.2 Show that the vectors $X_1 = (1, -1, -2, -4)$, $X_2 = (2, 3, -1, -1)$, $X_3 = (3, 1, 3, -2)$ and $X_4 = (6, 3, 0, -7)$ are linearly dependent. Find also the relationship among them.

$$A = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \left(R_2' = R_2 - 2R_1, R_3' = R_3 - 3R_1, R_4' = R_4 - 6R_1\right)$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \left(R_2'' = \frac{1}{5}R_2';\ R_3'' = R_3';\ R_4'' = R_4'\right)$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \end{bmatrix} = \left(R_3''' = R_3'' - 4R_2'';\ R_4''' = R_4'' - 9R_2''\right)$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(R_4'''' = R_4''' - R_3'''\right)$$

Number of non-zero vectors in echelon form of the matrix $A = 3$.

$\therefore$ The vectors X_1, X_2, X_3, X_4 are linearly dependent.

Now
$$\begin{aligned} 0 = R_4'''' = R_4''' - R_3''' \\ &= \left(R_4'' - 9R_2''\right) - \left(R_3'' - 4R_2''\right) \\ &= R_4' - R_3' - R_2' \\ &= \left(R_4 - 6R_1\right) - \left(R_3 - 3R_1\right) - \left(R_2 - 2R_1\right) \\ &= -R_1 - R_2 - R_3 + R_4 \end{aligned}$$

$\therefore$ The relation among X_1, X_2, X_3, X_4 is

$$-X_1 - X_2 - X_3 + X_4 = 0 \quad \text{or} \quad X_1 + X_2 + X_3 - X_4 = 0.$$

Example 1.3 Show that the vectors $X_1 = (2, -2, 1)$, $X_2 = (1, 4, -1)$ and $X_3 = (4, 6, -3)$ are linearly independent.

Method 1

Let $\quad k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$

i.e. $k_1 (2, -2, 1) + k_2 (1, 4, -1) + k_3 (4, 6, -3) = (0, 0, 0)$

$\therefore$
$$2k_1 + k_2 + 4k_3 = 0 \tag{1}$$
$$-2k_1 + 4k_2 + 6k_3 = 0 \tag{2}$$
$$k_1 - k_2 - 3k_3 = 0 \tag{3}$$

From (1) and (2), $\quad k_2 + 2k_3 = 0 \quad (4)$

From (2) and (3), $\quad k_2 = 0 \quad (5)$

$\therefore \quad k_1 = 0 = k_2 = k_3.$

$\therefore$ The vectors X_1, X_2, X_3 are linearly independent.

Method 2

$$A = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 4 & -1 \\ 4 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 \\ 2 & -2 & 1 \\ 4 & 6 & -3 \end{bmatrix} \left(R_1' = R_2\,;\, R_2' = R_1\right)$$

$$\sim \begin{bmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & -10 & 1 \end{bmatrix} \left(R_2'' = R_2' - 2R_1'\,;\, R_3'' = R_3' - 4R_1'\right)$$

$$\sim \begin{bmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & 0 & -2 \end{bmatrix} \left(R_3''' = R_3'' - R_2''\right)$$

Number of non-zero vectors in the echelon form of A = number of given vectors,
$\therefore X_1, X_2, X_3$ are linearly independent.

Example 1.4 Show that the vectors $X_1 = (1, -1, -1, 3)$, $X_2 = (2, 1, -2, -1)$ and $X_3 = (7, 2, -7, 4)$ are linearly independent.

$$A = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 3 \\ 2 & 1 & -2 & -1 \\ 7 & 2 & -7 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 3 & 0 & -7 \\ 0 & 9 & 0 & -17 \end{bmatrix} \left(\begin{aligned} R_2' &= R_2 - 2R_1\,; \\ R_3' &= R_3 - 7R_1 \end{aligned}\right)$$

$$\sim\begin{bmatrix}1 & -1 & -1 & 3\\0 & 3 & 0 & -7\\0 & 0 & 0 & 4\end{bmatrix}\left(R_3'' = R_3' - 3R_2'\right)$$

Number of non-zero vectors in the echelon form of A = number of given vectors.
$\therefore X_1, X_2, X_3$ are linearly independent.

Example 1.5 Test for the consistency of the following system of equations:

$$\begin{aligned}x_1 + 2x_2 + 3x_3 + 4x_4 &= 5\\6x_1 + 7x_2 + 8x_3 + 9x_4 &= 10\\11x_1 + 12x_2 + 13x_3 + 14x_4 &= 15\\16x_1 + 17x_2 + 18x_3 + 19x_4 &= 20\\21x_1 + 22x_2 + 23x_3 + 24x_4 &= 25\end{aligned}$$

The given equations can be put as

$$\begin{bmatrix}1 & 2 & 3 & 4\\6 & 7 & 8 & 9\\11 & 12 & 13 & 14\\16 & 17 & 18 & 19\\21 & 22 & 23 & 24\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}5\\10\\15\\20\\25\end{bmatrix}$$

i.e. $AX = B$ (say)

Let us find the rank of the augmented matrix $[A, B]$ by reducing it to the normal form

$$[A, B]=\begin{bmatrix}1 & 2 & 3 & 4 & 5\\6 & 7 & 8 & 9 & 10\\11 & 12 & 13 & 14 & 15\\16 & 17 & 18 & 19 & 20\\21 & 22 & 23 & 24 & 25\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 3 & 4 & 5\\5 & 5 & 5 & 5 & 5\\10 & 10 & 10 & 10 & 10\\15 & 15 & 15 & 15 & 15\\20 & 20 & 20 & 20 & 20\end{bmatrix}\begin{matrix}(R_2 \to R_2 - R_1\\R_3 \to R_3 - R_1\\R_4 \to R_4 - R_1\\R_5 \to R_5 - R_1)\end{matrix}$$

Note ☑ If two matrices A and B are equivalent, i.e. are of the same rank, it is denoted as $A \sim B$.

$$\sim\begin{bmatrix}1 & 2 & 3 & 4 & 5\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\end{bmatrix}\left(R_2 \to \frac{1}{5}R_2,\ R_3 \to \frac{1}{10}R_3,\ R_4 \to \frac{1}{15}R_4,\ R_5 \to \frac{1}{20}R_5\right)$$

$$\sim\begin{bmatrix}1 & 2 & 3 & 4 & 5\\0 & -1 & -2 & -3 & -4\\0 & -1 & -2 & -3 & -4\\0 & -1 & -2 & -3 & -4\\0 & -1 & -2 & -3 & -4\end{bmatrix}\left(R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_1,\ R_5 \to R_5 - R_1\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -2 & -3 & -4 \end{bmatrix} \begin{array}{l} (C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_1, \\ \ C_4 \to C_4 - 4C_1, C_5 \to C_5 - 5C_1) \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} \left[C_2 \to -C_2, C_3 \to C_3 \div (-2), \right. \\ \left. C_4 \to C_4 \div (-3), C_5 \to C_5 \div (-4)\right] \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \left(R_3 \to R_3 - R_2, R_4 \to R_4 - R_2, \right. \\ \left. \ R_5 \to R_5 - R_2\right) \end{array}$$

$$\sim \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \left(C_3 \to C_3 - C_2, C_4 \to C_4 - C_2, \right. \\ \left. \ C_5 \to C_5 - C_2\right) \end{array}$$

Now $[A, B]$ has been reduced to the normal form $\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$

The order of the unit matrix present in the normal form = 2.
Hence the rank of $[A, B] = 2$.
The rank of the coefficient matrix A can be found as 2, in a similar manner.
Thus $R(A) = R[A, B] = 2$
$\therefore$ The given system of equations is consistent and possesses many solutions.

Example 1.6 Test for the consistency of the following system of equations:

$$x_1 - 2x_2 - 3x_3 = 2;\ 3x_1 - 2x_2 = -1;\ -2x_2 - 3x_3 = 2;\ x_2 + 2x_3 = 1.$$

The system can be put as

$$\begin{bmatrix} 1 & -2 & -3 \\ 3 & -2 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

i.e. $AX = B$ (say)

$$[A, B] = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 3 & -2 & 0 & -1 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} (R_2 \to R_2 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 9 & -7 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} (C_2 \to C_2 + 2C_1, C_3 \to C_3 + 3C_1, C_4 \to C_4 - 2C_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \end{bmatrix} (R_2 \to R_4,\ R_4 \to R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & -11 \end{bmatrix} (R_3 \to R_3 + 2R_2, R_4 \to R_4 - 4R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & -11 \end{bmatrix} (C_3 \to C_3 - 2C_2, C_4 \to C_4 - C_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -15 \end{bmatrix} (R_4 \to R_4 - R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -15 \end{bmatrix} (C_4 \to C_4 - 4C_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(R_4 \to -\frac{1}{15} R_4\right)$$

$\therefore R[A, B] = 4$

But $R(A) \neq 4$, as A is a (4×3) matrix.

In fact $R(A) = 3$, as the value of the minor

$$\begin{vmatrix} 3 & -2 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} \neq 0$$

Thus $R(A) \neq R[A, B]$

$\therefore$ The given system is inconsistent.

Example 1.7 Test for the consistency of the following system of equations and solve them, if consistent, by matrix inversion.

$$x - y + z + 1 = 0;\ x - 3y + 4z + 6 = 0;\ 4x + 3y - 2z + 3 = 0;$$
$$7x - 4y + 7z + 16 = 0.$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 4 \\ 4 & 3 & -2 \\ 7 & -4 & 7 \end{bmatrix}$$

$$[A, B] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -3 & 4 & -6 \\ 4 & 3 & -2 & -3 \\ 7 & -4 & 7 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & -2 & 3 & -5 \\ 0 & 7 & -6 & 1 \\ 0 & 3 & 0 & -9 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - 4R_1, \\ R_4 \to R_4 - 7R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & -5 \\ 0 & 7 & -6 & 1 \\ 0 & 3 & 0 & -9 \end{bmatrix} (C_2 \to C_2 + C_1, C_3 \to C_3 - C_1, C_4 \to C_4 + C_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & 7 \\ 0 & -5 & 3 & -2 \\ 0 & -9 & 0 & 3 \end{bmatrix} (R_2 \leftrightarrow R_3 \text{ and then } C_2 \leftrightarrow C_4)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -27 & 33 \\ 0 & 0 & -54 & 66 \end{bmatrix} \begin{matrix} (R_3 \to R_3 + 5R_2, R_4 \to R_4 + 9R_2) \\ \text{and then} \\ (C_3 \to C_3 + 6C_2, C_4 \to C_4 - 7C_2) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \left(C_3 \to -\frac{1}{27} C_3, C_4 \to \frac{1}{33} C_4 \right)$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (R_4 \to R_4 - 2R_3 \text{ and then} \\ \quad C_4 \to C_4 - C_3) \end{array}$$

$\therefore$ $R\,[A, B] = 3$. Also $R\,(A) = 3$

$\therefore$ The given system is consistent and has a unique solution.

To solve the system, we take any three, say the first three, of the given equations.

i.e.
$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 4 \\ 4 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ -3 \end{bmatrix}$$

i.e.
$$AX = B, \text{ say}$$

$\therefore$
$$X = A^{-1} B \tag{1}$$

Let
$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 4 \\ 4 & 3 & -2 \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now A_{11} = co-factor of a_{11} in $|A| = -6$

$A_{12} = 18$; $A_{13} = 15$; $A_{21} = 1$; $A_{22} = -6$; $A_{23} = -7$;

$A_{31} = -1$; $A_{32} = -3$; $A_{33} = -2$.

$\therefore$
$$\text{Adj}\,(A) = \begin{bmatrix} -6 & 1 & -1 \\ 18 & -6 & -3 \\ 15 & -7 & -2 \end{bmatrix}$$

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} = -9$$

$\therefore$
$$A^{-1} = \frac{1}{|A|} \text{adj } A = -\frac{1}{9} \begin{bmatrix} -6 & 1 & -1 \\ 18 & -6 & -3 \\ 15 & -7 & -2 \end{bmatrix} \tag{2}$$

Using (2) in (1),

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} -6 & 1 & -1 \\ 18 & -6 & -3 \\ 15 & -7 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ -6 \\ -3 \end{bmatrix}$$

$$= -\frac{1}{9}\begin{bmatrix} 3 \\ 27 \\ 33 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -3 \\ -\frac{11}{3} \end{bmatrix}$$

$\therefore$ Solution of the system is $x = -\frac{1}{3}, y = -3, z = -\frac{11}{3}$

Example 1.8 Test for the consistency of the following system of equations and solve them, if consistent:

$$3x + y + z = 8; -x + y - 2z = -5; x + y + z = 6; -2x + 2y - 3z = -7.$$

Note ☑ As the solution can be found out by any method, when the system is consistent, we may prefer the triangularisation method (also known as Gaussian elimination method) to reduce the augmented matrix $[A, B]$ to an equivalent matrix. Using the equivalent matrix, we can test the consistency of the system and also find the solution easily when it exists. In this method, we use *only elementary row operations* and convert the elements below the principal diagonal of A as zeros.

$$[A, B] = \begin{bmatrix} 3 & 1 & 1 & 8 \\ -1 & 1 & -2 & -5 \\ 1 & 1 & 1 & 6 \\ -2 & 2 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ -1 & 1 & -2 & -5 \\ 3 & 1 & 1 & 8 \\ -2 & 2 & -3 & -7 \end{bmatrix} (R_1 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & -2 & -10 \\ 0 & 4 & -1 & 5 \end{bmatrix} (R_2 \to R_2 + R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 + 2R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 1 & 3 \end{bmatrix} (R_3 \to R_3 + R_2, R_4 \to R_4 - 2R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(R_4 \to R_4 + \frac{1}{3}R_3\right) \qquad (1)$$

Now, Determinant of $[A, B]$ = − Determinant of the equivalent matrix = 0. ($\because$ Two rows interchanged in the first operation)

$$\therefore \qquad R[A, B] \le 3$$

Now $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{vmatrix} = -6 \neq 0$

$\therefore$ $R\,[A, B] = R\,(A) = 3$ = the number of unknowns.

$\therefore$ The system is consistent and has a unique solution.

A system of equations equivalent to the given system is also obtained from the equivalent matrix in (1).

The equivalent equations are

$$x + y + z = 6, \qquad 2y - z = 1 \qquad \text{and} \qquad -3z = -9$$

Solving them backwards, we get

$$x = 1, y = 2, z = 3.$$

Example 1.9 Examine if the following system of equations is consistent and find the solution if it exists.

$$x + y + z = 1, 2x - 2y + 3z = 1; x - y + 2z = 5; 3x + y + z = 2.$$

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & 3 & 1 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & -2 & 1 & 4 \\ 0 & -2 & -2 & -1 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 - R_1, \\ R_4 \to R_4 - 3R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & -\frac{5}{2} & -\frac{1}{2} \end{bmatrix} \left(R_3 \to R_3 - \frac{1}{2} R_2, R_4 \to R_4 - \frac{1}{2} R_2 \right)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 22 \end{bmatrix} (R_4 \to R_4 + 5R_3)$$

It is obvious that det $[A, B] = 4$ and det $[A] = 3$

$\therefore$ $$R\,[A, B] \neq R\,[A].$$

$\therefore$ The system is inconsistent.

Note ☑ The last row of the equivalent matrix corresponds to the equation $0 \cdot x + 0 \cdot y + 0 \cdot z = 22$, which is absurd. From this also, we can conclude that the system is inconsistent.

Example 1.10 Solve the following system of equations, if consistent:

$$x+y+z=3,\ x+y-z=1;\ 3x+3y-5z=1.$$

$$[A, B]=\begin{bmatrix}1 & 1 & 1 & 3\\ 1 & 1 & -1 & 1\\ 3 & 3 & -5 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1 & 3\\ 0 & 0 & -2 & -2\\ 0 & 0 & -8 & -8\end{bmatrix} \left(R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1\right)$$

$$\sim \begin{bmatrix}1 & 1 & 1 & 3\\ 0 & 0 & -2 & -2\\ 0 & 0 & 0 & 0\end{bmatrix} \left(R_3 \to R_3 - 4R_2\right)$$

$\therefore$ All the third order determinants vanish

$\therefore$ $$R\,[A, B] \neq 3$$

Consider $\begin{vmatrix}1 & 1\\ 0 & -2\end{vmatrix}$, which is a minor of both A and $[A, B]$.

The value of this minor $= -2 \neq 0$

$\therefore$ $R\,(A) = R\,[A, B]$ < the number of unknowns.

$\therefore$ The system is consistent with many solutions.

From the first two rows of the equivalent matrix, we have $x + y + z = 3$ and $-2z = -2$

i.e. $z = 1$ and $x + y = 2$.

$\therefore$ The system has a one parameter family of solutions, namely $x = k$, $y = 2 - k$, $z = 1$, where k is the parameter.

Giving various values for k, we get infinitely many solutions.

Example 1.11 Solve the following system of equations, if consistent:

$$x_1 + 2x_2 - x_3 - 5x_4 = 4;\ x_1 + 3x_2 - 2x_3 - 7x_4 = 5;\ 2x_1 - x_2 + 3x_3 = 3.$$

$$[A, B]=\begin{bmatrix}1 & 2 & -1 & -5 & 4\\ 1 & 3 & -2 & -7 & 5\\ 2 & -1 & 3 & 0 & 3\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 2 & -1 & -5 & 4\\ 0 & 1 & -1 & -2 & 1\\ 0 & -5 & 5 & 10 & -5\end{bmatrix} \left(R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1\right)$$

$$\sim \begin{bmatrix}1 & 2 & -1 & -5 & 4\\ 0 & 1 & -1 & -2 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} \left(R_3 \to R_3 + 5R_2\right)$$

$\therefore$ $R\,[A, B] \neq 3$ ($\because$ the last row contains only zeros)

Similarly $R\,(A) \neq 3$.

Since $\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix} \neq 0$, $R\,(A) = R\,[A, B] = 2$ < the number of unknowns.

$\therefore$ The given system is consistent with many solutions.

From the first two rows of the equivalent matrix, we have

$$x_1 + 2x_2 - x_3 - 5x_4 = 4 \qquad (1)$$

and $$x_2 - x_3 - 2x_4 = 1 \qquad (2)$$

As there are only 2 equations, we can solve for only 2 unknowns.

Hence the other 2 unknowns are to be treated as parameters.

Taking $x_3 = k$ and $x_4 = k'$, we get

$$x_2 = 1 + k + 2k' \qquad \text{[from (2)]}$$

and $$x_1 = 4 - 2\,(1 + k + 2k') + k + 5k' \qquad \text{[from (1)]}$$

i.e. $$x_1 = 2 - k + k'$$

∴ The given system possesses a two parameter family of solutions.

Note ☑ From the Examples (10) and (11), we note that the number of parameters in the solution equals the difference between the number of unknowns and the common rank of A and $[A, B]$.

Example 1.12 Find the values of k, for which the equations $x + y + z = 1$, $x + 2y + 3z = k$ and $x + 5y + 9z = k^2$ have a solution. For these values of k, find the solutions also.

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & k \\ 1 & 5 & 9 & k^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & k-1 \\ 0 & 4 & 8 & k^2-1 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & k-1 \\ 0 & 0 & 0 & k^2 - 4k + 3 \end{bmatrix} (R_3 \to R_3 - 4R_2) \qquad (1)$$

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \therefore\ R(A) = 2$$

If the system possesses a solution, $R\,[A, B]$ must also be 2.

∴ The last row of the matrix in (1) must contain only zeros.

∴ $$k^2 - 4k + 3 = 0 \qquad \text{i.e. } k = 1 \text{ or } 3.$$

For these values of k, $R\,(A) = R\,[A, B] = 2 <$ the number of unknowns.

∴ The given system has many solutions.

Case (i) $k = 1$

The first two rows of (1) give the equivalent equations as

$$x + y + z = 1 \qquad (2)$$

and $$y + 2z = 0 \qquad (3)$$

Puting $z = \lambda$, the one-parameter family of solutions of the given system is

$$x = \lambda + 1,\ y = -2\lambda \quad \text{and} \quad z = \lambda$$

Case (ii) $k = 3$

The equivalent equations are

$$x + y + z = 1 \qquad (2)$$

and $$y + 2z = 2 \tag{4}$$

Putting $z = \mu$, the one-parameter family of solutions of the given system is

$$x = \mu - 1, y = 2 - 2\mu, z = \mu.$$

Example 1.13 Find the condition satisfied by $a, b, c,$ so that the following system of equations may have a solution:

$$x + 2y - 3z = a;\ 3x - y + 2z = b;\ x - 5y + 8z = c.$$

$$[A, B] = \begin{bmatrix} 1 & 2 & -3 & a \\ 3 & -1 & 2 & b \\ 1 & -5 & 8 & c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & -7 & 11 & c-a \end{bmatrix} \left(R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - R_1\right)$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & 0 & 0 & 2a-b+c \end{bmatrix} \left(R_3 \to R_3 - R_2\right) \tag{1}$$

$$A \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore\ R(A) = 2$$

If the given system possesses a solution, $R[A, B] = 2$.

$\therefore$ The last row of (1) should contain only zeros.

$\therefore\ 2a - b + c = 0$. Only when this condition is satisfied by $a, b, c,$ the system will have a solution.

Example 1.14 Find the value of k such that the following system of equations has (i) a unique solution, (ii) many solutions and (iii) no solution.

$$kx + y + z = 1;\ x + ky + z = 1;\ x + y + kz = 1.$$

$$\therefore \quad A = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$$

$$\begin{aligned} |A| &= k\left(k^2 - 1\right) + (1 - k) + (1 - k) \\ &= (k - 1)\left(k^2 + k - 2\right) \\ &= (k - 1)^2 (k + 2) \end{aligned}$$

$|A| = 0$, when $k = 1$ or $k = -2$

When $k \neq 1$ and $k \neq -2$, $|A| \neq 0$ $\quad \therefore R(A) = 3$

Then the system will have a unique solution.

When $k = 1$, the system reduces to the single equation $x + y + z = 1$.
In this case, $R(A) = R[A, B] = 1$.
$\therefore$ The system will have many solutions.
(i.e. a two parameter family of solutions)
When $k = -2$,

$$[A, B] = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} (R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & 0 \end{bmatrix} (R_2 \to R_2 + 2R_1, R_3 \to R_3 - R_1)$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} (R_3 \to R_3 + R_2)$$

Now

$$\begin{vmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad \therefore\ R(A) < 3$$

$$\begin{vmatrix} 1 & -2 \\ 0 & -3 \end{vmatrix} \neq 0 \quad \therefore R(A) = 2$$

$$\begin{vmatrix} -2 & 1 & 1 \\ -3 & 3 & 3 \\ 0 & 0 & 3 \end{vmatrix} = \text{a minor of } [A, B] \neq 0$$

$\therefore$ $R[A, B] = 3$. Thus $R(A) \neq R[A, B]$.
$\therefore$ The system has no solution.

Example 1.15 Investigate for what values of λ, μ, the equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$ have (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions.

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix} (R_3 \to R_3 - R_2)$$

$$\therefore\ A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \text{ and } |A| = \lambda - 3$$

If $\lambda \neq 3$, $|A| \neq 0$ $\quad \therefore R(A) = 3$

$\therefore$ When $\lambda \neq 3$ and μ takes any value, the system has a unique solution.

If $\lambda = 3$, $|A| = 0$ and a second order minor of A, i.e. $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \neq 0$

$\therefore R(A) = 2$.

When $\lambda = 3$,

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & \mu - 10 \end{bmatrix} \qquad (1)$$

When $\lambda = 3$ and $\mu = 10$, the last row of (1) contains only zeros.

$\therefore \quad R[A, B] \neq 3$ and clearly $R[A, B] = 2$.

Thus, when $\lambda = 3$ and $\mu = 10$, $R(A) = R[A, B] = 2$.

$\therefore$ The system has an infinite number of solutions.

When $\lambda = 3$ and $\mu \neq 10$, a third order minor of $[A, B]$, i.e.

$$\begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 4 \\ 0 & 0 & \mu - 10 \end{vmatrix} = \mu - 10 \neq 0$$

$\therefore \quad R[A, B] = 3$

Thus, when $\lambda = 3$ and $\mu \neq 10$, $R(A) \neq R[A, B]$.

$\therefore$ The given system has no solution.

Example 1.16 Test whether the following system of equations possess a non-trivial solution.

$$x_1 + x_2 + 2x_3 + 3x_4 = 0; \quad 3x_1 + 4x_2 + 7x_3 + 10x_4 = 0;$$

$$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0; \quad 6x_1 + 8x_2 + 13x_3 + 16x_4 = 0.$$

The given system is a homogeneous linear system of the form $AX = 0$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & -2 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 5R_1, \\ R_4 \to R_4 - 6R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -4 \end{bmatrix} (R_3 \to R_3 - 2R_2, R_4 \to R_4 - 2R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} (R_4 \to R_4 - R_3)$$

$\therefore |A| = 4$ i.e. A is non-singular

$$R(A) = R[A, 0] = 4$$

$\therefore$ The system has a unique solution, namely, the trivial solution.

Example 1.17 Find the non-trivial solution of the equations $x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 11z = 0$, if it exists.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -10 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 7R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} (R_3 \to R_3 - 2R_2) \tag{1}$$

$\therefore |A| = 0$ and $\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} \neq 0 \quad \therefore R(A) = 2$

$\therefore$ The system has non-trivial solution. From the first two rows of (1), we see that the given equations are equivalent to

$$x + 2y + 3z = 0 \tag{2}$$

and
$$-2y - 5z = 0 \tag{3}$$

Putting $z = k$, we get $y = -\dfrac{5}{2}k$ from (3) and $x = 2k$.

Thus the non-trivial solution is $x = 4k$, $y = -5k$ and $z = 2k$.

Example 1.18 Find the non-trivial solution of the equations $x - y + 2z - 3w = 0$, $3x + 2y - 4z + w = 0$, $5x - 3y + 2z + 6w = 0$, $x - 9y + 14z - 2w = 0$, if it exists.

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 5 & -3 & 2 & 6 \\ 1 & -9 & 14 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -10 & 10 \\ 0 & 2 & -8 & 21 \\ 0 & -8 & 12 & 1 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 5R_1, \\ R_4 \to R_4 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -8 & 21 \\ 0 & -8 & 12 & 1 \end{bmatrix} \left(R_2 \to \frac{1}{5} R_2\right)$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -4 & 17 \\ 0 & 0 & -4 & 17 \end{bmatrix} (R_3 \to R_3 - 2R_2, R_4 \to R_4 + 8R_2)$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -4 & 17 \\ 0 & 0 & 0 & 0 \end{bmatrix} (R_4 \to R_4 - R_3)$$

$\therefore$ $|A| = 0$ i.e. $R(A) < 4$

$\therefore$ The system has a non-trivial solution.

The system is equivalent to

$$x - y + 2z - 3w = 0 \quad (1)$$

$$y - 2z + 2w = 0 \quad (2)$$

$$-4z + 17w = 0 \quad (3)$$

Putting $w = 4k$, we get $z = 17k$ from (3), $y = 26k$ from (2) and $x = 4k$.
Thus the non-trivial solution is $x = 4k$, $y = 26k$, $z = 17k$ and $w = 4k$.

Example 1.19 Find the values of λ for which the equations $x + (\lambda + 4)\,y + (4\lambda + 2)z = 0$, $x + 2(\lambda + 1)\,y + (3\lambda + 4)\,z = 0$, $2x + 3\lambda y + (3\lambda + 4)\,z = 0$ have a non-trivial solution. Also find the solution in each case.

$$A = \begin{bmatrix} 1 & \lambda + 4 & 4\lambda + 2 \\ 1 & 2\lambda + 2 & 3\lambda + 4 \\ 2 & 3\lambda & 3\lambda + 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \lambda + 4 & 4\lambda + 2 \\ 0 & \lambda - 2 & -\lambda + 2 \\ 0 & \lambda - 8 & -5\lambda \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - 2R_1 \end{pmatrix} \quad (1)$$

For non-trivial solution, $|A| = 0$

i.e. $$-5\lambda\,(\lambda - 2) - (\lambda - 8)\,(2 - \lambda) = 0$$

i.e. $$-4\lambda^2 + 16 = 0$$

$\therefore$ $$\lambda = \pm\, 2$$

When $\lambda = 2$, the system is equivalent to

$$x + 6y + 10z = 0$$

$$-6y - 10z = 0, \quad \text{from (1)}$$

Putting $z = 3k$, we get $y = -5k$ and $x = 0$
i.e. the solution is $x = 0$, $y = -5k$ and $z = 3k$.
When $\lambda = -2$, the system is equivalent to

$$x + 2y - 6z = 0$$

$$-4y + 4z = 0, \quad \text{from (1)}$$

Putting $z = k$, we get $y = k$ and $x = 4k$.
i.e. the solution is $x = 4k$, $y = k$ and $z = k$.

EXERCISE 1(a)

Part A

(Short Answer Questions)

1. Define the linear dependence of a set of vectors.
2. Define the linear independence of a set of vectors.
3. If a set of vectors is linearly dependent, show that at least one member of the set can be expressed as a linear combination of the other members.
4. Show that the vectors $X_1 = (1, 2)$, $X_2 = (2, 3)$ and $X_3 = (4, 5)$ are linearly dependent.
5. Show that the vectors $X_1 = (0, 1, 2)$, $X_2 = (0, 3, 5)$ and $X_3 = (0, 2, 5)$ are linearly dependent.
6. Express $X_1 = (1, 2)$ as a linear combination of $X_2 = (2, 3)$ and $X_3 = (4, 5)$.
7. Show that the vectors (1, 1, 1), (1, 2, 3) and (2, 3, 8) are linearly independent.
8. Find the value of a if the vectors $(2, -1, 0)$, $(4, 1, 1)$ and $(a, -1, 1)$ are linearly dependent.
9. What do you mean by consistent and inconsistent systems of equations. Give examples.
10. State Rouche's theorem.
11. State the condition for a system of equations in n unknowns to have (i) one solution, (ii) many solutions and (iii) no solution.
12. Give an example of 2 equations in 2 unknowns that are (i) consistent with only one solution and (ii) inconsistent.
13. Give an example of 2 equations in 2 unknowns that are consistent with many solutions.
14. Find the values of a and b, if the equations $2x - 3y = 5$ and $ax + by = -10$ have many solutions.
15. Test if the equations $x + y + z = a$, $2x + y + 3z = b$, $5x + 2y + z = c$ have a unique solution, where a, b, c are not all zero.
16. Find the value of λ, if the equations $x + y - z = 10$, $x - y + 2z = 20$ and $\lambda x - y + 4z = 30$ have a unique solution.
17. If the augmented matrix of a system of equations is equivalent to $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$, find the value of λ, for which the system has a unique solution.
18. If the augmented matrix of a system of equations is equivalent to $\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & \lambda+1 & \mu-3 \end{bmatrix}$, find the values of λ and μ for which the system has only one solution.

19. If the augmented matrix of a system of equations is equivalent to $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 5 & 4 & 2 \\ 0 & 0 & \lambda-2 & \mu-3 \end{bmatrix}$, find the values of λ and μ for which the system has many solutions.
20. If the augmented matrix of a system of equations is equivalent to $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & \lambda-8 & \mu-11 \end{bmatrix}$, find the values of λ and μ for which the system has no solution.
21. Do the equations $x - 3y - 8z = 0$, $3x + y = 0$ and $2x + 5y + 6z = 0$ have a non-trivial solution? Why?
22. If the equations $x + 2y + z = 0$, $5x + y - z = 0$ and $x + 5y + \lambda z = 0$ have a non-trivial solution, find the value of λ.
23. Given that the equations $x + 2y - z = 0$, $3x + y - z = 0$ and $2x - y = 0$ have non-trivial solution, find it.

Part B

Show that the following sets of vectors are linearly dependent. Find their relationship in each case:

24. $X_1 = (1, 2, 1)$, $X_2 = (4, 1, 2)$, $X_3 = (6, 5, 4)$, $X_4 = (-3, 8, 1)$.
25. $X_1 = (3, 1, -4)$, $X_2 = (2, 2, -3)$, $X_3 = (0, -4, 1)$, $X_4 = (-4, -4, 6)$
26. $X_1 = (1, 2, -1, 3)$, $X_2 = (0, -2, 1, -1)$, $X_3 = (2, 2, -1, 5)$
27. $X_1 = (1, 0, 4, 3)$, $X_2 = (2, 1, -1, 1)$, $X_3 = (3, 2, -6, -1)$
28. $X_1 = (1, -2, 4, 1)$, $X_2 = (1, 0, 6, -5)$, $X_3 = (2, -3, 9, -1)$ and $X_4 = (2, -5, 7, 5)$.
29. Determine whether the vector $x_5 = (4, 2, 1, 0)$ is a linear combination of the set of vectors $X_1 = (6, -1, 2, 1)$, $X_2 = (1, 7, -3, -2)$, $X_3 = (3, 1, 0, 0)$ and $X_4 = (3, 3, -2, -1)$.

    Show that each of the following sets of vectors is linearly independent.
30. $X_1 = (1, 1, 1)$; $X_2 = (1, 2, 3)$; $X_3 = (2, -1, 1)$.
31. $X_1 = (1, -1, 2, 3)$; $X_2 = (1, 0, -1, 2)$; $X_3 = (1, 1, -4, 0)$
32. $X_1 = (1, 2, -1, 0)$ $X_2 = (1, 3, 1, 2)$; $X_3 = (4, 2, 1, 0)$; $X_4 = (6, 1, 0, 1)$.
33. $X_1 = (1, -2, -3, -2, 1)$; $X_2 = (3, -2, 0, -1, -7)$; $X_3 = (0, 1, 2, 1, -6)$; $X_4 = (0, 2, 2, 1, -5)$.
34. Test for the consistency of the following system of equations:

$$\begin{bmatrix} 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 \\ 15 & 16 & 17 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 14 \\ 19 \end{bmatrix}$$

Test for the consistency of the following systems of equations and solve, if consistent:

35. $2x - 5y + 2z = -3; -x - 3y + 3z = -1; x + y - z = 0; -x + y = 1.$
36. $3x + 5y - 2z = 1; x - y + 4z = 7; -6x - 2y + 5z = 9; 7x - 3y + z = 4.$
37. $2x + 2y + 4z = 6; 3x + 3y + 7z = 10; 5x + 7y + 11z = 17; 6x + 8y + 13z = 16.$

Test for the consistency of the following systems of equations and solve, if consistent:

38. $x + 2y + z = 3; 2x + 3y + 2z = 5; 3x - 5y + 5z = 2; 3x + 9y - z = 4.$
39. $2x + 6y - 3z = 18; 3x - 4y + 7z = 31; 5x + 3y + 3z = 48; 8x - 3y + 2z = 21.$
40. $x + 2y + 3z = 6; 5x - 3y + 2z = 4; 2x + 4y - z = 5; 3x + 2y + 4z = 9.$
41. $x + 2y = 4; 10y + 3z = -2; 2x - 3y - z = 5; 3x + 3y + 2z = 1.$
42. $2x_1 + x_2 + 2x_3 + x_4 = 6; x_1 - x_2 + x_3 + 2x_4 = 6; 4x_1 + 3x_2 + 3x_3 - 3x_4 = -1; 2x_1 + 2x_2 - x_3 + x_4 = 10$
43. $2x + y + 5z + w = 5; x + y + 3z - 4w = -1; 3x + 6y - 2z + w = 8; 2x + 2y + 2z - 3w = 2.$

Test for the consistency of the following systems of equations and solve, if consistent:

44. $x - 3y - 8z = -10; 3x + y = 4z; 2x + 5y + 6z = 13.$
45. $5x + 3y + 7z = 4; 3x + 26y + 2z = 9; 7x + 2y + 10z = 5.$
46. $x - 4y - 3z + 16 = 0; 2x + 7y + 12z = 48; 4x - y + 6z = 16; 5x - 5y + 3z = 0.$
47. $x - 2y + 3w = 1; 2x - 3y + 2z + 5w = 3; 3x - 7y - 2z + 10w = 2.$
48. $x_1 + 2x_2 + 2x_3 - x_4 = 3; x_1 + 2x_2 + 3x_3 + x_4 = 1; 3x_1 + 6x_2 + 8x_3 + x_4 = 5.$
49. Find the values of k, for which the equations $x + y + z = 1, x + 2y + 4z = k$ and $x + 4y + 10z = k^2$ have a solution. For these values of k, find the solutions also.
50. Find the values of λ, for which the equtions $x + 2y + z = 4, 2x - y - z = 3\lambda$ and $4x - 7y - 5z = \lambda^2$ have a solution. For these values of λ, find the solutions also.
51. Find the condition on a, b, c, so that the equations $x + y + z = a, x + 2y + 3z = b, 3x + 5y + 7z = c$ may have a one-parameter family of solutions.
52. Find the value of k for which the equations $kx - 2y + z = 1, x - 2ky + z = -2$ and $x - 2y + kz = 1$ have (i) no solution, (ii) one solution and (iii) many solutions.
53. Investigate for what values of λ, μ the equations $x + y + 2z = 2, 2x - y + 3z = 2$ and $5x - y + \lambda z = \mu$ have (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions.
54. Find the values of a and b for which the equations $x + y + 2z = 3, 2x - y + 3z = 4$ and $5x - y + az = b$ have (i) no solution, (ii) a unique solution, (iii) many solutions.
55. Find the non-trivial solution of the equations $x + 2y + z = 0; 5x + y - z = 0$ and $x + 5y + 3z = 0$, if it exists.
56. Find the non-trivial solution of the equations $x + 2y + z + 2w = 0; x + 3y + 2z + 2w = 0; 2x + 4y + 3z + 6w = 0$ and $3x + 7y + 4z + 6w = 0$, if it exists.
57. Find the values of λ for which the equations $3x + y - \lambda z = 0, 4x - 2y - 3z = 0$ and $2\lambda x + 4y + \lambda z = 0$ possess a non-trivial solution. For these values of λ, find the solution also.
58. Find the values of λ for which the equations $(11 - \lambda) x - 4y - 7z = 0, 7x - (\lambda + 2) y - 5z = 0, 10x - 4y - (6 + \lambda) z = 0$ possess a non-trivial solution. For these values of λ, find the solution also.

1.6 EIGENVALUES AND EIGENVECTORS

1.6.1 Definition

Let $A = [a_{ij}]$ be a square matrix of order n. If there exists a *non-zero* column vector X and a scalar λ, such that

$$AX = \lambda X$$

then λ is called an eigenvalue of the matrix A and X is called the eigenvector corresponding to the eigenvalue λ.

To find the eigenvalues and the corresponding eigenvectors of a square matrix A, we proceed as follows:

Let λ be an eigenvalue of A and X be the corresponding eigenvector. Then, by definition,

$AX = \lambda X = \lambda IX$, where I is the unit matrix of order n.

i.e. $$(A - \lambda I)\, X = 0 \tag{1}$$

i.e. $$\left\{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \hdashline a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ \hdashline 0 & 0 & 0 & \dots & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e. $$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots (a_{nn} - \lambda)\, x_n &= 0 \end{aligned} \tag{2}$$

Equations (2) are a system of homogeneous linear equations in the unknowns x_1, $x_2, \dots, x_n$.

Since $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is to be a non-zero vector,

$x_1, x_2, \dots, x_n$ should not be all zeros. In other words, the solution of the system (2) should be a non-trivial solution.

The condition for the system (2) to have a non-trivial solution is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \hdashline a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \tag{3}$$

i.e. $$|A - \lambda I| = 0 \tag{4}$$

The determinant $|A - \lambda I|$ is a polynomial of degree n in λ and is called *the characteristic polynomial* of A.

The equation $|A - \lambda I| = 0$ or the equation (3) is called *the characteristic equation* of A.

When we solve the characteristic equation, we get n values for λ. These n roots of the characteristic equation are called the *characteristic roots* or *latent roots* or *eigenvalues* of A.

Corresponding to each value of λ, the equations (2) possess a non-zero (non-trivial) solution X. X is called *the invariant vector* or *latent vector* or *eigenvector* of A corresponding to the eigenvalue λ.

Notes ☑

1. Corresponding to an eigenvalue, the non-trivial solution of the system (2) will be a one-parameter family of solutions. Hence the eigenvector corresponding to an eigenvalue is not unique.
2. If all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix A are distinct, then the corresponding eigenvectors are linearly independent.
3. If two or more eigenvalues are equal, then the eigenvectors may be linearly independent or linearly dependent.

1.6.2 Properties of Eigenvalues

1. A square matrix A and its transpose A^T have the same eigenvalues.
   Let $A = (a_{ij})$; $i, j = 1, 2, \ldots, n$.
   The characteristic polynomial of A is

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \hdashline a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} \qquad (1)$$

   The characteristic polynomial of A^T is

$$|A^T-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22}-\lambda & \ldots & a_{n2} \\ \hdashline a_{1n} & a_{2n} & \ldots & a_{nn}-\lambda \end{vmatrix} \qquad (2)$$

   Determinant (2) can be obtained by changing rows into columns of determinant (1).

   $\therefore \qquad |A-\lambda I| = |A^T - \lambda I|$

   $\therefore$ The characteristic equations of A and A^T are identical.

   $\therefore$ The eigenvalues of A and A^T are the same.

2. The sum of the eigenvalues of a matrix A is equal to the sum of the principal diagonal elements of A. (The sum of the principal diagonal elements is called *the Trace* of the matrix.)

   The characteristic equation of an n^{th} order matrix A may be written as

$$\lambda^n - D_1\lambda^{n-1} + D_2\lambda^{n-2} - \cdots + (-1)^n D_n = 0, \qquad (1)$$

   where D_r is the sum of all the r^{th} order minors of A whose principal diagonals lie along the principal diagonal of A.

(*Note* ☑ $D_n = |A|$). We shall verify the above result for a third order matrix.

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The characteristic equation of A is given by

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = 0 \qquad (2)$$

Expanding (2), the characteristic equation is

$$(a_{11}-\lambda)\left\{\lambda^2 - (a_{22}+a_{33})\lambda + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right\}$$
$$-a_{12}\left\{-a_{21}\lambda + \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right\} + a_{13}\left\{a_{31}\lambda + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right\} = 0$$

i.e.
$$-\lambda^3 + (a_{11}+a_{22}+a_{33})\lambda^2$$
$$-\left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right\}\lambda + |A| = 0$$

i.e. $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$, using the notation given above.
This result holds good for a matrix of order n.

Note ☑ This form of the characteristic equation provides an alternative method for getting the characteristic equation of a matrix.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A.
∴ They are the roots of equation (1).

$$\therefore \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_n = \frac{-(-D_1)}{1} = D_1$$
$$= a_{11} + a_{22} + \cdots + a_{nn}$$
$$= \text{Trace of the matrix } A.$$

3. The product of the eigenvalues of a matrix A is equal to $|A|$.
   If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, they are the roots of

$$\lambda^n - D_1\lambda^{n-1} + D_2\lambda^{n-2} - \cdots + (-1)^n D_n = 0.$$

$$\therefore \text{ Product of the roots } = \frac{(-1)^n \cdot (-1)^n D_n}{1}$$

i.e. $\lambda_1, \lambda_2 \ldots \lambda_n = D_n = |A|$.

1.6.3 Aliter

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of $|A - \lambda I| = 0$

$\therefore$ $|A - \lambda I| \equiv (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$, since L.S. is a n^{th} degree polynomial in λ whose leading term is $(-1)^n \lambda^n$.

Putting $\lambda = 0$ in the above identity, we get $|A| = (-1)^n (-\lambda_1)(-\lambda_2) \ldots (-\lambda_n)$ i.e. $\lambda_1 \lambda_2 \ldots \lambda_n = |A|$.

1.6.4 Corollary

If $|A| = 0$, i.e. A is a singular matrix, at least one of the eigenvalues of A is zero and conversely.

4. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a matrix A, then
   (i) $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ are the eigenvalues of the matrix kA, where k is a non-zero scalar.
   (ii) $\lambda_1^p, \lambda_2^p, \ldots, \lambda_n^p$ are the eigenvalues of the matrix A^p, where p is a positive integer.
   (iii) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigenvalues of the inverse matrix A^{-1}, provided $\lambda_r \neq 0$ i.e. A is non-singular.

(i) Let λ_r be an eigenvalue of A and X_r the corresponding eigenvector. Then, by definition,

$$AX_r = \lambda_r x_r \tag{1}$$

Multiplying both sides of (1) by k,

$$(kA)X_r = (k\lambda_r) X_r \tag{2}$$

From (2), we see that $k\lambda_r$ is an eigenvalue of kA and the corresponding eigenvector is the same as that of λ_r, namely X_r.

(ii) Premultiplying both sides of (1) by A,

$$\begin{aligned} A^2 X_r &= A(AX_r) \\ &= A(\lambda_r X_r) \\ &= \lambda_r (AX_r) \\ &= \lambda_r^2 X_r \end{aligned}$$

Similarly $A^3 X_r = \lambda_r^3 X_r$ and so on.

In general, $A^p X_r = \lambda_r^p X_r$

From, (3), we see that λ_r^p is an eigenvalue of A^p with the corresponding eigenvector equal to X_r, which is the same for λ_r.

(iii) Premultiplying both sides of (1) by A^{-1},

$$A^{-1}(AX_r) = A^{-1}(\lambda_r X_r)$$

i.e. $$X_r = \lambda_r (A^{-1} X_r)$$

$\therefore$ $$A^{-1} X_r = \frac{1}{\lambda_r} X_r \tag{4}$$

From (4), we see that $\frac{1}{\lambda_r}$ is an eigenvalue of A^{-1} with the corresponding eigenvector equal to X_r, which is the same for λ_r.

5. The eigenvalues of a real symmetric matrix (i.e. a symmetric matrix with real elements) are real.

Let λ be an eigenvalue of the real symmetric matrix and X be the corresponding eigenvector.

Then $$AX = \lambda \quad (1)$$

Premultiplying both sides of (1) by $\bar{X}^T$ (the transpose of the conjugate of X), we get

$$\bar{X}^T AX = \lambda \bar{X}^T X \quad (2)$$

Taking the complex conjugate on both sides of (2),

$$X^T \bar{A} \bar{X} = \bar{\lambda} X^T \bar{X} \text{ (assuming that } \lambda \text{ may be complex)}$$

i.e. $$X^T A \bar{X} = \bar{\lambda} X^T \bar{X} \ (\because \bar{A} = A, \text{ as } A \text{ is real}) \quad (3)$$

Taking transpose on both sides of (3),

$$\bar{X}^T A^T X = \bar{\lambda} \bar{X}^T X \qquad \left[\because (AB)^T = B^T A^T\right]$$

i.e. $$\bar{X}^T AX = \bar{\lambda} \bar{X}^T X \qquad \left[\because (A)^T = A, \text{ as } A \text{ is symmetric}\right] \quad (4)$$

From (2) and (4), we get

$$\lambda \bar{X}^T X = \bar{\lambda} \bar{X}^T X$$

i.e. $$(\lambda - \bar{\lambda}) \bar{X}^T X = 0$$

$\bar{X}^T X$ is an 1×1 matrix, i.e. a single element which is positive

$\therefore$ $$\lambda - \bar{\lambda} = 0$$

i.e. λ is real.

Hence all the eigenvalues are real.

6. The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

Note ☑ Two column vectors $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ are said to be orthogonal, if their inner product $(x_1 y_1 + x_2 y_2 + \cdots x_n y_n) = 0$ i.e. if $X^T Y = 0$.

Let λ_1, λ_2 be any two distinct eigenvalues of the real symmetric matrix A and X_1, X_2 be the corresponding eigenvectors respectively.

Then $$AX_1 = \lambda_1 X_1 \quad (1)$$

and $$AX_2 = \lambda_2 X_2 \quad (2)$$

Premultiplying both sides of (1) by, X_2^T we get

$$X_2^T AX_1 = \lambda_1 X_2^T X_1$$

Taking the transpose on both sides,

$$X_1^T AX_2 = \lambda_1 X_1^T X_2 \qquad (\because A^T = A) \tag{3}$$

Premultiplying both sides of (2) by X_1^T, we get

$$X_1^T AX_2 = \lambda_2 X_1^T X_2 \tag{4}$$

From (3) and (4), we have

$$\lambda_1 X_1^T X_2 = \lambda_2 X_1^T X_2$$

i.e. $$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$$

Since $$\lambda_1 \neq \lambda_2, X_1^T X_2 = 0$$

i.e. the eigenvectors X_1 and X_2 are orthogonal.

WORKED EXAMPLE 1(b)

Example 1.1 Given that $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, verify that the eigenvalues of A^2 are the squares of those of A.

Verify also that the respective eigenvectors are the same.

The characteristic equation of A is $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

i.e. $$(5 - \lambda)(2 - \lambda) - 4 = 0$$

i.e. $$\lambda^2 - 7\lambda + 6 = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 1, 6$.

The eigenvector corresponding to any λ is given by $(A - \lambda I) X = 0$

i.e. $$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

When $\lambda = 1$, the eigenvector is given by the equations

$4x_1 + 4x_2 = 0$ and

$x_1 + x_2 = 0$, which are one and the same.

Solving, $x_1 = -x_2$. Taking $x_1 = 1, x_2 = -1$.

$\therefore$ The eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

When $\lambda = 6$, the eigenvector is given by

$$-x_1 + 4x_2 = 0$$

and $$x_1 - 4x_2 = 0$$

Solving, $x_1 = 4x_2$

Taking $x_2 = 1, x_1 = 4$

$\therefore$ The eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Now $$A^2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix}$$

The characteristic equation of A^2 is $\begin{vmatrix} 29-\lambda & 28 \\ 7 & 8-\lambda \end{vmatrix} = 0$

i.e. $$(29 - \lambda)(8 - \lambda) - 196 = 0$$

i.e. $$\lambda^2 - 37\lambda + 36 = 0$$

i.e. $$(\lambda - 1)(\lambda - 36) = 0$$

$\therefore$ The eigenvalues of A^2 are 1 and 36, that are the squares of the eigenvalues of A, namely 1 and 6. When $\lambda = 1$, the eigenvector of A^2 is given by

$$\begin{bmatrix} 28 & 28 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \quad \text{i.e.} \quad 28x_1 + 28x_2 = 0 \quad \text{and} \quad 7x_1 + 7x_2 = 0$$

Solving, $x_1 = -x_2$. Taking $x_1 = 1, x_2 = -1$.

When $\lambda = 36$, the eigenvector of A^2 is given by

$$\begin{bmatrix} -7 & 28 \\ 7 & -28 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \text{ i.e.} -7x_1 + 28x_2 = 0 \quad \text{and} \quad 7x_1 - 28x_2 = 0.$$

Solving, $x_1 = 4x_2$. Taking $x_2 = 1, x_1 = 4$.

Thus the eigenvectors of A^2 are

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, which are the same as the respective eigenvectors of A.

Example 1.2 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

i.e. $(1 - \lambda)\{\lambda^2 - 6\lambda + 4\} - (1 - \lambda - 3) + 3(1 - 15 + 3\lambda) = 0$

i.e. $-\lambda^3 + 7\lambda^2 - 36 = 0$ or $\lambda^3 - 7\lambda^2 + 36 = 0$ (1)

i.e. $(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$ [$\therefore \lambda = -2$ satisfies (1)]

i.e. $(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$

$\therefore$ The eigenvalues of A are $\lambda = -2, 3, 6$.

Case (i) $\lambda = -2$.

The eigenvector is given by

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad (2)$$

i.e.

$$x_1 + 7x_2 + x_3 = 0$$
$$3x_1 + x_2 + 3x_3 = 0$$

Solving these equations by the rule of cross-multiplication, we have

$$\frac{x_1}{21-1} = \frac{x_2}{3-3} = \frac{x_3}{1-21}$$

i.e.

$$\frac{x_1}{20} = \frac{x_2}{0} = \frac{x_3}{-20} \qquad (3)$$

Note ☑ To solve for x_1, x_2, x_3, we have taken the equations corresponding to the second and third rows of the matrix in step (2). The proportional values of x_1, x_2, x_3 obtained in step (3) are the co-factors of the elements of the first row of the determinant of the matrix in step (2). This provides an alternative method for finding the eigenvector.

From step (3), $x_1 = k$, $x_2 = 0$ and $x_3 = -k$.

Usually the eigenvector is expressed in terms of the simplest possible numbers, corresponding to $k = 1$ or -1.

$\therefore \qquad x_1 = 1, \qquad x_2 = 0, \qquad x_3 = -1$

Thus the eigenvector corresponding to $\lambda = -2$ is

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$

Values of x_1, x_2, x_3 are proportional to the co-factors of –2, 1, 3 (elements of the first row i.e. –5, 5, –5.

i.e.

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \quad \text{or} \quad \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\therefore \qquad X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = 6$.

The eigenvector is given by $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore \qquad \dfrac{x_1}{4} = \dfrac{x_2}{8} = \dfrac{x_3}{4}$

or $\qquad \dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{1}$

$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Note ☑ Since the eigenvalues of A are distinct, the eigenvectors X_1, X_2, X_3 are linearly independent, as can be seen from the fact that the equation $k_1X_1 + k_2X_2 + k_3X_3 = 0$ is satisfied only when $k_1 = k_2 = k_3 = 0$.

Example 1.3 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is given by

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0, \text{ where}$$

D_1 = the sum of the first order minors of A that lie along the main diagonal of A
$= 0 + 0 + 0$
$= 0$

D_2 = the sum of the second order minors of A whose principal diagonals lie along the principal diagonal of A.

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= -3$

$D_3 = |A| = 2$

Thus the characteristic equation of A is

$$\lambda^3 - 3\lambda - 2 = 0$$

i.e. $\qquad (\lambda + 1)^2 (\lambda - 2) = 0$

$\therefore$ The eigenvalues of A are $\lambda = -1, -1, 2$.

Case (i) $\qquad \lambda = -1.$

The eigenvector is given by

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

All the three equations reduce to one and the same equation $x_1 + x_2 + x_3 = 0$. There is one equation in three unknowns.

$\therefore$ Two of the unknowns, say, x_1 and x_2 are to be treated as free variables (parameters). Taking $x_1 = 1$ and $x_2 = 0$, we get $x_3 = -1$ and taking $x_1 = 0$ and $x_2 = 1$, we get $x_3 = -1$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 2.$

The eigenvector is given by

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Values of x_1, x_2, x_3 are proportional to the co-factors of elements in the first row.

i.e. $$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

or $$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ $$x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Note ☑ Though two of the eigenvalues are equal, the eigenvectors X_1, X_2, X_3 are found to be linearly independent.

Example 1.4 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(\lambda^2 - 4) + 2(-1-\lambda-1) + 2(3-1+\lambda) = 0$$

i.e. $$(2-\lambda)(\lambda-2)(\lambda+2) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = -2, 2, 2.$

Case (i) $\lambda = -2$

The eigenvector is given by

$$\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore \dfrac{x_1}{-8} = \dfrac{x_2}{-2} = \dfrac{x_3}{14}$ (by taking the co-factors of elements of the third row)

i.e.
$$\frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7}$$

$\therefore$
$$X_1 = \begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}$$

Case (ii) $\lambda = 2$.

The eigenvector is given by
$$\begin{vmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$
$$\frac{x_1}{0} = \frac{x_2}{4} = \frac{x_3}{4} \quad \text{or} \quad \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$
$$X_2 = X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Note ☑ Two eigenvalues are equal and the eigenvectors are linearly dependent.

Example 1.5 Find the eigenvalues and eigenvectors of the matrix
$$A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$

Can you guess the nature of A from the eigenvalues? Verify your answer.

The characteristic equation of A is
$$\begin{vmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{vmatrix} = 0$$

i.e.
$$(11-\lambda)(\lambda^2 + 8\lambda - 8) + 4(8 - 7\lambda) - 7(10\lambda - 8) = 0$$

i.e.
$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 0, 1, 2$.

Case (i) $\lambda = 0$.

The eigenvector is given by
$$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$
$$\frac{x_1}{-8} = \frac{x_2}{-8} = \frac{x_3}{-8}$$

or $$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore$ $$X_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Case (iii) $\lambda = 2$.

The eigenvector is given by $\begin{bmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{12} = \frac{x_2}{6} = \frac{x_3}{12}$$

or $$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{2}$$

$\therefore$ $$X_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Since one of the eigenvalues of A is zero, product of the eigenvalues = $|A| = 0$, i.e. A is non-singular. It is verified below:

$$\begin{vmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{vmatrix} = 11(12 - 20) + 4(-42 + 50) - 7(-28 + 20) = 0.$$

Example 1.6 Verify that the sum of the eigenvalues of A equals the trace of A and that their product equals $|A|$, for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

i.e. $$(1-\lambda)(\lambda^2 - 6\lambda + 8) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 1, 2, 4$.
Sum of the eigenvalues = 7.
Trace of the matrix = 1 + 3 + 3 = 7
Product of the eigenvalues = 8.

$$|A| = 1 \times (9-1) = 8.$$

Hence the properties verified.

Example 1.7 Verify that the eigenvalues of A^2 and A^{-1} are respectively the squares and reciprocals of the eigenvalues of A, given that

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

i.e. $$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 3, 2, 5$.

Now $$A^2 = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{bmatrix}$$

The characteristic equation of A^2 is

$$\begin{vmatrix} 9-\lambda & 5 & 38 \\ 0 & 4-\lambda & 42 \\ 0 & 0 & 25-\lambda \end{vmatrix} = 0$$

i.e. $$(9-\lambda)(4-\lambda)(25-\lambda) = 0$$

$\therefore$ The eigenvalues of A^2 are 9, 4, 25, which are the squares of the eigenvalues of A.

Let $$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A_{11} = Co-factor of $a_{11} = 10$; $A_{12} = 0$; $A_{13} = 0$;

$A_{21} = -5$; $A_{22} = 15$; $A_{23} = 0$; $A_{31} = -2$; $A_{32} = -18$; $A_{33} = 6$

$$|A| = 30.$$

$$\therefore \qquad A^{-1} = \frac{1}{30}\begin{bmatrix} 10 & -5 & -2 \\ 0 & 15 & -18 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{15} \\ 0 & \frac{1}{2} & -\frac{3}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

The characteristic equation of A^{-1} is

$$\begin{vmatrix} \frac{1}{3}-\lambda & -\frac{1}{6} & -\frac{1}{15} \\ 0 & \frac{1}{2}-\lambda & -\frac{3}{5} \\ 0 & 0 & \frac{1}{5}-\lambda \end{vmatrix} = 0$$

i.e.
$$\left(\frac{1}{3}-\lambda\right)\left(\frac{1}{2}-\lambda\right)\left(\frac{1}{5}-\lambda\right) = 0$$

$\therefore$ The eigenvalues of A^{-1} are $\frac{1}{3}, \frac{1}{2}, \frac{1}{5}$, which are the reciprocals of the eigenvalues of A.

Hence the properties verified.

Example 1.8 Find the eigenvalues and eigenvectors of (adj A), given that the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

i.e.
$$(2-\lambda)^3 - (2-\lambda) = 0$$

i.e.
$$(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 1, 2, 3$.

Case (i) $\quad \lambda = 1$.

The eigenvector is given by
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \qquad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore \qquad X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 2$.

The eigenvector is given by $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

i.e. $-x_3 = 0$ and $-x_1 = 0$

$\therefore$ $x_1 = 0$, $x_3 = 0$ and x_2 is arbitrary. Let $x_2 = 1$

$$\therefore \qquad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$

$$\therefore \qquad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The eigenvalues of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$ with the eigenvectors X_1, X_2, X_3.

Now $$\frac{\text{adj } A}{|A|} = A^{-1}$$

i.e. adj $A = |A| \cdot A^{-1} = 6A^{-1}$ ($\because |A| = 6$ for the given matrix A)

$\therefore$ The eigenvalues of (adj A) are equal to 6 times those of A^{-1}, namely, 6, 3, 2. The corresponding eigenvectors are X_1, X_2, X_3 respectively.

Example 1.9 Verify that the eigenvectors of the real symmetric matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

are orthogonal in pairs.

The characteristic equation of A is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

i.e. $$(3-\lambda)(\lambda^2 - 8\lambda + 14) + (\lambda - 3 + 1) + (1 + \lambda - 5) = 0$$

i.e. $$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

i.e. $$(\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 2, 3, 6$.

Case (i) $\lambda = 2$.

The eigenvector is given by $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{-2} \quad \text{or} \quad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$\therefore$ $$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = 6$.

The eigenvector is given by $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Now

$$X_1^T\, X_2 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$X_2^T\, X_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$X_3^T\, X_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$$

Hence the eigenvectors are orthogonal in pairs.

Example 1.10 Verify that the matrix

$$A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

is an orthogonal matrix. Also verify that $\frac{1}{\lambda}$ is an eigenvalue of A, if λ is an eigenvalue and that the eigenvalues of A are of unit modulus.

Note ☑ A square matrix A is said to be orthogonal if $AA^T = A^TA = I$.

Now

$$AA^T = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly we can prove that $A^TA = I$.
Hence A is an orthogonal matrix.
The characteristic equation of $3A$ is

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ -2 & 1-\lambda & 2 \\ 1 & -2 & 2-\lambda \end{vmatrix} = 0$$

i.e. $(2 - \lambda)\,(\lambda^2 - 3\lambda + 6) - 2(2\lambda - 4 - 2) + (4 - 1 + \lambda) = 0$

i.e. $\lambda^3 - 5\lambda^2 + 15\lambda - 27 = 0$

i.e. $(\lambda - 3)\,(\lambda^2 - 2\lambda + 9) = 0$

$\therefore$ The eigenvalues of $3A$ are given by

$$\lambda = 3 \qquad \text{and} \qquad \lambda = \frac{2 \pm \sqrt{4-36}}{2} = 1 \pm i\,2\sqrt{2}$$

$\therefore$ The eigenvalues of A are

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1+i\,2\sqrt{2}}{3}, \quad \lambda_3 = \frac{1-i\,2\sqrt{2}}{3}$$

Now $$\frac{1}{\lambda_1} = 1 = \lambda_1$$

$$\frac{1}{\lambda_2} = \frac{3}{1+i2\sqrt{2}} = \frac{3\,(1-i2\sqrt{2})}{(1+i\,2\sqrt{2})\,(1-i\,2\sqrt{2})} = \frac{1-i\,2\sqrt{2}}{3} = \lambda_3$$

and similarly $\dfrac{1}{\lambda_3} = \lambda_2$.

Thus, if λ is an eigenvalue of an orthogonal matrix, $\dfrac{1}{\lambda}$ is also an eigenvalue.

Also $|\lambda_1| = |1| = 1$.

$$|\lambda_2| = \left|\frac{1}{3} + \frac{i\,2\sqrt{2}}{3}\right| = \sqrt{\frac{1}{9} + \frac{8}{9}} = 1$$

Similarly, $|\lambda_3| = 1$.

Thus the eigenvalues of an orthogonal matrix are of unit modulus.

EXERCISE 1(b)

Part A

(Short Answer Questions)

1. Define eigenvalues and eigenvectors of a matrix.
2. Prove that A and A^T have the same eigenvalues.
3. Find the eigenvalues of $2A^2$, if $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.
4. Prove that the eigenvalues of $(-3A^{-1})$ are the same as those of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.
5. Find the sum and product of the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$.
6. Find the sum of the squares of the eigenvalues of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

7. Find the sum of the eigenvalues of $2A$, if $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

8. Two eigenvalues of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find the third eigenvalue.

9. If the sum of two eigenvalues and trace of a 3×3 matrix A are equal, find the value of $|A|$.

10. Find the eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

11. Find the sum of the eigenvalues of the inverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$.

12. The product of two eigenvalues of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigenvalue.

Part B

13. Verify that the eigenvalues of A^{-1} are the reciprocals of those of A and that the respective eigenvectors are the same with respect to the matrix

$$A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}.$$

14. Show that the eigenvectors of the matrix $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ are $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Find the eigenvalues and eigenvectors of the following matrices:

15. $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ 16. $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ 17. $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

18. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ 19. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ 20. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ 22. $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ 23. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$

25. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

    What can you infer about the matrix A from the eigenvalues. Verify your answer.

26. Given that $A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$, verify that the sum and product of the eigenvalues of A are equal to the trace of A and $|A|$ respectively.

27. Verify that the eigenvalues of A^2 and A^{-1} are respectively the squares and reciprocals of the eigenvalues of A, given that $A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$.

28. Find the eigenvalues and eigenvectors of (adj A), when $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

29. Verify that the eigenvectors of the real symmetric matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$.

    are orthogonal in pairs.

30. Verify that the matrix $A = \dfrac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is orthogonal and that its eigenvalues are of unit modulus.

1.7 CAYLEY-HAMILTON THEOREM

This theorem is an interesting one that provides an alternative method for finding the inverse of a matrix A. Also any positive integral power of A can be expressed, using this theorem, as a linear combination of those of lower degree. We give below the statement of the theorem without proof:

1.7.1 Statement of the Theorem

Every square matrix satisfies its own characteristic equation.

This means that, if $c_0 \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n = 0$ is the characteristic equation of a square matrix A of order n, then

$$c_0 A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I = 0 \tag{1}$$

Note: ☑ When λ is replaced by A in the characteristic equation, the constant term c_n should be replaced by $c_n I$ to get the result of Cayley-Hamilton theorem, where I is the unit matrix of order n.

Also 0 in the R.S. of (1) is a null matrix of order n.

1.7.2 Corollary

(1) If A is non-singular, we can get A^{-1}, using the theorem, as follows:
Multiplying both sides of (1) by A^{-1} we have

$$c_0 A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I + c_n A^{-1} = 0$$

$$\therefore \qquad A^{-1} = -\frac{1}{c_n}\left(c_0 A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I\right).$$

(2) If we multiply both sides of (1) by A, $c_0 A^{n+1} + c_1 A^n + \cdots + c_{n-1} A^2 + c_n A = 0$

$$\therefore \qquad A^{n+1} = -\frac{1}{c_0}\left(c_1 A^n + c_2 A^{n-1} + \cdots + c_{n+1} A^2 + c_n A\right)$$

Thus higher positive integral powers of A can be computed, if we know powers of A of lower degree.

1.7.3 Similar Matrices

Two matrices A and B are said to be similar, if there exists a non-singular matrix P such that $B = P^{-1} AP$.

When A and B are connected by the relation $B = P^{-1} AP$, B is said to be obtained from A by a similarity transformation.

When B is obtained from A by a similarity transformation, A is also obtained from B by a similarity transformation as explained below:

$$B = P^{-1} AP$$

Premultiplying both sides by P and postmultiplying by P^{-1}, we get

$$PBP^{-1} = PP^{-1} APP^{-1}$$
$$= A$$

Thus $\qquad A = PBP^{-1}$

Now taking $P^{-1} = Q$, we get $A = Q^{-1} BQ$.

1.8 PROPERTY

Two similar matrices have the same eigenvalues.

Let A and B be two similar matrices.

Then, by definition, $B = P^{-1} AP$

$$\therefore \qquad B - \lambda I = P^{-1}AP - \lambda I$$
$$= P^{-1} AP - P^{-1} \lambda IP$$

$$= P^{-1}(A - \lambda I)P$$

$$\therefore \quad |B - \lambda I| = |P^{-1}|\,|A - \lambda I|\,|P|$$
$$= |A - \lambda I|\,|P^{-1}P|$$
$$= |A - \lambda I|\,|I|$$
$$= |A - \lambda I|$$

Thus A and B have the same characteristic polynomials and hence the same characteristic equations.

$\therefore$ A and B have the same eigenvalues.

1.8.1 Diagonalisation of a Matrix

The process of finding a matrix M such that $M^{-1}AM = D$, where D is a diagonal matrix, is called diagonalisation of the matrix A. As $M^{-1}AM = D$ is a similarity transformation, the matrices A and D are similar and hence A and D have the same eigenvalues.

The eigenvalues of D are its diagonal elements. Thus, if we can find a matrix M such that $M^{-1}AM = D$, D is not any arbitrary diagonal matrix, but it is a diagonal matrix whose diagonal elements are the eigenvalues of A.

The following theorem provides the method of finding M for a given square matrix whose eigenvectors are distinct and hence whose eigenvectors are linearly independent.

1.8.2 Theorem

If A is a square matrix with distinct eigenvalues and M is the matrix whose columns are the eigenvectors of A, then A can be diagonalised by the similarity transformation $M^{-1}AM = D$, where D is the diagonal matrix whose diagonal elements are the eigenvalues of A.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigenvalues of A and $X_1, X_2, \ldots, X_n$ be the corresponding eigenvectors.

Let $M = [X_1, X_2, \ldots, X_n]$, which is an $n \times n$ matrix, called the Modal matrix.

$\therefore$ $AM = [AX_1, AX_2, \ldots, AX_n]$ [*Note* ☑ Each AX_r is a $(n \times 1)$ column vector]

Since X_r is the eigenvector of A corresponding to the eigenvalue λ_r,

$$AX_r = \lambda X_r \quad (r = 1, 2, \ldots n)$$

$$\therefore \quad AM = [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n]$$

$$= [X_1, X_2, \ldots, X_n]\begin{bmatrix} \lambda_1 & 0 & 0 & -- & 0 \\ 0 & \lambda_2 & 0 & -- & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & -- & \lambda_n \end{bmatrix}$$

$$= MD \qquad (1)$$

As $X_1, X_2, \ldots, X_n$ are linearly independent column vectors, M is a non-singular matrix Premultiplying both sides of (1) by M^{-1}, we get $M^{-1}AM = M^{-1}MD = D$.

Note ☑ For this diagonalisation process, A need not necessarily have distinct eigenvalues. Even if two or more eigenvalues of A are equal, the process holds good, provided the eigenvectors of A are linearly independent.

1.9 CALCULATION OF POWERS OF A MATRIX *A*

Assuming A satisfies the conditions of the previous theorem,

$$D = M^{-1} A M$$

$$\therefore \quad A = M D M^{-1}$$

$$A^2 = (M D M^{-1})(M D M^{-1})$$

$$= MD(M^{-1} M)DM^{-1}$$

$$= MD^2M^{-1}$$

Similarly, $\quad A^3 = MD^3 M^{-1}$

Extending, $\quad A^k = M D^k M^{-1}$

$$= M \begin{bmatrix} \lambda_1^k & 0 & 0 & -- & 0 \\ 0 & \lambda_2^k & 0 & -- & 0 \\ - & - & - & -- & - \\ 0 & 0 & 0 & -- & \lambda_n^k \end{bmatrix} M^{-1}$$

1.10 DIAGONALISATION BY ORTHOGONAL TRANSFORMATION OR ORTHOGONAL REDUCTION

If A is a real symmetric matrix, then the eigenvectors of A will be not only linearly independent but also pairwise orthogonal. If we normalise each eigenvector X_r, i.e. divide each element of X_r by the square-root of the sum of the squares of all the elements of X_r and use the normalised eigenvectors of A to form the normalised modal matrix N, then it can be proved that N is an orthogonal matrix. By a property of orthogonal matrix, $N^{-1} = N^T$.

$\therefore$ The similarity transformation $M^{-1} A M = D$ takes the form $N^T AN = D$.

Transforming A into D by means of the transformation $N^T AN = D$ is known as orthogonal transformation or orthogonal reduction.

Note: ☑ Diagonalisation by orthogonal transformation is possible only for a real symmetric matrix.

WORKED EXAMPLE 1(c)

Example 1.1 Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and also use it to find A^{-1}.

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

i.e. $(1-\lambda)(\lambda^2-3\lambda-4)-3(4-4\lambda-3)+7(8-2+\lambda)=0$

i.e. $\lambda^3-4\lambda^2-20\lambda-35=0$

Cayley-Hamilton theorem states that

$$A^3-4A^2-20A-35I=0 \tag{1}$$

which is to be verified.

Now, $$A^2=\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}=\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}$$

$$A^3=A\cdot A^2=\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}=\begin{bmatrix}135&152&232\\140&163&208\\60&76&111\end{bmatrix}$$

Substituting these values in (1), we get,

$$\text{L.S.}=\begin{bmatrix}135&152&232\\140&163&208\\60&76&111\end{bmatrix}-\begin{bmatrix}80&92&92\\60&88&148\\40&36&56\end{bmatrix}-\begin{bmatrix}20&60&140\\80&40&60\\20&40&20\end{bmatrix}-\begin{bmatrix}35&0&0\\0&35&0\\0&0&35\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$=\text{R.S.}$$

Thus Cayley-Hamilton theorem is verified. Premultiplying (1) by A^{-1},

$$A^2-4A-20I-35A^{-1}=0$$

$$\therefore \qquad A^{-1}=\frac{1}{35}\left(A^2-4A-20I\right)$$

$$=\frac{1}{35}\left(\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}-\begin{bmatrix}4&12&28\\16&8&12\\4&8&4\end{bmatrix}-\begin{bmatrix}20&0&0\\0&20&0\\0&0&20\end{bmatrix}\right)$$

$$=\frac{1}{35}\begin{bmatrix}-4&11&-5\\-1&-6&25\\6&1&-10\end{bmatrix}$$

Example 1.2 Verify that the matrix $A=\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}$ satisfies its characteristic equation and hence find A^4.

The characteristic equation of A is

$$\begin{vmatrix}2-\lambda&-1&2\\-1&2-\lambda&-1\\1&-1&2-\lambda\end{vmatrix}=0$$

i.e. $(2-\lambda)(\lambda^2-4\lambda+3)+(\lambda-2+1)+2(1-2+\lambda)=0$

i.e. $\lambda^3-6\lambda^2+8\lambda-3=0$ (1)

According to Cayley-Hamilton theorem, A satisfies (1), i.e.

$$A^3-6A^2+8A-3I=0 \qquad (2)$$

which is to be verified.

Now $$A^2=\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}=\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}$$

$$A^3=A\cdot A^2=\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}=\begin{bmatrix}29&-28&38\\-22&23&-28\\22&-22&29\end{bmatrix}$$

Substituting these values in (2),

$$\text{L.S.}=\begin{bmatrix}29&-28&38\\-22&23&-28\\22&-22&29\end{bmatrix}-\begin{bmatrix}42&-36&54\\-30&36&-36\\30&-30&42\end{bmatrix}+\begin{bmatrix}16&-8&16\\-8&16&-8\\8&-8&16\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=\text{R.S.}$$

Thus A satisfies its characteristic equation.

Multiplying both sides of (2) by A, we have,

$$A^4-6A^3+8A^2-3A=0$$

$\therefore$ $$A^4=6A^3-8A^2+3A \qquad (3)$$

$$=6(6A^2-8A+3I)-8A^2+3A, \text{ using (2)}$$

$$=28A^2-45A+18I \qquad (4)$$

A^4 can be computed by using either (3) or (4).

From (4),

$$A^4=\begin{bmatrix}196&-168&252\\-140&168&-168\\140&-140&196\end{bmatrix}-\begin{bmatrix}90&-45&90\\-45&90&-45\\45&-45&90\end{bmatrix}+\begin{bmatrix}18&0&0\\0&18&0\\0&0&18\end{bmatrix}$$

$$=\begin{bmatrix}124&-123&162\\-95&96&-123\\95&-95&124\end{bmatrix}$$

Example 1.3 Use Cayley-Hamilton theorem to find the value of the matrix given by $(A^8-5A^7+7A^6-3A^5+8A^4-5A^3+8A^2-2A+I)$, if the matrix

$$A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}.$$

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

i.e. $(2-\lambda)(\lambda^2 - 3\lambda + 2) + \lambda - 1 = 0$

i.e. $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$\therefore$ $A^3 - 5A^2 + 7A - 3I = 0$, by Cayley-Hamilton theorem (1)

Now the given polynomial in A

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 8A - 2I) + I$

$= 0 + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$, by (1)

$= A^2 + A + I$, again using (1) (2)

Now $$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

Substituting in (2), the given polynomial

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 1.4 Find the eigenvalues of A and hence find A^n (n is a positive integer), given that $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

i.e. $\lambda^2 - 4\lambda - 5 = 0$

$\therefore$ The eigenvalues of A are $\lambda = -1, 5$

When λ^n is divided by $(\lambda^2 - 4\lambda - 5)$, let the quotient be $Q(\lambda)$ and the remainder be $(a\lambda + b)$.

Then $\lambda^n \equiv (\lambda^2 - 4\lambda - 5)\, Q(\lambda) + (a\lambda + b)$ (1)

Put $\lambda = -1$ in (1). $-a + b = (-1)^n$ (2)

Put $\lambda = 5$ in (l). $5a + b = 5^n$ (3)

Solving (2) and (3), we get

$$a = \frac{5^n - (-1)^n}{6} \qquad \text{and} \qquad b = \frac{5^n + 5(-1)^n}{6}$$

Replacing λ by the matrix A in (1), we have

$$\begin{aligned} A^n &= (A^2 - 4A - 5I)\,Q(A) + aA + bI \\ &= 0 \times Q(A) + aA + bI \text{ (by Cayley-Hamilton theorem)} \\ &= \left\{\frac{5^n - (-1)^n}{6}\right\}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \left\{\frac{5^n + 5(-1)^n}{6}\right\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

For example, when $n = 3$,

$$\begin{aligned} A^3 &= \left(\frac{125+1}{6}\right)\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \left(\frac{125-5}{6}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 42 \\ 84 & 63 \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \\ &= \begin{bmatrix} 41 & 42 \\ 84 & 83 \end{bmatrix} \end{aligned}$$

Example 1.5 Diagonalise the matrix $A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ by similarity transformation and hence find A^4.

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 2 & -7 \\ 2 & 1-\lambda & 2 \\ 0 & 1 & -3-\lambda \end{vmatrix} = 0$$

i.e. $(2 - \lambda)(\lambda^2 + 2\lambda - 5) - 2(-6 - 2\lambda + 7) = 0$

i.e. $\lambda^3 - 13\lambda + 12 = 0$

i.e. $(\lambda - 1)(\lambda - 3)(\lambda + 4) = 0$

$\therefore$ Eigenvalues of A are $\lambda = 1, 3, -4$.

Case (i) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 1 & 2 & -7 \\ 2 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \quad \frac{x_1}{-2} = \frac{x_2}{8} = \frac{x_3}{2}$$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} -1 & 2 & -7 \\ 2 & -2 & 2 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore \qquad \dfrac{x_1}{10} = \dfrac{x_2}{12} = \dfrac{x_3}{2}$

$\therefore \qquad X_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$

Case (iii) $\lambda = -4$.

The eigenvector is given by $\begin{bmatrix} 6 & 2 & -7 \\ 2 & 5 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore \qquad \dfrac{x_1}{3} = \dfrac{x_2}{-2} = \dfrac{x_3}{2}$

$\therefore \qquad X_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

Hence the modal matrix is $M = \begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$

Let $\qquad M \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then the co-factors are given by

$$A_{11} = 14, \quad A_{12} = 10, \quad A_{13} = 2, \quad A_{21} = -7, \quad A_{22} = 5, \quad A_{23} = -6,$$
$$A_{31} = -28, \quad A_{32} = -10, \quad A_{33} = 26.$$

and $\qquad |M| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 70.$

$\therefore \qquad M^{-1} = \dfrac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}$

The required similarity transformation is

$$M^{-1} A\, M = D(1, 3, -4) \tag{1}$$

which is verified as follows:

$$AM = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 15 & -12 \\ -4 & 18 & 8 \\ -1 & 3 & -8 \end{bmatrix}$$

$$M^{-1}\,A\,M = \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}\begin{bmatrix} 1 & 15 & -12 \\ -4 & 18 & 8 \\ -1 & 3 & -8 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 70 & 0 & 0 \\ 0 & 210 & 0 \\ 0 & 0 & -280 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

A^4 is given by $$A^4 = M\,D^4\,M^{-1} \tag{2}$$

$$D^4\;M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix} \times \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 810 & 405 & -810 \\ 512 & -1536 & 6656 \end{bmatrix}$$

$$M\,D^4\;M^{-1} = \begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix} \times \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 810 & 405 & -810 \\ 512 & -1536 & 6656 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 5600 & -2590 & 15890 \\ 3780 & 5530 & -18060 \\ 1820 & -2660 & 12530 \end{bmatrix}$$

i.e. $$A^4 = \begin{bmatrix} 80 & -37 & 227 \\ 54 & 79 & -258 \\ 26 & -38 & 179 \end{bmatrix}$$

Example 1.6 Find the matrix M that diagonalises the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ by means of a similarity transformation. Verify your answer. The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(\lambda^2 - 5\lambda + 4) - 2(1-\lambda) + (\lambda - 1) = 0$$

i.e. $$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

i.e. $$(\lambda - 1)^2 (\lambda - 5) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 5, 1, 1$.

Case (i) $\lambda = 5$.

The eigenvector is given by $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

All the three equations are one and the same, namely, $x_1 + 2x_2 + x_3 = 0$

Two independent solutions are obtained as follows:

Putting $x_2 = -1$ and $x_3 = 0$, we get $x_1 = 2$

Putting $x_2 = 0$ and $x_3 = -1$, we get $x_1 = 1$

$\therefore$ $$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Hence the modal matrix is

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the co-factors are given by

$A_{11} = 1,\quad A_{12} = 1,\quad A_{13} = 1,\quad A_{21} = 2,\quad A_{22} = -2,\quad A_{23} = 2$

$A_{31} = 1,\quad A_{32} = 1,\quad A_{33} = -3$ and

$$|M| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 4$$

$$\therefore \qquad M^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

The required similarity transformation is

$$M^{-1}\, A\, M = D\left(5, 1, 1\right) \tag{1}$$

We shall now verify (1).

$$AM = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 1 \\ 5 & -2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$M^{-1}\, A\, M = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 2 & 1 \\ 5 & -2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= D\left(5, 1, 1\right).$$

Example 1.7 Diagonalise the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of an orthogonal transformation. The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$$

i.e. $$\left(2-\lambda\right)\left(\lambda^2 - 2\lambda - 3\right) - \left(-\lambda - 1\right) - \left(-\lambda - 1\right) = 0$$

i.e. $$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

i.e. $$\left(\lambda + 1\right)\left(\lambda - 1\right)\left(\lambda - 4\right) = 0$$

$\therefore$ The eigenvalues of A are 1 = –1, 1, 4.

Case (i) $\lambda = -1$.

The eigenvector is given by $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$\therefore \qquad X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{-4} = \frac{x_2}{2} = \frac{x_3}{-2}$$

$$\therefore \qquad X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = 4$.

The eigenvector is given by $\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Hence the modal matrix $M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Normalising each column vector of *M*, i.e. dividing each element of the first column by $\sqrt{2}$, that of the second column by $\sqrt{6}$ and that of the third column by $\sqrt{3}$, we get the normalised modal matrix *N*.

Thus
$$N = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

The required orthogonal transformation that diagonalises A is

$$N^T A\, N = D\,(-1, 1, 4) \tag{1}$$

which is verified below:

$$A\,N = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}$$

$$N^T\ A\,N = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= D\left(-1, 1, 4\right).$$

Example 1.8 Diagonalise the matrix $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by means of an orthogonal transformation.

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(6-\lambda)(2-\lambda) - 16(6-\lambda) = 0$$

i.e. $$(6-\lambda)(\lambda^2 - 4\lambda - 12) = 0$$

i.e. $$(6-\lambda)(\lambda-6)(\lambda+2) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = -2, 6, 6$.

Case (i) $\lambda = -2$.

The eigenvector is given by $\begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{32} = \frac{x_2}{0} = \frac{x_3}{-32}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 6$.

The eigenvector is given by $\begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

We get only one equation,

i.e. $$x_1 - x_3 = 0 \qquad (1)$$

From this we get, $x_1 = x_3$ and x_2 is arbitrary.

x_2 must be so chosen that X_2 and X_3 are orthogonal among themselves and also each is orthogonal with X_1.

Let us choose X_2 arbitrarily as $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Note ☑ This assumption of X_2 satisfies (1) and x_2 is taken as 0.

Let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

X_3 is orthogonal to X_1

$\therefore$ $$a - c = 0 \qquad (2)$$

X_3 is orthogonal to X_2

$\therefore$ $$a + c = 0 \qquad (3)$$

Solving (2) and (3), we get $a = c = 0$ and b is arbitrary.

Taking $b = 1, X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Note ☑ Had we assumed X_2 in a different form, we should have got a different X_3.

For example, if $X_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, then $X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

The modal matrix is $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

The normalised model matrix is

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

The required orthogonal transformation that diagonalises A is

$$N^T AN = D(-2, 6, 6) \tag{1}$$

which is verified below:

$$AN = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{bmatrix}$$

$$N^T AN = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= D(-2, 6, 6)$$

Note ☑ From the above problem, it is clear that diagonalisation of a real symmetric matrix is possible by orthogonal transformation, even if two or more eigenvalues are equal.

EXERCISE 1(c)

Part A

(Short Answer Questions)

1. State Cayley-Hamilton theorem.
2. Give two uses of Cayley-Hamilton theorem.
3. When are two matrices said to be similar? Give a property of similar matrices.
4. What do you mean by diagonalising a matrix?
5. Explain how you will find A^k, using the similarity transformation $M^{-1} AM = D$.
6. What is the difference between diagonalisation of a matrix by similarity and orthogonal transformations?
7. What type of matrices can be diagonalised using (i) similarity transformation and (ii) orthogonal transformation?
8. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.
9. Use Cayley-Hamilton theorem to find the inverse of $A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$.
10. Use Cayley-Hamilton theorem to find A^3, given that $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$.

11. Use Cayley-Hamilton theorem to find $(A^4 - 4A^3 - 5A^2 + A + 2I)$, when $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

12. Find the modal matrix that will diagonalise the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.

Part B

13. Show that the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies its own characteristic equation and hence find A^{-1}.

14. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ and hence find A^{-1}

15. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ and hence find A^{-1}.

16. Verify that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ satisfies its own characteristic equation and hence find A^4.

17. Verify that the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ satisfies its a own characteristic equation and hence find A^4.

18. Find A^n, using Cayley-Hamilton theorem, when $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$. Hence find A^4.

19. Find A^n, using Cayley-Hamilton theorem, when $A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$. Hence find A^3.

20. Given that $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, compute the value of $(A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 + 31A - 36I)$, using Caylay-Hamilton theorem.

Diagonalise the following matrices by similarity transformation:

21. $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$; find also the fourth power of this matrix.

23. $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ 24. $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ 25. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

26. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Diagonalise the following matrices by orthogonal transformation:

27. $\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$ 28. $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ 29. $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

30. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

1.11 QUADRATIC FORMS

A homogeneous polynomial of the second degree in any number of variables is called a quadratic form.

For example, $x_1^2 + 2x_2^2 - 3x_3^2 + 5x_1x_2 - 6x_1x_3 + 4x_2x_3$ is a quadratic form in three variables.

The general form of a quadratic form, denoted by Q in n variables is

$$\begin{aligned} Q = {} & c_{11}x_1^2 + c_{12}x_1x_2 + \cdots + c_{1n}x_1x_n \\ & + c_{21}x_2x_1 + c_{22}x_2^2 + \cdots + c_{2n}x_2x_n \\ & + c_{31}x_3x_1 + c_{32}x_3x_2 + \cdots + c_{3n}x_3x_n \\ & + (\text{- - - - - - - - - - - - - - - -}) \\ & + c_{n1}x_nx_1 + c_{n2}x_nx_2 + \cdots + c_{nn}x_n^2 \end{aligned}$$

i.e.
$$Q = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_ix_j$$

In general, $c_{ij} \neq c_{ji}$. The coefficient of $x_i x_j = c_{ij} + c_{ji}$.

Now if we define $a_{ij} = \frac{1}{2}\left(c_{ij} + c_{ji}\right)$, for all i and j, then $a_{ii} = c_{ii}$, $a_{ij} = a_{ji}$ and $a_{ij} + a_{ji} = 2a_{ij} = c_{ij} + c_{ji}$.

$\therefore Q = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$, where $a_{ij} = a_{ji}$ and hence the matrix $A = [a_{ij}]$ is a symmetric matrix. In matrix notation, the quadratic form Q can be represented as $Q = X^T AX$, where

$$A = [a_{ij}],\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad X^T = [x_1, x_2, \dots, x_n].$$

The symmetric matrix $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ is called *the matrix of the quadratic form Q*.

Note ☑ To find the symmetric matrix A of a quadratic form, the coefficient of x_i^2 is placed in the a_{ii} position and $\left(\frac{1}{2} \times \text{coefficient } x_i\ x_j\right)$ is placed in each of the a_{ij} and a_{ji} positions.

For example, (i) if $Q = 2x_1^2 - 3x_1x_2 + 4x_2^2$, then

$$A = \begin{bmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & 4 \end{bmatrix}$$

(ii) if $Q = x_1^2 + 3x_2^2 + 6x_3^2 - 2x_1x_2 + 6x_1x_3 + 5x_2x_3$,

then $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 3 & \frac{5}{2} \\ 3 & \frac{5}{2} & 6 \end{bmatrix}$

Conversely, the quadratic form whose matrix is

$$\begin{bmatrix} 3 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 6 \\ 0 & 6 & -7 \end{bmatrix} \text{ is } Q = 3x_1^2 - 7x_3^2 + x_1x_2 + 12x_2x_3$$

1.11.1 Definitions

If A is the matrix of a quadratic form Q, $|A|$ is called *the determinant* or *modulus of Q.*

The rank r of the matrix A is called *the rank of the quadratic form.*

If $r < n$ (the order of A) or $|A| = 0$ or A is singular, the quadratic form is called *singular*. Otherwise it is non-singular.

1.11.2 Linear Transformation of a Quadratic Form

Let $Q = X^T AX$ be a quadratic form in the n variables $x_1, x_2, \ldots, x_n$.

Consider the transformation $X = PY$, that transforms the variable set $X = [x_1, x_2, \ldots, x_n]^T$ to a new variable set $Y = [y_1, y_2, \ldots, y_n]^T$, where P is a non-singular matrix.

We can easily verify that the transformation $X = PY$ expresses each of the variables $x_1, x_2, \ldots, x_n$ as homogeneous linear expressions in $y_1, y_2, \ldots, y_n$. Hence $X = PY$ is called a non-singular linear trans formation.

By this transformation, $Q = X^T AX$ is transformed to

$$\begin{aligned} Q &= (PY)^T A(PY) \\ &= Y^T \left(P^T AP\right)Y \\ &= Y^T BY, \text{ where } B = P^T AP \end{aligned}$$

Now
$$\begin{aligned} B^T &= \left(P^T AP\right)^T = P^T A^T P \\ &= P^T AP \qquad (\because A \text{ is sysmmetric}) \\ &= B \end{aligned}$$

$\therefore$ B is also a symmetric matrix.

Hence B is the matrix of the quadratic form $Y^T BY$ in the variables $y_1, y_2, \ldots, y_n$. Thus $Y^T B\ Y$ is the linear transform of the quadratic form $X^T AX$ under the linear transformation $X = PY$, where $B = P^T AP$.

1.11.3 Canonical Form of a Quadratic Form

In the linear transformation $X = PY$, if P is chosen such that $B = P^T\ A\ P$ is a diagonal

matrix of the form $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, then the quadratic form Q gets reduced as

$$\begin{aligned} Q &= Y^T BY \\ &= [y_1, y_2, \ldots, y_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \end{aligned}$$

This form of Q is called *the sum of the squares form of Q* or *the canonical form of Q*.

1.11.4 Orthogonal Reduction of a Quadratic Form to the Canonical Form

If, in the transformation $X = PY$, P is an orthogonal matrix and if $X = PY$ transforms the quadratic form Q to the canonical form then Q is said to be reduced to the canonical form by an orthogonal transformation.

We recall that if A is a real symmetric matrix and N is the normalised modal matrix of A, then N is an orthogonal matrix such that $N^T AN = D$, where D is a diagonal matrix with the eigenvalues of A as diagonal elements.

Hence, to reduce a quadratic form $Q = X^T AX$ to the canonical form by an orthogonal transformation, we may use the linear transformation $X = NY$, where N is the normalised modal matrix of A. By this orthogonal transformation, Q gets transformed into $Y^T DY$, where D is the diagonal matrix with the eigenvalues of A as diagonal elements.

1.11.5 Nature of Quadratic Forms

When the quadratic form $X^T AX$ is reduced to the canonical form, it will contain only r terms, if the rank of A is r.

The terms in the canonical form may be positive, zero or negative.

The number of positive terms in the canonical form is called *the index* (p) of the quadratic form.

The excess of the number of positive terms over the number of negative terms in the canonical form i.e. $p - (r - p) = 2p - r$ is called the *signature(s)* of the quadratic form i.e. $s = 2p - r$.

The quadratic form $Q = X^T A X$ in n variables is said to be

(i) positive definite, if $r = n$ and $p = n$ or if all the eigenvalues of A are positive.

(ii) negative definite, if $r = n$ and $p = 0$ or if all the eigenvalues of A are negative.

(iii) positive semidefinite, if $r < n$ and $p = r$ or if all the eigenvalues of $A \geq 0$ and at least one eigenvalue is zero.

(iv) negative semidefinite, if $r < n$ and $p = 0$ or if all the eigenvalues of $A \leq 0$ and at least one eigenvalue is zero.

(v) indefinite in all other cases or if A has positive as well as negative eigenvalues.

WORKED EXAMPLE 1(d)

Example 1.1 Reduce the quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form by an orthogonal transformation. Also find the rank, index, signature and nature of the quadratic form.

$$\text{Matrix of the Q.F. is } A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

Refer to the worked example (7) in section 1(c).

The eigenvalues of A are $-1, 1, 4$.

The corresponding eigenvectors are $[0, 1, 1]^T$ $[2, -1, 1]^T$ and $[1, 1, -1]^T$ respectively.

The modal matrix $M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

The normalised modal matrix $N = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$

Hence $N^T AN = D\,(-1, 1, 4)$, where D is a diagonal matrix with $-1, 1, 4$ as the principal diagonal elements.

$\therefore$ The orthogonal transformation $X = NY$ will reduce the Q.F. to the canonical form $-y_1^2 + y_2^2 + 4y_3^2$

Rank of the Q.F. = 3.

Index = 2

Signature = 1

Q.F. is indefinite in nature, as the canonical form contains both positive and negative terms.

Example 1.2 Reduce the quadratic form $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to canonical form by orthogonal reduction. Find also the nature of the quadratic form.

Matrix of the Q.F. is $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$

Refer to worked example (8) in section 1(c).

The eigenvalues of A are $-2, 6, 6$.

The corresponding eigenvectors are $[1, 0, -1]^T$, $[1, 0, 1]^T$ and $[0, 1, 0]^T$ respectively.

Note ☑ Though two of the eigenvalues are equal, the eigenvectors have been so chosen that all the three eigenvectors are pairwise orthogonal.

The modal matrix $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

The normalised modal matrix is given by

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Hence $N^T AN$ = Diag (−2, 6, 6)

∴ The orthogonal transformation $X = NY$

i.e.
$$x_1 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$$
$$x_2 = y_2$$
$$x_3 = -\frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$$

will reduce the given Q.F. to the canonical form $-2y_1^2 + 6y_2^2 + 6y_3^2$.

The Q.F. is indefinite in nature, as the canonical form contains both positive and negative terms.

Example 1.3 Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive semidefinite. Give also a non-zero set of values (x_1, x_2, x_3) which makes this quadratic form zero.

Matrix of the Q.F. is $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$

i.e. $$(1-\lambda)\{(2-\lambda)(1-\lambda)-1\}-(1-\lambda)=0$$

i.e. $$(1-\lambda)(\lambda^2 - 3\lambda) = 0$$

∴ The eigenvalues of A are $\lambda = 0, 1, 3$.

When $\lambda = 0$, the elements of the eigenvector are given by $x_1 - x_2 = 0, - x_1 + 2x_2 + x_3 = 0$ and $x_2 + x_3 = 0$.

Solving these equations, $x_1 = 1, x_2 = 1, x_3 = -1$

∴ The eigenvector corresponding to $\lambda = 0$ is

$$[1, 1, -1]^T$$

When $\lambda = 1$, the elements of the eigenvector are given by $-x_2 = 0, - x_1 + x_2 + x_3 = 0$ and $x_2 = 0$.

Solving these equations, $x_1 = 1, x_2 = 0, x_3 = 1$.

∴ When $\lambda = 1$, the eigenvector is

$$[1, 0, 1]^T$$

When $\lambda = 3$, the elements of the eigenvector are given by $-2x_1 - x_2 = 0, - x_1 - x_2 + x_3 = 0$ and $x_2 - 2x_3 = 0$

Solving these equation, $x_1 = -1, x_2 = 2, x_3 = 1$.

∴ When $\lambda = 3$, the eigenvector is $[-1, 2, 1]^T$.

Now the modal matrix is $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

The normalised modal matrix is

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Hence $N^T AN$ = Diag (0, 1, 3)

$\therefore$ The orthogonal transformation $X = NY$.

i.e.

$$x_1 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3$$

$$x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_3$$

$$x_3 = -\frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$$

will reduce the given Q.F. to the canonical form $0 \cdot y_1^2 + y_2^2 + 3y_3^2 = y_2^2 + 3y_3^2$.

As the canonical form contains only two terms, both of which are positive, the Q.F. is positive semi-definite.

The canonical form of the Q.F. is zero, when $y_2 = 0$, $y_3 = 0$ and y_1 is arbitrary.

Taking $y_1 = \sqrt{3}$, $y_2 = 0$ and $y_3 = 0$, we get $x_1 = 1$, $x_2 = 1$ and $x_3 = -1$.

These values of x_1, x_2, x_3 make the Q.F. zero.

Example 1.4 Determine the nature of the following quadratic forms without reducing them to canonical forms:

(i) $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

(ii) $5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3\,x_1$

(iii) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1\,x_2 - 8x_2\,x_3 - 4x_3\,x_1$.

Note ☑ We can find the nature of a Q.F. without reducing it to canonical form. The alternative method uses the principal sub-determinants of the matrix of the Q.F., as explained below:

Let $A = (a_{ij})_{n \times n}$ be the matrix of the Q.F.

Let $D_1 = |a_{11}| = a_{11}$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$,

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ etc. and } D_n = |A|$$

$D_1, D_2, D_3, \ldots D_n$ are called the principal sub-determinants or principal minors of A.

(i) The Q.F. is positive definite, if $D_1, D_2, \ldots, D_n$ are all positive i.e. $D_n > 0$ for all n.
(ii) The Q.F. is negative definite, if $D_1, D_3, D_5, \ldots$ are all negative and $D_2, D_4, D_6, \ldots$ are all positive i.e. $(-1)^n D_n > 0$ for all n.
(iii) The Q.F. is positive semidefinite, if $D_n \geq 0$ and least one $D_i = 0$.
(iv) The Q.F. is negative semidefinite, if $(-1)^n\ D_n \geq 0$ and at least one $D_i = 0$.
(v) The Q.F. is indefinite in all other cases.

(i) $Q = x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

Matrix of the Q.F. is $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}$

Now $\quad D_1 = |1| = 1;\ D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2\ ;$

$$D_3 = 1 \cdot (18 - 1) - 1 \cdot (6 - 2) + 2(1 - 6) = 3.$$

D_1, D_2, D_3 are all positive.

$\therefore$ The Q.F. is positive definite.

(ii) $Q = 5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3x_1.$

$$A = \begin{bmatrix} 5 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 14 \end{bmatrix}$$

Now $D_1 = 5;\ D_2 = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 24;$

$$\begin{aligned} D_3 = |A| &= 5 \cdot (70 - 64) - 1 \cdot (14 - 32) - 4 \cdot (-8 + 20) \\ &= 30 + 18 - 48 = 0 \end{aligned}$$

D_1 and D_2 are > 0, but $D_3 = 0$

$\therefore$ The Q.F. is positive semidefinite.

(iii) $Q = 2x_1^2 + x_2^2 - 3x_3^2 + 12x_1\ x_2 - 8x_2\ x_3 - 4x_3\ x_1$

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

Now $D_1 = |2| = 2;\ D_2 = \begin{vmatrix} 2 & 6 \\ 6 & 1 \end{vmatrix} = -34;$

$$D_3 = |A| = 2\cdot(-3-16) - 6\cdot(-18-8) - 2(-24+2)$$
$$= -38 + 156 + 44 = 162$$

∴ The Q.F. is indefinite.

Example 1.5 Reduce the quadratic forms $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 4x_2x_3 + 18x_3x_1$ and $2x_1^2 + 5x_2^2 + 4x_1x_2 + 2x_3x_1$ simultaneously to canonical forms by a real non-singular transformation.

Note ☑ We can reduce two quadratic forms $X^T AX$ and $X^T BX$ to canonical forms simultaneously by the same linear transformation using the following theorem, (stated without proof):

If A and B are two symmetric matrices such that the roots of $|A - \lambda B| = 0$ are all distinct, then there exists a matrix P such that $P^T AP$ and $P^T BP$ are both diagonal matrices.

The procedure to reduce two quadratic forms simultaneously to canonical forms is given below:

(1) Let A and B be the matrices of the two given quadratic forms.
(2) Form the characteristic equation $|A - \lambda B| = 0$ and solve it. Let the eigenvalues (roots of this equation) be $\lambda_1, \lambda_2, \ldots, \lambda_n$.
(3) Find the eigenvectors X_i ($i = 1, 2, \ldots, n$) corresponding to the eigenvalues λ_i, using the equation $(A - \lambda_i B) X_i = 0$.
(4) Construct the matrix P whose column vectors are $X_1, X_2, \ldots, X_n$. Then $X = PY$ is the required linear transformation.
(5) Find $P^T AP$ and $P^T BP$, which will be diagonal matrices.
(6) The quadratic forms corresponding to these diagonal matrices are the required canonical forms.

The matrix of the first quadratic form is

$$A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$$

The matrix of the second quadratic form is

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic equation is $|A - \lambda B| = 0$

i.e.
$$\begin{vmatrix} 6-2\lambda & 2-2\lambda & 9-\lambda \\ 2-2\lambda & 3-5\lambda & 2 \\ 9-\lambda & 2 & 14 \end{vmatrix} = 0$$

Simplifying, $$5\lambda^3 - \lambda^2 - 5\lambda + 1 = 0$$
i.e. $$(\lambda - 1)(5\lambda - 1)(\lambda + 1) = 0$$

$$\therefore\ \lambda = -1, \frac{1}{5}, 1$$

When $\lambda = -1$, $(A - \lambda B)\,X = 0$ gives the equations.
$8x_1 + 4x_2 + 10x_3 = 0$; $4x_1 + 8x_2 + 2x_3 = 0$; $10x_1 + 2x_2 + 14x_3 = 0$.

Solving these equations, $\dfrac{x_1}{-72} = \dfrac{x_2}{24} = \dfrac{x_3}{48}$

$$\therefore \qquad X_1 = [-3, 1, 2]^T$$

When $\lambda = \dfrac{1}{5}, (A - \lambda B)\,X = 0$ gives the equations.

$28x_1 + 8x_2 + 44x_3 = 0$; $\quad 8x_1 + 10x_2 + 10x_3 = 0$; $\quad 44x_1 + 10x_2 + 70x_3 = 0$.

Solving these equations, $\dfrac{x_1}{-360} = \dfrac{x_2}{72} = \dfrac{x_3}{216}$

$$\therefore \qquad X_2 = [-5, 1, 3]^T$$

When $\lambda = 1, (A - \lambda B)\,X = 0$ gives the equations

$$4x_1 + 8x_3 = 0; \quad -2x_2 + 2x_3 = 0; \quad 8x_1 + 2x_2 + 14x_3 = 0$$

$$\therefore \qquad X_3 = [2, -1, -1]^T.$$

Now $P = [X_1, X_2, X_3] = \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

Now $$P^T AP = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ -1 & -1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the Q.F. $X^T AX$ is reduced to the canonical form $y_1^2 + y_2^2 + y_3^2$.

Now $$P^T B\,P = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & -3 \\ -5 & -5 & -5 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the Q.F. $X^T B\,X$ is reduced to the canonical form $-y_1^2 + 5y_2^2 + y_3^2$.
Thus the transformation $X = PY$ reduces both the Q.F.'s to canonical forms.

Note ☑ $X = PY$ is not an orthogonal transformation, but only a linear non-singular transformation.

EXERCISE 1(d)

Part A

(Short answer questions)

1. Define a quadratic form and give an example for the same in three variables:
2. Write down the matrix of the quadratic form $3x_1^2 + 5x_2^2 + 5x_3^2 - 2x_1x_2 + 2x_2x_3 + 6x_3x_1$.
3. Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$.
4. When is a Q.F. said to be singular? What is its rank then?
5. If the Q.F. $X^T AX$ gets transformed to $Y^T BY$ under the transformation $X = PY$, prove that B is a symmetric matrix.
6. What do you mean by canonical form of a quadratic form? State the condition for $X = PY$ to reduce the Q.F. $X^T AX$ into the canonical form.
7. How will you find an orthogonal transformation to reduce a Q.F. $X^T AX$ to the canonical form?
8. Define index and signature of a quadratic form.
9. Find the index and signature of the Q.F. $x_1^2 + 2x_2^2 - 3x_3^2$.
10. State the conditions for a Q.F. to be positive definite and positive semidefinite.

Part B

11. Reduce the quadratic form $2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1x_2$ to canonical form by an orthogonal transformation. Also find the rank, index and signature of the Q.F.
12. Reduce the Q.F. $3x_1^2 - 3x_2^2 - 5x_3^2 - 2x_1x_2 - 6x_2x_3 - 6x_3x_1$ to canonical form by an orthogonal transformation. Also find the rank, index and signature of the Q.F.
13. Reduce the Q.F. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ to canonical form by an orthogonal transformation. Also state its nature.
14. Obtain an orthogonal transformation which will transform the quadratic form $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ into sum of squares form and find also the reduced form.
15. Find an orthogonal transformation which will reduce the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ into the canonical form and hence find its nature.
16. Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ to the canonical form through an orthogonal transformation and hence show that it is positive definite. Find also a non-zero set of values for x_1, x_2, x_3 that will make the Q.F. zero.
17. Reduce the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ to a canonical form by orthogonal reduction. Find also a set of non-zero values of x_1, x_2, x_3, which will make the Q.F. zero.
18. Reduce the quadratic form $5x_1^2 + 26x_2^2 + 10x_3^2 + 6x_1x_2 + 4x_2x_3 + 14x_3x_1$ to a canonical form by orthogonal reduction. Find also a set of non-zero values of x_1, x_2, x_3, which will make the Q.F. zero.

19. Determine the nature of the following quadratic forms without reducing them to canonical forms:

    (i) $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_2x_3 + 18x_1x_3 + 4x_1x_2$

    (ii) $x_1^2 - 2x_1x_2 + x_2^2 + x_3^2$

    (iii) $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_3x_1$

20. Find the value of λ so that the quadratic form $\lambda(x_1^2 + x_2^2 + x_3^2) + 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ may be positive definite.
21. Find real non-singular transformations that reduce the following pairs of quadratic forms simultaneously to the canonical forms.

    (i) $6x_1^2 + 2x_2^2 + 3x_3^2 - 4x_1x_2 + 8x_3x_1$ and $5x_1^2 + x_2^2 + 5x_3^2 - 2x_1x_2 + 8x_3x_1$.

    (ii) $3x_1^2 + 3x_2^2 - 3x_3^2 + 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ and $4x_1x_2 + 2x_2x_3 - 2x_3x_1$.

    (iii) $2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_2x_3 - 4x_3x_1$ and $2x_2x_3 - 2x_1x_2 - x_2^2$.

    (iv) $3x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_2 - 4x_2x_3$ and $5x_1^2 + 5x_2^2 + x_3^2 - 8x_1x_2 - 2x_2x_3$.

ANSWERS

Exercise 1(a)

Part A

(6) $X_1 = -\frac{1}{2}X_2 + \frac{3}{2}X_3$
(8) $a = 8$.
(12) $x + 2y = 3$ and $2x - y = 1$; $x + 2y = 3$ and $2x + 4y = 5$.
(13) $x + 2y = 3$ and $2x + 4y = 6$.
(14) $a = -4$, $b = 6$.
(15) Have a unique solution.
(16) $\lambda \neq 5$.
(17) No unique solution for any value of λ.
(18) $\lambda \neq -1$ and μ = any value.
(19) $\lambda = 2$ and $\mu = 3$.
(20) $\lambda = 8$ and $\mu \neq 11$.
(21) No, as $|A| \neq 0$.
(22) $\lambda = 3$
(23) $x = k, y = 2k, z = 5k$.

Part B

(24) $-7X_1 + X_2 + X_3 + X_4 = 0$;
(25) $2X_1 - X_2 - X_3 + X_4 = 0$;
(26) $2X_1 + X_2 - X_3 = 0$;
(27) $X_1 - 2X_2 + X_3 = 0$;
(28) $X_1 - X_2 + X_3 - X_4 = 0$;
(29) Yes. $X_5 = 2X_1 + X_2 - 3X_3 + 0.X_4$.
(34) $R(A) = R[A, B] = 2$; Consistent with many solutions.

(35) $R(A) = 3$ and $R[A, B] = 4$; Inconsistent.
(36) $R(A) = 3$ and $R[A, B] = 4$; Inconsistent.
(37) $R(A) = 3$ and $R[A, B] = 4$; Inconsistent.
(38) Consistent; $x = -1, y = 1, z = 2$. (39) Consistent; $x = 3, y = 5, z = 6$.
(40) Consistent; $x = 1, y = 1, z = 1$. (41) Consistent; $x = 2, y = 1, z = -4$.
(42) Consistent; $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$.
(43) Consistent; $x = 2,\ y = \frac{1}{5},\ z = 0,\ w = \frac{4}{5}$.
(44) Consistent; $x = 2k - 1, y = 3 - 2k, z = k$.
(45) Consistent; $x = \frac{7-16k}{11},\ y = \frac{k+3}{11},\ z = k$.
(46) Consistent; $x = \frac{16}{3} - \frac{9}{5}k,\ y = \frac{16}{3} - \frac{6}{5}k,\ z = k$.
(47) Consistent; $x = 3 - 4k - k', y = 1 - 2k + k', z = k, w = k'$.
(48) Consistent; $x_1 = -2k + 5k' + 7, x_2 = k, x3 = -2k' - 2, x_4 = k'$.
(49) $k = 1, 2$: When $k = 1, x = 2\lambda + 1, y = -3\lambda, z = \lambda$.
When $k = 2, x = 2\mu, y = 1 - 3\mu, z = \mu$.
(50) $\lambda = 1, 8$: When $\lambda = 1, x = k + 2, y = 1 - 3k, z = 5k$.
When $\lambda = 8,\ x = \frac{1}{5}(k+52),\quad y = -\frac{1}{5}(3k+16),\ z = k$.

(51) a $+ 2b - c = 0$.
(52) No solution, when $k = 1$; one solution, when $k \neq 1$ and -2; Many solutions, when $k = -2$.
(53) No solution when $\lambda = 8$; and $\mu \neq 6$; unique solution, when $\lambda \neq 8$ and $\mu =$ any value; many solutions when $\lambda = 8$ and $\mu = 6$.
(54) If $a = 8$, $b \neq 11$ no solution, ; If $a \neq 8$ and $b =$ any value, unique solution; If $a = 8$ and $b = 11$, many solutions.
(55) $x = k, y = -2k, z = 3k$. (56) $x = -4k, y = 2k, z = -2k, w = k$.
(57) $\lambda = 1, -9$; When $\lambda = 1, x = k, y = -k, z = 2k$ and when $\lambda = -9, x = 3k, y = 9k, z = -2k$.
(58) $\lambda = 0, 1, 2$; When $\lambda = 0$, solution is (k, k, k); When $\lambda = 1$, solution is $(k, -k, 2k)$; When $\lambda = 2$, solution is $(2k, k, 2k)$.

Exercise 1(b)

(3) 2, 50. (5) $-2, -1$.
(6) 38. (7) 36.
(8) 5. (9) 0.
(10) $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$. (11) $\frac{47}{60}$.
(12) 2.

(15) $1, 3 -4; (-2, 1, 4)^T, (2, 1, -2)^T, (1, -3, 13)^T$
(16) $1, \sqrt{5}, -\sqrt{5}; (1, 0, -1)^T, (\sqrt{5} - 1, 1, -1)^T, (\sqrt{5} + 1,\ -1, 1)^T$.
(17) $1, 3, -4; (-1, 4, 1)^T, (5, 6, 1)^T, (3, -2, 2)^T$

(18) 5, −3, −3; $(1, 2, -1)^T$, $(2, -1, 0)^T$, $(3, 0, 1)^T$
(19) 5, 1, 1, ; $(1, 1, 1)^T$, $(2, -1, 0)^T$, $(1, 0, -1)^T$
(20) 8, 2, 2; $(2, -1, 1)^T$, $(1, 2, 0)^T$, $(1, 0, -2)^T$
(21) 3,.2, 2; $(1, 1, -2)^T$, $(5, 2, -5)^T$
(22) −2, 2, 2 ; $(4, 1, -7)^T$, $(0, 1, 1)^T$
(23) 2, 2, 2; $(1, 0, 0)^T$.
(24) 1, 1, 6, 6; $(0, 0, 1, 2)^T$, $(1, -2, 0, 0)^T$, $(0, 0, 2, -1)^T$ and $(2, 1, 0, 0)^T$
(25) 0, 3, 15; $(1, 2, 2)^T$, $(2, 1, -2)^T$, $(2, -2, 1)^T$; A is singular
(26) Eigenvalues are 5, −10, −20; Trace = −25; $|A|$ = 1000
(28) 1, 4, 4; $(1, -1, 1)^T$, $(2, -1, 0)^T$, $(1, 0, -1)^T$
(29) −1, 1, 4; $(0, 1, 1)^T$, $(2, -1, 1)^T$, $(1, 1, -1)^T$

Exercise 1(c)

(9) $\dfrac{1}{36}\begin{bmatrix} 6 & -3 \\ -2 & 7 \end{bmatrix}$

(10) $\begin{bmatrix} -19 & 57 \\ 38 & 76 \end{bmatrix}$

(11) $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

(12) $M = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$

(13) $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(14) $\dfrac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

(15) $-\dfrac{1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$

(16) $\begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$

(17) $\begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$

(18) $A^n = \left(\dfrac{6^n - 2^n}{4}\right)\cdot\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} + \left(\dfrac{3.2^n - 6^n}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix}$

(19) $A^n = \left(\dfrac{9^n - 4^n}{5}\right)\begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} + \left(\dfrac{9.4^n - 4.9^n}{5}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 463 & 399 \\ 266 & 330 \end{bmatrix}$

(20) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(21) $D(1, 3, -4);\quad M = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & -3 \\ -4 & -2 & 13 \end{bmatrix}$

(22) $D(1, 2, 3);\quad M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix};\quad A^4 = \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & 160 & 81 \end{bmatrix}$

(23) $D(2, 3, 6)$; $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$

(24) $D(4, -2, -2)$; $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

(25) $D(8, 2, 2)$; $M = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

(26) $D(2, -1, -1)$; $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

(27) $D(0, 3, 14)$; $N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}$

(28) $D(1, 3, 4)$; $N = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

(29) $D(4, 1, 1)$; $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

(30) $D(5, -3, -3)$; $N = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$

Exercise 1(d)

(2) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 5 \end{bmatrix}$

(3) $2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_2x_3 - 4x_3x_1$.

(4) Singular, when $|A| = 0$; Rank $r < n$.

(6) $P^T AP$ must be a diagonal matrix.

(9) index = 2 and signature = 1.

(11) $N = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}; Q = y_1^2 + 3y_2^2 + 6y_3^2; r = 3;\ p = 3; s = 3$

(12) $N = \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{35}} & \frac{1}{\sqrt{14}} \\ 0 & -\frac{5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}} & \frac{3}{\sqrt{14}} \end{bmatrix}; Q = 4y_1^2 - y_2^2 - 8y_3^2; r = 3;\ p = 1; s = -1$

(13) $N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}; Q = 8y_1^2 + 2y_2^2 + 2y_3^2;$ Q is positive definite

(14) $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}; Q = 4y_1^2 + y_2^2 + y_3^2$

(15) $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}; Q = 2y_1^2 - y_2^2 - y_3^2;$ Q is indefinite

(16) $N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}; Q = 3y_2^2 + 15y_3^2;\ x_1 = 1,\ x_2 = 2,\ x_3 = 2$

(17) $N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}; Q = 3y_2^2 + 14y_3^2;\ x_1 = 1,\ x_2 = -5,\ x_3 = 4.$

(18) $N = \begin{bmatrix} \frac{16}{\sqrt{378}} & -\frac{2}{\sqrt{14}} & \frac{1}{\sqrt{27}} \\ -\frac{1}{\sqrt{378}} & \frac{1}{\sqrt{14}} & \frac{5}{\sqrt{27}} \\ -\frac{11}{\sqrt{378}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{27}} \end{bmatrix}; Q = 14y_2^2 + 27y_3^2;\ x_1 = 16,\ x_2 = -1,\ x_3 = -11.$

(19) (i) positive definite; (ii) positive semidefinite; (iii) indefinite.

(20) $\lambda > 2$.

(21) (i) $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}; Q_1 = -y_1^2 + 4y_2^2 + 2y_3^2; Q_2 = y_1^2 + 4y_2^2 + y_3^2$

(ii) $P = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -2 & 0 & 0 \end{bmatrix}; Q_1 = -16y_1^2 + 4y_2^2 + 8y_3^2; Q_2 = 4y_1^2 - 4y_2^2 + 4y_3^2$

(iii) $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; Q_1 = y_1^2 + y_2^2 + y_3^2; Q_2 = y_2^2 - y_3^2.$

(iv) $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}; Q_1 = 2y_1^2 + 4y_2^2 - y_3^2; Q_2 = y_1^2 + 4y_2^2 + y_3^2.$

Chapter 2

Sequences and Series

2.1 DEFINITIONS

If $u_1, u_2, u_3, \ldots u_n \ldots$ be an ordered set of quantities formed according to a certain law (called a *sequence*), then $u_1 + u_2 + u_3 + \ldots u_n + \ldots$ is called a *series*. If the number of terms in a series is limited, then it is called a *finite series*. If the series consists of an infinite number of terms, then it is called an *infinite series*.

For example

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \text{ to } \infty \text{ and}$$

$$\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \ldots \text{ to } \infty$$

are infinite series.

The terms of an infinite series may be constants or variables. The infinite series $u_1 + u_2 + \cdots + u_n + \cdots$ to ∞ is denoted by $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$. The sum of its first n terms, namely, $(u_1 + u_2 + \cdots + u_n)$ is called the n^{th} partial sum and is denoted by s_n.

If s_n tends to a finite limit s as n tends to infinity, then the series $\sum u_n$ is said to be *convergent* and s is called the sum to infinity (or simply the sum) of the series. If $s_n \to \pm \infty$ as $n \to \infty$, then the series $\sum u_n$ is said to be *divergent*.

If s_n neither tends to a finite limit nor to $\pm \infty$ as $n \to \infty$, then the series $\sum u_n$ is said to be *oscillatory*. When $\sum u_n$ oscillates, s_n may tend to more than one limit as $n \to \infty$.

To understand the ideas of convergence, divergence and oscillation of infinite series, let us consider the familiar geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \text{ to } \infty \tag{1}$$

For the geometric series, s_n is given by.

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$= \frac{a(1 - r^n)}{1 - r} \quad \text{or} \quad \frac{a(r^n - 1)}{r - 1}$$

Case (i) Let $|r| < 1$ or $-1 < r < 1$.

$$s_n = \frac{a}{1-r} - \frac{a}{1-r} \cdot r^n$$

$$\therefore \quad \lim_{n\to\infty} (s_n) = \frac{a}{1-r} - \frac{a}{1-r} \cdot \lim_{n\to\infty} (r^n)$$

$$= \frac{a}{1-r} - \frac{a}{1-r} \times 0, \text{ since } |r| < 1$$

$$= \frac{a}{1-r} = a \text{ finite quantity.}$$

$\therefore$ The geometric series (1) converges and its sum is $\dfrac{a}{1-r}$.

Case (ii) Let $r > 1$.

$$s_n = \frac{a(r^n - 1)}{r-1} = \frac{a}{r-1} . r^n - \frac{a}{r-1}$$

$$\therefore \quad \lim_{n\to\infty} (s_n) = \frac{a}{r-1} \times \lim_{n\to\infty} (r^n) - \frac{a}{r-1}$$

$$= \frac{a}{r-1} \times \infty$$

$= \pm\,\infty$, according as a is positive or negative.

$\therefore$ Series (1) is divergent.

Case (iii) Let $r = 1$.

Then $\quad s_n = a + a + a + \cdots + a$ (n terms)

$= na$

$$\therefore \quad \lim_{n\to\infty} (s_n) = a \lim_{n\to\infty} (n)$$

$= \pm\,\infty$, according as a is positive or negative.

$\therefore$ Series (1) is divergent.

Case (iv) Let $r < -1$ and put $r = -k$

Then $k > 1$

$$s_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$= \frac{a}{1+k} - a\frac{(-k)^n}{1+k}$$

$$= \frac{a}{1+k} + a\frac{(-1)^{n+1} \cdot k^n}{1+k}$$

Now $\lim_{n\to\infty} (k^n) = \infty$, since $k > 1$

$\therefore$ $\lim_{n\to\infty} (s_n) = \infty$, if n is odd and $= -\infty$, if n is even.

i.e. s_n oscillates between $-\infty$ and $+\infty$.

$\therefore$ Series (1) is oscillatory, oscillating between $-\infty$ and ∞.

Case (v) Let $r = -1$

Then
$$s_n = a - a + a - a + \ldots \text{ to } n \text{ terms}$$
$$= a \text{ or } 0, \text{ according as } n \text{ is odd or even.}$$

$\therefore$ Series (1) oscillates between a and 0.

Thus the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ is convergent, if $|r| < 1$, divergent if $|r| \geq 1$ and oscillatory if $|r| \leq -1$.

2.2 GENERAL PROPERTIES OF SERIES

1. If a finite number of terms are added to or deleted from a series, the convergence or divergence or oscillation of the series is unchanged.
2. The convergence or divergence of an infinite series is not affected when each of its terms is multiplied by a finite quantity.
3. If the two series $\sum u_n$ and $\sum v_n$ are convergent to s and s', then $\sum (u_n + v_n)$ is also convergent and its sum is $(s + s')$.

Note ☑ Form the geometric series example, it is clear that, to find the convergence or divergence of a series, we have to find s_n and its limit. In many situations, it may not be possible to find s_n and hence the definition of convergence cannot be applied directly in such cases. Tests have been devised to determine whether a given series is convergent or not, without finding s_n. Some important tests of convergence of series of positive terms are described below without proof.

2.2.1 Necessary Condition for Convergence

If a series of positive terms $\sum u_n$ is convergent, then $\lim_{n\to\infty} (u_n) = 0$.

Since $\sum u_n$ is convergent, $\lim_{n\to\infty} (s_n) = s$, where $s_n = u_1 + u_2 + \cdots + u_n$.

Now
$$\lim_{n\to\infty} (s_{n-1}) = \lim_{n\to\infty} (u_1 + u_2 + \cdots + u_{n-1})$$
$$= \lim_{m\to\infty} (u_1 + u_2 + \cdots + u_m), \text{ putting } m = n - 1$$
$$= \lim_{m\to\infty} s_m$$
$$= s$$

$$\therefore \quad \lim_{n\to\infty}(u_n) = \lim_{n\to\infty}(s_n - s_{n-1})$$
$$= \lim_{n\to\infty}(s_n) - \lim_{n\to\infty}(s_{n-1})$$
$$= s - s$$
$$= 0$$

Note ☑ The condition is only necessary but not sufficient, i.e. $\lim_{n\to\infty}(u_n) = 0$ does not imply that $\sum u_n$ is convergent.

For example, if $u_n = \frac{1}{n}$, then $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$, but $\sum u_n$ is known to be divergent.

2.2.2 A Simple Test for Divergence

If $\lim_{n\to\infty}(u_n) \neq 0$, then $\sum u_n$ is not convergent. Since a series of positive terms either converges or diverges, we conclude that $\sum u_n$ is divergent, when $\lim_{n\to\infty}(u_n) \neq 0$.

2.2.3 Simplified Notation

When a series is convergent, it is written as **Series is (C)**.
When a series is divergent, it is written as **series is (D)**.

2.2.4 Comparison Test (Form I)

1. If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $u_n \leq v_n$ for all n $(= 1, 2, 3, \ldots)$ and if $\sum v_n$ is (C), then $\sum u_n$ is also (C).
2. If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $u_n \geq v_n$ for all n and if $\sum v_n$ (D), then $\sum u_n$ is also (D).

2.2.5 Comparison Test (Form II or Limit Form)

If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = l$, a finite number $\neq 0$, then $\sum u_n$ and $\sum v_n$ converge together or diverge together.

Note ☑ Using comparison test, we can test the convergence of $\sum u_n$, provided we know another series $\sum v_n$ (known as auxiliary series) whose convergence or divergence is known beforehand.

In most situations, one of the following series is chosen as the auxiliary series for the application of comparison test.

1. The geometric series $1 + r + r^2 + \cdots$, which is (C), when $|r| < 1$ and (D), when $r \geq 1$.

2. The factorial series $\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ which is (C) as discussed below.

3. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ which is (C), when $p > 1$ and is (D), when $p \le 1$.

2.2.6 Convergence of the Series $\sum_{n=1}^{\infty} \frac{1}{n!}$

Let $$\sum u_n = \sum \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \text{ to } \infty$$

Consider $$\sum v_n = \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 2} + \frac{1}{1\cdot 2\cdot 2\cdot 2} + \cdots \text{ to } \infty$$

We note that $u_1 = v_1$ and $u_2 = v_2$

Since $3! > 1\cdot 2\cdot 2$, $\frac{1}{3!} < \frac{1}{1\cdot 2\cdot 2}$, i.e., $u_3 < v_3$

Similarly $u_4 < v_4$ and so on.

Thus each term of $\sum u_n$ after the second is less then the corresponding term of $\sum v_n$.

But $\sum v_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ is a geometric series in which $|r| = \frac{1}{2} < 1$.

Hence $\sum v_n$ is (C).

$\therefore$ By the comparison test, $\sum u_n$ is also (C).

2.2.7 Convergence of the *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Let $$\sum u_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \infty$$

Case (i) Let $p > 1$.

$\sum u_n$ can be rewritten as

$$\sum u_n = \left(\frac{1}{1^p}\right) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots,$$

such that the n^{th} group contains 2^{n-1} terms of $\sum u_n$.

Consider the auxiliary series

$$\sum v_n = \left(\frac{1}{1^p}\right) + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \cdots$$

We note that $u_1 = v_1$, $u_2 = v_2$, $u_4 = v_4$, $u_8 = v_8$ and so on.
Since $p > 1$, $3^p > 2^p$

$\therefore$ $\quad \dfrac{1}{3^p} < \dfrac{1}{2^p}$, i.e. $u_3 < v_3$

Similarly $\dfrac{1}{5^p} < \dfrac{1}{4^p}$, i.e. $u_5 < v_5$, $u_6 < v_6$, $u_7 < v_7$ and so on.

Hence in the two series $\sum u_n$ and $\sum v_n$, $u_n \le v_n$ for all n.

$\therefore$ By comparison test, $\sum u_n$ is (C), provided $\sum v_n$ is (C).

Now

$$\begin{aligned}\sum v_n &= \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \cdots \\ &= \frac{1}{1^p} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots \\ &= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots\end{aligned}$$

This series is a geometric series with $r = \dfrac{1}{2^{p-1}}$

Since $\quad p > 1, p - 1 > 0$

$\therefore$ $\quad 2^{p-1} > 1$ and so $\dfrac{1}{2^{p-1}} < 1$

$\therefore$ $\quad \sum v_n$ is (C).

Hence $\sum u_n$ is also (C).

Case (ii) $p = 1$

Now $\sum u_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \text{to } \infty$ (This series is called the *harmonic series*).

$\sum u_n$ can be rewritten as

$$\sum u_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots \text{ to } \infty$$

Consider the auxiliary series

$$\sum v_n = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \text{ to } \infty$$

We note that $u_1 = v_1$, $u_2 = v_2$, $u_4 = v_4$, $u_8 = v_8$, $u_{16} = v_{16}$ and so on.

Also since $\frac{1}{3} > \frac{1}{4}$, $u_3 > v_3$; since $\frac{1}{5} > \frac{1}{8}$, $u_5 > v_5$;

Similarly $u_6 > v_6$, $u_7 > v_7$ and so on.

$\therefore$ In the two series $\sum u_n$ and $\sum v_n$, $u_n \geq v_n$ for all n.

$\therefore$ By comparison test, $\sum u_n$ is (D), provided $\sum v_n$ is (D).

Now $$\sum v_n = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \infty$$

$$s_n = n^{\text{th}} \text{ partial sum of } \sum v_n = 1 + \frac{n-1}{2} = \frac{n+1}{2}$$

$$\lim_{n \to \infty} (s_n) = \infty$$

$\therefore$ $\sum v_n$ is (D)

$\therefore$ $\sum u_n$ is also (D)

Case (iii) Let $p < 1$.

$$\sum u_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \text{ to } \infty$$

Consider the auxiliary series

$$\sum v_n = 1 + \frac{1}{2} + \frac{1}{3} \cdots \text{ to } \infty$$

Since $p < 1$, $n^p < n$ (except when $n = 1$)

$\therefore$ $$\frac{1}{n^p} \geq \frac{1}{n}, \text{ for all values of } n$$

i.e. $$u_n \geq v_n, \text{ for all values of } n$$

But, by case (ii), $\sum v_n$ is (D).

$\therefore$ By comparison test, $\sum u_n$ is also (D).

Cauchy's Root Test

If $\sum u_n$ is a series of positive terms such that $\lim_{n \to \infty} \left(u_n^{1/n}\right) = l$, then the series $\sum u_n$ is (C), when $l < 1$ and (D) when $l > 1$. When $l = 1$, the test fails.

WORKED EXAMPLE 2(a)

Example 2.1 Test the convergence of the series

(i) $\sum_{n=1}^{\infty}\frac{1}{1+3^n}$; (ii) $\sum_{n=1}^{\infty}\frac{\cos^2 n}{2^n}$.

(i) Let $$\sum u_n = \sum\frac{1}{1+3^n}$$

Let $$\sum v_n = \sum\frac{1}{3^n}$$

Now $1 + 3^n > 3^n$ for all n

$\therefore$ $\frac{1}{1+3^n} < \frac{1}{3^n}$ for all n.

i.e. $u_n < v_n$ for all n

$\therefore$ $\sum u_n$ is (C), if $\sum v_n$ is (C).

Now $\sum v_n = \sum\frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$ is a geometric series with $r = \frac{1}{3} < 1$.

$\therefore$ $\sum v_n$ is (C)

Hence $\sum u_n$ is also (C).

(ii) Let $$\sum u_n = \sum\frac{\cos^2 n}{2^n}$$

Let $$\sum v_n = \sum\frac{1}{2^n}$$

Now $|\cos n| < 1$ or $-1 < \cos n < 1$

$\therefore$ $\cos^2 n < 1$ for all n

Hence $\frac{\cos^2 n}{2^n} < \frac{1}{2^n}$ for all n,

i.e. $u_n < v_n$ for all n

$\therefore$ $\sum u_n$ is (C), if $\sum v_n$ is (C).

Now $\sum v_n = \sum\frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is a geometric series with $r = 1/2 < 1$.

$\therefore$ $\sum v_n$ is (C)

Hence $\sum u_n$ is also (C).

Example 2.2 Test the convergence of the following series

(i) $\frac{1}{1\cdot 2}+\frac{2}{3\cdot 4}+\frac{3}{5\cdot 6}+\cdots$ to ∞

(ii) $\frac{1\cdot 2}{3^2\cdot 4^2}+\frac{3\cdot 4}{5^2\cdot 6^2}+\frac{5\cdot 6}{7^2\cdot 8^2}+\cdots$ to ∞

(iii) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\cdots$ to ∞.

(i) The given series is $\sum u_n = \sum \frac{n}{(2n-1)\cdot 2n}$

$$= \frac{1}{2}\sum \frac{1}{2n-1}$$

Let $$\sum v_n = \sum \frac{1}{n}.$$

Note ☑ If the numerator and denominator of u_n are expressions of degree p and q in n, then we choose $v_n = \frac{n^p}{n^q} = \frac{1}{n^{q-p}}$

Then $$\frac{u_n}{v_n} = \frac{1}{2}\cdot\frac{\left(\frac{1}{2n-1}\right)}{\left(\frac{1}{n}\right)} = \frac{1}{2}\cdot\frac{n}{2n-1}$$

$$= \frac{1}{2}\cdot\frac{1}{2-\frac{1}{n}}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{1}{2}\cdot\lim_{n\to\infty}\left(\frac{1}{2-\frac{1}{n}}\right)$$

$$= \frac{1}{4} \neq 0.$$

$\therefore$ By the limit form of comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Now $\sum v_n = \sum \frac{1}{n}$, which is the harmonic series is (D).

$\therefore$ $\sum u_n$ is also (D).

(ii) The given series is $\sum u_n = \dfrac{1\cdot 2}{3^2\cdot 4^2} + \dfrac{3\cdot 4}{5^2\cdot 6^2} + \ldots$ to ∞

i.e. $$\sum u_n = \sum \frac{(2n-1)\,(2n)}{(2n+1)^2\,(2n+2)^2}$$

Let $\sum v_n = \sum \dfrac{n^2}{n^4}$ ($\because$ the numerator is degree 2 and the denominator is of degree 4)

$$= \sum \frac{1}{n^2}$$

Then $$\frac{u_n}{v_n} = \frac{(2n-1)(2n)\cdot n^2}{(2n+1)^2\,(2n+2)^2}$$

$$= \frac{\left(2-\dfrac{1}{n}\right)\cdot 2}{\left(2+\dfrac{1}{n}\right)^2\left(2+\dfrac{2}{n}\right)^2}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{2^2}{2^2\cdot 2^2} = \frac{1}{4} \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Now $\sum v_n = \sum \dfrac{1}{n^2}$ is (C) $\left[\because \sum \dfrac{1}{n^p} \text{ is (C), when } p>1\right]$

$\therefore$ $\sum u_n$ is also (C).

(iii) The given series is $\sum u_n = \dfrac{1}{2^2} + \dfrac{2^2}{3^3} + \dfrac{3^3}{4^4} + \ldots$ to ∞ (omitting the first term)

i.e. $$\sum u_n = \sum \frac{n^n}{(n+1)^{n+1}}$$

Let $$\sum v_n = \sum \frac{n^n}{n^{n+1}} \text{ or } \sum \frac{1}{n}$$

Then $$\frac{u_n}{v_n} = \frac{n^n}{(n+1)^{n+1}}\cdot n$$

$$= \left(\frac{n}{n+1}\right)^{n+1} = \left(\frac{1}{1+\dfrac{1}{n}}\right)^{n+1}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \left(\frac{1}{1+\frac{1}{n}}\right)$$

$$\therefore \quad \lim_{x\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e} \neq 0.$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Now $\quad \sum v_n = \sum \frac{1}{n}$ is (D)

$\therefore \quad \sum u_n$ is also (D).

Note ☑ Omission of the first term (= 1) of the given series does not alter the convergence or divergence of the series.

Example 2.3 Examine the convergence of the following series:

(i) $\sum\limits_{n=1}^{\infty} \frac{\sqrt[3]{2n^2-1}}{\sqrt[4]{3n^3+2n+5}}$; $\quad$ (ii) $\sum\limits_{n=1}^{\infty}\left(\frac{3^n+4^n}{4^n+5^n}\right)$;

(iii) $\sum\limits_{n=1}^{\infty} n\sin^2\left(\frac{1}{n}\right)$; $\quad$ (iv) $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right)$;

(i) $$\sum u_n = \sum \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$$

Let $$\sum v_n = \sum \frac{n^{2/3}}{n^{3/4}} \text{ or } \sum \frac{1}{n^{1/12}}$$

Then $$\frac{u_n}{v_n} = \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}} \times n^{1/12}$$

$$= \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}} \times \frac{n^{3/4}}{n^{2/3}}$$

$$= \frac{\left(\frac{2n^2-1}{n^2}\right)^{1/3}}{\left(\frac{3n^3+2n+5}{n^3}\right)^{1/4}}$$

$$= \frac{\left(2 - \frac{1}{n^2}\right)^{1/3}}{\left(3 + \frac{2}{n^2} + \frac{5}{n^3}\right)^{1/4}}$$

$$\therefore \qquad \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{2^{1/3}}{3^{1/4}} \neq 0 .$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Now $\qquad \sum v_n = \sum \frac{1}{n^{1/12}}$ is (D) $\qquad \left[\because \sum \frac{1}{n^p} \text{ is (D) when } p < 1\right]$

$\therefore \sum u_n$ is (D).

(ii) $\qquad \sum u_n = \sum\left(\frac{3^n + 4^n}{4^n + 5^n}\right)$

Let $\qquad \sum v_n = \sum\left(\frac{4}{5}\right)^n$

Then $\qquad \frac{u_n}{v_n} = \frac{3^n + 4^n}{4^n + 5^n} \times \frac{5^n}{4^n}$

$$= \frac{\left(\frac{3}{4}\right)^n + 1}{\left(\frac{4}{5}\right)^n + 1}$$

$$\therefore \qquad \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{0+1}{0+1} = 1 \neq 0 .$$

$\therefore \sum u_n$ and $\sum v_n$ converge or diverge together, by comparison test.

Now $\sum v_n = \sum\left(\frac{4}{5}\right)^n$ is a geometric series with $r = \frac{4}{5} < 1$ and hence is convergent.

Hence $\sum u_n$ is also (C).

(iii) $\qquad \sum u_n = \sum_{n=1}^{\infty} n \sin^2\left(\frac{1}{n}\right)$

Let $\qquad \sum v_n = \sum \frac{1}{n}$

Then $$\frac{u_n}{v_n} = n^2 \sin^2\left(\frac{1}{n}\right)$$

$$= \left\{\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right\}^2$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \left\{\lim_{n\to\infty}\left[\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right]\right\}^2$$

$$= 1\left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1\right]$$

$$\neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n} \text{ is (D).}$$

$\therefore$ $\sum u_n$ is also (D).

(iv) $$\sum u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$$

Let $$\sum v_n = \sum \frac{1}{n^{3/2}}$$

Then $$\frac{u_n}{v_n} = \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right) \times n^{3/2}$$

$$= \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left[\frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right]$$

$$= 1\left[\because \lim_{\theta\to\infty}\left(\frac{\tan\theta}{\theta}\right) = 1\right]$$

$$\neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n^{3/2}} \text{ is (C)} \qquad \left[\because \sum \frac{1}{n^p} \text{ is (C), when } p > 1\right]$$

Hence $\sum u_n$ is also (C).

Example 2.4 Determine whether the following series are (C) or (D).

(i) $\sum\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$; (ii) $\sum\left(\dfrac{\sqrt{n+1}-\sqrt{n}}{n^\alpha}\right)$;

(iii) $\sum\left(\sqrt[3]{n^3+1}-n\right)$; (iv) $\sum\left(\dfrac{1}{\sqrt{n}+\sqrt{n+1}}\right)$.

(i)
$$\sum u_n = \sum\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right);$$

$$= \sum\left\{\frac{\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)}{\sqrt{n^4+1}+\sqrt{n^4-1}}\right\}$$

$$= \sum\left(\frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}\right)$$

Let
$$\sum v_n = \sum \frac{1}{n^2}$$

Then
$$\frac{u_n}{v_n} = \frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}$$

$\therefore$
$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{2}{1+1} = 1 \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n^2} \text{ is (C).}$$

Hence $\sum u_n$ is also (C).

(ii)
$$\sum u_n = \sum \left(\frac{\sqrt{n+1} - \sqrt{n}}{n^{\alpha}} \right)$$
$$= \sum \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{n^{\alpha}\left(\sqrt{n+1} + \sqrt{n}\right)}$$
$$= \sum \frac{1}{n^{\alpha}\left(\sqrt{n+1} + \sqrt{n}\right)} \tag{1}$$

Let
$$\sum v_n = \sum \frac{1}{n^{\alpha + \frac{1}{2}}}$$

Then
$$\frac{u_n}{v_n} = \frac{n^{\alpha + \frac{1}{2}}}{n^{\alpha}\left(\sqrt{n+1} + \sqrt{n}\right)}$$
$$= \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}$$

$\therefore$
$$\lim_{n \to \infty} \left(\frac{u_n}{v_n} \right) = \frac{1}{2} \neq 0.$$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

$\sum v_n = \sum \frac{1}{n^{\alpha + \frac{1}{2}}}$ is (C) when $\alpha + \frac{1}{2} > 1$ or $\alpha > \frac{1}{2}$, and it is (D) when $\alpha + \frac{1}{2} \leq 1$ or $\alpha \leq \frac{1}{2}$.

$\therefore$ $\sum u_n$ is (C) when $\alpha > \frac{1}{2}$ and (D) when $\alpha \leq \frac{1}{2}$.

Note ☑ Keeping $u_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^{\alpha}}$, if we choose $v_n = \frac{n^{1/2}}{n^{\alpha}} = \frac{1}{n^{\alpha - 1/2}}$ we will get $\lim_{n \to \infty} \left(\frac{u_n}{v_n} \right) = 0$ and so comparison test fails.

(iii)
$$u_n = (n^3 + 1)^{1/3} - (n^3)^{1/3}$$
$$= \frac{(n^3 + 1) - n^3}{(n^3 + 1)^{2/3} + (n^3 + 1)^{1/3} n + (n^3)^{2/3}} \quad \left[\because a - b = \frac{a^3 - b^3}{a^2 + ab + b^2} \right]$$

$$= \frac{1}{(n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2}$$

Let $$v_n = \frac{1}{n^2}$$

Then $$\frac{u_n}{v_n} = \frac{n^2}{(n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2}$$

$$= \frac{1}{\left(1+\frac{1}{n^3}\right)^{2/3} + \left(1+\frac{1}{n^3}\right)^{1/3} + 1}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{1}{3} \neq 0$$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n^2} \text{ is (C)}$$

Hence $\sum u_n$ is also (C).

(iv) $$\sum u_n = \sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

Let $$\sum v_n = \sum \frac{1}{\sqrt{n}}$$

$$\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{1}{2} \neq 0$$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n^{1/2}} \text{ is (D)} \left[\because \text{In } \sum \frac{1}{n^p},\ p = \frac{1}{2} < 1\right]$$

$\therefore$ $\sum u_n$ is also (D).

Example 2.5 Examine the convergence of the following series:

(i) $\left(\frac{1}{4}\right) + \left(\frac{2}{7}\right)^2 + \left(\frac{3}{10}\right)^3 + \ldots \text{to } \infty$

(ii) $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots$ to ∞;

(iii) $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$ to ∞;

(iv) $a+b+a^2+b^2+a^3+b^3+\ldots$ to ∞, given $a>0$, $b>0$.

(i) Given series is $\frac{1}{4}+\left(\frac{2}{7}\right)^2+\left(\frac{3}{10}\right)^3+\ldots$ to ∞

$$\therefore \quad u_n = n^{\text{th}} \text{ term of the given series}$$

$$= \left(\frac{n}{3n+1}\right)^n$$

$$\therefore \quad \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{n}{3n+1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{3+\frac{1}{n}}\right)$$

$$= \frac{1}{3} < 1.$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C).

(ii) $\sum u_n = \frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots$ (omitting the first term)

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

Then $$u_n^{1/n} = \left(\frac{n+1}{n+2}\right)x \text{ or } \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x$$

$$\therefore \quad \lim_{n\to\infty}(u_n)^{1/n} = x$$

$\therefore$ By Cauchy's root test,

$\sum u_n$ is (C) if $x<1$ and it is (D) if $x>1$

If $x = 1$, Cauchy's root test fails.

In this case, $$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}$$

$$= \frac{e}{e^2} \text{ or } \frac{1}{e} \neq 0$$

i.e. the necessary condition for convergence of $\sum u_n$ is not satisfied.

$\therefore$ $\sum u_n$ is (D)

(iii) $$\sum u_n = \left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$$

$\therefore$ $$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$$

$\therefore$ $$(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$$

$$= \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n} - 1\right]^{-1}$$

Then $$\lim_{n\to\infty} (u_n)^{1/n} = (e-1)^{-1} = \frac{1}{e-1} < 1$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C)

(iv) $$\sum u_n = a + b + a^2 + b^2 + a^3 + b^3 + \cdots$$

Then $$u_n = a^{\frac{n+1}{2}}, \text{ if } n \text{ odd}$$

$$= b^{n/2}, \text{ if } n \text{ is even}$$

$\therefore$ $$u_n^{1/n} = a^{\frac{1}{2}+\frac{1}{2n}}, \text{ if } n \text{ is odd}$$

$$= b^{1/2}, \text{ if } n \text{ is even}$$

$$\lim_{n\to\infty} (u_n^{1/n}) = \sqrt{a}, \text{ if } n \text{ is odd}$$

$$= \sqrt{b}, \text{ if } n \text{ is even}$$

$\therefore$ Whether n is odd or even, $\sum u_n$ is (C) if $a < 1$ and $b < 1$, and (D) if $a > 1$ and $b > 1$, by Cauchy's root test.

When $a = 1 = b$, the series becomes

$1 + 1 + 1 \cdots$ to ∞, which is (D).

Example 2.6 Test the convergence of the following series:

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$; (ii) $\displaystyle\sum \left(\frac{n}{n+1}\right)^{n^2}$;

(iii) $\displaystyle\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{n^{3/2}}$; (iv) $\displaystyle\sum \frac{[(n+1)x]^n}{n^{n+1}}$, $x > 0$.

(i)
$$\sum u_n = \sum \frac{1}{(\log n)^n} \therefore u_n = \frac{1}{(\log n)^n}, n \geq 2$$

$$\lim_{n\to\infty} (u_n^{1/n}) = \lim_{n\to\infty} \left(\frac{1}{\log n}\right)$$
$$= 0 < 1$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C).

(ii)
$$\sum u_n = \sum \left(\frac{n}{n+1}\right)^{n^2}$$

$\therefore$
$$u_n = \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}} \quad \text{and so} \quad u_n^{1/n} = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$\therefore$
$$\lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C).

(iii)
$$\sum u_n = \sum \left(1 + \frac{1}{\sqrt{n}}\right)^{n^{3/2}}$$

$\therefore$
$$u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{n^{3/2}} \quad \text{and so} \quad u_n^{1/n} = \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}$$

Hence
$$\lim_{n\to\infty} \left(u_n^{\frac{1}{n}}\right) = \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = e > 1$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (D).

(iv) $$\sum u_n = \sum \frac{[(n+1)x]^n}{n^{n+1}}$$

$\therefore$ $$u_n = \frac{[(n+1)x]^n}{n^{n+1}} \quad \text{and so} \quad u_n^{1/n} = \frac{(n+1)x}{n^{\frac{n+1}{n}}}$$

$$= \frac{(n+1)x}{n^{1+1/n}}$$

$$= \left(\frac{n+1}{n}\right) \cdot \frac{x}{n^{1/n}}$$

$\therefore$ $$\lim_{n\to\infty} (u_n^{1/n}) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot \frac{x}{\lim_{n\to\infty} (n^{1/n})}$$

$$= \frac{x}{\lim_{n\to\infty} (n^{1/n})} \qquad (1)$$

Now let $v = n^{1/n}$ $\therefore$ $\log v = \dfrac{1}{n} \log n$

$$\lim_{n\to\infty} (\log v) = \lim_{n\to\infty} \left(\frac{\log n}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right), \text{by L'Hospital's rule}$$

i.e. $$\log\left[\lim_{n\to\infty} (v)\right] = 0 \text{ or } \lim_{n\to\infty} v = e^0 = 1 \qquad (2)$$

Using (2) in (1), we have

$$\lim_{n\to\infty} (u_n^{1/n}) = x$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C) if $x < 1$, and (D) if $x > 1$

If $x = 1$, the series becomes $u_n = \sum \dfrac{(n+1)^n}{n^{n+1}}$

Let $$\sum v_n = \sum \frac{1}{n}$$

Then $$\frac{u_n}{v_n} = \frac{(n+1)^n}{n^{n+1}} \times n = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$$

$$\therefore \qquad \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = e \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$\sum v_n$ is (D). Hence $\sum u_n$ is (D) if $x = 1$.

EXERCISE 2(a)

Part A

(Short Answer Questions)

1. Define convergence of an infinite series with an example.
2. Define divergence of an infinite series with an example.
3. Show that the series $1 + x + x^2 + \cdots$ to ∞ oscillates when $x = -1$.
4. Show that the series $1 + x + x^2 + \cdots$ to ∞ oscillates between $-\infty$ and ∞, when $x < -1$.
5. Give an example to shown that $\sum u_n$ is not (C), even though $\lim_{n\to\infty}(u_n) = 0$.
6. Prove that the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ to ∞ is (D).
7. State two forms of comparison test for the convergence of $\sum u_n$.
8. State two forms of comparison test for the divergence of $\sum u_n$.
9. State Cauchy's root test.
10. Test the convergence of the series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$ to ∞.
11. Test the convergence of the series $\sum \sin\left(\frac{1}{n}\right)$.
12. Test the convergence of the series $\sum e^{-n^2}$.

Part B

Examine the convergence of the following series:

13. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{3^n}$

14. $\displaystyle\sum_{1}^{\infty} \frac{1}{1+4^n}$

15. $\dfrac{1\cdot 2}{3\cdot 4\cdot 5} + \dfrac{2\cdot 3}{4\cdot 5\cdot 6} + \dfrac{3\cdot 4}{5\cdot 6\cdot 7} + \cdots$

16. $\dfrac{1}{1+\sqrt{2}} + \dfrac{2}{1+2\sqrt{3}} + \dfrac{3}{1+3\sqrt{4}} + \cdots$

17. $\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \cdots$

18. $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \dfrac{1}{3\cdot 4\cdot 5} + \cdots$

19. $\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\cdots$

20. $\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n+5^n}$

21. $\sum\left(\frac{n^3-4n+2}{n^5+2n-3}\right)^{\frac{4}{3}}$

22. $\left(\frac{n^3-5n^2+7}{n^5+4n^4-n}\right)^{\frac{1}{3}}$

23. $\sum\sin^2\left(\frac{1}{n}\right)$

24. $\sum\sqrt{n\tan^{-1}\left(\frac{1}{n^3}\right)}$ $\left[\textit{Hint}\text{: }\lim_{x\to 0}\left(\frac{\tan^{-1}x}{x}\right)=1\right]$

25. $\sum\sin\left(\frac{n}{n^2+1}\right)$

26. $\sum\sqrt{\frac{2^n-1}{3^n-1}}$

27. $\sum\frac{(n+1)^3}{n^k+(n+2)^k}$

28. $\sum\frac{(n+1)}{n^p}$

29. $\sum\frac{1}{n^{1+1/n}}$ $\left[\textit{Hint}\text{: Choose } v_n=\frac{1}{n}\text{ and }\lim_{n\to\infty}n^{1/n}=1\right]$

30. $\sum\left(\sqrt{n^2+1}-n^2\right)$

31. $\sum\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$

32. $\sum\left(\sqrt{n^4+1}-n^2\right)$

33. $\sum\left(\frac{\sqrt{n+1}-\sqrt{n}}{n}\right)$

34. $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\cdots$

35. $\frac{3x}{4}+\left(\frac{4}{5}\right)^2x^2+\left(\frac{5}{6}\right)^3x^3+\cdots$

36. $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\cdots\ (x>0)$

37. $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\ (x>0)$

38. $1+2\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)^2+2\left(\frac{1}{3}\right)^3+\left(\frac{1}{3}\right)^4+2\left(\frac{1}{3}\right)^5+\cdots$

39. $\sum\left(1+\frac{1}{n}\right)^{-n^2}$

40. $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{n^{-3/2}}$

D'Alambert's Ratio Test

If $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)=l$, then the series $\sum u_n$ is (C) when $l<1$ and is (D) when $l>1$. When $l=1$, the test fails.

Raabe's Test

If $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]=l$, then the series $\sum u_n$ is (C) when $l > 1$ and is (D) when $l < 1$. When $l = 1$, the test fails.

WORKED EXAMPLE 2(b)

Example 2.1 Test the convergence of the following series:

(i) $\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+\cdots$ to ∞;

(ii) $\frac{3}{4}+\frac{3\cdot 4}{4\cdot 6}+\frac{3\cdot 4\cdot 5}{4\cdot 6\cdot 8}+\cdots$ to ∞;

(iii) $\frac{1}{1+3}+\frac{2}{1+3^2}+\frac{3}{1+3^3}+\cdots$ to ∞;

(iv) $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots\infty$.

(i) $$\sum u_n = \frac{2}{1}+\frac{2^2}{2}+\frac{2^2}{3}+\cdots+\frac{2^n}{n}+\cdots$$

$\therefore$ $$u_n = \frac{2^n}{n} \quad \text{and} \quad u_{n+1} = \frac{2^{n+1}}{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{n+1}\times\frac{n}{2^n} = \frac{1}{\left(1+\frac{1}{n}\right)}\times 2$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 2 > 1$$

$\therefore$ By ratio test, $\sum u_n$ is (D).

(ii) $$\sum u_n = \frac{3}{4}+\frac{3\cdot 4}{4\cdot 6}+\frac{3\cdot 4\cdot 5}{4\cdot 6\cdot 8}+\cdots$$

$\therefore$ $$u_n = \frac{3\cdot 4\cdot 5\ldots(n+2)}{4\cdot 6\cdot 8\ldots(2n+2)}$$

[*Note* ☑ There are n factors each in the numerator and denominator of the n^{th} term. The factors are in A.P.]

$$u_{n+1} = \frac{3\cdot 4\cdot 5 \ldots (n+2)\,(n+3)}{4\cdot 6\cdot 8 \ldots (2n+2)\,(2n+4)}$$

$$\therefore \qquad \frac{u_{n+1}}{u_n} = \frac{n+3}{2n+4} = \frac{1+\dfrac{3}{n}}{2+\dfrac{4}{n}}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{2} < 1.$$

$\therefore$ By ratio test, $\sum u_n$ is (C).

(iii) $$\sum u_n = \frac{1}{1+3} + \frac{2}{1+3^2} + \frac{3}{1+3^3} + \cdots$$

$$\therefore \qquad u_n = \frac{n}{1+3^n} \quad \text{and} \quad u_{n+1} = \frac{n+1}{1+3^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{1+3^{n+1}} \times \frac{1+3^n}{n}$$

$$= \left(1+\frac{1}{n}\right)\cdot\left(\frac{\dfrac{1}{3^n}+1}{\dfrac{1}{3^n}+3}\right)$$

$$\therefore \qquad \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{3} < 1.$$

$\therefore$ By ratio test, $\sum u_n$ is (C).

(iv) $$\sum u_n = 1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!}$$

$$\therefore \qquad u_n = \frac{n^p}{n!} \quad \text{and} \quad u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\therefore \qquad \frac{u_{n+1}}{u_n} = \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p}$$

$$= \left(\frac{n+1}{n}\right)^p \cdot \frac{1}{n+1}$$

$$= \left(1+\frac{1}{n}\right)^p \cdot \frac{1}{n+1}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 0 < 1.$$

$\therefore$ By ratio test, $\sum u_n$ is (C).

Example 2.2 Test the convergence of the following series:

(i) $\displaystyle\sum_{n+1}^{\infty}\frac{(a+1)\,(2a+1)\,(3a+1)\ldots(na+1)}{(b+1)(2b+1)\,(3b+1)\ldots(nb+1)}; a, b, > 0.$

(ii) $\displaystyle\sum\frac{3^n\; n!}{n^n};$

(iii) $\displaystyle\sum\frac{a^n\; x^n}{1+n^2}(x>0);$

(iv) $\displaystyle\sum\frac{x^n}{1+x^{2n}}(x>0);$

(i) $$\sum u_n = \sum\frac{(a+1)\,(2a+1)\ldots(na+1)}{(b+1)\,(2b+1)\ldots(nb+1)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(a+1)\,(2a+1)\ldots(na+1)\,(\overline{n+1}\cdot a+1)}{(b+1)\,(2b+1)\ldots(nb+1)\,(\overline{n+1}\cdot b+1)}\times\frac{(b+1)\,(2b+1)\ldots(nb+1)}{(a+1)\,(2a+1)\ldots(na+1)}$$

$$= \frac{(n+1)\,a+1}{(n+1)b+1} = \frac{a+\dfrac{1}{n+1}}{b+\dfrac{1}{n+1}}$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{a}{b}$$

$\therefore$ By Ratio test, $\sum u_n$ is (C) if $\dfrac{a}{b}<1$ or $a<b$, and $\sum u_n$ is (D) if $\dfrac{a}{b}>1$ or $a>b$.

If $a = b$, the ratio test fails.
But in this case, the series becomes $1 + 1 + 1 + \cdots$ to ∞, which is (D).
Thus $\sum u_n$ is (C) when $0 < a < b$, and it is (D) when $0 < b \le a$.

(ii) $$\sum u_n = \sum\frac{3^n\; n!}{n^n}$$

$$\therefore \quad u_n = \frac{3^n\cdot n!}{n^n} \quad\text{and}\quad y_{n+1} + \frac{3^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{3^n \cdot n!}$$

$$= 3 \cdot \left(\frac{n}{n+1}\right)^n \text{ or } 3\left(\frac{1}{1+\frac{1}{n}}\right)^n$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{3}{e} > 1 \quad (\because e = 2.71828 \cdots)$$

$\therefore$ By ratio test, $\sum u_n$ is (D).

(iii) $$\sum u_n = \sum \frac{a^n x^n}{1+n^2}$$

$\therefore$ $$u_n = \frac{a^n x^n}{1+n^2} \text{ and } u_{n+1} = \frac{a^{n+1} x^{n+1}}{1+(n+1)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{a^{n+1} x^{n+1}}{1+(n+1)^2} \times \frac{1+n^2}{a^n x^n}$$

$$= ax \cdot \frac{(1+n^2)}{2+2n+n^2}$$

$$= ax \cdot \left(\frac{\frac{1}{n^2}+1}{\frac{2}{n^2}+\frac{2}{n}+1}\right)$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = ax$$

$\therefore$ By ratio test, $\sum u_n$ is (C), if $ax < 1$ or $x > \frac{1}{a}$.

$\sum u_n$ = is (D), if $ax > 1$ or $x < \frac{1}{a}$

Ratio test fails, when $x = \frac{1}{a}$.

But when $x = \frac{1}{a}$, $\sum u_n = \sum \frac{1}{1+n^2}$

By choosing $\sum v_n = \frac{1}{n^2}$ and using comparison test, we can prove that $\sum u_n$ is (C).

Thus $\sum u_n$ is (C) when $x \leq \frac{1}{a}$ and (D) when $x > \frac{1}{a}$.

(iv) $$\sum u_n = \sum \frac{x^n}{1+x^{2n}}$$

$$u_n = \frac{x^n}{1+x^{2n}} \quad \text{and} \quad u_{n+1} = \frac{x^{n+1}}{1+x^{2n+2}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+x^{2n+2}} \times \frac{1+x^{2n}}{x^n}$$

$$= \frac{x+x^{2n+1}}{1+x^{2n+2}}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x, \text{ if } x<1 \left[\because \lim_{n\to\infty}(x^{2n+1}) = 0 = \lim_{n\to\infty}(x^{2n+2})\right]$$

$$\frac{u_{n+1}}{u_n} = \left(\frac{\dfrac{1}{x^{2n+1}} + \dfrac{1}{x}}{\dfrac{1}{x^{2n+2}} + 1}\right)$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{x}, \text{ if } x>1$$

Thus when $x < 1$ and $x > 1$, $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) < 1$ and hence $\sum u_n$ is (C).

But when $x = 1$, the ratio test fails.
In this case, the series becomes

$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ to ∞, which is (D).

Example 2.3 Test the convergence of the following series:

(i) $\sqrt{\frac{1}{2}}x + \sqrt{\frac{2}{5}}x^2 + \sqrt{\frac{3}{10}}x^3 + \ldots + \sqrt{\frac{n}{n^2+1}}x^n + \ldots (x>0)$

(ii) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots (x>0)$

(iii) $\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \ldots (x>0)$

(iv) $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \ldots (x>0)$

(i) $$\sum u_n = \sqrt{\frac{1}{2}}x + \sqrt{\frac{2}{5}}x^2 + \sqrt{\frac{3}{10}}x^3 + \ldots$$

$\therefore$ $$u_n = \sqrt{\frac{n}{n^2+1}}x^n \text{ and } u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}}x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \sqrt{\left(\frac{n+1}{n}\right).\left\{\frac{n^2+1}{(n+1)^2+1}\right\}}x$$

$$= \sqrt{\left(1+\frac{1}{n}\right)\left\{\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}\right\}}x$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x.$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $x < 1$ and it is (D) if $x > 1$.

When $x = 1$, ratio test fails.

In this case, the series becomes $\sum u_n = \sum\sqrt{\frac{n}{n^2+1}}$.

Choosing $\sum v_n = \frac{1}{\sum n^{1/2}}$ and using comparison test, we can prove that $\sum u_n$ is (D).

Thus $\sum u_n$ is (C) when $x < 1$ and (D) when $x \geq 1$.

(ii) $$\sum u_n = \frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \ldots$$

$\therefore$ $$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \text{ and } u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n}}{x^{2n-2}}\cdot\frac{(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}}$$

$$= \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\sqrt{\left(1+\frac{1}{n}\right)}}\cdot x^2$$

$\therefore \qquad \lim_{n\to\infty}\left(\dfrac{u_{n+1}}{u_n}\right) = x^2$

$\therefore$ By ratio test $\sum u_n$ is (C) if $x^2 < 1$ and (D) if $x^2 > 1$.

When $x^2 = 1$, $\sum u_n = \dfrac{1}{2\sqrt{1}} + \dfrac{1}{3\sqrt{2}} + \dfrac{1}{4\sqrt{3}} + \ldots + \dfrac{1}{(n+1)\sqrt{n}}$. Choosing

$\sum v_n = \dfrac{1}{\sum n^{3/2}}$ and using comparison test, we can prove that $\sum u_n$ is (C).

Thus $\sum u_n$ is (C) when $x^2 \le 1$ and it is (D) when $x^2 > 1$.

(iii) $$\sum u_n = \sum \frac{x^n}{1+x^n}$$

$$\therefore \qquad \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+x^{n+1}} \times \frac{1+x^n}{x^n}$$

$$= \frac{x+x^{n+1}}{1+x^{n+1}}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x, \text{ if } x<1 \ \left[\because x^{n+1} \to 0 \text{ as } n \to \infty\right]$$

Also $$\frac{u_{n+1}}{u_n} = \frac{\dfrac{x}{x^{n+1}}+1}{\dfrac{1}{x^{n+1}}+1}$$

$$\lim_{n\to\infty}\left(\frac{u_n+1}{u_n}\right) = 1, \text{ if } x > 1$$

$\therefore$ By ratio test, $\sum u_n$ is (C). If $x < 1$ and ratio test fails if $x > 1$.

Also when $x = 1$, the ratio test fails.

In this case, $\sum u_n = \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + \ldots$, which is (D).

(iv) $$\sum u_n = x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \ldots + \frac{n^n x^n}{n!} + \ldots$$

$$\therefore \qquad \frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} \times \frac{n!}{n^n x^n}$$

$$= \left(\frac{n+1}{n}\right)^n x \text{ or } \left(1+\frac{1}{n}\right)^n \cdot x$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)=e.x$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $ex < 1$ or $x < \frac{1}{e}$ and it is (D) if $ex > 1$ or $x > \frac{1}{e}$.

When $x = \frac{1}{e}$, ratio test fails.

In this case, $\sum u_n = \sum \dfrac{\left(\frac{n}{e}\right)^n}{n!}$.

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left[\frac{\left(\frac{n}{e}\right)^n}{n!}\right] \neq 0. \;\therefore\; \sum u_n \text{ is (D), when } x = \frac{1}{e}.$$

Example 2.4 Test the convergence of the following series:

(i) $1+\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\ldots$

(ii) $\frac{1^2}{4^2}+\frac{1^2\cdot 5^2}{4^2\cdot 8^2}+\frac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2}+\ldots$

(iii) $\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\ldots(x>0)$

(iv) $1+\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+\ldots(x>0)$.

(i) $$\sum u_n = \frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\ldots \text{ (omitting the first term)}$$

$\therefore$ $$u_n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n} \text{ and}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots 2n\cdot(2n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2} \text{ or } \frac{2+\frac{1}{n}}{2+\frac{2}{n}}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 1.$$

Hence ratio test fails.

Let us try now Raabe's test.

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} = \lim_{n\to\infty}\left[n\left(\frac{2n+2}{2n+1}-1\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{n}{2n+1}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2+\frac{1}{n}}\right] = \frac{1}{2} < 1$$

$\therefore$ By Rabbe's test $\sum u_n$ is (D).

(ii) $$\sum u_n = \frac{1^2}{4^2} + \frac{1^2\cdot 5^2}{4^2\cdot 8^2} + \frac{1^2\cdot 5^2\cdot 9^2}{7^2\cdot 8^2\cdot 12^2} + \ldots$$

$\therefore$ $$u_n = \frac{1^2\cdot 5^2\cdot 9^2\ \ldots(4n-3)^2}{4^2\cdot 8^2\cdot 12^2\ \ldots(4n)^2}$$

and $$u_{n+1} = \frac{1^2\cdot 5^2\ \ldots(4n-3)^2\,(4n+1)^2}{4^2\cdot 8^2\ \ldots(4n)^2\,(4n+4)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{(4n+1)^2}{(4n+4)^2} = \frac{\left(4+\frac{1}{n}\right)^2}{\left(4+\frac{4}{n}\right)^2}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 1.$$

Hence ratio test fails.

Now $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left\{\frac{(4n+4)^2}{(4n+1)^2}-1\right\}$$

$$= n\left\{\frac{(8n+5)\cdot 3}{(4n+1)^2}\right\}$$

$\therefore$ $$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} 3\left\{\frac{8+\frac{5}{n}}{\left(4+\frac{1}{n}\right)^2}\right\}$$

$$= \frac{3\times 8}{4^2} = \frac{3}{2} > 1.$$

$\therefore$ By Raabe's test $\sum u_n$ is (C).

(iii) $$\sum u_n = \frac{3}{7}x + \frac{3\cdot 6}{7\cdot 10}x^2 + \frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3 + \ldots$$

$\therefore$ $$u_n = \frac{3\cdot 6\cdot 9\ldots(3n)}{7\cdot 10\cdot 13\ldots(3n+4)}x^n$$

and $$u_{n+1} = \frac{3\cdot 6\cdot 9\ldots(3n)(3n+3)}{7\cdot 10\cdot 13\ldots(3n+4)(3n+7)}x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+7}x = \left(\frac{3+\frac{3}{n}}{3+\frac{7}{n}}\right).x$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $x < 1$ and $\sum u_n$ is (D) if $x > 1$.
When $x = 1$, ratio test fails.

In the case, $$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+7}$$

$\therefore$ $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{3n+7}{3n+3} - 1\right) = \frac{4n}{3n+3}$$

$\therefore$ $$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \lim_{n\to\infty}\left(\frac{4}{3+\frac{3}{n}}\right)$$

$$= \frac{4}{3} > 1.$$

$\therefore$ By Raabe's test, $\sum u_n$ is (C), when $x = 1$.

Thus $\sum u_n$ is (C) if $x \le 1$ and it is (D) if $x > 1$.

(iv) $$\sum u_n = \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \ldots \text{ (omitting the first term)}$$

$\therefore$ $$u_n = \frac{x^n}{(n+1)^2} \text{ and } u_{n+1} = \frac{x^{n+1}}{(n+2)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+2)^2} \times \frac{(n+1)^2}{x^n} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{2}{n}\right)^2} \cdot x$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x.$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $x < 1$ and is (D) if $x > 1$.

Ratio test fails, when $x = 1$.

When $x = 1$, $\dfrac{u_n}{u_{n+1}} = \dfrac{(n+2)^2}{(n+1)^2}$

$$\therefore \quad \lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \lim_{n\to\infty}\left[\frac{n(2n+3)}{(n+1)^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{2+\frac{3}{n}}{\left(1+\frac{1}{n}\right)^2}\right]$$

$$= 2 > 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is (C).

Note ☑ The convergence of $\sum u_n$ can be proved, when $x = 1$, by comparison test also.

$\therefore$ $\sum u_n$ is (C) if $x \le 1$ and is (D) if $x > 1$.

Example 2.5 Examine the convergence of the following series:

(i) $\sum \dfrac{a\,(a+1)\,(a+2)\ldots(a+\overline{n-1})}{b(b+1)\,(b+2)\ldots(b+\overline{n-1})}$; (ii) $\sum \dfrac{x^{n-1}}{n\cdot 3^n}$;

(iii) $\sum \dfrac{(2n)!}{(n!)^2}\,x^n$.

(i) $$\sum u_n = \sum \frac{a\,(a+1)\,(a+2)\ldots(a+\overline{n-1})}{b(b+1)\,(b+2)\ldots(b+\overline{n-1})}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{a+n}{b+n}$$

$$\lim_{n \to \infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n \to \infty}\left(\frac{\frac{a}{n}+1}{\frac{b}{n}+1}\right) = 1$$

$\therefore$ Ratio test fails

Now $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{b+n}{a+n} - 1\right)$$

$$= \frac{n(b-a)}{a+n} \text{ or } \frac{(b-a)}{\frac{a}{n}+1}$$

$$\lim_{n \to \infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = b - a$$

$\therefore$ By Raabe's test, $\sum u_n$ (C) if $b - a > 1$ or $b > a + 1$ and it is (D) if $b - a < 1$ or $b < a + 1$.

If $b = a + 1$, Raabe's test fails.

In this case,

$$u_n = \frac{a\,(a+1)\,(a+2)\ldots(a+\overline{n-1})}{(a+1)\,(a+2)\,(a+3)\ldots(a+\overline{n-1})(a+n)}$$

$$= \frac{a}{a+n}$$

Let $$\sum v_n = \sum \frac{1}{n}$$

$$\frac{u_n}{v_n} = \frac{na}{a+n} = \frac{a}{\frac{a}{n}+1}$$

$\therefore$ $$\lim_{n \to \infty}\left(\frac{u_n}{v_n}\right) = a \neq 0.$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n} \text{ is (D).}$$

$\therefore$ $\sum u_n$ is also (D).

Thus $\sum u_n$ is (C) if $b > (a + 1)$ and is (D) if $b \leq (a + 1)$.

(ii) $$\sum u_n = \sum \frac{x^{n-1}}{n \cdot 3^n}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{x^n}{(n+1)\, 3^{n+1}} \times \frac{n \cdot 3^n}{x^{n-1}}$$

$$= \frac{x}{3} \cdot \frac{1}{1 + \frac{1}{n}}$$

$$\therefore \quad \lim_{n \to \infty} \left(\frac{u_{n+1}}{u_n} \right) = \frac{x}{3}$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $\frac{x}{3} < 1$ or $x > 3$ and it is (D) if $\frac{x}{3} > 1$ or $x > 3$.

If $x = 3$, ratio test fails.

When $x = 3$, $\sum u_n = \sum \frac{1}{3n}$ or $\frac{1}{3} \sum \frac{1}{n}$ is (D)

$\therefore$ $\sum u_n$ is (C) if $x < 3$ and it is (D), if $x \geq 3$.

(iii) $$\sum u_n = \sum \frac{(2n)!}{(n!)^2} x^n$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{(2n+2)!}{[(n+1)!]^2} x^{n+1} \times \frac{(n!)^2}{(2n)!\, x^n}$$

$$= \frac{(2n+1)\,(2n+2)}{(n+1)^2} x$$

$$= \frac{\left(2 + \frac{1}{n}\right)\left(2 + \frac{2}{n}\right)}{\left(1 + \frac{1}{n}\right)^2} \cdot x$$

$$\therefore \quad \lim_{n \to \infty} \left(\frac{u_{n+1}}{u_n} \right) = 4x$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $4x < 1$ or $x < \frac{1}{4}$ and is (D) if $4x > 1$ or $x < \frac{1}{4}$.

If $x = \frac{1}{4}$, ratio test fails.

When $x = 1/4$,

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\frac{4(n+1)^2}{(2n+1)\,(2n+2)}-1\right]$$

$$= \frac{n}{2n+1} \quad \text{or} \quad \frac{1}{2+\dfrac{1}{n}}$$

$$\therefore \quad \lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \frac{1}{2}<1$$

$\therefore$ By Raabe's test, $\sum u_n$ is (D).

Thus $\sum u_n$ is (C), if $x < 1/4$ and is (D) if $x \geq 1/4$.

EXERCISE 2(b)

Part A

(Short Answer Questions)

1. State D'Alembert's ratio test.
2. State Raabe's test.
3. For the series $\sum u_n = \sum \frac{1}{n}$, show that both the ratio test and Raabe's test fail.
4. Use Raabe's test to establish the convergence of $\sum \frac{1}{n^2}$.
5. Prove that series $\sum (n+1)\, x^n$ is (C) if $0 < x < 1$.

Part B

Examine the convergence or divergence of the following series:

6. $\dfrac{3}{2^4}+\dfrac{3^2}{2^5}+\dfrac{3^3}{2^6}+\cdots$ to ∞.
7. $1+\dfrac{1\cdot 2}{1\cdot 3}+\dfrac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\cdots$ to ∞.
8. $\dfrac{1}{1+2}+\dfrac{2}{1+2^2}+\dfrac{3}{1+2^3}+\cdots$ to ∞.
9. $\dfrac{1!}{2}+\dfrac{2!}{4}+\dfrac{3!}{8}+\dfrac{4!}{16}+\cdots$ to ∞.

10. $\dfrac{3^2}{2\cdot 2}+\dfrac{3^3}{3\cdot 2^2}+\dfrac{3^4}{4\cdot 2^3}+\cdots$ to ∞.

11. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3+k}{2^n+k}$.

12. $\displaystyle\sum\frac{n!}{n^n}$.

13. $\sum n^4\, e^{-n^2}$.

14. $\sum(2n+1)\, x^n\cdot(x>0)$.

15. $\displaystyle\sum\frac{x^{2n-2}}{(n+1)\sqrt{n}}\,(x>0)$.

16. $\dfrac{x}{1\cdot 2}+\dfrac{x^2}{3\cdot 4}+\dfrac{x^3}{5\cdot 6}+\cdots$ to ∞. $(x>0)$

17. $2+\dfrac{3}{2}x+\dfrac{4}{3}x^2+\dfrac{5}{4}x^3+\cdots$ to ∞. $(x>0)$

18. $1+2^2x+3^2x^2+4^2x^3+\cdots$ to ∞. $(x>0)$

19. $\dfrac{2}{1\cdot 3}x+\dfrac{3}{2\cdot 4}x^2+\dfrac{4}{3\cdot 5}x^3+\cdots$ to ∞. $(x>0)$

20. $\dfrac{2x}{1}+\dfrac{3x^2}{8}+\dfrac{4x^3}{27}+\cdots$ to ∞. $(x>0)$

21. $\dfrac{1}{2}\cdot\dfrac{1}{4}+\dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{1}{6}+\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{1}{8}+\cdots$ to ∞.

22. $\dfrac{3^2}{5^2}+\dfrac{3^2\cdot 5^2}{5^2\cdot 7^2}+\dfrac{3^2\cdot 5^2\cdot 7^2}{5^2\cdot 7^2\cdot 9^2}+\cdots$ to ∞.

23. $x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{x^5}{5}+\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{x^7}{7}+\cdots$ to ∞. $(x>0)$

24. $x+\dfrac{2^2\cdot x^2}{2!}+\dfrac{3^3\cdot x^3}{3!}+\dfrac{4^4\cdot x^4}{4!}+\cdots$ to ∞. $(x>0)$

25. $\dfrac{1}{3}x+\dfrac{1\cdot 2}{3\cdot 5}x^2+\dfrac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}x^3+\cdots$ to ∞. $(x>0)$

26. $\displaystyle\sum\frac{(n!)^2}{(n+1)!}x^n\ (x>0)$.

27. $\sum \dfrac{4 \cdot 7 \ldots (3n+1)}{n!} x^n$

28. $\sum \dfrac{2 \cdot 5 \cdot 8 \ldots (3n-1)}{7 \cdot 10 \cdot 13 \ldots (3n+4)}$.

29. $\sum \left\{ \dfrac{1 \cdot 4 \cdot 7 \ldots (3n-2)}{3 \cdot 6 \cdot 9 \ldots 3n} \right\}^2$.

30. $\sum \left\{ \dfrac{2^2 \cdot 4^2 \cdot 6^2 \ldots (2n)^2}{3 \cdot 4 \cdot 5 \ldots (2n+2)} \right\}$.

2.3 ALTERNATING SERIES

A series in which the terms are alternately positive and negative is called an alternating series.

An alternating series is of the form

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} u_{n_1}$$

where all the u's are positive.

2.3.1 Leibnitz Test for Convergence of an Alternating Series

The alternating series $u_1 - u_2 + u_3 - u_4 + \cdots \; (-1)^{n-1} u_n + \cdots$, in which $u_1, u_2, u_3, \ldots$ are all positive, is convergent if (i) each term is numerically less than the preceding term, i.e. $u_{n+1} < u_n$, for all n and (ii) $\lim_{n \to \infty} (u_n) = 0$.

Note ☑ If $\lim_{n \to \infty} (u_n) \neq 0$, then $\sum (-1)^{n-1} u_n$ is not convergent, but oscillating.

For example, let us consider the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1} \cdot \frac{1}{n} + \cdots \text{ to } \infty.$$

Here $u_n = \dfrac{1}{n}$ and $u_{n+1} = \dfrac{1}{n+1}$

Since $n + 1 > n, \dfrac{1}{n+1} < \dfrac{1}{n}$

i.e. $u_{n+1} < u_n$ for all n.

Also $\lim_{n \to \infty} (u_n) = \lim_{n \to \infty} \left(\dfrac{1}{n} \right) = 0$

$\therefore$ By Leibnitz test, $\sum (-1)^{n-1} \cdot \dfrac{1}{n}$ is (C).

2.3.2 Absolute and Conditional Convergence

A series $\sum u_n$, in which any term is either positive or negative, is said to be *absolutely convergent* if the series $\sum |u_n|$ is convergent.

A series $\sum u_n$ consisting of positive and negative terms is said to be *conditionally convergent*, if $\sum u_n$ is (C), but $\sum |u_n|$ is (D). For example, let us consider the series.

$\sum u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$, which is a series of +ve and –ve terms. (In fact, it is an alternating series).

Now $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is (C), since it is a geometric series with $r = 1/2 < 1$.

$\therefore$ The given series $\sum u_n$ is absolutely (C).

Let us now consider $\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ By Leibnitz test, we have proved that $\sum u_n$ is (C).

Now $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is known to be divergent.

$\therefore$ $\sum u_n$ is conditionally (C).

Note ☑ 1. We can prove that an absolutely convergent series is (ordinarily) convergent. The converse of this result is not true i.e. series which is convergent need not be absolutely convergent, as in the case of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \infty$.

2. To prove the absolute convergence of $\sum u_n$, we have to prove the convergence of $\sum |u_n|$. Since $\sum |u_n|$ is a series of positive terms, we may use any of the standard tests (comparison, Cauchy's root, Ratio and Raabie's tests) to prove its convergence.

2.3.3 Convergence of the Binomial Series

The series $1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots$ is called the *Binomial series*. The sum to which this series converges is $(1+x)^n$.

Let us now find the values of x for which the binomial series is (C) for any n.

Omitting the first term 1 in the binomial series,

let $\sum u_r = \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \cdots$

$\therefore$ The general term $u_r = \dfrac{n(n-1)\ldots(n-r+1)}{r!}x^r$

Note ☑ As 'n' is now a given constant occurring in the given series, the r^{th} term u_r is taken as the general term.

$$u_{r+1} = \frac{n(n-1)\ldots(n-r+1)(n-r)}{(r+1)!}x^{r+1}$$

$$\therefore \qquad \frac{u_{r+1}}{u_r} = \frac{n-r}{r+1}\cdot x = \frac{\frac{n}{r}-1}{1+\frac{1}{r}}\cdot x$$

$$\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty}\frac{\left|\frac{n}{r}-1\right|}{1+\frac{1}{r}}|x|$$

$$= |-1|\cdot|x|$$

$$= |x|$$

$\therefore$ By ratio test, $\sum|u_r|$ is (C) if $|x|<1$ and it is (D) if $|x|>1$.

$\therefore$ $\sum u_r$, i.e. the given binomial series is absolutely convergent and hence (C) if $|x| < 1$ and not (C) if $|x| > 1$.

Note ☑ When $|x| = 1$, the convergence or divergence of $\sum u_r$ can be established with further analysis. If $x = -1$, $\sum u_r$ is (C) when $n > 0$ and is (D) when $n < 0$.

If $x = 1$, $\sum u_r$ is (C) when $n > -1$ and oscillatory when $n \le -1$.

2.3.4 Convergence of the Exponential Series

The series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ to ∞ is called the *Exponential series*. The sum to which the series converges is e^x.

Let us now consider the convergence of the series $\sum u_n = \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ (omitting the first term)

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}$$

$$= \frac{x}{n+1}$$

$$\therefore \qquad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 < 1.$$

$\therefore$ By ratio test, $\sum |u_n|$ is (C), for all x.

$\therefore$ The given exponential series $\sum u_n$ is absolutely (C) and hence (C) for all values of x.

2.3.5 Convergence of the Logarithmic Series

The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$ is called the *Logarithmic series*. The sum to which the series converges is log $(1 + x)$.

Let us now consider the convergence of the series

$$\sum u_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

$$u_n = \frac{(-1)^{n-1}}{n}x^n \quad \text{and} \quad u_{n+1} = \frac{(-1)^n\, x^{n+1}}{n+1}$$

$$\therefore \qquad \frac{u_{n+1}}{u_n} = -x\cdot\frac{n}{n+1} = -x\cdot\frac{1}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|-x|}{1+\frac{1}{n}} = |x|$$

$\therefore$ By ratio test, $\sum |u_n|$ is (C) if $|x| < 1$ and is (D) if $|x| > 1$.

$\therefore$ The logarithmic series $\sum u_n$ is absolutely (C) and hence (C) if $|x| < 1$ and not (C) if $|x| > 1$.

If $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. It is an alternating series which has been proved to be (C) by Leibnitz test.

If $x = -1$, the series becomes $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots \infty$. which is (D).

Thus the logarithmic series is (C), if $-1 < x \le 1$.

WORKED EXAMPLE 2(c)

Example 2.1 Examine the convergence of the series:

(i) $\frac{1}{2}-\frac{2}{5}+\frac{3}{10}-\cdots+(-1)^{n-1}\cdot\frac{n}{n^2+1}+\cdots$

(ii) $\frac{1}{1}-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\cdots$

(i) The given series is $\sum(-1)^{n-1}\frac{n}{n^2+1}=\sum(-1)^{n-1}u_n$, say.

$$\therefore \qquad u_n = \frac{n}{n^2+1}=\frac{\frac{1}{n}}{1+\frac{1}{n^2}}$$

$$\therefore \qquad \lim_{n\to\infty}(u_n)=0$$

Now

$$u_n-u_{n+1} = \frac{n}{n^2+1}-\frac{n+1}{(n+1)^2+1}$$

$$= \frac{n\{(n+1)^2+1\}-(n+1)(n^2+1)}{(n^2+1)\{(n+1)^2+1\}}$$

$$= \frac{n(n^2+2n+2)-(n+1)(n^2+1)}{(n^2+1)(n^2+2n+2)}$$

$$= \frac{n^2+n-1}{(n^2+1)(n^2+2n+2)}$$

$$= \frac{n(n+1)-1}{(n^2+1)(n^2+2n+2)}$$

$= \text{positive, for } n\ge 1$

$\therefore$ $u_n+1<u_n$ for all n.

$\therefore$ By Leibnitz test, $\sum(-1)^{n-1}u_n$ is (C) .

(ii)

$$\sum(-1)^{n-1}u_n = \frac{1}{1}-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\cdots$$

$$\therefore \qquad u_n = \frac{n}{2n-1}$$

$$u_n - u_{n+1} = \frac{n}{2n-1} - \frac{n+1}{2n+1}$$

$$= \frac{n(2n+1)-(n+1)(2n-1)}{(2n-1)(2n+1)}$$

$$= \frac{1}{4n^2-1} > 0, \text{ for all } n.$$

$\therefore$ $\qquad u_{n+1} < u_n$ for all n.

But
$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{n}{2n-1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{2-\frac{1}{n}}\right)$$

$$= \frac{1}{2} \neq 0$$

$\therefore$ The given series is not (C). It is oscillating.

Example 2.2 Examine the convergence of the series:

(i) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$ to ∞

(ii) $\sum_{n=1}^{\infty}(-1)^{n-1}\sqrt{n+1}-\sqrt{n})$.

(i) Let
$$\sum(-1)^{n-1}u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots + (-1)^{n-1}\cdot\frac{1}{n\sqrt{n}} + \cdots$$

$$u_n = \frac{1}{n\sqrt{n}}$$

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{1}{n\sqrt{n}}\right) = 0$$

Now
$$u_n - u_{n+1} = \frac{1}{n\sqrt{n}} - \frac{1}{(n+1)\sqrt{n+1}}$$

$$= \frac{(n+1)\sqrt{n+1} - n\sqrt{n}}{n(n+1)\sqrt{n(n+1}} > 0, \text{ for all } n \geq 1$$

$\therefore$ $\qquad u_{n+1} < u_n$, for all n.

$\therefore$ The given series is (C) by Leibnitz test.

(ii) Let $\sum(-1)^{n-1} u_n = \sum(-1)^{n-1} (\sqrt{n+1} - \sqrt{n})$

$\therefore$ $$u_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n\to\infty} (u_n) = \lim_{n\to\infty} \left\{\frac{1}{\sqrt{n+1} + \sqrt{n}}\right\}$$

$$= 0$$

Now $$u_n - u_{n+1} = \frac{1}{\sqrt{n+1} + \sqrt{n}} - \frac{1}{\sqrt{n+2} + \sqrt{n+1}}$$

$$= \frac{(\sqrt{n+2} + \sqrt{n+1}) - (\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})\ (\sqrt{n+2} + \sqrt{n+1})}$$

$$= \frac{\sqrt{n+2} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+2} + \sqrt{n+1})} > 0, \text{ for all } n \geq 1.$$

$\therefore$ $$u_{n+1} < u_n \text{ for all } n.$$

$\therefore$ By Leibnitz test, the given series is (C).

Example 2.3 Examine the convergence of the series:

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ to ∞;

(ii) $\frac{1}{1^2} - \frac{1}{4^2} + \frac{1}{7^2} - \frac{1}{10^2} + \cdots$ to ∞;

(i) Let $\sum(-1)^{n-1} u_n = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots + (-1)^{n-1} \frac{1}{n^2} + \cdots$

$\therefore$ $$u_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \left(\frac{1}{n^2}\right) = 0$$

Also $$u_n - u_{n+1} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$= \frac{(n+1)^2 - n^2}{n^2(n+1)^2}$$

$$= \frac{2n+1}{n^2(n+1)^2} > 0, \text{ for all } n \geq 1$$

$\therefore$ $u_{n+1} < u_n$, for all n.

$\therefore$ By Leibnitz test, the given series is (C).

(ii) Let $$\sum(-1)^{n-1}u_n = \frac{1}{1^2}-\frac{1}{4^2}+\frac{1}{7^2}-\cdots+(-1)^{n-1}\cdot\frac{1}{(3n-2)^2}+\cdots$$

$\therefore$ $$u_n = \frac{1}{(3n-2)^2}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{(3n-2)^2} = 0$$

Also $$u_n - u_{n+1} = \frac{1}{(3n-2)^2}-\frac{1}{(3n+1)^2}$$

$$= \frac{(3n+1)^2-(3n-2)^2}{(3n-2)^2\,(3n+1)^2}$$

$$= \frac{3(6n-1)}{(3n-2)^2\,(3n+1)^2} > 0 \text{ for all } n \geq 1$$

i.e. $u_{n+1} < u_n$ for all n.

$\therefore$ The given series is (C), by Leibnitz test.

Example 2.4 Examine the convergence of the series:

(i) $\dfrac{1}{1\cdot 2}-\dfrac{1}{3\cdot 4}+\dfrac{1}{5\cdot 6}-\dfrac{1}{7\cdot 8}+\cdots;$

(ii) $\dfrac{1}{1\cdot 2\cdot 3}-\dfrac{1}{2\cdot 3\cdot 4}+\dfrac{1}{3\cdot 4\cdot 5}-\cdots;$

(i) Let $$\sum(-1)^{n-1}u_n = \frac{1}{1\cdot 2}-\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}-\cdots$$

$\therefore$ $$u_n = \frac{1}{(2n-1)\cdot 2n}$$

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\frac{1}{(2n-1)\cdot 2n} = 0$$

Also $$u_n - u_{n+1} = \frac{1}{(2n-1)2n}-\frac{1}{(2n+1)\,(2n+2)}$$

$$= \frac{(2n+1)(2n+2)-2n(2n-1)}{(2n-1)\cdot 2n\cdot(2n+1)\,(2n+2)}$$

$$= \frac{8n+2}{(2n-1)\cdot 2n\cdot(2n+1)\,(2n+2)} > 0, \text{ for all } n \geq 1$$

$\therefore$ $u_{n+1} < u_n$ for all n.

$\therefore$ By Leibnitz test, the given series is (C).

(ii) Let
$$\sum (-1)^{n-1} u_n = \frac{1}{1\cdot 2\cdot 3} = \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} - \cdots$$

$$u_n = \frac{1}{n(n+1)(n+2)}$$

$$\lim_{n\to\infty} (u_n) = \lim_{n\to\infty} \left\{\frac{1}{n(n+1)\,(n+2)}\right\} = 0$$

Also
$$u_n - u_{n+1} = \frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)\,(n+3)}$$

$$= \frac{(n+3)-n}{n(n+1)(n+2)(n+3)} = \frac{3}{n(n+1)(n+2)(n+3)}$$

$$u_n - u_{n+1} > 0, \text{ for all } n \geq 1$$

or $u_{n+1} < u_n$, for all n.

∴ By Leibnitz test, the given series is (C).

Example 2.5 Examine the convergence of the following series:

(i) $\dfrac{1}{2!} - \dfrac{2}{3!} + \dfrac{3}{4!} - \cdots;$

(ii) $\dfrac{x}{1+x} - \dfrac{x^2}{1+x^2} + \dfrac{x^3}{1+x^3} - \cdots \ (0 < x < 1).$

(i) Let
$$\sum (-1)^{n-1} u_n = \frac{1}{2!} - \frac{2}{3!} + \frac{3}{4!} - \cdots$$

∴
$$u_n = \frac{n}{(n+1)!}$$

$$\lim_{n\to\infty} (u_n) = \lim_{n\to\infty} \left[\frac{n}{1\cdot 2\cdot 3\cdots n\cdot(n+1)}\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{1\cdot 2\cdot 3\cdots (n-1)(n+1)}\right]$$

$$= 0$$

Also
$$u_n - u_{n+1} = \frac{n}{(n+1)!} - \frac{n+1}{(n+2)!}$$

$$= \frac{n(n+2)-(n+1)}{(n+2)!}$$

$$= \frac{n^2 + n - 1}{(n+2)!}$$

$$= \frac{n(n+1) - 1}{(n+2)!} > 0, \text{ for all } n \geq 1$$

$\therefore$ $u_{n+1} < u_n$, for all n.

$\therefore$ By Leibnitz test, the given series is (C).

(ii) Let $\sum (-1)^{n-1} u_n = \frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \cdots (0 < x < 1).$

$$\therefore \quad u_n = \frac{x^n}{1+x^n}$$

$$\lim_{n \to \infty} (u_n) = \lim_{n \to \infty} \left(\frac{x^n}{1+x^n} \right)$$

$$= \lim_{n \to \infty} \left\{ \frac{1}{\left(\frac{1}{x}\right)^n + 1} \right\}$$

$$= 0, \text{ since } \left(\frac{1}{x}\right)^n \to \infty, \text{ as } \frac{1}{x} > 1$$

Also $$u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$= \frac{x^n (1 + x^{n+1}) - x^{n+1} (1 + x^n)}{(1 + x^n)(1 + x^{n+1})}$$

$$= \frac{x^n (1 - x)}{(1 + x^n)(1 + x^{n+1})} > 0, \text{ for all } n, \text{ since } 0 < x < 1$$

i.e. $u_{n+1} < u_n$ for all n.

$\therefore$ By Leibnitz test, the given series is (C).

EXERCISE 2(c)

Part A

(Short answer questions)

1. State Leibnitz test for the convergence of an alternating series.
2. Show that the series $\sum (-1)^{n-1} \cdot \frac{1}{n}$ is (C).

3. What is meant by absolute convergence?
4. Give an example for a series which is absolutely convergent.
5. What do you mean by conditional convergence?
6. Give an example for a series that is conditionally convergent.
7. Give the values of x for which the binomial series and the logarithmic series are convergent.
8. Show that the series $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots$ is convergent.

Part B

Examine the convergence of the following alternating series.

9. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots$
10. $1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \frac{9}{16} \ldots$
11. $\frac{3}{4} - \frac{5}{7} + \frac{7}{10} - \frac{9}{13} + \ldots$
12. $1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} - \frac{\sqrt{4}}{4} + \ldots$
13. $\sum_{n=1}^{\infty} (-1)^{n-1} . \frac{(n^2+1)}{n^3+1}$
14. $\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$
15. $\frac{1}{1^2+1} - \frac{1}{2^2+1} + \frac{1}{3^2+1} - \frac{1}{4^2+1} + \ldots$
16. $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots$
17. $\frac{1}{3.4.5} - \frac{1}{4.5.6} + \frac{1}{5.6.7} - \ldots$
18. $\frac{1}{1.1.3} - \frac{1}{2.3.5} + \frac{1}{3.5.7} - \frac{1}{4.7.9} + \ldots$
19. $\frac{3}{2!} - \frac{5}{4!} + \frac{7}{6!} - \ldots$
20. $\frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} - \ldots$

2.4 SEQUENCES AND SERIES

Definition

If to each positive integer n, a quantity a_n is assigned, then the quantities a_1, a_2, …, a_n, ... are said to form an *infinite sequence* or simply *sequence*, denoted by $\{a_n\}$. The individual quantities a_n are called the *terms* of the sequence.

If the terms of a sequence are real, then it is called *a real sequence*.

Limit of a Sequence

A sequence $\{a_n\}$ is said to be *convergent* to the limit 'l', if there exists an integer N, such that

$|a_n - l| < \in$ for all $n > N$, where $\in$ is a positive real quantity, however small it may be, but not zero.

This is denoted as $\lim\limits_{n\to\infty} (a_n) = l$ or $a_n \to l$, as $n \to \infty$

Note ☑ If a sequence converges, the limit is unique.

Note ☑ For all $n > N$ 'in the definition means for infinitely many n'. A sequence, that is not *convergent*, is said to be *divergent.*

Examples

1. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$, viz., $\left\{\frac{n}{n+1}\right\}$ is convergent to the limit 1, as

   $$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \in, \text{ when } n+1 > \frac{1}{\in} \text{ or } n > \frac{1}{\in} - 1.$$

   We note that $n > 99$, if we chose $\varepsilon = 0.01$.

2. The sequence $\left|2 + \frac{3}{n}\right|$, viz., 5, $\frac{5}{7}$, 3, $\frac{11}{4}$, $\frac{13}{5}$, $\cdots$ converges to the limit 2, as

   $$\left|2 + \frac{3}{n} - 2\right| = \frac{3}{n} < 6, \text{ when } n > \frac{3}{6}$$

It is noted that $n > 300$, if we choose $\in = 0.01$.

Definitions

1. The sequence $\{a_n\}$ is said to be *bounded*, if there is a positive number K such that $|a_n| < K$, for all n. A sequence that is not bounded is said to be *unbounded*.
2. A real sequence $\{a_n\}$ is said to be monotonic increasing or monotonic decreasing, according as

   $$a_1 \le a_2 \le a_3 \le \ldots \quad \text{or} \quad a_1 \ge a_2 \ge a_3 \ge \ldots$$

A sequence that is either *monotonic increasing* or *monotonic decreasing* is called a *monotonic sequence*.

We give below three theorems regarding convergence of sequences without proof:

Theorems

1. A sequence $\{a_n\}$ is convergent, if and only if for every position number $\in$, we can find a number N (which may depend on $\in$) such that
$$|a_m - a_n| < \in, \text{ when } m > N \text{ and } n > N$$
2. Every convergent sequence is bounded. Hence if a sequence is unbounded, it diverges.

Example The sequences 1, 2, 3, 4,… and $\frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots$ are unbounded and hence diverge.

Note ☑ Boundedness is *not sufficient* for convergence.

Example The sequence $\frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \ldots$ is bounded since $|a_n| < 1$, but it is divergent, since $\lim\limits_{n\to a}(a_n) = 0$ or 1.

3. If a real sequence is bounded and monotonic, it is convergent.

Examples

1. Thought the sequence 1, 2, 3, … is monotonic increasing, it is divergent, as it is unbounded.
2. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ is both monotonic increasing and bounded and hence it converges to the limit 1.

EXERCISE 2(d)

Test the convergence of the following sequences: If convergent find the limit also.

1. $\left\{\frac{2n-1}{n}\right\} \equiv 1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots$
2. $\left\{\frac{n+2}{n}\right\} \equiv 3, 2, \frac{5}{3}, \frac{3}{2}, \ldots$
3. $\frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots$
4. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
5. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
6. $1, -2, 3, -4, \ldots$
7. $\left\{\frac{1}{n}\log_{e^n}\right\}\ldots$
8. $\{n^{1/n}\}\ldots$

9. $\left\{\frac{a^n}{n!}\right\}....(a<0)$

10. $a, 2a^2, 3a^3, ...(|a|<1)$

More Tests of Convergence for Series of Positive Terms

1. Cauchy's Integral Test

If $\sum_{n=1}^{\infty} u_n$ is a series of decreasing positive terms, so that *u(x)* is a decreasing function of x, for $x \geq 1$, then the given series is convergent, if $\int_1^{\infty} u(x)\,dx$ exists and divergent if $\int_1^{\infty} u(x)\,dx$ does not exists.

2. Cauchy's Condensation Test

If $f(n)$ is a decreasing positive function of n and 'a' is any positive integer > 1, then the two series $\sum f(n)$ and $\sum a^n . f(a^n)$ are both divergent.

3. Logarithmic Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right)=l$, then $\sum u_n$ is convergent if $l > 1$ and divergent if $l < 1$.

4. Gauss's test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that $\frac{u_n}{u_{n+1}}=1+\frac{h}{n}+\frac{A(n)}{n^2}$, where $A(n)$ is a bounded function of n as $n \to \infty$, then the series is convergent if $h > 1$ and divergent if $h \leq 1$.

5. Kummer's Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms and $\{a_n\}$ is a sequence of positive terms such that $\left(a_n.\frac{u_n}{u_{n+1}}-a_{n+1}\right)\geq r>0$, for $n \geq m$, then $\sum u_n$ is convergent. If $\left(a_n.\frac{u_n}{u_{n+1}}-a_{n+1}\right)\leq 0$, then $\sum u_n$ is divergent, provided that $\sum_{n=1}^{\infty}\frac{1}{a_n}$ is divergent.

Note ☑ Raabe's test is a particular case of Kummer's test corresponding to $a_n = n$.

WORKED EXAMPLE 2(d)

Example 2.1 Test the convergence of the p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$, by using the integral test.

$$\sum_{n=1}^{\infty}\frac{1}{np} = \sum_{n=1}^{\infty}u_n \qquad \therefore\ u(x) = \frac{1}{x^p}$$

$$\int_1^{\infty}\frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^{\infty} = -\frac{1}{p-1}\cdot\left(\frac{1}{x^{p-1}}\right)_1^{\infty} = \begin{cases}\frac{1}{p-1}, \text{if } p>1\\ \infty, \text{if } p\le 1\end{cases}$$

$\therefore\ \sum_{n=1}^{\infty}\frac{1}{np}$ is (c), when $p > 1$ and (D), when $p \le 1$, by Cauchy's integral test.

Example 2.2 Test the convergence of the series $\sum\frac{1}{n\log n}$.

Let $\quad \sum\frac{1}{n\log n} = \sum u_n \qquad \therefore\ u(x) = \frac{1}{x\log x}$

Now $\quad \int_1^{\infty}\frac{1}{x\log x}\,dx = [\log\log x]_1^{\infty} = \infty$

$\therefore$ By integral test, $\sum u_n$ is (D).

Example 2.3 Test the convergence of the series $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}$

Let $\quad f(n) = \frac{1}{n(\log n)^p}$

$$\therefore\quad a^n f(a^n) = a^n\frac{1}{a^n(\log a^n)^p} = \frac{1}{(n\log a)^p} = \frac{1}{(\log a)^p}\cdot\frac{1}{n^p}$$

Now $\quad \sum\frac{1}{(\log a)^p . n^p} = \frac{1}{(\log a)^p}\sum\frac{1}{n^p}$

is (C), if $p > 1$ and (D), if $p \le 1$, by Cauchy's condensation test,

Example 2.4 Test the convergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots\infty$.

$$\frac{u_n}{u_{n+1}} = \frac{n^nx^n}{n!}\times\frac{(n+1)!}{(n+1)^{n+1}x^{n+1}} = \left(\frac{n}{n+1}\right)^n\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right)=\lim_{n\to\infty}\frac{1}{(1+1/n)^n}\cdot\frac{1}{x}=\frac{1}{ex}$$

$\therefore$ By Ratio test, $\sum u_n$ is (C), when $x<\frac{1}{e}$ and (D) when $x>\frac{1}{e}$.

When $x=\frac{1}{e}$, the ratio test fails.

$$\log\left(\frac{u_n}{u_{n+1}}\right)=\log\left\{\frac{e}{(1+1/n)^n}\right\}=1-n\log\left(1+\frac{1}{n}\right)$$

$$=1-n\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right]=\frac{1}{2n}-\frac{1}{3n^2}+\ldots$$

Now $\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\}=\frac{1}{2}<1.$

$\therefore$ By the Logarithmic test, $\sum u_n$ is (D).

Example 2.5 Test the convergence of the series $\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}.\frac{1}{5}+\frac{1.3.5}{2.4.6}.\frac{1}{7}+\cdots\infty.$

$$u_n=\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots(2n)}\cdot\frac{1}{2n+1}$$

$$\therefore\quad \frac{u_n}{u_{n+1}}=\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots(2n)}\cdot\frac{1}{2n+1}\times\frac{2\cdot4\cdot6\ldots(2n)(2n+2)}{1\cdot3\cdot5\ldots(2n-1)(2n+1)}\cdot(2n+3)$$

$$=\frac{(2n+2)(2n+3)}{(2n+1)^2}$$

$$=\frac{(1+1/n)(1+3/2n)}{(1+1/2n)^2}=\left\{1+\frac{(5/2)}{n}+\frac{(3/2)}{n^2}\right\}\left(1+\frac{1}{2n}\right)^{-2}$$

$$=\left\{1+\frac{(5/2)}{n}+\frac{(3/2)}{n^2}\right\}\left\{1-\frac{2}{2n}+\frac{3}{4n^2}\cdots\right\}^{-2}$$

$$=1+\frac{(3/2)}{n}+\text{terms containing}\frac{1}{n^2}\text{and higher power of }\frac{1}{n}$$

$$=1+\frac{(3/2)}{n}+0\left(\frac{1}{n^2}\right)\equiv1+\frac{h}{n}+0\left(\frac{1}{n^2}\right)$$

$\therefore$ $h>1$ and so $\sum u_n$ is (C), by Gauss's test.

EXERCISE 2(e)

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$, using integral test.

2. $\sum_{n=1}^{\infty} \frac{1}{n}$, using integral test.

3. $\sum_{n=1}^{\infty} \frac{n}{1+n^4}$, using integral test.

4. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, using integral test.

5. $\sum_{n=1}^{\infty} \frac{1}{n^p}$, using condensation test.

6. $\sum_{n=2}^{\infty} \frac{1}{n \log n}$, using condensation test.

7. $1 + \frac{2x}{2!} + \frac{3^2 x^2}{3!} + \frac{4^3 x^3}{4!} + \cdots \infty$, using logarithmic test.

8. $\frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots \infty$, using logarithmic test.

9. $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \infty$, using Gauss's test.

10. $\frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5}x^3 + \cdots \infty$, using Gauss's test.

ANSWERS

Exercise 2(a)

(5) $\sum \frac{1}{n}$	(10) Dgt.	(11) Dgt.	(12) Cgt.	(13) Cgt.	(14) Cgt.
(15) Dgt.	(16) Dgt.	(17) Dgt.	(18) Cgt.	(19) Cgt.	(20) Cgt.
(21) Cgt.	(22) Dgt.	(23) Cgt.	(24) Dgt.	(25) Dgt.	(26) Cgt.

(27) Cgt. if $k > 4$ and dgt. if $k \geq 4$. (28) Cgt. if $p > 2$ and dgt. if $p \leq 2$
(29) Dgt. (30) Dgt. (31) Cgt. (32) Cgt. (33) Cgt. (34) Cgt.
(35) Cgt. if $x > 4$ and dgt. if $x \geq 1$. (36) Cgt.
(37) Cgt. if $x > 4$ and dgt. if $x \geq 1$. (38) Cgt. (39) Cgt. (40) Cgt.

Exercise 2(b)

(6) Dgt. (7) Cgt. (8) Cgt. (9) Dgt. (10) Dgt. (11) Cgt.
(12) Cgt. (13) Cgt. (14) Cgt. if $x < 1$ and dgt. if $x \geq 1$.
(15) Cgt. if $x^2 \leq 1$ and dgt. if $x^2 > 1$. (16) Cgt. if $x \leq 1$ and dgt. if $x > 1$.
(17) Cgt. if $x < 1$ and dgt. if $x \geq 1$. (18) Cgt. if $x < 1$ and dgt. if $x \geq 1$.
(19) Cgt. if $x < 1$ and dgt. if $x \geq 1$. (20) Cgt. if $x \leq 1$ and dgt. if $x > 1$.
(21) Cgt. (22) Cgt. (23) Cgt. if $x^2 \leq 1$ and dgt. if $x^2 > 1$.
(24) Cgt. if $x < \frac{i}{e}$ and dgt. if $x \geq \frac{i}{e}$. (25) Cgt. if $x < 2$ and dgt. if $x \geq 2$.
(26) Cgt. if $x < 4$ and dgt. if $x \geq 4$. (27) Cgt. if $x < \frac{1}{3}$ and dgt. if $x \geq \frac{1}{3}$.
(28) Cgt. (29) Cgt. (30) Cgt.

Exercise 2(c)

(4) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ (6) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ (9) Cgt.
(10) Cgt. (11) Oscillatory (12) Cgt. (13) Cgt. (14) Cgt.
(15) Cgt. (16) Cgt. (17) Cgt. (18) Cgt. (19) Cgt. (20) Cgt.

Exercise 2(d)

(1) Cgt. to 2 (2) Cgt. to 1 (3) Dgt. (4) Cgt. to 1 (5) Cgt. to 0 (6) Dgt.
(7) Cgt. to 0 (8) Cgt. to 1 (9) Cgt. to 0 (10) Cgt. to 0, when $0 \leq 1/2$

Exercise 2(e)

(1) Cgt. (2) Dgt. (3) Cgt. (4) Cgt. if $p > 1$ and dgt. if $p \leq 1$
(5) Cgt. if $p > 1$ and dgt, if $p \leq 1$. (6) Dgt.
(7) Cgt., if $x \leq \frac{1}{e}$ and dgt., if $x > \frac{1}{e}$ (8) Cgt., if $x < 1$ and dgt. If $x \geq 1$
(9) Dgt. (10) Cgt.

Chapter 3

Application of Differential Calculus

3.1 CURVATURE AND RADIUS OF CURVATURE

Consider the two circles shown in the Fig. 3.1. It is obvious that the ways in which the two circles bend or 'curve' at the point P are not the same. The smaller circle 'curves' or changes its direction more rapidly than the bigger circle. In other words the smaller circle is said to have greater curvature than the other. This concept of curvature which holds good for any curve is formally defined as follows:

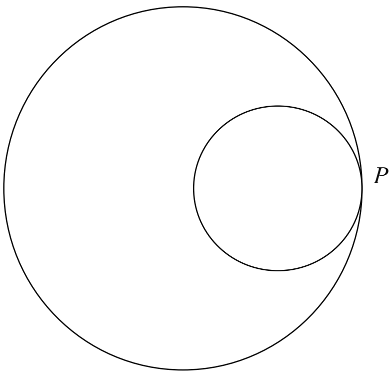


Fig. 3.1

3.1.1 Definition of Curvature

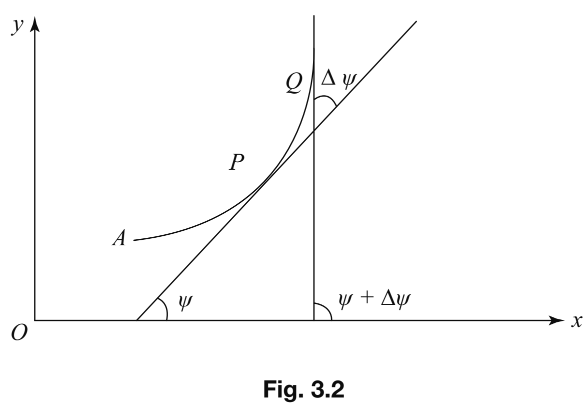


Fig. 3.2

Let P and Q be any two close points on a plane curve. Let the arcual distances of P and Q measured from a fixed point A on the given curve be s and $s + \Delta s$, so that $\overparen{PQ}$ (the arcual length of PQ) is Δs. [Refer to Fig. 3.2]

Let the tangents at P and Q to the curve make angles ψ and $\psi + \Delta\psi$ with a fixed line in the plane of the curve, say, the x-axis.
Then the angle between the tangents at P and $Q = \Delta\psi$.

Thus for a change of Δs in the arcual length of the curve, the direction of the tangent to the curve changes by $\Delta\psi$.

Hence $\frac{\Delta\psi}{\Delta s}$ is the average rate of bending of the curve (or average rate of change of direction of the tangent to the curve in the arcual interval $\overparen{PQ}$) or average curvature of the arc PQ.

$\therefore \lim\limits_{\Delta s \to 0}\left(\frac{\Delta\psi}{\Delta s}\right) = \frac{d\psi}{ds}$ is the rate of bending of the curve with respect to arcual distance at P or *the curvature* of the curve at the point P. The curvature is denoted by k.

For example, let us find the curvature of a circle of radius at any point on it. [Refer to Fig. 3.3]

Let the arcual distances of points on the circle be measured from A, the lowest point of the circle and let the tangent at A be chosen as the x-axis. Let $AP = s$ and let the tangent at P make an angle ψ with the x-axis.

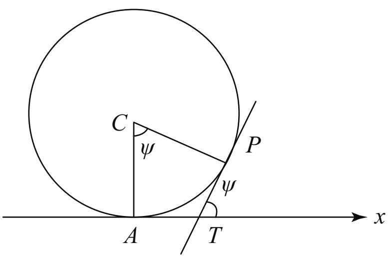


Fig. 3.3

Then $s = a\, A\hat{C}P$

$= a\ \psi$ [$\because$ the angle between CA and CP equals the angle between the respective perpendiculars AT and PT.]

or $\quad \psi = \frac{1}{a}s$

$\therefore \frac{d\psi}{ds} = \frac{1}{a}$

Thus the curvature of a circle at any point on it equals the reciprocal of its radius. Equivalently, the radius of a circle equals the reciprocal of the curvature at any point on it. It is this property of the circle that has led to the definition of radius of curvature.

Radius of curvature of a curve at any point on it is defined as the reciprocal of the curvature of the curve at that point and denoted by ρ. Thus $\rho = \frac{1}{k} = \frac{ds}{d\psi}$.

Note ☑ To find k or ρ of a curve at any point on it, we should know the relation between s and ψ for that curve, which is not easily derivable in most cases.

Generally curves will be defined by means of their cartesian, parametric or polar equations. Hence formulas for ρ in terms of cartesian, parametric or polar co-ordinates are necessary, which are derived below:

Some Basic Results: Let $P\,(x, y)$ and $Q\,(x + \Delta x,\, y + \Delta y)$ be any two close points on a curve $y = f(x)$. [Refer to Fig. 3.4.] Let $\overparen{AP} = s$ and $\overparen{AQ} = s + \Delta s$ where A is a fixed point on the curve. Let the chord PQ make an angle θ with the x-axis.

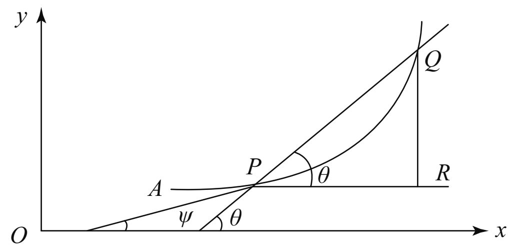


Fig. 3.4

From ΔPQR, $\sin\theta = \dfrac{RQ}{PQ} = \dfrac{RQ}{\Delta s}\cdot\dfrac{\Delta s}{PQ}$, where $\widehat{PQ} = \Delta s$ (1)

$$= \frac{\Delta y}{\Delta s}\cdot\frac{\Delta s}{PQ}$$

and $\cos\theta = \dfrac{PR}{PQ} = \dfrac{PR}{\Delta s}\cdot\dfrac{\Delta s}{PQ}$

$$= \frac{\Delta x}{\Delta s}\cdot\frac{\Delta s}{PQ} \tag{2}$$

When Q approaches P, chord $PQ \to$ tangent at P and hence $\theta \to \psi$. Also $\dfrac{\Delta s}{PQ} \to 1$.

Thus in the limiting case when $Q \to P$, (1) and (2) become $\sin\psi = \dfrac{dy}{ds}$ and $\cos\psi = \dfrac{dx}{ds}$.

$\therefore$ $$\tan\psi = \frac{dy}{dx}.$$

3.1.2 Formula for Radius of Curvature in Cartesian Co-ordinates

Let ψ be the angle made by the tangent at any point (x, y) on the curve $y = f(x)$.

Then $$\tan\psi = \frac{dy}{dx} \tag{1}$$

Differentiating both sides of (1) w.r.t. x, we get,

$$\sec^2\psi\,\frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

i.e. $$\sec^2\psi\,\frac{d\psi}{ds}\cdot\frac{ds}{dx} = \frac{d^2y}{dx^2}$$

i.e. $$\sec^2\psi\cdot\frac{1}{\rho}\cdot\sec\psi = \frac{d^2y}{dx^2} \qquad \left[\because \cos\psi = \frac{dx}{ds}\right]$$

$\therefore$ $$\rho = \frac{\sec^3\psi}{\dfrac{d^2y}{dx^2}}$$

$$= \frac{(1+\tan^2\psi)^{3/2}}{\dfrac{d^2 y}{dx^2}} = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2 y}{dx^2}}, \text{ by (1).}$$

Note ☑ As curvature (and hence radius of curvature) of a curve at any point is independent of the choice of x and y-axis, x and y can be interchanged in the formula for ρ derived above. Thus ρ is also given by

$$\rho = \frac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}{\dfrac{d^2 x}{dy^2}}$$

This formula will be of use, when $\dfrac{dy}{dx}$ is infinite at a point.

3.1.3 Formula for Radius of Curvature in Parametric Co-ordinates

Let the parametric equations of the curve be

$$x = f(t) \quad \text{and} \quad y = g(t).$$

Then $\quad \dot{x} = \dfrac{dx}{dt} = f'(t) \quad$ and $\quad \dot{y} = \dfrac{dy}{dt} = g'(t).$

$$\therefore \quad \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \times \frac{dt}{dx}$$

$$= \left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}\right) \cdot \frac{1}{\dot{x}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Now $$\rho = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\left(\dfrac{d^2 y}{dx^2}\right)}$$

$$= \frac{\left\{1+\left(\dfrac{\dot{y}}{\dot{x}}\right)^2\right\}^{3/2}}{\left(\dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}\right)}$$

$$= \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}^3} \times \frac{\dot{x}^3}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

3.1.4 Formula for Radius of Curvature in Polar Co-ordinates

Students are familiar with the following transformations from cartesian co-ordinates (x, y) to polar co-ordinates (r, θ):

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{1}$$

We shall make use of (1) and the formula for the radius of curvature in cartesian co-ordinates, namely,

$$\rho = \frac{\left\{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2}}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}} \tag{2}$$

and derive the corresponding formula for ρ at the point (r, θ) which lies on the curve $r = f(\theta)$

Now
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$$

where $r' = \dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ (3)

$[\because r \text{ is a function of } \theta]$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}\theta} \div \frac{\mathrm{d}x}{\mathrm{d}\theta}$$

$$= \frac{[(-r\sin\theta + r'\cos\theta)(-r\sin\theta + 2r'\cos\theta + r''\sin\theta) - (r\cos\theta + r'\sin\theta)(-r\cos\theta - 2r'\sin\theta + r''\cos\theta)]}{(-r\sin\theta + r'\cos\theta)^3}$$

$$= \frac{[(r^2\sin^2\theta - 3rr'\sin\theta\cos\theta - rr''\sin^2\theta + 2r'^2\cos^2\theta + r'r''\sin\theta\cos\theta) + (r^2\cos^2\theta + 3rr'\sin\theta\cos\theta - rr''\cos^2\theta + 2r'^2\sin^2\theta - r'r''\sin\theta\cos\theta)]}{(-r\sin\theta + r'\cos\theta)^3}$$

$$= \frac{r^2 - rr'' + 2r'^2}{(-r\sin\theta + r'\cos\theta)^3}, \text{where } r'' = \frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} \tag{4}$$

Also
$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \left(\frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}\right)^2$$

$$= \frac{r^2 + r'^2}{(-r\sin\theta + r'\cos\theta)^2} \tag{5}$$

Using (4) and (5) in (2), we get

$$\rho = \frac{(r^2 + r'^2)^{3/2}}{r^2 - rr'' + 2r'^2}$$

3.2 CENTRE AND CIRCLE OF CURVATURE

Let $P(x, y)$ be a point on the curve $y = f(x)$. On the inward drawn normal to the curve at P, cut off a length PC = radius of curvature of the curve at P (namely ρ). The point C is called *the centre of curvature* at P for the curve. [Refer to Fig. 3.5]

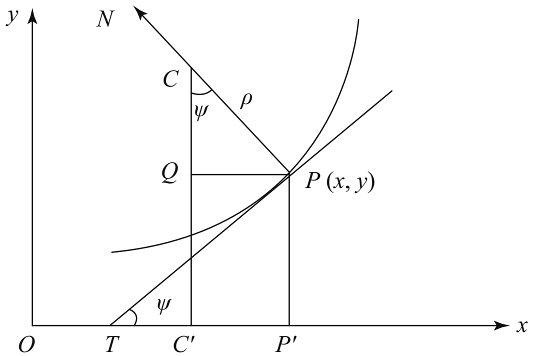


Fig. 3.5

The circle whose centre is C and radius ρ is called *the circle of curvature* at P for the curve.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of C.

Then $\bar{x} = OC'$

$= OP' - QP$

$= x - \rho \sin\psi$ ($\because$ angle between CP and CQ = angle between the respective perpendiculars PT and OP')

$$= x - \frac{\rho}{\operatorname{cosec}\psi}$$

$$= x - \frac{\rho}{\sqrt{1 + \cot^2\psi}}$$

$$= x - \frac{\rho y'}{\sqrt{1 + y'^2}} \qquad \left(\because \cot\psi = \frac{dx}{dy} = \frac{1}{y'}\right)$$

$$= x - \frac{(1 + y'^2)^{3/2}}{y''} \cdot \frac{y'}{\sqrt{1 + y'^2}} \qquad \left(\text{where } y' = \frac{dy}{dx} \text{ and } y'' = \frac{d^2y}{dx^2}\right)$$

i.e. $$\bar{x} = x - \frac{y'}{y''}(1 + y'^2)$$

Now $\bar{y} = C'C$

$= P'P + QC$

$$= y + \rho \cos \psi$$

$$= y + \frac{\rho}{\sec \psi}$$

$$= y + \frac{\rho}{\sqrt{1 + \tan^2 \psi}}$$

$$= y + \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} \cdot \frac{1}{\sqrt{1 + y'^2}}$$

$$= y + \frac{(1 + y'^2)}{y''}$$

Having found out the co-ordinates of the centre of curvature, the equation of the circle of curvature is written as $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

WORKED EXAMPLE 3(a)

Example 3.1 Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$.

Differentiating the equation of the curve with respect to x,

$$3\left(x^2 + y^2 \frac{dy}{dx}\right) = 3a\left(x \frac{dy}{dx} + y\right)$$

i.e. $$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$\therefore$ $$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \tag{1}$$

Again differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2} \tag{2}$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1 \quad \text{and}$$

$$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{\frac{3}{4}a^2(-a-3a) - \left(-\frac{3a^2}{4}\right)(-3a-a)}{\left(\frac{9a^4}{16}\right)}$$

Note ☑ It is not necessary to express $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of x and y from (1) and then evaluate $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$. When $x = \frac{3a}{2}$ and $y = \frac{3a}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, which may be used in (2).

i.e.
$$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -\frac{6a^3}{9a^4} \times 16 = -\frac{32}{3a}$$

$$\rho = \frac{\left\{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2}}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}$$

∴
$$(\rho)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{(1+1)^{3/2}}{\left(-\frac{32}{3a}\right)}$$

∴
$$|\rho| = \frac{3\sqrt{2}a}{16}.$$

Example 3.2 Find the radius of curvature at $(a, 0)$ on the curve $xy^2 = a^3 - x^3$.

The equation of the curve is

$$y^2 = \frac{a^3 - x^3}{x} \tag{1}$$

Differentiating w.r.t. x,

$$2yy' = \frac{x(-3x^2) - (a^3 - x^3)}{x^2} = \frac{-(2x^3 + a^3)}{x^2}$$

i.e.
$$y' = -\frac{(2x^3 + a^3)}{2x^2 y} \tag{2}$$

Now $(y')_{(a,0)} = \infty$.

∴ The formula $\rho = \dfrac{\left\{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2\right\}^{3/2}}{\left(\frac{\mathrm{d}^2 x}{\mathrm{d}y^2}\right)}$ should be used.

From (2), $$\frac{dx}{dy} = -\frac{2x^2 y}{2x^3 + a^3} \tag{3}$$

Differentiating (3) w.r.t. y,

$$\frac{d^2x}{dy^2} = -2\left[\frac{\left(2x^3 + a^3\right)\left(x^2 + y \cdot 2x\,\dfrac{dx}{dy}\right) - x^2\,y \cdot 6x^2\,\dfrac{dx}{dy}}{\left(2x^3 + a^3\right)^2}\right] \tag{4}$$

From (3), we get $\left(\dfrac{dx}{dy}\right)_{(a,\,0)} = 0$

From(4), we get $\left(\dfrac{d^2x}{dy^2}\right)_{(a,\,0)} = -\dfrac{2 \times 3a^5}{9a^6} = -\dfrac{2}{3a}$

$\therefore$ $$|\rho| = \frac{(1+0)^{3/2}}{2/3a} = \frac{3a}{2}$$

Example 3.3 If ρ is the radius of curvature at any point (x, y) on the curve $y = \dfrac{ax}{a+x}$, show that $\left(\dfrac{2\rho}{a}\right)^{2/3} = \left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2$.

$$y = \frac{ax}{a+x} \tag{1}$$

Differentiating w.r.t. x,

$$y' = \frac{a\left(a + x - x\right)}{(a+x)^2} = \frac{a^2}{(a+x)^2} \tag{2}$$

Differentiating again w.r.t. x,

$$y'' = \frac{-2a^2}{(a+x)^3} \tag{3}$$

$$\rho = \frac{\left(1 + y'^2\right)^{3/2}}{|y''|} \quad \text{(only the numerical value of } \rho \text{ is considered)}$$

$$= \frac{\left\{1 + \dfrac{a^4}{(a+x)^4}\right\}^{3/2} (a+x)^3}{2a^2}$$

$\therefore$ $$\frac{2\rho}{a} = \frac{\left\{(a+x)^4 + a^4\right\}^{3/2}}{a^3\,(a+x)^3}$$

$\therefore$ $$\left(\frac{2\rho}{a}\right)^{2/3} = \frac{(a+x)^4 + a^4}{a^2\,(a+x)^2} = \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

$$=\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2 \quad \left[\because \text{ the point } (x,y) \text{ lies on the curve } \frac{y}{x}=\frac{a}{a+x}\right]$$

Example 3.4 Show that the measure of curvature of the curve $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ at any point (x, y) on it is $\dfrac{ab}{2(ax+by)^{\frac{3}{2}}}$.

The equation of the curve is

$$\frac{1}{\sqrt{a}}\sqrt{x}+\frac{1}{\sqrt{b}}\sqrt{y}=1 \tag{1}$$

Differentiating w.r.t. x,

$$\frac{1}{2\sqrt{a}\sqrt{x}}+\frac{1}{2\sqrt{b}\sqrt{y}}y'=0$$

$\therefore$

$$y'=-\frac{\sqrt{b}\sqrt{y}}{\sqrt{a}\sqrt{x}} \tag{2}$$

Differentiating further w.r.t. x,

$$y''=-\frac{\sqrt{b}}{\sqrt{a}}\left\{\frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}y'-\sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}\right\}$$

$$=-\frac{\sqrt{b}}{\sqrt{a}}\cdot\frac{1}{x}\left\{-\frac{\sqrt{b}}{2\sqrt{a}}-\frac{\sqrt{y}}{2\sqrt{x}}\right\}, \text{ using } (2)$$

$$=\frac{\sqrt{b}}{2ax^{\frac{3}{2}}}\left\{\sqrt{bx}+\sqrt{ay}\right\}$$

$$=\frac{\sqrt{b}}{2ax^{\frac{3}{2}}}\times\sqrt{ab}, \text{ using } (1)$$

$$=\frac{b}{2\sqrt{a}\,x^{\frac{3}{2}}}$$

Now,

$$\rho=\frac{\left(1+y'^2\right)^{\frac{3}{2}}}{y''}=\frac{\left(1+\frac{by}{ax}\right)^{\frac{3}{2}}}{b}\times 2\sqrt{a}\,x^{\frac{3}{2}}$$

$$=\frac{2}{ab}(ax+by)^{\frac{3}{2}}$$

$\therefore$

$$\text{Curvature } k=\frac{1}{\rho}=\frac{ab}{2(ax+by)^{3/2}}$$

Example 3.5 Find the co-ordinates of the real points on the curve $y^2 = 2x(3 - x^2)$, the tangents at which are parallel to the x-axis. Show that the radius of curvature at each of these point is $\frac{1}{3}$.

$$y^2 = 2x(3 - x^2) \tag{1}$$

Differentiating w.r.t. x,

$$2yy' = 2[3 - 3x^2]$$

i.e.

$$yy' = 3(1 - x^2) \tag{2}$$

The points at which the tangents are parallel to the x-axis are given by $y' = 0$.

i.e. $3(1 - x^2) = 0$, from (2)

i.e. $x = \pm 1$.

Putting $x = -1$ in (1), we get y^2 = negative

i.e. y is imaginary.

$\therefore$ The real points are given by $x = 1$.

Putting $x = 1$ in (1), we get $y^2 = 4$. $\therefore$ $y = \pm 2$.

$\therefore$ The points, the tangents at which are parallel to the x-axis, are $(1, 2)$ and $(1, -2)$.

From(2),

$$y' = \frac{3(1 - x^2)}{y}$$

$$= \frac{3(1 - x^2)}{\sqrt{2}\sqrt{3x - x^3}}$$

Differentiating w.r.t. x,

$$y'' = \frac{3}{\sqrt{2}}\left\{\frac{\sqrt{3x - x^3}(-2x) - (1 - x^2)\frac{d}{dx}\sqrt{3x - x^3}}{3x - x^3}\right\}$$

$\therefore$

$$(y'')_{(1,\pm 2)} = \frac{3}{\sqrt{2}}\cdot\frac{(-2\sqrt{2})}{2} = -3$$

$$\rho = \frac{(1 + y'^2)^{3/2}}{|y''|}$$

$\therefore$

$$(\rho)_{(1,\pm 2)} = \frac{(1 + 0)^{3/2}}{|-3|} = \frac{1}{3}$$

Example 3.6 Show that the curves $y = c\cosh\frac{x}{c}$ and $x^2 = 2c(y - c)$ have the same curvature at the points where they cross the y-axis.

The point at which the curve $y = c\cosh\frac{x}{c}$ crosses the y-axis is got by solving the equation of the curve with $x = 0$.

Thus the point is (0, c).

Similarly the point of intersection of the second curve with the y-axis is also found to be (0, c)

Equation of the first curve is $y = c\cosh\dfrac{x}{c}$.

Differentiating w.r.t. x twice, we get

$$y' = c\sinh\frac{x}{c}.\frac{1}{c} = \sinh\frac{x}{c}$$

$$y'' = \frac{1}{c}\cosh\frac{x}{c}$$

$$\rho = \frac{(1+y'^2)^{3/2}}{y''} = \frac{\left(1+\sinh^2\dfrac{x}{c}\right)^{3/2}}{\dfrac{1}{c}\cosh\dfrac{x}{c}}$$

$$\therefore \qquad (\rho)_{(0,c)} = c.$$

Equation of the second curve is $y = \dfrac{x^2}{2c} + c$

Differentiating w.r.t. x twice, we get

$$y' = \frac{x}{c} \quad \text{and} \quad y'' = \frac{1}{c}$$

$$\therefore \qquad (\rho)_{(0,c)} = \left[\frac{\left(1+\dfrac{x^2}{c^2}\right)^{3/2}}{\dfrac{1}{c}}\right]_{x=0} = c$$

Thus $(\rho)_{(0,c)}$ is the same for both curves.

$\therefore$ $(k)_{(0,c)}$ is the same $\left(=\dfrac{1}{c}\right)$ for both curves.

Example 3.7 Find the radius of curvature at the point ($a\cos^3\theta$, $a\sin^3\theta$) on the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

The parametric equations of the given curve are $x = a\cos^3\theta$ and $y = a\sin^3\theta$. Differentiating twice w.r.t. θ,

$$\dot{x} = \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta);\ \dot{y} = \frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\ddot{x} = \frac{d^2x}{d\theta^2} = -3a\left(\cos^3\theta - 2\cos\theta\sin^2\theta\right)$$

$$\ddot{y} = \frac{d^2y}{d\theta^2} = 3a\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)$$

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{(9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta)^{3/2}}{9a^2\left\{-\cos^2\theta\sin\theta\left(2\sin\theta\cos^2\theta-\sin^3\theta\right)+\sin^2\theta\cos\theta\left(\cos^3\theta-2\cos\theta\sin^2\theta\right)\right\}}$$

$$= \frac{27a^3\sin^3\theta\cos^3\theta\left(\cos^2\theta+\sin^2\theta\right)^{3/2}}{9a^2\sin^2\theta\cos^2\theta\left\{-\left(2\cos^2\theta-\sin^2\theta\right)+\left(\cos^2\theta-2\sin^2\theta\right)\right\}}$$

$$= \frac{3a\sin\theta\cos\theta}{-\left(\cos^2\theta+\sin^2\theta\right)} = -3a\sin\theta\cos\theta$$

$\therefore \quad |\rho| = 3a\sin\theta\cos\theta\,.$

Example 3.8 Show that the radius of curvature at the point 'θ' on the curve $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$ is $3a\sin\theta$.

$$x = 3a\cos\theta - a\cos 3\theta;\ y = 3a\sin\theta - a\sin 3\theta.$$

Differentiating w.r.t. θ,

$$\dot{x} = 3a\sin 3\theta - 3a\sin\theta;\ \dot{y} = 3a\cos\theta - 3a\cos 3\theta;$$

Now,
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3a(\cos\theta - \cos 3\theta)}{3a(\sin 3\theta - \sin\theta)}$$

$$= \frac{2\sin 2\theta\cdot\sin\theta}{2\cos 2\theta\cdot\sin\theta} = \tan 2\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\tan 2\theta\right)\cdot\frac{d\theta}{dx}$$

$$= 2\sec^2 2\theta\cdot\frac{1}{3a(\sin 3\theta - \sin\theta)}$$

$$= \frac{2\sec^2 2\theta}{6a\cos 2\theta\sin\theta}$$

$$= \frac{\sec^3 2\theta}{3a\sin\theta}$$

$$\rho = \frac{(1+y'^2)^{3/2}}{y''} = \frac{(1+\tan^2 2\theta)^{3/2}}{\sec^3 2\theta}\cdot 3a\sin\theta$$

$$= 3a\sin\theta.$$

Example 3.9 Find the radius of curvature of the curve $r = a(1+\cos\theta)$ at the point $\theta = \frac{\pi}{2}\cdot$

$$r = a(1+\cos\theta)$$

$\therefore \quad r' = -a\sin\theta \quad$ and $\quad r'' = -a\cos\theta$

$$\rho = \frac{(r^2+r'^2)^{3/2}}{r^2-r''+2r'^2}$$

$$= \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a^2(1+\cos\theta)^2 + a^2\cos\theta(1+\cos\theta) + 2a^2\sin^2\theta}$$

$$= \frac{a^3\left[2(1+\cos\theta)\right]^{3/2}}{a^2\left[3(1+\cos\theta)\right]}$$

$$= \frac{2\sqrt{2}}{3}a(1+\cos\theta)^{1/2} = \frac{4a}{3}\cos\frac{\theta}{2}$$

$$\therefore \quad (\rho)_{\theta=\frac{\pi}{2}} = \frac{4a}{3}\cos\frac{\pi}{4} = \frac{2\sqrt{2}}{3}a.$$

Example 3.10 Show that the radius of curvature of the curve $r^n = a^n \sin n\theta$ at the pole is $\dfrac{a^n}{(n+1)r^{n-1}}$.

$$r^n = a^n \sin n\,\theta \tag{1}$$

$$\therefore \quad n\log r = n\log a + \log\sin n\,\theta.$$

Differentiating w.r.t. θ,

$$\frac{n}{r}r' = n\cot n\theta$$

$$\therefore \quad r' = r\cot n\theta \tag{2}$$

$$r'' = r'\cot n\theta - nr\,\mathrm{cosec}^2\, n\theta.$$

$$= r\cot^2 n\theta - nr\,\mathrm{cosec}^2\, n\theta. \tag{3}$$

$$(\rho) = \frac{(r^2 + r'^2)^{3/2}}{r^2 - rr'' + 2r'^2}$$

$$= \frac{\left(r^2 + r^2\cot^2 n\theta\right)^{3/2}}{r^2 - r^2\cot^2 n\theta + nr^2\,\mathrm{cosec}^2\, n\theta + 2r^2\cot^2 n\theta}, \text{ using (2) and (3)}$$

$$= \frac{\left(r^2\,\mathrm{cosec}^2 n\theta\right)^{3/2}}{r^2\,\mathrm{cosec}^2 n\theta(n+1)}$$

$$= \frac{r\,\mathrm{cosec}\, n\theta}{n+1}$$

$$= \frac{a^n}{(n+1)r^{n-1}}$$

Example 3.11 Find the radius of curvature at the point (r, θ) on the curve $r^2\cos 2\theta = a^2$.

$$r^2 = a^2\sec 2\theta \tag{1}$$

$$\therefore \quad 2\log r = 2\log a + \log\sec 2\,\theta.$$

Differentiating w.r.t. θ,

$$\frac{2}{r}r' = 2\tan 2\theta.$$

i.e. $\qquad r' = r\tan 2\theta \qquad (2)$

$\therefore \qquad r'' = r'\tan 2\theta + 2r\sec^2 2\theta$

$$= r\tan^2 2\theta + 2r\sec^2 2\theta, \text{ using (2).}$$

$$\rho = \frac{(r^2 + r'^2)^{3/2}}{r^2 - rr'' + 2r'^2}$$

$$= \frac{(r^2 + r^2\tan^2 2\theta)^{3/2}}{r^2 - r^2\tan^2 2\theta - 2r^2\sec^2 2\theta + 2r^2\tan^2 2\theta}, \text{ using (1) and (2),}$$

$$= \frac{r^3\sec^3 2\theta}{-r^2\sec^2 2\theta}$$

$\therefore \qquad |\rho| = r\sec 2\theta = \dfrac{r^3}{a^2}$

Example 3.12 Show that at the points of intersection of the curves $r = a\theta$ and $r = \dfrac{a}{\theta}$, their curvatures are in the ratio 3:1.

$$r = a\theta \qquad (1)$$

and $\qquad r = \dfrac{a}{\theta} \qquad (2)$

Solving (1) and (2), $a\theta = \dfrac{a}{\theta}$

i.e. $\qquad \theta = \pm 1.$

$\therefore$ The points of intersection of the two curves are given by $\theta = \pm 1$.

For curve (1), $r' = a$ and $r'' = 0$.

$\therefore$
$$\rho_1 = \frac{(a^2\theta^2 + a^2)^{3/2}}{a^2\theta^2 + 2a^2}$$

$\therefore$
$$(\rho_1)_{\theta=\pm 1} = \frac{(2a^2)^{3/2}}{3a^2} = \frac{2\sqrt{2}}{3}a.$$

For curve (2), $r' = -\dfrac{a}{\theta^2}$ and $r'' = \dfrac{2a}{\theta^3}$

$\therefore$
$$\rho_2 = \frac{\left(\dfrac{a^2}{\theta^2} + \dfrac{a^2}{\theta^4}\right)^{3/2}}{\dfrac{a^2}{\theta^2} - \dfrac{2a^2}{\theta^4} + 2\dfrac{a^2}{\theta^4}}$$

$$\therefore \qquad (\rho_2)_{\theta=\pm 1} = \frac{(2a^2)^{3/2}}{a^2} = 2\sqrt{2a}$$

$$\therefore \qquad \rho_1 : \rho_2 = 1 : 3$$

$\therefore$ Ratio of their curvatures = 3:1.

Example 3.13 If the centre of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at one end of the minor axis lies at the other end, prove that the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$.

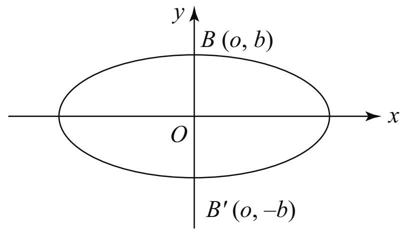


Fig. 3.6

The centre of curvature of the ellipse at $B(0, b)$ lies at B' $(0, -b)$. [Refer to Fig. 3.6]

We recall that if the centre of curvature of any curve at a point P is C, then PC equals the radius of curvature of the curve at P.

$\therefore$ Radius of curvature of the ellipse at

$$B = BB' = 2b. \tag{1}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiating w.r.t. x,

$$\frac{x}{a^2}+\frac{y}{b^2}\cdot y'=0$$

$$\therefore \qquad y' = -\frac{b^2x}{a^2y} \tag{2}$$

Differentiating again w.r.t. x,

$$y'' = -\frac{b^2}{a^2}\left(\frac{y-xy'}{y^2}\right) \tag{3}$$

From (2) and (3), we get

$$(y')_{(o,b)} = 0 \quad \text{and} \quad (y'')_{(o,b)} = -\frac{b}{a^2}$$

Now,
$$\rho = \frac{(1+y'^2)^{\frac{3}{2}}}{|y''|}$$

$$\therefore \qquad (\rho)_{(0,b)} = \frac{1}{\left|-\frac{b}{a^2}\right|} = \frac{a^2}{b}$$

From (1)
$$\frac{a^2}{b} = 2b \text{ i.e. } a^2 = 2b^2 \tag{4}$$

The eccentricity e of the ellipse is given by

$$b^2 = a^2(1 - e^2) \quad \text{or} \quad e^2 = \frac{a^2 - b^2}{a^2}$$

Using (4), we get, $e^2 = \dfrac{b^2}{2b^2} = \dfrac{1}{2}$ $\qquad \therefore\ e = \dfrac{1}{\sqrt{2}}$

Example 3.14 Find the centre of curvature at $\theta = \dfrac{\pi}{2}$ on the curve

$$x = 2\cos t + \cos 2t,\ y = 2\sin t + \sin 2t.$$

$$x = 2\cos t + \cos 2t;\quad y = 2\sin t + \sin 2t$$

Differentiating w.r.t. t,

$$\dot{x} = -\left(2\sin t + 2\sin 2t\right);\ \dot{y} = \left(2\cos t + 2\cos 2t\right)$$

$$\therefore \qquad y' = \frac{-2\left(\cos 2t + \cos t\right)}{2\left(\sin 2t + \sin t\right)}$$

$$= -\frac{2\cos\dfrac{3t}{2}\cos\dfrac{t}{2}}{2\sin\dfrac{3t}{2}\cos\dfrac{t}{2}}$$

$$= -\cot\frac{3t}{2}$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\cot\frac{3t}{2}\right)\cdot\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$= \frac{3}{2}\operatorname{cosec}^2\frac{3t}{2}\cdot\frac{1}{-2\left(\sin 2t + \sin t\right)}$$

$$= -\frac{3}{8\sin^3\left(\dfrac{t}{2}\right)\cdot\cos\left(\dfrac{t}{2}\right)}$$

$$\left(x\right)_{\theta=\frac{\pi}{2}} = -1;\ \left(y\right)_{\theta=\frac{\pi}{2}} = 2;\ \left(y'\right)_{\theta=\frac{\pi}{2}} = -\cot\frac{3\pi}{4} = 1;$$

$$\left(y''\right)_{\theta=\frac{\pi}{2}} = -\frac{3}{8\sin^3\dfrac{\pi}{4}\cos\dfrac{\pi}{4}} = -\frac{3}{2}$$

Now $$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$

$$\therefore \qquad \left(\bar{x}\right)_{\theta=\frac{\pi}{2}} = -1 - \frac{\left(1+1\right)}{\left(-\dfrac{3}{2}\right)} = -1 + \frac{4}{3} = \frac{1}{3}$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$$\therefore \qquad (\bar{y})_{\theta=\frac{\pi}{2}} = 2 + \frac{(1+1)}{\left(-\frac{3}{2}\right)} = 2 - \frac{4}{3} = \frac{2}{3}$$

$\therefore$ Required centre of curvature is $\left(\frac{1}{3}, \frac{2}{3}\right)$.

Example 3.15 Find the equation of the circle of curvature of the parabola $y^2 = 12x$ at the point (3, 6).

$$y^2 = 12x$$

Differentiating w.r.t. x,

$$2yy' = 12 \qquad \therefore\ y' = \frac{6}{y}$$

Differentiating again w.r.t. x,

$$y'' = -\frac{6}{y^2}\, y'$$

$$(y')_{(3,\,6)} = 1 \quad \text{and} \quad (y'')_{(3,\,6)} = -\frac{1}{6}$$

$$\rho = \frac{(1+y'^2)^{3/2}}{|y''|} \qquad \therefore\ (\rho)_{(3,6)} = \frac{2\sqrt{2}}{\frac{1}{6}} = 12\sqrt{2}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$

$$\therefore \qquad (\bar{x})_{(3,\,6)} = 3 - \frac{1}{\left(-\frac{1}{6}\right)}(1+1) = 15.$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$$= 6 + \frac{1}{\left(-\frac{1}{6}\right)}(1+1) = -6$$

The equation of the circle of curvature is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

$\therefore$ The equation of the circle of curvature at the point (3, 6) is

$$(x-15)^2 + (y+6)^2 = (12\sqrt{2})^2$$

i.e. $\qquad x^2 - 30x + 225 + y^2 + 12y + 36 = 288$

i.e. $\qquad x^2 + y^2 - 30x + 12y - 27 = 0.$

Example 3.16 Find the equation of the circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4},\frac{a}{4}\right)$.

$$\sqrt{x}+\sqrt{y}=\sqrt{a}$$

Differentiating w.r.t. x, $\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\,y'=0$

$$\therefore \qquad y'=-\frac{\sqrt{y}}{\sqrt{x}}$$

Differentiating again w.r.t. x,

$$y''=-\left\{\frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\,y'-\sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}\right\}$$

$$\therefore \qquad (y')_{\left(\frac{a}{4},\frac{a}{4}\right)}=-1 \quad \text{and}$$

$$(y'')_{\left(\frac{a}{4},\frac{a}{4}\right)}=-\left\{\frac{-\frac{1}{2}-\frac{1}{2}}{\frac{a}{4}}\right\}=\frac{4}{a}$$

$$\rho=\frac{\left(1+y'^2\right)^{3/2}}{y''} \qquad \therefore \quad (\rho)_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{2\sqrt{2}}{\left(\frac{4}{a}\right)}=\frac{a}{\sqrt{2}}$$

$$\bar{x}=x-\frac{y'}{y''}\left(1+y'^2\right)$$

$$\therefore \qquad (\bar{x})_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{a}{4}+\frac{1}{\left(\frac{4}{a}\right)}(1+1)=\frac{3a}{4}$$

$$\bar{y}=y+\frac{1}{y''}\left(1+y'^2\right);(\bar{y})_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{a}{4}+\frac{1}{\left(\frac{4}{a}\right)}(1+1)=\frac{3a}{4}$$

The equation of the circle of curvature is

$$(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$$

$\therefore$ The equation of the circle of curvature at $\left(\frac{a}{4},\frac{a}{4}\right)$ is

$$\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}$$

EXERCISE 3(a)

Part A

(Short Answer Questions)

1. Define curvature and radius of curvature.
2. Prove that the radius of curvature of a circle is its radius.
3. Find the curvature of the curve given by $s = c \tan \psi$ at $\psi = 0$.

   Find the radius of curvature of each of the following curves at the points indicated:
4. $y = e^x$ at $x = 0$.
5. $y = e^{\sqrt{3}x}$ at $x = 0$.
6. $y = \log \sec x$ at any point on it.
7. $y = \log \sin x$ at $x = \frac{\pi}{2}$.
8. $xy = c^2$ at (c, c).
9. $y^2 = 4ax$ at $y = 2a$.
10. $x = t^2, y = t$ at $t = 1$.
11. $r = a\theta$ at the pole.
12. $r\theta = a$ at any point on it.
13. $r = a \cos \theta$ at any point on it.
14. $r = e^\theta$ at any point on it.

Part B

Find the radius of curvature of the following curves at the points specified:

15. $x^3 + xy^2 - 6y^2 = 0$ at $(3, 3)$.
16. $4ay^2 = (2a - x)^3$ at $\left(a, \frac{a}{2}\right)$.
17. $x^3 + y^3 = (y - x)(y - 2x)$ at $(0, 0)$.
18. $xy^2 = a^2 (a - x)$ at $(a, 0)$
19. $4ay^2 = 27 (x - 2a)^3$ at $\left(\frac{7}{3} a, \frac{a}{2}\right)$
20. $y = x^2 (x - 3)$ at the points where the tangent is parallel to the x-axis.
21. $y = c \cosh \frac{x}{c}$ at the point where it is minimum.
22. $x^2 = 4ay$ at the point where the slope of the tangent is $\tan \theta$.
23. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \cos \theta, b \sin \theta)$

Find the radius of curvature of the following curves at the points specified:

24. $x = a (\cos t + t \sin t), y = a (\sin t - t \cos t)$ at 't'.
25. $x = a (\theta - \sin \theta), y = a (1 - \cos \theta)$ at 'θ'.
26. $x = e^t \cos t, y = e^t \sin t$ at $(1, 0)$.

27. $x = a \log\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$, $y = a \sec\theta$ at 'θ'.
28. $x = a \log\left(\cot\dfrac{\theta}{2} - \cos\theta\right)$, $y = a \sin\theta$ at 'θ'.
29. Find the radius of curvature at any point on the equiangular spiral $r = a\,e^{\theta \cot \alpha}$.
30. Find the radius of curvature of the curve $r = a(1 - \cos\theta)$ at any point on it.
31. Find the radius of curvature of the curve $r^n = a^n \cos n\theta$ at any point (r, θ). Hence prove that the radius of curvature of the lemniscate $r^2 = a^2 \cos 2\theta$ is $\dfrac{a^2}{3r}$.
32. Find the radius of curvature at any point (r, θ) on the curve $\sqrt{r}\cos\dfrac{\theta}{2} = \sqrt{a}$.
33. Find the radius of curvature at any point (r, θ) on the curve $r(1 + \cos\theta) = a$.
34. If ρ_1 and ρ_2 be the radii of curvature at the ends of any chord of the cardioid $r = a(1 + \cos\theta)$, that passes through the pole, prove that $\rho_1^2 + \rho_2^2 = \dfrac{16a^2}{9}$.

    [*Hint:* The ends of any chord that passes through the pole are given by $\theta = \theta_1$ and $\theta = \pi + \theta_1$. Use the result $(\rho)_\theta = \dfrac{4a}{3}\cos\dfrac{\theta}{2}$.]
35. Find the centre of curvature of the curve $y = x^3 - 6x^2 + 3x + 1$ at the point $(1, -1)$
36. Find the centre of curvature of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $(a \sec\theta, b \tan\theta)$.
37. Show that the line joining any point 't' on the cycloid $x = a(t + \sin t)$. $y = a(1 - \cos t)$ and its centre of curvature is bisected by the line $y = 2a$.
38. Find the equation of the circle of curvature of the parabola $y^2 = 4ax$ at the positive end of the latus rectum.
39. Find the equation of the circle of curvature of the rectangular hyperbola $xy = 12$ at the point $(3, 4)$.
40. Find the equation of the circle of curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$.

3.3 EVOLUTES AND ENVELOPES

Let Q be the centre of curvature of a given curve C at the point P on it. When P moves on the curve C and takes different positions, Q will also take different positions and move on another curve C'. This curve C' is called *the evolute* of the curve C. Thus evolute can be defined as the locus of the centre of curvature.

When C' is the evolute of the curve C, C is called *the involute* of the curve C'.

The procedure to find the equation of the evolute of a given curve is given below:

Let the equation of the given curve be $y = f(x)$ (1)

If $(\bar{x}, \bar{y})$ is the centre of curvature corresponding to the point (x, y) on (1), then

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right) \tag{2}$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right) \tag{3}$$

By eliminating x and y from (1), (2), (3), we get a relation between $\bar{x}$ and $\bar{y}$, which is the equation of the evolute.

Note ☑ If the parametric co-ordinates of any point on the given curve are assumed, then we have to eliminate the parameter from Equations (2) and (3), which will simplify the procedure.

Evolute of a given curve can also be defined in a different manner, using the concept of envelope of a family of curves, which is discussed below:

Consider the equation $f(x, y, c) = 0$, where c is a constant. If c takes a particular value, the equation represents a single curve. If c is an arbitrary constant or parameter which takes different values, then the equation $f(x, y, c) = 0$ represents a family of similar curves.

If we assign two consecutive values for c, we get two close curves of the family. The locus of the limiting positions of the points of intersection of consecutive members of a family of curves is called *the envelope* of the family.

It can be proved that the envelope of a family of curves touches every member of the family of curves.

3.3.1 Method of Finding the Equation of the Envelope of a Famly of Curves

Let $f(x, y, c) = 0$ be the equation of the given family of curves, where c is the parameter. Two consecutive members of the family (corresponding to two close values of c) are given by

$$f(x, y, c) = 0 \tag{1}$$

and

$$f(x, y, c + \Delta c) = 0 \tag{2}$$

The co-ordinates of the points of intersection of (1) and (2) will satisfy (1) and (2) and hence satisfy $\dfrac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c} = 0$

Hence the co-ordinates of the limiting positions of the points of intersection of (1) and (2) will satisfy the equation

$$\lim_{\Delta c \to 0}\left\{\frac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c}\right\} = 0$$

i.e.

$$\frac{\partial f}{\partial c}(x, y, c) = 0 \tag{3}$$

These limiting points will continue to lie on (1) and satisfy

$$f(x, y, c) = 0$$

If we eliminate c between (1) and (3), we get the equation of a curve, which is the locus of the limiting positions of the points of intersection of consecutive members of the given family, i.e. we get the equation of the envelope.

Thus the equation of the envelope of the family of curves $f(x, y, c) = 0$ (c is the parameter) is obtained by eliminating c between the equations

$$f(x, y, c) = 0 \quad \text{and} \quad \frac{\partial f}{\partial c}(x, y, c) = 0\cdot$$

Equation of the envelope of the family $A\alpha^2 + B\alpha + C = 0$, where α is the parameter and A, B, C are functions of x and y:

Very often the equation of the family of curves will be a quadratic equation in the parameter. In such cases, the equation of the envelope may be remembered as a formula.

Let the equation of the family of curves be

$$A\alpha^2 + B\alpha + C = 0 \tag{1}$$

Differentiating partially w.r.t. α,

$$2A\alpha + B = 0 \tag{2}$$

From (2), $\alpha = -\dfrac{B}{2A}$

Substituting this values of α in (1), we get the eliminant of α as

$$A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C = 0$$

i.e.

$$\frac{B^2}{4A} - \frac{B^2}{2A} + C = 0$$

i.e. $B^2 - 4AC = 0$, which is the equation of the envelope of the family (1).

3.3.2 Evolute as the Envelope of Normals

The normals to a curve form a family of straight lines. The envelope of this family of normals is the locus of the limiting position of the point of intersection of consecutive normals. But the point of intersection of consecutive normals of a curve is the centre of curvature of the curve. Hence the locus of centre of curvature is the same as the envelope of normals.

Thus the evolute of a curve is the envelope of the normals of that curve.

WORKED EXAMPLE 3(b)

Example 3.1 Find the evolute of the parabola $x^2 = 4ay$.

The parametric co-ordinates of any point on the parabola $x^2 = 4ay$ are

$$x = 2\,at \quad \text{and} \quad y = a\,t^2$$

$$\dot{x} = 2a\,;\ \dot{y} = 2\ at \qquad \therefore\ \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(t) \times \frac{dt}{dx} = \frac{1}{2a}$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature at the point 't'

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$

$$= 2at - \frac{t}{\left(\frac{1}{2a}\right)}\left(1 + t^2\right)$$

$$= -2\,at^3 \tag{1}$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$$= at^2 + \frac{1}{\left(\frac{1}{2a}\right)}\left(1 + t^2\right) = 3at^2 + 2a \tag{2}$$

To get the relation between $\bar{x}$ and $\bar{y}$, we have to eliminate t from (1) and (2).

From (1), we get $t^3 = -\dfrac{\bar{x}}{2a}$ (3)

From (2), we get $t^2 = \dfrac{\bar{y} - 2a}{3a}$ (4)

From (3) and (4), we get

$$\left(-\frac{\bar{x}}{2a}\right)^2 = \left(\frac{\bar{y} - 2a}{3a}\right)^3$$

i.e.

$$27a\,\bar{x}^2 = 4\left(\bar{y} - 2a\right)^3.$$

$\therefore$ Locus of $(\bar{x}, \bar{y})$, i.e. the equation of the evolute is $27a\,x^2 = 4(y - 2a)^3$.

Example 3.2 Find the evolute of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

The parametric co-ordinates of any point on the hyperbola are

$$x = a\sec\theta \quad \text{and} \quad y = b\tan\theta$$

$$\therefore \quad \dot{x} = a\sec\theta\tan\theta; \quad \dot{y} = b\sec^2\theta$$

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{b\sec\theta}{a\tan\theta} = \frac{b}{a\sin\theta}$$

$$\therefore \quad y'' = -\frac{b}{a\sin^2\theta}\cos\theta \cdot \frac{d\theta}{dx} = -\frac{b\cos\theta}{a\sin^2\theta} \cdot \frac{\cos^2\theta}{a\sin\theta}$$

$$= -\frac{b\cos^3\theta}{a^2\sin^3\theta}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^{\,2}\right)$$

$$= a \sec\theta + \frac{b}{a\sin\theta}\cdot\frac{a^2\sin^3\theta}{b\cos^3\theta}\left(1 + \frac{b^2}{a^2\sin^2\theta}\right)$$

$$= \frac{a}{\cos\theta} + \frac{1}{a\cos^3\theta}\left(a^2\sin^2\theta + b^2\right)$$

$$= \frac{a^2\cos^2\theta + a^2\sin^2\theta + b^2}{a\cos^3\theta}$$

$$= \frac{a^2 + b^2}{a\cos^3\theta} \tag{1}$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$$= b\tan\theta - \frac{a^2\sin^3\theta}{b\cos^3\theta}\left(1 + \frac{b^2}{a^2\sin^2\theta}\right)$$

$$= \frac{b\sin\theta}{\cos\theta} - \frac{\sin\theta}{b\cos^3\theta}\left(a^2\sin^2\theta + b^2\right)$$

$$= \frac{\sin\theta}{b\cos^3\theta}\left(b^2\cos^2\theta - a^2\sin^2\theta - b^2\right)$$

$$= -\frac{\left(a^2 + b^2\right)}{b}\tan^3\theta \tag{2}$$

From (1), $\quad \sec^3\theta = \dfrac{a\bar{x}}{a^2 + b^2}$ and

From (2), $\quad \tan^3\theta = -\dfrac{b\bar{y}}{a^2 + b^2}$

To eliminate θ, we use the identity $\sec^2\theta - \tan^2\theta = 1$

$$\therefore \quad \left(\frac{a\bar{x}}{a^2 + b^2}\right)^{2/3} - \left(\frac{-b\bar{y}}{a^2 + b^2}\right)^{2/3} = 1$$

$\therefore$ The locus of $\left(\bar{x}, \bar{y}\right)$ i.e. the evolute of the hyperbola is

$$(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3} \qquad [\because (-by)^{2/3} = (by)^{2/3}]$$

Example 3.3 Find the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

The parametric co-ordinates of any point on the curve are $x = a\cos^3\theta$ and $y = a\sin^3\theta$

$$\therefore \qquad \dot{x} = -3a\cos^2\theta\sin\theta; \qquad \dot{y} = 3a\sin^2\theta\cos\theta$$

$$\therefore \qquad y' = \frac{\dot{y}}{\dot{x}} = -\tan\theta$$

$$y'' = -\sec^2\theta\cdot\frac{d\theta}{dx} = -\sec^2\theta\left(\frac{1}{-3a\cos^2\theta\sin\theta}\right)$$

$$= \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\overline{x} = x - \frac{y'}{y''}\left(1+y'^2\right)$$

$$= a\cos^3\theta + \frac{\tan\theta\left(1+\tan^2\theta\right)}{\left(\dfrac{1}{3a\cos^4\theta\sin\theta}\right)}$$

$$= a\cos^3\theta + \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos^2\theta}\cdot 3a\cos^4\theta\sin\theta$$

$$= a\cos^3\theta + 3a\cos\theta\sin^2\theta \tag{1}$$

$$\overline{y} = y + \frac{1}{y''}\left(1+y'^2\right)$$

$$= a\sin^3\theta + \sec^2\theta\cdot 3a\cos^4\theta\sin\theta$$

$$= a\sin^3\theta + 3a\sin\theta\cos^2\theta \tag{2}$$

Now $$\overline{x}+\overline{y} = a\left(\cos^3\theta+\sin^3\theta+3\cos^2\theta\sin\theta+3\cos\theta\sin^2\theta\right)$$

$$= a\left(\cos\theta+\sin\theta\right)^3 \tag{3}$$

and $$\overline{x}-\overline{y} = a\left(\cos^3\theta-3\cos^2\theta\sin\theta+3\cos\theta\sin^2\theta-\sin^3\theta\right)$$

$$= a\left(\cos\theta-\sin\theta\right)^3 \tag{4}$$

Now $$\left(\frac{\overline{x}+\overline{y}}{a}\right)^{2/3} + \left(\frac{\overline{x}-\overline{y}}{a}\right)^{2/3} = \left(\cos\theta+\sin\theta\right)^2 + \left(\cos\theta-\sin\theta\right)^2$$

$$= 2$$

i.e. $$\left(\overline{x}+\overline{y}\right)^{2/3} + \left(\overline{x}-\overline{y}\right)^{2/3} = 2a^{2/3}$$

$\therefore$ The equation of the evolute is

$$\left(x+y\right)^{2/3} + \left(x-y\right)^{2/3} = 2a^{2/3}.$$

Example 3.4 Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ is another cycloid.

Any point on the cycloid is given by

$$x = a\,(\theta - \sin\theta) \quad \text{and} \quad y = a(1-\cos\theta)$$

$$\therefore \qquad \dot{x} = a\left(1-\cos\theta\right); \qquad \dot{y} = a\sin\theta$$

$$y' = \frac{a \sin\theta}{a\left(1-\cos\theta\right)} = \frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

$$y'' = -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{d\theta}{dx} = -\frac{1}{2\sin^2\frac{\theta}{2}}\cdot\frac{1}{2a\sin^2\frac{\theta}{2}}$$

$$= -\frac{1}{4a\sin^4\frac{\theta}{2}}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1+y'^2\right)$$

$$= a\left(\theta-\sin\theta\right)+\cot\frac{\theta}{2}\cdot\operatorname{cosec}^2\frac{\theta}{2}\cdot 4a\sin^4\frac{\theta}{2}$$

$$= a\left(\theta-\sin\theta\right)+4a\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$$

$$= a\left(\theta-\sin\theta\right)+2a\sin\theta$$

$$= a\left(\theta+\sin\theta\right) \qquad (1)$$

$$\bar{y} = y + \frac{1}{y''}\left(1+y'^2\right)$$

$$= a\left(1-\cos\theta\right)-\operatorname{cosec}^2\frac{\theta}{2}\cdot 4a\sin^4\frac{\theta}{2}$$

$$= a\left(1-\cos\theta\right)-2a\left(1-\cos\theta\right)$$

$$= -a\left(1-\cos\theta\right) \qquad (2)$$

Elimination of θ from (1) and (2) is not easy.

$\therefore$ The locus of $\left(\bar{x},\bar{y}\right)$ is given by the parametric equations $x = a\,(\theta + \sin\theta)$ and $y = -a\,(1 - \cos\theta)$, which represent another cycloid.

Example 3.5 Find the equation of the evolute of the curve $x = a\,(\cos t + t\sin t)$, $y = a\,(\sin t - t\cos t)$

$$x = a(\cos t + t\sin t);\ y = a\,(\sin t - t\cos t)$$

$$\therefore \quad \dot{x} = a\left(-\sin t+\sin t+t\cos t\right);\ \dot{y} = a\left(\cos t-\cos t+t\sin t\right)$$

$$\therefore \quad y' = \frac{at\sin t}{at\cos t} = \tan t$$

$$y'' = \sec^2 t\cdot\frac{dt}{dx} = \sec^2 t\cdot\frac{1}{at\cos t} = \frac{1}{at\cos^3 t}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1+y'^2\right)$$

$$= a\left(\cos t+t\sin t\right)-\tan t\cdot\sec^2 t\cdot at\cos^3 t$$

$$= a\left(\cos t+t\sin t\right)-a\,t\sin t$$

$$= a\cos t \qquad (1)$$

$$\bar{y} = y + \frac{1}{y''}(1 + y'^2)$$
$$= a(\sin t - t\cos t) + \sec^2 t \cdot at\cos^3 t$$
$$= a\sin t \tag{2}$$

Eliminating t between (1) and (2), we get

$$\bar{x}^2 + \bar{y}^2 = a^2$$

$\therefore$ The evolute of the given curve is $x^2 + y^2 = a^2$.

Example 3.6 Prove that the evolute of the curve $x = a\left(\cos\theta + \log\tan\dfrac{\theta}{2}\right)$, $y = a\sin\theta$ is the catenary $y = a\cosh\dfrac{x}{a}$.

$$x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right);\ y = a\sin\theta.$$

$$\dot{x} = a\left(-\sin\theta + \frac{1}{\tan\dfrac{\theta}{2}}\cdot\sec^2\frac{\theta}{2}\cdot\frac{1}{2}\right)$$

$$= a\left(-\sin\theta + \frac{1}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}\right)$$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right) = \frac{a\cos^2\theta}{\sin\theta}$$

and $$\dot{y} = a\cos\theta$$

$\therefore$ $$y' = a\cos\theta\cdot\frac{\sin\theta}{a\cos^2\theta} = \tan\theta$$

$$y'' = \sec^2\theta\cdot\frac{d\theta}{dx} = \sec^2\theta\cdot\frac{\sin\theta}{a\cos^2\theta} = \frac{\sin\theta}{a\cos^4\theta}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$
$$= a\left(\cos\theta + \log\tan\frac{\theta}{2}\right) - \tan\theta\cdot\sec^2\theta\cdot\frac{a\cos^4\theta}{\sin\theta}$$
$$= a\left(\cos\theta + \log\tan\frac{\theta}{2}\right) - a\cos\theta$$
$$= a\log\tan\frac{\theta}{2} \tag{1}$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$$= a \sin\theta + \sec^2\theta \cdot \frac{a\cos^4\theta}{\sin\theta}$$

$$= a\sin\theta + \frac{a\cos^2\theta}{\sin\theta}$$

$$= \frac{a}{\sin\theta} \tag{2}$$

Now $$\bar{y} = \frac{a\left(1+\tan^2\frac{\theta}{2}\right)}{2\tan\frac{\theta}{2}} \tag{3}$$

and $$\tan\frac{\theta}{2} = e^{\bar{x}/a} \quad \text{(4) [from (1)]}$$

From (3) and (4), we get,

$$\bar{y} = \frac{a}{2}\left\{\frac{1+e^{2x/a}}{e^{x/a}}\right\}$$

$$= \frac{a}{2}\left\{e^{\bar{x}/a} + e^{-\bar{x}/a}\right\} = a\cosh\frac{\bar{x}}{a}$$

$\therefore$ The evolute is $y = a\cosh\dfrac{x}{a}$

Example 3.7 Find the envelope of the family of straight lines given by (i) $y = mx \pm \sqrt{a^2m^2 - b^2}$, where m is the parameter, (ii) x cos α+ y sin α= a sec α, where α is the parameter, (iii) the family of parabolas given by $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$, where α is the parameter.

(i) Rewriting the given equation, we have

$$a^2 m^2 - b^2 = (y - mx)^2$$
$$= y^2 - 2xym + m^2x^2$$

i.e. $$(x^2 - a^2)\, m^2 - 2xym + (y^2 + b^2) = 0$$

This is a quadratic equation in 'm'. $\therefore$ The envelope is given by the equation '$B^2 - 4AC = 0$'

i.e. $$4x^2y^2 - 4(x^2 - a^2)(y^2 + b^2) = 0$$

i.e. $$x^2y^2 - (x^2y^2 + b^2x^2 - a^2y^2 - a^2b^2) = 0$$

i.e. $$b^2x^2 - a^2y^2 = a^2b^2$$

i.e. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$ which is the standard hyperbola.

Note ☑ The envelope touches every member of the given family of straight lines and vice versa. This is, in fact, obvious as the given family represents the family of tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$·

(ii) $x \cos \alpha + y \sin \alpha = a \sec \alpha$

Dividing throughout by $\cos \alpha$, we have

$$x + y \tan \alpha = a \sec^2 \alpha$$

i.e. $x + yt = a(1 + t^2)$, where $t = \tan \alpha$ can be treated as the new parameter.

i.e. $$at^2 - yt + (a - x) = 0$$

This is a quadratic equation in 't'.

∴ The envelope is given by

$$y^2 - 4a(a - x) = 0 \qquad \text{i.e.} \quad y^2 = -4a(x - a)$$

(iii) $y = x \tan \alpha - \dfrac{gx^2}{2u^2 \cos^2 \alpha}$

Putting $t = \tan \alpha$, we get

$$\frac{gx^2}{2u^2}\left(1+t^2\right) - xt + y = 0$$

i.e. $$gx^2t^2 - 2u^2xt + (gx^2 + 2u^2y) = 0$$

If we treat 't' as the parameter, we see that this equation is a quadratic equation in the parameter

∴ The envelope is given by

$$4u^4x^2 - 4gx^2(gx^2 + 2u^2y) = 0$$

i.e. $$g^2x^2 + 2u^2gy - u^4 = 0$$

i.e. $$x^2 = -\frac{2u^2}{g}\left(y - \frac{u^2}{2g}\right).$$

Example 3.8 Find the envelope of the family of straight lines (i) $y \cos \theta - x \sin \theta = a \cos 2\theta$, θ being the parameter, (ii) $x \cos \alpha + y \sin \alpha = c \sin \alpha \cos \alpha$, α being the parameter, (iii) $x \sec^2 \theta + y \operatorname{cosec}^2 \theta = c$, θ being the parameter.

(i) $y \cos \theta - x \sin \theta = a \cos 2\theta$ (1)

Differentiating (1) partially w.r.t. θ,

$$-y \sin \theta - x \cos \theta = -2a \sin 2\theta \tag{2}$$

(1) × $\cos \theta$ − (2) × $\sin \theta$ gives

$$\begin{aligned} y &= a(\cos 2\theta \cos \theta + 2 \sin 2\theta \sin \theta) \\ &= a(\cos \theta + \sin 2\theta \sin \theta) \end{aligned} \tag{3}$$

(1) × $\sin \theta$ + (2) × $\cos \theta$ gives

$$\begin{aligned} x &= -a(\cos 2\theta \sin \theta - 2 \sin 2\theta \cos \theta) \\ &= -a(-\sin \theta - \sin 2\theta \cos \theta) \\ &= a(\sin \theta + \sin 2\theta \cos \theta) \end{aligned} \tag{4}$$

Adding (3) and (4), we get

$$\begin{aligned} x + y &= a\{(\sin\theta + \cos\theta) + \sin 2\theta \,.\, (\sin\theta + \cos\theta)\} \\ &= a(\sin\theta + \cos\theta)(1 + \sin 2\theta) \\ &= a(\sin\theta + \cos\theta)(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta) \\ &= a(\sin\theta + \cos\theta)(\sin\theta + \cos\theta)^2 \\ &= a(\sin\theta + \cos\theta)^3 \qquad (5) \end{aligned}$$

Subtracting (3) from (4), we get

$$\begin{aligned} x - y &= a\{(\sin\theta - \cos\theta) - \sin 2\theta(\sin\theta - \cos\theta)\} \\ &= a(\sin\theta - \cos\theta)^3 \qquad (6) \end{aligned}$$

From (5) and (6), we get

$$\begin{aligned} \left(\frac{x+y}{a}\right)^{2/3} + \left(\frac{x-y}{a}\right)^{2/3} &= (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 \\ &= 2 \end{aligned}$$

$\therefore$ The envelope is

$$(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$$

Note ☑ This is the evolute of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$

[Refer to Example (3.3) above].
In this problem, we have found out the envelope of the normals of the astroid.

(ii) $x\cos\alpha + y\sin\alpha = c\sin\alpha\cos\alpha$.
Dividing by $\sin\alpha\cos\alpha$, we get

$$\frac{x}{\sin\alpha} + \frac{y}{\cos\alpha} = c \qquad (1)$$

Differentiating (1) w.r.t. α,

$$-\frac{x}{\sin^2\alpha}\cos\alpha + \frac{y}{\cos^2\alpha}\sin\alpha = 0 \qquad (2)$$

From (2), $\dfrac{x\cos\alpha}{\sin^2\alpha} = \dfrac{y\sin\alpha}{\cos^2\alpha}$

i.e. $$\frac{x}{\sin^3\alpha} = \frac{y}{\cos^3\alpha} = k \text{ say.}$$

$\therefore$ $$\sin^3\alpha = \frac{x}{k} \quad \text{and} \quad \cos^3\alpha = \frac{y}{k} \qquad (3)$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$\therefore$ $$\left(\frac{x}{k}\right)^{2/3} + \left(\frac{y}{k}\right)^{2/3} = 1$$

i.e. $$k^{2/3} = x^{2/3} + y^{2/3} \qquad (4)$$

$\therefore$ From (3), we have

$$\sin\alpha=\frac{x^{1/3}}{k^{1/3}} \quad \text{and} \quad \cos\alpha=\frac{y^{1/3}}{k^{1/3}} \tag{5}$$

Using (5) in (1), the equation of the envelope is $k^{1/3}(x^{2/3}+y^{2/3})=c$

i.e. $$\left(x^{2/3}+y^{2/3}\right)^{1/2}\cdot\left(x^{2/3}+y^{2/3}\right)=c, \qquad \text{from (4)}$$

i.e. $$\left(x^{2/3}+y^{2/3}\right)^{3/2}=c$$

i.e. $$x^{2/3}+y^{2/3}=c^{2/3}$$

(iii) $x\sec^2\theta+y\operatorname{cosec}^2\theta=c$ (1)

Differentiating (1) partially w.r.t. θ,

$$2x\sec^2\theta\tan\theta-2y\operatorname{cosec}^2\theta\cot\theta=0$$

i.e. $$\frac{x\sin\theta}{\cos^3\theta}-\frac{y\cos\theta}{\sin^3\theta}=0 \tag{2}$$

From (2), $\dfrac{x}{\cos^4\theta}=\dfrac{y}{\sin^4\theta}=k$. say.

$\therefore$ $$\cos^4\theta=\frac{x}{k} \quad \text{and} \quad \sin^4\theta=\frac{y}{k} \tag{3}$$

Using the identity $\cos^2\theta+\sin^2\theta=1$, we have

$$\sqrt{\frac{x}{k}}+\sqrt{\frac{y}{k}}=1$$

i.e. $$k=(\sqrt{x}+\sqrt{y})^2 \tag{4}$$

Using (3) and (4) in (1), we get

$$x\cdot\frac{\left(\sqrt{x}+\sqrt{y}\right)}{\sqrt{x}}+y\cdot\frac{\left(\sqrt{x}+\sqrt{y}\right)}{\sqrt{y}}=c$$

i.e. $\left(\sqrt{x}+\sqrt{y}\right)^2=c$

i.e. $\sqrt{x}+\sqrt{y}=\sqrt{c}$, which is the equation of the required envelope.

Example 3.9 Find the envelope of the straight line $\dfrac{x}{a}+\dfrac{y}{b}=1$, where a and b are parameters that are connected by the relation $a+b=c$.

$$\frac{x}{a}+\frac{y}{b}=1 \tag{1}$$

$$a+b=c \tag{2}$$

From (2), $b=c-a$.

Using in (1), $\dfrac{x}{a}+\dfrac{y}{c-a}=1$, where a is the only parameter. (3)

Differentiating (3) w.r.t. a, we get

$$\frac{-x}{a^2}+\frac{y}{(c-a)^2}=0 \tag{4}$$

From (3) $\quad \dfrac{x}{a^2}=\dfrac{y}{(c-a)^2}$

$\therefore \quad \dfrac{\sqrt{x}}{a}=\dfrac{\sqrt{y}}{c-a}=\dfrac{\sqrt{x}+\sqrt{y}}{c}$

$\therefore \quad \dfrac{1}{a}=\dfrac{\sqrt{x}+\sqrt{y}}{c\sqrt{x}}$ and $\dfrac{1}{c-a}=\dfrac{\sqrt{x}+\sqrt{y}}{c\sqrt{y}}$ (5)

Using (5) in (3), the equation of the envelope is $\dfrac{\sqrt{x}}{c}\left(\sqrt{x}+\sqrt{y}\right)+\dfrac{\sqrt{y}}{c}\left(\sqrt{x}+\sqrt{y}\right)=1$

i.e. $\quad \left(\sqrt{x}+\sqrt{y}\right)^2=c$

or $\quad \sqrt{x}+\sqrt{y}=\sqrt{c}$.

3.4 ALITER

Without eliminating one of the parameters, we may treat both a and b as functions of a third parameter t and proceed as follows:
Differentiating (1) w.r.t. t,

$$-\frac{x}{a^2}\frac{\mathrm{d}a}{\mathrm{d}t}-\frac{y}{b^2}\frac{\mathrm{d}b}{\mathrm{d}t}=0$$

i.e.
$$\frac{x}{a^2}\frac{\mathrm{d}a}{\mathrm{d}t}=-\frac{y}{b^2}\frac{\mathrm{d}b}{\mathrm{d}t} \tag{3}$$

Differentiating $a + b = c$ w.r.t. t

$$\frac{\mathrm{d}a}{\mathrm{d}t}=-\frac{\mathrm{d}b}{\mathrm{d}t} \tag{4}$$

Dividing (3) by (4), we have

$$\frac{x}{a^2}=\frac{y}{b^2}$$

$\therefore$
$$\frac{\sqrt{x}}{a}=\frac{\sqrt{y}}{b}=\frac{\sqrt{x}+\sqrt{y}}{c} \tag{5}$$

$(\because\ a+b=c)$

Using (5) in (1), we get

$$\frac{\sqrt{x}}{c}\left(\sqrt{x}+\sqrt{y}\right)+\frac{\sqrt{y}}{c}\left(\sqrt{x}+\sqrt{y}\right)=1$$

i.e. $\left(\sqrt{x}+\sqrt{y}\right)^2=c$

or $\sqrt{x}+\sqrt{y}=\sqrt{c}$.

Example 3.10 Find the envelope of the system of lines $\frac{x}{l}+\frac{y}{m}=1,$, where l and m are connected by the relation $\frac{l}{a}+\frac{m}{b}=1$ (l and m are the parameters).

$$\frac{x}{l}+\frac{y}{m}=1 \tag{1}$$

and
$$\frac{l}{a}+\frac{m}{b}=1 \tag{2}$$

Differentiating (1) and (2) w.r.t. t,

$$-\frac{x}{l^2}\frac{\mathrm{d}l}{\mathrm{d}t}-\frac{y}{m^2}\frac{\mathrm{d}m}{\mathrm{d}t}=0 \tag{3}$$

and
$$\frac{1}{a}\frac{\mathrm{d}l}{\mathrm{d}t}+\frac{1}{b}\frac{\mathrm{d}m}{\mathrm{d}t}=0 \tag{4}$$

From (3) and (4), we have

$$\frac{x}{l^2/a}=\frac{y}{m^2/b}$$

i.e.
$$\frac{\left(\frac{x}{l}\right)}{\left(\frac{l}{a}\right)}=\frac{\left(\frac{y}{m}\right)}{\left(\frac{m}{b}\right)}=\frac{\frac{x}{l}+\frac{y}{m}}{\frac{l}{a}+\frac{m}{b}}=\frac{1}{1}$$

or
$$\frac{ax}{l^2}=\frac{by}{m^2}=1$$

$\therefore$
$$l=\sqrt{ax} \quad \text{and} \quad m=\sqrt{by} \tag{5}$$

Using (5) in (1), we get the envelope as $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$.

Example 3.11 Find the envelope of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are connected by the relation $a^2 + b^2 = c^2$, c being a constant.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

and
$$a^2 + b^2 = c^2 \tag{2}$$

Eliminating b from (1) and (2), we get

$$\frac{x^2}{a^2}+\frac{y^2}{c^2-a^2}=1$$

i.e. $(c^2 - a^2)x^2 + a^2y^2 = a^2(c^2 - a^2)$

i.e. $a^4 - a^2(c^2 + x^2 - y^2) + c^2x^2 = 0$ (3)

(3) is a quadratic equation in a^2, which may be regarded as the parameter.

$\therefore$ The envelope is given by $B^2 - 4AC = 0$

i.e. $(c^2 + x^2 - y^2)^2 - 4c^2x^2 = 0$

i.e. $[(c^2 + x^2 - y^2) + 2cx]\,[c^2 + x^2 - y^2 - 2cx] = 0$

i.e. $(x + c)^2 - y^2 = 0;\quad (x - c)^2 - y^2 = 0$

$\therefore$ $x + c = \pm y$ and $x - c = \pm y$

i.e. $x = -c \pm y$ and $x = c \pm y$

i.e. $x \pm y = \pm c.$

Example 3.12 Find the envelope of a system of concentric ellipses with their axes along the co-ordinate axes and of constant area.

The equation of the system of ellipses is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

The condition satisfied by a and b is $\pi\, ab = c$ (2)

Differentiating (1) and (2) w.r.t. 't',

$$-\frac{2x^2}{a^3}\frac{\mathrm{d}a}{\mathrm{d}t}-\frac{2y^2}{b^3}\frac{\mathrm{d}b}{\mathrm{d}t}=0 \tag{3}$$

$$\pi b\frac{\mathrm{d}a}{\mathrm{d}t}+\pi a\frac{\mathrm{d}b}{\mathrm{d}t}=0 \tag{4}$$

From (3) and (4), we have

$$\frac{x^2}{a^3 b}=\frac{y^2}{ab^3} \quad \text{or} \quad \frac{x^2}{a^2}=\frac{y^2}{b^2}$$

or $$\frac{x}{a}=\frac{y}{b}=k, \text{ say} \tag{5}$$

From (5), $a = \dfrac{x}{k}$ and $b = \dfrac{y}{k}$

Using in (2), $\dfrac{\pi xy}{k^2}=c$ or $k=\sqrt{\dfrac{\pi xy}{c}}$ (6)

Using (5) and (6) in (1), the equation of the envelope is $\dfrac{\pi xy}{c}+\dfrac{\pi xy}{c}=1$

i.e. $2\,\pi\, xy = c.$

Example 3.13 Find the evolute of the parabola $y^2 = 4ax$, considering it as the envelope of its normals.

The normal at any point $(at^2, 2at)$ on the parabola $y^2 = 4ax$ is

$$y + xt = 2at + at^3 \tag{1}$$

(1) represents the family of normals, where t is the parameter.
Differentiating (1) w.r.t. 't',

$$x = 2a + 3at^2 \tag{2}$$

From (2),

$$t = \left(\frac{x-2a}{3a}\right)^{\frac{1}{2}} \tag{3}$$

Substituting (3) in (1), we get

$$y = -(x-2a)\left(\frac{x-2a}{3a}\right)^{\frac{1}{2}} + a\left(\frac{x-2a}{3a}\right)^{\frac{3}{2}}$$

$$= \frac{-(x-2a)^{\frac{3}{2}}}{(3a)^{\frac{1}{2}}} + \frac{1}{3}\cdot\frac{(x-2a)^{\frac{3}{2}}}{(3a)^{\frac{1}{2}}}$$

$$= -\frac{2}{3}\cdot\frac{(x-2a)^{\frac{3}{2}}}{(3a)^{\frac{1}{2}}}$$

i.e.

$$y^2 = \frac{4}{27a}(x-2a)^3$$

∴ The evolute of the parabola is

$$27ay^2 = 4(x-2a)^3$$

Example 3.14 Find the envelope of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, treating it as the envelope of its normals.

The normal at any point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \tag{1}$$

where θ is the parameter.
Differentiating (1) w.r.t. 'θ',

$$\frac{ax}{\cos^2\theta}\sin\theta + \frac{by\cos\theta}{\sin^2\theta} = 0 \tag{2}$$

From (2), $\frac{ax}{\cos^3\theta} = -\frac{by}{\sin^3\theta} = k$, say

∴

$$\cos\theta = \left(\frac{ax}{k}\right)^{\frac{1}{3}} \quad \text{and} \quad \sin\theta = \left(-\frac{by}{k}\right)^{\frac{1}{3}} \tag{3}$$

Using the identity $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{ax}{k}\right)^{\frac{2}{3}} + \left(\frac{by}{k}\right)^{\frac{2}{3}} = 1$$

$\therefore$ $$k^{\frac{2}{3}} = (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}$$

i.e. $$k^{\frac{1}{3}} = \left\{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\right\}^{\frac{1}{2}} \tag{4}$$

Using (3) and (4) in (1), we have

$$\left\{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\right\} k^{\frac{1}{3}} = a^2 - b^2$$

i.e. $$\left\{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\right\}^{\frac{3}{2}} = a^2 - b^2$$

i.e. $$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = \left(a^2 - b^2\right)^{\frac{2}{3}}$$

Example 3.15 Find the evolute of the tractrix $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$, $y = a\sin\theta$, treating it as the envelope of its normals.

$$x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right),\ y = a\sin\theta.$$

Differentiating w.r.t. θ,

$$\dot{x} = a\left(-\sin\theta + \frac{1}{\tan\frac{\theta}{2}\cdot\cos^2\frac{\theta}{2}\cdot\frac{1}{2}}\right)$$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right) = \frac{a\cos^2\theta}{\sin\theta}$$

and $$\dot{y} = a\cos\theta$$

$\therefore$ $$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \tan\theta.$$

$\therefore$ Slope of the normal at 'θ' $= -\cot\theta$

Now the equation of the normal at 'θ' is

$$y - a\sin\theta = -\cot\theta\left\{x - a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)\right\} \tag{1}$$

(1) represents the family of normals of the tractrix, where θ is the parameter.

The evolute of the tractrix is the envelope of (1).
Differentiating (1) w.r.t. 'θ',

$$-a\cos\theta = x\operatorname{cosec}^2\theta + a\cot\theta\cdot\frac{\cos^2\theta}{\sin\theta} - a\operatorname{cosec}^2\theta\times\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$$

$$= \frac{x}{\sin^2\theta} + \frac{a\cos^3\theta}{\sin^2\theta} - \frac{a\cos\theta}{\sin^2\theta} - a\operatorname{cosec}^2\theta\times\log\tan\frac{\theta}{2}$$

$$= \frac{x}{\sin^2\theta} - a\cos\theta - \frac{a}{\sin^2\theta}\log\tan\frac{\theta}{2}$$

$\therefore$ $$x = a\log\tan\frac{\theta}{2} \tag{2}$$

Rewriting (1), we have

$$y = a\sin\theta - x\cot\theta + a\frac{\cos^2\theta}{\sin\theta} + a\cot\theta\cdot\log\tan\frac{\theta}{2} \tag{3}$$

Using (2) in (3), we get

$$y = a\sin\theta + \frac{a\cos^2\theta}{\sin\theta} - a\cot\theta\log\tan\frac{\theta}{2} + a\cot\theta\log\tan\frac{\theta}{2}$$

i.e. $$y = \frac{a}{\sin\theta}$$

$$= \frac{a\left(1+\tan^2\frac{\theta}{2}\right)}{2\tan\frac{\theta}{2}}$$

$$= \frac{a}{2}\left\{\tan\frac{\theta}{2} + \frac{1}{\tan\frac{\theta}{2}}\right\}$$

$$= \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}, \text{ again using (2)}$$

i.e. $y = a\cosh\dfrac{x}{a}$, which is the equation of the evolute of the tractrix.

EXERCISE 3(b)

Part A

(Short Answer Questions)

1. Define evolute and involute.
2. Explain briefly the procedure to find the evolute of a given curve $y = f(x)$.

3. Define envelope of a family of curves.
4. Give the working rule to find the equation of the envelope of the family $f(x, y, \alpha) = 0$, α being the parameter.
5. Obtain the equation of the envelope of the family $f_1(x, y)\alpha^2 + f_2(x, y)\alpha + f_3(x, y) = 0$, where α is the parameter.
6. Define evolute of a curve as an envelope.
7. If the centre of curvature of a curve at a variable point 't' on it is $(2a + 3at^2, -2at^3)$, find the evolute of the curve.
8. If the centre of curvature of a curve at a variable point 't' on it is $\left(\frac{c}{a}\cos^3 t, -\frac{c}{b}\sin^3 t\right)$, find the evolute of the curve.
9. If the centre of curvature of curve at a variable point 'θ' on it is $\left(a \log \cot \frac{\theta}{2}, \frac{a}{\sin\theta}\right)$, find the evolute of the curve.
10. Find the envelope of the family of lines $y = mx \pm a\sqrt{1+m^2}$, m being the parameter.
11. Find the envelope of the family of lines $y = mx + \frac{a}{m}$, m being the parameter.
12. Find the envelope of the family of lines $y = mx + am^2$, m being the parameter.
13. Find the envelope of the family of lines $y = mx \pm \sqrt{a^2m^2 + b^2}$, m being the parameter.
14. Find the envelope of the family of lines $\frac{x}{t} + yt = 2c$, t being the parameter.
15. Find the envelope of the lines $x \cos \alpha + y \sin \alpha = p$, α being the parameter.
16. Find the envelope of the lines $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$, θ being the parameter.
17. Find the envelope of the lines $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$, θ being the parameter.
18. Find the envelope of the lines $x \sec \theta - y \tan \theta = a$, θ being the parameter.
19. Find the envelope of the lines $x \operatorname{cosec} \theta - y \cot \theta = a$, θ being the parameter.
20. Show that the family of circles $(x - a)^2 + y^2 = a^2$ (a is the parameter) has no envelope.

Part B

21. Find the evolute of the parabola $y^2 = 4ax$.
22. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
23. Find the evolute of the rectangular hyperbola $xy = c^2$.
24. Show that the evolute of the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ is another cycloid, given by $x = a(\theta - \sin \theta)$, $y - 2a = a(1 + \cos \theta)$.
25. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are connected by the relation $a^2 + b^2 = c^2$.

26. Find the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$, where the parameters a and b are connected by the relation $ab = c^2$.
27. Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $a + b = c$.
28. From a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, perpendiculars are drawn to the axis and the feet of these perpendiculars are joined. Find the envelope of the line thus formed.
29. Find the evolute of the parabola $x^2 = 4ay$, treating it as the envelope of its normals.
30. Find the evolute of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, treating it as the envelope of its normals.

ANSWERS

Exercise 3(a)

(5) $\frac{1}{c}$. (4) $2\sqrt{2}$. (5) $\frac{8}{3}$.

(6) $\sec x$. (7) 1. (8) $c\sqrt{2}$.

(9) $4a\sqrt{2}$. (10) $\frac{5\sqrt{5}}{2}$. (11) $\frac{a}{2}$.

(12) $\frac{a\left(1+\theta^2\right)^{\frac{3}{2}}}{\theta^4}$. (13) $\frac{a}{2}$. (14) $\sqrt{2}\, r$.

(15) $5\sqrt{5}$. (16) $\frac{125a}{24}$. (17) $\frac{5\sqrt{5}}{18}$.

(18) $\frac{a}{2}$. (19) $\frac{97\sqrt{97}\, a}{216}$. (20) $\frac{1}{6}$.

(21) c. (22) $2a\sec^3\theta$.

(23) $\left(a^2\sin^2\theta+b^2\cos^2\theta\right)^{\frac{3}{2}}/ab$. (24) at.

(25) $4a\sin\frac{\theta}{2}$. (26) $\sqrt{2}$. (27) $a\sec^2\theta$.

(28) $a\cot\theta$. (29) r cosec α. (30) $\frac{2}{3}\sqrt{2ar}$.

(31) $\frac{a^n}{(n+1)r^{n-1}}$. (32) $\frac{2r^{\frac{3}{2}}}{\sqrt{a}}$. (33) $\frac{(2r)^{\frac{3}{2}}}{\sqrt{a}}$.

(35) $\left(-36, -\frac{43}{6}\right)$. (36) $\left[\frac{(a^2+b^2)}{a}\sec^3\theta, \frac{-(a^2+b^2)}{b}\tan^3\theta\right]$.

(37) $x^2+y^2-10ax+4ay-3a^2=0$

(39) $\left(x-\frac{43}{6}\right)^2+\left(y-\frac{57}{8}\right)^2=\left(\frac{125}{24}\right)^2$.

(40) $\left(x-\frac{21a}{16}\right)^2+\left(y-\frac{21a}{16}\right)^2=\frac{9a^2}{128}$.

Exercise 3(b)

(5) $f_2^2-4f_1f_3=0$. (7) $27ay^2=4(x-2a)^3$.

(8) $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=c^{\frac{2}{3}}$. (9) $y=a\cosh\frac{x}{a}$.

(10) $x^2+y^2=a^2$. (11) $y^2=4ax$. (12) $x^2+4ay=0$.

(13) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (14) $xy=c^2$. (15) $x^2+y^2=p^2$.

(16) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (17) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. (18) $x^2-y^2=a^2$.

(19) $x^2-y^2=a^2$. (21) $4(x-2a)^3=27ay^2$.

(22) $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

(23) $(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}}=(4c)^{\frac{2}{3}}$. (25) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$.

(26) $4xy=c^2$. (27) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$

(28) $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$. (29) $27ax^2=4(y-2a)^3$.

(30) $(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}$.

Chapter 4

Differential Calculus of Several Variables

4.1 INTRODUCTION

The students have studied in the lower classes the concept of partial differentiation of a function of more than one variable. They were also exposed to Homogeneous functions of several variables and Euler's theorem associated with such functions. In this chapter, we discuss some of the applications of the concept of partial differentiation, which are frequently required in engineering problems.

4.2 TOTAL DIFFERENTIATION

In partial differentiation of a function of two or more variables, it is assumed that only one of the independent variables varies at a time. In total differentiation, all the independent variables concerned are assumed to vary and so to take increments simultaneously.

Let $z = f(x, y)$, where x and y are continuous functions of another variable t.

Let Δt be a small increment in t. Let the corresponding increments in x, y, z be Δx, Δy and Δz respectively.

Then
$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$$
$$= \{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)\} + \{f(x, y+\Delta y) - f(x, y)\}$$

$$\therefore \quad \frac{\Delta z}{\Delta t} = \left\{\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right\}\cdot\frac{\Delta x}{\Delta t} + \left\{\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right\}\cdot\frac{\Delta y}{\Delta t} \qquad (1)$$

We note that Δx and $\Delta y \to 0$ as $\Delta t \to 0$ and hence $\Delta z \to 0$ as $\Delta t \to 0$

Taking limits on both sides of (1) as $\Delta t \to 0$, we have $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}t}$

($\because$ x, y and z are functions of t only and f is a function of x and y).

i.e., $$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \text{ [since } f(x, y) \equiv z(x, y)]. \quad (2)$$

$\frac{\mathrm{d}z}{\mathrm{d}t}\left(\text{and also } \frac{\mathrm{d}x}{\mathrm{d}t} \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t}\right)$ is called the *total differential coefficient* of z.

This name is given to distinguish it from the partial differential coefficients $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}\cdot$ Thus to differentiate z, which is directly a function of x and y, (where x and y are functions of t) with respect to t, we need not express z as a function of t by substituting for x and y. We can differentiate z with respect to t via x and y using the result (2).

Corollary 1: In the differential form, result (2) can be written as

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y \quad (3)$$

dz is called the *total differential* of z.

Corollary 2: If z is directly a function of two variables u and υ, which are in turn functions of two other variables x and y, clearly z is a function of x and y ultimately. Hence the total differentiation of z is meaningless. We can find only $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by using the following results which can be derived as result (2) given above.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial \upsilon}\cdot\frac{\partial \upsilon}{\partial x} \quad (4)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial \upsilon}\cdot\frac{\partial \upsilon}{\partial y} \quad (5)$$

We note that the partial differentiation of z is performed via the intermediate variables u and υ, which are functions of x and y. Hence $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called *partial derivatives of a function of two functions.*

Note ☑ Results (2), (3), (4) and (5) can be extended to a function z of several intermediate variables.

4.2.1 Small Errors and Approximations

Since $\lim\limits_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x}$ approximately or $\Delta y \simeq \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\Delta x$ (1)

If we assume that dx and dy are approximately equal to Δx and Δy respectively, result (1) can be derived from the differential relation.

$$dy = \left(\frac{dy}{dx}\right)dx \tag{2}$$

Though (2) is an exact relation, it can be made use of to get the approximate relation (1), by replacing dx and dy by Δx and Δy respectively.

Let $y = f(x)$. If we assume that the value of x is obtained by measurement, it is likely that there is a small error Δx in the measured value of x. This error in the value of x will contribute a small error Δy in the calculated value of y, as x and y are functionally related. The small increments Δx and Δy can be assumed to represent the small errors Δx and Δy. Thus the relation between the errors Δx and Δy can be taken as

$$\Delta y \simeq f'(x)\,\Delta x$$

This concept can be extended to a function of several variables.

If $u = u(x, y, z)$ or $f(x, y, z)$ and if the value of u is calculated on the measured values of x, y, z, the likely errors Δx, Δy, Δz will result in an error Δu in the calculated value of u, given by

$$\Delta u \simeq \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z,$$

which can be assumed as the approximate version of the total differential relation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

Note ☑ The error Δx in x is called *the absolute error* in x, while $\frac{\Delta x}{x}$ is called the *relative* or *proportional error* in x and $\frac{100\,\Delta x}{x}$ is called *the percentage error* in x.

4.2.2 Differentiation of Implicit Functions

When x and y are connected by means of a relation of the form $f(x, y) = 0$, x and y are said to be implicitly related or y is said to be an *implicit function* of x. When x and y are implicitly related, it may not be possible in many cases to express y as a single valued function of x explicitly. However $\frac{dy}{dx}$ can be found out in such cases as a mixed function of x and y using partial derivatives as explained below:

Since $\qquad f(x, y) = 0,\ df = 0$

i.e., $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$, by definition of total differential. Dividing by dx, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{\left(\dfrac{\partial f}{\partial x}\right)}{\left(\dfrac{\partial f}{\partial y}\right)} \tag{1}$$

If we denote $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y^2}$ by the letters p, q, r, s, t respectively, then

$$\frac{dy}{dx} = -\frac{p}{q} \tag{2}$$

We can express the second order derivative $\dfrac{d^2y}{dx^2}$ in terms of p, q, r, s, t as given below. Noting that p and q are functions of x and y and differentiating both sides of (2) with respect to x totally, we have

$$\frac{d^2y}{dx^2} = -\left(\frac{q\dfrac{dp}{dx} - p\dfrac{dq}{dx}}{q^2}\right)$$

$$= \frac{p\left\{\dfrac{\partial q}{\partial x} + \dfrac{\partial q}{\partial y}\cdot\dfrac{dy}{dx}\right\} - q\left\{\dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial y}\cdot\dfrac{dy}{dx}\right\}}{q^2}$$

$$= \frac{p\left\{s + t\left(\dfrac{-p}{q}\right)\right\} - q\left\{r + s\left(\dfrac{-p}{q}\right)\right\}}{q^2},$$

since $\qquad \dfrac{\partial p}{\partial x} = \dfrac{\partial^2 f}{\partial x^2} = r;\ \dfrac{\partial p}{\partial y} = \dfrac{\partial q}{\partial x} = \dfrac{\partial^2 f}{\partial x\,\partial y} = s;\ \dfrac{\partial q}{\partial y} = \dfrac{\partial^2 f}{\partial y^2} = t.$

$$= \frac{p(qs - pt) - q(qr - ps)}{q^3}$$

$$= -\frac{(p^2t - 2pqs + q^2r)}{q^3}$$

WORKED EXAMPLE 4(a)

Example 4.1

(i) If $u = xy + yz + zx$, where $x = e^t$, $y = e^{-t}$ and $z = \dfrac{1}{t}$, find $\dfrac{du}{dt}$

(ii) If $u = \sin^{-1}(x - y)$, where $x = 3t$ and $y = 4t^3$, show that $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1 - t^2}}$

(i) $u = xy + yz + zx$

$$\therefore \qquad \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= (y+z)e^t + (z+x)(-e^{-t}) + (x+y)\left(-\frac{1}{t^2}\right)$$

$$= \left(e^{-t} + \frac{1}{t}\right)e^t - \left(\frac{1}{t} + e^t\right)e^{-t} - \left(e^t + e^{-t}\right).\frac{1}{t^2}$$

$$= 1 + \frac{1}{t}e^t - \frac{1}{t}e^{-t} - 1 - \frac{1}{t^2}e^t - \frac{1}{t^2}e^{-t}$$

$$= \frac{2}{t}\sinh t - \frac{2}{t^2}\cosh t.$$

(ii) $u = \sin^{-1}(x - y)$

$$\therefore \qquad \frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}\frac{dx}{dt} + \frac{1}{\sqrt{1-(x-y)^2}}\left(-\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}(3-12t^2) \qquad (1)$$

Now $1-(x-y)^2 = 1-(3t-4t^3)^2$

$$= 1 - 9t^2 + 24t^4 - 16t^6$$

$$= (1-t^2)(1-8t^2+16t^4)$$

$$= (1-t^2)(1-4t^2)^2 \qquad (2)$$

Using (2) in (1), we get

$$\frac{du}{dt} = \frac{1}{(1-4t^2)\sqrt{1-t^2}} \times 3(1-4t^2)$$

$$= \frac{3}{\sqrt{1-t^2}}\cdot$$

Example 4.2 If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

Let $$r = \frac{x}{y},\ s = \frac{y}{z} \text{ and } t = \frac{z}{x} \qquad (1)$$

$\therefore\ u = f(r, s, t)$, where r, s, t are functions of x, y, z as assumed in (1)

$\therefore$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{1}{y}\cdot\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\cdot 0 - \frac{z}{x^2}\cdot\frac{\partial u}{\partial t} \tag{2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= -\frac{x}{y^2}\cdot\frac{\partial u}{\partial r} + \frac{1}{z}\cdot\frac{\partial u}{\partial s} \tag{3}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= -\frac{y}{z^2}\cdot\frac{\partial u}{\partial s} + \frac{1}{x}\cdot\frac{\partial u}{\partial t} \tag{4}$$

From (2), (3) and (4), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \left(\frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t}\right)$$

$$+ \left(-\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s}\right) + \left(-\frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}\right)$$

$$= 0.$$

Example 4.3 If z be a function of x and y, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \tag{1}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= -e^{-v}\frac{\partial z}{\partial x} - e^v\cdot\frac{\partial z}{\partial y} \tag{2}$$

From (1) and (2), we have

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Example 4.4 If $u = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\theta\cdot\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \tag{1}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\cdot\frac{\partial u}{\partial x} + r\cos\theta\cdot\frac{\partial u}{\partial y}$$

i.e.,
$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\cdot\frac{\partial u}{\partial y} \tag{2}$$

Squaring both sides of (1) and (2) and adding, we get

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Example 4.5 Find the equivalent of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in polar co-ordinates.

$$u = u(x, y), \text{ where } x = r\cos\theta \text{ and } y = r\sin\theta$$

$\therefore$ u can also be considered as $u(r, \theta)$, where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now we proceed to find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ via r and θ.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{x}{\sqrt{x^2 + y^2}}\frac{\partial u}{\partial r} + \frac{1}{1 + \frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)\cdot\frac{\partial u}{\partial \theta}$$

$$= \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \tag{1}$$

From (1), we can infer that

$$\frac{\partial}{\partial x} \equiv \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{2}$$

Now
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$
$$= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$
$$= \cos^2\theta \cdot \frac{\partial^2 u}{\partial r^2} - \sin\theta\cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)$$
$$- \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r}\right) + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

$\left(\because r \text{ and } \theta \text{ are independent and } \dfrac{\partial u}{\partial r} \text{ and } \dfrac{\partial u}{\partial \theta} \text{ are functions of } r \text{ and } \theta\right)$.

$$= \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \sin\theta\,\cos\theta\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)$$
$$- \frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 u}{\partial\theta\,\partial r} - \sin\theta\frac{\partial u}{\partial r}\right) + \frac{\sin\theta}{r^2}\left(\sin\theta\frac{\partial^2 u}{\partial\theta^2} + \cos\theta\frac{\partial u}{\partial\theta}\right) \tag{3}$$

Now
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$
$$= \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\left(\frac{1}{x}\right)\frac{\partial u}{\partial\theta}$$
$$= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} \tag{4}$$

From (4) we infer that

$$\frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \tag{5}$$

$\therefore$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$
$$= \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$
$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \sin\theta\cos\theta\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)$$
$$+ \frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 u}{\partial\theta\,\partial r} + \cos\theta\frac{\partial u}{\partial r}\right) + \frac{\cos\theta}{r^2}\left(\cos\theta\frac{\partial^2 u}{\partial\theta^2} - \sin\theta\frac{\partial u}{\partial\theta}\right) \tag{6}$$

Adding (3) and (6), we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Example 4.6 Given the transformations $u = e^x \cos y$ and $\upsilon = e^x \sin y$ and that f is a function of u and υ and also of x and y, prove that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} = (u^2+\upsilon^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial \upsilon^2}\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial \upsilon}\cdot\frac{\partial \upsilon}{\partial x}$$

$$= e^x \cos y\cdot\frac{\partial f}{\partial u}+e^x \sin y\cdot\frac{\partial f}{\partial \upsilon}$$

$$= u\frac{\partial f}{\partial u} + \upsilon\frac{\partial f}{\partial \upsilon} \tag{1}$$

$$\frac{\partial}{\partial x} \equiv u\frac{\partial}{\partial u}+\upsilon\frac{\partial}{\partial \upsilon} \tag{2}$$

$$\frac{\partial^2 f}{\partial x^2}=\left(u\frac{\partial}{\partial u}+\upsilon\frac{\partial}{\partial \upsilon}\right)\left(u\frac{\partial f}{\partial u}+\upsilon\frac{\partial f}{\partial \upsilon}\right)$$

$$=u\left(u\cdot\frac{\partial^2 f}{\partial u^2}+\frac{\partial f}{\partial u}\right)+u\upsilon\frac{\partial^2 f}{\partial u\,\partial \upsilon}+u\upsilon\frac{\partial^2 f}{\partial \upsilon\,\partial u}$$

$$+\upsilon\left(\upsilon\frac{\partial^2 f}{\partial \upsilon^2}+\frac{\partial f}{\partial \upsilon}\right) \tag{3}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial \upsilon}\cdot\frac{\partial \upsilon}{\partial y}$$

$$=-e^x \sin y\cdot\frac{\partial f}{\partial u}+e^x \cos y\frac{\partial f}{\partial \upsilon}$$

$$=-\upsilon\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial \upsilon} \tag{4}$$

$\therefore$

$$\frac{\partial}{\partial y} \equiv -\upsilon\frac{\partial}{\partial u}+u\frac{\partial}{\partial \upsilon} \tag{5}$$

$$\frac{\partial^2 f}{\partial y^2} = \left(-\upsilon\frac{\partial}{\partial u} + u\frac{\partial}{\partial \upsilon}\right)\left(-\upsilon\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial \upsilon}\right)$$

$$= \upsilon^2\frac{\partial^2 f}{\partial u^2}-\upsilon\left(u\frac{\partial^2 f}{\partial u\,\partial \upsilon} + \frac{\partial f}{\partial \upsilon}\right)-u\left(\upsilon\frac{\partial^2 f}{\partial \upsilon\,\partial u} + \frac{\partial f}{\partial u}\right)$$

$$+ u^2\frac{\partial^2 f}{\partial \upsilon^2} \tag{6}$$

Adding (3) and (6), we get

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} = (u^2+\upsilon^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial \upsilon^2}\right)$$

Example 4.7 If $z = f(u,\upsilon)$, where $u = \cosh x \cos y$ and $\upsilon = \sinh x \sin y$, prove that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = \left(\sinh^2 x+\sin^2 y\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial \upsilon^2}\right)$$

$$z_x = z_u \cdot u_x + z_\upsilon \cdot \upsilon_x, \text{ where } z_x \equiv \frac{\partial z}{\partial x} \text{ etc.}$$

$$= \sinh x \cdot \cos y \cdot z_u + \cosh x \sin y \cdot z_\upsilon$$

Since z is a function of u and υ, z_u and z_υ are also functions of u and υ. Hence to differentiate z_u and z_υ with respect to x or y, we have to do it via u and υ.

$$\therefore \quad z_{xx} = \cos y\left[\cosh x \cdot z_u + \sinh x\left\{z_{uu} \cdot \sinh x \cos y + z_{u\upsilon} \cosh x \sin y\right\}\right]$$

$$+\sin y\left[\sinh x \cdot z_\upsilon + \cosh x\left\{z_{\upsilon u} \cdot \sinh x \cos y + z_{\upsilon\upsilon} \cdot \cosh x \sin y\right\}\right]$$

i.e.,

$$z_{xx} = \cosh x \cos y \cdot z_u + \sinh x \cdot \sin y\, z_\upsilon + \sinh^2 x \cos^2 y \cdot z_{uu}$$

$$+2\sinh x \cosh x \sin y \cos y \cdot z_{u\upsilon} + \cosh^2 x \sin^2 y \cdot z_{\upsilon\upsilon} \qquad (1)$$

$$z_y = -z_u \cdot \cosh x \sin y + z_\upsilon \sinh x \cos y$$

$$z_{yy} = -\cosh x[\cos y \cdot z_u + \sin y\{z_{uu} \cdot (-\cosh x \sin y)$$

$$+z_{u\upsilon} \cdot \sinh x \cdot \cos y\} + \sinh x[-\sinh y \cdot z_\upsilon$$

$$+ \cos y\{-z_{\upsilon u} \cdot \cosh x \sin y + z_{\upsilon\upsilon} \cdot \sinh x \cos y\}]$$

i.e.,

$$z_{yy} = -\cosh x \cos y \cdot z_u - \sinh x \cdot \sin y \cdot z_\upsilon$$

$$+\cosh^2 x \sin^2 y \cdot z_{uu} - 2\sinh x \cosh x \sin y \cos y \cdot z_{u\upsilon}$$

$$+\sinh^2 x \cos^2 y \cdot z_{\upsilon\upsilon} \qquad (2)$$

Adding (1) and (2), we get

$$z_{xx} + z_{yy} = (\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y)(z_{uu} + z_{\upsilon\upsilon})$$

$$= \{\sinh^2 x(1-\sin^2 y) + (1+\sinh^2 x)\sin^2 y\}(z_{uu} + z_{\upsilon\upsilon})$$

$$= (\sinh^2 x + \sin^2 y)(z_{uu} + z_{\upsilon\upsilon})$$

Example 4.8 Find $\dfrac{dy}{dx}$, when (i) $x^3 + y^3 = 3ax^2y$ and (ii) $x^y + y^x = c$.

(i) $f(x, y) = x^3 + y^3 - 3ax^2y$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 6axy$$

$$q = \frac{\partial f}{\partial y} = 3y^2 - 3ax^2$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{3(x^2 - 2axy)}{3(y^2 - ax^2)} = \frac{x(2ay - x)}{y^2 - ax^2}$$

(ii) $f(x, y) = x^y + y^x - c$

$$p = \frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$q = \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{yx^{y-1} + y^x \log x}{xy^{x-1} + x^y \log x}.$$

Example 4.9 If $ax^2 + 2hxy + by^2 = 1$, show that $\dfrac{d^2y}{dx^2} = \dfrac{h^2 - ab}{(hx+by)^3}$.

$f(x, y) = ax^2 + 2hxy + by^2 - 1$

$$p = \frac{\partial f}{\partial x} = 2(ax+hy);\ q = \frac{\partial f}{\partial y} = 2(hx+by)$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2a;\ s = \frac{\partial^2 f}{\partial x\,\partial y} = 2h;\ t = \frac{\partial^2 f}{\partial y^2} = 2b$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{(ax+hy)}{hx+by}$$

$$\frac{d^2y}{dx^2} = \frac{-(p^2t - 2pqs + q^2r)}{q^3}$$

(Refer to differentiation of implicit functions)

$$= \frac{-\left\{8b(ax+hy)^2 - 16h(ax+hy)(hx+by) + 8a(hx+by)^2\right\}}{8(hx+by)^3}$$

$$= \frac{1}{(hx+by)^3}[2h\{ahx^2 + (ab+h^2)xy + bhy^2\}$$
$$-\{a^2bx^2 + 2abhxy + h^2by^2\} - \{ah^2x^2 + 2abhxy + ab^2y^2\}]$$

$$= \frac{1}{(hx+by)^3}[a(h^2-ab)x^2+2h(h^2-ab)xy+b(h^2-ab)y^2]$$

$$= \frac{(h^2-ab)}{(hx+by)^3}(ax^2+2hxy+by^2) = \frac{(h^2-ab)\cdot 1}{(hx+by)^3} = \frac{h^2-ab}{(hx+by)^3}\cdot$$

Example 4.10 Find $\frac{du}{dx}$ if (i) $u = \sin(x^2+y^2)$, where $a^2x^2+b^2y^2=c^2$ (i), $u=\tan^{-1}\left(\frac{y}{x}\right)$ where $x^2+y^2=a^2$, by treating u as function of x and y only.

(i) $u = \sin(x^2+y^2)$

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2x\cos\left(x^2+y^2\right) + 2y\cos\left(x^2+y^2\right)\frac{dy}{dx} \tag{1}$$

Now $a^2x^2+b^2y^2=c^2$

Differentiating with respect to x,

$$2a^2x + 2b^2y\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{a^2x}{b^2y} \tag{2}$$

Using (2) in (1), we get

$$\frac{du}{dx} = 2x\cos\left(x^2+y^2\right) + 2y\cos\left(x^2+y^2\right)\times\left(\frac{-a^2x}{b^2y}\right)$$

$$= 2x\cos\left(x^2+y^2\right)\left(b^2-a^2\right)/b^2$$

(ii)

$$u = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)\cdot\frac{dy}{dx}$$

$$= -\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}\cdot\frac{dy}{dx} \tag{3}$$

$$x^2+y^2 = a^2$$

$$\therefore \quad 2x + 2y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{x}{y} \tag{4}$$

Using (4) in (3), we get

$$\begin{aligned}\frac{du}{dx} &= -\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}\left(-\frac{x}{y}\right) \\ &= -\frac{1}{y}.\end{aligned}$$

Example 4.11 If $u = x^2 - y^2$ and $\upsilon = xy$, find the values of $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial \upsilon}, \frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial \upsilon}$.

x and y cannot be easily expressed as single valued functions or u and υ.

Given $$x^2 - y^2 = u \tag{1}$$

and $$xy = \upsilon \tag{2}$$

Nothing that x and y are functions of u and υ and differentiating both sides of (1) and (2) partially with respect to u, we have

$$2x\frac{\partial x}{\partial u} - 2y\frac{\partial y}{\partial u} = 1 \tag{3}$$

$$y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u} = 0 \tag{4}$$

Solving (3) and (4), we get

$$\frac{\partial x}{\partial u} = \frac{x}{2\left(x^2+y^2\right)} \text{ and } \frac{\partial y}{\partial u} = -\frac{y}{2\left(x^2+y^2\right)}$$

Differentiating both sides of (1) and (2) partially with respect to υ, we have

$$2x\frac{\partial x}{\partial \upsilon} - 2y\frac{\partial y}{\partial \upsilon} = 0 \tag{5}$$

$$y\frac{\partial x}{\partial \upsilon} + x\frac{\partial y}{\partial \upsilon} = 1 \tag{6}$$

Solving (5) and (6), we get

$$\frac{\partial x}{\partial \upsilon} = \frac{y}{x^2+y^2} \text{ and } \frac{\partial y}{\partial \upsilon} = \frac{x}{x^2+y^2}.$$

Example 4.12 If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dz}{\sqrt{1-z^2}} = 0.$

Let
$$\phi \equiv x^2 + y^2 + z^2 - 2xyz - 1 = 0 \tag{1}$$

$\therefore$
$$d\phi = 0$$

i.e.,
$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \tag{2}$$

i.e.,
$$2(x - yz)dx + 2(y - zx)dy + 2(z - xy)dz = 0$$

Now
$$\begin{aligned}(x - yz)^2 &= x^2 - 2xyz + y^2z^2 \\ &= 1 - y^2 - z^2 + y^2z^2, \text{ from (1)} \\ &= (1 - y^2)(1 - z^2)\end{aligned}$$

$\therefore$
$$x - yz = \sqrt{(1-y^2)(1-z^2)}$$

Similarly,
$$y - zx = \sqrt{(1-z^2)(1-x^2)}$$

and
$$z - xy = \sqrt{(1-x^2)(1-y^2)}$$

Using these values in (2), we have

$$\sqrt{(1-y^2)(1-z^2)}\,dx + \sqrt{(1-z^2)(1-x^2)}\,dy + \sqrt{(1-x^2)(1-y^2)}\,dz = 0$$

Dividing by $\sqrt{(1-x^2)(1-y^2)(1-z^2)}$, we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

Example 4.13 The specific gravity s of a body is given by $s = \dfrac{W_1}{W_1 - W_2}$ where W_1 and W_2 are the weights of the body in air and in water respectively. Show that if there is an error of 1% in each weighing, s is not affected. But if there is an error of 1% in W_1 and 2% in W_2, show that the percentage error in s is $\dfrac{W_2}{W_1 - W_2}$.

$$s = \frac{W_1}{W_1 - W_2}$$

$\therefore$
$$\log s = \log W_1 - \log(W_1 - W_2)$$

Taking differentials on both sides,

$$\frac{1}{s}\mathrm{d}s = \frac{1}{W_1}\mathrm{d}W_1 - \frac{1}{W_1 - W_2}\left(dW_1 - dW_2\right)$$

$\therefore$ The relation among the errors is nearly

$$\frac{1}{s}\Delta s = \frac{1}{W_1}\Delta W_1 - \frac{1}{W_1 - W_2}\left(\Delta W_1 - \Delta W_2\right) \tag{1}$$

or
$$\frac{100\,\Delta s}{s} = \frac{100\,\Delta W_1}{W_1} - \frac{1}{W_1 - W_2}\left(100\,\Delta W_1 - 100\,\Delta W_2\right) \tag{2}$$

(i) Given that $\dfrac{100\,\Delta W_1}{W_1} = 1$ and $\dfrac{100\,\Delta W_2}{W_2} = 1$

Using these values in (2), we have

$$\frac{100\,\Delta s}{s} = 1 - \frac{1}{W_1 - W_2}\left(W_1 - W_2\right) = 0$$

$\therefore$ s is not affected, viz., there is no error in s.

(ii) Given that $\dfrac{100\,\Delta W_1}{W_1} = 1$ and $\dfrac{100\,\Delta W_2}{W_2} = 2.$ Using these values in (2), we have

$$\frac{100\,\Delta s}{s} = 1 - \frac{1}{W_1 - W_2}\left(W_1 - 2W_2\right)$$
$$= \frac{W_2}{W_1 - W_2}$$

i.e., % error in $s = \dfrac{W_2}{W_1 - W_2}.$

Example 4.14 The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $s^2 D^{3/2} \div t^2$. Find approximately the percentage increase of work necessary when the distance is increased by 1%, the time is diminished by 1% and the displacement of the ship is diminished by 3%.

Given that $W = ks^2 D^{3/2}/t^2$, where k is the constant of proportionality.

$\therefore$
$$\log W = \log k + 2\log s + \frac{3}{2}\log D - 2\log t.$$

Taking differentials on both sides,

$$\frac{\mathrm{d}W}{W} = 2\frac{\mathrm{d}s}{s} + \frac{3}{2}\frac{\mathrm{d}D}{D} - 2\frac{\mathrm{d}t}{t}$$

$\therefore$ The relation among the percentage errors is approximately,

$$\frac{100\,\Delta W}{W} = 2\times\frac{100\,\Delta s}{s}+\frac{3}{2}\cdot\frac{100\,\Delta D}{D}-2\times\frac{100\,\Delta t}{t} \tag{1}$$

Given that $$\frac{100\,\Delta s}{s} = 1,\ \frac{100\,\Delta t}{t} = -1 \text{ and } \frac{100\,\Delta D}{D} = -3.$$

Using these values in (1), we have

$$\begin{aligned}\frac{100\,\Delta W}{W} &= 2\times 1+\frac{3}{2}\times(-3)-2\times(-1)\\ &= -0.5\end{aligned}$$

i.e., percentage decrease of work = 0.5.

Example 4.15 The period T of a simple pendulum with small oscillations is given by $T = 2\pi\sqrt{\frac{l}{g}}$. If T is computed using $l = 6$ cm and $g = 980$ cm/sec^2, find approximately the error in T, if the values are $l = 5.9$ cm and $g = 981$ cm/sec^2. Find also the percentage error.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$\therefore$ $$\log T = \log 2+\log\pi+\frac{1}{2}\log l-\frac{1}{2}\log g$$

Taking differentials on both sides,

$$\frac{1}{T}\,\mathrm{d}T=\frac{1}{2l}\,\mathrm{d}l-\frac{1}{2g}\,\mathrm{d}g \tag{1}$$

$\therefore$ $$\begin{aligned}\mathrm{d}T &= 2\pi\sqrt{\frac{l}{g}}\left\{\frac{1}{2l}\,\mathrm{d}l-\frac{1}{2g}\,\mathrm{d}g\right\}\\ &=\pi\left\{\frac{1}{\sqrt{l\,g}}\,\mathrm{d}l-\frac{\sqrt{l}}{g\sqrt{g}}\,\mathrm{d}g\right\}\\ &= \pi\left\{\frac{0\cdot 1}{\sqrt{5.9\times 981}}-\frac{\sqrt{5\cdot 9}}{981\sqrt{981}}\times(-1)\right\}\end{aligned}$$

i.e., Error in $T = 0.0044$ sec.

$$\% \text{ error in } T = \frac{100\,\mathrm{d}T}{T}$$

$$= 50\left\{\frac{\mathrm{d}l}{l} - \frac{\mathrm{d}g}{g}\right\}, \text{by (1)}$$

$$= 50\left\{\frac{0\cdot 1}{5\cdot 9} - \frac{(-1)}{981}\right\}$$

$$= 0.8984$$

Example 4.16 The base diameter and altitude of a right circular cone are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the value computed for the volume and lateral surface.

Volume of the right circular cone is given by $V = \frac{1}{3}\pi\cdot\left(\frac{D}{2}\right)^2 h$

$$\therefore \qquad \mathrm{d}V = \frac{\pi}{12}(D^2\cdot \mathrm{d}h + 2\,Dh\cdot \mathrm{d}D)$$

$$= \frac{\pi}{12}\{16\times 0.1 + 2\times 4\times 6\times 0.1\}$$

i.e., Error in $V = 1.6755$ cm^3.

Lateral surface area of the right circular cone is given by

$$S = \pi\frac{D}{2}l$$

$$= \frac{\pi}{4}D\sqrt{D^2+4h^2}$$

$$\therefore \qquad \mathrm{d}S = \frac{\pi}{4}\left[D\cdot\frac{1}{2\sqrt{D^2+4h^2}}(2D\,\mathrm{d}D + 8h\,\mathrm{d}h + \sqrt{D^2+4h^2}\ \mathrm{d}D\right]$$

$$= \frac{\pi}{4}\left[\frac{4}{\sqrt{16+144}}\{4\times 0\cdot 1 + 24\times 0\cdot 1\} + \sqrt{16+144}\times 0\cdot 1\right]$$

$$= 1.6889\,\text{cm}^2.$$

Example 4.17 The side c of a triangle ABC is calculated by using the measured values of its sides a, b and the angle C. Show that the error in the side c is given by

$$\Delta c = \cos B \cdot \Delta a + \cos A \cdot \Delta b + a \sin B \cdot \Delta C.$$

The side c is given by the formula

$$c^2 = a^2 + b^2 - 2ab \cos C \tag{1}$$

Taking the differentials on both sides of (1),

$$2c\,\Delta c = 2a\,\Delta a + 2b\,\Delta b - 2\{b \cos C \cdot \Delta a + a \cos C \cdot \Delta b - ab \sin C \cdot \Delta C\}, \text{ nearly}$$

i.e.,
$$\Delta c = \frac{(a - b\cos C)\Delta a + (b - a\cos C)\Delta b + ab\sin C \cdot \Delta C}{c} \tag{2}$$

Now $b \cos C + c \cos B = a$

$$\therefore \quad \frac{a - b\cos C}{c} = \cos B \tag{3}$$

$$a \cos C + c \cos A = b$$

$$\therefore \quad \frac{b - a\cos C}{c} = \cos A \tag{4}$$

Also
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad b \sin C = c \sin B$$

$$\therefore \quad \frac{ab\sin C}{c} = a \sin B \tag{5}$$

Using (3), (4) and (5) in (2), we get

$$\Delta c = \cos B \cdot \Delta a + \cos A \cdot \Delta b + a \sin B \cdot \Delta C$$

Example 4.18 The angles of a triangle ABC are calculated from the sides a, b, c. If small changes δa, δb, δc are made in the measurement of the sides, show that

$$\delta A = \frac{a}{2\Delta}(\delta a - \delta b \cos C - \delta c \cos B)$$

and δB and δC are given by similar expressions, where Δ is the area of the triangle. Verify that $\delta A + \delta B + \delta C = 0$.

In triangle ABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1}$$

Taking differentials on both sides of (1),

$$-2\sin A\cdot\delta A=\left[bc\left\{2b\,\delta b+2c\delta c-2a\delta a\right\}-\left(b^2+c^2-a^2\right)\left(b\delta c+c\delta b\right)\right]\div b^2c^2$$

$$=\frac{\left(b^2c-c^3+a^2c\right)\delta b+\left(bc^2-b^3+a^2b\right)\delta c-2abc\,\delta a}{b^2c^2}$$

$$=\frac{c\left(a^2+b^2-c^2\right)\delta b+b\left(c^2+a^2-b^2\right)\delta c-2abc\delta a}{b^2c^2}$$

$$=\frac{c\left(2\,ab\cos C\right)\delta b+b\left(2\,ca\cos B\right)\delta c-2abc\,\delta a}{b^2c^2},$$

by formulas similar to (1)

$$=\frac{2a}{bc}\left(\cos\,C\cdot\delta b+\cos B\cdot\delta c-\delta a\right)$$

$$\therefore\qquad \delta A=\frac{a}{bc\sin A}\left(\delta a-\cos C\cdot\delta b-\cos B\cdot\delta c\right)$$

$$=\frac{a}{2\Delta}\left(\delta a-\cos C\cdot\delta b-\cos B\cdot\delta c\right),\text{ since }\Delta=\frac{1}{2}bc\sin A \qquad (2)$$

Similarly,

$$\delta B=\frac{b}{2\Delta}\left(\delta b-\cos A\cdot\delta c-\cos C\cdot\delta a\right) \qquad (3)$$

$$\delta C=\frac{c}{2\Delta}\left(\delta c-\cos B\cdot\delta a-\cos A\cdot\delta b\right) \qquad (4)$$

Adding (2), (3) and (4), we get

$$\begin{aligned}2\Delta\left(\delta A+\delta B+\delta C\right) &=\left(a-b\cos C-c\cos B\right)\delta a\\ &\quad+\left(b-a\cos C-c\cos A\right)\delta b+\left(c-a\cos B-b\cos A\right)\delta c\\ &=\left(a-a\right)\delta a+\left(b-b\right)\delta b+\left(c-c\right)\delta c\\ &\qquad\qquad\left(\because\ b\cos C+c\cos B=a\,\text{etc.}\right)\\ &=0\end{aligned}$$

$$\therefore\qquad \delta A+\delta B+\delta C=0.$$

Example 4.19 The area of a triangle *ABC* is calculated from the lengths of the sides *a*, *b*, *c*. If *a* is diminished and *b* is increased by the same small amount *k*, prove that the consequent change in the area is given by

$$\frac{\delta\Delta}{\Delta} = \frac{2(a-b)k}{c^2-(a-b)^2}$$

The area of triangle ABC is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\ , \text{ where } 2s = a+b+c$$

$$\therefore \qquad \log\Delta = \frac{1}{2}\{\log s + \log(s-a) + \log(s-b) + \log(s-c)\}$$

Taking differentials on both sides, we get

$$\frac{\delta\Delta}{\Delta} = \frac{1}{2}\left\{\frac{\delta s}{s} + \frac{\delta s-\delta a}{s-a} + \frac{\delta s-\delta b}{s-b} + \frac{\delta s-\delta c}{s-c}\right\} \tag{1}$$

Since $\qquad 2s = a+b+c,\ 2\delta s = \delta a+\delta b+\delta c$

i.e.; $2\delta s = -k + k + 0 = 0$, by the given data.

$$\therefore \qquad \delta s = 0 \tag{2}$$

Using (2) in (1), we have

$$\begin{aligned}\frac{\delta\Delta}{\Delta} &= \frac{1}{2}\left(\frac{k}{s-a} - \frac{k}{s-b}\right)\\ &= \frac{k}{2}\left\{\frac{2}{b+c-a} - \frac{2}{c+a-b}\right\} \ (\because 2s = a+b+c)\\ &= \frac{k}{2}\times 2\left\{\frac{(c+a-b)-(b+c-a)}{[c-(a-b)][c+(a-b)]}\right\}\\ &= \frac{2k(a-b)}{c^2-(a-b)^2}\end{aligned}$$

EXERCISE 4(a)

Part A

(Short Answer Questions)

1. What is meant by total differential? Why it is called so?
2. If $u = \sin(xy^2)$, express the total differential of u in terms of those of x and y.
3. If $u = x^y \cdot y^x$, express du in terms of dx and dy.
4. If $u = xy \log xy$, express du in terms of dx and dy.
5. If $u = a^{xy}$, express du in terms of dx and dy.

6. Find $\dfrac{\mathrm{d}u}{\mathrm{d}t}$, if $u = x^3y^2 + x^2y^3$, where $x = at^2$, $y = 2at$.

7. Find $\dfrac{\mathrm{d}u}{\mathrm{d}t}$, if $u = e^{xy}$, where $x = \sqrt{a^2 - t^2}$, $y = \sin^3 t$.

8. Find $\dfrac{\mathrm{d}u}{\mathrm{d}t}$, if $u = \log(x + y + z)$, where $x = e^{-t}$, $y = \sin t$, $z = \cos t$.

9. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, using partial differentiation, if $x^3 + 3x^2y + 6xy^2 + y^3 = 1$.

10. If $x \sin(x - y) - (x + y) = 0$, use partial differentiation to prove that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y + x^2 \cos(x - y)}{x + x^2 \cos(x - y)}.$$

11. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, when $u = \sin(x^2 + y^2)$, where $x^2 + 4y^2 = 9$.

12. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, if $u = x^2y$, where $x^2 + xy + y^2 = 1$.

13. Define absolute, relative and percentage errors.
14. Using differentials, find the approximate value of $\sqrt{15}$.
15. Using differentials, find the approximate value of $2x^4 + 7x^3 - 8x^2 + 3x + 1$ when $x = 0.999$.
16. What error in the common logarithm of a number will be produced by an error of 1% in the number?
17. The radius of a sphere is found to be 10 cm with a possible error of 0.02 cm. Find the relative errors in computing the volume and surface area.
18. Find the percentage error in the area of an ellipse, when an error of 1% is made in measuring the lengths of its axes.
19. Find the approximate error in the surface of a rectangular parallelopiped of sides a, b, c if an error of k is made in measuring each side.
20. If the measured volume of a right circular cylinder is 2% too large and the measured length is 1% too small, find the percentage error in the calculated radius.

Part B

21. If $u = f(x - y, y - z, z - x)$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

22. If f is a function of u, v, w, where $u = \sqrt{yz}$, $v = \sqrt{zx}$, and $w = \sqrt{xy}$ show that

$$\sum u \frac{\partial f}{\partial u} = \sum x \frac{\partial f}{\partial x}.$$

23. If $f = f\left(\dfrac{y-x}{xy}, \dfrac{z-x}{zx}\right)$, show that $x^2\dfrac{\partial f}{\partial x} + y^2\dfrac{\partial f}{\partial y} + z^2\dfrac{\partial f}{\partial z} = 0.$

24. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z} = 0.$$

25. If $f(cx - az, cy - bz) = 0$, where z is a function of x and y, prove that

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c.$$

26. If $z = f(u, \upsilon)$, where $u = x + y$ and $\upsilon = x - y$, show that $2\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}.$

27. If $z = f(x, y)$, where $x = u^2 + \upsilon^2$, $y = 2u\upsilon$, prove that

$$u\frac{\partial z}{\partial u} - \upsilon\frac{\partial z}{\partial \upsilon} = 2\sqrt{(x^2 - y^2)}\,\frac{\partial z}{\partial x}.$$

28. If $z = f(x, y)$, where $x = u + \upsilon$, $y = u\upsilon$, prove that

$$u\frac{\partial z}{\partial u} + \upsilon\frac{\partial z}{\partial \upsilon} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}.$$

29. If $x = r\cos\theta$, $y = r\sin\theta$, prove that the equation $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$ is equivalent to $\dfrac{\partial u}{\partial r} + \dfrac{1}{r}\tan\left(\dfrac{\pi}{4} - \theta\right)\dfrac{\partial u}{\partial \theta} = 0.$

30. If $z = f(u, \upsilon)$, where $u = x^2 - 2xy - y^2$ and $\upsilon = y$, show that the equation $(x + y)\dfrac{\partial z}{\partial x} + (x - y)\dfrac{\partial z}{\partial y} = 0$ is equivalent to $\dfrac{\partial z}{\partial \upsilon} = 0.$

31. If $z = f(u, \upsilon)$, where $u = x^2 - y^2$ and $\upsilon = 2xy$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x^2 + y^2)\left\{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial \upsilon}\right)^2\right\}.$$

32. If $z = f(u, \upsilon)$, where $u = x^2 - y^2$ and $\upsilon = 2xy$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial \upsilon^2}\right).$$

33. If $z = f(x, y)$ where $x = X\cos a - Y\sin\alpha$ and $y = X\sin\alpha + Y\cos\alpha$, show that $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial X^2}+\dfrac{\partial^2 z}{\partial Y^2}$.

34. If $z = f(u, \upsilon)$, where $u = lx + my$ and $\upsilon = ly - mx$, show that
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial \upsilon^2}\right).$$

35. By changing the independent variables x and t to u and υ by means of the transformations $u = x - at$ and $\upsilon = x + at$, show that $a^2\dfrac{\partial^2 y}{\partial x^2}-\dfrac{\partial^2 y}{\partial t^2} = 4a^2\dfrac{\partial^2 y}{\partial u\,\partial \upsilon}$.

36. By using the transformations $u = x + y$ and $\upsilon = x - y$, change the independent variables x and y in the equation $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2} = 0$ to u and υ.

37. Transform the equation $\dfrac{\partial^2 z}{\partial x^2}+2\dfrac{\partial^2 z}{\partial x\,\partial y}+\dfrac{\partial^2 z}{\partial y^2} = 0$ by changing the independent variables using $u = x - y$ and $\upsilon = x + y$.

38. Transform the equation $x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\,\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2} = 0$, by changing the independent variables using $u = x$ and $\upsilon = \dfrac{y^2}{x}$.

39. Transform the equation $\dfrac{\partial^2 z}{\partial x^2}-5\dfrac{\partial^2 z}{\partial x\,\partial y}+6\dfrac{\partial^2 z}{\partial y^2} = 0$, by changing the independent variables using $u = 2x + y$ and $\upsilon = 3x + y$.

40. Transform the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = 0$, by changing the independent variables using $z = x + iy$ and $z^* = x - iy$, where $i = \sqrt{-1}$.

41. Use partial differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, when (i) $x^y = y^x$; (ii) $x^m y^n = (x+y)^{m+n}$; (iii) $(\cos x)^y = (\sin y)^x$; (iv) $(\sec x)^y = (\cot y)^x$; (v) $x^y = e^{x-y}$.

42. Use partial differentiation to find $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, when $x^3 + y^3 - 3axy = 0$.

43. Use partial differentiation to find $\dfrac{d^2y}{dx^2}$, when $x^4 + y^4 = 4a^2xy$.

44. Use partial differentiation to prove that $\dfrac{d^2y}{dx^2} = \dfrac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$, when $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

45. Use partial differentiation to prove that $\dfrac{d^2y}{dx^2} = \dfrac{b^2 - ac}{(ay + b)^3}$, when $ay^2 + 2by + c = x^2$.

46. If $x^2 - y^2 + u^2 + 2\upsilon^2 = 1$ and $x^2 + y^2 - u^2 - \upsilon^2 = 2$, prove that $\dfrac{\partial u}{\partial x} = \dfrac{3x}{u}$ and $\dfrac{\partial \upsilon}{\partial x} = -\dfrac{2x}{\upsilon}$.

47. The deflection at the centre of a rod of length l and diameter d supported at its ends and loaded at the centre with a weight w is proportional to wl^3/d^4. What is the percentage increase in the deflection, if the percentage increases in w, l and d are 3, 2 and 1 respectively.

48 The torsional rigidity of a length of wire is obtained from the formula $N = \dfrac{8\pi Il}{t^2r^4}$. If l is decreased by 2%, t is increased by 1.5% and r is increased by 2%, show that the value of N is decreased by 13% approximately.

49. The Current C measured by a tangent galvanometer is given by the relation $C = k \tan \theta$, where θ is the angle of deflection. Show that the relative error in C due to a given error in θ is minimum when $\theta = 45°$.

50. The range R of a projectile projected with velocity υ at an elevation θ is given by $R = \dfrac{\upsilon^2}{g} \sin 2\theta$. Find the percentage error in R due to errors of 1% in υ and $\dfrac{1}{2}$% in θ, when $\theta = \dfrac{\pi}{6}$.

51. The velocity υ of a wave is given by $\upsilon^2 = \dfrac{g\lambda}{2\pi} + \dfrac{2\pi T}{\rho\lambda}$, where g and λ are constants and ρ and T are variables. Prove that, if ρ is increased by 1% and T is decreased by 2%, then the percentage decrease in υ is approximately $\dfrac{3\pi T}{\lambda\rho\upsilon^2}$.

52. The focal length of a mirror is given by the formula $\dfrac{1}{f} = \dfrac{1}{\upsilon} - \dfrac{1}{u}$. If equal errors k are made in the determination of u and υ, show that the percentage error in f is $100k\left(\dfrac{1}{u} + \dfrac{1}{\upsilon}\right)$.

53. A closed rectangular box of dimensions a, b, c has the edges slightly altered in length by amounts Δa, Δb and Δc respectively, so that both its volume and surface area remain unaltered. Show that $\dfrac{\Delta a}{a^2(b-c)} = \dfrac{\Delta b}{b^2(c-a)} = \dfrac{\Delta c}{c^2(a-b)}$.

    [**Hint:** Solve the equations $dV = 0$ and $dS = 0$ for Δa, Δb, Δc using the method of cross-multiplication]

54. If a triangle ABC is slightly disturbed so as to remain inscribed in the same circle, prove that

$$\frac{\Delta a}{\cos A} + \frac{\Delta b}{\cos B} + \frac{\Delta c}{\cos C} = 0.$$

55. The area of a triangle ABC is calculated using the formula $\Delta = \dfrac{1}{2}\, bc \sin A$. Show that the relative error in Δ is given by

$$\frac{\delta\Delta}{\Delta} = \frac{\delta b}{b} + \frac{\delta c}{c} + \cot A\,\delta A.$$

    If an error of $5'$ is made in the measurement of A which is taken as $60°$, find the percentage error in Δ.

56. Prove that the error in the area Δ of a triangle ABC due to a small error in the measurement of c is given by

$$\delta\Delta = \frac{\Delta}{4}\left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right)\delta c.$$

57. The area of a triangle ABC is determined from the side a and the two angles B and C. If there are small errors in the values of B and C, show that the resulting error in the calculated value of the area Δ will be $\dfrac{1}{2}(c^2\Delta B + b^2\Delta C)$.

$$\left[\textbf{Hint}: \Delta = \frac{1}{2}\,\frac{a^2 \sin B \sin C}{\sin(B+C)}\right]$$

4.2.3 Taylor's Series Expansion of a Function of Two Variables

Students are familiar with Taylor's series of a function of one variable viz. $f(x + h) = f(x) + \dfrac{h}{1!} f'(x) + \dfrac{h^2}{2!} f''(x) + \cdots \infty$, which is an infinite series of powers of h. This idea can be extended to expand $f(x + h, y + k)$ in an infinite series of powers of h and k.

Statement

If $f(x, y)$ and all its partial derivatives are finite and continuous at all points (x, y), then

$$f(x+h, y+k) = f(x, y)+\frac{1}{1!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f+\cdots\infty$$

Proof:

If we assume y to be a constant, $f(x + h, y + k)$ can be treated as a function of x only.

Then $f(x+h, y+k) = f(x, y+k) + \dfrac{h}{1!}\dfrac{\partial f(x, y+k)}{\partial x} + \dfrac{h^2}{2!}\dfrac{\partial^2 f(x, y+k)}{\partial x^2}+\cdots$ (1)

Now treating x as a constant,

$$f\left(x, y+k\right) = f\left(x, y\right)+\frac{k}{1!}\frac{\partial f\left(x, y\right)}{\partial y}+\frac{k^2}{2!}\frac{\partial^2 f\left(x, y\right)}{\partial y^2}+\cdots \tag{2}$$

Using (2) in (1), we have

$$f(x+h, y+k) = \left[f(x, y)+\frac{k}{1!}\frac{\partial f(x, y)}{\partial y}+\frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2}+\cdots\right]$$

$$+\frac{h}{1!}\frac{\partial}{\partial x}\left\{f(x, y)+\frac{k}{1!}\frac{\partial f(x, y)}{\partial y}+\frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2}+\cdots\right\}$$

$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y)+\frac{k}{1!}\frac{\partial f(x, y)}{\partial y}+\frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2}+\cdots\right\}+\cdots\infty$$

$$= f(x, y)+\frac{1}{1!}\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\,\partial y}+k^2\frac{\partial^2 f}{\partial y^2}+\cdots\infty\right)$$

$$= f(x, y)+\frac{1}{1!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f+\cdots\infty \tag{3}$$

Interchanging x and h and also y and k in (3) and then putting $h = k = 0$, we have

$$f(x, y) = f(0,0)+\frac{1}{1!}\left\{x\frac{\partial f(0,0)}{\partial x}+y\frac{\partial f(0,0)}{\partial y}\right\}+\frac{1}{2!}\left\{x^2\frac{\partial^2 f(0,0)}{\partial x^2}\right.$$

$$\left.+2xy\frac{\partial^2 f(0,0)}{\partial x\,\partial y}+y^2\frac{\partial^2 f(0,0)}{\partial y^2}\right\}+\cdots \tag{4}$$

Series in (4) is the *Maclarin's series* of the function $f(x, y)$ in powers of x and y.
Another form of Taylor's series of $f(x, y)$

$$
\begin{aligned}
f(x,y) &= f(a+\overline{x-a},\ b+\overline{y-b}) \\
&= f(a+h),\ (b+k),\ \text{say} \\
&= f(a,b)+\frac{1}{1!}\left\{h\frac{\partial f(a,b)}{\partial x}+k\frac{\partial f(a,b)}{\partial y}\right\} \\
&\quad +\frac{1}{2!}\left\{h^2\frac{\partial^2 f(a,b)}{\partial x^2}+2kh\frac{\partial^2 f(a,b)}{\partial x\,\partial y}+k^2\frac{\partial^2 f(a,b)}{\partial y^2}\right\}+\cdots,\ \text{by (3)} \\
&= f(a,b)+\frac{1}{1!}\left[(x-a)\frac{\partial f(a,b)}{\partial x}+(y-b)\frac{\partial f(a,b)}{\partial y}\right] \\
&\quad +\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2}+2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x\,\partial y}+(y-b)^2\frac{\partial^2 f(a,b)}{\partial y^2}\right]+\cdots \qquad (5)
\end{aligned}
$$

(5) is called the Taylor's series of $f(x, y)$ at or near the point (a, b).

Thus the Taylor's series of $f(x, y)$ at or near the point (0, 0) is Maclaurins series of $f(x, y)$.

4.3 JACOBIANS

If u and υ are functions of two independent variables x and y, then the determinant

$$
\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial \upsilon}{\partial x} & \dfrac{\partial \upsilon}{\partial y} \end{vmatrix}
$$

is called *the Jacobian* or *functional determinant* of u, υ with respect to x and y and is written as

$$
\frac{\partial(u,\upsilon)}{\partial(x,y)} \text{ or } J\left(\frac{u,\upsilon}{x,y}\right).
$$

Similarly the Jacobian of u, υ, w with respect to x, y, z is defined as

$$
\frac{\partial(u,\upsilon,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial \upsilon}{\partial x} & \dfrac{\partial \upsilon}{\partial y} & \dfrac{\partial \upsilon}{\partial z} \\[2ex] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}
$$

Note ☑

1. To define the Jacobian of n dependent variables, each of these must be a function of n independent variables.
2. The concept of Jacobians is used when we change the variables in multiple integrals. (See property 4 given below)

4.3.1 Properties of Jacobians

1. If u and υ are functions of x and y, then $\dfrac{\partial(u, \upsilon)}{\partial(x, y)} \times \dfrac{\partial(x, y)}{\partial(u, \upsilon)} = 1.$

Proof:

Let $u = f(x, y)$ and $\upsilon = g(x, y)$. When we solve for x and y, let $x = \phi(u, \upsilon)$ and $y = \psi(u, \upsilon)$.

Then

$$\left.\begin{aligned}
\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} &= \frac{\partial u}{\partial u} = 1 \\
\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \upsilon} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \upsilon} &= \frac{\partial u}{\partial \upsilon} = 0 \\
\frac{\partial \upsilon}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial \upsilon}{\partial y}\cdot\frac{\partial y}{\partial u} &= \frac{\partial \upsilon}{\partial u} = 0 \\
\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \upsilon} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \upsilon} &= \frac{\partial \upsilon}{\partial \upsilon} = 1
\end{aligned}\right\} \tag{1}$$

Now

$$\frac{\partial(u,\upsilon)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,\upsilon)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial \upsilon}{\partial x} & \dfrac{\partial \upsilon}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial \upsilon} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial \upsilon} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial \upsilon}{\partial x} & \dfrac{\partial \upsilon}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial \upsilon} & \dfrac{\partial y}{\partial \upsilon} \end{vmatrix}, \quad \text{by interchanging the rows and columns of the second determinant.}$$

$$= \begin{vmatrix} \left(\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial u}\right) & \left(\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial \upsilon} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial \upsilon}\right) \\ \left(\dfrac{\partial \upsilon}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial \upsilon}{\partial y}\cdot\dfrac{\partial y}{\partial u}\right) & \left(\dfrac{\partial \upsilon}{\partial x}\cdot\dfrac{\partial x}{\partial \upsilon} + \dfrac{\partial \upsilon}{\partial y}\cdot\dfrac{\partial y}{\partial \upsilon}\right) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{[by (1)]}$$

$$= 1$$

2. If u and υ functions of r and s, where r and s are functions of x and y, then

$$\frac{\partial(u,\upsilon)}{\partial(x,y)} = \frac{\partial(u,\upsilon)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

Proof:

$$\frac{\partial(u,\upsilon)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial \upsilon}{\partial r} & \frac{\partial \upsilon}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial \upsilon}{\partial r} & \frac{\partial \upsilon}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix}$$, by rewriting the second determinant.

$$= \begin{vmatrix} \left(\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}\right) & \left(\frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}\right) \\ \left(\frac{\partial \upsilon}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial \upsilon}{\partial s}.\frac{\partial s}{\partial x}\right) & \left(\frac{\partial \upsilon}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial \upsilon}{\partial s}.\frac{\partial s}{\partial y}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial \upsilon}{\partial x} & \frac{\partial \upsilon}{\partial y} \end{vmatrix} = \frac{\partial(u,\upsilon)}{\partial(x,y)}.$$

Note ☑ The two properties given above hold good for more than two variables too.

3. If u, υ, w are functionally dependent functions of three independent variables

x, y, z then $\frac{\partial(u,\upsilon,w)}{\partial(x,y,z)} = 0.$

Note ☑ The functions u, υ, w are said to be functionally dependent, if each can be expressed in terms of the others or equivalently $f(u, \upsilon, w) = 0$. Linear dependence of functions is a particular case of functional dependence.

Proof:

Since u, υ, w are functionally dependent, $f(u, \upsilon, w) = 0$ (1)

Differentiating (1) partially with respect to x, y and z, we have

$$f_u \,.\, u_x + f_\upsilon \,.\, \upsilon_x + f_w \,.\, w_x = 0 \quad (2)$$

$$f_u \,.\, u_y + f_\upsilon \,.\, \upsilon_y + f_w \,.\, w_y = 0 \quad (3)$$

$$f_u \,.\, u_z + f_\upsilon \,.\, \upsilon_z + f_w \,.\, w_z = 0 \quad (4)$$

Equations (2), (3) and (4) are homogeneous equations in the unknowns f_u, f_υ, f_w. At least one of f_u, f_υ and f_w is not zero, since if all of them are zero, then $f(u, \upsilon, w) \equiv$ constant, which is meaningless.

Thus the homogeneous equations (2), (3) and (4) possess a non-trivial solution.

$\therefore$ Matrix of the coefficients of f_u, f_υ, f_w is singular.

i.e., $$\begin{vmatrix} u_x & \upsilon_x & w_x \\ u_y & \upsilon_y & w_y \\ u_z & \upsilon_z & w_z \end{vmatrix} = 0$$

i.e., $$\frac{\partial(u, \upsilon, w)}{\partial(x, y, z)} = 0$$

Note ☑ The converse of this property is also true. viz., if u, υ, w are functions of x, y, z such that $\dfrac{\partial(u, \upsilon, w)}{\partial(x, y, z)} = 0$ then u, υ, w are functionally dependent. i.e., there exists a relationship among them.

4. If the transformations $x = x(u, \upsilon)$ and $y = y(u, \upsilon)$ are made in the double integral $\iint f(x, y)\,\mathrm{d}x\,\mathrm{d}y$, then $f(x, y) = F(u, \upsilon)$ and $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}\upsilon$, where $J = \dfrac{\partial(x, y)}{\partial(u, \upsilon)}$.

Proof:

$\mathrm{d}x\ \mathrm{d}y$ = Elemental area of a rectangle with vertices (x, y), $(x + \mathrm{d}x, y)$, $(x + \mathrm{d}x, y + \mathrm{d}y)$ and $(x, y + \mathrm{d}y)$

This elemental area can be regarded as equal to the area of the parallelogram with vertices (x, y), $\left(x + \dfrac{\partial x}{\partial u}\mathrm{d}u, y + \dfrac{\partial y}{\partial u}\mathrm{d}u\right)$, $\left(x + \dfrac{\partial x}{\partial u}\mathrm{d}u + \dfrac{\partial x}{\partial \upsilon}\mathrm{d}\upsilon, y + \dfrac{\partial y}{\partial u}\mathrm{d}u + \dfrac{\partial y}{\partial \upsilon}\mathrm{d}\upsilon\right)$ and $\left(x + \dfrac{\partial x}{\partial \upsilon}\mathrm{d}\upsilon, y + \dfrac{\partial y}{\partial \upsilon}\mathrm{d}\upsilon\right)$, since $\mathrm{d}x$ and $\mathrm{d}y$ are infinitesimals.

Now the area of this parallelogram is equal to 2× area of the triangle with vertices (x, y), $\left(x + \dfrac{\partial x}{\partial u}\mathrm{d}u, y + \dfrac{\partial y}{\partial u}\mathrm{d}u\right)$ and $\left(x + \dfrac{\partial x}{\partial \upsilon}\mathrm{d}\upsilon, y + \dfrac{\partial y}{\partial \upsilon}\mathrm{d}\upsilon\right)$

$$\therefore \qquad \mathrm{d}x\ \mathrm{d}y = 2 \times \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x + \dfrac{\partial x}{\partial u}\mathrm{d}u & y + \dfrac{\partial y}{\partial u}\mathrm{d}u & 1 \\ x + \dfrac{\partial x}{\partial \upsilon}\mathrm{d}\upsilon & y + \dfrac{\partial y}{\partial \upsilon}\mathrm{d}\upsilon & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & y & 1 \\ \frac{\partial x}{\partial u}\mathrm{d}u & \frac{\partial y}{\partial u}\mathrm{d}u & 0 \\ \frac{\partial x}{\partial \upsilon}\mathrm{d}\upsilon & \frac{\partial y}{\partial \upsilon}\mathrm{d}\upsilon & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u}\mathrm{d}u & \frac{\partial y}{\partial u}\mathrm{d}u \\ \frac{\partial x}{\partial \upsilon}\mathrm{d}\upsilon & \frac{\partial y}{\partial \upsilon}\mathrm{d}\upsilon \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial \upsilon} & \frac{\partial y}{\partial \upsilon} \end{vmatrix} \mathrm{d}u\,\mathrm{d}\upsilon$$

i.e., $$\mathrm{d}x\,\mathrm{d}y = \frac{\partial(x, y)}{\partial(u, \upsilon)}\mathrm{d}u\,\mathrm{d}\upsilon.$$

Since both dx dy and du dυ are positive, $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}\upsilon$, where $J = \dfrac{\partial(x, y)}{\partial(u, \upsilon)}$
Similarly, if we make the transformations

$$x = x(u, \upsilon, w),\ y = y(u, \upsilon, w) \text{ and } z = z(u, \upsilon, w)$$

in the triple integral $\iiint f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, then $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}\upsilon\,\mathrm{d}w$, where $J = \dfrac{\partial(x, y, z)}{\partial(u, \upsilon, w)}$.

4.4 DIFFERENTIATION UNDER THE INTEGRAL SIGN

When a function $f(x, y)$ of two variables is integrated with respect to y partially, viz., treating x as a parameter, between the limits a and b, then $\int_a^b f(x, y)\,\mathrm{d}y$ will be a function of x.

Let it be denoted by $F(x)$.

Now to find $F'(x)$, if it exists, we need not find $F(x)$ and then differentiate it with respect to x. $F'(x)$ can be found out without finding $F(x)$, by using *Leibnitz's* rules, given below:

1. Leibnitz's rule for constant limits of integration

If $f(x, y)$ and $\dfrac{\partial f(x, y)}{\partial x}$ are continuous functions of x and y, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^b f(x, y)\mathrm{d}y\right] = \int_a^b \frac{\partial f(x, y)}{\partial x}\mathrm{d}y, \text{ where}$$

a and b are constants independent of x.

Proof:

Let $$\int_a^b f(x,y)\,\mathrm{d}y = F(x).$$

Then $$F(x+\Delta x) - F(x) = \int_a^b f(x+\Delta x,\, y)\mathrm{d}y - \int_a^b f(x,y)\mathrm{d}y$$

$$= \int_a^b [f(x+\Delta x,\, y) - f(x,y)]\mathrm{d}y$$

$$= \Delta x \int_a^b \frac{\partial f(x+\theta\Delta\, x,\, y)}{\partial x}\mathrm{d}y,\ 0<\theta<1,$$

$$\left[\text{by Mean Value theorem, viz.,} f(x+h) - f(x) = h\frac{\mathrm{d}f(x+\theta h)}{\mathrm{d}x},\ 0<\theta<1\right]$$

$\therefore$ $$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \int_a^b \frac{\partial f(x+\theta\cdot\Delta x, y)}{\partial x}\mathrm{d}y \tag{1}$$

Taking limits on both sides of (1) as $\Delta x \to 0$,

$$F'(x) = \int_a^b \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y$$

i.e., $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^b f(x,y)\mathrm{d}y\right] = \int_a^b \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y$$

2. Leibnitz's rule for variable limits of integration

If $f(x, y)$ and $\dfrac{\partial f(x,y)}{\partial x}$ are continuous functions of x and y, then $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\displaystyle\int_{a(x)}^{b(x)} f(x,y)\mathrm{d}y\right]$ $= \displaystyle\int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\mathrm{d}y + f\{x,b(x)\}\frac{\mathrm{d}b}{\mathrm{d}x} - f\{x,a(x)\}\frac{\mathrm{d}a}{\mathrm{d}x}$, provided $a(x)$ and $b(x)$ possess continuous first order derivatives.

Proof:

Let $\int f(x,y)\mathrm{d}y = F(x,y)$, so that $\dfrac{\partial}{\partial y}F(x,y) = f(x,y)$ (1)

$$\therefore \qquad \int_{a(x)}^{b(x)} f(x,y)\mathrm{d}y = F\{x,b(x)\} - F\{x,a(x)\}$$

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,y)dy\right] = \frac{d}{dx}F\{x,b(x)\} - \frac{d}{dx}F\{x,a(x)\}$$

$$= \left[\frac{d}{dx}F(x,y)\right]_{y=b(x)} - \left[\frac{d}{dx}F(x,y)\right]_{y=a(x)}$$

$$= \left[\frac{\partial}{\partial x}F(x,y) + \frac{\partial}{\partial y}F(x,y)\cdot\frac{dy}{dx}\right]_{y=b(x)}$$

$$- \left[\frac{\partial}{\partial x}F(x,y) + \frac{\partial}{\partial y}F(x,y)\frac{dy}{dx}\right]_{y=a(x)}$$

by differentiation of implicit functions

$$= \left[\frac{\partial}{\partial x}F(x,y)\right]_{y=a(x)}^{y=b(x)} + \left[f(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{y=b(x)}$$

$$- \left[f(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{y=a(x)} \qquad \text{by (1)}$$

$$= \left[\frac{\partial}{\partial x}\int f(x,y)\,\mathrm{d}y\right]_{y=a(x)}^{y=b(x)} + f\{x,\, b(x)\}b'(x) - f\{x,\, a(x)\}a'(x)$$

$$= \left[\int \frac{\partial f(x,y)}{\partial x}\mathrm{d}y\right]_{y=a(x)}^{y=b(x)} + f\{x,\, b(x)\}b'(x) - f\{x,\, a(x)\}a'(x)$$

$$= \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\mathrm{d}y + f\{x,\, b(x)\}b'(x) - f\{x,\, a(x)\}a'(x)$$

WORKED EXAMPLE 4(b)

Example 4.1 Expand $e^x \cos y$ in powers of x and y as far as the terms of the third degree.

$$f(x,y) = e^x \cos y; \ \frac{\partial f(x,y)}{\partial x} = f_x(x,y) = e^x \cos y;$$

$$\frac{\partial f(x,y)}{\partial y} = f_y(x,y) = -e^x \sin y.$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = f_{xx}(x,y) = e^x \cos y; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = f_{yx}(x,y) = -e^x \sin y$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = f_{yy}(x,y) = -e^x \cos y.$$

Similarly $\quad f_{xxx}(x,y) = e^x \cos y;\ f_{xxy}(x,y) = -\,e^x \sin y;$

$$f_{xyy}(x,y) = -\,e^x \cos y;\ f_{yyy}(x,y) = e^x \sin y$$

$\therefore \quad f(0,0) = 1;\ f_x(0,0) = 1;\ f_y(0,0) = 0;$

$$f_{xx}(0,0) = 1;\ f_{xy}(0,0) = 0;\ f_{yy}(0,0) = -1;$$

$$f_{xxx}(0,0) = 1;\ f_{xxy}(0,0) = 0;\ f_{xyy}(0,0) = -1;\ f_{yyy}(0,0) = 0$$

Taylor's series of $f(x, y)$ in powers of x and y is

$$f(x,y) = f(0,0) + \frac{1}{1!}\{xf_x(0,0) + yf_y(0,0)\} +$$
$$\frac{1}{2!}\{x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\} +$$
$$\frac{1}{3!}\{x^3 f_{xxx}(0,0) + 3x^2 yf_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\} + \cdots$$

$$\therefore\ e^x \cos y = 1 + \frac{1}{1!}\{x\cdot 1 + y\cdot 0\} + \frac{1}{2!}\{x^2\cdot 1 + 2xy\cdot 0 + y^2(-1)\}$$
$$+ \frac{1}{3!}\{x^3\cdot 1 + 3x^2 y\cdot 0 + 3xy^2\cdot(-1) + y^3\cdot 0\} + \cdots$$
$$= 1 + \frac{x}{1!} + \frac{1}{2!} + (x^2 - y^2) + \frac{1}{3!}(x^3 - 3xy^2) + \cdots$$

4.4.1 Verification

$e^x \cos y =$ Real part of e^{x+iy}

$$= \text{R.P. of}\left[1 + \frac{x+iy}{1!} + \frac{(x+iy)^2}{2!} + \frac{(x+iy)^3}{3!} + \cdots\right],$$

by exponential theorem

$$= 1 + \frac{x}{1!} + \frac{1}{2!}(x^2 - y^2) + \frac{1}{3!}(x^3 - 3y^2) + \cdots$$

Example 4.2 Expand $\dfrac{(x+h)(y+k)}{x+h+y+k}$ in a series of powers of h and k upto the second degree terms.

Let $$f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$$

$\therefore$ $$f(x, y) = \frac{xy}{x+y}.$$

Taylor's series of $f(x + h, y + k)$ in powers of h and k is

$$f(x+h, y+k) = f(x, y) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots \quad (1)$$

Now $$f_x = y\left\{\frac{(x+y) - x}{(x+y)^2}\right\} = \frac{y^2}{(x+y)^2}$$

$$f_y = x\left\{\frac{(x+y) - y}{(x+y)^2}\right\} = \frac{x^2}{(x+y)^2}$$

$$f_{xx} = -\frac{2y^2}{(x+y)^3};\ f_{xy} = \frac{(x+y)^2 \cdot 2y - y^2 \cdot 2(x+y)}{(x+y)^4}$$

$$= \frac{2\{y(x+y) - y^2\}}{(x+y)^3} = \frac{2xy}{(x+y)^3}$$

$$f_{yy} = -\frac{2x^2}{(x+y)^3}.$$

Using these values in (1), we have

$$\frac{(x+h)(y+k)}{x+h+y+k} = \frac{xy}{x+y} + \frac{hy^2}{(x+y)^2} + \frac{kx^2}{(x+y)^2} - \frac{h^2 y^2}{(x+y)^3} + \frac{2hkxy}{(x+y)^3} - \frac{k^2 x^2}{(x+y)^3} + \cdots$$

Example 4.3 Find the Taylor's series expansion of x^y near the point (1, 1) upto the second degree terms.

Taylor's series of $f(x, y)$ near the point (1, 1) is $f(x,y) = f(1,1)+\frac{1}{1!}$

$$\{(x-1)f_x(1,1)+(y-1)f_y(1,1)\}+\frac{1}{2!}\{(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)f_{xy}(1,1)$$
$$+(y-1)^2 f_{yy}(1,1)\}+\cdots \tag{1}$$

$$f(x,y) = x^y;\ f_x(x,y) = yx^{y-1};\ f_y(x,y) = x^y \log x;$$
$$f_{xx}(x,y) = y(y-1)x^{y-2};\ f_{xy}(x,y) = x^{y-1}+yx^{y-1}\log x$$
$$f_{yy}(x,y) = x^y \cdot (\log x)^2 \cdot$$
$$f_{xx}(1,1) = 1;\ f_x(1,1) = 1;\ f_y(1,1) = 0;$$
$$f_{xx}(1,1) = 0;\ f_{xy}(1,1) = 1;\ f_{yy}(1,1) = 0$$

Using these values in (1), we get

$$x^y = 1+(x-1)+(x-1)(y-1)+\cdots$$

Example 4.4 Find the Taylor's series expansion of $e^x \sin y$ near the point $\left(-1,\frac{\pi}{4}\right)$ upto the third degree terms.

Taylor's series of $f(x, y)$ near the point $\left(-1,\frac{\pi}{4}\right)$ is

$$f(x,y) = f\left(-1,\frac{\pi}{4}\right)+\frac{1}{1!}\left\{(x+1)f_x\left(-1,\frac{\pi}{4}\right)+\left(y-\frac{\pi}{4}\right)f_y\left(-1,\frac{\pi}{4}\right)\right\}$$
$$+\frac{1}{2!}\left\{(x+1)^2 f_{xx}\left(-1,\frac{\pi}{4}\right)+2(x+1)\left(y-\frac{\pi}{4}\right)f_{xy}\left(-1,\frac{\pi}{4}\right)\right.$$
$$\left.+\left(y-\frac{\pi}{4}\right)^2 f_{yy}\left(-1,\frac{\pi}{4}\right)\right\}+\cdots \tag{1}$$

$$f(x,y) = e^x \sin y;\ f_x = e^x \sin y;\ f_y = e^x \cos y;$$
$$f_{xx} = e^x \sin y;\ f_{xy} = e^x \cos y;\ f_{yy} = -e^x \sin y;$$
$$f_{xxx} = e^x \sin y;\ f_{xxy} = e^x \cos y;\ f_{xyy} = -e^x \sin y;$$
$$f_{yyy} = -e^x \cos y.$$

$\therefore$

$$f\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_x\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_y\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};$$
$$f_{xx}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_{xy}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\ f_{yy}\left(-1,\frac{\pi}{4}\right) = -\frac{1}{e\sqrt{2}};$$

$$f_{xxx}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_{xxy}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_{xyy}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_{yyy}\left(-1,\frac{\pi}{4}\right)=-\frac{1}{e\sqrt{2}}.$$

Using these values in (1), we get

$$e^x \sin y=\frac{1}{e\sqrt{2}}\left[1+\frac{1}{1!}\left\{(x+1)+\left(y-\frac{\pi}{4}\right)\right\}\right.$$
$$+\frac{1}{2!}\left\{(x+1)^2+2(x+1)\left(y-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)^2\right\}$$
$$\left.+\frac{1}{3!}\left\{(x+1)^3+3(x+1)^2\left(y-\frac{\pi}{4}\right)-3(x+1)\left(y-\frac{\pi}{4}\right)^2-\left(y-\frac{\pi}{4}\right)^3\right\}+\cdots\right.$$

Example 4.5 Find the Taylor's series expansion of $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x + 2)$ and $(y - 1)$ upto the third powers.

Taylor's series of $f(x, y)$ in powers of $(x + 2)$ and $(y - 1)$ or near $(-2, 1)$ is

$$f(x,y)=f(-2,1)+\frac{1}{1!}\{(x+2)f_x(-2,1)+(y-1)f_y(-2,1)\}$$
$$+\frac{1}{2!}\{(x+2)^2 f_{xx}(-2,1)+2(x+2)(y-1)f_{xy}(-2,1)$$
$$+(y-1)^2 f_{yy}(-2,1)\}+\cdots \qquad (1)$$

$f(x,y)=x^2y^2+2x^2y+3xy^2$	$f(-2,1)=6$
$f_x=2xy^2+4xy+3y^2$	$f_x(-2,1)=-9$
$f_y=2x^2y+2x^2+6xy$	$f_y(-2,1)=4$
$f_{xx}=2y^2+4y$	$f_{xx}(-2,1)=6$
$f_{xy}=4xy+4x+6y$	$f_{xy}(-2,1)=-10$
$f_{yy}=2x^2+6x$	$f_{yy}(-2,1)=-4$
$f_{xxx}=0$	$f_{xxx}(-2,1)=0$
$f_{xxy}=4y+4$	$f_{xxy}(-2,1)=8$
$f_{xyy}=4x+6$	$f_{xyy}(-2,1)=-2$
$f_{yyy}=0$	$f_{yyy}(-2,1)=0$

Using these values in (1), we have

$$\begin{aligned}x^2y^2+2x^2y+3xy^2 &= 6+\frac{1}{1!}\{-9(x+2)+4(y-1)\}\\&+\frac{1}{2!}\{6(x+2)^2-20(x+2)(y-1)-4(y-1)^2\}\\&+\frac{1}{3!}\{24(x+2)^2(y-1)-6(x+2)(y-1)^2\}+\cdots\end{aligned}$$

Example 4.6 Using Taylor's series, verify that

$$\log(1+x+y)=(x+y)-\frac{1}{2}(x+y)^2+\frac{1}{3}(x+y)^3-\cdots$$

The series given in the R.H.S. is a series of powers of x and y.

So let us expand $f(x, y) = \log(1 + x + y)$ as a Taylor's series near (0, 0) or Maclaurin's series.

$$f_x=\frac{1}{1+x+y};\ f_y=\frac{1}{1+x+y}$$

$$f_{xx}=-\frac{1}{(1+x+y)^2}=f_{xy}=f_{yy}$$

$$f_{xxx}=\frac{2}{(1+x+y)^3}=f_{xxy}=f_{xyy}=f_{yyy}$$

$$f(0,0)=0;\ f_x(0,0)=f_y(0,0)=1;$$

$$f_{xx}(0,0)=f_{xy}(0,0)=f_{yy}(0,0)=-1;$$

$$f_{xxx}(0,0)=f_{xxy}(0,0)=f_{xyy}(0,0)=f_{yyy}(0,0)=2\,.$$

Maclaurin's series of $f(x, y)$ is given by

$$\begin{aligned}f(x,y)&=f(0,0)+\frac{1}{1!}\{xf_x(0,0)+yf_y(0,0)\}\\&+\frac{1}{2!}\{x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\}+\cdots \quad (1)\end{aligned}$$

Using the relevant values in (1), we have

$$\begin{aligned}\log(1+x+y)&=(x+y)+\frac{1}{2}\{-x^2-2xy-y^2\}+\\&+\frac{1}{6}\{2x^3+6x^2y+6xy^2+2y^3\}+\cdots\end{aligned}$$

$$= (x+y) - \frac{1}{2}(x+y)^2 + \frac{1}{3}(x+y)^3 - \cdots$$

Example 4.7 If $x = r\cos\theta$, $y = r\sin\theta$, verify that $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$.

$$x = r\cos\theta;\ y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Now $$r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}\frac{y}{x}$$

$\therefore$ $$2r\frac{\partial r}{\partial x} = 2x$$

$\therefore$ $$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2}$$

$$= -\frac{y}{x^2+y^2} = \frac{-y}{r^2}$$

Similarly $\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$\therefore$ $$\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \times \frac{1}{r} = 1.$$

Example 4.8 If we transform from three dimensional cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), show that the Jacobian of x, y, z with respect to r, θ, ϕ is $r^2 \sin\theta$.

The transformation equations are

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta.$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi, \frac{\partial y}{\partial r} = \sin\theta\sin\phi, \frac{\partial z}{\partial r} = \cos\theta$$
$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \frac{\partial z}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi, \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi, \frac{\partial z}{\partial \phi} = 0.$$

Now $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= r^2[\sin\theta\cos\phi\left(0+\sin^2\theta\cos\phi\right)-\sin\theta\sin\phi$$
$$\times(0-\sin^2\theta\sin\phi)+\cos\theta(\sin\theta\cos\theta\cos^2\phi+\sin\theta\cos\theta\sin^2\phi)]$$

$$= r^2[\sin^3\theta\cos^2\phi+\sin^3\theta\sin^2\phi+\sin\theta\cos^2\theta]$$

$$= r^2(\sin^3\theta+\sin\theta\cos^2\theta)$$

$$= r^2\sin\theta.$$

Example 4.9 If $u = 2xy$, $\upsilon = x^2 - y^2$, $x = r\cos\theta$ and $y = r\sin\theta$, compute $\dfrac{\partial(u,\upsilon)}{\partial(r,\theta)}$.

By the property of Jacobians,

$$\frac{\partial(u,\upsilon)}{\partial(r,\theta)} = \frac{\partial(u,\upsilon)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} u_x & u_y \\ \upsilon_x & \upsilon_y \end{vmatrix} \times \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= -4(y^2 + x^2) \times r(\cos^2\theta + \sin^2\theta)$$
$$= -4r^3.$$

Example 4.10 Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3, if

$$y_1 = \frac{x_2x_3}{x_1},\ y_2 = \frac{x_3x_1}{x_2},\ y_3 = \frac{x_1x_2}{x_3}$$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

$$= \frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = 4$$

Example 4.11 Express $\iiint \sqrt{xyz(1-x-y-z)}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in terms of u, υ, w given that $x + y + z = u$, $y + z = u\upsilon$ and $z = u\upsilon w$.

The given transformations are

$$x + y + z = u \tag{1}$$

$$y + z = u\upsilon \tag{2}$$

and

$$z = u\upsilon w \tag{3}$$

Using (3) in (2), we have $y = u\upsilon(1 - w)$
Using (2) in (1), we have $x = u(1 - \upsilon)$

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}\upsilon\,\mathrm{d}w, \text{ where}$$

$$J = \frac{\partial(x,y,z)}{\partial(u,\upsilon,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \upsilon} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \upsilon} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial \upsilon} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-\upsilon & -u & 0 \\ \upsilon(1-w) & u(1-w) & -u\upsilon \\ \upsilon w & wu & u\upsilon \end{vmatrix}$$

$$= (1-\upsilon)\{u^2\upsilon\,(1-w)+u^2\upsilon w\}+u\{u\upsilon^2\,(1-w)+u\upsilon^2 w\}$$

$$= u^2\upsilon\,(1-\upsilon)+u^2\upsilon^2$$

$$= u^2\upsilon$$

$$\therefore \qquad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = u^2\upsilon\,\mathrm{d}u\,\mathrm{d}\upsilon\,\mathrm{d}w \tag{4}$$

Using (1), (2), (3) and (4) in the given triple integral I, we have

$$I = \iiint \sqrt{u^3\upsilon^2\,w(1-\upsilon)(1-w)(1-u)}\;u^2\upsilon\,\mathrm{d}u\,\mathrm{d}\upsilon\,\mathrm{d}w$$

$$= \iiint u^{7/2}\,\upsilon^2\,w^{1/2}\,(1-u)^{\frac{1}{2}}\,(1-\upsilon)^{\frac{1}{2}}\,(1-w)^{\frac{1}{2}}\,\mathrm{d}u\,\mathrm{d}\upsilon\,\mathrm{d}w$$

Example 4.12 Examine if the following functions are functionally dependent. If they are, find also the functional relationship.

(i) $u = \sin^{-1} x+\sin^{-1} y;\ \upsilon = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

(ii) $u = y+z;\ \upsilon = x+2z^2;\ w = x-4yz-2y^2$

(i) $u = \sin^{-1} x+\sin^{-1} y;\ \upsilon = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$$\frac{\partial u}{\partial x} = \left(\frac{1}{\sqrt{1-x^2}}\right); \frac{\partial u}{\partial y} = \left(\frac{1}{\sqrt{1-y^2}}\right); \frac{\partial \upsilon}{\partial x} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}};$$

$$\frac{\partial \upsilon}{\partial y} = -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

Now $$\frac{\partial(u,\upsilon)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= \frac{-xy}{\sqrt{(1-x^2)(1-y^2)}} + 1 - 1 + \frac{xy}{\sqrt{(1-x^2)(1-y^2)}}$$

$$= 0.$$

$\therefore$ u and υ are functionally dependent by property (3).

Now $$\sin u = \sin\,(\sin^{-1} x + \sin^{-1} y)$$

$$= \sin\,(\sin^{-1} x)\cos\,(\sin^{-1} y) + \cos\,(\sin^{-1} x)\sin\,(\sin^{-1} y)$$

$$= x\cdot\cos\left\{\cos^{-1}(\sqrt{1-y^2})\right\} + \cos\left\{\cos^{-1}(\sqrt{1-x^2})\right\}\cdot y$$

$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$= \upsilon.$$

$\therefore$ The functional relationship between u and υ is $\upsilon = \sin u$.

(ii) $u = y+z;\ \upsilon = x+2z^2;\ w = x-4yz-2y^2$

$$\frac{\partial u}{\partial x} = 0; \frac{\partial \upsilon}{\partial x} = 1; \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 1; \frac{\partial \upsilon}{\partial y} = 0; \frac{\partial w}{\partial y} = -4y-4z$$

$$\frac{\partial u}{\partial z} = 1; \frac{\partial \upsilon}{\partial z} = 4z; \frac{\partial w}{\partial z} = -4y$$

Now $$\frac{\partial(u,\upsilon,w)}{\partial(x,y,z)} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4y-4z & -4y \end{vmatrix}$$

$$= -\{-4y+4y+4z\}+4z=0.$$

$\therefore$ u, υ and w are functionally dependent

Now $$\upsilon - w = 2z^2 + 4yz + 2y^2$$
$$= 2(y+z)^2 = 2u^2$$

$\therefore$ The functional relationship among u, υ and w is $2u^2 = \upsilon - w$.

Example 4.13 Given that $\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}} (a > b)$, find

$$\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} \text{ and } \int_0^{\pi} \frac{\cos x \, dx}{(a+b\cos x)^2}$$

$$\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}} \tag{1}$$

Differentiating both sides of (1) with respect to a, we get

$$\int_0^{\pi} \frac{\partial}{\partial a}\left(\frac{1}{a+b\cos x}\right) dx = \frac{\partial}{\partial a} \cdot \frac{\pi}{\sqrt{a^2-b^2}},$$ since the limits of integration are constants

i.e., $$\int_0^{\pi} \frac{-dx}{(a+b\cos x)^2} = \pi \times -1/2\left(a^2-b^2\right)^{-3/2} 2a$$

i.e., $$\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{\left(a^2-b^2\right)^{3/2}}$$

Differentiating both sides of (1) with respect to b, we get

$$\int_0^{\pi} \frac{\partial}{\partial b}\left(\frac{1}{a+b\cos x}\right) dx = \frac{\partial}{\partial b} \cdot \frac{\pi}{\sqrt{a^2-b^2}}$$

i.e., $$\int_0^{\pi} -\frac{1}{(a+b\cos x)^2} \times \cos x \, dx = \pi \times -1/2\left(a^2-b^2\right)^{-3/2} (-2b)$$

i.e., $$\int_0^{\pi} \frac{\cos x}{(a+b\cos x)^2} dx = -\frac{\pi b}{\left(a^2-b^2\right)^{3/2}}.$$

Example 4.14 By differentiating inside the integral, find the value of $\int_0^x \frac{\log(1+xy)}{1+y^2}\,dy$.

Hence find the value of $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$.

Let

$$f(x) = \int_0^x \frac{\log(1+xy)}{1+y^2}\,dy \tag{1}$$

Differentiating both sides of (1) with respect to x, we have

$$f'(x) = \frac{d}{dx}\int_0^x \frac{\log(1+xy)}{1+y^2}\,dx$$

$$= \int_0^x \frac{\partial}{\partial x}\left\{\frac{\log(1+xy)}{1+y^2}\right\}dx + \frac{\log\left(1+x^2\right)}{1+x^2}\cdot\frac{d(x)}{dx}$$

$$(\text{by Leibnitz's rule})$$

$$= \int_0^x \frac{y}{(1+xy)\left(1+y^2\right)}\,dy + \frac{\log\left(1+x^2\right)}{1+x^2}$$

$$= \int_0^x \left[\frac{-x}{\left(1+x^2\right)(1+xy)} + \frac{1}{1+x^2}\left(\frac{y+x}{1+y^2}\right)\right]dy + \frac{\log\left(1+x^2\right)}{1+x^2},$$

by resolving the integrand in the first term into partial fractions

$$= \left[-\frac{1}{1+x^2}\log(1+xy) + \frac{1}{2}\cdot\frac{1}{1+x^2}\log\left(1+y^2\right) + \frac{x}{1+x^2}\tan^{-1} y\right]_0^x + \frac{\log\left(1+x^2\right)}{1+x^2}$$

$$= \frac{1}{2}\cdot\frac{1}{1+x^2}\log\left(1+x^2\right) + \frac{x}{1+x^2}\tan^{-1} x \tag{2}$$

Integrating both sides of (2) with respect to x, we have

$$f(x) = \frac{1}{2}\int \log\left(1+x^2\right)d\left(\tan^{-1} x\right) + \int \frac{x}{1+x^2}\tan^{-1} x\,dx + c$$

$$= \frac{1}{2}\left[\tan^{-1} x\log\left(1+x^2\right) - \int \tan^{-1} x\cdot\frac{2x}{1+x^2}\,dx\right]$$

$$+ \int \frac{x\tan^{-1} x}{1+x^2}\,dx + c$$

$$= \frac{1}{2} \tan^{-1} x \cdot \log\left(1+x^2\right) + c \tag{2}$$

Now putting $x = 0$ in (2), we get

$$c = f(0) = 0, \text{ by } (1)$$

$\therefore$

$$f(x) = \int_0^x \frac{\log(1+xy)}{1+y^2} \, dy = \frac{1}{2} \tan^{-1} x \cdot \log\left(1+x^2\right) \tag{3}$$

Putting $x = 1$ in (3), we get

$$\int_0^1 \frac{\log(1+y)}{1+y^2} \, dy = \frac{1}{2} \tan^{-1}(1) \cdot \log 2$$

$$= \frac{\pi}{8} \log 2$$

Since y is only a dummy variable,

$$\int_0^1 \frac{\log(1+x)}{1+x^2} \, dx = \frac{\pi}{8} \log 2$$

Example 4.15 Show that $\dfrac{d}{da} \displaystyle\int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right) dx = 2a \tan^{-1}(a) - \frac{1}{2} \log\left(a^2+1\right)$.

$$\frac{d}{da} \int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right) dx = \int_0^{a^2} \frac{\partial}{\partial a} \tan^{-1}\left(\frac{x}{a}\right) dx + \tan^{-1}\left(\frac{a^2}{a}\right) \cdot \frac{d}{da}\left(a^2\right),$$

by Leibnitz's rule

$$= \int_0^{a^2} \frac{1}{1+\dfrac{x^2}{a^2}} \cdot \left(\frac{-x}{a^2}\right) dx + 2a \tan^{-1} a$$

$$= -\int_0^{a^2} \frac{x}{x^2+a^2} \, dx + 2a \tan^{-1} a$$

$$= -\frac{1}{2} \left[\log\left(x^2+a^2\right)\right]_0^{a^2} + 2a \tan^{-1} a$$

$$= -\frac{1}{2}\log\left(\frac{a^4+a^2}{a^2}\right)+2a\tan^{-1}a$$

$$= 2a\tan^{-1}a-\frac{1}{2}\log\left(a^2+1\right)$$

Example 4.16 If $I = \int_0^\infty e^{-x^2-\left(\frac{a}{x}\right)^2}\,\mathrm{d}x$, prove that $\frac{\mathrm{d}I}{\mathrm{d}a} = -2I$. Hence find the value of I.

$$I = \int_0^\infty e^{-x^2-\left(\frac{a}{x}\right)^2}\,\mathrm{d}x \tag{1}$$

Differentiating both sides of (1) with respect to a, we have

$$\frac{\mathrm{d}I}{\mathrm{d}a} = \int_0^\infty \frac{\partial}{\partial a}\left\{e^{-x^2-\frac{a^2}{x^2}}\right\}\mathrm{d}x$$

$$= \int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\cdot\left(-\frac{2a}{x^2}\right)\mathrm{d}x$$

$$= \int_\infty^0 e^{-\frac{a^2}{y^2}-y^2}\,2\,\mathrm{d}y, \text{ on putting } x = \frac{a}{y} \text{ or } y = \frac{a}{x}$$

$$= -2\int_0^\infty e^{-y^2-\left(\frac{a}{y}\right)^2}\,\mathrm{d}y$$

i.e.,
$$\frac{\mathrm{d}I}{\mathrm{d}a} = -2I \tag{2}$$

$\therefore$
$$\frac{\mathrm{d}I}{I} = -2\,\mathrm{d}a$$

Solving, we get $\log I = \log c - 2a$

$\therefore$
$$I = ce^{-2a} \tag{3}$$

When
$$a = 0,\ I = \int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2} \tag{4}$$

Using (4) in (3), we get $c = \frac{\sqrt{\pi}}{2}$

Hence $$I = \frac{\sqrt{\pi}}{2} e^{-2a}.$$

EXERCISE 4(b)

Part A

(Short Answer Question)

1. Write down the Taylor's series expansion of $f(x + h, y + k)$ in a series of (i) powers of h and k (ii) power of x and y.
2. Write down the Maclaurin's series expansion of (i) $f(x, y)$, (ii) $f(x + h, y + k)$.
3. Write down the Taylor's series expansion of $f(x, y)$ near the point (a, b).
4. Write down the Maclaurin's series for e^{x+y}.
5. Write down the Maclaurin's series for $\sin(x + y)$.
6. Define Jacobian.
7. State any three properties of Jacobians.
8. State the condition for the functional dependence of three functions $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$.
9. Prove that $\iint f(x,y)\,dx\,dy = \iint f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta.$
10. Show that $\iint f(x,y)\,dx\,dy = \iint f\{u(1-v),\, uv\}\cdot u\,du\,dv.$
11. If $x = u(1 + v)$ and $y = v(1 + u)$, find the Jacobian of x, y with respect to u, v.
12. State the Leibnitz's rule for differentiation under integral sign, when both the limits of integration are variables.
13. Write down the Leibnitz's formula for $\frac{d}{dx}\int_a^{b(x)} f(x,y)\,dy,$ where a is a constant.
14. Write down the Leibnitz's formula for $\frac{d}{dx}\int_{a(x)}^{b} f(x,y)\,dy,$ where b is a constant.
15. Evaluate $\frac{d}{dy}\int_0^1 \log(x^2 + y^2)\,dx$, without integrating the given function.

Part B

16. Expand $e^x \sin y$ in a series of powers of x and y as far as the terms of the third degree.
17. Find the Taylor's series expansion of $e^x \cos y$ in the neighbourhood of the point $\left(1, \frac{\pi}{4}\right)$ upto the second degree terms.

18. Find the Maclaurin's series expansion of $e^x \log(1 + y)$ upto the terms of the third degree.
19. Find the Taylor's series expansion of $\tan^{-1}\left(\dfrac{y}{x}\right)$ in powers of $(x-1)$ and $(y-1)$ upto the second degree terms.
20. Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ upto the third degree terms.
21. Expand $xy^2 + 2x - 3y$ in powers of $(x + 2)$ and $(y - 1)$ upto the third degree terms.
22. Find the Taylor's series expansion of y^x at (1, 1) upto the second degree terms.
23. Find the Taylor's series expansion of e^{xy} at (1, 1) upto the third degree terms.
24. Using Taylor's series, verify that

$$\cos(x+y) = 1-\frac{(x+y)^2}{2!}+\frac{(x+y)^4}{4!}-\cdots\infty$$

25. Using Taylor's series, verify that

$$\tan^{-1}(x+y) = (x+y)+\frac{1}{3}(x+y)^3\cdots\infty$$

26. If $x = u(1 - \upsilon)$, $y = u\upsilon$, verify that

$$\frac{\partial(x,y)}{\partial(u,\upsilon)}\times\frac{\partial(u,\upsilon)}{\partial(x,y)}=1$$

27. (i) if $x = u^2 - \upsilon^2$ and $y = 2u\upsilon$, find the Jacobian of x and y with respect to u and υ.
    (ii) if $u = x^2$ and $\upsilon = y^2$, find $\dfrac{\partial(u,\upsilon)}{\partial(x,y)}$
28. If $x = a\cosh u\cos\upsilon$ and $y = a\sinh u\cdot\sin\upsilon$, show that

$$\frac{\partial(x,y)}{\partial(u,\upsilon)}=\frac{a^2}{2}(\cosh 2u-\cos 2\upsilon).$$

29. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$
30. If $F = xu + \upsilon - y$, $G = u^2 + \upsilon y + w$ and $H = zu - \upsilon + \upsilon w$, compute $\dfrac{\partial(F,G,H)}{\partial(u,\upsilon,w)}$.
31. If $u = xyz$, $\upsilon = xy + yz + zx$ and $w = x + y + z$, find $\dfrac{\partial(u,\upsilon,w)}{\partial(x,y,z)}$.
32. Examine the functional dependence of the functions $u = \dfrac{x+y}{x-y}$ and $\upsilon = \dfrac{xy}{(x-y)^2}\cdot$ If they are dependent, find the relation between them.

33. Are the functions $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$ functionally dependent? If so, find the relation between them.

34. Are the functions $f_1 = x + y + z, f_2 = x^2 + y^2 + z^2$ and $f_3 = xy + yz + zx$ functionally dependent? If so, find the relation among f_1, f_2 and f_3.

35. If $\int_0^x \lambda e^{-\lambda(x-y)} f(y)\mathrm{d}y = \lambda^2 \cdot xe^{-\lambda x}$, prove that $f(x) = \lambda e^{-\lambda x}$. [**Hint:** Differentiate both sides with respect to x].

Use the concept of differentiation under integral sign to evaluate the following:

36. $\displaystyle\int_0^x \frac{\mathrm{d}x}{(x^2+a^2)^2}$ $\left[\textbf{Hint:} \text{ Use} \displaystyle\int_0^x \frac{\mathrm{d}x}{x^2+a^2}\right]$

37. $\displaystyle\int_0^1 x^m (\log x)^n \,\mathrm{d}x$ $\left[\textbf{Hint:} \text{ Use} \displaystyle\int_0^1 x^m \,\mathrm{d}x\right]$

38. $\displaystyle\int_0^\infty e^{-x^2} \cos 2ax \,\mathrm{d}x$

39. $\displaystyle\int_0^\infty \frac{e^{-ax}\sin x}{x}\mathrm{d}x$ and hence $\displaystyle\int_0^\infty \frac{\sin x}{x}\mathrm{d}x$

40. $\displaystyle\int_0^1 \frac{x^m - 1}{\log x}\mathrm{d}x,\ m \geq 0\cdot.$

4.5 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Students are familiar with the concept of maxima and minima of a function of one variable. Now we shall consider the maxima and minima of a function of two variables.

A function $f(x, y)$ is said to have a *relative maximum* (or simply maximum) at $x = a$ and $y = b$, if $f(a, b) > f(a + h, b + k)$ for all small values of h and k.

A function $f(x, y)$ is said to have a *relative maximum* (or simply maximum) at $x = a$ and $y = b$, if $f(a, b) < f(\mathrm{a} + h, b + k)$ for all small values of h and k.

A maximum or a minimum value of a function is called its *extreme value.* We give below *the working rule to find the extreme values of a function* $f(x, y)$:

(1) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(2) Solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously. Let the solutions be (a, b); (c, d);…

Note ☑ The points like (a, b) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are called *stationary points* of the function $f(x, y)$. The values of $f(x, y)$ at the stationary points are called *stationary values* of $f(x, y)$.

(3) For each solution in step (2), find the values of $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial x\, \mathrm{d}y}$, $C = \frac{\partial^2 f}{\partial y^2}$ and $\Delta = AC - B^2$.

(4) (i) If $\Delta > 0$ and A (or C) < 0 for the solution (a, b) then $f(x, y)$ has a maximum value at (a, b).
 (ii) If $\Delta > 0$ and A (or C) > 0 for the solution (a, b) then $f(x, y)$ has a minimum value at (a, b).
 (iii) If $\Delta < 0$ for the solution (a, b), then $f(x, y)$ has neither a maximum nor a minimum value at (a, b). In this case, the point (a, b) is called *a saddle point* of the function $f(x, y)$.
 (iv) If $\Delta = 0$ or $A = 0$, the case is doubtful and further investigations are required to decide the nature of the extreme values of the function $f(x, y)$.

4.5.1 Constrained Maxima and Minima

Sometimes we may require to find the extreme values of a function of three (or more) variables say $f(x, y, z)$ which are not independent but are connected by some given relation $\phi(x, y, z) = 0$. The extreme values of $f(x, y, z)$ in such a situation are called *constrained extreme values*.

In such situations, we use $\phi(x, y, z) = 0$ to eliminate one of the variables, say z from the given function, thus converting the function as a function of only two variables and then find the unconstrained extreme values of the converted function. [Refer to examples (4.8), (4.9), (4.10)].

When this procedure is not practicable, we use Lagrange's method, which is comparatively simpler.

4.5.2 Lagrange's Method of Undetermined Multipliers

Let
$$u = f(x, y, z) \tag{1}$$
be the function whose extreme values are required to be found subject to the constraint
$$\phi(x, y, z) = 0 \tag{2}$$
The necessary conditions for the extreme values of u are $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$

$$\therefore \qquad \frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y+\frac{\partial f}{\partial z}\mathrm{d}z = 0 \tag{3}$$

From (2), we have

$$\frac{\partial \phi}{\partial x}\mathrm{d}x+\frac{\partial \phi}{\partial y}\mathrm{d}y+\frac{\partial \phi}{\partial z}\mathrm{d}z = 0 \tag{4}$$

Now (3) + λ × (4), where λ is an unknown multiplier, called *Langrange multiplier*, gives

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)\mathrm{d}y+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right) = 0 \tag{5}$$

Equation (5) holds good, if

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x} = 0 \tag{6}$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y} = 0 \tag{7}$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z} = 0 \tag{8}$$

Solving the Equations (2), (6), (7) and (8), we get the values of x, y, z, λ, which give the extreme values of u.

Note ☑

(1) The Equations (2), (6), (7) and (8) are simply the necessary conditions for the extremum of the auxiliary function $(f + \lambda\phi)$, where λ is also treated as a variable.

(2) Lagrange's method does not specify whether the extreme value found out is a maximum value or a minimum value. It is decided from the physical consideration of the problem.

WORKED EXAMPLE 4(c)

Example 4.1 Examine $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ for extreme values.

$$f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$$

$$f_x = 3x^2+3y^2-30x+72$$

$$f_y = 6xy-30y$$

$$f_{xx} = 6x-30;\ f_{xy} = 6y;\ f_y = 6x-30$$

The stationary points are given by $f_x = 0$ and $f_y = 0$

i.e., $$3(x^2 + y^2 - 10x + 24) = 0 \quad (1)$$

and $$6y(x-5) = 0 \quad (2)$$

From (2), $x = 5$ or $y = 0$
When $x = 5$, from (1), we get $y^2 - 1 = 0$; $\therefore\ y = \pm 1$
When $y = 0$, from (1), we get $x^2 - 10x + 24 = 0$

$\therefore$ $$x = 4, 6.$$

The stationary points are (5, 1), (5, –1), (4, 0) and (6, 0)
At the point (5, ±1), $A = f_{xx} = 0$; $B = f_{xy} = \pm 6$; $C = f_{yy} = 0$
Though $AC - B^2 < 0$, $A = 0$
$\therefore$ Nothing can be said about the maxima or minima of $f(x, y)$ at (5, ±1).
At the point (4, 0), $A = -6$, $B = 0$, $C = -6$

$\therefore$ $$AC - B^2 = 36 > 0 \quad \text{and} \quad A < 0$$

$\therefore$ $f(x, y)$ is maximum at (4, 0) and maximum value of $f(x, y) = 112$.
At point (6, 0), $A = 6$, $B = 0$, $C = 6$

$\therefore$ $$AC - B^2 = 36 > 0 \text{ and } A > 0.$$

$\therefore$ $f(x, y)$ is minimum at (6, 0) and the minimum value of $f(x, y) = 108$.

Example 4.2 Examine the function $f(x, y) = x^3y^2(12 - x - y)$ for extreme values.

$$\begin{aligned} f(x,y) &= 12x^3y^2 - x^4y^2 - x^3y^3 \\ f_x &= 36x^2y^2 - 4x^3y^2 - 3x^2y^3 \\ f_y &= 24x^3y - 2x^4y - 3x^3y^2 \\ f_{xx} &= 72xy^2 - 12x^2y^2 - 6xy^3 \\ f_{xy} &= 72x^2y - 8x^3y - 9x^2y^2 \\ f_{yy} &= 24x^3 - 2x^4 - 6x^3y \end{aligned}$$

The stationary points are given by $f_x = 0$; $f_y = 0$

i.e., $$x^2 y^2 (36 - 4x - 3y) = 0 \quad (1)$$

and $$x^3 y (24 - 2x - 3y) = 0 \quad (2)$$

Solving (1) and (2), the stationary points are (0, 0), (0, 8), (0, 12), (12, 0), (9, 0) and (6, 4).

At the first five points, $AC - B^2 = 0$.

$\therefore$ Further investigation is required to investigate the extremum at these points. At the point (6, 4), $A = -2304$, $B = -1728$, $C = -2592$ and $AC - B^2 > 0$.

Since $AC - B^2 > 0$ and $A < 0$, $f(x, y)$ has a maximum at the point (6, 4).

Maximum value of $f(x, y) = 6912$.

Example 4.3 Discuss the maxima and minima of the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$$

$$f_x = 4(x^3 - x + y)$$

$$f_y = 4(y^3 + x - y)$$

$$f_{xx} = 4(3x^2 - 1);\ f_{xy} = 4;\ f_{yy} = 4(3y^2 - 1)$$

The possible extreme points are given by

$$f_x = 0 \text{ and } f_y = 0$$

i.e.,
$$x^3 - x + y = 0 \tag{1}$$

and
$$y^3 + x - y = 0 \tag{2}$$

Adding (1) and (2),
$$x^3 + y^3 = 0 \therefore y = -x \tag{3}$$

Using (3) in (1):
$$x^3 - 2x = 0$$

i.e.,
$$x(x^2 - 2) = 0 \therefore x = 0, +\sqrt{2}, -\sqrt{2}$$

and the corresponding values of y are 0, $-\sqrt{2}, +\sqrt{2}$.

$\therefore$ The possible extreme points of $f(x, y)$ are (0, 0), $(+\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

At the point (0, 0), $A = -4$, $B = 4$ and $C = -4$

$$AC - B^2 = 0$$

$\therefore$ The nature of $f(x, y)$ is undecided at (0, 0). At the points $(\pm\sqrt{2}, \mp\sqrt{2})$, $A = 20$, $B = 4$, $C = 20$

$$AC - B^2 > 0$$

$\therefore f(x, y)$ is minimum at the points $(\pm\sqrt{2}, \mp\sqrt{2})$, and minimum value of $f(x, y) = 8$.

Example 4.4 Examine the extrema of $f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$.

$$f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$$

$$f_x = 2x + y - \frac{1}{x^2}$$

$$f_y = x + 2y - \frac{1}{y^2}$$

$$f_{xx} = 2+\frac{2}{x^3};\ f_{xy} = 1;\ f_{yy} = 2+\frac{2}{y^3}$$

The possible extreme points are given by $f_x = 0$ and $f_y = 0$.

i.e., $$2x+y-\frac{1}{x^2} = 0 \qquad (1)$$

and $$x+2y-\frac{1}{y^2} = 0 \qquad (2)$$

(1) – (2) gives $$x-y+\frac{1}{y^2}-\frac{1}{x^2} = 0$$

i.e., $$x-y+\frac{x^2-y^2}{x^2y^2} = 0$$

i.e., $$(x-y)(x^2y^2+x+y) = 0$$

$\therefore$ $$x = y \qquad (3)$$

Using (3) in (1), $3x^3 - 1 = 0$

$\therefore$ $$x = \left(\frac{1}{3}\right)^{\frac{1}{3}} = y$$

At the point $\left\{\left(\frac{1}{3}\right)^{\frac{1}{3}}, \left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}$, $A = 8$, $B = 1$ and $C = 8$

$\therefore$ $$AC - B^2 > 0$$

$\therefore$ $f(x, y)$ is minimum at $\left\{\left(\frac{1}{3}\right)^{\frac{1}{3}}, \left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}$ and minimum value of $f(x, y) = 3^{\frac{4}{3}}$.

Example 4.5 Discuss the extrema of the function $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^4$ at the origin

$$f(x,y) = x^2-2xy+y^2+x^3-y^3+x^4.$$

$$f_x = 2x-2y+3x^2+4x^3$$

$$f_y = -2x+2y-3y^2$$

$$f_{xx} = 2+6x+12x^2$$

$$f_{xy} = -2;\quad f_{yy} = 2-6y$$

The origin (0, 0) satisfies the equations $f_x = 0$ and $f_y = 0$.
$\therefore$ (0, 0) is a stationary point of $f(x, y)$.
At the origin, $A = 2$, $B = -2$ and $C = 2$

$$\therefore \qquad AC - B^2 = 0$$

Hence further investigation is required to find the nature of the extrema of $f(x, y)$ at the origin.

Let us consider the values of $f(x, y)$ at three points close to (0, 0), namely at $(h, 0)$, $(0, k)$ and (h, h) which lie on the x-axis, the y-axis and the line $y = x$ respectively.

$$f(h,0) = h^2 + h^3 + h^4 > 0.$$
$$f(0,k) = k^2 - k^3 = k^2(1-k) > 0, \text{ when } 0 < k < 1$$
$$f(h,h) = h^4 > 0$$

Thus $f(x, y) > f(0, 0)$ in the neighbourhood of (0, 0).
$\therefore$ (0, 0) is a minimum point of $f(x, y)$ and minimum value of $f(x, y) = 0$.

Example 4.6 Find the maximum and minimum values of

$$f(x, y) = \sin x \sin y \sin(x+y); 0 < x, y < \pi.$$

$$f(x, y) = \sin x \sin y \sin(x+y)$$
$$f_x = \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y)$$
$$f_y = \sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y)$$

i.e., $$f_x = \sin y \sin(2x+y)$$

and $$f_y = \sin x \cdot \sin(x+2y)$$
$$f_{xx} = 2\sin y \cos(2x+y)$$
$$f_{xy} = \sin y \cos(2x+y) + \cos y \cdot \sin(2x+y)$$
$$= \sin(2x+2y)$$
$$f_{yy} = 2\sin x \cos(x+2y)$$

For maximum or minimum values of $f(x, y)$, $f_x = 0$ and $f_y = 0$
i.e., $\sin y \sin(2x + y) = 0$ and $\sin x \cdot \sin(x + 2y) = 0$

i.e., $\frac{1}{2}[\cos 2x - \cos(2x+2y)] = 0$ and $\frac{1}{2}[\cos 2y - \cos(2x+2y)] = 0$

i.e., $$\cos 2x - \cos(2x+2y) = 0 \qquad (1)$$

and $$\cos 2y - \cos(2x+2y) = 0 \qquad (2)$$

From (1) and (2), $\cos 2x = \cos 2y$. Hence $x = y$ (3)
Using (3) in (1), $\cos 2x - \cos 4x = 0$
i.e., $2 \sin x \sin 3x = 0$
$\therefore$ $\sin x = 0$ or $\sin 3x = 0$
$\therefore$ $x = 0, \pi$ and $3x = 0, \pi, 2\pi$ i.e., $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore$ The admissible values of x are $0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$.

Thus the maxima and minima of $f(x, y)$ are given by $(0, 0)$ $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$ and $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$

At the point (0, 0), $A = B = C = 0$

$\therefore \qquad AC - B^2 = 0$

Thus the extremum of $f(x, y)$ at (0, 0) is undecided.

At the point $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, $A = -\sqrt{3},\ B = -\dfrac{\sqrt{3}}{2}$ and $C = -\sqrt{3}$ and $AC - B^2 = 3 - \dfrac{3}{4} > 0\cdot$

As $AC - B^2 > 0$ and $A < 0$, $f(x, y)$ is maximum at $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$.

Maximum value of $f(x, y) = \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{8}$.

At the point $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$, $A = \sqrt{3}, B = \dfrac{\sqrt{3}}{2}$ and $C = \sqrt{3}$ and $AC - B^2 = 3 - \dfrac{3}{4} > 0$.

As $AC - B^2 > 0$ and $A > 0$, $f(x, y)$ is maximum at $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$.

Minimum value of $f(x, y) = -\dfrac{3\sqrt{3}}{8}$.

Example 4.7 Identify the saddle point and the extremum points of

$$f(x, y) = x^4 - y^4 - 2x^2 + 2y^2.$$

$$f(x, y) = x^4 - y^4 - 2x^2 + 2y^2$$

$$f_x = 4x^3 - 4x;\ f_y = 4y - 4y^3$$

$$f_{xx} = 12x^2 - 4;\ f_{xy} = 0; \quad f_{yy} = 4 - 12y^2.$$

The stationary points of $f(x, y)$ are given by $f_x = 0$ and $f_y = 0$

i.e., $\qquad 4(x^3 - x) = 0$ and $4(y - y^3) = 0$

i.e., $\qquad 4x(x^2 - 1) = 0$ and $4y(1 - y^2) = 0$

$\therefore$ $x = 0$ or ± 1 and $y = 0$ or ± 1.

At the points (0, 0), $(\pm 1, \pm 1)$, $AC - B^2 < 0$

$\therefore$ The points (0, 0), (1, 1), (1, –1), (–1, 1) and (–1, –1) are saddle points of the function $f(x, y)$.

At the point $(\pm 1, 0)$, $AC - B^2 > 0$ and $A > 0$

$\therefore$ $f(x, y)$ attains its minimum at $(\pm 1, 0)$ and the minimum value is -1.

At the point $(0, \pm 1)$, $AC - B^2 > 0$ and $A < 0$

$\therefore f(x, y)$ attains its maximum at $(0, \pm 1)$ and the maximum value is $+ 1$.

Example 4.8 Find the minimum value of $x^2 + y^2 + z^2$, when $x + y + z = 3a$.

Here we try to find the conditional minimum of $x^2 + y^2 + z^2$, subject to the condition

$$x + y + z = 3a \tag{1}$$

Using (1), we first express the given function as a function of x and y.

From (1), $z = 3a - x - y$.

Using this in the given function, we get

$$f(x, y) = x^2 + y^2 + (3a - x - y)^2$$

$$f_x = 2x - 2(3a - x - y)$$

$$f_y = 2y - 2(3a - x - y)$$

$$f_{xx} = 4;\ f_{xy} = 2;\ f_{yy} = 4$$

The possible extreme points are given by $f_x = 0$ and $f_y = 0$.

i.e., $$2x + y = 3a \tag{2}$$

and $$x + 2y = 3a \tag{3}$$

Solving (2) and (3), we get the only extreme point as (a, a)

At the point (a, a), $AC - B^2 > 0$ and $A > 0$

$\therefore f(x, y)$ is minimum at (a, a) and the minimum value of $f(x, y) = 3a^2$.

Example 4.9 Show that, if the perimeter of a triangle is constant, its area is maximum when it is equilateral.

Let the sides of the triangle be a, b, c.

Given that $$a + b + c = \text{constant}$$

$$= 2k, \text{ say} \tag{1}$$

Area of the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{2}$$

where $$s = \frac{a+b+c}{2}$$

Using (1) in (2),

$$A = \sqrt{k(k-a)(k-b)(k-c)} \tag{3}$$

A is a function of three variables a, b, c

Again using (1) in (3), we get

$$A = \sqrt{k(k-a)(k-b)(a+b-k)}$$

A is maximum or minimum, when $f(a, b) = \dfrac{A^2}{k} = (k - a)(k - b)(a + b - k)$ is maximum or minimum.

$$\begin{aligned} f_a &= (k-b)\{(k-a)\cdot 1+(a+b-k)\cdot(-1)\} \\ &= (k-b)(2k-2a-b) \\ f_b &= (k-a)\{(k-b)\cdot 1+(a+b-k)\cdot(-1)\} \\ &= (k-a)(2k-a-2b) \\ f_{aa} &= -2(k-b);\ f_{ab} = -3k+2a+2b; \\ f_{ab} &= -2(k-a) \end{aligned}$$

The possible extreme points of $f(a, b)$ are given by

$f_a = 0$ and $f_b = 0$

i.e., $(k - b)(2k - 2a - b) = 0$ and $(k - a)(2k - a - 2b) = 0$

$\therefore\ b = k$ or $2a + b = 2k$ and $a = k$ or $a + 2b = 2k$

Thus the possible extreme points are given by

(i) $a = k, b = k$; (ii) $b = k, a + 2b = 2k$; (iii) $a = k, 2a + b = 2k$ and (iv) $2a + b = 2k$, $a + 2b = 2k$.

(i) gives $a = k$, $b = k$ and hence $c = 0$.
(ii) gives $a = 0$, $b = k$ and hence $c = k$.
(iii) gives $a = k$, $b = 0$ and hence $c = k$.

All these lead to meaningless results.

Solving $2a + b = 2k$ and $a + 2b = 2k$, we get

$$a = \frac{2k}{3} \text{ and } b = \frac{2k}{3}$$

At the point $\left(\dfrac{2k}{3}, \dfrac{2k}{3}\right)$,

$$A = f_{aa} = -\frac{2k}{3};\ B = f_{ab} = -\frac{k}{3};\ C = f_{bb} = -\frac{2k}{3}$$

$AC - B^2 > 0$ and $A < 0$

$\therefore\ f(a, b)$ is maximum at $\left(\dfrac{2k}{3}, \dfrac{2k}{3}\right)$

Hence the area of the triangle is maximum when $a = \dfrac{2k}{3}$ and $b = \dfrac{2k}{3}$.

When $a = \dfrac{2k}{3}, b = \dfrac{2k}{3};\ c = 2k-(a+b) = \dfrac{2k}{3}$

Thus the area of the triangle is maximum, when $a = b = c = \dfrac{2k}{3}$, i.e., when the triangle is equilateral.

Example 4.10 In a triangle ABC, find the maximum value of $\cos A \cos B \cos C$. In triangle ABC, $A + B + C = \pi$.

Using this condition, we express the given function as a function of A and B

Thus $\cos A \cos B \cos C = \cos A \cos B \cos\{\pi - (A+B)\}$

$$= -\cos A \cos B \cos(A+B)$$

Let
$$f(A,B) = -\cos A \cos B \cos(A+B)$$
$$f_A = -\cos B\{-\sin A \cos(A+B) - \cos A \sin(A+B)\}$$
$$= \cos B \sin(2A+B)$$
$$f_B = -\cos A\{-\sin B \cos(A+B) - \cos B \sin(A+B)\}$$
$$= \cos A \sin(A+2B)$$
$$f_{AA} = 2\cos B \cos(2A+B)$$
$$f_{AB} = \cos B \cos(2A+B) - \sin B \sin(2A+B)$$
$$= \cos(2A+2B)$$
$$f_{BB} = 2\cos A \cos(A+2B)$$

The possible extreme points are given by

$$f_A = 0 \text{ and } f_B = 0$$

i.e.,
$$\cos B \sin(2A+B) = 0 \tag{1}$$

and
$$\cos A \sin(A+2B) = 0$$

Thus the possible values of A and B are given by (i) $\cos B = 0$, $\cos A = 0$; (ii) $\cos B = 0$, $\sin(A + 2B) = 0$; (iii) $\sin(2A + B) = 0$, $\cos A = 0$ and (iv) $\sin(2A + B) = 0$, $\sin(A + 2B) = 0$

i.e., (i) $A = \frac{\pi}{2}$, $B = \frac{\pi}{2}$; (ii) $B = \frac{\pi}{2}$, $A = 0$ **or** π, (iii) $A = \frac{\pi}{2}$, $B = 0$ **or** π and (iv) $2A+B = \pi$, $A+2B = \pi$ or

$$A = \frac{\pi}{3},\ B = \frac{\pi}{3}$$

The first three sets of values of A and B lead to meaningless results.

Hence $A = \frac{\pi}{3}$, $B = \frac{\pi}{3}$ give the extreme point.

At this point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $A = f_{AA} = -1$; $B = f_{AB} = -\frac{1}{2}$; $f_{BB} = -1$ and $AC = B^2 > 0$.

Also $A < 0$

$\therefore f(A, B)$ is maximum at $A = B = \frac{\pi}{3}$ and the maximum value

$$= -\cos\frac{\pi}{3} \cdot \cos\cdot\frac{\pi}{3}\cos\frac{2\pi}{3} = \frac{1}{8}.$$

Example 4.11 Find the maximum value of $x^m y^n z^p$, when $x + y + z = a$.

Let $f = x^m y^n z^p$ and $\phi = x + y + z - a$.

Using the Lagrange multiple λ, the auxiliary function is $g = (f + \lambda\phi)$.
This stationary points of $g = (f + \lambda\phi)$ are given by $g_x = 0$, $g_y = 0$, $g_z = 0$ and $g_\lambda = 0$

i.e.,
$$mx^{m-1} y^n z^p + \lambda = 0 \quad (1)$$
$$nx^m y^{n-1} z^p + \lambda = 0 \quad (2)$$
$$px^m y^n z^{p-1} + \lambda = 0 \quad (3)$$
$$x + y + z - a = 0 \quad (4)$$

From (1), (2) and (3), we have
$$-\lambda = mx^{m-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}.$$

i.e.,
$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z}$$
$$= \frac{m+n+p}{a}, \text{ by (4)}$$

$\therefore$ Maximum value of f occurs,

when $x = \dfrac{am}{m+n+p}$, $y = \dfrac{an}{m+n+p}$, $z = \dfrac{ap}{m+n+p}$

Thus maximum value of $f = \dfrac{a^{m+n+p} \cdot m^m \cdot n^n \cdot p^p}{(m+n+p)^{m+n+p}}$

Example 4.12 A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box, that requires the least material for its construction.

Let, x, y, z be the length, breadth and height of the respectively.
The material for the construction of the box is least, when the area of surface of the box is least.
Hence we have to minimise
$$S = xy + 2yz + 2zx,$$
subject to the condition that the volume of the box, i.e., $xyz = 32$.
Here $f = xy + 2yz + 2zx$; $\phi = xyz - 32$.
The auxiliary function is $g = f + \lambda\phi$, where λ is the Lagrange multiplier. The stationary points of g are given by $g_x = 0$, $g_y = 0$, $g_z = 0$ and $g_\lambda = 0$

i.e.,
$$y + 2z + \lambda yz = 0 \quad (1)$$
$$x + 2x + \lambda zx = 0 \quad (2)$$
$$2x + 2y + \lambda xy = 0 \quad (3)$$
$$xyz - 32 = 0 \quad (4)$$

From (1), (2) and (3), we have

$$\frac{1}{z}+\frac{2}{y} = -\lambda \tag{5}$$

$$\frac{1}{z}+\frac{2}{x} = -\lambda \tag{6}$$

$$\frac{2}{y}+\frac{2}{x} = -\lambda \tag{7}$$

Solving (5), (6) and (7), we get

$$x = -\frac{4}{\lambda},\ y = -\frac{4}{\lambda} \text{ and } z = -\frac{2}{\lambda}$$

Using these values in (4), we get

$$-\frac{32}{\lambda^3}-32 = 0$$

i.e., $\lambda = -1$

∴ $x = 4,\ y = 4,\ z = 2.$

Thus the dimensions of the box and 4 cm; 4 cm and 2 cm.

Example 4.13 Find the volume of the greatest rectangular parallelopiped inscribed in the ellipsoid whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$

Let $2x$, $2y$, $2z$ be the dimensions of the required rectangular parallelopiped.

By symmetry, the centre of the parallelopiped coincides with that of the ellipsoid, namely, the origin and its faces are parallel to the co-ordinate planes.

Also one of the vertices of the parallelopiped has co-ordinates (x, y, z), which satisfy the equation of the ellipsoid.

Thus, we have to maximise $V = 8xyz$, subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Here $f = 8xyz$ and $\phi = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

The auxiliary function is $g = f + \lambda\phi$, where λ is the Lagrange multiplier. The stationary points of g are given by

$$g_x = 0,\ g_y = 0,\ g_z = 0 \text{ and } g_\lambda = 0$$

i.e.,

$$8yz+\frac{2\lambda x}{a^2} = 0 \tag{1}$$

$$8zx+\frac{2\lambda y}{b^2} = 0 \tag{2}$$

$$8xy + \frac{2\lambda z}{c^2} = 0 \tag{3}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{4}$$

Multiplying (1) by x, $\dfrac{2\lambda x^2}{a^2} = -8xyz$

Similarly $\dfrac{2\lambda y^2}{b^2} = \dfrac{2\lambda z^2}{c^2} = -8xyz$ from (2) and (3)

Thus $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2} = k$ say

Using in (4), $3k = 1 \quad \therefore\ k = \dfrac{1}{3}$

$$\therefore x = \frac{a}{\sqrt{3}},\ y = \frac{b}{\sqrt{3}} \text{ and } z = \frac{c}{\sqrt{3}}$$

$$\therefore \text{Maximum volume} = \frac{8abc}{3\sqrt{3}}.$$

Example 4.14 Find the shortest and the longest distances from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$.

Let (x, y, z) be any point on the sphere. Distance of the point (x, y, z) from $(1, 2, -1)$ is given by $d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$.

We have to find the maximum and minimum values of d or equivalently

$$d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2,$$

subject to the constant $x^2 + y^2 + z^2 - 24 = 0$

Here $\quad f = (x-1)^2 + (y-2)^2 + (z+1)^2$ and

$\quad \phi = x^2 + y^2 + z^2 - 24$

The auxiliary function is $g = f + \lambda\phi$, where λ is the Lagrange multiplier. The stationary points of g are given by $g_x = 0$, $g_y = 0$, $g_z = 0$ and $g_\lambda = 0$.

i.e.,

$$2(x-1) + 2\lambda x = 0 \tag{1}$$

$$2(y-2) + 2\lambda y = 0 \tag{2}$$

$$2(z+1) + 2\lambda z = 0 \tag{3}$$

$$x^2 + y^2 + z^2 = 24 \tag{4}$$

From (1), (2) and (3), we get

$$x = \frac{1}{1+\lambda},\ y = \frac{2}{1+\lambda},\ z = \frac{1}{1+\lambda}$$

Using these values in (4), we get

$$\frac{6}{(1+\lambda)^2} = 24 \text{ i.e., } (1+\lambda)^2 = \frac{1}{4}$$

$\therefore$ $$\lambda = -\frac{1}{2} \text{ or } -\frac{3}{2}.$$

When $\lambda = -\frac{1}{2}$, the point on the sphere is (2, 4, –2)

When $\lambda = -\frac{3}{2}$, the point on the sphere is (–2, –4, 2)

When the point is (2, 4, –2), $d = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}$

When the point is (–2, –4, 2), $d = \sqrt{(-3)^2 + (-6)^2 + 3^2} = 3\sqrt{6}$

$\therefore$ Shortest and longest distances are $\sqrt{6}$ and $3\sqrt{6}$ respectively.

Example 4.15 Find the point on the curve of intersection of the surfaces $z = xy + 5$ and $x + y + z = 1$ which is nearest to the origin.

Let (x, y, z) be the required point.

It lies on both the given surfaces.

$\therefore$ $$xy - z + 5 = 0 \quad \text{and} \quad x + y + z = 1$$

Distance of the point (x, y, z) from the origin is given by $d = \sqrt{x^2 + y^2 + z^2}$.

We have to minimize d or equivalently

$$d^2 = x^2 + y^2 + z^2,$$

subject to the constraints $xy - z + 5 = 0$ and $x + y + z - 1 = 0$.

Note ☑ Here we have two constraint conditions. To find the extremum of $f(x, y, z)$ subject to the conditions $\phi_1(x, y, z) = 0$ and $\phi_2(x, y, z) = 0$, we form the auxiliary function

$g = f + \lambda_1\phi_1 + \lambda_2\phi_2$, where λ_1 and λ_2 are two Lagrange multipliers.

The stationary points of g are given by $g_x = 0, g_y = 0, g_z = 0, g_{\lambda 1} = 0$ and $g_{\lambda 2} = 0$.

In this problem, $f = x^2 + y^2 + z^2$, $\phi_1 = xy - z + 5$ and $\phi_2 = x + y + z - 1$.

The auxiliary function is $g = f + \lambda_1\phi_1 + \lambda_2\phi_2$, where λ_1, λ_2 are Lagrange multipliers.

The stationary points of g are given by

$$2x + \lambda_1 y + \lambda_2 = 0 \tag{1}$$

$$2y + \lambda_1 x + \lambda_2 = 0 \tag{2}$$

$$2z - \lambda_1 + \lambda_2 = 0 \tag{3}$$

$$xy - z + 5 = 0 \qquad (4)$$

$$x + y + z - 1 = 0 \qquad (5)$$

Eliminating λ_1, λ_2 from (1), (2), (3), we have

$$\begin{vmatrix} 2x & y & 1 \\ 2y & x & 1 \\ 2z & -1 & 1 \end{vmatrix} = 0$$

i.e., $x(x+1) - y(y-2) - (y + zx) = 0$

i.e., $x^2 - y^2 + x - y - z(x-y) = 0$

i.e., $(x-y)(x+y-z+1) = 0$

$\therefore$ $x = y$ or $x + y - z + 1 = 0$

Using $x = y$ in (4) and (5), we have

$$z = x^2 + 5 \qquad (6)$$

and $z = 1 - 2x$ (7)

From (6) and (7), $x^2 + 2x + 4 = 0$, which gives only imaginary values for x.

Hence $x + y - z + 1 = 0$ (8)

Solving (5) and (8), we get $x + y = 0$ (9)

and $z = 1$ (10)

Using (10) in (4), we get $xy = -4$ (11)

Solving (9) and (11), we get $x = \pm 2$ and $y = \pm 2$.

$\therefore$ The required points are (2, –2, 1) and (–2, 2 ,1) and the shortest distance is 3.

EXERCISE 4(c)

Part A

(Short Answer Questions)

1. Define relative maximum and relative minimum of a function of two variables.
2. State the conditions for the stationary point (a, b) of $f(x, y)$ to be (i) a maximum point (ii) a minimum point and (iii) a saddle point.
3. Define saddle point of a function $f(x, y)$.
4. Write down the conditions to be satisfied by $f(x, y, z)$ and $\phi(x, y, z)$, when we extremise $f(x, y, z)$ subject to the condition $\phi(x, y, z) = 0$.
5. Find the minimum point of $f(x, y) = x^2 + y^2 + 6x + 12$.
6. Find the stationary point of $f(x, y) = x^2 - xy + y^2 - 2x + y$.
7. Find the stationary point of $f(x, y) = 4x^2 + 6xy + 9y^2 - 8x - 24y + 4$.

8. Find the possible extreme point of $f(x, y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$.
9. Find the nature of the stationary point (1, 1) of the function $f(x, y)$, if $f_{xx} = 6xy^3$, $f_{xy} = 9x^2 y^2$ and $f_{yy} = 6x^3 y$.
10. Given $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 6y$, find the nature of the stationary point (1, 2) of the function $f(x, y)$.

Part B

Examine the following functions for extreme values:

11. $x^3 + y^3 - 3axy$
12. $x^3 + y^3 - 12x - 3y + 20$
13. $x^4 + 2x^2 y - x^2 + 3y^2$
14. $x^3 \text{ y} - 3x^2 - 2y^2 - 4y - 3$
15. $x^4 + x^2 y + y^2$ at the origin
16. $x^3 y^2 (a - x - y)$
17. $x^3 y^2 (12 - 3x - 4y)$
18. $xy + 27\left(\frac{1}{x} + \frac{1}{y}\right)$
19. $\sin x + \sin y + \sin(x + y), 0 \le x, y \le \frac{\pi}{2}$.
20. Identify the saddle points and extreme points of the function $xy(3x + 2y + 1)$.
21. Find the minimum value of $x^2 + y^2 + z^2$, when (i) $xyz = a^3$ and (ii) $xy + yz + zx = 3a^2$.
22. Find the minimum value of $x^2 + y^2 + z^2$, when $ax + by + cz = p$.
23. Show that the minimum value of $(a^3 x^2 + b^3 y^2 + c^3 z^2)$, when $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{k}$, is $k^2 (a + b + c)^3$.
24. Split 24 into three parts such that the continued product of the first, square of the second and cube of the third may be minimum.
25. The temperature at any point (x, y, z) in space is given by $T = k\,x\,y\,z^2$, where k is a constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
26. Find the dimensions of a rectangular box, without top, of maximum capacity and surface area 432 square meters.
27. Show that, of all rectangular parallelopipeds of given volume, the cube has the least surface.
28. Show that, of all rectangular parallelopipeds with given surface area, the cube has the greatest volume.
29. Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.
30. Find the points on the surface $z^2 = xy + 1$ whose distance from the origin is minimum.
31. If the equation $5x^2 + 6xy + 5y^2 = 8$ represents an ellipse with centre at the origin, find the lengths of its major and minor axes.

(**Hint**: The longest distance of a point on the ellipse from its centre gives the length of the semi-major axis. The shortest distance of a point on the ellipse from its centre gives the length of the semi-minor axis).

32. Find the point on the surface $z = x^2 + y^2$, that is nearest to the point (3, –6, 4).
33. Find the minimum distance from the point (3, 4, 15) to the cone $x^2 + y^2 = 4z^2$.
34. Find the points on the ellipse obtained as the curve of intersection of the surfaces $x + y = 1$ and $x^2 + 2y^2 + z^2 = 1$, which are nearest to and farthest from the origin.
35. Find the greatest and least values of z, where (x, y, z) lies on the ellipse formed by the intersection of the plane $x + y + z = 1$ and the ellipsoid $16x^2 + 4y^2 + z^2 = 16$.

ANSWERS

Exercise 4(a)

(2) $du = \cos(xy^2)(y^2 dx + 2xy\, dy)$

(3) $du = x^{y-1} \cdot y^x (y + x \log y)\, dx + x^y y^{x-1}(x + y \log x)\, dy$

(4) $du = y(1 + \log xy)\, dx + x(1 + \log xy)\, dy$

(5) $du = (y \log a)\, a^{xy}\, dx + (x \log a)\, a^{xy}\, dx$

(6) $8\, a^5 t^6 (4t + 7)$;

(7) $e^{\sqrt{a^2-t^2}} \sin^3 t\left\{3\sqrt{a^2-t^2}\sin^2 t\cos t - t\sin^3 t/\sqrt{a^2-t^2}\right\}$

(8) $(\cos t - e^{-t} - \sin t)/(e^{-t} + \sin t + \cos t)$

(9) $-\dfrac{x^2+2xy+2y^2}{x^2+4xy+y^2}$;

(11) $\dfrac{3}{2}x\cos(x^2+y^2)$

(12) $x(xy + 4y^2 - 2x^2)/(x + 2y)$;

(14) 3.875

(15) 4.984

(16) 0.0043

(17) 0.006 cm^3; 0.004 cm^2

(18) 2

(19) $4(a + b + c)k$

(20) 1.5

(36) $\dfrac{\partial^2 z}{\partial u\, \partial v} = 0$;

(37) $\dfrac{\partial^2 z}{\partial v^2} = 0$

(38) $\dfrac{\partial^2 z}{\partial u^2} = 0$

(39) $\dfrac{\partial^2 z}{\partial u \partial v} = 0$;

(40) $\dfrac{\partial^2 u}{\partial z \cdot \partial z^*} = 0$

(41) (i) $\dfrac{y(y - x\log y)}{x(x - y\log x)}$

(ii) $\dfrac{y}{x}$

(iii) $\dfrac{y\tan x + \log\sin y}{\log\cos x - x\cot y}$

(iv) $\dfrac{\log\cot y - y\tan x}{\log\sec x + x\sec y\,\mathrm{cosec}\, y}$

(v) $\dfrac{x-y}{x(1+\log x)}$

(42) $2a^3xy/(ax-y^2)^3$

(43) $2a^2xy\,(3a^4+x^2y^2)/(a^2x-y^3)^3$

(47) 5%

(50) 2.3%

(55) $\dfrac{5\sqrt{3}\pi}{324}$.

Exercise 4(b)

(4) $1+(x+y)+\dfrac{(x+y)^2}{2}+\cdots$

(5) $(x+y)-\dfrac{1}{3!}(x+y)^3+\ldots$

(11) $u+\upsilon+1$;

(15) $2\tan^{-1}\left(\dfrac{1}{y}\right)$

(16) $y+xy+\dfrac{x^2y}{2}-\dfrac{y^3}{6}+\cdots$

(17) $\dfrac{e}{\sqrt{2}}\left\{1+(x-1)-\left(y-\dfrac{\pi}{4}\right)+\dfrac{(x-1)^2}{2}-(x-1)\left(y-\dfrac{\pi}{4}\right)-\dfrac{1}{2}\left(y-\dfrac{\pi}{4}\right)^2+\cdots\right\}$

(18) $y+xy-\dfrac{y^2}{2}+\dfrac{1}{2}x^2y-\dfrac{1}{2}xy^2+\dfrac{1}{3}y^3+\cdots$

(19) $\dfrac{\pi}{4}-\dfrac{1}{2}(x-1)+\dfrac{1}{2}(y-1)+\dfrac{1}{4}(x-1)^2-\dfrac{1}{4}(y-1)^2+\cdots$

(20) $-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2)$

(21) $-9+3(x+2)-7(y-1)+2(x+2)(y-1)-2(y-1)^2+(x+2)(y-1)^2$

(22) $1+(y-1)+(x-1)(y-1)+\cdots$

(23) $$e\left[1+(x-1)+(y-1)+\frac{1}{2}(x-1)^2+2(x-1)(y-1)+(y-1)^2+\frac{1}{6}(x-1)^3\right.$$
$$\left.+\frac{3}{2}(x-1)^2(y-2)+\frac{3}{2}(x-1)(y-2)^2+\frac{1}{6}(y-2)^3\right]$$

(27) (i) $4(u^2+\upsilon^2)$

(ii) $4xy$

(28) r

(30) $x(y\upsilon+1-w)+z-2u\upsilon$

(31) $(x-y)(y-z)(z-x)$

(32) $u^2=\upsilon+1$

(33) $u\tan\upsilon$

(34) $f_1^2 = f_2 + 2f_3$

(36) $\frac{1}{2a^3}\{\tan^{-1}\frac{x}{a} + (ax)/(x^2+a^2)\}$

(37) $\frac{(-1)^n\, n!}{(m+1)^{n+1}}$

(38) $\frac{1}{2}\sqrt{\pi}e^{-a^2}$

(39) $\tan^{-1}\left(\frac{1}{a}\right); \frac{\pi}{2}$

(40) log (1 + m)

Exercise 4(c)

(5) (–3, 0). (6) (1, 0). (7) $\left(0, \frac{4}{3}\right)$

(8) (1, 1). (9) Saddle point. (10) Minimum point.

(11) Maximum at (a, a) if $a < 0$ and minimum at (a, a) if $a > 0$.

(12) Minimum at (2, 1) and maximum at (–2, –1).

(13) Minimum at $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{4}\right)$

(14) Maximum at (0, –1). (15) Minimum at (0, 0).

(16) Maximum at $\left(\frac{a}{2}, \frac{a}{3}\right)$ (17) maximum at (2, 1). (18) Minimum at (3, 3).

(19) Maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and minimum at $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$.

(20) Saddle point are (0, 0), $\left(-\frac{1}{3}, 0\right)$ and $\left(0, -\frac{1}{2}\right)$; maximum at $\left(-\frac{1}{9}, -\frac{1}{6}\right)$.

(21) $3a^2$; $3a^2$. (22) $\frac{p^2}{a^2+b^2+c^2}$ (24) 4, 8, 12.

(25) $\frac{ka^4}{8}$. (26) 12, 12 and 6 metres.

(30) (0, 0, 1) and (0, 0, –1). (31) 4, 2. (32) (1, –2, 5).

(33) $5\sqrt{5}$. (34) $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$; (1, 0, 0). (35) $\frac{8}{3}; -\frac{8}{7}$.

Chapter 5

Multiple Integrals

5.1 INTRODUCTION

When a function $f(x)$ is integrated with respect to x between the limits a and b, we get

the difinite integral $\int_a^b f(x)\mathrm{d}x$.

If the integrand is a funtion $f(x,y)$ and if it is integrated with respect to x and y repeatedly between the limits x_0 and x_1 (for x) and between the limits y_0 and y_1 (for y),

we get a *double integral* that is denoted by the symbol $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y)\mathrm{d}x\,\mathrm{d}y$.

Extending the concept of double integral one step further, we get the *triple integral*

$$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

5.2 EVALUATION OF DOUBLE AND TRIPLE INTEGRALS

To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y)\,\mathrm{d}x\,\mathrm{d}y,$ we first integrate $f(x, y)$ with respect to x partially,

i.e. treating y as a constant temporarily, between x_0 and x_1. The resulting function got after the inner integration and substitution of limits will be a function of y. Then we integrate this function of y with respect to y between the limits y_0 and y_1 as usual.

The order in which the integrations are performed in the double integral is illustrated in Fig. 5.1.

$$\boxed{\int_{y_0}^{y_1} \boxed{\int_{x_0}^{x_1} f(x, y)\, \mathrm{d}x} \; \mathrm{d}y}$$

Fig. 5.1

Note ☑ Since the resulting function got after evaluating the inner integral is to be a function of y, the limits x_0 and x_1 may be either constants or functions of y.

The order in which the integrations are performed in a triple integral is illustrated in Fig. 5.2.

$$\boxed{\int_{z_0}^{z_1} \boxed{\int_{y_0}^{y_1} \boxed{\int_{x_0}^{x_1} f(x, y, z)\, \mathrm{d}x} \; \mathrm{d}y} \; \mathrm{d}z}$$

Fig. 5.2

When we first perform the innermost integration with respect to x, we treat y and z as constants temporarily. The limits x_0 and x_1 may be constants or functions of y and z, so that the resulting function got after the innermost integration may be a function of y and z. Then we perform the middle integration with respect to y, treating z as a constant temporarily. The limits y_0 and y_1 may be constants or functions of z, so that the resulting function got after the middle integration may be a function of z only. Finally we perform the outermost integration with respect to z between the constant limits z_0 and z_1.

Note ☑ Sometimes $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y)\mathrm{d}x\, \mathrm{d}y$ is also denoted as $\int_{y_0}^{y_1} \mathrm{d}y \int_{x_0}^{x_1} f(x, y)\mathrm{d}x$ and $\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x, y, z)\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ is also denoted as $\int_{z_0}^{z_1} \mathrm{d}z \int_{y_0}^{y_1} \mathrm{d}y \int_{x_0}^{x_1} f(x, y, z)\, \mathrm{d}x$. If these notations are used to denote the double and triple integrals, the integrations are performed from right to left in order.

5.3 REGION OF INTEGRATION

Consider the double integral $\int_{c}^{d}\int_{\phi_1(y)}^{\phi_2(y)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$. As stated above x varies from $\phi_1(y)$ to $\phi_2(y)$ and y varies from c to d.

i.e. $\phi_1(y) \le x \le \phi_2(y)$ and $c \le y \le d$.

These inequalities determine a region in the xy-plane, whose boundaries are the curves $x = \phi_1(y)$, $x = \phi_2(y)$ and the lines $y = c$, $y = d$ and which is shown in Fig. 5.3. This region $ABCD$ is known as the region of integration of the above double integral.

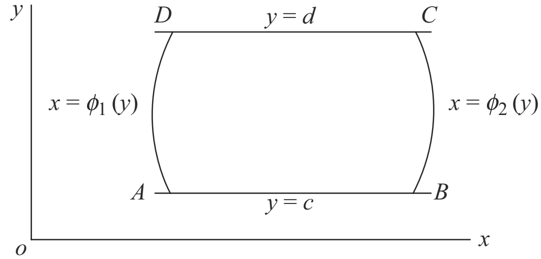


Fig. 5.3

Similarly, for the double integral $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\,dy\,dx$, the region of integration $ABCD$, whose boundaries are the curves $y = \phi_1(x)$, $y = \phi_2(x)$ and the lines $x = a$, $x = b$, is shown in Fig. 5.4.

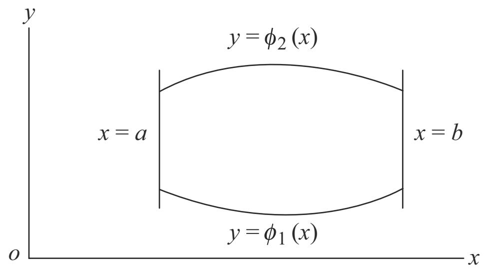


Fig. 5.4

For the triple integral $\int_{z_1}^{z_2} \int_{\psi_1(z)}^{\psi_2(z)} \int_{\phi_1(y,z)}^{\phi_2(y,z)} f(x, y, z)\,dx\,dy\,dz$, the inequalities $\phi_1(y, z) \le x \le \phi_2(y, z)$; $\psi_1(z) \le y \le \psi_2(z)$; $z_1 \le z \le z_2$ hold good. These inequalities determine a domain in space whose boundaries are the surfaces $x = \phi_1(y, z)$, $x = \phi_2(y, z)$, $y = \psi_1(z)$, $y = \psi_2(z)$, $z = z_1$ and $z = z_2$. This domain is called the domain of integration of the above triple integral.

WORKED EXAMPLE 5(a)

Example 5.1 Verify that $\int_1^2 \int_0^1 (x^2 + y^2)\,dx\,dy = \int_0^1 \int_1^2 (x^2 + y^2)\,dy\,dx.$

$$\text{L.S.} = \int_1^2 \left[\int_0^1 (x^2 + y^2)\,dx \right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} + y^2 x \right]_{x=0}^{x=1} \mathrm{d}y$$

Note ☑ y is treated a constant during inner integration with respect to x.

$$= \int_1^2 \left(\frac{1}{3} + y^2 \right) \mathrm{d}y = \left(\frac{y}{3} + \frac{y^3}{3} \right)_1^2 = \frac{8}{3}$$

$$\text{R.S.} = \int_0^1 \left[\int_1^2 (x^2 + y^2)\,\mathrm{d}y \right] \mathrm{d}x$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_{y=1}^{y=2} \mathrm{d}x$$

Note ☑ x is treated a constant during inner integration with respect to y.

$$= \int_0^1 \left(x^2 + \frac{7}{3} \right) \mathrm{d}x = \left(\frac{x^3}{3} + \frac{7}{3} x \right)_0^1 = \frac{8}{3}$$

Thus the two double integrals are equal.

Note ☑ From the above problem we note the following fact: If the limits of integration in a double integral are constants, then the order of integration is immaterial, provided the relevant limits are taken for the concerned variable and the integrand is continuous in the region of integration. This result holds good for a triple integral also.

Example 5.2 Evaluate $\int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^4 \sin\phi \,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta$.

$$\text{The given integral} = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \mathrm{d}\phi \int_0^{a} r^4 \sin\phi \,\mathrm{d}r$$

$$= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \left(\frac{r^5}{5} \right)_0^a \sin\phi \,\mathrm{d}\phi$$

$$= \frac{a^5}{5} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \sin\phi \,\mathrm{d}\phi$$

$$= \frac{a^5}{5} \int_0^{2\pi} (-\cos\phi)_0^{\pi} \,\mathrm{d}\theta$$

$$= \frac{2}{5} a^5 \int_0^{2\pi} \mathrm{d}\theta$$

$$= \frac{4}{5} \pi a^5$$

Example 5.3 Evaluate $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{1+x^2+y^2}$

$$\text{The given integral} = \int_0^1 \left[\int_0^{\sqrt{1+y^2}} \frac{1}{(1+y^2)+x^2}\,dx \right] dy$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+y^2}} \tan^{-1} \frac{x}{\sqrt{1+y^2}} \right]_{x=0}^{x=\sqrt{1+y^2}} dy$$

$$= \frac{\pi}{4} \int_0^1 \frac{dy}{\sqrt{1+y^2}}$$

$$= \frac{\pi}{4} \left[\log\left(y + \sqrt{1+y^2} \right) \right]_0^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

Example 5.4 Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\,(x+y)\,dx\,dy$.

Since the limits for the inner integral are functions of x, the variable of inner integration should be y. Effecting this change, the given integral I becomes

$$I = \int_0^1 \left[\int_x^{\sqrt{x}} xy\,(x+y)\,dy \right] dx$$

$$= \int_0^1 \left(x^2 \frac{y^2}{2} + x\frac{y^3}{3} \right)_{y=x}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(\frac{x^3}{2} + \frac{1}{3} x^{5/2} \right) - \left(\frac{x^4}{2} + \frac{x^4}{3} \right) \right] dx$$

$$= \left(\frac{x^4}{8} + \frac{2}{21} x^{7/2} - \frac{x^5}{6} \right)_0^1$$

$$= \frac{1}{8} + \frac{2}{21} - \frac{1}{6} = \frac{3}{56}$$

Example 5.5 Evaluate $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xyz \, dx \, dy \, dz$.

$$\text{The given integral } = \int_0^1 \int_0^{1-z} yz \left(\frac{x^2}{2}\right)_0^{1-y-z} dy \, dz$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-z} yz \, (1-y-z)^2 \, dy \, dz$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-z} yz \, \{(1-z)^2 - 2\,(1-z)y + y^2\} \, dy \, dz$$

$$= \frac{1}{2} \int_0^1 \left[z(1-z)^2 \, \frac{y^2}{2} - 2z(1-z) \, \frac{y^3}{3} + z \, \frac{y^4}{4} \right]_{y=0}^{y=1-z} dz$$

$$= \frac{1}{2} \int_0^1 \left[\frac{1}{2} z(1-z)^4 - \frac{2}{3} z(1-z)^4 + \frac{1}{4} z(1-z)^4 \right] dz$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) \int_0^1 z(1-z)^4 \, dz$$

$$= \frac{1}{24} \int_0^1 \{1-(1-z)\} \, (1-z)^4 \, dz$$

$$= \frac{1}{24} \left[\frac{(1-z)^5}{-5} + \frac{(1-z)^6}{6} \right]_0^1$$

$$= \frac{1}{24}\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{720}$$

Example 5.6 Evaluate $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dx \, dy \, dz$.

Since the upper limit for the innermost integration is a function of x, y, the corresponding variable of integration should be z. Since the upper limit for the middle integration is a function of x, the corresponding variable of integration should be y. The variable of integration for the outermost integration is then x. Effecting these changes, the given triple integral I becomes,

$$I = \int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx$$

$$= \int_0^{\log 2} dx \int_0^{x} dy \, e^{x+y} \, (e^z)_{z=0}^{z=x+y}$$

$$= \int_0^{\log 2} dx \int_0^{x} (e^{2x+2y} - e^{x+y}) \, dy$$

$$= \int_0^{\log 2} \mathrm{d}x \left(e^{2x} \cdot \frac{e^{2y}}{2} - e^x \cdot e^y \right)_{y=0}^{y=x}$$

$$= \int_0^{\log 2} \left(\frac{1}{2} e^{4x} - \frac{3}{2} e^{2x} + e^x \right) \mathrm{d}x$$

$$= \left(\frac{1}{8} e^{4x} - \frac{3}{4} e^{2x} + e^x \right)_0^{\log 2}$$

$$= \frac{5}{8}$$

Example 5.7 Evaluate $\iint_R xy \, \mathrm{d}x \, \mathrm{d}y$, where R is the region bounded by the line $x + 2y = 2$, lying in the first quadrant.

We draw a rough sketch of the boundaries of R and identify R.

The boundaries of R are the lines $x = 0$, $y = 0$ and the segment of the line $\frac{x}{2} + \frac{y}{1} = 1$ lying in the first quadrant.

Now R is the region as shown in Fig. 5.5.

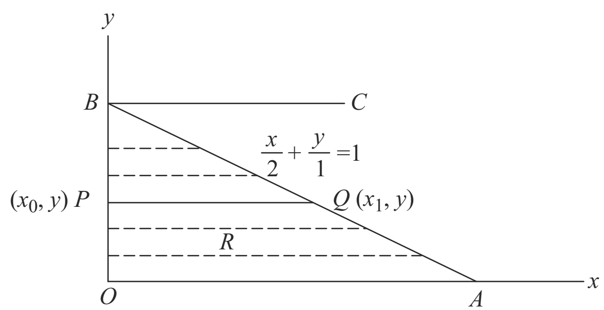


Fig. 5.5

Since the limits of the variables of integration are not given in the problem and to be fixed by us, we can choose the order of integration arbitrarily.

Let us integrate with respect to x first and then with respect to y. Then the integral I becomes

$$\mathrm{I} = \int \left[\int_R xy \, \mathrm{d}x \right] \mathrm{d}y$$

When we perform the inner integration with respect to x, we have to treat y as a constant temporarily and find the limits for x.

Geometrically, treating y = constant is equivalent to drawing a line parallel to the x-axis arbitrarily lying within the region of integration R as shown in the figure.

Finding the limits for x (while y is a constant) is equivalent to finding the variation of the x co-ordinate of any point on the line PQ. We assume that the y co-ordinates of all points on PQ are y each (since y is constant on PQ) and $P \equiv (x_0, y)$ and $Q \equiv (x_1, y)$.

Thus x varies from x_0 to x_1.

Wherever the line PQ has been drawn, the left end P lies on the y-axis and hence $x_0 = 0$ and the right end Q lies on the line $x + 2y = 2$, and hence $x_1 + 2y = 2$ i.e. $x_1 = 2 - 2y$.

Thus the limits for the variable x of inner integration are 0 and $2 - 2y$. When we go to the outer integration, we have to find the limits for y.

Geometrically we have to find the variation of the line PQ, so that the region R is fully covered. To sweep the entire area of the region R, PQ has to start from the position OA where $y = 0$, move parallel to itself and go up to the position BC where $y = 1$.

Thus the limits for y are 0 and 1.

$$\therefore \qquad I = \int_0^1 \int_0^{2-2y} xy \, dx \, dy$$

$$= \int_0^1 y \left(\frac{x^2}{2} \right)_0^{2-2y} dy$$

$$= \int_0^1 \frac{y}{2} (2 - 2y)^2 \, dy$$

$$= 2 \int_0^1 y (1 - y)^2 \, dy$$

$$= 2 \left(\frac{y^2}{2} - 2 \frac{y^3}{3} + \frac{y^4}{4} \right)_0^1$$

$$= \frac{1}{6}$$

5.3.1 Aliter

Let us integrate with respect to y first and then with respect to x.

Then $I = \int_R \left[\int xy \, dy \right] dx$

As explained above, to find the limits for y, we draw a line parallal to the y-axis (x = constant) in the region of integration and note the variation of y on this line

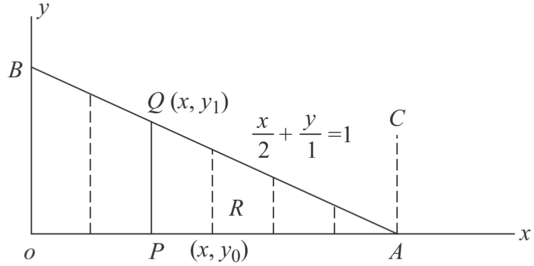


Fig. 5.6

$P(x, y_0)$ lies on the x-axis. $\therefore\ y_0 = 0$

$Q(x, y_1)$ lies on the line $x + 2y = 2$. $\therefore\ y_1 = \frac{1}{2}(2-x)$

i.e., the limits for y are 0 and $\frac{1}{2}(2-x)$.

To cover the region of integration OAB, the line PQ has to vary from OB ($x = 0$) to AC ($x = 2$)

$\therefore$ The limits for x are 0 and 2.

$$\therefore \qquad I = \int_0^2 \int_0^{\frac{1}{2}(2-x)} xy\,dy\,dx$$

$$= \int_0^2 x \left(\frac{y^2}{2}\right)_0^{\frac{1}{2}(2-x)} dx$$

$$= \frac{1}{8}\int_0^2 x(2-x)^2\,dx$$

$$= \frac{1}{8}\left(4\frac{x^2}{2} - 4\frac{x^3}{3} + \frac{x^4}{4}\right)_0^2$$

$$= \frac{1}{6}$$

Example 5.8 Evaluate $\iint_R \frac{e^{-y}}{y}\,dx\,dy$, by choosing the order of integration suitably, given that R is the region bounded by the lines $x = 0$, $x = y$ and $y = \infty$.

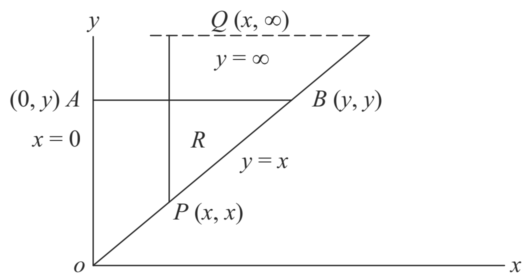


Fig. 5.7

Let $$I = \iint_R \frac{e^{-y}}{y}\,dx\,dy$$

Suppose we wish to integrate with respect to y first.

Then $$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

We note that the choice of order of integration is wrong, as the inner integration cannot be performed. Hence we try to integrate with respect to x first.

Then
$$\mathrm{I} = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^\infty \frac{e^{-y}}{y}(x)_0^y\,\mathrm{d}y$$
$$= \int_0^\infty e^{-y}\,\mathrm{d}y$$
$$= \left(e^{-y}\right)_\infty^0 = 1$$

Note ☑ From this example, we note that the choice of order of integration sometimes depends on the function to be integrated.

Example 5.9 Evaluate $\iint_R xy\,\mathrm{d}x\,\mathrm{d}y$, where R is the region bounded by the parabola $y^2 = x$ and the lines $y = 0$ and $x + y = 2$, lying in the first quadrant.

R is the region $OABCDE$ shown in Fig. 5.8.

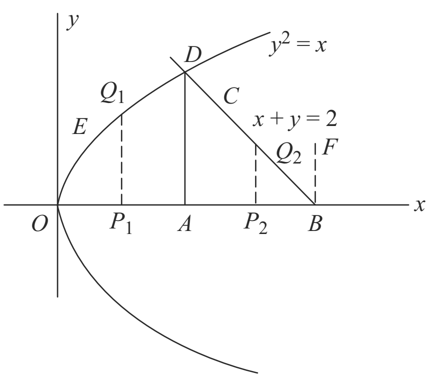


Fig. 5.8

Suppose we wish to integrate with respect to y first. Then we will draw an arbitrary line parallel to y-axis (x = constant). We note that such a line does not intersect the region of integration in the same fashion throughout.

If the line is drawn in the region $OADE$, the upper end of the line will lie on the parabola $y^2 = x$; on the other hand, if it is drawn in the region $ABCD$, the upper end of the line will lie on the line $x + y = 2$.

Hence in order to cover the entire region R, it should be divided into two, namely, $OADE$ and $ABCD$ and the line $P_1 Q_1$ should move from the y-axis to AD and the line $P_2 Q_2$ should move from AD to BF.

Accordingly, the given integral I is given by

$$\mathrm{I} = \int_0^1 \int_0^{\sqrt{x}} xy\,\mathrm{d}y\,\mathrm{d}x + \int_1^2 \int_0^{2-x} xy\,\mathrm{d}y\,\mathrm{d}x$$

[$\because$ the co-ordinates of D are (1, 1) and so the equation of AD is $x = 1$]

$$\text{I} = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

Note ☑ This approach results in splitting the double integral into two and evaluating two double integrals. On the other hand, had we integrated with respect to x first, the problem would have been solved in a simpler way as indicated below. [Refer to Fig. 5.9]

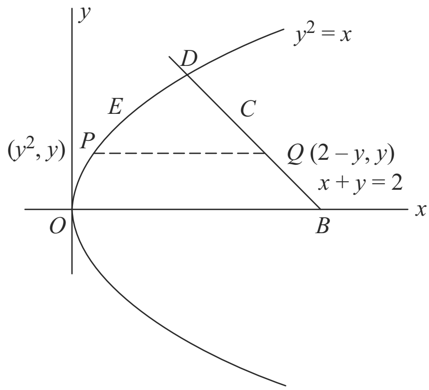


Fig. 5.9

$$\begin{aligned}
\text{I} &= \int_0^1 \int_{y^2}^{2-y} xy\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 y\{(2-y)^2 - y^4\}\,\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 (4y - 4y^2 + y^3 - y^5)\,\mathrm{d}y \\
&= \frac{3}{8}
\end{aligned}$$

Note ☑ From this example, we note that the choice of order of integration is to be made by considering the region of integration so as to simplify the evaluation.

Example 5.10 Evaluate $\iiint_V (x + y + z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the volume of the rectangular parallelopiped bounded by $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$ and $z = c$.

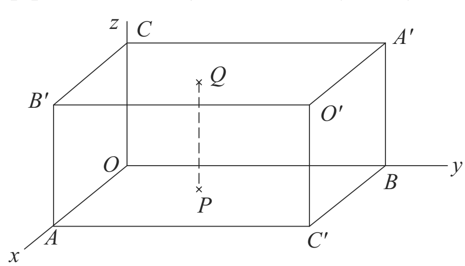


Fig. 5.10

The region of integration is the volume of the parallelopiped shown in Fig. 5.10, in which OA = a, OB = b, OC = c. Since the limits of the variables of integration are not given, we can choose the order of integration arbitrarily.

Let us take the given integral I as

$$I = \iiint_V (x + y + z)\, dz\, dy\, dx$$

The innermost integration is to be done with respect to z, treating x and y as constants.

Geometrically, x = constant and y = constant jointly represent a line parallel to the z-axis.

Hence we draw an arbitrary line PQ in the region of integration and we note the variation of z on this line so as to cover the entire volume. In this problem, z varies from 0 to c. since $P \equiv (x, y, 0)$ and $Q \equiv (x, y, c)$

Having performed the innermost integration with respect to z between the limits 0 and c, we get a double integral.

As P take all positions inside the rectangle OAC′B in the xy-plane, the line PQ covers the entire voulme of the parallelopiped. Hence, the double integral got after the innermost integration is to be evaluated over the plane region OAC′B.

The limits for the double integral can be easily seen to be 0 and b (for y) and 0 and a (for x).

$$\therefore \qquad I = \int_0^a \int_0^b \int_0^c (x + y + z)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^b \left\{ (x + y)z + \frac{z^2}{2} \right\}_{z=0}^{c} dy\, dx$$

$$= \int_0^a \int_0^b \left\{ c\,(x + y) + \frac{c^2}{2} \right\} dy\, dx$$

$$= \int_0^a \left\{ \left(cx + \frac{c^2}{2} \right) y + c\frac{y^2}{2} \right\}_0^b dx$$

$$= \int_0^a \left(bcx + \frac{bc^2}{2} + \frac{b^2c}{2} \right) dx$$

$$= \left[bc\frac{x^2}{2} + \frac{bc}{2}(b + c)x \right]_0^a$$

$$= \frac{abc}{2}(a + b + c)$$

Example 5.11 Evaluate $\iiint_V dx\, dy\, dz$, where V is the finite region of space (terra- hedron) formed by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 3y + 4z = 12$.

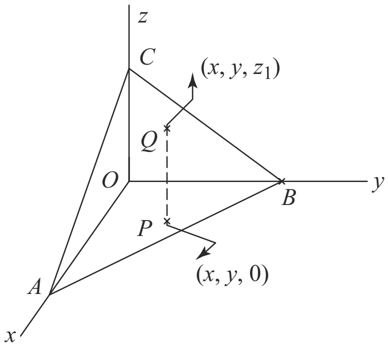


Fig. 5.11

Let I = the given integral.

Let $\mathrm{I} = \iiint_V \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$

The limits for z, the variable of the innermost integral, are 0 and z_1, where (x, y, z_1) lies on the plane $2x + 3y + 4z = 12$. [Refer to Fig. 5.11]

$$\therefore \qquad z_1 = \frac{1}{4}(12 - 2x - 3y)$$

After performing the innermost integration, the resulting double integral is evaluated over the orthogonal projection of the plane ABC on the xy-plane, i.e. over the triangular region OAB in the xy-plane as shown in Fig.5. 12.

In the double integral, the limits for y are 0 and $\frac{1}{3}(12 - 2x)$ and those for x are 0 and 6.

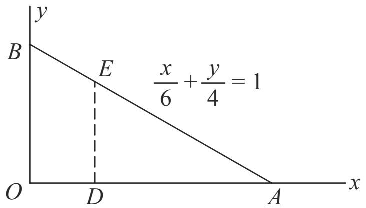


Fig. 5.12

$$\therefore \quad \mathrm{I} = \int_0^6 \mathrm{d}x \int_0^{\frac{1}{3}(12-2x)} \mathrm{d}y \int_0^{\frac{1}{4}(12-2x-3y)} \mathrm{d}z$$

$$= \frac{1}{4}\int_0^6 \mathrm{d}x \int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y)\, \mathrm{d}y$$

$$= \frac{1}{4}\int_0^6 \mathrm{d}x \left[(12 - 2x)y - \frac{3y^2}{2} \right]_{y=0}^{y=\frac{1}{3}(12-2x)}$$

$$= \frac{1}{24}\int_0^6 (12 - 2x)^2\, \mathrm{d}x$$

$$= \frac{1}{6}\left\{ \frac{(6 - x)^3}{-3} \right\}_0^6$$

$$= 12$$

Example 5.12 Evaluate $\iiint\limits_V \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$, where V is the region of space bounded by the co-ordinate planes and the sphere $x^2 + y^2 + z^2 = 1$ and contained in the positive octant.

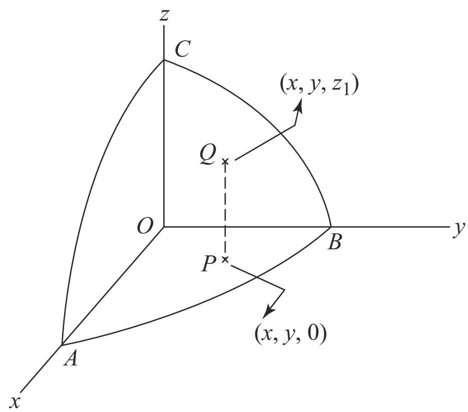


Fig. 5.13

Note ☑ In two dimensions, the x and y-axes divide the entire xy-plane into 4 quadrants. The quadrant containing the positive x and the positive y-axes is called the positive quadrant.

Similarly in three dimensions the xy, yz and zx-planes divide the entire space into 8 parts, called octants. The octant containing the positive x, y and z-axes is called the positive octant.

The region of space V given in this problem is shown in Fig. 5.13.

Let $$I = \iiint\limits_V \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$$

To find the limits for z, we draw a line PQ parallel to the z-axis cutting the voulme of integration.

The limits for z and 0 and z_1, where (x, y, z_1) lies on the sphere $x^2 + y^2 + z^2 = 1$

$\therefore$ $$z_1 = \sqrt{1-x^2-y^2} \quad (\because \text{ the point } Q \text{ lies in the positive octant})$$

After performing the innermost integration, the resulting double integral is evaluated over the orthogonal projection of the spherical surface on the xy-plane, i.e. over the circular region lying in the positive quadrant as shown in Fig. 5.14.

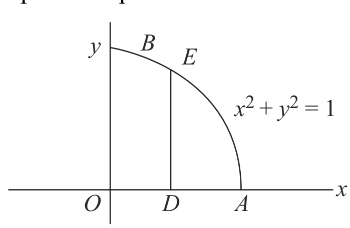


Fig. 5.14

In the double integral, the limits for y are 0 and $\sqrt{1-x^2}$ and those for x are 0 and 1.

$\therefore$

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{(1-x^2-y^2)-z^2}}$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \left(\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}}\right)_{z=0}^{z=\sqrt{1-x^2-y^2}}$$

$$= \frac{\pi}{2}\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right)_0^1$$

$$= \frac{\pi}{2}\times\frac{\pi}{4} = \frac{\pi^2}{8}$$

EXERCISE 5(a)

Part A

(Short Answer Questions)

1. Evaluate $\int_0^2 \int_0^1 4xy \, dx \, dy$
2. Evaluate $\int_1^b \int_1^a \frac{dx\,dy}{xy}$
3. Evaluate $\int_0^{\pi/2} \int_0^{\pi/2} \sin(\theta+\phi)\, d\theta\, d\phi$
4. Evaluate $\int_0^1 \int_0^x dx\, dy$.
5. Evaluate $\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta$
6. Evaluate $\int_0^1 \int_0^2 \int_0^3 xyz\, dx\, dy\, dz$
7. Evaluate $\int_0^1 \int_0^z \int_0^{y+z} dz\, dy\, dx$

Sketch roughly the region of integration for the following double integrals:

8. $\int_{-b}^{b}\int_{-a}^{a} f(x, y)\,dx\,dy.$

9. $\int_{0}^{1}\int_{0}^{x} f(x, y)\,dx\,dy.$

10. $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} f(x, y)\,dx\,dy.$

11. $\int_{0}^{b}\int_{0}^{\frac{a}{b}(b-y)} f(x, y)\,dx\,dy.$

Find the limits of integration in the double integral $\iint_R f(x, y)\,dx\,dy$, where R is in the first quadrant and bounded by

12. $x = 0, y = 0, x + y = 1.$
13. $x = 0,\ y = 0,\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
14. $x = 0, x = y, y = 1$
15. $x = 1, y = 0, y^2 = 4x$

Part B

16. Evaluate $\int_{0}^{4}\int_{y^2/4}^{y} \dfrac{y\,dx\,dy}{x^2 + y^2}$ and also sketch the region of integration roughly.

17. Evaluate $\int_{0}^{a}\int_{a-x}^{\sqrt{a^2-x^2}} y\,dx\,dy$ and also sketch the region of integration roughly.

18. Evaluate $\int_{0}^{1}\int_{x}^{1} \dfrac{y\,dx\,dy}{x^2 + y^2}$ and also sketch the region of integration roughly.

19. Evaluate $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dx\,dy.$

20. Evaluate $\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-y} xyz\,dx\,dy\,dz.$

21. Evaluate $\int_{0}^{\log 2}\int_{0}^{x}\int_{0}^{x+\log y} e^{x+y+z}\,dz\,dy\,dx.$

22. Evaluate $\iint x e^{-\frac{x^2}{y}}\,\mathrm{d}x\,\mathrm{d}y$, over the region bounded by $x = 0$, $x = \infty$, $y = 0$ and $y = x$.
23. Evaluate $\iint xy\,\mathrm{d}x\,\mathrm{d}y$, over the region in the positive quadrant bounded by the line $2x + 3y = 6$.
24. Evaluate $\iint x\,\mathrm{d}x\,\mathrm{d}y$, over the region in the positive quadrant bounded by the circle $x^2 - 2ax + y^2 = 0$.
25. Evaluate $\iint (x+y)\,\mathrm{d}x\,\mathrm{d}y$, over the region in the positive quadrant bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
26. Evaluate $\iint (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, over the area bounded by the parabola $y^2 = 4x$ and its latus rectum.
27. Evaluate $\iint_R x^2\,\mathrm{d}x\,\mathrm{d}y$, where R is the region bounded by the hyerbola $xy = 4$, $y = 0$, $x = 1$ and $x = 2$.
28. Evaluate $\iiint_V (xy+yz+zx)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space bounded by $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$ and $z = 3$.
29. Evaluate $\iiint_V \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(x+y+z+1)^3}$, where V is the region of space bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.
30. Evaluate $\iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space bounded by the co-ordinate planes and the sphere $x^2 + y^2 + z^2 = 1$ and contained in the positive octant.

5.4 CHANGE OF ORDER OF INTEGRATION IN A DOUBLE INTEGRAL

In worked example (1) of the previous section, we have observed that if the limits of integration in a double integral are constants, then the order of integration can be changed, provided the relevant limits are taken for the concerned variables.

But when the limits for inner integration are functions of a variable, the change in the order of integration will result in changes in the limits of integration.

i.e. the double integral $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ will take the form $\int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$, when the order of integration is changed. This process of converting a given double integral into its equivalent double integral by changing the order of integration is often called *change of order of integration*. To effect the change of order of integration, the region of integration is identified first, a rough sketch of the region is drawn and then the new limits are fixed, as illustrated in the following worked examples.

5.5 PLANE AREA AS DOUBLE INTEGRAL

Plane area enclosed by one or more curves can be expressed as a double integral both in Cartesian coordinates and in polar coordinates. The formulas for plane areas in both the systems are derived below:

(*i*) *Cartesian System*

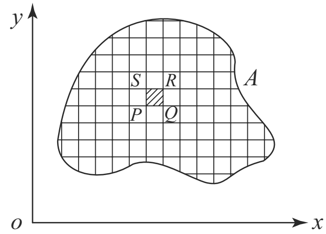


Fig. 5.15

Let R be the plane region, the area of which is required. Let us divide the area into a large number of elemental areas like $PQRS$ (shaded) by drawing lines parallel to the y-axis at intervals of Δx and lines parallel to the x-axis at intervals of Δy (Fig. 5.15).

Area of the elemental rectangle $PQRS = \Delta x.\Delta y$. Required area A of the region R is the sum of elemental areas like $PQRS$.

viz.,
$$A = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Sigma\Sigma\, \Delta x\, \Delta y)$$
$$= \iint_R \mathrm{d}x\, \mathrm{d}y$$

(*ii*) *Polar System*

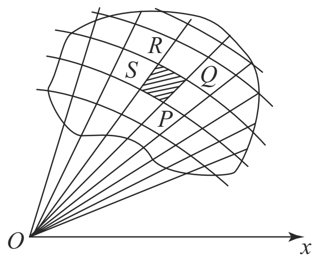


Fig. 5.16

We divide the area A of the given region R into a large number of elemental curvilinear rectangular areas like $PQRS$ (shaded) by drawing radial lines and concentric circular arcs, where P and R have polar coordinates (r, θ) and $(r + \Delta r, \theta + \Delta\theta)$ (Fig. 5.16)

Area of the element $PQRS = r\, \Delta r\, \Delta\theta$
($\because$ $PS = r\, \Delta\theta$ and $PQ = \Delta r$)

$\therefore$ Required area
$$A = \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} (\Sigma\Sigma\, r\, \Delta r\, \Delta\theta)$$
$$= \iint_R r\, \mathrm{d}r\, \mathrm{d}\theta.$$

5.5.1 Change of Variables

(*i*) *From Cartesian Coordinates to Plane Polar Coordinates*

If the transformations $x = x(u, v)$ and $y = y(u, v)$ are made in the double integral $\iint f(x, y)\, \mathrm{d}x\, \mathrm{d}y$, then $f(x, y) \equiv g(u, v)$ and $\mathrm{d}x\, \mathrm{d}y = |J|\, \mathrm{d}u\, \mathrm{d}v$, where $J = \dfrac{\partial(x, y)}{\partial(u, v)}$.

[Refer to properties of Jacobians in the Chapter 4, “Functions of Several Variables” in Part I] .

When we transform from cartesian system to plane polar system,

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

In this case, $$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Hence $\displaystyle\iint_R f(x, y)\, dx\, dy = \iint_R g(r, \theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta$

In particular,

Area A of the plane region R is given by

$$A = \iint_R \mathrm{d}x\,\mathrm{d}y = \iint_R r\,\mathrm{d}r\,\mathrm{d}\theta$$

(*ii*) *From Three Dimensional Cartesians to Cylindrical Coordinates*

Let us first define cylindrical coordinates of a point in space and derive the relations between cartesian and cylindrical coordinates (Fig. 5.17).

Let P be the point (x, y, z) in Cartesian coordinate system. Let PM be drawn $\perp r$ to the xoy-plane and MN parallel to Oy. Let $\underline{|NOM} = \theta$ and $OM = r$. The triplet (r, θ, z) are called the cylindrical coordinates of P.

Clearly, $ON = x = r\cos\theta$; $NM = y = r\sin\theta$ and $MP = z$.

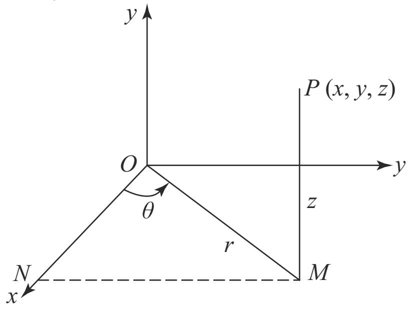


Fig. 5.17

Thus the transformations from three dimensional cartesians to cylindrical coordinates are $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

In this case,

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r$$

Hence $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

and $\displaystyle\iiint_V f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_V g(r, \theta, z)\, r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

In particular, the volume of a region of space V is given by

$$\iiint_V \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_V r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z$$

Note ☑ Whenever $\iiint f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ is to be evaluated throughout the volume of a right circular cylinder, it will be advantageous to evaluate the corresponding triple integral in cylindrical coordinates.

(*iii*) *From Three Dimensional Cartesians to Spherical Polar Coordinates*

Let us first define spherical polar coordinates of a point in space and derive the relations between Cartesian and spherical polar coordinates (Fig. 5.18).

Let P be the point whose Cartesian coordinates are (x, y, z). Let PM be drawn $\perp$ r to the xOy-plane. Let MN be parallel to y-axis. Let $OP = r$, the angle made by OP with the positive z-axis $= \theta$ and the angle made by OM with x-axis $= \phi$.

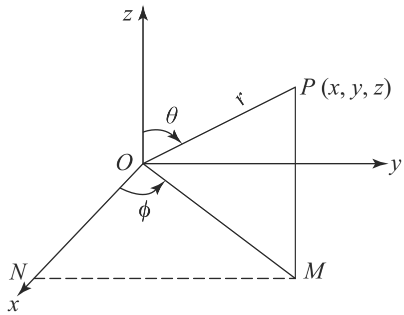


Fig. 5.18

The triplet (r, θ, ϕ) are called the spherical polar coordinates of P.

Since $\underline{|OMP} = 90°$, $MP = z = r \cos \theta$, $OM = r \sin \theta$, $ON = x = r \sin \theta \cos \phi$ and $NM = y = r \sin \theta \sin \phi$.

Thus the transformations from three dimensional cartesians to spherical polar coordinates are

$$x = r \sin \theta \cos \phi,\ y = r \sin \theta \sin \phi,\ z = r \cos \theta$$

In this case,

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$$

[Refer to example (4.8) of Worked example set 4(b) in the Chapter 4 "Functions of Several Variables." in Part I]

Hence $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = r^2 \sin\theta\, \mathrm{d}r\, d\theta\, \mathrm{d}\phi$ and $\iiint_V f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_V g(r, \theta, \phi)\, r^2$ $\sin \theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$.

In particular, the volume of a region of space V is given by

$$\iiint_V \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_V r^2 \sin \theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi .$$

Note ☑ Whenever $\iiint f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ is to be evaluated throughout the volume of a sphere, hemisphere or octant of a sphere, it will be advantageous to use spherical polar coordinates.)

WORKED EXAMPLE 5(b)

Example 5.1 Change the order of integration in $\int_0^a \int_y^a \frac{x}{\sqrt{x^2 + y^2}}\, dx\, dy$ and then evaluate it.

The region of integration R is defined by $y \le x \le a$ and $0 \le y \le a$.
i.e. it is bounded by the lines $x = y$, $x = a$, $y = 0$ and $y = a$.
The rough sketch of the boundaries and the region R is given in Fig. 5.19.

After changing the order of integration, the given integral I becomes

$$I = \iint_R \frac{x}{\sqrt{x^2 + y^2}}\, dy\, dx$$

The limits of inner integration are found by treating x as a constant, i.e. by drawing a line parallel to the y-axis in the region of integration as explained in the previous section.

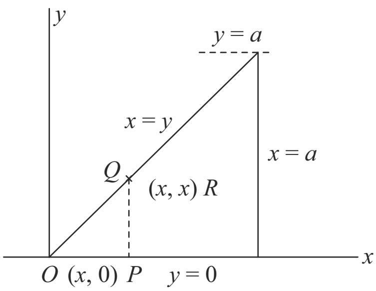


Fig. 5.19

Thus
$$I = \int_0^a \int_0^x \frac{x}{\sqrt{x^2 + y^2}}\, dy\, dx$$

$$= \int_0^a x \left\{ \log \left(y + \sqrt{y^2 + x^2} \right) \right\}_{y=0}^{y=x} dx$$

$$= \int_0^a x \left[\log (x + x\sqrt{2}) - \log x \right] dx$$

$$= \log (1 + \sqrt{2}) \cdot \left(\frac{x^2}{2} \right)_0^a = \frac{a^2}{2} \log \; (1 + \sqrt{2})$$

Example 5.2 Change the order of integration in $\int_0^1 \int_x^1 \frac{x\, dx\, dy}{x^2 + y^2}$ and then evaluate it.

Note ☑ Since the limits of inner integration are x and 1, the corresponding variable of integration should be y. So we rewrite the given integral I in the corrected form first.

$$I = \int_0^1 \int_x^1 \frac{x\, dy\, dx}{x^2 + y^2}$$

The region of integration R is bounded by the lines $x = 0$, $x = 1$, $y = x$ and $y = 1$ and is given in Fig. 5.20.

The limits for the inner integration (after changing the order of integration) with respect to x are fixed as usual, by drawing a line parallel to x-axis (y = constant)

$$\therefore \qquad I = \int_0^1 \int_0^y \frac{x\, dx\, dy}{x^2 + y^2}$$

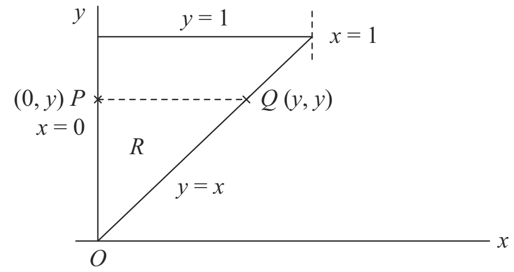


Fig. 5.20

$$= \int_0^1 \left[\frac{1}{2} \log \ (x^2 + y^2) \right]_{x=0}^{x=y} \mathrm{d}y$$

$$= \frac{1}{2} \int_0^1 \log \left(\frac{2y^2}{y^2} \right) \mathrm{d}y$$

$$= \frac{1}{2} \log 2.$$

Example 5.3 Change the order of integration in $\int_0^b \int_0^{\frac{a}{b}(b-y)} xy \, \mathrm{d}x \, \mathrm{d}y$ and then evaluate it.

The region of integration R is bounded by the lines $x = 0$, $x = \frac{a}{b}(b-y)$ or $\frac{x}{a} + \frac{y}{b} = 1$, $y = 0$ and $y = b$ and is shown in Fig. 5.21.

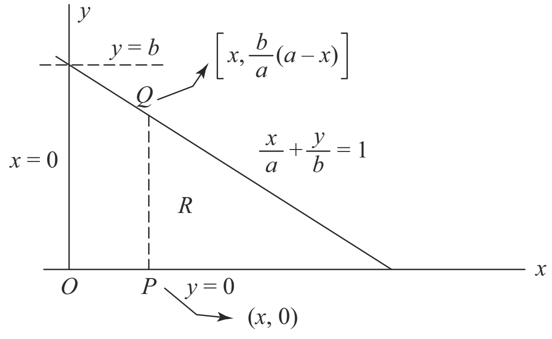


Fig. 5.21

After changing the order of integration, the integral becomes $\mathrm{I} = \iint_R xy \, \mathrm{d}y \, \mathrm{d}x$. The limits are fixed as usual.

$$\mathrm{I} = \int_0^a \int_0^{\frac{b}{a}(a-x)} xy \, \mathrm{d}y \, \mathrm{d}x$$

$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{\frac{b}{a}(a-x)} dx$$

$$= \frac{b^2}{2a^2}\int_0^a x\,(a-x)^2\,dx$$

$$= \frac{b^2}{2a^2}\left[a^2\frac{x^2}{2} - 2a\frac{x^3}{3} + \frac{x^4}{4}\right]_0^a$$

$$= \frac{a^2b^2}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{a^2b^2}{24}$$

Example 5.4 Change the order of integration in $\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\,dy\,dx$ and then integrate it.

The region of integration R is bounded by the lines $x = 0$, $x = a$, $y = 0$ and the curve $y = \frac{b}{a}\sqrt{a^2 - x^2}$ i.e. the curve $\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$, i.e. the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and is shown in Fig. 5.22.

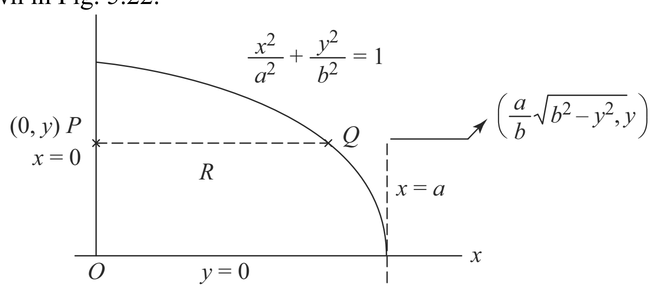


Fig. 5.22

After changing the order of integration, the integral becomes

$$\text{I} = \iint_R x^2\,dx\,dy$$

The limits are fixed as usual.

$$\text{I} = \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} x^2\,dx\,dy$$

$$= \int_0^b \left(\frac{x^3}{3}\right)_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \frac{a^3}{3b^3} \int_0^b (b^2 - y^2)^{\frac{3}{2}}\, \mathrm{d}y$$

$$= \frac{a^3}{3b^3} \int_0^{\pi/2} b^4 \cos^4 \theta\, \mathrm{d}\theta \qquad \text{(on putting } y = b \sin \theta\text{)}$$

$$= \frac{a^3 b}{3} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{16} a^3 b$$

Example 5.5 Change the order of integration in $\displaystyle\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\, \mathrm{d}x\, \mathrm{d}y$ and then evaluate it.

The region of integration R is bounded by the line $x = a - y$, the curve $x = \sqrt{a^2 - y^2}$, the lines $y = 0$ and $y = a$.
i.e. the line $x + y = a$, the circle $x^2 + y^2 = a^2$ and the lines $y = 0$, $y = a$. R is shown in Fig. 5.23.

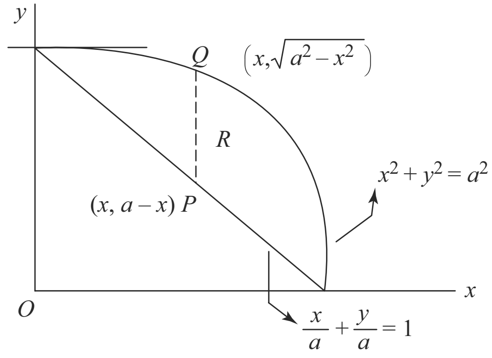


Fig. 5.23

After changing the order of integration, the integral I becomes,

$$\mathrm{I} = \iint_R y\, \mathrm{d}y\, \mathrm{d}x$$

$$= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, \mathrm{d}y\, \mathrm{d}x$$

$$= \int_0^a \left(\frac{y^2}{2} \right)_{a-x}^{\sqrt{a^2-x^2}} \mathrm{d}x$$

$$= \frac{1}{2} \int_0^a (2ax - 2x^2)\, \mathrm{d}x$$

$$= \left(a\frac{x^2}{2} - \frac{x^3}{3}\right)_0^a$$

$$= \frac{a^3}{6}.$$

Example 5.6 Change the order of integration in $\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} \mathrm{d}y\,\mathrm{d}x$ and then evaluate it.

The region of integration R is bounded by the curve $y = \frac{x^2}{4}$ i.e. the parabola $x^2 = 4y$, the curve $y = 2\sqrt{x}$ i.e. the parabola $y^2 = 4x$ and the lines $x = 0$, $x = 4$. R is shown in Fig. 5.24.

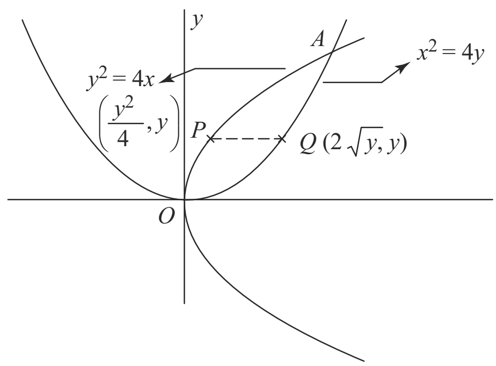


Fig. 5.24

The points of intersection of the two parabolas are obtained by solving the equations $x^2 = 4y$ and $y^2 = 4x$.

Solving them, we get $\left(\frac{x^2}{4}\right)^2 = 4x$

i.e $\quad x\,(x^3 - 64) = 0$

$\therefore \quad x = 0, \quad x = 4$

and $\quad y = 0, \quad y = 4$

i.e. the points of intersection are $O(0, 0)$ and $A(4, 4)$.

After changing the order of integration, the given integral

$$I = \iint_R \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^4 \left(2\sqrt{y} - \frac{y^2}{4}\right)\mathrm{d}y$$

$$= \left(\frac{4}{3}y^{\frac{3}{2}} - \frac{y^3}{12}\right)_0^4$$

$$= \frac{32}{3} - \frac{16}{3}$$

$$= \frac{16}{3}$$

Example 5.7 Change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2+y^2}} xy\,\mathrm{d}x\,\mathrm{d}y$ and then evaluate it.

The region of integration R is bounded by the curve $x = a \mp \sqrt{a^2 - y^2}$, i.e. the circle $(x - a)^2 + y^2 = a^2$ and the lines $y = 0$ and $y = a$. The region R is shown in Fig. 5.25.

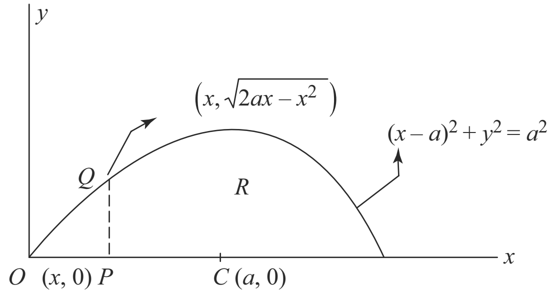


Fig. 5.25

After changing the order of integration, the integral I becomes

$$\mathrm{I} = \iint_R xy\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xy\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^{2a} x\left(\frac{y^2}{2}\right)_0^{\sqrt{2ax-x^2}} \mathrm{d}x$$

$$= \frac{1}{2}\int_0^{2a} (2ax^2 - x^3)\mathrm{d}x$$

$$= \frac{1}{2}\left(2a\frac{x^3}{3} - \frac{x^4}{4}\right)_0^{2a}$$

$$= \frac{2}{3}a^4 \cdot$$

Example 5.8 Change the order of integration in $\int_0^1 \int_y^{2-y} xy \,\mathrm{d}x\,\mathrm{d}y$ and then evaluate it.

The region of integration R is bounded by the lines $x = y$, $x + y = 2$, $y = 0$ and $y = 1$. It is shown in Fig. 5.26.

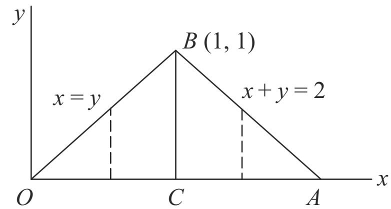


Fig. 5.26

After changing the order of integration, the integral I becomes

$$\mathrm{I} = \iint_R xy \,\mathrm{d}y\,\mathrm{d}x$$

To fix the limits for y in the inner integration, we have to draw a line parallel to y-axis (since x = constant). The line drawn parallel to the y-axis does not intersect the region R in the same fashion. If the line segment is drawn in the region OCB, its upper end lies on the line $y = x$; on the other hand, if it is drawn in the region BCA, its upper end lies on the line $x + y = 2$. In such situations, we divide the region into two sub-regions and fix the limits for each sub-region as illustrated below:

$$\mathrm{I} = \iint_{\Delta OCB} xy \,\mathrm{d}y\,\mathrm{d}x + \iint_{\Delta BCA} xy \,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \int_0^x xy \,\mathrm{d}y\,\mathrm{d}x + \int_1^2 \int_0^{2-x} xy \,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^x \mathrm{d}x + \int_1^2 x\left(\frac{y^2}{2}\right)_0^{2-x} \mathrm{d}x$$

$$= \int_0^1 \frac{x^3}{2}\,\mathrm{d}x + \int_1^2 \frac{x}{2}(2-x)^2\,\mathrm{d}x$$

$$= \left(\frac{x^4}{8}\right)_0^1 + \frac{1}{2}\left(2x^2 - \frac{4}{3}x^3 + \frac{x^4}{4}\right)_1^2$$

$$= \frac{1}{8} + \frac{5}{24}$$

$$= \frac{1}{3}$$

Example 5.9 Change the order of integration in $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy \,\mathrm{d}y\,\mathrm{d}x$ and then evaluate it.

The region of integration R is bounded by the curve $y = \dfrac{x^2}{a}$, i.e. the parabola $x^2 = ay$, the line $y = 2a - x$, i.e. $x + y = 2a$ and the lines $x = 0$ and $x = a$. It is shown in Fig. 5.27.

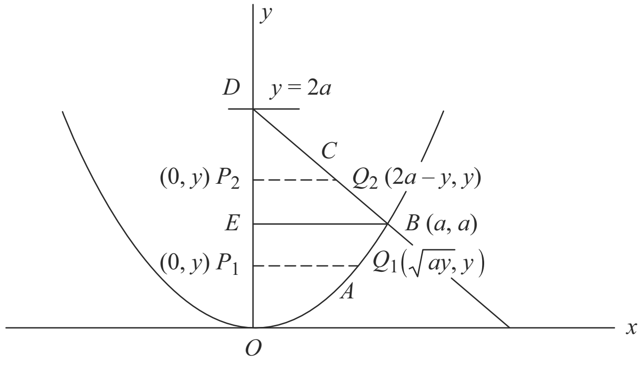


Fig. 5.27

After changing the order of integration, the integral I becomes

$$\mathrm{I} = \iint_R xy \, \mathrm{d}x \, \mathrm{d}y$$

When we draw a line parallel to x-axis for fixing the limits for the inner integration with respect to x, it does not intersect the region of integration in the same fashion. Hence the region R is divided into two sub-regions $OABE$ and $EBCD$ and then the limits are fixed as given below:

$$\mathrm{I} = \iint_{OABE} xy \, \mathrm{d}x \, \mathrm{d}y + \iint_{EBCD} xy \, \mathrm{d}x \, \mathrm{d}y$$

$$= \int_0^a \int_0^{\sqrt{ay}} xy \, \mathrm{d}x \, \mathrm{d}y + \int_a^{2a} \int_0^{2a-y} xy \, \mathrm{d}x \, \mathrm{d}y$$

Note ☑ The co-ordinates of the point B are obtained by solving the equations $x + y = 2a$ and $x^2 = ay$.

$B \equiv (a, a)$ and the equation of EB is $y = a$.

$$\mathrm{I} = \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} \mathrm{d}y + \int_a^{2a} y\left(\frac{x^2}{2}\right)_0^{2a-y} \mathrm{d}y$$

$$= \frac{1}{2}\left[\int_0^a ay^2 \, \mathrm{d}y + \int_a^{2a} y\,(2a-y)^2 \, \mathrm{d}y\right]$$

$$= \frac{1}{2}\left[a\left(\frac{y^3}{3}\right)_0^a + \left(2a^2\, y^2 - \frac{4a}{3}\, y^3 + \frac{y^4}{4}\right)_a^{2a}\right] = \frac{3}{8}a^4.$$

Example 5.10 Change the order of integration in each of the double integrals $\int_0^1 \int_1^2 \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$ and $\int_1^2 \int_y^2 \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2}$ and hence express their sum as one double integral and evaluate it.

The region of integration R_1 for the first double integral I_1 is bounded by the lines $x = 1$, $x = 2$, $y = 0$ and $y = 1$.

The region of integration R_2 for the second double integral I_2 is bounded by the lines $x = y$, $x = 2$, $y = 1$ and $y = 2$.

R_1 and R_2 are shown in Fig. 5.28.

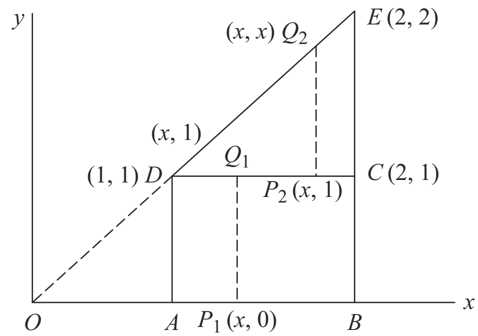


Fig. 5.28

After changing the order of integration,

$$I_1 = \int_1^2 \int_0^1 \frac{dy\,dx}{x^2 + y^2}$$

and
$$I_2 = \int_1^2 \int_1^x \frac{dy\,dx}{x^2 + y^2}$$

Adding the integrals I_1 and I_2, we get

$$I = \int_1^2 dx \left(\int_0^1 \frac{dy}{x^2 + y^2} + \int_1^x \frac{dy}{x^2 + y^2} \right)$$

$$= \int_1^2 dx \int_0^x \frac{dy}{x^2 + y^2}$$

$$= \int_1^2 \left(\frac{1}{x} \tan^{-1} \frac{y}{x} \right)_{y=0}^{y=x} dx$$

$$= \int_1^2 \frac{\pi}{4} \frac{dx}{x} = \frac{\pi}{4} \log 2 .$$

Example 5.11 Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = x$ by double integration.

The region, the area of which is required is bounded by the parabolas $(y - 0)^2 = -(x - 4)$ and $y^2 = x$ and is shown in Fig. 5.29.

Required area $= \int_{OC} \int_{AB} dx\,dy$

$$= 2 \int_{o} \int_{AB} \mathrm{d}x\, \mathrm{d}y, \text{ by symmetry}$$

$$= 2 \int_{0}^{\sqrt{2}} \int_{y^2}^{4-y^2} \mathrm{d}x\, \mathrm{d}y$$

$$= 2 \int_{0}^{\sqrt{2}} (4 - y^2 - y^2)\, \mathrm{d}y$$

$$= 2 \left(4y - \frac{2}{3} y^3 \right)_0^{\sqrt{2}}$$

$$= 2 \left(4\sqrt{2} - \frac{4}{3} \sqrt{2} \right)$$

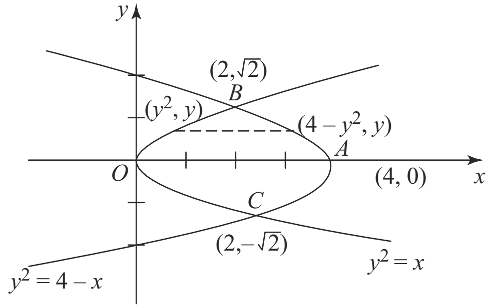


Fig. 5.29

$$= \frac{16}{3} \sqrt{2} \text{ square units}$$

Example 5.12 Find the area between the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ lying in the first quadrant, by double integration.

The plane region, the area of which is required, is shown in Fig. 5.30.

$$\text{Required area } = \iint_{ABC} \mathrm{d}x\, \mathrm{d}y$$

$$= \int_{0}^{a} \int_{a-y}^{\sqrt{a^2 - y^2}} \mathrm{d}x\, \mathrm{d}y$$

$$= \int_{0}^{a} \left(\sqrt{a^2 - y^2} - a + y \right) \mathrm{d}y$$

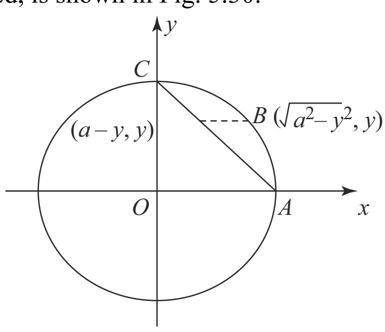


Fig. 5.30

$$= \left(\frac{y}{2} \sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} - ay + \frac{y^2}{2} \right)_0^a$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2} - a^2 + \frac{a^2}{2} = (\pi - 2) \frac{a^2}{4} \cdot$$

Example 5.13 Find the area enclosed by the lemniscate $r^2 = a^2 \cos 2\theta$, by double integration.

As the equation $r^2 = a^2 \cos 2\theta$ remains unaltered on changing θ to $-\theta$, the curve is symmetrical about the initial line.

The points of intersection of the curve with the initial line $\theta = 0$ are given by $r^2 = a^2$ or $r = \pm a$.

Since $r^2 = a^2 \cos 2\alpha = a^2 \cos 2(\pi - \alpha)$, the curve is symmetrical about the line $\theta = \dfrac{\pi}{2}$.

On putting $r = 0$, we get $\cos 2\theta = 0$. Hence $\theta = \pm\dfrac{\pi}{4}, \pm\dfrac{3\pi}{4}$. Hence there is a loop of the curve between $\theta = -\dfrac{\pi}{4}$ and $\theta = \dfrac{\pi}{4}$ and another loop between $\theta = -\dfrac{3\pi}{4}$ and $\theta = \dfrac{3\pi}{4}$.

Based on the observations given above the lemniscate is drawn in Fig. 5.31.

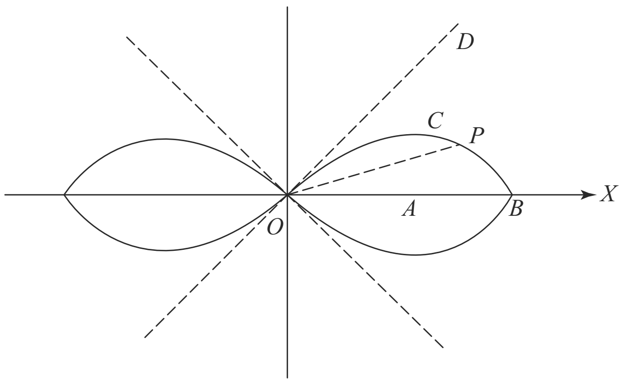


Fig. 5.31

Required area = 4 × area *OABC* (by symmetry)

$$= 4 \iint_{OAB} r\,\mathrm{d}r\,\mathrm{d}\theta$$

When we perform the inner integration with respect to r, we have to treat θ as a constant temporarily and find the limits for r.

Geometrically, treating θ = constant means drawing a line *OP* arbitrarily through the pole lying within the region of integration as shown in the figure.

Finding the limits for r (while θ is a constant) is equivalent to finding the variation of the r coordinate of any point on the line *OP*. Assuming that the θ coordinates of all points on *OP* are θ each (since θ is constant on *OP*), we take $O \equiv (0, \theta)$ and $P \equiv (r_1, \theta)$; viz., r varies from 0 to r_1. Now wherever *OP* be drawn, the point $P(r_1, \theta)$ lies on the lemniscate.

Hence $r_1^2 = a^2 \cos 2\theta$ or $r_1 = a\sqrt{\cos 2\theta}$ (since r coordinate of any point is + ve)

Thus the limits for inner integration are 0 and $a\sqrt{\cos 2\theta}$.

When we perform the outer integration, we have to find the limits for θ. Geometrically, we have to find the variation of the line *OP* so that it sweeps the area of the region, namely *OABC*. To cover this area, the line *OP* has to start from the position OA ($\theta = 0$) and move in the anticlockwise direction and go up to $OD\left(\theta = \frac{\pi}{4}\right)$. Thus the limits for θ are 0 and $\frac{\pi}{4}$.

$$\therefore \quad \text{Required area} = 4\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\int_0^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} \mathrm{d}\theta$$

$$= 2a^2\int_0^{\frac{\pi}{4}} \cos 2\theta\,\mathrm{d}\theta$$

$$= a^2\,(\sin 2\theta)_0^{\frac{\pi}{4}} = a^2$$

Example 5.14 Find the area that lies inside the cardioid $r = a\,(1 + \cos\theta)$ and outside the circle $r = a$, by double integration.

The cardioid $r = a\,(1 + \cos\theta)$ is symmetrical about the initial line. The point of intersection of the line $\theta = 0$ with the cardioid is given by $r = 2a$, viz., the point $(2a, 0)$.

Putting $r = 0$ in the equation, we get $\cos\theta = -1$ and $\theta = \pm\pi$. Hence the cardioid lies between the lines $\theta = -\pi$ and $\theta = \pi$.

The point of intersection of the line $\theta = \frac{\pi}{2}$ is $\left(a, \frac{\pi}{2}\right)$.

Noting the above properties, the cardioid is drawn as shown in Fig. 5.32. All the points on the curve $r = a$ have the same r coordinate a, viz., they are at the same distance a from the pole. Hence the equation $r = a$ represents a circle with centre at the pole and radius equal to a.

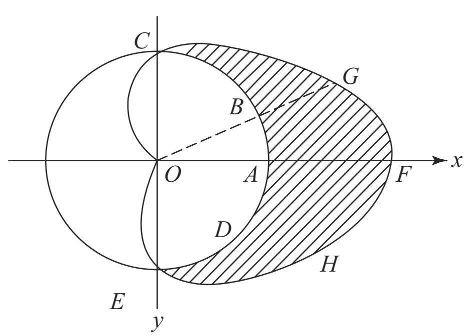


Fig. 5.32

Noting the above points, the circle $r = a$ is drawn as shown in Fig. 5.32. The area that lies outside the circle $r = a$ and inside the cardioid is shaded in the figure.

Both the curves are symmetric about the initial line. Hence the required area

$$= 2 \times AFGCB$$

$$= 2\int_0^{\frac{\pi}{2}}\int_{r_1}^{r_2} r\,dr\,d\theta$$, where (r_1, θ) lies on the circle $r = a$ and (r_2, θ) lies on the cardioid $r = a\,(1 + \cos\theta)$

$$= 2\int_0^{\frac{\pi}{2}}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}[(1+\cos\theta)^2 - 1]\,d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}\left(2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= a^2\left[2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= a^2\left(2 + \frac{\pi}{4}\right) = \frac{a^2}{4}(\pi + 8)$$

Example 5.15 Express $\int_0^a\int_y^a \frac{x^2\,dx\,dy}{\left(x^2+y^2\right)^{3/2}}$ in polar coordinates and then evaluate it.

The region of integration is bounded by the lines $x = y$, $x = a$, $y = 0$ and $y = a$, whose equations in polar system are $\theta = \frac{\pi}{4}$, $r = a\sec\theta$, $\theta = 0$ and $r = a\operatorname{cosec}\theta$ respectively. The region is shown in Fig. 5.33.

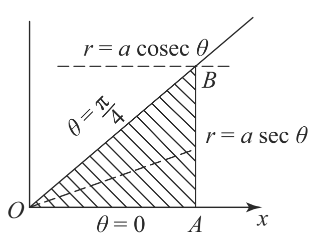


Fig. 5.33

Putting $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$ in the given double integral I, we get

$$I = \iint_{OAB} \frac{r^3\cos^2\theta}{r^3}\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \cos^2\theta\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \int_0^{\pi/4} \cos^2\theta \cdot [r]_0^{a\sec\theta}\, \mathrm{d}\theta$$

$$= a\int_0^{\pi/4} \cos\theta \mathrm{d}\theta = a\,[\sin\theta]_0^{\pi/4} = \frac{a}{\sqrt{2}}$$

Example 5.16 Transform the double integral $\displaystyle\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{\mathrm{d}x\, \mathrm{d}y}{\sqrt{a^2-x^2-y^2}}$ in polar coordinates and then evaluate it.

The region of integration is bounded by the curves $y=\sqrt{ax-x^2}$, $y=\sqrt{a^2-x^2}$ and the lines $x = 0$ and $x = a$.

$$y=\sqrt{ax-x^2} \text{ is the curve } x^2 + y^2 - ax = 0$$

i.e.,

$$\left(x-\frac{a}{2}\right)^2 + (y-0)^2 = \left(\frac{a}{2}\right)^2$$

i.e. the circle with centre at $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$

$$y=\sqrt{a^2-x^2} \text{ is the curve } x^2 + y^2 = a^2$$

i.e. the circle with centre at the origin and radius a.

The polar equations of the boundaries of the region of integration are $r^2 - ar\cos\theta = 0$ or $r = a\cos\theta$, $r = a$, $r = a\sec\theta$ and $\theta=\frac{\pi}{2}$. The region of integration is shown in Fig. 5.34.

Putting $x = r\cos\theta$, $y = r\sin\theta$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ in the given double integral I, we get

$$I = \int_0^{\pi/2} \int_{a\cos\theta}^{a} \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{a^2-r^2}}$$

$$= \int_0^{\pi/2} \left\{-\frac{1}{2}\times 2\sqrt{a^2-r^2}\right\}_{a\cos\theta}^{a} \mathrm{d}\theta \text{, on putting } a^2 - r^2 = t$$

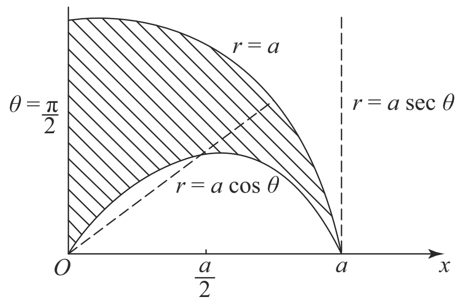


Fig. 5.34

$$= \int_0^{\pi/2} a \sin \theta \, d\theta = -a \left[\cos \theta\right]_0^{\frac{\pi}{2}} = a$$

Example 5.17 By transforming into cylindrical coordinates, evaluate the integral $\iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz$ taken over the region of space defined by $x^2 + y^2 \le 1$ and $0 \le z \le 1$.

The region of space is the region enclosed by the cylinder $x^2 + y^2 = 1$ whose base radius is 1 and axis is the z-axis and the planes $z = 0$ and $z = 1$. The equation of the cylinder in cylindrical coordinates is $r = 1$. The region of space is shown in Fig. 5.35.

Putting $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ and $dx\, dy\, dz = r\, dr\, d\theta\, dz$ in the given triple integral I, we get

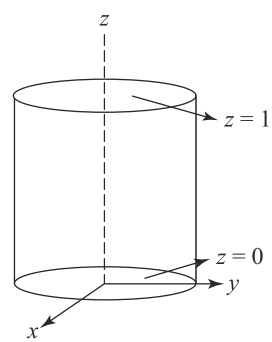


Fig. 5.35

$$I = \iiint_V (r^2 + z^2)\, r\, dr\, d\theta\, dz,$$

where V is the volume of the region of space.

$$= \int_0^1 \int_0^{2\pi} \int_0^1 (r^2 + z^2)\, r\, dr\, d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{r^4}{4} + z^2 \frac{r^2}{2} \right)_0^1 d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{1}{4} + \frac{1}{2} z^2 \right) d\theta\, dz$$

$$= 2\pi \int_0^1 \left(\frac{1}{4} + \frac{1}{2} z^2 \right) dz$$

$$= 2\pi\left[\frac{z}{4} + \frac{z^3}{6}\right]_0^1$$

$$= \frac{5}{6}\pi$$

Note ☑ The intersection of z = constant c and the cylinder $x^2 + y^2 = 1$ is a circle with centre at $(0, 0, c)$ and radius 1. The limits for r and θ have been fixed to cover the area of this circle and then the variation of z has been used so as to cover the entire volume.]

Example 5.18 Find the volume of the portion of the cylinder $x^2 + y^2 = 1$ intercepted between the plane $z = 0$ and the paraboloid $x^2 + y^2 = 4 - z$.

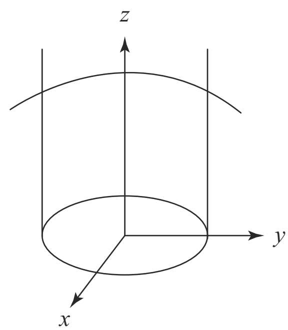


Fig. 5.36

Using cylindrical coordinates, the required volume V is given by

$$V = \iiint r \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z, \text{ taken throughout the region of space.}$$

Since the variation of z is not between constant limits, we first integrate with respect to z and then with respect to r and θ.

Changing to cylindrical coordinates, the boundaries of the region of space are $r = 1$, $z = 0$ and $z = 4 - r^2$.

$\therefore$
$$V = \int_0^{2\pi}\int_0^1\int_0^{4-r^2} \mathrm{d}z \, r \,\mathrm{d}r \,\mathrm{d}\theta$$

$$= \int_0^{2\pi}\int_0^1 r(4 - r^2) \,\mathrm{d}r \,\mathrm{d}\theta$$

$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^1 \mathrm{d}\theta = \frac{7}{4}\int_0^{2\pi} \mathrm{d}\theta = \frac{7}{2}\pi$$

Example 5.19 Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} xyz \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x$, by transforming to spherical polar coordinates.

The boundaries of the region of integration are $z = 0$, $z=\sqrt{a^2-x^2-y^2}$ or $x^2 + y^2 + z^2 = a^2$, $y = 0$, $y=\sqrt{a^2-x^2}$ or $x^2 + y^2 = a^2$, $x = 0$ and $x = a$. From the boundaries, we note that the region of integration is the volume of the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

By putting $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$, the given triple integral I becomes

$$I=\iiint_V r^3\sin^2\theta\cos\theta\sin\phi\cos\phi . r^2\sin\theta\,dr\,d\theta\,d\phi$$

where V is the volume of the positive octant of the sphere $r = a$, which is shown in Fig. 5.37.

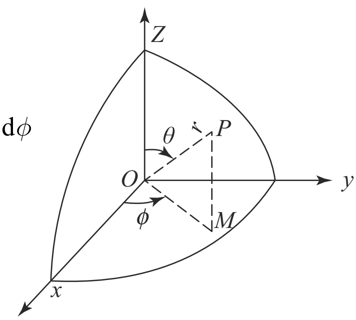


Fig. 5.37

To cover the volume V, r has to vary from 0 to a, θ has to vary from 0 to $\frac{\pi}{2}$ and ϕ has to vary from 0 to $\frac{\pi}{2}$.

Thus $$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a} r^5\sin^3\theta\cos\theta\sin\phi\cos\phi\,dr\,d\theta\,d\phi$$

$$=\int_0^{\frac{\pi}{2}}\sin\phi\cos\phi\,d\phi.\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta.\int_0^{a} r^5\,dr$$

[∵ the limits are constants]

$$=\left[\frac{\sin^2\phi}{2}\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{\sin^4\theta}{4}\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{r^6}{6}\right]_0^{a}$$

$$=\frac{1}{48}a^6.$$

Example 5.20 Evaluate $\iiint\sqrt{1-x^2-y^2-z^2}\,dx\,dy\,dz$, taken throughout the volume of the sphere $x^2 + y^2 + z^2 = 1$, by transforming to spherical polar coordinates.

Changing to spherical polar coordinates, the given triple integral I becomes

$$I=\iiint_V\sqrt{1-r^2}\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

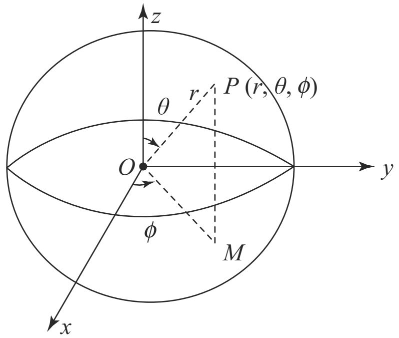


Fig. 5.38

To cover the entire volume V of the sphere, r has to vary from 0 to 1, θ has to vary from 0 to π and ϕ has to vary from 0 to 2π.

Thus
$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sqrt{1-r^2}\, r^2 \,\mathrm{d}r \cdot \sin\theta \,\mathrm{d}\theta \cdot \mathrm{d}\phi$$

$$= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \,\mathrm{d}\theta \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t \,\mathrm{d}t \text{, by putting}$$

$r = \sin t$ in the innermost integral

$$= 2\pi \times (-\cos\theta)_0^{\pi} \times \left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= 4\pi \times \frac{\pi}{4} \times \frac{1}{4} = \frac{1}{4}\pi^2$$

EXERCISE 5(b)

Part A

(Short Answer Questions)

1. Change the order of integration in $\int_0^a \int_0^x f(x, y)\,\mathrm{d}y\,\mathrm{d}x$.
2. Change the order of integration in $\int_0^1 \int_y^1 f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.
3. Change the order of integration in $\int_0^a \int_x^a f(x, y)\,\mathrm{d}y\,\mathrm{d}x$.
4. Change the order of integration in $\int_0^1 \int_0^y f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.
5. Change the order of integration in $\int_0^1 \int_0^{1-y} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

6. Change the order of integration in $\int_0^a \int_0^{a-x} f(x,y)\,dy\,dx$.

7. Change the order of integration in $\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$.

8. Change the order of integration in $\int_0^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$.

9. Change the order of integration in $\int_0^1 \int_0^{2\sqrt{x}} f(x,y)\,dy\,dx$.

10. Change the order of integration in $\int_0^\infty \int_0^{1/y} f(x,y)\,dx\,dy$.

Part B

Change the order of integration in the following integrals and then evaluate them:

11. $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$

12. $\int_0^2 \int_x^2 (x^2+y^2)\,dy\,dx$

13. $\int_0^\infty \int_0^x x\,e^{-\frac{x^2}{y}}\,dy\,dx$

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

15. $\int_0^1 \int_0^{1-x} e^{2x+y}\,dy\,dx$

16. $\int_0^2 \int_0^{\sqrt{4-y^2}} xy\,dx\,dy$

17. $\int_0^{2a} \int_{\frac{x^2}{4a}}^{a} (x+y)\,dy\,dx$

18. $\int_0^1 \int_{y^2}^{y} \frac{y\,dx\,dy}{x^2+y^2}$

19. $\int_0^3 \int_1^{\sqrt{4-x}} (x+y)\,dy\,dx$

20. $\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} y^2\,dx\,dy$

21. $\int_0^3 \int_{\frac{5}{9}x^2}^{\frac{5}{\sqrt{3}}x} dx\,dy$

22. $\int_1^2 \int_0^{4/x} xy\,dy\,dx$.

23. $\int_0^1 \int_x^{\sqrt{2-x^2}} dy\,dx$

24. Change the order of integration in each of the double integrals $\int_0^1 \int_0^{\sqrt{x}} xy\,dy\,dx$ and $\int_1^2 \int_0^{2-x} xy\,dy\,dx$ and hence express their sum as one double integral and evaluate it.

25. Change the order of integration in each of the double integrals $\int_{-1}^{0}\int_{-x}^{1}(x^2+y^2)\,dy\,dx$ and $\int_{0}^{1}\int_{x}^{1}(x^2+y^2)\,dy\,dx$ and hence express their sum as one double integral and evaluate it.

Find the area specified in the following problems (26–35), using double integration:

26. The area bounded by the parabola $y = x^2$ and the straight line $2x - y + 3 = 0$.
27. The area included between the parabolas $y^2 = 4a(x + a)$ and $y^2 = 4a(a - x)$.
28. The area bounded by the two parabolas $y^2 = 4ax$ and $x^2 = 4by$.
29. The area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$.
30. The area bounded by the curve $y^2 = \dfrac{x^3}{2-x}$ and its asymptote.
31. The area of the cardioid $r = a(1 + \cos\theta)$.
32. The area common to the two circles $r = a$ and $r = 2a\cos\theta$.
33. The area common to the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.
34. The area that lies inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$.
35. The area that lies outside the circle $r = a\cos\theta$ and inside the circle $r = 2a\cos\theta$.

Change the following integrals (36 − 40), into polar coordinates and then evaluate them:

36. $\displaystyle\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dx\,dy$

37. $\displaystyle\int_{0}^{a}\int_{y}^{a}\frac{x\,dx\,dy}{(x^2+y^2)}$

38. $\displaystyle\int_{0}^{a}\int_{0}^{x}\frac{x^3\,dx\,dy}{\sqrt{x^2+y^2}}$

39. $\displaystyle\int_{0}^{2a}\int_{0}^{\sqrt{2ax-x^2}}\frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$

40. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx\,dy}{(a^2+x^2+y^2)^{3/2}}$

Evaluate the following integrals (41− 45) after transforming into cylindrical coordinates:

41. $\displaystyle\iiint_V (x+y+z)\,dx\,dy\,dz$, where V is the region of space inside the cylinder $x^2 + y^2 = a^2$, that is bounded by the planes $z = 0$ and $z = h$.

42. $\displaystyle\iiint (x^2+y^2)\,dx\,dy\,dz$, taken throughout the volume of the cylinder $x^2 + y^2 = 1$ that is bounded by the planes $z = 0$ and $z = 4$.

43. $\displaystyle\iiint dx\,dy\,dz$, taken throughout the volume of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ and $y + z = 3$.

44. $\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, taken throughout the volume of the cylinder $x^2 + y^2 = 4$ bounded by the plane $z = 0$ and the surface $z = x^2 + y^2 + 2$.

45. $\iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, taken throughout the volume bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylindrical surface $x^2 + y^2 - 2ay = 0$.

Evaluate the following integrals (46-50) after transforming into spherical polar coordinates:

46. $\iiint \dfrac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{x^2+y^2+z^2}$, taken throughout the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

47. $\iiint \dfrac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\sqrt{1-x^2-y^2-z^2}}$, taken throughout the volume contained in the positive octant of the sphere $x^2 + y^2 + z^2 = 1$.

48. $\iiint\limits_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space bounded by the sphere $x^2 + y^2 + z^2 = a^2$ above the xOy-plane.

49. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

50. $\displaystyle\int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\left(a^2+x^2+y^2+z^2\right)^{5/2}}$

5.6 LINE INTEGRAL

The concept of a line integral is a generalisation of the concept of a definite integral $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

In the definite integral, we integrate along the x-axis from a to b and the integrand $f(x)$ is defined at each point in (a, b). In a line integral, we shall integrate along a curve C in the plane (or space) and the integrand will be defined at each point of C. The formal definition of a line integral is as follows.

Definition Let C be the segment of a continuous curve joining $A(a, b)$ and $B(c, d)$ (Fig. 5.39).

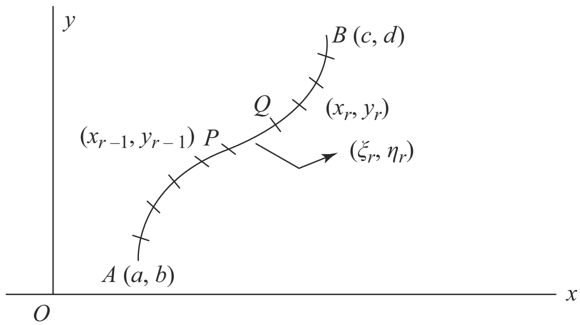


Fig. 5.39

Let $f(x, y)$, $f_1(x, y)$, $f_2(x, y)$ be single-valued and continuous functions of x and y, defined at all points of C.
Divide C into n arcs at (x_i, y_i) $[i = 1, 2, \ldots (n - 1)]$
Let $x_0 = a$, $x_n = c$, $y_0 = b$, $y_n = d$.
Let $x_r - x_{r-1} = \Delta x_r$, $y_r - y_{r-1} = \Delta y_r$ and the arcual length of PQ (i.e. $\widehat{PQ}) = \Delta s_r$, where P is (x_{r-1}, y_{r-1}) and $Q(x_r, y_r)$.
Let (ξ_r, η_r) be any point on C between P and Q.

Then
$$\lim_{n\to\infty}\sum_{r=1}^{n} f(\xi_r, \eta_r)\Delta s_r$$

or
$$\lim_{n\to\infty}\sum_{r=1}^{n}\left[f_1(\xi_r, \eta_r)\Delta x_r + f_2(\xi_r, \eta_r)\Delta y_r\right]$$

is defined as a line integral along the curve C and denoted respectively as

$$\int_C f(x, y)\,\mathrm{d}s \quad \text{or} \quad \int_C [f_1(x, y)\,\mathrm{d}x + f_2(x, y)\,\mathrm{d}y]$$

5.6.1 Evaluation of a Line Integral

Using the equation $y = \phi(x)$ or $x = \psi(y)$ of the curve C, we express $\int_C [f_1(x, y)\,\mathrm{d}x + f_2(x, y)\,\mathrm{d}y]$ either in the form $\int_a^c g(x)\,\mathrm{d}x$ or in the form $\int_b^d h(y)\,\mathrm{d}y$ and evaluate it, which is only a definite integral.

If the line integral is in the form $\int_C f(x, y)\,\mathrm{d}s$, it is first rewritten as $\int_C f(x, y)\dfrac{\mathrm{d}s}{\mathrm{d}x}\mathrm{d}x = \int_C f(x, y)\sqrt{1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ or as $\int_C f(x, y)\dfrac{\mathrm{d}s}{\mathrm{d}y}\mathrm{d}y = \int_C f(x, y)\sqrt{1+\left(\dfrac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$ and then evaluated after expressing it as a definite integral.

5.6.2 Evaluation when C is a Curve in Space

The definition of the line integral given above can be extended when C is a curve in space. In this case, the line integral will take either the form $\int_C [f_1(x,y,z)\,\mathrm{d}x + f_2(x,y,z)\,\mathrm{d}y + f_3(x,y,z)\,\mathrm{d}z]$ or the form $\int_C f(x,y,z)\,\mathrm{d}s$. When C is a curve in space, very often the parametric equations of C will be known in the form $x = \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$. Using the parametric equations of C, the line integral can be expressed as a definite integral. In the case of $\int_C f(x,y,z)\,\mathrm{d}s$, it is rewritten as $\int_C f(x,y,z)\dfrac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t$, where

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\ .$$

5.7 SURFACE INTEGRAL

The concept of a surface integral is a generalisation of the concept of a double integral. While a double integral is evaluated over the area of a plane surface, a surface integral is evaluated over the area of a curved surface in general. The formal definition of a surface integral is given below.

Definition Let S be a portion of a regular two-sided surface. Let $f(z, y, z)$ be a function defined and continuous at all points on S. Divide S into n sub-regions Δs_1, $\Delta S_2, \ldots, \Delta S_n$. Let $P(\xi_r, \eta_r, \zeta_r)$ be any point in ΔS_r. Then $\lim\limits_{\substack{n\to\infty \\ \Delta S_r \to 0}} \sum\limits_{r=1}^{n} f(\xi_r, \eta_r, \zeta_r)\,\Delta S_r$ is called the surface integral of $f(x, y, z)$ over the surface S and denoted as $\int\limits_S f(x, y, z)\,\mathrm{d}S$ or $\iint\limits_S f(x, y, z)\,\mathrm{d}S$.

5.7.1 Evaluation of a Surface Integral

Let the surface integral be $\iint\limits_S f(x, y, z)\,\mathrm{d}S$, where S is the portion of the surface whose equation is $\phi\,(x, y, z) = c$ (Fig. 5.40).

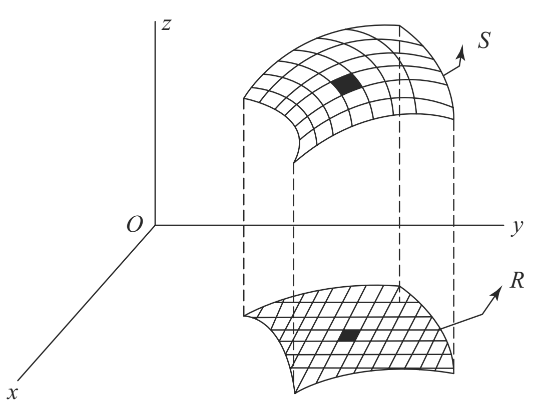


Fig. 5.40

Project the surface S orthogonally on xoy-plane (or any one of the co-ordinate planes) so that the projection is a plane region R.

The projection of the typical elemental surface ΔS (shaded in the figure) is the typical elemental plane area ΔA (shaded in the figure).

We can divide the area of the region R into elemental areas by drawing lines parallel to x and y axes at intervals of Δy and Δx respectively. Then $\Delta A = \Delta x \cdot \Delta y$.

Then $\Delta x \cdot \Delta y = \Delta S \cos\theta$, where θ is the angle between the surface S and the plane R (xoy-plane), i.e. θ is the angle between the normal to the surface S at the typical point (x, y, z) and the normal to the xoy-plane (z-axis). From Calculus, it is known that the direction ratios of the normal at the point (x, y, z) to the surface $\phi(x, y, z) = c$ are $\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$. The direction cosines of the z-axis are (0, 0, 1)

$$\therefore \qquad \cos\theta = \frac{\frac{\partial\phi}{\partial z}}{\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}}$$

Thus
$$\Delta S = \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z}\Delta x\,\Delta y.$$

$$\therefore \qquad \iint_S f(x, y, z)\,\mathrm{d}S = \iint_R f(x, y, z)\cdot\frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z}\,\mathrm{d}x\,\mathrm{d}y$$

Thus the surface integral is converted into a double integral by using the above relation, in which the limits for the double integration on the right side are fixed so as to cover the entire region R and the integrand is converted into a function of x and y, using the equation of S.

Note ☑ Had we projected the curved surface S on the *yoz*-plane or *zox*-plane then the conversion formula would have been

$$\iint_S f(x, y, z)\mathrm{d}S = \iint_R f(x, y, z)\cdot\frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_x}\,\mathrm{d}y\,\mathrm{d}z$$

or
$$\iint_S f(x, y, z)\mathrm{d}S = \iint_R f(x, y, z)\cdot\frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_y}\,\mathrm{d}z\,\mathrm{d}x, \qquad \text{respectively.}$$

5.8 VOLUME INTEGRAL

Definition Let V be a region of space, bounded by a closed surface. Let $f(x, y, z)$ be a continuous function defined at all points of V. Divide V into n sub-regions ΔV_r by drawing planes parallel to the *yoz*, *zox* and *xoy*-planes at intervals of Δx, Δy and Δz respectively. Then ΔV_r is a rectangular parallelopiped with dimensions Δx, Δy, Δz.
Let $P(\xi_r, \eta_r, \zeta_r)$ be any point in ΔV_r.

Then $\lim\limits_{\substack{n\to\infty \\ \Delta V_r \to 0}} \sum_{r=1}^{n} f(\xi_r, \eta_r, \zeta_r)\,\Delta V_r$ is called the volume integral of $f(x, y, z)$ over the region V (or throughout the volume V) and denoted as

$$\int_V f(x, y, z)\,\mathrm{d}v \quad \text{or} \quad \iiint_V f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

5.8.1 Triple Integral Versus Volume Integral

A triple integral discussed earlier is a three times repeated integral in which the limits of integration are given, whereas a volume integral is a triple integral in which the limits of integration will not be explicitly given, but the region of space in which it is to be evaluated will be specified. The limits of integration in a volume integral are fixed so as to cover the entire volume of the region of space V.

Note ☑ Though the line integral and surface integral have been defined in the scalar form in this chapter, they are also defined in the vector form. The vector form of the line and surface integrals will be discussed in Part II, Chapter 2.

WORKED EXAMPLE 5(c)

Example 5.1 Evaluate $\int_C [(3xy^2 + y^3)\,\mathrm{d}x + (x^3 + 3xy^2)\,\mathrm{d}y]$ where C is the parabola $y^2 = 4ax$ from the point (0, 0) to the point $(a, 2a)$

The given integral

$$\mathrm{I} = \int_{y^2 = 4ax} [(3xy^2 + y^3)\,\mathrm{d}x + (x^3 + 3\,xy^2)\,\mathrm{d}y]$$

In order to use the fact that the line integral is evaluated along the parabola $y^2 = 4ax$, we use this equation and the relation between dx and dy derived from it, namely, $2y\,\mathrm{d}y = 4a\,\mathrm{d}x$ and convert the body of the integral either to the form $f(x)\,\mathrm{d}x$ or to the form $\phi(y)\,\mathrm{d}y$. Then the resulting definite integral is evaluated between the concerned limits, got from the end points of C.

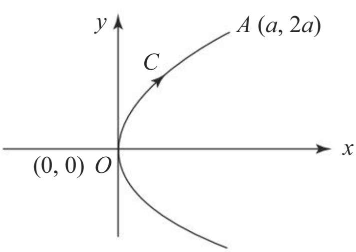


Fig. 5.41

The choice of the form $f(x)\,\mathrm{d}x$ or $\phi(y)\,\mathrm{d}y$ for the body of the integral depends on convenience. In this problem, x is expressed as $\frac{1}{4a}y^2$ more easily than expressing y as $2\sqrt{ax}$.

Note ☑ From $y^2 = 4ax$, we get $y = \pm 2\sqrt{ax}$. Since the arc C lies in the first quadrant, y is positive and hence $y = 2\sqrt{ax}$.

Thus $\mathrm{I} = \int_0^{2a} \left[\left(3\cdot\frac{1}{4a}y^2\cdot y^2 + y^3\right)\frac{y}{2a}\mathrm{d}y + \left(\frac{1}{64a^3}y^6 + 3\cdot\frac{1}{4a}y^2\cdot y^2\right)\mathrm{d}y\right]$

(As the integration is done with respect to y, the limits for y are the y co-ordinates of the terminal points of the arc C).

$$I = \int_0^{2a} \left(\frac{5}{4a} y^4 + \frac{3}{8a^2} y^5 + \frac{1}{64a^3} y^6 \right) dy$$

$$= \left(\frac{1}{4a} y^5 + \frac{1}{16a^2} y^6 + \frac{1}{448a^3} y^7 \right)_0^{2a}$$

$$= \frac{86}{7} a^4$$

Example 5.2 Evaluate $\int_C [(2x - y)\,dx + (x + y)\,dy]$, where C is the circle $x^2 + y^2 = 9$.

In this problem the line integral is evaluated around a closed curve. In such a situation the line integral is denoted as

$\oint_C [(2x - y)\,dx + (x + y)\,dy]$, where a small circle is put across the integral symbol.

When a line integral is evaluated around a closed curve, it is assumed to be described in the anticlockwise sense, unless specified otherwise. (Fig. 5.42)

In the case of a line integral around a closed curve C, any point on C can be assumed to be the initial point, which will also be the terminal point.

Further if we take x or y as the variable of integration, the limits of integration will be the same, resulting in the value ‘zero’ of the line integral, which is meaningless. Hence whenever a line integral is evaluated around a closed curve, the parametric equations of the curve are used and hence the body of integral is converted to the form $f(t)\,dt$ or $f(\theta)\,d\theta$.

In this problem, the parametric equations of the circle $x^2 + y^2 = 9$ are $x = 3\cos\theta$ and $y = 3\sin\theta$.

$\therefore$ $$dx = -3\sin\theta\, d\theta \quad \text{and} \quad dy = 3\cos\theta\, d\theta.$$

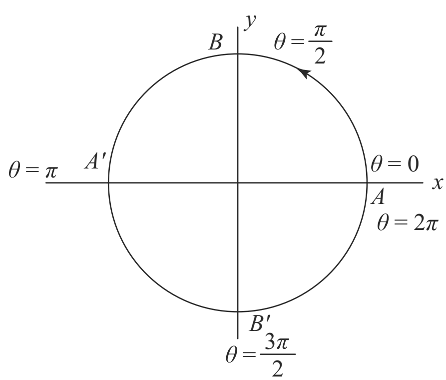


Fig. 5.42

The given integral $= \int_0^{2\pi} [(6\cos\theta - 3\sin\theta)\,(-3\sin\theta\, d\theta)$

$$+ (3\cos\theta + 3\sin\theta)(3\cos\theta\, d\theta)]$$

$$= 9\int_0^{2\pi} (1 - \sin\theta\cos\theta)\, d\theta$$

$$= 9\left(\theta - \frac{\sin^2\theta}{2}\right)_0^{2\pi}$$

$$= 18\pi$$

Example 5.3 Evaluate $\int_C xy\, ds$, where C is the arc of the parabola $y^2 = 4x$ between the vertex and the positive end of the latus rectum.

Given integral $$I = \int_C xy \frac{ds}{dx} dx$$

Equation of the parabola is $y^2 = 4x$

Differentiating with respect to x, $\frac{dy}{dx} = \frac{2}{y}$

$\therefore$ $$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{4}{y^2}}$$

$\therefore$ $$I = \int_C xy \frac{\sqrt{y^2 + 4}}{y} dx$$

$$= \int_0^1 x\sqrt{4x + 4}\, dx$$

$$= 2\int_1^{\sqrt{2}} (t^2 - 1) \cdot t \cdot 2t\, dt, \text{ on putting } x + 1 = t^2$$

$$= 4\int_1^{\sqrt{2}} (t^4 - t^2)\, dt$$

$$= 4\left(\frac{t^5}{5} - \frac{t^3}{3}\right)_1^{\sqrt{2}}$$

$$= \frac{8}{15}\left(1 + \sqrt{2}\right)$$

Example 5.4 Evaluate $\int_C (y^2 dx - x^2 dy)$, where C is the boundary of the triangle whose vertices are $(-1, 0)$, $(1, 0)$ and $(0, 1)$ (Fig. 5.43).
C is made up of the lines BC, CA and AB.

Equations of BC, CA and AB are respectively $y = 0$, $x + y = 1$ and $-x + y = 1$.

$$\text{Given integral} = \int\limits_{\substack{BC\\ y=0\\ dy=0}} + \int\limits_{\substack{CA\\ x+y=1\\ dy=-dx}} + \int\limits_{\substack{AB\\ -x+y=1\\ dy=dx}} (y^2 dx - x^2 dy)$$

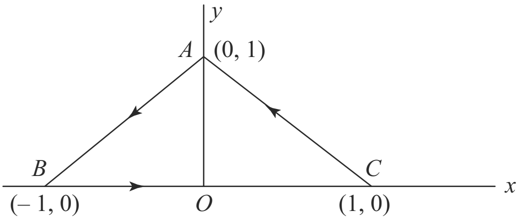


Fig. 5.43

$$= 0 + \int_1^0 [(1-x)^2 + x^2] dx + \int_0^{-1} [(1+x)^2 - x^2] dx$$

$$= \int_1^0 (1 - 2x + 2x^2) dx + \int_0^{-1} (1 + 2x) dx$$

$$= \left(x - x^2 + \frac{2x^3}{3}\right)_1^0 + (x + x^2)_0^{-1}$$

$$= -\frac{2}{3}$$

Example 5.5 Evaluate $\int\limits_C [x^2 y\, dx + (x - z)\, dy + xyz\, dz]$, where C is the arc of the parabola $y = x^2$ in the plane $z = 2$ from (0, 0, 2) to (1, 1, 2).

$$\text{Given integral} = \int\limits_{\left(\substack{y=x^2\\ z=2}\right)} [x^2 y\, dx + (x - z)\, dy + xyz\, dz]$$

$$= \int\limits_{y=x^2} [x^2 y\, dx + (x - 2)\, dy]$$

[$\because$ $dz = 0$, when $z = 2$]

$$= \int_0^1 [x^4 + (x-2)\, 2x] dx = \left(\frac{x^5}{5} + \frac{2x^3}{3} - 2x^2\right)_0^1$$

$$= -\frac{17}{15}$$

Example 5.6 Evaluate $\int\limits_C (x\, dx + xy\, dy + xyz\, dz)$, where C is the arc of the twisted curve $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$.

The parametric equations of C are $x = t$, $y = t^2$, $z = t^3$

$\therefore$ $dx = dt$, $dy = 2t\, dt$, $dz = 3t^2\, dt$ on C.

Using these values in the given integral I,

$$\mathrm{I}=\int_0^1 (t+t^3\cdot 2t+t^6\cdot 3t^2)\,\mathrm{d}t$$

$$=\left(\frac{t^2}{2}+2\frac{t^5}{5}+3\frac{t^9}{9}\right)_0^1$$

$$=\frac{17}{30}$$

Example 5.7 Evaluate $\int_C (x^2+y^2+z^2)\,\mathrm{d}s$, where C is the arc of the circular helix $x = \cos t$, $y = \sin t$, $z = 3t$ from (1, 0, 0) to (1, 0, 6π)

The parametric of equations of C are

$x = \cos t$, $y \sin t$, $z = 3t$.

$$\therefore \qquad \frac{\mathrm{d}x}{\mathrm{d}t}=-\sin t, \qquad \frac{\mathrm{d}y}{\mathrm{d}x}=\cos t, \qquad \frac{\mathrm{d}z}{\mathrm{d}t}=3 \text{ on } C.$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}$$

$$=\sqrt{\sin^2 t+\cos^2 t+9}=\sqrt{10}$$

Given integral $$\mathrm{I}=\int_0^{2\pi} (\cos^2 t+\sin^2 t+9t^2)\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t$$

Note ☑ The point (1, 0, 0) corresponds to $t=0$ and $(1, 0, 6\pi)$ corresponds to $t=2\pi$.

$$\mathrm{I}=\left(t+3t^3\right)_0^{2\pi}\times\sqrt{10}$$

$$=2\sqrt{10}\pi(1+12\pi^2)$$

Example 5.8 Evaluate $\iint_S xyz\,\mathrm{d}S$, where S is the surface of the rectangular parallelopiped formed by x = 0, y = 0, z = 0, x = a, y = b and z = c (Fig. 5.44).

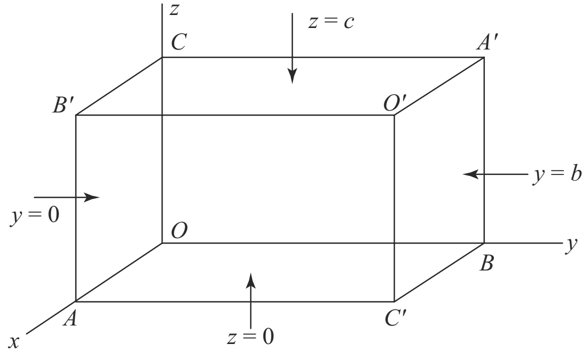


Fig. 5.44

Since S is made up of 6 plane faces, the given surface integral I is expressed as

$$I = \iint_{x=0} + \iint_{x=a} + \iint_{y=0} + \iint_{y=b} + \iint_{z=0} + \iint_{z=c} (xyz\,\mathrm{d}S)$$

Since all the faces are planes, the elemental curved surface area dS becomes the elemental plane surface area dA.

On the planes $x = 0$ and $x = a$, dA = dy dz.
On the planes $y = 0$ and $y = b$, dA = dz dx.
On the planes $z = 0$ and $z = c$, dA = dx dy.

$$\therefore \quad I = \iint_{x=0} + \iint_{x=a} (xyz\,\mathrm{d}y\,\mathrm{d}z) + \iint_{y=0} + \iint_{y=b} (xyz\,\mathrm{d}z\,\mathrm{d}x)$$
$$+ \iint_{z=0} + \iint_{z=c} (xyz\,\mathrm{d}x\,\mathrm{d}y)$$

Simplifying the integrands using the equations of the planes over which the surface integrals are evaluated, we get

$$I = a\int_0^c\int_0^b yz\,\mathrm{d}y\,\mathrm{d}z + b\int_0^a\int_0^c zx\,\mathrm{d}z\,\mathrm{d}x + c\int_0^b\int_0^a xy\,\mathrm{d}x\,\mathrm{d}y$$

Note ☑ On the plane face $O'A'CB'$ ($z = c$), the limits for x and y are easily found to be 0, a and 0, b. Similarly the limits are found on the faces $O'B'AC'$ ($x = a$) and $O'C'BA'$ ($y = b$).]

Now
$$I = a\frac{b^2}{2}\cdot\frac{c^2}{2} + b\cdot\frac{c^2}{2}\cdot\frac{a^2}{2} + c\cdot\frac{a^2}{2}\cdot\frac{b^2}{2}$$
$$= \frac{abc}{4}(ab + bc + ca)$$

Example 5.9 Evaluate $\iint_S (y + 2z - 2)\,\mathrm{d}S$, where S is the part of the plane $2x + 3y + 6z = 12$, that lies in the positive octant (Fig. 5.45).

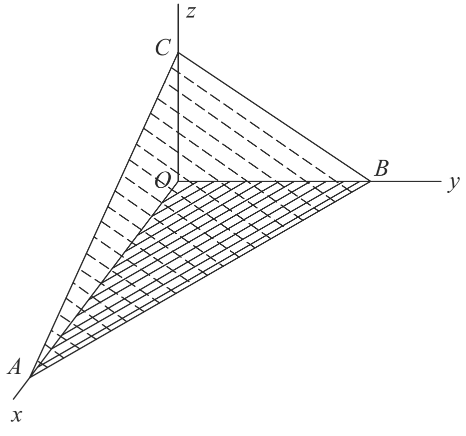


Fig. 5.45

Rewriting the equation of the (plane) surface S in the intercept form, we get

$$\frac{x}{6}+\frac{y}{4}+\frac{z}{2}=1$$

$\therefore$ S is the plane that cuts off intercepts of lengths 6, 4 and 2 on the x, y and z-axes respectively and lies in the positive octant.

We note that the projection of the given plane surface S on the xoy-plane is the triangular region OAB shown in the two-dimensional Fig. 5.46.

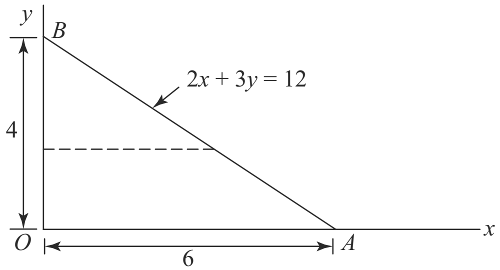


Fig. 5.46

Converting the given surface integral I as a double integral,

$$\mathrm{I}=\iint_{\Delta OAB}(y+2z-2)\frac{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}{\phi_z}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\phi = c$ is the equation of the given surface S.

Here $\phi = 2x + 3y + 6z$

$\therefore$ $\phi_x = 2;\ \phi_y = 3;\ \phi_z = 6.$

$\therefore$
$$\mathrm{I}=\iint_{\Delta OAB}(y+2z-2)\frac{\sqrt{4+9+36}}{6}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{7}{6}\iint_{\Delta OAB}(y+2z-2)\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

Now the integrand is expressed as a function of x and y, by using the value of z (as a function of x and y) got from the equation of S, i.e. from the equation $2x + 3y + 6z = 12$

Thus
$$z=\frac{1}{6}(12-2x-3y) \tag{2}$$

Using (2) in (1), we get

$$\mathrm{I}=\frac{7}{6}\iint_{\Delta OAB}\frac{1}{3}(6-2x)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{7}{18}\int_0^4 \int_0^{6-\frac{3}{2}y} (6-2x)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{7}{18}\int_0^4 \left(9y - \frac{9}{4}y^2\right)\mathrm{d}y = \frac{28}{3}$$

Example 5.10 Evaluate $\iint_S z^3\, dS$, where S is the positive octant of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ (Fig. 5.47)

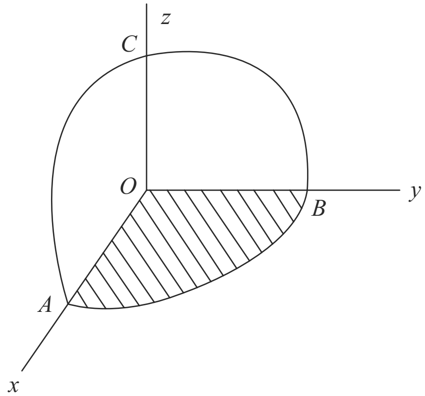


Fig. 5.47

The projection of the given surface of the sphere $x^2 + y^2 + z^2 = a^2$ (lying in the positive octant) in the xoy - plane is the quadrant of the circular region OAB, shown in the two-dimensional Fig. 5.48.

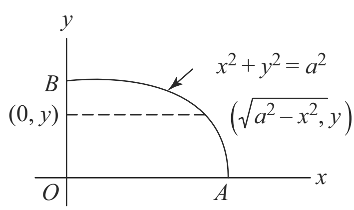


Fig. 5.48

Converting the given surface integral I as a double integral.

$$\mathrm{I} = \iint_{OAB} z^3 \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\phi \equiv x^2 + y^2 + z^2 = a^2$ is the equation of the given spherical surface.

$\phi_x = 2x;\ \phi_y = 2y;\ \phi_z = 2z.$

$$\therefore \qquad \mathrm{I} = \iint_{OAB} z^3 \frac{\sqrt{4(x^2 + y^2 + z^2)}}{2z}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_{OAB} z^2 \sqrt{x^2 + y^2 + z^2}\ \mathrm{d}x\,\mathrm{d}y$$

$$= a \iint\limits_{OAB} (a^2 - x^2 - y^2)\,dx\,dy \qquad [\because (x, y, z) \text{ lies on } x^2 + y^2 + z^2 = a^2]$$

$$= a \int_0^a \int_0^{\sqrt{a^2-y^2}} (a^2 - y^2 - x^2)\,dx\,dy$$

$$= a \int_0^a \left[(a^2 - y^2)x - \frac{x^3}{3} \right]_{x=0}^{x=\sqrt{a^2-y^2}} dy$$

$$= \frac{2}{3} a \int_0^a (a^2 - y^2)^{\frac{3}{2}}\,dy$$

$$= \frac{2}{3} a^5 \int_0^{\pi/2} \cos^4\theta\,d\theta, \text{ on putting } x = a\sin\theta.$$

$$= \frac{2}{3} a^5 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{8} a^5 \cdot$$

Example 5.11 Evaluate $\iint\limits_S y(z + x)\,dS$, where S is the curved surface of the cylinder $x^2 + y^2 = 16$, that lies in the positive octant and that is included between the planes $z = 0$ and $z = 5$ (Fig. 5.49).

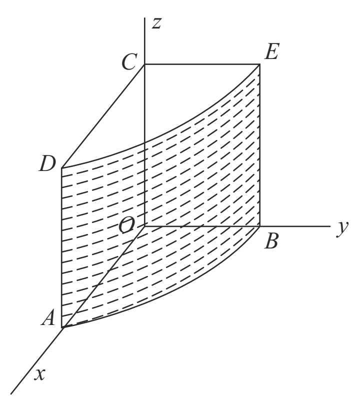


Fig. 5.49

We note that the projection of S on the *xoy*-plane is not a plane (region) surface, but only the arc AB of the circle whose centre is O and radius equal to 4.

For converting the given surface integral into a double integral, the projection of S must be a plane region. Hence we project S on the *zox*-plane (or *yoz*-plane). The projection of S in this case is the rectangular region $OCDA$, which is shown in Fig. 5.50.

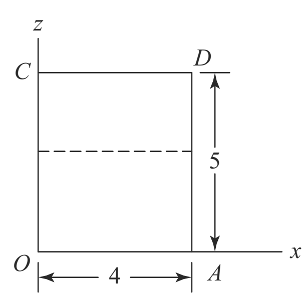


Fig. 5.50

Converting the given surface integral I as a double integral,

$$\mathrm{I} = \iint_{OADC} y(z+x)\frac{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}{\phi y}\,\mathrm{d}z\,\mathrm{d}x,$$

where $\phi \equiv x^2 + y^2 = 16$ is the equation of the given cylindrical surface. $\phi_x = 2x$; $\phi_y = 2y$; $\phi_z = 0$.

$$\therefore \qquad \mathrm{I} = \iint_{OADC} y(z+x)\frac{\sqrt{4(x^2+y^2)}}{2y}\,\mathrm{d}z\,\mathrm{d}x$$

$$= 4\iint_{OADC} (z+x)\,\mathrm{d}z\,\mathrm{d}x \qquad [\because (x, y, z) \text{ lies on } x^2 + y^2 = 16]$$

$$= 4\int_0^5\int_0^4 (z+x)\,\mathrm{d}x\,\mathrm{d}z\,.$$

$$= 4\int_0^5 (4z+8)\,\mathrm{d}z$$

$$= 8(z^2+4z)_0^5$$

$$= 360.$$

Example 5.12 Evaluate $\iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space inside the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

Vide worked Example 5.11 in the section on 'Double and triple integrals' for fixing the limits of the volume integral.

$$\mathrm{I} = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} xyz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} xy\left(\frac{z^2}{2}\right)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} \mathrm{d}y\,\mathrm{d}x$$

$$= \frac{c^2}{2}\int_0^a\int_0^{bt} xy\left(t-\frac{y}{b}\right)^2 \mathrm{d}y\,\mathrm{d}x\,, \text{ where } t = 1-\frac{x}{a}$$

$$= \frac{c^2}{2}\int_0^a x\left(t^2\frac{y^2}{2}-\frac{2t}{b}\frac{y^3}{3}+\frac{1}{b^2}\frac{y^4}{4}\right)_0^{bt} \mathrm{d}x$$

$$= \frac{c^2}{2}\int_0^a \left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) b^2\,xt^4\,\mathrm{d}x$$

$$= \frac{b^2 c^2}{24} \int_0^a x\left(1 - \frac{x}{a}\right)^4 \mathrm{d}x$$

$$= \frac{b^2 c^2}{24} \int_0^a a\left[1 - \left(1 - \frac{x}{a}\right)\right] \cdot \left(1 - \frac{x}{a}\right)^4 \mathrm{d}x$$

$$= \frac{ab^2 c^2}{24} \left[\frac{\left(1 - \frac{x}{a}\right)^5}{-\frac{5}{a}} + \frac{\left(1 - \frac{x}{a}\right)^6}{\frac{6}{a}} \right]_0^a$$

$$= \frac{a^2 b^2 c^2}{24} \left(\frac{1}{5} - \frac{1}{6}\right)$$

$$= \frac{1}{720} a^2 b^2 c^2 \cdot$$

Example 5.13 Express the volume of the sphere $x^2 + y^2 + z^2 = a^2$ as a volume integral and hence evaluate it. [Refer to Fig. 5.51]

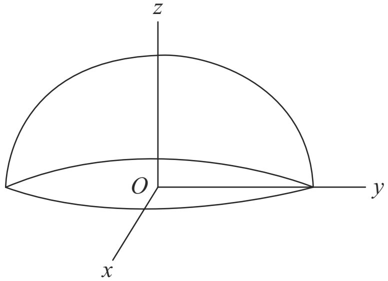


Fig. 5.51

Required volume = 2 × volume of the hemisphere above the xoy-plane. Vide worked example 5.12 in the section on 'Double and Triple Integrals'.

$$\text{Required volume } = 2\int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \int_{0}^{\sqrt{a^2 - x^2 - y^2}} \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

$$= 2\int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \sqrt{(a^2 - x^2) - y^2}\, \mathrm{d}y\, \mathrm{d}x$$

Taking $a^2 - x^2 = b^2$, when integration with respect to y is performed,

$$V = 2\int_{-a}^{a} \int_{-b}^{b} \sqrt{b^2 - y^2}\, \mathrm{d}y\, \mathrm{d}x$$

$$= 4\int_{-a}^{a}\int_{0}^{b}\sqrt{b^2-y^2}\,\mathrm{d}y\,\mathrm{d}x \qquad [\because \sqrt{b^2-y^2} \text{ is an even function of } y]$$

$$= 4\int_{-a}^{a}\left(\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}\sin^{-1}\frac{y}{b}\right)_0^b \mathrm{d}x$$

$$= \pi\int_{-a}^{a}(a^2-x^2)\,\mathrm{d}x$$

$$= 2\pi\left(a^2x-\frac{x^3}{3}\right)_0^a$$

$$= \frac{4}{3}\pi a^3$$

Example 5.14 Evaluate $\iiint\limits_V (x+y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where V is the region of space inside the cylinder $x^2 + y^2 = a^2$ that is bounded by the planes $z = 0$ and $z = h$ [Refer to Fig. 5.52].

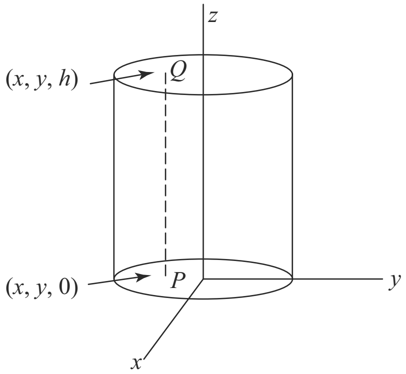


Fig. 5.52

Note ☑ The equation $x^2 + y^2 = a^2$ (in three dimensions) represents the right circular cylinder whose axis is the z-axis and base circle is the one with centre at the origin and radius equal to a.

$$\mathrm{I} = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{h}(x+y+z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[(x+y)h+\frac{h^2}{2}\right]\mathrm{d}y\,\mathrm{d}x$$

$$= 2h\cdot\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\left(x+\frac{h}{2}\right)\mathrm{d}y\,\mathrm{d}x$$

[by using properties of odd and even functions]

$$= 2h \int_{-a}^{a} \left(x + \frac{h}{2} \right) \sqrt{a^2 - x^2}\, \mathrm{d}x$$

$$= 2h^2 \int_{0}^{a} \sqrt{a^2 - x^2}\, \mathrm{d}x \qquad [\because\ x\sqrt{a^2 - x^2} \text{ is odd and } \sqrt{a^2 - x^2} \text{ is even}]$$

$$= 2h^2 \left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right)_0^a$$

$$= \frac{\pi}{2} a^2 h^2.$$

EXERCISE 5(c)

Part A

(Short Answer Questions)

1. Define a line integral.
2. What is the difference between a definite integral and a line integral?
3. Define a surface integral.
4. What is the difference between a double integral and a surface integral?
5. Define a volume integral.
6. What is the difference between a triple integral and a volume integral?
7. Write down the formula that converts a surface integral into a double integral.
8. Evaluate $\int_C (x^2\, \mathrm{d}y + y^2\, \mathrm{d}x)$ where C is the path $y = x$ from (0, 0) to (1, 1).
9. Evaluate $\int_C \sqrt{(x^2 + y^2)}\, \mathrm{d}s$, where C is the path $y = -x$ from (0, 0) to (−1, 1).
10. Evaluate $\int_C (x\, \mathrm{d}y - y\, \mathrm{d}x)$, where C is the circle $x^2 + y^2 = 1$ from (1, 0) to (0, 1) in the counterclockwise sense.
11. Evaluate $\iint_S \mathrm{d}S$, where S is the surface of the parallelopiped formed by $x = \pm 1, y = \pm 2, z = \pm 3$.
    [*Hint:* $\iint_S \mathrm{d}S$ gives the area of the surface S]
12. Evaluate $\iint_S \mathrm{d}S$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
13. Evaluate $\iint_S \mathrm{d}S$, where S is the curved surface of the right circular cylinder $x^2 + y^2 = a^2$, included between $z = 0$ and $z = h$.
14. Evaluate $\iiint_V \mathrm{d}V$, where V is the region of space bounded by the planes $x = 0, x = a, y = 0, y = 2b, z = 0$ and $z = 3c$.

[*Hint:* $\iiint\limits_V \mathrm{d}V$ gives the volume of the region V]

15. Evaluate $\iiint\limits_V \mathrm{d}V$, where V is the region of space bounded by $x^2 + y^2 + z^2 = 1$.
16. Evaluate $\iiint\limits_V \mathrm{d}V$, where V is the region of space bounded by $x^2 + y^2 = a^2$, $z = -h, z = h$.

Part B

17. Evaluate $\int\limits_{(0,0)}^{(1,3)} [x^2 y\,\mathrm{d}x + (x^2 - y^2)\,\mathrm{d}y]$ along the (i) curve $y = 3x^2$, (ii) line $y = 3x$.
18. Evaluate $\int\limits_C [(x^2 - y^2 + x)\,\mathrm{d}x - (2x\,y + y)\,\mathrm{d}y]$ from (0, 0) to (1, 1), when C is (i) $y^2 = x$, (ii) $y = x$.
19. Evaluate $\int\limits_{(-a,0)}^{(a,0)} (y^2\,\mathrm{d}x - x^2\mathrm{d}y)$ along the upper half of the circle $x^2 + y^2 = a^2$.
20. Evaluate $\int\limits_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$, where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and described in the anticlockwise sense.
21. Evaluate $\int\limits_C [(x^2 - y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y]$, where C is the boundary of the rectangle formed by the lines $x = 0, x = 2, y = 0, y = 1$ and described in the anticlockwise sense.
22. Evaluate $\int\limits_C [(3x^2 - 8y^2)\,\mathrm{d}x + (4y - 6xy)\,\mathrm{d}y]$, where C is the boundary of the region enclosed by $y^2 = x$ and $x^2 = y$ and described in the anticlockwise sense.
23. Evaluate $\int\limits_C (x - y^2)\,\mathrm{d}s$, where C is the arc of the circle $x = a\cos\theta$, $y = a\sin\theta$; $0 \le \theta \le \frac{\pi}{2}$.
24. Evaluate $\int\limits_C x\,\mathrm{d}s$, where C is the arc of the parabola $x^2 = 2y$ from (0, 0) to $\left(1, \frac{1}{2}\right)$.
25. Evaluate $\int\limits_C [xy\,\mathrm{d}x + (x^2 + z)\,\mathrm{d}y + (y^2 + x)\,\mathrm{d}z]$ from (0, 0, 0) to (1, 1, 1) along the curve C given by $y = x^2$ and $z = x^3$.
26. Evaluate $\int\limits_C [(3x^2 + 6y)\,\mathrm{d}x - 14yz\,\mathrm{d}y + 20xz^2\,\mathrm{d}z]$, where C is the segment of the straight line joining (0, 0, 0) and (1, 1, 1).

27. Evaluate $\int_C [3x^2 \, \mathrm{d}x + (2xy - y) \, \mathrm{d}y - z \, \mathrm{d}z]$ from $t = 0$ to $t = 1$ along the curve C given by $x = 2t^2$, $y = t$, $z = 4t^3$.
28. Evaluate $\int xy \, \mathrm{d}s$ along the arc of the curve given by the equations $x = a \tan \theta$, $y = a \cot \theta$, $z = \sqrt{2}\, a \log \tan \theta$ from the point $\theta = \frac{\pi}{4}$ to the point $\theta = \frac{\pi}{3}$.
29. Evaluate $\int_C (xy + z^2) \mathrm{d}s$, where C is the arc of the helix $x = \cos t$, $y = \sin t$, $z = t$ from $(1, 0, 0)$ to $(-1, 0, \pi)$.
30. Find the area of that part of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ that lies in the positive octant. $\left[\textit{Hint}\text{: Area of the surface} = \iint_S \mathrm{d}s \right]$
31. Evaluate $\iint_S z \, \mathrm{d}S$, where S is the positive octant of the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
32. Evaluate $\iint xy \, \mathrm{d}S$, where S is the curved surface of the cylinder $x^2 + y^2 = a^2$, $0 \le z \le k$, included in the positive octant.
33. Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
34. Evaluate $\iiint_V z \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$, where V is the region of space bounded by the sphere $x^2 + y^2 + z^2 = a^2$ above the xoy-plane.
35. Evaluate $\iiint_V (x^2 + y^2) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$, where V is the region of space inside the cylinder $x^2 + y^2 = a^2$ that is bounded by the planes $z = 0$ and $z = h$.

5.9 GAMMA AND BETA FUNCTIONS

Definitions The definite integral $\int_0^\infty e^{-x} x^{n-1} \, \mathrm{d}x$ exists only when $n > 0$ and when it exists, it is a function of n and called *Gamma function* and denoted by $\Gamma(n)$ [read as "Gamma n"].

Thus
$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, \mathrm{d}x$$

The definite integral $\int_0^1 x^{m-1} (1 - x)^{n-1} \, \mathrm{d}x$ exists only when $m > 0$ and $n > 0$ and when it exists, it is a function of m and n and called *Beta function* and denoted by $\beta(m, n)$ [read as "Beta m, n"].

Thus $\beta(m, n) = \int_0^1 x^{m-1} (1 - x)^{n-1} \mathrm{d}x$:

Note ☑ $$\Gamma(1) = \int_0^\infty e^{-x}\, dx = (-e^{-x})_0^\infty = 1.$$

$$\beta(1,1) = \int_0^1 dx = 1.$$

5.9.1 Recurrence Formula for Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$$

$$= -(x^{n-1}\, e^{-x})_0^\infty + \int_0^\infty (n-1)\, e^{-x}\, x^{n-2}\, dx \quad \text{[integrating by parts]}$$

$$= (n-1)\,\Gamma(n-1), \text{ since } \lim_{n\to\infty}\left(\frac{x^{n-1}}{e^x}\right) = 0$$

This recurrence formula $\Gamma(n) = (n-1)\,\Gamma\,(n-1)$ is valid only when $n > 1$, as $\Gamma(n-1)$ exists only when $n > 1$.

Cor.

$\Gamma(n+1) = n\,!$, where n is a positive integer.

$$\begin{aligned}\Gamma(n+1) &= n\,\Gamma(n)\\ &= n\,(n-1)\,\Gamma(n-1)\\ &= n\,(n-1)\,(n-2)\,\Gamma\,(n-2)\\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &= n\,(n-1)\,(n-2)\,...\,3.2.1\,\Gamma(1)\\ &= n\,!\ (\because \Gamma(1) = 1)\end{aligned}$$

Note ☑

1. $\Gamma(n)$ does not exist (i.e. $= \infty$), when n is 0 or a negative integer.
2. When n is a negative fraction, $\Gamma(n)$ is defined by using the recurrence formula. i.e. when $n < 0$, but not an integer,

$$\Gamma(n) = \frac{1}{n}\,\Gamma(n+1)$$

For example, $$\Gamma(-3.5) = \frac{1}{(-3.5)}\,\Gamma(-2.5)$$

$$= \frac{1}{(-3.5)}\cdot\frac{1}{(-2.5)}\,\Gamma(-1.5)$$

$$= \frac{1}{(3.5)\,(2.5)\,(-1.5)}\,\Gamma(-.5)$$

$$= \frac{\Gamma\,(0.5)}{(3.5)\,(2.5)\,(1.5)\,(0.5)}$$

The value of $\Gamma(0.5)$ can be obtained from the table of Gamma functions, though its value can be found out mathematically as given below.

Value of $\Gamma\left(\frac{1}{2}\right)$

By definition, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt$

$$= \int_0^\infty e^{-x^2} \cdot \frac{1}{x} \cdot 2x\, dx \qquad \text{(on putting } t = x^2\text{)}$$

$$= 2\int_0^\infty e^{-x^2}\, dx$$

Now $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 2\int_0^\infty e^{-x^2}\, dx \cdot 2\int_0^\infty e^{-y^2} dy$ [∵ the variable in a definite integral is only a dummy variable]

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy \qquad (1)$$

[∵ the product of two definite integrals can be expressed as a double integral, when the limits are constants].

Now the region of the double integral in (1) is given by $0 \le x < \infty$ and $0 \le y < \infty$, i.e. the entire first quadrant of the xy-plane.

Let us change over to polar co-ordinates through the transformations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Then $dx\, dy = |J|\, d\, r\, d\, \theta = r\, d\, r\, d\, \theta$

The region of the double integration is now given by $0 \le r < \infty$ and $0 \le \theta \le \frac{\pi}{2}$. Then, from (1), we have

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} d\theta \left(-\frac{1}{2}e^{-r^2}\right)_0^\infty$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= \pi$$

$$\therefore \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

5.9.2 Symmetry of Beta Function

$$\beta\,(m, n) = \beta\,(n, m)$$

By definition, $\beta\,(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \qquad (1)$

Using the property $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ in (1),

$$\beta(m, n) = \int_0^1 (1-x)^{m-1} \{1-(1-x)\}^{n-1}\, dx$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1}\, dx$$

$$= \beta(n, m).$$

5.9.3 Trigonometric Form of Beta Function

By definition, $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

Put $x = \sin^2\theta$ $\quad\therefore\ dx = 2\sin\theta\cos\theta\, d\theta$

The limits for θ are 0 and $\frac{\pi}{2}$.

$$\therefore \qquad \beta(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

Note ☑ $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m, n)$

The first argument of the Beta function is obtained by adding 1 to the exponent of $\sin\theta$ and dividing the sum by 2. The second argument is obtained by adding 1 to the exponent of $\cos\theta$ and dividing the sum by 2.

Thus
$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

5.9.4 Relation Between Gamma and Beta Functions

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Consider
$$\Gamma(m)\,\Gamma(n) = \int_0^\infty e^{-t}\, t^{m-1}\, dt \cdot \int_0^\infty e^{-s} \cdot s^{n-1}\, ds$$

In the first integral, put $t = x^2$ and in the second, put $s = y^2$.

$$\therefore \qquad \Gamma(m)\cdot\Gamma(n) = 2\int_0^\infty e^{-x^2}\, x^{2m-1}\, dx \cdot 2\int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

$$= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, x^{2m-1} \cdot y^{2n-1}\, \mathrm{d}x\, \mathrm{d}y$$

$$= 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, (r\cos\theta)^{2m-1} \cdot (r\sin\theta)^{2n-1}\, r\, \mathrm{d}r\, \mathrm{d}\theta$$

[changing over to polar co-ordinates]

$$= 4 \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, \mathrm{d}\theta \cdot \int_0^\infty e^{-r^2}\, r^{2m+2n-2} \cdot r\, \mathrm{d}r$$

$$= \beta(m, n) \int_0^\infty e^{-r^2}\, r^{2(m+n-1)} \cdot 2r\, \mathrm{d}r$$

$$= \beta(m, n) \cdot \int_0^\infty e^{-u} \cdot u^{m+n-1}\, \mathrm{d}u \qquad \text{[putting } r^2 = u]$$

$$= \beta(m, n) \cdot \Gamma(m, n)$$

$$\therefore \qquad \beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Cor.

Putting $m = n = \frac{1}{2}$ in the above relation, $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \dfrac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2}{\Gamma(1)}$

$$\therefore \qquad \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= 2\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta\, \mathrm{d}\theta$$

$$= \pi$$

$$\therefore \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

WORKED EXAMPLE 5(d)

Example 5.1 Prove that $\int_0^\infty e^{-ax}\, x^{n-1}\, \mathrm{d}x = \dfrac{\Gamma(n)}{a^n}$, where a and n are positive.

Hence find the value of $\int_0^1 x^{q-1} \left[\log\left(\frac{1}{x}\right)\right]^{p-1} \mathrm{d}x$.

In $\int_0^\infty e^{-ax}\, x^{n-1}\, \mathrm{d}x$, put $ax = t$, so that $\mathrm{d}x = \dfrac{\mathrm{d}t}{a}$

$$\therefore \quad \int_0^\infty e^{-ax}\, x^{n-1}\, dx = \int_0^\infty e^{-t} \left(\frac{t}{a}\right)^{n-1} \cdot \frac{dt}{a}$$

$$= \frac{1}{a^n} \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

$$= \frac{1}{a^n}\, \Gamma(n) \qquad (1)$$

In $I = \int_0^1 x^{q-1} \log\left(\frac{1}{x}\right)^{p-1} dx$,

put $\quad \frac{1}{x} = e^y$

i.e. $\quad x = e^{-y}$

Then $\quad dx = -e^{-y}\, dy$

Also the limits for y are ∞ and 0.

$$\therefore \quad I = \int_\infty^0 e^{-(q-1)y} \cdot y^{p-1} \cdot (-e^{-y})\, dy$$

$$= \int_0^\infty e^{-qy}\, y^{p-1}\, dy$$

$$= \frac{1}{q^p} \cdot \Gamma(p) \text{ [by (1)]}.$$

Example 5.2 Prove that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx.$

Hence deduce that $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$.

By definition, $\beta(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt \qquad (1)$

In (1), put $t = \frac{x}{1+x}$. Then $dt = \frac{1}{(1+x)^2}\, dx$

When $t = 0$, $x = 0$; when $t = 1$, $x = \infty$ $\qquad \left(\because x = \frac{t}{1-t}\right)$

Then (1) becomes,

$$\beta(m,n) = \int_0^\infty \left(\frac{x}{1+x}\right)^{m-1} \cdot \left(\frac{1}{1+x}\right)^{n-1} \cdot \frac{1}{(1+x)^2}\, dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \qquad (2)$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \qquad (3)$$

In $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$, put $x=\frac{1}{y}$. Then $dx=-\frac{1}{y^2}\,dy$

When $x = 1$, $y = 1$; when $x = \infty$, $y = 0$

$$\therefore \qquad \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=\int_1^0 \frac{\frac{1}{y^{(m-1)}}}{\left(1+\frac{1}{y}\right)^{m+n}}\cdot\left(-\frac{1}{y^2}\right)dy$$

$$=\int_0^1 \frac{y^{m+n}}{(1+y)^{m+n}\cdot y^{m+1}}\,dy$$

$$=\int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$=\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

[changing the dummy variable] (4)

Using (4) in (3), we have

$$\beta(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$

Example 5.3 Evaluate $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of Gamma functions and hence find $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$.

In $\qquad I=\int_0^1 x^m(1-x^n)^p\,dx,$

put $\qquad x^n=t;$

then $\qquad nx^{n-1}\,dx=dt$

$$\therefore \qquad dx=\frac{1}{n}\cdot\frac{dt}{t^{1-\frac{1}{n}}}$$

When $x = 0$, $t = 0$; when $x = 1$, $t = 1$.

$$I=\int_0^1 t^{\frac{m}{n}}(1-t)^p\cdot\frac{1}{n}\cdot t^{\frac{1}{n}-1}\,dt$$

$$=\frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}\cdot(1-t)^p\,dt$$

$$=\frac{1}{n}\beta\left(\frac{m+1}{n},\,p+1\right)$$

$$= \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \cdot \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p + 1\right)} \tag{1}$$

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^n}} = \int_0^1 x^0 (1-x^n)^{-\frac{1}{2}} \, \mathrm{d}x$$

Here $m = 0$, $n = n$, $p = -\frac{1}{2}$.

Using (1); we have

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^n}} = \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$$

Example 5.4 Prove that $\beta(n,n) = \dfrac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)}$

(or) $$\beta(n,n) = \frac{1}{2^{2n-1}} \cdot \beta\left(n, \frac{1}{2}\right)$$

$$\beta(n,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cdot \cos^{2n-1}\theta \, \mathrm{d}\theta \qquad \text{[using trigonometric form]}$$

$$= 2\int_0^{\pi/2} (\sin\theta \cos\theta)^{2n-1} \mathrm{d}\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2n-1} \mathrm{d}\theta$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} \sin^{2n-1} 2\theta \, \mathrm{d}\theta$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi} \sin^{2n-1}\phi \frac{\mathrm{d}\phi}{2}, \text{ putting } 2\theta = \phi$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} \sin^{2n-1}\phi \, d\phi \quad \left[\because \int_0^{\pi} f(\sin\phi)\, d\phi = 2\int_0^{\pi/2} f(\sin\phi)\, d\phi\right]$$

$$= \frac{1}{2^{2n-2}} \cdot \frac{1}{2}\,\beta\left(n, \frac{1}{2}\right)$$

$$= \frac{1}{2^{2n-1}}\,\beta\left(n, \frac{1}{2}\right)$$

$$= \frac{1}{2^{2n-1}} \cdot \frac{\Gamma(n)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}\cdot\Gamma(n)}{2^{2n-1}\cdot\Gamma\left(n+\frac{1}{2}\right)}$$

Example 5.5 Show that $\int_0^{\infty} x^n e^{-a^2x^2}\, dx = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$.

Deduce that $\int_0^{\infty} e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{2a}$. Hence show that

$$\int_0^{\infty} \cos(x^2)\, dx = \int_0^{\infty} \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

In $I = \int_0^{\infty} x^n e^{-a^2x^2}\, dx$, put $ax = \sqrt{t}$; then $dx = \frac{dt}{2a\sqrt{t}}$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$.

$\therefore$
$$I = \int_0^{\infty} \left(\frac{\sqrt{t}}{a}\right)^n e^{-t}\, \frac{dt}{2a\sqrt{t}}$$

$$= \frac{1}{2a^{n+1}} \int_0^{\infty} t^{\frac{n-1}{2}} \cdot e^{-t} dt$$

$$= \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right) \tag{1}$$

In (1), put $n = 0$.

Then
$$\int_0^{\infty} e^{-a^2x^2}\, dx = \frac{\Gamma\left(\frac{1}{2}\right)}{2a} = \frac{\sqrt{\pi}}{2a} \tag{2}$$

In (2), put $a = \frac{1-i}{\sqrt{2}}$; then $a^2 = -i$

$$\therefore \quad \int_0^\infty e^{ix^2}\,dx = \frac{\sqrt{\pi}}{\sqrt{2}(1-i)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i)$$

Equating the real parts on both sides,

$$\int_0^\infty \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,.$$

Equating the imaginary parts on both sides,

$$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,.$$

Example 5.6 Evaluate

(i) $\int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx$ and

(ii) $\int_{-a}^a (a+x)^{m-1}\cdot(a-x)^{n-1}\,dx$ in terms of Beta function.

(i) In $I_1 = \int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx$,

put $x - a = y$; then $dx = dy$

When $x = a$, $y = 0$; when $x = b$, $y = b - a$

$$\therefore \quad I_1 = \int_0^{b-a} y^{m-1}\left\{(b-a)-y\right\}^{n-1}\,dy$$

$$= (b-a)^{n-1}\int_0^{b-a} y^{m-1}\left(1-\frac{y}{b-a}\right)^{n-1}\,dy \qquad (1)$$

In (1), put $\frac{y}{b-a} = t$; then $dy = (b-a)\,dt$

When $y = 0$, $t = 0$; when $y = b - a$, $t = 1$.

$$\therefore \quad I_1 = (b-a)^{m+n-1}\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$= (b-a)^{m+n-1}\,\beta(m,n)$$

(ii) In $I_2 = \int_{-a}^{a} (a+x)^{m-1}\,(a-x)^{n-1}\,dx$,

put $a + x = y$; then $dx = dy$

When $x = -a$, $y = 0$; when $x = a$, $y = 2a$.

$$\therefore \quad I_2 = \int_0^{2a} y^{m-1}(2a-y)^{n-1}\,dy$$

$$= (2a)^{n-1} \int_0^{2a} y^{m-1} \left(1 - \frac{y}{2a}\right)^{n-1} \mathrm{d}y \qquad (2)$$

In (2), put $\frac{y}{2a} = t$; then $\mathrm{d}y = 2a\,\mathrm{d}t$.

When $y = 0$, $t = 0$; when $y = 2a$, $t = 1$.

$$\therefore \qquad \mathrm{I}_2 = (2a)^{m+n-1} \cdot \int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t \cdot$$

$$= (2a)^{m+n-1} \beta\,(m,n)$$

Example 5.7 Prove that $\displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,\mathrm{d}x \times \int_0^\infty x^2 e^{-x^4}\,\mathrm{d}x = \frac{\pi}{4\sqrt{2}}$

In $\mathrm{I}_1 = \displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,\mathrm{d}x$, put $x^2 = t$; then $\mathrm{d}x = \dfrac{\mathrm{d}t}{2x} = \dfrac{\mathrm{d}t}{2\sqrt{t}}$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$

$$\therefore \qquad \mathrm{I}_1 = \int_0^\infty \frac{e^{-t}}{t^{1/4}} \cdot \frac{\mathrm{d}t}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{-\frac{3}{4}}\,\mathrm{d}t$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

In $\mathrm{I}_2 = \displaystyle\int_0^\infty x^2\, e^{-x^4}\,\mathrm{d}x$, put $x^4 = s$; then $\mathrm{d}x = \dfrac{\mathrm{d}s}{4x^3} = \dfrac{\mathrm{d}s}{4s^{3/4}}$

When $x = 0$, $s = 0$; when $x = \infty$, $s = \infty$.

$$\therefore \qquad \mathrm{I}_2 = \int_0^\infty \sqrt{s}\, e^{-s} \cdot \frac{\mathrm{d}s}{4s^{3/4}} = \frac{1}{4}\int_0^\infty s^{-\frac{1}{4}}\, e^{-s}\mathrm{d}s$$

$$= \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\therefore \qquad \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,\mathrm{d}x \times \int_0^\infty x^2 e^{-x^4}\,\mathrm{d}x = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \qquad (1)$$

From Example 5.4;

$$\beta\,(n, n) = \frac{1}{2^{\,2n-1}} \cdot \beta\left(n, \frac{1}{2}\right)$$

i.e.
$$\frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \frac{1}{2^{\,2n-1}} \cdot \frac{\Gamma(n) \cdot \sqrt{\pi}}{\Gamma\left(n + \frac{1}{2}\right)}$$

$$\therefore \qquad \Gamma(n) \cdot \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}\;\Gamma(2n)}{2^{2n-1}}$$

Putting $n = \dfrac{1}{4}$, we get

$$\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}\cdot\Gamma\left(\frac{1}{2}\right)}{2^{-\frac{1}{2}}} = \pi\sqrt{2} \tag{2}$$

Using (2) in (1);

$$\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,\mathrm{d}x \times \int_0^\infty x^2 e^{-x^4}\,\mathrm{d}x = \frac{\pi\sqrt{2}}{8} = \frac{\pi}{4\sqrt{2}}.$$

Example 5.8 Evaluate $\displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x^n)^p}\,\mathrm{d}x$ and deduce that $\displaystyle\int_0^\infty \frac{x^{m-1}}{1+x^n}\,\mathrm{d}x = \frac{\pi}{n\sin\left(\frac{m\pi}{n}\right)}$. Hence show that $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

In $\mathrm{I} = \displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x^n)^p}\,\mathrm{d}x$, put $t = \dfrac{1}{1+x^n}$

Then $x^n = \dfrac{1-t}{t} \quad \therefore \quad nx^{n-1}\,\mathrm{d}x = -\dfrac{1}{t^2}\,\mathrm{d}t$

When $x = 0$, $t = 1$; when $x = \infty$, $t = 0$

$$\therefore \qquad \mathrm{I} = \int_0^1 \frac{t^{-\left(\frac{m-1}{n}\right)}\cdot(1-t)^{\frac{m-1}{n}}}{t^{-p}}\cdot\frac{\mathrm{d}t}{nt^2\cdot t^{-\frac{(n-1)}{n}}(1-t)^{\frac{n-1}{n}}}$$

$$= \frac{1}{n}\int_0^1 t^{p-\frac{m}{n}-1}\cdot(1-t)^{\frac{m}{n}-1}\,\mathrm{d}t$$

$$= \frac{1}{n}\beta\left(p-\frac{m}{n}, \frac{m}{n}\right)$$

$$= \frac{1}{n}\frac{\Gamma\left(p-\frac{m}{n}\right)\cdot\Gamma\left(\frac{m}{n}\right)}{\Gamma(p)} \tag{1}$$

Putting $p = 1$ in (1), we get

$$\int_0^\infty \frac{x^{m-1}\,\mathrm{d}x}{1+x^n} = \frac{1}{n}\Gamma\left(1-\frac{m}{n}\right)\Gamma\left(\frac{m}{n}\right)$$

$$= \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi m}{n}\right) \qquad \left[H\,int: \text{Use}\,\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}\right] \tag{2}$$

Taking $m = 1$ and $n = 4$ in (2), we get

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{4}\cdot \operatorname{cosec}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2\sqrt{2}}$$

Example 5.9 Find the value of $\iint x^{m-1} y^{n-1}\, dx\, dy$, over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, in terms of Gamma functions.

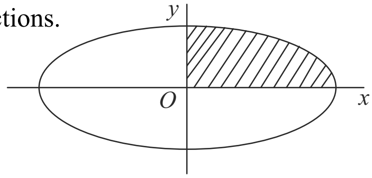


Fig. 5.53

Put $\frac{x}{a} = \sqrt{X}$ and $\frac{y}{b} = \sqrt{Y}$

Then $dx = \frac{a}{2\sqrt{X}}\, dX$ and $dy = \frac{b}{2\sqrt{Y}}\, dY$.

The region of double integration in the xy-plane is given by $x \geq 0, y \geq 0$ and

$\frac{x^2}{a^2}+\frac{y^2}{b^2} \leq 1$, shown in Fig. 5.53.

$\therefore$ The region of integration in the XY-plane is given by $X \geq 0$, $Y \geq 0$ and $X + Y \leq 1$, shown in Fig. 5.54.

The given integral

$$\mathrm{I} = \iint_{\Delta OAB} \left(a\sqrt{X}\right)^{m-1}\cdot\left(b\sqrt{Y}\right)^{n-1} \frac{ab}{4\sqrt{X}\sqrt{Y}}\, dX\, dY$$

$$= \frac{a^m b^n}{4} \iint_{\Delta OAB} X^{\frac{m}{2}-1} Y^{\frac{n}{2}-1}\, dX\, dY$$

$$= \frac{a^m b^n}{4} \int_0^1 \int_0^{1-Y} X^{\frac{m}{2}-1} Y^{\frac{n}{2}-1}\, dX\, dY$$

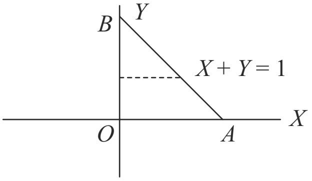


Fig. 5.54

$$\mathrm{I} = \frac{a^m b^n}{4} \int_0^1 Y^{\frac{n}{2}-1} \cdot \frac{2}{m}\left(X^{m/2}\right)_0^{1-Y} dY$$

$$= \frac{a^m b^n}{2m} \int_0^1 Y^{\frac{n}{2}-1} \cdot (1-Y)^{m/2}\, dY$$

$$= \frac{a^m b^n}{2m} \beta\cdot\left(\frac{n}{2}, \frac{m}{2}+1\right)$$

$$= \frac{a^m b^n}{2m}\cdot \frac{\Gamma\left(\frac{n}{2}\right)\cdot\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)}$$

$$= \frac{a^m b^n}{2m} \cdot \frac{\frac{m}{2}\Gamma\left(\frac{m}{2}\right)\cdot\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)}$$

$$= \frac{a^m b^n}{4} \cdot \frac{\Gamma\left(\frac{m}{2}\right)\cdot\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m+n}{2}+1\right)}$$

Example 5.10 Find the erea of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, using Gamma functions.

By symmetry of the astroid, required area $A =$

$4 \times$ area of $OACB = 4 \iint\limits_{OACB} \mathrm{d}x\,\mathrm{d}y$

Put $\left(\frac{x}{a}\right)^{2/3} = X$ and $\left(\frac{y}{a}\right)^{2/3} = Y$

i.e. $x = aX^{3/2}$ and $y = aY^{3/2}$

$\therefore\ \mathrm{d}x = \frac{3}{2}aX^{1/2}$ and $\mathrm{d}y = \frac{3}{2}aY^{1/2}\,\mathrm{d}Y$

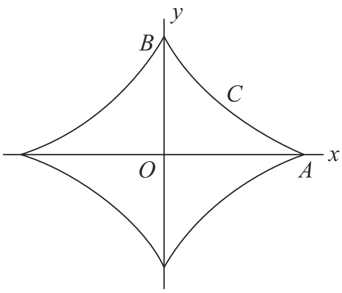


Fig. 5.55

The region of integration in the xy-plane is given by $x \ge 0, y \ge 0$ and $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} \le 1$, as shown in Fig. 5.55.

$\therefore$ The region of integration in the XY-plane is given by $X \ge 0,\ Y \ge 0$ and $X + Y \le 1$ as shown in Fig. 5.56.

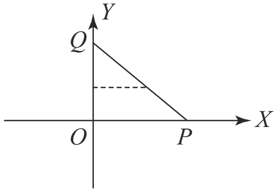


Fig. 5.56

$$\therefore \quad A = 4 \times \frac{9}{4} a^2 \iint\limits_{\Delta OPQ} X^{1/2}\, Y^{1/2}\, \mathrm{d}X\,\mathrm{d}Y$$

$$= 9a^2 \cdot \int_0^1 \int_0^{1-Y} X^{1/2}\, Y^{1/2}\, \mathrm{d}X\,\mathrm{d}Y$$

$$= 9a^2 \int_0^1 Y^{1/2} \left(\frac{2}{3} X^{3/2}\right)_0^{1-Y} \mathrm{d}Y$$

$$= 6a^2 \int_0^1 Y^{1/2} (1-Y)^{3/2}\, \mathrm{d}Y$$

$$= 6a^2 \times \beta\left(\frac{3}{2}, \frac{5}{2}\right)$$

$$= 6a^2 \times \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(4)}$$

$$= a^2 \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \qquad [\because \Gamma(4) = 3!]$$

$$= \frac{3}{8}\pi a^2 \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 5.11 Evaluate $\iint [xy(1-x-y)]^{1/2}\, dx\, dy$, over the area enclosed by the lines $x = 0$, $y = 0$ and $x + y = 1$ in the positive quadrant.

Given Intergral $I = \int_0^1 \int_0^{1-x} x^{1/2}\, y^{1/2}\, (1-x-y)^{1/2}\, dy\, dx,$

$$= \int_0^1 x^{1/2}\, dx \int_0^a y^{1/2}\, (a-y)^{1/2}\, dy$$

where $a = 1 - x$. (1)

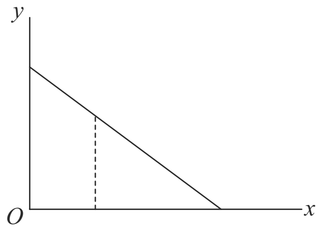


Fig. 5.57

Consider $\int_0^a y^{m-1}\, (a-y)^{n-1}\, dy$

$$= a^{n-1} \int_0^a y^{m-1} \left(1 - \frac{y}{a}\right)^{n-1} dy$$

$$= a^{n-1} \int_0^1 a^{m-1}\, z^{m-1}\, (1-z)^{n-1}\, a\, dz \qquad \left(\text{putting } \frac{y}{a} = z\right)$$

$$= a^{m+n-1} \cdot \beta(m, n) \qquad (2)$$

Note ☑ This result (2) will be of use in the following worked examples also.

Using (2) in (1) $\left[\text{note that } m = n = \frac{3}{2}\right]$,

$$I = \int_0^1 x^{1/2}\, (1-x)^2\, \beta\left(\frac{3}{2}, \frac{3}{2}\right) dx$$

$$= \beta\left(\frac{3}{2}, \frac{3}{2}\right) \times \beta\left(\frac{3}{2}, 3\right)$$

$$= \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} \times \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma(3)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{\frac{1}{2}\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi} \times \Gamma\left(\frac{3}{2}\right)}{\frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{2\pi}{105}$$

Example 5.12 Show that the volume of the region of space bounded by the coordinate planes and the surface $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$ is $\frac{abc}{90}$.

Required volume is given by

$$\text{Vol} = \iiint_V \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x, \text{ where } V \text{ is the region of space given.}$$

Put $\sqrt{\frac{x}{a}} = X, \sqrt{\frac{y}{b}} = Y, \sqrt{\frac{z}{c}} = Z$

i.e. $x = aX^2, y = bY^2, z = cZ^2$

$\therefore$ $\mathrm{d}x = 2aX\,\mathrm{d}X$, $\mathrm{d}y = 2bY\,\mathrm{d}Y$, $\mathrm{d}z = 2cZ\,\mathrm{d}Z$

$\therefore$ $\text{Vol} = \int\int\int_{V'} 8abc\, XYZ\, \mathrm{d}Z\, \mathrm{d}Y\, \mathrm{d}X$, where V' is the region of space in XYZ-space defined by $X \geq 0$, $Y \geq 0$, $Z \geq 0$, $X + Y + Z \leq 1$ [Refer to Fig. 5.58]

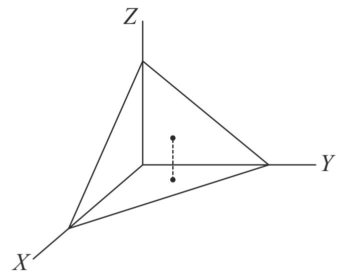


Fig. 5.58

$$\therefore \quad \text{Vol} = 8\,abc \int_0^1 \mathrm{d}X \int_0^{1-X} \mathrm{d}Y \int_0^{1-X-Y} XYZ\, \mathrm{d}Z$$

$$= 8\,abc \int_0^1 X\,\mathrm{d}X \int_0^{1-X} Y\,\mathrm{d}Y \left(\frac{Z^2}{2}\right)_0^{1-X-Y}$$

$$= 4\,abc \int_0^1 X\,\mathrm{d}X \int_0^{1-X} Y\cdot(1-X-Y)^2\,\mathrm{d}Y$$

$$= 4\,abc \int_0^1 X\,(1-X)^4 \cdot \beta(2,3)\,\mathrm{d}X \quad \text{[by step (2) of Example (5.11)]}$$

$$= 4\,abc \cdot \frac{\Gamma(2)\,\Gamma(3)}{\Gamma(5)} \cdot \beta(2,5)$$

$$= 4\,abc \times \frac{1\times 2}{24} \times \frac{\Gamma(2)\,\Gamma(5)}{\Gamma(7)}$$

$$= \frac{abc}{3} \times \frac{1\times 24}{720}$$

$$= \frac{abc}{90}$$

Example 5.13 Evaluate $\int\int\int \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\sqrt{1-x^2-y^2-z^2}}$, taken over the region of space in the positive octant bounded by the sphere $x^2 + y^2 + z^2 = 1$.

Put $x^2 = X$, $y^2 = Y$, $z^2 = Z$

$$\therefore \qquad \mathrm{d}x = \frac{\mathrm{d}X}{2\sqrt{X}}, \quad \mathrm{d}y = \frac{\mathrm{d}Y}{2\sqrt{Y}}, \quad \mathrm{d}z = \frac{\mathrm{d}Z}{2\sqrt{Z}}$$

The region of integration in *xyz*-space is defined by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x^2 + y^2 + z^2 \leq 1$.

$\therefore$ The region of integration V in the *XYZ*-space is defined by $X \geq 0$, $Y \geq 0$, $Z \geq 0$ and $X + Y + Z \leq 1$.

$\therefore$ Given intergral

$$\mathrm{I} = \iiint\limits_V \frac{1}{8} X^{-\frac{1}{2}} Y^{-\frac{1}{2}} Z^{-\frac{1}{2}} (1 - X - Y - Z)^{-\frac{1}{2}} \, \mathrm{d}X \cdot \mathrm{d}Y \cdot \mathrm{d}Z$$

$$= \frac{1}{8} \int_0^1 X^{-\frac{1}{2}} \, \mathrm{d}X \int_0^{1-X} Y^{-\frac{1}{2}} \, \mathrm{d}Y \int_0^{1-X-Y} Z^{-\frac{1}{2}} (1 - X - Y - Z)^{-\frac{1}{2}} \, \mathrm{d}Z$$

$$= \frac{1}{8} \int_0^1 X^{-\frac{1}{2}} \, \mathrm{d}X \int_0^{1-X} Y^{-\frac{1}{2}} \, \mathrm{d}Y \, (1 - X - Y)^{\frac{1}{2}+\frac{1}{2}-1} \cdot \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

[by step (2) of Example (5.11)]

$$= \frac{\pi}{8} \int_0^1 X^{-\frac{1}{2}} \, \mathrm{d}X \left(2Y^{\frac{1}{2}}\right)_0^{1-X} \qquad \left\{\because \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi\right\}$$

$$= \frac{\pi}{4} \int_0^1 X^{-\frac{1}{2}} (1 - X)^{1/2} \, \mathrm{d}X \cdot$$

$$= \frac{\pi}{4} \beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$= \frac{\pi}{4} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}$$

$$= \frac{\pi^2}{8}$$

Example 5.14 Evaluate $\displaystyle\iiint\limits_V \sqrt{a^2 b^2 c^2 - b^2 c^2 x^2 - c^2 a^2 y^2 - a^2 b^2 z^2} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$,

where V is the region defind by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$.

Put $\left(\dfrac{x}{a}\right)^2 = X, \left(\dfrac{y}{b}\right)^2 = Y, \left(\dfrac{z}{c}\right)^2 = Z$

i.e. $\quad x = a\sqrt{X}, \; y = b\sqrt{Y}, \; z = c\sqrt{Z}$

$$\therefore \qquad \mathrm{d}x = \frac{a}{2\sqrt{X}} \, \mathrm{d}X, \; \mathrm{d}y = \frac{b}{2\sqrt{Y}} \, \mathrm{d}Y, \; \mathrm{d}z = \frac{c}{2\sqrt{Z}} \, \mathrm{d}Z$$

The region V' of integration in the *XYZ*-space is defined by $X \geq 0$, $Y \geq 0$, $Z \geq 0$, $X + Y + Z \leq 1$.

$\therefore$ $$\text{Integral} = abc \iiint_{V'} \sqrt{1-X-Y-Z}\;\frac{abc\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z}{8\sqrt{X}\sqrt{Y}\sqrt{Z}}$$

$$= \frac{a^2b^2c^2}{8}\int_0^1 X^{-\frac{1}{2}}\,\mathrm{d}X\int_0^{1-X} Y^{-\frac{1}{2}}\,\mathrm{d}Y\int_0^{1-X-Y} Z^{-\frac{1}{2}}(1-X-Y-Z)^{\frac{1}{2}}\,\mathrm{d}Z$$

$$= \frac{a^2b^2c^2}{8}\int_0^1 X^{-\frac{1}{2}}\,\mathrm{d}X\int_0^{1-X} Y^{-\frac{1}{2}}\,\mathrm{d}Y\,(1-X-Y)\cdot\beta\left(\frac{1}{2},\frac{3}{2}\right)$$

[by step 2 of Example (5.11)]

$$= \frac{a^2b^2c^2}{8}\cdot\beta\left(\frac{1}{2},\frac{3}{2}\right)\cdot\int_0^1 X^{-\frac{1}{2}}\,\mathrm{d}X\,(1-X)^{\frac{3}{2}}\cdot\beta\left(\frac{1}{2},2\right)$$

[by step 2 of Example 5.11]

$$= \frac{a^2b^2c^2}{8}\cdot\beta\left(\frac{1}{2},\frac{3}{2}\right)\cdot\beta\left(\frac{1}{2},2\right)\cdot\beta\left(\frac{1}{2},\frac{5}{2}\right)$$

$$= \frac{a^2b^2c^2}{8}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)}$$

$$= \frac{\pi^2a^2b^2c^2}{32}.$$

Example 5.15 Find the value of $\iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,(1-x-y-z)^{p-1}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, taken over the interior of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$, in terms of Gamma functions.

$$\text{Given integral} = \int_0^1 x^{l-1}\,\mathrm{d}x\int_0^{1-x} y^{m-1}\,\mathrm{d}y\int_0^{1-x-y} z^{n-1}\,(1-x-y-z)^{p-1}\,\mathrm{d}z,$$

$$= \int_0^1 x^{l-1}\,\mathrm{d}x\int_0^{1-x} y^{m-1}\,(1-x-y)^{n+p-1}\cdot\beta(n,p)\,\mathrm{d}y,$$

[by step (2) of Example (5.11)]

$$= \beta(n,p)\int_0^1 x^{l-1}\,(1-x)^{m+n+p-1}\,\beta(m,n+p)\,\mathrm{d}x,$$

[by step (2) of Example (5.11)]

$$= \beta(n,p)\cdot\beta(m,n+p)\cdot\beta(l,m+n+p)$$

$$= \frac{\Gamma(n)\,\Gamma(p)}{\Gamma(n+p)}\cdot\frac{\Gamma(m)\,\Gamma(n+p)}{\Gamma(m+n+p)}\cdot\frac{\Gamma(l)\cdot\Gamma(m+n+p)}{\Gamma(l+m+n+p)}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\,\Gamma(p)}{\Gamma(l+m+n+p)}$$

EXERCISE 5(d)

Part A

(Short Answer Questions)

1. Prove that $\int_0^\infty x^4 e^{-x^2}\,dx = \frac{3}{8}\sqrt{\pi}$.

2. Evaluate $\int_0^\infty x\, e^{-x^3}\,dx$, given that $\Gamma\left(\frac{5}{3}\right) = 0.902$.

3. Find the value of $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\,dx$.

4. Find the value of $\int_0^{\pi/2} \sin^5\theta \cos^7\theta\,d\theta$.

5. Find the value of $\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$ in terms of Gamma functions.

6. Prove that $\int_0^{\pi/2} \sqrt{\cot\theta}\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$.

7. Find the value of $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$.

8. Prove that $\int_0^{\pi/2} \sqrt{\cos x}\,dx \times \int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}} = \pi$.

9. Prove that $\int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx = \Gamma(n)$.

10. Find the value of $\int_0^\infty \frac{x^n}{n^x}\,dx\ (n > 1)$.

11. Assuming that $\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin\alpha\pi}$, prove that $\Gamma(\alpha)\cdot\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}$, where α is neither zero nor an integer. [*Hint:* put $x = \tan^2\theta$]

12 Find the value of $\int_{-\infty}^{\infty} e^{-kx^2}\,dx$.

13. Prove that $\frac{\beta(m+1,n)}{\beta(m,n+1)} = \frac{m}{n}$.

14. Prove that $\frac{\beta(m+1,n)}{m}\,\frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n}$.

15. Find the value of $\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$ in terms of a Beta function.

$\left[\mathit{Hint}\text{: put } x = \frac{a}{b}t\right]$

16. Prove that $\beta(m + 1, n) + \beta(m, n + 1) = \beta(m, n)$.
17. Find the value of $\int_0^{2} (8-x^3)^{-\frac{1}{3}}\,dx$ in terms of Gamma functions.
18. Prove that $\int_0^{a} x^m (a-x)^n\,dx = a^{m+n+1} \cdot \beta(m+1, n+1)$.
19. Define Gamma and Beta functions.
20. Derive the recurrence formula for the Gamma function.
21. When n is a positive integer, prove that $\Gamma(n + 1) = n!$
22. State the relation between Gamma and Beta functions and use it to find the value of $\Gamma\left(\frac{1}{2}\right)$.

Part B

23. Prove that $\int_0^{\infty}\int_0^{\infty} e^{-(ax^2+by^2)}\, x^{2m-1}\, y^{2n-1}\,dx\,dy = \frac{1}{4a^m b^n}\Gamma(m)\Gamma(n)$.
24. When n is a positive integer and $m > -1$, prove that $\int_0^{1} x^m (\log x)^n\,dx = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$.
25. Prove that $\int_0^{1} \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n (a+b)^m}$.

$\left[\mathit{Hint}\text{: Put } \frac{x}{a+bx} = \frac{z}{a+b}\right]$

26. Express $\beta\left(n+\frac{1}{2}, n+\frac{1}{2}\right)$ in terms of Gamma functions in two different ways and hence prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}$.
27. Prove that $\int_0^{\infty} \sqrt{x}\, e^{-x^2}\,dx \times \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}}\,dx = \frac{\pi}{2\sqrt{2}}$.
28. Prove that $\int_0^{\infty} x\, e^{-x^8}\,dx \times \int_0^{\infty} x^2 e^{-x^4}\,dx = \frac{\pi}{16\sqrt{2}}$.
29. Prove that $\int_0^{1} \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^{1} \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4}$.

30. Evaluate (i) $\int_0^\infty \frac{dx}{1+x^4}$, (ii) $\int_0^\infty \frac{x^2\,dx}{(1+x^4)^2}$ and (iii) $\int_0^\infty \frac{x^2\,dx}{(1+x^4)^3}$

[*Hint:* put $x^2 = \tan^2\theta$]

31. Find the value of $\iint x^m y^n\,dx\,dy$, taken over the area $x \ge 0, y \ge 0, x + y \le 1$ in terms of Gamma functions, if $m, n > 0$.
32. Find the value, in terms of Gamma functions, of $\iiint x^m y^n z^p\,dx\,dy\,dz$ taken over the volume of the tetrahedron given by $x \ge 0$, $y \ge 0$, $z \ge 0$ and $x + y + z \le 1$.
33. Find the area in the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ and the co-ordinate axes.
34. Evaluate $\iint x^{m-1} y^{n-1}(1-x-y)^{p-1}\,dx\,dy$, taken over the area in the first quadrant enclosed by the lines $x = 0$, $y = 0$, $x + y = 1$.
35. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B and C. Find the volume of the tatrahedron $OABC$.
36. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.
37. Find the volume of the region of the space bounded by the co-ordinate planes and the surface $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n + \left(\frac{z}{c}\right)^n = 1$ and lying in the first octant.
38. Evaluate $\iiint \sqrt{xyz(1-x-y-z)}\,dx\,dy\,dz$, taken over the tetrahedral volume in the first octant enclosed by the plane $x=0, y=0, z=0$ and $x+y+z=1$.
39. Evaluate $\iiint x^2 yz\,dx\,dy\,dz$, taken throughout the volume in the first octant bounded by $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.
40. Evaluate $\iiint xyz\,dx\,dy\,dz$, taken over the space defined by $x \ge 0, y \ge 0, z \ge 0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$.

ANSWERS

Exercise 5(a)

(1) 4	(2) $\log a \log b$	(3) 2
(4) $\frac{1}{2}$	(5) $\frac{\pi}{4}$	(6) $\frac{9}{2}$

(7) $\frac{1}{2}$

(8)

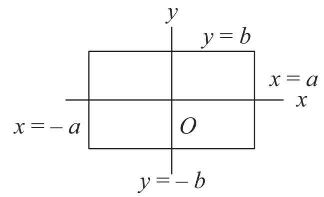


Fig. 5.59

(9)

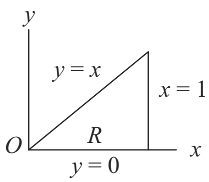


Fig. 5.60

(10)

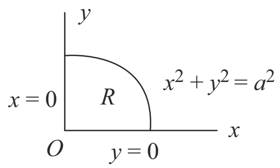


Fig. 5.61

(11)

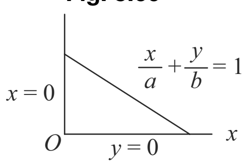


Fig. 5.62

(12) $\int_0^1 \int_0^{1-x} f(x,y)\,dy\;dx$ (or) $\int_0^1 \int_0^{1-y} f(x,y)\,dx\;dy$

(13) $\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$ (or) $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} f(x,y)\,dx\,dy$

(14) $\int_0^1 \int_0^y f(x,y)\,dx\,dy$ (or) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$

(15) $\int_0^2 \int_{\frac{y^2}{4}}^1 f(x,y)\,dx\,dy$ (or) $\int_0^1 \int_0^{2\sqrt{x}} f(x,y)\,dy\,dx$

(16) 2 log 2 (17) $\frac{a^3}{6}$ (18) $\frac{\pi}{4}$

(19) $\frac{\pi a^3}{6}$ (20) $\frac{1}{720}$

(21) $\frac{8}{3}\log 2 - \frac{19}{9}$ (22) 1 (23) $\frac{3}{2}$

(24) $\frac{\pi}{2}a^3$ (25) $\frac{1}{3}ab(a+b)$ (26) $\frac{344}{105}$

(27) 6 (28) $\frac{33}{2}$

(29) $\frac{1}{16}(8\log 2 - 5)$ (30) $\frac{1}{48}$

Exercise 5(b)

(1) $\int_0^a \int_y^a f(x,y)\,dx\,dy$ (2) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$

(3) $\int_0^a \int_0^y f(x,y)\,dx\,dy$ (4) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$

(5) $\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$ (6) $\int_0^a \int_0^{a-y} f(x,y)\,dx\,dy$

(7) $\int_0^1 \int_0^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$ (8) $\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

(9) $\int_0^2 \int_{\frac{y^2}{4}}^1 f(x,y)\,dx\,dy$ (10) $\int_0^\infty \int_0^{1/x} f(x,y)\,dy\,dx$

(11) $\frac{\pi a}{4}$ (12) $\frac{16}{3}$ (13) 1

(14) 1 (15) $\frac{1}{2}(e-1)^2$ (16) 2

(17) $\frac{9}{5}a^3$ (18) $\frac{1}{2}\log 2$ (19) $\frac{241}{60}$

(20) $\frac{\pi}{8}a^4$ (21) 3 (22) $8\log 2$

(23) $\frac{\pi}{4}$ (24) $\frac{3}{8}$ (25) $\frac{2}{3}$

(26) $\frac{32}{3}$ (27) $\frac{16}{3}a^2$ (28) $\frac{16}{3}ab$

(29) $\frac{\pi}{2}+\frac{1}{3}$ (30) 3π (31) $\frac{3}{2}\pi a^2$

(32) $a^2\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ (33) $\frac{a^2}{2}(3\pi-8)$ (34) πa^2

(35) $\frac{3}{4}\pi a^2$ (36) $\frac{\pi}{4}\left(1-e^{-a^2}\right)$ (37) $\frac{\pi a}{4}$

(38) $\frac{a^4}{4}\log\left(1+\sqrt{2}\right)$ (39) $\frac{4}{3}a^3$ (40) $\frac{2\pi}{a}$

(41) $\frac{\pi}{2}a^2h^2$ (42) 2π (43) 12π

(44) 16π (45) $\frac{16}{9}a^3(3\pi-4)$ (46) $4\pi a$

(47) $\frac{\pi^2}{8}$ (48) $\frac{\pi}{4}a^4$ (49) $\frac{\pi}{16}a^4$

(50) $\frac{\pi}{6a^2}$

Exercise 5(c)

(8) $\frac{2}{3}$ (9) 1 (10) $\frac{\pi}{2}$

(11) 88 (12) $4\pi a^2$ (13) $2\pi ah$

(14) 6 abc (15) $\frac{4}{3}\pi$ (16) $2\pi a^2 h$

(17) $-\frac{69}{10};-\frac{29}{4}$ (18) $-\frac{2}{3};-\frac{2}{3}$ (19) $\frac{4}{3}a^3$

(20) $2\pi ab$ (21) 4 (22) $\frac{3}{2}$

(23) $a^2\left(1-\frac{\pi a}{4}\right)$ (24) $\frac{1}{3}\left(2\sqrt{2}-1\right)$. (25) $\frac{163}{70}$

(26) $\frac{13}{3}$ (27) $\frac{13}{6}$ (28) $\frac{2a^3}{\sqrt{3}}$

(29) $\frac{\sqrt{2}}{3}\pi^3$ (30) $\frac{1}{2}\sqrt{a^2 b^2+b^2 c^2+c^2 a^2}$

(31) $\frac{\pi a^3}{4}$ (32) $\frac{ka^3}{2}$ (33) $\frac{abc}{6}$

(34) $\frac{\pi}{4}a^4$ (35) $\frac{\pi}{2}a^4 h$

Exercise 5(d)

(2) 0.456 (3) $\frac{8}{77}$ (4) $\frac{1}{120}$

(5) $\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (7) π (10) $\frac{1}{(\log n)^{n+1}}\Gamma(n+1)$

(12) $\sqrt{\frac{\pi}{k}}$ (15) $\frac{1}{a^n b^m}\beta(m,n)$ (17) $\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)$

(22) $\sqrt{\pi}$ (30) $\frac{\pi}{2\sqrt{2}};\frac{\pi}{8\sqrt{2}};\frac{5\pi\sqrt{2}}{128}$

(31) $\Gamma(m+1)\,\Gamma(n+1)\,/\,\Gamma(m+n+3)$.

(32) $\Gamma(m+1)\cdot\Gamma(n+1)\,\Gamma(p+1)\,/\,\Gamma(m+n+p+4)$

(33) $\dfrac{3\pi ab}{32}$ (34) $\Gamma(m)\,\Gamma(n)\,\Gamma(p)\,/\,\Gamma(m+n+p)$

(35) $\dfrac{abc}{6}$ (36) $\dfrac{4}{3}\pi\, abc$

(37) $abc\left\{\Gamma\left(\dfrac{1}{n}\right)\right\}^3 \Big/ 3n^2\ \Gamma\left(\dfrac{3}{n}\right)$ (38) $\pi^2/1920$.

(39) $a^3b^2c^2/2520$. (40) $a^2b^2c^2/48$.

Appendix **A**

Partial Differentiation

A.1 INTRODUCTION

In many situations we come across a quantity whose value depends on the values of more than one variable. For example, (i) the volume of a right circular cylinder is a function of the base radius and height; (ii) the volume of a cuboid depends on its length, breadth and height. If the value of the variable quantity u depends on the values of several other variable quantities $x, y, z, \ldots$, we say that u is a function of $x, y, z, \ldots$ and it is denoted as $u = f(x, y, z, \ldots)$; $x, y, z, \ldots$ are called independent variables and u is called the dependent variable.

A.2 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two independent variables. Let Δx be a small increment given to x and let Δz be the corresponding increment in z.
Then $\quad z + \Delta z = f(x + \Delta x, y)$

Note ☑ (We do not make any change in y, viz., y is kept constant)

$$\Delta z = f(x + \Delta x, y) - f(x, y)$$

Then $\lim\limits_{\Delta x \to 0} \left(\dfrac{\Delta z}{\Delta x} \right) = \lim\limits_{\Delta x \to 0} \left\{ \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \right\}$

is called *the partial derivative* of z with respect to x and denoted as $\dfrac{\partial z}{\partial x}$ (if the limit exists)

Smilarly,

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \left(\frac{\Delta z}{\Delta y} \right) = \lim_{\Delta y \to 0} \left(\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \right)$$

The process of finding the partial derivative of z with respect to x is similar to that of finding the ordinary derivative with respect to x, but with the only difference that we treat the other independent variables as constants temporarily.

Note ☑ All elementary rules of ordinary differentiation hold good for partial differentiation too. For example, if u and v are both functions of x and y, then

$$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

Partial derivatives of higher order

When $u = f(x, y)$, the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ will be, in general, functions of x and y. Hence each of them may be further differentiated partially with respect to x or y, giving partial derivatives of the second order.

Thus $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$ is a second order partial derivative of u

Similarly $$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

and $$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

Note ☑ The ways by which the two mixed second order partial derivatives $\frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x \partial y}$ are different, but they are equal when they are continuous. In almost all situations we may assume that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

Further partial differentiations will lead to third and higher order partial derivatives.

Alternative Notations

$\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are denoted also as u_x and u_y. $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ is denoted by u_{xy}, indicating that u_x is first found out and then it is differentiated with respect to y.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \text{ is denoted by } u_{yx}.$$

$\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$ are denoted by u_{xx} and u_{yy}.

A.3 HOMOGENEOUS FUNCTIONS

When a function of x and y can be rewritten as $x^n f\left(\dfrac{y}{x}\right)$, then the function is called a homogeous function of the nth degree in x and y.

The homogeneous polynomial in x, y of the nth degree, viz., $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n$, with which the students are familiar, may be considered a homogeneous function of the nth degree in x and y, as $a_0x^n + a_1x^{n-1}y + \cdots + a_{n-1}xy^{n-1} + a_ny^n$

$$\equiv x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\right]$$

$$= x^n f\left(\frac{y}{x}\right)$$

A.4 EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If u is a homogeneous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof

Since u is a homogeneous function of the nth degree in x, y, we may assume that

$$u = x^n f\left(\frac{y}{x}\right) \tag{1}$$

Differentiating (1) partially with respect to x,

$$\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\times\left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -x^{n-1}y\, f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) \tag{2}$$

Differentiating (1) partially with respect to y,

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\times\frac{1}{x}$$

$$\therefore \quad y\frac{\partial u}{\partial y} = x^{n-1}y\, f'\left(\frac{y}{x}\right) \tag{3}$$

Adding (2) and (3), we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n f\left(\frac{y}{x}\right)$$
$$=nu, \text{ by (1)}$$

Note ☑ If u is a function of several variables x, y, z,…, such that $u = x^n f\left(\frac{y}{x},\frac{z}{x},\ldots\right)$, then u is said to be a homogeneous function of the nth degree in x, y, z,…. In this case, Euler's theorem will be $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial x}+\cdots=nu.$

A.5 EULER'S THEOREM FOR SECOND DERIVATIVES

If u is a homogeneous function of degree n in x and y, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Proof

By Euler's theorem for first derivatives, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \tag{1}$$

Differentiating (1) partially with respect to x,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}$$

i.e.,
$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \tag{2}$$

Differentiating (1) partially with respect to y,

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}$$

i.e.,
$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \tag{3}$$

(2) × x + (3) × y gives

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$
$$=(n-1)\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\}$$
$$= n(n-1)\,u, \text{ by Euler's theorem}$$

WORKED EXAMPLE

Example 1 If $u = \log(\tan x + \tan y + \tan z)$, prove that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \cdot \dfrac{\partial u}{\partial y} + \sin 2z \cdot \dfrac{\partial u}{\partial z} = 2$.

$$u = \log(\tan x + \tan y + \tan z)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\sec^2 x}{\Sigma \tan x}$$

$$\therefore \quad \sin 2x \frac{\partial u}{\partial x} = \frac{2 \sin x \cos x \sec^2 x}{\Sigma \tan x}$$

$$= \frac{2 \tan x}{\Sigma \tan x} \qquad (1)$$

Similarly,
$$\sin 2y \frac{\partial u}{\partial y} = \frac{2 \tan y}{\Sigma \tan x} \qquad (2)$$

and
$$\sin 2z \frac{\partial u}{\partial z} = \frac{2 \tan z}{\Sigma \tan x} \qquad (3)$$

Adding (1), (2) and (3), we get

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{\Sigma \tan x}$$

$$= 2$$

Example 2 If $u = \log(x^3 + y^3 + z^3 - 3xyz)$,

show that (i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x + y + z}$

(ii) $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = -\dfrac{9}{(x + y + z)^2}$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz}$$

Similarly
$$\frac{\partial u}{\partial y} = \frac{3(y^2 - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

and $$\frac{\partial u}{\partial z} = \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz}$$

Adding, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3}{x+y+z} \quad (\because \text{ the denominator}$$
$$= (x+y+z)\{\Sigma x^2 - \Sigma xy\})$$

i.e., $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u = \frac{3}{x+y+z} \qquad (1)$$

$$\therefore \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ by (1)}$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= -\frac{9}{(x+y+z)^2}$$

Example 3 If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(i) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$ (ii) $r\dfrac{\partial \theta}{\partial x} = \dfrac{1}{r}\dfrac{\partial x}{\partial \theta}$

(iii) $\dfrac{\partial^2}{\partial x^2}(\log r) = -\dfrac{\partial^2}{\partial y^2}(\log r) = \dfrac{\partial^2 \theta}{\partial x \partial y} = -\dfrac{\cos 2\theta}{r^2}$

(iv) $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0$.

$$x = r\cos\theta \qquad (1)$$

and $$y = r\sin\theta \qquad (2)$$

$\therefore$ $$r^2 = x^2 + y^2 \qquad (3)$$

and $$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad (4)$$

Differentiating (3) partially with respect to x, we have

$$2r\frac{\partial r}{\partial x} = 2x$$

$\therefore$ $$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta$$

$$\therefore \qquad \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r},$$

which is the required result (i)
Differentiating (4) partially w.r.t. *x*,

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y^2}{x^2}\right)} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$= -\frac{r\sin\theta}{r^2} = -\frac{1}{r}\sin\theta$$

i.e.,
$$r\frac{\partial \theta}{\partial x} = -\sin\theta \tag{5}$$

Differentiating (1) partially w.r.t. θ,

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\therefore \qquad \frac{1}{r}\frac{\partial x}{\partial \theta} = -\sin\theta \tag{6}$$

From (5) and (6), $r\dfrac{\partial \theta}{\partial x} = \dfrac{1}{r}\dfrac{\partial x}{\partial \theta}$, which is required in (ii)

From (3), 2 log r = log $(x^2 + y^2)$

$$\therefore \qquad \frac{\partial}{\partial x}(\log r) = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2}{\partial x^2}(\log r) = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{r^2(\sin^2\theta - \cos^2\theta)}{r^4}$$

$$= -\frac{1}{r^2}\cos 2\theta \tag{7}$$

Now
$$\frac{\partial}{\partial y}(\log r) = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}$$

$$\therefore \qquad \frac{\partial^2}{\partial y^2}(\log r) = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{1}{r^2} \cos 2\theta \tag{8}$$

From (4), we get $\dfrac{\partial\theta}{\partial y} = \dfrac{1}{1+\dfrac{y^2}{x^2}} \cdot \dfrac{1}{x} = \dfrac{x}{x^2+y^2}$

$$\therefore \qquad \frac{\partial^2\theta}{\partial x\partial y} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = -\frac{1}{r^2}\cos 2\,\theta \tag{9}$$

From (7), (8), (9), the required result (iii) follows.

From (4), we have $\dfrac{\partial\theta}{\partial x} = \dfrac{1}{1+\dfrac{y^2}{x^2}} \cdot \left(-\dfrac{y}{x^2}\right)$

$$= -\frac{y}{x^2+y^2}$$

$$\therefore \qquad \frac{\partial^2\theta}{\partial y^2} = -y \cdot -\frac{1}{(x^2+y^2)^2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2} \tag{10}$$

From (4) again, we have

$$\frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2}$$

$$\therefore \qquad \frac{\partial^2\theta}{\partial y^2} = x \cdot -\frac{1}{(x^2+y^2)^2} \cdot 2y = -\frac{2xy}{(x^2+y^2)^2} \tag{11}$$

From (10) and (11), the required result (iv) follows

Example 4 If $u = (x^2 + y^2 + z^2)^{-1/2}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Putting $\qquad x^2 + y^2 + z^2 = r^2, \qquad (1)$

we get $\qquad u = \dfrac{1}{r}$

$$\therefore \qquad \frac{\partial u}{\partial x} = -\frac{1}{r^2} \cdot \frac{\partial r}{\partial x} = -\frac{1}{r^2} \cdot \frac{x}{r}, \text{ from (1)}$$

$$\therefore \qquad \frac{\partial^2 u}{\partial x^2} = -\left[\frac{r^3 \cdot 1 - x \cdot 3r^2 \dfrac{\partial r}{\partial x}}{r^6}\right]$$

$$= -\left[\frac{r^3 - 3xr^2 \cdot \dfrac{x}{r}}{r^6}\right], \text{ from (1)}$$

$$= \frac{3x^2 - r^2}{r^5}$$

Since u is a symmetric function in x, y, z,

we get $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{3y^2 - r^2}{r^5}$ and $\dfrac{\partial^2 u}{\partial z^2} = \dfrac{3z^2 - r^2}{r^5}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3(x^2 + y^2 + z^2) - 3r^2}{r^5}$$

$$= \frac{3r^2 - 3r^2}{r^5} = 0, \text{ by (1)}$$

Example 5 If $u^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$$

$$u^2 = (x - a)^2 + (y - b)^2 + (z - c)^2 \tag{1}$$

$$\therefore \quad 2u\frac{\partial u}{\partial x} = 2(x - a)$$

i.e.,
$$\frac{\partial u}{\partial x} = \frac{x - a}{u} \tag{2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{u - (x - a)\dfrac{\partial u}{\partial x}}{u^2}$$

$$= \frac{u - \left(\dfrac{x - a}{u}\right)^2}{u^2}, \text{ by (2)}$$

$$= \frac{u^2 - (x - a)^2}{u^3} \tag{3}$$

Similarly, by symmetry, we have

$$\frac{\partial^2 u}{\partial y^2} = \frac{u^2 - (y - b)^2}{u^3} \tag{4}$$

and
$$\frac{\partial^2 u}{\partial z^2} = \frac{u^2 - (z - c)^2}{u^3} \tag{5}$$

Adding (3), (4) and (5), we get

$$\sum \frac{\partial^2 u}{\partial x^2} = \frac{3u^2 - \{(x-a)^2 + (y-b)^2 + (z-c)^2\}}{u^3}$$

$$= \frac{3u^2 - u^2}{u^3}, \quad \text{by (1)}$$

$$= \frac{2}{u}.$$

Example 6 If $V = \dfrac{xz}{x^2 + y^2}$, prove that V satisfies

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V = \frac{xz}{x^2 + y^2}$$

$\therefore$

$$\frac{\partial V}{\partial x} = z\left\{\frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2}\right\} = \frac{z(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 V}{\partial x^2} = z\left[\frac{(x^2 + y^2)^2 \cdot (-2x) - (y^2 - x^2) \cdot 2(x^2 + y^2) \cdot 2x}{(x^2 + y^2)^4}\right]$$

$$= -\frac{2xz(3y^2 - x^2)}{(x^2 + y^2)^3} \tag{1}$$

Note ☑ V is not symmetric in x, y, z

$$\frac{\partial V}{\partial y} = xz \cdot \frac{-1}{(x^2 + y^2)^2} \cdot 2y = -\frac{2xyz}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 V}{\partial y^2} = -2xz\left[\frac{(x^2 + y^2)^2 \cdot 1 - y \cdot 2(x^2 + y^2) \cdot 2y}{(x^2 + y^2)^4}\right]$$

$$= \frac{-2xz(x^2 - 3y^2)}{(x^2 + y^2)^3} \tag{2}$$

Now $$\frac{\partial V}{\partial z} = \frac{x}{x^2 + y^2} \quad \text{and} \quad \frac{\partial^2 V}{\partial z^2} = 0 \tag{3}$$

Adding (1), (2), (3), the required result follows.

Example 7 If $x^x \cdot y^y \cdot z^z = 1$, find the value of $\dfrac{\partial^2 z}{\partial x \partial y}$ when $x = y = z = 1$.

$$x^x \cdot y^y \cdot z^z = 1$$

Taking logarithms, we have

$$z \log z = -x \log x - y \log y \tag{1}$$

Differentiating (1) partially w.r.t. y,

$$z \cdot \frac{1}{z}\frac{\partial z}{\partial y} + \log z \frac{\partial z}{\partial y} = -\left\{y \cdot \frac{1}{y} + \log y\right\}$$

i.e.,
$$(1 + \log z)\frac{\partial z}{\partial y} = -(1 + \log y) \tag{2}$$

Differentiating (2) partially w.r.t. x,

$$(1 + \log z)\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x} = 0$$

$\therefore$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-\frac{1}{z}\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}}{(1 + \log z)} \tag{3}$$

Differentiating (1) partially w.r.t. x, we can get $(1 + \log z)\dfrac{\partial z}{\partial x} = -(1 + \log x)$ (4)

Using (2) and (4) in (3), we have

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1 + \log x)(1 + \log y)}{z(1 + \log z)^3}$$

$\therefore$
$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)_{x=y=z=1} = -\frac{1 \cdot 1}{1 \cdot 1^3} = -1$$

Example 8 If $f(x, y) = (1 - 2xy + y^2)^{-1/2}$, show that

$$\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial f}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2 \frac{\partial f}{\partial y}\right] = 0$$

$$f(x, y) = (1 - 2xy + y^2)^{-1/2} \tag{1}$$

$\therefore$
$$\frac{\partial f}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2} \times (-2y)$$

$$= \frac{y}{(1 - 2xy + y^2)^{3/2}}$$

$\therefore$
$$\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{(1 - x^2)y}{(1 - 2xy + y^2)^{3/2}}\right]$$

$$= y\left[\frac{(1-2xy+y^2)^{3/2}\cdot(-2x)-(1-x^2)\cdot\frac{3}{2}(1-2xy+y^2)^{1/2}\cdot(-2y)}{(1-2xy+y^2)^3}\right]$$

$$= y\left[\frac{-2x(1-2xy+y^2)+3(1-x^2)y}{(1-2xy+y^2)^{5/2}}\right]$$

$$= y\left[\frac{-2x+3y+x^2y-2xy^2}{(1-2xy+y^2)^{5/2}}\right] \quad (2)$$

Differentiating (1) partially w.r.t. y,

$$\frac{\partial f}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}\times(-2x+2y)$$

$$\therefore \qquad y^2\frac{\partial f}{\partial y} = y^2(x-y)(1-2xy+y^2)^{-3/2}$$

$$\therefore \quad \frac{\partial}{\partial y}\left[y^2\frac{\partial f}{\partial y}\right] = \frac{\left[\begin{array}{l}(1-2xy+y^2)^{3/2}(2xy-3y^2)- \\ \qquad (y^2x-y^3)\cdot\frac{3}{2}(1-2xy+y^2)^{1/2}\cdot(2y-2x)\end{array}\right]}{(1-2xy+y^2)^3}$$

$$= \frac{(1-2xy+y^2)(2xy-3y^2)+3(y^2x-y^3)(x-y)}{(1-2xy+y^2)^{5/2}}$$

$$= \frac{-y\{-2x+3y+x^2y-2xy^2\}}{(1-2xy+y^2)^{5/2}} \quad (3)$$

Adding (2) and (3), the required result follows:

Example 9 Verify that $\dfrac{\partial^2 u}{\partial y\partial x} = \dfrac{\partial^2 u}{\partial x\partial y}$,

when $\qquad u = x^2\tan^{-1}\left(\dfrac{y}{x}\right) - y^2\tan^{-1}\left(\dfrac{x}{y}\right).$

$$u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right) \quad (1)$$

$$\frac{\partial u}{\partial x} \doteq x^2\cdot\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)+2x\tan^{-1}\left(\frac{y}{x}\right)-y^2\cdot\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}$$

$$= -\frac{x^2 y}{x^2+y^2} + 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{y^3}{x^2+y^2}$$

$$= -y + 2x\tan^{-1}\left(\frac{y}{x}\right) \tag{2}$$

$$\therefore \qquad \frac{\partial^2 u}{\partial y \partial x} = -1 + 2x \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$= -1 + \frac{2x^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \tag{3}$$

Differentiating (1) partially w.r.t. y,

$$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} - \left\{ y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) + 2y\tan^{-1}\left(\frac{x}{y}\right)\right\}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\therefore \qquad \frac{\partial^2 u}{\partial x \partial y} = 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \tag{4}$$

From (3) and (4), we see that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Example 10 Verify that $\dfrac{\partial^2 V}{\partial y \partial x} = \dfrac{\partial^2 V}{\partial x \partial y}$,

when $\qquad V = x^y \cdot y^x$

$V = x^y \cdot y^x$

Taking logarithms,

$$\log V = y\log x + x\log y \tag{1}$$

$$\therefore \qquad \frac{1}{V}\frac{\partial V}{\partial y} = \log x + \frac{x}{y} \tag{2}$$

and
$$\frac{1}{V}\frac{\partial V}{\partial x} = \frac{y}{x} + \log y \tag{3}$$

Differentiating (2) partially w.r.t. x;

$$\frac{1}{V}\frac{\partial^2 V}{\partial x \partial y} - \frac{1}{V^2}\frac{\partial V}{\partial x}\cdot\frac{\partial V}{\partial y} = \frac{1}{x} + \frac{1}{y}$$

i.e.,
$$\frac{\partial^2 V}{\partial x \partial y} = V\left\{\frac{1}{V^2}\frac{\partial V}{\partial x}\cdot\frac{\partial V}{\partial y} + \frac{1}{x} + \frac{1}{y}\right\} \tag{4}$$

Differentiating (3) partially w.r.t. y;

$$\frac{1}{V}\frac{\partial^2 V}{\partial y \partial x} - \frac{1}{V^2}\frac{\partial V}{\partial y}\frac{\partial V}{\partial x} = \frac{1}{x} + \frac{1}{y}$$

$$\therefore \qquad \frac{\partial^2 V}{\partial y \cdot \partial x} = V\left\{\frac{1}{V^2}\frac{\partial V}{\partial x}\cdot\frac{\partial V}{\partial y} + \frac{1}{x} + \frac{1}{y}\right\} \tag{5}$$

From (4) and (5), we see that $\dfrac{\partial^2 V}{\partial x \partial y} = \dfrac{\partial^2 V}{\partial y \partial x}$.

Example 11 Verify Euler's theorem, when (i) $u = ax^2 + 2hxy + by^2$ and (ii) $u = e^{x^3+y^3}$

(i) $u = ax^2 + 2hxy + by^2$

$$\frac{\partial u}{\partial x} = 2ax + 2hy; \quad \frac{\partial u}{\partial y} = 2hx + 2by$$

$$\therefore \qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (2ax^2 + 2hxy) + (2hxy + 2by^2)$$

$$= 2\,(ax^2 + 2hxy + by^2)$$
$$= 2u \tag{1}$$

$$u = x^2\left\{a^2 + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2\right\} = x^2 f\left(\frac{y}{x}\right)$$

$\therefore$ u is a homogeneous function of degree 2.

$\therefore$ Step (1) verifies Euler's theorem.

(ii) $u = e^{x^3+y^3}$

$$\therefore \qquad v = \log u = x^3 + y^3$$

is a homogeneous functions of degree 3.

$$\frac{\partial v}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} = 3x^2$$

$$\frac{\partial v}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y} 3y^2$$

$$\therefore \qquad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3\left(x^3 + y^3\right)$$

$$= 3v \tag{2}$$

Step (2) verifies Euler's theorem

Note ☑ (2) can be rewritten as

$$x\cdot\frac{1}{u}\frac{\partial u}{\partial x} + y\cdot\frac{1}{u}\frac{\partial u}{\partial y} = 3\log u$$

i.e.,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u$$

Example 12 If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$, (i), by using Euler's theorem, (ii) without using Euler's theorem

(i) $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$

i.e.,
$$\sin u = \frac{x^2 + y^2}{x + y} = \frac{x^2\left\{1 + \left(\frac{y}{x}\right)\right\}^2}{x\left\{1 + \left(\frac{y}{x}\right)\right\}} = xf\left(\frac{y}{x}\right)$$

$\therefore$ $\sin u$ is a homogeneous function of order (degree) 1.

$\therefore$ By Euler's theorem, we have

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 1\times\sin u$$

i.e.,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

(ii) $\sin u = \frac{x^2 + y^2}{x + y}$ or $(x + y)\sin u = x^2 + y^2$ (1)

Differentiating (1) partially w.r.t. x,

$$(x + y)\cos u\frac{\partial u}{\partial x} + \sin u = 2x \tag{2}$$

Differentiating (1) partially w.r.t. y,

$$(x + y)\cos u\frac{\partial u}{\partial y} + \sin u = 2y \tag{3}$$

(2) × x + (3) × y gives,

$$(x+y)\cos u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]+(x+y)\sin u=2\left(x^2+y^2\right)$$

i.e., $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\tan u=\frac{2\left(x^2+y^2\right)}{(x+y)\cos u}$$

$$=2\frac{\sin u}{\cos u}\quad \text{or}\quad 2\tan u$$

$$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$$

Example 13 If $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$, find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

$$u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$$

$$=x^2\left[\tan^{-1}\left(\frac{y}{x}\right)-\left(\frac{y}{x}\right)^2\tan^{-1}\left(\frac{y}{x}\right)^{-1}\right]$$

$$=x^2 f\left(\frac{y}{x}\right)$$

∴ u is a homogenous function of degree 2.

∴ By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$

Note ☑ From example (9), $\frac{\partial u}{\partial x}=-y+2x\tan^{-1}\left(\frac{y}{x}\right)$

and $$\frac{\partial u}{\partial y}=x-2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-xy+2x^2\tan^{-1}\left(\frac{y}{x}\right)+xy-2y^2\tan^{-1}\left(\frac{y}{x}\right)$$

$$=2\left\{x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)\right\}$$

$$=2u$$

Example 14 If $u = \tan^{-1}\left(\dfrac{x^2+y^2}{x-y}\right)$, find the value of

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

$$u = \tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$$

$\therefore$ $\tan u = \dfrac{x^2+y^2}{x-y}$ is a homogeneous function of degree 1.

$\therefore$ By Euler's theorem,

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u$$

i.e.,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} = \sin u \cos u$$

$$= \frac{1}{2}\sin 2u \tag{1}$$

Differentiating (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \cos 2u\frac{\partial u}{\partial x}$$

$\therefore$
$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} = x\cos 2u\frac{\partial u}{\partial x} \tag{2}$$

Differentiating (1) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \cos 2u\frac{\partial u}{\partial y}$$

$\therefore$
$$xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = y\cos 2u\frac{\partial u}{\partial y} \tag{3}$$

Adding (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$
$$= \cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\therefore \qquad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\sin 2u$$

$$= \frac{1}{2}\sin 2u \cos 2u, \text{ by (1)}$$

i.e.,
$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = \frac{1}{2}\sin 2u(\cos 2u - 1)$$

EXERCISE

Part A

(Short answer questions)

1. If $u = (x - y)^4 + (y - z)^4 + (z - x)^4$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$
2. If $u = (x - y)(y - z)(z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial u}{\partial z} = 0$.
3. If $u = x^3y - xy^3$, show that the value of $\left(\frac{\partial u}{\partial x}\right)^{-1} + \left(\frac{\partial u}{\partial y}\right)^{-1}$ at the point (1, 2) is $-\frac{13}{22}$.
4. If $z^3 - 3yz - 3x = 0$, show that $z\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$.
5. If $u = x \cos y + y \sin x$, verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$
6. If $u = \log\left(\frac{x^2 + y^2}{xy}\right)$, verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.
7. Verify Euler's theorem when $u = \sqrt{x} + \sqrt{y}$.
8. Verify Euler's theorem, when $u = x^3 \sin\left(\frac{y}{x}\right)$.
9. If $u = f\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$
10. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$
11. Define a homogeneous function and state Euler's theorem on homogeneous functions.

12. State Euler's theorem on homogeneous functions for second order derivatives.

13. If $u = xf\left(\dfrac{y}{x}\right)$, show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$$

14. If $u = \dfrac{xy}{x+y}$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0$

15. If $x = r \cos \theta$ and $y = r \sin \theta$, prove that $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1$

Part B

16. If $z(x + y) = x^2 + y^2$, prove that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$

17. If $u = \log (x^2 + y^2)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

18. If $u = \log (x^2 + y^2 + z^2)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{x^2 + y^2 + z^2}$$

19. If $u = \dfrac{1}{r}$, where $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0 \cdot$

20. If $V = \dfrac{1}{r}$, where $r^2 = x^2 + y^2$, prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = \dfrac{1}{r^3}$.

21. If $V = \log r$, where $r^2 = (x - a)^2 + (y - b)^2$, prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0.$

22. If $y = f(x - at) + \phi(x + at)$, show that $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$.

23. If $z = \tan (y + ax) + (y - ax)^{3/2}$, prove that $\dfrac{\partial^2 z}{\partial x^2} - a^2 \dfrac{\partial^2 z}{\partial y^2} = 0$.

24. If $x^x y^y z^z = c$, show that, when $x = y = z$, $\dfrac{\partial^2 z}{\partial x \partial y} = -\{x \cdot (1 + \log x)\}^{-1}$

25. If $x = r\cos\theta$ and $y = r\sin\theta$, prove that

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$$

26. If $V = e^{a\theta}\cos(a\log r)$, prove that

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}=0\cdot$$

27. Verify that $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$, when $u = x^y + y^x$.

28. Verify that $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$, when

$$u = 3xy - y^2 + (y^2 - 2x)^{3/2}.$$

29. If $u=\tan^{-1}\dfrac{xy}{1+x^2+y^2}$, show that

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{(1+x^2+y^2)^{3/2}}$$

30. Without using Euler's theorem, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1,\text{ when }u=\log\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

31. If $u=\cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0.$$

32. If $u=\sin^{-1}\left(\dfrac{x^3-y^3}{x+y}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u.$$

33. If $u=e^{x^3+y^3}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=3u\log u$

34. If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u \cos 2u}{4\cos^3 u}$$

35. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$$

Appendix **B**

Solutions to the January 2012 Question Paper

B.E./B.Tech. DEGREE EXAMINATIONS, JANUARY 2012
(First Semester)

MA2111 MATHEMATICS - I
(Common to all Branches)
(Regulations 2008)

Times: 3 Hours **Maximum: 100 marks**

Answer ALL questions.

PART-A **(10 × 2 = 20 marks)**

1. The product of two eigenvalues of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigenvalue of A.

2. Can $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be diagonalized? Why?

3. Find the equation of the sphere concentric with $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$ and passing through the point (1, 2, 3).

4. Find the equation of the right circular cone with vertex at the origin, whose axis is $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$ and with a semi-vertical angle 30°.

5. Find the radius of curvature for $y = e^x$ at the point where it cuts the y-axis.

6. Find the envelope of the family of lines $\frac{x}{t} + yt = 2c$, where t is the parameter.

7. If $u = x^y$, show that $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$.

8. If $x = u^2 - v^2$ and $y = 2uv$, find the Jacobian of x and y with respect to u by v.

9. Express $\int_0^\infty \int_0^\infty f(x, y)\, dxdy$ in polar co-ordinates.

10. Evaluate $\int_0^1 \int_0^y \int_0^{x+y} dxdydz$.

Part-B **(5 × 16 = 80 marks)**

11. (a) (i) Find the eigenvalues and eigenvectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. (8)

(ii) Find A^n using Cayley Hamilton theorem, taking $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Hence find A^3. (8)

Or

(b) Reduce the quadratic form $2x^2 + 5y^2 + 3z^2 + 4xy$ to canonical form by orthogonal reduction and state its nature. (16)

12. (a) (i) Obtain the equation of the sphere having the circle $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$, $x + y + z = 3$ as the greatest circle. (8)

(ii) Find the equation of the cone formed by rotating the line $2x + 3y = 6$, $z = 0$ about the y-axis. (8)

Or

(b) (i) Obtain the equation of the tangent planes to the sphere $x^2 + y^2 + z^2 + 2x - 4y + 6z - 7 = 0$ which intersect in the line $6x - 3y - 23 = 0 = 3z + 2$. (8)

(ii) Find the equation of the right circular cylinder of radius 2 and whose axis is the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$. (8)

13. (a) (i) If $y = \frac{ax}{a+x}$, prove that $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$, where ρ is the radius of curvature. (8)

(ii) Find the circle of curvature of $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$. (8)

Or

(b) (i) Find the evolute of the parabola $y^2 = 4ax$. (8)

(ii) Find the envelope of $\frac{x}{l}+\frac{y}{m}=1$, where the parameters l and m are connected by the relation $\frac{l}{a}+\frac{m}{b}=1$ (a and b are constants). (8)

14. (a) (i) If $z = f(x, y)$, where $x = u^2 - v^2$, $y = 2uv$, prove that $\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$. (8)

(ii) Find the Taylor's series expansion of $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x + 2)$ and $(y - 1)$ upto 3^{rd} degree terms. (8)

Or

(b) **(i)** If $x + y + z = u$, $y + z = uv$, $z = uvw$, prove that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2v$. (8)

(ii) Find the extreme values of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$. (8)

15. (a) (i) Change the order of integration in $\int_0^a \int_0^{b(\sqrt{a^2-x^2})} x^2\, dydx$ and then evaluate it. (8)

(ii) Transform the double integral $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$ into polar co-ordinates and then evaluate it. (8)

Or

(b) **(i)** Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$. (8)

(ii) Find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$ (8)

SOLUTIONS

PART-A

1. Product of all the three eigenvalues $= |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$

Product of 2 eigenvalues = 16; $\therefore$ the third eigenvalues = 2.

2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ cannot be further diagonalised, as it is already in the diagonal form.

3. Equation of a sphere concentric with the given sphere is of the form $x^2 + y^2 + z^2 - 4x + 6y - 8z + k = 0$.
Since it passes though the point (1, 2, 3), $1 + 4 + 9 - 4 + 12 - 24 + k = 0$
i.e., $k = 2$
$\therefore$ The required sphere is $\sum x^2 - 4x + 6y - 8z + 2 = 0$.

4. The equation of the right circular cone is

$$\{1^2 + (-1)^2 + 2^2\}\,(x^2 + y^2 + z^2)\,\frac{3}{4} = (x - y + 2z)^2$$

i.e., $9(x^2 + y^2 + z^2) = 2(x^2 + y^2 + 4z^2 - 2xy - 4yz + 4zx)$

i.e., $7x^2 + 7y^2 + z^2 + 4xy + 8yz - 8zx = 0.$

5. The point where $y = e^x$ cuts the y-axis is (0, 1)

$$y' = e^x = y'';\ \rho = \frac{(1 + y'^2)\,3/2}{y''} = \frac{(1 + e^{2x})\,3/2}{e^x}$$

$\therefore \quad [\rho]_{(0,\,1)} = 2\sqrt{2}$.

6. The equation of the family is $yt^2 - 2ct + x = 0$

Equation of the envelope is $B^2 = 4AC$; i.e., $4c^2 = 4xy$ or $xy = c^2$

7. $x = x^y;\ \dfrac{\partial u}{\partial y} = x^y \log x\,;\ \dfrac{\partial^2 u}{\partial x \partial y} = y \cdot x^{y-1} \log x + x^{y-1}$

$$\frac{\partial u}{\partial x} = y\,x^{y-1};\ \frac{\partial^2 u}{\partial y \partial x} = x^{y-1} + y\,x^{y-1} \log x; \qquad \therefore\ u_{yx} = u_{xy}$$

8. $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4\,(u^2 + v^2)$

9. $\displaystyle\int_0^\infty \int_0^\infty f(x, y)\,dx\,dy = \int_0^{\pi/2} \int_0^\infty F(r, \theta)\,r\,dr\,d\theta\,.$

10. $\displaystyle I = \int_0^1 \int_0^y \int_0^{x+y} dz\,dy\,dx = \int_0^1 \int_0^y (x + y)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} + yx\right]_0^y dy$

$$= \int_0^1 \frac{3}{2} y^2\,dy = \frac{1}{2}\,(y^3)_0^1 = \frac{1}{2}$$

11. (a) (i) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$; d.E. of A is $\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} = 0;$

i.e.; $(\lambda - 5)(\lambda - 1)^2 = 0$

$\therefore$ Eigenvalues of A are 5, 1, 1.

When $\lambda = 5$, the eigenvector is given by $-3x_1 + 2x_2 + x_3 = 0$ and $x_1 - 2x_2 + x_3 = 0$

Solving, $X_1 = (1, 1, 1)^T$.

When $\lambda = 1$, the eigenvector is given by $x_1 + 2x_2 + x_3 = 0$

Taking $x_1 = 1, x_2 = 0$, we get $x_3 = -1$ $\quad \therefore X_2 = (1, 0, -1)^T$

Taking $x_1 = 1, x_3 = 0$, we get $x_2 = -1/2$ $\quad \therefore X_3 = (2, -1, 0)^T$.

(ii) $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$; The C.E. of A is $\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} = 0$; i.e., $\lambda^2 - 4\lambda - 5 = 0$

$\therefore$ The eigenvalues are $\lambda = -1, 5$

When A^n is divided by $\lambda^2 - 4\lambda - 5$, let the quotient and remainder be $\theta(\lambda)$ and $(a\lambda + b)$ respy.

Then $\lambda^n \equiv (\lambda^2 - 4\lambda - 5)\,\theta(\lambda) + (a\lambda + b)$ (1)

When $\lambda = -1$, from (1), $-a + b = (-1)^n$

When $\lambda = 5$, from (1), $5a + b = 5^n$

Solving, we get

$a = \dfrac{5^n - (-1)^n}{6}$ and $b = \dfrac{5^n + 5(-1)^n}{6}$

Replacing λ by A in (1);

$$\begin{aligned} A^n &= (A^2 - 4A - 5I)\,\theta(A) + (aA + bI) \\ &= aA + bI, \text{ by C.H. Theorem} \end{aligned}$$

$$\therefore \quad A^n = \left\{\frac{5^n - (-1)^n}{6}\right\}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \left\{\frac{5^n + 5(-1)}{6}\right\}^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Putting $n = 3$, we have $A^3 = 21\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix}$

11. (b) $\theta = 2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1x_2$

$$\therefore \quad A_\theta = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \text{ C.E. of } A \text{ is } \begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

i.e., C.E. of A is $(3 - \lambda)\{\lambda^2 - 7\lambda + 6\} = 0$ or $(\lambda - 3)(\lambda - 1)(\lambda - 6) = 0$

$\therefore$ Eigenvalues of A are 1, 3, 6

When $\lambda = 1$, the eigenvector is given by $x_1 + 2x_2 = 0$ and $2x_3 = 0$

$\therefore$ $X_1 \equiv (2, -1, 0)^T$

When $\lambda = 3$, the eigenvector is given by $-x_1 + 2x_2 = 0$ and $x_1 + x_2 = 0$; x_3 is arbitrary.

$\therefore$ $X_2 \equiv (0, 0, 1)^T$

When $\lambda = 6$, the eigenvector is given by $2x_1 - x_2 = 0$ and $x_3 = 0$

$\therefore$ $X_3 = (1, 2, 0)^T$

$$\therefore \quad \text{Modal matrix } M = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{The normalised modal matrix } N = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ The orthogonal transformation required for reduction is $X = NY$

$$\text{i.e., } x_1 = \frac{2}{\sqrt{5}} y_1 + \frac{1}{\sqrt{5}} y_3 ; x_2 = \frac{-1}{\sqrt{5}} y_1 + \frac{2}{\sqrt{5}} y_3 \text{ and } x_3 = y_3$$

The canonical form of θ is $y_1^2 + 3y_2^2 + 6y_3^2$.

θ is positive definite.

12. (a) (i) The problem is the same as the worked example 2.11 in page I-2.57 of the book "Engg. Maths for sem I and II-Third edition"

(ii) Let (l', m', n') be the DR's of the rotating line.

Then $2l' + 3m' = 0$ and $n' = 0$ $\quad \therefore \dfrac{l'}{3} = \dfrac{m'}{-2} = \dfrac{n'}{0}$

DC's of the y-axis are (0, 1, 0)

$\therefore$ Semivertical angle θ of the required right circular cone is given by

$$\cos\theta = \frac{-2}{\sqrt{1.3} \cdot \sqrt{1}} \text{ or } \cos^2\theta = \frac{4}{13}$$

The vertex of the right circular cone is the point of intersection of the two lines $2x + 3y = 6$, $z = 0$ and $x = 0$, $z = 0$; i.e., $v \equiv (0, 2, 0)$

The equation of the right circular cone is given by

$$\Sigma l^2 \cdot \{\Sigma (x - \alpha)^2\} \cos^2\theta = [\Sigma\lambda (x - \alpha)]^2$$

In this problem, $l = 0$, $m = 1$, $n = 0$; $\alpha = 0$, $\beta = 2$, $\gamma = 0$ and $\cos^2 \theta = \dfrac{4}{13}$

$\therefore$ The equation of the cone is $\{x^2 + (y-2)^2 + z^2\} \times \dfrac{4}{13} = (y-2)^2$

i.e., $4[x^2 + y^2 + z^2 - 4y + 4] = 13\,(y^2 - 4y + 4)$

i.e., $4x^2 - 9y^2 + 4z^2 + 36y - 36 = 0$

12. (b) (i) Any plane that intersects (or passes through) the line $6x - 3y - 23 = 0 = 3z + 2$ is

$$6x - 3y - 23 + \lambda\,(3z + 2) = 0 \text{ or } 6x - 3y + 3\lambda z + (2\lambda - 23) = 0 \quad (1)$$

Centre of the given sphere $C \equiv (-1, 2, -3)$ and
radius $r = \sqrt{1 + 4 + 9 + 7} = \sqrt{21}$

If plane (1) is to be the tangent plane to the given sphere,
Length of the $\perp r$ from C on plane (1) $= r$

i.e., $$\frac{|-6 - 6 - 9\lambda + 2\lambda - 23|}{\sqrt{36 + 9 + 9\lambda^2}} = \sqrt{21}$$

i.e., $2\lambda^2 - 7\lambda - 4 = 0$ or $(2\lambda + 1)\,(\lambda - 4) = 0$

$\therefore$ $\lambda = \dfrac{-1}{2}$; 4.

When $\lambda = \dfrac{-1}{2}$, the equation of one tangent plane is

$$6x - 3y - 23 - \frac{1}{2}(3z + 2) = 0 \text{ or } 4x - 2y - z - 16 = 0$$

When $\lambda = 4$, the equation of the second tangent plane is

$$6x - 3y - 23 + 4\,(3z + 2) = 0 \text{ or } 2x - y + 4z - 5 = 0.$$

(b) (ii) This problem is similar to the worked example 2.4 in page I-2.77 of the book, except for the change in the radius. Instead of 5 in the W.E., 2 is given as the radius of the cylinder.

Proceeding as in the W.E., the equation of the cylinder is

$$\frac{1}{9}\{2x + y + 2z - 10\}^2 + 4 = (x-1)^2 + (y-2)^2 + (z-3)^2$$

i.e., $$4x^2 + y^2 + 4z^2 + 100 + 4xy + 4yz + 8zx - 40x - 20y - 40z + 36$$
$$= 9(x^2 + y^2 + z^2 - 2x - 4y - 6z + 14)$$

i.e., $5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8zx + 22x - 16y - 10 = 0$

13. (a) (i) This problem is the same as the worked example 3.3, given in page I-3.9 of the book.

(ii) This problem is the same as the worked example 3.16, given in page I-3.19 of the book.

(b) (i) This problem is the same as the worked example 3.13, given in page I-3.35 of the book.

(ii) This problem is the same as the worked example 3.10, given in page I-3.34 of the book.

14. (a) (i) $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = 2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}$$

$\therefore \quad z_{uu} = 2z_x + 2u\,[z_{xx}\,2u + z_{xy}\,2v] + 2v\,[z_{xy}\,2u + z_{yy}\,2v]$,

since z_x and z_y are also functions of x and y, since z is a function of x and y

$$= 2z_x + 4u^2\,z_{xx} + 8uv\,z_{xy} + 4v^2\,z_{yy} \quad (1)$$

$$z_v = z_x \cdot (-2v) + z_y \cdot (2x)$$

$\therefore \quad z_{vv} = -2z_x - 2v\,[z_{xx}\,(-2v) + z_{xy}\,2u] + 2u\,[z_{xy}\,(-2v) + z_{yy}\,2u]$

$$= -2z_x + 4v^2\,z_{xx} - 8uv\,z_{xy} + 4u^2\,z_{yy} \quad (2)$$

Adding (1) and (2), we get

$$z_{uu} + z_{vv} = 4(u^2 + v^2)\,z_{xx} + 4(u^2 + v^2)\,z_{yy}$$
$$= 4(u^2 + v^2)\,(z_{xx} + z_{yy})$$

(ii) This problem is the same as the worked example 4.5 given in page I-4.37 of the book.

(b) (i) The solution to this problem is given as the major part of the solution of the worked example 4.11, given in page I-4.41 of the book.

(ii) $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3;\ f_y = 3y^2 - 12;\ f_{xx} = 6x;\ f_{xy} = 0;\ f_{yy} = 6y$

The stationary points are given by $f_x = 0$ and $f_y = 0$

i.e., $x = \pm 1$ and $y = \pm 2$

The possible stationary points are (1, 2), (1, –2), (–1, 2) and (–1, –2)

At the point (1, 2), $A = f_{xx} = 6;\ B = 0;\ C = f_{yy} = 12$

$AC - B^2 = 72 > 0$ and A and $C > 0$

$\therefore f(x, y)$ is minimum at (1, 2)

At the point (–1, –2), $A = -6;\ B = 0;\ C = -12$

$AC - B^2 = 72 > 0$ and A and $C < 0$.

$\therefore f(x, y)$ is maximum at $(-1, -2)$

At the points $(1, -2)$ and $(-1, 2)$, $AC - B^2 < 0$

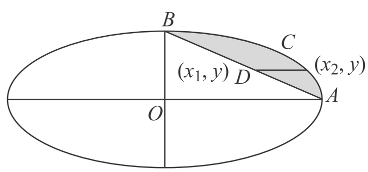
$\therefore f(x, y)$ is neither maximum nor minimum at these points

15 (a) (i) This problem is the same as the worked example 5.4, given in page I-5.23 of the book.

(ii) This problem is the same as the worked example 5.16, given in page I-5.34 of the book.

(b) (i) The solution of this problem is available as the latter part of the solution of the worked example (in which I is evaluated) 5.12 given in page I-5.14 of the book.

(ii) The given ellipse is $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$ and the given line is $\frac{x}{3} + \frac{y}{2} = 1$



The required area is the shaded part of the diagram.

The required area $= \iint\limits_{ABCD} dx\,dy = \int_0^2 \int_{x_1}^{x_2} dx\,dy$

(x_1, y) lies on $2x + 3y = 6$ $\quad\therefore\quad x_1 = \frac{1}{2}(6 - 3y)$

(x_2, y) lies on $4x^2 + 9y^2 = 36$ $\quad\therefore\quad x_2 = \frac{1}{2}\sqrt{36 - 9y^2}$

$$\therefore \text{Area} = \int_0^2 \int_{\frac{1}{2}(6-3y)}^{\frac{1}{2}\sqrt{36-9y^2}} dx\,dy = \int_0^2 \left[\frac{1}{2}\sqrt{36 - 9y^2} - \frac{1}{2}(6 - 3y)\right] dy$$

$$= \frac{3}{2}\left[\int_0^2 \sqrt{2^2 - y^2}\,dy - \int_0^2 (2 - y)\,dy\right]$$

$$= \frac{3}{2}\left[\frac{y}{2}\sqrt{2^2 - y^2} + \frac{2^2}{2}\sin^{-1}\frac{y}{2} - 2y + \frac{y^2}{2}\right]_0^2$$

$$= \frac{3}{2}[\pi - 4 + 2] = \frac{3}{2}(\pi - 2)$$

Appendix **C**

Solutions to the January 2013 Question Paper

B.E./B.Tech. DEGREE EXAMINATIONS, JANUARY 2013
(First Semester)
(Common to all Branches)

MA2111/MA 12/080030001—MATHEMATICS - I
(Regulations 2008/2010)

Times: 3 Hours **Maximum: 100 marks**

Answer ALL questions.

PART-A **($10 \times 2 = 20$ marks)**

1. Find the symmetric matrix A, whose eigenvalues are 1 and 3 with corresponding eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
2. Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{bmatrix}$.
3. Find the equation of the sphere whose centre is $(1, 2, -1)$ and which touches the plane $2x - y + z + 3 = 0$.
4. Find the radius of curvature of the curve $x^2 + y^2 - 4x + 2y - 8 = 0$.
5. Find the equation of the right circular cylinder whose axis is z-axis and radius is 'a'.
6. Find the envelope of the lines $x \operatorname{cosec} \theta - y \cot \theta = a$, θ being the parameter.
7. If $u = f(y - z, z - x, x - y)$, find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.
8. If $r = \frac{yz}{x}, s = \frac{zx}{y}, t = \frac{xy}{z}$, find $\frac{\partial(r, s, t)}{\partial(x, y, z)}$.

9. Plot the region of integration to evaluate the integral $\iint_D f(x, y)dxdy$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 - 2x + 6$.

10. Evaluate $\int_0^2 \int_0^{\pi} r \sin^2 \theta d\theta dr$.

Part-B **(5 × 16 = 80 marks)**

11. (a) (i) Find the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$. (8)

(ii) If the eigenvalues of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are 0, 3, 15, find the eigenvectors of A and diagonalize the matrix A. (8)

Or

(b) (i) Reduce the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ into canonical form. (8)

(ii) Show that the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ satisfies its own characteristic equation. Find also its inverse. (8)

12. (a) (i) Find the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x - 2y + 6z + 5 = 0$ which are parallel to the plane $x + 4y + 8z = 0$. Find also their points of contact. (8)

(ii) Find the equation of the right circular cone whose vertex is (2, 1, 0) semiverticle angle is 30° and the axis is the line $\frac{x-2}{3} = \frac{y-1}{1} = \frac{z}{2}$. (8)

Or

(b) (i) Find the equation of the cylinder whose generators are parallel to $\frac{x}{2} = \frac{y}{2} = \frac{z}{-3}$ and whose guiding curve is the ellipse $3x^2 + y^2 = 3$, $z = 2$. (8)

(ii) Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z = 3$ and also find the point of contact. (8)

13. (a) (i) Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$, where the parameters are related by the equation $a^2 + b^2 = c^2$. (8)

(ii) Find the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$. (8)

Or

(b) (i) Find the radius of curvature and centre of curvature of the parabola $y^2 = 4x$ at the point t. Also find the equation of the evolute. (10)

(ii) Find the envelope of the circles drawn upon the radius vectors of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as diameter. (8)

14. (a) (i) If $u = e^{xy}$, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{u}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$. (8)

(ii) Test for the maxima and minima of the function $f(x, y) = x^3y^3\,(6 - x - y)$. (8)

Or

(b) (i) If F is a function of x and y and if $x = e^u \sin v$, $y = e^u \cos v$, prove that $\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=e^{-2u}\left[\frac{\partial^2 F}{\partial u^2}+\frac{\partial^2 F}{\partial v^2}\right]$. (8)

(ii) If $x^2 + y^2 + z^2 = r^2$, show that the maximum value of $yz + zx + xy$ is r^2 and the minimum value is $-\frac{r^2}{2}$. (8)

15. (a) (i) Change the order of integration in the integral $\int_0^a \int_{x^2/2}^{2a-x} xy\,dx\,dy$ and evaluate it. (8)

(ii) Evaluate $\iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$ for all positive values of x, y, z for which the integral is real. (8)

Or

(b) (i) By transforming into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, $(b > a)$. (8)

(ii) Find the area which is inside the circler $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$. (8)

SOLUTIONS

PART-A

1. $A = MDM^{-1}$, where D is the diagonal matrix and M is the model matrix of the required matrix A

$$\therefore \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

2. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{bmatrix}$; The quadratic form corresponding to A is given by

$$\theta_A = 2x_1^2 + 2x_2^2 - 2x_3^2 - 4x_1x_3 + 2x_2x_3 .$$

3. Equation of the sphere, whose center is (1, 2, – 1) and radius is r, is given by

$$(x-1)^2 + (y-2)^2 + (z+1)^2 = r^2 \qquad (1)$$

Sphere (1) touches the plane $2x - y + z + 3 = 0$ (2)

$\therefore$ r = the length of the $\perp r$ drawn from (1, 2, –1) on the plane (2)

$$= \frac{|2-2-1+3|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{2}{\sqrt{6}}$$

$\therefore$ Equation of the required sphere is $(x-1)^2 + (y-2)^2 + (z+1)^2 = \frac{2}{3}$.

4. The given curve $x^2 + y^2 - 4x + 2y - 8 = 0$ is the circle $(x-2)^2 + (y+1)^2 = (\sqrt{13})^2$.

Radius of curvature of a circle = its radius = $\sqrt{13}$ at any point on it.

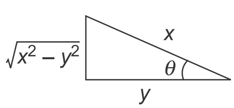
5. The right circular cylinder whose axis is the z-axis and radius is 'a' is that whose guiding curve is a circle of radius 'a' in the xy-plane

$\therefore$ Its equation is $x^2 + y^2 = a^2$

6. $x \operatorname{cosec}\theta - y\cot\theta = a$ (1)

Diffg.(1) w. r. t. 'θ', $-x\operatorname{cosec}\theta\cot\theta + y\operatorname{cosec}^2\theta = 0$

i.e., $\quad -x\cos\theta + y = 0$ or $\cos\theta = \dfrac{y}{x}$ (2)



using (2) in (1), we have

$$\frac{x^2}{\sqrt{x^2-y^2}} - \frac{y^2}{\sqrt{x^2-y^2}} = a \text{ or } \sqrt{x^2-y^2} = a$$

$\therefore$ Equation of the required envelope is $x^2 - y^2 = a^2$

7. $u = f(r, s, t)$, where $r = y - z$, $s = z - x$ and $t = x - y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial n}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial n}{\partial t}$$

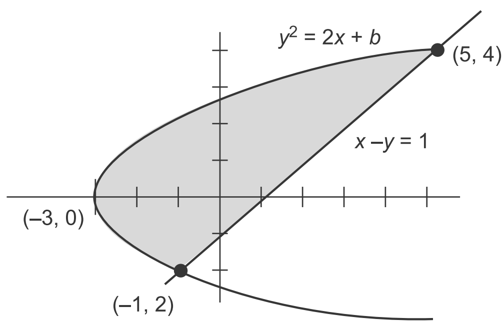
$$|||\text{by}, \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial n}{\partial t} \text{ and } \frac{\partial n}{\partial z} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$\therefore \quad \sum \frac{\partial u}{\partial x} = 0$$

8. The problem is the same as the worked example 4.10 in page I–4.41 of the book "Engineering Mathematics (For semesters I and II) – Third Edition".

The letters x_1, x_2, x_3 must be changed as x, y, z and the letters y_1, y_2, y_3 must be changed as r, s, t. Answer = 4.

9. $\iint f(x, y)\, dx\, dy$, where D is the region bounded by the line $y = x - 1$ or $x - y = 1$ and the parabola $(y - 0^2 = 2(x + 3)$



The shaded region is the region of integration D.

10. $$\int_0^2 \int_0^\pi r\sin^2\theta\, d\theta\, dr = \int_0^2 r\, dr \int_0^\pi \sin^2\theta\, d\theta = \left(\frac{r^2}{2}\right)^2 \times 2 \times \frac{1}{2} \times \frac{\pi}{2} = \pi$$

PART-B

11. (a) (i) $C, E,$ of the given matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \text{ is } \begin{bmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{bmatrix} = 0$$

i.e., $\lambda^3 + \lambda^2 - 12\lambda = 0$ or $\lambda(\lambda + 4)(\lambda - 3) = 0$

$\therefore$ The eigenvalues of the A are $-4, 0, 3$.

$\therefore$ When $\lambda = -4$, the eigenvector is given by $5x_1 + 2x_2 + x_3 = 0$ and $6x_1 + 3x_2 = 0$

i.e., $X_1 = (-1, 2, 1)^T$

When $\lambda = 0$, the eigenvector is given by $x_1 + 2x_2 + x_3 = 0$ and $6x_1 - x_2 = 0$

i.e., $X_2 = (1, 6, -13)^T$

When $\lambda = 3$, the eigenvector is given by $-2x_1 + 2x_2 + x_3 = 0$ and $-6x_1 - 4x_2 = 0$

i.e., $X_3 = (2, 3, -2)^T$

(ii) When $\lambda = 0$ the eigenvector is given by $8x_1 - 6x_2 + 2x_3 = 0$ and $-6x_1 + 7x_2 - 4x_3 = 0$

i.e., $X_1 = (1, 2, 2)^T$

When $\lambda = 3$, the eigenvector is given by $5x_1 - 6x_2 + 2x_3 = 0$ and $-6x_1 + 4x_2 - 4x_3 = 0$

i.e., $X_2 = (2, 1, -2)^T$

When $\lambda = 15$, the eigenvector is given by $-7x_1 - 6x_2 + 2x_3 = 0$ and $-6x_1 - 8x_2 - 4x_3 = 0$

i.e., $X_3 = (2, -2, 1)^T$

Diagonalisation of A is given by $M^{-1}AM = D$ or $N^TAN = D$, where N is the normalized model matrix given by

$$N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}; \text{ Verification can be done.}$$

11. (b) (i) $Q = 2x_1x_2 + 2x_2x_3 + 2x_3x_1$.

Matrix of the $Q.F.$ is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$;

$C.E.$ of A is $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

i.e., the C.E. is $(\lambda + 1)^2 (\lambda - 2) = 0$;

$\therefore$ the eigenvalues of A are $2, -1, -1$

When $\lambda = -1$, the eigenvector is given by $-2x_1 + x_2 + x_3 = 0$ and $x_1 - 2x_2 + x_3 = 0$

i.e., $X_1 = (1, 1, 1)^T$

When $\lambda = -1$, the eigenvector is given by the single equation $x_1 + x_2 + x_3 = 0$.

Let us choose $x_3 = 0$ and $x_1 = 1$, $x_2 = -1$. i.e., $X_2 = (1, -1, 0)^T$
Let $X_3 = (a, b. c)^T$. X_3 is orthogonal to both X_1 and X_2
$\therefore \quad a + b + c = 0$ and $a - b = 0$
Let a = b = 1, so that c = –2 $\quad \therefore \quad X_3 = (1, 1, -2)^T$

$$\therefore \quad \text{Model matrix } M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}; \text{ Normalised model matrix N}$$

$$\text{is given by } N = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

The orthogonal transformation $X = NY$ i.e., $x_1 = \dfrac{1}{\sqrt{3}} y_1 + \dfrac{1}{\sqrt{2}} y_2 + \dfrac{1}{\sqrt{6}} y_3$,

$x_2 = \dfrac{1}{\sqrt{3}} y_1 - \dfrac{1}{\sqrt{2}} y_2 + \dfrac{1}{\sqrt{6}} y_3$ and $x_3 = \dfrac{1}{\sqrt{3}} y_1 - \dfrac{2}{\sqrt{6}} y_3$ will reduce the given $Q.F.$ to the Canonical form $2y_1^2 - y_2^2 - y_3^2$.

(ii) The $C.E.$ of $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ is $\begin{vmatrix} 1-\lambda & -1 & 1 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0.$

i.e., $\lambda^3 - 5\lambda^2 + 5\lambda - 1 = 0$

We have to verify that $A^3 - 5A^2 + 5A - I = 0$. (1)

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 4 \\ 0 & 0 & 0 \\ 8 & -2 & 11 \end{bmatrix};$$

$$A^3 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 & 4 \\ 0 & 0 & 0 \\ 8 & -2 & 11 \end{bmatrix} = \begin{bmatrix} 11 & -5 & 15 \\ 0 & 1 & 0 \\ 30 & -10 & 41 \end{bmatrix}$$

L. S. of (1)

$$= \begin{bmatrix} 11 & -5 & 15 \\ 0 & 1 & 0 \\ 30 & -10 & 41 \end{bmatrix} - \begin{bmatrix} 15 & -10 & 20 \\ 0 & 5 & 0 \\ 40 & -10 & 55 \end{bmatrix} + \begin{bmatrix} 5 & -5 & 5 \\ 0 & 5 & 0 \\ 10 & 0 & 15 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.S. of (1)}$$

Hence Cayley-Hamilton theorem is verified

From (1), $A^2 - 5A + 5I - A^{-1} = 0$ $\quad\therefore A^{-1} = A^2 - 5A + 5I$

$$\text{i.e., } A^{-1} = \begin{bmatrix} 3 & -2 & 4 \\ 0 & 1 & 0 \\ 8 & -2 & 11 \end{bmatrix} - \begin{bmatrix} 5 & -5 & 5 \\ 0 & 5 & 0 \\ 10 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -1 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$$

12. (a) (i) This problem is the same as the worked example 2.6 in page I – 2.53 of the book "Engg. Maths (For Sem I and II) Third edition".

(ii) The equation of the right circular cone whose vertex is (α, β, γ), axis is $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ and semi-vertical angle θ is given by

$$(l^2 + m^2 + n^2)\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}\cos^2\theta$$
$$= \{l(x-\alpha) + m(y-\beta) + n(z-\gamma)\}^2$$

In this problem $(\alpha, \beta, \gamma) \equiv (2, 1, 0)$, $(l, m, n) \equiv (3, 1, 2)$ and $\theta = 30°$

$\therefore$ Required equation of the cone is

$$14\ \{(x-2)^2 + (y-1)^2 + (z-0)^2\} \times \frac{3}{4}$$
$$= \{3(x-2)^2 + 1(y-1)^2 + 2(z-0)^2\}$$

i.e., $21\ \{x^2 + y^2 + z^2 - 4x - 2y + 5\} = 2(3x + y + 2z - 7)^2$

i.e., $3x^2 + 19y^2 + 13z^2 - 12xy - 8yz - 24zx - 14y + 56z + 7 = 0$

(b) (i) Let $P\,(x_1, y_1, z_1)$ be any point on the required cylinder.

Then the equations of the generator through P are

$$\frac{x-x_1}{2} = \frac{y-y_1}{2} = \frac{z-z_1}{-3} = r$$

Any point Q on this generator is $(x_1 + 2r, y_1 + 2r, z_1 - 3r)$

Since the generator intersects the guiding curve, for somes,

We have $3(x_1 + 2r)^2 + (y_1 + 2r)^2 = 3$ (1)

and $z_1 - 3r = 2$ (2)

From (2), $r = \dfrac{1}{3}(z_1 - 2)$. Using the value of r in (1), we have

$$3\left\{x_1 + \frac{2}{3}(z_1 - 2)\right\}^2 + \left\{y_1 + \frac{2}{3}(z_1 - 2)\right\}^2 = 3$$

i.e., $$\frac{1}{3}(3x_1 + 2z_1 - 4)^2 + \frac{1}{9}(3y_1 + 2z_1 - 4)^2 = 3$$

i.e., $$3(3x_1 - 2z_1 - 4)^2 + (3y_1 + 2z_1 - 4)^2 = 27$$

(ii) This problem is the same as the worked example 2.5 in page I – 2.53 of the book "Engg. Maths (For Sem I and II) Third edition".

13. (a) (i) $\frac{x}{a}+\frac{y}{b}=1$, where $a^2+b^2=c^2$ or $b=\sqrt{c^2-a^2}$

$\therefore$ The family of straight lines is $\frac{x}{a}+\frac{y}{\sqrt{c^2-a^2}}=1 \ldots(1)$, where '$a$' is the parameter.

Diffg.(1) w.r.t. 'a' $-\frac{x}{a^2}+\frac{ay}{(c^2-a^2)^{3/2}}=0$ (2)

From (2), $\frac{x}{a^3}=\frac{y}{(c^2-a^2)^{3/2}}$ or $\frac{x^{2/3}}{a^2}=\frac{y^{2/3}}{c^2-a^2}=\frac{x^{2/3}+y^{2/3}}{c^2}$ (3)

From (3), $\frac{1}{a}=\frac{\sqrt{x^{2/3}+y^{2/3}}}{c\,x^{1/3}}$ and $\frac{1}{c^2-a^2}=\frac{\sqrt{x^{2/3}+y^{2/3}}}{cy^{1/3}}$ (3)

Using these values in (1), the equation of the envelope is

$$\frac{x^{2/3}\sqrt{x^{2/3}+y^{2/3}}}{c}+\frac{y^{2/3}\sqrt{x^{2/3}+y^{2/3}}}{c}=1$$

i.e., $(x^{2/3}+y^{2/3})^{3/2}=c$ or $x^{2/3}+y^{2/3}=c^{2/3}$

(ii) $x=a(\theta+\sin\theta)$; $\dot{x}=a\,(1+\cos\theta)$ and $\ddot{x}=-a\sin\theta$

$y=a\,(1-\cos\theta)$; $\dot{y}=a\sin\theta$ and $\ddot{y}=a\cos\theta$

$$\rho=\frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}=\frac{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}^{3/2}}{|a^2\cos\theta(1+\cos\theta)+a^2\sin^2\theta|}=\frac{a^3(2+2\cos\theta)^{3/2}}{a^2(1+\cos\theta)}$$

$$=\frac{2\sqrt{2}\,a(1+\cos\theta)^{3/2}}{1+\cos\theta}=2\sqrt{2}\,a\sqrt{1+\cos\theta}\ \text{ or }\ 4a\cos\frac{\theta}{2}$$

(b) (i) The parameter equations of the parabola $y^2=4ax$ are

$x=at^2$; $y=2at$;

$\therefore$ $\dot{x}=2at$; $\dot{y}=2a$ $\therefore \frac{dx}{dy}=\frac{\dot{y}}{\dot{x}}=\frac{1}{t}$

$$\frac{d^2y}{d^2x}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\div\frac{dx}{dt}=-\frac{1}{t^2}\times\frac{1}{2at}=-\frac{1}{2at^3}$$

$$\therefore \qquad \rho=\frac{(1+y'^2)^{3/2}}{|y''|}=\frac{\left(1+\frac{1}{t^2}\right)^{3/2}}{\frac{1}{2at^3}}=2a(1+t^2)^{3/2}$$

If $(\overline{x}, \overline{y})$ is the centre of curvature at'*t*'

$$\overline{x} = x - \frac{y'}{y''}(1 + y'^2) = at^2 + \frac{1}{t} \times 2at^3\left(1 + \frac{1}{t^2}\right)$$

$$= 2a + 3at^2 \tag{1}$$

$$\overline{y} = y + \frac{1}{y''}(1 + y'^2) = 2at - 2at^3\left(1 + \frac{1}{t^2}\right) = -2at^3 \tag{2}$$

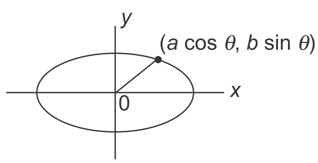
From (1), $t^2 = \dfrac{\overline{x} - 2a}{3a}$; From (2), $t^3 = -\dfrac{\overline{y}}{2a}$

Eliminating t form these equations, we get $\left(-\dfrac{\overline{y}}{2a}\right)^2 = \left(\dfrac{\overline{x} - 2a}{3a}\right)^3$.

$\therefore$ Locus of $(\overline{x}, \overline{y})$ or the evaluate of the parabola is

$27\,ay^2 = 4\,(x - 2a)^3$

(ii) The family of the circles drawn on the radius vectors of the ellipse as diameter is given by



$x(x - a\cos\theta) + y\,(y - b\sin\theta) = 0$

i.e., $ax\cos\theta + by\sin\theta = x^2 + y^2$ … (1), where θ is the parameter

Diffg. (1) w. r. t. 'θ', we get $-ax\sin\theta$ + by $\cos\theta = 0$ (2)

From (2), $\dfrac{\sin\theta}{by} = \dfrac{\cos\theta}{ax} = \dfrac{1}{\sqrt{(ax)^2 + (by)^2}}$

i.e., $\sin\theta = \dfrac{by}{\sqrt{(ax)^2 + (by)^2}}$ and $\cos\theta = \dfrac{ax}{\sqrt{(ax)^2 + (by)^2}}$

Using these values in (1), we get

$$\frac{(ax)^2 + (by)^2}{\sqrt{(ax)^2 + (by)^2}} = x^2 + y^2 \text{ i.e., } \sqrt{(ax)^2 + (by)^2} = x^2 + y^2$$

i.e., $(ax)^2 + (by)^2 = (x^2 + y^2)^2$

14. (a) (i) $u = e^{xy}$; $u_x = ye^{xy}$; $u_{xx} = y^2e^{xy}$

$u_y = xe^{xy}$; $\quad u_{yy} = x^2e^{xy}$

LS $= u_{xx} + u_{yy} = (x^2 + y^2)e^{xy}$

$$\text{RS} = \frac{1}{u}[u_x{}^2 + y^2] = \frac{1}{e^{xy}}[y^2e^{2xy} + x^2e^{2xy}] = e^{xy}(x^2 + y^2)$$

$\therefore \quad \text{RS} = \text{LS}$

(ii) $f(x, y) = x^3y^3(6 - x - y)$; $f_x = y^2(18x^2 - 4x^3 - 3x^2y)$; $f_y = x^3(12y - 2xy - 3y^2)$

$f_{xx} = y^2(36x - 12x^2 - 6xy)$; $f_{xy} = 36x^2y - 8\,x^3y - 9x^2y^2$;

$f_{yy} = x^3\,(12 - 2x - 6y)$

Stationary points are given by $f_x = 0$ and $f_y = 0$

i.e., $x^2y^2\,(18 - 4x - 3y) = 0$ and $x^3y\,(12 - 2x - 3y) = 0$

i.e., they are (0, 0), (0, 4), (6, 0), (0, 6), (9/2, 0) and (3, 2)

(3, 2) is the only stationary point which requires examination.

At (3, 2), $A = f_{xx} = -144$; $B = f_{xy} = -108$; $C = f_{yy} = -162$

$AC - B^2 = 11664 > 0$ and $A < 0$

$\therefore f(x, y)$ is maximum at the point (3, 2)

(b) **(i)** $x = e^u \sin v$; $y = e^u \cos v$

$F_u = F_x.\, e^u \sin v + F_y\,.\, e^u \cos v = xF_x + yF_y$; $\therefore \dfrac{\partial}{\partial x} = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$

$$F_{uu} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(xF_x + yF_y)$$

$$= (xF_x + yF_y) + (x^2\,F_{xx} + 2xyF_{xy} + y^2F_{yy}) \qquad (1)$$

$F_v = yF_x - xF_y \quad \therefore \dfrac{\partial}{\partial v} = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$

$$F_{vv} = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)(yF_x - yF_y)$$

$$= -(xF_x + yF_y) + (y^2F_{xx} - 2xyF_{xy} + x^2F_{yy}) \qquad (2)$$

Adding (1) and (2):

$$F_{uu} + F_{vv} = (x^2 + y^2)(F_{xx} + F_{yy}) \text{ or } e^{2x}\,(F_{xx} + F_{yy})$$

$$\therefore \quad F_{xx} + F_{yy} = e^{-2x}\,(F_{uu} + F_{vv})$$

(ii) Let $f = yz + zx + xy$ and $\phi = x^2 + y^2 + z^2 - r^2$

$\therefore$ The Lagrange's auxiliary function is $g = f + \lambda\phi$

The stationary points of g are given by $g_x = 0$; $g_y = 0$; $g_z = 0$ and $g_\lambda = 0$.

i.e., $(y + z) + \lambda \cdot 2x = 0 \qquad (1)$

$(z + x) + \lambda \cdot 2y = 0 \qquad (2)$

$(x + y) + \lambda \cdot 2z = 0 \qquad (3)$

and $x^2 + y^2 + z^2 = r^2$ (4)

From (1), (2) and (3), $-2\lambda = \dfrac{y+z}{x} = \dfrac{z+x}{y} = \dfrac{x+y}{z} = \dfrac{2(x+y+z)}{x+y+z}$

i.e., $\sum x + \lambda \sum x = 0$; i.e., $\left(\sum x\right)(\lambda + 1) = 0$

$\therefore \sum x = 0$ or $\lambda = -1$

When $\sum x = 0$, $(x + y + z)^2 = 0$, i.e., $x^2 + y^2 + z^2 + 2\,(xy + yz + zx) = 0$

i.e., $xy + yz + zx = -r^2/2$, which corresponds to the minimum value of f.

When $\lambda = -1$, $y + z = 2x$, $z + x = 2y$, $x + y = 2z$

$\therefore y - x = 2(x - y)$; i.e., $3(x - y) = 0$ $\therefore x = y$

Similarly $y = z$ and $z = x$

$\therefore x = y = z$

$\therefore$ When $\lambda = -1$, $3x^2 = 3y^2 = 3z^2 = r^2$ or $x^2 = y^2 = z^2 = \dfrac{r^2}{3}$

$\therefore$ When $\lambda = -1$, the value $f = r^2$, which corresponds to the maximum value of f.

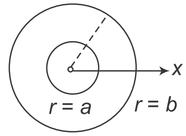
15. (a) (i) $I = \displaystyle\int_0^a \int_{x^2/2}^{2a-x} xy\, dy\, dx$

The problem is wrongly given. It ought to have been given as

$I = \displaystyle\int_0^a \int_{x^2/2}^{2a-x} xy\, dy\, dx$. The corrected problem is the same as the worked example 5.9, given in page I-5.27 of the book.

(ii) $I = \displaystyle\iiint \frac{dx\, dy\, dz}{\sqrt{1 - x^2 - y^2 - z^2}}$. The integral is real, when the region of space is bounded by the co-ordinate planes and the sphere $x^2 + y^2 + z^2 = 1$. This problem is the same as the worked example 5.12, given in page I-5.14 of the book.

(b) (i) $I = \displaystyle\iint \frac{x^2 y^2}{x^2 + y^2}\, dx\, dy$, over the region.



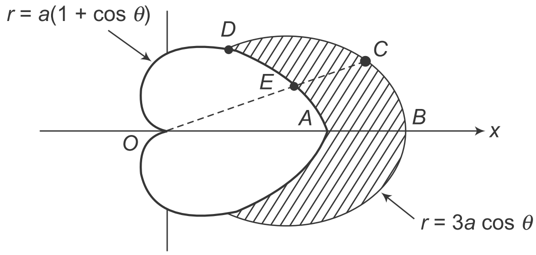
Putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = r\, dr\, d\theta$, we get

$$I = \iint\limits_R \frac{x^4 \cos^2\theta \sin\theta}{r^2} r\, dr\, d\theta \text{, where } R \text{ is the annular region shown}$$

$$= \int_a^b \int_0^{2\pi} r^3 \cos^2\theta \sin^2\theta\, dr\, d\theta = \int_a^b r^3\, dr \int_0^{2\pi} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{16}(b^4 - a^4) \int_0^{2\pi} \left(\frac{1 - \cos 4\theta}{2}\right) d\theta = \frac{\pi}{16}(b^2 - a^4)$$

(ii)



By symmetry, the required area $= 2 \times$ area $ABCDE$

$$\text{area} = 2 \int_0^{\pi/3} \int_{a(1+\cos\theta)}^{3a\cos\theta} r\, dr\, d\theta$$

$$= 2 \int_0^{\pi/3} \left(\frac{r^2}{2}\right)_{a(1+\cos\theta)}^{3a\cos\theta} d\theta$$

$$= \int_0^{\pi/3} [9a^2 \cos^2\theta - a^2 (1 + \cos\theta)^2]\, d\theta$$

i.e., Area $= a^2 \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)\, d\theta$

$$= a^2 \int_0^{\pi/3} \{4(1 + \cos 2\theta) - 2\cos\theta - 1\}\, d\theta$$

$$= a^2 \left[2\sin 2\theta - 2\sin\theta + 3\theta\right]_0^{\pi/3} d\theta$$

$$= a^2 \left[\left(2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3} + \pi\right) - 0\right]$$

$$= a^2 \left[\sqrt{3} - \sqrt{3} + \pi\right] = \pi a^2$$

Appendix **D**

Model Question Paper I

B.E./B.Tech. DEGREE EXAMINATIONS
(First Semester)
(Regulations: 2013)

MATHEMATICS - I
(Common to all Branches)

Times: 3 Hours | **Maximum: 100 marks**

INSTRUCTIONS

Answer ALL Questions

PART-A

($10 \times 2 = 20$ Marks)

1. If $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$, find the eigenvalues of A^{-1} and A^3.

2. Use Cayley-Hamilton theorem to find the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

3. Use the definition to show that the sequence $1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \cdots$ converges to the limit 2.

4. Give one example for each of absolutely convergent and conditionally convergent series.

5. Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point (1, 1).

6. Find the envelope of the line $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$, where θ is the parameter.

7. If $u = x\cos y + y\sin x$, verify that $\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$.

8. If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial(u, v)}{\partial(r, \theta)}$.

9. Express the area bounded by $x = 0, y = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant as a double integral.

10. Change the order of integration in $\int_0^\infty \int_-^x x\, e^{-x^2/y}\, dy\, dx$.

PART-B

(5 × 16 = 80 marks)

11. (a) (i) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix} \tag{8}$$

(ii) Verify cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and also use it to find A^{-1} (8)

Or

(b) Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation. Give also a non-zero set of values (x_1, x_2, x_3) which makes this quadratic form zero. (16)

12. (a) (i) Examine the convergence of the series (i) $\sum \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$ and (ii) $\sum \frac{n}{1+3^n}$ (5 + 5)

(ii) Examine the convergence of the series $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \cdots$ (6)

Or

(b) (i) Test the convergence of the series (i) $\sum \frac{x^{n-1}}{n\cdot 2^n}$ and (ii) $\sum \frac{3^n}{2^{n+3}}$ (5 + 5)

(ii) Test the convergence of the series $\frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} \cdots$ (6)

13. (a) (i) Find the radius of curvature of the curve $xy^2 = a^2(a - x)$ at the point $(a, 0)$. (8)

(ii) Find the evolute of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, treating it as the envelope of its normals. (8)

Or

(b) (i) Show that the measure of curvature of the curve $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$ at any point on it is $ab/2(ax + by)^{3/2}$. (8)

(ii) Find the equation of the evolute of the curve $x = a(\cos t + t \sin t)$, $(y = a (\sin t - t \cos t)$. (8)

14. (a) (i) Find the equivalent of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ in polar co-ordinates. (8)

(ii) Given that $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$, find the value of $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$. (8)

(b) (i) If $z = f(u, \text{v})$ where $u = x^2 - y^2$ and $v = 2\,xy$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) \quad (8)$$

(ii) Show that the minimum value of $x^2 + y^2 + z^2$ when $ax + by + cz = p$. (8)

15. (a) (i) Evaluate $\iiint_v \dfrac{dz\, dy\, dx}{\sqrt{1 - x^2\; y^2 - z^2}}$, where V is the region of space bounded by the co-ordinate planes and the sphere $x^2 + y^2 + z^2 = 1$ and contained in the positive octant. (8)

(ii) Find the area that lies inside the cardioid $r = a\,(1 + \cos \theta)$ and outside the circle $r = a$ by double integration. (8)

Or

(b) (i) Change the order of integration in $\displaystyle\int_{-a}^{a} \int_{0}^{\sqrt{a^2 - y^2}} y^2\, dx\, dy$ and then evaluate it. (8)

(ii) Find the value of $\iint_S z\, dS$, where S is the positive octant of the surface of the sphere $x^2 + y^2 + z^2 = a^2$. (8)

SOLUTIONS

PART-A

1. C.E. of A is $\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$; i.e., $\lambda^2 - 5\lambda - 6 = 0$ i.e., $(\lambda - 6)(\lambda + 1) = 0$.

$\therefore$ Eigenvalues of A are 6 and 2 – 1.

$\therefore$ Eigenvalues of A^{-1} and A^3 are respectively $\frac{1}{6}$ and –1 and 216 and –1.

2. C.E. of A is $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$; i.e., $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

By Cayley-Hamilton theorem $A^2 - (a+d)A + (ad - bc)\, I = 0$

$\therefore A - (a+d)\, I + (ad - bc)\, A^{-1} = 0$

$$\therefore \quad A^{-1} = \frac{1}{ad - bc}\{(a+d)\, I - A)\}$$

$$\text{i.e., } A^{-1} = \frac{1}{ad-bc}\left[\begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3. $\{a_n\} = 1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \cdots \;\therefore\; a_n = \frac{2n-1}{n}$

$|a_n - 2| = \left|2 - \frac{1}{n} - 2\right| = \frac{1}{n} < \in$, if $n > \frac{1}{\in}$; If we choose $\in = 0.01$, n can be found as 101, 102, … $\therefore \{a_n\}$ is cgt. to the limit 2.

4. **(i)** $\sum u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$ is also cgt, since $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\infty$,

which is a geometric series with $r = \frac{1}{2}$

(ii) $\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally cgt;

since $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is dgt.

5. $x^4 + y^4 = 2;\ 4x^3 + 4y^3y' = 0 \quad \therefore\ y' = -\dfrac{x^3}{y^3};\ y'' = \dfrac{3x^2(xy' - y)}{y^4}$

$(y')_{(1,\,1)} = -1;\ (y'')_{(1,\,1)} = -6;\ r = \dfrac{(1 + y'^2)^{3/2}}{|y''|} = \dfrac{\sqrt{2}}{3}.$

6. $\dfrac{x}{a}\sec\theta - \dfrac{y}{b}\tan\theta = 1;\ \therefore\ \dfrac{x}{a}\sec\theta - \tan\theta - \dfrac{y}{b}\sec^2\theta = 0$ *i.* $\sin\theta = \dfrac{ay}{bx}$

Using in (1); $\dfrac{bx^2}{a} - \dfrac{ay^2}{b} = \sqrt{(bx)^2 * - (ay)^2}$; i.e., $\sqrt{(bx)^2 - (ay)^2} = ab$

i.e., $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

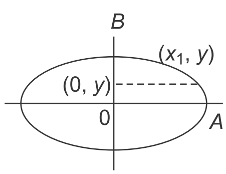
7. $u = x\cos y + y\sin x;\ \dfrac{\partial u}{\partial x} = \cos y + y\cos x;\ \dfrac{\partial^2 u}{\partial y\,\partial x} = -\sin y + \cos x$

$\dfrac{\partial u}{\partial y} = -x\sin y + \sin x;\ \dfrac{\partial^2 u}{\partial x\,\partial y} = -\sin y + \cos x \quad \therefore\ \dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial^2 u}{\partial x\,\partial y}$

8. $\dfrac{\partial(u, v)}{\partial(r, \theta)} = \dfrac{\partial(u, v)}{\partial(x, y)} \cdot \dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \cdot \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$

$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \cdot \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

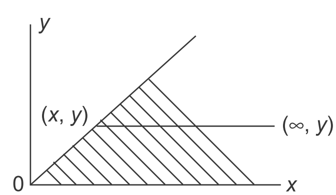
$= -4(x^2 + y^2) \cdot r(\cos^2\theta + \sin^2\theta) = -4r^3$

9. 

$A = \displaystyle\iint_{OAB} dx\,dy = \int_0^b\int_0^{x_1} dx\,dy = \int_0^b\int_0^{\frac{a}{b}} dx\,dy$

10. $\displaystyle\int_0^\infty\int_0^x x\,e^{-x^2/y}\,dy\,dx = \int_0^\infty\int_0^{x_1} x\,e^{-x^2/y}\,dx\,dy$

$= \displaystyle\int_0^\infty\int_y^\infty x\,e^{-x^2/y}\,dx\,dy$



PART-B

11. (a) (i) C.E. of A is $\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0$; i.e., $\lambda^3 - 13\lambda + 12 = 0$;

i.e. $(\lambda + 4)(\lambda - 1)(\lambda - 3) = 0$

$\therefore$ Eigenvalues of A are –4, 1, 3.

When $\lambda = -4$; the eigenvector is given by $6x_1 + 2x_2 = 0$ and $2x_1 + 5x_2 + x_3 = 0$

Solving, we get $X_1 = (1, -3, 13)^T$

When $\lambda = 1$; the eigenvector is given by $x_1 + 2x_2 = 0$ and $2x_1 + x_3 = 0$

Solving, we get $X_2 = (2, -1, -4)^T$

When $\lambda = 3$; the eigenvector is given by $-x_1 + 2x_2 = 0$ and $2x_1 - 2x_2 + x_3 = 0$

Solving, we get $X_3 = (2, 1, -2)^T$.

(ii) Worked example in the book.

(b) Worked example in the book.

12. (a) (i) (1) worked example in the book.

(2) worked example in the book.

(ii) Worked example in the book.

(b) (i) (1) $\sum u_n = \sum \dfrac{x^{n-1}}{n \cdot 2^n}$; $\dfrac{u_{n+1}}{u_n} \dfrac{u^n}{(n+1)2^{n+1}} \cdot \dfrac{n \cdot 2^n}{x^{n-1}} = \dfrac{x}{2} \cdot \dfrac{1}{1 + \dfrac{1}{r^n}}$

1. $\lim\limits_{n \to \infty} \left(\dfrac{u_{n+1}}{u_n} \right) = \dfrac{x}{2}$. $\therefore$ By ratio test, $\sum u_n$ is cgt. if $x < 2$ and dgt. if $x > 2$.

When $x = 2$, $\sum u_n = \sum \dfrac{1}{2n}$ or $\dfrac{1}{2} \sum \dfrac{1}{n}$ is (D)

(2) $\sum u_n = \sum \dfrac{3^n}{2^{n+3}}$; $\dfrac{u_{n+1}}{u_n} = \dfrac{3^{n+1}}{2^{n+4}} \times \dfrac{2^{n+3}}{3^n} = \dfrac{3}{2}$

$\therefore \lim\limits_{n \to \infty} \left(\dfrac{y_{n+1}}{u_n} \right) = \dfrac{3}{2} > 1$ $\therefore \sum u_n$ is dgt. by Ratio test.

(ii) Let the given series be $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)\log(n+1)} \equiv \sum (-1)^{n+1} \cdot u_n$

$$\lim_{n\to\infty} (u_n) = \lim_{n\to\infty} \left[\frac{1}{(n+1)\log(n+1)}\right] = 0$$

$$u_{n+1} - u_n = \frac{1}{(n+2)\log(n+2)} - \frac{1}{(n+1)\log(n+1)} < 0, \text{ for all } n.$$

$\therefore$ By Leibnitz test, the given series is Cgt.

13. (a) (i) $y^2 = a^2\left(\frac{a}{x} - 1\right); \quad \therefore 2y\,y' = -\frac{a^3}{x^2}$

i.e., $y' = -\frac{a^3}{2x^2 y} \quad \therefore \ (y')_{(a,0)} = \infty \ \alpha \left(\frac{dx}{dy}\right)_{(a,0} = 0$

$$\therefore \frac{dx}{dy} = -\frac{2x^2 y}{a^3}; \frac{d^2x}{dy^2} = -\frac{2}{a^3}\left\{x^2 + 2xy\frac{dx}{dy}\right\} = -\frac{2}{a}$$

$$\rho = \frac{(1 + x'^2)^{3/2}}{|x''|} = \frac{1}{\left|-\frac{2}{a}\right|} = \frac{a}{2}$$

(ii) Equation of normal at (a sec θ, b tan θ) is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} - a^2 + b^2$ (1)

Diffg w.r.t θ; $-ax \sin\theta - by \operatorname{cosec}^2\theta = 9$ (2)

From (2), $\sin^3\theta = -\frac{by}{ax}$ or $\sin\theta = \frac{-(by)^{1/3}}{(ax)^{1/3}}$

$$\therefore \cos\theta = \frac{\sqrt{(ax)^{2/3} - (by)^{2/3}}}{(ax)^{1/3}} \text{ and } \cot\theta = \frac{\sqrt{(ax)^{2/3} - (by)^{2/3}}}{(by)^{1/3}}$$

Using these values in (1);

$$ax^{2/3} \cdot \sqrt{(ax)^{2/3} - (by)^{2/3}} - (by)^{2/3}\sqrt{(ax)^{2/3} - (by)^{2/3}} = a^2 + b^2$$

i.e., $[(ax)^{2/3} - (by)^{2/3})]^{3/2} = a^2 + b^2$
i.e., the equation of the evolute is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$

(b) (i) Worked example in the book.

(ii) Worked example in the book.

14. (a) (i) Worked example in the book.

(ii) Worked example in the book.

(b) (i) $z = f(u, v)$; $u = x^2 - y^2$; $v = 2xy$

$z_x = z_u \cdot 2x + z_v \cdot 2y$;

$z_{xx} = 2z_u + 2x\{z_{uu} \cdot 2x + z_{uv} \cdot 2y\} + 2y\{z_{uv} \cdot 2x + z_{vv} \cdot 2y\}$

$$= 2z_u + 4x^2 z_{uu} + 8xy\, z_{uv} + 4y^2\, z_{vv} \qquad (1)$$

$z_y = z_u \cdot (-2y) + z_v \cdot 2x$

$\therefore\quad z_{yy} = -2z_u \cdot\ -2y\{z_{uu} \cdot (-2y) + z_{uv} \cdot 2x\} + 2x\{z_{uv} \cdot (-2y) + z_{vv} \cdot 2x\}$

$$= -2z_u + 4y^2\, z_{uu} - 8xy\, z_{uv} + 4x^2 z_{vv} \qquad (2)$$

Adding (1) and (2);

$$z_{xx} + z_{yy} = 4(x^2 + y^2)z_{uu} + 4(x^2 + y^2)\, z_{vv} = 4(x^2 + y^2)\,(z_{uu} + z_{vv})$$

(ii) Consider $g = f + \lambda\,\phi$, where $f = x^2 + y^2 + z^2$ and $\phi = ax + by + cz - p$

The stationary points of g are given by $g_x = 0$, $g_y = 0$, $g_z = 0$, $g_\lambda = 0$

i.e., $2x + \lambda a = 0$; $2y + \lambda b = 0$; $2z + \lambda c = 0$ and $ax + by + cz = p$.

$$\therefore -\lambda = \frac{2x}{a} = \frac{2y}{b} = \frac{2z}{c} = \frac{2(ax + by + cz)}{a^2 + b^2 + c^2} = \frac{2p}{\sum a^2}$$

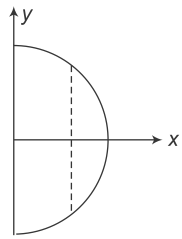
$$\therefore\quad x = \frac{pa}{\sum a^2},\ y = \frac{pb}{\sum a^2};\ z = \frac{pc}{\sum a^2}$$

Minimum value of $f = \dfrac{p^2 \sum a^2}{\left(\sum a^2\right)} = \dfrac{p^2}{a^2 + b^2 + c^2}$

15. (a) (i) Worked example in the book.

(ii) Worked example in the book.

(b) (i) $I = \displaystyle\int_{-a}^{a} \int_{0}^{\sqrt{a^2 - y^2}} y^2\, dx\, dy$; Area of integration is bounded by $x = 0$, $x = \sqrt{a^2 - y^2}$ or $x^2 + y^2 = a^2$, $y = -a$ and $y = a$. It is shown in the figure.



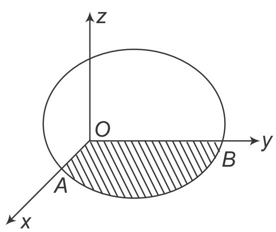
On changing the order of integration,

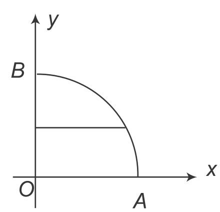
$$I = \int_0^a \int_{-\sqrt{z^2-x^2}}^{+\sqrt{a^2-x^2}} y^2 \, dy \, dx = \int_0^a \left[\frac{y^3}{3} \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{3} \int_0^a 2(a^2 - x^2)^{3/2} dx = \frac{2}{3} \int^{\pi/2} a^3 \cos^3 \theta (a \cos \theta) \, d\theta, \text{ on putting } x = a \sin \theta$$

$$= \frac{2}{3} a^4 \times \int_0^{\pi/2} \cos^4 \theta \, d\theta = \frac{2}{3} a^4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{8} a^4.$$

(ii)





Projection of the spherical surface lying in the +ve octant in the x_{oy}-plane is the quadrant of the circular region OAB.

Converting the surface integral into a double integral, we get

$$I = \iint_{OAB} z \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z} \, dx \, dy, \text{ where } \phi \equiv x^2 + y^2 + z^2 - a^2$$

$$= \iint_{OAB} z \cdot \frac{\sqrt{4(x^2 + y^2 + z^2)}}{2z} \, dx \, dy = \iint_{OAB} a \cdot dx \, dy = a \cdot \frac{\pi a^2}{4} = \frac{\pi}{4} a^3.$$

Appendix **E**

Model Question Paper II

B.E./B.Tech. DEGREE EXAMINATIONS
(First Semester)
(Regulations: 2013)

MATHEMATICS - I
(Common to all Branches)

Times: 3 Hours **Maximum: 100 Marks**

INSTRUCTIONS

Answer ALL Questions

PART-A

($10 \times 2 = 20$ Marks)

1. Find the sum of the eigenvalues of A^{-1}, if $A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$

2. Find the matrix $B = A^4 - 4A^3 - 5A^2 + A + 2I$, when $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, using Cayley-Hamilton theorem.

3. Give an example of a sequence which is bounded and monotonic. What is the limit to which it converges?

4. Show that the series $\sum (-1)^{n-1} \cdot \dfrac{1}{n}$ is convergent.

5. Find the curvature of the curve $y = \log \sec x$ at any point on it.

6. Find the envelope of the line $y = mx + \sqrt{a^2m^2 + b^2}$, where m is the parameter.

7. If $u = (x - y)\,(y - z) * (z - x)$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

8. Expand $e^x \sin y$ in a series of powers of x and y as far as the terms of the second degree.

9. Change the order of integration in $\int_0^{\infty}\int_x^{\infty} \frac{1}{y} e^{-y}\, dy\, dx$.

10. Express the area that lies outside the circle $r = 2\cos\theta$ and inside the circle $r = 4\cos\theta$ as a double integral.

PART-B

($5 \times 16 = 80$ Marks)

11. (a) (i) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \qquad (8)$$

(ii) Show that $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ satisfies the equation $A(A + 2I)(A - I) = 1$ (8)

Or

(b) Diagonalise the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of an orthogonal transformation. Verify your answer. (16)

12. (a) (i) Examine the convergence of the series

(i) $\sum \sin^2\left(\frac{1}{n}\right)$ and (ii) $\sum \frac{n}{1+n^4}$ (5 + 5)

(ii) For what values of x, the series $\sum (-1)^{n-1} \cdot \frac{x^n}{1+x^n}$ is convergent? (6)

Or

(b) (i) Test the convergence of the series

(i) $\sum \frac{n^n x^n}{n!}\ (x > 0)$ and (ii) $\sum \frac{1}{n(\log n)^2}$ (5 + 5)

(ii) Test the convergence of the series $\sum (-1)^{n-1} (\sqrt{n+1} - \sqrt{n})$. (6)

13. (a) (i) Find the equation of the circle of curvature of the parabola $y^2 = 12x$ at the point (3, 6). (8)

(ii) Find the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ (8)

Or

(b) (i) Find the radius of curvature of the curve $x = a\left[\log\left(\cot\frac{\theta}{2}\right) - \cos\theta\right]$, $y = a \sin\theta$ at the point 'θ'. (8)

(ii) Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are connected by the relation $ab = c^2$. (8)

14. (a) (i) If $f = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$, show that $x^2 \frac{\partial f}{\partial x} + y^2 \frac{\partial f}{\partial y} + z^2 \frac{\partial f}{\partial y} = 0$ (8)

(ii) Find the extreme value of $x^3 y^2 (12 - 3x - 4y)$. (8)

(b) (i) Find the Taylor's series expansion of $(x^2 y^2 + 2x^2 y + 3xy^2)$ in powers of $(x + 2)$ and $(y - 1)$ upto the third powers. (8)

(ii) A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box, that requires the fast material for its construction. (8)

15. (a) (i) Change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\, dx\, dy$ and then evaluate it. (8)

(ii) Express the volume of the sphere $x^2 + y^2 + z^2 = a^2$ as a volume integral and hence evaluate it, (8)

Or

(b) (i) Find the area included between the parabolas $y^2 = 4\, ax$ and $x^2 = 4\, by$, using double integration. (8)

(ii) Change the double integral $\int_0^a \int_y^a \frac{x\, dx\, dy}{x^2 + y^2}$ into polar co-ordinates and then evaluate it. (8)

SOLUTIONS

PART-A

1. $A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$; C.E. of A is $\begin{vmatrix} 3-\lambda & 0 & 0 \\ 8 & 4-\lambda & 0 \\ 6 & 2 & 5-\lambda \end{vmatrix} = 0$;

i.e., $(3-\lambda)(4-\lambda)(5-\lambda) = 0$.

$\therefore$ E' values of A are 3, 4, 5. $\therefore$ E' values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$.

$\therefore$ Sum of the eigenvalues of $A^{-1} = \frac{47}{60}$.

2. C.E. of A is $\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$; i.e., $\lambda^2 - 4\lambda - 5 = 0$

By C.H theorem, $A^2 - 4A - 5I = 0$

$$\therefore \quad A^4 - 4A^3 - 5A^2 + A + 2I = A^2(A^2 - 4A - 5I) + A + 2I$$

$$= 0 + \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$$

3. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$; The sequence in bounded, as $|a_n| \leq 1$; It is monotonic decreasing. The limit of the sequence is 0.

4. Let $\sum (-1)^{n-1} u_n = \sum (-1)^{n-1} \cdot \frac{1}{n}$;

$$u_{n+1} - u_n = \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)} < 0. \text{ for all, } \lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

$\therefore$ By Leibnitz test, $\sum (-1)^{n-1} \cdot \frac{1}{n}$ is cgt.

5. $y = \log \sec x$; $y' = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$; $y'' = \sec^2 x$

$$\rho = \frac{(1+y'^2)^{3/2}}{y''} = \frac{(1+\tan^2 x)^{3/2}}{\sec^2 x} = \sec x \qquad \therefore C = \cos x$$

6. $y = mx + \sqrt{a^2 m^2 + b^2}$, i.e. $(y - mx)^2 = a^2m^2 + b^2$; i.e., $(x^2 - a^2)m^2 - 2xym + (y^2 - b^2) = 0$. This is a quadratic equation in m.

$\therefore$ The envelope is $4xy^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$

i.e. $a^2y^2 + b^2x^2 = a^2b^2$ or $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

7. $u = (x - y)(y - z)(z - x);\ \dfrac{\partial u}{\partial x} = (y - z)(z - x - x + y) = (y - z)(y + z - 2x)$
$= y^2 - z^2 - 2x(y - z)$

Similarly, $\dfrac{\partial u}{\partial y} = z^2 - x^2 - 2y(z - x);\ \dfrac{\partial u}{\partial z} = x^2 - y^2 - 2z(x - y)$

$\therefore\ \sum \dfrac{\partial u}{\partial x} = 0 - 2 \times 0 = 0$

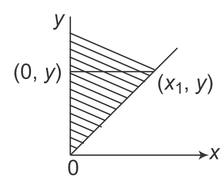
8. $f = e^x \sin y; f_x = e^x \sin y; f_y = e^x \cos y; f_{xx} = e^x \sin y; f_{xy} = e^x \cos y; f_{yy} = -e^x \sin y.$

$f(0, 0) = 0; f_x(0, 0) = 0; f_y(0, 0) = 1; f_{xx}(0, 0) = 0; f_{xy} = 1; f_{yy} = 0.$

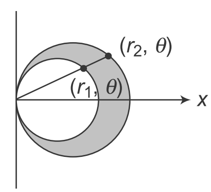
$$f(x, y) = f(0, 0) + \frac{1}{1!}\{xf_x(0, 0) = y \cdot f_y(0, 0)$$
$$+ \frac{1}{2!}\{x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\} + \cdots$$

$\therefore\ e^x \sin y = \dfrac{1}{1!} y + \dfrac{1}{2!} 2xy + \cdots$

9. $\displaystyle\int_0^\infty \int_x^\infty \frac{1}{y} e^{-y}\, dy\, dx = \int_0^\infty \int_0^y \frac{1}{y} e^{-y}\, dx\, dy$



10. $A = \displaystyle\int_0^{\pi/2} \int_{2\cos\theta}^{4\cos\theta} r\, dr\, d\theta$



PART-B

11. (a) (i) Worked example in the book.

(ii) C.E of A is $\begin{vmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 3 & -4-\lambda \end{vmatrix} = 0$; i.e. $\lambda^3 + \lambda^2 - 2\lambda = 0$;

i.e., $\lambda(\lambda + 2)(\lambda - 1) = 0$

By Cayley-Hamilton theorem, $A(\lambda + 2I)(A - I) = 0$.

(b) Worked example in the book.

12. (a) (i) **(1)** $\sum u_n = \sum \sin^2\left(\frac{1}{n}\right)$. Choose $\sum v_n = \sum \frac{1}{n^2}$

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left\{\frac{\sin(1/n)}{(1/n)}\right\}^2 = \lim_{\theta\to 0}\left\{\frac{\sin\theta}{\theta}\right\}^2 = 1 \neq 0.$$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

$\sum v_n = \sum \frac{1}{n^2}$ is cgt. $\therefore$ By comparison test, $\sum u_n$ is also egt.

(2) $\sum u_n = \sum \frac{n}{1+n^4}$; $\int_1^\infty u(x)dx = \int_1^\infty \frac{x\,dx}{1+x^4} = \frac{1}{2}\int_1^\infty \frac{dt}{1+t^2}$, on putting $x^2 = t$

$$= \frac{1}{2}(\tan^{-1} t)_1^\infty = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}$$

Since $\int_1^\infty u(x)dx$ exists, $\sum u_n$ is cgt, by integral test.

(ii) Worked example in the book.

(b) (i) **(1)** Worked example in the book.

(2) $\sum u_n = \sum \frac{1}{n(\log n)^2}$; $\int_1^\infty u(x)dx = \int_1^\infty \frac{dx}{x(\log x)^2} = \int_0^\infty \frac{dy}{y^2}$,

on putting $\log x = y$

$$= \left[\frac{1}{y}\right]_\infty^0 = \infty$$

Since $\int_1^\infty u(x)\,dx$ does not exist, $\sum u_n$ is dgt, by integral test.

(ii) Worked example in the book.

13. (a) (i) Worked example in the book.

(ii) Worked example in the book.

(b) (i) $x = a\left[\log\left(\cot\frac{\theta}{2}\right) - \cos\theta\right]$;

$$\dot{x} = a\left[-\frac{1}{2}\frac{\operatorname{cosec}^2 \theta/2}{\cot \theta/2} + \sin\theta\right] = a\left(-\frac{1}{\sin\theta} + \sin\theta\right) = -\frac{a\cos^2\theta}{\sin\theta}$$

$$y = a\sin\theta;\ \dot{y} = a\cos\theta;\ y' = \frac{\dot{y}}{\dot{x}} = -\tan\theta$$

$$y'' = \frac{d}{d\theta}(-\tan\theta)\cdot\frac{d\theta}{dx} = -\sec^2\theta\cdot\left(\frac{-\sin\theta}{a\cot^2\theta}\right) = \frac{\sin\theta}{a\cos^4\theta}$$

$$\therefore\ \rho = \frac{(1+y'^2)^{3/2}}{y''} = \sec^3\theta\cdot\frac{a\cos^4\theta}{\sin\theta} = a\cot\theta.$$

(ii) $\frac{x}{a} + \frac{y}{b} = 1$, where $ab = c^2$ i.e., $\frac{x}{a} + \frac{ay}{cz} = 1$

i.e., the given family is $y\,a^2 - c^2a + c^2x = 0$, where 'a' is the parameter.
This is of the form $A\,a^2 + Ba + C = 0$.
$\therefore$ The envelope is $B^2 - 4\,AC = 0$, i.e., $c^4 - 4c^2xy = 0$
i.e., $4xy = c^2$.

14. (a) (i) $f = f(r, s)$, where $r = \frac{x-y}{xy} = \frac{1}{y} - \frac{1}{x}$ and $s = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} = \frac{1}{x^2}\cdot f_r * -\frac{1}{x^2}\ f_s\ \ \therefore x^2\ f_x = f_r - f_s$$

$$\frac{\partial f}{\partial y} = f_r\cdot\left(-\frac{1}{y^2}\right); \frac{\partial f}{\partial z} = \frac{1}{z^2}\ f_s\ \ \therefore y^2 fy = -f_r \text{ and } z^2 f_z = f_s$$

$$\therefore\ \sum x^2 f_x = f_r - f_s - f_r + f_s = 0.$$

(ii) $f(x, y) = x^3\,y^2(12 - 3x - 4y) = 12x^3y^2 - 3x^4\,y^2 - 4x^3y^3$
$f_x = 36\,x^2y^2 - 12x^3y^2 - 12\,x^2y^3; f_y = 24x^3y - 6x^4y - 12\,x^3y^2$
$f_{xx} = 72x\,y^2 - 36x^2y^2 - 24xy^3; f_{xy} = 72x^2y - 24x^3y - 36\,x^2y^2$
$f_{yy} = 24x^3 - 6x^4 - 24x^3y.$

The stationary points are given by $f_x = 0$ and $f_y = 0$
i.e., $x^2y^2\,(36 - 12x - 12y) = 0$ and $6x^3y(4 - x - 2y) = 0$

Solving, we get the possible stationary points are (0, 0), (0, 2), (4, 0), (0, 3), (3, 0) and (2, 1).

At all points except (2, 1), $AC - B^2 = 0$, which requires further consideration.

At the point (2, 1), $A = -48$, $B = -48$, $C = -96$ $\therefore AC - B^2 > 0$ and $A < 0$

$\therefore$ $f(x, y)$ is maximum at the point (2, 1) and the maximum value of $f(x, y) = 16$.

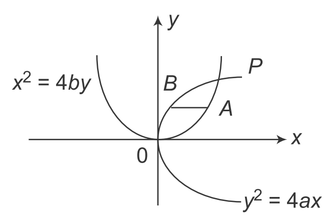
(b) (i) Worked example in the book.

(ii) Worked example in the book.

15. (a) (i) Worked example in the book.

(ii) Worked example in the book.

(b) (i)



The point of intersection of the two parabolas in given by $\dfrac{x^4}{16b^2} = 4\,ax$

i.e. $x = 0$ and $x^3 = 64\,ab^2$ or $x = 4a^{1/3}\,b^{2/3}$ and
$y = 0$ $\qquad$ $y = 4a^{2/3} \cdot b^{1/3}$

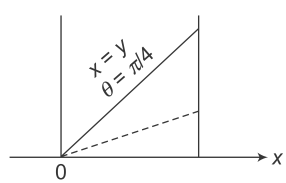
$$\text{Required area} = \iint\limits_{OAPB} dx\,dy = \int_0^{4a^{2/3}b^{2/3}} \int_{y^2/4a}^{\sqrt{4by}} dx\,dy$$

$$= \int_0^{4a^{2/3}b^{1/3}} (\sqrt{4by} - y^2/4a)\,dy$$

$$= \left(\sqrt{4b}\,.\,y^{3/2}\cdot\frac{2}{3} - \frac{y^3}{12a}\right)_0^{4a^{2/3}\,b^{1/3}}$$

$$= \frac{4}{3}\,b^{1/2}\cdot 8a\,b^{1/2} - \frac{64}{12a}\cdot a^2 b = \frac{1b}{3}\,ab.$$

(ii)



$x = a$ or $r = a\sec\theta$;

$$I = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r\cos\theta \cdot r\,dr\,d\theta}{r^2}$$

$$= \int_0^{\pi/4} \cos\theta \cdot \left[r\right]_0^{a\sec\theta} \int_0^{\pi/4} a\,d\theta = \frac{\pi a}{4}.$$