Third Edition

Abstract Algebra

An Introduction

Thomas W. Hungerford

NOTATIONS

The number after each entry refers to a page where the symbol is explained in the text.

Sets and Functions

$c \in B$	c is an element of the set B, 509
$c \notin B$	c is not an element of the set B, 509
$\varnothing$	Empty set [or null set], 510
$B \subseteq C$	B is a subset of C, 510
$B - C$	Relative complement of set C in set B, 511
$B \cap C$	Intersection of sets B and C, 511
$\bigcap_{i \in I} A_i$	Intersection of the sets A_i with $i \in I$, 511
$B \cup C$	Union of sets B and C, 511
$\bigcup_{i \in I} A_i$	Union of the sets A_i with $i \in I$, 511
$B \times C$	Cartesian product of sets B and C, 512
$f: B \to C$	Function [or mapping] from set B to set C, 512
$f(b)$	Image of b under the function $f: B \to C$, or the value of f at b, 512
$\iota_B: B \to B$	Identity map on the set B, 512
$g \circ f$	Composite function of $f: B \to C$ and $g: C \to D$, 512–513
$\operatorname{Im} f$	Image of the function $f: B \to C$, which is a subset of C, 517

Important Sets

$\mathbb{N}$	Nonnegative integers, 523
$\mathbb{Z}$	Integers, 3
$\mathbb{Q}$	Rational Numbers, 49, 191
$\mathbb{R}$	Real Numbers, 45, 191
$\mathbb{C}$	Complex numbers, 49, 191
$\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$	Nonzero elements of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ respectively, 178, 192
$\mathbb{Q}^{**}, \mathbb{R}^{**}$	Positive elements of $\mathbb{Q}$, $\mathbb{R}$ respectively, 178, 192

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Integers

$b \mid a$	b divides a [or b is a factor of a], 9
(a, b)	Greatest common divisor (gcd) of a and b, 10
$(a_1, a_2, \ldots, a_n)$	Greatest common divisor (gcd) of $a_1, a_2, \ldots, a_n$, 16
$[a, b]$	Least common multiple (lcm) of a and b, 16

$[a_1, a_2, \ldots, a_n]$	Least common multiple (lcm) of $a_1, a_2, \ldots, a_n$, 16
$a \equiv b \pmod{n}$	a is congruent to b modulo n, 25
$[a]$ or $[a]_n$	Congruence class of a modulo n, 27, 28
$\mathbb{Z}_n$	Set of congruence classes modulo n, 30

Rings and Ideals

1_R	Multiplicative identity element in a ring with identity, 44
$M(\mathbb{R})$	Ring of 2×2 matrices over the real numbers $\mathbb{R}$, 46
$M(\mathbb{Z})$, $M(\mathbb{Q})$, $M(\mathbb{C})$, $M(\mathbb{Z}_n)$	Ring of 2×2 matrices over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}_n$ respectively, 48
$\mathbf{0}$	Zero matrix in $M(\mathbb{R})$, 47
$M(R)$	Ring of 2×2 matrices over a commutative ring R with identity, 48
$R \cong S$	Ring R is isomorphic to ring S, 72
(c)	Principal ideal generated by c, 144
$(c_1, c_2, \ldots, c_n)$	Ideal generated by $c_1, c_2, \ldots, c_n$, 145
$a \equiv b \pmod{I}$	a is congruent to b modulo the ideal I, 145
$a + I$	Coset [congruence class] of a modulo the ideal I, 147
R/I	Quotient ring [or factor ring] of the ring R by the ideal I, 147, 154
$I + J$	Sum of ideals I and J (which is also an ideal), 149
IJ	Product of ideals I and J (which is also an ideal), 150
$\mathbb{Z}[\sqrt{d}]$	The subring $\{r + s\sqrt{d} \mid d, r, s \in \mathbb{Z}\}$ of $\mathbb{C}$, 322
$\mathbb{Z}[i]$ or $\mathbb{Z}[\sqrt{-1}]$	Ring of Gaussian integers, 322
$\mathbb{Q}_{\mathbb{Z}}[x]$	Ring of polynomials in $\mathbb{Q}[x]$ whose constant term is an integer, 336
$N: \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}$	Norm function, 346
$F(x)$	Field of quotients [or field of rational functions] of the polynomial ring $F[x]$ over the field F, 358

Polynomials

$R[x]$	Ring of polynomials with coefficients in the ring R, 86
$\deg f(x)$	Degree of the polynomial $f(x)$, 88
$f(x) \mid g(x)$	$f(x)$ divides [or is a factor of] $g(x)$, 96
$f(x) \equiv g(x) \pmod{p(x)}$	$f(x)$ is congruent to $g(x)$ modulo $p(x)$, 125
$[f(x)]$ or $[f(x)]_{p(x)}$	Congruence class [or residue class] of $f(x)$ modulo $p(x)$, 126
$F[x]/p(x)$	Ring of congruence classes modulo $p(x)$, 128, 131

List continues on inside back cover.

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ABSTRACT ALGEBRA

An Introduction

THIRD EDITION

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Dedicated to the memory of

Vincent O. McBrien

and

Raymond J. Swords, S.J.

College of the Holy Cross

TABLE OF CONTENTS

*Sections in the Core Course marked * may be omitted or postponed. See the beginning of each such section for specifics.

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PREFACE

This book is intended for a first undergraduate course in modern abstract algebra. Linear algebra is not a prerequisite. The flexible design makes the text suitable for courses of various lengths and different levels of mathematical sophistication, including (but not limited to) a traditional abstract algebra course, or one with a more applied flavor, or a course for prospective secondary school teachers. As in previous editions, the emphasis is on clarity of exposition and the goal is to produce a book that an average student can read with minimal outside assistance.

New in the Third Edition

Groups First Option Those who believe (as I do) that covering rings before groups is the better pedagogical approach to abstract algebra can use this edition exactly as they used the previous ones.

Nevertheless, anecdotal evidence indicates that some instructors have used the second edition for a "groups first" course, which presumably means that they liked other aspects of the book enough that they were willing to take on the burden of adapting it to their needs. To make life easier for them (and for anyone else who prefers "groups first")

It is now possible (though not necessary) to use this text for a course that covers groups before rings.

See the TO THE INSTRUCTOR section for details.

Much of the rewriting needed to make this option feasible also benefits the "rings first" users. A number of them have suggested that complete proofs were needed in parts of the group theory chapters instead of directions that said in effect "adapt the proof of the analogous theorem for rings". The full proofs are now there.

Proofs for Beginners Many students entering a first abstract algebra course have had little (or no) experience in reading and writing proofs. To assist such students (and better prepared students as well), a number of proofs (especially in Chapters 1 and 2) have been rewritten and expanded. They are broken into several steps, each of which is carefully explained and proved in detail. Such proofs take up more space, but I think it's worth it if they provide better understanding.

So that students can better concentrate on the essential topics, various items from number theory that play no role in the remainder of the book have been eliminated from Chapters 1 and 2 (though some remain as exercises).

More Examples and Exercises In the core course (Chapters 1–8), there are 35% more examples than in the previous edition and 13% more exercises. Some older exercises have been replaced, so 18% of the exercises are new. The entire text has about 350 examples and 1600 exercises. For easier reference, the examples are now numbered.

Coverage The breadth of coverage in this edition is substantially the same as in the preceding ones, with one minor exception. The chapter on Lattices and Boolean Algebra (which apparently was rarely used) has been eliminated. However, it is available at our website (www.CengageBrain.com) for those who want to use it.

The coverage of groups is much the same as before, but the first group theory chapter in the second edition (the longest one in the book by far) has been divided into two chapters of more manageable size. This arrangement has the added advantage of making the parallel development of integers, polynomials, groups, and rings more apparent.

Endpapers The endpapers now provide a useful catalog of symbols and notations.

Website The website (www.CengageBrain.com) provides several downloadable programs for TI graphing calculators that make otherwise lengthy calculations in Chapters 1 and 14 quite easy. It also contains a chapter on Lattices and Boolean Algebra, whose prerequisites are Chapter 3 and Appendices A and B.

Continuing Features

Thematic Development The Core Course (Chapters 1–8) is organized around two themes: Arithmetic and Congruence. The themes are developed for integers (Chapters 1 and 2), polynomials (Chapters 4 and 5), rings (Chapters 3 and 6), and groups (Chapters 7 and 8). See the **Thematic Table of Contents** in the TO THE STUDENT section for a fuller picture.

Congruence The Congruence theme is strongly emphasized hi the development of quotient rings and quotient groups. Consequently, students can see more clearly that ideals, normal subgroups, quotient rings, and quotient groups are simply an extension of familiar concepts in the integers, rather than an unmotivated mystery.

Useful Appendices These contain prerequisite material (e.g., logic, proof, sets, functions, and induction) and optional material that some instructors may wish to introduce (e.g., equivalence relations and the Binomial Theorem).

Acknowledgments

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T. W. H.

TO THE INSTRUCTOR

Here are some items that will assist you in making up your syllabus.

Course Planning

Using the chart on the opposite page, the Table of Contents (in which optional sections are marked), and the chapter introductions, you can easily plan courses of varying length, emphasis, and order of topics. If you plan to cover groups before rings, please note that Section 7.1 should be replaced by Section 7.1. A (which appears immediately after 7.1).

Appendices

Appendix A (Logic and Proof) is a prerequisite for the entire text. Prerequisites for various parts of the text are in Appendices B–F. Depending on the preparation of your students and your syllabus, you may want to incorporate some of this material into your course. Note the following.

- *Appendix B (Sets and Functions):* The middle part (Cartesian products and binary operations) is first used in Section 3.1 [7.1.A].* The last five pages (injective and surjective functions) are first used in Section 3.3 [7.4].
- *Appendix C (Induction):* Ordinary induction (Theorem C.1) is first used in Section 4.4. Complete Induction (Theorem C.2) is first used in Section 4.1 [9.2]. The equivalence of induction and well-ordering (Theorem C.4) is not needed in the body of the text.
- *Appendix D (Equivalence Relations):* Important examples of equivalence relations are presented in Sections 2.1, 5.1, 6.1, and 8.1, but the formal definition is not needed until Section 10.4 [9.4].
- *Appendix E (The Binomial Theorem):* This is used only in Section 11.6 and occasional exercises earlier.
- *Appendix F (Matrix Algebra):* This is a prerequisite for Chapter 16 but is not needed by students who have had a linear algebra course.

Finally, Appendix G presents a formal development of polynomials and indeterminates. I personally think it's a bit much for beginners, but some people like it.

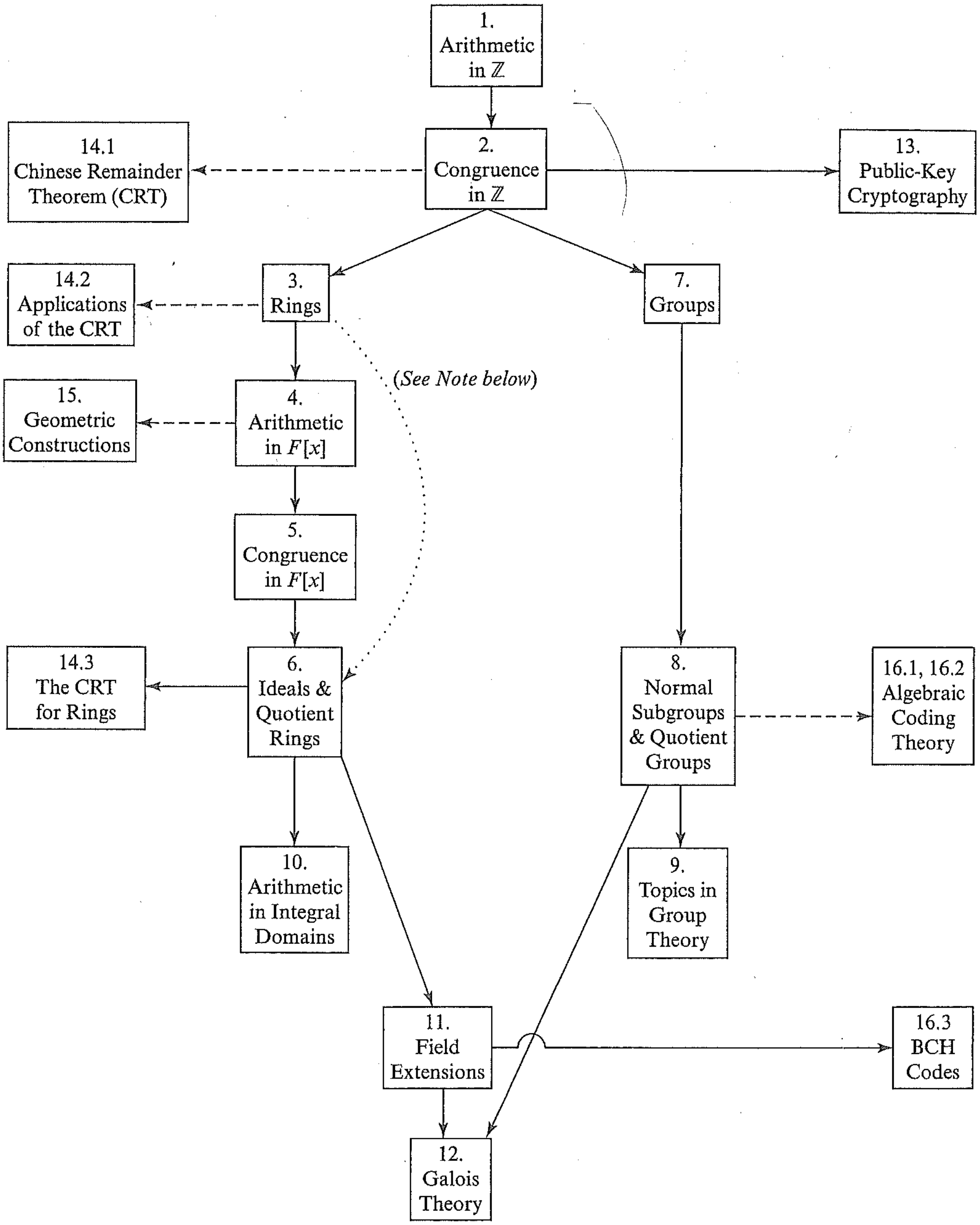
Exercises

The exercises in Group A involve routine calculations or short straightforward proofs. Those in Group B require a reasonable amount of thought, but the vast majority should be accessible to most students. Group C consists of difficult exercises.

Answers (or hints) for more than half of the odd-numbered exercises are given at the end of the book. Answers for the remaining exercises are in the Instructor's Manual available to adopters of the text.

*The section numbers in brackets are for groups-first courses.

CHAPTER INTERDEPENDENCE*



NOTE: To go quickly from Chapter 3 to Chapter 6, first cover Section 4.1 (except the proof of the Division Algorithm), then proceed to Chapter 6. If you plan to cover Chapter 11, however, you will need to cover Chapter 4 first.

---

*A solid arrow $A \rightarrow B$ means that A is a prerequisite for B; a dashed arrow $A \dashrightarrow B$ means that B depends only on parts of A (see the Table of Contents for specifics). For the dotted arrow 3···>6, see the Note at the bottom of the chart.

TO THE STUDENT

Overview

This book begins with grade-school arithmetic and the algebra of polynomials from high school (from a more advanced viewpoint, of course). In later chapters of the book, you will see how these familiar topics fit into a larger framework of abstract algebraic systems. This presentation is organized around these two themes:

Arithmetic You will see how the familiar properties of division, remainders, factorization, and primes in the integers carry over to polynomials, and then to more general algebraic systems.

Congruence You may be familiar with "clock arithmetic".* This is an example of congruence and leads to new finite arithmetic systems that provide a model for what can be done for polynomials and other algebraic systems. Congruence and the related concept of a quotient object are the keys to understanding abstract algebra.

Proofs

The emphasis in this course, much more than in high-school algebra, is on the rigorous logical development of the subject. If you have had little experience with reading or writing proofs, you would do well to read Appendix A, which summarizes the basic rules of logic and the proof techniques that are used throughout the book.

You should first concentrate on understanding the proofs in the text (which is quite different from constructing a proof yourself). Just as you can appreciate a new building without being an architect or a contractor, you can verify the validity of proofs presented by others, *even if you can't see how anyone ever thought of doing it this way in the first place.*

Begin by skimming through the proof to get an idea of its general outline before worrying about the details in each step. It's easier to understand an argument if you know approximately where it's headed. Then go back to the beginning and read the proof carefully, line by line. If it says "such and such is true by Theorem 5.18", check to see just what Theorem 5.18 says and be sure you understand why it applies here. If you get stuck, take that part on faith and finish the rest of the proof. Then go back and see if you can figure out the sticky point.

---

*When the hour hand of a clock moves 3 hours or 15 hours from 12, it ends in the same position, so $3 = 15$ on the clock. If the hour hand starts at 12 and moves 8 hours, then moves an additional 9 hours, it finishes at 5; so $8 + 9 = 5$ on the clock.

When you're really stuck, *ask your instructor*. He or she will welcome questions that arise from a serious effort on your part.

Exercises

Mathematics is not a spectator sport. You can't expect to learn mathematics without *doing* mathematics, any more than you could learn to swim without getting in the water. That's why there are so many exercises in this book.

The exercises in group A are usually straightforward. If you can't do almost all of them, you don't really understand the material. The exercises in group B often require a reasonable amount of thought—and for most of us, some trial and error as well. But the vast majority of them are within your grasp. The exercises in group C are usually difficult . . . a good test for strong students.

Many exercises will ask you to prove something. As you build up your skill in understanding the proofs of others (as discussed above), you will find it easier to make proofs of your own. The proofs that you will be asked to provide will usually be much simpler than proofs in the text (which can, nevertheless, serve as models).

Answers (or hints) for more than half of the odd-numbered exercises are given at the back of the book.

Keeping It All Straight

In the Core Course (Chapters 1–8), students often have trouble seeing how the various topics tie together, or even *if* they do. The *Thematic Table of Contents* on the next two pages is arranged according to the themes of arithmetic and congruence, so you *can* see how things fit together.

THEMATIC TABLE OF CONTENTS FOR THE CORE COURSE

Directions: Reading from left to right across these two pages shows how the theme or subtheme in the left-hand column is developed in the four algebraic systems listed in the top row. Each vertical column shows how the themes are carried out for the system listed at the top of the column.

*In the Arithmetic Theme, the sections of Chapters 3 (Rings) and 8 (Groups) do not correspond to the individual subthemes (as do the sections of Chapters 1 and 4). For integral domains, however, there is a correspondence, as you will see in Chapter 10 (Arithmetic in Integral Domains).

PART 1

THE CORE COURSE

CHAPTER 1

Arithmetic in $\mathbb{Z}$ Revisited

Algebra grew out of arithmetic and depends heavily on it. So we begin our study of abstract algebra with a review of those facts from arithmetic that are used frequently in the rest of this book and provide a model for much of the work we do. We stress primarily the underlying pattern and properties rather than methods of computation. Nevertheless, the fundamental concepts are ones that you have seen before.

1.1 The Division Algorithm

Our starting point is the set of all integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. We assume that you are familiar with the arithmetic of integers and with the usual order relation ($<$) on the set $\mathbb{Z}$. We also assume the

WELL-ORDERING AXIOM *Every nonempty subset of the set of nonnegative integers contains a smallest element.*

If you think of the nonnegative integers laid out on the usual number line, it is intuitively plausible that each subset contains an element that lies to the left of all the other elements in the subset—that is the smallest element. On the other hand, the Well-Ordering Axiom does not hold in the set $\mathbb{Z}$ of all integers (there is no smallest negative integer). Nor does it hold in the set of all nonnegative rational numbers (the subset of all positive rationals does not contain a smallest element because, for any positive rational number r, there is always a smaller positive rational—for instance, $r/2$).

NOTE: The rest of this chapter and the next require Theorem 1.1, which is stated below. Unfortunately, its proof is a bit more complicated than is desirable at the beginning of the course, since some readers may not have seen many (or any) formal mathematical proofs. To alleviate this situation, we shall first look at the origins of Theorem 1.1 and explain the idea of its proof. Unless you have a strong mathematical background, we suggest that you read this additional material carefully before beginning the proof.

To ease the beginner's way, the proof itself will be broken into several steps and given in more detail than is customary in most books. However, because the proof does not show how the theorem is actually used in practice, some instructors may wish to postpone the proof until the class has more experience in proving results. In any case, all students should at least read the outline of the proof (its first three lines and the statements of Steps 1–4).

So here we go. Consider the following grade-school division problem:

$$
\begin{array}{lcr}
\textit{Quotient} & \longrightarrow & 11 \\
\textit{Divisor} & \longrightarrow & 7\overline{)82} \\
\textit{Dividend} & \nearrow & \underline{7} \\
 & & 12 \\
 & & \underline{7} \\
\textit{Remainder} & \longrightarrow & 5
\end{array}
\qquad
\begin{array}{rcl}
\text{Check: } 11 & \longleftarrow & \textit{Quotient} \\
\underline{\times 7} & \longleftarrow & \textit{Divisor} \\
77 & & \\
\underline{+5} & \longleftarrow & \textit{Remainder} \\
82 & \longleftarrow & \textit{Dividend}
\end{array}
$$

The division process stops when we reach a remainder that is less than the divisor. All the essential facts are contained in the checking procedure, which may be verbally summarized like this:

$$\text{dividend} = (\text{divisor})\,(\text{quotient}) + (\text{remainder}).$$

Here is a formal statement of this idea, in which the dividend is denoted by a, the divisor by b, the quotient by q, and the remainder by r:

Theorem 1.1 The Division Algorithm

Let a, b be integers with $b > 0$. Then there exist unique integers q and r such that

$$a = bq + r \qquad \text{and} \qquad 0 \leq r < b.$$

Theorem 1.1 allows the possibility that the dividend a might be negative but requires that the remainder r must not only be less than the divisor b but also must be *nonnegative.* To see why this last requirement is necessary, suppose $a = -14$ is divided by $b = 3$, so that $-14 = 3q + r$. If we only require that the remainder be less than the divisor 3, then there are many possibilities for the quotient q and remainder r, including these three:

$$
\begin{array}{lll}
-14 = 3(-3) + (-5), & \text{with } -5 < 3 & [\textit{Here } q = -3 \textit{ and } r = -5.] \\
-14 = 3(-4) + (-2), & \text{with } -2 < 3 & [\textit{Here } q = -4 \textit{ and } r = -2.] \\
-14 = 3(-5) + 1, & \text{with } \ 1 < 3 & [\textit{Here } q = -5 \textit{ and } r = 1.].
\end{array}
$$

When the remainder is also required to be nonnegative as in Theorem 1.1, then there is exactly one quotient q and one remainder r, namely, $q = -5$ and $r = 1$, as will be shown in the proof.

The fundamental idea underlying the proof of Theorem 1.1 is that division is just repeated subtraction. For example, the division of 82 by 7 is just a shorthand method for repeatedly subtracting 7:

$$\begin{array}{l} 82 \\ \underline{-7} \\ 75 \longleftarrow 82 - 7 \cdot 1 \\ \underline{-7} \\ 68 \longleftarrow 82 - 7 \cdot 2 \\ \underline{-7} \\ 61 \longleftarrow 82 - 7 \cdot 3 \\ \underline{-7} \\ 54 \longleftarrow 82 - 7 \cdot 4 \\ \underline{-7} \\ 47 \longleftarrow 82 - 7 \cdot 5 \\ \underline{-7} \\ 40 \longleftarrow 82 - 7 \cdot 6 \end{array} \qquad \begin{array}{l} 40 \\ \underline{-7} \\ 33 \longleftarrow 82 - 7 \cdot 7 \\ \underline{-7} \\ 26 \longleftarrow 82 - 7 \cdot 8 \\ \underline{-7} \\ 19 \longleftarrow 82 - 7 \cdot 9 \\ \underline{-7} \\ 12 \longleftarrow 82 - 7 \cdot 10 \\ \underline{-7} \\ 5 \longleftarrow 82 - 7 \cdot 11 \end{array}$$

The subtractions continue until you reach a nonnegative number less than 7 (in this case 5). The number 5 is the remainder, and the *number* of multiples of 7 that were subtracted (namely, 11, as shown at the right of the subtractions) is the quotient.

In the preceding example we looked at the numbers

$$82 - 7 \cdot 1, \quad 82 - 7 \cdot 2, \quad 82 - 7 \cdot 3, \text{ and so on.}$$

In other words, we looked at numbers of the form $82 - 7x$ for $x = 1, 2, 3, \ldots$ and found the smallest nonnegative one (namely, 5). In the proof of Theorem 1.1 we shall do something very similar.

Proof of Theorem 1.1* ▸ Let a and b be fixed integers with $b > 0$. Consider the set S of all integers of the form

$$a - bx, \qquad \text{where } x \text{ is an integer and } a - bx \geq 0.$$

Note that x may be any integer—positive, negative, or 0—but $a - bx$ must be nonnegative. There are four main steps in the proof, as indicated below.

Step 1 *Show that S is nonempty by finding a value for x such that $a - bx \geq 0$.*

Proof of Step 1: We first show that $a + b|a| \geq 0$. Since b is a positive integer by hypothesis, we must have

$$\begin{aligned} b &\geq 1 \\ b|a| &\geq |a| && \text{[\textit{Multiply both sides of the preceding inequality by } |a|.]} \\ b|a| &\geq -a && \text{[\textit{Because } |a| \geq -a \textit{ by the definition of absolute value.}]} \\ a + b|a| &\geq 0. \end{aligned}$$

*For an alternate proof by induction of part of the theorem, see Example 2 in Appendix C.

Now let $x = -|a|$. Then

$$a - bx = a - b(-|a|) = a + b|a| \geq 0.$$

Hence, $a - bx$ is in S when $x = -|a|$, which means that S is nonempty.

Step 2 *Find q and r such that $a = bq + r$ and $r \geq 0$.*

Proof of Step 2: By the Well-Ordering Axiom, S contains a smallest element—call it r. Since $r \in S$, we know that $r \geq 0$ and $r = a - bx$ for some x, say $x = q$. Thus,

$$r = a - bq \text{ and } r \geq 0, \qquad \text{or, equivalently,} \qquad a = bq + r \text{ and } r \geq 0.$$

Step 3 *Show that $r < b$.*

Proof of Step 3: We shall use a "proof by contradiction" (which is explained on page 506 of Appendix A). We want to show that $r < b$. So suppose, on the contrary, that $r \geq b$. Then $r - b \geq 0$, so that

$$0 \leq r - b = (a - bq) - b = a - b(q + 1).$$

Since $a - b(q + 1)$ is nonnegative, it is an element of S by definition. But since b is positive, it is certainly true that $r - b < r$. Thus

$$a - b(q + 1) = r - b < r.$$

The last inequality states that $a - b(q + 1)$—which is an element of S—is less than r, the *smallest* element of S. This is a contradiction. So our assumption that $r \geq b$ is false, and we conclude that $r < b$. Therefore, we have found integers q and r such that

$$a = bq + r \qquad \text{and} \qquad 0 \leq r < b.$$

Step 4 *Show that r and q are the only numbers with these properties (that's what "unique" means in the statement of the theorem).*

Proof of Step 4: To prove uniqueness, we suppose that there are integers q_1 and r_1 such that $a = bq_1 + r_1$ and $0 \leq r_1 < b$, and prove that $q_1 = q$ and $r_1 = r$.

Since $a = bq + r$ and $a = bq_1 + r_1$, we have

$$bq + r = bq_1 + r_1$$

so that

$$(*) \qquad b(q - q_1) = r_1 - r.$$

Furthermore,

$$0 \leq r < b$$
$$0 \leq r_1 < b.$$

Multiplying the first inequality by -1 (and reversing the direction of the inequality), we obtain

$$-b < -r \leq 0$$
$$0 \leq r_1 < b.$$

Adding these two inequalities produces

$$-b < r_1 - r < b$$
$$-b < b(q - q_1) < b \qquad [\textit{By Equation } (*)]$$
$$-1 < q - q_1 < 1 \qquad [\textit{Divide each term by } b.]$$

But $q - q_1$ is an *integer* (because q and q_1 are integers) and the only integer strictly between -1 and 1 is 0. Therefore $q - q_1 = 0$ and $q = q_1$. Substituting $q - q_1 = 0$ in Equation $(*)$ shows that $r_1 - r = 0$ and hence $r = r_1$. Thus the quotient and remainder are unique, and the proof is complete. ■*

When both the dividend a and the divisor b in a division problem are positive, then the quotient and remainder are easily found either by long division (as on page 4) or with a calculator when the integers involved are larger.

EXAMPLE 1

Suppose $a = 4327$ is divided by $b = 281$. Entering a/b in a calculator produces $15.39857 \cdots$. The integer to the left of the decimal point (15 here) is the quotient q and the remainder is

$$r = a - bq = 4327 - 281 \cdot 15 = 112.$$

These calculations are shown on the graphing calculator screen in Figure 1.

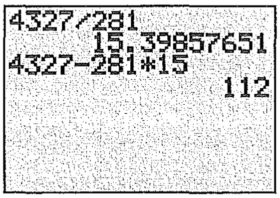


FIGURE 1

When the dividend a is negative, a slightly different procedure is needed so that the remainder will be nonnegative.

*The symbol ■ indicates the end of a proof.

EXAMPLE 2

Suppose $a = -7432$ is divided by $b = 453$. Entering a/b in a calculator produces $-16.40618 \cdots$. In this case the quotient q is *not* -16; instead,

$$q = (\text{the integer to the left of the decimal point}) - 1 = -16 - 1 = -17.$$

(Without this adjustment, you will end up with a negative remainder.) Now, as usual,

$$r = a - bq = -7432 - 453 \cdot (-17) = 269.$$

The preceding calculations are summarized in the calculator screen in Figure 2.

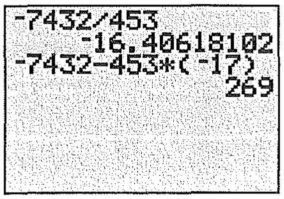


FIGURE 2

Exercises

A. *In Exercises 1 and 2, find the quotient q and remainder r when a is divided by b, without using technology. Check your answers.*

1. **(a)** $a = 17; b = 4$ **(b)** $a = 0; b = 19$ **(c)** $a = -17; b = 4$

2. **(a)** $a = -51; b = 6$ **(b)** $a = 302; b = 19$ **(c)** $a = 2000; b = 17$

In Exercises 3 and 4, use a calculator to find the quotient q and remainder r when a is divided by b.

3. **(a)** $a = 517; b = 83$ **(b)** $a = -612; b = 74$

(c) $a = 7{,}965{,}532; b = 127$

4. **(a)** $a = 8{,}126{,}493; b = 541$ **(b)** $a = -9{,}217{,}645; b = 617$

(c) $a = 171{,}819{,}920; b = 4321$

5. Let a be any integer and let b and c be positive integers. Suppose that when a is divided by b, the quotient is q and the remainder is r, so that

$$a = bq + r \qquad \text{and} \qquad 0 \le r < b.$$

If ac is divided by bc, show that the quotient is q and the remainder is rc.

B. **6.** Let a, b, c, and q be as in Exercise 5. Suppose that when q is divided by c, the quotient is k. Prove that when a is divided by bc, then the quotient is also k.

7. Prove that the square of any integer a is either of the form $3k$ or of the form $3k + 1$ for some integer k. [*Hint:* By the Division Algorithm, a must be of the form $3q$ or $3q + 1$ or $3q + 2$.]

8. Use the Division Algorithm to prove that every odd integer is either of the form $4k + 1$ or of the form $4k + 3$ for some integer k.
9. Prove that the cube of any integer a has to be exactly one of these forms: $9k$ or $9k + 1$ or $9k + 8$ for some integer k. [*Hint:* Adapt the hint in Exercise 7, and cube a in each case.]
10. Let n be a positive integer. Prove that a and c leave the same remainder when divided by n if and only if $a - c = nk$ for some integer k.
11. Prove the following version of the Division Algorithm, which holds for both positive and negative divisors.

    Extended Division Algorithm: Let a and b be integers with $b \neq 0$. Then there exist unique integers q and r such that $a = bq + r$ and $0 \leq r < |b|$.

    [*Hint:* Apply Theorem 1.1 when a is divided by $|b|$. Then consider two cases ($b > 0$ and $b < 0$).]

1.2 Divisibility

An important case of division occurs when the remainder is 0, that is, when the divisor is a factor of the dividend. Here is a formal definition:

Definition Let a and b be integers with $b \neq 0$. We say that **b divides a** (or that **b is a divisor of a**, or that **b is a factor of a**) if $a = bc$ for some integer c. In symbols, "b divides a" is written $b \mid a$ and "b does not divide a" is written $b \nmid a$.

EXAMPLE 1

$3 \mid 24$ because $24 = 3 \cdot 8$, but $3 \nmid 17$. Negative divisors are allowed: $-6 \mid 54$ because $54 = (-6)(-9)$, but $-6 \nmid (-13)$.

EXAMPLE 2

Every nonzero integer b divides 0 because $0 = b \cdot 0$. For every integer a, we have $1 \mid a$ because $a = 1 \cdot a$.

Remark If b divides a, then $a = bc$ for some c. Hence $-a = b(-c)$, so that $b \mid (-a)$. An analogous argument shows that every divisor of $-a$ is also a divisor of a. Therefore

a and $-a$ have the same divisors.

Remark Suppose $a \neq 0$ and $b \mid a$. Then $a = bc$, so that $|a| = |b||c|$. Consequently, $0 \leq |b| \leq |a|$. This last inequality is equivalent to $-|a| \leq b \leq |a|$. Therefore

(i) every divisor of the nonzero integer a is less than or equal to $|a|$;

(ii) a nonzero integer has only finitely many divisors.

All the divisors of the integer 12 are

$$1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 12, -12.$$

Similarly, all the divisors of 30 are

$$1, -1, 2, -2, 3, -3, 5, -5, 6, -6, 10, -10, 15, -15, 30, -30.$$

The **common divisors** of 12 and 30 are the numbers that divide both 12 and 30, that is, the numbers that appear on both of the preceding lists:

$$1, -1, 2, -2, 3, -3, 6, -6.$$

The largest of these common divisors, namely 6, is called the "greatest common divisor" of 12 and 30. This is an example of the following definition.

Definition

Let a and b be integers, not both 0. The **greatest common divisor (gcd)** of a and b is the largest integer d that divides both a and b. In other words, d is the gcd of a and b provided that

(1) $d \mid a$ and $d \mid b$;
(2) if $c \mid a$ and $c \mid b$, then $c \le d$.

The greatest common divisor of a and b is usually denoted (a, b).

If a and b are not both 0, then their gcd exists and is unique. The reason is that a nonzero integer has only finitely many divisors, and so there are only a finite number of common divisors. Hence there must be a unique largest one. Furthermore, the greatest common divisor of a and b satisfies the inequality

$$(a, b) \ge 1$$

because 1 is a common divisor of a and b.

EXAMPLE 3

$(12, 30) = 6$, as shown above. The only common divisors of 10 and 21 are 1 and -1. Hence $(10, 21) = 1$. Two integers whose greatest common divisor is 1, such as 10 and 21, are said to be **relatively prime.**

EXAMPLE 4

The common divisors of an integer a and 0 are just the divisors of a. If $a > 0$, then the largest divisor of a is clearly a itself. Hence, if $a > 0$, then $(a, 0) = a$.

Listing all the divisors of two integers in order to find their gcd can be quite time consuming. However, the Euclidean Algorithm (Exercise 15) is a relatively quick method for finding gcd's by hand. You can also use technology.

Technology Tip: To find a gcd on a TI-graphing calculator, select "gcd" in the NUM submenu of the MATH menu.

We have seen that $6 = (12, 30)$. A little arithmetic shows that something else is true here: 6 is a *linear combination* of 12 and 30. For instance,

$$6 = 12(-2) + 30(1) \qquad \text{and} \qquad 6 = 12(8) + 30(-3).$$

You can readily find other integers u and v such that $6 = 12u + 30v$. The following theorem shows that the same thing is possible for any greatest common divisor.

Theorem 1.2

Let a and b be integers, not both 0, and let d be their greatest common divisor. Then there exist (not necessarily unique) integers u and v such that $d = au + bv$.

CAUTION: Read the theorem carefully. The fact that $d = au + bv$ does *not* imply that $d = (a, b)$. See Exercise 25.

For the benefit of inexperienced readers, the proofs of Theorem 1.2 and Corollary 1.3 will be broken into several steps. The basic idea of the proof of Theorem 1.2 is to look at all possible linear combinations of a and b and find one that is equal to d.

Proof of Theorem 1.2 ▸ Let S be the set of all linear combinations of a and b, that is

$$S = \{am + bn \mid m, n \in \mathbb{Z}\}.$$

Step 1 *Find the smallest positive element of S.*

Proof of Step 1: Note that $a^2 + b^2 = aa + bb$ is in S and $a^2 + b^2 \geq 0$. Since a and b are not both 0, $a^2 + b^2$ must be positive. Therefore S contains positive integers and hence must contain a smallest positive integer by the Well-Ordering Axiom. Let t denote this smallest positive element of S. By the definition of S, we know that $t = au + bv$ for some integers u and v.

Step 2 *Prove that t is the gcd of a and b, that is, $t = d$.*

Proof of Step 2: We must prove that t satisfies the two conditions in the definition of the gcd:

(1) $t \mid a$ and $t \mid b$;

(2) If $c \mid a$ and $c \mid b$, then $c \leq t$.

Proof of (1): By the Division Algorithm, there are integers q and r such that $a = tq + r$, with $0 \leq r < t$. Consequently,

$$\begin{aligned} r &= a - tq, \\ r &= a - (au + bv)q = a - aqu - bvq, \\ r &= a(1 - qu) + b(-vq). \end{aligned}$$

Thus r is a linear combination of a and b, and hence $r \in S$. Since $r < t$ (the smallest positive element of S), we know that r is not positive. Since $r \geq 0$, the only possibility is that $r = 0$. Therefore, $a = tq + r = tq + 0 = tq$, so that $t \mid a$. A similar argument shows that $t \mid b$. Hence, t is a common divisor of a and b.

Proof of (2): Let c be any other common divisor of a and b, so that $c \mid a$ and $c \mid b$. Then $a = ck$ and $b = cs$ for some integers k and s. Consequently,

$$\begin{aligned} t = au + bv &= (ck)u + (cs)v \\ &= c(ku + sv). \end{aligned}$$

The first and last terms of this equation show that $c \mid t$. Hence, $c \leq |t|$ by the second Remark on page 9. But t is positive, so $|t| = t$. Thus $c \leq t$.

This shows that t is the greatest common divisor d and completes the proof of the theorem. ■

Technology Tip: To find the gcd of a and b and express it in the form $au + bv$ on a TI calculator, download the GCD program on our website (www.CengageBrain.com). Figure 1 shows the result when you enter $a = 2579$ and $b = 4321$. The gcd is 1 and you can easily verify that $2579 \cdot 826 + 4321 \cdot (-493) = 1$.

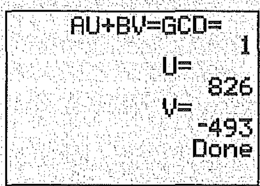


FIGURE 1

To do the same thing with Maple, use the command igcdex(a, b, 's', 't');.

Corollary 1.3

Let a and b be integers, not both 0, and let d be a positive integer. Then d is the greatest common divisor of a and b if and only if d satisfies these conditions:

(i) $d \mid a$ and $d \mid b$;

(ii) if $c \mid a$ and $c \mid b$, then $c \mid d$.

Proof ▸ The proof of an "if and only if" statement requires two steps (see page 507 in Appendix A).

Step 1 *Prove: If $d = (a, b)$, then d satisfies conditions (i) and (ii).*

Proof of Step 1: If $d = (a, b)$, then by the definition of the gcd, d divides both a and b. So d satisfies condition (i).

To verify that d satisfies condition (ii), suppose that c is an integer such that $c \mid a$ and $c \mid b$. Then $a = cr$ and $b = cs$ for some integers r and s, by the definition of "divides". By Theorem 1.2 there are integers u and v such that

$$d = au + bv$$

$$d = (cr)u + (cs)u \qquad [\textit{Because } a = cr \textit{ and } b = cs.]$$

$$d = c(ru + sv) \qquad [\textit{Factor } c \textit{ out of both terms.}]$$

But this last equation says that $c \mid d$. Therefore, d satisfies condition (ii).

Step 2 *Prove: If d is a positive integer that satisfies conditions (i) and (ii), then $d = (a, b)$.*

Proof of Step 2: To prove that $d = (a, b)$, we must show that d satisfies the requirements of the definition of the gcd, namely,

(1) $d \mid a$ and $d \mid b$;

(2) If $c \mid a$ and $c \mid b$, then $c \leq d$.

Obviously d satisfies (1) since requirement (1) and condition (i) are identical. To prove that d satisfies requirement (2), suppose c is an integer that divides both a and b, then $c \mid d$ by condition (ii). Consequently, by the second Remark on page 9, $c \leq |d|$. But d is positive, so $|d| = d$. Thus, $c \leq d$. Therefore, d satisfies requirement (2) and, hence, d is the gcd of a and b. ■

The answer to the following question will be needed on several occasions. If $a \mid bc$, then under what conditions is it true that $a \mid b$ or $a \mid c$? It is certainly not always true, as this example shows:

$$6 \mid 3 \cdot 4, \qquad \text{but} \qquad 6 \nmid 3 \qquad \text{and} \qquad 6 \nmid 4.$$

Note that 6 has a nontrivial factor in common with 3 and another in common with 4. When a divisor of bc has no common factors (except ± 1) with either b or c, then there is a useful answer to the question.

Theorem 1.4

If $a \mid bc$ and $(a, b) = 1$, then $a \mid c$.

Proof ▸ Since $(a, b) = 1$, Theorem 1.2 shows that $au + bv = 1$ for some integers u and v. Multiplying this equation by c shows that $acu + bcv = c$. But $a \mid bc$, so that $bc = ar$ for some r. Therefore

$$c = acu + bcv = acu + (ar)v = a(cu + rv).$$

The first and last parts of this equation show that $a \mid c$. ■

Exercises

1. Find the greatest common divisors. You should be able to do parts (a)–(c) by hand, but technology is OK for the rest.

(a) $(56, 72)$ **(b)** $(24, 138)$ **(c)** $(112, 57)$

(d) $(143, 231)$ **(e)** $(306, 657)$ **(f)** $(272, 1479)$

(g) $(4144, 7696)$ **(h)** $(12378, 3054)$

2. Prove that $b \mid a$ if and only if $(-b) \mid a$.

3. If $a \mid b$ and $b \mid c$, prove that $a \mid c$.

4. **(a)** If $a \mid b$ and $a \mid c$, prove that $a \mid (b + c)$.

(b) If $a \mid b$ and $a \mid c$, prove that $a \mid (br + ct)$ for any $r, t \in \mathbb{Z}$.

5. If a and b are nonzero integers such that $a \mid b$ and $b \mid a$, prove that $a = \pm b$.

6. If $a \mid b$ and $c \mid d$, prove that $ac \mid bd$.

7. If $a < 0$, find $(a, 0)$.

8. Prove that $(n, n + 1) = 1$ for every integer n.

9. If $a \mid c$ and $b \mid c$, must ab divide c? Justify your answer.

10. If $(a, 0) = 1$, what can a possibly be?

11. If $n \in \mathbb{Z}$, what are the possible values of

(a) $(n, n + 2)$ **(b)** $(n, n + 6)$

12. Suppose that $(a, b) = 1$ and $(a, c) = 1$. Are any of the following statements false? Justify your answers.

(a) $(ab, a) = 1$ **(b)** $(b, c) = 1$ **(c)** $(ab, c) = 1$

13. Suppose that a, b, q, and r are integers such that $a = bq + r$. Prove each of the following statements.

(a) Every common divisor c of a and b is also a common divisor of b and r.

[*Hint:* For some integers s and t, we have $a = cs$ and $b = ct$. Substitute these results into $a = bq + r$, and show that $c \mid r$.]

(b) Every common divisor of b and r is also a common divisor of a and b.

(c) $(a, b) = (b, r)$.

14. Find the smallest positive integer in the given set. [*Hint:* Theorem 1.2.]

(a) $\{6u + 15v \mid u, v \in \mathbb{Z}\}$ **(b)** $\{12r + 17s \mid r, s \in \mathbb{Z}\}$

15. The *Euclidean Algorithm* is an efficient way to find (a, b) for any positive integers a and b. It only requires you to apply the Division Algorithm several times until you reach the gcd, as illustrated here for $(524, 148)$.

(a) Verify that the following statements are correct.

$$
\begin{array}{rcll}
524 &=& 148\cdot 3 + 80 & 0 \le 80 < 148 \\
148 &=& 80\cdot 1 + 68 & 0 \le 68 < 80 \\
80 &=& 68\cdot 3 + 12 & 0 \le 12 < 68 \\
68 &=& 12\cdot 5 + 8 & 0 \le 8 < 12 \\
12 &=& 8\cdot 1 + 4 & 0 \le 4 < 8 \\
8 &=& 4\cdot 2 + 0 &
\end{array}
$$

[*The divisor in each line becomes the dividend in the next line, and the remainder in each line becomes the divisor in the next line.*]

[*As shown in part (b), the last nonzero remainder, namely 4, is the gcd (a, b).*]

(b) Use part (a) and Exercises 13 and Example 4 to prove that

$$(524, 148) = (148, 80) = (80, 68) = (68, 12) = (12, 8) = (8, 4) = (4, 0) = 4.$$

Use the Euclidean Algorithm to find

(c) $(1003, 456)$ **(d)** $(322, 148)$ **(e)** $(5858, 1436)$

The equations in part (a) can be used to express the gcd 4 as a linear combination of 524 and 148 as follows. First, rearrange the first 5 equations in part (a), as shown below.

$$
\begin{array}{rcll}
80 &=& 524 - 148\cdot 3 & (1) \\
68 &=& 148 - 80 & (2) \\
12 &=& 80 - 68\cdot 3 & (3) \\
8 &=& 68 - 12\cdot 5 & (4) \\
4 &=& 12 - 8 & (5)
\end{array}
$$

(f) Equation (1) expresses 80 as a linear combination of 524 and 148. Use this fact and Equation (2) to write 68 as a linear combination of 524 and 148.

(g) Use Equation (1), part (f), and Equation (3) to write 12 as a linear combination of 524 and 148.

(h) Use parts (f) and (g) to write 8 as a linear combination of 524 and 148.

(i) Use parts (g) and (h) to write the gcd 4 as a linear combination of 524 and 148, as desired.

(j) Use the method described in parts (f)–(i) to express the gcd in part (c) as a linear combination of 1003 and 456.

B. 16. If $(a, b) = d$, prove that $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$. [*Hint:* $a = dr$ and $b = ds$ for some integers r and s (Why?). So $a/d = r$ and $b/d = s$ and you must prove that $(r, s) = 1$. Apply Theorem 1.2 to (a, b) and divide the resulting equation by d.]

17. Suppose $(a, b) = 1$. If $a \mid c$ and $b \mid c$, prove that $ab \mid c$. [*Hint:* $c = bt$ (Why?), so $a \mid bt$. Use Theorem 1.4.]

18. If $c > 0$, prove that $(ca, cb) = c(a, b)$. [*Hint:* Let $(a, b) = d$ and $(ca, cb) = k$. Show that $cd \mid k$ and $k \mid cd$. See Exercise 5.]

19. If $a \mid (b + c)$ and $(b, c) = 1$, prove that $(a, b) = 1 = (a, c)$.

20. Prove that $(a, b) = (a, b + at)$ for every $t \in \mathbb{Z}$.

21. Prove that $(a, (b, c)) = ((a, b), c)$.

22. If $(a, c) = 1$ and $(b, c) = 1$, prove that $(ab, c) = 1$.

23. Use induction to show that if $(a, b) = 1$, then $(a, b^n) = 1$ for all $n \geq 1$.*

24. Let $a, b, c \in \mathbb{Z}$. Prove that the equation $ax + by = c$ has integer solutions if and only if $(a, b) \mid c$.

25. **(a)** If $a, b, u, v \in \mathbb{Z}$ are such that $au + bv = 1$, prove that $(a, b) = 1$.

(b) Show by example that if $au + bv = d > 1$, then (a, b) may not be d.

26. If $a \mid c$ and $b \mid c$ and $(a, b) = d$, prove that $ab \mid cd$.

27. If $c \mid ab$ and $(c, a) = d$, prove that $c \mid db$.

28. Prove that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3. [*Hint:* $10^3 = 999 + 1$ and similarly for other powers of 10.]

29. Prove that a positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9. [See Exercise 28.]

30. If $a_1, a_2, \ldots, a_n$ are integers, not all zero, then their **greatest common divisor (gcd)** is the largest integer d such that $d \mid a_i$ for every i. Prove that there exist integers u_i such that $d = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. [*Hint:* Adapt the proof of Theorem 1.2.]

31. The **least common multiple (lcm)** of nonzero integers $a_1, a_2, \ldots, a_k$ is the smallest positive integer m such that $a_i \mid m$ for $i = 1, 2, \ldots, k$ and is denoted $[a_1, a_2, \ldots, a_k]$.

(a) Find each of the following: $[6, 10]$, $[4, 5, 6, 10]$, $[20, 42]$, and $[2, 3, 14, 36, 42]$.

(b) If t is an integer such that $a_i \mid t$ for $i = 1, 2, \ldots, k$, prove that $[a_1, a_2, \ldots, a_k] \mid t$. [*Hint:* Denote $[a_1, a_2, \ldots, a_k]$ by m. By the Division Algorithm, $t = mq + r$, with $0 \leq r < m$. Show that $a_i \mid r$ for $i = 1, 2, \ldots, k$. Since m is the smallest positive integer with this property, what can you conclude about r?]

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*Induction is discussed in Appendix C.

32. Let a and b be integers, not both 0, and let t be a positive integer. Prove that t is the least common multiple of a and b if and only if t satisfies these conditions:

(i) $a \mid t$ and $b \mid t$;

(ii) If $a \mid c$ and $b \mid c$, then $t \mid c$.

C. 33. If $a > 0$ and $b > 0$, prove that $[a, b] = \dfrac{ab}{(a, b)}$. ($[a, b]$ is defined in Exercise 31.)

34. Prove that

(a) $(a, b) \mid (a + b, a - b)$;

(b) if a is odd and b is even, then $(a, b) = (a + b, a - b)$;

(c) if a and b are odd, then $2(a, b) = (a + b, a - b)$.

1.3 Primes and Unique Factorization

Every nonzero integer n except ± 1 has at least four distinct divisors, namely $1, -1, n, -n$. Integers that have *only* these four divisors play a crucial role.

Definition

An integer p is said to be **prime** if $p \neq 0, \pm 1$ and the only divisors of p are ± 1 and $\pm p$.

EXAMPLE 1

$3, -5, 7, -11, 13$, and -17 are prime, but 15 is not (because 15 has divisors other than ± 1 and ± 15, such as 3 and 5). The integer 4567 is prime, but proving this fact from the definition requires a tedious check of all its possible divisors. Fortunately, there are more efficient methods for determining whether an integer is prime, one of which is discussed at the end of this section.

It is not difficult to show that there are infinitely many distinct primes (Exercise 32). Because an integer p has the same divisors as $-p$, we see that

p is prime if and only if $-p$ is prime.

If p and q are both prime and $p \mid q$, then p must be one of $1, -1, q, -q$. But since p is prime, $p \neq \pm 1$. Hence,

if p and q are prime and $p \mid q$, then $p = \pm q$.

Under what conditions does a divisor of a product bc necessarily divide b or c? Theorem 1.4 gave one answer to this question. Here is another.

Theorem 1.5

Let p be an integer with $p \neq 0, \pm 1$. Then p is prime if and only if p has this property:

$$\text{whenever } p \mid bc, \text{ then } p \mid b \text{ or } p \mid c.$$

Proof ▸ Since this is an "if and only if" statement, there are two parts to the proof.

Step 1 *Assume that p is prime and prove that p has the property stated in the theorem.*

Proof of Step 1: If p is prime and divides bc, consider the gcd of p and b. Now (p, b) must be a positive divisor of the prime p. So the only possibilities are $(p, b) = 1$ and $(p, b) = \pm p$ (whichever is positive). If $(p, b) = \pm p$, then $p \mid b$. If $(p, b) = 1$, since $p \mid bc$, we must have $p \mid c$ by Theorem 1.4. In every case, therefore, $p \mid b$ or $p \mid c$. Hence, p has the property stated in the theorem.

Step 2 *Assume that p is an integer that has the property stated in the theorem and prove that p is prime.*

Proof of Step 2: This proof is left to the reader (Exercise 14). ■

Corollary 1.6

If p is prime and $p \mid a_1a_2 \cdots a_n$, then p divides at least one of the a_i.

Proof ▸ If $p \mid a_1(a_2a_3 \cdots a_n)$, then $p \mid a_1$ or $p \mid a_2a_3 \cdots a_n$ by Theorem 1.5. If $p \mid a_1$, we are finished. If $p \mid a_2(a_3a_4 \cdots a_n)$, then $p \mid a_2$ or $p \mid a_3a_4 \cdots a_n$ by Theorem 1.5 again. If $p \mid a_2$, we are finished; if not, continue this process, using Theorem 1.5 repeatedly. After at most n steps, there must be an a_i that is divisible by p. ■

Choose an integer other than $0, \pm 1$. If you factor it "as much as possible," you will find that it is a product of one or more primes. For example,

$$12 = 4 \cdot 3 = 2 \cdot 2 \cdot 3,$$
$$60 = 12 \cdot 5 = 2 \cdot 2 \cdot 3 \cdot 5,$$
$$113 = 113 \text{ (prime)}.$$

In this context, *we allow the possibility of a "product" with just one factor* in case the number we begin with is actually a prime. What was done in these examples can always be done:

Theorem 1.7

Every integer n except $0, \pm 1$ is a product of primes.

Proof ▸ First note that if n is a product of primes, say $n = p_1p_2 \cdots p_k$, then $-n = (-p_1)p_2 \cdots p_k$ is also a product of primes. Consequently, we need prove the theorem only when $n > 1$. The idea of the proof can be summarized like this:

> *Let S be the set of all integers greater than 1 that are* not *a product of primes. Show that S is the empty set. Then, since there are no integers in S, it must be the case that every integer greater than 1* is *a product of primes (otherwise, it would be in S).*

Proof that S is empty: The proof is by contradiction: We assume that S is not empty and use that assumption to reach a contradiction. So assume that S is not empty. Then S contains a smallest integer m by the Well-Ordering Axiom. Since $m \in S$, m is not itself prime. Hence m must have positive divisors other than 1 or m, say $m = ab$ with $1 < a < m$ and $1 < b < m$. Since both a and b are less than m (the smallest element of S), neither a nor b is in S. By the definition of S, both a and b *are* the product of primes, say

$$a = p_1p_2 \cdots p_r \qquad \text{and} \qquad b = q_1q_2 \cdots q_s,$$

with $r \geq 1$, $s \geq 1$, and each p_i, q_j prime. Therefore

$$m = ab = p_1p_2 \cdots p_rq_1q_2 \cdots q_s$$

is a product of primes, so that $m \notin S$. We have reached a contradiction: $m \in S$ and $m \notin S$. Therefore, S must be empty. ∎

Technology Tip: To find the prime factorization of integers as large as 10–12 digits on a TI graphing calculator, download the FACTOR program on our website (www.CengageBrain.com). The program uses Theorem 1.10, which is proved on page 21, to do the factorization. Maple and Mathematica can find the prime factorization of these and much larger integers very quickly.

An integer other than 0, ± 1 that is not prime is called **composite**. Although a composite integer may have several different prime factorizations, such as

$$45 = 3 \cdot 3 \cdot 5,$$
$$45 = (-3) \cdot 5 \cdot (-3),$$
$$45 = 5 \cdot 3 \cdot 3,$$
$$45 = (-5) \cdot (-3) \cdot 3,$$

these factorizations are essentially the same. The only differences are the order of the factors and the insertion of minus signs. You can readily convince yourself that every prime factorization of 45 has exactly three prime factors, say $q_1q_2q_3$. Furthermore, by rearranging and relabeling the q's, you will always have $3 = \pm q_1$, $3 = \pm q_2$, and $5 = \pm q_3$. This is an example of the following theorem.

Theorem 1.8 The Fundamental Theorem of Arithmetic

Every integer n except 0, ± 1 is a product of primes. This prime factorization is unique in the following sense: If

$$n = p_1 p_2 \cdots p_r \qquad \text{and} \qquad n = q_1 q_2 \cdots q_s$$

with each p_i, q_j prime, then $r = s$ (that is, the number of factors is the same) and after reordering and relabeling the q's,

$$p_1 = \pm q_1, \qquad p_2 = \pm q_2, \qquad p_3 = \pm q_3, \ldots, p_r = \pm q_r.$$

Proof▸ Every integer n except 0, ± 1 has at least one prime factorization by Theorem 1.7. Suppose that n has two prime factorizations, as listed in the statement of the theorem. Then

$$p_1(p_2 p_3 \cdots p_r) = q_1 q_2 q_3 \cdots q_s,$$

so that $p_1 \mid q_1 q_2 \cdots q_s$. By Corollary 1.6, p_1 must divide one of the q_j. By reordering and relabeling the q's if necessary, we may assume that $p_1 \mid q_1$. Since p_1 and q_1 are prime, we must have $p_1 = \pm q_1$. Consequently,

$$\pm q_1 p_2 p_3 \cdots p_r = q_1 q_2 q_3 \cdots q_s.$$

Dividing both sides by q_1 shows that

$$p_2(\pm p_3 p_4 \cdots p_r) = q_2 q_3 q_4 \cdots q_s,$$

so that $p_2 \mid q_2 q_3 \cdots q_s$. By Corollary 1.6, p_2 must divide one of the q_j; as before, we may assume $p_2 \mid q_2$. Hence, $p_2 = \pm q_2$ and

$$\pm q_2 p_3 p_4 \cdots p_r = q_2 q_3 q_4 \cdots q_s.$$

Dividing both sides by q_2 shows that

$$p_3(\pm p_4 \cdots p_r) = q_3 q_4 \cdots q_s.$$

We continue in this manner, repeatedly using Corollary 1.6 and eliminating one prime on each side at every step. If $r = s$, then this process leads to the desired conclusion: $p_1 = \pm q_1, p_2 = \pm q_2, \ldots, p_r = \pm q_r$. So to complete the proof of the theorem, we must show that $r = s$. The proof that $r = s$ is a proof by contradiction: We assume that $r \neq s$ (which means that $r > s$ or that $r < s$), and show that this assumption leads to a contradiction.

First, suppose that $r > s$. Then after s steps of the preceding process, all the q's will have been eliminated and the equation will read

$$\pm p_{s+1} p_{s+2} \cdots p_r = 1.$$

This equation says (among other things) that $p_r \mid 1$. Since the only divisors of 1 are ± 1, we have $p_r = \pm 1$. However, since p_r is prime, we know that $p_r \neq \pm 1$ by the definition of "prime". We have reached a contradiction ($p_r = \pm 1$ and $p_r \neq \pm 1$). So $r > s$ cannot occur. A similar argument shows that the assumption $r < s$ also leads to a contraction and, hence, cannot occur. Therefore, $r = s$ is the only possibility, and the theorem is proved. ■

Technology Tip: The FACTOR program for TI calculators on our website (www.CengageBrain.com) factors an integer n as a product of primes relatively quickly. For example, if $n = 94{,}017$, then $n = 3 \cdot 7 \cdot 11^2 \cdot 37$, as shown in Figure 1.

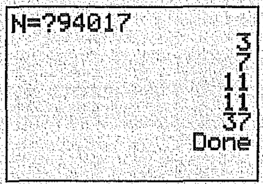

FIGURE 1

On Maple, the command ifactor(n); will produce the prime factorization of n.

If consideration is restricted to positive integers, then there is a stronger version of unique factorization:

Corollary 1.9

Every integer $n > 1$ can be written in one and only one way in the form $n = p_1p_2p_3 \cdots p_r$, where the p_i are positive primes such that $p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_r$.

Proof ▸ Exercise 12 ■

Primality Testing

In theory it is easy to determine if a positive integer n is prime. Just divide n by every integer between 1 and n to see if n has a factor other than 1 or n. Actually, you need only check prime divisors because any factor of n (except 1) is divisible by at least one prime. The following primality test greatly reduces the number of divisions that are necessary.

Theorem 1.10

Let $n > 1$. If n has no positive prime factor less than or equal to $\sqrt{n}$, then n is prime.

Before proving this theorem, it may be helpful to see how it is used.

EXAMPLE 2

To prove that 137 is prime, the theorem says that we must verify that 137 has no positive prime factors less than or equal to $\sqrt{137} \approx 11.7$; that is, we need only show that 2, 3, 5, 7, and 11 are not factors of 137. You can easily verify that none of them divide 137. Hence, 137 is prime by Theorem 1.10.

The proof of Theorem 1.10 (like several earlier in this chapter) is somewhat more detailed than is necessary. In particular, the underlined parts of the proof are normally omitted.

Proof of Theorem 1.10 ▶ <u>The proof is by contradiction</u>. Suppose that n is *not* prime. Then n has at least two positive prime factors, say p_1 and p_2, so that $n = p_1p_2k$ for some positive integer k. By hypothesis, <u>n has no positive prime divisors less than or equal to $\sqrt{n}$. Hence,</u> $p_1 > \sqrt{n}$ and $p_2 > \sqrt{n}$. Therefore,

$$n = p_1p_2k \geq p_1p_2 > \sqrt{n}\sqrt{n} = n,$$

which says that $n > n$, a contradiction. <u>Since the assumption that n is not prime has led to a contradiction, we conclude that n is prime</u>. ■

Theorem 1.10 is useful when working by hand with relatively small numbers. Testing very large integers for primality, however, requires a computer and techniques that are beyond the scope of this book.

■ Exercises

A. **1.** Express each number as a product of primes:

(a) 5040 **(b)** -2345

(c) 45,670 **(d)** 2,042,040

2. **(a)** Verify that $2^5 - 1$ and $2^7 - 1$ are prime.

(b) Show that $2^{11} - 1$ is not prime.

3. Which of the following numbers are prime:

(a) 701 **(b)** 1009

(c) 1949 **(d)** 1951

4. Primes p and q are said to be *twin primes* if $q = p + 2$. For example, 3 and 5 are twin primes; so are 11 and 13. Find all pairs of positive twin primes less than 200.

5. **(a)** List all the positive integer divisors of 3^s5^t, where $s, t \in \mathbb{Z}$ and $s, t > 0$.

(b) If $r, s, t \in \mathbb{Z}$ are positive, how many positive divisors does $2^r3^s5^t$ have?

6. If $p > 5$ is prime and p is divided by 10, show that the remainder is 1, 3, 7, or 9.

7. If a, b, c are integers and p is a prime that divides both a and $a + bc$, prove that $p \mid b$ or $p \mid c$.

8. (a) Verify that $x - 1$ is a factor of $x^n - 1$.

   (b) If n is a positive integer, prove that the prime factorization of $2^{2n} \cdot 3^n - 1$ includes 11 as one of the prime factors. [*Hint:* $(2^{2n} \cdot 3^n) = (2^2 \cdot 3)^n$.]

9. Let p be an integer other than $0, \pm 1$. Prove that p is prime if and only if it has this property: Whenever r and s are integers such that $p = rs$, then $r = \pm 1$ or $s = \pm 1$.

10. Let p be an integer other than $0, \pm 1$. Prove that p is prime if and only if for each $a \in \mathbb{Z}$ either $(a, p) = 1$ or $p \mid a$.

11. If a, b, c, d are integers and p is a prime factor of both $a - b$ and $c - d$, prove that p is a prime factor of $(a + c) - (b + d)$.

12. Prove Corollary 1.9.

13. Prove that every integer $n > 1$ can be written in the form $p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$, with the p_i distinct positive primes and every $r_i > 0$.

14. Let p be an integer other than $0, \pm 1$ with this property: Whenever b and c are integers such that $p \mid bc$, then $p \mid b$ or $p \mid c$. Prove that p is prime. [*Hint:* If d is a divisor of p, say $p = dt$, then $p \mid d$ or $p \mid t$. Show that this implies $d = \pm p$ or $d = \pm 1$.]

15. If p is prime and $p \mid a^n$, is it true that $p^n \mid a^n$? Justify your answer. [*Hint:* Corollary 1.6.]

16. Prove that $(a, b) = 1$ if and only if there is no prime p such that $p \mid a$ and $p \mid b$.

17. If p is prime and $(a, b) = p$, then $(a^2, b^2) = ?$

18. Prove or disprove each of the following statements:

    (a) If p is prime and $p \mid (a^2 + b^2)$ and $p \mid (c^2 + d^2)$, then $p \mid (a^2 - c^2)$.

    (b) If p is prime and $p \mid (a^2 + b^2)$ and $p \mid (c^2 + d^2)$, then $p \mid (a^2 + c^2)$.

    (c) If p is prime and $p \mid a$ and $p \mid (a^2 + b^2)$, then $p \mid b$.

B. 19. Suppose that $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and $b = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, where $p_1, p_2, \ldots, p_k$ are distinct positive primes and each $r_i, s_i \geq 0$. Prove that $a \mid b$ if and only if $r_i \leq s_i$ for every i.

20. If $a = p_1^{r_1} p_2^{r_2} p_3^{r_3} \cdots p_k^{r_k}$ and $b = p_1^{s_1} p_2^{s_2} p_3^{s_3} \cdots p_k^{s_k}$, where $p_1, p_2, \ldots, p_k$ are distinct positive primes and each $r_i, s_i \geq 0$, then prove that

    (a) $(a, b) = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$, where for each i, n_i = minimum of r_i, s_i.

    (b) $[a, b] = p_1^{t_1} p_2^{t_2} p_3^{t_3} \cdots p_k^{t_k}$, where t_i = maximum of r_i, s_i. [See Exercise 31 in Section 1.2.]

21. If $c^2 = ab$ and $(a, b) = 1$, prove that a and b are perfect squares.

22. Let $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and each $r_i \geq 0$. Prove that n is a perfect square if and only if each r_i is even.

23. Prove that $a \mid b$ if and only if $a^2 \mid b^2$. [*Hint:* Exercise 19.]

24. Prove that $a \mid b$ if and only if $a^n \mid b^n$.

25. Let p be prime and $1 \le k < p$. Prove that p divides the binomial coefficient $\binom{p}{k}$.
$\left[\text{Recall that } \binom{p}{k} = \dfrac{p!}{k!(p-k)!}.\right]$

26. If n is a positive integer, prove that there exist n consecutive composite integers. [*Hint:* Consider $(n+1)!+2, (n+1)!+3, (n+1)!+4, \ldots$.]

27. If $p > 3$ is prime, prove that $p^2 + 2$ is composite. [*Hint:* Consider the possible remainders when p is divided by 3.]

28. Prove or disprove: The sums

$$1+2+4, \qquad 1+2+4+8, \qquad 1+2+4+8+16, \ldots$$

are alternately prime and composite.

29. If $n \in \mathbb{Z}$ and $n \neq 0$, prove that n can be written uniquely in the form $n = 2^k m$, where $k \geq 0$ and m is odd.

30. **(a)** Prove that there are no nonzero integers a, b such that $a^2 = 2b^2$. [*Hint:* Use the Fundamental Theorem of Arithmetic.]

(b) Prove that $\sqrt{2}$ is irrational. [*Hint:* Use proof by contradiction (Appendix A). Assume that $\sqrt{2} = a/b$ (with $a, b \in \mathbb{Z}$) and use part (a) to reach a contradiction.]

31. If p is a positive prime, prove that $\sqrt{p}$ is irrational. [See Exercise 30.]

32. (Euclid) Prove that there are infinitely many primes. [*Hint:* Use proof by contradiction (Appendix A). Assume there are only finitely many primes $p_1, p_2, \ldots, p_k$, and reach a contradiction by showing that the number $p_1 p_2 \cdots p_k + 1$ is not divisible by any of $p_1, p_2, \ldots, p_k$.]

33. Let $p > 1$. If $2^p - 1$ is prime, prove that p is prime. [*Hint:* Prove the contrapositive: If p is composite, so is $2^p - 1$.]
Note: The converse is false by Exercise 2(b).

C. 34. Prove or disprove: If n is an integer and $n > 2$, then there exists a prime p such that $n < p < n!$.

35. **(a)** Let a be a positive integer. If $\sqrt{a}$ is rational, prove that $\sqrt{a}$ is an integer.

(b) Let r be a rational number and a an integer such that $r^n = a$. Prove that r is an integer. [Part (a) is the case when $n = 2$.]

36. Let p, q be primes with $p \geq 5, q \geq 5$. Prove that $24 \mid (p^2 - q^2)$.

CHAPTER 2

Congruence in $\mathbb{Z}$ and Modular Arithmetic

Basic concepts of integer arithmetic are extended here to include the idea of "congruence modulo n." Congruence leads to the construction of the set $\mathbb{Z}_n$ of all congruence classes of integers modulo n. This construction will serve as a model for many similar constructions in the rest of this book. It also provides our first example of a system of arithmetic that shares many fundamental properties with ordinary arithmetic and yet differs significantly from it.

2.1 Congruence and Congruence Classes

The concept of "congruence" may be thought of as a generalization of the equality relation. Two integers a and b are equal if their difference is 0 or, equivalently, if their difference is a multiple of 0. If n is a positive integer, we say that two integers are *congruent modulo n* if their difference is a multiple of n. To say that $a - b = nk$ for some integer k means that n divides $a - b$. So we have this formal definition:

Definition Let a, b, n be integers with $n > 0$. Then **a is congruent to b modulo n** [written "$a \equiv b \pmod{n}$"], provided that n divides $a - b$.

EXAMPLE 1

$17 \equiv 5 \pmod 6$ because 6 divides $17 - 5 = 12$. Similarly, $4 \equiv 25 \pmod 7$ because 7 divides $4 - 25 = -21$, and $6 \equiv -4 \pmod 5$ because 5 divides $6 - (-4) = 10$.

Remark In the notation "$a \equiv b \pmod{n}$," the symbols "$\equiv$" and "(mod n)" are really parts of a single symbol; "$a \equiv b$" by itself is meaningless. Some texts write "$a \equiv_n b$" instead of "$a \equiv b \pmod{n}$." Although this single-symbol notation is advantageous, we shall stick with the traditional "(mod n)" notation here.

The symbol used to denote congruence looks very much like an equal sign. This is no accident since the relation of congruence has many of the same properties as the relation of equality. For example, we know that equality is

reflexive: $a = a$ for every integer a;

symmetric: if $a = b$, then $b = a$;

transitive: if $a = b$ and $b = c$, then $a = c$.

We now see that congruence modulo n is also reflexive, symmetric, and transitive.

Theorem 2.1

Let n be a positive integer. For all $a, b, c \in \mathbb{Z}$,

(1) $a \equiv a \pmod{n}$;

(2) if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$;

(3) if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

Proof ▸ (1) To prove that $a \equiv a \pmod{n}$, we must show that $n \mid (a - a)$. But $a - a = 0$ and $n \mid 0$ (see Example 2 on page 9). Hence, $n \mid (a - a)$ and $a \equiv a \pmod{n}$.

(2) $a \equiv b \pmod{n}$ means that $a - b = nk$ for some integer k. Therefore, $b - a = -(a - b) = -nk = n(-k)$. The first and last parts of this equation say that $n \mid (b - a)$. Hence, $b \equiv a \pmod{n}$.

(3) If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then by the definition of congruence, there are integers k and t such that $a - b = nk$ and $b - c = nt$. Therefore,

$$\begin{aligned}(a - b) + (b - c) &= nk + nt \\ a - c &= n(k + t).\end{aligned}$$

Thus $n \mid (a - c)$ and, hence, $a \equiv c \pmod{n}$. ■

Several essential arithmetic and algebraic manipulations depend on this key fact:

$$\text{If } a = b \text{ and } c = d, \text{ then } a + c = b + d \text{ and } ac = bd.$$

We now show that the same thing is true for congruence.

Theorem 2.2

If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

(1) $a + c \equiv b + d \pmod{n}$;

(2) $ac \equiv bd \pmod{n}$.

Proof ▸ (1) To prove that $a + c \equiv b + d \pmod{n}$, we must show that n divides $(a + c) - (b + d)$. Since $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, we know that $n \mid (a - b)$ and $n \mid (b - d)$. Hence, there are integers k and t such that

$$(*) \qquad a - b = nk \quad \text{and} \quad c - d = nt.$$

We use these facts to show that n divides $(a + c) - (b + d)$:

$$\begin{aligned}
(a + c)-(b + d) &= a + c - b - d && [\textit{Arithmetic}]\\
&= (a - b) + (c - d) && [\textit{Rearrange terms.}]\\
&= nk + nt && [a - b = nk \textit{ and } c - d = nt.]\\
(a + c) - (b + d) &= n(k + t) && [\textit{Factor right side}]
\end{aligned}$$

The last equation says that n divides $(a + c) - (b + d)$. Hence, $a + c \equiv b + d \pmod{n}$.

(2) We must prove that n divides $ac - bd$.*

$$\begin{aligned}
ac - bd &= ac + 0 - bd\\
&= ac - bc + bc - bd && [-bc + bc = 0.]\\
&= (a - b)c + b(c - d) && [\textit{Factor first two terms and last two terms.}]\\
&= (nk)c + b(nt) && [a - b = nk \textit{ and } c - d = nt \textit{ by } (*) \textit{ above.}]\\
ac - bd &= n(kc + bt) && [\textit{Factor } n \textit{ from each term.}]
\end{aligned}$$

The last equation says that $n \mid (ac - bd)$. Therefore, $ac \equiv bd \pmod{n}$. ∎

With the equality relation, it's easy to see what numbers are equal to a given number a—just a itself. With congruence, however, the story is different and leads to some interesting consequences.

Definition Let a and n be integers with $n > 0$. The **congruence class of a modulo n** (denoted $[a]$) is the set of all those integers that are congruent to a modulo n, that is,

$$[a] = \{b \mid b \in \mathbb{Z} \quad \text{and} \quad b \equiv a \pmod{n}\}.$$

To say that $b \equiv a \pmod{n}$ means that $b - a = kn$ for some integer k or, equivalently, that $b = a + kn$. Thus

$$\begin{aligned}
[a] = \{b \mid b \equiv a \pmod{n}\} &= \{b \mid b = a + kn \text{ with } k \in \mathbb{Z}\}\\
&= \{a + kn \mid k \in \mathbb{Z}\}.
\end{aligned}$$

*The first two lines of this proof are a standard algebraic technique: Rewrite 0 in the form $-X + X$ for a suitable expression X.

EXAMPLE 2

In congruence modulo 5, we have

$$\begin{aligned}[9] &= \{9 + 5k \mid k \in \mathbb{Z}\} = \{9, 9 \pm 5, 9 \pm 10, 9 \pm 15, \dots\} \\ &= \{\dots, -11, -6, -1, 4, 9, 14, 19, 24, \dots\}.\end{aligned}$$

EXAMPLE 3

The meaning of the symbol "[ ]" depends on the context. In congruence modulo 3, for instance,

$$[2] = \{2 + 3k \mid k \in \mathbb{Z}\} = \{\dots, -7, -4, -1, 2, 5, 8, \dots\},$$

but in congruence modulo 5 the congruence class [2] is the set

$$\{2 + 5k \mid k \in \mathbb{Z}\} = \{\dots, -13, -8, -3, 2, 7, 12, \dots\}.$$

This ambiguity will not cause any difficulty when only one modulus is under discussion. On the few occasions when several moduli are discussed simultaneously, we avoid confusion by denoting the congruence class of a modulo n by $[a]_n$.

EXAMPLE 4

In congruence modulo 3, the congruence class

$$[2] = \{\dots, -7, -4, -1, 2, 5, 8, \dots\}.$$

Notice, however, that $[-1]$ is the same class because

$$[-1] = \{-1 + 3k \mid k \in \mathbb{Z}\} = \{\dots, -7, -4, -1, 2, 5, \dots\}.$$

Furthermore, $2 \equiv -1 \pmod 3$. This is an example of the following theorem.

Theorem 2.3

$a \equiv c \pmod n$ if and only if $[a] = [c]$.

Since Theorem 2.3 is an "if and only if" statement, we must prove two different things:

1. If $a \equiv c \pmod n$, then $[a] = [c]$.
2. If $[a] = [c]$, then $a \equiv c \pmod n$.

Neither of these proofs will use the definition of congruence. Instead, the proofs will use only the fact that congruence is reflexive, symmetric, and transitive (Theorem 2.1).

Proof of Theorem 2.3 ▸ First, assume that $a \equiv c \pmod{n}$. To prove that $[a] = [c]$, we first show that $[a] \subseteq [c]$. To do this, let $b \in [a]$. Then by definition $b \equiv a \pmod{n}$. Since $a \equiv c \pmod{n}$, we have $b \equiv c \pmod{n}$ by transitivity. Therefore, $b \in [c]$ and $[a] \subseteq [c]$. Reversing the roles of a and c in this argument and using the fact that $c \equiv a$ by symmetry, show that $[c] \subseteq [a]$. Therefore, $[a] = [c]$.

Conversely, assume that $[a] = [c]$. Since $a \equiv a \pmod{n}$ by reflexivity, we have $a \in [a]$ and, hence, $a \in [c]$. By the definition of $[c]$, we see that $a \equiv c \pmod{n}$. ■

If A and C are two sets, there are usually three possibilities: Either A and C are disjoint, or $A = C$, or $A \cap C$ is nonempty but $A \neq C$. With congruence classes, however, there are only two possibilities:

Corollary 2.4

Two congruence classes modulo n are either disjoint or identical.

Proof ▸ If $[a]$ and $[c]$ are disjoint, there is nothing to prove. Suppose that $[a] \cap [c]$ is nonempty. Then there is an integer b with $b \in [a]$ and $b \in [c]$. By the definition of congruence class, $b \equiv a \pmod{n}$ and $b \equiv c \pmod{n}$. Therefore, by symmetry and transitivity, $a \equiv c \pmod{n}$. Hence, $[a] = [c]$ by Theorem 2.3. ■

Corollary 2.5

Let $n > 1$ be an integer and consider congruence modulo n.

(1) If a is any integer and r is the remainder when a is divided by n, then $[a] = [r]$.

(2) There are exactly n distinct congruences classes, namely, $[0], [1], [2], \ldots, [n-1]$.

Proof ▸ (1) Let $a \in \mathbb{Z}$. By the Division Algorithm, $a = nq + r$, with $0 \leq r < n$. Thus $a - r = qn$, so that $a \equiv r \pmod{n}$. By Theorem 2.3, $[a] = [r]$.

(2) If $[a]$ is any congruence class, then (1) shows that $[a] = [r]$ with $0 \leq r < n$. Hence, $[a]$ must be one of $[0], [1], [2], \ldots, [n-1]$.

To complete the proof, we must show that these n classes are all distinct. To do this, we first show that no two of $0, 1, 2, \ldots, n-1$ are congruent modulo n. Suppose that s and t are distinct integers in the list $0, 1, 2, \ldots, n-1$. Then one is larger than the other, say t, so that $0 \leq s < t < n$. Consequently, $t - s$ is a positive integer that is less than n. Hence, n does not divide $t - s$, which means that $t \not\equiv s$. Thus, no two of $0, 1, 2, \ldots, n-1$ are congruent modulo n. Therefore, by Theorem 2.3, the classes $[0], [1], [2], \ldots, [n-1]$ are all distinct. ■

Definition The set of all congruence classes modulo n is denoted $\mathbb{Z}_n$ (which is read "$\mathbb{Z}$ mod n").

There are several points to be careful about here. The elements of $\mathbb{Z}_n$ are *classes*, not single integers. So the statement $[5] \in \mathbb{Z}_n$ is true, but the statement $5 \in \mathbb{Z}_n$ is not. Furthermore, every element of $\mathbb{Z}_n$ can be denoted in many different ways. For example, we know that

$$2 \equiv 5 \pmod 3 \qquad 2 \equiv -1 \pmod 3 \qquad 2 \equiv 14 \pmod 3.$$

Therefore, by Theorem 2.3, $[2] = [5] = [-1] = [14]$ in $\mathbb{Z}_3$. Even though each element of $\mathbb{Z}_n$ (that is, each congruence class) has infinitely many different labels, there are only finitely many distinct classes by Corollary 2.5, which says in effect that

The set $\mathbb{Z}_n$ has exactly n elements.

For example, the set $\mathbb{Z}_3$ consists of the three elements $[0], [1], [2]$.

Exercises

A. 1. Show that $a^{p-1} \equiv 1 \pmod p$ for the given p and a:

(a) $a = 2, p = 5$ (b) $a = 4, p = 7$ (c) $a = 3, p = 11$

2. (a) If $k \equiv 1 \pmod 4$, then what is $6k + 5$ congruent to modulo 4?

(b) If $r \equiv 3 \pmod{10}$ and $s \equiv -7 \pmod{10}$, then what is $2r + 3s$ congruent to modulo 10?

3. Every published book has a ten-digit ISBN-10 number (on the back cover or the copyright page) that is usually of the form $x_1–x_2x_3x_4–x_5x_6x_7x_8x_9–x_{10}$ (where each x_i is a single digit).* The first 9 digits identify the book. The last digit x_{10} is a *check digit*; it is chosen so that

$$10x_1 + 9x_2 + 8x_3 + 7x_4 + 6x_5 + 5x_6 + 4x_7 + 3x_8 + 2x_9 + x_{10} \equiv 0 \pmod{11}.$$

If an error is made when scanning or keying an ISBN number into a computer, the left side of the congruence will not be congruent to 0 modulo 11, and the number will be rejected as invalid.† Which of the following are apparently valid ISBN numbers?

(a) 3–540–90518–9 (b) 0–031–10559–5 (c) 0–385–49596–X

*Sometimes the last digit of an ISBN number is the letter X. In such cases, treat X as if it were the number 10.

†The procedures in Exercises 3 and 4 will detect every single digit substitution error (for instance, 3 is entered as 8 and no other error is made). They will detect about 90% of transposition errors (for instance, 74 is entered as 47 and no other error is made). However, they may not detect multiple errors.

4. Virtually every item sold in a store has a 12-digit UPC barcode which is scanned at the checkout counter. The first 11 digits of a UPC number $d_1d_2d_3 \cdots d_{11}d_{12}$ identify the manufacturer and product. The last digit d_{12} is a check digit which is chosen so that

$$3d_1 + d_2 + 3d_3 + d_4 + 3d_5 + d_6 + 3d_7 + d_8 + 3d_9 + d_{10} + 3d_{11} + d_{12} \equiv 0 \pmod{10}.$$

If the congruence does not hold, an error has been made and the item must be scanned again, or the UPC code entered by hand. Which of the following UPC numbers were scanned incorrectly?

(a) 037000356691 **(b)** 833732000625 **(c)** 040293673034

5. **(a)** Which of [0], [1], [2], [3] is equal to $[5^{2000}]$ in $\mathbb{Z}_4$? [*Hint:* $5 \equiv 1 \pmod 4$; use Theorems 2.2 and 2.3.]

   (b) Which of [0], [1], [2], [3], [4] is equal to $[4^{2001}]$ in $\mathbb{Z}_5$?

6. If $a \equiv b \pmod n$ and $k \mid n$, is it true that $a \equiv b \pmod k$? Justify your answer.

7. If $a \in \mathbb{Z}$, prove that a^2 is not congruent to 2 modulo 4 or to 3 modulo 4.

8. Prove that every odd integer is congruent to 1 modulo 4 or to 3 modulo 4.

9. Prove that

   (a) $(n - a)^2 \equiv a^2 \pmod n$ **(b)** $(2n - a)^2 \equiv a^2 \pmod{4n}$

10. If a is a nonnegative integer, prove that a is congruent to its last digit mod 10 [for example, $27 \equiv 7 \pmod{10}$].

B. 11. If a, b are integers such that $a \equiv b \pmod p$ for every positive prime p, prove that $a = b$.

12. If $p \geq 5$ and p is prime, prove that $[p] = [1]$ or $[p] = [5]$ in $\mathbb{Z}_6$. [*Hint:* Theorem 2.3 and Corollary 2.5.]

13. Prove that $a \equiv b \pmod n$ if and only if a and b leave the same remainder when divided by n.

14. **(a)** Prove or disprove: If $ab \equiv 0 \pmod n$, then $a \equiv 0 \pmod n$ or $b \equiv 0 \pmod n$.

    (b) Do part (a) when n is prime.

15. If $(a, n) = 1$, prove that there is an integer b such that $ab \equiv 1 \pmod n$.

16. If $[a] = [1]$ in $\mathbb{Z}_n$, prove that $(a, n) = 1$. Show by example that the converse may be false.

17. Prove that $10^n \equiv (-1)^n \pmod{11}$ for every positive n.

18. Use congruences (not a calculator) to show that $(125698)(23797) \neq 2891235306$. [*Hint:* See Exercise 21.]

19. Prove or disprove: If $[a] = [b]$ in $\mathbb{Z}_n$, then $(a, n) = (b, n)$.

20. **(a)** Prove or disprove: If $a^2 \equiv b^2 \pmod n$, then $a \equiv b \pmod n$ or $a \equiv -b \pmod n$.

    (b) Do part (a) when n is prime.

21. (a) Show that $10^n \equiv 1 \pmod{9}$ for every positive n.
    (b) Prove that every positive integer is congruent to the sum of its digits mod 9 [for example, $38 \equiv 11 \pmod{9}$].

22. (a) Give an example to show that the following statement is false: If $ab \equiv ac \pmod{n}$ and $a \not\equiv 0 \pmod{n}$, then $b \equiv c \pmod{n}$.
    (b) Prove that the statement in part (a) is true whenever $(a, n) = 1$.

EXCURSION: The Chinese Remainder Theorem (Section 14.1) may be covered at this point if desired.

2.2 Modular Arithmetic

The finite set $\mathbb{Z}_n$ is closely related to the infinite set $\mathbb{Z}$. So it is natural to ask if it is possible to define addition and multiplication in $\mathbb{Z}_n$ and do some reasonable kind of arithmetic there. To define addition in $\mathbb{Z}_n$, we must have some way of taking two classes in $\mathbb{Z}_n$ and producing another class—their sum. Because addition of integers *is* defined, the following *tentative* definition seems worth investigating:

The sum of the classes $[a]$ and $[c]$ is the class containing $a + c$ or, in symbols,

$$[a] \oplus [c] = [a + c],$$

where addition of classes is denoted by $\oplus$ to distinguish it from ordinary addition of integers.

We can try a similar *tentative* definition for multiplication:

The product of $[a]$ and $[c]$ is the class containing ac:

$$[a] \odot [c] = [ac],$$

where $\odot$ denotes multiplication of classes.

EXAMPLE 1

In $\mathbb{Z}_5$ we have $[3] \oplus [4] = [3 + 4] = [7] = [2]$ and $[3] \odot [2] = [3 \cdot 2] = [6] = [1]$.

Everything seems to work so far, but there is a possible difficulty. Every element of $\mathbb{Z}_n$ can be written in many different ways. In $\mathbb{Z}_5$, for instance, $[3] = [13]$ and $[4] = [9]$. In the preceding example, we saw that $[3] \oplus [4] = [2]$ in $\mathbb{Z}_5$. Do we get the same answer if we use $[13]$ in place of $[3]$ and $[9]$ in place of $[4]$? In this case the answer is "yes" because

$$[13] \oplus [9] = [13 + 9] = [22] = [2].$$

But how do we know that the answer will be the same no matter which way we write the classes?

To get some idea of the kind of thing that might go wrong, consider these five classes of integers:

$$A = \{\ldots, -14, -8, -2, 0, 6, 12, 18, \ldots\}$$

$$B = \{\ldots, -11, -7, -3, 1, 5, 9, 13, \ldots\}$$

$$C = \{\ldots, -9, -5, -1, 3, 7, 11, 15, \ldots\}$$

$$D = \{\ldots, -16, -10, -4, 2, 8, 14, 20, \ldots\}$$

$$E = \{\ldots, -18, -12, -6, 4, 10, 16, 22, \ldots\}.$$

These classes, like the classes in $\mathbb{Z}_5$, have the following basic properties: Every integer is in one of them, and any two of them are either disjoint or identical. Since 1 is in B and 7 is in C, we could define $B + C$ as the class containing $1 + 7 = 8$, that is, $B + C = D$. But B is also the class containing -3 and C the class containing 15, and so $B + C$ ought to be the class containing $-3 + 15 = 12$. But 12 is in A, so that $B + C = A$. Thus you get different answers, depending on which "representatives" you choose from the classes B and C. Obviously you can't have any meaningful concept of addition if the answer is one thing this time and something else another time.

In order to remove the word "tentative" from our definition of addition and multiplication in $\mathbb{Z}_n$, we must first prove that these operations do not depend on the choice of representatives from the various classes. Here is what's needed:

Theorem 2.6

If $[a] = [b]$ and $[c] = [d]$ in $\mathbb{Z}_n$, then

$$[a + c] = [b + d] \qquad \text{and} \qquad [ac] = [bd].$$

Proof ▸ Since $[a] = [b]$, we know that $a \equiv b \pmod{n}$ by Theorem 2.3. Similarly, $[c] = [d]$ implies that $c \equiv d \pmod{n}$. Therefore, by Theorem 2.2,

$$a + c \equiv b + d \pmod{n} \qquad \text{and} \qquad ac \equiv bd \pmod{n}.$$

Hence, by Theorem 2.3 again,

$$[a + c] = [b + d] \qquad \text{and} \qquad [ac] = [bd]. \quad \blacksquare$$

Because of Theorem 2.6, we know that the following formal definition of addition and multiplication of classes is independent of the choice of representatives from each class:

Definition

Addition and multiplication in $\mathbb{Z}_n$ are defined by

$$[a] \oplus [c] = [a + c] \qquad \text{and} \qquad [a] \odot [c] = [ac].$$

EXAMPLE 2

Here are the complete addition and multiplication tables for $\mathbb{Z}_5$ (verify that these calculations are correct):*

$\oplus$	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

$\odot$	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]
[2]	[0]	[2]	[4]	[1]	[3]
[3]	[0]	[3]	[1]	[4]	[2]
[4]	[0]	[4]	[3]	[2]	[1]

And here are the tables for $\mathbb{Z}_6$:

$\oplus$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	[3]	[4]	[5]	[0]
[2]	[2]	[3]	[4]	[5]	[0]	[1]
[3]	[3]	[4]	[5]	[0]	[1]	[2]
[4]	[4]	[5]	[0]	[1]	[2]	[3]
[5]	[5]	[0]	[1]	[2]	[3]	[4]

$\odot$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[3]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]

Properties of Modular Arithmetic

Now that addition and multiplication are defined in $\mathbb{Z}_n$, we want to compare the properties of these "miniature arithmetics" with the well-known properties of $\mathbb{Z}$. The key facts about arithmetic in $\mathbb{Z}$ (and the usual titles for these properties) are as follows. For all $a, b, c \in \mathbb{Z}$:

1. If $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$. [*Closure for addition*]
2. $a + (b + c) = (a + b) + c$. [*Associative addition*]
3. $a + b = b + a$. [*Commutative addition*]
4. $a + 0 = a = 0 + a$. [*Additive identity*]

*These tables are read like this: If $[a]$ appears in the left-hand vertical column and $[c]$ in the top horizontal row of the addition table, for example, then the sum $[a] \oplus [c]$ appears at the intersection of the horizontal row containing $[a]$ and the vertical column containing $[c]$.

5. For each $a \in \mathbb{Z}$, the equation $a + x = 0$ has a solution in $\mathbb{Z}$.
6. If $a, b \in \mathbb{Z}$, then $ab \in \mathbb{Z}$. [*Closure for multiplication*]
7. $a(bc) = (ab)c$. [*Associative multiplication*]
8. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$. [*Distributive laws*]
9. $ab = ba$ [*Commutative multiplication*]
10. $a \cdot 1 = a = 1 \cdot a$ [*Multiplicative identity*]
11. If $ab = 0$, then $a = 0$ or $b = 0$.

By using the tables in the preceding example, you can verify that the first ten of these properties hold in $\mathbb{Z}_5$ and $\mathbb{Z}_6$ and that Property 11 holds in $\mathbb{Z}_5$ and fails in $\mathbb{Z}_6$. But using tables is not a very efficient method of proof (especially for verifying associativity or distributivity). So the proof that Properties 1–10 hold for any $\mathbb{Z}_n$ is based on the definition of the operations in $\mathbb{Z}_n$ and on the fact that these properties are known to be valid in $\mathbb{Z}$.

Theorem 2.7

For any classes $[a]$, $[b]$, $[c]$ in $\mathbb{Z}_n$,

1. If $[a] \in \mathbb{Z}_n$ and $[b] \in \mathbb{Z}_n$, then $[a] \oplus [b] \in \mathbb{Z}_n$.
2. $[a] \oplus ([b] \oplus [c]) = ([a] \oplus [b]) \oplus [c]$.
3. $[a] \oplus [b] = [b] \oplus [a]$.
4. $[a] \oplus [0] = [a] = [0] \oplus [a]$.
5. For each $[a]$ in $\mathbb{Z}_n$, the equation $[a] \oplus X = [0]$ has a solution in $\mathbb{Z}_n$.
6. If $[a] \in \mathbb{Z}_n$ and $[b] \in \mathbb{Z}_n$, then $[a] \odot [b] \in \mathbb{Z}_n$.
7. $[a] \odot ([b] \odot [c]) = ([a] \odot [b]) \odot [c]$.
8. $[a] \odot ([b] \oplus [c]) = [a] \odot [b] \oplus [a] \odot [c]$ and $([a] \oplus [b]) \odot [c] = [a] \odot [c] \oplus [b] \odot [c]$.
9. $[a] \odot [b] = [b] \odot [a]$.
10. $[a] \odot [1] = [a] = [1] \odot [a]$.

Proof ▸ Properties 1 and 6 are an immediate consequence of the definition of $\oplus$ and $\odot$ in $\mathbb{Z}_n$.

To prove Property 2, note that by the definition of addition,

$$[a] \oplus ([b] \oplus [c]) = [a] \oplus [b + c] = [a + (b + c)].$$

In $\mathbb{Z}$ we know that $a + (b + c) = (a + b) + c$. So the classes of these integers must be the same in $\mathbb{Z}_n$; that is, $[a + (b + c)] = [(a + b) + c]$. By the definition of addition in $\mathbb{Z}_n$, we have

$$[(a + b) + c] = [a + b] \oplus [c] = ([a] \oplus [b]) \oplus [c].$$

This proves Property 2. The proofs of Properties 3, 7, 8, and 9 are analogous (Exercise 10).

Properties 4 and 10 are proved by a direct calculation; for instance, $[a] \odot [1] = [a \cdot 1] = [a]$.

For Property 5, it is easy to see that $X = [-a]$ is a solution of the equation since $[a] \oplus [-a] = [a + (-a)] = [0]$. ■

Exponents and Equations

The same exponent notation used in ordinary arithmetic is also used in $\mathbb{Z}_n$. If $[a] \in \mathbb{Z}_n$, and k is a positive integer, then $[a]^k$ denotes the product

$$[a] \odot [a] \odot [a] \odot \cdots \odot [a] \quad (k \text{ factors}).$$

EXAMPLE 3

In Z_5, $[3]^2 = [3] \odot [3] = [4]$ and $[3]^4 = [3] \odot [3] \odot [3] \odot [3] = [1]$.

As noted on page 9, the set $\mathbb{Z}_n$ has exactly n elements. Consequently, any equation in $\mathbb{Z}_n$ can be solved by substituting each of these n elements in the equation to see which ones are solutions.

EXAMPLE 4

To solve $x^2 \oplus [5] \odot x = [0]$ in $\mathbb{Z}_6$, substitute each of [0], [1], [2], [3], [4], and [5] in the equation to see if it is a solution:

x	$x^2 \oplus [5] \odot x$	Is $x^2 \oplus [5] \odot x = [0]$?
[0]	$[0]\odot[0] \oplus [5]\odot[0] = [0] \oplus [0] = [0]$	Yes; solution
[1]	$[1]\odot[1] \oplus [5]\odot[1] = [1] \oplus [5] = [0]$	Yes; solution
[2]	$[2]\odot [2] \oplus [5]\odot[2] = [4] \oplus [4] = [2]$	No
[3]	$[3]\odot[3] \oplus [5]\odot[3] = [3] \oplus [3] = [0]$	Yes; solution
[4]	$[4]\odot[4] \oplus [5]\odot[4] = [4] \oplus [2] = [0]$	Yes; solution
[5]	$[5]\odot[5] \oplus [5]\odot[5] = [1] \oplus [1] = [2]$	No

So the equation has four solutions: [0], [1], [3], and [4].

Example 4 shows that solving equations in $\mathbb{Z}_n$ may be quite different from solving equations in $\mathbb{Z}$. A quadratic equation in $\mathbb{Z}$ has at most two solutions, whereas the quadratic equation $x^2 \oplus [5]\odot x = [0]$ has four solutions in $\mathbb{Z}_6$.

Exercises

A. **1.** Write out the addition and multiplication tables for

(a) $\mathbb{Z}_2$ **(b)** $\mathbb{Z}_4$ **(c)** $\mathbb{Z}_7$ **(d)** $\mathbb{Z}_{12}$

In Exercises 2–8, solve the equation.

2. $x^2 \oplus x = [0]$ in $\mathbb{Z}_4$

3. $x^2 = [1]$ in $\mathbb{Z}_8$
4. $x^4 = [1]$ in $\mathbb{Z}_5$
5. $x^2 \oplus [3] \odot x \oplus [2] = [0]$ in $\mathbb{Z}_6$
6. $x^2 \oplus [8] \odot x = [0]$ in $\mathbb{Z}_9$
7. $x^3 \oplus x^2 \oplus x \oplus [1] = [0]$ in $\mathbb{Z}_8$
8. $x^3 + x^2 = [2]$ in $\mathbb{Z}_{10}$
9. **(a)** Find an element $[a]$ in $\mathbb{Z}_7$ such that every nonzero element of $\mathbb{Z}_7$ is a power of $[a]$.
   (b) Do part (a) in $\mathbb{Z}_5$. **(c)** Can you do part (a) in $\mathbb{Z}_6$?
10. Prove parts 3, 7, 8, and 9 of Theorem 2.7.
11. Solve the following equations.
    (a) $x \oplus x \oplus x = [0]$ in $\mathbb{Z}_3$
    (b) $x \oplus x \oplus x \oplus x = [0]$ in $\mathbb{Z}_4$
    (c) $x \oplus x \oplus x \oplus x \oplus x = [0]$ in $\mathbb{Z}_5$
12. Prove or disprove: If $[a] \odot [b] = [0]$ in $\mathbb{Z}_n$, then $[a] = [0]$ or $[b] = [0]$.
13. Prove or disprove: If $[a] \odot [b] = [a] \odot [c]$ and $[a] \neq [0]$ in $\mathbb{Z}_n$, then $[b] = [c]$.

B. 14. Solve the following equations.
    (a) $x^2 + x = [0]$ in $\mathbb{Z}_5$
    (b) $x^2 + x = [0]$ in $\mathbb{Z}_6$
    (c) If p is prime, prove that the only solutions of $x^2 + x = [0]$ in $\mathbb{Z}_p$ are $[0]$ and $[p - 1]$.
15. Compute the following products.
    (a) $([a] \oplus [b])^2$ in $\mathbb{Z}_2$
    (b) $([a] \oplus [b])^3$ in $\mathbb{Z}_3$ [*Hint:* Exercise 11(a) may be helpful.]
    (c) $([a] \oplus [b])^5$ in $\mathbb{Z}_5$ [*Hint:* See Exercise 11(c).]
    (d) Based on the results of parts (a)–(c), what do you think $([a] \oplus [b])^7$ is equal to in $\mathbb{Z}_7$?
16. **(a)** Find all $[a]$ in $\mathbb{Z}_5$ for which the equation $[a] \odot x = [1]$ has a solution. Then do the same thing for
    (b) $\mathbb{Z}_4$ **(c)** $\mathbb{Z}_3$ **(d)** $\mathbb{Z}_6$

2.3 The Structure of $\mathbb{Z}_p$ (p Prime) and $\mathbb{Z}_n$

We now present some facts about the structure of $\mathbb{Z}_n$ (particularly when n is prime) that will provide a model for our future work. First, however, we make a change of notation.

New Notation

We have been very careful to distinguish integers in $\mathbb{Z}$ and classes in $\mathbb{Z}_n$ and have even used different symbols for the operations in the two systems. By now, however, you should be reasonably comfortable with the fundamental ideas and familiar with arithmetic in $\mathbb{Z}_n$. So we shall adopt a new notation that is widely used in mathematics, even though it has the flaw that the same symbol represents two totally different entities.

Whenever the context makes clear that we are dealing with $\mathbb{Z}_n$, we shall abbreviate the class notation "$[a]$" and write simply "a." In $\mathbb{Z}_6$, for instance, we might say $6 = 0$, which is certainly true for classes in $\mathbb{Z}_6$ even though it is nonsense if 6 and 0 are ordinary integers. We shall use an ordinary plus sign for addition in $\mathbb{Z}_n$ and either a small dot or juxtaposition for multiplication. For example, in $\mathbb{Z}_5$ we may write things like

$$4 + 1 = 0 \qquad \text{or} \qquad 3 \cdot 4 = 2 \qquad \text{or} \qquad 4 + 4 = 3.$$

On those few occasions where this usage might cause confusion, we will return to the brackets notation for classes.

EXAMPLE 1

In this new notation, the addition and multiplication tables for $\mathbb{Z}_3$ are

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

CAUTION: *Exponents are ordinary integers*—not elements of $\mathbb{Z}_n$. In $\mathbb{Z}_3$, for instance, $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 1$ and $2^1 = 2$, so that $2^4 \neq 2^1$ even though $4 = 1$ in $\mathbb{Z}_3$.

The Structure of $\mathbb{Z}_p$ When p Is Prime

Some of the $\mathbb{Z}_n$ do not share all the nice properties of $\mathbb{Z}$. For instance, the product of nonzero integers in $\mathbb{Z}$ is always nonzero, but in $\mathbb{Z}_6$ we have $2 \cdot 3 = 0$ even though $2 \neq 0$ and $3 \neq 0$. On the other hand, the multiplication table on page 34 shows that the product of nonzero elements in $\mathbb{Z}_5$ is always nonzero. Indeed, $\mathbb{Z}_5$ has a much stronger property than $\mathbb{Z}$. When $a \neq 0$, the equation $ax = 1$ has a solution in $\mathbb{Z}$ if and only if $a = \pm 1$. But the multiplication table for $\mathbb{Z}_5$ shows that, for any $a \neq 0$, the equation $ax = 1$ has a solution in $\mathbb{Z}_5$; for example,

$$x = 3 \text{ is a solution of } 2x = 1$$
$$x = 4 \text{ is a solution of } 4x = 1.$$

More generally, whenever n is prime, $\mathbb{Z}_n$ has special properties:

Theorem 2.8

If $p > 1$ is an integer, then the following conditions are equivalent:*

(1) p is prime.

(2) For any $a \neq 0$ in $\mathbb{Z}_p$, the equation $ax = 1$ has a solution in $\mathbb{Z}_p$.

(3) Whenever $bc = 0$ in $\mathbb{Z}_p$, then $b = 0$ or $c = 0$.

The proof of this theorem illustrates the two basic techniques for proving statements that involve $\mathbb{Z}_n$:

(i) Translate equations in $\mathbb{Z}_n$ into equivalent congruence statements in $\mathbb{Z}$. Then the properties of congruence and arithmetic in $\mathbb{Z}$ can be used. The brackets notation for elements of $\mathbb{Z}_n$ may be necessary to avoid confusion.

(ii) Use the arithmetic properties of $\mathbb{Z}_n$ directly, without involving arithmetic in $\mathbb{Z}$. In this case, the brackets notation in $\mathbb{Z}_n$ isn't needed.

Proof of Theorem 2.8 ▸ (1) $\Rightarrow$ (2) We use the first technique. Suppose p is prime and $[a] \neq [0]$ in $\mathbb{Z}_p$. Then in $\mathbb{Z}$, $a \not\equiv 0 \pmod{p}$ by Theorem 2.3. Hence, $p \nmid a$ by the definition of congruence. Now the gcd of a and p is a positive divisor of p and thus must be either p or 1. Since (a, p) also divides a and $p \nmid a$, we must have $(a, p) = 1$. By Theorem 1.2, $au + pv = 1$ for some integers u and v. Hence, $au - 1 = p(-v)$, so that $au \equiv 1 \pmod{p}$. Therefore $[au] = [1]$ in $\mathbb{Z}_p$ by Theorem 2.3. Thus $[a][u] = [au] = [1]$, so that $x = [u]$ is a solution of $[a]x = [1]$.

(2) $\Rightarrow$ (3) We use the second technique. Suppose $ab = 0$ in $\mathbb{Z}_p$. If $a = 0$, there is nothing to prove. If $a \neq 0$, then by (2) there exists $u \in \mathbb{Z}_p$ such that $au = 1$. Then

$$0 = u \cdot 0 = u(ab) = (ua)b = (au)b = 1 \cdot b = b$$

In every case, therefore, we have $a = 0$ or $b = 0$.

(3) $\Rightarrow$ (1) Back to the first technique. Suppose that b and c are any integers and that $p \mid bc$. Then $bc \equiv 0 \pmod{p}$. So by Theorem 2.3,

$$[b][c] = [bc] = [0] \text{ in } \mathbb{Z}_p.$$

Hence, by (3), we have $[b] = [0]$ or $[c] = [0]$. Thus, $b \equiv 0 \pmod{p}$ or $c \equiv 0 \pmod{p}$ by Theorem 2.3, which means that $p \mid b$ or $p \mid c$ by the definition of congruence. Therefore, p is prime by Theorem 1.5. ■

The Structure of $\mathbb{Z}_n$

When n is not prime, the equation $ax = 1$ need not have a solution in $\mathbb{Z}_n$. For instance, the equation $2x = 1$ has no solution in $\mathbb{Z}_4$, as you can easily verify. The next result tells us exactly when $ax = 1$ *does* have a solution in $\mathbb{Z}_n$. For clarity, we use brackets notation.

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*See page 508 in Appendix A for the meaning of "the following conditions are equivalent" and what must be done to prove such a statement.

Theorem 2.9

Let a and n be integers with $n > 1$. Then

The equation $[a]x = [1]$ has a solution in $\mathbb{Z}_n$ if and only if $(a, n) = 1$ in $\mathbb{Z}$.

Proof ▸ Since this is an "if and only if" statement, the proof has two parts. First we assume that the equation has a solution and show that $(a, n) = 1$. If $[w]$ is a solution of $[a]x = [1]$, then

$$\begin{aligned} [a][w] &= [1] & & \\ [aw] &= [1] & & [\textit{Multiplication in } \mathbb{Z}_n] \\ aw &\equiv 1 \pmod{n} \text{ in } \mathbb{Z} & & [\textit{Theorem 2.3}] \\ aw - 1 &= kn \text{ for some integer } k & & [\textit{Definition of congruence}] \\ aw + n(-k) &= 1 & & [\textit{Rearrange terms}] \end{aligned}$$

Denote (a, n) by d. Since d is a common divisor of a and n, there are integers r and s such that $dr = a$ and $ds = n$. So we have

$$\begin{aligned} aw + n(-k) &= 1 \\ drw + ds(-k) &= 1 \\ d(rw - sk) &= 1. \end{aligned}$$

So $d \mid 1$. Since d is positive by definition, we must have $d = 1$, that is, $(a, n) = 1$.

Now we assume that $(a, n) = 1$ and show that $[a]x = [1]$ has a solution in $\mathbb{Z}_n$. Actually, we've already done this. In the proof of (1) $\Rightarrow$ (2) of Theorem 2.8, the primeness of p is used only to show that $(a, p) = 1$. From there on, the proof is valid in any $\mathbb{Z}_n$ when $(a, n) = 1$, and shows that $[a]x = [1]$ has a solution in $\mathbb{Z}_n$. ■

Units and Zero Divisors

Some special terminology is often used when dealing with certain equations. An element a in $\mathbb{Z}_n$ is called a **unit** if the equation $ax = 1$ has a solution. In other words, a is a unit if there is an element b in $\mathbb{Z}_n$ such that $ab = 1$. In this case, we say that b is the **inverse** of a. Note that $ab = 1$ also says that b is a unit (with inverse a).

EXAMPLE 2

Both 2 and 8 are units in $\mathbb{Z}_{15}$ because $2 \cdot 8 = 1$. 8 is the inverse of 2 and 2 is the inverse of 8. Similarly, 3 is a unit in $\mathbb{Z}_4$ because $3 \cdot 3 = 1$. So 3 is its own inverse.

EXAMPLE 3

Part (2) of Theorem 2.8 says that when p is prime, every nonzero element of $\mathbb{Z}_p$ is a unit.

Here is a restatement of Theorem 2.9 in the terminology of units.

Theorem 2.10

Let a and n be integers with $n > 1$. Then

$$[a] \text{ is a unit in } \mathbb{Z}_n \text{ if and only if } (a, n) = 1 \text{ in } \mathbb{Z}.$$

A nonzero element a of $\mathbb{Z}_n$ is called a **zero divisor** if the equation $ax = 0$ has a *nonzero* solution (that is, if there is a nonzero element c in $\mathbb{Z}_n$ such that $ac = 0$).

EXAMPLE 4

Both 3 and 5 are zero divisors in $\mathbb{Z}_{15}$ because $3 \cdot 5 = 0$. Similarly, 2 is a zero divisor in $\mathbb{Z}_4$ because $2 \cdot 2 = 0$.

EXAMPLE 5

Part (3) of Theorem 2.8 says that when p is prime, there are no zero divisors in $\mathbb{Z}_p$.

Exercises

A. **1.** Find all the units in

(a) $\mathbb{Z}_7$ **(b)** $\mathbb{Z}_8$ **(c)** $\mathbb{Z}_9$ **(d)** $\mathbb{Z}_{10}$.

2. Find all the zero divisors in

(a) $\mathbb{Z}_7$ **(b)** $\mathbb{Z}_8$ **(c)** $\mathbb{Z}_9$ **(d)** $\mathbb{Z}_{10}$.

3. Based on Exercises 1 and 2, make a conjecture about units and zero divisors in $\mathbb{Z}_n$.

4. How many solutions does the equation $6x = 4$ have in

(a) $\mathbb{Z}_7$? **(b)** $\mathbb{Z}_8$? **(c)** $\mathbb{Z}_9$? **(d)** $\mathbb{Z}_{10}$?

5. If a is a unit and b is a zero divisor in $\mathbb{Z}_n$, show that ab is a zero divisor.

6. If n is composite, prove that there is at least one zero divisor in $\mathbb{Z}_n$. (See Exercise 2.)

7. Without using Theorem 2.8, prove that if p is prime and $ab = 0$ in $\mathbb{Z}_p$, then $a = 0$ or $b = 0$. [*Hint:* Theorem 1.8.]

8. **(a)** Give three examples of equations of the form $ax = b$ in $\mathbb{Z}_{12}$ that have no nonzero solutions.

(b) For each of the equations in part (a), does the equation $ax = 0$ have a nonzero solution?

B. **9.** **(a)** If a is a unit in $\mathbb{Z}_n$, prove that a is not a zero divisor.

(b) If a is a zero divisor in $\mathbb{Z}_n$, prove that a is not a unit. [*Hint:* Think contrapositive in part (a).]

10. Prove that every nonzero element of $\mathbb{Z}_n$ is either a unit or a zero divisor, but not both. [*Hint:* Exercise 9 provides the proof of "not both".]

11. Without using Exercises 13 and 14, prove: If $a, b \in \mathbb{Z}_n$ and a is a unit, then the equation $ax = b$ has a unique solution in $\mathbb{Z}_n$. [*Note:* You must find a solution for the equation *and* show that this solution is the only one.]

12. Let a, b, n be integers with $n > 1$ and let $d = (a, n)$. If the equation $[a]x = [b]$ has a solution in $\mathbb{Z}_n$, prove that $d \mid b$. [*Hint:* If $x = [r]$ is a solution, then $[ar] = [b]$ so that $ar - b = kn$ for some integer k.]

13. Let a, b, n be integers with $n > 1$. Let $d = (a, n)$ and assume $d \mid b$. Prove that the equation $[a]x = [b]$ has a solution in $\mathbb{Z}_n$ as follows.

(a) Explain why there are integers u, v, a_1, b_1, n_1 such that $au + nv = d$, $a = da_1$, $b = db_1$, $n = dn_1$.

(b) Show that each of

$$[ub_1], [ub_1 + n_1], [ub_1 + 2n_1], [ub_1, + 3n_1], \dots, [ub_1 + (d-1)n_1]$$

is a solution of $[a]x = [b]$.

14. Let a, b, n be integers with $n > 1$. Let $d = (a, n)$ and assume $d \mid b$. Prove that the equation $[a]x = [b]$ has d distinct solutions in $\mathbb{Z}_n$ as follows.

(a) Show that the solutions listed in Exercise 13 (b) are all distinct. [*Hint:* $[r] = [s]$ if and only if $n \mid (r - s)$.]

(b) If $x = [r]$ is any solution of $[a]x = [b]$, show that $[r] = [ub_1 + kn_1]$ for some integer k with $0 \le k \le d - 1$. [*Hint:* $[ar] - [aub_1] = [0]$ (Why?), so that $n \mid (a(r - ub_1))$. Show that $n_1 \mid (a_1(r - ub_1))$ and use Theorem 1.4 to show that $n_1 \mid (r - ub_1)$.]

15. Use Exercise 13 to solve the following equations.s

(a) $15x = 9$ in $\mathbb{Z}_{18}$ (b) $25x = 10$ in $\mathbb{Z}_{65}$.

16. If $a \neq 0$ and b are elements of $\mathbb{Z}_n$ and $ax = b$ has no solutions in $\mathbb{Z}_n$, prove that a is a zero divisor.

17. Prove that the product of two units in $\mathbb{Z}_n$ is also a unit.

18. The usual ordering of $\mathbb{Z}$ by $<$ is transitive and behaves nicely with respect to addition. Show that there is *no* ordering of $\mathbb{Z}_n$ such that

(i) if $a < b$ and $b < c$, then $a < c$;

(ii) if $a < b$, then $a + c < b + c$ for every c in $\mathbb{Z}_n$.

[*Hint:* If there is such an ordering with $0 < 1$, then adding 1 repeatedly to both sides shows that $0 < 1 < 2 < \cdots < n - 1$ by (ii). Thus $0 < n - 1$ by (i). Add 1 to each side and get a contradiction. Make a similar argument when $1 < 0$.]

APPLICATION: Public Key Cryptography (Chapter 13) may be covered at this point if desired.

CHAPTER 3

Rings

ALTERNATE ROUTE: If you want to cover groups before studying rings, you should read Chapters 7 and 8 now.

We have seen that many rules of ordinary arithmetic hold not only in $\mathbb{Z}$ but also in the miniature arithmetics $\mathbb{Z}_n$. You know other mathematical systems, such as the real numbers, in which many of these same rules hold. Your high-school algebra courses dealt with the arithmetic of polynomials.

The fact that similar rules of arithmetic hold in different systems suggests that it might be worthwhile to consider the common features of such systems. In the long run, this might save a lot of work: If we can prove a theorem about one system using only the properties that it has in common with a second system, then the theorem is also valid in the second system. By "abstracting" the common core of essential features, we can develop a general theory that includes as special cases $\mathbb{Z}$, $\mathbb{Z}_n$, and the other familiar systems. Results proved for this general theory will apply simultaneously to all the systems covered by the theory. This process of abstraction will allow us to discover the real reasons a particular statement is true (or false, for that matter) without getting bogged down in nonessential details. In this way a deeper understanding of all the systems involved should result.

So we now begin the development of ***abstract*** algebra. This chapter is just the first step and consists primarily of definitions, examples, and terminology. Systems that share a minimal number of fundamental properties with $\mathbb{Z}$ and $\mathbb{Z}_n$ are called *rings.* Other names are applied to rings that may have additional properties, as you will see in Section 3.1. The elementary facts about arithmetic and algebra in arbitrary rings are developed in Section 3.2. In Section 3.3 we consider rings that appear to be different from one another but actually are "essentially the same" except for the labels on their elements.

3.1 Definition and Examples of Rings

We begin the process of abstracting the common features of familiar systems with this definition:

Definition

A **ring** is a nonempty set R equipped with two operations* (usually written as addition and multiplication) that satisfy the following axioms. For all a, b, $c \in R$:

1. If $a \in R$ and $b \in R$, then $a + b \in R$. [*Closure for addition*]
2. $a + (b + c) = (a + b) + c$. [*Associative addition*]
3. $a + b = b + a$. [*Commutative addition*]
4. There is an element 0_R in R such that $a + 0_R = a = 0_R + a$ for every $a \in R$. [*Additive identity or zero element*]
5. For each $a \in R$, the equation $a + x = 0_R$ has a solution in R.†
6. If $a \in R$ and $b \in R$, then $ab \in R$. [*Closure for multiplication*]
7. $a(bc) = (ab)c$. [*Associative multiplication*]
8. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$. [*Distributive laws*]

These axioms are the bare minimum needed for a system to resemble $\mathbb{Z}$ and $\mathbb{Z}_n$. But $\mathbb{Z}$ and $\mathbb{Z}_n$ have several additional properties that are worth special mention:

Definition

A **commutative ring** is a ring R that satisfies this axiom:

9. $ab = ba$ for all $a, b \in R$. [*Commutative multiplication*]

Definition

A **ring with identity** is a ring R that contains an element 1_R satisfying this axiom:

10. $a1_R = a = 1_R a$ for all $a \in R$. [*Multiplicative identity*]

---

*"Operation" and "closure" are defined in Appendix B.

†Those who have already read Chapter 7 should note that Axioms 1–5 simply say that a ring is an abelian group under addition.

In the following examples, the verification of most of the axioms is left to the reader.

EXAMPLE 1

With the usual addition and multiplication,

$\mathbb{Z}$ (the integers) and $\mathbb{R}$ (the real numbers)

are commutative rings with identity.

EXAMPLE 2

The set $\mathbb{Z}_n$, with the usual addition and multiplication of classes, is a commutative ring with identity by Theorem 2.7.

EXAMPLE 3

Let E be the set of even integers with the usual addition and multiplication. Since the sum or product of two even integers is also even, the closure axioms (1 and 6) hold. Since 0 is an even integer, E has an additive identity element (Axiom 4). If a is even, then the solution of $a + x = 0$ (namely $-a$) is also even, and so Axiom 5 holds. The remaining axioms (2, 3, 7, 8, and 9) hold for *all* integers and, therefore, are true whenever a, b, c are even. Consequently, E is a commutative ring. E does *not* have an identity, however, because no *even* integer e has the property that $ae = a = ea$ for every even integer a.

EXAMPLE 4

The set of odd integers with the usual addition and multiplication is *not* a ring. Among other things, Axiom 1 fails: The sum of two odd integers is not odd.

Although the definition of ring was constructed with $\mathbb{Z}$ and $\mathbb{Z}_n$ as models, there are many rings that aren't at all like these models. In these rings, the elements may not be numbers or classes of numbers, and their operations may have nothing to do with "ordinary" addition and multiplication.

EXAMPLE 5

The set $T = \{r, s, t, z\}$ equipped with the addition and multiplication defined by the following tables is a ring:

$+$	z	r	s	t
z	z	r	s	t
r	r	z	t	s
s	s	t	z	r
t	t	s	r	z

$\cdot$	z	r	s	t
z	z	z	z	z
r	z	z	r	r
s	z	z	s	s
t	z	z	t	t

You may take our word for it that associativity and distributivity hold (Axioms 2, 7, and 8). The remaining axioms can be easily verified from the operation tables above. In particular, they show that T is closed under both addition and multiplication (Axioms 1 and 6) and that addition is commutative (Axiom 3).

The element z is the additive identity—the element denoted 0_R in Axiom 4. It behaves in the same way the number 0 does in $\mathbb{Z}$ (that's why the notation 0_R is used in the axiom), but z is not the integer 0—in fact, it's not any kind of number. Nevertheless, we shall call z the "zero element" of the ring T.

In order to verify Axiom 5, you must show that each of the equations

$$r + x = z \qquad s + x = z \qquad t + x = z \qquad z + x = z$$

has a solution in T. This is easily seen to be the case from the addition table; for example, $x = r$ is the solution of $r + x = z$ because $r + r = z$.

Finally, note that T is *not* a commutative ring; for instance, $rs = r$ and $sr = z$, so that $rs \neq sr$.

EXAMPLE 6

Let $M(\mathbb{R})$ be the set of all 2×2 matrices over the real numbers, that is, $M(\mathbb{R})$ consists of all arrays

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{where } a, b, c, d \text{ are real numbers.}$$

Two matrices are equal provided that the entries in corresponding positions are equal; that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \qquad \text{if and only if} \qquad a = r, b = s, c = t, d = u.$$

For example,

$$\begin{pmatrix} 4 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 2+2 & 0 \\ 1-4 & 1 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} \neq \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}.$$

Addition of matrices is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ c + c' & d + d' \end{pmatrix}.$$

For example,

$$\begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 7 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} 3 + 4 & -2 + 7 \\ 5 + 6 & 1 + 0 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 11 & 1 \end{pmatrix}.$$

Multiplication of matrices is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}.$$

For example,

$$\begin{aligned} \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ 6 & 7 \end{pmatrix} &= \begin{pmatrix} 2 \cdot 1 + 3 \cdot 6 & 2(-5) + 3 \cdot 7 \\ 0 \cdot 1 + (-4)6 & 0(-5) + (-4)7 \end{pmatrix} \\ &= \begin{pmatrix} 20 & 11 \\ -24 & -28 \end{pmatrix}. \end{aligned}$$

Reversing the order of the factors in matrix multiplication *may* produce a different answer, as is the case here:

$$\begin{aligned} \begin{pmatrix} 1 & -5 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} &= \begin{pmatrix} 1 \cdot 2 + (-5)0 & 1 \cdot 3 + (-5)(-4) \\ 6 \cdot 2 + 7 \cdot 0 & 6 \cdot 3 + 7(-4) \end{pmatrix} \\ &= \begin{pmatrix} 2 & 23 \\ 12 & -10 \end{pmatrix}. \end{aligned}$$

So this multiplication is not commutative. With a bit of work, you can verify that $M(\mathbb{R})$ is a ring with identity. The zero element is the **zero matrix**

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which is denoted $\mathbf{0}$ and $X = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ is a solution of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We claim that the multiplicative identity element (Axiom 10) is the matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

To prove this claim, we first multiply a typical matrix in $M(\mathbb{R})$ on the right by I:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since multiplication is not commutative here, we also need to check left multiplication by I as well:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \cdot a + 0 \cdot c & 1 \cdot b + 0 \cdot d \\ 0 \cdot a + 1 \cdot c & 0 \cdot b + 1 \cdot d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This proves that I satisfies Axiom 10.* Consequently, I is called the **identity matrix.**

Note that the product of nonzero elements of $M(\mathbb{R})$ may be the zero element; for example,

$$\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -3 & -9 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 4(-3) + 6 \cdot 2 & 4(-9) + 6 \cdot 6 \\ 2(-3) + 3 \cdot 2 & 2(-9) + 3 \cdot 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

EXAMPLE 7

If R is a commutative ring with identity, then $M(R)$ denotes the set of all 2×2 matrices with entries in R. With addition and multiplication defined as in Example 6, $M(R)$ is a noncommutative ring with identity, as you can readily verify. For instance, $M(\mathbb{Z})$ is the ring of 2×2 matrices with integer entries, $M(\mathbb{Q})$ the ring of 2×2 matrices with rational number entries, and $M(\mathbb{Z}_n)$ the ring of 2×2 matrices with entries from $\mathbb{Z}_n$.

EXAMPLE 8

Let T be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. As in calculus, $f + g$ and fg are the functions defined by

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (fg)(x) = f(x)g(x).$$

You can readily verify that T is a commutative ring with identity. The zero element is the function h given by $h(x) = 0$ for all $x \in \mathbb{R}$. The identity element is the function e given by $e(x) = 1$ for all $x \in \mathbb{R}$. Once again the product of nonzero elements of T may turn out to be the zero element; see Exercise 36.

We have seen that some rings do *not* have the property that the product of two nonzero elements is always nonzero. But some of the rings that *do* have this property, such as $\mathbb{Z}$, occur frequently enough to merit a title.

Definition

An **integral domain** is a commutative ring R with identity $1_R \neq 0_R$ that satisfies this axiom:

11. Whenever $a, b \in R$ and $ab = 0_R$, then $a = 0_R$ or $b = 0_R$.

*Checking a possible identity element under both right and left multiplication is essential. There are rings in which an element acts like an identity when you multiply on the right, but not when you multiply on the left. See Exercise 11.

The condition $1_R \neq 0_R$ is needed to exclude the zero ring (that is, the single-element ring $\{0_R\}$) from the class of integral domains. Note that Axiom 11 is logically equivalent to its contrapositive.*

$$\text{Whenever } a \neq 0_R \text{ and } b \neq 0_R, \text{ then } ab \neq 0_R.$$

EXAMPLE 9

The ring $\mathbb{Z}$ of integers is an integral domain. If p is prime, then $\mathbb{Z}_p$ is an integral domain by Theorem 2.8. On the other hand, $\mathbb{Z}_6$ is not an integral domain because $4 \cdot 3 = 0$, even though $4 \neq 0$ and $3 \neq 0$.

You should be familiar with the set $\mathbb{Q}$ of rational numbers, which consists of all fractions a/b with $a, b \in \mathbb{Z}$ and $b \neq 0$. Equality of fractions, addition, and multiplication are given by the usual rules:

$$\frac{a}{b} = \frac{r}{s} \quad \text{if and only if} \quad as = br$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

It is easy to verify that $\mathbb{Q}$ is an integral domain. But $\mathbb{Q}$ has an additional property that does not hold in $\mathbb{Z}$: Every equation of the form $ax = 1$ (with $a \neq 0$) has a solution in $\mathbb{Q}$. Therefore, $\mathbb{Q}$ is an example of the next definition.

Definition

A **field** is a commutative ring R with identity $1_R \neq 0_R$ that satisfies this axiom:

12. For each $a \neq 0_R$ in R, the equation $ax = 1_R$ has a solution in R.

Once again the condition $1_R \neq 0_R$ is needed to exclude the zero ring. Note that Axiom 11 is not mentioned explicitly in the definition of a field. However, Axiom 11 does hold in fields, as we shall see in Theorem 3.8 below.

EXAMPLE 10

The set $\mathbb{R}$ of real numbers, with the usual addition and multiplication, is a field. If p is a prime, then $\mathbb{Z}_p$ is a field by Theorem 2.8.

EXAMPLE 11

The set $\mathbb{C}$ of complex numbers consists of all numbers of the form $a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. Equality in $\mathbb{C}$ is defined by

$$a + bi = r + si \quad \text{if and only if} \quad a = r \text{ and } b = s.$$

*See Appendix A for a discussion of contrapositives.

The set $\mathbb{C}$ is a field with addition and multiplication given by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

The field $\mathbb{R}$ of real numbers is contained in $\mathbb{C}$ because $\mathbb{R}$ consists of all complex numbers of the form $a + 0i$. If $a + bi \neq 0$ in $\mathbb{C}$, then the solution of the equation $(a + bi)x = 1$ is $x = c + di$, where

$$c = a/(a^2 + b^2) \in \mathbb{R} \qquad \text{and} \qquad d = -b/(a^2 + b^2) \in \mathbb{R} \text{ (verify!)}.$$

EXAMPLE 12

Let K be the set of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

where a and b are real numbers. We claim that K is a field. For any two matrices in K,

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} a + c & b + d \\ -b - d & a + c \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix}.$$

In each case the matrix on the right is in K because the entries along the **main diagonal** (upper left to lower right) are the same and the entries on the opposite diagonal (upper right to lower left) are negatives of each other. Therefore, K is closed under addition and multiplication. K is commutative because

$$\begin{pmatrix} c & d \\ -d & c \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix}.$$

Clearly, the zero matrix and the identity matrix I are in K. If

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

is not the zero matrix, then verify that the solution of $AX = I$ is

$$X = \begin{pmatrix} a/d & -b/d \\ b/d & a/d \end{pmatrix} \in K, \quad \text{where } d = a^2 + b^2.$$

Whenever the rings in the preceding examples are mentioned, you may assume that addition and multiplication are the operations defined above, unless there is some specific statement to the contrary. You should be aware, however, that a given set (such as $\mathbb{Z}$) may be made into a ring in many different ways by defining different addition and multiplication operations on it. See Exercises 17 and 22–26 for examples.

Now that we know a variety of different kinds of rings, we can use them to produce new rings in the following way.

EXAMPLE 13

Let T be the Cartesian product $\mathbb{Z}_6 \times \mathbb{Z}$, as defined in Appendix B. Define addition in T by the rule

$$(a, z) + (a', z') = (a + a', z + z').$$

The plus sign is being used in three ways here: In the first coordinate on the right-hand side of the equal sign, $+$ denotes addition in $\mathbb{Z}_6$; in the second coordinate, $+$ denotes addition in $\mathbb{Z}$; the $+$ on the left of the equal sign is the addition in T that is being defined. Since $\mathbb{Z}_6$ is a ring and $a, a' \in \mathbb{Z}_6$, the first coordinate on the right, $a + a'$, is in $\mathbb{Z}_6$. Similarly $z + z' \in \mathbb{Z}$. Therefore, addition in T is closed. Multiplication is defined similarly:

$$(a, z)(a', z') = (aa', zz').$$

For example, $(3, 5) + (4, 9) = (3 + 4, 5 + 9) = (1, 14)$ and $(3, 5)(4, 9) = (3 \cdot 4, 5 \cdot 9) = (0, 45)$. You can readily verify that T is a commutative ring with identity. The zero element is $(0, 0)$, and the multiplicative identity is $(1, 1)$. What was done here can be done for any two rings.

Theorem 3.1

Let R and S be rings. Define addition and multiplication on the Cartesian product $R \times S$ by

$$(r, s) + (r', s') = (r + r', s + s') \quad \text{and} \quad (r, s)(r', s') = (rr', ss').$$

Then $R \times S$ is a ring. If R and S are both commutative, then so is $R \times S$. If both R and S have an identity, then so does $R \times S$.

Proof ▸ Exercise 33. ■

Subrings

If R is a ring and S is a subset of R, then S may or may not itself be a ring under the operations in R. In the ring $\mathbb{Z}$ of integers, for example, the subset E of even integers is a ring, but the subset O of odd integers is not, as we saw in Examples 3 and 4. When a subset S of a ring R is itself a ring under the addition and multiplication in R, then we say that S is a **subring** of R.

EXAMPLE 14

$\mathbb{Z}$ is a subring of the ring $\mathbb{Q}$ of rational numbers and $\mathbb{Q}$ is a subring of the field $\mathbb{R}$ of all real numbers. Since $\mathbb{Q}$ is itself a field, we say that $\mathbb{Q}$ is a **subfield** of $\mathbb{R}$. Similarly, $\mathbb{R}$ is a subfield of the field $\mathbb{C}$ of complex numbers.

EXAMPLE 15

The matrix rings $M(\mathbb{Z})$ and $M(\mathbb{Q})$ in Example 7 are subrings of $M(\mathbb{R})$.

EXAMPLE 16

The ring K in Example 12 is a subring of $M(\mathbb{R})$.

EXAMPLE 17

Let T be the ring of all functions from $\mathbb{R}$ to $\mathbb{R}$ in Example 8. Then the subset S consisting of all *continuous* functions from $\mathbb{R}$ to $\mathbb{R}$ is a subring of T. To prove this, you need one fact proved in calculus: The sum and product of continuous functions are also continuous. So S is closed under addition and multiplication (Axioms 1 and 6). You can readily verify the other axioms.

Proving that a subset S of a ring R is actually a subring is easier than proving directly that S is a ring. For instance, since $a + b = b + a$ for all elements of R, this fact is also true when a, b happen to be in the subset S. Thus Axiom 3 (commutative addition) automatically holds in any subset S of a ring. In fact, to prove that a subset of a ring is actually a subring, you need only verify a few of the axioms for a ring, as the next theorem shows.

Theorem 3.2

Suppose that R is a ring and that S is a subset of R such that

(i) S is closed under addition (if $a, b \in S$, then $a + b \in S$);

(ii) S is closed under multiplication (if $a, b \in S$, then $ab \in S$);

(iii) $0_R \in S$;

(iv) If $a \in S$, then the solution of the equation $a + x = 0_R$ is in S.

Then S is a subring of R.

Note condition (iv) carefully. To verify it, you need not show that the equation $a + x = 0_R$ *has* a solution—we already know that it does because R is a ring. You need only show that this solution is an element of S (which implies that Axiom 5 holds for S).

Proof of Theorem 3.2 ▸ As noted before the theorem, Axioms 2, 3, 7, and 8 hold for *all* elements of R, and so they necessarily hold for the elements of the subset S. Axioms 1, 6, 4, and 5 hold by (i)–(iv). ■

EXAMPLE 18

The subset $S = \{0, 3\}$ of $\mathbb{Z}_6$ is closed under addition and multiplication ($0 + 0 = 0$; $0 + 3 = 3$; $3 + 3 = 0$; similarly, $0 \cdot 0 = 0 = 0 \cdot 3$; $3 \cdot 3 = 3$). By the definition of S we have $0 \in S$. Finally, the equation $0 + x = 0$ has solution $x = 0 \in S$, and the equation $3 + x = 0$ has solution $x = 3 \in S$. Therefore, S is a subring of $\mathbb{Z}_6$ by Theorem 3.2.

EXAMPLE 19

Let S be the subset of $M(\mathbb{R})$ consisting of all matrices of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$. Then S is closed under addition and multiplication because

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + \begin{pmatrix} r & 0 \\ s & t \end{pmatrix} = \begin{pmatrix} a+r & 0+0 \\ b+s & c+t \end{pmatrix} = \begin{pmatrix} a+r & 0 \\ b+s & c+t \end{pmatrix} \in S \quad \text{and}$$

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} r & 0 \\ s & t \end{pmatrix} = \begin{pmatrix} ar & 0 \\ br+cs & ct \end{pmatrix} \in S.$$

The identity matrix is in S (let $a = 1, b = 0, c = 1$) and the solution of

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{is} \quad x = \begin{pmatrix} -a & 0 \\ -b & -c \end{pmatrix} \in S.$$

Hence S is a subring by Theorem 3.2.

EXAMPLE 20

The set $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{R}$. You can easily verify that

$$\begin{aligned}(a + b\sqrt{2})(c + d\sqrt{2}) &= ac + ad\sqrt{2} + bc\sqrt{2} + bd\sqrt{2}\cdot\sqrt{2} \\ &= (ac + 2bd) + (ad + bc)\sqrt{2}) \in \mathbb{Z}[\sqrt{2}].\end{aligned}$$

So $\mathbb{Z}[\sqrt{2}]$ is closed under multiplication. See Exercise 13 for the rest of the proof.

Exercises

A. 1. The following subsets of $\mathbb{Z}$ (with ordinary addition and multiplication) satisfy all but one of the axioms for a ring. In each case, which axiom fails?

   (a) The set S of all odd integers and 0.

   (b) The set of nonnegative integers.

2. Let $R = \{0, e, b, c\}$ with addition and multiplication defined by the tables on page 54. Assume associativity and distributivity and show that R is a ring with identity. Is R commutative? Is R a field?

$+$	0	e	b	c
0	0	e	b	c
e	e	0	c	b
b	b	c	0	e
c	c	b	e	0

$\cdot$	0	e	b	c
0	0	0	0	0
e	0	e	b	c
b	0	b	b	0
c	0	c	0	c

3. Let $F = \{0, e, a, b\}$ with operations given by the following tables. Assume associativity and distributivity and show that F is a field.

$+$	0	e	a	b
0	0	e	a	b
e	e	0	b	a
a	a	b	0	e
b	b	a	e	0

$\cdot$	0	e	a	b
0	0	0	0	0
e	0	e	a	b
a	0	a	b	e
b	0	b	e	a

4. Find matrices A and C in $M(\mathbb{R})$ such that $AC = \mathbf{0}$, but $CA \neq \mathbf{0}$, where $\mathbf{0}$ is the zero matrix. [*Hint:* Example 6.]

5. Which of the following six sets are subrings of $M(\mathbb{R})$? Which ones have an identity?

(a) All matrices of the form $\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$ with $r \in \mathbb{Q}$.

(b) All matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, b, c \in \mathbb{Z}$.

(c) All matrices of the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a, b, c \in \mathbb{R}$.

(d) All matrices of the form $\begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix}$ with $a \in \mathbb{R}$.

(e) All matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with $a \in \mathbb{R}$.

(f) All matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ with $a \in \mathbb{R}$.

6. **(a)** Show that the set R of all multiples of 3 is a subring of $\mathbb{Z}$.

(b) Let k be a fixed integer. Show that the set of all multiples of k is a subring of $\mathbb{Z}$.

7. Let K be the set of all integer multiples of $\sqrt{2}$, that is, all real numbers of the form $n\sqrt{2}$ with $n \in \mathbb{Z}$. Show that K satisfies Axioms 1–5, but is not a ring.

8. Is the subset $\{1, -1, i, -i\}$ a subring of $\mathbb{C}$?

9. Let R be a ring and consider the subset R^* of $R \times R$ defined by $R^* = \{(r, r) \mid r \in R\}$.

(a) If $R = \mathbb{Z}_6$, list the elements of R^*.

(b) For any ring R, show that R^* is a subring of $R \times R$.

10. Is $S = \{(a, b) \mid a + b = 0\}$ a subring of $\mathbb{Z} \times \mathbb{Z}$? Justify your answer.

11. Let S be the subset of $M(\mathbb{R})$ consisting of all matrices of the form $\begin{pmatrix} a & a \\ b & b \end{pmatrix}$.

(a) Prove that S is a ring.

(b) Show that $J = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is a *right identity* in S (meaning that $AJ = A$ for every A in S).

(c) Show that J is not a *left identity* in S by finding a matrix B in S such that $JB \neq B$.

For more information about S, see Exercise 41.

12. Let $\mathbb{Z}[i]$ denote the set $\{a + bi \mid a, b \in \mathbb{Z}\}$. Show that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$.

13. Let $\mathbb{Z}[\sqrt{2}]$ denote the set $\{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. Show that $\mathbb{Z}[\sqrt{2}]$ is a subring of $\mathbb{R}$. [See Example 20.]

14. Let T be the ring in Example 8. Let $S = \{f \in T \mid f(2) = 0\}$. Prove that S is a subring of T.

15. Write out the addition and multiplication tables for

(a) $\mathbb{Z}_2 \times \mathbb{Z}_3$ (b) $\mathbb{Z}_2 \times \mathbb{Z}_2$ (c) $\mathbb{Z}_3 \times \mathbb{Z}_3$

16. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in $M(\mathbb{R})$. Let S be the set of all matrices B such that $AB = \mathbf{0}$.

(a) List three matrices in S. [Many correct answers are possible.]

(b) Prove that S is a subring of $M(\mathbb{R})$. [*Hint:* If B and C are in S, show that $B + C$ and BC are in S by computing $A(B + C)$ and $A(BC)$.]

17. Define a new multiplication in $\mathbb{Z}$ by the rule: $ab = 0$ for all $a, b, \in \mathbb{Z}$. Show that with ordinary addition and this new multiplication, $\mathbb{Z}$ is a commutative ring.

18. Define a new multiplication in $\mathbb{Z}$ by the rule: $ab = 1$ for all $a, b, \in \mathbb{Z}$. With ordinary addition and this new multiplication, is $\mathbb{Z}$ is a ring?

19. Let $S = \{a, b, c\}$ and let $P(S)$ be the set of all subsets of S; denote the elements of $P(S)$ as follows:

$$S = \{a, b, c\}; \quad D = \{a, b\}; \quad E = \{a, c\}; \quad F = \{b, c\};$$

$$A = \{a\}; \quad B = \{b\}; \quad C = \{c\}; \quad 0 = \varnothing.$$

Define addition and multiplication in $P(S)$ by these rules:

$$M + N = (M - N) \cup (N - M) \qquad \text{and} \qquad MN = M \cap N.$$

Write out the addition and multiplication tables for $P(S)$. Also, see Exercise 44.

B. 20. Show that the subset $R = \{0, 3, 6, 9, 12, 15\}$ of $\mathbb{Z}_{18}$ is a subring. Does R have an identity?

21. Show that the subset $S = \{0, 2, 4, 6, 8\}$ of $\mathbb{Z}_{10}$ is a subring. Does S have an identity?

22. Define a new addition $\oplus$ and multiplication $\odot$ on $\mathbb{Z}$ by

$$a \oplus b = a + b - 1 \qquad \text{and} \qquad a \odot b = a + b - ab,$$

where the operations on the right-hand side of the equal signs are ordinary addition, subtraction, and multiplication. Prove that, with the new operations $\oplus$ and $\odot$, $\mathbb{Z}$ is an integral domain.

23. Let E be the set of even integers with ordinary addition. Define a new multiplication $*$ on E by the rule "$a * b = ab/2$" (where the product on the right is ordinary multiplication). Prove that with these operations E is a commutative ring with identity.

24. Define a new addition and multiplication on $\mathbb{Z}$ by

$$a \oplus b = a + b - 1 \qquad \text{and} \qquad a \odot b = ab - (a + b) + 2.$$

Prove that with these new operations $\mathbb{Z}$ is an integral domain.

25. Define a new addition and multiplication on $\mathbb{Q}$ by

$$r \oplus s = r + s + 1 \qquad \text{and} \qquad r \odot s = rs + r + s.$$

Prove that with these new operations $\mathbb{Q}$ is a commutative ring with identity. Is it an integral domain?

26. Let L be the set of positive real numbers. Define a new addition and multiplication on L by

$$a \oplus b = ab \qquad \text{and} \qquad a \otimes b = a^{\log b}.$$

(a) Is L a ring under these operations?

(b) Is L a commutative ring?

(c) Is L a field?

27. Let S be the set of rational numbers that can be written with an odd denominator. Prove that S is a subring of $\mathbb{Q}$ but is not a field.

28. Let p be a positive prime and let R be the set of all rational numbers that can be written in the form r/p^i with $r, i \in \mathbb{Z}$, and $i \geq 0$. Note that $\mathbb{Z} \subseteq R$ because each $n \in \mathbb{Z}$ can be written as n/p^0. Show that R is a subring of $\mathbb{Q}$.

29. The addition table and part of the multiplication table for a three-element ring are given below. Use the distributive laws to complete the multiplication table.

$+$	r	s	t
r	r	s	t
s	s	t	r
t	t	r	s

$\cdot$	r	s	t
r	r	r	r
s	r	t	
t	r		

30. Do Exercise 29 for this four-element ring:

$+$	w	x	y	z
w	w	x	y	z
x	x	y	z	w
y	y	z	w	x
z	z	w	x	y

$\cdot$	w	x	y	z
w	w	w	w	w
x	w	y		
y	w		w	
z	w		w	y

31. A *scalar matrix* in $M(\mathbb{R})$ is a matrix of the form $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ for some real number k.

(a) Prove that the set of scalar matrices is a subring of $M(\mathbb{R})$.

(b) If K is a scalar matrix, show that $KA = AK$ for every A in $M(\mathbb{R})$.

(c) If K is a matrix in $M(\mathbb{R})$ such that $KA = AK$ for every A in $M(\mathbb{R})$, show that K is a scalar matrix. [*Hint:* If $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Use the fact that $KA = AK$ to show that $b = 0$ and $c = 0$. Then make a similar argument with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to show that $a = d$.]

32. Let R be a ring and let $Z(R) = \{a \in R \mid ar = ra \text{ for every } r \in R\}$. In other words, $Z(R)$ consists of all elements of R that commute with every other element of R. Prove that $Z(R)$ is a subring of R. $Z(R)$ is called the **center** of the ring R. [Exercise 31 shows that the center of $M(\mathbb{R})$ is the subring of scalar matrices.]

33. Prove Theorem 3.1.

34. Show that $M(\mathbb{Z}_2)$ (all 2×2 matrices with entries in $\mathbb{Z}_2$) is a 16-element noncommutative ring with identity.

35. Prove or disprove:

(a) If R and S are integral domains, then $R \times S$ is an integral domain.

(b) If R and S are fields, then $R \times S$ is a field.

36. Let T be the ring in Example 8 and let f, g be given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ x - 2 & \text{if } x > 2 \end{cases} \qquad g(x) = \begin{cases} 2 - x & \text{if } x \leq 2 \\ 0 & \text{if } x > 2. \end{cases}$$

Show that $f, g \in T$ and that $fg = 0_T$. Therefore T is not an integral domain.

37. **(a)** If R is a ring, show that the ring $M(R)$ of all 2×2 matrices with entries in R is a ring.

(b) If R has an identity, show that $M(R)$ also has an identity.

38. If R is a ring and $a \in R$, let $A_R = \{r \in R \mid ar = 0_R\}$. Prove that A_R is a subring of R. A_R is called the **right annihilator of *a*.** [For an example, see Exercise 16 in which the ring S is the right annihilator of the matrix A.]

39. Let $\mathbb{Q}(\sqrt{2}) = \{r + s\sqrt{2} \mid r, s \in \mathbb{Q}\}$. Show that $\mathbb{Q}(\sqrt{2})$ is a subfield of $\mathbb{R}$. [*Hint:* To show that the solution of $(r + s\sqrt{2})x = 1$ is actually in $\mathbb{Q}(\sqrt{2})$, multiply $1/(r + s\sqrt{2})$ by $(r - s\sqrt{2})/(r - s\sqrt{2})$.]

40. Let d be an integer that is not a perfect square. Show that $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{C}$. [*Hint:* See Exercise 39.]

41. Let S be the ring in Exercise 11.

(a) Verify that each of these matrices is a right identity in S:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} .7 & .7 \\ .3 & .3 \end{pmatrix}, \text{ and } \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}.$$

(b) Prove that the matrix $\begin{pmatrix} x & x \\ y & y \end{pmatrix}$ is a right identity in S if and only if $x + y = 1$.

(c) If $x + y = 1$, show that $\begin{pmatrix} x & x \\ y & y \end{pmatrix}$ is not a left identity in S.

42. A **division ring** is a (not necessarily commutative) ring R with identity $1_R \neq 0_R$ that satisfies Axioms 11 and 12 (pages 48 and 49). Thus a field is a commutative division ring. See Exercise 43 for a noncommutative example. Suppose R is a division ring and a, b are nonzero elements of R.

(a) If $bb = b$, prove that $b = 1_R$. [*Hint:* Let v be the solution of $bx = 1_R$ and note that $bv = b^2v$.]

(b) If u is the solution of the equation $ax = 1_R$, prove that u is also a solution of the equation $xa = 1_R$. (Remember that R may not be commutative.) [*Hint:* Use part (a) with $b = ua$.]

43. In the ring $M(\mathbb{C})$, let

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The product of a real number and a matrix is the matrix given by this rule:

$$r\begin{pmatrix} t & u \\ v & w \end{pmatrix} = \begin{pmatrix} rt & ru \\ rv & rw \end{pmatrix}$$

The set H of **real quaternions** consists of all matrices of the form

$$\begin{aligned} a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} &= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} bi & 0 \\ 0 & -bi \end{pmatrix} + \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} + \begin{pmatrix} 0 & di \\ di & 0 \end{pmatrix} \\ &= \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}, \end{aligned}$$

where a, b, c, and d are real numbers.

(a) Prove that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$
$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i} \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

(b) Show that H is a noncommutative ring with identity.

(c) Show that H is a division ring (defined in Exercise 42). [*Hint:* If $M = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, then verify that the solution of the equation $Mx = \mathbf{1}$ is the matrix $ta\mathbf{1} - tb\mathbf{i} - tc\mathbf{j} - td\mathbf{k}$, where $t = 1/(a^2 + b^2 + c^2 + d^2)$.]

(d) Show that the equation $x^2 = -1$ has infinitely many solutions in H. [*Hint:* Consider quaternions of the form $0\mathbf{1} + b\mathbf{i} + c\mathbf{j} - d\mathbf{k}$, where $b^2 + c^2 + d^2 = 1$.]

44. Let S be a set and let $P(S)$ be the set of all subsets of S. Define addition and multiplication in $P(S)$ by the rules

$$M + N = (M - N) \cup (N - M) \quad \text{and} \quad MN = M \cap N.$$

(a) Prove that $P(S)$ is a commutative ring with identity. [The verification of additive associativity and distributivity is a bit messy, but an informal discussion using Venn diagrams is adequate for appreciating this example. See Exercise 19 for a special case.]

(b) Show that every element of $P(S)$ satisfies the equations $x^2 = x$ and $x + x = 0_{P(S)}$.

C. 45. Let C be the set $\mathbb{R} \times \mathbb{R}$ with the usual coordinatewise addition (as in Theorem 3.1) and a new multiplication given by

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

Show that with these operations C is a field.

46. Let r and s be positive integers such that r divides $ks + 1$ for some k with $1 \le k < r$. Prove that the subset $\{0, r, 2r, 3r, \ldots, (s - 1)r\}$ of $\mathbb{Z}_{rs}$ is a ring with identity $ks + 1$ under the usual addition and multiplication in $\mathbb{Z}_{rs}$. Exercise 21 is a special case of this result.

APPLICATION: Applications of the Chinese Remainder Theorem (Section 14.2) may be covered at this point if desired.

3.2 Basic Properties of Rings

When you do arithmetic in $\mathbb{Z}$, you often use far more than the axioms for an integral domain. For instance, subtraction appears regularly, as do cancelation and the various rules for multiplying negative numbers. We begin by showing that many of these same properties hold in every ring.

Arithmetic in Rings

Subtraction is not mentioned in the axioms for a ring, and we cannot just assume that such an operation exists in an arbitrary ring. If we want to define a subtraction operation in a ring, we must do so in terms of addition, multiplication, and the ring axioms. The first step is

Theorem 3.3

For any element a in a ring R, the equation $a + x = 0_R$ has a unique solution.

Proof ▸ We know that $a + x = 0_R$ has at least one solution u by Axiom 5. If v is also a solution, then $a + u = 0_R$ and $a + v = 0_R$, so that

$$v = 0_R + v = (a + u) + v = (u + a) + v = u + (a + v) = u + 0_R = u.$$

Therefore, u is the only solution. ■

We can now define negatives and subtraction in any ring by copying what happens in familiar rings such as $\mathbb{Z}$. Let R be a ring and $a \in R$. By Theorem 3.3 the equation $a + x = 0_R$ has a unique solution. Using notation adapted from $\mathbb{Z}$, we denote this unique solution by the symbol "$-a$." Since addition is commutative,

$-a$ is the unique element of R such that

$$a + (-a) = 0_R = (-a) + a.$$

In familiar rings, this definition coincides with the known concept of the negative of an element. More importantly, it provides a meaning for "negative" in *any* ring.

EXAMPLE 1

In the ring $\mathbb{Z}_6$, the solution of the equation $2 + x = 0$ is 4, and so in this ring $-2 = 4$. Similarly, $-9 = 5$ in $\mathbb{Z}_{14}$ because 5 is the solution of $9 + x = 0$.

Subtraction in a ring is now defined by the rule

$b - a$ means $b + (-a)$.

In $\mathbb{Z}$ and other familiar rings, this is just ordinary subtraction. In other rings we have a new operation.

EXAMPLE 2

In $\mathbb{Z}_6$ we have $1 - 2 = 1 + (-2) = 1 + 4 = 5$.

In junior high school you learned many computational and algebraic rules for dealing with negatives and subtraction. The next two theorems show that these rules are valid in any ring. Although these facts are not particularly interesting in themselves, it is essential to establish their validity so that we may do arithmetic in arbitrary rings.

Theorem 3.4

If $a + b = a + c$ in a ring R, then $b = c$.

Proof ▸ Adding $-a$ to both sides of $a + b = a + c$ and then using associativity and negatives show that

$$\begin{aligned} -a + (a + b) &= -a + (a + c) \\ (-a + a) + b &= (-a + a) + c \\ 0_R + b &= 0_R + c \\ b &= c. \quad \blacksquare \end{aligned}$$

Theorem 3.5

For any elements a and b of a ring R,

(1) $a \cdot 0_R = 0_R = 0_R \cdot a$. In particular, $0_R \cdot 0_R = 0_R$.

(2) $a(-b) = -ab$ and $(-a)b = -ab$.

(3) $-(-a) = a$.

(4) $-(a + b) = (-a) + (-b)$.

(5) $-(a - b) = -a + b$.

(6) $(-a)(-b) = ab$.

If R has an identity, then

(7) $(-1_R)a = -a$.

Proof ▸ (1) Since $0_R + 0_R = 0_R$, the distributive law shows that

$$a \cdot 0_R + a \cdot 0_R = a(0_R + 0_R) = a \cdot 0_R = a \cdot 0_R + 0_R.$$

Applying Theorem 3.4 to the first and last parts of this equation shows that $a \cdot 0_R = 0_R$. The proof that $0_R \cdot a = 0_R$ is similar.

(2) By definition, $-ab$ is the *unique* solution of the equation $ab + x = 0_R$, and so any other solution of this equation must be equal to $-ab$. But $x = a(-b)$ is a solution because, by the distribution law and (1),

$$ab + a(-b) = a[b + (-b)] = a[0_R] = 0_R.$$

Therefore, $a(-b) = -ab$. The other part is proved similarly.

(3) By definition, $-(-a)$ is the unique solution of $(-a) + x = 0_R$. But a is a solution of this equation since $(-a) + a = 0_R$. Hence, $-(-a) = a$ by uniqueness.

(4) By definition, $-(a + b)$ is the unique solution of $(a + b) + x = 0_R$, but $(-a) + (-b)$ is also a solution, because addition is commutative, so that

$$\begin{aligned} (a + b) + [(-a) + (-b)] &= a + (-a) + b + (-b) \\ &= \quad 0_R \quad + \quad 0_R \quad = 0_R. \end{aligned}$$

Therefore, $-(a + b) = (-a) + (-b)$ by uniqueness.

(5) By the definition of subtraction and (4) and (3),

$$-(a - b) = -(a + (-b)) = (-a) + (-(-b)) = -a + b.$$

(6)
$$\begin{aligned}(-a)(-b) &= -(a(-b)) && \textit{[By the second equation in (2), with } -b \textit{ in place of } b\textit{]}\\ &= -(-ab) && \textit{[By the first equation in (2)]}\\ &= ab && \textit{[By (3), with } ab \textit{ in place of } a\textit{]}\end{aligned}$$

(7) By (2),

$$(-1_R)a = -(1_Ra) = -(a) = -a. \quad \blacksquare$$

When doing ordinary arithmetic, exponent notation is a definite convenience, as is its additive analogue (for instance, $a + a + a = 3a$). We now carry these concepts over to arbitrary rings. If R is a ring, $a \in R$, and n is a positive integer, then we *define*

$$a^n = aaa \cdots a \qquad (n \text{ factors}).$$

It is easy to verify that for any $a \in R$ and positive integers m and n,

$$a^m a^n = a^{m+n} \qquad \text{and} \qquad (a^m)^n = a^{mn}.$$

If R has an identity and $a \neq 0_R$, then we define a^0 to be the element 1_R. In this case, the exponent rules are valid for all $m, n \geq 0$.

If R is a ring, $a \in R$, and n is a positive integer, then we define

$$na = a + a + a + \cdots + a. \quad (n \text{ summands})$$

$$-na = (-a) + (-a) + (-a) + \cdots + (-a). \quad (n \text{ summands})$$

Finally, we define $0a = 0_R$. In familiar rings this is nothing new, but in other rings it gives a meaning to the "product" of an integer n and a ring element a.

EXAMPLE 3

Let R be a ring and $a, b \in R$. Then

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) = a(a + b) + b(a + b)\\ &= aa + ab + ba + bb = a^2 + ab + ba + b^2.\end{aligned}$$

Be careful here. If $ab \neq ba$, then you *can't* combine the middle terms. If R is a commutative ring, however, then $ab = ba$ and we have the familiar pattern

$$(a + b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

For a calculation of $(a + b)^n$ in a commutative ring, with $n > 2$, see the Binomial Theorem in Appendix E.

It's worth noting that subtraction provides a faster method than Theorem 3.2 for showing that a subset of a ring is actually a subring.

Theorem 3.6

Let S be a nonempty subset of a ring R such that

(1) S is closed under subtraction (if $a, b \in S$, then $a - b \in S$);

(2) S is closed under multiplication (if $a, b \in S$, then $ab \in S$).

Then S is a subring of R.

Proof ▸ We show that S satisfies conditions (i)–(iv) of Theorem 3.2 and hence is a subring. The conditions will be proved in this order: (ii), (iii), (iv), and (i).

(ii) Hypothesis (2) here is identical with condition (ii) of Theorem 3.2. Hence, S satisfies condition (ii).

(iii) Since S is nonempty, there is some element c with $c \in S$. Applying (1) (with $a = c$ and $b = c$), we see that $c - c = 0_R$ is in S. Therefore, S satisfies condition (iii) of Theorem 3.2.

(iv) If a is any element of S, then by (1), $0_R - a = -a$ is also in S. Since $-a$ is the solution of $a + x = 0_R$, condition (iv) of Theorem 3.2 is satisfied.

(i) If $a, b \in S$, then $-b$ is in S by the proof of (iv). By (1), $a - (-b) = a + b$ is in S. So S satisfies condition (i) of Theorem 3.2.

Therefore, S is a subring of R by Theorem 3.2. ■

Units and Zero Divisors

Units and zero divisors in $\mathbb{Z}_n$ were introduced in Section 2.3. We now carry these concepts over to arbitrary rings.

Definition

An element a in a ring R with identity is called a **unit** if there exists $u \in R$ such that $au = 1_R = ua$. In this case the element u is called the **(multiplicative) inverse** of a and is denoted a^{-1}.

EXAMPLE 4

The only units in $\mathbb{Z}$ are 1 and -1.

EXAMPLE 5

By Theorem 2.10, the units in $\mathbb{Z}_{15}$ are 1, 2, 4, 7, 8, 11, 13, and 14. For instance, $2 \cdot 8 = 1$, so $2^{-1} = 8$ and $8^{-1} = 2$.

EXAMPLE 6

Every nonzero element of the field $\mathbb{R}$ is a unit: If $a \neq 0$, then $a \cdot \frac{1}{a} = 1$. The same thing is true for every field F. By definition, F satisfies Axiom 12: If $a \neq 0_F$, then the equation $ax = 1_F$ has a solution in F. Hence,

Every nonzero element of a field is a unit.

EXAMPLE 7

A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $M(\mathbb{R})$ such that $ad - bc \neq 0$ is a unit because, as you can easily verify,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, each of these matrices is a unit:

$$A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 3 \\ -2 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 1/3 & 0 \\ 5 & 6 \end{pmatrix}.$$

Units in a matrix ring are called **invertible matrices.**

EXAMPLE 8

Let F be a field and $M(F)$ the ring of 2×2 matrices with entries in F. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(F)$ and $ad - bc \neq 0_F$, then $ad - bc$ is a unit in F by Example 6. The computations in Example 7, with $\frac{1}{ad - bc}$ replaced by $(ad - bc)^{-1}$, show that A is an invertible matrix [unit in $M(F)$] with inverse $\begin{pmatrix} d(ad-bc)^{-1} & -b(ad-bc)^{-1} \\ -c(ad-bc)^{-1} & a(ad-bc)^{-1} \end{pmatrix}$.

Definition

An element a in a ring R is a **zero divisor** provided that

(1) $a \neq 0_R$.

(2) There exists a nonzero element c in R such that $ac = 0_R$ or $ca = 0_R$.

Note that in requirement (2), the element c is not unique: Many elements in the ring may satisfy the equation $ax = 0_R$ or the equation $xa = 0_R$ (Exercise 6). Furthermore, in a noncommutative ring, it is possible to have $ac = 0_R$ and $ca \neq 0_R$ (Exercise 4 in Section 3.1).

EXAMPLE 9

Both 2 and 3 are zero divisors in $\mathbb{Z}_6$ because $2 \cdot 3 = 0$. Similarly, 4 and 9 are zero divisors in $\mathbb{Z}_{12}$ because $4 \cdot 9 = 0$.

For a zero divisor A in a matrix ring, it is possible to find a matrix C such that $AC = \mathbf{0}$ and $CA = \mathbf{0}$.

EXAMPLE 10

Let F be a field. A nonzero matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $M(F)$ such that $ad - bc = 0_F$ is a zero divisor because, as you can easily verify,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 0_F & 0_F \\ 0_F & 0_F \end{pmatrix} \text{ and } \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0_F & 0_F \\ 0_F & 0_F \end{pmatrix}.$$

In particular, each of these matrices is a zero divisor in the given ring:

$$A = \begin{pmatrix} 3 & 2 \\ 9 & 6 \end{pmatrix} \text{ in } M(\mathbb{R}), \quad B = \begin{pmatrix} 4/3 & -8 \\ -2 & 12 \end{pmatrix} \text{ in } M(\mathbb{Q}), \text{ and } C = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix} \text{ in } M(\mathbb{Z}_6).$$

EXAMPLE 11

Every integral R domain satisfies Axiom 11: If $ab = 0_R$, then $a = 0_R$ or $b = 0_R$. In other words, the product of two nonzero elements cannot be 0. Therefore,

An integral domain contains no zero divisors.

Finally, we present some useful facts about integral domains and fields.

Theorem 3.7

Cancelation is valid in any integral domain R: If $a \neq 0_R$ and $ab = ac$ in R, then $b = c$.

Cancelation may fail in rings that are not integral domains. In $\mathbb{Z}_{12}$, for instance, $2 \cdot 4 = 2 \cdot 10$, but $4 \neq 10$.

Proof of Theorem 3.7 ▸ If $ab = bc$, then $ab - bc = 0_R$, so that $a(b - c) = 0_R$. Since $a \neq 0_R$, we must have $b - c = 0_R$ (if not, then a is a zero divisor, contradicting Axiom 11). Therefore, $b = c$. ■

Theorem 3.8

Every field F is an integral domain.

Proof ▸ Since a field is a commutative ring with identity by definition, we need only show that F satisfies Axiom 11: If $ab = 0_F$, then $a = 0_F$ or $b = 0_F$. So suppose that $ab = 0_F$. If $b = 0_F$, there is nothing to prove. If $b \neq 0_F$, then b is a unit (Example 6). Consequently, by the definition of unit and part (1) of Theorem 3.5,

$$a = a1_F = abb^{-1} = 0_F b^{-1} = 0_F.$$

So in every case, $a = 0_F$ or $b = 0_F$. Hence, Axiom 11 holds and F is an integral domain. ■

The converse of Theorem 3.8 is false in general ($\mathbb{Z}$ is an integral domain that is not a field), but true in the finite case.

Theorem 3.9

Every finite integral domain R is a field.

Proof ▸ Since R is a commutative ring with identity, we need only show that for each $a \neq 0_R$, the equation $ax = 1_R$ has a solution. Let $a_1, a_2, \ldots, a_n$ be the distinct elements of R and suppose $a_t \neq 0_R$. To show that $a_t x = 1_R$ has a solution, consider the products $a_t a_1, a_t a_2, a_t a_3, \ldots, a_t a_n$. If $a_i \neq a_j$, then we must have $a_t a_i \neq a_t a_j$ (because $a_t a_i = a_t a_j$ would imply that $a_i = a_j$ by cancelation). Therefore, $a_t a_1, a_t a_2, \ldots, a_t a_n$ are n distinct elements of R. However, R has exactly n elements all together, and so these must be all the elements of R in some order. In particular, for some j, $a_t a_j = 1_R$. Therefore, the equation $a_t x = 1_R$ has a solution and R is a field. ■

Exercises

A. 1. Let R be a ring and $a, b \in R$.

(a) $(a + b)(a - b) = ?$ (b) $(a + b)^3 = ?$

(c) What are the answers in parts (a) and (b) if R is commutative?

2. Find the inverse of matrices A, B, and C in Example 7.

3. An element e of a ring R is said to be **idempotent** if $e^2 = e$.

(a) Find four idempotent elements in the ring $M(\mathbb{R})$.

(b) Find all idempotents in $\mathbb{Z}_{12}$.

4. For each matrix A find a matrix C such that $AC = 0$ or $CA = 0$:

$$A = \begin{pmatrix} 6 & 9 \\ 2 & 3 \end{pmatrix}; \quad A = \begin{pmatrix} 5 & -10 \\ -2 & 4 \end{pmatrix}; \quad A = \begin{pmatrix} 1/2 & 1/4 \\ 3 & 3/2 \end{pmatrix}.$$

5. **(a)** Show that a ring has only one zero element. [*Hint:* If there were more than one, how many solutions would the equation $0_R + x = 0_R$ have?]

   (b) Show that a ring R with identity has only one identity element.

   (c) Can a unit in a ring R with identity have more than one inverse? Why?

6. **(a)** Suppose A and C are nonzero matrices in $M(\mathbb{R})$ such that $AC = \mathbf{0}$. If k is any real number, show that $A(kC) = \mathbf{0}$, where kC is the matrix C with every entry multiplied by k. Hence the equation $AX = \mathbf{0}$ has infinitely many solutions.

   (b) If $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, find four solutions of the equation $AX = \mathbf{0}$.

7. Let R be a ring with identity and let $S = \{n1_R \mid n \in \mathbb{Z}\}$. Prove that S is a subring of R. [The definition of na with $n \in \mathbb{Z}$, $a \in R$ is on page 62. Also see Exercise 27.]

8. Let R be a ring and b a fixed element of R. Let $T = \{rb \mid r \in R\}$. Prove that T is a subring of R.

9. Show that the set S of matrices of the form $\begin{pmatrix} a & 4b \\ b & a \end{pmatrix}$, with a and b real numbers is a subring of $M(\mathbb{R})$.

10. Let R and S be rings and consider these subsets of $R \times S$:

$$\overline{R} = \{(r, 0_S) \mid r \in R\} \quad \text{and} \quad \overline{S} = \{(0_R, s) \mid s \in S\}.$$

   (a) If $R = \mathbb{Z}_3$ and $S = \mathbb{Z}_5$. What are the sets $\overline{R}$ and $\overline{S}$?

   (b) For any rings R and S, show that $\overline{R}$ is a subring of $R \times S$.

   (c) For any rings R and S, show that $\overline{S}$ is a subring of $R \times S$.

11. Let R be a ring and m a fixed integer. Let $S = \{r \in R \mid mr = 0_R\}$. Prove that S is a subring of R.

12. Let a and b be elements of a ring R.

    (a) Prove that the equation $a + x = b$ has a unique solution in R. (You must prove that there is a solution *and* that this solution is the only one.)

    (b) If R is a ring with identity and a is a unit, prove that the equation $ax = b$ has a unique solution in R.

13. Let S and T be subrings of a ring R. In (a) and (b), if the answer is "yes," prove it. If the answer is "no," give a counterexample.

    (a) Is $S \cap T$ a subring of R?

    (b) Is $S \cup T$ a subring of R?

14. Prove that the only idempotents in an integral domain R are 0_R and 1_R. (See Exercise 3.)

15. **(a)** If a and b are units in a ring R with identity, prove that ab is a unit whose inverse is $(ab)^{-1} = b^{-1}a^{-1}$.

(b) Give an example to show that if a and b are units, then $a^{-1}b^{-1}$ need not be the multiplicative inverse of ab.

16. Prove or disprove: The set of units in a ring R with identity is a subring of R.

17. If u is a unit in a ring R with identity, prove that u is not a zero divisor.

18. Let a be a nonzero element of a ring R with identity. If the equation $ax = 1_R$ has a solution u and the equation $ya = 1_R$ has a solution v, prove that $u = v$.

19. Let R and S be rings with identity. What are the units in the ring $R \times S$?

20. Let R and S be nonzero rings (meaning that each of them contains at least one nonzero element). Show that $R \times S$ contains zero divisors.

21. Let R be a ring and let a be a nonzero element of R that is not a zero divisor. Prove that cancelation holds for a; that is, prove that

(a) If $ab = ac$ in R, then $b = c$.

(b) If $ba = ca$ in R, then $b = c$.

22. **(a)** If ab is a zero divisor in a ring R, prove that a or b is a zero divisor.

(b) If a or b is a zero divisor in a commutative ring R and $ab \neq 0_R$, prove that ab is a zero divisor.

23. **(a)** Let R be a ring and $a, b \in R$. Let m and n be nonnegative integers and prove that

(i) $(m + n)a = ma + na$.

(ii) $m(a + b) = ma + mb$.

(iii) $m(ab) = (ma)b = a(mb)$.

(iv) $(ma)(nb) = mn(ab)$.

(b) Do part (a) when m and n are any integers.

24. Let R be a ring and $a, b \in R$. Let m and n be positive integers.

(a) Show that $a^m a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$.

(b) Under what conditions is it true that $(ab)^n = a^n b^n$?

25. Let S be a subring of a ring R with identity.

(a) If S has an identity, show by example that 1_S may not be the same as 1_R.

(b) If both R and S are integral domains, prove that $1_S = 1_R$.

B. 26. Let S be a subring of a ring R. Prove that $0_S = 0_R$. [*Hint:* For $a \in S$, consider the equation $a + x = a$.]

27. Let R be a ring with identity and b a fixed element of R and let $S = \{nb \mid n \in \mathbb{Z}\}$. Is S necessarily a subring of R? [Exercise 7 is the case when $b = 1_R$.]

28. Assume that $R = \{0_R, 1_R, a, b\}$ is a ring and that a and b are units. Write out the multiplication table of R.

29. Let R be a commutative ring with identity. Prove that R is an integral domain if and only if cancelation holds in R (that is, $a \neq 0_R$ and $ab = ac$ in R imply $b = c$).

30. Let R be a commutative ring with identity and $b \in R$. Let T be the subring of all multiples of b (as in Exercise 8). If u is a unit in R and $u \in T$, prove that $T = R$.

31. A **Boolean ring** is a ring R with identity in which $x^2 = x$ for every $x \in R$. For examples, see Exercises 19 and 44 in Section 3.1. If R is a Boolean ring, prove that

(a) $a + a = 0_R$ for every $a \in R$, which means that $a = -a$. [*Hint:* Expand $(a + a)^2$.]

(b) R is commutative. [*Hint:* Expand $(a + b)^2$.]

32. Let R be a ring without identity. Let T be the set $R \times \mathbb{Z}$. Define addition and multiplication in T by these rules:

$$(r, m) + (s, n) = (r + s, m + n).$$

$$(r, m)(s, n) = (rs + ms + nr, mn).$$

(a) Prove that T is a ring with identity.

(b) Let $\overline{R}$ consist of all elements of the form $(r, 0)$ in T. Prove that $\overline{R}$ is a subring of T.

33. Let R be a ring with identity. If ab and a are units in R, prove that b is a unit.

34. Let F be a field and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a matrix in $M(F)$.

(a) Prove that A is invertible if and only if $ad - bc \neq 0_F$. [*Hint:* Examples 7, 8, and 10 and Exercise 17.]

(b) Prove that A is a zero divisor if and only if $ad - bc = 0_F$.

35. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with integer entries.

(a) If $ad - bc = \pm 1$, show that A is invertible in $M(\mathbb{Z})$. [*Hint:* Example 7.]

(b) If $ad - bc \neq 0, 1$, or -1, show that A is neither a unit nor a zero divisor in $M(\mathbb{Z})$. [*Hint:* Show that A has an inverse in $M(\mathbb{R})$ that is not in $M(\mathbb{Z})$; see Exercise 5(c). For zero divisors, see Exercise 34(b) and Example 10.]

36. Let R be a commutative ring with identity. Then the set $M(R)$ of 2×2 matrices with entries in R) is a ring with identity by Exercise 37 of Section 3.1. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(R)$ and $ad - bc$ is a unit in R, show that A is invertible in $M(R)$. [*Hint:* Replace $\dfrac{1}{ad - bc}$ by $(ad - bc)^{-1}$ in Example 7.]

37. Let R be a ring with identity and $a, b \in R$. Assume that a is not a zero divisor. Prove that $ab = 1_R$, if and only if $ba = 1_R$. [*Hint:* Note that both $ab = 1_R$ and $ba = 1_R$ imply $aba = a$ (why?); use Exercise 21.]

38. Let R be a ring with identity and $a, b \in R$. Assume that neither a nor b is a zero divisor. If ab is a unit, prove that a and b are units. [*Hint:* Exercise 21.]

39. **(a)** If R is a finite commutative ring with identity and $a \in R$, prove that a is either a zero divisor or a unit. [*Hint:* If a is not a zero divisor, adapt the proof of Theorem 3.8, using Exercise 21.]

    (b) Is part (a) true if R is infinite? Justify your answer.

40. An element a of a ring is **nilpotent** if $a^n = 0_R$ for some positive integer n. Prove that R has no nonzero nilpotent elements if and only if 0_R is the unique solution of the equation $x^2 = 0_R$.

The following definition is needed for Exercises 41–43. Let R be a ring with identity. If there is a smallest positive integer n such that $n1_R = 0_R$, then R is said to have ***characteristic n.*** *If no such n exists, R is said to have* ***characteristic zero.***

41. **(a)** Show that $\mathbb{Z}$ has characteristic zero and $\mathbb{Z}_n$ has characteristic n.

    (b) What is the characteristic of $\mathbb{Z}_4 \times \mathbb{Z}_6$?

42. Prove that a finite ring with identity has characteristic n for some $n > 0$.

43. Let R be a ring with identity of characteristic $n > 0$.

    (a) Prove that $na = 0_R$ for every $a \in R$.

    (b) If R is an integral domain, prove that n is prime.

C. 44. **(a)** Let a and b be nilpotent elements in a commutative ring R (see Exercise 40). Prove that $a + b$ and ab are also nilpotent. [You will need the Binomial Theorem from Appendix E.]

    (b) Let N be the set of all nilpotent elements of R. Show that N is a subring of R.

45. Let R be a ring such that $x^3 = x$ for every $x \in R$. Prove that R is commutative.

46. Let R be a nonzero finite commutative ring with no zero divisors. Prove that R is a field.

3.3 Isomorphisms and Homomorphisms

If you were unfamiliar with roman numerals and came across a discussion of integer arithmetic written solely with roman numerals, it might take you some time to realize that this arithmetic was essentially the same as the familiar arithmetic in $\mathbb{Z}$ except for the labels on the elements. Here is a less trivial example.

EXAMPLE 1

Consider the subset $S = \{0, 2, 4, 6, 8\}$ of $\mathbb{Z}_{10}$. With the addition and multiplication of $\mathbb{Z}_{10}$, S is actually a commutative ring, as can be seen from these tables:*

*The reason the elements of S are listed in this order will become clear in a moment.

+	0	6	2	8	4
0	0	6	2	8	4
6	6	2	8	4	0
2	2	8	4	0	6
8	8	4	0	6	2
4	4	0	6	2	8

·	0	6	2	8	4
0	0	0	0	0	0
6	0	6	2	8	4
2	0	2	4	6	8
8	0	8	6	4	2
4	0	4	8	2	6

A careful examination of the tables shows that S is a field with five elements and that the multiplicative identity of this field is the element 6.

We claim that S is "essentially the same" as the field $\mathbb{Z}_5$ except for the labels on the elements. You can see this as follows. Write out addition and multiplication tables for $\mathbb{Z}_5$.* To avoid any possible confusion with elements of S, denote the elements of $\mathbb{Z}_5$ by $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}$. Then relabel the entries in the $\mathbb{Z}_5$ tables according to this scheme:

Relabel $\bar{0}$ as 0, relabel $\bar{1}$ as 6, relabel $\bar{2}$ as 2,
relabel $\bar{3}$ as 8, relabel $\bar{4}$ as 4.

Look what happens to the addition and multiplication tables for $\mathbb{Z}_5$:

+	$\cancel{\bar{0}}$ 0	$\cancel{\bar{1}}$ 6	$\cancel{\bar{2}}$ 2	$\cancel{\bar{3}}$ 8	$\cancel{\bar{4}}$ 4
$\cancel{\bar{0}}$ 0	$\cancel{\bar{0}}$ 0	$\cancel{\bar{1}}$ 6	$\cancel{\bar{2}}$ 2	$\cancel{\bar{3}}$ 8	$\cancel{\bar{4}}$ 4
$\cancel{\bar{1}}$ 6	$\cancel{\bar{1}}$ 6	$\cancel{\bar{2}}$ 2	$\cancel{\bar{3}}$ 8	$\cancel{\bar{4}}$ 4	$\cancel{\bar{0}}$ 0
$\cancel{\bar{2}}$ 2	$\cancel{\bar{2}}$ 2	$\cancel{\bar{3}}$ 8	$\cancel{\bar{4}}$ 4	$\cancel{\bar{0}}$ 0	$\cancel{\bar{1}}$ 6
$\cancel{\bar{3}}$ 8	$\cancel{\bar{3}}$ 8	$\cancel{\bar{4}}$ 4	$\cancel{\bar{0}}$ 0	$\cancel{\bar{1}}$ 6	$\cancel{\bar{2}}$ 2
$\cancel{\bar{4}}$ 4	$\cancel{\bar{4}}$ 4	$\cancel{\bar{0}}$ 0	$\cancel{\bar{1}}$ 6	$\cancel{\bar{2}}$ 2	$\cancel{\bar{3}}$ 8

·	$\cancel{\bar{0}}$ 0	$\cancel{\bar{1}}$ 6	$\cancel{\bar{2}}$ 2	$\cancel{\bar{3}}$ 8	$\cancel{\bar{4}}$ 4
$\cancel{\bar{0}}$ 0	$\cancel{\bar{0}}$ 0	$\cancel{\bar{0}}$ 0	$\cancel{\bar{0}}$ 0	$\cancel{\bar{0}}$ 0	$\cancel{\bar{0}}$ 0
$\cancel{\bar{1}}$ 6	$\cancel{\bar{0}}$ 0	$\cancel{\bar{1}}$ 6	$\cancel{\bar{2}}$ 2	$\cancel{\bar{3}}$ 8	$\cancel{\bar{4}}$ 4
$\cancel{\bar{2}}$ 2	$\cancel{\bar{0}}$ 0	$\cancel{\bar{2}}$ 2	$\cancel{\bar{4}}$ 4	$\cancel{\bar{1}}$ 6	$\cancel{\bar{3}}$ 8
$\cancel{\bar{3}}$ 8	$\cancel{\bar{0}}$ 0	$\cancel{\bar{3}}$ 8	$\cancel{\bar{1}}$ 6	$\cancel{\bar{4}}$ 4	$\cancel{\bar{2}}$ 2
$\cancel{\bar{4}}$ 4	$\cancel{\bar{0}}$ 0	$\cancel{\bar{4}}$ 4	$\cancel{\bar{3}}$ 8	$\cancel{\bar{2}}$ 2	$\cancel{\bar{1}}$ 6

By relabeling the elements of $\mathbb{Z}_5$, you obtain the addition and multiplication tables for S. Thus the operations in $\mathbb{Z}_5$ and S work in exactly the same way—the only difference is the way the elements are labeled. As far as ring structure goes, S is just the ring $\mathbb{Z}_5$ with new labels on the elements. In more technical terms, $\mathbb{Z}_5$ and S are said to be *isomorphic*.

In general, *isomorphic rings* are rings that have the same structure, in the sense that the addition and multiplication tables of one are the tables of the other with the elements suitably relabeled, as in Example 1. Although this intuitive idea is adequate for small finite systems, we need a rigorous mathematical definition of isomorphism that agrees with this intuitive idea *and* is readily applicable to large rings as well.

There are two aspects to the intuitive idea that rings R and S are isomorphic: relabeling the elements of R and comparing the resulting tables with those of S to verify that they are the same. Relabeling means that every element of R is paired with a unique element of S (its new label). In other words, there is a function $f: R \to S$ that

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*The $\mathbb{Z}_5$ tables (in congruence class notation) are shown in Example 2 of Section 2.2.

assigns to each $r \in R$ its new label $f(r) \in S$. In the preceding example, we used the relabeling function $f\colon \mathbb{Z}_5 \to S$, given by

$$f(\bar{0}) = 0 \qquad f(\bar{1}) = 6 \qquad f(\bar{2}) = 2 \qquad f(\bar{3}) = 8 \qquad f(\bar{4}) = 4.$$

Such a function must have these additional properties:

(i) Distinct elements of R must get distinct new labels:

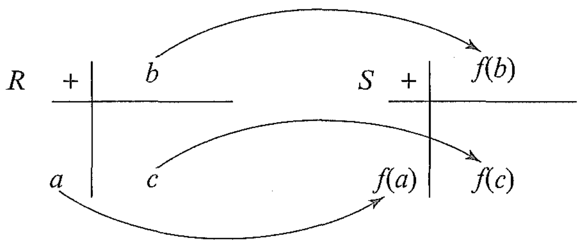
$$\text{If } r \neq r' \text{ in } R, \text{ then } f(r) \neq f(r') \text{ in } S.$$

(ii) Every element of S must be the label of some element in R:*

$$\text{For each } s \in S, \text{ there is an } r \in R \text{ such that } f(r) = s.$$

Statements (i) and (ii) simply say that the function f must be both injective and surjective, that is, *f must be a bijection.*†

In order for a bijection (relabeling scheme) f to be an isomorphism, applying f to the addition and multiplication tables of R must produce the addition and multiplication tables of S. So if $a + b = c$ in the R-table, we must have $f(a) + f(b) = f(c)$ in the S-table, as indicated in the diagram:



However, since $a + b = c$, we must also have $f(a + b) = f(c)$. Combining this with the fact that $f(a) + f(b) = f(c)$, we see that

$$f(a + b) = f(a) + f(b).$$

This is the condition that f must satisfy in order for f to change the addition tables of R into those of S. The analogous condition on f for the multiplication tables is $f(ab) = f(a)f(b)$. We now can state a formal definition of isomorphism:

Definition

A ring R is **isomorphic** to a ring S (in symbols, $R \cong S$) if there is a function $f\colon R \to S$ such that

(i) f is injective;
(ii) f is surjective;
(iii) $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$.

In this case the function f is called an **isomorphism**.

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*Otherwise, we couldn't possibly get the complete tables of S from those of R.

†Injective, surjective, and bijective functions are discussed in Appendix B.

CAUTION: In order to be an isomorphism, a function must satisfy *all three* of the conditions in the definition. It is quite possible for a function to satisfy any two of these conditions but not the third; see Exercises 4, 25, and 32.

EXAMPLE 2

In Example 12 on page 50, we considered the field K of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

where a and b are real numbers. We claim that K is isomorphic to the field $\mathbb{C}$ of complex numbers. To prove this, define a function $f:K \to \mathbb{C}$ by the rule

$$f\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a + bi.$$

To show that f is injective, suppose

$$f\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = f\begin{pmatrix} r & s \\ -s & r \end{pmatrix}.$$

Then by the definition of f, $a + bi = r + si$ in $\mathbb{C}$. By the rules of equality in $\mathbb{C}$, we must have $a = r$ and $b = s$. Hence, in K

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} r & s \\ -s & r \end{pmatrix},$$

so that f is injective. The function f is surjective because any complex number $a + bi$ is the image under f of the matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

in K. Finally, for any matrices A and B in K, we must show that $f(A + B) = f(A) + f(B)$ and $f(AB) = f(A)f(B)$. We have

$$\begin{aligned} f\left[\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix}\right] &= f\begin{pmatrix} a+c & b+d \\ -b-d & a+c \end{pmatrix} \\ &= (a + c) + (b + d)i \\ &= (a + bi) + (c + di) \\ &= f\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + f\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \end{aligned}$$

and

$$f\left[\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix}\right] = f\begin{pmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{pmatrix}$$
$$= (ac-bd)+(ad+bc)i$$
$$= (a+bi)(c+di)$$
$$= f\begin{pmatrix} a & b \\ -b & a \end{pmatrix} f\begin{pmatrix} c & d \\ -d & c \end{pmatrix}.$$

Therefore, f is an isomorphism.

It is quite possible to relabel the elements of a single ring in such a way that the ring is isomorphic to itself.

EXAMPLE 3

Let $f:\mathbb{C}\to\mathbb{C}$ be the complex conjugation map given by $f(a+bi)=a-bi$.* The function f satisfies

$$\begin{aligned} f[(a+bi)+(c+di)] &= f[(a+c)+(b+d)i] \\ &= (a+c)-(b+d)i = (a-bi)+(c-di) \\ &= f(a+bi)+f(c+di) \end{aligned}$$

and

$$\begin{aligned} f[(a+bi)(c+di)] &= f[(ac-bd)+(ad+bc)i] \\ &= (ac-bd)-(ad+bc)i = (a-bi)(c-di) \\ &= f(a+bi)f(c+di). \end{aligned}$$

You can readily verify that f is both injective and surjective (Exercise 17). Therefore f is an isomorphism.

EXAMPLE 4

If R is any ring and $\iota_R:R\to R$ is the identity map given by $\iota_R(r)=r$, then for any $a, b\in R$

$$\iota_R(a+b) = a+b = \iota_R(a)+\iota_R(b) \qquad \text{and} \qquad \iota_R(ab)=ab=\iota_R(a)\iota_R(b).$$

Since ι_R is obviously bijective, it is an isomorphism.

Our intuitive notion of isomorphism is symmetric: "R is isomorphic to S" means the same thing as "S is isomorphic to R". The formal definition of isomorphism is not

*The function f has a geometric interpretation in the complex plane, where $a+bi$ is identified with the point (a, b): It reflects the plane in the x-axis.

symmetric, however, since it requires a function from R onto S but no function from S onto R. This apparent asymmetry is easily remedied. If $f:R \to S$ is an isomorphism, then f is a bijective function of sets. Therefore, f has an inverse function $g:S \to R$ such that $g \circ f = \iota_R$ (the identity function on R) and $f \circ g = \iota_S$.* It is not hard to verify that the function g is actually an isomorphism (Exercise 29). Thus $R \cong S$ implies that $S \cong R$, and symmetry is restored.

Homomorphisms

Many functions that are not injective or surjective satisfy condition (iii) of the definition of isomorphism. Such functions are given a special name.

Definition Let R and S be rings. A function $f:R \to S$ is said to be a **homomorphism** if

$$f(a + b) = f(a) + f(b) \quad \text{and} \quad f(ab) = f(a)f(b) \quad \text{for all } a, b \in R.$$

Thus every isomorphism is a homomorphism, but as the following examples show, a homomorphism need not be an isomorphism because a homomorphism may fail to be injective or surjective.

EXAMPLE 5

For any rings R and S the **zero map** $z:R \to S$ given by $z(r) = 0_S$ for every $r \in R$ is a homomorphism because for any $a, b \in R$

$$z(a + b) = 0_S = 0_S + 0_S = z(a) + z(b)$$

and

$$z(ab) = 0_S = 0_S \cdot 0_S = z(a)z(b).$$

When both R and S contain nonzero elements, then the zero map is neither injective nor surjective.

EXAMPLE 6

The function $f:\mathbb{Z} \to \mathbb{Z}_6$ given by $f(a) = [a]$ is a homomorphism because of the way that addition and subtraction are defined in $\mathbb{Z}_6$: for any $a, b \in \mathbb{Z}$

$$f(a + b) = [a + b] = [a] + [b] = f(a) + f(b)$$

and

$$f(ab) = [ab] = [a][b] = f(a)f(b).$$

The homomorphism f is surjective, but not injective (Why?).

*See Appendix B for details.

EXAMPLE 7

The map $g:\mathbb{R} \to M(\mathbb{R})$ given by

$$g(r) = \begin{pmatrix} 0 & 0 \\ -r & r \end{pmatrix}$$

is a homomorphism because for any $r, s \in \mathbb{R}$

$$\begin{aligned} g(r) + g(s) &= \begin{pmatrix} 0 & 0 \\ -r & r \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -s & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -r-s & r+s \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -(r+s) & r+s \end{pmatrix} = g(r+s) \end{aligned}$$

and

$$g(r)g(s) = \begin{pmatrix} 0 & 0 \\ -r & r \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -s & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -rs & rs \end{pmatrix} = g(rs).$$

The homomorphism g is injective but not surjective (Exercise 26).

CAUTION: Not all functions are homomorphisms. The properties

$$f(a+b) = f(a) + f(b) \qquad \text{and} \qquad f(ab) = f(a)f(b)$$

fail for many functions. For example, if $f:\mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 2$, then

$$f(3+4) = f(7) = 9 \qquad \text{but} \qquad f(3) + f(4) = 5 + 6 = 11$$

so that $f(3+4) \neq f(3) + f(4)$. Similarly, $f(3 \cdot 4) \neq f(3)f(4)$ because

$$f(3 \cdot 4) = f(12) = 14, \qquad \text{but} \qquad f(3)f(4) = 5 \cdot 6 = 30.$$

Theorem 3.10

Let $f:R \to S$ be a homomorphism of rings. Then

(1) $f(0_R) = 0_S$.

(2) $f(-a) = -f(a)$ for every $a \in R$.

(3) $f(a - b) = f(a) - f(b)$ for all $a, b \in R$.

If R is a ring with identity and f is surjective, then

(4) S is a ring with identity $f(1_R)$.

(5) Whenever u is a unit in R, then $f(u)$ is a unit in S and $f(u)^{-1} = f(u^{-1})$.

Proof ▸ (1)

$$\begin{aligned}
f(0_R) + f(0_R) &= f(0_R + 0_R) && [f \text{ is a homomorphism.}]\\
f(0_R) + f(0_R) &= f(0_R) && [0_R + 0_R = 0_R \text{ in } R]\\
f(0_R) + f(0_R) &= f(0_R) + 0_S && [f(0_R) + 0_S = f(0_R) \text{ in } S]\\
f(0_R) &= 0_S && [\text{Subtract } f(0_R) \text{ from both sides.}].
\end{aligned}$$

(2) First, note that

$$\begin{aligned}
f(a) + f(-a) &= f(a + (-a)) && [f \text{ is a homomorphism.}]\\
&= f(0_R) && [a + (-a) = 0_R]\\
&= 0_S && [\text{Part } (1)].
\end{aligned}$$

Therefore, $f(-a)$ is a solution of the equation $f(a) + x = 0_S$. But the unique solution of this equation is $-f(a)$ by Theorem 3.3. Hence $f(-a) = -f(a)$ by uniqueness.

(3)

$$\begin{aligned}
f(a - b) &= f(a + (-b)) && [\text{Definition of subtraction}]\\
&= f(a) + f(-b)) && [f \text{ is a homomorphism.}]\\
&= f(a) + (-f(b)) && [\text{Part } (2)]\\
&= f(a) - f(b) && [\text{Definition of subtraction}].
\end{aligned}$$

(4) We shall show that $f(1_R) \in S$ is the identity element of S. Let s be any element of S. Then since f is surjective, $s = f(r)$ for some $r \in R$. Hence,

$$s \cdot f(1_R) = f(r)f(1_R) = f(r \cdot 1_R) = f(r) = s$$

and, similarly, $f(1_R) \cdot s = s$. Therefore, S has $f(1_R)$ as its identity element.

(5) Since u is a unit in R, there is an element v in R such that $uv = 1_R = vu$. Hence, by (4)

$$f(u)f(v) = f(uv) = f(1_R) = 1_S.$$

Similarly, $vu = 1_R$ implies that $f(v)f(u) = 1_S$. Therefore, $f(u)$ is a unit in S, with inverse $f(v)$. In other words, $f(u)^{-1} = f(v)$. Since $v = u^{-1}$, we see that $f(u)^{-1} = f(v) = f(u^{-1})$. ■

If $f{:}R \to S$ is a function, then the **image** of f is this subset of S:

$$Im\, f = \{s \in S \mid s = f(r) \text{ for some } r \in R\} = \{f(r) \mid r \in R\}.$$

If f is surjective, then $Im\, f = S$ by the definition of surjective. In any case we have:

Corollary 3.11

If $f{:}R \to S$ is a homomorphism of rings, then the image of f is a subring of S.

Proof ▸ Denote $Im\, f$ by I. I is nonempty because $0_S = f(0_R) \in I$ by (1) of Theorem 3.10. The definition of homomorphism shows that I is closed under multiplication: If $f(a), f(b) \in I$, then $f(a)f(b) = f(ab) \in I$. Similarly, I is closed under subtraction because $f(a) - f(b) = f(a - b) \in I$ by Theorem 3.10. Therefore, I is a subring of S by Theorem 3.6. ■

Existence of Isomorphisms

If you suspect that two rings are isomorphic, there are no hard and fast rules for finding a function that is an isomorphism between them. However the properties of homomorphisms in Theorem 3.10 can sometimes be helpful.

EXAMPLE 8

If there is an isomorphism f from $\mathbb{Z}_{12}$ to the ring $\mathbb{Z}_3 \times \mathbb{Z}_4$, then $f(1) = (1, 1)$ by part (4) of Theorem 3.10. Since f is a homomorphism, it has to satisfy

$$f(2) = f(1 + 1) = f(1) + f(1) = (1, 1) + (1, 1) = (2, 2)$$
$$f(3) = f(2 + 1) = f(2) + f(1) = (2, 2) + (1, 1) = (0, 3)$$
$$f(4) = f(3 + 1) = f(3) + f(1) = (0, 3) + (1, 1) = (1, 0).$$

Continuing in this fashion shows that *if* f is an isomorphism, then it must be this bijective function:

$$\begin{array}{llll} f(1) = (1, 1) & f(4) = (1, 0) & f(7) = (1, 3) & f(10) = (1, 2) \\ f(2) = (2, 2) & f(5) = (2, 1) & f(8) = (2, 0) & f(11) = (2, 3) \\ f(3) = (0, 3) & f(6) = (0, 2) & f(9) = (0, 1) & f(0) = (0, 0). \end{array}$$

All we have shown up to here is that this bijective function f is the only *possible* isomorphism. To show that this f actually is an isomorphism, we must verify that it is a homomorphism. This can be done either by writing out the tables (tedious) or by observing that the rule of f can be described this way:

$$f([a]_{12}) = ([a]_3, [a]_4),$$

where $[a]_{12}$ denotes the congruence class of the integer a in $\mathbb{Z}_{12}$, $[a]_3$ denotes the class of a in $\mathbb{Z}_3$, and $[a]_4$ the class of a in $\mathbb{Z}_4$. (*Verify that this last statement is correct.*) Then

$$\begin{aligned} f([a]_{12} + [b]_{12}) &= f([a + b]_{12}) && [\textit{Definition of addition in } \mathbb{Z}_{12}] \\ &= ([a + b]_3, [a + b]_4) && [\textit{Definition of } f] \\ &= ([a]_3 + [b]_3, [a]_4 + [b]_4) && [\textit{Definition of addition in } \mathbb{Z}_3 \textit{ and } \mathbb{Z}_4] \\ &= ([a]_3, [a]_4) + ([b]_3, [b]_4) && [\textit{Definition of addition in } \mathbb{Z}_3 \times \mathbb{Z}_4] \\ &= f([a]_{12}) + f([b]_{12}) && [\textit{Definition of } f]. \end{aligned}$$

An identical argument using multiplication in place of addition shows that $f([a]_{12}[b]_{12}) = f([a]_{12})f([b]_{12})$. Therefore, f is an isomorphism and $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$.

Up to now we have concentrated on showing that various rings are isomorphic, but sometimes it is equally important to demonstrate that two rings are *not* isomorphic. To do this, you must show that there is *no* possible function from one to the other satisfying the three conditions of the definition.

EXAMPLE 9

$\mathbb{Z}_6$ is not isomorphic to $\mathbb{Z}_{12}$ or to $\mathbb{Z}$ because it is not possible to have a surjective function from a six-element set to a larger set (or an injective one from a larger set to $\mathbb{Z}_6$).

To show that two infinite rings or two finite rings with the same number of elements are not isomorphic, it is usually best to proceed indirectly.

EXAMPLE 10

The rings $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic. To show this, suppose on the contrary that $f: \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is an isomorphism. Then $f(0) = (0, 0)$ and $f(1) = (1, 1)$ by Theorem 3.10. Consequently,

$$f(2) = f(1 + 1) = f(1) + f(1) = (1, 1) + (1, 1) = (0, 0).$$

Since f is injective and $f(0) = f(2)$, we have a contradiction. Therefore, no isomorphism is possible.

Suppose that $f: R \to S$ is an isomorphism and the elements $a, b, c, \ldots$ of R have a particular property. If the elements $f(a), f(b), f(c), \ldots$ of S have the same property, then we say that the property is **preserved by isomorphism.** According to parts (1), (4), and (5) of Theorem 3.10, for example, the property of being the zero element or the identity element or a unit is preserved by isomorphism. A property that is preserved by isomorphism can sometimes be used to prove that two rings are *not* isomorphic, as in the following examples.

EXAMPLE 11

In the ring $\mathbb{Z}_8$ the elements 1, 3, 5, and 7 are units by Theorem 2.10. Since being a unit is preserved by isomorphism, any isomorphism from $\mathbb{Z}_8$ to another ring with identity will map these four units to four units in the other ring. Consequently, $\mathbb{Z}_8$ is not isomorphic to any ring with less than four units. In particular, $\mathbb{Z}_8$ is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ because there are only two units in this latter ring, namely (1, 1) and (3, 1) as you can readily verify.

EXAMPLE 12

None of $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ is isomorphic to $\mathbb{Z}$ because every nonzero element in the fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ is a unit, whereas $\mathbb{Z}$ has only two units (1 and -1).

EXAMPLE 13

Suppose R is a commutative ring and $f: R \to S$ is an isomorphism. Then for any $a, b \in R$, we have $ab = ba$ in R. Therefore, in S

$$f(a)f(b) = f(ab) = f(ba) = f(b)f(a).$$

Hence, S is also commutative because any two elements of S are of the form $f(a)$, $f(b)$ (since f is surjective). In other words, the property of being a commutative ring is preserved by isomorphism. Therefore, no commutative ring can be isomorphic to a noncommutative ring.

Exercises

A. 1. Let $f: \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$ be the bijection given by

$0 \to (0, 0)$, $1 \to (1, 1)$, $2 \to (0, 2)$, $3 \to (1, 0)$,
$4 \to (0, 1)$, $5 \to (1, 2)$.

Use the addition and multiplication tables of $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ to show that f is an isomorphism.

2. Use tables to show that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the ring R of Exercise 2 in Section 3.1.

3. Let R be a ring and let R^* be the subring of $R \times R$ consisting of all elements of the form (a, a). Show that the function $f: R \to R^*$ given by $f(a) = (a, a)$ is an isomorphism.

4. Let S be the subring $\{0, 2, 4, 6, 8\}$ of $\mathbb{Z}_{10}$ and let $\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4},\}$ (notation as in Example 1). Show that the following bijection from $\mathbb{Z}_5$ to S is *not* an isomorphism:

$$\bar{0} \longrightarrow 0 \quad \bar{1} \longrightarrow 2 \quad \bar{2} \longrightarrow 4 \quad \bar{3} \longrightarrow 6 \quad \bar{4} \longrightarrow 8.$$

5. Prove that the field $\mathbb{R}$ of real numbers is isomorphic to the ring of all 2×2 matrices of the form $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$, with a $\in \mathbb{R}$. [*Hint:* Consider the function f given by $f(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$.]

6. Let R and S be rings and let $\overline{R}$ be the subring of $R \times S$ consisting of all elements of the form $(a, 0_S)$. Show that the function $f: R \to \overline{R}$ given by $f(a) = (a, 0_S)$ is an isomorphism.

7. Prove that $\mathbb{R}$ is isomorphic to the ring S of all 2×2 matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, where $a \in \mathbb{R}$.

8. Let $\mathbb{Q}(\sqrt{2})$ be as in Exercise 39 of Section 3.1. Prove that the function $f: \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ given by $f(a + b\sqrt{2}) = a - b\sqrt{2}$ is an isomorphism.

9. If $f: \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, prove that f is the identity map. [*Hint:* What are $f(1), f(1 + 1), \ldots$?]

10. If R is a ring with identity and $f: R \to S$ is a homomorphism from R to a ring S, prove that $f(1_R)$ is an idempotent in S. [Idempotents were defined in Exercise 3 of Section 3.2.]

11. State at least one reason why the given function is *not* a homomorphism.

(a) $f:\mathbb{R} \to \mathbb{R}$ and $f(x) = \sqrt{x}$.

(b) $g:E \to E$, where E is the ring of even integers and $f(x) = 3x$.

(c) $h:\mathbb{R} \to \mathbb{R}$ and $f(x) = 2^x$.

(d) $k:\mathbb{Q} \to \mathbb{Q}$, where $k(0) = 0$ and $k\left(\frac{a}{b}\right) = \frac{b}{a}$ if $a \neq 0$.

12. Which of the following functions are homomorphisms?

(a) $f:\mathbb{Z} \to \mathbb{Z}$, defined by $f(x) = -x$.

(b) $f:\mathbb{Z}_2 \to \mathbb{Z}_2$, defined by $f(x) = -x$.

(c) $g:\mathbb{Q} \to \mathbb{Q}$, defined by $g(x) = \dfrac{1}{x^2 + 1}$.

(d) $h:\mathbb{R} \to M(\mathbb{R})$, defined by $h(a) = \begin{pmatrix} -a & 0 \\ a & 0 \end{pmatrix}$.

(e) $f:\mathbb{Z}_{12} \to \mathbb{Z}_4$, defined by $f([x]_{12}) = [x]_4$, where $[u]_n$ denotes the class of the integer u in $\mathbb{Z}_n$.

13. Let R and S be rings.

(a) Prove that $f:R \times S \to R$ given by $f((r, s)) = r$ is a surjective homomorphism.

(b) Prove that $g:R \times S \to S$ given by $g((r, s)) = s$ is a surjective homomorphism.

(c) If both R and S are nonzero rings, prove that the homomorphisms f and g are not injective.

14. Let $f:\mathbb{Z} \to \mathbb{Z}_6$ be the homomorphism in Example 6. Let $K = \{a \in \mathbb{Z} \mid f(a) = [0]\}$. Prove that K is a subring of $\mathbb{Z}$.

15. Let $f:R \to S$ be a homomorphism of rings. If r is a zero divisor in R, is $f(r)$ a zero divisor in S?

B. 16. Let T, R, and F be the four-element rings whose tables are given in Example 5 of Section 3.1 and in Exercises 2 and 3 of Section 3.1. Show that no two of these rings are isomorphic.

17. Show that the complex conjugation function $f:\mathbb{C} \to \mathbb{C}$ (whose rule is $f(a + bi) = a - bi$) is a bijection.

18. Show that the isomorphism of $\mathbb{Z}_5$ and S in Example 1 is given by the function whose rule is $f([x]_5) = [6x]_{10}$ (notation as in Exercise 12(e)). Give a direct proof (without using tables) that this map is a homomorphism.

19. Show that $S = \{0, 4, 8, 12, 16, 20, 24\}$ is a subring of $\mathbb{Z}_{28}$. Then prove that the map $f:\mathbb{Z}_7 \to S$ given by $f([x]_7) = [8x]_{28}$ is an isomorphism.

20. Let E be the ring of even integers with the $*$ multiplication defined in Exercise 23 of Section 3.1. Show that the map $f:E \to \mathbb{Z}$ given by $f(x) = x/2$ is an isomorphism.

21. Let $\mathbb{Z}^*$ denote the ring of integers with the $\oplus$ and $\odot$ operations defined in Exercise 22 of Section 3.1. Prove that $\mathbb{Z}$ is isomorphic to $\mathbb{Z}^*$.

22. Let $\overline{\mathbb{Z}}$ denote the ring of integers with the $\oplus$ and $\odot$ operations defined in Exercise 24 of Section 3.1. Prove that $\overline{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}$.

23. Let C be the field of Exercise 45 of Section 3.1. Show that C is isomorphic to the field $\mathbb{C}$ of complex numbers.

24. **(a)** Let R be the set $\mathbb{R} \times \mathbb{R}$ with the usual coordinatewise addition, as in Theorem 3.1. Define a new multiplication by the rule $(a, b)(c, d) = (ac, bc)$. Show that R is a ring.

    (b) Show that the ring of part (a) is isomorphic to the ring of all matrices in $M(\mathbb{R})$ of the form $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$.

25. Let L be the ring of all matrices in $M(\mathbb{Z})$ of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$. Show that the function $f{:}L \to \mathbb{Z}$ given by $f\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = a$ is a surjective homomorphism but not an isomorphism.

26. Show that the homomorphism g in Example 7 is injective but not surjective.

27. **(a)** If $g{:}R \to S$ and $f{:}S \to T$ are homomorphisms, show that $f \circ g{:}R \to T$ is a homomorphism.

    (b) If f and g are isomorphisms, show that $f \circ g$ is also an isomorphism.

28. **(a)** Give an example of a homomorphism $f{:}R \to S$ such that R has an identity but S does not. Does this contradict part (4) of Theorem 3.10?

    (b) Give an example of a homomorphism $f{:}R \to S$ such that S has an identity but R does not.

29. Let $f{:}R \to S$ be an isomorphism of rings and let $g{:}S \to R$ be the inverse function of f (as defined in Appendix B). Show that g is also an isomorphism. [*Hint:* To show $g(a + b) = g(a) + g(b)$, consider the images of the left- and right-hand side under f and use the facts that f is a homomorphism and $f \circ g$ is the identity map.]

30. Let $f{:}R \to S$ be a homomorphism of rings and let $K = \{r \in R \mid f(r) = 0_S\}$. Prove that K is a subring of R.

31. Let $f{:}R \to S$ be a homomorphism of rings and T a subring of S. Let $P = \{r \in R \mid f(r) \in T\}$. Prove that P is a subring of R.

32. Assume $n \equiv 1 \pmod{m}$. Show that the function $f{:}\mathbb{Z}_m \to \mathbb{Z}_{mn}$ given by $f([x]_m) = [nx]_{mn}$ is an injective homomorphism but not an isomorphism when $n \geq 2$ (notation as in Exercise 12(e)).

33. **(a)** Let T be the ring of functions from $\mathbb{R}$ to $\mathbb{R}$, as in Example 8 of Section 3.1. Let $\theta{:}T \to \mathbb{R}$ be the function defined by $\theta(f) = f(5)$. Prove that θ is a surjective homomorphism. Is θ an isomorphism?

    (b) Is part (a) true if 5 is replaced by any constant $c \in \mathbb{R}$?

34. If $f{:}R \to S$ is an isomomorphism of rings, which of the following properties are preserved by this isomorphism? Justify your answers.

    (a) $a \in R$ is a zero divisor.

(b) $a \in R$ is idempotent.*

(c) R is an integral domain.

35. Show that the first ring is not isomorphic to the second.

(a) E and $\mathbb{Z}$

(b) $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $M(\mathbb{R})$

(c) $\mathbb{Z}_4 \times \mathbb{Z}_{14}$ and $\mathbb{Z}_{16}$

(d) $\mathbb{Q}$ and $\mathbb{R}$

(e) $\mathbb{Z} \times \mathbb{Z}_2$ and $\mathbb{Z}$

(f) $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_{16}$

36. (a) If $f{:}R \to S$ is a homomorphism of rings, show that for any $r \in R$ and $n \in \mathbb{Z}$, $f(nr) = nf(r)$.

(b) Prove that isomorphic rings with identity have the same characteristic. [See Exercises 41–43 of Section 3.2.]

(c) If $f{:}R \to S$ is a homomorphism of rings with identity, is it true that R and S have the same characteristic?

37. (a) Assume that e is a nonzero idempotent in a ring R and that e is not a zero divisor.* Prove that e is the identity element of R. [*Hint:* $e^2 = e$ (Why?). If $a \in R$, multiply both sides of $e^2 = e$ by a.]

(b) Let S be a ring with identity and T a ring with no zero divisors. Assume that $f{:}S \to T$ is a nonzero homomorphism of rings (meaning that at least one element of S is not mapped to 0_T). Prove that $f(1_S)$ is the identity element of T. [*Hint:* Show that $f(1_S)$ satisfies the hypotheses of part (a).]

38. Let F be a field and $f{:}F \to R$ a homomorphism of rings.

(a) If there is a nonzero element c of F such that $f(c) = 0_R$, prove that f is the zero homorphism (that is, $f(x) = 0_R$ for every $x \in F$). [*Hint:* c^{-1} exists (Why?). If $x \in F$, consider $f(xcc^{-1})$.]

(b) Prove that f is either injective or the zero homomorphism. [*Hint:* If f is not the zero homomorphism and $f(a) = f(b)$, then $f(a - b) = 0_R$.]

39. Let R be a ring without identity. Let T be the ring with identity of Exercise 32 in Section 3.2. Show that R is isomorphic to the subring $\overline{R}$ of T. Thus, if R is identified with $\overline{R}$, then R is a subring of a ring with identity.

C. 40. For each positive integer k, let $k\mathbb{Z}$ denote the ring of all integer multiples of k (see Exercise 6 of Section 3.1). Prove that if $m \neq n$, then $m\mathbb{Z}$ is not isomorphic to $n\mathbb{Z}$.

41. Let $m, n \in \mathbb{Z}$ with $(m, n) = 1$ and let $f{:}\,\mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ be the function given by $f([a]_{mn}) = ([a]_m, [a]_n)$. (Notation as in Exercise 12(e). Example 8 is the case $m = 3, n = 4$.)

(a) Show that the map f is well defined, that is, show that if $[a]_{mn} = [b]_{mn}$ in $\mathbb{Z}_{mn}$, then $[a]_m = [b]_m$ in $\mathbb{Z}_m$ and $[a]_n = [b]_n$ in $\mathbb{Z}_n$.

(b) Prove that f is an isomorphism. [*Hint:* Adapt the proof in Example 8; the difference is that proving f is a bijection takes more work here.]

42. If $(m, n) \neq 1$, prove that $\mathbb{Z}_{mn}$ is not isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.

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*Idempotents are defined in Exercise 3 of Section 3.2.

CHAPTER 4

Arithmetic in $F[x]$

In Chapter 1 we examined grade-school arithmetic from an advanced standpoint and developed some important properties of the ring $\mathbb{Z}$ of integers. In this chapter we follow a parallel path, but the starting point here is high-school algebra—specifically, polynomials with coefficients in the field $\mathbb{R}$ of real numbers, such as

$$x^2 - 3x - 5, \qquad 6x^3 - 3x^2 + 7x + 4, \qquad x^{12} - 1.$$

Dealing with polynomials means dealing with the mysterious symbol "x", which is used in three different ways in high-school algebra. First, x often "stands for" a number, as in the equation $12x - 8 = 0$, where x is the number $\frac{2}{3}$. Second, x sometimes doesn't seem to stand for any particular number but is treated as if it were a number in simplification exercises such as this one:

$$\frac{x^3 + x}{x^2 + 1} = \frac{x(x^2 + 1)}{x^2 + 1} = x.$$

Third, x is also used as the variable in the rules of functions such as $f(x) = 3x + 5$.

Now that you know what rings and fields are, we shall consider polynomials with coefficients in any ring and attempt to clear up some of the mystery about the nature of x. In Sections 4.1–4.3, we shall see that when x is given a meaning similar to the second way it is used in high school, then the polynomials with coefficients in a field F form a ring (denoted $F[x]$) whose structure is remarkably similar to that of the ring $\mathbb{Z}$ of integers. In many cases the proofs for $\mathbb{Z}$ given in Chapter 1 carry over almost verbatim to $F[x]$.

In Sections 4.4–4.6 we consider tests to determine whether a polynomial is irreducible (the analogue of testing an integer for primality). Here the development is not an exact copy of what was done in the integers. The reason is that the polynomial ring $F[x]$ has features that have no analogues in the ring of integers, namely, the concepts of the root of a polynomial and of a polynomial function (which correspond to the first and third uses of x in high school).

4.1 Polynomial Arithmetic and the Division Algorithm

The underlying idea here is to define "polynomial" in a way that is the obvious extension of polynomials with real-number coefficients. Let R be any ring. A *polynomial with coefficients in R* is an expression of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where n is a nonnegative integer and $a_i \in R$.

This informal definition raises several questions: What is x? Is it an element of R? If not, what does it mean to multiply x by a ring element? In order to answer these questions, note that an expression of the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ makes sense, provided that the a_i and x are all elements of some larger ring. An analogy might be helpful here. The number π is not in the ring $\mathbb{Z}$ of integers, but expressions such as $3 - 4\pi + 12\pi^2 + \pi^3$ and $8 - \pi^2 + 6\pi^5$ make sense in the real numbers. Furthermore, it is not difficult to verify that the set of all numbers of the form

$$a_0 + a_1\pi + a_2\pi^2 + \cdots + a_n\pi^n, \qquad \text{with } n \geq 0 \text{ and } a_i \in \mathbb{Z}$$

is a subring of $\mathbb{R}$ that contains both $\mathbb{Z}$ and π (Exercise 2).

For the present we shall think of polynomials with coefficients in a ring R in much the same way, as elements of a larger ring that contains both R and a special element x that is not in R. This is analogous to the situation in the preceding paragraph with R in place of $\mathbb{Z}$ and x in place of π, except that here we don't know anything about the element x or even if such a larger ring exists. The following theorem provides the answer, as well as a definition of "polynomial".

Theorem 4.1

If R is a ring, then there exists a ring T containing an element x that is not in R and has these properties:

(i) R is a subring of T.

(ii) $xa = ax$ for every $a \in R$.

(iii) The set $R[x]$ of all elements of T of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \qquad (\text{where } n \geq 0 \text{ and } a_i \in R)$$

is a subring of T that contains R.

(iv) The representation of elements of $R[x]$ is unique: If $n \leq m$ and

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = b_0 + b_1x + b_2x^2 + \cdots + b_nx^m,$$

then $a_i = b_i$ for $i = 1, 2, \ldots, n$ and $b_i = 0_R$ for each $i > n$.

(v) $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0_R$ if and only if $a_i = 0_R$ for every i.

Proof ▸ See Appendix G. We shall assume Theorem 4.1 here. ■

The elements of the ring $R[x]$ in Theorem 4.1 (iii) are called **polynomials with coefficients in R** and the elements a_i are called **coefficients.** The special element x is sometimes called an **indeterminate.*** To avoid any misunderstandings in Theorem 4.1, please note the following facts.

1. Property (ii) of Theorem 4.1 does *not* imply that the ring T is commutative, but only that the special element x commutes with each element of the subring R (whose elements may not necessarily commute with each other).
2. Property (v) is the special case of property (iv) when each $b_i = 0_R$.
3. The first expression in property (v) is *not* an equation to be solved for x. In this context, asking what value of x makes $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0_R$ is as meaningless as asking what value of π makes $3 + 5\pi - 7\pi^2 = 0$ because x (like π) is a specific element of a ring, not a variable that can be assigned values.†

EXAMPLE 1

The rings $\mathbb{Z}[x]$, $\mathbb{Q}[x]$, and $\mathbb{R}[x]$ are the rings you are familiar with from high school. For instance, $3 + 5x - 7x^2$ is in all three of these rings, but $3 + 7.5x^2$ is only in $\mathbb{Q}[x]$ and $\mathbb{R}[x]$ because the coefficient 7.5 is not an integer. Similarly, $4.2 + 3x + \sqrt{5}x^4$ is in $\mathbb{R}[x]$ but not in the other two rings since $\sqrt{5}$ is not a rational number. Terms with zero coefficents are usually omitted, as they were in the preceding sentence.

EXAMPLE 2

Let E be the ring of even integers. Then $4 - 6x + 4x^3 \in E[x]$. However, the polynomial x is *not* in $E[x]$, because it cannot be written with even coefficients.

Polynomial Arithmetic

The rules for adding and multiplying polynomials follow directly from the fact that $R[x]$ is a ring.

EXAMPLE 3

If $f(x) = 1 + 5x - x^2 + 4x^3 + 2x^4$ and $g(x) = 4 + 2x + 3x^2 + x^3$ in $\mathbb{Z}_7[x]$, then the commutative, associative, and distributive laws show that

$$\begin{aligned} f(x) + g(x) &= (1 + 5x - x^2 + 4x^3 + 2x^4) + (4 + 2x + 3x^2 + x^3 + 0x^4) \\ &= (1 + 4) + (5 + 2)x + (-1 + 3)x^2 + (4 + 1)x^3 + (2 + 0)x^4 \\ &= 5 + 0x + 2x^2 + 5x^3 + 2x^4 = 5 + 2x^2 + 5x^3 + 2x^4. \end{aligned}$$

*Although in common use, the term "indeterminate" is misleading. As shown in Appendix G, there is nothing undetermined or ambiguous about x. It is a specific element of the larger ring T and is not an element of R.

†Variables and equations will be dealt with in Section 4.4.

EXAMPLE 4

The product of $1 - 7x + x^2$ and $2 + 3x$ in $\mathbb{Q}[x]$ is found by using the distributive law repeatedly:

$$\begin{aligned}(1 - 7x + x^2)(2 + 3x) &= 1(2 + 3x) - 7x(2 + 3x) + x^2(2 + 3x)\\ &= 1(2) + 1(3x) - 7x(2) - 7x(3x) + x^2(2) + x^2(3x)\\ &= 2 + 3x - 14x - 21x^2 + 2x^2 + 3x^3\\ &= 2 - 11x - 19x^2 + 3x^3.\end{aligned}$$

The preceding examples are typical of the general case. You add polynomials by adding the corresponding coefficients, and you multiply polynomials by using the distributive laws and collecting like powers of x. Thus **polynomial addition** is given by the rule:*

$$\begin{aligned}&(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n)\\ &\quad = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n\end{aligned}$$

and **polynomial multiplication** is given by the rule:

$$\begin{aligned}&(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)(b_0 + b_1x + b_2x^2 + \cdots + b_mx^m)\\ &\quad = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots + a_nb_mx^{n+m}.\end{aligned}$$

For each $k \geq 0$, the coefficient of x^k in the product is

$$a_0b_k + a_1b_{k-1} + a_2b_{k-2} + \cdots + a_{k-2}b_2 + a_{k-1}b_1 + a_kb_0 = \sum_{i=0}^{k} a_ib_{k-i},$$

where $a_i = 0_R$ if $i > n$ and $b_j = 0_R$ if $j > m$.

It follows readily from this description of multiplication in $R[x]$ that if R is commutative, then so is $R[x]$ (Exercise 7). Furthermore, if R has a multiplicative identity 1_R, then 1_R is also the multiplicative identity of $R[x]$ (Exercise 8).

Definition

Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial in $R[x]$ with $a_n \neq 0_R$. Then a_n is called the **leading coefficient** of $f(x)$. The **degree** of $f(x)$ is the integer n; it is denoted "deg $f(x)$". In other words, deg $f(x)$ is the largest exponent of x that appears with a nonzero coefficient, and this coefficient is the leading coefficient.

EXAMPLE 5

The degree of $3 - x + 4x^2 - 7x^3 \in \mathbb{R}[x]$ is 3, and its leading coefficient is -7. Similarly, deg $(3 + 5x) = 1$ and deg $(x^{12}) = 12$. The degree of $2 + x + 4x^2 - 0x^3 + 0x^5$ is 2 (the largest exponent of x with a *nonzero* coefficient); its leading coefficient is 4.

*We may assume that the same powers of x appear by inserting zero coefficients where necessary.

The ring R that we start with is a subring of the polynomial ring $R[x]$. The elements of R, considered as polynomials in $R[x]$, are called **constant polynomials.** The polynomials of degree 0 in $R[x]$ are precisely the nonzero constant polynomials. Note that

the constant polynomial 0_R does not have a degree

(because no power of x appears with *nonzero* coefficient).

Theorem 4.2

If R is an integral domain and $f(x)$, $g(x)$ are nonzero polynomials in $R[x]$, then

$$\deg[f(x)g(x)] = \deg f(x) + \deg g(x).$$

Proof ▸ Suppose $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ with $a_n \neq 0_R$ and $b_m \neq 0_R$, so that $\deg f(x) = n$ and $\deg g(x) = m$. Then

$$f(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots + a_nb_mx^{n+m}.$$

The largest exponent of x that can *possibly* have a nonzero coefficient is $n + m$. But $a_nb_m \neq 0_R$ because R is an integral domain and $a_n \neq 0_R$ and $b_m \neq 0_R$. Therefore, $f(x)g(x)$ is nonzero and $\deg[f(x)g(x)] = n + m = \deg f(x) + \deg g(x)$. ■

Corollary 4.3

If R is an integral domain, then so is $R[x]$.

Proof ▸ Since R is a commutative ring with identity, so is $R[x]$ (Exercises 7 and 8). The proof of Theorem 4.2 shows that the product of nonzero polynomials in $R[x]$ is nonzero. Therefore, $R[x]$ is an integral domain. ■

The first five lines of the proof of Theorem 4.2 are valid in any ring and lead to this conclusion.

Corollary 4.4

Let R be a ring. If $f(x)$, $g(x)$, and $f(x)g(x)$ are nonzero in $R[x]$, then

$$\deg [f(x)g(x)] \leq \deg f(x) + \deg g(x).$$

EXAMPLE 6

In $\mathbb{Z}_6[x]$, let $f(x) = 2x^4$ and $g(x) = 5x$. Then $f(x)g(x) = (2x^4)(5x) = 4x^5$, so $\deg [f(x)g(x)] = \deg f(x) + \deg g(x)$. However, if $g(x) = 1 + 3x^2$, then

$$f(x)g(x) = 2x^4(1 + 3x^3) = 2x^4 + 2 \cdot 3x^6 = 2x^4 + 0x^6 = 2x^4,$$

which has degree 4. But $\deg f(x) + \deg g(x) = 6$. So $\deg [f(x)g(x)] < \deg f(x) + \deg g(x)$.

For information on the degree of the *sum* of polynomials, see Exercises 4 and 12.

Corollary 4.5

Let R be an integral domain and $f(x) \in R[x]$. Then

$f(x)$ is a unit in $R[x]$ if and only if $f(x)$ is a constant polynomial that is a unit in R.

In particular, if F is a field, the units in $F[x]$ are the nonzero constants in F.

Remember that the proof of an "if and only if" statement requires two separate proofs.

Proof of Corollary 4.5 ▸ First, assume that $f(x)$ is a unit in $R[x]$. Then $f(x)g(x) = 1_R$ for some $g(x)$ in $R[x]$. By Theorem 4.2,

$$\deg f(x) + \deg g(x) = \deg [f(x)g(x)] = \deg 1_R = 0.$$

Since the degrees of polynomials are nonnegative, we must have $\deg f(x) = 0$ and $\deg g(x) = 0$. Therefore, $f(x)$ and $g(x)$ are constant polynomials, that is, constants in R. Since $f(x)g(x) = 1_R$, $f(x)$ is a unit in R.

Conversely, assume that $f(x)$ is a constant polynomial that is a unit in R, say $f(x) = b$, with b a unit in R. Let $h(x) = b^{-1}$. Then $f(x)h(x) = bb^{-1} = 1_R$. Therefore, $f(x)$ is a unit in $R[x]$.

The last statement of the corollary follows immediately since every nonzero element of a field is a unit in the field (see Example 6 in Section 3.2). ■

EXAMPLE 7

The only units in $\mathbb{Z}[x]$ are 1 and -1, since these are the only units in $\mathbb{Z}$. The units in $\mathbb{R}[x]$ (or in $\mathbb{Q}[x]$ or in $\mathbb{C}[x]$) are all nonzero constants, since $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{C}$ are fields.

Corollary 4.5 may be false if R is not an integral domain (Exercise 11).

EXAMPLE 8

$5x + 1$ is a unit in $\mathbb{Z}_{25}[x]$ that is not a constant because (as you should verify) $(5x + 1)(20x + 1) = 1$.

The Division Algorithm in $F[x]$

Our principal interest in the rest of this chapter will be polynomials with coefficients in a field F (such as $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{Z}_5$). As noted in the chapter introduction, the domain $F[x]$ has many of the same properties as the domain $\mathbb{Z}$ of integers, including the Division Algorithm (Theorem 1.1), which states that for any integers a and b with b positive, there exist unique integers q and r such that

$$a = bq + r \qquad \text{and} \qquad 0 \leq r < b.$$

For polynomials, the only changes are to require the divisor to be nonzero and to replace the statement "$0 \le r < b$" by a statement involving degrees. Here is the formal statement (with $f(x)$ in place of a, $g(x)$ in place of b, and $q(x)$, $r(x)$ in place of q, r respectively).

Theorem 4.6 The Division Algorithm in $F[x]$

Let F be a field and $f(x), g(x) \in F[x]$ with $g(x) \neq 0_F$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x) \qquad \text{and either} \quad r(x) = 0_F \quad \text{or} \quad \deg r(x) < \deg g(x).$$

Example 9 shows how polynomial division works and why the Division Algorithm is valid in one particular case.

EXAMPLE 9

We shall divide $f(x) = 3x^5 + 2x^4 + 2x^3 + 4x^2 + x - 2$ by $g(x) = 2x^3 + 1$. The italic column on the right keeps track of what happens at each step.*

divisor $g(x)$

$$
\begin{array}{rrl}
 & \frac{3}{2}x^2 + x + 1 & \leftarrow \textit{quotient } q(x) \\
2x^3 + 1 \,\big) & 3x^5 + 2x^4 + 2x^3 + 4x^2 + x - 2 & \leftarrow \textit{dividend } f(x) \\
 & 3x^5 + \qquad\qquad \frac{3}{2}x^2 \qquad\quad & \leftarrow \left(\frac{3}{2}x^2\right)g(x) \\
\hline
 & 2x^4 + 2x^3 + \frac{5}{2}x^2 + x - 2 & \leftarrow f(x) - \left(\frac{3}{2}x^2\right)g(x) \\
 & 2x^4 \qquad\qquad\qquad + x \quad\;\; & \leftarrow xg(x) \\
\hline
 & 2x^3 + \frac{5}{2}x^2 \qquad - 2 & \leftarrow f(x) - \left(\frac{3}{2}x^2\right)g(x) - xg(x) \\
 & 2x^3 \qquad\qquad\quad + 1 & \leftarrow 1g(x) \\
\hline
\textit{remainder } r(x) \longrightarrow & \frac{5}{2}x^2 \qquad - 3 & \leftarrow f(x) - \left(\frac{3}{2}x^2\right)g(x) - xg(x) - 1g(x) = \\
 & & \quad f(x) - g(x)\left(\frac{3}{2}x^2 + x + 1\right) = \\
 & & \quad f(x) - g(x)q(x)
\end{array}
$$

The last line on the left side and the last three lines on the right side show that

$$f(x) - g(x)q(x) = r(x) \quad \text{or equivalently,} \quad f(x) = g(x)q(x) + r(x).$$

So the Division Algorithm holds for the polynomials $f(x)$ and $g(x)$.

**Division Refesher:* The first term of the quotient $\left(\frac{3}{2}x^2\right)$ is obtained by dividing the leading term of the dividend ($3x^5$) by the leading term of the divisor ($2x^3$): $3x^5/2x^3 = \frac{3}{2}x^2$. The product of this term and the divisor $\left(\left(\frac{3}{2}x^2\right)g(x)\right)$ is then subtracted from the dividend resulting in $2x^4 + 2x^3 + \frac{5}{2}x^2 + x - 2$, as shown. The process is repeated, using this last expression as the dividend and the same divisor, and continues until you reach a polynomial with degree smaller than the degree of the divisor.

Of course, an example is *not* a proof, even though you can readily convince yourself that the same procedure works with other divisors and dividends (Exercise 5). Consequently, skipping the proof until you are familiar with mathematical induction, would be quite reasonable. That's why the proof of Theorem 4.6 is marked optional.

Proof of Theorem 4.6 The Division Algorithm (Optional) ▸

We first prove the existence of the polynomials $q(x)$ and $r(x)$.

Case 1: If $f(x) = 0_F$ or if $\deg f(x) < \deg g(x)$, then the theorem is true with $q(x) = 0_F$ and $r(x) = f(x)$ because $f(x) = g(x)0_F + f(x)$.

Case 2: If $f(x) \neq 0_F$ and $\deg g(x) \leq \deg f(x)$, then the proof of existence is by induction on the degree of the dividend $f(x)$.* If $\deg f(x) = 0$, then $\deg g(x) = 0$ also. Hence, $f(x) = a$ and $g(x) = b$ for some nonzero $a, b \in F$. Since F is a field, b is a unit and $a = b(b^{-1}a) + 0_F$. Thus the theorem is true with $q(x) = b^{-1}a$ and $r(x) = 0_F$.

Assume inductively that the theorem is true whenever the dividend has degree less than n. This part of the proof is presented in two columns. The left-hand column is the formal proof, while the right-hand column refers to Example 9. The example will help you understand what's being done in the proof.

PROOF	EXAMPLE 9
We must show that the theorem is true whenever the dividend $f(x)$ has degree n, say	$n = 5$
$f(x) = a_nx^n + \cdots + a_1x + a_0$	$f(x) = \underbrace{3x^5}_{a_nx^n} + 2x^4 + 2x^3 + 4x^2 + x - 2$
with $a_n \neq 0_F$. The divisor $g(x)$ must have the form	$m = 3$
$g(x) = b_mx^m + \cdots + b_1x + b_0$	$g(x) = \underbrace{2x^3}_{b_mx^m} + 1$
with $b_m \neq 0_F$ and $m \leq n$. We begin as we would in the long division of $g(x)$ into $f(x)$. Since F is a field and $b_m \neq 0_F$, b_m is a unit. Multiply the divisor $g(x)$ by $a_nb_m^{-1}x^{n-m}$ to obtain	$a_nb_m^{-1}x^{n-m} = 3 \cdot 2^{-1}x^{5-3} = \underbrace{\frac{3}{2}x^2}_{\text{first term of the quotient}}$
$a_nb_m^{-1}x^{n-m}g(x)$ $= a_nb_m^{-1}x^{n-m}(b_mx^m + \cdots + b_1x + b_0)$ $= a_nx^n + a_nb_m^{-1}b_{m-1}x^{n-1} + \cdots + a_nb_m^{-1}b_0x^{n-m}$	$\frac{3}{2}x^2g(x) = \frac{3}{2}x^2(2x^3 + 1)$ $= 3x^5 + \frac{3}{2}x^2$

* We use the Principle of Complete Induction; see Appendix C.

Since $a_n b_m^{-1} x^{n-m} g(x)$ and $f(x)$ have the same degree and the same leading coefficient, the difference

$$f(x) - a_n b_m^{-1} x^{n-m} g(x)$$

is a polynomial of degree *less than n* (or possibly the zero polynomial). Now apply the induction hypothesis with $g(x)$ as divisor and the polynomial $f(x) - a_n b_m^{-1} x^{n-m} g(x)$ as dividend (or use Case 1 if this dividend is zero). By induction there exist polynomials $q_1(x)$ and $r(x)$ such that

$$f(x) - a_n b_m^{-1} x^{n-m} g(x) = g(x)q_1(x) + r(x) \quad \text{and}$$
$$r(x) = 0_F \quad \text{or} \quad \deg r(x) < \deg g(x).$$

$$\begin{aligned} f(x) - \frac{3}{2}x^2 g(x) &= f(x) - \left(3x^5 + \frac{3}{2}x^2\right) \\ &= \underbrace{2x^4 + 2x^3 + \frac{5}{2}x^2 + x - 2}_{\textit{fourth line of long division}} \end{aligned}$$

$$\underbrace{q_1(x) = x + 1}_{\substack{\textit{last part of} \\ \textit{the quotient}}} \quad \underbrace{r(x) = \frac{5}{2}x^2 - 3}_{\textit{remainder}}$$

Therefore,

$$f(x) = g(x)[a_n b_m^{-1} x^{n-m} + q_1(x)] + r(x) \quad \text{and}$$
$$r(x) = 0_F \quad \text{or} \quad \deg r(x) < \deg g(x).$$

Thus the theorem is true with $q(x) = a_n b_m^{-1} x^{n-m} + q_1(x)$ when $\deg f(x) = n$. This completes the induction and shows that $q(x)$ and $r(x)$ always exist for any divisor and dividend.

To prove that $q(x)$ and $r(x)$ are unique, suppose that $q_2(x)$ and $r_2(x)$ are polynomials such that

$$f(x) = g(x)q_2(x) + r_2(x) \quad \text{and} \quad r_2(x) = 0_F \text{ or } \deg r_2(x) < \deg g(x).$$

Then

$$g(x)q(x) + r(x) = f(x) = g(x)q_2(x) + r_2(x),$$

so that

$$g(x)[q(x) - q_2(x)] = r_2(x) - r(x).$$

If $q(x) - q_2(x)$ is nonzero, then by Theorem 4.2 the degree of the left side is $\deg g(x) + \deg[q(x) - q_2(x)]$, a number greater than or equal to $\deg g(x)$. However, both $r_2(x)$ and $r(x)$ have degree strictly less than $\deg g(x)$, and so the right-hand side of the equation must also have degree strictly less than $\deg g(x)$ (Exercise 12). This is a contradiction. Therefore $q(x) - q_2(x) = 0_F$, or, equivalently, $q(x) = q_2(x)$. Since the left side is zero, we must have $r_2(x) - r(x) = 0_F$, so that $r_2(x) = r(x)$. Thus the polynomials $q(x)$ and $r(x)$ are unique. ∎

Exercises

NOTE: *R denotes a ring and F a field.*

A. 1. Perform the indicated operation and simplify your answer:

(a) $(3x^4 + 2x^3 - 4x^2 + x + 4) + (4x^3 + x^2 + 4x + 3)$ in $\mathbb{Z}_5[x]$

(b) $(x + 1)^3$ in $\mathbb{Z}_3[x]$

(c) $(x - 1)^5$ in $\mathbb{Z}_5[x]$

(d) $(x^2 - 3x + 2)(2x^3 - 4x + 1)$ in $\mathbb{Z}_7[x]$

2. Show that the set of all real numbers of the form

$$a_0 + a_1\pi + a_2\pi^2 + \cdots + a_n\pi^n, \quad \text{with } n \geq 0 \text{ and } a_i \in \mathbb{Z}$$

is a subring of $\mathbb{R}$ that contains both $\mathbb{Z}$ and π.

3. **(a)** List all polynomials of degree 3 in $\mathbb{Z}_2[x]$.

(b) List all polynomials of degree less than 3 in $\mathbb{Z}_3[x]$.

4. In each part, give an example of polynomials $f(x), g(x) \in \mathbb{Q}[x]$ that satisfy the given condition:

(a) The deg of $f(x) + g(x)$ is less than the maximum of $\deg f(x)$ and $\deg g(x)$.

(b) $\operatorname{Deg}[f(x) + g(x)] = \max\{\deg f(x), \deg g(x)\}$.

5. Find polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)q(x) + r(x)$, and $r(x) = 0$ or $\deg r(x) < \deg g(x)$:

(a) $f(x) = 3x^4 - 2x^3 + 6x^2 - x + 2$ and $g(x) = x^2 + x + 1$ in $\mathbb{Q}[x]$.

(b) $f(x) = x^4 - 7x + 1$ and $g(x) = 2x^2 + 1$ in $\mathbb{Q}[x]$.

(c) $f(x) = 2x^4 + x^2 - x + 1$ and $g(x) = 2x - 1$ in $\mathbb{Z}_5[x]$.

(d) $f(x) = 4x^4 + 2x^3 + 6x^2 + 4x + 5$ and $g(x) = 3x^2 + 2$ in $\mathbb{Z}_7[x]$.

6. Which of the following subsets of $R[x]$ are subrings of $R[x]$? Justify your answer:

(a) All polynomials with constant term 0_R.

(b) All polynomials of degree 2.

(c) All polynomials of degree $\leq k$, where k is a fixed positive integer.

(d) All polynomials in which the odd powers of x have zero coefficients.

(e) All polynomials in which the even powers of x have zero coefficients.

7. If R is commutative, show that $R[x]$ is also commutative.

8. If R has multiplicative identity 1_R, show that 1_R is also the multiplicative identity of $R[x]$.

9. If $c \in R$ is a zero divisor in a commutative ring R, then is c also a zero divisor in $R[x]$?

10. If F is a field, show that $F[x]$ is not a field. [*Hint:* Is x a unit in $F[x]$?]

B. 11. Show that $1 + 3x$ is a unit in $\mathbb{Z}_9[x]$. Hence, Corollary 4.5 may be false if R is not an integral domain.

12. If $f(x), g(x) \in R[x]$ and $f(x) + g(x) \neq 0_R$, show that

$$\deg[f(x) + g(x)] \leq \max\{\deg f(x), \deg g(x)\}.$$

13. Let R be a commutative ring. If $a_n \neq 0_R$ and $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ (with $a_n \neq 0_R$) is a zero divisor in $R[x]$, prove that a_n is a zero divisor in R.

14. **(a)** Let R be an integral domain and $f(x), g(x) \in R[x]$. Assume that the leading coefficient of $g(x)$ is a unit in R. Verify that the Division Algorithm holds for $f(x)$ as dividend and $g(x)$ as divisor. [*Hint:* Adapt the proof of Theorem 4.6. Where is the hypothesis that F is a field used there?]

(b) Give an example in $\mathbb{Z}[x]$ to show that part (a) may be false if the leading coefficient of $g(x)$ is not a unit. [*Hint:* Exercise 5(b) with $\mathbb{Z}$ in place of $\mathbb{Q}$.]

15. Let R be a commutative ring with identity and $a \in R$.

(a) If $a^3 = 0_R$, show that $1_R + ax$ is a unit in $R[x]$. [*Hint:* Consider $1 - ax + a^2x^2$.]

(b) If $a^4 = 0_R$, show that $1_R + ax$ is a unit in $R[x]$.

16. Let R be a commutative ring with identity and $a \in R$. If $1_R + ax$ is a unit in $R[x]$, show that $a^n = 0_R$ for some integer $n > 0$. [*Hint:* Suppose that the inverse of $1_R + ax$ is $b_0 + b_1x + b_2x^2 + \cdots + b_kx^k$. Since their product is 1_R, $b_0 = 1_R$ (Why?) and the other coefficients are all 0_R.]

17. Let R be an integral domain. Assume that the Division Algorithm always holds in $R[x]$. Prove that R is a field.

18. Let $\varphi: R[x] \to R$ be the function that maps each polynomial in $R[x]$ onto its constant term (an element of R). Show that φ is a surjective homomorphism of rings.

19. Let $\varphi: \mathbb{Z}[x] \to \mathbb{Z}_n[x]$ be the function that maps the polynomial $a_0 + a_1x + \cdots + a_kx^k$ in $\mathbb{Z}[x]$ onto the polynomial $[a_0] + [a_1]x + \cdots + [a_k]x^k$, where $[a]$ denotes the class of the integer a in $\mathbb{Z}_n$. Show that φ is a surjective homomorphism of rings.

20. Let $D: \mathbb{R}[x] \to \mathbb{R}[x]$ be the derivative map defined by

$$D(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}.$$

Is D a homomorphism of rings? An isomorphism?

C. 21. Let $h: R \to S$ be a homomorphism of rings and define a function $\bar{h}: R[x] \to S[x]$ by the rule

$$\bar{h}(a_0 + a_1x + \cdots + a_nx^n) = h(a_0) + h(a_1)x + h(a_2)x^2 + \cdots + h(a_n)x^n.$$

Prove that

(a) $\bar{h}$ is a homomorphism of rings.

(b) $\bar{h}$ is injective if and only if h is injective.

(c) $\bar{h}$ is surjective if and only if h is surjective.

(d) If $R \cong S$, then $R[x] \cong S[x]$.

22. Let R be a commutative ring and let $k(x)$ be a fixed polynomial in $R[x]$. Prove that there exists a unique homomorphism $\varphi: R[x] \to R[x]$ such that

$$\varphi(r) = r \text{ for all } r \in R \qquad \text{and} \qquad \varphi(x) = k(x).$$

4.2 Divisibility in $F[x]$

All the results of Section 1.2 on divisibility and greatest common divisors in $\mathbb{Z}$ now carry over, with only minor modifications, to the ring of polynomials over a field. *Throughout this section, F always denotes a field.*

Definition

Let F be a field and $a(x), b(x) \in F[x]$ with $b(x)$ nonzero. We say that $b(x)$ **divides** $a(x)$ [or that $b(x)$ is a **factor** of $a(x)$], and write $b(x) \mid a(x)$ if $a(x) = b(x)h(x)$ for some $h(x) \in F[x]$.

EXAMPLE 1

$(2x + 1) \mid (6x^2 - x - 2)$ in $\mathbb{Q}[x]$ because $6x^2 - x - 2 = (2x + 1)(3x - 2)$. Furthermore, every constant multiple of $2x + 1$ also divides $6x^2 - x - 2$. For instance, $5(2x + 1) = 10x + 5$ divides $6x^2 - x - 2$ because $6x^2 - x - 2 = 5(2x + 1)\left[\frac{1}{5}(3x - 2)\right]$.

Example 1 illustrates the first part of the following result.

Theorem 4.7

Let F be a field and $a(x), b(x) \in F[x]$ with $b(x)$ nonzero.

(1) If $b(x)$ divides $a(x)$, then $cb(x)$ divides $a(x)$ for each nonzero $c \in F$.

(2) Every divisor of $a(x)$ has degree less than or equal to $\deg a(x)$.

Proof ▸ (1) If $b(x) \mid a(x)$, then $a(x) = b(x)h(x)$ for some $h(x) \in F[x]$. Hence,

$$a(x) = 1_F \cdot b(x)h(x) = cc^{-1}b(x)h(x) = cb(x)[c^{-1}h(x)].$$

Therefore, $cb(x) \mid a(x)$.

(2) Suppose $b(x) \mid a(x)$, say $a(x) = b(x)h(x)$. By Theorem 4.2, $\deg a(x) = \deg b(x) + \deg h(x)$.

Since degrees are nonnegative, we must have $0 \le \deg b(x) \le \deg a(x)$. ■

As we learned earlier, the greatest common divisor of two integers is the largest integer that divides both of them. By analogy, the greatest common divisor of two polynomials $a(x), b(x) \in F[x]$ ought to be the polynomial of highest degree that divides both of them. But such a greatest common divisor would not be unique because each constant multiple of it would have the same degree and would also divide both $a(x)$ and $b(x)$. In order to guarantee a unique gcd, we modify this definition slightly by introducing a new concept. A polynomial in $F[x]$ is said to be **monic** if its leading coefficient is 1_F. For instance, $x^3 + x + 2$ is monic in $\mathbb{Q}[x]$, but $2x + 1$ is not.

Definition

Let F be a field and $a(x), b(x) \in F[x]$, not both zero. The **greatest common divisor (gcd)** of $a(x)$ and $b(x)$ is the monic polynomial of highest degree that divides both $a(x)$ and $b(x)$.

In other words, $d(x)$ is the gcd of $a(x)$ and $b(x)$ provided that $d(x)$ is monic and

(1) $d(x) \mid a(x)$ and $d(x) \mid b(x)$;

(2) If $c(x) \mid a(x)$ and $c(x) \mid b(x)$, then $\deg c(x) \le \deg d(x)$.

Polynomials $a(x)$ and $b(x)$ have at least one monic common divisor (namely 1_F). Since the degree of a common divisor of $a(x)$ and $b(x)$ cannot exceed either $\deg a(x)$ or $\deg b(x)$ by Theorem 4.7, there must be at least one monic common divisor of highest degree. In Theorem 4.8 below we shall show that there is only one monic common divisor of highest degree, thus justifying the definition's reference to *the* greatest common divisor.

EXAMPLE 2

To find the gcd of $3x^2 + x + 6$ and 0 in $\mathbb{Q}[x]$, we note that the common divisors of highest degree are just the divisors of $3x^2 + x + 6$ of degree 2. These include $3x^2 + x + 6$ itself and *all nonzero constant multiples* of this polynomial—in particular, the monic polynomial

$$\frac{1}{3}(3x^2 + x + 6) = x^2 + \frac{1}{3}x + 2.$$

Hence, $x^2 + \frac{1}{3}x + 2$ is a gcd of $3x^2 + x + 6$ and 0.

EXAMPLE 3

You can easily verify these factorizations in $\mathbb{Q}[x]$:

$$a(x) = 2x^4 + 5x^3 - 5x - 2 = (2x + 1)(x + 2)(x + 1)(x - 1),$$
$$b(x) = 2x^3 - 3x^2 - 2x = (2x + 1)(x - 2)x.$$

It appears that $2x + 1$ is a common divisor of highest degree of $a(x)$ and $b(x)$. In this case, the constant multiple $\frac{1}{2}(2x + 1) = x + \frac{1}{2}$ is a *monic* common divisor of highest degree. For a proof that $x + \frac{1}{2}$ actually is the greatest common divisor, see Exercise 5(g).

The remainder of this section, which is referred to only a few times in the rest of the book, may be skimmed if time is short—read the theorems and corollaries, but skip the proofs.

Theorem 4.8

Let F be a field and $a(x), b(x) \in F[x]$, not both zero. Then there is a unique greatest common divisor $d(x)$ of $a(x)$ and $b(x)$. Furthermore, there are (not necessarily unique) polynomials $u(x)$ and $v(x)$ such that $d(x) = a(x)u(x) + b(x)v(x)$.

Steps 1 and 2 of the proof are patterned after the proof of Theorem 1.2.

Proof of Theorem 4.8 ▸ Let S be the set of all linear combinations of $a(x)$ and $b(x)$, that is,

$$S = \{a(x)m(x) + b(x)n(x) \mid m(x),\ n(x) \in F[x]\}.$$

Step 1 *Find a monic polynomial of smallest degree in S.*

Proof of Step 1: S contains nonzero polynomials (for instance, at least one of $a(x) \cdot 1_F + b(x) \cdot 0_F$ or $a(x) \cdot 0_F, + b(x) \cdot 1_F$). So the set of all *degrees* of polynomials in S is a nonempty set of nonnegative integers, which has a smallest element by the Well-Ordering Axiom. Hence, there is a polynomial $w(x)$ of smallest degree in S. If d is the leading coefficient of $w(x)$, then $t(x) = d^{-1}w(x)$ is a monic polynomial of smallest degree in S. By the definition of S,

$$t(x) = a(x)u(x) + b(x)v(x) \text{ for some } u(x), v(x) \in F[x].$$

Step 2 *Prove that $t(x)$ is a gcd of $a(x)$ and $b(x)$.*

Proof of Step 2: We must prove that t satisfies the two conditions in the definition of the gcd:

(1) $t(x) \mid a(x)$ and $t(x) \mid b(x)$;

(2) If $c(x) \mid a(x)$ and $c(x) \mid b(x)$, then $\deg c(x) \leq \deg t(x)$.

Proof of (1): In the proof of Step 2 of Theorem 1.2, replace $a, b, c, t, q, r, u, v, k$, and s with $a(x), b(x), c(x), t(x), q(x), r(x), u(x), v(x), k(x)$, and $s(x)$, respectively, to show that $t(x)$ is a common divisor of $a(x)$ and $b(x)$.

Proof of (2): With the same replacements as in the proof of (1), repeat the proof of Step 2 of Theorem 1.2, until you reach this statement:

$$\begin{aligned} t(x) = a(x)u(x) + b(x)v(x) &= [c(x)k(x)]u(x) + [c(x)s(x)]v(x) \\ &= c(x)[k(x)u(x) + s(x)v(x)]. \end{aligned}$$

The first and last terms of this equation show that $c(x) \mid t(x)$. By Theorem 4.7, $\deg c(x) \leq \deg t(x)$.

This shows that $t(x)$ is *a* greatest common divisor of $f(x)$ and $g(x)$.

Step 3 *Prove that $t(x)$ is the unique gcd of $a(x)$ and $b(x)$.*

Proof of Step 3: Suppose that $d(x)$ is any gcd of $a(x)$ and $b(x)$. To prove uniqueness, we must show that $d(x) = t(x)$. Since $d(x)$ is a common divisor, we have $a(x) = d(x)f(x)$ and $b(x) = d(x)g(x)$ for some $f(x), g(x) \in F[x]$. Therefore,

$$\begin{aligned} t(x) = a(x)u(x) + b(x)v(x) &= [d(x)f(x)]u(x) + [d(x)g(x)]v(x) \\ &= d(x)[f(x)u(x) + g(x)v(x)]. \end{aligned}$$

By Theorem 4.2,

$$\deg t(x) = \deg d(x) + \deg [f(x)u(x) + g(x)v(x)].$$

Since they are gcd's, $t(x)$ and $d(x)$ have the same degree. Hence,

$$\deg [f(x)u(x) + g(x)v(x)] = 0,$$

so that $f(x)u(x) + g(x)v(x) = c$ for some constant $c \in F$. Therefore, $t(x) = d(x)c$. Since both $t(x)$ and $d(x)$ are monic, the leading coefficient on the left side is 1_F and the leading coefficient on the right side is c. So we must have $c = 1_F$. Therefore, $d(x) = t(x) = a(x)u(x) + b(x)v(x)$ is the unique gcd of $a(x)$ and $b(x)$. ■

Corollary 4.9

Let F be a field and $a(x)$, $b(x) \in F[x]$, not both zero. A monic polynomial $d(x) \in F[x]$ is the greatest common divisor of $a(x)$ and $b(x)$ if and only if $d(x)$ satisfies these conditions.

(i) $d(x) \mid a(x)$ and $d(x) \mid b(x)$.

(ii) if $c(x) \mid a(x)$ and $c(x) \mid b(x)$, then $c(x) \mid d(x)$.

Proof ▸ Adapt the proof of Corollary 1.3 to $F[x]$. ■

Polynomials $f(x)$ and $g(x)$ are said to be **relatively prime** if their greatest common divisor is 1_F.

Theorem 4.10

Let F be a field and $a(x)$, $b(x)$, $c(x) \in F[x]$. If $a(x) \mid b(x)c(x)$ and $a(x)$ and $b(x)$ are relatively prime, then $a(x) \mid c(x)$.

Proof ▸ Adapt the proof of Theorem 1.4 to $F[x]$. ■

Exercises

NOTE: *F denotes a field.*

A. 1. If $f(x) \in F[x]$, show that every nonzero constant polynomial divides $f(x)$.

2. If $f(x) = c_n x^n + \cdots + c_0$ with $c_n \neq 0_F$, what is the gcd of $f(x)$ and 0_F?

3. If $a, b \in F$ and $a \neq b$, show that $x + a$ and $x + b$ are relatively prime in $F[x]$.

4. **(a)** Let $f(x)$, $g(x) \in F[x]$. If $f(x) \mid g(x)$ and $g(x) \mid f(x)$, show that $f(x) = cg(x)$ for some nonzero $c \in F$.

(b) If $f(x)$ and $g(x)$ in part (a) are monic, show that $f(x) = g(x)$.

5. The *Euclidean Algorithm* for finding gcd's is described for integers in Exercise 15 of Section 1.2. The process given there also works for polynomials over a field, with one minor adjustment. For integers, the last nonzero remainder is the gcd. For polynomials the last nonzero remainder is a common divisor of highest degree, but it may not be monic. In that case, multiply it by the inverse of its leading coefficient to obtain the gcd. Use the Euclidean Algorithm to find the gcd of the given polynomials:

(a) $x^4 - x^3 - x^2 + 1$ and $x^3 - 1$ in $\mathbb{Q}[x]$

(b) $x^5 + x^4 + 2x^3 - x^2 - x - 2$ and $x^4 + 2x^3 + 5x^2 + 4x + 4$ in $\mathbb{Q}[x]$

(c) $x^4 + 3x^3 + 2x + 4$ and $x^2 - 1$ in $\mathbb{Z}_5[x]$

(d) $4x^4 + 2x^3 + 6x^2 + 4x + 5$ and $3x^3 + 5x^2 + 6x$ in $\mathbb{Z}_7[x]$

(e) $x^3 - ix^2 + 4x - 4i$ and $x^2 + 1$ in $\mathbb{C}[x]$

(f) $x^4 + x + 1$ and $x^2 + x + 1$ in $\mathbb{Z}_2[x]$

(g) $2x^4 + 5x^3 - 5x - 2$ and $2x^3 - 3x^2 - 2x$ in $\mathbb{Q}[x]$.

6. Express each of the gcd's in Exercise 5 as a linear combination of the two polynomials.

B. 7. Let $f(x) \in F[x]$ and assume that $f(x) \mid g(x)$ for every nonconstant $g(x) \in F[x]$. Show that $f(x)$ is a constant polynomial. [*Hint:* $f(x)$ must divide both $x + 1$ and x.]

8. Let $f(x), g(x) \in F[x]$, not both zero, and let $d(x)$ be their gcd. If $h(x)$ is a common divisor of $f(x)$ and $g(x)$ of highest possible degree, then prove that $h(x) = cd(x)$ for some nonzero $c \in F$.

9. If $f(x) \neq 0_F$ and $f(x)$ is relatively prime to 0_F, what can be said about $f(x)$?

10. Find the gcd of $x + a + b$ and $x^3 - 3abx + a^3 + b^3$ in $\mathbb{Q}[x]$.

11. Fill in the details of the proof of Theorem 4.8.

12. Prove Corollary 4.9.

13. Prove Theorem 4.10.

14. Let $f(x), g(x), h(x) \in F[x]$, with $f(x)$ and $g(x)$ relatively prime. If $f(x) \mid h(x)$ and $g(x) \mid h(x)$, prove that $f(x)g(x) \mid h(x)$.

15. Let $f(x), g(x), h(x) \in F[x]$, with $f(x)$ and $g(x)$ relatively prime. If $h(x) \mid f(x)$, prove that $h(x)$ and $g(x)$ are relatively prime.

16. Let $f(x), g(x), h(x) \in F[x]$, with $f(x)$ and $g(x)$ relatively prime. Prove that the gcd of $f(x)h(x)$ and $g(x)$ is the same as the gcd of $h(x)$ and $g(x)$.

4.3 Irreducibles and Unique Factorization

Throughout this section F always denotes a field. Before carrying over the results of Section 1.3 on unique factorization in $\mathbb{Z}$ to the ring $F[x]$, we must first examine an area in which $\mathbb{Z}$ differs significantly from $F[x]$. In $\mathbb{Z}$ there are only two units,* namely ± 1, but a polynomial ring may have many more units (see Corollary 4.5).

An element a in a commutative ring with identity R is said to be an **associate** of an element b of R if $a = bu$ for some unit u. In this case b is also an associate of a because u^{-1} is a unit and $b = au^{-1}$. In the ring $\mathbb{Z}$, the only associates of an integer n are n and $-n$ because ± 1 are the only units. If F is a field, then by Corollary 4.5, the units in $F[x]$ are the nonzero constants. Therefore,

$f(x)$ is an associate of $g(x)$ in $F[x]$ if and only if $f(x) = cg(x)$ for some nonzero $c \in F$.

Recall that a nonzero integer p is prime in $\mathbb{Z}$ if it is not ± 1 (that is, p is not a unit in $\mathbb{Z}$) and its only divisors are ± 1 (the units) and $\pm p$ (the associates of p). In $F[x]$ the units are the nonzero constants, which suggests the following definition.

*"Unit" is defined just before Example 4 in Section 3.2.

Definition

Let F be a field. A nonconstant polynomial $p(x) \in F[x]$ is said to be **irreducible*** if its only divisors are its associates and the nonzero constant polynomials (units). A nonconstant polynomial that is not irreducible is said to be **reducible**.

EXAMPLE 1

The polynomial $x + 2$ is irreducible in $\mathbb{Q}[x]$ because, by Theorem 4.2, all its divisors must have degree 0 or 1. Divisors of degree 0 are nonzero constants. If $f(x) \mid (x + 2)$, say $x + 2 = f(x)g(x)$, and if $\deg f(x) = 1$, then $g(x)$ has degree 0, so that $g(x) = c$. Thus $c^{-1}(x + 2) = f(x)$, and $f(x)$ is an associate of $x + 2$. A similar argument in the general case shows that

every polynomial of degree 1 in $F[x]$ is irreducible in $F[x]$.

The definition of irreducibility is a natural generalization of the concept of primality in $\mathbb{Z}$. In most high-school texts, however, a polynomial is defined to be irreducible if it is *not* the product of polynomials of lower degree. The next theorem shows that these two definitions are equivalent.

Theorem 4.11

Let F be a field. A nonzero polynomial $f(x)$ is reducible in $F[x]$ if and only if $f(x)$ can be written as the product of two polynomials of lower degree.

Proof ▸ First, assume that $f(x)$ is reducible. Then it must have a divisor $g(x)$ that is neither an associate nor a nonzero constant, say $f(x) = g(x)h(x)$. If either $g(x)$ or $h(x)$ has the same degree as $f(x)$, then the other must have degree 0 by Theorem 4.2. Since a polynomial of degree 0 is a nonzero constant in F, this means that either $g(x)$ is a constant or an associate of $f(x)$, contrary to hypothesis. Therefore, both $g(x)$ and $h(x)$ have lower degree than $f(x)$.

Now assume that $f(x)$ can be written as the product of two polynomials of lower degree, and see Exercise 8. ■

Various other tests for irreducibility are presented in Sections 4.4 to 4.6. For now, we note that the concept of irreducibility is not an absolute one. For instance, $x^2 + 1$ is reducible in $\mathbb{C}[x]$ because $x^2 + 1 = (x + i)(x - i)$ and neither factor is a constant or an associate of $x^2 + 1$. But $x^2 + 1$ is irreducible in $\mathbb{Q}[x]$ (Exercise 6).

The following theorem shows that irreducibles in $F[x]$ have essentially the same divisibility properties as do primes in $\mathbb{Z}$. Condition (3) in the theorem is often used to prove that a polynomial is irreducible; in many books, (3) is given as the definition of "irreducible".

*You could just as well call such a polynomial "prime", but "irreducible" is the customary term with polynomials.

Theorem 4.12

Let F be a field and $p(x)$ a nonconstant polynomial in $F[x]$. Then the following conditions are equivalent:*

(1) $p(x)$ is irreducible.

(2) If $b(x)$ and $c(x)$ are any polynomials such that $p(x) \mid b(x)c(x)$, then $p(x) \mid b(x)$ or $p(x) \mid c(x)$.

(3) If $r(x)$ and $s(x)$ are any polynomials such that $p(x) = r(x)s(x)$, then $r(x)$ or $s(x)$ is a nonzero constant polynomial.

Proof ▸ (1) $\Rightarrow$ (2) Adapt the proof of Theorem 1.5 to $F[x]$. Replace statements about $\pm p$ by statements about the associates of $p(x)$; replace statements about ± 1 by statements about units (nonzero constant polynomials) in $F[x]$; use Theorem 4.10 in place of Theorem 1.4.

(2) $\Rightarrow$ (3) If $p(x) = r(x)s(x)$, then $p(x) \mid r(x)$ or $p(x) \mid s(x)$, by (2). If $p(x) \mid r(x)$, say $r(x) = p(x)v(x)$, then $p(x) = r(x)s(x) = p(x)v(x)s(x)$. Since $F[x]$ is an integral domain, we can cancel $p(x)$ by Theorem 3.7 and conclude that $1_F = v(x)s(x)$. Thus $s(x)$ is a unit, and hence by Corollary 4.5, $s(x)$ is a nonzero constant. A similar argument shows that if $p(x) \mid s(x)$, then $r(x)$ is a nonzero constant.

(3) $\Rightarrow$ (1) Let $c(x)$ be any divisor of $p(x)$, say $p(x) = c(x)d(x)$. Then by (3), either $c(x)$ is a nonzero constant or $d(x)$ is a nonzero constant. If $d(x) = d \neq 0_F$, then multiplying both sides of $p(x) = c(x)d(x) = dc(x)$ by d^{-1} shows that $c(x) = d^{-1}p(x)$. Thus in every case, $c(x)$ is a nonzero constant or an associate of $p(x)$. Therefore, $p(x)$ is irreducible. ■

Corollary 4.13

Let F be a field and $p(x)$ an irreducible polynomial in $F[x]$. If $p(x) \mid a_1(x)a_2(x) \cdots a_n(x)$, then $p(x)$ divides at least one of the $a_i(x)$.

Proof ▸ Adapt the proof of Corollary 1.6 to $F[x]$. ■

Theorem 4.14

Let F be a field. Every nonconstant polynomial $f(x)$ in $F[x]$ is a product of irreducible polynomials in $F[x]$.† This factorization is unique in the following sense: If

$$f(x) = p_1(x)p_2(x) \cdots p_r(x) \qquad \text{and} \qquad f(x) = q_1(x)q_2(x) \cdots q_s(x)$$

---

*For the meaning of "the following conditions are equivalent" and what must be done to prove Theorem 4.12, see page 508 of Appendix A. Example 2 there is the integer analogue of Theorem 4.12.

†We allow the possibility of a product with just one factor in case $f(x)$ is itself irreducible.

with each $p_i(x)$ and $q_j(x)$ irreducible, then $r = s$ (that is, the number of irreducible factors is the same). After the $q_j(x)$ are reordered and relabeled, if necessary,

$$p_i(x) \text{ is an associate of } q_i(x) \qquad (i = 1, 2, 3, \ldots, r).$$

Proof ▸ To show that $f(x)$ is a product of irreducibles, adapt the proof of Theorem 1.7 to $F[x]$: Let S be the set of all nonconstant polynomials that are *not* the product of irreducibles, and use a proof by contradiction to show that S is empty. To prove that this factorization is unique up to associates, suppose $f(x) = p_1(x)p_2(x) \cdots p_1(x) = q_1(x)q_2(x) \cdots q_s(x)$ with each $p_i(x)$ and $q_j(x)$ irreducible. Then $p_1(x)[p_2(x) \cdots p_r(x)] = q_1(x)q_2(x) \cdots q_s(x)$, so that $p_1(x)$ divides $q_1(x)q_2(x) \cdots q_s(x)$. Corollary 4.13 shows that $p_1(x) \mid q_j(x)$ for some j. After rearranging and relabeling the $q(x)$'s if necessary, we may assume that $p_1(x) \mid q_1(x)$. Since $q_1(x)$ is irreducible, $p_1(x)$ must be either a constant or an associate of $q_1(x)$. However, $p_1(x)$ is irreducible, and so it is not a constant. Therefore, $p_1(x)$ is an associate of $q_1(x)$, with $p_1(x) = c_1q_1(x)$ for some constant c_1. Thus

$$q_1(x)[c_1p_2(x)p_3(x) \cdots p_r(x)] = p_1(x)p_2(x) \cdots p_r(x) = q_1(x)q_2(x) \cdots q_s(x).$$

Canceling $q_1(x)$ on each end, we have

$$p_2(x)[c_1p_3(x) \cdots p_r(x)] = q_2(x)q_3(x) \cdots q_s(x).$$

Complete the argument by adapting the proof of Theorem 1.8 to $F[x]$, replacing statements about $\pm q_j$ with statements about associates of $q_j(x)$. ■

Exercises

NOTE: *F denotes a field and p a positive prime integer.*

A. **1.** Find a monic associate of

(a) $3x^3 + 2x^2 + x + 5$ in $\mathbb{Q}[x]$ **(b)** $3x^5 - 4x^2 + 1$ in $\mathbb{Z}_5[x]$

(c) $ix^3 + x - 1$ in $\mathbb{C}[x]$

2. Prove that every nonzero $f(x) \in F[x]$ has a unique monic associate in $F[x]$.

3. List all associates of

(a) $x^2 + x + 1$ in $\mathbb{Z}_5[x]$ **(b)** $3x + 2$ in $\mathbb{Z}_7[x]$

4. Show that a nonzero polynomial in $\mathbb{Z}_p[x]$ has exactly $p - 1$ associates.

5. Prove that $f(x)$ and $g(x)$ are associates in $F[x]$ if and only if $f(x) \mid g(x)$ and $g(x) \mid f(x)$.

6. Show that $x^2 + 1$ is irreducible in $\mathbb{Q}[x]$. [*Hint:* If not, it must factor as $(ax + b)(cx + d)$ with $a, b, c, d \in \mathbb{Q}$; show that this is impossible.]

7. Prove that $f(x)$ is irreducible in $F[x]$ if and only if each of its associates is irreducible.

8. If $f(x) \in F[x]$ can be written as the product of two polynomials of lower degree, prove that $f(x)$ is reducible in $F[x]$. (This is the second part of the proof of Theorem 4.11.)

9. Find all irreducible polynomials of

   (a) degree 2 in $\mathbb{Z}_2[x]$ (b) degree 3 in $\mathbb{Z}_2[x]$

   (c) degree 2 in $\mathbb{Z}_3[x]$

10. Is the given polynomial irreducible:

   (a) $x^2 - 3$ in $\mathbb{Q}[x]$? In $\mathbb{R}[x]$?

   (b) $x^2 + x - 2$ in $\mathbb{Z}_3[x]$? In $\mathbb{Z}_7[x]$?

11. Show that $x^3 - 3$ is irreducible in $\mathbb{Z}_7[x]$.

12. Express $x^4 - 4$ as a product of irreducibles in $\mathbb{Q}[x]$, in $\mathbb{R}[x]$, and in $\mathbb{C}[x]$.

13. Use unique factorization to find the gcd in $\mathbb{C}[x]$ of $(x-3)^3(x-4)^4(x-i)^2$ and $(x-1)(x-3)(x-4)^3$.

14. Show that $x^2 + x$ can be factored in two ways in $\mathbb{Z}_6[x]$ as the product of nonconstant polynomials that are not units and not associates of x or $x + 1$.

B. 15. (a) By counting products of the form $(x + a)(x + b)$, show that there are exactly $(p^2 + p)/2$ monic polynomials of degree 2 that are *not* irreducible in $\mathbb{Z}_p[x]$.

   (b) Show that there are exactly $(p^2 - p)/2$ monic irreducible polynomials of degree 2 in $\mathbb{Z}_p[x]$.

16. Prove that $p(x)$ is irreducible in $F[x]$ if and only if for every $g(x) \in F[x]$, either $p(x) \mid g(x)$ or $p(x)$ is relatively prime to $g(x)$.

17. Prove (1) $\Rightarrow$ (2) in Theorem 4.12.

18. Without using statement (2), prove directly that statement (1) is equivalent to statement (3) in Theorem 4.12.

19. Prove Corollary 4.13.

20. If $p(x)$ and $q(x)$ are nonassociate irreducibles in $F[x]$, prove that $p(x)$ and $q(x)$ are relatively prime.

21. (a) Find a polynomial of positive degree in $\mathbb{Z}_9[x]$ that is a unit.

   (b) Show that every polynomial (except the constant polynomials 3 and 6) in $\mathbb{Z}_9[x]$ can be written as the product of two polynomials of positive degree.

22. (a) Show that $x^3 + a$ is reducible in $\mathbb{Z}_3[x]$ for each $a \in \mathbb{Z}_3$.

   (b) Show that $x^5 + a$ is reducible in $\mathbb{Z}_5[x]$ for each $a \in \mathbb{Z}_5$.

23. (a) Show that $x^2 + 2$ is irreducible in $\mathbb{Z}_5[x]$.

   (b) Factor $x^4 - 4$ as a product of irreducibles in $\mathbb{Z}_5[x]$.

24. Prove Theorem 4.14.

25. Prove that every nonconstant $f(x) \in F[x]$ can be written in the form $cp_1(x)p_2(x) \cdots p_n(x)$, with $c \in F$ and each $p_i(x)$ monic irreducible in $F[x]$. Show further that if $f(x) = dq_1(x)q_2(x) \cdots q_m(x)$ with $d \in F$ and each $q_j(x)$ monic irreducible in $F[x]$, then $m = n$, $c = d$, and after reordering and relabeling if necessary, $p_i(x) = q_i(x)$ for each i.

4.4 Polynomial Functions, Roots, and Reducibility

In the parallel development of $F[x]$ and $\mathbb{Z}$, the next step is to consider criteria for irreducibility of polynomials (the analogue of primality testing for integers). Unlike the situation in the integers, there are a number of such criteria for polynomials whose implementation does not depend on a computer. Most of them are based on the fact that every polynomial in $F[x]$ induces a function from F to F. The properties of this function (in particular, the places where it is zero) are closely related to the reducibility or irreducibility of the polynomial.

Throughout this section, R is a commutative ring. Associated with each polynomial $a_nx^n + \cdots + a_2x^2 + a_1x + a_0$ in $R[x]$ is a function $f:R \to R$ whose rule is

$$\text{for each } r \in R, \qquad f(r) = a_nr^n + \cdots + a_2r^2 + a_1r + a_0.$$

The function f induced by a polynomial in this way is called a **polynomial function.**

EXAMPLE 1

The polynomial $x^2 + 5x + 3 \in \mathbb{R}[x]$ induces the function $f:\mathbb{R} \to \mathbb{R}$ whose rule is $f(r) = r^2 + 5r + 3$ for each $r \in \mathbb{R}$.

EXAMPLE 2

The polynomial $x^4 + x + 1 \in \mathbb{Z}_3[x]$ induces the function $f:\mathbb{Z}_3 \to \mathbb{Z}_3$ whose rule is $f(r) = r^4 + r + 1$. Thus

$$f(0) = 0^4 + 0 + 1 = 1, \qquad f(1) = 1^4 + 1 + 1 = 0,$$
$$f(2) = 2^4 + 2 + 1 = 1.$$

The polynomial $x^3 + x^2 + 1 \in \mathbb{Z}_3[x]$ induces the function $g:\mathbb{Z}_3 \to \mathbb{Z}_3$ given by

$$g(0) = 0^3 + 0^2 + 1 = 1, \qquad g(1) = 1^3 + 1^2 + 1 = 0,$$
$$g(2) = 2^3 + 2^2 + 1 = 1.$$

Thus f and g are the *same* function on $\mathbb{Z}_3$, even though they are induced by *different* polynomials in $\mathbb{Z}_3[x]$.*

Although the distinction between a polynomial and the polynomial function it induces is clear, the customary notation is quite ambiguous. For example, you will see a

*Remember that functions f and g are equal if $f(r) = g(r)$ for every r in the domain.

statement such as $f(x) = x^2 - 3x + 2$. Depending on the context, $f(x)$ might denote the polynomial $x^2 - 3x + 2 \in \mathbb{R}[x]$ or the rule of its induced function $f\colon \mathbb{R} \to \mathbb{R}$. The symbol x is being used in two different ways here. In the polynomial $x^2 - 3x + 2$, x is an indeterminate (transcendental element) of the ring $R[x]$.* But in the polynomial function $f\colon\mathbb{R} \to \mathbb{R}$, the symbol x is used as a variable to describe the rule of the function. It might be better to use one symbol for an indeterminate and another for a variable, but the practice of using x for both is so widespread you may as well get used to it.

The use of the same notation for both the polynomial and its induced function also affects the language that is used. For instance, one says "evaluate the polynomial $3x^2 - 5x + 4$ at $x = 2$" or "substitute $x = 2$ in $3x^2 - 5x + 4$" when what is really meant is "find $f(2)$ when f is the function induced by the polynomial $3x^2 - 5x + 4$".

The truth or falsity of certain statements depends on whether x is treated as an indeterminate or a variable. For instance, in the ring $\mathbb{R}[x]$, where x is an indeterminate (special element of the ring), the statement $x^2 - 3x + 2 = 0$ is *false* because, by Theorem 4.1, a polynomial is zero if and only if all its coefficients are zero. When x is a variable, however, as in the rule of the polynomial function $f(x) = x^2 - 3x + 2$, things are different. Here it is perfectly reasonable to ask which elements of $\mathbb{R}$ are mapped to 0 by the function f, that is, for which values of the variable x is it true that $x^2 - 3x + 2 = 0$. It may help to remember that statements about the variable x occur in the ring R, whereas statements about the indeterminate x occur in the polynomial ring $R[x]$.

Roots of Polynomials

Questions about the reducibility of a polynomial can sometimes be answered by considering its induced polynomial function. The key to this analysis is the concept of a root.

Definition

Let R be a commutative ring and $f(x) \in R[x]$. An element a of R is said to be a **root** (or **zero**) of the polynomial $f(x)$ if $f(a) = 0_R$, that is, if the induced function $f\colon R \to R$ maps a to 0_R.

EXAMPLE 3

The roots of the polynomial $f(x) = x^2 - 3x + 2 \in \mathbb{R}[x]$ are the values of the variable x for which $f(x) = 0$, that is, the solutions of the equation $x^2 - 3x + 2 = 0$. It is easy to see that the roots are 1 and 2.

EXAMPLE 4

The polynomial $x^2 + 1 \in \mathbb{R}[x]$ has no roots in $\mathbb{R}$ because there are no real-number solutions of the equation $x^2 + 1 = 0$. However, if $x^2 + 1$ is considered as a polynomial in $\mathbb{C}[x]$, then it has i and $-i$ as roots because these are the solutions in $\mathbb{C}$ of $x^2 + 1 = 0$.

*See page 550 in Appendix G for more information.

Theorem 4.15 The Remainder Theorem

Let F be a field, $f(x) \in F[x]$, and $a \in F$. The remainder when $f(x)$ is divided by the polynomial $x - a$ is $f(a)$.

EXAMPLE 5

To find the remainder when $f(x) = x^{79} + 3x^{24} + 5$ is divided by $x - 1$, we apply the Remainder Theorem with $a = 1$. The remainder is

$$f(1) = 1^{79} + 3 \cdot 1^{24} + 5 = 1 + 3 + 5 = 9.$$

EXAMPLE 6

To find the remainder when $f(x) = 3x^4 - 8x^2 + 11x + 1$ is divided by $x + 2$, we apply the Remainder Theorem *carefully*. The divisor in the theorem is $x - a$, not $x + a$. So we rewrite $x + 2$ as $x - (-2)$ and apply the Remainder Theorem with $a = -2$. The remainder is

$$f(-2) = 3(-2)^4 - 8(-2)^2 + 11(-2) + 1 = 48 - 32 - 22 + 1 = -5.$$

Proof of Theorem 4.15 ▸ By the Division Algorithm, $f(x) = (x - a)q(x) + r(x)$, where the remainder $r(x)$ either is 0_F or has smaller degree than the divisor $x - a$. Thus $\deg r(x) = 0$ or $r(x) = 0_F$. In either case, $r(x) = c$ for some $c \in F$. Hence, $f(x) = (x - a)q(x) + c$, so that $f(a) = (a - a)q(a) + c = 0_F + c = c$. ■

Theorem 4.16 The Factor Theorem

Let F be a field, $f(x) \in F[x]$, and $a \in F$. Then a is a root of the polynomial $f(x)$ if and only if $x - a$ is a factor of $f(x)$ in $F[x]$.

Proof ▸ First assume that a is a root of $f(x)$. Then we have

$$\begin{aligned} f(x) &= (x - a)q(x) + r(x) && [\textit{Division Algorithm}] \\ f(x) &= (x - a)q(x) + f(a) && [\textit{Remainder Theorem}] \\ f(x) &= (x - a)q(x) && [a \textit{ is a root of } f(x), \textit{ so } f(a) = 0_F.] \end{aligned}$$

Therefore, $x - a$ is a factor of $f(x)$.

Conversely, assume that $x - a$ is a factor of $f(x)$, say $f(x) = (x - a)g(x)$. Then a is a root of $f(x)$ because $f(a) = (a - a)g(a) = 0_F g(a) = 0_F$. ■

EXAMPLE 7

To show that $x^7 - x^5 + 2x^4 - 3x^2 - x + 2$ is reducible in $\mathbb{Q}[x]$, note that 1 is a root of this polynomial. Therefore, $x - 1$ is a factor.

Corollary 4.17

Let F be a field and $f(x)$ a nonzero polynomial of degree n in $F[x]$. Then $f(x)$ has at most n roots in F.

Proof* ▸ If $f(x)$ has a root a_1 in F, then by the Factor Theorem, $f(x) = (x - a_1)h_1(x)$ for some $h_1(x) \in F[x]$. If $h_1(x)$ has a root a_2 in F, then by the Factor Theorem

$$f(x) = (x - a_1)(x - a_2)h_2(x) \quad \text{for some } h_2(x) \in F[x].$$

If $h_2(x)$ has a root a_3 in F, repeat this procedure and continue doing so until you reach one of these situations:

(1) $f(x) = (x - a_1)(x - a_2) \cdots (x - a_n)h_n(x)$
(2) $f(x) = (x - a_1)(x - a_2) \cdots (x - a_k)h_k(x)$ and $h_k(x)$ has no root in F.

In Case (1), by Theorem 4.2, we have

$$\begin{aligned}\deg f(x) &= \deg(x - a_1) + \deg(x - a_2) + \cdots + \deg(x - a_n) + \deg h_n(x)\\ n &= 1 + 1 + \cdots + 1 + \deg h_n(x)\\ n &= n + \deg h_n(x)\end{aligned}$$

Thus, $\deg h_n(x) = 0$, so $h_n(x) = c$ for some constant $c \in F$ and $f(x)$ factors as

$$f(x) = c(x - a_1)(x - a_2) \cdots (x - a_n).$$

Clearly, the n numbers $a_1, a_2, \ldots, a_n$ are the only roots of $f(x)$.

The argument in Case (2) is essentially the same (just replace n by k) and leads to this conclusion: $n = \deg f(x) = k + \deg h_k(x)$. So the number of roots is k and $k \leq n$. ■

Corollary 4.18

Let F be a field and $f(x) \in F[x]$, with $\deg f(x) \geq 2$. If $f(x)$ is irreducible in $F[x]$, then $f(x)$ has no roots in F.

Proof ▸ If $f(x)$ is irreducible, then it has no factor of the form $x - a$ in $F[x]$. Therefore, $f(x)$ has no roots in F by the Factor Theorem. ■

*If you prefer a proof by induction, see Exercise 29.

The converse of Corollary 4.18 is false in general. For example, $x^4 + 2x^2 + 1 = (x^2 + 1)(x^2 + 1)$ has no roots in $\mathbb{Q}$ but is reducible in $\mathbb{Q}[x]$. However, the converse is true for degrees 2 and 3.

Corollary 4.19

Let F be a field and let $f(x) \in F[x]$ be a polynomial of degree 2 or 3. Then $f(x)$ is irreducible in $F[x]$ if and only if $f(x)$ has no roots in F.

Proof ▸ Suppose $f(x)$ is irreducible. Then $f(x)$ has no roots in F by Corollary 4.18. Conversely, suppose that $f(x)$ has no roots in F. Then $f(x)$ has no first-degree factor in $F[x]$ because every first-degree polynomial $cx + d$ in $F[x]$ has a root in F, namely $-c^{-1}d$. Therefore, if $f(x) = r(x)s(x)$, neither $r(x)$ nor $s(x)$ has degree 1. By Theorem 4.2, $\deg f(x) = \deg r(x) + \deg s(x)$. Since $f(x)$ has degree 2 or 3, the only possibilities for $(\deg r(x), \deg s(x))$ are (2, 0) or (0, 2) and (3, 0) or (0, 3). So either $r(x)$ or $s(x)$ must have degree 0, that is, either $r(x)$ or $s(x)$ is a nonzero constant. Hence, $f(x)$ is irreducible by Theorem 4.12. ■

EXAMPLE 7

To show that $x^3 + x + 1$ is irreducible in $\mathbb{Z}_5[x]$, you need only verify that none of $0, 1, 2, 3, 4 \in \mathbb{Z}_5$ is a root.

We close this section by returning to its starting point, polynomial functions. Example 2 shows that two different polynomials in $F[x]$ may induce the same function from F to F. We now see that this cannot occur if F is infinite.

Corollary 4.20

Let F be an infinite field and $f(x), g(x) \in F[x]$. Then $f(x)$ and $g(x)$ induce the same function from F to F if and only if $f(x) = g(x)$ in $F[x]$.

Proof ▸ Suppose that $f(x)$ and $g(x)$ induce the same function from F to F. Then $f(a) = g(a)$, so that $f(a) - g(a) = 0_F$, for every $a \in F$. This means that every element of F is a root of the polynomial $f(x) - g(x)$. Since F is infinite, this is impossible by Corollary 4.17 unless $f(x) - g(x)$ is the zero polynomial, that is, $f(x) = g(x)$. The converse is obvious. ■

Exercises

NOTE: *F denotes a field.*

A. **1. (a)** Find a nonzero polynomial in $\mathbb{Z}_2[x]$ that induces the zero function on $\mathbb{Z}_2$.

(b) Do the same in $\mathbb{Z}_3[x]$.

2. Find the remainder when $f(x)$ is divided by $g(x)$:

(a) $f(x) = x^{10} + x^8$ and $g(x) = x - 1$ in $\mathbb{Q}[x]$

(b) $f(x) = 2x^5 - 3x^4 + x^3 - 2x^2 + x - 8$ and $g(x) = x - 10$ in $\mathbb{Q}[x]$

(c) $f(x) = 10x^{75} - 8x^{65} + 6x^{45} + 4x^{37} - 2x^{15} + 5$ and $g(x) = x + 1$ in $\mathbb{Q}[x]$

(d) $f(x) = 2x^5 - 3x^4 + x^3 + 2x + 3$ and $g(x) = x - 3$ in $\mathbb{Z}_5[x]$

3. Determine if $h(x)$ is a factor of $f(x)$:

(a) $h(x) = x + 2$ and $f(x) = x^3 - 3x^2 - 4x - 12$ in $\mathbb{R}[x]$

(b) $h(x) = x - \dfrac{1}{2}$ and $f(x) = 2x^4 + x^3 + x - \dfrac{3}{4}$ in $\mathbb{Q}[x]$

(c) $h(x) = x + 2$ and $f(x) = 3x^5 + 4x^4 + 2x^3 - x^2 + 2x + 1$ in $\mathbb{Z}_5[x]$

(d) $h(x) = x - 3$ and $f(x) = x^6 - x^3 + x - 5$ in $\mathbb{Z}_7[x]$

4. (a) For what value of k is $x - 2$ a factor of $x^4 - 5x^3 + 5x^2 + 3x + k$ in $\mathbb{Q}[x]$?

(b) For what value of k is $x + 1$ a factor of $x^4 + 2x^3 - 3x^2 + kx + 1$ in $\mathbb{Z}_5[x]$?

5. Show that $x - 1_F$ divides $a_nx^n + \cdots + a_2x^2 + a_1x + a_0$ in $F[x]$ if and only if $a_0 + a_1 + a_2 + \cdots + a_n = 0_F$.

6. (a) Verify that every element of $\mathbb{Z}_3$ is a root of $x^3 - x \in \mathbb{Z}_3[x]$.

(b) Verify that every element of $\mathbb{Z}_5$ is a root of $x^5 - x \in \mathbb{Z}_5[x]$.

(c) Make a conjecture about the roots of $x^p - x \in \mathbb{Z}_p[x]$ (p prime).

7. Use the Factor Theorem to show that $x^7 - x$ factors in $\mathbb{Z}_7[x]$ as $x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$, without doing any polynomial multiplication.

8. Determine if the given polynomial is irreducible:

(a) $x^2 - 7$ in $\mathbb{R}[x]$ **(b)** $x^2 - 7$ in $\mathbb{Q}[x]$

(c) $x^2 + 7$ in $\mathbb{C}[x]$ **(d)** $2x^3 + x^2 + 2x + 2$ in $\mathbb{Z}_5[x]$

(e) $x^3 - 9$ in $\mathbb{Z}_{11}[x]$ **(f)** $x^4 + x^2 + 1$ in $\mathbb{Z}_3[x]$

9. List all monic irreducible polynomials of degree 2 in $\mathbb{Z}_3[x]$. Do the same in $\mathbb{Z}_5[x]$.

10. Find a prime $p > 5$ such that $x^2 + 1$ is reducible in $\mathbb{Z}_p[x]$.

11. Find an odd prime p for which $x - 2$ is a divisor of $x^4 + x^3 + 3x^2 + x + 1$ in $\mathbb{Z}_p[x]$.

B. 12. If $a \in F$ is a nonzero root of $c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 \in F[x]$, show that a^{-1} is a root of $c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$.

13. (a) If $f(x)$ and $g(x)$ are associates in $F[x]$, show that they have the same roots in F.

(b) If $f(x), g(x) \in F[x]$ have the same roots in F, are they associates in $F[x]$?

14. (a) Suppose $r, s \in F$ are roots of $ax^2 + bx + c \in F[x]$ (with $a \neq 0_F$). Use the Factor Theorem to show that $r + s = -a^{-1}b$ and $rs = a^{-1}c$.

(b) Suppose $r, s, t \in F$ are roots of $ax^3 + bx^2 + cx + d \in F[x]$ (with $a \neq 0_F$). Show that $r + s + t = -a^{-1}b$ and $rs + st + rt = a^{-1}c$ and $rst = -a^{-1}d$.

15. Prove that $x^2 + 1$ is reducible in $\mathbb{Z}_p[x]$ if and only if there exist integers a and b such that $p = a + b$ and $ab \equiv 1 \pmod p$.

16. Let $f(x), g(x) \in F[x]$ have degree $\leq n$ and let $c_0, c_1, \dots, c_n$ be distinct elements of F. If $f(c_i) = g(c_i)$ for $i = 0, 1, \dots, n$, prove that $f(x) = g(x)$ in $F[x]$.

17. Find a polynomial of degree 2 in $\mathbb{Z}_6[x]$ that has four roots in $\mathbb{Z}_6$. Does this contradict Corollary 4.17?

18. Let $\varphi:\mathbb{C} \to \mathbb{C}$ be an isomorphism of rings such that $\varphi(a) = a$ for each $a \in \mathbb{Q}$. Suppose $r \in \mathbb{C}$ is a root of $f(x) \in \mathbb{Q}[x]$. Prove that $\varphi(r)$ is also a root of $f(x)$.

19. We say that $a \in F$ is a **multiple root** of $f(x) \in F[x]$ if $(x - a)^k$ is a factor of $f(x)$ for some $k \geq 2$.

(a) Prove that $a \in \mathbb{R}$ is a multiple root of $f(x) \in \mathbb{R}[x]$ if and only if a is a root of both $f(x)$ and $f'(x)$, where $f'(x)$ is the derivative of $f(x)$.

(b) If $f(x) \in \mathbb{R}[x]$ and if $f(x)$ is relatively prime to $f'(x)$, prove that $f(x)$ has no multiple root in $\mathbb{R}$.

20. Let R be an integral domain. Then the Division Algorithm holds in $R[x]$ whenever the divisor is monic, by Exercise 14 in Section 4.1. Use this fact to show that the Remainder and Factor Theorems hold in $R[x]$.

21. If R is an integral domain and $f(x)$ is a nonzero polynomial of degree n in $R[x]$, prove that $f(x)$ has at most n roots in R. [*Hint:* Exercise 20.]

22. Show that Corollary 4.20 holds if F is an infinite integral domain. [*Hint:* See Exercise 21.]

23. Let $f(x), g(x), h(x) \in F[x]$ and $r \in F$.

(a) If $f(x) = g(x) + h(x)$ in $F[x]$, show that $f(r) = g(r) + h(r)$ in F.

(b) If $f(x) = g(x)h(x)$ in $F[x]$, show that $f(r) = g(r)h(r)$ in F.

Where were these facts used in this section?

24. Let a be a fixed element of F and define a map $\varphi_a:F[x] \to F$ by $\varphi_a[f(x)] = f(a)$. Prove that φ_a is a surjective homomorphism of rings. The map φ_a is called an **evaluation homomorphism**; there is one for each $a \in F$.

25. Let $\mathbb{Q}[\pi]$ be the set of all real numbers of the form

$$r_0 + r_1\pi + r_2\pi^2 + \cdots + a_n\pi^n, \quad \text{with } n \geq 0 \text{ and } r_i \in \mathbb{Q}.$$

(a) Show that $\mathbb{Q}[\pi]$ is a subring of $\mathbb{R}$.

(b) Show that the function $\theta:\mathbb{Q}[x] \to \mathbb{Q}[\pi]$ defined by $\theta(f(x)) = f(\pi)$ is an isomorphism. You may assume the following nontrivial fact: π is not the root of any nonzero polynomial with rational coefficients. Therefore, Theorem 4.1 is true with $R = \mathbb{Q}$ and π in place of x. However, see Exercise 26.

26. Let $\mathbb{Q}[\sqrt{2}]$ be the set of all real numbers of the form

$$r_0 + r_1\sqrt{2} + r_2(\sqrt{2})^2 + \cdots + r_n(\sqrt{2})^n, \text{ with } n \geq 0 \text{ and } r_i \in \mathbb{Q}.$$

(a) Show that $\mathbb{Q}[\sqrt{2}]$ is a subring of $\mathbb{R}$.

(b) Show that the function $\theta:\mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$ defined by $\theta(f(x)) = f(\sqrt{2})$ is a surjective homomorphism, but not an isomorphism. Thus Theorem 4.1 is *not* true with $R = \mathbb{Q}$ and $\sqrt{2}$ in place of x. Compare this with Exercise 25.

27. Let T be the set of all polynomial functions from F to F. Show that T is a commutative ring with identity, with operations defined as in calculus: For each $r \in F$,

$$(f + g)(r) = f(r) + g(r) \qquad \text{and} \qquad (fg)(r) = f(r)g(r).$$

[*Hint:* To show that T is closed under addition and multiplication, use Exercise 23 to verify that $f + g$ and fg are the polynomial functions induced by the sum and product polynomials $f(x) + g(x)$ and $f(x)g(x)$, respectively.]

28. Let T be the ring of all polynomial functions from $\mathbb{Z}_3$ to $\mathbb{Z}_3$ (see Exercise 27).

(a) Show that T is a finite ring with zero divisors. [*Hint:* Consider $f(x) = x + 1$ and $g(x) = x^2 + 2x$.]

(b) Show that T cannot possibly be isomorphic to $\mathbb{Z}_3[x]$. Then see Exercise 30.

29. Use mathematical induction to prove Corollary 4.17.

C.30. If F is an infinite field, prove that the polynomial ring $F[x]$ is isomorphic to the ring T of all polynomial functions from F to F (Exercise 27). [*Hint:* Define a map $\varphi:F[x] \to T$ by assigning to each polynomial $f(x) \in F[x]$ its induced function in T; φ is injective by Corollary 4.20.]

31. Let $\varphi:F[x] \to F[x]$ be an isomorphism such that $\varphi(a) = a$ for every $a \in F$. Prove that $f(x)$ is irreducible in $F[x]$ if and only if $\varphi(f(x))$ is.

32. **(a)** Show that the map $\varphi:F[x] \to F[x]$ given by $\varphi(f(x)) = f(x + 1_F)$ is an isomorphism such that $\varphi(a) = a$ for every $a \in F$.

(b) Use Exercise 31 to show that $f(x)$ is irreducible in $F[x]$ if and only if $f(x + 1_F)$ is.

4.5 Irreducibility in $\mathbb{Q}[x]$*

The central theme of this section is that factoring in $\mathbb{Q}[x]$ can be reduced to factoring in $\mathbb{Z}[x]$. Then elementary number theory can be used to check polynomials with integer coefficients for irreducibility. We begin by noting a fact that will be used frequently:

If $f(x) \in \mathbb{Q}[x]$, then $cf(x)$ has integer coefficients for some nonzero integer c.

*This section is used only in Chapters 11, 12, and 15. It may be omitted until then, if desired. Section 4.6 is independent of this section.

For example, consider

$$f(x) = x^5 + \frac{2}{3}x^4 + \frac{3}{4}x^3 - \frac{1}{6}.$$

The least common denominator of the coefficients of $f(x)$ is 12, and $12f(x)$ has integer coefficients:

$$12f(x) = 12\left[x^5 + \frac{2}{3}x^4 + \frac{3}{4}x^3 - \frac{1}{6}\right] = 12x^5 + 8x^4 + 9x^3 - 2.$$

According to the Factor Theorem, finding first-degree factors of a polynomial $g(x) \in \mathbb{Q}[x]$ is equivalent to finding the roots of $g(x)$ in $\mathbb{Q}$. Now, $g(x)$ has the same roots as $cg(x)$ for any nonzero constant c. When c is chosen so that $cg(x)$ has integer coefficients, we can find the roots of $g(x)$ by using

Theorem 4.21 Rational Root Test

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. If $r \neq 0$ and the rational number r/s (in lowest terms) is a root of $f(x)$, then $r \mid a_0$ and $s \mid a_n$.

Proof ▸ First consider the case when $s = 1$, that is, the case when the integer r is a root of $f(x)$, which means that $a_nr^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0$. Hence,

$$\begin{aligned} a_0 &= -a_nr^n - a_{n-1}r^{n-1} - \cdots - a_1r \\ a_0 &= r(-a_nr^{n-1} - a_{n-1}r^{n-2} - \cdots - a_1), \end{aligned}$$

which says that r divides a_0.

In the general case, we use essentially the same strategy. Since r/s is a root of $f(x)$, we have

$$a_n\left(\frac{r^n}{s^n}\right) + a_{n-1}\left(\frac{r^{n-1}}{s^{n-1}}\right) + \cdots + a_1\left(\frac{r}{s}\right) + a_0 = 0.$$

We need an equation involving only integers (as in the case when $s = 1$). So multiply both sides by s^n, rearrange, and factor as before:

$$\begin{aligned} a_nr^n + a_{n-1}sr^{n-1} + \cdots + a_1s^{n-1}r + a_0s^n &= 0 \\ (*) \qquad a_0s^n &= -a_nr^n - a_{n-1}sr^{n-1} - \cdots - a_1s^{n-1}r \\ a_0s^n &= r[-a_nr^{n-1} - a_{n-1}sr^{n-2} - \cdots - a_1s^{n-1}]. \end{aligned}$$

This last equation says that r divides a_0s^n, which is not quite what we want. However, since r/s is in lowest terms, we have $(r, s) = 1$. It follows that $(r, s^n) = 1$ (a prime that divides s^n also divides s, by Corollary 1.6). Since $r \mid a_0s^n$ and $(r, s^n) = 1$, Theorem 1.4 shows that $r \mid a_0$. A similar argument proves that $s \mid a_n$ (just rearrange Equation $(*)$ so that a_nr^n is on one side and everything else is on the other side). ■

EXAMPLE 1

The possible roots in $\mathbb{Q}$ of $f(x) = 2x^4 + x^3 - 21x^2 - 14x + 12$ are of the form r/s, where r is one of $\pm1, \pm2, \pm3, \pm4, \pm6$, or ±12 (the divisors of the constant term, 12) and s is ±1 or ±2 (the divisors of the leading coefficient, 2). Hence, the Rational Root Test reduces the search for roots of $f(x)$ to this finite list of possibilities:

$$1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 12, -12, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}.$$

It is tedious but straightforward to substitute each of these in $f(x)$ to find that -3 and $\frac{1}{2}$ are the only roots of $f(x)$ in $\mathbb{Q}$.* By the Factor Theorem, both $x - (-3) = x + 3$ and $x - \frac{1}{2}$ are factors of $f(x)$. Division shows that

$$f(x) = (x+3)\left(x - \frac{1}{2}\right)(2x^2 - 4x - 8).$$

The quadratic formula shows that the roots of $2x^2 - 4x - 8$ are $1 \pm \sqrt{5}$, neither of which is in $\mathbb{Q}$. Therefore, $2x^2 - 4x - 8$ is irreducible in $\mathbb{Q}[x]$ by Corollary 4.19. Hence, we have factored $f(x)$ as a product of irreducible polynomials in $\mathbb{Q}[x]$.

EXAMPLE 2

The only possible roots of $g(x) = x^3 + 4x^2 + x - 1$ in $\mathbb{Q}$ are 1 and -1 (Why?). Verify that neither 1 nor -1 is a root of $g(x)$. Hence $g(x)$ is irreducible in $\mathbb{Q}[x]$ by Corollary 4.19.

If $f(x) \in \mathbb{Q}[x]$, then $cf(x)$ has integer coefficients for some nonzero integer c. Any factorization of $cf(x)$ in $\mathbb{Z}[x]$ leads to factorization of $f(x)$ in $\mathbb{Q}[x]$. So it appears that tests for irreducibility in $\mathbb{Q}[x]$ can be restricted to polynomials with integer coefficients. However, we must first rule out the possibility that a polynomial with integer coefficients could factor in $\mathbb{Q}[x]$ but not in $\mathbb{Z}[x]$. In order to do this, we need

Lemma 4.22

Let $f(x), g(x), h(x) \in \mathbb{Z}[x]$ with $f(x) = g(x)h(x)$. If p is a prime that divides every coefficient of $f(x)$, then either p divides every coefficient of $g(x)$ or p divides every coefficient of $h(x)$.

*A graphing calculator will reduce the amount of computation significantly. Since the x-intercepts of the graph of $y = f(x)$ are the roots of $f(x)$, you can eliminate any numbers from the list that aren't near an intercept. In this case, the graph indicates that you need only check $-3, \frac{1}{2}$, and $-\frac{3}{2}$.

Proof ▸ Let $f(x) = a_0 + a_1x + \cdots + a_kx^k$, $g(x) = b_0 + b_1x + \cdots + b_mx^m$, and $h(x) = c_0 + c_1x + \cdots + c_nx^n$. We use a proof by contradiction. If the lemma is false, then p does not divide some coefficient of $g(x)$ and some coefficient of $h(x)$. Let b_r be the *first* coefficient of $g(x)$ that is *not* divisible by p, and let c_t be the *first* coefficient of $h(x)$ that is *not* divisible by p. Then $p \mid b_i$ for $i < r$ and $p \mid c_j$ for $j < t$. Consider the coefficient a_{r+t} of $f(x)$. Since $f(x) = g(x)h(x)$,

$$a_{r+t} = b_0c_{r+t} + \cdots + b_{r-1}c_{t+1} + b_rc_t + b_{r+1}c_{t-1} + \cdots + b_{r+t}c_0.$$

Consequently,

$$b_rc_t = a_{r+t} - [b_0c_{r+t} + \cdots + b_{r-1}c_{t+1}] - [b_{r+1}c_{t-1} + \cdots + b_{r+t}c_0].$$

Now, $p \mid a_{r+t}$ by hypothesis. Also, p divides each term in the first pair of brackets because r was chosen so that $p \mid b_i$ for each $i < r$. Similarly, p divides each term in the second pair of brackets because $p \mid c_j$ for each $j < t$. Since p divides every term on the right side, we see that $p \mid b_rc_t$. Therefore, $p \mid b_r$ or $p \mid c_t$ by Theorem 1.5. This contradicts the fact that neither b_r nor c_t is divisible by p. ■

Theorem 4.23

Let $f(x)$ be a polynomial with integer coefficients. Then $f(x)$ factors as a product of polynomials of degrees m and n in $\mathbb{Q}[x]$ if and only if $f(x)$ factors as a product of polynomials of degrees m and n in $\mathbb{Z}[x]$.

Proof ▸ Obviously, if $f(x)$ factors in $\mathbb{Z}[x]$, it factors in $\mathbb{Q}[x]$. Conversely, suppose $f(x) = g(x)h(x)$ in $\mathbb{Q}[x]$. Let c and d be nonzero integers such that $cg(x)$ and $dh(x)$ have integer coefficients. Then $cdf(x) = [cg(x)][dh(x)]$ in $\mathbb{Z}[x]$ with deg $cg(x) =$ deg $g(x)$ and deg $dh(x) =$ deg $h(x)$. Let p be any prime divisor of cd, say $cd = pt$. Then p divides every coefficient of the polynomial $cdf(x)$. By Lemma 4.22, p divides either every coefficient of $cg(x)$ or every coefficient of $dh(x)$, say the former. Then $cg(x) = pk(x)$ with $k(x) \in \mathbb{Z}[x]$ and deg $k(x) =$ deg $g(x)$. Therefore, $ptf(x) = cdf(x) = [cg(x)][dh(x)] = [pk(x)][dh(x)]$. Canceling p on each end, we have $tf(x) = k(x)[dh(x)]$ in $\mathbb{Z}[x]$.

Now repeat the same argument with any prime divisor of t and cancel that prime from both sides of the equation. Continue until every prime factor of cd has been canceled. Then the left side of the equation will be $\pm f(x)$, and the right side will be a product of two polynomials in $\mathbb{Z}[x]$, one with the same degree as $g(x)$ and one with the same degree as $h(x)$. ■

EXAMPLE 3

We claim that $f(x) = x^4 - 5x^2 + 1$ is irreducible in $\mathbb{Q}[x]$. The proof is by contradiction. If $f(x)$ is reducible, it can be factored as the product of two nonconstant polynomials in $\mathbb{Q}[x]$. If either of these factors has degree 1, then $f(x)$ has a root in $\mathbb{Q}$. But the Rational Root Test shows that $f(x)$ has no roots in $\mathbb{Q}$. (The only possibilities are ± 1, and neither is a root.) Thus if $f(x)$ is reducible, the only possible factorization is as a product of two quadratics, by Theorem 4.2. In this case Theorem 4.23 shows that there is such a factorization in $\mathbb{Z}[x]$. Furthermore, there is a factorization as a product of *monic* quadratics in $\mathbb{Z}[x]$ by Exercise 10, say

$$(x^2 + ax + b)(x^2 + cx + d) = x^4 - 5x^2 + 1$$

with $a, b, c, d \in \mathbb{Z}$. Multiplying out the left-hand side, we have

$$\begin{aligned} x^4 + (a + c)x^3 + (ac + b + d)x^2 + (bc + ad)x + bd \\ = x^4 + 0x^3 - 5x^2 + 0x + 1. \end{aligned}$$

Equal polynomials have equal coefficients; hence,

$$a + c = 0 \qquad ac + b + d = -5 \qquad bc + ad = 0 \qquad bd = 1.$$

Since $a + c = 0$, we have $a = -c$, so that

$$-5 = ac + b + d = -c^2 + b + d,$$

or, equivalently,

$$5 = c^2 - b - d.$$

However, $bd = 1$ in $\mathbb{Z}$ implies that $b = d = 1$ or $b = d = -1$, and so there are only these two possibilities:

$$\begin{aligned} 5 &= c^2 - 1 - 1 & \quad\text{or}\quad && 5 &= c^2 + 1 + 1 \\ 7 &= c^2 & && 3 &= c^2. \end{aligned}$$

There is no integer whose square is 3 or 7, and so a factorization of $f(x)$ as a product of quadratics in $\mathbb{Z}[x]$, and, hence in $\mathbb{Q}[x]$, is impossible. Therefore, $f(x)$ is irreducible in $\mathbb{Q}[x]$.

The brute-force methods of the preceding example are less effective for polynomials of high degree because the system of equations that must be solved is complicated and difficult to handle in a systematic way. However, the irreducibility of certain polynomials of high degree is easily established by

Theorem 4.24 Eisenstein's Criterion

Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a nonconstant polynomial with integer coefficients. If there is a prime p such that p divides each of $a_0, a_1, \ldots, a_{n-1}$ but p does not divide a_n and p^2 does not divide a_0, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Proof ▸ The proof is by contradiction. If $f(x)$ is reducible, then by Theorem 4.23 it can be factored in $\mathbb{Z}[x]$, say

$$f(x) = (b_0 + b_1 x + \cdots + b_r x^r)(c_0 + c_1 x + \cdots + c_s x^s),$$

where each $b_i, c_j \in \mathbb{Z}$, $r \geq 1$, and $s \geq 1$. Note that $a_0 = b_0c_0$. By hypothesis, $p \mid a_0$ and, hence, $p \mid b_0$ or $p \mid c_0$ by Theorem 1.5, say $p \mid b_0$. Since p^2 does not divide a_0, we see that c_0 is not divisible by p. We also have $a_n = b_rc_s$. Consequently, p does not divide b_r (otherwise a_n would be divisible by p, contrary to hypothesis). There may be other b_i not divisible by p as well. Let b_k be the first of the b_i not divisible by p; then $0 < k \leq r < n$ and

$$p \mid b_i \text{ for } i < k \qquad \text{and} \qquad p \nmid b_k.$$

By the rules of polynomial multiplication,

$$a_k = b_0c_k + b_1c_{k-1} + \cdots + b_{k-1}c_1 + b_kc_0,$$

so that

$$b_kc_0 = a_k - b_0c_k - b_1c_{k-1} - \cdots - b_{k-1}c_1.$$

Since $p \mid a_k$ and $p \mid b_i$ for $i < k$, we see that p divides every term on the right-hand side of this equation. Hence, $p \mid b_kc_0$. By Theorem 1.5, p must divide b_k or c_0. This contradicts the fact that neither b_k nor c_0 is divisible by p. Therefore, $f(x)$ is irreducible in $\mathbb{Q}[x]$. ∎

EXAMPLE 4

The polynomial $x^{17} + 6x^{13} - 15x^4 + 3x^2 - 9x + 12$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion with $p = 3$.

EXAMPLE 5

The polynomial $x^9 + 5$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion with $p = 5$. Similarly, $x^n + 5$ is irreducible in $\mathbb{Q}[x]$ for each $n \geq 1$. Thus

there are irreducible polynomials of every degree in $\mathbb{Q}[x]$.

Although Eisenstein's Criterion is very efficient, there are many polynomials to which it cannot be applied. In such cases other techniques are necessary. One such method involves reducing a polynomial mod p, in the following sense. Let p be a positive prime. For each integer a, let $[a]$ denote the congruence class of a in $\mathbb{Z}_p$. If $f(x) = a_kx^k + \cdots + a_1x + a_0$ is a polynomial with integer coefficients, let $\bar{f}(x)$ denote the polynomial $[a_k]x^k + \cdots + [a_1]x + [a_0]$ in $\mathbb{Z}_p[x]$. For instance, if $f(x) = 2x^4 - 3x^2 + 5x + 7$ in $\mathbb{Z}[x]$, then in $\mathbb{Z}_3[x]$,

$$\begin{aligned}\bar{f}(x) &= [2]x^4 - [3]x^2 + [5]x + [7] \\ &= [2]x^4 - [0]x^2 + [2]x + [1] = [2]x^4 + [2]x + [1].\end{aligned}$$

Notice that $f(x)$ and $\bar{f}(x)$ have the same degree. This will always be the case when the leading coefficient of $f(x)$ is not divisible by p (so that the leading coefficient of $\bar{f}(x)$ will not be the zero class in $\mathbb{Z}_p$).

Theorem 4.25

Let $f(x) = a_k x^k + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients, and let p be a positive prime that does not divide a_k. If $\overline{f}(x)$ is irreducible in $\mathbb{Z}_p[x]$, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Proof ▸ Suppose, on the contrary, that $f(x)$ is reducible in $\mathbb{Q}[x]$. Then by Theorem 4.23, $f(x) = g(x)h(x)$ with $g(x)$, $h(x)$ nonconstant polynomials in $\mathbb{Z}[x]$. Since p does not divide a_k, the leading coefficient of $f(x)$, it cannot divide the leading coefficients of $g(x)$ or $h(x)$ (whose product is a_k). Consequently, $\deg \overline{g}(x) = \deg g(x)$ and $\deg \overline{h}(x) = \deg h(x)$. In particular, neither $\overline{g}(x)$ nor $\overline{h}(x)$ is a constant polynomial in $\mathbb{Z}_p[x]$.

Verify that $f(x) = g(x)h(x)$ in $\mathbb{Z}[x]$ implies that $\overline{f}(x) = \overline{g}(x)\overline{h}(x)$ in $\mathbb{Z}_p[x]$ (Exercise 20). This contradicts the irreducibility of $\overline{f}(x)$ in $\mathbb{Z}_p[x]$. Therefore, $f(x)$ must be irreducible in $\mathbb{Q}[x]$. ■

The usefulness of Theorem 4.25 depends on this fact: For each nonnegative integer k, there are only finitely many polynomials of degree k in $\mathbb{Z}_p[x]$ (Exercise 17). Therefore, it is always possible, in theory, to determine whether a given polynomial in $\mathbb{Z}_p[x]$ is irreducible by checking the finite number of possible factors. Depending on the size of p and on the degree of $f(x)$, this can often be done in a reasonable amount of time.

EXAMPLE 6

To show that $f(x) = x^5 + 8x^4 + 3x^2 + 4x + 7$ is irreducible in $\mathbb{Q}[x]$, we reduce mod 2. In $\mathbb{Z}_2[x]$, $\overline{f}(x) = x^5 + x^2 + 1$.* It is easy to see that $\overline{f}(x)$ has no roots in $\mathbb{Z}_2$ and hence no first-degree factors in $\mathbb{Z}_2[x]$. The only quadratic polynomials in $\mathbb{Z}_2[x]$ are x^2, $x^2 + x$, $x^2 + 1$, and $x^2 + x + 1$. However, if x^2, $x^2 + x = x(x + 1)$, or $x^2 + 1 = (x + 1)(x + 1)$ were a factor, then $\overline{f}(x)$ would have a first-degree factor, which it doesn't. You can use division to show that the remaining quadratic, $x^2 + x + 1$, is not a factor of $\overline{f}(x)$. Finally, $\overline{f}(x)$ cannot have a factor of degree 3 or 4 (if it did, the other factor would have degree 2 or 1, which is impossible). Therefore, $\overline{f}(x)$ is irreducible in $\mathbb{Z}_2[x]$. Hence, $f(x)$ is irreducible in $\mathbb{Q}[x]$.

CAUTION: If a polynomial in $\mathbb{Z}[x]$ reduces mod p to a polynomial that is *reducible* in $\mathbb{Z}_p[x]$, then no conclusion can be drawn from Theorem 4.25. Unfortunately, there may be many p for which the reduction of $f(x)$ is reducible in $\mathbb{Z}_p[x]$, even when $f(x)$ is actually irreducible in $\mathbb{Q}[x]$. Consequently, it may take more time to apply Theorem 4.25 than is first apparent.

*When no confusion is likely, we omit the brackets for elements of $\mathbb{Z}_2$.

Exercises

A. **1.** Use the Rational Root Test to write each polynomial as a product of irreducible polynomials in $\mathbb{Q}[x]$:

(a) $-x^4 + x^3 + x^2 + x + 2$ (b) $x^5 + 4x^4 + x^3 - x^2$

(c) $3x^5 + 2x^4 - 7x^3 + 2x^2$ (d) $2x^4 - 5x^3 + 3x^2 + 4x - 6$

(e) $2x^4 + 7x^3 + 5x^2 + 7x + 3$ (f) $6x^4 - 31x^3 + 25x^2 + 33x + 7$

2. Show that $\sqrt{p}$ is irrational for every positive prime integer p. [*Hint:* What are the roots of $x^2 - p$? Do you prefer this proof to the one in Exercises 30 and 31 of Section 1.3?]

3. If a monic polynomial with integer coefficients has a root in $\mathbb{Q}$, show that this root must be an integer.

4. Show that each polynomial is irreducible in $\mathbb{Q}[x]$, as in Example 3.

(a) $x^4 + 2x^3 + x + 1$ (b) $x^4 - 2x^2 + 8x + 1$

5. Use Eisenstein's Criterion to show that each polynomial is irreducible in $\mathbb{Q}[x]$:

(a) $x^5 - 4x + 22$ (b) $10 - 15x + 25x^2 - 7x^4$

(c) $5x^{11} - 6x^4 + 12x^3 + 36x - 6$

6. Show that there are infinitely many integers k such that $x^9 + 12x^5 - 21x + k$ is irreducible in $\mathbb{Q}[x]$.

7. Show that each polynomial $f(x)$ is irreducible in $\mathbb{Q}[x]$ by finding a prime p such that $f(x)$ is irreducible in $\mathbb{Z}_p[x]$

(a) $7x^3 + 6x^2 + 4x + 6$ (b) $9x^4 + 4x^3 - 3x + 7$

8. Give an example of a polynomial $f(x) \in \mathbb{Z}[x]$ and a prime p such that $f(x)$ is reducible in $\mathbb{Q}[x]$ but $\bar{f}(x)$ is irreducible in $\mathbb{Z}_p[x]$. Does this contradict Theorem 4.25?

9. Give an example of a polynomial in $\mathbb{Z}[x]$ that is irreducible in $\mathbb{Q}[x]$ but factors when reduced mod 2, 3, 4, and 5.

10. If a monic polynomial with integer coefficients factors in $\mathbb{Z}[x]$ as a product of polynomials of degrees m and n, prove that it can be factored as a product of monic polynomials of degrees m and n in $\mathbb{Z}[x]$.

B. **11.** Prove that $30x^n - 91$ (where $n \in \mathbb{Z}, n > 1$) has no roots in $\mathbb{Q}$.

12. Let F be a field and $f(x) \in F[x]$. If $c \in F$ and $f(x + c)$ is irreducible in $F[x]$, prove that $f(x)$ is irreducible in $F[x]$. [*Hint:* Prove the contrapositive.]

13. Prove that $f(x) = x^4 + 4x + 1$ is irreducible in $\mathbb{Q}[x]$ by using Eisenstein's Criterion to show that $f(x + 1)$ is irreducible and applying Exercise 12.

14. Prove that $f(x) = x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$. [*Hint:* Use the hint for Exercise 21 with $p = 5$.]

15. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients. If p is a prime such that $p \mid a_1, p \mid a_2, \ldots, p \mid a_n$ but $p \nmid a_0$ and

$p^2 \nmid a_n$, prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$. [*Hint:* Let $y = 1/x$ in $f(x)/x^n$; the resulting polynomial is irreducible, by Theorem 4.24.]

16. Show by example that this statement is false: If $f(x) \in \mathbb{Z}[x]$ and there is no prime p satisfying the hypotheses of Theorem 4.24, then $f(x)$ is reducible in $\mathbb{Q}[x]$.

17. Show that there are $n^{k+1} - n^k$ polynomials of degree k in $\mathbb{Z}_n[x]$.

18. Which of these polynomials are irreducible in $\mathbb{Q}[x]$:

(a) $x^4 - x^2 + 1$ **(b)** $x^4 + x + 1$

(c) $x^5 + 4x^4 + 2x^3 + 3x^2 - x + 5$ **(d)** $x^5 + 5x^2 + 4x + 7$

19. Write each polynomial as a product of irreducible polynomials in $\mathbb{Q}[x]$.

(a) $x^5 + 2x^4 - 6x^2 - 16x - 8$ **(b)** $x^7 - 2x^6 - 6x^4 - 15x^2 - 33x - 9$

20. If $f(x) = a_nx^n + \cdots + a_1x + a_0$, $g(x) = b_rx^r + \cdots + b_1x + b_0$, and $h(x) = c_sx^s + \cdots + c_1x + c_0$ are polynomials in $\mathbb{Z}[x]$ such that $f(x) = g(x)h(x)$, show that in $\mathbb{Z}_n[x]$, $\bar{f}(x) = \bar{g}(x)\bar{h}(x)$. Also, see Exercise 19 in Section 4.1.

C.21. Prove that for p prime, $f(x) = x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$. [*Hint:* $(x-1)f(x) = x^p - 1$, so that $f(x) = (x^p - 1)/(x-1)$ and $f(x+1) = [(x+1)^p - 1]/x$. Expand $(x+1)^p$ by the Binomial Theorem (Appendix E) and note that p divides $\binom{p}{k}$ when $k > 0$. Use Eisenstein's Criterion to show that $f(x+1)$ is irreducible; apply Exercise 12.]

EXCURSION: Geometric Constructions (Chapter 15) may be covered at this point if desired.

4.6 Irreducibility in $\mathbb{R}[x]$ and $\mathbb{C}[x]$*

Unlike the situation in $\mathbb{Q}[x]$, it is possible to give an explicit description of all the irreducible polynomials in $\mathbb{R}[x]$ and $\mathbb{C}[x]$. Consequently, you can immediately tell if a polynomial in $\mathbb{R}[x]$ or $\mathbb{C}[x]$ is irreducible without any elaborate tests or criteria. These facts are a consequence of the following theorem, which was first proved by Gauss in 1799:

Theorem 4.26 The Fundamental Theorem of Algebra

Every nonconstant polynomial in $\mathbb{C}[x]$ has a root in $\mathbb{C}$.

This theorem is sometimes expressed in other terminology by saying that the field $\mathbb{C}$ is **algebraically closed.** Every known proof of the theorem depends significantly on facts from analysis and/or the theory of functions of a complex variable. For this reason, we shall consider only some of the implications of the Fundamental Theorem on irreducibility in $\mathbb{C}[x]$ and $\mathbb{R}[x]$. For a proof, see Hungerford [5].

*This section is used only in Chapters 11 and 12. It may be omitted until then, if desired.

Corollary 4.27

A polynomial is irreducible in $\mathbb{C}[x]$ if and only if it has degree 1.

Proof ▸ A polynomial $f(x)$ of degree ≥ 2 in $\mathbb{C}[x]$ has a root in $\mathbb{C}$ by Theorem 4.26 and hence a first-degree factor by the Factor Theorem. Therefore $f(x)$ is reducible in $\mathbb{C}[x]$, and every irreducible polynomial in $\mathbb{C}[x]$ must have degree 1. Conversely, every first-degree polynomial is irreducible (Example 1 in Section 4.3). ■

Corollary 4.28

Every nonconstant polynomial $f(x)$ of degree n in $\mathbb{C}[x]$ can be written in the form $c(x - a_1)(x - a_2) \cdots (x - a_n)$ for some $c, a_1, a_2, \ldots, a_n \in \mathbb{C}$. This factorization is unique except for the order of the factors.

Proof ▸ By Theorem 4.14, $f(x)$ is a product of irreducible polynomials in $\mathbb{C}[x]$. Each of them has degree 1 by Corollary 4.27, and there are exactly n of them by Theorem 4.2. Therefore,

$$\begin{aligned} f(x) &= (r_1x + s_1)(r_2x + s_2) \cdots (r_nx + s_n) \\ &= r_1(x - (-r_1^{-1}s_1))r_2(x - (-r_2^{-1}s_2)) \cdots r_n(x - (-r_n^{-1}s_n)) \\ &= c(x - a_1)(x - a_2) \cdots (x - a_n), \end{aligned}$$

where $c = r_1r_2 \cdots r_n$ and $a_i = r_i^{-1}s_i$. Uniqueness follows from Theorem 4.14; see Exercise 25 in Section 4.3. ■

To obtain a description of all the irreducible polynomials in $\mathbb{R}[x]$, we need

Lemma 4.29

If $f(x)$ is a polynomial in $\mathbb{R}[x]$ and $a + bi$ is a root of $f(x)$ in $\mathbb{C}$, then $a - bi$ is also a root of $f(x)$.

Proof ▸ If $c = a + bi \in \mathbb{C}$ (with $a, b \in \mathbb{R}$), let $\bar{c}$ denote $a - bi$. Verify that for any $c, d \in \mathbb{C}$,

$$\overline{(c + d)} = \bar{c} + \bar{d} \qquad \text{and} \qquad \overline{cd} = \bar{c}\,\bar{d}.$$

Also note that $\bar{c} = c$ *if and only if c is a real number.* Now, if $f(x) = a_nx^n + \cdots + a_1x + a_0$ and c is a root of $f(x)$, then $f(c) = 0$, so that

$$\begin{aligned} 0 = \bar{0} = \overline{f(c)} &= \overline{a_nc^n + \cdots + a_1c + a_0} \\ &= \bar{a}_n\bar{c}^n + \cdots + \bar{a}_1\bar{c} + \bar{a}_0 \\ &= a_n\bar{c}^n + \cdots + a_1\bar{c} + a_0 \quad [\textit{Because each } a_i \in \mathbb{R}.] \\ &= f(\bar{c}). \end{aligned}$$

Therefore $\bar{c} = a - bi$ is also a root of $f(x)$. ■

Theorem 4.30

A polynomial $f(x)$ is irreducible in $\mathbb{R}[x]$ if and only if $f(x)$ is a first-degree polynomial or

$$f(x) = ax^2 + bx + c \qquad \text{with } b^2 - 4ac < 0.$$

Proof ▶ The proof that the two kinds of polynomials mentioned in the theorem are in fact irreducible is left to the reader (Exercise 7). Conversely, suppose $f(x)$ has degree ≥ 2 and is irreducible in $\mathbb{R}[x]$. Then $f(x)$ has a root w in $\mathbb{C}$ by Theorem 4.26. Lemma 4.29 shows that $\overline{w}$ is also a root of $f(x)$. Furthermore, $w \neq \overline{w}$ (otherwise w would be a real root of $f(x)$, contradicting the irreducibility of $f(x)$). Consequently, by the Factor Theorem, $x - w$ and $x - \overline{w}$ are factors of $f(x)$ in $\mathbb{C}[x]$; that is, $f(x) = (x - w)(x - \overline{w})h(x)$ for some $h(x)$ in $\mathbb{C}[x]$. Let $g(x) = (x - w)(x - \overline{w})$; then $f(x) = g(x)h(x)$ in $\mathbb{C}[x]$. Furthermore, if $w = r + si$ (with $r, s \in \mathbb{R}$), then

$$\begin{aligned} g(x) &= (x - w)(x - \overline{w}) = (x - (r + si))(x - (r - si)) \\ &= x^2 - 2rx + (r^2 + s^2). \end{aligned}$$

Hence, the coefficients of $g(x)$ are real numbers.

We now show that $h(x)$ also has real coefficients. The Division Algorithm in $\mathbb{R}[x]$ shows that there are polynomials $q(x)$, $r(x)$ in $\mathbb{R}[x]$ such that $f(x) = g(x)q(x) + r(x)$, with $r(x) = 0$ or $\deg r(x) < \deg g(x)$. In $\mathbb{C}[x]$, however, we have $f(x) = g(x)h(x) + 0$. Since $q(x)$ and $r(x)$ can be considered as polynomials in $\mathbb{C}[x]$, the uniqueness part of the Division Algorithm in $\mathbb{C}[x]$ shows that $q(x) = h(x)$ and $r(x) = 0$. Thus $h(x) = q(x) \in \mathbb{R}[x]$. Since $f(x) = g(x)h(x)$ and $f(x)$ is irreducible in $\mathbb{R}[x]$ and $\deg g(x) = 2$, $h(x)$ must be a constant $d \in \mathbb{R}$. Consequently, $f(x) = dg(x)$ is a quadratic polynomial in $\mathbb{R}[x]$ and hence has the form $ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$. Since $f(x)$ has no roots in $\mathbb{R}$, the quadratic formula (Exercise 6) shows that $b^2 - 4ac < 0$. ■

Corollary 4.31

Every polynomial $f(x)$ of odd degree in $\mathbb{R}[x]$ has a root in $\mathbb{R}$.

Proof ▶ By Theorem 4.14, $f(x) = p_1(x)p_2(x) \cdots p_k(x)$ with each $p_i(x)$ irreducible in $\mathbb{R}[x]$. Each $p_i(x)$ has degree 1 or 2 by Theorem 4.30. Theorem 4.2 shows that

$$\deg f(x) = \deg p_1(x) + \deg p_2(x) + \cdots + \deg p_k(x).$$

Since $f(x)$ has odd degree, at least one of the $p_i(x)$ must have degree 1. Therefore, $f(x)$ has a first-degree factor in $\mathbb{R}[x]$ and, hence, a root in $\mathbb{R}$. ■

It may seem that the Fundamental Theorem and its corollaries settle all the basic questions about polynomial equations. Unfortunately, things aren't quite that simple. None of the known proofs of the Fundamental Theorem provides a constructive way to find the roots of a specific polynomial.* Therefore, even though we know that every polynomial equation has a solution in $\mathbb{C}$, we may not be able to solve a particular equation.

Polynomial equations of degree less than 5 are no problem. The quadratic formula shows that the solutions of any second-degree polynomial equation can be obtained from the coefficients of the polynomials by taking sums, differences, products, quotients, and square roots. There are analogous, but more complicated, formulas involving cube and fourth roots for third- and fourth-degree polynomial equations (see page 423 for one version of the cubic formula). However, there are no such formulas for finding the roots of all fifth-degree or higher-degree polynomials. This remarkable fact, which was proved nearly two centuries ago, is discussed in Section 12.3.

Exercises

A. 1. Find all the roots in $\mathbb{C}$ of each polynomial (one root is already given):

(a) $x^4 - 3x^3 + x^2 + 7x - 30$; root $1 - 2i$

(b) $x^4 - 2x^3 - x^2 + 6x - 6$; root $1 + i$

(c) $x^4 - 4x^3 + 3x^2 + 14x + 26$; root $3 + 2i$

2. Find a polynomial in $\mathbb{R}[x]$ that satisfies the given conditions:

(a) Monic of degree 3 with 2 and $3 + i$ as roots

(b) Monic of least possible degree with $1 - i$ and $2i$ as roots

(c) Monic of least possible degree with 3 and $4i - 1$ as roots

3. Factor each polynomial as a product of irreducible polynomials in $\mathbb{Q}[x]$, in $\mathbb{R}[x]$, and in $\mathbb{C}[x]$:

(a) $x^4 - 2$ (b) $x^3 + 1$ (c) $x^3 - x^2 - 5x + 5$

4. Factor $x^2 + x + 1 + i$ in $\mathbb{C}[x]$.

B. 5. Show that a polynomial of odd degree in $\mathbb{R}[x]$ with no multiple roots must have an odd number of real roots.

*It may seem strange that it is possible to prove that a root exists without actually exhibiting one, but such "existence theorems" are quite common in mathematics. A very rough analogy is the situation that occurs when a person is killed by a sniper's bullet. The police know that there *is* a killer, but actually *finding* the killer may be difficult or impossible.

6. Let $f(x) = ax^2 + bx + c \in \mathbb{R}[x]$ with $a \neq 0$. Prove that the roots of $f(x)$ in $\mathbb{C}$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

[*Hint:* Show that $ax^2 + bx + c = 0$ is equivalent to $x^2 + (b/a)x = -c/a$; then complete the square to find x.]

7. Prove that every $ax^2 + bx + c \in \mathbb{R}[x]$ with $b^2 - 4ac < 0$ is irreducible in $\mathbb{R}[x]$. [*Hint:* See Exercise 6].

8. If $a + bi$ is a root of $x^3 - 3x^2 + 2ix + i - 1 \in \mathbb{C}[x]$, then is it true that $a - bi$ is also a root?

CHAPTER 5

Congruence in $F[x]$ and Congruence-Class Arithmetic

In this chapter we continue to explore the analogy between the ring $\mathbb{Z}$ of integers and the ring $F[x]$ of polynomials with coefficients in a field F. We shall see that the concepts of congruence and congruence-class arithmetic carry over from $\mathbb{Z}$ to $F[x]$ with practically no changes. Because of the additional features of the polynomial ring $F[x]$ (polynomial functions and roots), these new congruence-class rings have a much richer structure than do the rings $\mathbb{Z}_n$. This additional structure leads to a striking result: Given any polynomial over any field, we can find a root of that polynomial in some larger field.

5.1 Congruence in $F[x]$ and Congruence Classes

The concept of congruence of integers depends only on some basic facts about divisibility in $\mathbb{Z}$. If F is a field, then the polynomial ring $F[x]$ has essentially the same divisibility properties as does $\mathbb{Z}$. So it is not surprising that the concept of congruence in $\mathbb{Z}$ and its basic properties (Section 2.1) can be carried over to $F[x]$ almost verbatim.

Definition Let F be a field and $f(x), g(x), p(x) \in F[x]$ with $p(x)$ nonzero. Then **$f(x)$ is congruent to $g(x)$ modulo $p(x)$**—written $f(x) \equiv g(x) \pmod{p(x)}$—provided that $p(x)$ divides $f(x) - g(x)$.

EXAMPLE 1

In $\mathbb{Q}[x]$, $x^2 + x + 1 \equiv x + 2 \pmod{x + 1}$ because

$$(x^2 + x + 1) - (x + 2) = x^2 - 1 = (x + 1)(x - 1).$$

EXAMPLE 2

In $\mathbb{R}[x]$, $3x^4 + 4x^2 + 2x + 2 \equiv x^3 + 3x^2 + 3x + 4 \pmod{x^2 + 1}$ because division shows that

$$\begin{aligned}(3x^4 + 4x^2 + 2x + 2) - (x^3 + 3x^2 + 3x + 4) &= 3x^4 - x^3 + x^2 - x - 2 \\ &= (x^2 + 1)(3x^2 - x - 2).\end{aligned}$$

Theorem 5.1

Let F be a field and $p(x)$ a nonzero polynomial in $F[x]$. Then the relation of congruence modulo $p(x)$ is

(1) reflexive: $f(x) \equiv f(x) \pmod{p(x)}$ for all $f(x) \in F[x]$;

(2) symmetric: if $f(x) \equiv g(x) \pmod{p(x)}$, then $g(x) \equiv f(x) \pmod{p(x)}$;

(3) transitive: if $f(x) \equiv g(x) \pmod{p(x)}$ and $g(x) \equiv h(x) \pmod{p(x)}$, then $f(x) \equiv h(x) \pmod{p(x)}$.

Proof ▸ Adapt the proof of Theorem 2.1 with $p(x), f(x), g(x), h(x)$ in place of n, a, b, c. ■

Theorem 5.2

Let F be a field and $p(x)$ a nonzero polynomial in $F[x]$. If $f(x) \equiv g(x) \pmod{p(x)}$ and $h(x) \equiv k(x) \pmod{p(x)}$, then

(1) $f(x) + h(x) \equiv g(x) + k(x) \pmod{p(x)}$,

(2) $f(x)h(x) \equiv g(x)k(x) \pmod{p(x)}$.

Proof ▸ Adapt the proof of Theorem 2.2 with $p(x), f(x), g(x), h(x), k(x)$ in place of n, a, b, c, d. ■

Definition Let F be a field and $f(x), p(x) \in F[x]$ with $p(x)$ nonzero. The **congruence class** (or **residue class**) of $f(x)$ modulo $p(x)$ is denoted $[f(x)]$ and consists of all polynomials in $F[x]$ that are congruent to $f(x)$ modulo $p(x)$, that is,

$$[f(x)] = \{g(x) \mid g(x) \in F[x] \text{ and } g(x) \equiv f(x) \pmod{p(x)}\}.$$

Since $g(x) \equiv f(x) \pmod{p(x)}$ means that $g(x) - f(x) = k(x)p(x)$ for some $k(x) \in F[x]$ or, equivalently, that $g(x) = f(x) + k(x)p(x)$, we see that

$$\begin{aligned}[f(x)] &= \{g(x) \mid g(x) \equiv f(x) \pmod{p(x)}\} \\ &= \{f(x) + k(x)p(x) \mid k(x) \in F[x]\}.\end{aligned}$$

EXAMPLE 3

Consider congruence modulo $x^2 + 1$ in $\mathbb{R}[x]$. The congruence class of $2x + 1$ is the set

$$\{(2x + 1) + k(x)(x^2 + 1) \mid k(x) \in \mathbb{R}[x]\}.$$

The Division Algorithm shows that the elements of this set are the polynomials in $\mathbb{R}[x]$ that leave remainder $2x + 1$ when divided by $x^2 + 1$.

EXAMPLE 4

Consider congruence modulo $x^2 + x + 1$ in $\mathbb{Z}_2[x]$. To find the congruence class of x^2, we note that $x^2 \equiv x + 1 \pmod{x^2 + x + 1}$ because $x^2 - (x + 1) = x^2 - x - 1 = (x^2 + x + 1)1$ (remember that $1 + 1 = 0$ in $\mathbb{Z}_2$, so that $1 = -1$). Therefore, $x + 1$ is a member of the congruence class $[x^2]$. In fact, the next theorem shows that $[x + 1] = [x^2]$.

Theorem 5.3

$f(x) \equiv g(x) \pmod{p(x)}$ if and only if $[f(x)] = [g(x)]$.

Proof ▸ Adapt the proof of Theorem 2.3 with $f(x)$, $g(x)$, $p(x)$, and Theorem 5.1 in place of a, c, n, and Theorem 2.1. ■

Corollary 5.4

Two congruence classes modulo $p(x)$ are either disjoint or identical.

Proof ▸ Adapt the proof of Corollary 2.4. ■

Under congruence modulo n in $\mathbb{Z}$, there are exactly n distinct congruence classes (Corollary 2.5). These classes are $[0], [1], \ldots, [n - 1]$. Note that there is a class for each possible remainder under division by n. In $F[x]$ the possible remainders under division by a polynomial of degree n are all the polynomials of degree less than n (and, of course, 0). So the analogue of Corollary 2.5 is

Corollary 5.5

Let F be a field and $p(x)$ a polynomial of degree n in $F[x]$, and consider congruence modulo $p(x)$.

(1) If $f(x) \in F[x]$ and $r(x)$ is the remainder when $f(x)$ is divided by $p(x)$, then $[f(x)] = [r(x)]$.

(2) Let S be the set consisting of the zero polynomial and all the polynomials of degree less than n in $F[x]$. Then every congruence class modulo $p(x)$ is the class of some polynomial in S, and the congruence classes of different polynomials in S are distinct.

Proof▸ (1) By the Division Algorithm, $f(x) = p(x)q(x) + r(x)$, with $r(x) = 0_F$ or $\deg r(x) < n$. Thus, $f(x) - r(x) = p(x)q(x)$, so that $f(x) \equiv r(x) \pmod{p(x)}$. By Theorem 5.3, $[f(x)] = [r(x)]$.

(2) Since $r(x) = 0_F$ or $\deg r(x) < n$, we see that $r(x) \in S$. Hence, every congruence class is equal to the congruence class of a polynomial in S. Two different polynomials in S cannot be congruent modulo $p(x)$ because their difference has degree less than n, and hence is not divisible by $p(x)$. Therefore, different polynomials in S must be in distinct congruence classes by Theorem 5.3. ∎

The set of all congruence classes modulo $p(x)$ is denoted

$$F[x]/(p(x)),$$

which is the notational analogue of $\mathbb{Z}_n$.

EXAMPLE 5

Consider congruence modulo $x^2 + 1$ in $\mathbb{R}[x]$. There is a congruence class for each possible remainder on division by $x^2 + 1$. Now, the possible remainders are polynomials of the form $rx + s$ (with $r, s \in \mathbb{R}$; one or both of r, s may possibly be 0). Therefore, $\mathbb{R}[x]/(x^2 + 1)$ consists of infinitely many distinct congruence classes, including

$$[0], [x], [x + 1], [5x + 3], \left[\frac{7}{9}x + 2\right], [x - 7], \ldots.$$

Corollary 5.5 states that $[rx + s] = [cx + d]$ if and only if $rx + s$ is *equal* (not just congruent) to $cx + d$. By the definition of polynomial equality, $rx + s = cx + d$ if and only if $r = c$ and $s = d$. Therefore, every element of $\mathbb{R}[x]/(x^2 + 1)$ can be written *uniquely* in the form $[rx + s]$.

EXAMPLE 6

Consider congruence modulo $x^2 + x + 1$ in $\mathbb{Z}_2[x]$. The possible remainders on division by $x^2 + x + 1$ are the polynomials of the form $ax + b$ with $a, b \in \mathbb{Z}_2$. Thus there are only four possible remainders: 0, 1, x, and $x + 1$. Therefore, $\mathbb{Z}_2[x]/(x^2 + x + 1)$ consists of four congruence classes: [0], [1], [x], and [$x + 1$].

EXAMPLE 7

The pattern in Example 6 works in the general case. Let n be a prime integer, so that $\mathbb{Z}_n$ is a field and the Division Algorithm holds in $\mathbb{Z}_n[x]$. If $p(x) \in \mathbb{Z}_n[x]$ has degree k, then the possible remainders on division by $p(x)$ are of the form

$a_0 + a_1x + \cdots + a_{k-1}x^{k-1}$, with $a_i \in \mathbb{Z}_n$. There are n possibilities for each of the k coefficients $a_0, \ldots, a_{k-1}$, and so there are n^k different polynomials of this form. Consequently, by Corollary 5.5, there are exactly n^k distinct congruence classes modulo $p(x)$ in $\mathbb{Z}_n[x]/(p(x))$.

Exercises

NOTE: *F denotes a field and p(x) a nonzero polynomial in F[x].*

A. 1. Let $f(x), g(x), p(x) \in F[x]$, with $p(x)$ nonzero. Determine whether $f(x) \equiv g(x) \pmod{p(x)}$. Show your work.

(a) $f(x) = x^5 - 2x^4 + 4x^3 + x + 1$; $g(x) = 3x^4 + 2x^3 - 5x^2 - 9$; $p(x) = x^2 + 1$; $F = \mathbb{Q}$

(b) $f(x) = x^4 + x^2 + x + 1$; $g(x) = x^4 + x^3 + x^2 + 1$; $p(x) = x^2 + x$; $F = \mathbb{Z}_2$

(c) $f(x) = 3x^5 + 4x^4 + 5x^3 - 6x^2 + 5x - 7$; $g(x) = 2x^5 + 6x^4 + x^3 + 2x^2 + 2x - 5$; $p(x) = x^3 - x^2 + x - 1$; $F = \mathbb{R}$

2. If $p(x)$ is a nonzero constant polynomial in $F[x]$, show that any two polynomials in $F[x]$ are congruent modulo $p(x)$.

3. How many distinct congruence classes are there modulo $x^3 + x + 1$ in $\mathbb{Z}_2[x]$? List them.

4. Show that, under congruence modulo $x^3 + 2x + 1$ in $\mathbb{Z}_3[x]$, there are exactly 27 distinct congruence classes.

5. Show that there are infinitely many distinct congruence classes modulo $x^2 - 2$ in $\mathbb{Q}[x]$. Describe them.

6. Let $a \in F$. Describe the congruence classes in $F[x]$ modulo the polynomial $x - a$.

7. Describe the congruence classes in $F[x]$ modulo the polynomial x.

B. 8. Prove or disprove: If $p(x)$ is relatively prime to $k(x)$ and $f(x)k(x) \equiv g(x)k(x) \pmod{p(x)}$, then $f(x) \equiv g(x) \pmod{p(x)}$.

9. Prove that $f(x) \equiv g(x) \pmod{p(x)}$ if and only if $f(x)$ and $g(x)$ leave the same remainder when divided by $p(x)$.

10. Prove or disprove: If $p(x)$ is irreducible in $F[x]$ and $f(x)g(x) \equiv 0_F \pmod{p(x)}$, then $f(x) \equiv 0_F \pmod{p(x)}$ or $g(x) \equiv 0_F \pmod{p(x)}$.

11. If $p(x)$ is reducible in $F[x]$, prove that there exist $f(x), g(x) \in F[x]$ such that $f(x) \not\equiv 0_F \pmod{p(x)}$ and $g(x) \not\equiv 0_F \pmod{p(x)}$ but $f(x)g(x) \equiv 0_F \pmod{p(x)}$.

12. If $f(x)$ is relatively prime to $p(x)$, prove that there is a polynomial $g(x) \in F[x]$ such that $f(x)g(x) \equiv 1_F \pmod{p(x)}$.

13. Suppose $f(x), g(x) \in \mathbb{R}[x]$ and $f(x) \equiv g(x) \pmod{x}$. What can be said about the graphs of $y = f(x)$ and $y = g(x)$?

5.2 Congruence-Class Arithmetic

Congruence in the integers led to the rings $\mathbb{Z}_n$. Similarly, congruence in $F[x]$ also produces new rings and fields. These turn out to be much richer in structure than the rings $\mathbb{Z}_n$. The development here closely parallels Section 2.2.

Theorem 5.6

Let F be a field and $p(x)$ a nonconstant polynomial in $F[x]$. If $[f(x)] = [g(x)]$ and $[h(x)] = [k(x)]$ in $F[x]/(p(x))$, then,

$$[f(x) + h(x)] = [g(x) + k(x)] \qquad \text{and} \qquad [f(x)h(x)] = [g(x)k(x)].$$

Proof ▸ Copy the proof of Theorem 2.6, with Theorems 5.2 and 5.3 in place of Theorems 2.2 and 2.3. ■

Because of Theorem 5.6 we can now define addition and multiplication of congruence classes just as we did in the integers and be certain that these operations are independent of the choice of representatives in each congruence class.

Definition

Let F be a field and $p(x)$ a nonconstant polynomial in $F[x]$. Addition and multiplication in $F[x]/(p(x))$ are defined by

$$[f(x)] + [g(x)] = [f(x) + g(x)],$$
$$[f(x)][g(x)] = [f(x)g(x)].$$

EXAMPLE 1

Consider congruence modulo $x^2 + 1$ in $\mathbb{R}[x]$. The sum of the classes $[2x + 1]$ and $[3x + 5]$ is the class

$$[(2x + 1) + (3x + 5)] = [5x + 6].$$

The product is

$$[2x + 1][3x + 5] = [(2x + 1)(3x + 5)] = [6x^2 + 13x + 5].$$

As noted in Example 5 of Section 5.1, every congruence class in $\mathbb{R}[x]/(x^2 + 1)$ can be written in the form $[ax + b]$. To express the class $[6x^2 + 13x + 5]$ in this form, we divide $6x^2 + 13x + 5$ by $x^2 + 1$ and find that

$$6x^2 + 13x + 5 = 6(x^2 + 1) + (13x - 1).$$

It follows that $6x^2 + 13x + 5 \equiv 13x - 1 \pmod{x^2 + 1}$, and hence $[6x^2 + 13x + 5] = [13x - 1]$.

EXAMPLE 2

In Example 6 of Section 5.1, we saw that $\mathbb{Z}_2[x]/(x^2 + x + 1)$ consists of four classes: [0], [1], [x], and [$x + 1$]. Using the definition of addition of classes, we see that $[x + 1] + [1] = [x + 1 + 1] = [x]$ (remember that $1 + 1 = 0$ in $\mathbb{Z}_2$). Similar calculations produce the following addition table for $\mathbb{Z}_2[x]/(x^2 + x + 1)$:

+	[0]	[1]	[x]	[$x + 1$]
[0]	[0]	[1]	[x]	[$x + 1$]
[1]	[1]	[0]	[$x + 1$]	[x]
[x]	[x]	[$x + 1$]	[0]	[1]
[$x + 1$]	[$x + 1$]	[x]	[1]	[0]

Most of the multiplication table for $\mathbb{Z}_2[x]/(x^2 + x + 1)$ is easily obtained from the definition:

·	[0]	[1]	[x]	[$x + 1$]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[x]	[$x + 1$]
[x]	[0]	[x]		
[$x + 1$]	[0]	[$x + 1$]		

To fill in the rest of the table, note, for example, that

$$[x] \cdot [x + 1] = [x(x + 1)] = [x^2 + x].$$

Now division or simple addition in $\mathbb{Z}_2[x]$ shows that $x^2 + x = (x^2 + x + 1) + 1$. Therefore, $x^2 + x \equiv 1 \pmod{x^2 + x + 1}$, so that $[x^2 + x] = [1]$. A similar calculation shows that $[x] \cdot [x] = [x^2] = [x + 1]$ (because $x^2 = (x^2 + x + 1) + (x + 1)$ in $\mathbb{Z}_2[x]$). Verify that $[x + 1][x + 1] = [x]$.

If you examine the tables in the preceding example, you will see that $\mathbb{Z}_2[x]/(x^2 + x + 1)$ is a commutative ring with identity (in fact, a field). In view of our experience with $\mathbb{Z}$ and $\mathbb{Z}_n$, this is not too surprising. What is unexpected is the upper left-hand corners of the two tables (the sums and products of [0] and [1]). It is easy to see that the subset $F^* = \{[0], [1]\}$ is actually a *subring* of $\mathbb{Z}_2[x]/(x^2 + x + 1)$ and that F^* is isomorphic to $\mathbb{Z}_2$ (the tables for the two systems are identical except for the brackets in F^*). These facts illustrate the next theorem.

Theorem 5.7

Let F be a field and $p(x)$ a nonconstant polynomial in $F[x]$. Then the set $F[x]/(p(x))$ of congruence classes modulo $p(x)$ is a commutative ring with identity. Furthermore, $F[x]/(p(x))$ contains a subring F^* that is isomorphic to F.

Proof ▸ To prove that $F[x]/(p(x))$ is a commutative ring with identity, adapt the proof of Theorem 2.7 to the present case. Let F^* be the subset of $F[x]/(p(x))$ consisting of the congruence classes of all the constant polynomials; that is, $F^* = \{[a] \mid a \in F\}$. Verify that F^* is a subring of $F[x]/(p(x))$ (Exercise 10). Define a map $\varphi: F \to F^*$ by $\varphi(a) = [a]$. This definition shows that φ is surjective. The definitions of addition and multiplication in $F[x]/(p(x))$ show that

$$\varphi(a+b) = [a+b] = [a] + [b] = \varphi(a) + \varphi(b) \qquad \text{and}$$
$$\varphi(ab) = [ab] = [a] \cdot [b] = \varphi(a) \cdot \varphi(b).$$

Therefore, φ is a homomorphism.

To see that φ is injective, suppose $\varphi(a) = \varphi(b)$. Then $[a] = [b]$, so that $a \equiv b \pmod{p(x)}$. Hence, $p(x)$ divides $a - b$. However, $p(x)$ has degree ≥ 1, and $a - b \in F$. This is impossible unless $a - b = 0$. Therefore, $a = b$ and φ is injective. Thus $\varphi: F \to F^*$ is an isomorphism. ■

We began with a field F and a polynomial $p(x)$ in $F[x]$. We have now constructed a ring $F[x]/(p(x))$ that contains an isomorphic *copy* of F. What we would really like is a ring that contains the field F *itself*. There are two possible ways to accomplish this, as illustrated in the following example.

EXAMPLE 3

In Example 2, we used the polynomial $x^2 + x + 1$ in $\mathbb{Z}_2[x]$ to construct the ring $\mathbb{Z}_2[x]/(x^2 + x + 1)$, which contains a subset $F^* = \{[0], [1]\}$ that is isomorphic to $\mathbb{Z}_2$. Suppose we *identify* $\mathbb{Z}_2$ with its isomorphic copy F^* inside $\mathbb{Z}_2[x]/(x^2 + x + 1)$ and write the elements of F^* as if they were in $\mathbb{Z}_2$. Then the tables in Example 2 become

$+$	0	1	$[x]$	$[x+1]$
0	0	1	$[x]$	$[x+1]$
1	1	0	$[x+1]$	$[x]$
$[x]$	$[x]$	$[x+1]$	0	1
$[x+1]$	$[x+1]$	$[x]$	1	0

$\cdot$	0	1	$[x]$	$[x+1]$
0	0	0	0	0
1	0	1	$[x]$	$[x+1]$
$[x]$	0	$[x]$	$[x+1]$	1
$[x+1]$	0	$[x+1]$	1	$[x]$

We now have a ring that has $\mathbb{Z}_2$ as a subset. If this procedure makes you a bit uneasy (is $\mathbb{Z}_2$ really a subset?), you can use the following alternate route to the same end. Let E be any four-element set that actually contains $\mathbb{Z}_2$ as a subset, say $E = \{0, 1, r, s\}$. Define addition and multiplication in E by

$+$	0	1	r	s
0	0	1	r	s
1	1	0	s	r
r	r	s	0	1
s	s	r	1	0

$\cdot$	0	1	r	s
0	0	0	0	0
1	0	1	r	s
r	0	r	s	1
s	0	s	1	r

A comparison of the tables for $\mathbb{Z}_2[x]/(x^2 + x + 1)$ and those for E shows that these two rings are isomorphic (replacing $[x]$ by r and $[x + 1]$ by s changes one set of tables into the other). Therefore, E is essentially the same ring we obtained before. However, E does contain $\mathbb{Z}_2$ as an honest-to-goodness subset, without any identification.

What was done in the preceding example can be done in the general case. Given a field F and a polynomial $p(x)$ in $F[x]$, we can construct a ring that contains F as a subset. The customary way to do this is to identify F with its isomorphic copy F^* inside $F[x]/(p(x))$ and to consider F to be a subset of $F[x]/(p(x))$. If doing this makes you uncomfortable, keep in mind that you can always build a ring isomorphic to $F[x]/(p(x))$ that genuinely contains F as a subset, as in the preceding example. Because this latter approach tends to get cumbersome, we shall follow the usual custom and identify F with F^* hereafter. Consequently, when $a, b \in F$, we shall write $b[x]$ instead of $[b][x]$ and $a + b[x]$ instead of $[a] + [b][x] = [a + bx]$. Then Theorem 5.7 can be reworded:

Theorem 5.8

Let F be a field and $p(x)$ a nonconstant polynomial in $F[x]$. Then $F[x]/(p(x))$ is a commutative ring with identity that contains F.

If a and n are integers such that $(a, n) = 1$, then by Theorem 2.10, $[a]$ is a unit in $\mathbb{Z}_n$. Here is the analogue for polynomials. ■

Theorem 5.9

Let F be a field and $p(x)$ a nonconstant polynomial in $F[x]$. If $f(x) \in F[x]$ and $f(x)$ is relatively prime to $p(x)$, then $[f(x)]$ is a unit in $F[x]/(p(x))$.

Proof ▸ By Theorem 4.8 there are polynomials $u(x)$ and $v(x)$ such that $f(x)u(x) + p(x)v(x) = 1$. Hence, $f(x)u(x) - 1 = -p(x)v(x) = p(x)(-v(x))$, which implies that $[f(x)u(x)] = [1]$ by Theorem 5.3. Therefore, $[f(x)][u(x)] = [f(x)u(x)] = [1]$, so that $[f(x)]$ is a unit in $F[x]/(p(x))$. ■

EXAMPLE 4

Since $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$, $2x + 5$ and $x^2 - 2$ are relatively prime in $\mathbb{Q}[x]$. (Why?) Hence, $[2x + 5]$ is a unit in the ring $\mathbb{Q}[x]/(x^2 - 2)$. The proof of Theorem 5.9 shows that its inverse is $[u(x)]$, where $(2x + 5)u(x) + (x^2 - 2)v(x) = 1$. Using the Euclidean Algorithm as in Exercise 15 of Section 1.2, we find that

$$(2x + 5)\left(-\frac{2}{17}x + \frac{5}{17}\right) + (x^2 - 2)\left(\frac{4}{17}\right) = 1.$$

Therefore, $\left[-\frac{2}{17}x + \frac{5}{17}\right]$ is the inverse of $[2x + 5]$ in $\mathbb{Q}[x]/(x^2 - 2)$.

Exercises

A. *In Exercises 1–4, write out the addition and multiplication tables for the congruence-class ring $F[x]/(p(x))$. In each case, is $F[x]/(p(x))$ a field?*

1. $F = \mathbb{Z}_2$; $p(x) = x^3 + x + 1$

2. $F = \mathbb{Z}_3$; $p(x) = x^2 + 1$

3. $F = \mathbb{Z}_2$; $p(x) = x^2 + 1$

4. $F = \mathbb{Z}_5$; $p(x) = x^2 + 1$

B. *In Exercises 5–8, each element of the given congruence-class ring can be written in the form $[ax + b]$ (Why?). Determine the rules for addition and multiplication of congruence classes. (In other words, if the product $[ax + b][cx + d]$ is the class $[rx + s]$, describe how to find r and s from a, b, c, d, and similarly for addition.)*

5. $\mathbb{R}[x]/(x^2 + 1)$ [*Hint:* See Example 1.]

6. $\mathbb{Q}[x]/(x^2 - 2)$

7. $\mathbb{Q}[x]/(x^2 - 3)$

8. $\mathbb{Q}[x]/(x^2)$

9. Show that $\mathbb{R}[x]/(x^2 + 1)$ is a field by verifying that every nonzero congruence class $[ax + b]$ is a unit. [*Hint:* Show that the inverse of $[ax + b]$ is $[cx + d]$, where $c = -a/(a^2 + b^2)$ and $d = b/(a^2 + b^2)$.]

10. Let F be a field and $p(x) \in F[x]$. Prove that $F^* = \{[a] \mid a \in F\}$ is a subring of $F[x]/(p(x))$.

11. Show that the ring in Exercise 8 is *not* a field.

12. Write out a complete proof of Theorem 5.6 (that is, carry over to $F[x]$ the proof of the analogous facts for $\mathbb{Z}$).

13. Prove the first statement of Theorem 5.7.

14. In each part explain why $[f(x)]$ is a unit in $F[x]/(p(x))$ and find its inverse. [*Hint:* To find the inverse, let $u(x)$ and $v(x)$ be as in the proof of Theorem 5.9. You may assume that $u(x) = ax + b$ and $v(x) = cx + d$. Expanding $f(x)u(x) + p(x)v(x)$ leads to a system of linear equations in a, b, c, d. Solve it.]

(a) $[f(x)] = [2x - 3] \in \mathbb{Q}[x]/(x^2 - 2)$

(b) $[f(x)] = [x^2 + x + 1] \in \mathbb{Z}_3[x]/(x^2 + 1)$

C. 15. Find a fourth-degree polynomial in $\mathbb{Z}_2[x]$ whose roots are the four elements of the field $\mathbb{Z}_2[x]/(x^2 + x + 1)$, whose tables are given in Example 3. [*Hint:* The Factor Theorem may be helpful.]

16. Show that $\mathbb{Q}[x]/(x^2 - 2)$ is a field.

5.3 The Structure of $F[x]/(p(x))$ When $p(x)$ Is Irreducible

When p is a prime integer, then Theorem 2.8 states, in effect, that $\mathbb{Z}_p$ is a field (and, of course, an integral domain). Here is the analogous result for $F[x]$ and an irreducible polynomial $p(x)$.

Theorem 5.10

Let F be a field and $p(x)$ a nonconstant polynomial in $F[x]$. Then the following statements are equivalent:

(1) $p(x)$ is irreducible in $F[x]$.

(2) $F[x]/(p(x))$ is a field.

(3) $F[x]/(p(x))$ is an integral domain.

Theorem 5.10 and most of its proof are a copy of Theorem 2.8 and its proof, with $\mathbb{Z}$ replaced by $F[x]$ and $\mathbb{Z}_p$ by $F(x)/(p(x))$, and the necessary adjustments made for the differences between prime integers and irreducible polynomials.

Proof of Theorem 5.10 ▸ (1) $\Rightarrow$ (2) By Theorem 5.7, $F(x)/(p(x))$ is a commutative ring with identity, and thus satisfies Axioms 1–10. To prove that $F(x)/(p(x))$ is a field, we must verify that every nonzero element in $F(x)/(p(x))$ is a unit (Axiom 12, page 49). Suppose that $[a(x)] \neq [0]$ in $F(x)/(p(x))$. We must find $[u(x)]$ such that $[a(x)]\,[u(x)] = [1_F]$. Since $[a(x)] \neq [0]$, we know that $a(x) \not\equiv 0 \pmod{p(x)}$ by Theorem 5.3. Hence, $p(x) \nmid a(x)$ by the definition of congruence. Now the gcd of $a(x)$ and $p(x)$ is a monic polynomial that divides both $a(x)$ and $p(x)$. Since $p(x)$ is irreducible, the gcd is either 1_F or a monic associate of $p(x)$ (the only monic divisors of $p(x)$). As explained on page 100, an associate of $p(x)$ is a polynomial of the form $cp(x)$, with $0_F \neq c \in F$. Consequently, $a(x)$ is not divisible by any associate of $p(x)$ (because $a(x)$ is not divisible by $p(x)$). Since the gcd also divides $a(x)$ and $p(x) \nmid a(x)$, the gcd of $a(x)$ and $p(x)$ must be 1_F. By Theorem 4.8, there are polynomials $u(x)$ and $v(x)$ so that $a(x)u(x) + p(x)v(x) = 1_F$. Hence, $a(x)u(x) - 1_F = p(x)(-v(x))$, so that $a(x)u(x) \equiv 1_F \pmod{p(x)}$. Therefore, $[a(x)u(x)] = [1_F]$ in $F(x)/(p(x))$ by Theorem 5.3. Thus, $[a(x)][u(x)] = [a(x)u(x)] = [1_F]$, so that $[a(x)]$ is a unit. Hence, $F(x)/(p(x))$ satisfies Axiom 12 and $F(x)/(p(x))$ is a field.

(2) $\Rightarrow$ (3) This is an immediate consequence of Theorem 3.8.

(3) $\Rightarrow$ (1) We shall verify statement (2) of Theorem 4.12 to show that $p(x)$ is irreducible. Suppose that $b(x)$ and $c(x)$ are any polynomials in $F[x]$ and $p(x) \mid b(x)c(x)$. Then $b(x)c(x) \equiv 0_F \pmod{p(x)}$. So by Theorem 5.3,

$$[b(x)][c(x)] = [b(x)c(x)] = [0_F] \text{ in } F(x)/(p(x)).$$

Because $F(x)/(p(x))$ is an integral domain by (3), we have $[a(x)] = [0_F]$ or $[b(x)] = [0_F]$. Thus, $b(x) \equiv 0_F \pmod{p(x)}$ or $c(x) \equiv 0_F \pmod{p(x)}$ by Theorem 5.3, which means that $p(x) \mid b(x)$ or $p(x) \mid c(x)$ by the definition of congruence. Therefore, $p(x)$ is irreducible by Theorem 4.12. ■

Theorem 5.10 can be used to construct finite fields. If p is prime and $f(x)$ is irreducible in $\mathbb{Z}_p[x]$ of degree k, then $\mathbb{Z}_p[x]/(f(x))$ is a field by Theorem 5.10. Example 7 in Section 5.1 shows that this field has p^k elements. Finite fields are discussed further in Section 11.6, where it is shown that there are irreducible polynomials of every positive degree in $\mathbb{Z}_p[x]$ and, hence, finite fields of all possible prime power orders. See Exercise 9 for an example.

Let F be a field and $p(x)$ an irreducible polynomial in $F[x]$. Let K denote the field of congruence classes $F[x]/(p(x))$. By Theorems 5.8 and 5.10, F is a subfield of the field K. One also says that K is an **extension field** of F. Polynomials in $F[x]$ can be considered to have coefficients in the larger field K, and we can ask about the roots of such polynomials in K. In particular, what can be said about the roots of the polynomial $p(x)$ that we started with? Even though $p(x)$ is irreducible in $F[x]$, it may have roots in the extension field K.

EXAMPLE 1

The polynomial $p(x) = x^2 + x + 1$ has no roots in $\mathbb{Z}_2$ and is, therefore, irreducible in $\mathbb{Z}_2[x]$ by Corollary 4.19. Consequently, $K = \mathbb{Z}_2[x]/(x^2 + x + 1)$ is an extension field of $\mathbb{Z}_2$ by Theorem 5.10. Using the tables for K in Example 3 of Section 5.2, we see that

$$[x]^2 + [x] + 1 = [x + 1] + [x] + 1 = 1 + 1 = 0.$$

This result may be a little easier to absorb if we use a different notation. Let $\alpha = [x]$. Then the calculation above says that $\alpha^2 + \alpha + 1 = 0$; that is, α is a root in K of $p(x) = x^2 + x + 1$. It's important to note here that you don't really need the tables for K to prove that α is a root of $p(x)$ because we know that $x^2 + x + 1 \equiv 0 \pmod{x^2 + x + 1}$. Consequently, $[x^2 + x + 1] = 0$ in K, and by the definition of congruence-class arithmetic,

$$\alpha^2 + \alpha + 1 = [x]^2 + [x] + 1 = [x^2 + x + 1] = 0.$$

For the general case we have

Theorem 5.11

Let F be a field and $p(x)$ an irreducible polynomial in $F[x]$. Then $F[x]/(p(x))$ is an extension field of F that contains a root of $p(x)$.

Proof ▸ Let $K = F[x]/(p(x))$. Then K is an extension field of F by Theorems 5.8 and 5.10. Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$, where each a_i is in F and, hence, in K. Let $\alpha = [x]$ in K. We shall show that α is a root of $p(x)$. By the definition of congruence-class arithmetic in K,

$$\begin{aligned} a_n \alpha^n + \cdots + a_1\alpha + a_0 &= a_n[x]^n + \cdots + a_1[x] + a_0 \\ &= [a_n x^n + \cdots + a_1 x + a_0] \\ &= [p(x)] = 0_F \quad [\textit{Because } p(x) \equiv 0_F \ (\textit{mod } p(x)).]. \end{aligned}$$

Therefore, $\alpha \in K$ is a root of $p(x)$. ■

Corollary 5.12

Let F be a field and $f(x)$ a nonconstant polynomial in $F[x]$. Then there is an extension field K of F that contains a root of $f(x)$.

Proof ▸ By Theorem 4.14, $f(x)$ has an irreducible factor $p(x)$ in $F[x]$. By Theorem 5.11, $K = F[x]/(p(x))$ is an extension field of F that contains a root of $p(x)$. Since every root of $p(x)$ is a root of $f(x)$, K contains a root of $f(x)$. ■

The implications of Theorem 5.11 run much deeper than might first appear. Throughout the history of mathematics, the passage from a known number system to a new, larger system has often been greeted with doubt and distrust. In the Middle Ages, some mathematicians refused to acknowledge the existence of negative numbers. When complex numbers were introduced in the seventeenth century, there was uneasiness—which extended for nearly a century—because some mathematicians would not accept the idea that there could be a number whose square is -1, that is, a root of $x^2 + 1$. One cause for these difficulties was the lack of a suitable framework in which to view the situation. Abstract algebra provides such a framework. Theorem 5.11 and its corollary, then, take care of the doubt and uncertainty.

It is instructive to consider the complex numbers from this point of view. Instead of asking about a *number* whose square is -1, we ask, "Is there a field containing $\mathbb{R}$ in which the polynomial $x^2 + 1$ has a root?" Since $x^2 + 1$ is irreducible in $\mathbb{R}[x]$, Theorem 5.11 tells us that the answer is yes: $K = \mathbb{R}[x]/(x^2 + 1)$ is an extension field of $\mathbb{R}$ that contains a root of $x^2 + 1$, namely $\alpha = [x]$. In the field K, α is an element whose square is -1. But how is the field K related to the field of complex numbers introduced earlier in the book?

As is noted in Example 5 of Section 5.1, every element of $K = \mathbb{R}[x]/(x^2 + 1)$ can be written uniquely in the form $[ax + b]$ with $a, b \in \mathbb{R}$. Since we are identifying each element $r \in \mathbb{R}$ with the element $[r]$ in K, we see that every element of K can be written uniquely in the form

$$[a + bx] = [a] + [b][x] = a + b\alpha.$$

Addition in K is given by the rule

$$\begin{aligned} (a + b\alpha) + (c + d\alpha) &= [a + bx] + [c + dx] = [(a + bx) + (c + dx)] \\ &= [(a + c) + (b + d)x] = [a + c] + [b + d][x]. \end{aligned}$$

so that

$$(a + b\alpha) + (c + d\alpha) = (a + c) + (b + d)\alpha.$$

Multiplication in K is given by the rule

$$\begin{aligned}(a + b\alpha)(c + d\alpha) &= [a + bx][c + dx] = [(a + bx)(c + dx)] \\ &= [ac + (ad + bc)x + bdx^2] \\ &= ac + (ad + bc)\alpha + bd\alpha^2.\end{aligned}$$

However, α is a root of $x^2 + 1$, and so $\alpha^2 = -1$. Therefore, the rule for multiplication in K becomes

$$(a + b\alpha)(c + d\alpha) = (ac - bd) + (ad + bc)\alpha.$$

If the symbol α is replaced by the symbol i, then these rules become the usual rules for adding and multiplying complex numbers. In formal language, the field K is isomorphic to the field $\mathbb{C}$, with the isomorphism f being given by $f(a + b\alpha) = a + bi$.

Up to now we have taken the position that the field $\mathbb{C}$ of complex numbers was already known. The field K constructed above then turns out to be isomorphic to the known field $\mathbb{C}$. A good case can be made, however, for not assuming any previous knowledge of the complex numbers and using the preceding example as a *definition* instead. In other words, we can define $\mathbb{C}$ to be the field $\mathbb{R}[x]/(x^2 + 1)$. Such a definition is obviously too sophisticated to use on high-school students, but for mature students it has the definite advantage of removing any lingering doubts about the validity of the complex numbers and their arithmetic.* Had this definition been available several centuries ago, the introduction of the complex numbers might have caused no stir whatsoever.

Exercises

NOTE: *F always denotes a field.*

A. 1. Determine whether the given congruence-class ring is a field. Justify your answer.

(a) $\mathbb{Z}_3[x]/(x^3 + 2x^2 + x + 1)$

(b) $\mathbb{Z}_5[x]/(2x^3 - 4x^2 + 2x + 1)$

(c) $\mathbb{Z}_2[x]/(x^4 + x^2 + 1)$

B. 2. (a) Verify that $\mathbb{Q}(\sqrt{2}) = \{r + s\sqrt{2} \mid r, s \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$.

(b) Show that $\mathbb{Q}(\sqrt{2})$ is isomorphic to $\mathbb{Q}[x]/(x^2 - 2)$. [*Hint*: Exercise 6 in Section 5.2 may be helpful.]

*Only a minor rearrangement of this book is needed to accommodate such a definition. A few examples in Chapter 3 would have to be omitted, and the discussion of irreducibility in $\mathbb{C}[x]$ and $\mathbb{R}[x]$ (Section 4.6) would have to be postponed. All the intervening material in Chapter 5 is independent of any formal knowledge of the complex numbers.

3. If $a \in F$, describe the field $F[x]/(x - a)$.

4. Let $p(x)$ be irreducible in $F[x]$. Without using Theorem 5.10, prove that if $[f(x)][g(x)] = [0_F]$ in $F[x]/(p(x))$, then $[f(x)] = [0_F]$ or $[g(x)] = [0_F]$. [*Hint:* Exercise 10 in Section 5.1.]

5. (a) Verify that $\mathbb{Q}(\sqrt{3}) = \{r + s\sqrt{3} \mid r, s \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$.

   (b) Show that $\mathbb{Q}(\sqrt{3})$ is isomorphic to $\mathbb{Q}[x]/(x^2 - 3)$.

6. Let $p(x)$ be irreducible in $F[x]$. If $[f(x)] \neq [0_F]$ in $F[x]/(p(x))$ and $h(x) \in F[x]$, prove that there exists $g(x) \in F[x]$ such that $[f(x)][g(x)] = [h(x)]$ in $F[x]/(p(x))$. [*Hint:* Theorem 5.10 and Exercise 12(b) in Section 3.2.]

7. If $f(x) \in F[x]$ has degree n, prove that there exists an extension field E of F such that $f(x) = c_0(x - c_1)(x - c_2) \cdots (x - c_n)$ for some (not necessarily distinct) $c_i \in E$. In other words, E contains all the roots of $f(x)$.

8. If $p(x)$ is an irreducible quadratic polynomial in $F[x]$, show that $F[x]/(p(x))$ contains all the roots of $p(x)$.

9. (a) Show that $\mathbb{Z}_2[x]/(x^3 + x + 1)$ is a field.

   (b) Show that the field $\mathbb{Z}_2[x]/(x^3 + x + 1)$ contains all three roots of $x^3 + x + 1$.

10. Show that $\mathbb{Q}[x]/(x^2 - 2)$ is not isomorphic to $\mathbb{Q}[x]/(x^2 - 3)$. [*Hint:* Exercises 2 and 5 may be helpful.]

11. Let K be a ring that contains $\mathbb{Z}_6$ as a subring. Show that $p(x) = 3x^2 + 1 \in \mathbb{Z}_6[x]$ has no roots in K. Thus, Corollary 5.12 may be false if F is not a field. [*Hint:* If u were a root, then $0 = 2 \cdot 3$ and $3u^2 + 1 = 0$. Derive a contradiction.]

12. Show that $2x^3 + 4x^2 + 8x + 3 \in \mathbb{Z}_{16}[x]$ has no roots in any ring K that contains $\mathbb{Z}_{16}$ as a subring. [See Exercise 11.]

C. 13. Show that every polynomial of degree 1, 2, or 4 in $\mathbb{Z}_2[x]$ has a root in $\mathbb{Z}_2[x]/(x^4 + x + 1)$.

CHAPTER 6

Ideals and Quotient Rings

Congruence in the integers led us to the finite arithmetics $\mathbb{Z}_n$ and helped motivate the definition of a ring. Congruence in the polynomial ring $F[x]$ resulted in a new class of rings consisting of the various $F[x]/(p(x))$. These rings enabled us to construct extension fields of F that contained roots of the polynomial $p(x)$. In this chapter the concept of congruence is extended to arbitrary rings, producing additional rings and a deeper understanding of algebraic structure.

You will see that much of the discussion is an exact parallel of the development of congruence in $\mathbb{Z}$ (Chapter 2) and in $F[x]$ (Chapter 5). Nevertheless, the results here are considerably broader than the earlier ones.

6.1 Ideals and Congruence

Our goal is to develop a notion of congruence in arbitrary rings that includes as special cases congruence modulo n in $\mathbb{Z}$ and congruence modulo $p(x)$ in $F[x]$. We begin by taking a second look at some examples of congruence in $\mathbb{Z}$ and $F[x]$ from a somewhat different viewpoint than before.

EXAMPLE 1

In the ring $\mathbb{Z}$, $a \equiv b \pmod 3$ means that $a - b$ is a multiple of 3. Let I be the set of all multiples of 3, so that

$$I = \{0, \pm 3, \pm 6, \dots\}.$$

Then congruence modulo 3 may be characterized like this:

$$a \equiv b \pmod 3 \qquad \text{means} \qquad a - b \in I.$$

Observe that the subset I is actually a *subring* of $\mathbb{Z}$ (sums and products of multiples of 3 are also multiples of 3). Furthermore, the product of any integer and a multiple of 3 is itself a multiple of 3. Thus the subring I has this property:

$$\text{Whenever } k \in \mathbb{Z} \text{ and } i \in I, \text{ then } ki \in I.$$

EXAMPLE 2*

The notation $f(x) \equiv g(x) \pmod{x^2 - 2}$ in the polynomial ring $\mathbb{Q}[x]$ means that $f(x) - g(x)$ is a multiple of $x^2 - 2$. Let I be the set of all multiples of $x^2 - 2$ in $\mathbb{Q}[x]$, that is, $I = \{h(x)(x^2 - 2) \mid h(x) \in \mathbb{Q}[x]\}$. Once again, it is not difficult to check that I is a subring of $\mathbb{Q}[x]$ with this property:

$$\text{Whenever } k(x) \in \mathbb{Q}[x] \text{ and } t(x) \in I, \text{ then } k(x)t(x) \in I$$

(the product of any polynomial with a multiple of $x^2 - 2$ is itself a multiple of $x^2 - 2$). Congruence modulo $x^2 - 2$ may be described in terms of I:

$$f(x) \equiv g(x) \pmod{x^2 - 2} \qquad \text{means} \qquad f(x) - g(x) \in I.$$

These examples suggest that congruence in a ring R might be defined in terms of certain subrings. If I were such a subring, we might define $a \equiv b \pmod{I}$ to mean $a - b \in I$. The subring I might consist of all multiples of a fixed element, as in the preceding examples, but there is no reason for restricting to this situation. The examples indicate that the key property for such a subring I is that it "absorbs products": Whenever you multiply an element of I by *any* element of the ring (either inside or outside I), the resulting product is an element of I. The set of all multiples of a fixed element has this absorption property. We shall see that many other subrings have it as well. Because such subrings play a crucial role in what follows, we pause to give them a name and to consider their basic properties.

Definition A subring I of a ring R is an **ideal** provided:

$$\text{Whenever } r \in R \text{ and } a \in I, \text{ then } ra \in I \text{ and } ar \in I.$$

The double absorption condition that $ra \in I$ and $ar \in I$ is necessary for noncommutative rings. When R is commutative, as in the preceding examples, this condition reduces to $ra \in I$.

EXAMPLE 3

The zero ideal in a ring R consists of the single element 0_R. This is a subring that absorbs all products since $r0_R = 0_R = 0_Rr$ for every $r \in R$. The entire ring R is also an ideal.

*Skip this example if you have not read Chapter 5.

EXAMPLE 4

In the ring $\mathbb{Z}[x]$ of all polynomials with integer coefficients, let I be the set of polynomials whose constant terms are even integers. Thus $x^3 + x + 6$ is in I, but $4x^2 + 3$ is not. Verify that I is an ideal in $\mathbb{Z}[x]$ (Exercise 2).

EXAMPLE 5

Let T be the ring of all functions from $\mathbb{R}$ to $\mathbb{R}$, as described in Example 8 of Section 3.1. Let I be the subset consisting of those functions g such that $g(2) = 0$. Then I is a subring of T (Exercise 14 of Section 3.1). If f is any function in T and if $g \in I$, then

$$(fg)(2) = f(2)g(2) = f(2) \cdot 0 = 0.$$

Therefore, $fg \in I$. Similarly, $gf \in I$, so that I is an ideal in T.

EXAMPLE 6

The subring $\mathbb{Z}$ of the rational numbers is *not* an ideal in $\mathbb{Q}$ because $\mathbb{Z}$ fails to have the absorption property. For instance, $\frac{1}{2} \in \mathbb{Q}$ and $5 \in \mathbb{Z}$, but their product, $\frac{5}{2}$, is not in $\mathbb{Z}$.

EXAMPLE 7

Verify that the set I of all matrices of the form $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ with $a, b \in \mathbb{R}$ forms a subring of the ring $M(\mathbb{R})$ of all 2×2 matrices over the reals. It is easy to see that I absorbs products on the *left*:

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} ra + sb & 0 \\ ta + ub & 0 \end{pmatrix} \in I.$$

But I is not an ideal in $M(\mathbb{R})$ because it may not absorb products on the right—for instance,

$$\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \notin I.$$

One sometimes says that I is a **left ideal,** but not a two-sided ideal, in $M(\mathbb{R})$.

The following generalization of Theorem 3.6 often simplifies the verification that a particular subset of a ring is an ideal.

Theorem 6.1

A nonempty subset I of a ring R is an ideal if and only if it has these properties:

(i) if $a, b \in I$, then $a - b \in I$;

(ii) if $r \in R$ and $a \in I$, then $ra \in I$ and $ar \in I$.

Proof ▸ Every ideal certainly has these two properties. Conversely, suppose I has properties (i) and (ii). Then I absorbs products by (ii), so we need only verify that I is a subring. Property (i) states that I is closed under subtraction. Since I is a subset of R, the product of any two elements of I must be in I by (ii). In other words, I is closed under multiplication. Therefore, I is a subring of R by Theorem 3.6. ■

Finitely Generated Ideals

In the first example of this section we saw that the set I of all multiples of 3 is an ideal in $\mathbb{Z}$. This fact is a special case of

Theorem 6.2

Let R be a commutative ring with identity, $c \in R$, and I the set of all multiples of c in R, that is, $I = \{rc \mid r \in R\}$. Then I is an ideal.

Proof ▸ If $r_1, r_2, r \in R$ and $r_1c, r_2c \in I$, then

$$r_1c - r_2c = (r_1 - r_2)c \in I \qquad \text{and} \qquad r(r_1c) = (rr_1)c \in I$$

because $r_1 - r_2$ and rr_1 are elements of R. Similarly, since R is commutative, $(r_1c)r = (rr_1)c \in I$. Therefore, I is an ideal by Theorem 6.1. ■

The ideal I in Theorem 6.2 is called the **principal ideal generated by** $\boldsymbol{c}$ and hereafter will be denoted by (c). In the ring $\mathbb{Z}$, for example, (3) indicates the ideal of all multiples of 3. In any commutative ring R with identity, the principal ideal (1_R) is the entire ring R because $r = r1_R$ for every $r \in R$. It can be shown that every ideal in $\mathbb{Z}$ is a principal ideal (Exercise 40). However, there are ideals in other rings that are not principal, that is, ideals that do not consist of all the multiples of a particular element of the ring.

EXAMPLE 8

We have seen that the set I of all polynomials with even constant terms is an ideal in the ring $\mathbb{Z}[x]$. We claim that I is *not* a principal ideal. To prove this, suppose, on the contrary, that I consists of all multiples of some polynomial $p(x)$. Since the constant polynomial 2 is in I, 2 must be a multiple of $p(x)$. By Theorem 4.2, this is possible only if $p(x)$ has degree 0, that is, if $p(x)$ is a constant, say $p(x) = c$. Since $p(x) \in I$, the constant c must be an even integer. Since 2 is a multiple of $p(x) = c$, the only possibility is $c = \pm 2$. On the other hand, $x \in I$ because it has even constant term 0. Therefore, x must be a multiple of $p(x) = \pm 2$. However, if $\pm 2g(x) = x$, then $g(x)$ has degree 1 by Theorem 4.2, say $g(x) = ax + b$. But $\pm 2(ax + b) = x$ implies that $\pm 2a = 1$ because the coefficient of x must be the same on both sides. This is impossible because a is an integer. Therefore, I does not consist of all multiples of $p(x)$ and is not a principal ideal.

In a commutative ring with identity, a principal ideal consists of all multiples of a fixed element. Here is a generalization of that idea.

Theorem 6.3

Let R be a commutative ring with identity and $c_1, c_2, \ldots, c_n \in R$. Then the set $I = \{r_1c_1 + r_2c_2 + \cdots + r_nc_n \mid r_1, r_2, \ldots, r_n \in R\}$ is an ideal in R.

Proof ▸ Exercise 14. ■

The ideal I in Theorem 6.3 is called the **ideal generated by $c_1, c_2, \ldots, c_n$** and is sometimes denoted by $(c_1, c_2, \ldots, c_n)$. Such an ideal is said to be **finitely generated.** A principal ideal is the special case $n = 1$, that is, an ideal generated by a single element.* The generators of a finitely generated ideal need not be unique, that is, the ideal generated by $c_1, c_2, \ldots, c_n$ might be the same set as the ideal generated by $d_1, d_2, \ldots, d_k$, even though no c_i is equal to any d_j (Exercise 16).

EXAMPLE 9

In the ring $\mathbb{Z}[x]$, the ideal generated by the polynomial x and the constant polynomial 2 consists of all polynomials of the form

$$f(x)x + g(x)2, \qquad \text{with } f(x), g(x) \in \mathbb{Z}[x].$$

It can be shown that this ideal is the ideal I of all polynomials with even constant term, which was discussed in Example 8 (Exercise 15).

Congruence

Now that you are familiar with ideals, we can define congruence in an arbitrary ring:

Definition

Let I be an ideal in a ring R and let $a, b \in R$. Then **a is congruent to b modulo I** [written $a \equiv b \pmod{I}$] provided that $a - b \in I$.

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*When a commutative ring does not have an identity, the ideal generated by $c_1, c_2, \ldots, c_n$ is defined somewhat differently (see Exercise 33).

Example 1 shows that congruence modulo 3 in the integers is the same thing as congruence modulo the ideal I, where I is the principal ideal (3) of all multiples of 3. Similarly, Example 2 shows that congruence modulo $x^2 - 2$ in $\mathbb{Q}[x]$ is the same as congruence modulo the principal ideal $(x^2 - 2)$. Thus congruence modulo an ideal includes as a special case the concepts of congruence in $\mathbb{Z}$ and $F[x]$ used earlier in this book.

EXAMPLE 10

Let T be the ring of all functions from $\mathbb{R}$ to $\mathbb{R}$ and let I be the ideal of all functions g such that $g(2) = 0$. If $f(x) = x^2 + 6$ and $h(x) = 5x$, then the function $f - h$ is in I because

$$(f - h)(2) = f(2) - h(2) = (2^2 + 6) - (5 \cdot 2) = 0.$$

Therefore, $f \equiv h \pmod{I}$.

Theorem 6.4

Let I be an ideal in a ring R. Then the relation of congruence modulo I is

(1) reflexive: $a \equiv a \pmod{I}$ for every $a \in R$;

(2) symmetric: if $a \equiv b \pmod{I}$, then $b \equiv a \pmod{I}$;

(3) transitive: if $a \equiv b \pmod{I}$ and $b \equiv c \pmod{I}$, then $a \equiv c \pmod{I}$.

This theorem generalizes Theorems 2.1 and 5.1. Observe that the proof is virtually identical to that of Theorem 2.1—just replace statements like "k is divisible by n" or "$n \mid k$" or "$k = nt$" with the statement "$k \in I$".

Proof of Theorem 6.4 ▸ (1) $a - a = 0_R \in I$; hence, $a \equiv a \pmod{I}$.

(2) $a \equiv b \pmod{I}$ means that $a - b = i$ for some $i \in I$. Therefore, $b - a = -(a - b) = -i$. Since I is an ideal, the negative of an element of I is also in I, and so $b - a = -i \in I$. Hence, $b \equiv a \pmod{I}$.

(3) If $a \equiv b \pmod{I}$ and $b \equiv c \pmod{I}$, then by the definition of congruence, there are elements i and j in I such that $a - b = i$ and $b - c = j$. Therefore, $a - c = (a - b) + (b - c) = i + j$. Since the ideal I is closed under addition, $i + j \in I$ and, hence, $a \equiv c \pmod{I}$. ■

Theorem 6.5

Let I be an ideal in a ring R. If $a \equiv b \pmod{I}$ and $c \equiv d \pmod{I}$, then

(1) $a + c \equiv b + d \pmod{I}$;

(2) $ac \equiv bd \pmod{I}$.

This theorem generalizes Theorems 2.2 and 5.2. Its proof is quite similar to theirs once you make the change to the language of ideals.

Proof of Theorem 6.5 ▶ (1) By the definition of congruence, there are $i, j \in I$ such that $a - b = i$ and $c - d = j$. Therefore, $(a + c) - (b + d) = (a - b) + (c - d) = i + j \in I$. Hence, $a + c \equiv b + d \pmod{I}$.

(2) $ac - bd = ac - bc + bc - bd = (a - b)c + b(c - d) = ic + bj$. Since the ideal I absorbs products on both left and right, $ic \in I$ and $bj \in I$. Hence, $ac - bd = ic + bj \in I$. Therefore, $ac \equiv bd \pmod{I}$. ■

If I is an ideal in a ring R and $a \in R$, then the **congruence class of a modulo I** is the set of all elements of R that are congruent to a modulo I, that is, the set

$$\begin{aligned}\{b \in R \mid b \equiv a \pmod{I}\} &= \{b \in R \mid b - a \in I\} \\ &= \{b \in R \mid b - a = i, \text{ with } i \in I\} \\ &= \{b \in R \mid b = a + i, \text{ with } i \in I\} \\ &= \{a + i \mid i \in I\}.\end{aligned}$$

Consequently, we shall denote the congruence class of a modulo I by the symbol $a + I$ rather than the symbol $[a]$ that was used in $\mathbb{Z}$ and $F[x]$. The plus sign in $a + I$ is just a formal symbol; we have not defined the sum of an element and an ideal. In this context, the congruence class $a + I$ is usually called a (left) **coset** of I in R.

Theorem 6.6

Let I be an ideal in a ring R and let $a, c \in R$. Then $a \equiv c \pmod{I}$ if and only if $a + I = c + I$.

Proof ▶ With only minor notational changes, the proof of Theorem 2.3 carries over almost verbatim to the present case. Simply replace "mod n" by "mod I" and "$[a]$" by "$a + I$"; use Theorem 6.4 in place of Theorem 2.1. ■

Corollary 6.7

Let I be an ideal in a ring R. Then two cosets of I are either disjoint or identical.

Proof ▶ Copy the proof of Corollary 2.4 with the obvious notational changes. ■

If I is an ideal in a ring R, then the set of all cosets of I (congruence classes modulo I) is denoted R/I.

EXAMPLE 11

Let I be the principal ideal (3) in the ring $\mathbb{Z}$. Then the cosets of I are just the congruence classes modulo 3, and so there are three distinct cosets: $0 + I = [0]$, $1 + I = [1]$, and $2 + I = [2]$. The set $\mathbb{Z}/I$ of all cosets is precisely the set $\mathbb{Z}_3$ in our previous notation.

EXAMPLE 12

Let I be the ideal in $\mathbb{Z}[x]$ consisting of all polynomials with even constant terms. We claim that $\mathbb{Z}[x]/I$ consists of exactly two distinct cosets, namely, $0 + I$ and $1 + I$. To see this, consider any coset $f(x) + I$. The constant term of $f(x)$ is either even or odd. If it is even, then $f(x) \in I$, so that $f(x) \equiv 0 \pmod{I}$. Therefore, $f(x) + I = 0 + I$ by Theorem 6.6. If $f(x)$ has odd constant term, then $f(x) - 1$ has even constant term, so that $f(x) \equiv 1 \pmod{I}$. Thus $f(x) + I = 1 + I$ by Theorem 6.6.

EXAMPLE 13

Let T be the ring of functions from $\mathbb{R}$ to $\mathbb{R}$ and let I be the ideal of all functions g such that $g(2) = 0$. Note that for each real number r, the constant function f_r (whose rule is $f_r(x) = r$) is an element of T. Let $h(x)$ be any element of T. Then $h(2)$ is some real number, say $h(2) = c$, and

$$(h - f_c)(2) = h(2) - f_c(2) = c - c = 0.$$

Thus $h - f_c \in I$, so that $h \equiv f_c \pmod{I}$ and, hence, $h + I = f_c + I$. Consequently, every coset of I can be written in the form $f_r + I$ for some real number r. Furthermore, if $c \neq d$, then $f_c(2) \neq f_d(2)$, so that $[f_c - f_d](2) \neq 0$ and $f_c - f_d \notin I$. Hence, $f_c \not\equiv f_d \pmod{I}$ and $f_c + I \neq f_d + I$. Therefore, there are infinitely many distinct cosets of I, one for each real number r.

Exercises

NOTE: *R denotes a ring.*

A. 1. Show that the set K of all constant polynomials in $\mathbb{Z}[x]$ is a subring but not an ideal in $\mathbb{Z}[x]$.

2. Show that the set I of all polynomials with even constant terms is an ideal in $\mathbb{Z}[x]$.

3. **(a)** Show that the set $I = \{(k, 0) \mid k \in \mathbb{Z}\}$ is an ideal in the ring $\mathbb{Z} \times \mathbb{Z}$.

   (b) Show that the set $T = \{(k, k) \mid k \in \mathbb{Z}\}$ is not an ideal in $\mathbb{Z} \times \mathbb{Z}$.

4. Is the set $J = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix} \mid r \in \mathbb{R} \right\}$ an ideal in the ring $M(\mathbb{R})$ of 2×2 matrices over $\mathbb{R}$?

5. Show that the set $K = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ is a subring of $M(\mathbb{R})$ that absorbs products on the right. Show that K is not an ideal because it may fail to absorb products on the left. Such a set K is sometimes called a **right ideal.**

6. **(a)** Show that the set of nonunits in $\mathbb{Z}_8$ is an ideal.

   (b) Do part (a) for $\mathbb{Z}_9$. [Also, see Exercise 24.]

7. Let $c \in R$ and let $I = \{rc \mid r \in R\}$.

   (a) If R is commutative, prove that I is an ideal (that is, Theorem 6.2 is true even when R does not have an identity).

   (b) If R is commutative but has no identity, is c an element of the ideal I? [*Hint:* Consider the ideal $\{2k \mid k \in E\}$ in the ring E of even integers. Also see Exercise 33.]

   (c) Give an example to show that if R is not commutative, then I need not be an ideal.

8. If I is an ideal in R and J is an ideal in the ring S, prove that $I \times J$ is an ideal in the ring $R \times S$.

9. Let R be a ring with identity and let I be an ideal in R.

   (a) If $1_R \in I$, prove that $I = R$.

   (b) If I contains a unit, prove that $I = R$.

10. If I is an ideal in a field F, prove that $I = (0_F)$ or $I = F$. [*Hint:* Exercise 9.]

11. List the distinct principal ideals in each ring:

    (a) $\mathbb{Z}_5$ (b) $\mathbb{Z}_9$ (c) $\mathbb{Z}_{12}$

12. List the distinct principal ideals in $\mathbb{Z}_2 \times \mathbb{Z}_3$.

13. If R is a commutative ring with identity and (a) and (b) are principal ideals such that $(a) = (b)$, is it true that $a = b$? Justify your answer.

14. Prove Theorem 6.3.

15. Show that the ideal generated by x and 2 in the ring $\mathbb{Z}[x]$ is the ideal I of all polynomials with even constant terms (see Example 9).

16. (a) Show that $(4, 6) = (2)$ in $\mathbb{Z}$, where $(4, 6)$ is the ideal generated by 4 and 6 and (2) is the principal ideal generated by 2.

    (b) Show that $(6, 9, 15) = (3)$ in $\mathbb{Z}$.

17. (a) If I and J are ideals in R, prove that $I \cap J$ is an ideal.

    (b) If $[I_k]$ is a (possibly infinite) family of ideals in R, prove that the intersection of all the I_k is an ideal.

18. Give an example in $\mathbb{Z}$ to show that the set theoretic union of two ideals may not be an ideal (in fact, it may not even be a subring).

19. If I is an ideal in R and S is a subring of R, prove that $I \cap S$ is an ideal in S.

20. Let I and J be ideals in R. Prove that the set $K = \{a + b \mid a \in I, b \in J\}$ is an ideal in R that contains both I and J. K is called the **sum** of I and J and is denoted $I + J$.

21. If d is the greatest common divisor of a and b in $\mathbb{Z}$, show that $(a) + (b) = (d)$. (The sum of ideals is defined in Exercise 20.)

22. Let I and J be ideals in R. Is the set $K = \{ab \mid a \in I, b \in J\}$ an ideal in R? Compare Exercise 20.

23. **(a)** Verify that $I = \{0, 3\}$ is an ideal in $\mathbb{Z}_6$ and list all its distinct cosets.

(b) Verify that $I = \{0, 3, 6, 9, 12\}$ is an ideal in $\mathbb{Z}_{15}$ and list all its distinct cosets.

B. 24. Let R be a commutative ring with identity, and let N be the set of nonunits in R. Give an example to show that N need not be an ideal.

25. Let J be an ideal in R. Prove that I is an ideal, where

$$I = \{r \in R \mid rt = 0_R \text{ for every } t \in J\}.$$

26. Let I be an ideal in R. Prove that K is an ideal, where

$$K = \{a \in R \mid ra \in I \text{ for every } r \in R\}.$$

27. Let $f{:}R \to S$ be a homomorphism of rings and let

$$K = \{r \in R \mid f(r) = 0_S\}.$$

Prove that K is an ideal in R.

28. If I is an ideal in R, prove that $I[x]$ (polynomials with coefficients in I) is an ideal in the polynomial ring $R[x]$.

29. If $(m, n) = 1$ in $\mathbb{Z}$, prove that $(m) \cap (n)$ is the ideal (mn).

30. Prove that the set of nilpotent elements in a commutative ring R is an ideal. [*Hint:* See Exercise 44 in Section 3.2.]

31. Let R be an integral domain and $a, b \in R$. Show that $(a) = (b)$ if and only if $a = bu$ for some unit $u \in R$.

32. **(a)** Prove that the set J of all polynomials in $\mathbb{Z}[x]$ whose constant terms are divisible by 3 is an ideal.

(b) Show that J is not a principal ideal.

33. Let R be a commutative ring without identity and let $a \in R$. Show that $A = \{ra + na \mid r \in R, n \in \mathbb{Z}\}$ is an ideal containing a and that every ideal containing a also contains A. A is called the **principal ideal generated by a.**

34. If M is an ideal in a commutative ring R with identity and if $a \in R$ with $a \notin M$, prove that the set

$$J = \{m + ra \mid r \in R \text{ and } m \in M\}$$

is an ideal such that $M \subsetneq J$.

35. Let I be an ideal in $\mathbb{Z}$ such that $(3) \subseteq I \subseteq \mathbb{Z}$. Prove that either $I = (3)$ or $I = \mathbb{Z}$.

36. Let I and J be ideals in R. Let IJ denote the set of all possible finite sums of elements of the form ab (with $a \in I$, $b \in J$), that is,

$$IJ = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid n \geq 1, a_k \in I, b_k \in J\}.$$

Prove that IJ is an ideal, IJ is called the **product** of I and J.

37. Let R be a commutative ring with identity $1_R \neq 0_R$ whose only ideals are (0_R) and R. Prove that R is a field. [*Hint:* If $a \neq 0_R$, use the ideal (a) to find a multiplicative inverse for a.]

38. Let I be an ideal in a commutative ring R and let

$$J = \{r \in R \mid r^n \in I \text{ for some positive integer } n\}.$$

Prove that J is an ideal that contains I. [*Hint:* You will need the Binomial Theorem from Appendix E. Exercise 30 is the case when $I = (0_R)$.]

39. **(a)** Show that the ring $M(\mathbb{R})$ is not a division ring by exhibiting a matrix that has no multiplicative inverse. (Division rings are defined in Exercise 42 of Section 3.1.)

(b) Show that $M(\mathbb{R})$ has no ideals except the zero ideal and $M(\mathbb{R})$ itself. [*Hint:* If J is a nonzero ideal, show that J contains a matrix A with a nonzero entry c in the upper left-hand corner. Verify that

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot A \cdot \begin{pmatrix} c^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and that this matrix is in J. Similarly,

show that $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is in J. What is their sum? See Exercise 9.]

40. Prove that every ideal in $\mathbb{Z}$ is principal. [*Hint:* If I is a nonzero ideal, show that I must contain positive elements and, hence, must contain a smallest positive element c (Why?). Since $c \in I$, every multiple of c is also in I; hence, $(c) \subseteq I$. To show that $I \subseteq (c)$, let a be any element of I. Then $a = cq + r$ with $0 \leq r < c$ (Why?). Show that $r = 0$ so that $a = cq \in (c)$.]

41. **(a)** Prove that the set S of rational numbers (in lowest terms) with odd denominators is a subring of $\mathbb{Q}$.

(b) Let I be the set of elements of S with even numerators. Prove that I is an ideal in S.

(c) Show that S/I consists of exactly two distinct cosets.

42. **(a)** Let p be a prime integer and let T be the set of rational numbers (in lowest terms) whose denominators are not divisible by p. Prove that T is a ring.

(b) Let I be the set of elements of T whose numerators are divisible by p. Prove that I is an ideal in T.

(c) Show that T/I consists of exactly p distinct cosets.

43. Let J be the set of all polynomials with zero constant term in $\mathbb{Z}[x]$.

(a) Show that J is the principal ideal (x) in $\mathbb{Z}[x]$.

(b) Show that $\mathbb{Z}[x]/J$ consists of an infinite number of distinct cosets, one for each $n \in \mathbb{Z}$.

44. **(a)** Prove that the set T of matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a, b \in \mathbb{R}$ is a subring of $M(\mathbb{R})$.

(b) Prove that the set I of matrices of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ with $b \in \mathbb{R}$ is an ideal in the ring T.

(c) Show that every coset in T/I can be written in the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + I$.

45. (a) Prove that the set S of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, b, c \in \mathbb{R}$ is a subring of $M(\mathbb{R})$.

(b) Prove that the set I of matrices of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ with $b \in \mathbb{R}$ is an ideal in the ring S.

(c) Show that there are infinitely many distinct cosets in S/I, one for each pair in $\mathbb{R} \times \mathbb{R}$.

C. 46. Let F be a field. Prove that every ideal in $F[x]$ is principal. [*Hint:* Use the Division Algorithm to show that the nonzero ideal I in $F[x]$ is $(p(x))$, where $p(x)$ is a polynomial of smallest possible degree in I.]

47. Prove that a subring S of $\mathbb{Z}_n$ has an identity if and only if there is an element u in S such that $u^2 = u$ and S is the ideal (u).

6.2 Quotient Rings and Homomorphisms

We now show that the set of congruence classes modulo an ideal is itself a ring. As you might expect, this is a straightforward generalization of what we did with congruence classes in $\mathbb{Z}$ and $F[x]$. However, you may not have expected these rings of congruence classes to have close connections with some topics studied in Chapter 3, isomorphisms and homomorphisms. These connections are explored in detail and provide new insight into the structure of rings.

Let I be an ideal in a ring R. The elements of the set R/I are the cosets of I (congruence classes modulo I), that is, all sets of the form $a + I = \{a + i \mid i \in I\}$. In order to define addition and multiplication of cosets as we did with congruence classes in $\mathbb{Z}$ and $F[x]$, we need

Theorem 6.8

Let I be an ideal in a ring R. If $a + I = b + I$ and $c + I = d + I$ in R/I, then

$$(a + c) + I = (b + d) + I \quad \text{and} \quad ac + I = bd + I.$$

Proof ▸ This is a generalization of Theorem 2.6, in slightly different notation. Replace "$[a]$" by "$a + I$" and copy the proof of Theorem 2.6, using Theorems 6.5 and 6.6 in place of Theorems 2.2 and 2.3. ■

We can now define addition and multiplication in R/I just as we did in $\mathbb{Z}_n$ and $F[x]/(p(x))$: The sum of the coset $a + I$ (congruence class of a) and the coset $c + I$ (congruence class of c) is the coset $(a + c) + I$ (congruence class of $a + c$). In symbols,

$$(a + I) + (c + I) = (a + c) + I.$$

This statement may be a bit confusing because the plus sign is used with three entirely different meanings:

as a formal symbol to denote a coset: $a + I$;

as an operation on elements of R: $a + c$;

as the addition operation on cosets that is being defined.*

The important thing is that, because of Theorem 6.8, coset addition is independent of the choice of representative elements in each coset. Even if we replace $a + I$ by an equal coset $b + I$ and replace $c + I$ by an equal coset $d + I$, the resulting coset sum, namely $(b + d) + I$, is the same as $(a + c) + I$.

Multiplication of cosets is defined similarly and is independent of the choice of representatives by Theorem 6.8:

$$(a + I)(c + I) = ac + I.$$

EXAMPLE 1

If I is the principal ideal (3) in $\mathbb{Z}$, then addition and multiplication of cosets is the same as addition and multiplication of congruence classes in Section 2.2. Thus $\mathbb{Z}/I$ is just the ring $\mathbb{Z}_3$.

EXAMPLE 2†

If F is a field, $p(x)$ is a polynomial in $F[x]$, and I is the principal ideal $(p(x))$, then cosets of I are precisely congruence classes modulo $p(x)$, so that addition and multiplication of cosets are done exactly as they were in Section 5.2. Thus $F[x]/I$ is the congruence-class ring $F[x]/(p(x))$.

EXAMPLE 3

Let I be the ideal of polynomials with even constant terms in $\mathbb{Z}[x]$. As we saw in Example 12 of Section 6.1, $\mathbb{Z}[x]/I$ consists of just two distinct cosets, $0 + I$ and $1 + I$. We have $(1 + I) + (1 + I) = (1 + 1) + I = 2 + I$, but $2 \in I$, so that $2 \equiv 0 \pmod{I}$ and, hence, $2 + I = 0 + I$. Similar calculations produce the following tables for $\mathbb{Z}[x]/I$. It is easy to see that $\mathbb{Z}[x]/I$ is a ring (in fact, a field) isomorphic to $\mathbb{Z}_2$:

$+$	$0+I$	$1+I$
$0+I$	$0+I$	$1+I$
$1+I$	$1+I$	$0+I$

$\cdot$	$0+I$	$1+I$
$0+I$	$0+I$	$0+I$
$1+I$	$0+I$	$1+I$

*This ambiguity can be avoided by using a different notation for cosets, such as $[a]$, and a different symbol for coset addition, such as $\oplus$. The notation above is customary, however, and once you're used to it, there should be no confusion.

†Skip this example if you have not read Chapter 5.

These examples illustrate the following theorem, which should not be very surprising in view of your previous experience with $\mathbb{Z}$ and $F[x]$.

Theorem 6.9

Let I be an ideal in a ring R. Then

(1) R/I is a ring, with addition and multiplication of cosets as defined previously.

(2) If R is commutative, then R/I is a commutative ring.

(3) If R has an identity, then so does the ring R/I.

Proof ▸ (1) With the usual change of notation ("$a + I$" instead of "$[a]$"), the proof of Theorem 2.7 carries over to the present situation since that proof depends only on the fact that $\mathbb{Z}$ is a ring. Don't take our word for it, though; write out the proof in detail for yourself.

(2) If R is commutative and $a, c \in R$, then $ac = ca$. Consequently, in R/I we have $(a + I)(c + I) = ac + I = ca + I = (c + I)(a + I)$. Hence, R/I is commutative.

(3) The identity in R/I is the coset $1_R + I$ because $(a + I)(1_R + I) = a1_R + I = a + I$ and similarly $(1_R + I)(a + I) = a + I$. ■

The ring R/I is called the **quotient ring** (or **factor ring**) of R by I. One sometimes speaks of factoring out the ideal I to obtain the quotient ring R/I.

Homomorphisms

Quotient rings are the natural generalization of congruence-class arithmetic in $\mathbb{Z}$ and $F[x]$. As is often the case in mathematics, however, a concept developed with one idea in mind may have unexpected linkages with other important mathematical concepts. That is precisely the situation here. We shall now see that the concept of homomorphism that arose in our study of isomorphism of rings in Chapter 3 is closely related to ideals and quotient rings.

Definition

Let $f: R \to S$ be a homomorphism of rings. Then the **kernel** of f is the set

$$K = \{r \in R \mid f(r) = 0_S\}.$$

Thus, the kernel of f is the subset of R consisting of those elements of R that f maps to 0_S in S. Note that 0_R is in the kernel since $f(0_R) = 0_S$ by Theorem 3.10. However, the kernel may also contain nonzero elements.

EXAMPLE 4

In Example 6 of Section 3.3 we saw that the function $f:\mathbb{Z} \to \mathbb{Z}_6$ defined by $f(r) = [r] \in \mathbb{Z}_6$ is a homomorphism of rings. Its kernel K contains many nonzero integers. For instance, $12 \in K$ because $f(12) = [12] = [0]$ in $\mathbb{Z}_6$. In fact *every* multiple of 6 is in the kernel because

$$\begin{aligned} K = \{r \in \mathbb{Z} \mid f(r) = [0]\} &= \{r \in \mathbb{Z} \mid [r] = [0]\} && [\textit{Definition of } f] \\ &= \{r \in \mathbb{Z} \mid r \equiv 0 \pmod 6\} && [\textit{Theorem } 2.3] \\ &= \{r \in \mathbb{Z} \mid 6 \mid r\} && [\textit{Definition of congruence mod } 6] \\ &= \{\text{all multiples of } 6\} && [6 \mid r \textit{ means } r \textit{ is a multiple of } 6]. \end{aligned}$$

So the kernel K is the principal ideal (6) in $\mathbb{Z}$.

EXAMPLE 5

The function $\theta:\mathbb{R}[x] \to \mathbb{R}$ that sends each polynomial in $\mathbb{R}[x]$ to its constant term in $\mathbb{R}$ is a ring homomorphism (Exercise 1). Its kernel consists of all polynomials with constant term 0. But every polynomial with 0 constant term is divisible by x. So the kernel is the principal ideal (x) in $\mathbb{R}[x]$.

Examples 4 and 5 provide examples of the following theorem.

Theorem 6.10

Let $f:R \to S$ be a homomorphism of rings. Then the kernel K of f is an ideal in the ring R.

Proof ▸ We shall use Theorem 6.1 to show that $K = \{r \in R \mid f(r) = 0_S\}$ is an ideal. We must verify that is a nonempty subset of R that is closed under subtraction and absorbs products. First, K is nonempty because $0_R \in K$ as noted before Example 4. To prove that K is closed under subtraction, we must show that for $a, b \in K$, the element $a - b$ is also in K. To show $a - b \in K$, we must show that $f(a - b) = 0_S$. This follows from the fact that f is a homomorphism and that $f(a) = 0_S$ and $f(b) = 0_S$ (because a, $b \in K$):

$$f(a - b) = f(a) - f(b) = 0_S - 0_S = 0_S.$$

To prove that K absorbs products we must first verify that $ra \in K$ for any $r \in R$ and $a \in K$, that is, that $f(ra) = 0_S$; here's the proof:

$$f(ra) = f(r)f(a) = f(r)\, 0_S = 0_S.$$

A similar argument shows that $ar \in K$. Therefore K is an ideal by Theorem 6.1. ■

In Examples 4 and 5, the kernel of the homomorphism contained many nonzero elements. Sometimes, however, the kernel of a homomorphism contains only 0_R, in which case we have an interesting result.

Theorem 6.11

Let $f:R \to S$ be a homomorphism of rings with kernel K. Then $K = (0_R)$ if and only if f is injective.

Proof ▸ Suppose that $K = (0_R)$. We must show that f is injective, so assume that $a, b \in R$ and $f(a) = f(b)$. Because f is a homomorphism, $f(a - b) = f(a) - f(b) = 0_S$. Hence, $a - b$ is in the kernel $K = (0_R)$, which means that $a - b = 0_R$ and $a = b$. Therefore f is injective.

Conversely, suppose f is injective. If $c \in K$, we must show that $c = 0_R$. By the definition of the kernel, $f(c) = 0_S$. By Theorem 3.10, $f(0_R) = 0_S = f(c)$. Therefore, $c = 0_R$ because f is injective. Hence, the kernel consists of the single element 0_R, that is, $K = (0_R)$. ■

EXAMPLE 6

In Example 7 of Section 3.3 we saw that the function $g:\mathbb{R} \to M(\mathbb{R})$ given by $g(r) = \begin{pmatrix} 0 & 0 \\ -r & r \end{pmatrix}$ is a ring homomorphism. Its kernel of g consists of all real numbers r such that $g(r) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, that is, such that $\begin{pmatrix} 0 & 0 \\ -r & r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. This can only occur when $r = 0$. So the kernel is the zero ideal (0). Hence, g is injective by Theorem 6.11.

Theorem 6.10 states that every kernel is an ideal. Conversely, every ideal is the kernel of a homomorphism:

Theorem 6.12

Let I be an ideal in a ring R. Then the map $\pi:R \to R/I$ given by $\pi(r) = r + I$ is a surjective homomorphism with kernel I.

The map π is called the **natural homomorphism** from R to R/I.

Proof of Theorem 6.12 ▸ The map π is surjective because given any coset $r + I$ in R/I, $\pi(r) = r + I$. The definition of addition and multiplication in R/I shows that π is a homomorphism:

$$\pi(r + s) = (r + s) + I = (r + I) + (s + I) = \pi(r) + \pi(s);$$
$$\pi(rs) = rs + I = (r + I)(s + I) = \pi(r)\,\pi(s).$$

The kernel of π is the set of elements $r \in R$ such that $\pi(r) = 0_R + I$ (the zero element in R/I). However, $\pi(r) = 0_R + I$ if and only if $r + I = 0_R + I$, which occurs if and only if $r \equiv 0_R \pmod I$, that is, if and only if $r \in I$. Therefore, I is the kernel of π. ∎

The natural homomorphism π in Theorem 6.12 is a special case of a more general situation. If $f:R \to S$ is a surjective homomorphism of rings, we say that S is a **homomorphic image** of R. If f is actually an isomorphism (so that S is an isomorphic image of R), then we know that R and S have identical structure. Whenever one of them has a particular algebraic property, the other one has it too. If f is not an isomorphism, then properties of one ring may not hold in the other. However, the properties of S and the homomorphism f often give us some useful information about R. An analogy with sculpture and photography may be helpful: If $f:R \to S$ is an isomorphism, then S is an exact, three-dimensional replica of R. If f is only a surjective homomorphism, then S is a two-dimensional photographic image of R in which some features of R are accurately reflected but others are distorted or missing. The next theorem tells us precisely how R, S, and the kernel of f are related in these circumstances.

Theorem 6.13 First Isomorphism Theorem

Let $f:R \to S$ be a surjective homomorphism of rings with kernel K. Then the quotient ring R/K is isomorphic to S.

The theorem states that every homomorphic image of a ring R is isomorphic to a quotient ring R/K for some ideal K. Thus if you know all the quotient rings of R, then you know all the possible homomorphic images of R. The ideal K measures how much information is lost in passing from the ring R to the homomorphic image R/K. When $K = (0_R)$, then f is an isomorphism by Theorem 6.11, and no information is lost. But when K is large, quite a bit may be lost.

Proof of Theorem 6.13 ▸ We shall define a function φ from R/K to S and then show that it is an isomorphism. To define φ, we must associate with each coset $r + K$ of R/K an element of S. A natural choice for such an element would be $f(r) \in S$; in other words, we would like to define $\varphi:R/K \to S$ by the rule $\varphi(r + K) = f(r)$. The only possible problem is that a coset can be labeled by many different elements of R. So we must show that the value of φ depends only on the coset and not on the particular representative r chosen to name it. If $r + K = t + K$, then $r \equiv t \pmod K$ by Theorem 6.6, which means that $r - t \in K$ by the definition of congruence. Consequently, since f is a homomorphism, $f(r) - f(t) = f(r - t) = 0_S$. Therefore, $r + K = t + K$ implies that $f(r) = f(t)$. It follows that the map $\varphi:R/K \to S$ given by the rule $\varphi(r + K) = f(r)$ is a well-defined function, independent of how the coset is written.

If $s \in S$, then $s = f(r)$ for some $r \in R$ because f is surjective. Thus $s = f(r) = \varphi(r + K)$, and φ is surjective. To show that φ is injective, we assume that $\varphi(r + K) = \varphi(c + K)$ and show that $r + K = c + K$, as follows:

$$\begin{aligned} \varphi(r + K) &= \varphi(c + K) & & \\ f(r) &= f(c) & & [\textit{Definition of } \varphi] \\ f(r) - f(c) &= 0_S & & \\ f(r - c) &= 0_S. & & [f \textit{ is a homomorphism.}] \end{aligned}$$

Thus, $r - c \in K$ and hence, $r \equiv c \pmod K$. So $r + K = c + K$ by Theorem 6.6. Therefore, φ is injective.

Finally, φ is a homomorphism because f is

$$\begin{aligned} \varphi[(c + K)(d + K)] &= \varphi(cd + K) = f(cd) = f(c)f(d) \\ &= \varphi(c + K)\varphi(d + K) \end{aligned}$$

and

$$\begin{aligned} \varphi[(c + K) + (d + K)] &= \varphi[(c + d) + K] = f(c + d) = f(c) + f(d) \\ &= \varphi(c + K) + \varphi(d + K). \end{aligned}$$

Therefore, $\varphi: R/K \to S$ is an isomorphism. ■

The First Isomorphism Theorem is a useful tool for determining the structure of quotient rings, as illustrated in the following examples.

EXAMPLE 7

In the ring $\mathbb{Z}[x]$, the principal ideal (x) consists of all multiples of x, that is, all polynomials with constant term 0. What does the quotient ring $\mathbb{Z}[x]/(x)$ look like? We can answer the question by using the function $\theta: \mathbb{Z}[x] \to \mathbb{Z}$, which maps each polynomial to its constant term. The function θ is certainly surjective because each $k \in \mathbb{Z}$ is the image of the polynomial $x + k$ in $\mathbb{Z}[x]$. Furthermore, θ is a homomorphism of rings (Exercise 1). The kernel of θ consists of all those polynomials that are mapped to 0, that is, all polynomials with constant term 0. Thus the kernel of θ is the ideal (x). By Theorem 6.13 the quotient ring $\mathbb{Z}[x]/(x)$ is isomorphic to $\mathbb{Z}$.

EXAMPLE 8

Let T be the ring of functions from $\mathbb{R}$ to $\mathbb{R}$ and I the ideal of all functions g such that $g(2) = 0$. In Example 13 of Section 6.1 we saw that T/I consists of the cosets $f_r + I$, one for each real number r, where $f_r: \mathbb{R} \to \mathbb{R}$ is the constant function given by $f_r(x) = r$ for every x. This suggests the possibility that the quotient ring T/I might be isomorphic to the field $\mathbb{R}$. We shall use

Theorem 6.13 to show that this is indeed the case by constructing a surjective homomorphism from T to $\mathbb{R}$ whose kernel is the ideal I. Let $\varphi: T \to \mathbb{R}$ be the function defined by $\varphi(f) = f(2)$. Then φ is surjective because for every real number r, $r = f_r(2) = \varphi(f_r)$. Furthermore, φ is a homomorphism of rings:

$$\varphi(f + h) = (f + h)(2) = f(2) + h(2) = \varphi(f) + \varphi(h)$$
$$\varphi(fh) = (fh)(2) = f(2)h(2) = \varphi(f)\varphi(h).$$

By definition, the kernel of φ is the set

$$\{g \in T \mid \varphi(g) = 0\} = \{g \in T \mid g(2) = 0\}.$$

Thus the kernel is precisely the ideal I. By Theorem 6.13, T/I is isomorphic to $\mathbb{R}$.

EXAMPLE 9

What do the homomorphic images of the ring $\mathbb{Z}$ look like? To answer this question, suppose that $f: \mathbb{Z} \to S$ is a surjective homomorphism. If f is actually an isomorphism, then S looks exactly like $\mathbb{Z}$, of course (in terms of algebraic structure). If f is surjective, but not an isomorphism (that is, not injective), then the kernel K of f is a nonzero ideal in $\mathbb{Z}$ by Theorem 6.11. Since K is an ideal in $\mathbb{Z}$, K must be a principal ideal, say $K = (n)$ for some $n \neq 0$, by Exercise 40 in Section 6.1. By Theorem 6.13, S is isomorphic to $\mathbb{Z}/K = \mathbb{Z}/(n) = \mathbb{Z}_n$. Thus every homomorphic image of $\mathbb{Z}$ is isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}_n$ for some n.

Exercises

A. 1. Show that the map $\theta: \mathbb{R}[x] \to \mathbb{R}$ that sends each polynomial $f(x)$ to its constant term is a surjective homomorphism.

2. Show that every homomorphic image of a field F is isomorphic either to F itself or to the zero ring. [*Hint:* See Exercise 10 in Section 6.1 and Exercise 7 below.]

3. If F is a field, R a nonzero ring, and $f: F \to R$ a surjective homomorphism, prove that f is an isomorphism.

4. Let $[a]_n$ denote the congruence class of the integer a modulo n.

   (a) Show that the map $f: \mathbb{Z}_{12} \to \mathbb{Z}_4$ that sends $[a]_{12}$ to $[a]_4$ is a well-defined, surjective homomorphism.

   (b) Find the kernel of f.

5. Let I be an ideal in an integral domain R. Is it true that R/I is also an integral domain?

6. The function $\varphi: \mathbb{R}[x] \to \mathbb{R}$ given by $\varphi(f(x)) = f(2)$ is a homomorphism of rings by Exercise 24 of Section 4.4 (with $a = 2$). Find the kernel of φ. [*Hint:* Theorem 4.16.]

7. If R is a ring, show that $R/(0_R) \cong R$.

8. Let R and S be rings. Show that $\pi: R \times S \to R$ given by $\pi(r, s) = r$ is a surjective homomorphism whose kernel is isomorphic to S.

9. $R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ is a ring with identity by Example 19 in Section 3.1.

   (a) Show that the map $f: R \to \mathbb{Z}$ given by $f\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = a$ is a surjective homomorphism.

   (b) What is the kernel of f?

10. (a) Let $f: R \to S$ be a surjective homomorphism of rings and let I be an ideal in R. Prove that $f(I)$ is an ideal in S, where $f(I) = \{s \in S \mid s = f(a)$ for some $a \in I\}$.

    (b) Show by example that part (a) may be false if f is not surjective.

11. $\mathbb{Z}[\sqrt{2}]$ is a ring by Exercise 13 of Section 3.1. Let $f: \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$ be the function defined by $f(a + b\sqrt{2}) = a - b\sqrt{2}$.

    (a) Show that f is a surjective homomorphism of rings.

    (b) Use Theorem 6.11 to show that f is also injective and hence is an isomorphism. [You may assume that $\sqrt{2}$ is irrational.]

12. Let I be an ideal in a noncommutative ring R such that $ab - ba \in I$ for all $a, b \in R$. Prove that R/I is commutative.

13. Let I be an ideal in a ring R. Prove that every element in R/I has a square root if and only if for every $a \in R$, there exists $b \in R$ such that $a - b^2 \in I$.

14. Let I be an ideal in a ring R. Prove that every element in R/I is a solution of $x^2 = x$ if and only if for every $a \in R$, $a^2 - a \in I$.

15. Let I be an ideal in a commutative ring R. Prove that R/I has an identity if and only if there exists $e \in R$ such that $ea - a \in I$ for every $a \in R$.

16. Let $I \neq R$ be an ideal in a commutative ring R with identity. Prove that R/I is an integral domain if and only if whenever $ab \in I$, either $a \in I$ or $b \in I$.

17. Suppose I and J are ideals in a ring R and let $f: R \to R/I \times R/J$ be the function defined by $f(a) = (a + I, a + J)$.

    (a) Prove that f is a homomorphism of rings.

    (b) Is f surjective? [*Hint:* Consider the case when $R = \mathbb{Z}$, $I = (2)$, $J = (4)$.]

    (c) What is the kernel of f?

18. Let R be a commutative ring with identity with the property that every ideal in R is principal. Prove that every homomorphic image of R has the same property.

19. Let I and K be ideals in a ring R, with $K \subseteq I$. Prove that $I/K = \{a + K \mid a \in I\}$ is an ideal in the quotient ring R/K.

20. Let $f:R \to S$ be a homomorphism of rings with kernel K. Let I be an ideal in R such that $I \subseteq K$. Show that $\bar{f}:R/I \to S$ given by $\bar{f}(r + I) = f(r)$ is a well-defined homomorphism.

21. Use the First Isomorphism Theorem to show that $\mathbb{Z}_{20}/(5) \cong \mathbb{Z}_5$.

22. Let $f:R \to S$ be a homomorphism of rings. If J is an ideal in S and $I = \{r \in R \mid f(r) \in J\}$, prove that I is an ideal in R that contains the kernel of f.

23. **(a)** Let R be a ring with identity. Show that the map $f:\mathbb{Z} \to R$ given by $f(k) = k1_R$ is a homomorphism.

(b) Show that the kernel of f is the ideal (n), where n is the characteristic of R. [*Hint:* "Characteristic" is defined immediately before Exercise 41 of Section 3.2. Also see Exercise 40 in Section 6.1.]

24. Find at least three idempotents in the quotient ring $\mathbb{Q}[x]/(x^4 + x^2)$. [See Exercise 3 in Section 3.2.]

25. Let R be a commutative ring and J the ideal of all nilpotent elements of R (as in Exercise 30 of Section 6.1). Prove that the quotient ring R/J has no nonzero nilpotent elements.

26. Let S and I be as in Exercise 41 of Section 6.1. Prove that $S/I \cong \mathbb{Z}_2$.

27. Let T and I be as in Exercise 42 of Section 6.1. Prove that $T/I \cong \mathbb{Z}_p$.

28. Let T and I be as in Exercise 44 of Section 6.1. Prove that $T/I \cong \mathbb{R}$.

29. Let S and I be as in Exercise 45 of Section 6.1. Prove that $S/I \cong \mathbb{R} \times \mathbb{R}$.

C. 30. **(The Second Isomorphism Theorem)** Let I and J be ideals in a ring R. Then $I \cap J$ is an ideal in I, and J is an ideal in $I + J$ by Exercises 19 and 20 of Section 6.1. Prove that $\dfrac{I}{I \cap J} \cong \dfrac{I + J}{J}$. [*Hint:* Show that $f:I \to (I + J)/J$ given by $f(a) = a + J$ is a surjective homomorphism with kernel $I \cap J$.]

31. **(The Third Isomorphism Theorem)** Let I and K be ideals in a ring R such that $K \subseteq I$. Then I/K is an ideal in R/K by Exercise 19. Prove that $(R/K)/(I/K) \cong R/I$. [*Hint:* Show that the map $f:R/K \to R/I$ given by $f(r + K) = r + I$ is a well-defined surjective homomorphism with kernel I/K.]

32. **(a)** Let K be an ideal in a ring R. Prove that every ideal in the quotient ring R/K is of the form I/K for some ideal I in R. [*Hint:* Exercises 19 and 22.]

(b) If $f:R \to S$ is a surjective homomorphism of rings with kernel K, prove that there is a bijective function from the set of all ideals of S to the set of all ideals of R that contain K. [*Hint:* Part (a) and Exercise 10.]

EXCURSION: The Chinese Remainder Theorem for Rings (Section 14.3) may be covered at this point if desired.

6.3 The Structure of R/I When I Is Prime or Maximal*

Quotient rings were developed as a natural generalization of the rings $\mathbb{Z}_p$ and $F[x]/(p(x))$. When p is prime and $p(x)$ irreducible, then $\mathbb{Z}_p$ and $F[x]/(p(x))$ are fields. In this section we explore the analogue of this situation for quotient rings of commutative rings. We shall determine the conditions necessary for a quotient ring to be either an integral domain or a field.

Primes in $\mathbb{Z}$ and irreducibles in $F[x]$ play essentially the same role in the structure of the congruence class rings. Our first task in arbitrary commutative rings is to find some reasonable way of describing this role in terms of ideals. According to Theorem 1.5, a nonzero integer p (other than ± 1) is prime if and only if p has this property: Whenever $p \mid bc$, then $p \mid b$ or $p \mid c$. To say that $p \mid a$ means that a is a multiple of p, that is, a is an element of the principal ideal (p) of all multiples of p. Thus this property of primes can be rephrased in terms of ideals:

If $p \neq 0, \pm 1$, then p is prime if and only if
whenever $bc \in (p)$, then $b \in (p)$ or $c \in (p)$.

The condition $p \neq \pm 1$ guarantees that 1 is not a multiple of p and, hence, that the ideal (p) is not all of $\mathbb{Z}$. Using this situation as a model, we have this

Definition An ideal P in a commutative ring R is said to be **prime** if $P \neq R$ and whenever $bc \in P$, then $b \in P$ or $c \in P$.

EXAMPLE 1

As shown above, the principal ideal (p) is prime in $\mathbb{Z}$ whenever p is a prime integer. On the other hand, the ideal $P = (6)$ is not prime in $\mathbb{Z}$ because $2 \cdot 3 \in P$ but $2 \notin P$ and $3 \notin P$.

EXAMPLE 2

The zero ideal in any integral domain R is prime because $ab = 0_R$ implies $a = 0_R$ or $b = 0_R$.

EXAMPLE 3

The implication (1) $\Rightarrow$ (2) of Theorem 4.12 shows that if F is a field and $p(x)$ is irreducible in $F[x]$, then the principal ideal $(p(x))$ is prime in $F[x]$.

*This section is not used in the sequel and may be omitted if desired.

EXAMPLE 4

Let I be the ideal of polynomials with even constant terms in $\mathbb{Z}[x]$. Then I is not principal (Example 8 of Section 6.1) and clearly $I \neq \mathbb{Z}[x]$. Let $f(x) = a_nx^n + \cdots + a_0$ and $g(x) = b_mx^m + \cdots + b_0$ be polynomials in $\mathbb{Z}[x]$ such that $f(x)g(x) \in I$. Then the constant term of $f(x)g(x)$, namely a_0b_0, must be even. Since the product of two odd integers is odd, we conclude that either a_0 is even (that is, $f(x) \in I$) or b_0 is even (that is, $g(x) \in I$). Therefore, I is a prime ideal.

The ideal I in Example 4 is prime, and the quotient ring $\mathbb{Z}[x]/I$ is a field (see Example 3 of Section 6.2). Similarly, $\mathbb{Z}/(p) = \mathbb{Z}_p$ is a field when p is prime. However, the next example shows that R/P may *not* always be a field when P is prime.

EXAMPLE 5

The principal ideal (x) in the ring $\mathbb{Z}[x]$ consists of polynomials that are multiples of x, that is, polynomials with zero constant terms. Hence, $(x) \neq \mathbb{Z}[x]$. If $f(x) = a_nx^n + \cdots + a_0$ and $g(x) = b_mx^m + \cdots + b_0$ and $f(x)g(x) \in I$, then the constant term of $f(x)g(x)$, namely a_0b_0, must be 0. This can happen only if $a_0 = 0$ or $b_0 = 0$, that is, only if $f(x) \in (x)$ or $g(x) \in (x)$. Therefore, (x) is a prime ideal. However, Example 7 of Section 6.2 shows that the quotient ring $\mathbb{Z}[x]/(x)$ is isomorphic to $\mathbb{Z}$. Therefore, $\mathbb{Z}[x]/(x)$ is an integral domain but not a field.

In light of Example 5, the next theorem is the best we can do with prime ideals.

Theorem 6.14

Let P be an ideal in a commutative ring R with identity. Then P is a prime ideal if and only if the quotient ring R/P is an integral domain.

Proof ▸ If P is any ideal in R, then by Theorem 6.6, $a + P = 0_R + P$ in R/P if and only if $a \equiv 0_R \pmod{P}$. Furthermore, $a \equiv 0_R \pmod{P}$ if and only if $a \in P$. So we have this useful fact:

$$(*) \qquad a + P = 0_R + P \text{ in } R/P \qquad \text{if and only if} \qquad a \in P.$$

Suppose P is prime. By Theorem 6.9, R/P is a commutative ring with identity. In order to prove that R/P is an integral domain, we must show that its identity is not the zero element and that it has no zero divisors. Since P is prime, $P \neq R$. Consequently, $1_R \notin P$ because any ideal containing 1_R must be the whole ring. However, $1_R \notin P$ implies that $1_R + P \neq 0_R + P$ in R/P by $(*)$. Now we show that R/P has no zero divisors. If $(b + P)(c + P) = 0_R + P$, then $bc + P = 0_R + P$ and $bc \in P$ by $(*)$. Hence $b \in P$ or $c \in P$. Thus $b + P = 0_R + P$ or $c + P = 0_R + P$, so that R/P has no zero divisors. Therefore R/P is an integral domain.

Now assume that R/P is an integral domain. Then by definition $1_R + P \neq 0_R + P$ and hence $1_R \notin P$ by (*). Therefore $P \neq R$. To complete the proof that P is prime we assume that $bc \in P$ and show that $b \in P$ or $c \in P$. Now if $bc \in P$, then in R/P we have $(b + P)(c + P) = bc + P = 0_R + P$ by (*). Thus $b + P = 0_R + P$ or $c + P = 0_R + P$ because R/P has no zero divisors. Hence $b \in P$ or $c \in P$ by (*). Therefore P is prime. ■

Since the quotient ring modulo a prime ideal is not necessarily a field, it is natural to ask what conditions an ideal must satisfy in order for the quotient ring to be a field.

EXAMPLE 6

Consider the ideal (3) in $\mathbb{Z}$. We know that $\mathbb{Z}/(3) = \mathbb{Z}_3$ is a field. Now consider the ideal (3). Suppose J is an ideal such that $(3) \subseteq J \subseteq \mathbb{Z}$. If $J \neq (3)$, then there exists $a \in J$ with $a \notin (3)$. In particular, $3 \nmid a$, so that 3 and a are relatively prime. Hence, there are integers u and v such that $3u + av = 1$. Since 3 and a are in the ideal J, it follows that $1 \in J$. Therefore $J = \mathbb{Z}$ by Exercise 9 of Section 6.1, and so *there are no ideals strictly between (3) and* $\mathbb{Z}$.

EXAMPLE 7

The quotient ring $\mathbb{Z}[x]/(x)$ is not a field (Example 5). Furthermore, the ideal I of polynomials with even constant terms lies strictly between (x) and $\mathbb{Z}[x]$, that is, $(x) \subsetneqq I \subsetneqq \mathbb{Z}[x]$.

Here is a formal definition of the property suggested by these examples:

Definition

An ideal M in a ring R is said to be **maximal** if $M \neq R$ and whenever J is an ideal such that $M \subseteq J \subseteq R$, then $M = J$ or $J = R$.

Example 6 shows that the ideal (3) is maximal in $\mathbb{Z}$ and Example 7 shows that the ideal (x) is not maximal in $\mathbb{Z}[x]$. Note that a ring may have more than one maximal ideal. The ideal $\{0, 2, 4\}$ is maximal in $\mathbb{Z}_6$, and so is the ideal $\{0, 3\}$. There are infinitely many maximal ideals in $\mathbb{Z}$ (Exercise 3). Maximal ideals provide the following answer to the question posed above:

Theorem 6.15

Let M be an ideal in a commutative ring R with identity. Then M is a maximal ideal if and only if the quotient ring R/M is a field.

Proof ▸ We shall use the same fact that was used in the proof of Theorem 6.14:

$$(*) \qquad a + M = 0_R + M \text{ in } R/M \quad \text{if and only if} \quad a \in M.$$

Suppose R/M is a field. Then by definition $1_R + M \neq 0_R + M$ and hence $1_R \notin M$ by (*). Therefore $M \neq R$. To show that M is maximal, we assume that J is an ideal with $M \subseteq J \subseteq R$ and show that $M = J$ or $J = R$. If $M = J$, there is nothing to prove. If $M \neq J$, then there exists $a \in J$ with $a \notin M$. Hence $a + M \neq 0_R + M$ in the field R/M, and $a + M$ has an inverse $b + M$ such that $(a + M)(b + M) = ab + M = 1_R + M$. Then $ab \equiv 1_R \pmod{M}$ by Theorem 6.6, so that $ab - 1_R = m$ for some $m \in M$. Thus $1_R = ab - m$. Since a and m are in the ideal J, it follows that $1_R \in J$ and $J = R$. Therefore M is a maximal ideal.

Now assume M is a maximal ideal in R. By Theorem 6.9, R/M is a commutative ring with identity. In order to prove that R/M is a field, we first show that its identity is not the zero element. Since M is maximal, $M \neq R$. Consequently, $1_R \notin M$ because any ideal containing 1_R must be the whole ring. However, $1_R \notin M$ implies that $1_R + M \neq 0_R + M$ in R/M by (*).

Next we show that every nonzero element of R/M has a multiplicative inverse. If $a + M$ is a nonzero element of R/M, then $a \notin M$ (otherwise $a + M$ would be the zero coset). The set

$$J = \{m + ra \mid r \in R \text{ and } m \in M\}$$

is an ideal in R that contains M by Exercise 34 of Section 6.1. Furthermore, $a = 0_R + 1_R a$ is in J, so that $M \neq J$. By maximality we must have $J = R$. Hence $1_R \in J$, which implies that $1_R = m + ca$ for some $m \in M$ and $c \in R$. Note that $ca - 1_R = -m \in M$, so that $ca \equiv 1_R \pmod{M}$, and hence $ca + M = 1_R + M$ by Theorem 6.6. Consequently, the coset $c + M$ is the inverse of $a + M$ in R/M:

$$(c + M)(a + M) = ca + M = 1_R + M.$$

So every nonzero element of R/M is a unit (Axiom 12 is satisfied). Therefore, R/M is a field. ∎

Corollary 6.16

In a commutative ring R with identity, every maximal ideal is prime.

Proof ▸ If M is a maximal ideal, then R/M is a field by Theorem 6.15. Hence, R/M is an integral domain by Theorem 3.8. Therefore, M is prime by Theorem 6.14. ∎

Theorem 6.15 can be used to show that several familiar ideals are maximal.

EXAMPLE 8

The ideal I of polynomials with even constant terms in $\mathbb{Z}[x]$ is maximal because $\mathbb{Z}[x]/I$ is a field (see Example 3 of Section 6.2).

EXAMPLE 9

Let T be the ring of functions from $\mathbb{R}$ to $\mathbb{R}$ and let I be the ideal of all functions g such that $g(2) = 0$. In Example 8 of Section 6.2 we saw that T/I is a field isomorphic to $\mathbb{R}$. Therefore, I is a maximal ideal in T.

Exercises

A. 1. If n is a composite integer, prove that (n) is not a prime ideal in $\mathbb{Z}$.

2. If R is a finite commutative ring with identity, prove that every prime ideal in R is maximal. [*Hint:* Theorem 3.9.]

3. **(a)** Prove that a nonzero integer p is prime if and only if the ideal (p) is maximal in $\mathbb{Z}$.

   (b) Let F be a field and $p(x) \in F[xl$. Prove that $p(x)$ is irreducible if and only if the ideal $(p(x))$ is maximal in $F[x]$.

4. Let R be a commutative ring with identity. Prove that R is an integral domain if and only if (0_R) is a prime ideal.

5. List all maximal ideals in $\mathbb{Z}_6$. Do the same in $\mathbb{Z}_{12}$.

6. **(a)** Show that there is exactly one maximal ideal in $\mathbb{Z}_8$. Do the same for $\mathbb{Z}_9$. [*Hint:* Exercise 6 in Section 6.1.]

   (b) Show that $\mathbb{Z}_{10}$ and $\mathbb{Z}_{15}$ have more than one maximal ideal.

7. Let R be a commutative ring with identity. Prove that R is a field if and only if (0_R) is a maximal ideal.

8. Give an example to show that the intersection of two prime ideals need not be prime. [*Hint:* Consider (2) and (3) in $\mathbb{Z}$.]

9. Let R be an integral domain in which every ideal is principal. If (p) is a nonzero prime ideal in R, prove that p has this property: Whenever p factors, $p = cd$, then c or d is a unit in R.

B. 10. Let p be a fixed prime and let J be the set of polynomials in $\mathbb{Z}[x]$ whose constant terms are divisible by p. Prove that J is a maximal ideal in $\mathbb{Z}[x]$.

11. Show that the principal ideal $(x - 1)$ in $\mathbb{Z}[x]$ is prime but not maximal.

12. If p is a prime integer, prove that M is a maximal ideal in $\mathbb{Z} \times \mathbb{Z}$, where $M = \{(pa, b) \mid a, b \in \mathbb{Z}\}$.

13. If I is an ideal in a ring R, then $I \times I$ is an ideal in $R \times R$ by Exercise 8 of Section 6.1. Prove that $(R \times R)/(I \times I)$ is isomorphic to $R/I \times R/I$. [*Hint:* Show that the function $f: R \times R \to R/I \times R/I$ given by $f((a, b)) = (a + I, b + I)$ is a surjective homomorphism of rings with kernel $I \times I$.]

14. If P is a prime ideal in a commutative ring R, is the ideal $P \times P$ a prime ideal in $R \times R$? [*Hint:* Exercise 13.]

15. **(a)** Let R be the set of integers equipped with the usual addition and multiplication given by $ab = 0$ for all $a, b \in R$. Show that R is a commutative ring.

(b) Show that $M = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$ is a maximal ideal in R that is *not prime*. Explain why this result does not contradict Corollary 6.16.

16. Show that $M = \{0, \pm 4, \pm 8, \ldots\}$ is a maximal ideal in the ring E of even integers but E/M is *not* a field. Explain why this result does not contradict Theorem 6.15.

17. Let $f: R \to S$ be a surjective homomorphism of commutative rings. If J is a prime ideal in S, and $I = \{r \in R \mid f(r) \in J\}$, prove that I is a prime ideal in R.

18. Let P be an ideal in a commutative ring R with $P \neq R$. Prove that P is prime if and only if it has this property: Whenever A and B are ideals in R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. [AB is defined in Exercise 36 of Section 6.1. This property is used as a definition of prime ideal in noncommutative rings.]

19. Assume that when R is a nonzero ring with identity, then every ideal of R except R itself is contained in a maximal ideal (the proof of this fact is beyond the scope of this book). Prove that a commutative ring R with identity has a unique maximal ideal if and only if the set of nonunits in R is an ideal. Such a ring is called a **local ring.** (See Exercise 6 of Section 6.1 for examples of local rings.)

20. Find an ideal in $\mathbb{Z} \times \mathbb{Z}$ that is prime but not maximal.

C. **21.** **(a)** Prove that $R = \{a + bi \mid a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$ and that

$$M = \{a + bi \mid 3 \mid a \text{ and } 3 \mid b\}$$

is a maximal ideal in R. [*Hint:* If $r + si \notin M$, then $3 \nmid r$ or $3 \nmid s$. Show that 3 does not divide $r^2 + s^2 = (r + si)(r - si)$. Then show that any ideal containing $r + si$ and M also contains 1.]

(b) Show that R/M is a field with nine elements.

22. Let R be as in Exercise 21. Show that J is not a maximal ideal in R, where $J = \{a + bi \mid 5 \mid a \text{ and } 5 \mid b\}$. [*Hint:* Consider the principal ideal $K = (2 + i)$ in R.]

23. If R and J are as in Exercise 22, show that $R/J \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.

24. If R and K are as in Exercise 22, show that $R/K \cong \mathbb{Z}_5$.

25. Prove that $T = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{R}$ and $M = \{a + b\sqrt{2} \mid 5 \mid a \text{ and } 5 \mid b\}$ is a maximal ideal in T.

ALTERNATIVE ROUTES: At this point there are three possibilities. You may explore a new algebraic concept, groups (Chapter7)—if you have not already done so—or continue further with either integral domains (Chapter 10) or fields (Chapter 11).

CHAPTER 7

Groups

The algebraic systems with which you are familiar, such as $\mathbb{Z}$, $\mathbb{Z}_n$, the rational numbers, the real numbers, and other rings all have two operations: addition and multiplication. In this chapter, we introduce a different kind of algebraic structure—called a group—that uses a single operation. Groups arise naturally in the study of symmetry, geometric transformations, algebraic coding theory, and in the analysis of the solutions of polynomial equations.

> **ALTERNATE ROUTE:** If you have not read Chapter 3 (Rings), you should replace Section 7.1 with Section 7.1.A, which begins on page 183.

7.1 Definition and Examples of Groups

A group is an algebraic system with one operation. Some groups arise from rings by ignoring one of their operations and concentrating on the other. As we shall see, for example, the integers form a group under addition (but not multiplication) and the nonzero rational numbers form a group under multiplication (but not addition). But many groups do not arise from a system with two operations. The most important of these latter groups (the ones that were the historical starting point of group theory) developed from the study of permutations.* Consequently, we begin with a consideration of permutations.

Informally, a permutation of a set T is just an ordering of its elements. For example, there are six possible permutations of $T = \{1, 2, 3\}$:

$$1\,2\,3 \qquad 1\,3\,2 \qquad 2\,1\,3 \qquad 2\,3\,1 \qquad 3\,1\,2 \qquad 3\,2\,1.$$

*In the early nineteenth century, permutations played a key role in the attempt to find formulas for solving higher-degree polynomial equations similar to the quadratic formula. For more information, see Chapter 12.

Each such ordering determines a bijective function from T to T: map 1 to the first element of the ordering, 2 to the second, and 3 to the third.* For instance, 2 3 1 determines the function $f: T \to T$ whose rule is $f(1) = 2; f(2) = 3; f(3) = 1$. Conversely, every bijective function from T to T defines an ordering of the elements, namely, $f(1)$, $f(2), f(3)$. Consequently, we *define* a **permutation of a set T** to be a bijective function from T to T. This definition preserves the informal idea of ordering and has the advantage of being applicable to infinite sets. For now, however, we shall concentrate on finite sets and develop a convenient notation for dealing with their permutations.

EXAMPLE 1

Let $T = \{1, 2, 3\}$. The permutation f whose rule is $f(1) = 2, f(2) = 3, f(3) = 1$ may be represented by the array $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, in which the image under f of an element in the first row is listed immediately below it in the second row. Using this notation, the six permutations of T are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Since the composition of two bijective functions is itself bijective, the composition of any two of these permutations is one of the six permutations on the list above. For instance, if $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, then $f \circ g$ is the function given by

$$\begin{aligned} (f \circ g)(1) &= f(g(1)) = f(2) = 2 \\ (f \circ g)(2) &= f(g(2)) = f(1) = 3 \\ (f \circ g)(3) &= f(g(3)) = f(3) = 1. \end{aligned}$$

Thus $f \circ g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. It is usually easier to make computations like this by visually tracing an element's progress as we first apply g and then f; for example,

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & & \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ & \downarrow & \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

---

*Bijective functions are discussed in Appendix B.

If we denote the set of permutations of T by S_3, then composition of functions ($\circ$) is an operation on the set S_3 with this property:

$$\text{If } f \in S_3 \text{ and } g \in S_3, \text{ then } f \circ g \in S_3.$$

Since composition of functions is associative,* we see that

$$(f \circ g) \circ h = f \circ (g \circ h) \qquad \text{for all } f, g, h \in S_3.$$

Verify that the identity permutation $I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ has this property:

$$I \circ f = f \qquad \text{and} \qquad f \circ I = f \qquad \text{for every } f \in S_3.$$

Every bijection has an inverse function;* consequently,

$$\text{if } f \in S_3, \text{ then there exists } g \in S_3 \text{ such that}$$
$$f \circ g = I \qquad \text{and} \qquad g \circ f = I.$$

For instance, if $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, then $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ because

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

You should determine the inverses of the other permutations in S_3 (Exercise 1). Finally, note that $f \circ g$ may not be equal to $g \circ f$; for instance,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

but

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

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*See Appendix B.

By abstracting the key properties of S_3 under the operation $\circ$, we obtain this

Definition

A **group** is a nonempty set G equipped with a binary operation $*$ that satisfies the following axioms[†]:

1. Closure: If $a \in G$ and $b \in G$, then $a * b \in G$.
2. Associativity: $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.
3. There is an element $e \in G$ (called the **identity element**) such that $a * e = a = e * a$ for every $a \in G$.
4. For each $a \in G$, there is an element $d \in G$ (called the **inverse** of a) such that $a * d = e$ and $d * a = e$.

A group is said to be **abelian**[‡] if it also satisfies this axiom:

5. Commutativity: $a * b = b * a$ for all $a, b \in G$.

A group G is said to be **finite** (or of **finite order**) if it has a finite number of elements. In this case, the number of elements in G is called the **order of** G and is denoted $|G|$. A group with infinitely many elements is said to have **infinite order.**

EXAMPLE 2

The discussion preceding the definition shows that S_3 is a nonabelian group of order 6, with the operation $*$ being composition of functions.

EXAMPLE 3

The permutation group S_3 is just a special case of a more general situation. Let n be a fixed positive integer and let T be the set $\{1, 2, 3, \ldots, n\}$. Let S_n be the set of all permutations of T (that is, all bijections $T \to T$). We shall use the same notation for such functions as we did in S_3. In S_6, for instance, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 3 & 5 & 1 \end{pmatrix}$ denotes the permutation that takes 1 to 4, 2 to 6, 3 to 2, 4 to 3, 5 to 5, and 6 to 1. Since the composite of two bijective functions is bijective,[§] S_n is closed under the operation of composition. For example, in S_6

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 1 & 6 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 3 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

(Remember that in composition of functions, we apply the right-hand function first and then the left-hand one. In this case, for instance, $4 \to 3 \to 2$, as shown

[†] Binary operations are defined in Appendix B.

[‡] In honor of the Norwegian mathematician N. H. Abel (1802–1829).

[§] See Appendix B.

by the arrows.) We claim that S_n is a group under this operation. Composition of functions is known to be associative, and every bijection has an inverse function under composition.[†] It is easy to verify that the identity permutation $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$ is the identity element of S_n. S_n is called the **symmetric group** on n symbols. The order of S_n is $n! = n(n-1)(n-2)\dots 2.1$ (Exercise 20).

EXAMPLE 4

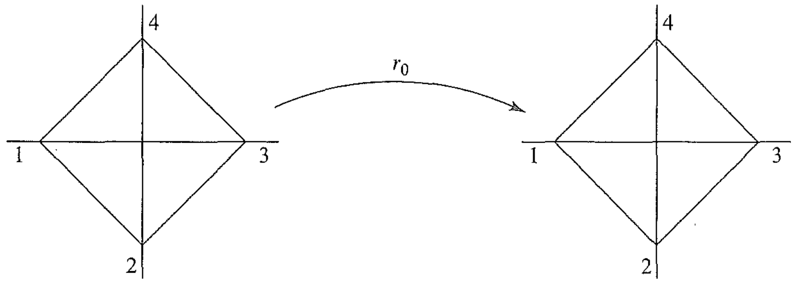
The preceding example is easily generalized. Let T be any nonempty set, possibly infinite. Let $A(T)$ be the set of all permutations of T (all bijective functions $T \to T$). The arguments given above for S_n carry over to $A(T)$ and show that $A(T)$ is a group under the operation of composition of functions (Exercise 12).

EXAMPLE 5

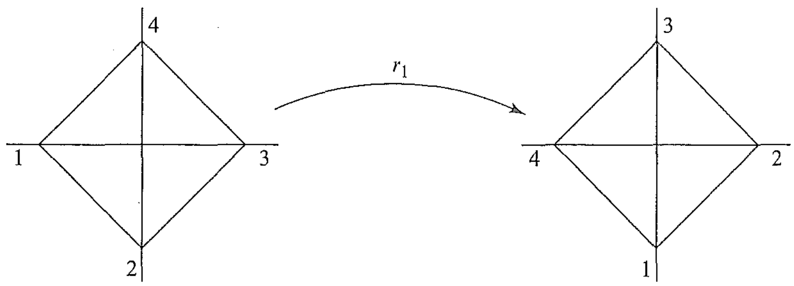
Think of the plane as a sheet of thin, rigid plastic. Suppose you cut out a square, pick it up, and move it around,[‡] then replace it so that it fits exactly in the cut-out space. Eight ways of doing this are shown below (where the square is centered at the origin and its corners numbered for easy reference). We claim that any motion of the square that ends with the square fitting exactly in the cut-out space has the same result as one of these eight motions (Exercise 14).

All Rotations Are Taken Counterclockwise Around the Center:

r_0 = rotation of 0°



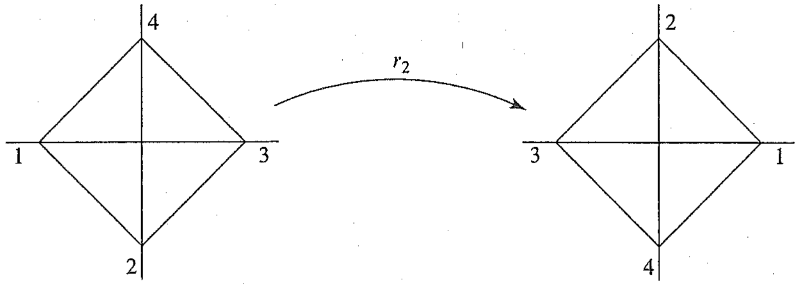
r_1 = rotation of 90°



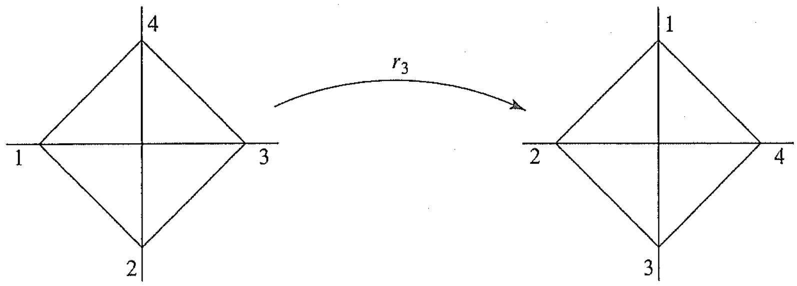
[†]See Appendix B for details.

[‡]Flip it, rotate it, turn it over, spin it, do whatever you want, as long as you don't bend, break, or distort it.

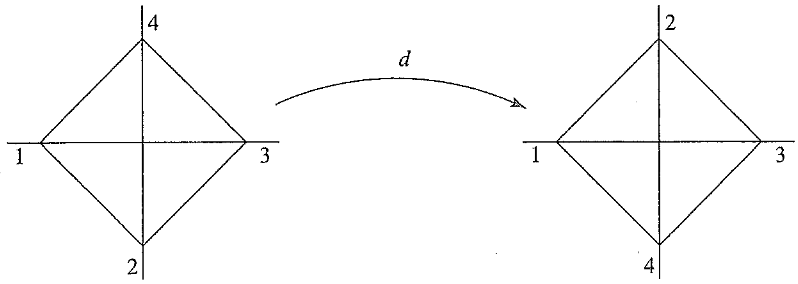
r_2 = rotation of 180°



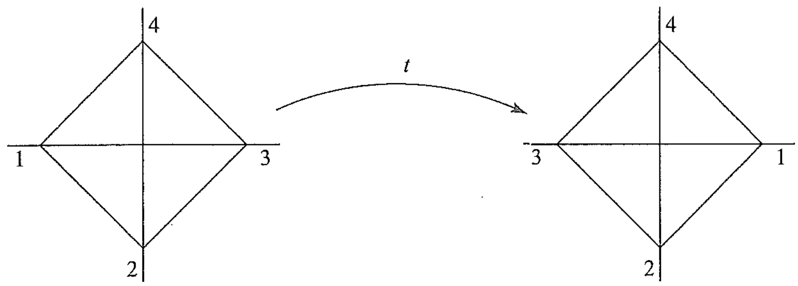
r_3 = rotation of 270°



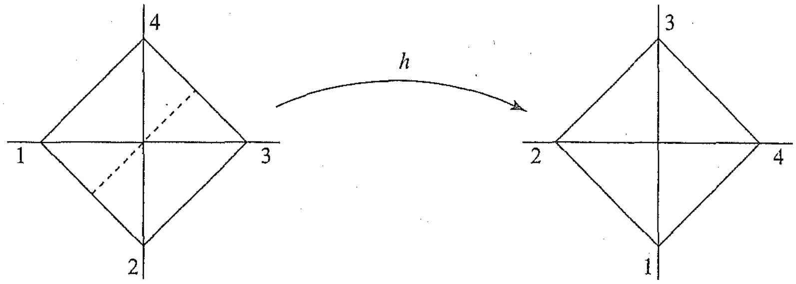
d = reflection in the x-axis



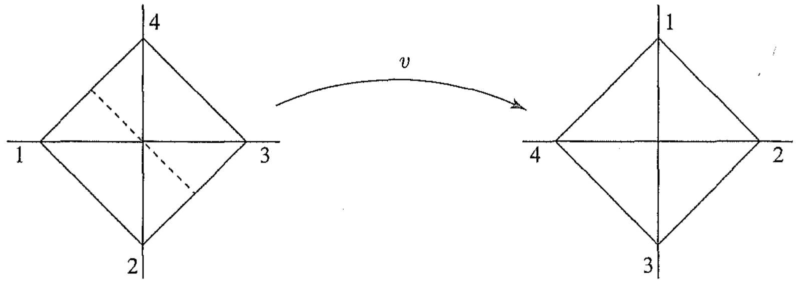
t = reflection in the y-axis



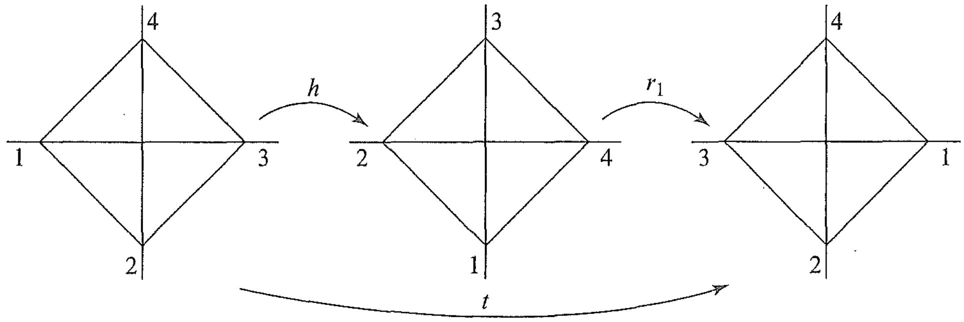
h = reflection in line $y = x$



v = reflection in line $y = -x$



If you perform one of these motions and follow it by another, the result will be one of the eight listed above; for example,



If you think of a motion as a function from the square to itself, then the idea of following one motion by another is just composition of functions. In the illustration above (h followed by r_1 is t), we can write $r_1 \circ h = t$ (remember $r_1 \circ h$ means first apply h, then apply r_1). Verify that the set

$$D_4 = \{r_0, r_1, r_2, r_3, h, v, d, t\}$$

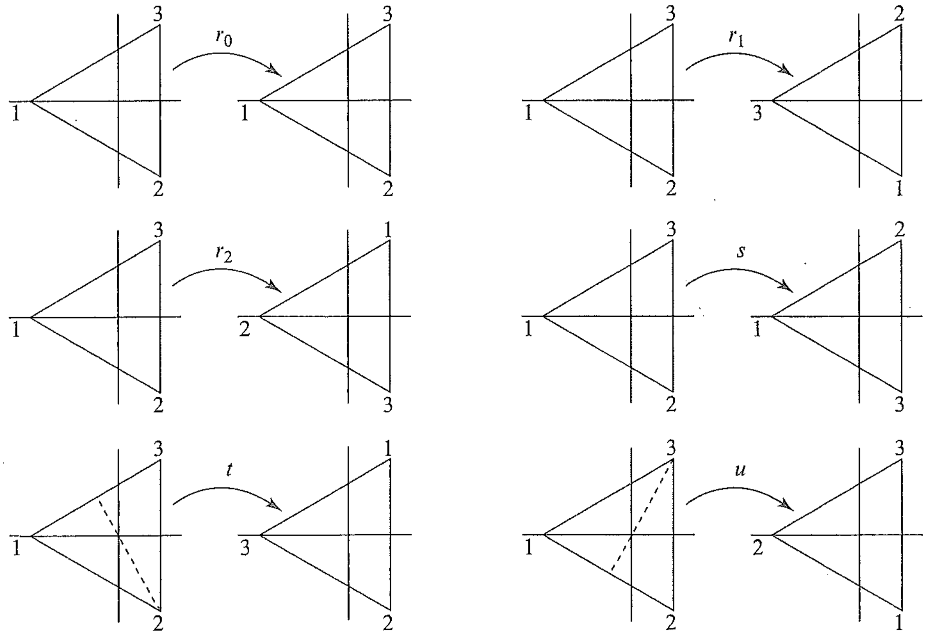
equipped with the composition operation has this table:

$\circ$	r_0	r_1	r_2	r_3	d	h	t	v
r_0	r_0	r_1	r_2	r_3	d	h	t	v
r_1	r_1	r_2	r_3	r_0	h	t	v	d
r_2	r_2	r_3	r_0	r_1	t	v	d	h
r_3	r_3	r_0	r_1	r_2	v	d	h	t
d	d	v	t	h	r_0	r_3	r_2	r_1
h	h	d	v	t	r_1	r_0	r_3	r_2
t	t	h	d	v	r_2	r_1	r_0	r_3
v	v	t	h	d	r_3	r_2	r_1	r_0

Clearly D_4 is closed under $\circ$, and composition of functions is known to be associative. The table shows that r_0 is the identity element and that every element of D_4 has an inverse. For instance, $r_3 \circ r_1 = r_0 = r_1 \circ r_3$ .Therefore, D_4 is a group. It is not abelian because, for example, $h \circ d \neq d \circ h$. D_4 is called the **dihedral group of degree 4** or the **group of symmetries of the square.**

EXAMPLE 6

The group of symmetries of the square is just one of many symmetry groups. An analogous procedure can be carried out with any regular polygon of n sides. The resulting group D_n is called the **dihedral group of degree n.** The group D_3, for example, consists of the six symmetries of an equilateral triangle (counterclockwise rotations about the center of 0°, 120°, and 240°; and the three reflections shown here), with composition of functions as the operation:



Symmetry groups arise frequently in art, architecture, and science. Crystallography and crystal physics use groups of symmetries of various three-dimensional shapes. The first accurate model of DNA (which led to the Nobel Prize for its creators) could not have been constructed without a recognition of the symmetry of the DNA molecule. Symmetry groups have been used by physicists to predict the existence of certain elementary particles that were later found experimentally.

Groups and Rings

A ring R has two associative operations, and it is natural to ask if R is a group under either one. For addition the answer is yes:

Theorem 7.1

Every ring is an abelian group under addition.

Proof ▸ An examination of the first five axioms for a ring (in Section 3.1) shows that they are identical to the five axioms for an abelian group, with the operation $*$ being $+$, the identity element e being 0_R, and the inverse of a being $-a$. ■

EXAMPLE 7

By Theorem 7.1, each of the following familiar rings is an abelian group under addition:

$$\mathbb{Z},\quad \mathbb{Z}_n,\quad \mathbb{Q},\quad \mathbb{R},\quad \mathbb{C};$$

Matrix rings, such as $M(\mathbb{R})$ and $M(\mathbb{Z}_2)$;

Polynomial rings such as $\mathbb{Z}[x]$, $\mathbb{R}[x]$, and $\mathbb{Z}_n[x]$.

Hereafter, when we use the word "group" without any qualification in referring to these or other rings, it is understood that the operation is addition.

Multiplication, however, is a different story:

A nonzero ring R is *never* a group under multiplication.

If R has no identity, Axiom 3 fails. If R has an identity, then 0_R has no inverse and Axiom 4 fails. Nevertheless, certain *subsets* of a ring with identity may be groups under multiplication.

Theorem 7.2

The nonzero elements of a field F form an abelian group under multiplication.

Hereafter we shall denote the set of nonzero elements in a field F by F^*.

Proof of Theorem 7.2 ▸ Multiplication in F^* satisfies the following ring axioms: 6 and 11 (closure), 7 (associativity), 10 (identity), 12 (inverses), and 9 (commutativity)—see pages 44, 48, and 49. So F^* satisfies group axioms 1–5 and, therefore, is an abelian group under multiplication. ■

EXAMPLE 8

Theorem 7.2 shows that each of the following is an abelian group under multiplication:

$\mathbb{Q}^*$ the nonzero rational numbers; $\mathbb{R}^*$ the nonzero real numbers;

$\mathbb{C}^*$ the nonzero complex numbers.

EXAMPLE 9

If p is prime, then $\mathbb{Z}_p$ is a field by Theorems 2.7 and 2.8. Therefore, $\mathbb{Z}_p{}^*$ is a group under multiplication by Theorem 7.2.

EXAMPLE 10

The positive rational numbers $\mathbb{Q}^{**}$ form an infinite abelian group under multiplication, because the product of positive numbers is positive, 1 is the identity element, and the inverse of a is $1/a$. Similarly, the positive reals $\mathbb{R}^{**}$ form an abelian group under multiplication.

EXAMPLE 11

The subset $\{1, -1, i, -i\}$ of the complex numbers forms an abelian group of order 4 under multiplication. You can easily verify closure, and 1 is the identity element. Since $i(-i) = 1$, i and $-i$ are inverses of each other; -1 is its own inverse since $(-1)(-1) = 1$. Hence, Axiom 4 holds.

EXAMPLE 12

Neither the nonzero integers nor the positive integers form a group under multiplication. Although 1 is the multiplicative identity for each system, no integers except for ± 1 have a multiplicative inverse, so Axiom 4 fails. For example, the equation $2x = 1$ has no *integer* solution, so 2 has no inverse under multiplication in the integers.

EXAMPLE 13

When n is composite, the nonzero elements of $\mathbb{Z}_n$ do not form a group under multiplication because (among other things) closure fails. In $\mathbb{Z}_6$, for instance, $2 \cdot 3 = 0$ and in $\mathbb{Z}_{20}$, $4 \cdot 5 = 0$. Similarly if $n = rs$, then in $\mathbb{Z}_n$, $rs = 0$.

A ring R with identity always has at least one subset that is a group under multiplication. Recall that a *unit* in R is an element a that has a multiplicative inverse, that is, an element u such that $au = 1_R = ua$.

Theorem 7.3

If R is a ring with identity, then the set U of all units in R is a group under multiplication.*

Proof ▸ The product of units is a unit (Exercise 15 in Section 3.2), so U is closed under multiplication (Axiom 1). Multiplication in R is associative, so Axiom 2 holds. Since 1_R is obviously a unit, U has an identity element (Axiom 3). Axiom 4 holds in U by the definition of unit. Therefore, U is a group. ■

EXAMPLE 14

Denote the multiplicative group of units in $\mathbb{Z}_n$ by U_n. According to Theorem 2.10, U_n consists of all $a \in \mathbb{Z}_n$ such that $(a, n) = 1$ (when a is considered as an ordinary integer). Thus the group of units in $\mathbb{Z}_8$ is $U_8 = \{1, 3, 5, 7\}$, and the group of units in $\mathbb{Z}_{15}$ is $U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$. Here is the operation table for U_8:

$\cdot$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

EXAMPLE 15

Examples 7 and 10 of Section 3.2, and Exercise 17 of Section 3.2 show that the group of units in $M(\mathbb{R})$ is

$$GL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \text{ where } a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\},$$

which is called the **general linear group** of degree 2 over $\mathbb{R}$. It is an infinite nonabelian group (Exercise 7).

EXAMPLE 16

Examples 8 and 10 of Section 3.2, and Exercise 17 of Section 3.2 show that the group of units in $M(\mathbb{Z}_2)$ is

$$GL(2, \mathbb{Z}_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \text{ where } a, b, c, d \in \mathbb{Z}_2 \text{ and } ad - bc \neq 0 \right\},$$

the general linear group of degree 2 over $\mathbb{Z}_2$. It is a nonabelian finite group of order 6 (Exercise 7).

*Theorem 7.2 is a special case of Theorem 7.3 because the units in a field are the nonzero elements.

New Groups from Old

The Cartesian product, with operations defined coordinatewise, allowed us to construct new rings from known ones. The same is true for groups.

Theorem 7.4

Let G (with operation $*$) and H (with operation $\diamond$) be groups. Define an operation $\blacksquare$ on $G \times H$ by

$$(g, h) \blacksquare (g', h') = (g * g', h \diamond h').$$

Then $G \times H$ is a group. If G and H are abelian, then so is $G \times H$. If G and H are finite, then so is $G \times H$ and $|G \times H| = |G||H|$.

Proof ▸ Exercise 26. ■

EXAMPLE 17

Both $\mathbb{Z}$ and $\mathbb{Z}_6$ are groups under addition. In $\mathbb{Z} \times \mathbb{Z}_6$ we have $(3, 5) \blacksquare (7, 4) = (3 + 7, 5 + 4) = (10, 3)$. The identity is $(0, 0)$, and the inverse of $(7, 4)$ is $(-7, 2)$.

EXAMPLE 18

Consider $\mathbb{R}^* \times D_4$, where $\mathbb{R}^*$ is the multiplicative group of nonzero real numbers. The table in Example 5 shows that

$$(2, r_1) \blacksquare (9, v) = (2 \cdot 9, r_1 \circ v) = (18, d).$$

The identity element is $(1, r_0)$, and the inverse of $(8, r_3)$ is $(1/8, r_1)$.

Exercises

A. **1.** Find the inverse of each permutation in S_3.

2. Find the multiplicative inverse of each nonzero element in

(a) $\mathbb{Z}_3$ **(b)** $\mathbb{Z}_5$ **(c)** $\mathbb{Z}_7$

3. What is the order of each group:

(a) $\mathbb{Z}_{18}$ **(b)** D_4 **(c)** S_4 **(d)** S_5 **(e)** U_{18}

4. Determine whether the set G is a group under the operation $*$.

(a) $G = \{2, 4, 6, 8\}$ in $\mathbb{Z}_{10}$; $a * b = ab$

(b) $G = \mathbb{Z}$; $a * b = a - b$

(c) $G = \{n \in \mathbb{Z} \mid n \text{ is odd}\}$; $a * b = a + b$

(d) $G = \{2^x \mid x \in \mathbb{Q}\}$; $a * b = ab$

5. Find the inverse of the given group element. [*Hint:* Example 8 in Section 3.2—or Example 16 in Section 7.1.A—and Exercise 2.]

   (a) $\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ in $\mathbb{Z}_3$ (b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in $\mathbb{Z}_5$ (c) $\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$ in $\mathbb{Z}_7$

6. Give an example of an abelian group of order 4 in which every nonidentity element a satisfies $a * a = e$. [*Hint:* Consider Theorem 7.4.]

7. (a) Show that the group $GL(2, \mathbb{Z}_2)$ has order 6 by listing all its elements.

   (b) Show by example that the groups $GL(2, \mathbb{R})$ and $GL(2, \mathbb{Z}_2)$ are nonabelian.

8. Use Theorem 2.10 to list the elements of each of these groups: U_4, U_6, U_{10}, U_{20}, U_{30}.

9. Write out the operation table for the group D_3 described in Example 6.

10. Show that $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \, a, b \in \mathbb{R}, \text{ not both } 0 \right\}$ is an abelian group under matrix multiplication.

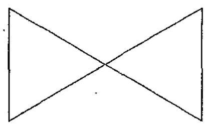
11. Consider the additive group $\mathbb{Z}_2$ and the multiplicative group $L = \{\pm 1, \pm i\}$ of complex numbers. Write out the operation table for the group $\mathbb{Z}_2 \times L$.

12. Let T be a nonempty set and $A(T)$ the set of all permutations of T. Show that $A(T)$ is a group under the operation of composition of functions.

13. Give examples of nonabelian groups of orders 12, 16, 30, and 48. [*Hint:* Theorem 7.4 may be helpful.]

B. 14. Show that every rigid motion of the square (as described in the footnote at the beginning of Example 5) has the same result as an element of D_4. [*Hint:* The position of the square after any motion is completely determined by the location of corner 1 and by the orientation of the square—face up or face down.]

15. Write out the operation table for the symmetry groups of the following figures:

   (a)  (b)  (c) 

16. Let $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ be the following matrices with complex entries:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

   (a) Prove that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1} \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$
$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i} \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

   (b) Show that set $Q = \{\mathbf{1}, \mathbf{i}, -\mathbf{1}, -\mathbf{i}, \mathbf{j}, \mathbf{k}, -\mathbf{j}, -\mathbf{k}\}$ is a group under matrix multiplication by writing out its multiplication table. Q is called the **quaternion group.**

17. If G is a group under the stated operation, prove it; if not, give a counterexample:

    (a) $G = \mathbb{Q}$; $a * b = a + b + 3$

    (b) $G = \{r \in \mathbb{Q} \mid r \neq 0\}$; $a * b = ab/3$

18. Let $K = \{r \in \mathbb{R} \mid r \neq 0, r \neq 1\}$. Let G consist of these six functions from K to K:

$$f(x) = \frac{1}{1 - x} \qquad g(x) = \frac{x - 1}{x} \qquad h(x) = \frac{1}{x}$$

$$i(x) = x \qquad j(x) = 1 - x \qquad k(x) = \frac{x}{x - 1}$$

    Is G a group under the operation of function composition?

19. Do the nonzero real numbers form a group under the operation given by $a * b = |a|b$, where $|a|$ is the absolute value of a?

20. Prove that S_n has order $n!$. [*Hint:* There are n possible images for 1; after one has been chosen, there are $n - 1$ possible images for 2; etc.]

21. Suppose G is a group with operation $*$. Define a new operation # on G by $a \# b = b * a$. Prove that G is a group under #.

22. List the elements of the group D_5 (the symmetries of a regular pentagon). [*Hint:* The group has order 10.]

23. Let $SL(2, \mathbb{R})$ be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Prove that $SL(2, \mathbb{R})$ is a group under matrix multiplication. It is called the **special linear group.**

24. Prove that the set of nonzero real numbers is a group under the operation $*$ defined by

$$a * b = \begin{cases} ab & \text{if } a > 0 \\ a/b & \text{if } a < 0. \end{cases}$$

25. Prove that $\mathbb{R}^* \times \mathbb{R}$ is a group under the operation $*$ defined by $(a, b) * (c, d) = (ac, bc + d)$.

26. Prove Theorem 7.4.

27. If $ab = ac$ in a group G, prove that $b = c$.

28. Prove that each element of a finite group G appears exactly once in each row and exactly once in each column of the operation table. [*Hint:* Exercise 27.]

29. Here is part of the operation table for a group G whose elements are a, b, c, d. Fill in the rest of the table. [*Hint:* Exercises 27 and 28.]

	a	b	c	d
a	a	b	c	d
b	b	a		
c	c		a	
d	d			

30. A partial operation table for a group $G = \{e, a, b, c, d, f\}$ is shown below. Complete the table. [*Hint:* Exercises 27 and 28.]

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d		
b	b					
c	c	f				a
d	d					
f	f					

31. Let T be a set with at least three elements. Show that the permutation group $A(T)$ (Exercise 12) is nonabelian.

32. Let T be an infinite set and let $A(T)$ be the group of permutations of T (Exercise 12). Let $M = \{f \in A(T) \mid f(t) \neq t \text{ for only a finite number of } t \in T\}$. Prove that M is a group.

33. If $a, b \in \mathbb{R}$ with $a \neq 0$, let $T_{a,b}: \mathbb{R} \to \mathbb{R}$ be the function given by $T_{a,b}(x) = ax + b$. Prove that the set $G = \{T_{a,b} \mid a, b \in \mathbb{R} \text{ with } a \neq 0\}$ forms a nonabelian group under composition of functions.

34. Let $H = \{T_{1,b} \mid b \in \mathbb{R}\}$ (notation as in Exercise 33). Prove that H is an abelian group under composition of functions.

C. 35. If $f \in S_n$, prove that $f^k = I$ for some positive integer k, where f^k means $f \circ f \circ f \circ \cdots \circ f$ (k times) and I is the identity permutation.

36. Let $G = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and assume G is a group under an operation $*$ with these properties:

(i) $a * b \leq a + b$ for all $a, b \in G$;

(ii) $a * a = 0$ for all $a \in G$.

Write out the operation table for G. [*Hint:* Exercises 27 and 28 may help.]

7.1.A Definition and Examples of Groups

NOTE: If you have read Section 7.1, omit this section and begin Section 7.2.

A group is an algebraic system with one operation. Some groups arise from familiar systems, such as $\mathbb{Z}$, $\mathbb{Z}_n$, the rational numbers, and the real numbers, by ignoring one of their operations and concentrating on the other. As we shall see, for example, the integers form a group under addition (but not multiplication) and the nonzero rational numbers form a group under multiplication (but not addition). But many groups do not arise from a system with two operations. The most important of these latter groups (the ones that were the historical starting point of group theory) developed from the study of permutations.* Consequently, we begin with a consideration of permutations.

Informally, a permutation of a set T is just an ordering of its elements. For example, there are six possible permutations of $T = \{1, 2, 3\}$:

$$1\,2\,3 \qquad 1\,3\,2 \qquad 2\,1\,3 \qquad 2\,3\,1 \qquad 3\,1\,2 \qquad 3\,2\,1.$$

Each such ordering determines a bijective function from T to T: map 1 to the first element of the ordering, 2 to the second, and 3 to the third.† For instance, 2 3 1 determines the function $f: T \to T$ whose rule is $f(1) = 2; f(2) = 3; f(3) = 1$. Conversely, every bijective function from T to T defines an ordering of the elements, namely, $f(1), f(2), f(3)$. Consequently, we *define* a **permutation of a set T** to be a bijective function from T to T. This definition preserves the informal idea of ordering and has the advantage of being applicable to infinite sets. For now, however, we shall concentrate on finite sets and develop a convenient notation for dealing with their permutations.

EXAMPLE 1

Let $T = \{1, 2, 3\}$. The permutation f whose rule is $f(1) = 2, f(2) = 3, f(3) = 1$ may be represented by the array $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, in which the image under f of an element in the first row is listed immediately below it in the second row. Using this notation, the six permutations of T are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Since the composition of two bijective functions is itself bijective, the composition of any two of these permutations is one of the six permutations on the list above. For instance, if $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, then $f \circ g$ is the function given by

$$(f \circ g)(1) = f(g(1)) = f(2) = 2$$
$$(f \circ g)(2) = f(g(2)) = f(1) = 3$$
$$(f \circ g)(3) = f(g(3)) = f(3) = 1.$$

*In the early nineteenth century, permutations played a key role in the attempt to find formulas for solving higher-degree polynomial equations similar to the quadratic formula. For more information, see Chapter 12.

†Bijective functions are discussed in Appendix B.

Thus $f \circ g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. It is usually easier to make computations like this by visually tracing an element's progress as we first apply g and then f; for example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

If we denote the set of permutations of T by S_3, then composition of functions ($\circ$) is an operation on the set S_3 with this property:

$$\text{If } f \in S_3 \text{ and } g \in S_3, \text{ then } f \circ g \in S_3.$$

Since composition of functions is associative,* we see that

$$(f \circ g) \circ h = f \circ (g \circ h) \qquad \text{for all } f, g, h \in S_3.$$

Verify that the identity permutation $I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ has this property:

$$I \circ f = f \qquad \text{and} \qquad f \circ I = f \qquad \text{for every } f \in S_3.$$

Every bijection has an inverse function;* consequently,

$$\text{if } f \in S_3, \text{ then there exists } g \in S_3 \text{ such that}$$
$$f \circ g = I \qquad \text{and} \qquad g \circ f = I.$$

For instance, if $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, then $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ because

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

You should determine the inverses of the other permutations in S_3 (Exercise 1). Finally, note that $f \circ g$ may not be equal to $g \circ f$; for instance,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

but

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

---

*See Appendix B.

By abstracting the key properties of S_3 under the operation $\circ$, we obtain this

Definition

> A **group** is a nonempty set G equipped with a binary operation $*$ that satisfies the following axioms[†]:
>
> 1. Closure: If $a \in G$ and $b \in G$, then $a * b \in G$.
> 2. Associativity: $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.
> 3. There is an element $e \in G$ (called the **identity element**) such that $a * e = a = e * a$ for every $a \in G$.
> 4. For each $a \in G$, there is an element $d \in G$ (called the **inverse** of a) such that $a * d = e$ and $d * a = e$.
>
> A group is said to be **abelian**[‡] if it also satisfies this axiom:
>
> 5. Commutativity: $a * b = b * a$ for all $a, b \in G$.

A group G is said to be **finite** (or of **finite order**) if it has a finite number of elements. In this case, the number of elements in G is called the **order of G** and is denoted $|G|$. A group with infinitely many elements is said to have **infinite order.**

EXAMPLE 2

The discussion preceding the definition shows that S_3 is a nonabelian group of order 6, with the operation $*$ being composition of functions.

EXAMPLE 3

The permutation group S_3 is just a special case of a more general situation. Let n be a fixed positive integer and let T be the set $\{1, 2, 3, \ldots, n\}$. Let S_n be the set of all permutations of T (that is, all bijections $T \to T$). We shall use the same notation for such functions as we did in S_3. In S_6, for instance, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 3 & 5 & 1 \end{pmatrix}$ denotes the permutation that takes 1 to 4, 2 to 6, 3 to 2, 4 to 3, 5 to 5, and 6 to 1. Since the composite of two bijective functions is bijective,[§] S_n is closed under the operation of composition. For example, in S_6

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 1 & 6 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 3 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

[†]Binary operations are defined in Appendix B.
[‡]In honor of the Norwegian mathematician N. H. Abel (1802–1829).
[§]See Appendix B for details.

(Remember that in composition of functions, we apply the right-hand function first and then the left-hand one. In this case, for instance, $4 \to 3 \to 2$, as shown by the arrows.) We claim that S_n is a group under this operation. Composition of functions is known to be associative, and every bijection has an inverse function under composition.† It is easy to verify that the identity permutation $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & & n \end{pmatrix}$ is the identity element of S_n. S_n is called the **symmetric group** on n symbols. The order of S_n is $n! = n(n-1)(n-2)\ldots 2.1$ (Exercise 20).

EXAMPLE 4

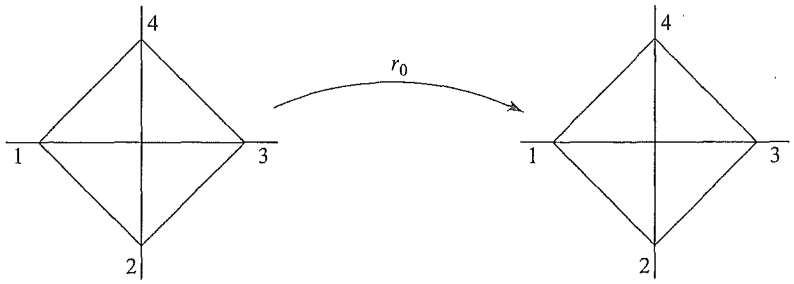
The preceding example is easily generalized. Let T be any nonempty set, possibly infinite. Let $A(T)$ be the set of all permutations of T (all bijective functions $T \to T$). The arguments given above for S_n carry over to $A(T)$ and show that $A(T)$ is a group under the operation of composition of functions (Exercise 12).

EXAMPLE 5

Think of the plane as a sheet of thin, rigid plastic. Suppose you cut out a square, pick it up, and move it around,‡ then replace it so that it fits exactly in the cut-out space. Eight ways of doing this are shown below (where the square is centered at the origin and its corners numbered for easy reference). We claim that any motion of the square that ends with the square fitting exactly in the cut-out space has the same result as one of these eight motions (Exercise 14).

All Rotations Are Taken Counterclockwise Around the Center:

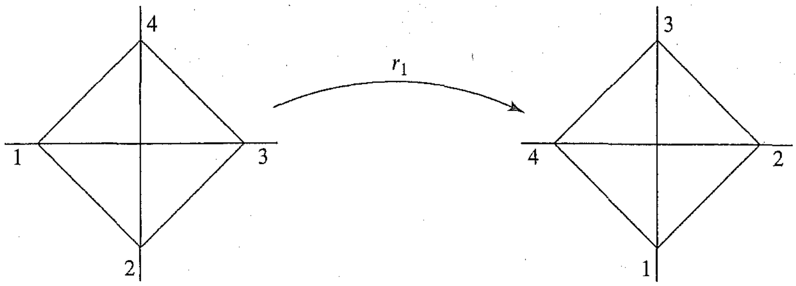
r_0 = rotation of 0°



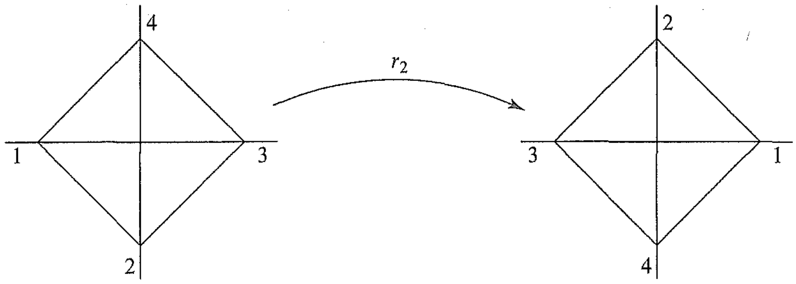
†See Appendix B for details.

‡Flip it, rotate it, turn it over, spin it, do whatever you want, as long as you don't bend, break, or distort it.

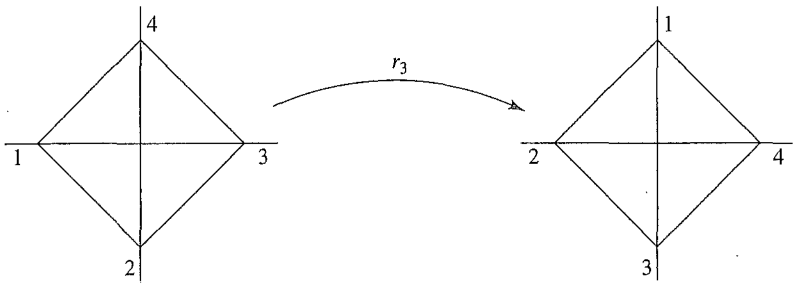
r_1 = rotation of 90°



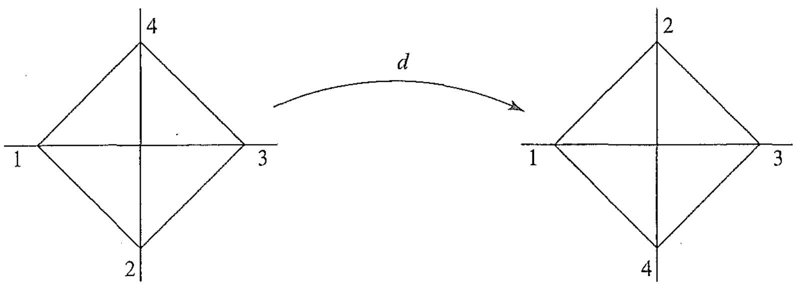
r_2 = rotation of 180°



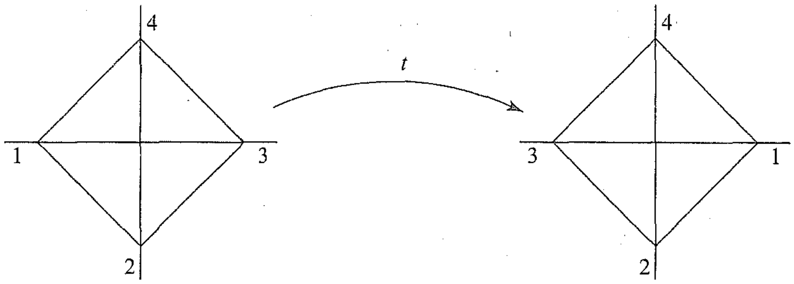
r_3 = rotation of 270°



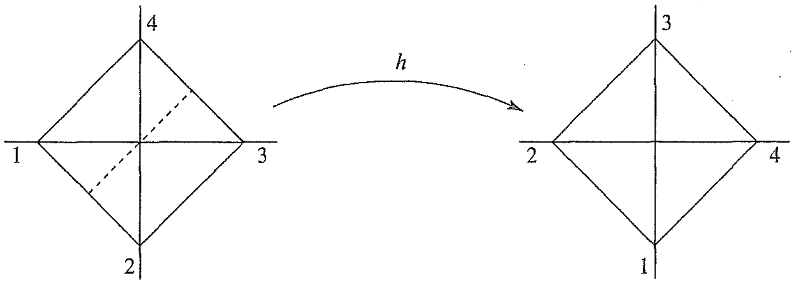
d = reflection in the x-axis



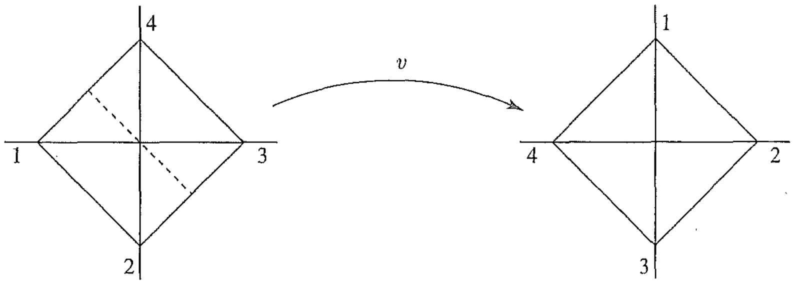
t = reflection in the y-axis



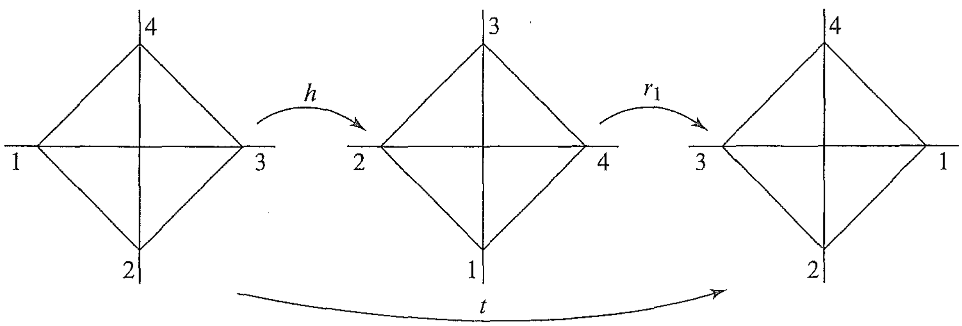
h = reflection in line $y = x$



v = reflection in line $y = -x$



If you perform one of these motions and follow it by another, the result will be one of the eight listed above; for example,



If you think of a motion as a function from the square to itself, then the idea of following one motion by another is just composition of functions. In the illustration above (h followed by r_1 is t), we can write $r_1 \circ h = t$ (remember $r_1 \circ h$ means first apply h, then apply r_1). Verify that the set

$$D_4 = \{r_0, r_1, r_2, r_3, h, v, d, t\}$$

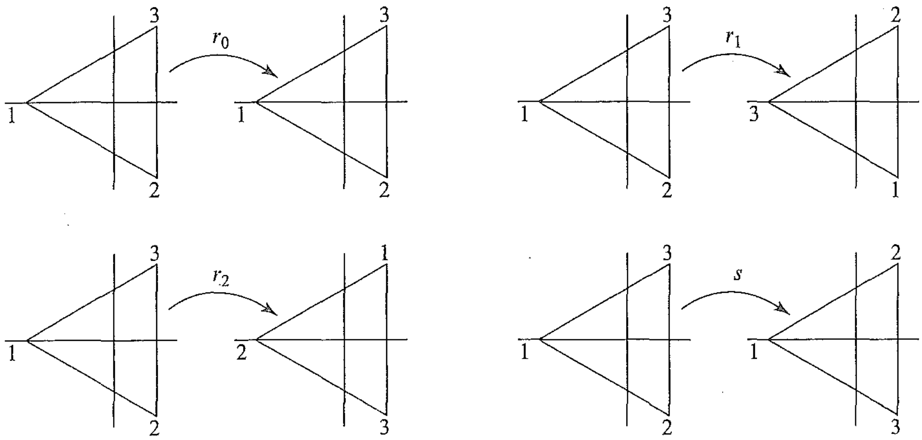
equipped with the composition operation has this table:

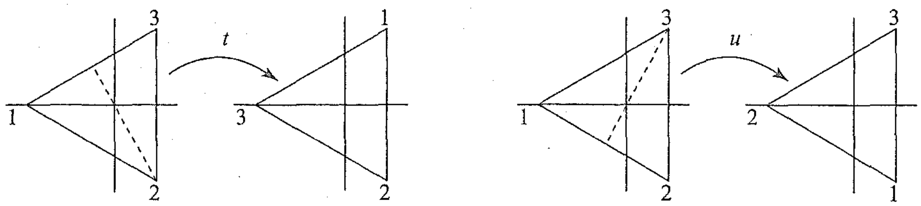
$\circ$	r_0	r_1	r_2	r_3	d	h	t	v
r_0	r_0	r_1	r_2	r_3	d	h	t	v
r_1	r_1	r_2	r_3	r_0	h	t	v	d
r_2	r_2	r_3	r_0	r_1	t	v	d	h
r_3	r_3	r_0	r_1	r_2	v	d	h	t
d	d	v	t	h	r_0	r_3	r_2	r_1
h	h	d	v	t	r_1	r_0	r_3	r_2
t	t	h	d	v	r_2	r_1	r_0	r_3
v	v	t	h	d	r_3	r_2	r_1	r_0

Clearly D_4 is closed under $\circ$, and composition of functions is known to be associative. The table shows that r_0 is the identity element and that every element of D_4 has an inverse. For instance, $r_3 \circ r_1 = r_0 = r_1 \circ r_3$ .Therefore, D_4 is a group. It is not abelian because, for example, $h \circ d \neq d \circ h$. D_4 is called the **dihedral group of degree 4** or the **group of symmetries of the square.**

EXAMPLE 6

The group of symmetries of the square is just one of many symmetry groups. An analogous procedure can be carried out with any regular polygon of n sides. The resulting group D_n is called the **dihedral group of degree n.** The group D_3, for example, consists of the six symmetries of an equilateral triangle (counterclockwise rotations about the center of 0°, 120°, and 240°; and the three reflections shown here and on the next page), with composition of functions as the operation:





Symmetry groups arise frequently in art, architecture, and science. Crystallography and crystal physics use groups of symmetries of various three-dimensional shapes. The first accurate model of DNA (which led to the Nobel Prize for its creators) could not have been constructed without a recognition of the symmetry of the DNA molecule. Symmetry groups have been used by physicists to predict the existence of certain elementary particles that were later found experimentally.

Systems with Two Operations

We now examine some familiar systems with two operations to see what groups arise when only one of the operations is considered.

EXAMPLE 7

We now show that each of the following is an abelian group under addition, that is, with the operation $*$ in the definition of a group being $+$:

$\mathbb{Z}$ the integers; $\mathbb{Z}_n$ the integers mod n;

$\mathbb{Q}$ the rational numbers; $\mathbb{R}$ the real numbers; $\mathbb{C}$ the complex numbers.

That each system is closed under addition is a fact from basic arithmetic (Axiom 1). Likewise, addition in each of these systems is associative: For any three numbers a, b, c,

$$a + (b + c) = (a + b) + c \qquad [\textit{Additive form of Axiom 2}]$$

In each system, the identity element is 0 because

$$a + 0 = a = 0 + a \qquad [\textit{Additive form of Axiom } 3]$$

Similarly, the inverse of a is $-a$ because

$$a + (-a) = 0 \qquad \text{and} \qquad -a + a = 0 \qquad [\textit{Additive form of Axiom } 4]$$

Finally, each group is abelian because for any two numbers a and b,

$$a + b = b + a \qquad [\textit{Additive form of Axiom } 5]$$

Hereafter, when we use the word "group" without any qualification in referring to $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, it is understood that the operation is addition. When it comes to multiplication, we have this basic fact:

None of $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ is a group under multiplication.

To be sure, each has 1 as its multiplicative identity element, but unfortunately 0 has no inverse—the equation $0x = 1$ has no solutions—so Axiom 4 fails. Nevertheless, certain subsets of these systems may be groups under multiplication.

EXAMPLE 8

Each of the following is an abelian group under multiplication:

$\mathbb{Q}^*$ the nonzero rational numbers; $\mathbb{R}^*$ the nonzero real numbers;

$\mathbb{C}^*$ the nonzero complex numbers.

Each system is closed under multiplication because the product of nonzero numbers is nonzero (Axiom 1). Basic arithmetic tells us that multiplication is associative and commutative (Axioms 2 and 5). The identity element in each system is 1 because $a \cdot 1 = a = 1 \cdot a$ (Axiom 3). The inverse of a is $1/a$ (Axiom 4).

EXAMPLE 9

Let p be a prime, and consider the nonzero elements of $\mathbb{Z}_p$ under multiplication. If $a \neq 0$ and $b \neq 0$, then $ab \neq 0$ by condition (3) of Theorem 2.8, so closure holds (Axiom 1). The identity element is 1 (Axiom 3) and inverses exist by condition (2) of Theorem 2.8 (Axiom 4). Multiplication is associative and commutative by Theorem 2.7 (Axioms 2 and 5). So the nonzero elements of $\mathbb{Z}_p$ form an abelian group under multiplication.

EXAMPLE 10

Each of

$\mathbb{Q}^{**}$ the positive rational numbers and $\mathbb{R}^{**}$ the positive real numbers

is an abelian group under multiplication. Both systems are closed under multiplication since the product of positive numbers is positive. The identity element is 1 and the inverse of a is $1/a$.

EXAMPLE 11

The subset $L = \{1, -1, i, -i\}$ of the complex numbers forms an abelian group under multiplication. You can easily verify that closure holds and that 1 is the identity element. Since $i(-i) = -i^2 = -(-1) = 1$, we see that i and $-i$ are inverses of each other; -1 is its own inverse since $(-1)(-1) = 1$. Hence, Axiom 4 holds.

EXAMPLE 12

Neither the nonzero integers nor the positive integers form a group under multiplication. Although 1 is the multiplicative identity for each system, no integers except for ± 1 have a multiplicative inverse, so Axiom 4 fails. For example, the equation $2x = 1$ has no *integer* solution, so 2 has no inverse under multiplication in the integers.

EXAMPLE 13

When n is composite, the nonzero elements of $\mathbb{Z}_n$ do not form a group under multiplication because (among other things) closure fails. In $\mathbb{Z}_6$, for instance, $2 \cdot 3 = 0$ and in $\mathbb{Z}_{20}$, $4 \cdot 5 = 0$. Similarly if $n = rs$, then in $\mathbb{Z}_n$, $rs = 0$.

EXAMPLE 14

Let U_n be the set of units in $\mathbb{Z}_n$.* By Exercise 17 of Section 2.3, the product of two units is a unit, so U_n is closed under multiplication (which is known to be associative and commutative). The identity 1 is a unit since $1 \cdot 1 = 1$. So U_n is an abelian group under multiplication. By Theorem 2.10, U_n consists of all $a \in \mathbb{Z}_n$ such that $(a, n) = 1$ (when a is considered as an ordinary integer). Thus, the group of units in $\mathbb{Z}_8$ is $U_8 = \{1, 3, 5, 7\}$, and the group of units in $\mathbb{Z}_{15}$ is $U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$. Here is the multiplication table for U_8:

$\cdot$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

The next example involves matrices.† A 2×2 matrix over the real numbers, is an array of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{where } a, b, c, d \text{ are real numbers.}$$

Two matrices are equal provided that the entries in corresponding positions are equals, that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \qquad \text{if and only if} \quad a = r, b = s, c = t, d = u.$$

For example,

$$\begin{pmatrix} 4 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 2+2 & 0 \\ 1-4 & 1 \end{pmatrix}, \quad \text{but} \quad \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} \neq \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}.$$

Matrix multiplication is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}.$$

*Recall that an element a in $\mathbb{Z}_n$ is a unit if the equation $ax = 1$ has a solution (that is, if a has an inverse under multiplication).

†If you have taken a course in linear algebra, you can skip this paragraph.

For example,

$$\begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 + 3\cdot 6 & 2(-5) + 3\cdot 7 \\ 0\cdot 1 + (-4)6 & 0(-5) + (-4)7 \end{pmatrix} = \begin{pmatrix} 20 & 11 \\ -24 & -28 \end{pmatrix}.$$

Reversing the order of the factors in this product produces

$$\begin{pmatrix} 1 & -5 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + (-5)0 & 1\cdot 3 + (-5)(-4) \\ 6\cdot 2 + 7\cdot 0 & 6\cdot 3 + 7(-4) \end{pmatrix} = \begin{pmatrix} 2 & 23 \\ 12 & -10 \end{pmatrix}.$$

So matrix multiplication is not commutative. A straightforward (but tedious) computation shows that matrix multiplication is associative. It's easy to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element.

EXAMPLE 15

We shall show that the set of matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \text{ where } a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}$$

is a group under multiplication, called the **general linear group** of degree 2 over $\mathbb{R}$ and denoted $GL(2, \mathbb{R})$. The discussion before the example shows that $GL(2, \mathbb{R})$ has associative multiplication and an identity element (Axioms 2 and 3). You can readily verify that when $ad - bc \neq 0$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So every matrix in $GL(2, \mathbb{R})$ has an inverse (Axiom 4).

To finish the proof, we need only show that $GL(2, \mathbb{R})$ is closed under multiplication (Axiom 1). Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ are in $GL(2, \mathbb{R})$, so that $ad - bc \neq 0$ and $wz - xy \neq 0$, and hence, $(ad - bc)(wz - xy) \neq 0$. To prove that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}$$

is in $GL(2, \mathbb{R})$, we must prove that $(aw + by)(cx + dz) - (ax + bz)(cw + dy) \neq 0$. Verify that

$$(aw + by)(cx + dz) - (ax + bz)(cw + dy) = (ad - bc)(wz - xy) \neq 0.$$

So the product matrix is in $GL(2, \mathbb{R})$. Therefore, $GL(2, \mathbb{R})$ is closed under multiplication and is a group, which is nonabelian (Exercise 7).

The discussion preceding Example 15 carries over to matrices whose entries are in systems other than the real numbers, such as $\mathbb{Q}$, $\mathbb{C}$, and $\mathbb{Z}_p$ (with p prime).

EXAMPLE 16

We shall show that

$$GL(2, \mathbb{Z}_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \text{ where } a, b, c, d \in \mathbb{Z}_2 \text{ and } ad - bc \neq 0 \right\},$$

the general linear group of degree 2 over $\mathbb{Z}_2$, is a group under multiplication. Matrix multiplication is associative, and the identity matrix is obviously in $GL(2, \mathbb{Z}_2)$. The proof that $GL(2, \mathbb{Z}_2)$ is closed under multiplication is identical to the one for $GL(2, \mathbb{R})$ in Example 15. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}_2)$, then $ad - bc \neq 0$ in $\mathbb{Z}_2$, so $ad - bc$ has an inverse by Example 9. Verify that the inverse of A is $\begin{pmatrix} d(ad - bc)^{-1} & -b(ad - bc)^{-1} \\ -c(ad - bc)^{-1} & a(ad - bc)^{-1} \end{pmatrix}$, which is the same inverse matrix given in Example 15, with a change of notation: $(ad - bc)^{-1}$ in place of $\dfrac{1}{ad - bc}$. Hence, $GL(2, \mathbb{Z}_2)$ is a group. It is a finite nonabelian group of order 6 (Exercise 7).

New Groups from Old

The Cartesian product $G \times H$ of sets G and H is defined on page 512 of Appendix B. Theorem 7.4 on the next page shows that the Cartesian product can be used to produce new groups from known ones.*

*Theorems 7.1–7.3 appear in Section 7.1 and assume that you have read Chapter 3, so they are not included in Section 7.1.A. However, many of the preceding examples are special cases of these theorems: Example 1 is a special case of Theorem 7.1; Examples 8 and 9 are special cases of Theorem 7.2; and Examples 14–16 are special cases of Theorem 7.3. So you haven't missed anything crucial for this chapter. You may wish to read Theorems 7.1–7.3 at a later date, after you have read Chapter 3.

Theorem 7.4

Let G (with operation $*$) and H (with operation $\diamond$) be groups. Define an operation $\blacksquare$ on $G \times H$ by

$$(g, h) \blacksquare (g', h') = (g * g', h \diamond h').$$

Then $G \times H$ is a group. If G and H are abelian, then so is $G \times H$. If G and H are finite, then so is $G \times H$ and $|G \times H| = |G||H|$.

Proof ▸ Exercise 26. ■

EXAMPLE 17

Both $\mathbb{Z}$ and $\mathbb{Z}_6$ are groups under addition. In $\mathbb{Z} \times \mathbb{Z}_6$ we have $(3, 5) \blacksquare (7, 4) = (3 + 7, 5 + 4) = (10, 3)$. The identity is $(0, 0)$, and the inverse of $(7, 4)$ is $(-7, 2)$.

EXAMPLE 18

Consider $\mathbb{R}^* \times D_4$, where $\mathbb{R}^*$ is the multiplicative group of nonzero real numbers. The table in Example 5 shows that

$$(2, r_1) \blacksquare (9, v) = (2 \cdot 9, r_1 \circ v) = (18, d).$$

The identity element is $(1, r_0)$, and the inverse of $(8, r_3)$ is $(1/8, r_1)$.

Exercises

The exercises for this section are the same as those for Section 7.1—see page 180.

7.2 Basic Properties of Groups

Before exploring the deeper concepts of group theory, we must develop some additional terminology and establish some elementary facts. We begin with a change in notation.

Now that you are comfortable with groups, we can switch to the **standard multiplicative notation.** Instead of $a * b$, we shall write ab when discussing abstract groups. However, particular groups in which the operation is addition (such as $\mathbb{Z}$) will still be written additively.

Although we have spoken of *the* inverse of an element or *the* identity element of a group, the definition of a group says nothing about inverses or identities being unique. Our first theorem settles the question, however.

Theorem 7.5

Let G be a group and let $a, b, c \in G$. Then

(1) G has a unique identity element.

(2) Cancelation holds in G:

$$\text{If } ab = ac, \text{ then } b = c; \qquad \text{if } ba = ca, \text{ then } b = c.$$

(3) Each element of G has a unique inverse.

Proof ▶ (1) The group G has at least one identity by the definition of a group. If e and e' are each identity elements of G, then

$$\begin{aligned} ee' &= e \quad &&[\textit{Because } e' \textit{ is an identity element.}] \\ ee' &= e' \quad &&[\textit{Because } e \textit{ is an identity element.}] \end{aligned}$$

Therefore,

$$e = ee' = e',$$

so that there is exactly one identity element.

(2) By the definition of a group, the element a has at least one inverse d such that $da = e = ad$. If $ab = ac$, then $d(ab) = d(ac)$. By associativity and the properties of inverses and identities,

$$\begin{aligned} (da)b &= (da)c \\ eb &= ec \\ b &= c. \end{aligned}$$

The second statement is proved similarly.

(3) Suppose that d and d' are both inverses of $a \in G$. Then $ad = e = ad'$, so that $d = d'$ by (2). Therefore a has exactly one inverse. ■

Hereafter the unique inverse of an element a in a group will be denoted a^{-1}. The uniqueness of a^{-1} means that

whenever $ay = e = ya$, then $y = a^{-1}$.

Corollary 7.6

If G is a group and $a, b \in G$, then

(1) $(ab)^{-1} = b^{-1}a^{-1}$;

(2) $(a^{-1})^{-1} = a$.

Note the order of the elements in statement (1). A common mistake is to write the inverse of ab as $a^{-1}b^{-1}$, which may not be true in nonabelian groups. See Exercise 2 for an example.

Proof of Corollary 7.6 ▶ (1) We have

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$$

and, similarly, $(b^{-1}a^{-1})(ab) = e$. Since the inverse of ab is unique by Theorem 7.5, $b^{-1}a^{-1}$ must be this inverse, that is, $(ab)^{-1} = b^{-1}a^{-1}$.

(2) By definition, $a^{-1}a = e$ and $(a^{-1})(a^{-1})^{-1} = e$, so that $a^{-1}a = a^{-1}(a^{-1})^{-1}$. Canceling a^{-1} by Theorem 7.5 shows that $a = (a^{-1})^{-1}$. ■

Let G be a group and let $a \in G$. We define $a^2 = aa$, $a^3 = aaa$, and for any positive integer n,

$$a^n = aaa \cdots a \qquad (n \text{ factors}).$$

We also define $a^0 = e$ and

$$a^{-n} = a^{-1}a^{-1}a^{-1} \cdots a^{-1} \qquad (n \text{ factors}).$$

These definitions are obviously motivated by the usual exponent notation in $\mathbb{R}$ and other familiar rings. But be careful in the nonabelian case when, for instance, $(ab)^n$ may not be equal to a^nb^n. Some exponent rules, however, *do* hold in groups:

Theorem 7.7

Let G be a group and let $a \in G$. Then for all m, n in $\mathbb{Z}$,

$$a^m a^n = a^{m+n} \qquad \text{and} \qquad (a^m)^n = a^{mn}.$$

Proof ▸ The proof consists of a verification of each statement in each possible case ($m \geq 0, n \geq 0$; $m \geq 0, n < 0$; etc.) and is left to the reader (Exercise 21). ■

NOTE ON ADDITIVE NOTATION: To avoid confusion, the operation in certain groups must be written as addition (for example, the additive group of real numbers since multiplication there has a completely different meaning). Here is a dictionary for translating multiplicative statements into additive ones:

	Multiplicative Notation	Additive Notation
Operation:	ab	$a + b$
Identity:	e	0
Inverse:	a^{-1}	$-a$
Exponents:	$a^n = aa \cdots a$ (n factors) $a^{-n} = a^{-1} \cdots a^{-1}$	$na = a + a + \cdots + a$ (n summands) $(-n)a = -a - a - \cdots - a$
Theorem 7.7:	$a^m a^n = a^{m+n}$ $(a^m)^n = a^{mn}$	$(ma) + (na) = (m + n)a$ $n(ma) = (mn)a$

Order of an Element

We return now to multiplicative notation for abstract groups. An element a in a group is said to have **finite order** if $a^k = e$ for some positive integer k.* In this case, the **order of the element** a is the *smallest* positive integer n such that $a^n = e$. The order of a is

*In additive notation, the condition is $ka = 0$.

denoted $|a|$. An element a is said to have **infinite order** if $a^k \neq e$ for every positive integer k.

EXAMPLE 1

In the multiplicative group of nonzero real numbers, 2 has infinite order because $2^k \neq 1$ for all $k \geq 1$. In the group $L = \{\pm 1, \pm i\}$ under multiplication of complex numbers, the order of i is 4 because $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Similarly, $|-i| = 4$. The element $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ in S_3 has order 3 because

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

The identity element in a group has order 1.

EXAMPLE 2

In the additive group $\mathbb{Z}_{12}$, the element 8 has order 3 because $8 + 8 = 4$ and $8 + 8 + 8 = 0$.

In the multiplicative group of nonzero real numbers, the element 2 has infinite order and all the powers of 2 (2^{-3}, 2^0, 2^5, etc.) are distinct. On the other hand, in the multiplicative group $L = \{\pm 1, \pm i\}$, the element i has order 4 and its powers are not distinct; for instance,

$$i^4 = 1 = i^0 \qquad \text{and} \qquad i^{10} = (i^4)^2 i^2 = i^2.$$

Observe that $i^{10} = i^2$ and $10 \equiv 2 \pmod 4$. These examples are illustrations of

Theorem 7.8

Let G be a group and let $a \in G$.

(1) If a has infinite order, then the elements a^k, with $k \in \mathbb{Z}$, are all distinct.

(2) If $a^i = a^j$ with $i \neq j$, then a has finite order.

Proof ▸ Note first that statement (1) is true if and only if statement (2) is true, because each statement is the contrapositive of the other, as explained on pages 503–504 of Appendix A. So we need only prove one of them. We shall prove statement (2):

Suppose that $a^i = a^j$, with $i > j$. Then multiplying both sides by a^{-j} shows that $a^{i-j} = a^{j-j} = a^0 = e$. Since $i - j > 0$, this says that a has finite order. ∎

Theorem 7.9

Let G be a group and $a \in G$ an element of finite order n. Then:

(1) $a^k = e$ if and only if $n \mid k$;

(2) $a^i = a^j$ if and only if $i \equiv j \pmod{n}$;

(3) If $n = td$, with $d \geq 1$, then a^t has order d.

Proof ▸ (1) If n divides k, say $k = nt$, then $a^k = a^{nt} = (a^n)^t = e^t = e$. Conversely, suppose that $a^k = e$. By the Division Algorithm, $k = nq + r$ with $0 \leq r < n$. Consequently,

$$e = a^k = a^{nq+r} = a^{nq}a^r = (a^n)^q a^r = e^q a^r = ea^r = a^r.$$

By the definition of order, n is the smallest positive integer with $a^n = e$. Since $r < n$, $a^r = e$ can occur only when $r = 0$. Thus, $k = nq + 0$ and n divides k.

(2) First, note that $a^i = a^j$ if and only if $a^{i-j} = e$. [*Proof*: if $a^i = a^j$, then $a^{i-j} = e$ by the proof of Theorem 7.8(2). Conversely, if $a^{i-j} = e$, then multiplying both sides by a^j shows that $a^i = a^j$.] But by (1), with $k = i - j$, we have $a^{i-j} = e$ if and only if $n \mid (i - j)$, that is, if and only if $i \equiv j \pmod{n}$. Therefore, $a^i = a^j$ if and only if $i \equiv j \pmod{n}$.

(3) Since $|a| = n$, we have $(a^t)^d = a^{td} = a^n = e$. We must show that d is the smallest positive integer with this property. If k is any positive integer such that $(a^t)^k = e$, then $a^{tk} = e$. Therefore, $n \mid tk$ by part (1), say $tk = nr = (td)r$. Hence, $k = dr$. Since k and d are positive and $d \mid k$, we have $d \leq k$. ■

Corollary 7.10

Let G be an abelian group in which every element has finite order. If $c \in G$ is an element of largest order in G (that is, $|a| \leq |c|$ for all $a \in G$), then the order of every element of G divides $|c|$. ■

For example, $(1, 0)$ has order 4 in the additive abelian group $\mathbb{Z}_4 \times \mathbb{Z}_2$ and every other element has order 1, 2, or 4 (Exercise 10(b)). Thus $(1, 0)$ is an element of largest possible order, and the order of every element of the group divides 4, the order of $(1, 0)$.

Proof of Corollary 7.10 ▸ Suppose, on the contrary, that $a \in G$ and $|a|$ does not divide $|c|$. Then there must be a prime p in the prime factorization of the integer $|a|$ that appears to a higher power than it does in the prime factorization of $|c|$. By prime factorization we can write $|a|$ as the product of a power of p and an integer that is not divisible by p and similarly for c. Thus there are integers m, n, r, s such that $|a| = p^r m$ and $|c| = p^s n$, with $(p, m) = 1 = (p, n)$ and $r > s$. By part (3) of Theorem 7.9, the element a^m has order p^r and c^{p^s} has order n. Exercise 33 shows that $a^m c^{p^s}$ has order $p^r n$. Hence, $|a^m c^{p^s}| = p^r n > p^s n = |c|$, contradicting the fact that c is an element of largest order. Therefore, $|a|$ divides $|c|$. ■

Exercises

NOTE: *Unless stated otherwise, G is a group with identity element e.*

A. 1. If $c^2 = c$ in a group, prove that $c = e$.

2. Let $a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ in S_3. Verify that $(ab)^{-1} \neq a^{-1}b^{-1}$.

3. If $a, b, c, d \in G$, then $(abcd)^{-1} = ?$

4. If $a, b \in G$ and $ab = e$, prove that $ba = e$.

5. Let $f: G \to G$ be given by $f(a) = a^{-1}$. Prove that f is a bijection.

6. Give an example of a group in which the equation $x^2 = e$ has more than two solutions.

7. Find the order of the given element.

   (a) 5 in U_8

   (b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 7 & 5 & 1 & 4 & 6 \end{pmatrix}$ in S_7

   (c) $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ in $GL(2, \mathbb{R})$

   (d) $\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ in $GL(2, \mathbb{R})$

8. Give an example of a group that contains nonidentity elements of finite order and of infinite order.

9. (a) Find the order of the groups U_{10}, U_{12}, and U_{24}.

   (b) List the order of each element of the group U_{20}.

10. Find the order of every element in each group:

    (a) $\mathbb{Z}_4$ (b) $\mathbb{Z}_4 \times \mathbb{Z}_2$ (c) S_3 (d) D_4 (e) $\mathbb{Z}$

11. Let G be an additive group. Write statement (2) of Theorem 7.8 and statements (1)–(3) of Theorem 7.9 in additive notation.

12. If $a, b \in G$ and n is any integer, show that $(aba^{-1})^n = ab^na^{-1}$.

13. If G is a finite group of order n and $a \in G$, prove that $|a| \leq n$. [*Hint:* Consider the $n + 1$ elements $e = a^0$ $a, a^2, a^3, \ldots, a^n$. Are they all distinct?] Thus every element in a finite group has finite order. The converse, however, is false; see Exercise 25 in Section 8.3 for an infinite group in which every element has finite order.

14. True or false: A group of order n contains an element of order n. Justify your answer.

15. (a) If $a \in G$ and $a^{12} = e$, what order can a possibly have?

    (b) If $e \neq b \in G$ and $b^p = e$ for some prime p, what is $|b|$?

16. (a) If $a \in G$ and $|a| = 12$, find the orders of each of the elements $a, a^2, a^3, \ldots, a^{11}$.

    (b) Based on the evidence in part (a), make a conjecture about the order of a^k when $|a| = n$.

17. (a) Let $a, b \in G$. Prove that the equations $ax = b$ and $ya = b$ each have a unique solution in G. [*Hint:* Two things must be done for each equation: First find a solution and then show that it is the only solution.]

    (b) Show by example that the solution of $ax = b$ may not be the same as the solution of $ya = b$. [*Hint:* Consider S_3.]

18. Let $G = \{a_1, a_2, \ldots, a_n\}$ be a finite abelian group of order n. Let $x = a_1a_2 \cdots a_n$. Prove that $x^2 = e$.

19. If $a, b \in G$, prove that $|bab^{-1}| = |a|$.

20. (a) Show that $a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ has order 3 in $GL(2, \mathbb{R})$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order 4.

    (b) Show that ab has infinite order.

B. 21. Prove Theorem 7.7.

22. Let $G = \{e, a, b\}$ be a group of order 3. Write out the operation table for G. [*Hint:* Exercise 28 in Section 7.1.]

23. Let G be a group with this property: If $a, b, c \in G$ and $ab = ca$, then $b = c$. Prove that G is abelian.

24. If $(ab)^2 = a^2b^2$ for all $a, b, \in G$, prove that G is abelian.

25. Prove that G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

26. Prove that every nonabelian group G has order at least 6; hence, every group of order 2, 3, 4, or 5 is abelian. [*Hint:* If $a, b \in G$ and $ab \neq ba$, show that the elements of the subset $H = \{e, a, b, ab, ba\}$ are all distinct. Show that either $a^2 \notin H$ or $a^2 = e$; in the latter case, verify that $aba \notin H$.]

27. If every nonidentity element of G has order 2, prove that G is abelian. [*Hint:* $|a| = 2$ if and only if $a \neq e$ and $a = a^{-1}$. Why?]

28. If $a \in G$, prove that $|a| = |a^{-1}|$.

29. If $a, b, c \in G$, prove that there is a unique element $x \in G$ such that $axb = c$.

30. If $a, b \in G$, prove that $|ab| = |ba|$.

31. (a) If $a, b \in G$ and $ab = ba$, prove that $(ab)^{|a||b|} = e$.

    (b) Show that part (a) may be false if $ab \neq ba$.

32. If $|G|$ is even, prove that G contains an element of order 2. [*Hint:* The identity element is its own inverse. See the hint for Exercise 27.]

33. Assume that $a, b \in G$ and $ab = ba$. If $|a|$ and $|b|$ are relatively prime, prove that ab has order $|a||b|$. [*Hint:* See Exercise 31.]

34. Suppose G has order 4, but contains no element of order 4.

    (a) Prove that no element of G has order 3. [*Hint:* If $|g| = 3$, then G consists of four distinct elements $g, g^2, g^3 = e, d$. Now gd must be one of these four elements. Show that each possibility leads to a contradiction.]

    (b) Explain why every nonidentity element of G has order 2.

    (c) Denote the elements of G by e, a, b, c and write out the operation table for G.

35. If $a, b \in G$, $b^6 = e$, and $ab = b^4a$, prove that $b^3 = e$ and $ab = ba$.

36. Suppose $a, b \in G$ with $|a| = 5$, $b \neq e$, and $aba^{-1} = b^2$. Find $|b|$.

37. If $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5$ for all $a, b \in G$, prove that G is abelian.

C. 38. If $(ab)^i = a^ib^i$ for three consecutive integers i and all $a, b \in G$, prove that G is abelian.

39. **(a)** Let G be a nonempty finite set equipped with an associative operation such that for all $a, b, c, d \in G$:

$$\text{if } ab = ac, \text{ then } b = c \text{ and if } bd = cd, \text{ then } b = c.$$

Prove that G is a group.

(b) Show that part (a) may be false if G is infinite.

40. Let G be a nonempty set equipped with an associative operation with these properties:

(i) There is an element $e \in G$ such that $ea = a$ for every $a \in G$.

(ii) For each $a \in G$, there exists $d \in G$ such that $da = e$.

Prove that G is a group.

41. Let G be a nonempty set equipped with an associative operation such that, for all $a, b \in G$, the equations $ax = b$ and $ya = b$ have solutions. Prove that G is a group.

7.3 Subgroups

We continue our discussion of the basic properties of groups, with special attention to subgroups.

Definition A subset H of a group G is a **subgroup** of G if H is itself a group under the operation in G.

Every group G has two subgroups: G itself and the one-element group $\{e\}$, which is called the **trivial subgroup.** All other subgroups are said to be **proper subgroups.**

EXAMPLE 1

The set $\mathbb{R}^*$ of nonzero real numbers is a group under multiplication. The group $\mathbb{R}^{**}$ of positive real numbers is a proper subgroup of $\mathbb{R}^*$.

EXAMPLE 2

The set $\mathbb{Z}$ of integers is a group under addition and is a subgroup of the additive group $\mathbb{Q}$ of rational numbers.

EXAMPLE 3

The subset $L = \{1, -1, i, -i\}$ of the complex numbers is a group under multiplication.* So it is a subgroup of $\mathbb{C}^*$, the multiplicative group of nonzero complex numbers.

EXAMPLE 4

Recall that the multiplicative group of units in $\mathbb{Z}_8$ is $U_8 = \{1, 3, 5, 7\}$. The upper-left quarter of its operation table in Example 14 of Section 7.1 or Section 7.1.A shows that the subset $\{1, 3\}$ is a subgroup of U_8.

EXAMPLE 5

The upper-left quarter of the operation table for D_4 in Example 5 of Section 7.1 or 7.1.A shows that $H = \{r_0, r_1, r_2, r_3\}$ is a subgroup of D_4.

EXAMPLE 6

In the additive group $\mathbb{Z}_6 \times \mathbb{Z}_4$, let $H = \{(0, 0), (3, 0), (0, 2), (3, 2)\}$. Verify that H is a subgroup by writing out its addition table.

When proving that a subset of a group is a subgroup, it is never necessary to check associativity. Since the associative law holds for *all* elements of the group, it automatically holds when the elements are in some subset H. In fact, you need only verify two group axioms:

Theorem 7.11

A nonempty subset H of a group G is a subgroup of G provided that

(i) if $a, b \in H$, then $ab \in H$; and

(ii) if $a \in H$, then $a^{-1} \in H$.

Proof ▸ Properties (i) and (ii) are the closure and inverse axioms for a group. Associativity holds in H, as noted above. Thus we need only verify that $e \in H$. Since H is nonempty, there exists an element $c \in H$. By (ii), $c^{-1} \in H$, and by (i) $cc^{-1} = e$ is in H. Therefore H is a group. ■

EXAMPLE 7

Let H consist of all 2×2 matrices of the form $b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in \mathbb{R}$. Since $1 \cdot 1 - b \cdot 0 = 1$, H is a nonempty subset of the group $GL(2, \mathbb{R})$, which was

*See Example 11 of Section 7.1 or Section 7.1.A.

defined in Example 15 of Section 7.1 or 7.1.A. The product of two matrices in H is in H because

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+c \\ 0 & 1 \end{pmatrix}$$

The inverse of $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$, which is also in H. Therefore, H is a subgroup of $GL(2, \mathbb{R})$ by Theorem 7.11.

When H is finite, just one axiom is sufficient to guarantee that H is a subgroup.

Theorem 7.12

Let H be a nonempty finite subset of a group G. If H is closed under the operation in G, then H is a subgroup of G.

Proof ▶ By Theorem 7.11, we need only verify that the inverse of each element of H is also in H. If $a \in H$, then closure implies that $a^k \in H$ for every positive integer k. Since H is finite, these powers cannot all be distinct. So a has finite order n by Theorem 7.8 and $a^n = e$. Since $n - 1 \equiv -1 \pmod{n}$, we have $a^{n-1} = a^{-1}$ by Theorem 7.9. If $n > 1$, then $n - 1$ is positive and $a^{-1} = a^{n-1}$ is in H. If $n = 1$, then $a = e$ and $a^{-1} = e = a$, so that a^{-1} is in H. ■

EXAMPLE 8

Let H consist of all permutations in S_5 that fix the element 1. In other words, $H = \{f \in S_5 \mid f(1) = 1\}$. H is a finite set since S_5 is a finite group. If $g, h \in H$, then $g(1) = 1$ and $h(1) = 1$. Hence, $(g \circ h)(1) = g(h(1)) = g(1) = 1$. Thus $g \circ h \in H$ and H is closed. Therefore, H is a subgroup of S_5 by Theorem 7.12.

The Center of a Group

If G is a group, then the **center** of G is the subset denoted $Z(G)$ and defined by

$$Z(G) = \{a \in G \mid ag = ga \text{ for every } g \in G\}.$$

In other words, an element of G is in $Z(G)$ if and only if it commutes with every element of G. If G is an abelian group, then $Z(G) = G$ because all elements commute with each other. When G is nonabelian, however, $Z(G)$ is not all of G.

EXAMPLE 9

The center of S_3 consists of the identity element alone because this is the only element that commutes with every element of S_3 (Exercise 25).

EXAMPLE 10

The operation table for D_4 in Example 5 of Section 7.1 or 7.1.A shows that r_1 commutes with some elements of D_4 (for instance, $r_1 \circ r_3 = r_3 \circ r_1$). However, it does not commute with *every* element of D_4 because $r_1 \circ d \neq d \circ r_1$. Hence, r_1 is not in $Z(D_4)$ nor is d. Careful examination of the table shows that $Z(D_4) = \{r_0, r_2\}$ since these are the only elements that commute with every element of D_4. It is easy to verify that $\{r_0, r_2\}$ is a subgroup of D_4. This is an example of the following result.

Theorem 7.13

The center $Z(G)$ of a group G is a subgroup of G.

Proof ▹ For every $g \in G$, we have $eg = g = ge$. Hence, $e \in Z(G)$ and $Z(G)$ is nonempty. If $a, b \in Z(G)$, then for any $g \in G$ we have $ag = ga$ and $bg = gb$, so that

$$(ab)g = a(bg) = a(gb) = (ag)b = (ga)b = g(ab).$$

Therefore, $ab \in Z(G)$. Finally, if $a \in Z(G)$ and $g \in G$, then $ag = ga$. Multiplying both sides of this equation on the left and right by a^{-1} shows that

$$\begin{aligned} a^{-1}(ag)a^{-1} &= a^{-1}(ga)a^{-1} \\ ga^{-1} &= a^{-1}g \end{aligned}$$

Therefore, $a^{-1} \in Z(G)$ and $Z(G)$ is a subgroup by Theorem 7.11. ■

Cyclic Groups

An important type of subgroup can be constructed as follows. If G is a group and $a \in G$, let $\langle a \rangle$ denote the set of all powers of a:

$$\langle a \rangle = \{\ldots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, \ldots\} = \{a^n \mid n \in \mathbb{Z}\}.$$

Theorem 7.14

If G is a group and $a \in G$, then $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G.

Proof ▹ The product of any two elements of $\langle a \rangle$ is also in $\langle a \rangle$ because $a^i a^j = a^{i+j}$. The inverse of a^k is a^{-k}, which is also in $\langle a \rangle$. By Theorem 7.11, $\langle a \rangle$ is a subgroup of G. ■

The group $\langle a \rangle$ is called the **cyclic subgroup generated by a.** If the subgroup $\langle a \rangle$ is the entire group G, we say that G is a **cyclic group.** Note that every cyclic group is abelian since $a^i a^j = a^{i+j} = a^j a^i$.

EXAMPLE 11

The multiplicative group of units in the ring $\mathbb{Z}_{15}$ is $U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$ by Theorem 2.10. In order to determine the cyclic subgroup generated by 7, we compute

$$7^1 = 7 \qquad 7^2 = 4 \qquad 7^3 = 13 \qquad 7^4 = 1 = 7^0.$$

Therefore, the element 7 has order 4 in U_{15}. We claim that the cyclic subgroup $\langle 7\rangle$ consists of $\{7^0, 7^1, 7^2, 7^3\} = \{1, 7, 4, 13\}$. [*Proof:* By definition, every element of $\langle 7\rangle$ is of the form 7^i for some integer i. Since every integer is congruent modulo 4 to one of 0, 1, 2, 3, the element 7^i must be one of $7^0, 7^1, 7^2$ or 7^3 by Theorem 7.9(2).] Hence, $\langle 7\rangle = \{1, 7, 4, 13\}$. Thus, the cyclic subgroup $\langle 7\rangle$ has order 4—the order of the element 7 that generates the group.

EXAMPLE 12

Different elements of a group may generate the same cyclic subgroup. For instance, verify that 13 has order 4 in U_{15}. Then the same argument used in Example 11 shows that the cyclic subgroup $\langle 13\rangle = \{13^0, 13^1, 13^2, 13^3\} = \{1, 13, 4, 7\} = \langle 7\rangle$.

The argument used in Examples 11 and 12 works in general and provides the connection between the two uses of the word "order". It states, in effect, that the order of an element a is the same as the order of the cyclic subgroup generated by a.

Theorem 7.15

Let G be a group and let $a \in G$.

(1) If a has infinite order, then $\langle a\rangle$ is an infinite subgroup consisting of the distinct elements a^k, with $k \in \mathbb{Z}$.

(2) If a has finite order n, then $\langle a\rangle$ is a subgroup of order n and $\langle a\rangle = \{e = a^0, a^1, a^2, a^3, \ldots, a^{n-1}\}$.

Proof ▸ (1) This is an immediate consequence of part (1) of Theorem 7.8.

(2) Let a^i be any element of $\langle a\rangle$. Then i is congruent modulo n to one of $0, 1, 2, \ldots, n-1$. Consequently, by part (2) of Theorem 7.9, a^i must be equal to one of $a^0, a^1, a^2, \ldots, a^{n-1}$. Furthermore, no two of these powers of a are equal since no two of the integers $0, 1, 2, \ldots, n-1$ are congruent modulo n. Therefore, $\langle a\rangle = \{a^0, a^1, a^2, \ldots, a^{n-1}\}$ is a group of order n. ■

NOTE ON ADDITIVE NOTATION: When the group operation is addition, then, as shown in the dictionary on page 198, we write ka in place of a^k. So the cyclic subgroup $\langle a\rangle = \{na \mid n \in \mathbb{Z}\}$. Theorem 7.15 in additive notation is shown on the next page.

Theorem 7.15 (Additive Version)

Let G be an additive group and let $a \in G$.

(1) If a has infinite order, then $\langle a \rangle$ is an infinite subgroup consisting of the distinct elements ka, with $k \in \mathbb{Z}$.

(2) If a has finite order n, then $\langle a \rangle$ is a subgroup of order n and

$$\langle a \rangle = \{0, 1a, 2a, 3a, 4a, \ldots, (n-1)a\}.$$

EXAMPLE 13

Since $\mathbb{Z} = \{n1 \mid n \in \mathbb{Z}\}$, we see that the additive group $\mathbb{Z}$ is an infinite cyclic group with generator 1, that is $\mathbb{Z} = \langle 1 \rangle$. The set E of even integers is a cyclic subgroup of the additive group $\mathbb{Z}$ because $E = \{n2 \mid n \in \mathbb{Z}\}$.

EXAMPLE 14

Each of the additive groups $\mathbb{Z}_n$ is a cyclic group of order n generated by 1 because $\mathbb{Z}_n$ consists of the "powers" of 1, namely, $1, 2 = 1 + 1, 3 = 1 + 1 + 1$, etc. For instance, $\mathbb{Z}_4 = \{1, 2, 3, 0\}$, that is, $\{1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1\}$.

The subgroup $\{1, -1, i, -i\}$ of the multiplicative group of nonzero elements of $\mathbb{C}$ is the cyclic subgroup $\langle i \rangle$ because $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Similarly, the multiplicative group of nonzero elements of $\mathbb{Z}_7$ is the cyclic group $\langle 3 \rangle$, as you can easily verify. These examples are special cases of the following theorem.

Theorem 7.16

Let F be any one of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_p$ (with p prime), and let F^* be the multiplicative group of nonzero elements of F.† If G is a finite subgroup of F^*, then G is cyclic.‡

Proof ▸ Let $c \in G$ be an element of largest order (there must be one since G is finite), say $|c| = m$. If $a \in G$, then $|a|$ divides m by Corollary 7.10, so that $a^m = 1$ by part (1) of Theorem 7.9. Thus every element of G is a solution of the equation $x^m - 1 = 0$. Since a polynomial equation of degree m has at most m solutions in F (by Corollary 4.17§), we must have $|G| \leq m$. But $\langle c \rangle$ is a subgroup of G of order m by Theorem 7.15. Therefore, $\langle c \rangle$ must be all of G, that is, G is cyclic. ∎

†See Examples 8 and 9 of Section 7.1 or 7.1. A.
‡For those who have read Chapter 3: The theorem and its proof are valid when F is any field.
§If you haven't read Section 4.4, you'll have to take this on faith for now.

Now that we know what cyclic groups look like, the next step is to examine the possible subgroups of a cyclic group.

Theorem 7.17

Every subgroup of a cyclic group is itself cyclic.

Proof ▸ Suppose $G = \langle a \rangle$ and H is a subgroup of G. If $H = \langle e \rangle$, then H is the cyclic subgroup generated by e (all of whose powers are just e). If $H \neq \langle e \rangle$, then H contains a nonidentity element of G, say a^i with $i \neq 0$. Since H is a subgroup, the inverse element a^{-i} is also in H. One of i or $-i$ is positive, and so H contains positive powers of a. Let k be the smallest positive integer such that $a^k \in H$. We claim that H is the cyclic subgroup generated by a^k. To prove this, we must show that every element of H is a power of a^k. If $h \in H$, then $h \in G$, so that $h = a^m$ for some m. By the Division Algorithm, $m = kq + r$ with $0 \leq r < k$. Consequently, $r = m - kq$ and

$$a^r = a^{m-kq} = a^m a^{-kq} = a^m (a^k)^{-q}.$$

Both a^m and a^k are in H. Therefore, $a^r \in \mathrm{H}$ by closure. Since a^k is the *smallest* positive power of a in H and since $r < k$, we must have $r = 0$. Therefore, $m = kq$ and $h = a^m = a^{kq} = (a^k)^q \in \langle a^k \rangle$. Hence, $H = \langle a^k \rangle$. ■

For additional information on the structure of cyclic groups and their subgroups, see Exercises 44–46.

Generators of a Group

Suppose G is a group and $a \in G$. Think of the cyclic subgroup $\langle a \rangle$ as being constructed from the one-element set $S = \{a\}$ in this way: Form all possible products of a and a^{-1} in every possible order. Of course, each such product reduces to a single element of the form a^n. We want to generalize this procedure by beginning with a set S that may contain more than one element.

Theorem 7.18

Let S be a nonempty subset of a group G. Let $\langle S \rangle$ be the set of all possible products, in every order, of elements of S and their inverses.* Then

(1) $\langle S \rangle$ is a subgroup of G that contains set S.

(2) If H is a subgroup of G that contains the set S, then H contains the entire subgroup $\langle S \rangle$.

---

*We allow the possibility of a product with one element so that elements of S will be in $\langle S \rangle$.

This theorem shows that $\langle S\rangle$ is the smallest subgroup of G that contains the set S. In the special case when $S = \{a\}$, the group $\langle S\rangle$ is just the cyclic subgroup $\langle a\rangle$, which is the smallest subgroup of G that contains a. The group $\langle S\rangle$ is called the **subgroup generated by S.** If $\langle S\rangle$ is the entire group G, we say that S **generates** G and refer to the elements of S as the **generators** of the group.

Proof of Theorem 7.18 ▸ (1) $\langle S\rangle$ is nonempty because the set S is nonempty and every element of S (considered as a one-element product) is an element of $\langle S\rangle$. If $a, b \in \langle S\rangle$, then a is of the form $a_1a_2\cdots a_k$, where $k \geq 1$ and each a_i is either an element of S or the inverse of an element of S. Similarly, $b = b_1b_2\cdots b_t$, with $t \geq 1$ and each b_i either an element of S or the inverse of an element of S. Therefore, the product $ab = a_1a_2\cdots a_kb_1b_2\cdots b_t$ consists of elements of S or inverses of elements of S. Hence, $ab \in \langle S\rangle$, and $\langle S\rangle$ is closed. The inverse of the element $a = a_1a_2\cdots a_k$ of $\langle S\rangle$ is $a^{-1} = a_k^{-1}\cdots a_2^{-1}a_1^{-1}$ by Corollary 7.6. Since each a_i is either an element of S or the inverse of an element of S, the same is true of a_i^{-1}. Therefore, $a^{-1} \in \langle S\rangle$. Hence, $\langle S\rangle$ is a subgroup of G by Theorem 7.11.

(2) Any subgroup that contains the set S must include the inverse of every element of S. By closure, this subgroup must also contain all possible products, in every order, of elements of S and their inverses. Therefore, every subgroup that contains S must also contain the entire group $\langle S\rangle$. ■

EXAMPLE 15

The group $U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$ is generated by the set $S = \{7, 11\}$ since

$$\begin{array}{cccc} 7^1 = 7 & 7^2 = 4 & 7^3 = 13 & 7^4 = 1 \\ 11^1 = 11 & 7 \cdot 11 = 2 & 7^2 \cdot 11 = 14 & 7^3 \cdot 11 = 8. \end{array}$$

Different sets of elements may generate the same group. For instance, you can readily verify that U_{15} is also generated by the set $\{2, 13\}$ (Exercise 9).

EXAMPLE 16

Using the operation table in Example 5 of Section 7.1 or 7.1.A, we see that in the group D_4,

$$\begin{array}{cccc} (r_1)^1 = r_1 & (r_1)^2 = r_2 & (r_1)^3 = r_3 & (r_1)^4 = r_0 \\ h^1 = h & r_1 \circ h = t & (r_1)^2 \circ h = v & (r_1)^3 \circ h = d. \end{array}$$

Therefore, D_4 is generated by $\{r_1, h\}$. Note that the representation of group elements in terms of the generators is not unique; for instance,

$$(r_1)^3 \circ h = d \quad \text{and} \quad r_1 \circ h \circ (r_1)^2 = d.$$

Exercises

A. 1. List all the cyclic subgroups of

(a) U_{15} (b) U_{30}

2. (a) List all the cyclic subgroups of D_4.

(b) List at least one subgroup of D_4 that is not cyclic.

3. List the elements of the subgroup $\langle a \rangle$, of S_7, where

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 7 & 6 & 5 & 1 & 4 \end{pmatrix}.$$

In Exercises 4–8, list (if possible) or describe the elements of the given cyclic subgroup.

4. $\langle 2 \rangle$ in the additive group $\mathbb{Z}_{12}$.
5. $\langle 2 \rangle$ in the additive group $\mathbb{Z}$.
6. $\langle 2 \rangle$ in the multiplicative group of nonzero elements of $\mathbb{Z}_{11}$.
7. $\langle 2 \rangle$ in the multiplicative group $\mathbb{Q}^*$ of nonzero rational numbers.
8. $\langle 3 \rangle$ in the multiplicative group of nonzero elements of $\mathbb{Z}_{11}$.
9. Show that U_{15} is generated by the set $\{2, 13\}$.
10. Show that $(1, 0)$ and $(0, 2)$ generate the additive group $\mathbb{Z} \times \mathbb{Z}_7$.
11. Show that the additive group $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.
12. Show that the additive group $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not cyclic but is generated by two elements.
13. Let H be a subgroup of a group G. If e_G is the identity element of G and e_H is the identity element of H, prove that $e_G = e_H$.
14. Let H and K be subgroups of a group G.

(a) Show by example that $H \cup K$ need not be a subgroup of G.

(b) Prove that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

15. (a) Let H and K be subgroups of a group G. Prove that $H \cap K$ is a subgroup of G.

(b) Let $\{H_i\}$ be any collection of subgroups of G. Prove that $\cap H_i$ is a subgroup of G.

16. Let G_1 be a subgroup of a group G and H_1 a subgroup of a group H. Prove that $G_1 \times H_1$ is a subgroup of $G \times H$.
17. Show that the only generators of the additive cyclic group $\mathbb{Z}$ are 1 and -1.
18. Show that $(3, 1)$, $(-2, -1)$, and $(4, 3)$ generate the additive group $\mathbb{Z} \times \mathbb{Z}$.
19. Let G be an abelian group and let T be the set of elements of G with finite order. Prove that T is a subgroup of G; it is called the **torsion subgroup.** (This result may not hold if G is nonabelian; see Exercise 20 of Section 7.2.)
20. Let G be an abelian group, k a fixed positive integer, and $H = \{a \in G \mid |a| \text{ divides } k\}$. Prove that H is a subgroup of G.

21. **(a)** If G is a group and $ab \in Z(G)$, is it true that a and b are in $Z(G)$? [*Hint:* D_4.]

    (b) If G is a group and $ab \in Z(G)$, prove that $ab = ba$.

22. If a is the only element of order 2 in a group G, prove that $a \in Z(G)$.

23. Let G be a group and let $a \in G$. Prove that $\langle a \rangle = \langle a^{-1} \rangle$.

24. Show that $\mathbb{Q}^{**}$, the multiplicative group of positive rational numbers, is not a cyclic group. [*Hint:* if $1 \neq r \in \mathbb{Q}^{**}$, then there must be a rational between r and r^2.]

25. Show that the center of S_3 is the identity subgroup.

26. **(a)** Let H and K be subgroups of an abelian group G and let $HK = \{ab \mid a \in H, b \in K\}$. Prove that HK is a subgroup of G.

    (b) Show that part (a) may be false if G is not abelian.

27. Let H be a subgroup of a group G and, for $x \in G$, let $x^{-1}Hx$ denote the set $\{x^{-1}ax \mid a \in H\}$. Prove that $x^{-1}Hx$ is a subgroup of G.

28. Let G be an abelian group and n a fixed positive integer.

    (a) Prove that $H = \{a \in G \mid a^n = e\}$ is a subgroup of G.

    (b) Show by example that part (a) may be false if G is nonabelian. [*Hint:* S_3.]

29. Prove that a nonempty subset H of a group G is a subgroup of G if and only if whenever $a, b \in H$, then $ab^{-1} \in H$.

30. Let $A(T)$ be the group of permutations of the set T and let T_1 be a nonempty subset of T. Prove that $H = \{f \in A(T) \mid f(t) = t \text{ for every } t \in T_1\}$ is a subgroup of $A(T)$.

31. Let T and T_1 be as in Exercise 30. Prove that $K = \{f \in A(T) \mid f(T_1) = T_1\}$ is a subgroup of $A(T)$ that contains the subgroup H of Exercise 30. Verify that if T_1 has more than one element, then $K \neq H$.

32. Let H be a subgroup of a group G and assume that $x^{-1}Hx \subseteq H$ for every $x \in G$ (notation as in Exercise 27). Prove that $x^{-1}Hx = H$ for each $x \in G$.

33. Let G be a group and $a \in G$. The **centralizer of a** is the set $C(a) = \{g \in G \mid ga = ag\}$. Prove that $C(a)$ is a subgroup of G.

34. If G is a group, prove that $Z(G) = \bigcap_{a \in G} C(a)$ (notation as in Exercise 33).

35. Prove that an element a is in the center of a group G if and only if $C(a) = G$ (notation as in Exercise 33).

36. True or false: If every proper subgroup of a group G is cyclic, then G is cyclic. Justify your answer.

37. Suppose that H is a subgroup of a group G and that $a \in G$ has order n. If $a^k \in H$ and $(k, n) = 1$, prove that $a \in H$.

B. 38. **(a)** Let p be prime and let b be a nonzero element of $\mathbb{Z}_p$. Show that $b^{p-1} = 1$. [*Hint:* Theorem 7.16.]

    (b) Prove **Fermat's Little Theorem:** If p is a prime and a is any integer, then $a^p \equiv a \pmod{p}$. [*Hint:* Let b be the congruence class of a in $\mathbb{Z}_p$ and use part (a).]

39. If H is a subgroup of a group G, then the **normalizer** of H is the set $N(H) = \{x \in G \mid x^{-1}Hx = H\}$ (notation as in Exercise 27). Prove that $N(H)$ is a subgroup of G that contains H.

40. Prove that $H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \; a = 1 \text{ or } -1, b \in \mathbb{Z} \right\}$ is a subgroup of $GL(2, \mathbb{Q})$.

41. Let G be an abelian group and n a fixed positive integer. Prove that $H = \{a^n \mid a \in G\}$ is a subgroup of G.

42. Let k be a positive divisor of the positive integer n. Prove that $H_k = \{a \in U_n \mid a \equiv 1 \pmod{k}\}$ is a subgroup of U_n.

43. List all the subgroups of $\mathbb{Z}_{12}$. Do the same for $\mathbb{Z}_{20}$.

44. Let $G = \langle a \rangle$ be a cyclic group of order n.
    (a) Prove that the cyclic subgroup generated by a^m is the same as the cyclic subgroup generated by a^d, where $d = (m, n)$. [*Hint:* It suffices to show that a^d is a power of a^m and vice versa. (Why?) Note that by Theorem 1.2, there are integers u and v such that $d = mu + nv$.]
    (b) Prove that a^m is a generator of G if and only if $(m, n) = 1$.

45. Let $G = \langle a \rangle$ be a cyclic group of order n. If H is a subgroup of G, show that $|H|$ is a divisor of n. [*Hint:* Exercise 44 and Theorem 7.17.]

46. Let $G = \langle a \rangle$ be a cyclic group of order n. If k is a positive divisor of n, prove that G has a unique subgroup of order k. [*Hint:* Consider the subgroup generated by $a^{n/k}$.]

47. Let G be an abelian group of order mn where $(m, n) = 1$. Assume that G contains an element a of order m and an element b of order n. Prove that G is cyclic with generator ab.

48. Show that the multiplicative group $\mathbb{R}^*$ of nonzero real numbers is not cyclic.

49. If G is an infinite additive cyclic group with generator a. Prove that the equation $x + x = a$ has no solution in G.

50. Show that the additive group $\mathbb{Q}$ is not cyclic. [*Hint:* Exercise 49.]

51. Let G and H be groups. If $G \times H$ is a cyclic group, prove that G and H are both cyclic. (Exercise 12 shows that the converse is false.)

52. Prove that $\left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| \; n \in \mathbb{Z} \right\}$ is a cyclic subgroup of $GL(2, \mathbb{R})$.

53. Prove that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $(m, n) = 1$.

54. If $G \neq \langle e \rangle$ is a group that has no proper subgroups, prove that G is a cyclic group of prime order.

55. Is the additive group $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ cyclic?

56. Show that the group U_{20} of units in $\mathbb{Z}_{20}$ is not cyclic.

57. Show that the group U_{18} of units in $\mathbb{Z}_{18}$ is cyclic.

58. If S is a nonempty subset of a group G, show that $\langle S \rangle$ is the intersection of the family of all subgroups H such that $S \subseteq H$.

7.4 Isomorphisms and Homomorphisms*

If you were unfamiliar with roman numerals and came across a discussion of integer arithmetic written solely with roman numerals, it might take you some time to realize that this arithmetic was essentially the same as the familiar arithmetic in $\mathbb{Z}$ except for the labels on the elements. Here is a less obvious example of the same situation.

EXAMPLE 1

Recall the multiplicative subgroup $L = \{\mathbf{1}, \boldsymbol{i}, \boldsymbol{-i}, \mathbf{-1}\}$ of the complex numbers and the multiplicative group $U_5 = \{1, 2, 3, 4\}$ of units in $\mathbb{Z}_5$, whose operation tables are shown below.†

U_5

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

L

$\cdot$	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{-i}$	$\mathbf{-1}$
$\mathbf{1}$	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{-i}$	$\mathbf{-1}$
$\boldsymbol{i}$	$\boldsymbol{i}$	$\mathbf{-1}$	$\mathbf{1}$	$\boldsymbol{-i}$
$\boldsymbol{-i}$	$\boldsymbol{-i}$	$\mathbf{1}$	$\mathbf{-1}$	$\boldsymbol{i}$
$\mathbf{-1}$	$\mathbf{-1}$	$\boldsymbol{-i}$	$\boldsymbol{i}$	$\mathbf{1}$

At first glance, these groups don't seem the same. But we claim that they are "essentially the same", except for the lablels on the elements. To see this clearly, relabel the elements of U_5 according to this scheme:

Relabel 1 as $\mathbf{1}$; Relabel 2 as $\boldsymbol{i}$; Relabel 3 as $\boldsymbol{-i}$; Relabel 4 as $\mathbf{-1}$.

Now look what happens to the table for U_5—it becomes the table for L!

$\cdot$	$\cancel{1}$ $\mathbf{1}$	$\cancel{2}$ $\boldsymbol{i}$	$\cancel{3}$ $\boldsymbol{-i}$	$\cancel{4}$ $\mathbf{-1}$
$\cancel{1}$ $\mathbf{1}$	$\cancel{1}$ $\mathbf{1}$	$\cancel{2}$ $\boldsymbol{i}$	$\cancel{3}$ $\boldsymbol{-i}$	$\cancel{4}$ $\mathbf{-1}$
$\cancel{2}$ $\boldsymbol{i}$	$\cancel{2}$ $\boldsymbol{i}$	$\cancel{4}$ $\mathbf{-1}$	$\cancel{1}$ $\mathbf{1}$	$\cancel{3}$ $\boldsymbol{-i}$
$\cancel{3}$ $\boldsymbol{-i}$	$\cancel{3}$ $\boldsymbol{-i}$	$\cancel{1}$ $\mathbf{1}$	$\cancel{4}$ $\mathbf{-1}$	$\cancel{2}$ $\boldsymbol{i}$
$\cancel{4}$ $\mathbf{-1}$	$\cancel{4}$ $\mathbf{-1}$	$\cancel{3}$ $\boldsymbol{-i}$	$\cancel{2}$ $\boldsymbol{i}$	$\cancel{1}$ $\mathbf{1}$

The rewritten table shows that the operations in U_5 and L work in exactly the same way—the only difference is the way the elements are labeled. As far as

*The first few pages of this section explain the concept of isomorphism for groups, which is essentially the same as the explanation for rings in Section 3.3. If you have read that section, feel free to begin this one at the Definition on page 216.

†To make the elements of the two groups easily distinguishable, the elements of L are in boldface.

group structure goes L_5 is just the group U_5 with new labels on the elements. In more technical terms, U_5 and L are said to be *isomorphic*

In general, *isomorphic groups* are groups that have the *same structure*, in the sense that the operation table for one is the operation table of the other with the elements suitably relabeled. Although this intuitive idea is adequate for small finite groups, we need to develop a rigorous mathematical definition of isomorphism that agrees with this intuitive idea *and* is readily applicable to large groups as well.

There are two aspects to the intuitive idea that groups G and H are isomorphic: relabeling the elements of G, and comparing the new operation table with that of H. Relabeling means that every element of G is paired with a unique element of H (its new label). In other words, there is a function $f: G \to H$ that assigns to each $r \in G$ its new label $f(r) \in H$. In the preceding example, we used the relabeling function $f: U_5 \to K$ given by

$$f(1) = 1 \qquad f(2) = i \qquad f(3) = -i \qquad f(4) = -1.$$

The function $f: G \to H$ must have these properties:

(1) Distinct elements of G get distinct labels in H:

$$\text{If } r \neq r' \text{ in } G, \text{ then } f(r) \neq f(r') \text{ in } H.$$

(2) Every element of H is the label of some element of G:*

$$\text{For each } h \in H, \text{ there is an } r \in G \text{ such that } f(r) = h.$$

Properties (1) and (2) simply say that the function f must be both injective and surjective, that is, f is a *bijection*.†

In order to be an isomorphism, however, the table of G must become the table of H when f is applied. If this is the case, then for two elements a and b of G, the situation must look like this:

G

$*$	b
a	c

H

$*$	$f(b)$
$f(a)$	$f(c)$

As indicated in the two tables,

$$a * b = c \text{ in } G \quad \text{and} \quad f(a) * f(b) = f(c) \text{ in } H$$

Since $a * b = c$ in G, we must have $f(a * b) = f(c)$ in H. Combining this with the fact that $f(c) = f(a) * f(b)$ in H we see that

$$f(a * b) = f(a) * f(b).$$

This is the condition that f must satisfy in order for f to change the operation tables of G into those of H. We can now state a formal definition of isomorphism.

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*Otherwise we could not get the complete table of H from that of G.

†Injective, surjective, and bijective functions are discussed in Appendix B.

Definition

Let G and H be groups with the group operation denoted by $*$. G is **isomorphic** to a group H (in symbols, $G \cong H$) if there is a function $f: G \to H$ such that

(i) f is injective;
(ii) f is surjective;
(iii) $f(a * b) = f(a) * f(b)$ for all $a, b \in G$.

In this case, the function f is called an **isomorphism**.

It can be shown that $G \cong H$ if and only if $H \cong G$ (Exercise 53).

NOTE: In the preceding discussion, we have temporarily reverted to the $*$ notation for group operations to remind you that in a specific group, the operation might be addition, multiplication, or something else. In such cases, condition (iii) of the definition may take a different form; for instance,

Condition (iii)	$f(a * b) = f(a) * f(b)$
G and H additive:	$f(a + b) = f(a) + f(b)$
G and H multiplicative:	$f(ab) = f(a)f(b)$
G additive, H multiplicative:	$f(a + b) = f(a)f(b)$
G multiplicative, H additive:	$f(ab) = f(a) + f(b)$

EXAMPLE 2

The multiplicative group $U_8 = \{1, 3, 5, 7\}$ of units in $\mathbb{Z}_8$ is isomorphic to the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2$. To prove this, let $f: U_8 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be defined by

$$f(1) = (0, 0) \quad f(3) = (1, 0) \quad f(5) = (0, 1) \quad f(7) = (1, 1).$$

Clearly f is a bijection. Showing that $f(ab) = f(a) + f(b)$ for $a, b \in U_8$ is equivalent to showing that the operation table for $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be obtained from that of U_8 simply by replacing each $a \in U_8$ by $f(a) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Use the tables below to verify that this is indeed the case. Therefore, f is an isomorphism:

U_8

$\circ$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$\mathbb{Z}_2 \times \mathbb{Z}_2$

$+$	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(0, 0)	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(1, 0)	(1, 0)	(0, 0)	(1, 1)	(0, 1)
(0, 1)	(0, 1)	(1, 1)	(0, 0)	(1, 0)
(1, 1)	(1, 1)	(0, 1)	(1, 0)	(0, 0)

EXAMPLE 3

Let E be the additive group of even integers. We claim that $f{:}\mathbb{Z} \to E$ given by $f(a) = 2a$ is an isomorphism. Since $\mathbb{Z}$ and E are infinite, comparing tables is not an option. However, the formal definition of isomorphism will do the job. We begin by showing that f is injective.* Suppose $a, b \in \mathbb{Z}$ and $f(b) = f(b)$ in E. Then

$$\begin{aligned} f(b) &= f(b) & & \\ 2a &= 2b & & [\textit{Definition of } f\,] \\ a &= b & & [\textit{Divide both sides by } 2.] \end{aligned}$$

Hence, f is injective. Now suppose $n \in E$. Since n is an even integer, $n = 2k$ for some integer k. Therefore, $f(k) = 2k = n$, and f is surjective. Finally, for all a, $b \in \mathbb{Z}$,

$$f(a + b) = 2(a + b) = 2a + 2b = f(a) + f(b).$$

Hence, f is an isomorphism of additive groups.

EXAMPLE 4

The additive group $\mathbb{R}$ of real numbers is isomorphic to the multiplicative group $\mathbb{R}^{**}$ of positive real numbers. To prove this, let $f{:}\mathbb{R} \to \mathbb{R}^{**}$ be given by $f(r) = 10^r$. To show that f is injective, suppose that

$$f(r) = f(s).$$

Then

$$\begin{aligned} 10^r &= 10^s & & [\textit{Definition of } f\,] \\ \log 10^r &= \log 10^s & & [\textit{Take logarithms of both sides.}] \\ r &= s & & [\textit{Basic property of logarithms}] \end{aligned}$$

So f is injective. To prove that f is surjective, let $k \in \mathbb{R}$. Then $r = \log k$ is a real number, and by the definition of logarithm,

$$f(r) = 10^r = 10^{\log k} = k.$$

Thus, f is also surjective. Finally,

$$f(r + s) = 10^{r+s} = 10^r 10^s = f(r)f(s).$$

Therefore, f is an isomorphism and $\mathbb{R} \cong \mathbb{R}^{**}$.

---

*Injective, surjective, and bijective functions are discussed in Appendix B.

EXAMPLE 5

Two finite groups with different numbers of elements (such as $\mathbb{Z}_5$ and $\mathbb{Z}_{10}$) cannot be isomorphic, because no function from one to the other can be a bijection.

Example 1 presented two groups with the same number of elements that were isomorphic. However, this is not always the case.

EXAMPLE 6

S_3 and the additive group $\mathbb{Z}_6$ each have order 6, but are *not* isomorphic. There is no way to relabel the addition table of $\mathbb{Z}_6$ to obtain the table of S_3 because the operation in S_3 is not commutative, but addition in $\mathbb{Z}_6$ is. A similar argument in the general case (see Exercise 16) shows that for groups G and H,

If G is abelian and H is nonabelian, then G and H are *not* isomorphic.

EXAMPLE 7

The additive groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ each have order 4 but are not isomorphic because every nonzero element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ has order 2, but $\mathbb{Z}_4$ has two elements of order 4 (namely, 1 and 3). So relabeling the addition table of one cannot produce the table of the other. More generally by Exercise 29,

If f is an isomorphism, then a and $f(a)$ have the same order.

If G is a group, then an isomorphism $G \to G$ is called an **automorphism** of the group G.

EXAMPLE 8

If G is a group, then the **identity map** $\iota_G: G \to G$ given by $\iota_G(r) = r$ is an automorphism of G. It is clear that ι_G is bijective, and for any $a, b \in G$,

$$\iota_G(a * b) = a * b = \iota_G(a) * \iota_G(b).$$

EXAMPLE 9

Let c be a fixed element of a group G. Define $f: G \to G$ by $f(g) = c^{-1}gc$. Then

$$f(b)f(b) = (c^{-1}ac)(c^{-1}bc) = c^{-1}a(cc^{-1})bc = c^{-1}abc = f(ab).$$

If $g \in G$, then $cgc^{-1} \in G$ and

$$f(cgc^{-1}) = c^{-1}(cgc^{-1})c = \{c^{-1}c)g(c^{-1}c) = ege = g.$$

So f is surjective. To show that f is injective, suppose $f(a) = f(b)$. Then $c^{-1}ac = c^{-1}bc$. Canceling c on the right side and c^{-1} on the left side by Theorem 7.5, we have $a = b$. Hence, f is injective. Therefore, f is an isomorphism, called the **inner automorphism of** G induced by c. For more about automorphisms, see Exercises 36, 37, 58, and 59.

The next theorem completely characterizes all cyclic groups.

Theorem 7.19

Let G be a cyclic group.

(1) If G is infinite, then G is isomorphic to the additive group $\mathbb{Z}$.

(2) If G is finite of order n, then G is isomorphic to the additive group $\mathbb{Z}_n$.

Proof ▸ (1) Suppose that $G = \langle a \rangle$ is an infinite cyclic group. By Theorem 7.15 G consists of the elements a^k with $k \in \mathbb{Z}$, all of which are distinct (meaning that $a^i = a^j$ if and only if $i = j$). The function $f: G \to \mathbb{Z}$ defined by $f(a^k) = k$ is easily seen to be a bijection (Exercise 17). Since

$$f(a^i a^j) = f(a^{i+j}) = i + j = f(a^i) + f(a^j),$$

f is an isomorphism. Therefore, $G \cong \mathbb{Z}$.

(2) Now suppose that $G = \langle b \rangle$ and b has order n. By Theorem 7.15, $G = \{b^0, b^1, b^2, \ldots, b^{n-1}\}$, and by Corollary 2.5, $\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$. Define $g: G \to \mathbb{Z}_n$ by $g(b^i) = [i]$. Clearly g is a bijection. Finally,

$$g(b^i b^j) = g(b^{i+j}) = [i + j] = [i] + [j] = g(b^i) + g(b^j).$$

Hence, g is an isomorphism and $G \cong \mathbb{Z}_n$. ■

EXAMPLE 10

In multiplicative group $\mathbb{Q}^*$ of nonzero rational numbers, the cyclic subgroup generated by 2 is $\langle 2 \rangle = \left\{\ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\right\}$.* The group $\langle 2 \rangle$ is isomorphic to the additive group $\mathbb{Z}$ by Theorem 7.19.

EXAMPLE 11

The upper left-hand quadrant of the operation table for D_4 in Example 5 of Section 7.1 or 7.1.A and Theorem 7.12 show that $G = \{r_0, r_1, r_2, r_3\}$ is a subgroup of D_4. Verify that both G and $U_5 = \{1, 2, 3, 4\}$ are cyclic. By Theorem 7.19 each is isomorphic to the additive group $\mathbb{Z}_4$. Hence, they are isomorphic to each other: $G \cong U_5$ (Exercise 21).

*Exercise 7 of Section 7.3.

Homomorphisms

Many functions that are not injective or surjective satisfy condition (iii) of the definition of isomorphism. Such functions are given a special name and play an important role in later sections of this chapter.

Definition Let G and H be groups (with operation $*$). A function $f: G \to H$ is said to be a **homomorphism** if

$$f(a * b) = f(a) * f(b) \text{ for all } a, b \in G.$$

Every isomorphism is a homomorphism, but a homomorphism need not be an isomorphism.

EXAMPLE 12

The function $f: \mathbb{R}^* \to \mathbb{R}^*$ given by $f(x) = x^2$ is a homomorphism of multiplicative groups because

$$f(ab) = (ab)^2 = a^2b^2 = f(b)f(b).$$

However, f is not injective because $f(1) = f(-1)$ and is not surjective because $f(x) = x^2 \geq 0$ for all x, so no negative number is an image under f.

EXAMPLE 13

The function $f: \mathbb{Z} \to \mathbb{Z}_5$ given by $f(a) = [a]$ is a homomorphism of additive groups because

$$f(a + b) = [a + b] = [a] + [b] = f(b) = f(b).$$

The homomorphism f is surjective, but not injective (Why?).

EXAMPLE 14

If G and H are groups, the function $f: G \times H \to G$ given by $f((x, y)) = x$ is a surjective homomorphism (Exercise 9). If H is not the identity group, g is not injective. For instance, if $e_H \neq a \in H$, then $(e_G, a) \neq (e_G, e_H)$ in $G \times H$, but $f((e_G, a)) = e_G$ and $f((e_G, e_H)) = e_G$.

Recall that the **image** of a function $f: G \to H$ is a subset of H, namely $\operatorname{Im} f = \{h \in H \mid h = f(a) \text{ for some } a \in G\}$. The function f can be considered as a surjective map from G to $\operatorname{Im} f$.

Theorem 7.20

Let G and H be groups with identity elements e_G and e_H, respectively. If $f: G \to H$ is a homomorphism, then

(1) $f(e_G) = e_H$.

(2) $f(a^{-1}) = f(a)^{-1}$ for every $a \in G$.

(3) $\operatorname{Im} f$ is a subgroup of H.

(4) If f is injective, then $G \cong \operatorname{Im} f$.

Proof ▶ (1) Since f is a homomorphism, e_G is the identity in G, and e_H is the identity in H, we have

$$\begin{aligned} f(e_G)f(e_G) &= f(e_Ge_G) && [f \textit{ is a homomorphism.}] \\ f(e_G)f(e_G) &= f(e_G) && [e_G \textit{ is the identity in } G.] \\ f(e_G)f(e_G) &= e_Hf(e_G) && [f(e_G) \in H \textit{ and } e_H \textit{ is the identity in } H.] \end{aligned}$$

Canceling $f(e_G)$ on the right (by Theorem 7.5) produces $f(e_G) = e_H$.

(2) By (1) we have

$$f(a^{-1})f(a) = f(a^{-1}a) = f(e_G) = e_H = f(a)^{-1}f(a).$$

Canceling $f(a)$ on each end shows that $f(a^{-1}) = f(a)^{-1}$.

(3) The identity $e_H \in \operatorname{Im} f$ by (1), and so $\operatorname{Im} f$ is nonempty. Since $f(a)f(b) = f(ab)$, $\operatorname{Im} f$ is closed. The inverse of each $f(a) \in \operatorname{Im} f$ is also in $\operatorname{Im} f$ because $f(a)^{-1} = f(a^{-1})$ by (2). Therefore, $\operatorname{Im} f$ is a subgroup of H by Theorem 7.11.

(4) As noted before the theorem, f can be considered as a surjective function from G to $\operatorname{Im} f$. If f is also an injective homomorphism, then f is an isomorphism. ■

Group theory began with the study of permutations and groups of permutations. The abstract definition of a group came later and may appear to be far more general than the concept of a group of permutations. The next theorem shows that this is not the case, however.

Theorem 7.21 Cayley's Theorem

Every group G is isomorphic to a group of permutations.

Proof ▶ Consider the group $A(G)$ of all permutations of the *set* G. Recall that $A(G)$ consists of all bijective *functions* from G to G with composition as the group operation. These functions need not be homomorphisms.

To prove the theorem, we find a subgroup of $A(G)$ that is isomorphic to G.* We do this by constructing an injective homomorphism of groups $f: G \to A(G)$; then G is isomorphic to the subgroup $\operatorname{Im} f$ of $A(G)$ by Theorem 7.20.

If $a \in G$, then we claim that the map $\varphi_a: G \to G$ defined by $\varphi_a(x) = ax$ is a bijection of sets [that is, an element of $A(G)$]. This follows from the fact that if $b \in G$, then $\varphi_a(a^{-1}b) = a(a^{-1}b) = b$; hence, φ_a is surjective. If $\varphi_a(b) = \varphi_a(c)$, then $ab = ac$. Canceling a by Theorem 7.5, we conclude that $b = c$. Therefore, φ_a is injective and, hence, a bijection. Thus $\varphi_a \in A(G)$.

Now define $f: G \to A(G)$ by $f(a) = \varphi_a$. For any $a, b \in G, f(ab) = \varphi_{ab}$ is the map from G to G given by $\varphi_{ab}(x) = abx$. On the other hand, $f(a) \circ f(b) = \varphi_a \circ \varphi_b$ is the map given by $(\varphi_a \circ \varphi_b)(x) = \varphi_a(\varphi_b(x)) = \varphi_a(bx) = abx$. Therefore, $f(ab) = f(a) \circ f(b)$ and f is a homomorphism of groups. Finally, suppose $f(a) = f(c)$, so that $\varphi_a(x) = \varphi_c(x)$ for all $x \in G$. Then $a = ae = \varphi_a(e) = \varphi_c(e) = ce = c$. Hence, f is injective. Therefore, $G \cong \operatorname{Im} f$ by Theorem 7.20. ■

Corollary 7.22

Every finite group G of order n is isomorphic to a subgroup of the symmetric group S_n.

Proof ▹ The group G is isomorphic to a subgroup H of $A(G)$ by the proof of Theorem 7.21. Since G is a set of n elements, $A(G)$ is isomorphic to S_n by Exercise 38. Consequently, H is isomorphic to a subgroup K of S_n by Exercise 22. Finally, by Exercise 21, $G \cong H$ and $H \cong K$ imply that $G \cong K$. ■

Any homomorphism from a group G to a group of permutations is called a **representation** of G, and G is said to be **represented by a group of permutations.** The homomorphism $G \to A(G)$ in the proof of Theorem 7.21 is called the **left regular representation** of G. By the use of such representations, group theory can be reduced to the study of permutation groups. This approach is sometimes very advantageous because permutations are concrete objects that are readily visualized. Calculations with permutations are straightforward, which is not always the case in some groups. In certain situations, group representations are a very effective tool.

On the other hand, representation by permutations has some drawbacks. For one thing, a given group can be represented as a group of permutations in many ways—the homomorphism $G \to A(G)$ of Theorem 7.21 is just one of the possibilities (see Exercises 49, 51, and 54 for others). And many of these representations may be quite inefficient. According to Corollary 7.22, for example, every group of order 12 is isomorphic to a subgroup of S_{12}, but S_{12} has order $12! = 479{,}001{,}600$. Determining useful information about a subgroup of order 12 in a group that size is likely to be difficult at best.

Except for some special situations, then, the study of elementary group theory via the abstract definition (as we have been doing) rather than via concrete permutation representations is likely to be more effective. The abstract approach has the advantage of eliminating nonessential features and concentrating on the basic underlying structure. In the long run, this usually results in simpler proofs and better understanding.

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*The group $A(G)$ itself is usually far too large to be isomorphic to G. For instance, if G has order n, then $A(G)$ has order $n!$ by Exercise 20 of Section 7.1.

Exercises

A. 1. (a) Show that the function $f:\mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x$ is an isomorphism of additive groups.

(b) Let $\mathbb{R}^{**}$ be the multiplicative group of positive real numbers. Show that $f:\mathbb{R}^{**} \to \mathbb{R}^{**}$ given by $f(x) = 3x$ is not a homomorphism of groups.

2. Show that the function $g:\mathbb{R}^{**} \to \mathbb{R}^{**}$ given by $g(x) = \sqrt{x}$ is an isomorphism.

3. Show that $GL(2, \mathbb{Z}_2)$ is isomorphic to S_3 by writing out the operation tables for each group. [*Hint:* List the elements of $GL(2, \mathbb{Z}_2)$ in this order: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and the elements of S_3 in this order: $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.]

4. Prove that the function $f:\mathbb{R}^* \to \mathbb{R}^*$ defined by $f(x) = x^3$ is an isomorphism.

5. Prove that the function $g:\mathbb{Z}_9 \to \mathbb{Z}_9$ defined by $g(x) = 2x$ is an isomorphism.

6. Prove that the function $h:\mathbb{Z}_8 \to \mathbb{Z}_8$ defined by $h(x) = 2x$ is a homomorphism that is neither injective nor surjective.

7. Prove that the function $f:\mathbb{R}^* \to \mathbb{R}^{**}$ defined by $f(x) = |x|$ is a surjective homomorphism that is not injective.

8. Prove that the function $g:\mathbb{R} \to \mathbb{R}^*$ defined by $g(x) = 2^x$ is an injective homomorphism that is not surjective.

9. If G and H are groups, prove that the function $f:G \times H \to G$ given by $f((a, b)) = a$ is a surjective homomorphism.

10. Show that the function $f:\mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not a homorphism.

11. Prove that the function $g:\mathbb{R}^* \to GL(2, \mathbb{R})$ defined by $g(x) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ is an injective homomorphism.

12. Prove that the function $h:\mathbb{R} \to GL(2, \mathbb{R})$ defined by $h(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ is an injective homomorphism.

13. Show that U_5 is isomorphic to U_{10}.

14. Prove that the additive group $\mathbb{Z}_6$ is isomorphic to the multiplicative group of nonzero elements in $\mathbb{Z}_7$.

15. Let $f:G \to H$ be a homomorphism of groups. Prove that for each $a \in G$ and each integer n, $f(a^n) = f(a)^n$.

16. If $f:G \to H$ is a surjective homomorphism of groups and G is abelian, prove that H is abelian.

17. Prove that the function f in the proof of Theorem 7.19(1) is a bijection.

18. Let G, H, G_1, H_1 be groups such that $G \cong G_1$ and $H \cong H_1$. Prove that $G \times H \cong G_1 \times H_1$.

19. Prove that a group G is abelian if and only if the function $f: G \to G$ given by $f(x) = x^{-1}$ is a homomorphism of groups. In this case, show that f is an isomorphism.

20. Let N be a subgroup of a group G and let $a \in G$.

(a) Prove that $a^{-1}Na = \{a^{-1}na \mid n \in N\}$ is a subgroup of G.

(b) Prove that N is isomorphic to $a^{-1}Na$. [*Hint:* Define $f: N \to a^{-1}Na$ by $f(n) = a^{-1}na$.]

21. Let G, H, and K be groups. If $G \cong H$ and $H \cong K$, then prove that $G \cong K$. [*Hint:* If $f: G \to H$ and $g: H \to K$ are isomorphisms, prove that the composite function $g \circ f: G \to K$ is also an isomorphism.]

22. If $f: G \to H$ is an isomorphism of groups and if T is a subgroup of G, prove that T is isomorphic to the subgroup $f(T) = \{f(a) \mid a \in T\}$ of H.

23. **(a)** If G is an abelian group, prove that the function $f: G \to G$ given by $f(x) = x^2$ is a homomorphism.

(b) Prove that part (a) is false for every nonabelian group. [*Hint:* A counterexample is insufficient here (Why?). So try Exercise 24 of Section 7.2.]

B. 24. Let G be a multiplicative group. Let G^{op} be the set G equipped with a new operation $*$ defined by $a * b = ba$.

(a) Prove that G^{op} is a group.

(b) Prove that $G \cong G^{op}$. [*Hint:* Corollary 7.6 may be helpful.]

25. Assume that a and b are both generators of the cyclic group G, so that $G = \langle a \rangle$ and $G = \langle b \rangle$. Prove that the function $f: G \to G$ given by $f(a^i) = b^i$ is an automorphism of G.

26. If $G = \langle a \rangle$ is a cyclic group and $f: G \to H$ is a surjective homomorphism of groups, show that $f(a)$ is a generator of H, that is, H is the cyclic group $\langle f(a) \rangle$. [*Hint:* Exercise 15.]

27. Let G be a multiplicative group and c a fixed element of G. Let H be the *set* G equipped with a new operation $*$ defined by $a * b = acb$.

(a) Prove that H is a group.

(b) Prove that the map $f: G \to H$ given by $f(x) = c^{-1}x$ is an isomorphism.

28. Let $f: G \to H$ be a homomorphism of groups and suppose that $a \in G$ has finite order k.

(a) Prove that $f(a)^k = e$. [*Hint:* Exercise 15.]

(b) Prove that $|f(a)|$ divides $|a|$. [*Hint:* Theorem 7.9.]

29. If $f: G \to H$ is an injective homomorphism of groups and $a \in G$, prove that $|f(a)| = |a|$.

30. Let $f: G \to H$ be a homomorphism of groups and let K be a subgroup of H. Prove that the set $\{a \in G \mid f(a) \in K\}$ is a subgroup of G.

31. If $f: G \to G$ is a homomorphism of groups, prove that $F = \{a \in G \mid f(a) = a\}$ is a subgroup of G.

32. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix, the number $ad - bc$ is denoted det A and called the **determinant** of A. Prove that the function $f: GL(2, \mathbb{R}) \to \mathbb{R}^*$ given by $f(A) = \det A$ is a surjective homomorphism.

33. Let $f: G \to H$ be a homomorphism of groups and let $K_f = \{a \in G \mid f(a) = e_H\}$, that is, the set of elements of G that are mapped by f to the identity element of H. Prove that K_f is a subgroup of G. See Exercises 34 and 35 for examples.

34. The function $f: \mathbb{Z} \to \mathbb{Z}_5$ given by $f(x) = [x]$ is a homomorphism by Example 13. Find K_f (notation as in Exercise 33).

35. The function $f: U_5 \to U_5$ given by $f(x) = x^2$ is a homomorphism by Exercise 23. Find K_f (notation as in Exercise 33).

36. Let G be a group and let **Aut** G be the set of all automorphisms of G. Prove that Aut G is a group under the operation of composition of functions. [*Hint:* Exercise 21 may help.]

37. Let G be a group and let Aut G be as in Exercise 36. Let **Inn** G be the set of all inner automorphisms of G (that is, isomorphisms of the form $f(a) = c^{-1}ac$ for some $c \in G$, as in Example 9.). Prove that Inn G is a subgroup of Aut G. [*Note:* Two different elements of G may induce the same inner automorphism, that is, we may have $c^{-1}ac = d^{-1}ad$ for all $a \in G$. Hence, $|\text{Inn } G| \leq |G|$.]

38. Let T be a set n elements and let $A(T)$ be the group of permutations of T. Prove that $A(T) \cong S_n$. [*Hint:* If the elements of T in some order are relabeled as $1, 2, \ldots, n$, then every permutation of T becomes a permutation of $1, 2, \ldots, n$.]

39. Show that the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic.

In Exercises 40–44, explain why the given groups are not *isomorphic. (Exercises 16 and 29 may be helpful.)*

40. $\mathbb{Z}_6$ and S_3

41. $\mathbb{Z}_4 \times \mathbb{Z}_2$ and D_4

42. $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

43. U_8 and U_{10}

44. U_{10} and U_{12}

45. Is U_8 isomorphic to U_{12}? Justify your answer.

46. Prove that the additive group $\mathbb{R}$ of all real numbers is not isomorphic to the multiplicative group $\mathbb{R}^*$ of nonzero real numbers. [*Hint:* If there were an isomorphism $f: \mathbb{R} \to \mathbb{R}^*$, then $f(k) = -1$ for some k; use this fact to arrive at a contradiction.]

47. Show that D_4 is not isomorphic to the quaternion group of Exercise 16 of Section 7.1.

48. Prove that the additive group $\mathbb{Q}$ is *not* isomorphic to the multiplicative group $\mathbb{Q}^{**}$ of positive rational numbers, even though $\mathbb{R}$ and $\mathbb{R}^{**}$ *are* isomorphic.

49. Let G be a group and let $A(G)$ be the group of permutations of the set G. Define a function g from G to $A(G)$ by assigning to each $d \in G$ the inner automorphism induced by d^{-1} (as in Example 9 with $c = d^{-1}$). Prove that g is a homomorphism of groups.

50. Let G be a group and $h \in A(G)$. Assume that $h \circ \varphi_a = \varphi_a \circ h$ for all $a \in G$ (where φ_a is as in the proof of Theorem 7.21). Prove that there exists $b \in G$ such that $h(x) = xb^{-1}$ for all $x \in G$.

51. **(a)** Let G be a group and $c \in G$. Prove that the map $\theta_c: G \to G$ given by $\theta_c(x) = xc^{-1}$ is an element of $A(G)$.

    (b) Prove that $h: G \to A(G)$ given by $h(c) = \theta_c$ is an injective homomorphism of groups. Thus G is isomorphic to the subgroup Im h of $A(G)$. This is the **right regular representation of G.**

52. Find the left regular representation of each group (that is, express each group as a permutation group as in the proof of Theorem 7.21):

    (a) $\mathbb{Z}_3$ **(b)** $\mathbb{Z}_4$ **(c)** S_3

53. Let $f: G \to H$ be an isomorphism of groups. Let $g: H \to G$ be the inverse function of f as defined in Appendix B. Prove that g is also an isomorphism of groups. [*Hint:* To show that $g(ab) = g(a)g(b)$, consider the images of the left- and right-hand sides under f and use the facts that f is a homomorphism and $f \circ g$ is the identity map.]

54. **(a)** Show that $D_3 \cong S_3$. [*Hint:* D_3 is described in Example 6 of Section 7.1 or 7.1.A. Each motion in D_3 permutes the vertices; use this to define a function from D_3 to S_3.]

    (b) Show that D_4 is isomorphic to a subgroup of S_4. [*Hint:* See the hint for part (a). This isomorphism represents D_4, a group of order 8, as a subgroup of a permutation group of order $4! = 24$, whereas the left regular representation of Corollary 7.22 represents G as a subgroup of S_8, a group of order $8! = 40{,}320$.]

55. **(a)** Prove that $H = \left\{ \begin{pmatrix} 1-n & -n \\ n & 1+n \end{pmatrix} \middle| \; n \in \mathbb{Z} \right\}$ is a group under matrix multiplication.

    (b) Prove that $H \cong \mathbb{Z}$.

56. **(a)** Prove that $K = \left\{ \begin{pmatrix} 1-2n & n \\ -4n & 1+2n \end{pmatrix} \middle| \; n \in \mathbb{Z} \right\}$ is a group under matrix multiplication.

    (b) Is K isomorphic to $\mathbb{Z}$?

57. Prove that the additive group $\mathbb{Z}[x]$ is isomorphic to the multiplicative group $\mathbb{Q}^{**}$ of positive rationals. [*Hint:* Let $p_0, p_1, p_2, \ldots$ be the distinct positive primes in their usual order. Define $\varphi: \mathbb{Z}[x] \to \mathbb{Q}^{**}$ by

$$\varphi(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = p_0^{a_0}p_1^{a_1} \cdots p_n^{a_n}.]$$

58. Prove that G is an abelian group if and only if Inn G consists of a single element. [*Hint:* See Exercise 37.]

59. **(a)** Verify that the group Inn D_4 has order 4. [*Hint:* See Exercise 37.]

(b) Prove that Inn $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

60. Prove that Aut $\mathbb{Z} \cong \mathbb{Z}_2$. [*Hint:* What are the possible generators of the cyclic group $\mathbb{Z}$? See Exercises 25 and 26.]

61. Prove that Aut $\mathbb{Z}_n \cong U_n$. [*Hint:* See Exercise 25 above and Exercise 44 of Section 7.3.]

62. Prove that Aut $(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$.

> **APPLICATION:** Linear Codes (Section 16.1) may be covered at this point if desired.

7.5 The Symmetric and Alternating Groups*

The finite symmetric groups S_n are important because, as we saw in Corollary 7.22, every finite group is isomorphic to a subgroup of some S_n. In this section, we introduce a more convenient notation for permutations, and some important subgroups of the groups S_n. We begin with the new notation.

Consider the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 6 & 2 & 5 \end{pmatrix}$ in S_6. Note that 2 is mapped to 4, 4 is mapped to 6, 6 is mapped to 5, 5 is mapped back to 2, and the other two elements, 1 and 3, are mapped to themselves. All the essential information can be summarized by this diagram:

$$\begin{tikzcd} 2 \arrow[r, dashed] & 4 \arrow[r, dashed] & 6 \arrow[r, dashed] & 5 \arrow[lll, dashed, bend left] \end{tikzcd}$$

It isn't necessary to include the arrows here as long as we keep things in the same order. A complete description of this permutation is given by the symbol (2465), with the understanding that

each element is mapped to the element listed immediately to the right;

the last element in the string is mapped to the first;

elements not listed are mapped to themselves.

*Except for a few well-marked examples and exercises, this section is needed only in Sections 8.5, 9.3–9.5, and 12.3.

This is an example of *cycle notation.* Here is a formal definition.

Definition

Let $a_1, a_2, a_3, \ldots, a_k$ (with $k \geq 1$) be distinct elements of the set $\{1, 2, 3, \ldots, n\}$. Then $(a_1a_2a_3 \ldots a_k)$ denotes the permutation in S_n that maps a_1 to a_2, a_2 to $a_3, \ldots, a_{k-1}$ to a_k, and a_k to a_1, and maps every other element of $\{1, 2, 3, \ldots, n\}$ to itself. $(a_1a_2a_3 \ldots a_k)$ is called a **cycle of length** k or a k**-cycle.**

EXAMPLE 1

In S_4, (143) is the 3-cycle that maps 1 to 4, 4 to 3, 3 to 1, and 2 to itself; it was written $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ in the old notation. Note that (143) may also be denoted by (431) or (314) since each of these indicates the function that maps 1 to 4, 4 to 3, 3 to 1, and 2 to 2.

EXAMPLE 2

According to the definition above, the 1-cycle (3) in S_n is the permutation that maps 3 to 3 and maps every other element of $\{1, 2, \ldots, n\}$ to itself; in other words, (3) is the identity permutation. Similarly, for any k in $\{1, 2, \ldots, n\}$, the 1-cycle (k) is the identity permutation.

Strictly speaking, cycle notation is ambiguous since, for example, (163) might denote a permutation in S_6, in S_7, or in any S_n with $n \geq 6$. In context, however, this won't cause any problems because it will always be made clear which group S_n is under discussion.

Products in cycle notation can be visually calculated just as in the old notation. For example, we know that

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

(Remember that the product in S_n is composition of functions, and so the right-hand permutation is performed first.) In cycle notation, this product* becomes

$$(2 \quad 4 \quad 3)(1 \quad 2 \quad 4 \quad 3) = (1 \quad 4 \quad 2 \quad 3).$$

The arrows indicate the process: 1 is mapped to 2 and 2 is mapped to 4, so that the product maps 1 to 4. Similarly, 4 is mapped to 3 and 3 is mapped to 2, so that the product maps 4 to 2.

*Hereafter we shall omit the composition symbol $\circ$ and write the group operation in S_n multiplicatively.

EXAMPLE 3

In the old notation S_3 consists of

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \text{and} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

In the new notation, the elements of S_3 (in the same order) are

$$(1), (23), (13), (12), (123), \text{and } (132).$$

Two cycles are said to be **disjoint** if they have no elements in common. For instance, (13) and (2546) are disjoint cycles in S_6, but (13) and (345) are not since 3 appears in both cycles.

EXAMPLE 4

As shown before Example 3, $(243)(1243) = (1423)$. Verify that

$$(1243)(243) = (2341).$$

Hence, the cycles (243) and (1234) do not commute with each other. On the other hand, you can easily verify that the disjoint cycles (13) and (2546) *do* commute:

$$(13)(2546) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 6 & 4 & 2 \end{pmatrix} = (2546)(13).$$

This is an illustration of the following theorem.

Theorem 7.23

*If $\sigma = (a_1a_2 \cdots a_k)$ and $\tau = (b_1b_2 \cdots b_r)$ are disjoint cycles in S_n, then $\sigma\tau = \tau\sigma$.**

Proof▸ Exercise 18. ■

It is not true that every permutation is a cycle, but every permutation can be expressed as the product of disjoint cycles. Consider, for example, the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 7 & 2 & 4 & 6 & 3 \end{pmatrix}$ in S_7. Find an element that is not mapped to itself, say 1, and trace where it is sent by the permutation:

1 is mapped to 5, 5 is mapped to 4, 4 is mapped to 2, and 2 is mapped to 1 (the element with which we started).

*Greek letters are often used to denote permutations. We shall generally use the letters alpha (α), beta (β), delta (δ), sigma (σ), and tau (τ). For the entire Greek alphabet, see the inside back cover of this book.

Thus the given permutation has the same action as the cycle (1542) on these four elements. Now look at any element other than 1, 5, 4, 2 that is not mapped onto itself, say 3. Note that

3 is mapped to 7, and 7 is mapped to 3.

Thus the 2-cycle (37) has the same action on 7 and 3 as the given permutation. The only element now unaccounted for is 6, which is mapped to itself. You can now easily verify that the original permutation is the product of the two cycles we have found, that is,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 7 & 2 & 4 & 6 & 3 \end{pmatrix} = (1542)(37).$$

Although some care must be used and the notation is more cumbersome, essentially the same procedure works in the general case.

Theorem 7.24

Every permutation in S_n is the product of disjoint cycles.*

Proof ▸ Adapt the procedure in the preceding example; see Exercise 44. ■

Theorem 7.25

The order of a permutation τ in S_n is the least common multiple of the lengths of the disjoint cycles whose product is τ.†

Proof ▸ Exercise 19. ■

EXAMPLE 5

The permutation $\tau = (12)(34)(567)$ is a product of disjoint cycles of lengths 2, 2, and 3. The least common multiple of 2, 2, and 3 is 6. Theorem 7.25 tells us that τ has order 6. You can verify this directly by computing the powers of τ:

$$\tau = (12)(34)(567), \qquad \tau^2 = (576), \qquad \tau^3 = (12)(34),$$
$$\tau^4 = (567), \qquad \tau^5 = (12)(34)(576), \qquad \tau^6 = (1). \quad ■$$

The Alternating Groups

A 2-cycle is often called a **transposition.** Transpositions have some interesting properties.

EXAMPLE 6

If (ab) is a transposition, verify that $(ab)(ab) = (1)$. Hence,

Every transposition is its own inverse.

*As usual, we allow the possibility of a product with just one cycle in it.

†The least common multiple is defined in Exercise 31 of Section 1.2.

EXAMPLE 7

We claim that the inverse of the product (12)(34)(14)(13) is (13)(14)(34)(12) (*the same transpositions in reverse order*). To prove this claim, we use the fact that a transposition is its own inverse:

$$\begin{aligned}(12)(34)(14)(13)\cdot(13)(14)(34)(12) &= (12)(34)(14)\cdot(14)(34)(12)\\ &= (12)(34)\cdot(34)(12) = (12)(12) = (1).\end{aligned}$$

A similar argument works in the general case and shows that

If $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}$, and σ_n are transpositions, then

$$(\sigma_1\sigma_2\sigma_3\cdots\sigma_{n-1}\sigma_n)^{-1} = \sigma_n\sigma_{n-1}\cdots\sigma_3\sigma_2\sigma_1.$$

You can easily verify that

$$(1) = (12)(12), \quad (123) = (12)(23), \quad (1234) = (12)(23)(34).$$

These are examples of the following theorem.

Theorem 7.26

Every permutation in S_n is a product of (not necessarily disjoint) transpositions.

Proof ▸ Since every permutation is a product of cycles by Theorem 7.24, we need only verify that every cycle $(a_1a_2\cdots a_k)$ is a product of transpositions:

$$(a_1a_2\cdots a_k) = (a_1a_2)(a_2a_3)\cdots(a_{k-1}a_k). \quad ■$$

This corollary can also be proved directly by induction, without using Theorem 7.24 (Exercise 33).

A permutation in S_n is said to be **even** if it can be written as the product of an even number of transpositions, and **odd** if it can be written as the product of an odd number of transpositions.

EXAMPLE 8

(132) is even and (1243)(243) is odd because, as you can easily verify,

$$(132) = (12)(13) \qquad \text{and} \qquad (1243)(243) = (23)(34)(14).$$

Since no integer is both even and odd, the even-odd terminology for permutations *suggests* that no permutation is both even and odd. This is indeed the case, but it requires *proof*. The first step is to prove

Lemma 7.27

The identity permutation in S_n is even, but not odd.

Proof ▸ We write the identity permutation as (1). Verify that (12)(12) = (1). Hence, the identity permutation is even. To show that it is not odd, we use a proof by contradiction. Suppose that $(1) = \tau_k\cdots\tau_2\tau_1$ with each τ_i

a transposition and k odd. Let c be a symbol that appears in at least one of these transpositions. Let τ_r be the first transposition (reading from *right to left*) in which c appears, say $\tau_r = (cd)$. Then c does not appear in $\tau_{r-1}, \cdots \tau_1$ and is, therefore, left fixed by these transpositions. If $r = k$, then c is left fixed by all the τ's except τ_k, so that the product—the identity permutation—maps c to d, a contradiction. Hence, $r < k$.

Now consider the transposition τ_{r+1}. It must have one of the following forms (where x, y, c, d denote distinct elements of $\{1, 2, \cdots n\}$:

I. (xy) II. (xd) III. (cy) IV. (cd).

Consequently, there are four possibilities for the product $\tau_{r+1}\tau_r$:

I. $(xy)(cd)$ II. $(xd)(cd)$ III. $(cy)(cd)$ IV. $(cd)(cd)$.

In Case I, verify that $(xy)(cd) = (cd)(xy)$. Replace $(xy)(cd)$ by $(cd)(xy)$ in the product; this moves the first appearance of c one transposition to the left. In Case II, verify that $(xd)(cd) = (xc)(xd)$; if we replace $(xd)(cd)$ by $(xc)(xd)$, then once again the first appearance of c is one transposition farther left. Show that a similar conclusion holds in Case III by verifying that $(cy)(cd) = (cd)(dy)$.

Each repetition of the procedure in Cases I–III moves the first appearance of c one transposition farther left. Eventually Case IV must occur; otherwise, we could keep moving c until it first appears in the last permutation at the left, τ_k, which is impossible, as we saw in the first paragraph. In Case IV, however, we have $\tau_{r+1}\tau_r = (cd)(cd) = (1)$. So we can delete these two transpositions and write (1) as a product of two fewer transpositions than before. Obviously, we can carry out the same argument for any symbol that appears in a transposition in the product. If the original product contains an odd number of transpositions, eliminating two at a time eventually reduces it to a single transposition $(1) = (ab)$, which is a contradiction. Therefore, the identity permutation (1) cannot be written as the product of an odd number of transpositions. ■

Theorem 7.28

No permutation in S_n is both even and odd.

Proof ▸ Suppose $\alpha \in S_n$ can be written as $\sigma_1\sigma_2 \cdots \sigma_k$ and as $\tau_1\tau_2 \cdots \tau_r$ with each σ_i, τ_j a transposition, k odd, and r even. Since every transposition is its own inverse, Corollary 7.6 shows that

$$\begin{aligned}(1) = \alpha\alpha^{-1} &= (\sigma_1 \cdots \sigma_k)(\tau_1 \cdots \tau_r)^{-1} \\ &= \sigma_1 \cdots \sigma_k\tau_r^{-1} \cdots \tau_1^{-1} \\ &= \sigma_1 \cdots \sigma_k\tau_r \cdots \tau_1.\end{aligned}$$

Since k is odd and r is even, $k + r$ is odd, and we have written (1) as the product of an odd number of transpositions. This contradicts Lemma 7.27, and completes the proof of the theorem. ■

The set of all even permutations in S_n is denoted A_n and is called the **alternating group of degree** $\boldsymbol{n}$; the word "group" is justified by the following theorem.

Theorem 7.29

A_n is a subgroup of S_n of order $n!/2$.

Proof ▸ If α and β are in A_n, then $\alpha = \sigma_1\sigma_2\cdots\sigma_k$ and $\beta = \tau_1\tau_2\cdots\tau_r$, with each σ_i, τ_j a transposition and k, r even. Thus, $\alpha\beta = \sigma_1\sigma_2\cdots\sigma_k\tau_1\tau_2\cdots\tau_r$. Since $k + r$ is even, $\alpha\beta \in A_n$. So A_n is closed under multiplication. By Example 7, $\alpha^{-1} = \sigma_k\sigma_{k-1}\cdots\sigma_2\sigma_1$. Since k is even, $\alpha^{-1} \in A_n$. Therefore, A_n is a subgroup by Theorem 7.11. Exercise 24 shows that $|A_n| = n!/2$. ■

EXAMPLE 9

The elements of S_3 are listed in Example 3. Because $|S_3| = 3!$, we know that $|A_3| = \dfrac{3!}{2} = 3$. Since (12), (13), and (23) are obviously odd, A_3 must consist of (123), (132), and (1).

Exercises

A. **1.** Write each permutation in cycle notation:

(a) $\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 7&2&1&4&5&6&3&8&9 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 2&4&3&5&7&6&8&9&1 \end{pmatrix}$

(c) $\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 4&8&1&7&5&2&6&3&9 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 1&2&5&4&7&6&9&3&8 \end{pmatrix}$

2. Compute each product:

(a) (12)(23)(34) **(b)** (246)(147)(135)

(c) (12)(53214)(23) **(d)** (1234)(2345)

3. Express as a product of disjoint cycles:

(a) $\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 2&1&3&5&4&7&9&8&6 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 3&5&1&2&4&6&8&9&7 \end{pmatrix}$

(c) $\begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 3&5&1&2&4&9&8&7&6 \end{pmatrix}$ **(d)** (14)(27)(523)(34)(1472)

(e) (7236)(85)(571)(1537)(486)

4. Write each permutation in Exercise 3 as a product of transpositions.

5. Find the order of each permutation.

(a) (12) (b) (123) (c) (1234)

(d) What do you think the order of (123456789) is?

6. Find the order of each permutation.

(a) (13)(24) (b) (123)(456) (c) (123)(435)

(d) (1234)(4231) (e) (1234)(24)(43215)

7. Which of these permutations are even:

(a) (2468) (b) (246)(134) (c) (12)(123)(1234)

8. List the elements in each group:

(a) A_2 (b) A_4

9. What is the order of each group:

(a) A_4 (b) A_5 (c) A_{10}

10. Is the set B_n of odd permutations in S_n a group? Justify your answer.

11. List the order of each element of A_4.

12. Write (12)(34) as the product of two 3-cycles.

13. Show that $\alpha = (123)(234)(567)(78910)$ has order 10 in S_n $(n \geq 10)$. [*Hint:* Write α as a product of disjoint cycles and use Theorem 7.25.]

14. Show that $\beta = (1236)(5910)(465)(5678)$ has order 21 in S_n $(n \geq 10)$.

B. 15. Prove that the cycle $(a_1a_2 \cdots a_k)$ is even if and only if k is odd.

16. Show that the inverse of $(a_1a_2 \cdots a_k)$ in S_n is $(a_ka_{k-1} \cdots a_3a_2a_1)$.

17. Prove that a k-cycle in the group S_n has order k.

18. Let $\sigma = (a_1a_2 \cdots a_k)$ and $\tau = (b_1b_2 \cdots b_r)$ be disjoint cycles in S_n. Prove that $\sigma\tau = \tau\sigma$. [*Hint:* You must show that $\sigma\tau$ and $\tau\sigma$ agree as functions on each i in $\{1, 2, \ldots, n\}$. Consider three cases: i is one of the a's; i is one of the b's; i is neither.]

19. Prove Theorem 7.25: The order of a permutation τ in S_n is the least common multiple of the lengths of the disjoint cycles whose product is τ. [*Hint:* Theorem 7.23 and Exercise 17 may be helpful.]

20. Let α and β be permutations in S_n.

(a) Fill the blanks in the table.

α	β	$\alpha\beta\alpha^{-1}$	$\alpha\beta\alpha^{-1}\beta^{-1}$
even	even		
even	odd		even
odd	even		
odd	odd		

(b) What conclusions can you draw from the results in part (a).

21. Find the order of σ^{1000}, where σ is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 8 & 9 & 4 & 5 & 2 & 1 & 6 \end{pmatrix}$. [*Hint:* Write σ as a product of disjoint cycles.]

22. Show that S_{10} contains elements of orders 10, 20, and 30. Does it contain an element of order 40?

23. Prove that $\{(1), (12)(34), (13)(24), (14)(23)\}$ is a subgroup of A_4.

24. Let B_n denote the set of odd permutations in S_n. Define a function $f: A_n \to B_n$ by $f(\alpha) = (12)\alpha$.

    (a) Prove that f is injective.

    (b) Prove that f is surjective. [*Hint:* If $\beta \in B_n$, then $(12)\beta \in A_n$.] So f is bijective. Hence, A_n and B_n have the same number of elements.

    (c) Show that $|A_n| = n!/2$. [*Hint:* Every element of S_n is in A_n or B_n (but not both) and $|S_n| = n!$.]

    See Exercise 39(a) and (b) for a generalization of this exercise.

25. Show that the subgroup G of S_4 generated by the elements $\sigma = (1234)$ and $\tau = (24)$ has order 8.

26. Prove that the center of S_n $(n > 2)$ is the identity subgroup.

27. If σ is a k-cycle with k odd, prove that there is a cycle τ such that $\tau^2 = \sigma$.

28. Let σ be a k-cycle in S_n.

    (a) Prove that σ^2 is a cycle if and only if k is odd.

    (b) If $k = 2t$, prove that there are t-cycles τ and β such that $\sigma^2 = \tau\beta$.

29. Let σ and τ be transpositions in S_n with $n \geq 3$. Prove that $\sigma\tau$ is a product of (not necessarily disjoint) 3-cycles.

30. Prove that every element of A_n is a product of 3-cycles.

31. Let σ be a product of disjoint cycles of the same length. Prove that σ is a power of a cycle.

32. Prove that the decomposition of a permutation as a product of disjoint cycles is unique except for the order in which the cycles are listed.

33. Use induction on n to give an alternate proof of Theorem 7.26: Every element of S_n is a product of transpositions. [*Hint:* If the statement is true for $n = k - 1$ and if $\tau \in S_k$, consider the transposition (kr), where $r = \tau(k)$. Note that $(kr)\tau$ fixes k and hence may be considered as a permutation of $\{1, 2, \ldots, k - 1\}$.]

34. If $n \geq 3$; prove that every element of S_n can be written as a product of at most $n - 1$ transpositions.

35. Let τ be a transposition and let $\sigma \in S_n$. Prove that $\sigma\tau\sigma^{-1}$ is a transposition.

36. If τ is the k-cycle $(a_1 a_2 \cdots a_k)$ and if $\sigma \in S_n$, prove that $\sigma\tau\sigma^{-1} = (\sigma(a_1)\sigma(a_2) \cdots \sigma(a_k))$.

37. Let H consist of all permutations in S_n that fix 1 and n, that is,

$$H = \{\alpha \in S_n \mid \alpha(1) = 1 \text{ and } \alpha(n) = n\}.$$

Prove that H is a subgroup of S_n.

38. Show that D_4 is isomorphic to the group G in Exercise 25. [*Hint:* Note that every element of D_4 produces a permutation of the vertices of the square (see Example 5 in Section 7.1 or 7.1.A.). If the vertices are numbered 1, 2, 3, 4, then this permutation can be considered as an element of S_4. Define a function $f: D_4 \to S_4$ by mapping each element of D_4 to its permutation of the vertices. Verify that f is an injective homomorphism with image G.]

39. Let G be a subgroup of S_n that contains an odd permutation τ.

(a) Prove that the number of even permutations in G is the same as the number of odd permutations in G.

(b) Explain why 2 divides $|G|$.

(c) If K is a subgroup of S_n of odd order, prove that K is actually a subgroup of A_n.

C. 40. Prove that every element of A_n is a product of n-cycles.

41. Prove that the transpositions (12), (13), (14), . . . , $(1n)$ generate S_n.

42. Prove that (12) and $(123 \cdots n)$ generate S_n.

43. If f is an automorphism of S_3, prove that there exists $\sigma \in S_3$ such that $f(\tau) = \sigma\tau\sigma^{-1}$ for every $\tau \in S_3$.

44. Use the following steps to prove Theorem 7.24: Every permutation τ in S_n is a product of disjoint cycles.

(a) Let a_1 be any element of $\{1, 2, \ldots, n\}$ such that $\tau(a_1) \neq a_1$. Let $a_2 = \tau(a_1)$, $a_3 = \tau(a_2)$, $a_4 = \tau(a_3)$, and so on. Let k be the first index such that $\tau(a_k)$ is one of $a_1, \ldots, a_{k-1}$. Prove that $\tau(a_k) = a_1$. Conclude that τ has the same effect on $a_1, \ldots, a_k$ as the cycle $(a_1a_2 \cdots a_k)$.

(b) Let b_1 be any element of $\{1, 2, \ldots, n\}$ other than $a_1, \ldots, a_k$ that is not mapped to itself by τ. Let $b_2 = \tau(b_1)$, $b_3 = \tau(b_2)$, and so on. Show that $\tau(b_i)$ is never one of $a_1, \ldots, a_k$. Repeat the argument in part (a) to find a b_r such that $\tau(b_r) = b_1$ and τ agrees with the cycle $(b_1b_2 \cdots b_r)$ on the b's.

(c) Let c_1 be any element of $\{1, 2, \ldots, n\}$ other than the a's or b's above such that $\tau(c_1) \neq c_1$. Let $c_2 = \tau(c_1)$, and so on. As above, find c_s such that τ agrees with the cycle $(c_1c_2 \cdots c_s)$ on the c's.

(d) Continue in this fashion until the only elements unaccounted for are those that are mapped to themselves by τ. Verify that τ is the product of the cycles

$$(a_1 \cdots a_k)(b_1 \cdots b_r)(c_1 \cdots c_s) \cdots$$

and that these cycles are disjoint.

45. Prove that S_n is isomorphic to a subgroup of A_{n+2}.

CHAPTER 8

Normal Subgroups and Quotient Groups

Congruence in the integers led to the finite arithmetics $\mathbb{Z}_n$, which produced a number of interesting results. Now we shall extend the concept of congruence to groups, producing new groups and a deeper understanding of algebraic structure.

8.1 Congruence and Lagrange's Theorem

In this section we present the analogue for groups of the concept of congruence, which was introduced for integers in Chapter 2 and for rings in Chapter 6.* Except for some notational changes, the first three results of this section are virtually identical to those proved earlier for integers and rings. The following chart shows this parallel development.

INTEGERS	RINGS	GROUPS
Theorem 2.1	Theorem 6.4	Theorem 8.1
Theorem 2.3	Theorem 6.6	Theorem 8.2
Corollary 2.4	Corollary 6.7	Corollary 8.3

We begin by looking at an example of congruence in $\mathbb{Z}$ from a somewhat different viewpoint.

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*Chapter 6 is *not* a prerequisite for this section, but it will be mentioned occasionally. Section 2.1 will be the model for the presentation here.

EXAMPLE 1

In the integers, $a \equiv b \pmod 4$ means that 4 divides $a - b$, that is, that $a - b$ is a multiple of 4. Let K be the set of all multiples of 4, so that

$$K = \{0, \pm 4, \pm 8, \pm 12, \ldots\}.$$

Thus,

$$a \equiv b \pmod 4 \qquad \text{means} \qquad a - b \in K.$$

Note that K is actually a subgroup of $\mathbb{Z}$ (the additive cyclic subgroup generated by 4). Instead of thinking of congruence modulo the element 4, we can consider this as congruence modulo the subgroup K:

$$a \equiv b \pmod K \qquad \text{means} \qquad a - b \in K.$$

Now let G be any group and K a subgroup of G. The last line of the preceding example could be used as a definition of congruence modulo K. However, we normally use multiplicative notation for groups. So we must translate the proposed definition and results from Section 2.1 into equivalent statements in multiplicative notation.* The following dictionary may be helpful for this translation.

ADDITIVE NOTATION	MULTIPLICATIVE NOTATION
$a + b$	ab
0	e
$-c$	c^{-1}
$a - b = a + (-b)$	ab^{-1}

Thus, the additive statement $a - b \in K$ is equivalent to the multiplicative statement $ab^{-1} \in K$, and we have the following definition of congruence.

Definition Let K be a subgroup of a group G and let $a, b \in G$. Then a **is congruent to** b **modulo** K [written $a \equiv b \pmod K$] provided that $ab^{-1} \in K$.

EXAMPLE 2

Let K be the subgroup $\{r_0, r_1, r_2, r_3\}$ of D_4. Then the operation table in Example 5 of Section 7.1 or 7.1.A shows that $d^{-1} = d$ and $h \circ d^{-1} = h \circ d = r_1 \in K$. Therefore, $h \equiv d \pmod K$.

*There is a possibility of confusion here since integer multiplication is also defined. In carrying over congruence from integers to groups, we consider *only* the additive structure of the integers and ignore integer multiplication because the integers form an additive group, but not a multiplicative one.

Theorem 8.1

Let K be a subgroup of a group G. Then the relation of congruence modulo K is

(1) reflexive: $a \equiv a \pmod{K}$ for all $a \in G$;

(2) symmetric: if $a \equiv b \pmod{K}$, then $b \equiv a \pmod{K}$;

(3) transitive: if $a \equiv b \pmod{K}$ and $b \equiv c \pmod{K}$, then $a \equiv c \pmod{K}$.

The idea is to translate the proof of Theorem 2.1 to the present situation by changing congruence mod n to congruence mod K and replacing statements such as "x is divisible by n" or "$n \mid x$" or "$x = nt$" with the statement "$x \in K$". We must also change additive notation to multiplicative notation by using the dictionary above. It's straightforward for parts (1) and (3), but a bit trickier for part (2), since integer addition is commutative, but the multiplicative operation in G may not be.

Proof of Theorem 8.1 ▶ (1) $aa^{-1} = e$ and $e \in K$. Hence, $a \equiv a \pmod{K}$.

(2) $a \equiv b \pmod{K}$ means $ab^{-1} = k$ for some $k \in K$. Therefore, by Corollary 7.6,

$$k^{-1} = (ab^{-1})^{-1} = (b^{-1})^{-1}a^{-1} = ba^{-1}.$$

Since K is a group, the inverse of an element of K is also in K. Reading the preceding line from right to left, we see that $ba^{-1} = k^{-1} \in K$. Hence, $b \equiv a \pmod{K}$.

(3) If $a \equiv b \pmod{K}$ and $b \equiv c \pmod{K}$, then by the definition of congruence, there are $r, s \in K$ such that $ab^{-1} = r$ and $bc^{-1} = s$. Therefore,

$$\begin{aligned}(ab^{-1})(bc^{-1}) &= rs\\ ac^{-1} &= rs\end{aligned}$$

Thus, $ac^{-1} \in K$ (because r and s are in K). Hence, $a \equiv c \pmod{K}$. ∎

If K is a subgroup of a group G and if $a \in G$, then the **congruence class** of a modulo K is the set of all elements of G that are congruent to a modulo K, that is, the set

$$\begin{aligned}\{b \in G \mid b \equiv a \pmod{K}\} &= \{b \in G \mid ba^{-1} \in K\}\\ &= \{b \in G \mid ba^{-1} = k, \text{ with } k \in K\}.\end{aligned}$$

Right multiplication by a shows that the statement $ba^{-1} = k$ is equivalent to $b = ka$. Therefore, the congruence class of a modulo K is the set

$$\{b \in G \mid b = ka, \text{ with } k \in K\} = \{ka \mid k \in K\},$$

which is denoted Ka and called a **right coset** of K in G. In summary:

The congruence class of a modulo K is the right coset $Ka = \{ka \mid \text{with } k \in K\}$.

When the operation in the group G is addition, then a right coset is denoted $K + a$.*

*For those who have read Section 6.1: Cosets of an ideal I in a ring were denoted $a + I$ instead of $I + a$. It didn't make any difference there because addition in a ring is commutative, so $a + i = i + a$ for every $i \in I$. However, in Section 8.2 we shall see that when G is nonabelian, it is possible to have $Ka \neq aK$, where $aK = \{ak \mid \text{with } k \in K\}$.

Theorem 8.2

Let K be a subgroup of a group G and let $a, c \in G$. Then $a \equiv c \pmod{K}$ if and only if $Ka = Kc$.

Proof ▸ With minor notational changes, the proof is essentially the same as that of Theorem 2.3. Just replace "mod n" with "mod K" and "$[a]$" with "Ka" and use Theorem 8.1 in place of Theorem 2.1. ■

Corollary 8.3

Let K be a subgroup of a group G. Then two right cosets of K are either disjoint or identical.

Proof ▸ Copy the proof of Corollary 2.4 with the same notational changes as in the proof of Theorem 8.2. ■

Lagrange's Theorem

At this point we temporarily leave the parallel treatment of congruence in the integers and groups and use right cosets to develop some facts about finite groups that have no counterpart in the integers.

Theorem 8.4

Let K be a subgroup of a group G. Then

(1) G is the union of the right cosets of K: $G = \bigcup_{a \in G} Ka$.

(2) For each $a \in G$, there is a bijection $f: K \to Ka$. Consequently, if K is finite, any two right cosets of K contain the same number of elements.

Proof ▸ (1) Since every right coset consists of elements of G, we have $\bigcup_{a \in G} Ka \subseteq G$. If $b \in G$, then $b = eb \in Kb \subseteq \bigcup_{a \in G} Ka$, so that $G \subseteq \bigcup_{a \in G} Ka$. Hence, $G = \bigcup_{a \in G} Ka$.

(2) Define $f: K \to Ka$ by $f(x) = xa$. Then by the definition of Ka, f is surjective. If $f(x) = f(y)$, then $xa = ya$, so that $x = y$ by Theorem 7.5. Therefore, f is injective and, hence, a bijection. Consequently, if K is finite, every coset Ka has the same number of elements as K, namely $|K|$. ■

If H is a subgroup of a group G, then the number of distinct right cosets of H in G is called the **index of H in G** and is denoted $[G:H]$. If G is a finite group, then there can be only a finite number of distinct right cosets of H; hence, the index $[G:H]$ is finite. If G is an infinite group, then the index may be either finite or infinite.

EXAMPLE 3

Let H be the cyclic subgroup $\langle 3 \rangle$ of the additive group $\mathbb{Z}$. Then H consists of all multiples of 3, and the cosets of H are just the congruence classes modulo 3; for instance,

$$H + 2 = \{h + 2 \mid h \in H\} = \{3z + 2 \mid z \in \mathbb{Z}\} = [2].$$

Since there are exactly three distinct congruence classes modulo 3 (cosets of H), we have $[\mathbb{Z}:H] = 3$.

EXAMPLE 4

Under addition the group $\mathbb{Z}$ of integers is a subgroup of the group $\mathbb{Q}$ of rational numbers. By the definition of congruence and Theorem 8.2,

$$\mathbb{Z} + a = \mathbb{Z} + c \qquad \text{if and only if} \qquad a - c \in \mathbb{Z}.$$

Consequently, if $0 < c < a < 1$, then $\mathbb{Z} + a$ and $\mathbb{Z} + c$ are distinct cosets because $0 < a - c < 1$, which means that $a - c$ cannot be in $\mathbb{Z}$. Since there are infinitely many rationals between 0 and 1, there are an infinite number of distinct cosets of $\mathbb{Z}$ in $\mathbb{Q}$. Hence, $[\mathbb{Q}:\mathbb{Z}]$ is infinite.

Theorem 8.5 Lagrange's Theorem

If K is a subgroup of a finite group G, then the order of K divides the order of G. In particular, $|G| = |K|\,[G:K]$.

Proof ▸ It is convenient to adopt the following notation. If A is a finite set, then $|A|$ denotes the number of elements in A. Observe that if A and B are disjoint finite sets, then $|A \cup B| = |A| + |B|$. Now suppose that $[G:K] = n$ and denote the n distinct cosets of K in G by $Kc_1, Kc_2, \ldots, Kc_n$. By Theorem 8.4

$$G = Kc_1 \cup Kc_2 \cup \cdots \cup Kc_n.$$

Since these cosets are all distinct, they are mutually disjoint by Corollary 8.3. Consequently,

$$|G| = |Kc_1| + |Kc_2| + \cdots + |Kc_n|.$$

For each c_i, however, $|Kc_i| = |K|$ by Theorem 8.4. Therefore,

$$|G| = \underbrace{|K| + |K| + \cdots + |K|}_{n \text{ summands}} = |K|n = |K|[G:K]. \quad \blacksquare$$

Lagrange's Theorem shows that there are a limited number of possibilities for the subgroups of a finite group. For instance, a subgroup of a group of order 12 must have one of these orders: 1, 2, 3, 4, 6, or 12 (the only divisors of 12). Be careful, however, for these are only the *possible* orders of subgroups. Lagrange's Theorem does *not* say that a group G *must* have a subgroup of order k for every k that divides $|G|$. For instance, the alternating group A_4 has order 12 but has no subgroup of order 6 (Exercise 44). Lagrange's Theorem also puts limitations on the possible orders of elements in a group:

Corollary 8.6

Let G be a finite group.

(1) If $a \in G$, then the order of a divides the order of G.

(2) If $|G| = k$, then $a^k = e$ for every $a \in G$.

Proof ▸ (1) If $a \in G$ has order n, then the cyclic subgroup $\langle a \rangle$ of G has order n by Theorem 7.15. Consequently, n divides $|G|$ by Lagrange's Theorem.

(2) If $a \in G$ has order n, then $n \mid k$ by part (1), say $k = nt$. Therefore, $a^k = a^{nt} = (a^n)^t = e^t = e$. ■

The Structure of Finite Groups

A major goal of group theory is the classification of all finite groups up to isomorphism; that is, we would like to produce a list of groups such that every finite group is isomorphic to exactly one group on the list. This is a problem of immense difficulty, but a number of partial results have already been obtained. Theorem 7.19, for example, provides a classification of all cyclic groups; it says, in effect, that every nontrivial finite cyclic group is isomorphic to exactly one group on this list: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \ldots$. All finite abelian groups will be classified in Section 9.2.

We now use Lagrange's Theorem and its corollary to classify all groups of prime order and all groups of order less than 8. In the proofs below enough of the necessary calculations are included to show you how the argument goes, but you should take pencil and paper and supply all the missing computations.

Theorem 8.7

Let p be a positive prime integer. Every group of order p is cyclic and isomorphic to $\mathbb{Z}_p$.

Proof ▸ If G is a group of order p and a is any nonidentity element of G, then the cyclic subgroup $\langle a \rangle$ is a group of order greater than 1. Since the order of the group $\langle a \rangle$ must divide p and since p is prime, $\langle a \rangle$ must be a group of order p. Thus $\langle a \rangle$ is all of G, and G is a cyclic group of order p. Therefore, $G \cong \mathbb{Z}_p$ by Theorem 7.19. ■

Theorem 8.8

Every group of order 4 is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof ▸ Let G be a group of order 4. Either G contains an element of order 4 or it does not. If it does, then the cyclic subgroup generated by this element has order 4 by Theorem 7.15 and, hence, must be all of G. Therefore, G is a cyclic group of order 4, and $G \cong \mathbb{Z}_4$ by Theorem 7.19.

Now suppose that G does not contain an element of order 4. Let e, a, b, c be the distinct elements of G, with e the identity element. Since every element of G must have order dividing 4 by Corollary 8.6 and since e is the only element of order 1, each of a, b, c must have order 2. Thus the operation table of G must look like this:

	e	a	b	c
e	e	a	b	c
a	a	e		
b	b		e	
c	c			e

In order to fill in the missing entries, we first consider the product ab. If $ab = e$, then $ab = aa$ and, hence, $a = b$ by cancelation. This is a contradiction, and so $ab \neq e$. If $ab = a$, then $ab = ae$ and $b = e$ by cancelation, another contradiction. Similarly, $ab = b$ implies the contradiction $a = e$. Therefore, the only possibility is $ab = c$. Similar arguments show that there is only one possible operation table for G, namely,

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $f\colon G \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be given by $f(e) = (0, 0)$, $f(a) = (1, 0)$, $f(b) = (0, 1)$, and $f(c) = (1, 1)$. Show that f is an isomorphism by comparing the operation tables of the two groups. ∎

Theorem 8.9

Every group G of order 6 is isomorphic to either $\mathbb{Z}_6$ or S_3.

Proof ▸ If G contains an element of order 6, then G is a cyclic group of order 6 and, hence, is isomorphic to $\mathbb{Z}_6$ by Theorem 7.19. So suppose G contains no element of order 6. Then every nonidentity element of G has order 2 or 3 by Corollary 8.6. If every nonidentity element of G has order 2, then G is an abelian group by Exercise 27 of Section 7.2. If c and d are nonidentity elements of G, then the set $H = \{e, c, d, cd\}$ is closed under multiplication (because $c^2 = e = d^2$ and $cd = dc$). Hence, H is a subgroup of G by Theorem 7.12. This is a contradiction since no group of order 6 can have a subgroup of order 4 by Lagrange's Theorem. Therefore, the nonidentity elements of G cannot all have order 2, and G must contain an element a of order 3. Let N be the cyclic subgroup $\langle a \rangle = \{e, a, a^2\}$ and let b be any element of G that is not in N. The cosets $Ne = \{e, a, a^2\}$ and $Nb = \{b, ab, a^2b\}$ are not identical since $b \notin N = Ne$ and, hence, must be disjoint (Corollary 8.3). Therefore, G consists of the six elements e, a, a^2, b, ab, a^2b.

We now show that there is only one possible operation table for G. What are the possibilities for b^2? We claim that b^2 cannot be any of a, a^2, b, ab, or a^2b. For instance, if $b^2 = a$, then $b^4 = a^2$. However, b either has order 2 (in which case $a^2 = b^4 = b^2b^2 = ee = e$, a contradiction) or order 3 (in which case $a^2 = b^4 = b^3b = eb = b$, another contradiction since $b \notin N$). Similar arguments show that the only possibility is $b^2 = e$.

Next we determine the product ba. It is easy to see that ba cannot be any of b, e, a, or a^2 (for instance, $ba = a$ implies $b = e$). So the only possibilities are $ba = ab$ or $ba = a^2b$. If $ba = ab$, then verify that ba has order 6 by computing its powers. This contradicts our assumption that G has no element of order 6. Therefore, we must have $ba = a^2b$. Using these two facts:

$$b^2 = e \qquad \text{and} \qquad ba = a^2b,$$

we can now compute every product in G. For example, $ba^2 = (ba)a = (a^2b)a = a^2(ba) = a^2a^2b = a^4b = ab$.

Verify that the operation table for G must look like this:

	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

By comparing tables, show that G is isomorphic to S_3 under the correspondence

$$\begin{array}{cccccc} e & a & a^2 & b & ab & a^2b \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}. \end{array}$$ ■

The last three theorems provide a complete classification of all groups of order less than 8, as summarized in this table:

If G has order	then G is isomorphic to
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_3$
4	$\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$
5	$\mathbb{Z}_5$
6	$\mathbb{Z}_6$ or S_3
7	$\mathbb{Z}_7$

The classification of groups is discussed further in Chapter 9, particularly in Section 9.5 where the preceding chart is extended to order 15.

Exercises

A. 1. Let K be a subgroup of a group G and let $a \in G$. Prove that $Ka = K$ if and only if $a \in k$.

In Exercises 2–6, G is a group and K is a subgroup of G. List the distinct right cosets of K in G.

2. $K = \{r_0, v\}$: $G = D_4$ [The operation table for D_4 is in Example 5 of Section 7.1 or 7.1.A.]

3. $K = \{r_0, r_1, r_2, r_3\}$; $G = D_4$.

4. $K = \left\{ \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}, \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \right\}$; $G = S_3$.

5. $K = \{1, 17\}$; $G = U_{32}$.

6. $K = \langle 3 \rangle$; $G = U_{32}$.

In Exercises 7–11, G is a group and H is a subgroup of G. Find the index $[G{:}H]$.

7. $H = \{r_0, r_2\}$; $G = D_4$.

8. $H = \langle 3 \rangle$; $G = \mathbb{Z}_{12}$.

9. $H = \langle 3 \rangle$; $G = \mathbb{Z}_{20}$.

10. H is the subgroup generated by 12 and 20; $G = \mathbb{Z}_{40}$.

11. H is the cyclic subgroup generated by $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$; $G = S_4$.

12.* **(a)** Let $K = \{(1), (12)(34), (13)(24), (14)(23)\}$. Show that K is a subgroup of A_4, and hence, a subgroup of S_4. [*Hint:* Theorem 7.12.]

(b) State the number of cosets of K in A_4. Don't list them.

(c) State the number of cosets of K in S_4. Don't list them.

*Skip this exercise if you haven't read Section 7.5.

In Exercises 13–15, K is a subgroup of G. Determine whether the given cosets are disjoint or identical.

13. $G = \mathbb{Z}$; $K = \langle 7 \rangle$

(a) $K + 4$ and $K + 3$ **(b)** $K = 4$ and $K + 137$ **(c)** $K + (-4)$ and $K + 59$

14.* $G = S_4$; K is the subgroup of Exercise 12.

(a) $K(12)$ and $K(34)$ **(b)** $K(1234)$ and $K(1324)$

15. $G = U_{32}$; $K = \langle 9 \rangle$

(a) $K17$ and $K19$ **(b)** $K9$ and $K25$

16. Suppose G is the cyclic group $\langle a \rangle$ and $|a| = 15$. If $K = \langle a^3 \rangle$, list all the distinct cosets of K in G.

17. What are the possible orders of the subgroups of G when G is

(a) $\mathbb{Z}_{24}$ **(b)** S_4 **(c)** $D_4 \times \mathbb{Z}_{10}$

18. Give examples, other than those in the text, of infinite groups G and H such that

(a) $[G{:}H]$ is finite **(b)** $[G{:}H]$ is infinite

19. Let G be a finite group that has elements of every order from 1 through 12. What is the smallest possible value of $|G|$?

20. A group G has fewer than 100 elements and subgroups of orders 10 and 25. What is the order of G?

21. Let H and K, each of prime order p, be subgroups of a group G. If $H \neq K$, prove that $H \cap K = \langle e \rangle$.

22. If H and K are subgroups of a finite group G, prove that $|H \cap K|$ is a common divisor of $|H|$ and $|K|$.

B. 23. If G is a group with more than one element and G has no proper subgroups, prove that G is isomorphic to $\mathbb{Z}_p$ for some prime p.

24. If G is a group of order 25, prove that either G is cyclic or else every nonidentity element of G has order 5.

25. Let a be an element of order 30 in a group G. What is the index of $\langle a^4 \rangle$ in the group $\langle a \rangle$?

26. Prove that a group of order 8 must contain an element of order 2.

27. If $n > 2$, prove that $n - 1$ is an element of order 2 in U_n.

28. If $n > 2$, prove that the order of the group U_n is even.

29. Let H and K be subgroups of a finite group G such that $K \subseteq H$, $[G{:}H]$ is finite, and $[H{:}K]$ is finite. Prove that $[G{:}K] = [G{:}H][H{:}K]$. [*Hint:* Lagrange.]

30. Let H and K be subgroups of an infinite group G such that $K \subseteq H$, $[G{:}H]$ is finite, and $[H{:}K]$ is finite. Prove that $[G{:}K]$ is finite and $[G{:}K] = [G{:}H][H{:}K]$. [*Hint:* Let $Ha_1, Ha_2, \ldots, Ha_n$ be the distinct cosets of H in G and let $Kb_1, Kb_2, \ldots, Kb_m$ be the distinct cosets of K in H. Show that Kb_ia_j (with $1 \leq i \leq m$ and $1 \leq j \leq n$) are the distinct cosets of K in G.]

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*Skip this exercise if you haven't read Section 7.5.

31. If G is a group of even order, prove that G contains an element of order 2.

32. If G is an abelian group of order $2n$, with n odd, prove that G contains exactly one element of order 2.

33. **(a)** If a and b each have order 3 in a group and $a^2 = b^2$, prove that $a = b$. [*Hint:* What are a^{-1} and b^{-1}?]

(b) If G is a finite group, prove that there is an even number of elements of order 3 in G.

34. Let G be an abelian group of odd order. If $a_1, a_2, a_3, \ldots, a_n$ are the distinct elements of G (one of which is the identity e), prove that $a_1 a_2 a_3 \cdots a_n = e$.

35. If p and q are primes, show that every proper subgroup of a group of order pq is cyclic.

36. Let H and K be subgroups of a finite group G such that $[G{:}H] = p$ and $[G{:}K] = q$, with p and q distinct primes. Prove that pq divides $[G{:}H \cap K]$.

37. Let G be an abelian group of order n and let k be a positive integer. If $(k, n) = 1$, prove that the function $f{:}G \to G$ given by $f(a) = a^k$ is an isomorphism.

38. If G is a group of order n and G has 2^{n-1} subgroups, prove that $G = \langle e \rangle$ or $G \cong \mathbb{Z}_2$.

C. 39. Let G be a nonabelian group of order 10.

(a) Prove that G contains an element of order 5. [*Hint:* Exercise 27 of Section 7.2.]

(b) Prove that G contains five elements of order 2. [*Hint:* Use techniques similar to those in the proof of Theorem 8.9.]

40. If a prime p divides the order of a finite group G, prove that the number of elements of order p in G is a multiple of $p - 1$.

41. Prove that a group of order 33 contains an element of order 3.

42. Let G be a group generated by elements a and b such that $|a| = 4$, $|b| = 2$, and $ba = a^3b$. Show that G is a group of order 8 and that G is isomorphic to D_4.

43. Let G be a group generated by elements a and b such that $|a| = 4$, $b^2 = a^2$, and $ba = a^3b$. Show that G is a group of order 8 and that G is isomorphic to the quaternion group of Exercise 16 in Section 7.1.

44.* **(a)** Show that A_4 (which has order 12 by Theorem 7.29) has exactly three elements of order 2.

(b) Prove that the elements of order 2 and the identity element form a subgroup.

(c) Prove that A_4 has no subgroup of order 6. Hence, the converse of Lagrange's Theorem is false. [*Hint:* If N is a subgroup of order 6, use Theorem 8.9 to determine the structure of N and use part (b) to reach a contradiction.]

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*Skip this exercise if you haven't read Section 7.5.

8.2 Normal Subgroups

Suppose G is a group and K is a subgroup. Our goal in this section and the next is to create a new group (if possible), whose elements are the right cosets of K (that is, congruence classes mod K)—much as we created $\mathbb{Z}_n$, whose elements are congruence classes of integers.

Recall that the definition of addition of congruence classes of integers in Chapter 2 depended on part (1) of Theorem 2.2, which states

$$\text{If } a \equiv b \pmod{n} \text{ and } c \equiv d \pmod{n}, \text{ then } a + c \equiv b + d \pmod{n}.^*$$

If K is a subgroup of a multiplicative group G, then the translation of this statement to congruence mod K is

$$(*) \qquad \text{If } a \equiv b \pmod{K} \text{ and } c \equiv d \pmod{K}, \text{ then } ac \equiv bd \pmod{K}.$$

Unfortunately, however, statement $(*)$ is false for some subgroups. (see Exercise 2 for an example). Nevertheless, there is a class of subgroups for which statement $(*)$ is true. We shall identify these "special" subgroups in this section and define multiplication of their right cosets in Section 8.3.†

Recall that if K is a subgroup of G, then the right coset Ka is the set $Ka = \{ka \mid k \in K\}$. Similarly, the **left coset** aK is defined to be the set

$$aK = \{ak \mid k \in K\}.$$

EXAMPLE 1

Let K be the subgroup $\{r_0, v\}$ of D_4, whose operation table is shown below. The right coset Kd is the set $\{r_0 \circ d, v \circ d\} = \{d, r_3\}$ and the left coset dK is the set $\{d \circ r_0, d \circ v) = \{d, r_1\}$. So $Kd \neq dK$.

D_4

$\circ$	r_0	r_1	r_2	r_3	d	h	t	v
r_0	r_0	r_1	r_2	r_3	d	h	t	v
r_1	r_1	r_2	r_3	r_0	h	t	v	d
r_2	r_2	r_3	r_0	r_1	t	v	d	h
r_3	r_3	r_0	r_1	r_2	v	d	h	t
d	d	v	t	h	r_0	r_3	r_2	r_1
h	h	d	v	t	r_1	r_0	r_3	r_2
t	t	h	d	v	r_2	r_1	r_0	r_3
v	v	t	h	d	r_3	r_2	r_1	r_0

*We don't deal with integer multiplication here because the integers form a group under addition, but not under multiplication. Similarly in Chapter 6, when developing the basic facts about congruence and cosets in rings, we dealt only with the additive group of a ring and ignored its multiplication.

†Essentially the same thing was done in Chapter 6 when we needed to prove Theorem 6.5 (the analogue of Theorem 2.2 for rings)—the discussion did not apply to every subring, but only to ideals, each of which is a special kind of subring.

EXAMPLE 2

Let N be the subgroup $\{r_0, r_1, r_2, r_3\}$ of D_4. Then the right coset Nv is the set

$$Nv = \{r_0 \circ v, r_1 \circ v, r_2 \circ v, r_3 \circ v\} = \{v, d, h, t\}$$

and the left coset vN is the same set:

$$vN = \{v \circ r_0, v \circ r_1, v \circ r_2, v \circ r_3\} = \{v, t, h, d\}.$$

So in this case, $Nv = vN$.* Similar calculations (Exercise 3) show that *every* right coset of N is also a left coset, that is,

$$Nr_0 = r_0N, \quad Nr_1 = r_1N, \quad Nr_2 = r_2N, \quad Nr_3 = r_3N,$$
$$Nd = dN, \quad Nh = hN, \quad Nt = tN, \quad Nv = vN.$$

Subgroups with this property have a special name.

Definition A subgroup N of a group G is said to be **normal** if $Na = aN$ for every $a \in G$.

EXAMPLE 3

$N = \{r_0, r_1, r_2, r_3\}$ is a normal subgroup of D_4, but $K = \{r_0, v\}$ is not, as shown in Examples 1 and 2.

EXAMPLE 4

If N is a subgroup of an abelian group G and $a \in G$, then $na = an$ for every $n \in N$, so that the right coset Na is the same as the left coset aN. Hence,

Every subgroup of an abelian group is normal.

EXAMPLE 5

Let M be the subgroup $\{r_0, r_2\}$ of D_4. Then the operation table for D_4 in Example 1 shows that $r_0 \circ a = a \circ r_0$ and $r_2 \circ a = a \circ r_2$ for every $a \in D_4$. So it is certainly true that $Ma = aM$ for every $a \in D_4$. Hence, M is a normal subgroup of D_4.

In Example 5, the subgroup M is the center of D_4 (see Example 10 of Section 7.3). So the center of D_4 is a normal subgroup. The same thing is true in general.

*Remember that the elements of a set may be listed in any order.

EXAMPLE 6

The center $Z(G)$ of a group G is the subgroup

$$Z(G) = \{c \in G \mid cg = gc \text{ for every } g \in G\}$$

(Theorem 7.13). Since $ca = ac$ for every $c \in Z(G)$ and $a \in G$, we see that $Z(G)a = aZ(G)$ for every $a \in G$. Hence, $Z(G)$ is a normal subgroup of G.

Other examples of normal subgroups appear in Exercises 3–5, 7–9, 14, and 23. Examples 4–6, though important, are misleading in that the elements of the normal subgroup N commute with all the other elements of the group in each case. In the general case, however, this is not necessarily true. When N is a normal subgroup of G, then,

The condition $Na = aN$ does *not* imply that $na = an$ for every $n \in N$.

EXAMPLE 7

As we saw in the Example 2, $N = \{r_0, r_1 r_2, r_3\}$ is a normal subgroup of D_4. In particular, $Nv = vN$. However, v does not commute with *all* the elements of N. For instance, $r_3 \circ v \in Nv$ and $v \circ r_3 \in vN$, but the operation table for D_4 shows that

$$r_3 \circ v = t \quad \text{and} \quad v \circ r_3 = d, \qquad \text{so } r_3 \circ v \neq v \circ r_3,$$

even though $Nv = vN$.

Thus, if N is a normal subgroup of G, the elements of N may not commute with every element of G. Nevertheless, you can think of the normal subgroup N as providing a weak version of commutativity in the following sense.

If $n \in N$, and $a \in G$, then for some $n_1, n_2 \in N$,

$$na = an_1 \quad \text{and} \quad an = n_2 a,$$

because $na \in Na$ and $Na = aN$ and similarly, $an \in aN$ and $aN = Na$.

EXAMPLE 8

Once again, consider the normal subgroup $N = \{r_0, r_1, r_2, r_3)$ of D_4. The operation table for D_4 shows that $r_3 \circ v = t$ and $v \circ r_1 = t$. Hence,

$$r_3 \circ v = v \circ r_1.$$

This is the first part of the preceding boldface statement, with $n = r_3$, $a = v$, and $n_1 = r_1$.

Our goal at the beginning of this section was to find a class of subgroups for which statement (*) on page 248 (the group theory analogue of Theorem 2.2) is true. Normal subgroups are exactly what's needed.

Theorem 8.10

Let N be a normal subgroup of a group G.

If $a \equiv b \pmod{N}$ and $c \equiv d \pmod{N}$, then $ac \equiv bd \pmod{N}$.

The proof is essentially a translation into multiplicative notation of the proof of part (1) of Theorem 2.2, with commutativity of integers replaced by the weak commutativity in G provided by the normal subgroup N.

Proof of Theorem 8.10 ▸ By the definition of congruence, there are elements $m, n \in K$ such that $ab^{-1} = m$ and $cd^{-1} = n$. Then

$$\begin{aligned} (ac)(bd)^{-1} &= acd^{-1}b^{-1} && \textit{[Corollary 7.6]} \\ &= anb^{-1} && \textit{[Because } cd^{-1} = n.\textit{]} \end{aligned}$$

Now $an \in aN$ and $aN = Na$ by normality, so $an = n_2a$ for some $n_2 \in N$. Hence,

$$\begin{aligned} (ac)(bd)^{-1} &= anb^{-1} \\ &= n_2ab^{-1} \\ &= n_2m && \textit{[Because } ab^{-1} = m \in N.\textit{]} \end{aligned}$$

Therefore, $(ac)(bd)^{-1} = n_2m \in N$, and $ac \equiv bd \pmod{N}$. ■

We close this section with a theorem that provides alternate descriptions of normality. Verifying condition (2) or (3) in the theorem is often the easiest way to prove that a given subgroup is normal.

Theorem 8.11

The following conditions on a subgroup N of a group G are equivalent:

(1) N is a normal subgroup of G.
(2) $a^{-1}Na \subseteq N$ for every $a \in G$, where $a^{-1}Na = \{a^{-1}na \mid n \in N\}$.
(3) $aNa^{-1} \subseteq N$ for every $a \in G$, where $aNa^{-1} = \{ana^{-1} \mid n \in N\}$.
(4) $a^{-1}Na = N$ for every $a \in G$.
(5) $aNa^{-1} = N$ for every $a \in G$.

Note that in (4), $a^{-1}Na = N$ does *not* mean that $a^{-1}na = n$ for each $n \in N$; all it means is that $a^{-1}na = n_1$ for some $n_1 \in N$. Analogous remarks apply to (2), (3), and (5).

Proof of Theorem 8.11 ▸ (1) $\Rightarrow$ (2) Suppose $n \in N$ and $a^{-1}na \in a^{-1}Na$. We must show that $a^{-1}na \in N$. Note that na is an element of the right coset Na. Since N is normal by (1), $Na = aN$. Hence, $na = an_1$ for some $n_1 \in N$. Thus $a^{-1}na = a^{-1}an_1 = en_1 = n_1 \in N$. Therefore, $a^{-1}Na \subseteq N$.

(2) $\Leftrightarrow$ (3) If (2) holds for *every* element of G, then it holds with a^{-1} in place of a, that is,

$$(**) \qquad (a^{-1})^{-1}Na^{-1} \subseteq N.$$

But $(a^{-1})^{-1} = a$, so that (**) is statement (3): $aNa^{-1} \subseteq N$. Similarly, if (3) holds for every element of G, then it holds with a^{-1} in place of a, which implies statement (2).

(3) $\Rightarrow$ (4) Since (3) implies (2), we have $a^{-1}Na \subseteq N$. To prove $N \subseteq a^{-1}Na$, suppose $n \in N$. Then $n = a^{-1}(ana^{-1})a$. By (3) $ana^{-1} = n_2$ for some $n_2 \in N$. Thus $n = a^{-1}n_2a \in a^{-1}Na$, which proves that $N \subseteq a^{-1}Na$. Therefore, $a^{-1}Na = N$.

(4) $\Leftrightarrow$ (5) If (4) holds for *every* element of G, then it holds with a^{-1} in place of a, that is,

$$N = (a^{-1})^{-1}Na^{-1} = aNa^{-1}.$$

Similarly, if (5) holds for every element of G, then it holds with a^{-1} in place of a, which implies statement (4).

(5) $\Rightarrow$ (1) Suppose $n \in N$ and $an \in aN$. Then $ana^{-1} \in aNa^{-1} = N$ by (5), so that $ana^{-1} = n_3$ for some $n_3 \in N$. Multiplying this last equation on the right by a shows that $an = n_3a \in Na$. Therefore, $aN \subseteq Na$. Conversely, if $na \in Na$, then $a^{-1}na \in a^{-1}Na = N$ because (5) implies (4). Hence, $a^{-1}na = n_4$ for some $n_4 \in N$. Multiplying on the left by a shows that $na = an_4 \in aN$. Thus $Na \subseteq aN$. Therefore, $Na = aN$ for every $a \in G$ and N is a normal subgroup of G. ■

EXAMPLE 9

Verify that $A = \left\{\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}\right\}$ is a subgroup of S_3. You could show that A is a normal subgroup by calculating the right and left cosets, but that is cumbersome and time consuming. It's easier to proceed as follows. If $c \in S_3$, then by Exercise 20 of Section 7.4, $c^{-1}Ac$ is a subgroup of order 3. But A is the only subgroup of order 3 in S_3 (all the other nonidentity elements of S_3 have order 2, and hence, cannot be in a group of order 3 by Corollary 8.6). Therefore, we must have $c^{-1}Ac = A$. Thus, A is a normal subgroup by part (5) of Theorem 8.11.

Exercises

A. 1. Let K be a subgroup of a group G and let $a \in G$. Prove that $aK = K$ if and only if $a \in K$.

2. Let K be the subgroup $\{r_0, v\}$ of D_4. Show that $r_1 \equiv t \pmod K$ and $r_2 \equiv h \pmod K$, but $r_1 \circ r_2 \not\equiv t \circ h \pmod K$.

3. Prove that $N = \{r_0, r_1, r_2, r_3\}$ is a normal subgroup of D_4 by listing all its right and left cosets.

4. If G is a group, show that $\langle e \rangle$ and G are normal subgroups.

5. (a) Prove that $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \, a, b, d \in \mathbb{R} \text{ and } ad \neq 0 \right\}$ is a group under matrix multiplication and that $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{R} \right\}$ is a subgroup of G.

   (b) Use Theorem 8.11 to show that N is normal in G.

6. Prove that $\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}$ is a subgroup of S_3 but not normal.

7. Let G and H be groups. Prove that $G^* = \{(a, e) \mid a \in G\}$ is a normal subgroup of $G \times H$.

8. (a) List all the cyclic subgroups of the quaternion group (Exercise 16 of Section 7.1).

   (b) Show that each of the subgroups in part (a) is normal.

9. Let N be a subgroup of a group G. Suppose that, for each $a \in G$, there exists $b \in G$ such that $Na = bN$. Prove that N is a normal subgroup.

10. If G is a group, prove that every subgroup of $Z(G)$ is normal in G. [Compare with Exercise 14.]

11. A subgroup N of a group G is said to be **characteristic** if $f(N) \subseteq N$ for *every* automorphism f of G. Prove that every characteristic subgroup is normal. (The converse is false, but this is harder to prove.)

12. Prove that for any group G, the center $Z(G)$ is a characteristic subgroup.

13. Let N be a subgroup of a group G. Prove that N is normal if and only if $f(N) = N$ for every inner automorphism f of G.

14. Show by example that if M is a normal subgroup of N and if N is a normal subgroup of a group G, then M need *not* be a normal subgroup of G; in other words, normality isn't transitive. [*Hint:* Consider $M = \{v, r_0\}$ and $N = \{h, v, r_2, r_0\}$ in D_4.]

15.* Prove that A_n is a normal subgroup of S_n. [*Hint:* If $\sigma \in S_n$ and $\tau \in A_n$, is $\sigma^{-1}\tau\sigma$ even or odd? See Example 7 of Section 7.5.]

B. 16. If K is a normal subgroup of order 2 in a group G, prove that $K \subseteq Z(K)$. [*Hint:* If $K = \{e, k\}$ and $a \in G$, what are the possibilities for aka^{-1}?]

17. Let $f: G \to H$ be a homomorphism of groups and let $K = \{a \in G \mid f(a) = e_H\}$. Prove that K is a normal subgroup of G.

18. If K and N are normal subgroups of a group G, prove that $K \cap N$ is a normal subgroup of G.

19. Let N and K be subgroups of a group G. If N is normal in G, prove that $N \cap K$ is a normal subgroup of K.

20. (a) Let N and K be subgroups of a group G. If N is normal in G, prove that $NK = \{nk \mid n \in N, k \in K\}$ is a subgroup of G. [Compare Exercise 26(b) of Section 7.3.]

   (b) If both N and K are normal subgroups of G, prove that NK is normal.

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*Skip this exercise if you haven't read Section 7.5.

21. If K and N are normal subgroups of a group G such that $K \cap N = \langle e \rangle$, prove that $nk = kn$ for every $n \in N$, $k \in K$.

22. If $f:G \to H$ is a surjective homomorphism of groups and if N is a normal subgroup of G, prove that $f(N)$ is a normal subgroup of H.

23. Let N be a subgroup of a group G of index 2. Prove that N is a normal subgroup as follows.

(a) If $a \notin N$, prove that the coset Na consists of all elements of G that are not in N.

(b) For each $a \in G$, prove that $a^{-1}Na \subseteq N$ and apply Theorem 8.11. [*Hint:* If $a \notin N$ and $n \in N$, $a^{-1}na$ is either in N or in Na by part (a). Show that the latter possibility leads to a contradiction.]

24. Let $N = \{A \in GL(2, \mathbb{R}) \mid \det A \in \mathbb{Q}\}$. Prove that N is a normal subgroup of $GL(2, \mathbb{R})$. [*Hint:* Exercise 32 of Section 7.4.]

25. Prove that $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$. [*Hint:* $SL(2, \mathbb{R})$ is defined in Exercise 23 of Section 7.1 Use Exercise 17 above and Exercise 32 of Section 7.4.]

26. Let H be a subgroup of order n in a group G. If H is the only subgroup of order n, prove that H is normal. [*Hint:* Theorem 8.11 and Exercise 20 in Section 7.4.]

27. Prove that a subgroup N of a group G is normal if and only if it has this property: $ab \in N$ if and only if $ba \in N$, for all $a, b \in G$.

28. Prove that the cyclic subgroup $\langle a \rangle$ of a group G is normal if and only if for each $g \in G$, $ga = a^k g$ for some $k \in \mathbb{Z}$.

29. Let N be a cyclic normal subgroup of a group G, and H any subgroup of N. Prove that H is a normal subgroup of G. [Compare Exercise 14.]

30. Let A and B be normal subgroups of a group G such that $A \cap B = \langle e \rangle$ and $AB = G$ (see Exercise 20). Prove that $A \times B \cong G$. [*Hint:* Define $f:A \times B \to G$ by $f(a, b) = ab$ and use Exercise 21.]

31. Let H be a subgroup of a group G and let $N(H)$ be its normalizer (see Exercise 39 in Section 7.3). Prove that

(a) H is a normal subgroup of $N(H)$.

(b) If H is a normal subgroup of a subgroup K of G, then $K \subseteq N(H)$.

32. Prove that Inn G is a normal subgroup of Aut G. [See Exercise 37 of Section 7.4.]

33. Let T be a set with three or more elements and let $A(T)$ be the group of all permutations of T. If $a \in T$, let $H_a = \{f \in A(T) \mid f(a) = a\}$. Prove that H_a is a subgroup of $A(T)$ that is not normal.

34. Let G be a group that contains at least one subgroup of order n. Let $N = \cap K$, where the intersection is taken over all subgroups K of order n. Prove that N is a normal subgroup of G. [*Hint:* For each $a \in G$, verify that $a^{-1}Na = \cap a^{-1}Ka$, where the intersection is over all subgroups K of order n; use Exercise 20 of Section 7.4.]

35. Let H be a subgroup of a group G and let $N = \bigcap_{a \in G} a^{-1}Ha$. Prove that N is a normal subgroup of G.

36. If M is a characteristic subgroup of N and N is a normal subgroup of a group G, prove that M is a normal subgroup of G. [See Exercise 11.]

37. Let G be a group all of whose subgroups are normal. If $a, b \in G$, prove that there is an integer k such that $ab = ba^k$.

8.3 Quotient Groups

Let N be a normal subgroup of a group G. Then

G/N denotes the set of all right cosets of N in G.

Our first goal is to define an operation on right cosets so that G/N becomes a group. Since right cosets are congruence classes, our experience with $\mathbb{Z}$ and other rings suggests that it would be reasonable to define such an operation as follows: The product of the coset Na (the congruence class of a) and the coset Nb (the congruence class of b) is the coset Nab (the congruence class of ab). In symbols, this definition reads

$$(Na)(Nb) = Nab.$$

As in the past, we must verify that the definition does not depend on the elements chosen to represent the various cosets, and so we must prove

Theorem 8.12

Let N be a normal subgroup of a group G. If $Na = Nc$ and $Nb = Nd$ in G/N, then $Nab = Ncd$.

Proof ▸ $Na = Nc$ implies that $a \equiv c \pmod{N}$ by Theorem 8.2, similarly, $Nb = Nd$ implies that $b \equiv d$ (mod N). Therefore, $ab \equiv cd \pmod{N}$ by Theorem 8.10. Hence, $Nab = Ncd$ by Theorem 8.2. ■

Theorem 8.13

Let N be a normal subgroup of a group G. Then

(1) G/N is a group under the operation defined by $(Na)(Nc) = Nac$.

(2) If G is finite, then the order of G/N is $|G|/|N|$.

(3) If G is an abelian group, then so is G/N.

The group G/N is called the **quotient group** or **factor group** of G by N.

Proof of Theorem 8.13 ▸ (1) The operation in G/N is well defined by Theorem 8.12. The coset $N = Ne$ is the identity element in G/N since $(Na)(Ne) = Nae = Na$ and $(Ne)(Na) = Nea = Na$ for every Na in G/N. The inverse of Na is the coset Na^{-1} since $(Na)(Na^{-1}) = Naa^{-1} = Ne$ and, similarly, $(Na^{-1})(Na) = Ne$. Associativity in G/N follows from that in G:

$$\begin{aligned}[(Na)(Nb)](Nc) &= (Nab)(Nc) = N(ab)c = Na(bc) = (Na)(Nbc)\\ &= (Na)[(Nb)(Nc)].\end{aligned}$$

Therefore, G/N is a group.

(2) The order of G/N is the number of distinct right cosets of N, that is, the index $[G:N]$. By Lagrange's Theorem, $[G:N] = |G|/|N|$.

(3) Exercise 11. ■

EXAMPLE 1

In Example 2 of Section 8.2 we saw that $N = \{r_0, r_1, r_2, r_3\}$ is a normal subgroup of D_4. The operation table for D_4 in Example 1 of Section 8.2 shows that

$$Nr_0 = \{r_0 \circ r_0, r_1 \circ r_0, r_2 \circ r_0, r_3 \circ r_0\} = \{r_0, r_1, r_2, r_3\}$$

$$Nv = \{r_0 \circ v, r_1 \circ v, r_2 \circ v, r_3 \circ v\} = \{v, d, h, t\}.$$

Since every element of D_4 is in either Nr_0 or Nv and since any two cosets of N are either disjoint or identical (Corollary 8.3), every coset of N must be equal to Nr_0 or Nv. In other words, $D_4/N = \{Nr_0, Nv\}$. Since $r_0 \circ v = v = v \circ r_0$ and $v \circ v = r_0$, the operation table for the quotient group D_4/N is

	Nr_0	Nv
Nr_0	Nr_0	Nv
Nv	Nv	Nr_0

By Theorem 8.7, D_4/N is isomorphic to the additive group $\mathbb{Z}_2$.

EXAMPLE 2

In Example 5 of Section 8.2 we saw that $M = \{r_0, r_2\}$ is a normal subgroup of D_4. Using the operation table for D_4, we find that D_4/M consists of these four cosets:

$$Mr_0 = \{r_0, r_2\} = Mr_2 \qquad Mr_1 = \{r_1, r_3\} = Mr_3$$

$$Mh = \{h, v\} = Mv \qquad Md = \{d, t\} = Mt.$$

We shall choose one way of representing each coset and list the elements of D_4/M as Mr_0, Mr_1, Mh, and Md. When we compute products in D_4/M, we express the answers in terms of these four cosets. For instance, since $d \circ r_1 = v$ in D_4, we have

$(Md)(Mr_1) = M(d \circ r_1) = Mv$; but $Mv = Mh$, so we write $(Md)(Mr_1) = Mh$ in the table below. You should fill in the missing entries:

	Mr_0	Mr_1	Mh	Md
Mr_0	Mr_0	Mr_1	Mh	Md
Mr_1	Mr_1	Mr_0	Md	
Mh	Mh	Md	Mr_0	
Md	Md	Mh		

The completed tabel shows that D_4/M is an abelian group in which every nonidentity element has order 2 (Exercise 3). So D_4/M is not cyclic. Hence, D_4/M is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ by Theorem 8.8.

Examples 3–7 deal with abelian groups. So every subgroup is normal.

EXAMPLE 3

In the additive group $\mathbb{Z}_{12}$, let N be the cyclic group $\langle 4 \rangle = \{0, 4, 8\}$. These four cosets of N contain every element of $\mathbb{Z}_{12}$:

$$N + 0 = \{0, 4, 8\} = N$$
$$N + 1 = \{1, 5, 9\}$$
$$N + 2 = \{2, 6, 10\}$$
$$N + 3 = \{3, 7, 11\}.$$

Hence, every coset is one of these four. For instance, 5 is in $N + 1$ and 5 is also in $N + 5$ (Why?). So the two cosets are not disjoint. Hence, $N + 1 = N + 5$ by Corollary 8.3. Similarly,

$$N + 4 = N + 0 \quad \text{and} \quad N + 6 = N + 2.$$

Using these facts, we see that the addition table for $\mathbb{Z}_{12}/N$ is

	$N+0$	$N+1$	$N+2$	$N+3$
$N+0$	$N+0$	$N+1$	$N+2$	$N+3$
$N+1$	$N+1$	$N+2$	$N+3$	$N+0$
$N+2$	$N+2$	$N+3$	$N+0$	$N+1$
$N+3$	$N+3$	$N+0$	$N+1$	$N+2$

Verify that $N + 1$ has order 4. So $\mathbb{Z}_{12}/N$ is a cyclic group of order 4 and hence, is isomorphic to $\mathbb{Z}_4$ by Theorem 7.19.

EXAMPLE 4

Let N be the cyclic subgroup $\langle(1, 2)\rangle$ of the additive group $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. Since $(1, 2) + (1, 2) = (0, 0)$, we see that $N = \{(0, 0), (1, 2)\}$. Consequently, G/N consists of these four cosets

$$N + (0, 0) = \{(0, 0), (1, 2)\} = N + (1, 2)$$
$$N + (1, 0) = \{(1, 0), (0, 2)\} = N + (0, 2)$$
$$N + (0, 1) = \{(0, 1), (1, 3)\} = N + (1, 3)$$
$$N + (1, 1) = \{(1, 1), (0, 3)\} = N + (0, 3)$$

and has the following addition table:

	$N + (0, 0)$	$N + (1, 0)$	$N + (0, 1)$	$N + (1, 1)$
$N + (0, 0)$	$N + (0, 0)$	$N + (1, 0)$	$N + (0, 1)$	$N + (1, 1)$
$N + (1, 0)$	$N + (1, 0)$	$N + (0, 0)$	$N + (1, 1)$	$N + (0, 1)$
$N + (0, 1)$	$N + (0, 1)$	$N + (1, 1)$	$N + (1, 0)$	$N + (0, 0)$
$N + (1, 1)$	$N + (1, 1)$	$N + (0, 1)$	$N + (0, 0)$	$N + (1, 0)$

Use the table to verify that G/N is a cyclic group of order 4 generated by $N + (0, 1)$. Therefore, $G/N \cong \mathbb{Z}_4$ by Theorem 7.19.

It is not always necessary (or even possible) to write out the operation table for a quotient group G/N in order to determine its structure, as was done in Examples 1–4.

EXAMPLE 5

By Theorem 2.10, the group $U_{14} = \{1, 3, 5, 9, 11, 13\}$ and thus has order 6. Let M be the cyclic subgroup $\langle 13\rangle = \{1, 13\}$. Then $|U_{14}/M| = \dfrac{|U_{14}|}{|M|} = \dfrac{6}{2} = 3$ by Theorem 8.13. Therefore, U_{14}/M is isomorphic to $\mathbb{Z}_3$ by Theorem 8.7.

EXAMPLE 6

In the additive group $\mathbb{Z}$, let K be the cyclic subgroup

$$\langle 4\rangle = \{0, \pm 4, \pm 8, \pm 12, \ldots\}.$$

As we saw in Example 1 of Section 8.1, $a \equiv b \pmod 4$ means $a - b \in K$. Hence,

$$a \equiv b \pmod 4 \text{ if and only if } a \equiv b \pmod K.$$

So the set of integers that are congruent to a modulo 4 (the congruence class $[a]$) is exactly the same as the set of integers that are congruent to a modulo K (the coset $K + a$). In other words, $[a] = K + a$. Arithmetic is the same in either notation:

$$Ka + Kb = K(a + b) \quad \text{is the same as} \quad [a] + [b] = [a + b].$$

Therefore, $\mathbb{Z}/K$ is the group of congruence classes modulo 4, that is, $\mathbb{Z}/K = \mathbb{Z}_4$. The same argument works with any positive integer n in place of 4:

If K is the cyclic subgroup $\langle n \rangle$ of $\mathbb{Z}$, then $\mathbb{Z}/K = \mathbb{Z}_n$.

EXAMPLE 7

The subgroup $\mathbb{Z}$ of integers in the additive group $\mathbb{Q}$ of rational numbers is normal since $\mathbb{Q}$ is abelian. Example 4 of Section 8.1 shows that there are infinitely many distinct cosets of $\mathbb{Z}$ in $\mathbb{Q}$. Consequently, the quotient group $\mathbb{Q}/\mathbb{Z}$ is an infinite abelian group. Nevertheless, every element of $\mathbb{Q}/\mathbb{Z}$ has *finite* order (Exercise 25).

The Structure of Groups

If N is a normal subgroup of a group G, then the structure of each of the groups N, G, and G/N is related to the structure of the others. If we know enough information about two of these groups, we can often determine useful information about the third, as illustrated in the following theorems.

Theorem 8.14

Let N be a normal subgroup of a group G. Then G/N is abelian if and only if $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

Proof▸ G/N is abelian if and only if

$$Nab = NaNb = NbNa = Nba \quad \text{for all } a, b \in G.$$

But $Nab = Nba$ if and only if $(ab)(ba)^{-1} \in N$ by Theorem 8.21; and $(ab)(ba)^{-1} = aba^{-1}b^{-1}$ by Corollary 7.6. Therefore, G/N is abelian if and only if $aba^{-1}b^{-1} \in N$ for all $a, b \in G$. ■

If G is a group, Example 6 of Section 8.2 shows that its center $Z(G)$ is a normal subgroup of G.

Theorem 8.15

If G is a group such that the quotient group $G/Z(G)$ is cyclic, then G is abelian.

Proof ▸ For notational convenience, denote $Z(G)$ by C. Since G/C is cyclic, it has a generator Cd, and every coset in G/C is of the form $(Cd)^k = Cd^k$ for some integer k. Let a and b be any elements of G. Since $a = ea$ is in the coset Ca and since $Ca = Cd^i$ for some i, we have $a = c_1d^i$ for some $c_1 \in C$. Similarly, $b = c_2d^j$ for some $c_2 \in C$ and integer j. Now $d^id^j = d^{i+j} = d^{j+i} = d^jd^i$, and c_1 and c_2 commute with every element of G by the definition of the center. Consequently,

$$ab = (c_1d^i)(c_2d^j) = c_1c_2d^id^j = c_2c_1d^jd^i = (c_2d^j)(c_1d^i) = ba.$$

Therefore, G is abelian. ■

Exercises

1. Let N be the subgroup $\langle 4 \rangle$ of $\mathbb{Z}_{20}$. Find the order of $13 + N$ in the group $\mathbb{Z}_{20}/N$.
2. Let G be the subgroup $\langle 3 \rangle$ of $\mathbb{Z}$, and let N be the subgroup $\langle 15 \rangle$. Find the order of $6 + N$ in the group G/N.
3. Complete the table in Example 2 and verify that every nonidentity element of D_4/M has order 2.

A. 4. $N = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$ is a normal subgroup of S_3 by Example 9 of Section 8.2. Show that $S_3/N \cong \mathbb{Z}_2$.

5. Show that $\mathbb{Z}_{18}/M \cong \mathbb{Z}_6$, where M is the cyclic subgroup $\langle 6 \rangle$.
6. Show that $\mathbb{Z}_6/N \cong \mathbb{Z}_3$, where N is the subgroup $\{0, 3\}$.
7. Show that $U_{26}/\langle 5 \rangle$ is isomorphic to $\mathbb{Z}_3$.
8. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and let N be the cyclic subgroup generated by $(3, 2)$. Show that $G/N \cong \mathbb{Z}_4$.
9. Let $G = \mathbb{Z}_6 \times \mathbb{Z}_2$ and let N be the cyclic subgroup $\langle (1, 1) \rangle$. Describe the quotient group G/N.
10. **(a)** Let M be the cyclic subgroup $\langle (0, 2) \rangle$ of the additive group $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ and let N be the cyclic subgroup $\langle (1, 2) \rangle$, as in Example 4. Verify that M is isomorphic N.

    (b) Write out the operation table of G/M, using the four cosets $M + (0, 0)$, $M + (1, 0)$, $M + (0, 1)$, $M + (1, 1)$.

(c) Show that G/M is not isomorphic to G/N (the operation table for G/N is in Example 4). Thus for normal subgroups M and N, the fact that $M \cong N$ does not imply that G/M is isomorphic to G/N.

11. If N is a subgroup of an abelian group G, prove that G/N is abelian.

12. If N is a normal subgroup of a group G and if $x^2 \in N$ for every $x \in G$, prove that every nonidentity element of the quotient group G/N has order 2.

13. **(a)** Give an example of a nonabelian group G such that $G/Z(G)$ is abelian.

(b) Give an example of a group G such that $G/Z(G)$ is not abelian.

14. **(a)** Show that $V = \left\{\begin{pmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 4\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\2 & 1 & 4 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\4 & 3 & 2 & 1\end{pmatrix}\right\}$ is a normal subgroup of S_4.

(b) Write out the operation table for the group S_4/V.

B. *In Exercises 15 and 16, find an element of infinite order and an element of finite order in the given quotient group. There are many correct answers. Remember that $\mathbb{Z}$ is an additive group.*

15. $(\mathbb{Z} \times \mathbb{Z})/\langle(5, 5)\rangle$

16. $(\mathbb{Z} \times \mathbb{Z})/\langle(6, 9)\rangle$

17. Let E be the group of even integers and N the subgroup of all multiples of 8.

(a) Show that E/N has order 4.

(b) To what well-known group is E/N isomorphic? [*Hint:* Theorem 8.8.]

18. Show that $U_{32}/N \cong U_{16}$, where N is the subgroup $\{1, 17\}$.

19. An element b of a group is said to be a *square* if there is an element c in the group such that $b = c^2$. Let N be a subgroup of an abelian group G. If both N and G/N have the property that every element is a square, prove that every element of G is a square.

20. If G is a group and $[G{:}G/Z(G)] = 4$, prove that $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

21. Let G be an abelian group and T its torsion subgroup (see Exercise 19 of Section 7.3). Prove that G/T has no nonidentity elements of finite order.

22. Let $\mathbb{R}^*$ be the multiplicative group of nonzero real numbers and let N be the subgroup $\{1, -1\}$. Prove that $\mathbb{R}^*/N$ is isomorphic to the multiplicative group $\mathbb{R}^{**}$ of positive real numbers.

23. Describe the quotient group $\mathbb{R}^*/\mathbb{R}^{**}$, where $\mathbb{R}^*$ and $\mathbb{R}^{**}$ are as in Exercise 22.

24. If G is a cyclic group, prove that G/N is cyclic, where N is any subgroup of G.

25. **(a)** Find the order of $\frac{8}{9}$, $\frac{14}{5}$, and $\frac{48}{28}$ in the additive group $\mathbb{Q}/\mathbb{Z}$.

(b) Prove that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

(c) Prove that $\mathbb{Q}/\mathbb{Z}$ contains elements of every possible finite order.

26. Prove that the set of elements of finite order in the group $\mathbb{R}/\mathbb{Z}$ is the subgroup $\mathbb{Q}/\mathbb{Z}$.

27. Let G and H be groups and let G^* be the subset of $G \times H$ consisting of all (a, e) with $a \in G$.

(a) Show that G^* is isomorphic to G.

(b) Show that G^* is a normal subgroup of $G \times H$.

(c) Show that $(G \times H)/G^* \cong H$.

28. Let M and N be normal subgroups of a group G such that $M \cap N = \langle e \rangle$. Prove that G is isomorphic to a subgroup of $G/M \times G/N$.

29. If N is a normal subgroup of a group G and if every element of N and of G/N has finite order, prove that every element of G has finite order.

30. If N is a finite normal subgroup of a group G and if G/N contains an element of order n, prove that G contains an element of order n.

31. Let G be a group of order pq, with p and q (not necessarily distinct) primes. Prove that the center $Z(G)$ is either $\langle e \rangle$ or G.

32. A group H is said to be **finitely generated** if there is a finite subset S of H such that $H = \langle S \rangle$ (see Theorem 7.18). If N is a normal subgroup of a group G such that the groups N and G/N are finitely generated, prove that G is finitely generated.

33. Let G be a group and let S be the set of all elements of the form $aba^{-1}b^{-1}$ with $a, b \in G$. The subgroup G' generated by the set S (as in Theorem 7.18) is called the **commutator subgroup** of G. Prove

(a) G' is normal in G. [*Hint:* For any $g, a, b \in G$, show that $g^{-1}(aba^{-1}b^{-1})g = (g^{-1}ag)(g^{-1}bg)(g^{-1}a^{-1}g)(g^{-1}b^{-1}g)$ is in S.]

(b) G/G' is abelian.

34. Let G be the additive group $\mathbb{R} \times \mathbb{R}$.

(a) Show that $N = \{(x, y) \mid y = -x\}$ is a subgroup of G.

(b) Describe the quotient group G/N.

35. Let N be a normal subgroup of a group G and let G' be the commutator subgroup defined in Exercise 33. If $N \cap G' = \langle e \rangle$, prove that

(a) $N \subseteq Z(G)$ **(b)** The center of G/N is $Z(G)/N$.

36. If G is a group, prove that $G/Z(G)$ is isomorphic to the group Inn G of all inner automorphisms of G (see Exercise 37 in Section 7.4).

C. 37. Let A, B, N be normal subgroups of a group G such that $N \subseteq A$, $N \subseteq B$. If $G = AB$ and $A \cap B = N$, prove that $G/N \cong A/N \times B/N$. (The special case $N = \langle e \rangle$ is Exercise 30 in Section 8.2.)

8.4 Quotient Groups and Homomorphisms

There is a close connection between normal subgroups, quotient groups, and homomorphisms.* The following definition is crucial for developing this connection.

Definition Let $f:G \to H$ be a homomorphism of groups. Then the **kernel** of f is the set $\{a \in G \mid f(a) = e_H\}$.

Thus, the kernel is the set of elements in G that are mapped onto the identity element in H by the homomorphism f.

EXAMPLE 1

Let $\mathbb{R}^*$ be the multiplicative group of nonzero real numbers and $\mathbb{R}^{**}$ the multiplicative group of positive real numbers. The function $f:\mathbb{R}^* \to \mathbb{R}^{**}$ given by $f(x) = x^2$ is a homomorphism because $f(ab) = (ab)^2 = a^2b^2 = f(a)f(b)$. Its kernel is the set of real numbers x such that $x^2 = 1$, namely, $\{1, -1\}$.

EXAMPLE 2

Verify that the function $f:\mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}^*$ given by $f(a, b) = b$ is a homomorphism of multiplicative groups. Its kernel is the set of all pairs (a, b) such that $b = 1$, that is, $\{(a, 1) \mid a \in \mathbb{R}^*\}$.

EXAMPLE 3

In Example 13 of Section 7.4, we saw that the function $f:\mathbb{Z} \to \mathbb{Z}_5$ given by $f(a) = [a]$ is a homomorphism of additive groups. Its kernel is the set

$$K = \{a \in \mathbb{Z} \mid f(a) = [0]\} = \{a \in \mathbb{Z} \mid [a] = [0]\}.$$

But $[a] = [0]$ if and only if $a \equiv 0 \pmod 5$ by Theorem 2.3, and $a \equiv 0 \pmod 5$ if and only if $5 \mid a$ by the definition of congruence. Hence, K is the set of all integer multiples of 5, that is, the cyclic group $\langle 5 \rangle$.

You can easily verify that each of the kernels in Examples 1–3 is actually a (normal) subgroup. The same thing is true in the general case.

*If you have read Chapter 6, this should not come as a surprise. The first part of this section simply carries over to groups the facts about ideals, quotient rings, and ring homomorphisms that were developed at the end of Section 6.2. (pages 154–158).

Theorem 8.16

Let $f{:}G \to H$ be a homomorphism of groups with kernel K. Then K is a normal subgroup of G.

Proof ▸ If $c, d \in K$, then $f(c) = e_H$ and $f(d) = e_H$ by the definition of kernel. Hence, $f(cd) = f(c)f(d) = e_He_H = e_H$, so that $cd \in K$. If $c \in K$, then by Theorem 7.20 $f(c^{-1}) = f(c)^{-1} = (e_H)^{-1} = e_H$. Thus $c^{-1} \in K$. Therefore, K is a subgroup of G by Theorem 7.11. To show that K is normal, we must verify that for any $a \in G$ and $c \in K$, $a^{-1}ca \in K$ (Theorem 8.11). However,

$$f(a^{-1}ac) = f(a^{-1})f(c)f(a) = f(a)^{-1}e_Hf(a) = f(a)^{-1}f(a) = e_H.$$

Therefore, $a^{-1}ca \in K$ and K is normal. ■

EXAMPLE 4*

Define $f{:}S_n \to \mathbb{Z}_2$ as follows: $f(\sigma) = 0$ if σ is even and $f(\sigma) = 1$ if σ is odd. Then f is a homomorphism (Exercise 7). Clearly, the kernel of f consists of all even permutations, that is, the kernel is A_n. By Theorem 8.16, A_n is a normal subgroup of S_n.

The kernel of a homomorphism f measures how far f is from being injective.

Theorem 8.17

Let $f{:}G \to H$ be a homomorphism of groups with kernel K. Then

$$K = \langle e_G \rangle \text{ if and only if } f \text{ is injective.}$$

Proof† ▸ Suppose $K = \langle e_G \rangle$. If $f(a) = f(b)$, then

$$\begin{aligned} f(ab^{-1}) &= f(a)f(b^{-1}) && [f \textit{ is a homomorphism.}] \\ &= f(a)f(b)^{-1} && [\textit{Part (2) of Theorem 7.20}] \\ &= f(a)f(a)^{-1} = e_H && [f(a) = f(b) \textit{ by hypothesis.}] \end{aligned}$$

Thus, ab^{-1} is in the kernel, so that $ab^{-1} = e_G$ and hence, $a = b$. Therefore, f is injective.

Conversely, suppose f is injective. If c is any element in the kernel K, then $f(c) = e_H$. By part (1) of Theorem 7.20, $f(e_G) = e_H$. Hence, $f(c) = f(e_G)$, which implies that $c = e_G$ since f is injective. Therefore, e_G is the only element of K, so $K = \langle e_G \rangle$. ■

*Skip this example if you haven't read Section 7.5.

†The proofs of Theorems 8.17–8.20 are simply translations from rings to groups of the proofs of Theorems 6.11–6.13.

Theorem 8.16 states that every kernel is a normal subgroup. Conversely, every normal subgroup is a kernel:

Theorem 8.18

If N is a normal subgroup of a group G, then the map $\pi: G \to G/N$ given by $\pi(a) = Na$ is a surjective homomorphism with kernel N.

Proof ▸ The map π is surjective because given any coset Na in G/N, we have $\pi(a) = Na$. The definition of the group operation in G/N shows that π is a homomorphism:

$$\pi(ab) = Nab = NaNb = \pi(a)\pi(b).$$

The identity element of G/N is Ne. So the kernel of π is

$$\begin{aligned}
\{a \in G \mid \pi(a) = Ne\} &= \{a \in G \mid Na = Ne\} && [\textit{Definition of } \pi] \\
&= \{a \in G \mid a \equiv e \pmod N\} && [\textit{Theorem 8.2}] \\
&= \{a \in G \mid ae^{-1} \in N\} && [\textit{Definition of congruence}] \\
&= \{a \in G \mid a \in N\} = N && [ae^{-1} = ae = a.] \quad \blacksquare
\end{aligned}$$

In order to prove the First Isomorphism Theorem below, we need this lemma.

Lemma 8.19

Let $f: G \to H$ be a group homomorphism with kernel K. Let $a, b \in G$. Then

$$f(a) = f(b) \text{ if and only if } Ka = Kb.$$

Proof ▸ If $f(a) = f(b)$, then $f(a)f(b)^{-1} = e_H$. By Theorem 7.20,

$$f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} = e_H.$$

Hence, $ab^{-1} \in K$ and $a \equiv b \pmod K$. So $Ka = Kb$ by Theorem 8.2.

Conversely, suppose $Ka = Kb$. By Theorem 8.2, $a \equiv b \pmod K$, which means that $ab^{-1} \in K$. Hence, $f(ab^{-1}) = e_H$, and by Theorem 7.20,

$$f(a)f(b)^{-1} = f(a)f(b^{-1}) = f(ab^{-1}) = e_H.$$

Multiplying both ends on the right by $f(b)$ shows that $f(a) = f(b)$. ■

Theorem 8.20 First Isomorphism Theorem

Let $f:G \to H$ be a surjective homomorphism of groups with kernel K. Then the quotient group G/K is isomorphic to H.

Proof ▸ We would like to define $\varphi: G/K \to H$ by $\varphi(Ka) = f(a)$. However, a coset can be labeled by many different elements. We need to know that the value of φ depends only on the coset, and not on the particular representative element chosen to name it. So suppose that $Ka = Kb$. Then $f(a) = f(b)$ by Lemma 8.19, which means that $\varphi(Ka) = \varphi(Kb)$. Therefore, the map $\varphi: G/K \to H$ given by $\varphi(Ka) = f(a)$ is a well-defined function, independent of how cosets are written.

To prove that φ is surjective, suppose $h \in H$. Then $h = f(c)$ for some $c \in G$ because f is surjective. Thus, $\varphi(Kc) = f(c) = h$, and φ is surjective. To prove that φ is injective, suppose $\varphi(Ka) = \varphi(Kb)$. Then $f(a) = f(b)$, so that $Ka = Kb$ by Lemma 8.19. Hence, φ is injective. Finally, φ is a homomorphism because f is

$$\varphi(KaKb) = \varphi(Kab) = f(ab) = f(a)f(b) = \varphi(Ka)\,\varphi(Kb).$$

Therefore, $\varphi: G/K \to H$ is an isomorphism. ■

The First Isomorphism Theorem makes it easier to identify certain quotient groups.

EXAMPLE 5

Let G and H be groups and define $f: G \times H \to G$ by $f(a, b) = a$. Then f is a surjective homomorphism by Exercise 9 of Section 7.4. The kernel of f is

$$\overline{H} = \{(a, b) \mid f(a, b) = e_G\} = \{(a, b) \mid a = e_G\} = \{(e_G, b) \mid a \in H\}.$$

By the First Isomorphism Theorem, $(G \times H)/\overline{H} \cong G$, and it is easy to show that H is isomorphic to $\overline{H}$ (Exercise 15).

EXAMPLE 6

The function $f: \mathbb{C}^* \to \mathbb{R}^{**}$ given by $f(a + bi) = a^2 + b^2$ is a surjective homomorphism of multiplicative groups (Exercise 16). Since 1 is the identity in $\mathbb{R}^{**}$, the kernel of f is $N = \{a + bi \mid a^2 + b^2 = 1\}$. Then N is a normal subgroup by Theorem 8.16 and $\mathbb{C}^*/N \cong \mathbb{R}^{**}$ by the First Isomorphism Theorem.

EXAMPLE 7

As we saw in Example 1, the function $f: \mathbb{R}^* \to \mathbb{R}^{**}$ given by $f(x) = x^2$ is a homomorphism with kernel $K = \{1, -1\}$. Note that f is surjective because for any positive real number c, $f(\sqrt{c}) = (\sqrt{c})^2 = c$. By the First Isomorphism Theorem, $\mathbb{R}^*/K \cong \mathbb{R}^{**}$.

Subgroups of Quotient Groups

Let N be a normal subgroup of a group G. We now investigate the subgroups of the quotient group G/N.

Theorem 8.21

Let N be a normal subgroup of a group G and let K be any subgroup of G that contains N. Then K/N is a subgroup of G/N.

Proof ▸ N is obviously a subgroup of K. By normality, $Na = aN$ for every $a \in G$. In particular, $Na = aN$ for every $a \in K$. Hence, N is a normal subgroup of K and K/N is a group by Theorem 8.13. The elements of K/N are the cosets Na with $a \in K$. Since, every such coset is an element of G/N, we conclude that K/N is a subgroup of G/N. ■

When K is a normal subgroup of G, we get a stronger result.

Theorem 8.22 Third Isomorphism Theorem*

Let K and N be normal subgroups of a group G with $N \subseteq K \subseteq G$. Then K/N is a normal subgroup of G/N, and the quotient group $(G/N)/(K/N)$ is isomorphic to G/K.

Proof ▸ The basic idea of the proof is to define a surjective homomorphism from G/N to G/K whose kernel is K/N. Then the conclusion of the theorem will follow immediately from the First Isomorphism Theorem. First note that, if $Na = Nc$ in G/N, then $ac^{-1} \in N$ by Theorem 8.2 and the definition of congruence modulo N. Since $N \subseteq K$, this means that $ac^{-1} \in K$. Consequently, $Ka = Kc$ in G/K by Theorem 8.2 again. Therefore, the map $f: G/N \to G/K$ given by $f(Na) = Ka$ is a well-defined function, that is, independent of the coset representatives in G/N. Clearly f is surjective since any Ka in G/K is the image of Na in G/N. The definition of coset operation shows that

$$f(NaNb) = f(Nab) = Kab = KaKb = f(Na)f(Nb).$$

Hence, f is a homomorphism. Since the identity element of G/K is Ke, a coset Na is in the kernel of f if and only if $f(Na) = Ke$, that is, if and only if $Ka = Ke$. However, $Ka = Ke$ if and only if $a \in K$ by Theorem 8.2. Thus the kernel of f consists of all cosets Na with $a \in K$; in other words, K/N is the kernel of f. Therefore, K/N is a normal subgroup of G/N (Theorem 8.16), and by the First Isomorphism Theorem, $(G/N)/(K/N) = (G/N)/\text{kernel} f \cong G/K$. ■

*Yes, Virginia, there is a Second Isomorphism Theorem; see Exercise 40. For more about Virginia, go to *www.stormfax.com/bios.htm*

Corollary 8.23

Let N be a normal subgroup of a group G and let K be any subgroup of G that contains N. Then K is normal in G if and only if K/N is normal in G/N.

Proof ▸ If K is normal in G, then K/N is normal in G/N by Theorem 8.22. Conversely, suppose that K/N is normal in G/N. Let a be any element of G and k any element of K. We first prove that $a^{-1}ka \in K$. Since K/N is normal,

$$Na^{-1}ka = (Na^{-1})(Nk)(Na) = (Na)^{-1}(Nk)(Na) \in K/N.$$

Hence, $Na^{-1}ka = Nt$ for some $t \in K$, so that $a^{-1}ka = nt$ for some $n \in N$. Since $N \subseteq K$, we have $a^{-1}ka = nt \in K$, as desired. Since a and k were arbitrary, this proves that $a^{-1}Ka \subseteq K$. Therefore, K is normal in G by Theorem 8.11. ■

We now have complete information about subgroups of G/N that arise from subgroups of G that contain N. Are these the only subgroups of G/N? The next theorem answers this question in the affirmative.

Theorem 8.24

If T is any subgroup of G/N, then $T = H/N$, where H is a subgroup of G that contains N.

Proof ▸ Let $H = \{a \in G \mid Na \in T\}$. Exercise 23 shows that H is a subgroup of G. If $a \in N$, then $ae^{-1} = ae = a \in N$, so $a \equiv e \pmod{N}$. By Theorem 8.2, $Na = Ne \in T$. Hence, $a \in H$. Therefore, $N \subseteq H$. Finally, the quotient group H/N consists of all cosets Na with $a \in H$, that is, all $Na \in T$. Thus, $H/N = T$. ■

Simple Groups

In Section 8.1 we considered the classification problem for finite groups—the attempt to produce a list of groups such that every finite group is isomorphic to exactly one group on the list. We now introduce the groups that apparently are the key to solving the classification problem. Recall that a group G always has two normal subgroups, the trivial group $\langle e \rangle$ and G itself (Exercise 4 in Section 8.2). A group G is said to be **simple** if its only normal subgroups are $\langle e \rangle$ and G.

EXAMPLE 8

If p is prime, then any (normal) subgroup H of the additive group $\mathbb{Z}_p$ must have order dividing p by Lagrange's Theorem. So H must have order 1 or p, so that $H = \langle 0 \rangle$ or $H = \mathbb{Z}_p$. Therefore, $\mathbb{Z}_p$ is simple.

Theorem 8.25

G is a simple abelian group if and only if G is isomorphic to the additive group $\mathbb{Z}_p$ for some prime p.

Proof ▸ The preceding example shows that any group isomorphic to $\mathbb{Z}_p$ is simple. Conversely, suppose G is simple. Since every subgroup of an abelian group is normal, G has no subgroups at all, except $\langle e\rangle$ and G. So if a is any nonidentity element of G, then the cyclic subgroup $\langle a\rangle$ must be G itself. Since every infinite cyclic group is isomorphic to $\mathbb{Z}$ by Theorem 7.19 and $\mathbb{Z}$ has many proper subgroups, $G = \langle a\rangle$ must be a cyclic group of finite order n. We claim that n is prime. If n were composite, say $n = td$ with $1 < d < n$, then $\langle a^t\rangle$ would be a subgroup of G of order d by part (3) of Theorem 7.9, which is impossible since G is simple. Therefore, G is cyclic of prime order and, hence, is isomorphic to some $\mathbb{Z}_p$ by Theorem 7.19. ■

Nonabelian simple groups are relatively rare. There are only five of order less than 1000 and only 56 of order less than 1,000,000. A large class of nonabelian simple groups, the alternating groups, is considered in Section 8.5.

We now show why simple groups are the basic building blocks for all groups. If G is a finite group, then it has only finitely many normal subgroups other than itself (and there is at least one such subgroup since $\langle e\rangle$ is normal). Let G_1 be a normal subgroup (other than G) that has the largest possible order. We claim that G/G_1 is simple. If G/G_1 had a proper normal subgroup, then by Theorem 8.24 and Corollary 8.23 this subgroup would be of the form M/G_1, where M is a normal subgroup of G such that $G_1 \subsetneqq M \subsetneqq G$. In this case, M would be a normal subgroup other than G with order larger than $|G_1|$, a contradiction. Hence, G/G_1 is simple.

If $G_1 \neq \langle e\rangle$, let G_2 be a normal subgroup of G_1 (other than G_1) of largest possible order. (G_2 is normal in G_1, but need not be normal in G.) The argument in the preceding paragraph, with G_1 in place of G and G_2 in place of G_1, shows that G_1/G_2 is simple. Similarly, if $G_2 \neq \langle e\rangle$, there is a normal subgroup G_3 of G_2 such that $G_3 \neq G_2$ and G_2/G_3 is simple. This process can be continued until we reach some G_n that is the identity subgroup (and this must occur since the order of G_i gets smaller at each stage). Then we have a sequence of groups

$$G = G_0 \supsetneqq G_1 \supsetneqq G_2 \supsetneqq G_3 \supsetneqq \cdots \supsetneqq G_{n-1} \supsetneqq G_n = \langle e\rangle$$

such that each G_i is a normal subgroup of its predecessor and each quotient group G_i/G_{i+1} is simple. The simple groups $G_0/G_1, G_1/G_2, \ldots, G_{n-1}/G_n$ are called the **composition factors** of G.

It can be shown that the composition factors of a finite group G are independent of the choice of the subgroups G_i. In other words, if you made different choices of the G_i, the simple quotient groups you would obtain would be isomorphic to the ones obtained in the previous paragraph. This means that the composition factors of G are completely determined by the structure of G and suggests a strategy for solving the classification problem. If we could first classify all simple groups and then show how the composition factors of an arbitrary group determine the structure of the group, it would be possible to classify all groups.

The good news is that the first half of this plan has already succeeded. For more than four decades, a number of group theorists around the world worked on various aspects of the problem and eventually obtained a list of simple groups such that every finite simple group is isomorphic to exactly one group on the list.* The complete proof of this spectacular result runs some 10,000 pages! For a brief history of the search for simple groups, see Gallian [23] or Steen [25].

Exercises

NOTE: *The congruence class of a in* $\mathbb{Z}_n$ *is denoted* $[a]_n$ *whenever necessary to avoid confusion.*

A. *In Exercises 1–9, verify that the given function is a homomorphism and find its kernel.*

1. $f: \mathbb{C} \to \mathbb{R}$, where $f(a + bi) = b$.

2. $g: \mathbb{R}^* \to \mathbb{Z}_2$, where $g(x) = 0$ if $x > 0$ and $g(x) = 1$ if $x < 0$.

3. $h: \mathbb{R}^* \to \mathbb{R}^*$, where $h(x) = x^3$.

4. $f: \mathbb{Q}^* \to \mathbb{Q}^{**}$, where $f(x) = |x|$.

5. $g: \mathbb{Q} \times \mathbb{Z} \to \mathbb{Z}$, where $f((x, y)) = y$.

6. $h: \mathbb{C} \to \mathbb{C}$, where $h(x) = x^4$.

7.† $f: S_n \to \mathbb{Z}_2$, where $f(\sigma) = 0$ if σ is even and $f(\sigma) = 1$ if σ is odd.

8. $f: \mathbb{Z}_{12} \to \mathbb{Z}_{12}$, where $f(x) = 3x$.

9. $f: \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}_4$, where $f(a) = ([a]_2, [a]_4)$.

10. $\varphi: S_n \to S_{n+1}$, where for each $f \in S_n$, $\varphi(f) \in S_{n+1}$ is given by

$$\varphi(f)(k) = \begin{cases} f(k) & \text{if } 1 \leq k \leq n \\ n + 1 & \text{if } k = n + 1 \end{cases}$$

11. Suppose that k, n, and r are positive integers such that $k \mid n$. Show that the function $f: \mathbb{Z}_n \to \mathbb{Z}_k$ given by $f([a]_n) = [ra]_k$ is well defined (meaning that if $[a]_n = [b]_n$, then $[ra]_k = [rb]_k$).

*The proof was first announced in 1981, but a few years later a gap in the proof was discovered. It took until 2004 for this gap to be fixed.

†Skip this exercise if you haven't read Section 7.5.

In *Exercises 12–14, verify that the given function is a surjective homomorphism of additive groups. Then find its kernel and identify the cyclic group to which the kernel is isomorphic. [Exercise 11 may be helpful.]*

12. $h:\mathbb{Z}_{12} \to \mathbb{Z}_6$, where $h([a]_{12}) = [a]_6$.

13. $h:\mathbb{Z}_{16} \to \mathbb{Z}_4$, where $h([a]_{16}) = [3a]_6$.

14. $h:\mathbb{Z}_{18} \to \mathbb{Z}_3$, where $h([x]_{18}) = [2x]_3$.

15. If H and $\overline{H}$ are the groups in Example 5. Show that $H \cong \overline{H}$.

16. Prove that the function $f:\mathbb{C}^* \to \mathbb{R}^{**}$ given by $f(a + bi) = a^2 + b^2$ is a surjective homomorphism of groups.

17. **(a)** Produce a list of groups such that every homomorphic image of $\mathbb{Z}_{12}$ is isomorphic to exactly one group on the list. [*Hint:* See Exercise 26 in Section 7.4.]

(b) Do the same for $\mathbb{Z}_{20}$.

18. Find all homomorphic images of D_4.

19. Find all homomorphic images of S_3.

20. **(a)** List all subgroups of $\mathbb{Z}_{12}/H$, where $H = \{0, 6\}$.

(b) List all subgroups of $\mathbb{Z}_{20}/K$, where $K = \{0, 4, 8, 12, 16\}$.

21. Suppose that G is a simple group and $f:G \to H$ is a surjective homomorphism of groups. Prove that either f is an isomorphism or $H = \langle e \rangle$.

B. 22. Let G be an abelian group.

(a) Show that $K = \{a \in G \mid |a| \leq 2\}$ is a subgroup of G.

(b) Show that $H = \{x^2 \mid x \in G\}$ is a subgroup of G.

(c) Prove that $G/K \cong H$. [*Hint:* Define a surjective homomorphism from G to H with kernel K.]

23. If N is a normal subgroup of a group G and T is a subgroup of G/N, show that $H = \{a \in G \mid Na \in T\}$ is a subgroup of G.

24. If $k \mid n$ and $f:U_n \to U_k$ is given by $f([x]_n) = [x]_k$, show that f is a homomorphism and find its kernel.

25. Prove that $(\mathbb{Z} \times \mathbb{Z})/\langle(1, 1)\rangle \cong \mathbb{Z}$. [*Hint:* Show that $f:\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, given by $f((a, b)) = a - b$, is a surjective homomorphism.]

26. Prove that $(\mathbb{Z} \times \mathbb{Z})/\langle(2, 2)\rangle \cong \mathbb{Z} \times \mathbb{Z}_2$. [*Hint:* Show that $h:\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}_2$, given by $h((a, b)) = (a - b, [b]_2)$ is a surjective homomorphism.]

27. Let M be a normal subgroup of a group G and let N be a normal subgroup of a group H. Use the First Isomorphism Theorem to prove that $M \times N$ is a normal subgroup of $G \times H$ and that $(G \times H)/(M \times N) \cong G/M \times H/N$.

28. $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$ by Exercise 25 of Section 8.2. Prove that $GL(2, \mathbb{R})/SL(2, \mathbb{R})$ is isomorphic to the multiplicative group $\mathbb{R}^*$ of nonzero real numbers.

29. If $k \mid n$, prove that $\mathbb{Z}_n/\langle k \rangle \cong \mathbb{Z}_k$. [Exercise 11 may be helpful.]

30. If $f:G \to H$ is a homomorphism of finite groups, prove that $|\text{Im} f|$ divides $|G|$ and $|H|$. [Im f was defined just before Theorem 7.20.]

31. Prove that $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$. [Consider $f:\mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}_4$, given by $f(a) = ([a]_3, [a]_4)$.]

32. Let M be a normal subgroup of a group G and let N be a normal subgroup of a group H. If $f:G \to H$ is a homomorphism such that $f(M) \subseteq N$, prove that the map $g:G/M \to H/N$ given by $g(Ma) = Nf(a)$ is a well-defined homomorphism.

33. Let $f:G \to H$ be a surjective homomorphism of groups with kernel K. Prove that there is a bijection between the set of all subgroups of H and the set of subgroups of G that contain K.

34. (An exercise for those who know how to multiply 3×3 matrices.) Let G be the set of all matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{Q}$.

(a) Show that G is a group under matrix multiplication.

(b) Find the center C of G and show that C is isomorphic to the additive group $\mathbb{Q}$.

(c) Show that G/C is isomorphic to the additive group $\mathbb{Q} \times \mathbb{Q}$.

35. Let G and H be the groups in Exercises 33 and 34 of Section 7.1. Use the First Isomorphism Theorem to prove that H is normal in G and that G/H is isomorphic to the multiplicative group $\mathbb{R}^*$ of nonzero real numbers. [*Hint:* Consider the map $f:G \to \mathbb{R}^*$ given by $f(T_{a,b}) = a$.]

36. Let N be a normal subgroup of a group G and let $f:G \to H$ be a homomorphism of groups such that the restriction of f to N is an isomorphism $N \cong H$. Prove that $G \cong N \times K$, where K is the kernel of f. [*Hint:* Exercise 30 in Section 8.2.]

37. Prove that $\mathbb{Q}^* \cong \mathbb{Q}^{**} \times \mathbb{Z}_2$. [*Hint:* Exercises 4 and 36.]

38. Let N be a normal subgroup of a group G. Prove that G/N is simple if and only if there is no normal subgroup K such that $N \subsetneqq K \subsetneqq G$. [*Hint:* Corollary 8.23 and Theorem 8.24.]

39.* The additive group $\mathbb{Z}[x]$ contains $\mathbb{Z}$ (the set of constant polynomials) as a normal subgroup. Show that $\mathbb{Z}[x]/\mathbb{Z}$ is isomorphic to $\mathbb{Z}[x]$. This example shows that $G/N \cong G$ does not necessarily imply that $N = \langle e \rangle$. [*Hint:* Consider the map $T:\mathbb{Z}[x] \to \mathbb{Z}[x]/\mathbb{Z}$ given by $T(f(x)) = \mathbb{Z} + xf(x)$.]

C. 40. **(Second Isomorphism Theorem)** Let K and N be subgroups of a group G, with N normal in G. Then $NK = \{nk \mid n \in N, k \in K\}$ is a subgroup of G that contains both K and N by Exercise 20 of Section 8.2.

(a) Prove that N is a normal subgroup of NK.

*Skip this exercise if you have not read the first part of Section 4.1.

(b) Prove that the function $f:K \to NK/N$ given by $f(k) = Nk$ is a surjective homomorphism with kernel $K \cap N$.

(c) Conclude that $K/(N \cap K) \cong NK/N$.

41. Cayley's Theorem 7.21 represents a group G as a subgroup of the permutation group $A(G)$. A more efficient way of representing G as a permutation group arises from the following generalized Cayley's Theorem. Let K be a subgroup of G and let T be the set of all distinct right cosets of K.

(a) If $a \in G$, show that the map $f_a:T \to T$ given by $f_a(Kb) = Kba$ is a permutation of the set T.

(b) Prove that the function $\varphi:G \to A(T)$ given by $\varphi(a) = f_{a^{-1}}$, is a homomorphism of groups whose kernel is contained in K.

(c) If K is normal in G, prove that $K =$ kernel φ.

(d) Prove Cayley's Theorem by applying parts (b) and (c) with $K = \langle e \rangle$.

42. A group G is said to be **metabelian** if it has a subgroup N such that N is abelian, N is normal in G, and G/N is abelian.

(a) Show that S_3 is metabelian.

(b) Prove that every homomorphic image of a metabelian group is metabelian.

(c) Prove that every subgroup of a metabelian group is metabelian.

APPLICATION: Decoding Techniques (Section 16.2) may be covered at this point if desired.

8.5 The Simplicity of A_n*

As we saw at the end of Section 8.4, simple groups appear to be the key to solving the classification problem for finite groups. This fact and the following theorem are one reason that the alternating groups A_n are important.

Theorem 8.26

For each $n \neq 4$, the alternating group A_n is a simple group.

The group A_4 is not simple (Exercise 7). Although the entire proof of Theorem 8.26 is rather long, it requires only basic facts about the symmetric groups and normal subgroups. There will be many instances in the proof where we will deal with permutations such as $(abcd)$ or $(a2b)$ or $(ab)(cd)$. In all such cases,

distinct letters represent distinct elements of $\{1, 2, \ldots, n\}$.

The proof of the theorem requires two lemmas.

---

*Section 7.5 is a prerequisite. This section is not used in the sequel and may be omitted if desired.

Lemma 8.27

Every element of A_n (with $n \geq 3$) is a product of 3-cycles.

Proof ▸ Every element of A_n is by definition the product of *pairs* of transpositions. But every such pair must be of one of these forms: $(ab)(cd)$ or $(ab)(ac)$ or $(ab)(ab)$. In the first case verify that $(ab)(cd) = (adb)(adc)$, in the second that $(ab)(ac) = (acb)$, and in the last that $(ab)(ab) = (1) = (abc)(acb)$. Thus every pair of transpositions is either a 3-cycle or a product of two 3-cycles. Hence, every product of pairs of transpositions is a product of 3-cycles. ■

Lemma 8.28

If N is a normal subgroup of A_n (with $n \geq 3$) and N contains a 3-cycle, then $N = A_n$.

Proof ▸ For notational convenience, assume that $(123) \in N$ [the argument when $(rst) \in N$ is the same; just replace 1, 2, 3 by r, s, t, respectively]. Since $(123) \in N$, we see that $(123)(123) = (132)$ is also in N. For $k \geq 4$, let $x = (12)(3k)$ and verify that $x^{-1} = (3k)(12)$. The normality of N implies that $x(132)x^{-1} \in N$ by Theorem 8.11. But

$$x(132)x^{-1} = (12)(3k)(132)(3k)(12) = (12k).$$

Therefore,

(∗) **N contains all 3-cycles of the form $(12k)$ with $k \geq 3$.**

Verify that every other 3-cycle can be written in one of these forms:

$$(1a2), \qquad (1ab), \qquad (2ab), \qquad (abc)$$

where $a, b, c \geq 3$. By (∗) and closure in N,

$$(1a2) = (12a)(12a) \in N;$$
$$(1ab) = (12b)(12a)(12a) \in N;$$
$$(2ab) = (12b)(12b)(12a) \in N;$$
$$(abc) = (12a)(12a)(12c)(12b)(12b)(12a) \in N.$$

Thus N contains all 3-cycles, and, hence, N contains all products of 3-cycles by closure. Therefore, $N = A_n$ by Lemma 8.27. ■

We are now ready to prove Theorem 8.26. The following fact will be used frequently:

(∗∗) **The inverse of the cycle $(a_1a_2a_3 \cdots a_k)$ is the cycle $(a_1a_ka_{k-1} \cdots a_3a_2)$.**

For example, $(12345)^{-1} = (15432)$ and $(678)^{-1} = (687)$, as you can easily verify.

Proof of Theorem 8.26 ▸ A_2 and A_3 are simple abelian groups (Exercise 2). So assume $n \geq 5$. We must prove that A_n has no *proper* normal subgroups. Let N be any normal subgroup of A_n, with $N \neq (1)$. We need only show that $N = A_n$. When all the nonidentity elements of N are written as products of disjoint cycles, then there are three possibilities for the lengths of these cycles:

1. Some cycle has length ≥ 4.
2. Every cycle has length ≤ 3, and some have length 3.
3. Every cycle has length ≤ 2.

We shall show that in each of these cases, $N = A_n$.

Case 1 N contains an element σ that is the product of disjoint cycles, at least one of which has length $r \geq 4$. For notational convenience we assume that $\sigma = (1234 \cdots r)\tau$, where τ is a product of disjoint cycles, none of which involve the symbols $1, 2, 3, 4, \ldots, r$.† Let $\delta = (123) \in A_n$. Since N is a normal subgroup and $\sigma \in N$, we have $\sigma^{-1}(\delta\sigma\delta^{-1}) \in N$ by Theorem 8.11. An easy computation shows that

$$
\begin{aligned}
\sigma^{-1}(\delta\sigma\delta^{-1}) &= [(1234 \cdots r)\tau]^{-1}(123)[(1234 \cdots r)\tau](123)^{-1} \\
&= \tau^{-1}(1234 \cdots r)^{-1}(123)[(1234 \cdots r)\tau](123)^{-1} && [\textit{Corollary 7.6}] \\
&= \tau^{-1}(1r \cdots 432)(123)(1234 \cdots r)\tau(132) && [\textit{Statement } (**)] \\
&= \tau^{-1}\tau(1r \cdots 432)(123)(1234 \cdots r)(132) && [\textit{Theorem 7.23}] \\
&= (1)(13r) = (13r).
\end{aligned}
$$

Therefore, $(13r) \in N$, and hence, $N = A_n$ by Lemma 8.28.

Case 2A N contains an element σ that is the product of disjoint cycles, at least two of which have length 3. For convenience we assume that $\sigma = (123)(456)\tau$, where τ is a product of disjoint cycles, none of which involve the symbols $1, 2, \ldots, 6$. Let $\delta = (124) \in A_n$. Then, as in Case 1, N contains $\sigma^{-1}(\delta\sigma\delta^{-1})$, and we have a similar calculation:

$$
\begin{aligned}
\sigma^{-1}(\delta\sigma\delta^{-1}) &= [(123)(456)\tau]^{-1}(124)(123)(456)\tau(124)^{-1} \\
&= \tau^{-1}(456)^{-1}(123)^{-1}(124)(123)(456)\tau(124)^{-1} && [\textit{Corollary 7.6}] \\
&= \tau^{-1}(465)(132)(124)(123)(456)\tau\,(142) && [\textit{Statement } (**)] \\
&= \tau^{-1}\tau(465)(132)(124)(123)(456)(142) && [\textit{Theorem 7.23}] \\
&= (14263).
\end{aligned}
$$

Therefore, $(14263) \in N$, and $N = A_n$ by Case 1.

---

†The same argument works with an arbitrary r-cycle $(abcd \cdots t)$ in place of $(1234 \cdots r)$; just replace 1 by a, 2 by b, etc. *Analogous remarks apply in the other cases*, where specific cycles will also be used to make the argument easier to follow.

Case 2B N contains an element σ that is the product of one 3-cycle and some 2-cycles. We assume that $\sigma = (123)\tau$, where τ is a product of disjoint transpositions, none of which involve the symbols 1, 2, 3. Since a product of disjoint transpositions is its own inverse (Exercise 5), Theorem 7.23 shows that

$$\sigma^2 = (123)\tau(123)\tau = (123)(123)\tau\tau = (123)(123) = (132).$$

But $\sigma^2 \in N$ since $\sigma \in N$. Therefore, $(132) \in N$, and $N = A_n$ by Lemma 8.28.

Case 2C N contains a 3-cycle. Then $N = A_n$ by Lemma 8.28.

Case 3 Every element of N is the product of an even number of disjoint 2-cycles. Then a typical element σ of N has the form $(12)(34)\tau$, where τ is a product of disjoint transpositions, none of which involve the symbols 1, 2, 3, 4. Let $\delta = (123) \in A_n$. Then, as above, $\sigma^{-1}(\delta\sigma\delta^{-1}) \in N$. Using Corollary 7.6, Theorem 7.23, and statement (**), we see that

$$\sigma^{-1}(\delta\sigma\delta^{-1}) = \tau^{-1}(34)(12)(123)(12)(34)\tau(132) = (13)(24).$$

Since $n \geq 5$, there is an element k in $\{1, 2, \ldots, n\}$ distinct from 1, 2, 3, 4. Let $\alpha = (13k) \in A_n$. Let $\beta = (13)(24)$, which was just shown to be in N. Then by the normality of N and closure, $\beta(\alpha\beta\alpha^{-1}) \in N$. But

$$\beta(\alpha\beta\alpha^{-1}) = (13)(24)(13k)(13)(24)(1k3) = (13k).$$

Therefore, $(13k) \in N$, and $N = A_n$ by Lemma 8.28. ∎

Theorem 8.26 leads to an interesting fact about the normal subgroups of S_n:

Corollary 8.29

If $n \geq 5$, then (1), A_n, and S_n are the only normal subgroups of S_n.

Sketch of Proof ▸ Suppose that N is a normal subgroup of S_n. Then $N \cap A_n$ is a normal subgroup of A_n (Exercise 19 of Section 8.2). Theorem 8.26 shows that $N \cap A_n$ must either be A_n or (1). If $N \cap A_n = A_n$, then $N = A_n$ or S_n (Exercise 10). If $N \cap A_n = (1)$, then all the nonidentity elements of N are odd. Since the product of two odd permutations is even, that is, an element of A_n, and $N \cap A_n = (1)$, the product of any two elements of N is (1). Therefore, $N = (1)$ (Exercises 8 and 9). ∎

Exercises

A. 1. (a) List all the 3-cycles in S_4.

(b) List all the elements of A_4 and express each as a product of 3-cycles.

2. (a) Verify that $A_2 = (1)$.

(b) Show that A_3 is a cyclic group of order 3 and hence simple by Theorem 8.25.

3. Find the center of the group A_4.

4. If $n \geq 5$, what is the center of A_n?

B. 5. If $\sigma \in S_n$ is a product of disjoint transpositions, prove that $\sigma^2 = (1)$.

6. Prove that A_5 has no subgroup of order 30. [*Hint:* Exercise 23 of Section 8.2.]

7. Prove that $N = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of A_4. Hence, A_4 is not simple. [*Hint:* Exercise 23 of Section 7.5. For normality, use Exercise 1(a) and straightforward computations.]

8. Prove that no subgroup of order 2 in S_n ($n \geq 3$) is normal. [*Hint:* Exercises 26 of Section 7.5 and 16 of Section 8.2.]

9. Let N be a subgroup of S_n such that $\sigma\tau = (1)$ for all nonidentity elements $\sigma, \tau \in N$. Prove that $N = (1)$ or N is cyclic of order 2. [*Hint:* If $N \neq (1)$, let σ be a nonidentity element of N. Show that σ has order 2. If τ is any other nonidentity element of N, show that $\sigma = \tau$.]

10. If N is a normal subgroup of S_n and $N \cap A_n = A_n$, prove that $N = A_n$ or S_n. [*Hint:* Why is $A_n \subseteq N \subseteq S_n$? Use Theorem 7.29 and Lagrange's Theorem.]

11. Prove that A_n is the only subgroup of index 2 in S_n. [*Hint:* Exercise 23 of Section 8.2 and Corollary 8.29.]

12. If $f: S_n \to S_n$ is a homomorphism, prove that $f(A_n) \subseteq A_n$.

PART 2

ADVANCED TOPICS

CHAPTER 9

Topics in Group Theory

This chapter takes a deeper look at various aspects of the classification problem for finite groups, which was introduced in Section 8.1. After the necessary preliminaries are developed in Section 9.1, all finite abelian groups are classified up to isomorphism in Section 9.2. The basic tools for analyzing nonabelian groups are presented in Sections 9.3 and 9.4. Applications of these results and several other facts about the structure of finite groups are considered in Section 9.5, where groups of small order are classified.

Sections 9.3 and 9.4 are independent of Sections 9.1 and 9.2 and may be read first if desired. Sections 9.1–9.4 are prerequisites for Section 9.5.

9.1 Direct Products

If G and H are groups, then their Cartesian product $G \times H$ is also a group, with the operation defined coordinatewise (Theorem 7.4). In this section we extend this notion to more than two groups. Then we examine the conditions under which a group is (isomorphic to) a direct product of certain of its subgroups. When these subgroups are of a particularly simple kind, then the structure of the group can be completely determined, as will be demonstrated in Section 9.2. Throughout the general discussion, all groups are written multiplicatively, but specific examples of familiar additive groups are written additively as usual.

If $G_1, G_2, \ldots, G_n$ are groups, we define a coordinatewise operation on the Cartesian product $G_1 \times G_2 \times \cdots \times G_n$ as follows:

$$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1b_1, a_2b_2, \ldots, a_nb_n).$$

It is easy to verify that $G_1 \times G_2 \times \cdots \times G_n$ is a group under this operation: If e_i is the identity element of G_i, then $(e_1, e_2, \ldots, e_n)$ is the identity element of $G_1 \times G_2 \times \cdots \times G_n$ and $(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$ is the inverse of $(a_1, a_2, \ldots, a_n)$. This group is called the **direct product** of $G_1, G_2, \ldots, G_n$.*

*When each G_i is an additive abelian group, the direct product of $G_1, \ldots, G_n$ is sometimes called the **direct sum** and denoted $G_1 \oplus G_2 \oplus \cdots \oplus G_n$.

EXAMPLE 1

Recall that U_n is the multiplicative group of units in $\mathbb{Z}_n$ and that $U_4 = \{1, 3\}$ and $U_6 = \{1, 5\}$ (see Theorem 2.10). The direct product $U_4 \times U_6 \times \mathbb{Z}_3$ consists of the 12 triples

$$\begin{array}{cccccc} (1,1,0), & (1,1,1), & (1,1,2), & (1,5,0), & (1,5,1), & (1,5,2), \\ (3,1,0), & (3,1,1), & (3,1,2), & (3,5,0), & (3,5,1), & (3,5,2). \end{array}$$

Note that U_4 has order 2, U_6 has order 2, $\mathbb{Z}_3$ has order 3, and the direct product $U_4 \times U_6 \times \mathbb{Z}_3$ has order $2 \cdot 2 \cdot 3 = 12$. Similarly, in the general case,

if $G_1, G_2, \ldots, G_n$ are finite groups, then
$G_1 \times G_2 \times \cdots \times G_n$ has order $|G_1| \cdot |G_2| \cdots |G_n|$.

In the preceding example it is important to note that the groups U_4, U_6, and $\mathbb{Z}_3$ are *not* contained in the direct product $U_4 \times U_6 \times \mathbb{Z}_3$. For instance, 5 is an element of U_6, but 5 is *not* in $U_4 \times U_6 \times \mathbb{Z}_3$ because the elements of $U_4 \times U_6 \times \mathbb{Z}_3$ are *triples*. In general, for $1 \le i \le n$

G_i is *not* a subgroup of the direct product $G_1 \times G_2 \times \cdots \times G_n$.*

This situation is not entirely satisfactory, but by changing our viewpoint slightly we can develop a notion of direct product in which the component groups may be considered as subgroups.

EXAMPLE 2

It is easy to verify that $M = \{0, 3\}$ and $N = \{0, 2, 4\}$ are normal subgroups of $\mathbb{Z}_6$ (Do it!). Observe that every element of $\mathbb{Z}_6$ can be written as a sum of an element in M and an element in N in *one and only one* way:

$$\begin{array}{ccc} 0 = 0 + 0 & 1 = 3 + 4 & 2 = 0 + 2 \\ 3 = 3 + 0 & 4 = 0 + 4 & 5 = 3 + 2. \end{array}$$

Verify that, when the elements of $\mathbb{Z}_6$ are written as sums in this way, then the addition table for $\mathbb{Z}_6$ looks like this:

	0 + 0	3 + 4	0 + 2	3 + 0	0 + 4	3 + 2
0 + 0	0 + 0	3 + 4	0 + 2	3 + 0	0 + 4	3 + 2
3 + 4	3 + 4	0 + 2	3 + 0	0 + 4	3 + 2	0 + 0
0 + 2	0 + 2	3 + 0	0 + 4	3 + 2	0 + 0	3 + 4
3 + 0	3 + 0	0 + 4	3 + 2	0 + 0	3 + 4	0 + 2
0 + 4	0 + 4	3 + 2	0 + 0	3 + 4	0 + 2	3 + 0
3 + 2	3 + 2	0 + 0	3 + 4	0 + 2	3 + 0	0 + 4

*It is true, however, that an isomorphic copy of G_i is a subgroup of $G_1 \times G_2 \times \cdots \times G_n$ (see Exercise 12).

Compare the $\mathbb{Z}_6$ table with the operation table for the direct product $M \times N$:

	(0, 0)	(3, 4)	(0, 2)	(3, 0)	(0, 4)	(3, 2)
(0, 0)	(0, 0)	(3, 4)	(0, 2)	(3, 0)	(0, 4)	(3, 2)
(3, 4)	(3, 4)	(0, 2)	(3, 0)	(0, 4)	(3, 2)	(0, 0)
(0, 2)	(0, 2)	(3, 0)	(0, 4)	(3, 2)	(0, 0)	(3, 4)
(3, 0)	(3, 0)	(0, 4)	(3, 2)	(0, 0)	(3, 4)	(0, 2)
(0, 4)	(0, 4)	(3, 2)	(0, 0)	(3, 4)	(0, 2)	(3, 0)
(3, 2)	(3, 2)	(0, 0)	(3, 4)	(0, 2)	(3, 0)	(0, 4)

The only difference in these two tables is that elements are written $a + b$ in the first and (a, b) in the second. Among other things, the tables show that the direct product $M \times N$ is isomorphic to $\mathbb{Z}_6$ under the isomorphism that assigns each pair $(a, b) \in M \times N$ to the sum of its coordinates $a + b \in \mathbb{Z}_6$.

Consequently, we can express $\mathbb{Z}_6$ as a direct product in a purely *internal* fashion, without looking at the set $M \times N$, which is external to $\mathbb{Z}_6$: Write each element uniquely as a sum $a + b$, with $a \in M$ and $b \in N$. We now develop this same idea in the general case, with multiplicative notation in place of addition in $\mathbb{Z}_6$.

Theorem 9.1

Let $N_1, N_2 \ldots, N_k$ be normal subgroups of a group G such that every element in G can be written uniquely in the form $a_1a_2 \cdots a_k$, with $a_i \in N_i$.* Then G is isomorphic to the direct product $N_1 \times N_2 \times \cdots \times N_k$.

The proof depends on this useful fact:

Lemma 9.2

Let M and N be normal subgroups of a group G such that $M \cap N = \langle e \rangle$. If $a \in M$ and $b \in N$, then $ab = ba$.

Proof ▸ Consider $a^{-1}b^{-1}ab$. Since M is normal, $b^{-1}ab \in M$ by Theorem 8.11. Closure in M shows that $a^{-1}b^{-1}ab = a^{-1}(b^{-1}ab) \in M$. Similarly, the normality of N implies that $a^{-1}b^{-1}a \in N$ and, hence, $a^{-1}b^{-1}ab = (a^{-1}b^{-1}a)b \in N$. Thus $a^{-1}b^{-1}ab \in M \cap N = \langle e \rangle$. Multiplying both sides of $a^{-1}b^{-1}ab = e$ on the left by ba shows that $ab = ba$. ■

Proof of Theorem 9.1 ▸ Guided by the example preceding the theorem (but using multiplicative notation), we define a map

$$f: N_1 \times N_2 \times \cdots \times N_k \to G \quad \text{by} \quad f(a_1, a_2, \ldots, a_k) = a_1a_2 \cdots a_k.$$

*Uniqueness means that if $a_1a_2 \ldots a_k = b_1b_2 \ldots b_k$ with each $a_i, b_i \in N_i$, then $a_i = b_i$ for every i.

Since every element of G can be written in the form $a_1a_2\cdots a_k$ (with $a_i \in N_i$) by hypothesis, f is surjective. If $f(a_1, a_2, \ldots, a_k) = f(b_1, b_2, \ldots, b_k)$, then $a_1a_2\cdots a_k = b_1 b_2 \cdots b_k$. By the uniqueness hypothesis, $a_i = b_i$ for each $i(1 \le i \le k)$. Therefore,

$$(a_1, a_2, \ldots, a_k) = (b_1, b_2, \ldots, b_k) \text{ in } N_1 \times N_2 \times \cdots \times N_k,$$

and f is injective.

In order to prove that f is a homomorphism we must first show that the N's are mutually disjoint subgroups, that is, $N_i \cap N_j = \langle e \rangle$ when $i \neq j$. If $a \in N_i \cap N_j$, then a can be written as a product of elements of the N's in two different ways:

$$\begin{array}{cccccccc} ee\cdots & eae\cdots & e\cdots & e = a = & ee\cdots & e\cdots & eae\cdots & e. \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ N_1 & N_i & N_j & N_k & N_1 & N_i & N_j & N_k. \end{array}$$

The uniqueness hypothesis implies that the components in N_i must be equal: $a = e$. Therefore, $N_i \cap N_j = \langle e \rangle$ for $i \neq j$. In showing that f is a homomorphism, we shall make repeated use of this fact, which together with Lemma 9.2, implies that $a_ib_j = b_ja_i$ for $a_i \in N_i$ and $b_j \in N_j$:

$$\begin{aligned} f[(a_1, \ldots, a_k)(b_1, \ldots, b_k)] &= f(a_1b_1, \ldots, a_kb_k) \\ &= a_1b_1\, a_2b_2\, a_3b_3 \cdots a_kb_k \\ &= a_1a_2\, b_1b_2\, a_3b_3 \cdots a_kb_k \\ &= a_1a_2\, b_1a_3\, b_2b_3 \cdots a_kb_k \\ &= a_1a_2\, a_3b_1\, b_2b_3 \cdots a_kb_k. \end{aligned}$$

Continuing in this way we successively move $a_4, a_5, \ldots, a_k$ to the left until we obtain

$$\begin{aligned} f[(a_1, \ldots, a_k)(b_1, \ldots, b_k)] &= (a_1a_2\cdots a_k)(b_1b_2\cdots b_k) \\ &= f(a_1, \ldots, a_k)f(b_1, \ldots, b_k). \end{aligned}$$

Therefore, f is homomorphism and, hence, an isomorphism. ∎

Whenever G is a group and $N_1, \ldots, N_k$ are subgroups satisfying the hypotheses of Theorem 9.1 we shall say that **G is the direct product of $N_1, \ldots, N_k$** and write $G = N_1 \times \cdots \times N_k$. Each N_i is said to be a **direct factor** of G. Depending on the context, we can think of G as the *external* direct product of the N_i (each element a k-tuple $(a_1, \ldots, a_k) \in N_1 \times \cdots \times N_k$) or as an *internal* direct product (each element written uniquely in the form $a_1a_2\cdots a_k \in a_k \in G$).

The next theorem is often easier to use than Theorem 9.1 to prove that a group is the direct product of certain of its subgroups. The statement of the theorem uses the following notation. If M and N are subgroups of a group G, then MN denotes the set of all products mn, with $m \in M$ and $n \in N$.

Theorem 9.3

If M and N are normal subgroups of a group G such that $G = MN$ and $M \cap N = \langle e \rangle$, then $G = M \times N$.

For the case of more than two subgroups, see Exercise 25.

Proof of Theorem 9.3 ▸ By hypothesis every element of G is of the form mn, with $m \in M$, $n \in N$. Suppose that an element had two such representations, say $mn = m_1n_1$, with $m, m_1 \in M$ and $n, n_1 \in N$. Then

$$\begin{aligned} mn &= m_1n_1 \\ m_1^{-1}mn &= m_1^{-1}m_1n_1 && \textit{[Left multiply both sides by } m_1^{-1}.] \\ m_1^{-1}mn &= n_1 \\ m_1^{-1}mnn^{-1} &= n_1n^{-1} && \textit{[Right multiply both sides by } n^{-1}.] \\ m_1^{-1}m &= n_1n^{-1} \end{aligned}$$

But $m_1^{-1}m \in M$ and $n_1n^{-1} \in N$ and $M \cap N = \langle e \rangle$. Thus $m_1^{-1}m = e$ and $m = m_1$; similarly, $n = n_1$. Therefore, every element of G can be written *uniquely* in the form mn ($m \in M$, $n \in N$), and, hence, $G = M \times N$ by Theorem 9.1. ■

EXAMPLE 3

By Theorem 2.10, the multiplicative group of units in $\mathbb{Z}_{15}$ is $U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$. The groups $M = \{1, 11\}$ and $N = \{1, 2, 4, 8\}$ are normal subgroups whose intersection is $\langle 1 \rangle$. Every element of N is in MN (for instance, $2 = 1 \cdot 2$), and similarly for M. Since $11 \cdot 2 = 7$, $11 \cdot 8 = 13$, and $11 \cdot 4 = 14$, we see that $U_{15} = MN$. Therefore, $U_{15} = M \times N$ by Theorem 9.3. Since N is cyclic of order 2 and M cyclic of order 4 (2 is a generator), we conclude that U_{15} is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$ (see Exercise 10 and Theorem 7.19).

Exercises

NOTE: *Unless stated otherwise, $G_1, \ldots, G_n$ are groups.*

A. 1. Find the order of each element in the given group:

(a) $\mathbb{Z}_2 \times \mathbb{Z}_4$ (b) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ (c) $D_4 \times \mathbb{Z}_2$

2. What is the order of the group $U_5 \times U_6 \times U_7 \times U_8$?

3. (a) List all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$. (There are more than two.)

(b) Do the same for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

4. If G and H are groups, prove that $G \times H \cong H \times G$.

5. Give an example to show that the direct product of cyclic groups need not be cyclic.

6. (a) Write $\mathbb{Z}_{12}$ as a direct sum of two of its subgroups.

   (b) Do the same for $\mathbb{Z}_{15}$.

   (c) Write $\mathbb{Z}_{30}$ in three different ways as a direct sum of two or more of its subgroups. [*Hint:* Theorem 9.3.]

7. Let $G_1, \ldots, G_n$ be groups. Prove that $G_1 \times \cdots \times G_n$ is abelian if and only if every G_i is abelian.

8. Let i be an integer with $1 \leq i \leq n$. Prove that the function

$$\pi_i: G_1 \times G_2 \times \cdots \times G_n \to G_i$$

   given by $\pi_i(a_1, a_2, a_3, \ldots, a_n) = a_i$ is a surjective homomorphism of groups.

9. Is $\mathbb{Z}_8$ isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$?

B. 10. (a) If $f: G_1 \to H_1$ and $g: G_2 \to H_2$ are isomorphisms of groups, prove that the map $\theta: G_1 \times G_2 \to H_1 \times H_2$ given by $\theta(a, b) = (f(a), g(b))$ is an isomorphism.

   (b) If $G_i \cong H_i$ for $i = 1, 2, \ldots, n$, prove that

$$G_1 \times \cdots \times G_n \cong H_1 \times \cdots \times H_n.$$

11. Let H, K, M, N be groups such that $K \cong M \times N$. Prove that $H \times K \cong H \times M \times N$.

12. Let i be an integer with $1 \leq i \leq n$. Let $\overline{G_i}$ be the subset of $G_1 \times \cdots \times G_n$ consisting of those elements whose ith coordinate is any element of G_i and whose other coordinates are each the identity element, that is,

$$\overline{G_i} = \{(e_1, \ldots, e_{i-1}, a_i, e_{i+1}, \ldots, e_n) \mid a_i \in G_i\}.$$

   Prove that

   (a) $\overline{G_i}$ is a normal subgroup of $G_1 \times \cdots \times G_n$.

   (b) $\overline{G_i} \cong G_i$.

   (c) $G_1 \times \cdots \times G_n$ is the (internal) direct product of its subgroups $\overline{G_1}, \ldots, \overline{G_n}$. [*Hint:* Show that every element of $G_1 \times \cdots \times G_n$ can be written uniquely in the form $a_1 a_2 \cdots a_n$, with $a_i \in \overline{G_i}$; apply Theorem 9.1.]

13. Let G be a group and let $D = \{(a, a, a) \mid a \in G\}$.

   (a) Prove that D is a subgroup of $G \times G \times G$.

   (b) Prove that D is normal in $G \times G \times G$ if and only if G is abelian.

14. If $G_1, \ldots, G_n$ are finite groups, prove that the order of $(a_1, a_2, \ldots, a_n)$ in $G_1 \times \cdots \times G_n$ is the least common multiple of the orders $|a_1|, |a_2|, \ldots, |a_n|$.

15. Let $i_1, i_2, \ldots, i_n$ be a permutation of the integers $1, 2, \ldots, n$. Prove that

$$G_{i_1} \times G_{i_2} \times \cdots \times G_{i_n}$$

is isomorphic to

$$G_1 \times G_2 \times \cdots \times G_n.$$

[Exercise 4 is the case $n = 2$.]

16. If N, K are subgroups of a group G such that $G = N \times K$ and M is a normal subgroup of N, prove that M is a normal subgroup of G. [Compare this with Exercise 14 in Section 8.2.]

17. Let $\mathbb{Q}^*$ be the multiplicative group of nonzero rational numbers, $\mathbb{Q}^{**}$ the subgroup of positive rationals, and H the subgroup $\{1, -1\}$. Prove that $\mathbb{Q}^* = \mathbb{Q}^{**} \times H$.

18. Prove that U_{16} is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$ [*Hint:* Theorem 9.3.]

19. Let G be a group and $f_1: G \to G_1, f_2: G \to G_2, \ldots, f_n: G \to G_n$ homomorphisms. For $i = 1, 2, \ldots, n$, let π_i be the homomorphism of Exercise 8. Let $f^*: G \to G_1 \times \cdots \times G_n$ be the map defined by $f^*(a) = (f_1(a_1), f_2(a_2), \ldots, f_n(a_n))$.
    (a) Prove that f^* is a homomorphism such that $\pi_i \circ f^* = f_i$ for each i.
    (b) Prove that f^* is the unique homomorphism from G to $G_1 \times \cdots \times G_n$ such that $\pi_i \circ f^* = f_i$ for every i.

20. Let $N_1, \ldots, N_k$ be subgroups of an abelian group G. Assume that every element of G can be written in the form $a_1 \cdots a_n$ (with $a_i \in N_i$) and that whenever $a_1 a_2 \cdots a_n = e$, then $a_i = e$ for every i. Prove that $G = N_1 \times N_2 \times \cdots \times N_k$.

21. Let G be an additive abelian group with subgroups H and K. Prove that $G = H \times K$ if and only if there are homomorphisms

$$H \underset{\delta_1}{\overset{\pi_1}{\leftrightarrows}} G \underset{\delta_2}{\overset{\pi_2}{\rightleftarrows}} K$$

such that $\delta_1(\pi_1(x)) + \delta_2(\pi_2(x)) = x$ for every $x \in G$ and $\pi_1 \circ \delta_1 = \iota_H, \pi_2 \circ \delta_2 = \iota_K$, $\pi_1 \circ \delta_2 = 0$, and $\pi_2 \circ \delta_1 = 0$, where ι_X is the identity map on X, and 0 is the map that sends every element onto the zero (identity) element. [*Hint:* Let π_i be as in Exercise 8.]

22. Let G and H be finite cyclic groups. Prove that $G \times H$ is cyclic if and only if $(|G|, |H|) = 1$.

23. (a) Show by example that Lemma 9.2 may be false if N is not normal.
    (b) Do the same for Theorem 9.3.

24. Let N, K be subgroups of a group G, with N normal in G. If N and K are abelian groups and $G = NK$, is G the direct product of N and K?

25. Let $N_1, \ldots, N_k$ be normal subgroups of a group G. Let $N_1 N_2 \cdots N_k$ denote the set of all elements of the form $a_1 a_2 \cdots a_k$ with $a_j \in N_j$. Assume that $G = N_1 N_2 \cdots N_k$ and that

$$N_i \cap (N_1 \cdots N_{i-1} N_{i+1} \cdots N_k) = \langle e \rangle$$

for each i $(1 \leq i \leq n)$. Prove that $G = N_1 \times N_2 \times \cdots \times N_k$.

26. Let $N_1, \ldots, N_k$ be normal subgroups of a finite group G. If $G = N_1N_2 \cdots N_k$ (notation as in Exercise 25) and $|G| = |N_1| \cdot |N_2| \cdots |N_k|$, prove that $G = N_1 \times N_2 \times \cdots \times N_k$.

27. Let N, H be subgroups of a group G. G is called the **semidirect product** of N and H if N is normal in G, $G = NH$, and $N \cap H = \langle e \rangle$. Show that each of the following groups is the semidirect product of two of its subgroups:

(a) S_3 **(b)** D_4 **(c)** S_4

28. A group G is said to be **indecomposable** if it is *not* the direct product of two of its proper normal subgroups. Prove that each of these groups is indecomposable:

(a) S_3 **(b)** D_4 **(c)** $\mathbb{Z}$

29. If p is prime and n is a positive integer, prove that $\mathbb{Z}_{p^n}$ is indecomposable.

30. Prove that $\mathbb{Q}$ is an indecomposable group.

31. Show by example that a homomorphic image of an indecomposable group need not be indecomposable.

32. Prove that a group G is indecomposable if and only if whenever H and K are normal subgroups such that $G = H \times K$, then $H = \langle e \rangle$ or $K = \langle e \rangle$.

33. Let I be the set of positive integers and assume that for each $i \in I$, G_i is a group.* The **infinite direct product** of the G_i is denoted $\prod_{i \in I} G_i$ and consists of all sequences $(a_1, a_2, \ldots)$ with $a_i \in G_i$. Prove that $\prod_{i \in I} G_i$ is a group under the coordinatewise operation

$$(a_1, a_2, \ldots)(b_1, b_2, \ldots) = (a_1b_1, a_2b_2, \ldots).$$

C. 34. With the notation as in Exercise 33, let $\sum_{i \in I} G_i$ denote the subset of $\prod_{i \in I} G_i$ consisting of all sequences $(c_1, c_2, \ldots)$ such that there are at most a finite number of coordinates with $c_j \neq e_j$, where e_j is the identity element of G_j. Prove that $\sum_{i \in I} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$. $\sum_{i \in I} G_i$ is called the **infinite direct sum** of the G_i.

35. Let G be a group and assume that for each positive integer i, N_i is a normal subgroup of G. If every element of G can be written uniquely in the form $n_{i_1} \cdot n_{i_2} \cdots n_{i_k}$, with $i_1 < i_2 < \cdots < i_k$ and $n_{i_j} \in N_{i_j}$, prove that $G \cong \sum_{i \in I} N_i$ (see Exercise 34).† [*Hint:* Adapt the proof of Theorem 9.1 by defining $f(a_1, a_2, \ldots)$ to be the product of those a_i that are not the identity element.]

36. If $(m, n) = 1$, prove that $U_{mn} \cong U_m \times U_n$.

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*Any infinite index set I may be used here, but the restriction to the positive integers simplifies the notation.

†Uniqueness means that if $a_{i_1} \cdots a_{i_k} = b_{j_1} \cdots b_{j_t}$, with $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_t$, then $k = t$ and for $r = 1, 2, \ldots, k$: $i_r = j_r$ and $a_{i_r} b_{i_r}$.

37. Let H be a group and $\tau_1: H \to G_1$, $\tau_2: H \to G_2, \ldots, \tau_n: H \to G_n$ homomorphisms with this property: Whenever G is a group and $g_1: G \to G_1$, $g_2: G \to G_2, \ldots,$ $g_n: G \to G_n$ are homomorphisms, then there exists a unique homomorphism $g^*: G \to H$ such that $\tau_i \circ g^* = g_i$ for every i. Prove that $H \cong G_1 \times G_2 \times \cdots \times G_n$. [See Exercise 19.]

9.2 Finite Abelian Groups

All finite abelian groups will now be classified. We shall prove that every finite abelian group G is a direct sum of cyclic subgroups and that the orders of these cyclic subgroups are uniquely determined by G. The only prerequisites for the proof other than Section 9.1 are basic number theory (Section 1.2) and elementary group theory (Chapters 7 and 8, omitting Sections 7.5 and 8.5).

Following the usual custom with abelian groups, **all groups are written in additive notation** in this section. The following dictionary may be helpful for translating from multiplicative to additive notation:

MULTIPLICATIVE NOTATION	ADDITIVE NOTATION
ab	$a + b$
e	0
a^k	ka
$a^k = e$	$ka = 0$
$MN = \{mn \mid m \in M, n \in N\}$	$M + N = \{m + n \mid m \in M, n \in N\}$
direct product $M \times N$	direct sum $M \oplus N$
direct factor M	direct summand M

Here is a restatement in additive notation of several earlier results that will be used frequently here:

Theorem 7.9

Let G be an additive group and let $a \in G$.

(1) If a has order n, then $ka = 0$ if and only if $n \mid k$.

(3) If a has order td, with $d > 0$, then ta has order d. ■

Theorem 9.1

If $N_1, \ldots, N_k$ are normal subgroups of an additive group G such that every element of G can be written uniquely in the form $a_1 + a_2 + \cdots + a_k$ with $a_i \in N_i$, then $G \cong N_1 \oplus N_2 \oplus \cdots \oplus N_k$. ■

Theorem 9.3

If M and N are normal subgroups of an additive group G such that $G = M + N$ and $M \cap N = \langle 0 \rangle$, then $G = M \oplus N$. ∎

Finally we note that Exercise 11 of Section 9.1 will be used without explicit mention at several points.

If G is an abelian group and p is a prime, then $\boldsymbol{G(p)}$ denotes the set of elements in G whose order is some power of p; that is,

$$G(p) = \{a \in G \mid |a| = p^n \text{ for some } n \geq 0\}.$$

It is easy to verify that $G(p)$ is closed under addition and that the inverse of any element in $G(p)$ is also in $G(p)$ (Exercise 1). Therefore, $G(p)$ is a subgroup of G.

EXAMPLE 1

If $G = \mathbb{Z}_{12}$, then $G(2)$ is the set of elements having orders $2^0, 2^1, 2^2$, etc. Verify that $G(2)$ is the subgroup $\{0, 3, 6, 9\}$; similarly, $G(3) = \{0, 4, 8\}$. If $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3$, then $G(3) = G$ since every nonzero element in G has order 3.

The first step in proving that a finite abelian group G is the direct sum of cyclic subgroups is to show that G is the direct sum of its subgroups $G(p)$, one for each of the distinct primes dividing the order of G. In order to do this, we need

Lemma 9.4

Let G be an abelian group and $a \in G$ an element of finite order. Then $a = a_1 + a_2 + \cdots + a_t$, with $a_i \in G(p_i)$, where $p_1, \ldots, p_t$ are the distinct positive primes that divide the order of a.

Proof ▸ The proof is by induction on the number of distinct primes that divide the order of a. If $|a|$ is divisible only by the single prime p_1, then the order of a is a power of p_1 and, hence, $a \in G(p_1)$. So the lemma is true in this case. Assume inductively that the lemma is true for all elements whose order is divisible by at most $k - 1$ distinct primes and that $|a|$ is divisible by the distinct primes $p_1, \ldots, p_k$. Then $|a| = p_1^{r_1} \cdots p_k^{r_k}$, with each $r_i > 0$. Let $m = p_2^{r_2} \cdots p_k^{r_k}$ and $n = p_1^{r_1}$, so that $|a| = mn$. Then $(m, n) = 1$ and by Theorem 1.2 there are integers u, v such that $1 = mu + nv$. Consequently,

$$a = 1a = (mu + nv)a = mua + nva.$$

But $mua \in G(p_1)$ because a has order mn, and, hence, $p_1^{r_1}(mua) = (nm)ua = u(mna) = u0 = 0$. Similarly, $m(nva) = 0$ so that by Theorem 7.9 the order of nva divides m, an integer with only $k - 1$ distinct prime divisors. Therefore, by the induction assumption $nva = a_2 + a_3 + \cdots + a_k$, with $a_i \in G(p_i)$. Let $a_1 = mua$; then $a = mua + nva = a_1 + a_2 + \cdots + a_k$, with $a_i \in G(p_i)$. ∎

Theorem 9.5

If G is a finite abelian group, then

$$G = G(p_1) \oplus G(p_2) \oplus \cdots \oplus G(p_t),$$

where $p_1, \ldots, p_t$ are the distinct positive primes that divide the order of G.

Proof ▸ If $a \in G$, then its order divides $|G|$ by Corollary 8.6. Hence, $a = a_1 + \cdots + a_t$, with $a_i \in G(p_i)$ by Lemma 9.4 (where $a_j = 0$ if the prime p_j does not divide $|a|$). To prove that this expression is unique, suppose that $a_1 + a_2 + \cdots + a_t = b_1 + b_2 + \cdots + b_t$, with $a_i, b_i \in G(p_i)$. Since G is abelian

$$a_1 - b_1 = (b_2 - a_2) + (b_3 - a_3) + \cdots + (b_t - a_t).$$

For each i, $b_i - a_i \in G(p_i)$ and, hence, has order a power of p_i, say $p_i^{r_i}$. If $m = p_2^{r_2} \cdots p_t^{r_t}$, then $m(b_i - a_i) = 0$ for $i \geq 2$, so that

$$m(a_1 - b_1) = m(b_2 - a_2) + \cdots + m(b_t - a_t) = 0 + \cdots + 0 = 0.$$

Consequently, the order of $a_1 - b_1$ must divide m by Theorem 7.9. But $a_1 - b_1 \in G(p_1)$, so its order is a power of p_1. The only power of p_1 that divides $m = p_2^{r_2} \cdots p_t^{r_t}$ is $p_1^0 = 1$. Therefore, $a_1 - b_1 = 0$ and $a_1 = b_1$. Similar arguments for $i = 2, \ldots, t$ show that $a_i = b_i$ for every i. Therefore, every element of G can be written uniquely in the form $a_1 + \cdots + a_t$, with $a_i \in G(p_i)$ and, hence, $G = G(p_1) \oplus \cdots \oplus G(p_t)$ by Theorem 9.1. ■

If p is a prime, then a group in which every element has order a power of p is called a ***p*-group**. Each of the $G(p_i)$ in Theorem 9.5 is a p-group by its very definition. An element a of a p-group B is called an **element of maximal order** if $|b| \leq |a|$ for every $b \in B$. If $|a| = p^n$ and $b \in B$, then b has order p^j with $j \leq n$. Since $p^n = p^j p^{n-j}$ we see that $p^n b = p^{n-j}(p^j b) = 0$. Hence,

If a is an element of maximal order p^n in a p-group B, then $p^n b = 0$ for every $b \in B$.

Note that elements of maximal order always exist in a *finite* p-group.

The next step in classifying finite abelian groups is to prove that every finite abelian p-group has a cyclic direct summand, after which we will be able to prove that every finite abelian p-group is a direct sum of cyclic groups.

Lemma 9.6

Let G be a finite abelian p-group and a an element of maximal order in G. Then there is a subgroup K of G such that $G = \langle a \rangle \oplus K$.

The following proof is more intricate than most of the proofs earlier in the book. Nevertheless, it uses only elementary group theory, so if you read it carefully, you shouldn't have trouble following the argument.

Proof of Lemma 9.6 ▸ Consider those subgroups H of G such that $\langle a \rangle \cap H = \langle 0 \rangle$. There is at least one ($H = \langle 0 \rangle$), and since G is finite, there must be a largest subgroup K with this property. Then $\langle a \rangle \cap K = \langle 0 \rangle$, and by Theorem 9.3 we need only show that $G = \langle a \rangle + K$. If this is *not* the case, then there is a nonzero b such that $b \notin \langle a \rangle + K$. Let k be the smallest positive integer such that $p^k b \in \langle a \rangle + K$ (there must be one since G is a p-group and, hence, $p^j b = 0 = 0 + 0 \in \langle a \rangle + K$ for some positive j). Then

(1) $$c = p^{k-1}b \quad \text{is } \textit{not} \text{ in} \quad \langle a \rangle + K$$

and $pc = p^k b$ is in $\langle a \rangle + K$, say

(2) $$pc = ta + k \qquad (t \in \mathbb{Z}, k \in K).$$

If a has order p^n, then $p^n x = 0$ for all $x \in G$ because a has maximal order. Consequently, by **(2)**

$$p^{n-1}ta + p^{n-1}k = p^{n-1}(ta + k) = p^{n-1}(pc) = p^n c = 0.$$

Therefore, $p^{n-1}ta = -p^{n-1}k \in \langle a \rangle \cap K = \langle 0 \rangle$ and $p^{n-1}ta = 0$. Theorem 7.9 shows that p^n (the order of a) divides $p^{n-1}t$, and it follows that $p \mid t$, say $t = pm$. Therefore, $pc = ta + k = pma + k$, and consequently, $k = pc - pma = p(c - ma)$. Let

(3) $$d = c - ma.$$

Then $pd = p(c - ma) = k \in K$, but $d \notin K$ (since $c - ma = k' \in K$ would imply that $c = ma + k' \in \langle a \rangle + K$, contradicting (1)). Use Theorem 7.12 to verify that $H = \{x + zd \mid x \in K, z \in \mathbb{Z}\}$ is a subgroup of G with $K \subseteq H$. Since $d = 0 + 1d \in H$ and $d \notin K$, H is larger than K. But K is the largest group such that $\langle a \rangle \cap K = \langle 0 \rangle$, so we must have $\langle a \rangle \cap H \neq \langle 0 \rangle$. If w is a nonzero element of $\langle a \rangle \cap H$, then

(4) $$w = sa = k_1 + rd \qquad (k_1 \in K; r, s \in \mathbb{Z}).$$

We claim that $p \nmid r$; for if $r = py$, then since $pd \in K$, $0 \neq w = sa = k_1 + ypd \in \langle a \rangle \cap K$, a contradiction. Consequently, $(p, r) = 1$, and by Theorem 1.2 there are integers u, v with $pu + rv = 1$. Then

$$\begin{aligned} c = 1c = (pu + rv)c &= u(pc) + v(rc) \\ &= u(ta + k) + v(r(d + ma)) \quad [\textit{by (2) and (3)}] \\ &= u(ta + k) + v(rd + rma) \\ &= u(ta + k) + v(sa - k_1 + rma) \quad [\textit{by (4)}] \\ &= (ut + vs + rm)a + (uk - vk_1) \in \langle a \rangle + K. \end{aligned}$$

This contradicts **(1)**. Therefore, $G = \langle a \rangle + K$, and, hence, $G = \langle a \rangle \oplus K$ by Theorem 9.3. ■

Theorem 9.7 The Fundamental Theorem of Finite Abelian Groups

Every finite abelian group G is the direct sum of cyclic groups, each of prime power order.

Proof▸ By Theorem 9.5, G is the direct sum of its subgroups $G(p)$, one for each prime p that divides $|G|$. Each $G(p)$ is a p-group. So to complete the proof, we need only show that every finite abelian p-group H is a direct sum of cyclic groups, each of order a power of p. We prove this by induction on the order of H. The assertion is true when H has order 2 by Theorem 8.7. Assume inductively that it is true for all groups whose order is less than $|H|$ and let a be an element of maximal order p^n in H. Then $H = \langle a \rangle \oplus K$ by Lemma 9.6. By induction, K is a direct sum of cyclic groups, each with order a power of p. Therefore, the same is true of $H = \langle a \rangle \oplus K$. ■

EXAMPLE 2

The number 36 can be written as a product of prime powers in just four ways: $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2 \cdot 2 \cdot 3^2 = 2^2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$. Consequently, by Theorem 9.7 every abelian group of order 36 must be isomorphic to one of the following groups:

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3, \qquad \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9, \qquad \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3, \qquad \mathbb{Z}_4 \oplus \mathbb{Z}_9.$$

You can easily verify that no two of these groups are isomorphic (the number of elements of order 2 or 3 is different for each group). Thus we have a complete classification of all abelian groups of order 36 up to isomorphism.

You probably noticed that a familiar group of order 36, namely $\mathbb{Z}_{36}$, doesn't appear explicitly on the list in the preceding example. However, it is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_9$, as we now prove.

Lemma 9.8

If $(m, k) = 1$, then $\mathbb{Z}_m \oplus \mathbb{Z}_k \cong \mathbb{Z}_{mk}$.

Proof▸ The order of $(1, 1)$ in $\mathbb{Z}_m \oplus \mathbb{Z}_k$ is the smallest positive integer t such that $(0, 0) = t(1, 1) = (t, t)$. Thus $t \equiv 0 \pmod{m}$ and $t \equiv 0 \pmod{k}$, so that $m \mid t$ and $k \mid t$. But $(m, k) = 1$ implies that $mk \mid t$ by Exercise 17 in Section 1.2. Hence, $mk \leq t$. Since $mk(1, 1) = (mk, mk) = (0, 0)$ and t is the smallest positive integer with this property, we must have $mk = t = |(1, 1)|$. Therefore, $\mathbb{Z}_m \oplus \mathbb{Z}_k$ (a group of order mk) is the cyclic group generated by $(1, 1)$ and, hence, is isomorphic to $\mathbb{Z}_{mk}$ by Theorem 7.19. ■

Theorem 9.9

If $n = p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$, with $p_1, \ldots, p_t$ distinct primes, then

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{n_t}}.$$

Proof ▸ The theorem is true for groups of order 2. Assume inductively that it is true for groups of order less than n. Apply Lemma 9.8 with $m = p_1^{n_1}$ and $k = p_2^{n_2} \cdots p_t^{n_t}$. Then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_k$, and the induction hypothesis shows that $\mathbb{Z}_k \cong \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{n_t}}$. ■

Combining Theorems 9.7 and 9.9 yields a second way of expressing a finite abelian group as a direct sum of cyclic groups.

EXAMPLE 3

Consider the group

$$G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{25}.$$

Arrange the prime power orders of the cyclic factors by size, with one row for each prime:

$$\begin{array}{cccc} 2 & 2 & 2^2 & 2^3 \\ & 3 & 3 & 3 \\ & & 5 & 5^2 \end{array}$$

Now rearrange the cyclic factors of G using the *columns* of this array as a guide (see Exercise 15 of Section 9.1) and apply Theorem 9.9:

$$\begin{aligned} G &\cong (\mathbb{Z}_2) \oplus \underbrace{(\mathbb{Z}_2 \oplus \mathbb{Z}_3)} \oplus \underbrace{(\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5)} \oplus \underbrace{(\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25})} \\ G &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{600}. \end{aligned}$$

This last decomposition of G as a sum of cyclic groups is sometimes more convenient than the original prime power decomposition: There are fewer cyclic factors, and the order of each cyclic factor *divides* the order of the next one. Although the notation is a bit more involved, the same process works in the general case and proves the following Theorem.

Theorem 9.10

Every finite abelian group is the direct sum of cyclic groups of orders $m_1, m_2, \ldots, m_t$, where $m_1 \mid m_2$, $m_2 \mid m_3$, $m_3 \mid m_4, \ldots,$ and $m_{t-1} \mid m_t$.

We pause briefly here to present an interesting corollary that will be used in Chapter 11. A version of it was proved earlier as Theorem 7.16.

Corollary 9.11

If G is a finite subgroup of the multiplicative group of nonzero elements of a field F, then G is cyclic.*

Proof ▸ Since G is a finite abelian group, Theorem 9.10 implies that $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}$, where each m_i divides m_t. Every element b in $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}$ satisfies $m_t b = 0$ (Why?). Consequently, every element g of the multiplicative group G must satisfy $g^{m_t} = 1_F$ (that is, must be a solution of the equation $x^{m_t} - 1_F = 0$). Since G has order $m_1 m_2 \cdots m_t$ and $x^{m_t} - 1_F = 0$ has at most m_t distinct solutions in F by Corollary 4.17, we must have $t = 1$ and $G \cong \mathbb{Z}_{m_t}$. ■

If G is a finite abelian group, then the integers $m_1, \ldots, m_t$ in Theorem 9.10 are called the **invariant factors** of G. When G is written as a direct sum of cyclic groups of prime power orders, as in Theorem 9.7, the prime powers are called the **elementary divisors** of G. Theorems 9.7 and 9.10 show that the order of G is the product of its elementary divisors and also the product of its invariant factors.

EXAMPLE 4

All abelian groups of order 36 can be classified up to isomorphism in terms of their elementary divisors (as in Example 2) or in terms of their invariant factors (using the procedure in Example 3):

GROUP	ELEMENTARY DIVISORS	INVARIANT FACTORS	ISOMORPHIC GROUP
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	2, 2, 3, 3	6, 6	$\mathbb{Z}_6 \oplus \mathbb{Z}_6$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9$	2, 2, 3^2	2, 18	$\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$
$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	2^2, 3, 3	3, 12	$\mathbb{Z}_3 \oplus \mathbb{Z}_{12}$
$\mathbb{Z}_4 \oplus \mathbb{Z}_9$	2^2, 3^2	36	$\mathbb{Z}_{36}$

The Fundamental Theorem 9.7 can be used to obtain a list of all possible abelian groups of a given order. To complete the classification of such groups, we must show that no two groups on the list are isomorphic, that is, that the elementary divisors of a group are uniquely determined.†

Theorem 9.12

Let G and H be finite abelian groups. Then G is isomorphic to H if and only if G and H have the same elementary divisors.

*If you have not read Sections 3.1 and 4.4, skip this corollary until you have.

†The remainder of this section is optional. Theorem 9.12 is often considered to be part of the Fundamental Theorem of Finite Abelian Groups.

It is also true that $G \cong H$ if and only if G and H have the same invariant factors (Exercise 24).

Proof of Theorem 9.12 ▸ If G and H have the same elementary divisors, then both G and H are isomorphic to the same direct sum of cyclic groups and, hence, are isomorphic to each other. Conversely, if $f: G \to H$ is an isomorphism, then a and $f(a)$ have the same order for each $a \in G$. It follows that for each prime p, $f(G(p)) = H(p)$ and, hence, $G(p) \cong H(p)$. The elementary divisors of G that are powers of the prime p are precisely the elementary divisors of $G(p)$, and similarly for H. So we need only prove that isomorphic p-groups have the same elementary divisors. In other words, we need to prove this half of the theorem only when G and H are p-groups.

Assume G and H are isomorphic p-groups. We use induction on the order of G to prove that G and H have the same elementary divisors. All groups of order 2 obviously have the same elementary divisor, 2, by Theorem 8.7. So assume that the statement is true for all groups of order less than $|G|$. Suppose that the elementary divisors of G are

$$p^{n_1}, p^{n_2}, \ldots, p^{n_t}, \underbrace{p, p, \ldots, p}_{r \text{ copies}} \qquad \text{with } n_1 \geq n_2 \geq \cdots \geq n_t > 1$$

and that the elementary divisors of H are

$$p^{m_1}, p^{m_2}, \ldots, p^{m_k}, \underbrace{p, p, \ldots, p}_{s \text{ copies}} \qquad \text{with } m_1 \geq m_2 \geq \cdots \geq m_k > 1.$$

Verify that $pG = \{px \mid x \in G\}$ is a subgroup of G (Exercise 2). If G is the direct sum of groups C_i, verify that pG is the direct sum of the groups pC_i (Exercise 4). If C_i is cyclic with generator a of order p^n, then pC_i is the cyclic group generated by pa. Since pa has order p^{n-1} by part (3) of Theorem 7.9, pC_i is cyclic of order p^{n-1}. Note that when $n = 1$ (that is, when C_i is cyclic of order p), then $pC_i = \langle 0 \rangle$. Consequently, the elementary divisors of pG are

$$p^{n_1-1}, p^{n_2-1}, \ldots, p^{n_t-1}.$$

A similar argument shows that the elementary divisors of pH are

$$p^{m_1-1}, p^{m_2-1}, \ldots, p^{m_k-1}.$$

If $f: G \to H$ is an isomorphism, verify that $f(pG) = pH$ so that $pG \cong pH$. Furthermore, $pG \neq G$ (Exercise 9), so that $|pG| < |G|$. Hence pG and pH have the same elementary divisors by the induction hypothesis; that is, $t = k$ and

$$p^{n_i-1} = p^{m_i-1}, \qquad \text{so that } n_i - 1 = m_i - 1 \text{ for } i = 1, 2, \ldots, t.$$

Therefore, $n_i = m_i$ for each i. So the only possible difference in elementary divisors of G and H is the number of copies of p that appear on each list. Since $|G|$ is the product of its elementary divisors, and similarly for $|H|$, and since $G \cong H$, we have

$$p^{n_1}p^{n_2} \cdots p^{n_t}p^r = |G| = |H| = p^{m_1}p^{m_2} \cdots p^{m_k}p^s.$$

Since $m_i = n_i$ for each i, we must have $p^r = p^s$ and, hence, $r = s$. Thus G and H have the same elementary divisors. ■

Exercises

NOTE: *All groups are written additively, and p always denotes a positive prime, unless noted otherwise.*

A. **1.** If G is an abelian group, prove that $G(p)$ is a subgroup.

2. If G is an abelian group, prove that $pG = \{px \mid x \in G\}$ is a subgroup of G.

3. List all abelian groups (up to isomorphism) of the given order:

(a) 12 **(b)** 15 **(c)** 30 **(d)** 72

(e) 90 **(f)** 144 **(g)** 600 **(h)** 1160

4. If G and G_i $(1 \leq i \leq n)$ are abelian groups such that $G = G_1 \oplus \cdots \oplus G_n$, show that $pG = pG_1 \oplus \cdots \oplus pG_n$.

5. Find the elementary divisors of the given group:

(a) $\mathbb{Z}_{250}$ **(b)** $\mathbb{Z}_6 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$

(c) $\mathbb{Z}_{10} \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{40}$ **(d)** $\mathbb{Z}_{12} \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{100} \oplus \mathbb{Z}_{240}$

6. Find the invariant factors of each of the groups in Exercise 5.

B. **7.** Find the elementary divisors and the invariant factors of the given group. Note that the group operation is multiplication in the first three and addition in the last.

(a) U_8 **(b)** U_{17} **(c)** U_{15} **(d)** $M(\mathbb{Z}_2)$

8. If G is the additive group $\mathbb{Q}/\mathbb{Z}$, what are the elements of the subgroup $G(2)$? Of $G(p)$ for any positive prime p?

9. **(a)** If G is a finite abelian p-group, prove that $pG \neq G$.

(b) Show that part (a) may be false if G is infinite. [*Hint:* Consider the group $G(2)$ in Exercise 8.]

10. If G is an abelian p-group and $(n, p) = 1$ prove that the map $f: G \to G$ given by $f(a) = na$ is an isomorphism.

11. If G is a finite abelian p-group such that $pG = \langle 0 \rangle$, prove that $G \cong \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$ for some finite number of copies of $\mathbb{Z}_p$.

12. **(Cauchy's Theorem for Abelian Groups)** If G is a finite abelian group and p is a prime that divides $|G|$, prove that G contains an element of order p. [*Hint:* Use the Fundamental Theorem to show that G has a cyclic subgroup of order p^k; use Theorem 7.9 to find an element of order p.]

13. Prove that a finite abelian p-group has order a power of p.

14. If G is an abelian group of order $p^t m$, with $(p, m) = 1$, prove that $G(p)$ has order p^t.

15. If G is a finite abelian group and p is a prime such that p^n divides $|G|$, then prove that G has a subgroup of order p^n.

16. For which positive integers n is there exactly one abelian group of order n (up to isomorphism)?

17. Let G, H, K be finite abelian groups.

(a) If $G \oplus G \cong H \oplus H$, prove that $G \cong H$.

(b) If $G \oplus H \cong G \oplus K$, prove that $H \cong K$.

18. If G is an abelian group of order n and $k \mid n$, prove that there exist a group H of order k and a surjective homomorphism $G \to H$.

19. Let G be an abelian group and T the set of elements of finite order in G. Prove that

(a) T is a subgroup of G (called the **torsion subgroup**).

(b) Every nonzero element of the quotient group G/T has infinite order.

20. If G is an abelian group, do the elements of infinite order in G (together with 0) form a subgroup? [*Hint:* Consider $\mathbb{Z} \oplus \mathbb{Z}_3$.]

C. 21. If G is an abelian group and $f: G \to \mathbb{Z}$ a surjective homomorphism with kernel K, prove that G has a subgroup H such that $H \cong \mathbb{Z}$ and $G = K \oplus H$.

22. Let G and H be finite abelian groups with this property: For each positive integer m the number of elements of order m in G is the same as the number of elements of order m in H. Prove that $G \cong H$.

23. Let G be finite abelian group with this property: For each positive integer m such that $m \mid |G|$, there are exactly m elements in G with order dividing m. Prove that G is cyclic.

24. Let G and H be finite abelian groups. Prove that $G \cong H$ if and only if G and H have the same invariant factors.

25. If G is an infinite abelian **torsion group** (meaning that every element in G has finite order), prove that G is the infinite direct sum $\Sigma\, G(p)$, where the sum is taken over all positive primes p. [*Hint:* See Exercises 34 and 35 in Section 9.1 and adapt the proof of Theorem 9.5.]

9.3 The Sylow Theorems

Nonabelian finite groups are vastly more complicated than finite abelian groups, which were classified in the last section. The Sylow Theorems are the first basic step in understanding the structure of nonabelian finite groups. Since the proofs of these theorems are largely unrelated to the way the theorems are actually used to analyze groups, the proofs will be postponed to the next section.* In this section we shall try to give you a sound understanding of the meaning of the Sylow Theorems and some examples of their applications.

Throughout the general discussion in this section *all groups are written multiplicatively* and all integers are assumed to be nonnegative.

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*Puritans who believe that the work *must* come before the fun should read Section 9.4 before proceeding further.

Once again the major theme is the close connection between the structure of a group G and the arithmetical properties of the integer $|G|$. One of the most important results of this sort is Lagrange's Theorem, which states that if G has a subgroup H, then the integer $|H|$ divides $|G|$. The First Sylow Theorem provides a partial converse:

Theorem 9.13 First Sylow Theorem

Let G be a finite group. If p is a prime and p^k divides $|G|$, then G has a subgroup of order p^k. ■

EXAMPLE 1

The symmetric group S_6 has order $6! = 720 = 2^4 \cdot 3^2 \cdot 5$. The First Sylow Theorem (with $p = 2$) guarantees that S_6 has subgroups of orders 2, 4, 8, and 16. There may well be more than one subgroup of each of these orders. For instance, there are at least 60 subgroups of order 4 (Exercise 1). Applying the theorem with $p = 3$ shows that S_6 has subgroups of orders 3 and 9. Similarly, S_6 has at least one subgroup of order 5.

If p is a prime that divides the order of a group G, then G contains a subgroup K of order p by the First Sylow Theorem. Since K is cyclic by Theorem 8.7, its generator is an element of order p in G. This proves

Corollary 9.14 Cauchy's Theorem

If G is a finite group whose order is divisible by a prime p, then G contains an element of order p. ■

Let G be a finite group and p a prime. If p^n is the largest power of p that divides $|G|$, then a subgroup of G of order p^n is called a **Sylow p-subgroup.** The existence of Sylow p-subgroups is an immediate consequence of the First Sylow Theorem.

EXAMPLE 2

Since S_4 has order $4! = 24 = 2^3 \cdot 3$, every subgroup of order 8 is a Sylow 2-subgroup. You can readily verify that

$$\{(1), (1234), (13)(24), (1432), (24), (12)(34), (13), (14)(32)\}$$

is a subgroup of order 8 and, hence, a Sylow 2-subgroup. There are two other Sylow 2-subgroups (Exercise 2). Any subgroup of S_4 of order 3 is a Sylow 3-subgroup. Two of the four Sylow 3-subgroups are $\{(123), (132), (1)\}$ and $\{(134), (143), (1)\}$.

EXAMPLE 3*

Let p be a prime and G a finite *abelian* group of order $p^n m$, where $p \nmid m$. Then

$$G(p) = \{a \in G \mid |a| = p^k \text{ for some } k \geq 0\}$$

is a Sylow p-subgroup of G since $G(p)$ has order p^n by Exercise 14 of Section 9.2. As we shall see, $G(p)$ is the unique Sylow p-subgroup of G. Theorem 9.5 shows that G is the direct sum of all its Sylow subgroups (one for each of the distinct primes that divide $|G|$).

Let G be a group and $x \in G$. Example 9 of Section 7.4 shows that the map $f: G \to G$ given by $f(a) = x^{-1}ax$ is an isomorphism. If K is a subgroup of G, then the image of K under f is $x^{-1}Kx = \{x^{-1}kx \mid k \in K\}$. Hence, *$x^{-1}Kx$ is a subgroup of G that is isomorphic to K*. In particular, $x^{-1}Kx$ has the same order as K. Consequently,

if K is a Sylow p-subgroup of G, then so is $x^{-1}Kx$.

The next theorem shows that every Sylow p-subgroup of G can be obtained from K in this fashion.

Theorem 9.15 Second Sylow Theorem

If P and K are Sylow p-subgroups of a group G, then there exists $x \in G$ such that $P = x^{-1}Kx$. ■

Theorem 9.15, together with the italicized statement in the preceding paragraph, shows that

any two Sylow p-subgroups of G are isomorphic.

Corollary 9.16

Let G be a finite group and K a Sylow p-subgroup for some prime p. Then K is normal in G if and only if K is the only Sylow p-subgroup in G.

Proof ▸ We know that $x^{-1}Kx$ is a Sylow p-subgroup for every $x \in G$. If K is the only Sylow p-subgroup of G, then we must have $x^{-1}Kx = K$ for every $x \in G$. Therefore, K is normal by Theorem 8.11. Conversely, suppose K is normal and let P be any Sylow p-subgroup. By the Second Sylow Theorem there exists $x \in G$ such that $P = x^{-1}Kx$. Since K is normal, $P = x^{-1}Kx = K$. Therefore, K is the unique Sylow p-subgroup. ■

*Skip this example if you haven't read Section 9.2.

The preceding theorems establish the existence of Sylow p-subgroups and the relationship between any two such subgroups. The next theorem tells us how many Sylow p-subgroups a given group may have.

Theorem 9.17 Third Sylow Theorem

The number of Sylow p-subgroups of a finite group G divides $|G|$ and is of the form $1 + pk$ for some nonnegative integer k.

Applications of the Sylow Theorems

Simple groups (those with no proper normal subgroups) are the basic building blocks for all groups. So it is useful to be able to tell if there are any simple groups of a particular order. The Third Sylow Theorem, together with appropriate counting arguments and Corollary 9.16, can often be used to establish the existence of a proper normal subgroup of a group G, thus showing that G is not simple.

EXAMPLE 4

If G is a group of order $63 = 3^2 \cdot 7$, then each Sylow 7-subgroup has order 7 and the number of such subgroups is a divisor of 63 of the form $1 + 7k$ by the Third Sylow Theorem. The divisors of 63 are 1, 3, 7, 9, 21, 63 and the numbers of the form $1 + 7k$ (with $k \geq 0$) are 1, 8, 15, 22, 29, 36, 43, 50, 57, 64, etc. Since 1 is the only number on both lists, G has exactly one Sylow 7-subgroup. This subgroup is normal by Corollary 9.16. Consequently, no group of order 63 is simple.

EXAMPLE 5

We shall show that there is no simple group of order $56 = 2^3 \cdot 7$. The only divisors of 56 of the form $1 + 7k$ are 1 and 8. So G has either one or eight Sylow 7-subgroups, each of order 7. If there is just one Sylow 7-group, it has to be normal by Corollary 9.16. So G is not simple in that case. If G has eight Sylow 7-groups, then each of them has six nonidentity elements, and each nonidentity element has order 7 by Corollary 8.6. Furthermore, the intersection of any two of these subgroups is $\langle e \rangle$ by Exercise 21 of Section 8.1. Consequently, there are $8 \cdot 6 = 48$ elements of order 7 in G. Every Sylow 2-subgroup of G has order 8. Each element of a Sylow 2-subgroup must have order dividing 8 by Corollary 8.6 and, therefore, cannot be in the set of 48 elements of order 7. Thus there is room in G for only one group of order 8. In this case, therefore, the single Sylow 2-subgroup of order 8 is normal by Corollary 9.16, and G is not simple.

In the preceding examples, the Sylow Theorems were used to reach a negative conclusion (the group is not simple). But the same techniques can also lead to positive results. In particular, they allow us to classify certain finite groups.

Corollary 9.18

Let G be a group of order pq, where p and q are primes such that $p > q$. If $q \nmid (p-1)$, then $G \cong \mathbb{Z}_{pq}$.

Proof ▸ By the Third Sylow Theorem, the number of Sylow p-subgroups must divide $|G| = pq$, and hence, must be one of $1, p, q$, or pq. However, the number must also be of the form $1 + pk$ for some integer k. Since $p > q$, we cannot have $q = 1 + pk$. Furthermore, both $p = 1 + pk$ and $pq = 1 + pk$ imply that $p \mid 1$, which is impossible. Therefore, there is exactly one Sylow p-subgroup H of order p, which is normal by Corollary 9.16. A similar argument (using the fact that $q \nmid (p-1)$) shows that there is a unique Sylow q-subgroup K of order q, which is also normal. Since $H \cap K$ is a subgroup of both H and K, its order must divide both $|H| = p$ and $|K| = q$ by Lagrange's Theorem. Hence, $H \cap K = \langle e \rangle$. Exercise 15 shows that $G = HK$. Therefore, $G = H \times K$ by Theorem 9.3. But $H \cong \mathbb{Z}_p$ and $K \cong \mathbb{Z}_q$ by Theorem 8.7. Consequently, by Lemma 9.8, $G = H \times K \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$.* ■

EXAMPLE 6

It is now easy to classify all groups of order $15 = 5 \cdot 3$. Apply Corollary 9.18 with $p = 5$, $q = 3$ to conclude that every group of order 15 is isomorphic to $\mathbb{Z}_{15}$. Similarly, there is a single group (up to isomorphism) for each of these orders: $33 = 11 \cdot 3$, $35 = 7 \cdot 5$, $65 = 13 \cdot 5$, $77 = 11 \cdot 7$, and $91 = 13 \cdot 7$.

Other applications of the Sylow Theorems are given in Section 9.5.

Exercises

NOTE: *Unless stated otherwise, G is a finite group and p is a positive prime.*

A. 1. Show that S_6 has at least 60 subgroups of order 4. [*Hint:* Consider cyclic subgroups generated by a 4-cycle (such as $\langle(1234)\rangle$) or by the product of a 4-cycle and a disjoint transposition (such as $\langle(1234)(56)\rangle$); also look at noncyclic subgroups, such as $\{(1), (12), (34), (12)(34)\}$.]

2. (a) List three Sylow 2-subgroups of S_4.

   (b) List four Sylow 3-subgroups of S_4.

3. List the Sylow 2-subgroups and Sylow 3-subgroups of A_4.

4. List the Sylow 2-subgroups, Sylow 3-subgroups, and Sylow 5-subgroups of $\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{10}$. [Section 9.2 is a prerequisite for this exercise.]

*The proof of Lemma 9.8 is independent of the rest of Section 9.2 and may be read now if you skipped that section.

5. How many Sylow p-subgroups can G possibly have when

   (a) $p = 3$ and $|G| = 72$ (b) $p = 5$ and $|G| = 60$

6. Classify all groups of the given order:

   (a) 115 (b) 143 (c) 391

7. Prove that there are no simple groups of the given order:

   (a) 42 (b) 200 (c) 231 (d) 255

B. 8. Use Cauchy's Theorem to prove that a finite p-group has order p^n for some $n \geq 0$.

9. If N is a normal subgroup of a (not necessarily finite) group G and both N and G/N are p-groups, then prove that G is a p-group.

10. If H is a normal subgroup of G and $|H| = p^k$, show that H is contained in every Sylow p-subgroup of G. [You may assume Exercise 24 in Section 9.4.]

11. If f is an automorphism of G and K is a Sylow p-subgroup of G, is it true that $f(K) = K$?

12. Let K be a Sylow p-subgroup of G and H any subgroup of G. Is $K \cap H$ a Sylow p-subgroup of H? [*Hint:* Consider S_4.]

13. If every Sylow subgroup of G is normal, prove that G is the direct product of its Sylow subgroups (one for each prime that divides $|G|$). A group with this property is said to be **nilpotent.**

14. If p is prime, prove that there are no simple groups of order $2p$.

15. (a) If H and K are subgroups of G, then HK denotes the set $\{hk \in G \mid h \in H, k \in K\}$. If $H \cap K = \langle e \rangle$, prove that $|HK| = |H| \cdot |K|$. [*Hint:* If $hk = h_1k_1$, then $h_1^{-1}h = k_1k^{-1}$.]

    (b) If H and K are any subgroups of G, prove that

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

16. If G is a group of order 60 that has a normal Sylow 3-subgroup, prove that G also has a normal Sylow 5-subgroup.

17. If G is a noncyclic group of order 21, how many Sylow 3-subgroups does G have?

18. If G is a simple group of order 168, how many Sylow 7-subgroups does G have?

19. If p and q are distinct primes, prove that there are no simple groups of order pq.

20. If G has order $p^k m$ with $m < p$, prove that G is not simple.

21. Prove that there are no simple groups of order 30.

22. If p and q are distinct primes, prove that there is no simple group of order p^2q.

23. (a) If $|G| = 105$, prove that G has a subgroup of order 35.

    (b) If $|G| = 375$, prove that G has a subgroup of order 15.

24. Let K be a Sylow p-subgroup of G and N a normal subgroup of G. Prove that $K \cap N$ is a Sylow p-subgroup of N.

C. 25. If p, q, r are primes with $p < q < r$, prove that a group of order pqr has a normal Sylow r-subgroup and, hence, is not simple.

9.4 Conjugacy and the Proof of the Sylow Theorems

Appendix D (Equivalence Relations) is a prerequisite for this section. The proofs of the Sylow Theorems depend heavily on the concept of conjugacy, which we now develop.

Let G be a group and $a, b \in G$. We say that a is **conjugate** to b if there exists $x \in G$ such that $b = x^{-1}ax$. For example, (12) is conjugate to (13) in S_3 because

$$(123)^{-1}(12)(123) = (132)(12)(123) = (13).$$

The key fact about conjugation is

Theorem 9.19

Conjugacy is an equivalence relation on G.

Proof▸ We write $a \sim b$ if a is conjugate to b. *Reflexive:* $a \sim a$ since $a = eae = e^{-1}ae$. *Symmetric:* If $a \sim b$, then $b = x^{-1}ax$ for some x in G. Multiplying on the left by x and on the right by x^{-1} shows that $a = xbx^{-1} = (x^{-1})^{-1}bx^{-1}$. Hence, $b \sim a$. *Transitive:* If $a \sim b$ and $b \sim c$, then $b = x^{-1}ax$ and $c = y^{-1}by$ for some $x, y \in G$. Hence, $c = y^{-1}(x^{-1}ax)y = (y^{-1}x^{-1})\, a(xy) = (xy)^{-1}a(xy)$. Thus $a \sim c$; therefore, $\sim$ is an equivalence relation. ■

The equivalence classes in G under the relation of conjugacy are called **conjugacy classes.** The discussion of equivalence relations in Appendix D shows that

The conjugacy class of an element a consists of all the elements in G that are conjugate to a.

Two conjugacy classes are either disjoint or identical.

The group G is the union of its distinct conjugacy classes.

EXAMPLE 1

The conjugacy class of (12) in S_3 consists of all elements $x^{-1}(12)x$, with $x \in S_3$. A straightforward computation shows that for any $x \in S_3$, $x^{-1}(12)x$ is one of (12), (13), or (23); for instance,

$$(23)^{-1}(12)(23) = (23)(12)(23) = (13)$$
$$(132)^{-1}(12)(132) = (123)(12)(132) = (23).$$

Thus the conjugacy class of (12) is $\{(12), (13), (23)\}$. Similar computations show that there are three distinct conjugacy classes in S_3:

$$\{(1)\} \qquad \{(123), (132)\} \qquad \{(12), (13), (23)\}.$$

Although these conjugacy classes are of different sizes, note that the number of elements in any conjugacy class (1, 2, or 3) is a divisor of 6, the order of S_3. We shall see that this phenomenon occurs in the general case as well.

Let G be a group and $a \in G$. The **centralizer** of a is denoted $C(a)$ and consists of all elements in G that commute with a, that is,

$$C(a) = \{g \in G \mid ga = ag\}.$$

If $G = S_3$ and $a = (123)$, for example, you can readily verify that $C(a) = \{(1), (123), (132)\}$ and that $C(a)$ is a subgroup of S_3. If a is a nonzero rational number in the multiplicative group $\mathbb{Q}^*$, every element of $\mathbb{Q}^*$ commutes with a, so $C(a)$ is the entire group $\mathbb{Q}^*$. These examples are illustrations of

Theorem 9.20

If G is a group and $a \in G$, then $C(a)$ is a subgroup of G.

Proof▸ Since $ea = ae$, we have $e \in C(a)$, so that $C(a)$ is nonempty. If $g, h \in C(a)$, then

$$(gh)a = g(ha) = g(ah) = (ga)h = (ag)h = a(gh).$$

So $gh \in C(a)$, and $C(a)$ is closed. Multiplying $ga = ag$ on both the left and right by g^{-1} shows that $ag^{-1} = g^{-1}a$. Hence, $g \in C(a)$ implies that $g^{-1} \in C(a)$. Therefore, $C(a)$ is a subgroup by Theorem 7.11. ■

The centralizer leads to a very useful fact about the size of conjugacy classes:

Theorem 9.21

Let G be a finite group and $a \in G$. The number of elements in the conjugacy class of a is the index $[G:C(a)]$ and this number divides $|G|$.

Proof▸ For notational convenience, we shall sometimes denote $C(a)$ by C in this proof. Let S be the set of distinct right cosets of C in G, and let T be the conjugacy class of a in G (which consists of the distinct conjugates of a). Define a function $f: S \to T$ by the rule: $f(Cx) = x^{-1}ax$. We shall show below that f is a well-defined bijection of sets, which means that S and T have the same number of elements. The number of elements in S is the number of distinct right cosets of $C(a)$, namely $[G:C(a)]$, and the number of elements in T is the number of distinct conjugates of a. This proves the first part of the theorem. As for the final part, the number $[G:C(a)]$ divides $|G|$ by Lagrange's Theorem 8.5.

Now for the details: Reading each of the following "if and only if" statements in the direction $\Rightarrow$ shows that f is well defined (meaning that $Cx = Cy$ implies $f(Cx) = f(Cy)$):

$$\begin{aligned}
Cx = Cy &\Leftrightarrow xy^{-1} \in C && [\textit{Theorem 8.2}]\\
&\Leftrightarrow (xy^{-1})a = a(xy^{-1}) && [\textit{Definition of } C]\\
&\Leftrightarrow a = (xy^{-1})^{-1}a(xy^{-1}) && [\textit{Left multiply by } (xy^{-1})^{-1}.]\\
&\Leftrightarrow a = yx^{-1}axy^{-1} && [\textit{Corollary 7.6}]\\
&\Leftrightarrow y^{-1}ay = x^{-1}ax && [\textit{Left multiply by } y^{-1} \textit{ and}\\
& && \textit{right multiply by } y.]\\
&\Leftrightarrow f(Cy) = f(Cx) && [\textit{Definition of } f]
\end{aligned}$$

Reading these same statements in the direction $\Leftarrow$ from bottom to top shows that $f(Cx) = f(Cy)$ implies $Cx = Cy$, so that f is injective.* Finally, f is surjective because, given any conjugate $u^{-1}au$ of a, it is the image of the coset Cu. Therefore, f is bijective and the proof is complete. ■

Let G be a finite group and let $C_1, C_2, \ldots, C_t$ be the distinct conjugacy classes of G. Then $G = C_1 \cup C_2 \cup \cdots \cup C_t$. Since distinct conjugacy classes are mutually disjoint,

$$\text{(1)} \qquad |G| = |C_1 \cup C_2 \cup \cdots \cup C_t| = |C_1| + |C_2| + \cdots + |C_t|,$$

where $|C_i|$ denotes the number of elements in the class C_i. Now choose one element, say a_i, in each class C_i. Then C_i consists of all the conjugates of a_i. By Theorem 9.21, $|C_i|$ is precisely $[G:C(a_i)]$, a divisor of $|G|$. So equation **(1)** becomes

$$\text{(2)} \qquad |G| = [G:C(a_1)] + [G:C(a_2)] + \cdots + [G:C(a_t)].$$

This equation (in either version **(1)** or **(2)**) is called the **class equation** of the group G. It will be the basic tool for proving the Sylow Theorems. Other applications of the class equation are discussed in Section 9.5.

EXAMPLE 2

In Example 1 we saw that S_3 has three distinct conjugacy classes of sizes 1, 2, and 3. Since $|S_3| = 6$, the class equation of S_3 is $6 = 1 + 2 + 3$.

If c and x are elements of a group G, then $cx = xc$ if and only if $x^{-1}cx = c$. Thus c is in the center of G [$cx = xc$ for every $x \in G$] if and only if c has exactly one conjugate, itself [$x^{-1}cx = c$ for every $x \in G$]. Therefore, the center $Z(G)$ of G is the union of all the one-element conjugacy classes of G, so that the class equation can be written in a third form:

$$\text{(3)} \qquad |G| = |Z(G)| + |C_1| + |C_2| + \cdots + |C_r|,$$

where $C_1, \ldots, C_r$ are the distinct conjugacy classes of G that contain more than one element each and each $|C_i|$ divides $|G|$.

In addition to the class equation, one more result is needed for the proof of the Sylow Theorems.

*The reasons in the right-hand column above must be adjusted when reading from bottom to top (Exercise 4).

Lemma 9.22 Cauchy's Theorem for Abelian Groups

If G is a finite abelian group and p is a prime that divides the order of G, then G contains an element of order p.

The lemma is an immediate consequence of the Fundamental Theorem of Abelian Groups (Exercise 12 in Section 9.2). The following proof, however, depends only on Chapters 7 and 8.

Proof of Lemma 9.22 ▸ The proof is by induction on the order of G, using the Principle of Complete Induction.* To do this, we must first show that the theorem is true when $|G| = 2$. In this case, if p divides $|G|$, then $p = 2$. The nonidentity element of G must have order 2 by part (1) of Corollary 8.6, and so the theorem is true.

Now assume that the theorem is true for all abelian groups of order less than n and suppose $|G| = n$. Let a be any nonidentity element of G. Then the order of a is a positive integer and is therefore divisible by some prime q (Theorem 1.8), say $|a| = qt$. The element $b = a^t$ has order q by Theorem 7.9. If $q = p$, the theorem is proved. If $q \neq p$, let N be the cyclic subgroup $\langle b \rangle$. N is normal since G is abelian and N has order q by Theorem 7.15. By Theorem 8.13 the quotient group G/N has order $|G|/|N| = n/q < n$. Consequently, by the induction hypothesis, *the theorem is true for G/N.* The prime p divides $|G|$, and $|G| = |N|\,|G/N| = q\,|G/N|$. Since q is a prime other than p, p must divide $|G/N|$ by Theorem 1.5. Therefore, G/N contains an element of order p, say Nc. Since Nc has order p in G/N, we have $Nc^p = (Nc)^p = Ne$ and, hence, $c^p \in N$. Since N has order q, $c^{pq} = (c^p)^q = e$ by part (2) of Corollary 8.6.

Therefore, c must have order dividing pq by Theorem 7.9. However, c cannot have order 1 because then Nc would have order 1 instead of p in G/N. Nor can c have order q because then $(Nc)^q = Nc^q = Ne$ in G/N, so that p (the order of Nc) would divide q by Theorem 7.9. The only possibility is that c has order p or pq; in the latter case, c^q has order p by Theorem 7.9. In either case, G contains an element of order p. Therefore, the theorem is true for abelian groups of order n and, hence, by induction for all finite abelian groups. ■

Proofs of the Sylow Theorems

We now have all the tools needed to prove the Sylow Theorems.

Proof of the First Sylow Theorem 9.13 ▸ The proof is by induction on the order of G. If $|G| = 1$, then p^0 is the only prime power that divides $|G|$, and G itself is a subgroup of order p^0. Suppose $|G| > 1$ and assume inductively that the theorem is true for all groups of order less than $|G|$. Combining the second and third forms of the class equation of G shows that

$$|G| = |Z(G)| + [G{:}C(a_1)] + [G{:}C(a_2)] + \cdots + [G{:}C(a_r)],$$

*See Appendix C.

where for each i, $[G:C(a_i)] > 1$. Furthermore, $|Z(G)| \geq 1$ (since $e \in Z(G)$), and $|C(a_i)| < |G|$ (otherwise, $[G:C(a_i)] = 1$).

Suppose there is an index j such that p does *not* divide $[G:C(a_j)]$. Then by Theorem 1.5 p^k must divide $|C(a_j)|$ because p^k divides $|G|$ by hypothesis and $|G| = |C(a_j)| \cdot [G:C(a_j)]$ by Lagrange's Theorem. Since the subgroup $C(a_j)$ has order less than $|G|$, the induction hypothesis implies that $C(a_j)$, and, hence, G has a subgroup of order p^k.

On the other hand, if p divides $[G:C(a_i)]$ for every i, then since p divides $|G|$, p must also divide $|G| - [G:C(a_1)] - \cdots - [G:C(a_r)] = |Z(G)|$. Since $Z(G)$ is abelian, $Z(G)$ contains an element c of order p by Lemma 9.22. Let N be the cyclic subgroup generated by c. Then N has order p and is normal in G (Exercise 8). Consequently, the order of the quotient group G/N, namely $|G|/p$, is less than $|G|$ and divisible by p^{k-1}. By the induction hypothesis G/N has a subgroup T of order p^{k-1}. There is a subgroup H of G such that $N \subseteq H$ and $T = H/N$ by Theorem 8.24. Lagrange's Theorem shows that

$$|H| = |N| \cdot |H/N| = |N| \cdot |T| = pp^{k-1} = p^k.$$

So G has a subgroup of order p^k in this case, too. ■

The basic tools needed to prove the last two Sylow Theorems are very similar to those used above, except that we will now deal with conjugate subgroups rather than conjugate elements. More precisely, let H be a fixed subgroup of a group G and let A and B be any subgroups of G. We say that A is **H-conjugate** to B if there exists an $x \in H$ such that

$$B = x^{-1}Ax = \{x^{-1}ax \mid a \in A\}.$$

In the special case when H is the group G itself, we simply say that A is **conjugate** to B, or that B is a conjugate of A.

Theorem 9.23

Let H be a subgroup of a group G. Then H-conjugacy is an equivalence relation on the set of all subgroups of G.

Proof ▸ Copy the proof of Theorem 9.19, using subgroups A, B, C in place of elements a, b, c. ■

Let A be a subgroup of a group G. The **normalizer** of A is the set $N(A)$ defined by

$$N(A) = \{g \in G \mid g^{-1}Ag = A\}.$$

Theorem 9.24

If A is a subgroup of a group G, then $N(A)$ is a subgroup of G and A is a normal subgroup of $N(A)$.

Proof ▸ Exercise 7 shows that $A \subseteq N(A)$ and that $g \in N(A)$ if and only if $Ag = gA$. Using this fact, the proof of Theorem 9.20 can be readily adapted to prove that $N(A)$ is a subgroup. The definition of $N(A)$ shows that A is normal in $N(A)$. ■

Theorem 9.25

Let H and A be subgroups of a finite group G. The number of distinct H-conjugates of A (that is, the number of elements in the equivalence class of A under H-conjugacy) is $[H{:}H \cap N(A)]$ and, therefore, divides $|H|$.

Proof ▸ The proof of Theorem 9.21 carries over to the present situation if you replace G by H, a by A, and C by $H \cap N(A)$. ■

Lemma 9.26

Let Q be a Sylow p-subgroup of a finite group G. If $x \in G$ has order a power of p and $x^{-1}Qx = Q$, then $x \in Q$.

Proof ▸ Since Q is normal in $N(Q)$ by Theorem 9.24, the quotient group $N(Q)/Q$ is defined. By hypothesis, $x \in N(Q)$. Since $|x|$ is some power of p, the coset Qx in $N(Q)/Q$ also has order a power of p. Now Qx generates a cyclic subgroup T of $N(Q)/Q$ whose order is a power of p. By Theorem 8.24, $T = H/Q$, where H is a subgroup of G that contains Q. Since the orders of the groups Q and T are each powers of p and $|H| = |Q| \cdot |T|$ by Lagrange's Theorem, $|H|$ must be a power of p. But $Q \subseteq H$, and $|Q|$ is the largest power of p that divides $|G|$ by the definition of a Sylow p-subgroup. Therefore, $Q = H$, and, hence, $T = H/Q$ is the identity subgroup. So the generator Qx of T must be the identity coset Qe. The equality $Qx = Qe$ implies that $x \in Q$. ■

Proof of the Second Sylow Theorem 9.15 ▸ Since K is a Sylow p-subgroup, K has order p^n, where $|G| = p^n m$ and $p \nmid m$. Let $K = K_1, K_2, \ldots, K_t$ be the distinct conjugates of K in G. By Theorem 9.25 (with $H = G$ and $K = A$), $t = [G{:}N(K)]$. Note that *p does not divide t* [reason: $p^n m = |G| = |N(K)| \cdot [G{:}N(K)] = |N(K)| \cdot t$ and p^n divides $|N(K)|$ because K is a subgroup of $N(K)$]. We must prove that the Sylow p-subgroup P is conjugate to K, that is, that P is one of the K_i. To do so we use the relation of P-conjugacy.

Since each K_i is a conjugate of K_1 and conjugacy is transitive, every conjugate of K_i in G is also a conjugate of K_1. In other words, every conjugate of K_i is some K_j. Consequently, the equivalence class of K_i under P-conjugacy contains only various K_j. So the set $S = \{K_1, K_2, \ldots, K_t\}$ of all conjugates of K is a union of distinct equivalence classes under P-conjugacy. The number of subgroups in each of these equivalence classes is a power of p because by Theorem 9.25 the number of subgroups that are P-conjugate to K_i is $[P{:}\ P \cap N(K_i)]$, which is a divisor of $|P| = p^n$ by Lagrange's Theorem. Therefore, t (the number of subgroups in the set S) is the sum of various powers of p (each being the number of subgroups in one of the distinct equivalence classes whose union is S). Since p doesn't divide t, at least one of these powers of p must be $p^0 = 1$. Thus some K_i is in an equivalence class by itself, meaning that $x^{-1}K_ix = K_i$ for every $x \in P$. Lemma 9.26 (with $Q = K_i$) implies that $x \in K_i$ for every such x, so that $P \subseteq K_i$. Since both P and K_i are Sylow p-subgroups, they have the same order. Hence, $P = K_i$. ■

Proof of the Third Sylow Theorem 9.17 ▸ Let $S = \{K_1, \ldots, K_t\}$ be the set of all Sylow p-subgroups of G. By the Second Sylow Theorem, they are all the distinct conjugates of K_1. The proof of the Second Sylow Theorem shows that $t = [G: N(K_1)]$, which divides the order of G by Lagrange's Theorem.

Let P be one of the K_i and consider the relation of P-conjugacy. The only P-conjugate of P is P itself by closure. The proof of the Second Sylow Theorem shows that the only equivalence class consisting of a single subgroup is the class consisting of P itself. The proof also shows that S is the union of distinct equivalence classes and that the number of subgroups in each class is a power of p. Just one of these classes contains P, so the number of subgroups in each of the others is a *positive* power of p. Hence, the number t of Sylow p-subgroups is the sum of 1 and various positive powers of p and, therefore, can be written in the form $1 + kp$ for some integer k. ■

Exercises

NOTE: *Unless stated otherwise, G is a finite group and p is a positive prime.*

A. 1. List the distinct conjugacy classes of the given group.

(a) D_4 (b) S_4 (c) A_4

2. If $a \in G$, then show by example that $C(a)$ may not be abelian. [*Hint:* If $a = (12)$ in S_5, then (34) and (345) are in $C(a)$.]

3. If H is a subgroup of G and $a \in H$, show by example that the conjugacy class of a in H may not be the same as the conjugacy class of a in G.

4. Write out the part of the proof of Theorem 9.21 showing that f is injective, including the reasons for each step. Your answer should begin like this:

$$f(Cy) = f(C\,x) \Rightarrow y^{-1}ay = x^{-1}ax \quad [\textit{Definition of } f]$$
$$\Rightarrow a = yx^{-1}axy^{-1}. \quad [\textit{Left multiply by } y \textit{ and right multiply by } y^{-1}.]$$

5. List all conjugates of the Sylow 3-subgroup $\langle(123)\rangle$ in S_4.

6. If H and K are subgroups of G and H is normal in K, prove that K is a subgroup of $N(H)$. In other words, $N(H)$ is the largest subgroup of G in which H is a normal subgroup.

7. If A is a subgroup of G, prove that

(a) $A \subseteq N(A)$;

(b) $g \in N(A)$ if and only if $Ag = gA$.

8. If N is a subgroup of $Z(G)$, prove that N is a normal subgroup of G.

B. 9. If C is a conjugacy class in G and f is an automorphism of G, prove that $f(C)$ is also a conjugacy class of G.

10. Let G be an infinite group and H the subset of all elements of G that have only a finite number of distinct conjugates in G. Prove that H is a subgroup of G.

11. If G is a nilpotent group (see Exercise 13 of Section 9.3), prove that G has this property: If m divides $|G|$, then G has a subgroup of order m. [You may assume Exercise 22.]

12. Let K be a Sylow p-subgroup of G and N a normal subgroup of G. If K is a normal subgroup of N, prove that K is normal in G.

13. Prove Theorem 9.23.

14. Let N be a normal subgroup of G, $a \in G$, and C the conjugacy class of a in G.

(a) Prove that $a \in N$ if and only if $C \subseteq N$.

(b) If C_i is any conjugacy class in G, prove that $C_i \subseteq N$ or $C_i \cap N = \varnothing$.

(c) Use the class equation to show that $|N| = |C_1| + \cdots + |C_k|$, where $C_1, \ldots, C_k$ are all the conjugacy classes of G that are contained in N.

15. If $N \neq \langle e \rangle$ is a normal subgroup of G and $|G| = p^n$, prove that $N \cap Z(G) \neq \langle e \rangle$. [*Hint:* Exercise 14(c) may be helpful.]

16. Complete the proof of Theorem 9.24.

17. Prove Theorem 9.25.

18. If K is a Sylow p-subgroup of G and H is a subgroup that contains $N(K)$, prove that $[G{:}H] \equiv 1 \pmod{p}$.

19. If K is a Sylow p-subgroup of G, prove that $N(N(K)) = N(K)$.

20. If H is a proper subgroup of G, prove that G is *not* the union of all the conjugates of H. [*Hint:* Remember that H is a normal subgroup of $N(H)$; Theorem 9.25 may be helpful.]

21. If H is a normal subgroup of G and H is a subgroup of G with $|H| = p^k$, prove that H is contained in every Sylow p-subgroup of G. [You may assume Exercise 24.]

C. 22. If $|G| = p^n$, prove that G has a normal subgroup of order p^{n-1}. [*Hint:* You may assume Theorem 9.27 below. Use induction on n. Let $N = \langle a \rangle$, where $a \in Z(G)$ has order p (Why is there such an a?); then G/N has a subgroup of order p^{n-2}; use Theorem 8.24.]

23. If $|G| = p^n$, prove that every subgroup of G of order p^{n-1} is normal.

24. If H is a subgroup of G and H has order some power of p, prove that H is contained in a Sylow p-subgroup of G. [*Hint*: Proceed as in the proofs of the Second and Third Sylow Theorems but use the relation of H-conjugacy instead of P-conjugacy on the set $\{K_1, \ldots, K_t\}$ of all Sylow p-subgroups.]

9.5 The Structure of Finite Groups

The tools developed in Sections 9.1–9.4 are applied here to various aspects of the classification problem. In particular, all groups of orders ≤ 15 are classified. We begin with some useful facts about p-groups.

Theorem 9.27

If G is a group of order p^n, with p prime and $n \geq 1$, then the center $Z(G)$ contains more than one element. In particular, $|Z(G)| = p^k$ with $1 \leq k \leq n$.

Proof ▸ By Lagrange's Theorem, $|Z(G)| = p^k$ with $0 \leq k \leq n$. We now show that $k \geq 1$, that is, that $|Z(G)| \geq p$. Form (3) of the class equation (page 306) shows that

$$|Z(G)| = |G| - |C_1| - |C_2| - \cdots - |C_r|$$

where each $|C_i|$ is a number larger than 1 that divides $|G|$. Since $|G| = p^n$, the divisors of $|G|$ larger than 1 are positive powers of p. Therefore, each $|C_i|$ is divisible by p. Since $|G|$ is also divisible by p, it follows that p divides $|Z(G)|$ and, hence, $|Z(G)| \geq p$. ■

Corollary 9.28

If p is a prime and $n > 1$, then there is no simple group of order p^n.

Proof ▸ If G is a group of order p^n, then $Z(G)$ is a normal subgroup. If $Z(G) \neq G$, then G is not simple. If $Z(G) = G$, then G is abelian and not simple by Theorem 8.25. ■

Corollary 9.29

If G is a group of order p^2, with p prime, then G is abelian. Hence, G is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.

EXAMPLE 1

By Corollary 9.29, every group of order 9 is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Similarly, the only groups of order $169 = 13^2$ (up to isomorphism) are $\mathbb{Z}_{169}$ and $\mathbb{Z}_{13} \times \mathbb{Z}_{13}$.

Proof of Corollary 9.29 ▸ $Z(G)$ has order p or p^2 by Lagrange's Theorem and Theorem 9.27. If $Z(G)$ has order p^2, then $G = Z(G)$, which means that G is abelian. If $Z(G)$ has order p, then the quotient group $G/Z(G)$ has order $|G|/|Z(G)| = p^2/p = p$ by Theorem 8.13. Hence, $G/Z(G)$ is cyclic by Theorem 8.7. Therefore, G is abelian by Theorem 8.15. The last statement of the theorem now follows immediately from the Fundamental Theorem of Finite Abelian Groups. ■

In Corollary 9.18 certain groups of order pq (with p, q prime) were characterized. We can now extend that argument to some groups of order p^2q.

Theorem 9.30

Let p and q be distinct primes such that $q \not\equiv 1 \pmod p$ and $p^2 \not\equiv 1 \pmod q$. If G is a group of order p^2q, then G is isomorphic to $\mathbb{Z}_{p^2q}$ or $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$.

EXAMPLE 2

Theorem 9.30 allows us to classify all groups of order 45. Note that $45 = 3^2 \cdot 5$, and that $5 \not\equiv 1 \pmod 3$ and $3^2 \not\equiv 1 \pmod 5$. So if G is a group of order 45, then by Theorem 9.30 (with $p = 3$ and $q = 5$), G is isomorphic to $\mathbb{Z}_{45}$ or to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. Similar arguments may be used to classify groups of many different orders, including

$$99 = 9 \cdot 11, \quad 153 = 9 \cdot 17, \quad 175 = 25 \cdot 7, \quad 245 = 49 \cdot 5,$$
$$325 = 25 \cdot 13, \quad 539 = 49 \cdot 11.$$

Proof of Theorem 9.30 ▸ By the Third Sylow Theorem, the number of Sylow p-subgroups of G is congruent to 1 modulo p and divides $|G|$. Since the divisors of $|G|$ are $1, p, p^2, q, pq$, and p^2q, the only possibilities are 1 and q. There cannot be q of them because $q \not\equiv 1 \pmod p$. Hence, there is a unique Sylow p-subgroup H, which is normal by Corollary 9.16. Similarly, G has $1, p$, or p^2 Sylow q-subgroups, and neither p nor p^2 is possible since $p^2 \not\equiv 1 \pmod q$. Hence, there is a unique normal Sylow q-subgroup K. The order of the subgroup $H \cap K$ must divide both $|H| = p^2$ and $|K| = q$ by Lagrange's Theorem. Hence, $H \cap K = \langle e \rangle$. Furthermore, $HK = G$ by Exercise 15 in Section 9.3. Therefore, $G = H \times K$ by Theorem 9.3. Now H is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$ by Corollary 9.29 and $K \cong \mathbb{Z}_q$ by Theorem 8.7. Consequently, by Lemma 9.8, $G = H \times K \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q \cong \mathbb{Z}_{p^2q}$ or $G = H \times K \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$. ■

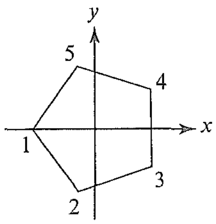
Corollary 9.31

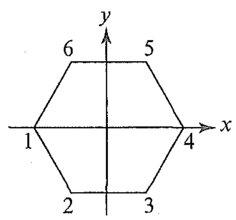
If p and q are distinct primes, then there is no simple group of order p^2q.

Proof ▶ Suppose G is a group of order p^2q. If either $p^2 \not\equiv 1 \pmod{q}$ or $q \not\equiv 1 \pmod{p}$, then the proof of Theorem 9.30 shows that G has a normal Sylow subgroup and, hence, is not simple. If both $p^2 \equiv 1 \pmod{q}$ and $q \equiv 1 \pmod{p}$, then $q \mid (p^2 - 1)$ and $p \mid (q - 1)$, which implies that $p \leq q - 1$ or, equivalently, $q \geq p + 1$. Since $p^2 - 1 = (p - 1)(p + 1)$, we know that $q \mid (p - 1)$ or $q \mid (p + 1)$ by Theorem 1.5. The former is impossible because $q \geq p + 1$, and the latter implies that $q \leq p + 1$, so that $q = p + 1$. Since p and q are primes, the only possibility is $p = 2$ and $q = 3$. Exercise 2 shows that no group of order $2^2 \cdot 3 = 12$ is simple. ■

Dihedral Groups

We now introduce a family of groups that play a crucial role in the classification of groups of order $2p$. Recall that the group D_4 consists of various rotations and reflections of the square (see Section 7.1 or 7.1.A). This idea can be generalized as follows. Let P be a regular polygon of n sides ($n \geq 3$).* For convenient reference, assume that P has its center at the origin and a vertex on the negative x-axis, with the other vertices numbered counterclockwise from this one, as illustrated here in the cases $n = 5$ and $n = 6$.





Think of the plane as a thin sheet of hard plastic. Cut out P, pick it up, and replace it, not necessarily in the same position, but so that it fits exactly in the cut-out space. Such a motion is called a **symmetry** of P.† By considering a symmetry as a function from P to itself and using composition of functions as the operation (gf means motion f followed by motion g), the set D_n of all symmetries of P forms a group, called the **dihedral group of degree n.**

Theorem 9.32

The dihedral group D_n is a group of order $2n$ generated by elements r and d such that

$$|r| = n, \qquad |d| = 2, \qquad \text{and} \qquad dr = r^{-1}d.$$

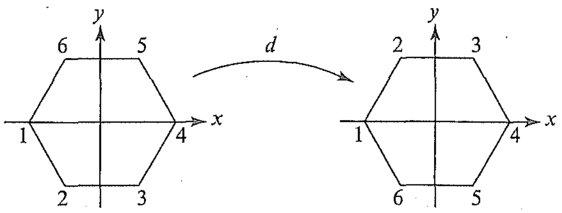
Proof ▶ The proof that D_n is a group is left to the reader. Let r be the counterclockwise rotation of $360/n$ degrees about the center of P; r sends vertex 1 to vertex 2, vertex 2 to vertex 3, and so on. Note that r has

---

* "Regular" means that all sides of P have the same length and all its vertex angles (each formed by two adjacent sides) are the same size. It can be shown that the perpendicular bisectors of the n sides all intersect at a single point, which is called the **center** of P.

†All motions that result in the same final position for P are considered to be the same.

order n because r^n is a 360° rotation that returns P to its initial position (the identity symmetry). Let d be the reflection in the x-axis. As shown in the following figure, d "reverses the orientation" of P: vertices that were formerly numbered counterclockwise from vertex 1 are now numbered clockwise:



The element d has order 2 because reflecting twice in the x-axis also returns P to its initial position.

Since adjacent vertices of P remain adjacent under any symmetry, the final position of P is completely determined by two factors: the new orientation of P (whether the vertices are numbered clockwise or counterclockwise from vertex 1) and the new location of vertex 1. Consequently, every symmetry is the same as either

r^i $\quad (0 \leq i < n)$ $\quad$ [*Counterclockwise rotation of i(360/n) degrees that preserves orientation and moves vertex 1 to the position originally occupied by vertex i + 1*]

or

$r^i d$ $\quad (0 \leq i < n)$ $\quad$ [*Reflection in the x-axis that reverses orientation followed by a counterclockwise rotation that moves vertex 1 to the position originally occupied by vertex i + 1*]

Therefore

$$D_n = \{e = r^0, r, r^2, \ldots, r^{n-1}; d = r^0 d, rd, r^2 d, \ldots, r^{n-1}d\}.$$

Furthermore, the $2n$ elements listed here are all distinct (r^i and r^j move vertex 1 to different positions and $r^i = r^j d$ is impossible since r^i preserves the vertex orientation, but $r^j d$ reverses it). Hence, D_n is a group of order $2n$.

Finally, verify that drd moves vertex 1 to the position originally occupied by vertex n and leaves the vertices in counterclockwise order. In other words, drd is the rotation that moves vertex 1 to vertex n, that is, $drd = r^{n-1}$. Since r has order n, $r^{-1} = r^{n-1}$ and, hence, $drd = r^{-1}$. Multiplying on the right by d shows that $dr = r^{-1}d$. ■

We can now classify another family of groups.

Theorem 9.33

If G is a group of order $2p$, where p is an odd prime, then G is isomorphic to the cyclic group $\mathbb{Z}_{2p}$ or the dihedral group D_p.

EXAMPLE 3

Theorem 9.33 can be used to classify all groups of orders 6, 10, 14, 22, 26, 34, etc. For instance, every group of order 22 is isomorphic either to $\mathbb{Z}_{22}$ or D_{11}, and every group of order 38 is isomorphic either to $\mathbb{Z}_{38}$ or D_{19}. Theorem 9.33 also provides a second proof that there are exactly two nonisomorphic groups of order 6. (See Theorem 8.9 for the first proof.)

Proof of Theorem 9.33 ▸ G contains an element a of order p and an element b of order 2 by Cauchy's Theorem (Corollary 9.14). Note that $b^2 = e$ implies $b^{-1} = b$. Let H be the cyclic group $\langle a \rangle$. Since $|G| = 2p$, the subgroup H has index 2 and is, therefore, normal by Exercise 23 of Section 8.2. Consequently, $bab = bab^{-1} \in H$. Since H is cyclic, $bab = a^t$ for some t. Using this and the fact that $b^2 = e$, we see that

$$a^{t^2} = (a^t)^t = (bab)^t = (bab)(bab)(bab)\cdots(bab) = ba^tb = b(bab)b = a$$

Hence, $t^2 \equiv 1 \pmod p$ by part (2) of Theorem 7.9. Consequently, p divides $t^2 - 1 = (t-1)(t+1)$, which implies that $p \mid (t-1)$ or $p \mid (t+1)$ by Theorem 1.5. Thus $t \equiv 1 \pmod p$ or $t \equiv -1 \pmod p$.

If $t \equiv 1 \pmod p$, then $bab = a^t = a$ by Theorem 7.9. Multiplying both sides by b shows that $ba = ab$. It follows that ab has order $2p = |G|$ (Exercise 33 of Section 7.2). Therefore, G is cyclic and isomorphic to $\mathbb{Z}_{2p}$ by Theorem 7.19.

If $t \equiv -1 \pmod p$, then $bab = a^{-1}$. Exercise 9 shows that the map $f: D_p \to G$ given by $f(r^id^j) = a^ib^j$ is a homomorphism. Let K be the subgroup $\langle b \rangle$. Since $|H| = p$ (with p odd) and $|K| = 2$, $H \cap K = \langle e \rangle$ by Lagrange's Theorem and $G = HK$ by Exercise 15 in Section 9.3. Thus every element of G can be written in the form a^ib^j, which implies that f is surjective. Since D_p and G have the same order, f must be injective and, hence, an isomorphism. ■

Groups of Small Order

We are now in a position to complete the classification of groups of small order that was begun in Section 8.1, where groups of orders ≤ 7 were classified. We already know three abelian groups of order 8 ($\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{Z}_8$) and one nonabelian one (D_4). Another nonabelian group of order 8, the quaternion group Q, was introduced in Exercise 16 of Section 7.1. It is not isomorphic to D_4 by Exercise 47 of Section 7.4. These five groups are the only ones:

Theorem 9.34

If G is a group of order 8, then G is isomorphic to one of the following groups: $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, the dihedral group D_4, or the quaternion group Q.

Proof ▸ If G is abelian, then G is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by the Fundamental Theorem of Finite Abelian Groups. So suppose G is a nonabelian group of order 8. The nonidentity elements of G must have order 2, 4, or 8 by Lagrange's Theorem. However, G cannot contain an element of order 8 (because then G would be cyclic and abelian), nor can all the nonidentity elements of G have order 2 (see Exercise 27 of Section 7.2). Hence, G contains an element a of order 4. Let b be any element of G such that $b \notin \langle a \rangle = \{e, a, a^2, a^3\}$. Then the eight elements $e, a, a^2, a^3, b, ab, a^2b, a^3b$ are all distinct because $|a| = 4$ and $a^i = a^jb$ implies $b = a^{i-j} \in \langle a \rangle$, contrary to the choice of b. Thus $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$.

The subgroup $\langle a \rangle$ has order 4 and index 2 in G. Hence, $\langle a \rangle$ is normal by Exercise 23 of Section 8.2. Now the element bab^{-1} has order 4 by Exercise 19 of Section 7.2 and $bab^{-1} \in \langle a \rangle$ by normality. Therefore, bab^{-1} is either a or a^3 (because e has order 1 and a^2 has order 2). If $bab^{-1} = a$, however, then $ba = ab$, which implies that G is abelian. Therefore, $bab^{-1} = a^3 = a^{-1}$ so that $ba = a^{-1}b$. This fact can be used to construct most of the multiplication table of G. For instance, $(ab)a^2 = a(ba)a = a(a^{-1}b)a = ba = a^{-1}b = a^3b$. You can use similar arguments to verify that the table must look like this:

	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	e	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	e	ab	a^2b	a^3b	b
a^2	a^2	a^3	e	a	a^2b	a^3b	b	ab
a^3	a^3	e	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab				
ab	ab	b	a^3b	a^2b				
a^2b	a^2b	ab	b	a^3b				
a^3b	a^3b	a^2b	ab	b				

In order to complete the table, we must find b^2. Since $b^2 = a^ib$ implies $b = a^i \in \langle a \rangle$, which is a contradiction, b^2 must be one of e, a, a^2, or a^3. If $b^2 = a$, however, then $ab = b^2b = bb^2 = ba$, which implies that G is abelian. Similarly, $b^2 = a^3$ implies that G is abelian (Exercise 15). Therefore, $b^2 = e$ or $b^2 = a^2$. Each of these possibilities leads to a different table for G. Completing the table when $b^2 = e$ and comparing it to the table for D_4 in Example 1 of Section 8.2 shows that $G \cong D_4$ under the correspondence

$$a^i \longrightarrow r_i, \quad b \longrightarrow d, \quad ab \longrightarrow h, \quad a^2b \longrightarrow t, \quad a^3b \longrightarrow v$$

(Exercise 4). Similarly, completing the table when $b^2 = a^2$ and comparing it to the table for the quaternion group Q shows that $G \cong Q$ (Exercise 5). ■

According to the Fundamental Theorem of Finite Abelian Groups there are two abelian groups of order 12: $\mathbb{Z}_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_{12}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. We have also seen two nonabelian groups of order 12: the alternating group A_4 and the dihedral group D_6. It can be shown that there is a third nonabelian group T of order 12, which is generated by elements a and b such that $|a| = 6$, $b^2 = a^3$, and $ba = a^{-1}b$ and that no two of these three nonabelian groups are isomorphic (Exercise 16).

Theorem 9.35

If G is a group of order 12, then G is isomorphic to one of the following groups: $\mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, the alternating group A_4, the dihedral group D_6, or the group T described in the preceding paragraph.

Proof ▶ An argument similar to the proof of Theorem 9.34 can be used to prove the theorem. See Theorem II.6.4 in Hungerford [5]. ■

The preceding results provide a complete classification of all groups of orders ≤ 15, that is, a list of groups such that every group of order ≤ 15 is isomorphic to exactly one group on the list.

ORDER	GROUPS	REFERENCE
2	$\mathbb{Z}_2$	Theorem 8.7
3	$\mathbb{Z}_3$	Theorem 8.7
4	$\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$	Theorem 8.8
5	$\mathbb{Z}_5$	Theorem 8.7
6	$\mathbb{Z}_6, S_3$	Theorem 8.9
7	$\mathbb{Z}_7$	Theorem 8.7
8	$\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_4, Q$	Theorem 9.34
9	$\mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3$	Corollary 9.29
10	$\mathbb{Z}_{10}, D_5$	Theorem 9.33
11	$\mathbb{Z}_{11}$	Theorem 8.7
12	$\mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, A_4, D_6, T$	Theorem 9.35
13	$\mathbb{Z}_{13}$	Theorem 8.7
14	$\mathbb{Z}_{14}, D_7$	Theorem 9.33
15	$\mathbb{Z}_{15}$	Corollary 9.18

This list could be continued to order 100 and beyond. For more than half of the orders between 2 and 100, the techniques presented above provide a complete classification of groups of that order (Exercise 6). For other orders, however, a great deal of additional work would be necessary. For instance, there are 14 different groups of order 16 and 267 of order 64. There is no known formula giving the number of distinct groups of order n.

Exercises

A. 1. If p and q are primes with $p < q$ and $q \not\equiv 1 \pmod{p}$ and G is a group of order p^2q, prove that G is abelian.

2. Prove that there is no simple group of order 12. [*Hint:* Show that one of the Sylow subgroups must be normal.]

3. Prove that D_3 is isomorphic to S_6.

4. **(a)** In the proof of Theorem 9.34, complete the operation table for the group G in the case when $b^2 = e$.

   (b) Show that $G \cong D_4$ under the correspondence

$$a^i \longrightarrow r_i, \quad b \longrightarrow d, \quad ab \longrightarrow h, \quad a^2b \longrightarrow t, \quad a^3b \longrightarrow v$$

   by comparing the table in part (a) with the table for D_4 in Example 1 of Section 8.2.

5. **(a)** In the proof of Theorem 9.34, complete the operation table for the group G in the case when $b^2 = a^2$.

   (b) Show that $G \cong Q$ under the correspondence

$$a^rb^s \longrightarrow i^rj^s \qquad (0 \le r \le 3, 0 \le s \le 1)$$

   by comparing the table in part (a) with the table for Q (see Exercise 16 in Section 7.1).

6. Theorems 8.7, 9.7, 9.30, and 9.33, and Corollaries 9.18 and 9.29 are sufficient to classify groups of many orders. List all such orders from 16 to 100.

B. 7. If G is a group such that every one of its Sylow subgroups (for every prime p) is cyclic and normal, prove that G is a cyclic group.

8. Let $n \ge 3$ be a positive integer and let G be the set of all matrices of the forms

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix} \quad \text{with } a \in \mathbb{Z}_n.$$

   (a) Prove that G is a group of order $2n$ under matrix multiplication.

   (b) Prove that G is isomorphic to D_n.

9. Complete the proof of Theorem 9.33 by showing that when $bab = a^{-1}$, the map $f{:}D_p \to G$ given by $f(r^id^j) = a^ib^j$ is a homomorphism. [*Hint:* $bab = a^{-1}$ is equivalent to $ba = a^{-1}b$. Use this fact and Theorem 9.32 to compute products in G and D_p.]

10. Prove that the dihedral group D_6 is isomorphic to $S_3 \times \mathbb{Z}_2$.

11. **(a)** If $n = 2k$, show that r^k is in the center of D_n.

    (b) If n is even, show that $Z(D_n) = \{e, r^k\}$.

    (c) If n is odd, show that $Z(D_n) = \{e\}$.

12. In Theorem 9.32, r is used to denote a rotation. To avoid confusion here, r will denote the 60° rotation in D_6 and $\bar{r}$ will denote the 120° rotation in D_3. The proof of Theorem 9.32 shows that the elements of D_6 can be written in the form r^id^j, and the elements of D_3 in the form $\bar{r}^id^j$.

    (a) Show that the function $\varphi{:}D_6 \to D_3$ given by $\varphi(r^id^j) = \bar{r}^id^j$ is a surjective homomorphism, with kernel $\{r^0, r^3\}$.

    (b) Prove that $D_6/Z(D_6)$ is isomorphic to D_3. [*Hint:* Exercise 11.]

13. What is the center of the quaternion group Q?

14. Show that every subgroup of the quaternion group Q is normal.

15. If G is a group of order 8 generated by elements a and b such that $|a| = 4$, $b \notin \langle a \rangle$, and $b^2 = a^3$, then G is abelian. [This fact is used in the proof of Theorem 9.34, so don't use Theorem 9.34 to prove it.]

16. Let G be the group $S_3 \times \mathbb{Z}_4$ and let $a = ((123), 2)$ and $b = ((12), 1)$.

(a) Show that $|a| = 6$, $b^2 = a^3$, and $ba = a^{-1}b$.

(b) Verify that the set $T = \{e = a^0, a^1, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$ consists of 12 distinct elements.

(c) Show that T is a nonabelian subgroup of G. [*Hint:* Use part (a) and Theorem 7.12.]

(d) Show that T is not isomorphic to D_6 or to A_4.

17. Let n be a composite positive integer and p a prime that divides n. Assume that 1 is the only divisor of n that is congruent to 1 modulo p. If G is a group of order n, prove that G is not simple.

18. If G is a simple group that has a subgroup K of index n, prove that $|G|$ divides $n!$. [*Hint:* Let T be the set of distinct right cosets of K and consider the homomorphism $\varphi: G \to A(T)$ of Exercise 41 in Section 8.4. Show that φ is injective and note that $A(T) \cong S_n$ (Why?).]

C. 19. Classify all groups of order 21 up to isomorphism.

20. Classify all groups of order 66 up to isomorphism.

21. Prove that there is no simple nonabelian group of order less than 60. [*Hint:* Exercise 18 may be helpful.]

CHAPTER 10

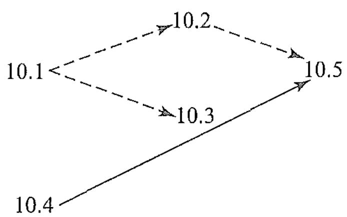
Arithmetic in Integral Domains

In Chapters 1 and 4 we saw that the ring $\mathbb{Z}$ of integers and the ring $F[x]$ of polynomials over a field F have very similar structures: both have division algorithms, greatest common divisors, and unique factorization into primes (irreducibles). In this chapter we find conditions under which these properties carry over to arbitrary integral domains, with particular emphasis on unique factorization.

Unique factorization turns out to be closely related to the ideals of a domain. On the one hand, unique factorization is not possible unless the principal ideals of the domain satisfy certain conditions (Section 10.2). On the other hand, ideals can be used to restore a kind of unique factorization to some domains that lack it. Indeed, ideals were originally invented just for this purpose, as we shall see in Section 10.3.

Section 10.4 (The Field of Quotients of an Integral Domain) is independent of the rest of the chapter and may be read at any point after Chapter 3. Sections 10.2 and 10.3 depend on Chapter 6, but the rest of the chapter may be read after Chapter 4.

The interdependence of the sections of this chapter is shown below. The dashed arrows indicate that Sections 10.2, 10.3, and 10.5 depend only on the first part of Section 10.1 (pages 322–324) and that Section 10.5 uses only three results in Section 10.2, all of which can be read independently of the rest of that section.



A shortened version of Sections 10.1 and 10.2 that contains all the basic information may be obtained by omitting the last parts of each of these sections (see the notes on pages 325 and 337).

10.1 Euclidean Domains

In early chapters we analyzed the structure of $\mathbb{Z}$ and the polynomial ring $F[x]$ by using divisibility, units, associates, and primes (irreducibles). We begin by defining these concepts in the more general setting of an integral domain.*

Throughout this chapter, R is an integral domain.

Let $a, b \in R$, with a nonzero. We say that **a divides b** (or **a is a factor of b**) and write $a \mid b$ if $b = ac$ for some $c \in R$. Recall that an element u in R is a **unit** provided that $uv = 1_R$ for some $v \in R$. Thus the units in R are precisely the divisors of 1_R.

EXAMPLE 1

The only units in $\mathbb{Z}$ are 1 and -1. If F is a field, then the units in the polynomial ring $F[x]$ are the nonzero constant polynomials (Corollary 4.5).

EXAMPLE 2

The set $\mathbb{Z}[\sqrt{2}] = \{r + s\sqrt{2} \mid r, s \in \mathbb{Z}\}$ is a subring of the real numbers (Exercise 1). The element $1 + \sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$ because

$$(1 + \sqrt{2})(-1 + \sqrt{2}) = 1.$$

The ring in the preceding example is one of many similar rings that will frequently be used as examples later. If d is a fixed integer, then it is easy to verify that the set $\mathbb{Z}[\sqrt{d}] = \{r + s\sqrt{d} \mid r, s \in \mathbb{Z}\}$ is an integral domain that is contained in the complex numbers. If $d \geq 0$, then $\mathbb{Z}[\sqrt{d}]$ is a subring of the real numbers (Exercise 1). When $d = -1$, then the ring $\mathbb{Z}[\sqrt{-1}]$ is usually denoted $\mathbb{Z}[i]$ and is called the **ring of Gaussian integers.**

Remark Let $u \in R$ be a unit with inverse v, so that $uv = 1_R$. For any $b \in R$ we have $u(vb) = (uv)b = 1_R b = b$. Therefore,

a unit divides every element of R

An element $a \in R$ is an **associate** of $b \in R$ provided $a = bu$ for some unit u. Now, u has an inverse, say $uv = 1_R$, and v is also a unit. Multiplying both sides of $a = bu$ by v shows that $av = buv = b1_R = b$. Use these facts to verify that

a* is an associate of *b* if and only if *b* is an associate of *a

and

a nonzero element of R is divisible by each of its associates.

*The basic definitions apply in any commutative ring with identity. We restrict our attention to integral domains because most of the theorems fail in nondomains.

EXAMPLE 3

Every nonzero integer n has exactly two associates in $\mathbb{Z}$, n and $-n$. If F is a field, the associates of $f(x) \in F[x]$ are the nonzero constant multiples of $f(x)$. In the ring $\mathbb{Z}[\sqrt{2}]$, the elements $\sqrt{2}$ and $2 - \sqrt{2}$ are associates because $\sqrt{2} = (2 - \sqrt{2})(1 + \sqrt{2})$ and $1 + \sqrt{2}$ is a unit by Example 2.

A nonzero element $p \in R$ is said to be **irreducible** provided that p is not a unit and the only divisors of p are its associates and the units of R.

EXAMPLE 4

The irreducible elements in $\mathbb{Z}$ are just the prime integers because the only divisors of a prime p are $\pm p$ (its associates) and ± 1 (the units in $\mathbb{Z}$). The definition of irreducible given above is identical to the definition of an irreducible polynomial in the integral domain $F[x]$, when F is a field (see Section 4.3). In Section 10.3 we shall see that $1 + i$ is irreducible in the ring $\mathbb{Z}[i]$.

The next theorem is usually the easiest way to prove that an element is irreducible and is sometimes used as a definition. Theorem 4.12 is the special case when $R = F[x]$.

Theorem 10.1

Let p be a nonzero, nonunit element in an integral domain R. Then p is irreducible if and only if

$$\text{whenever } p = rs, \text{ then } r \text{ or } s \text{ is a unit.}$$

Proof ▸ If p is irreducible and $p = rs$, then r is a divisor of p. So r must be either a unit or an associate of p. If r is a unit, there is nothing to prove. If r is an associate of p, say $r = pv$, then $p = rs = pvs$. Canceling p on the two ends (Theorem 3.7) shows that $1_R = vs$. Therefore, s is a unit.

To prove the converse, suppose p has the stated property. Let c be any divisor of p, say $p = cd$. Then by hypothesis either c or d is a unit. If d is a unit, then so is d^{-1}. Multiplying both sides of $p = cd$ by d^{-1} shows that $c = d^{-1}p$. Thus in every case c is either a unit or an associate of p. Therefore, p is irreducible. ■

Euclidean Domains

The Division Algorithm was a key tool in analyzing the arithmetic of both $\mathbb{Z}$ and $F[x]$. So we now look at domains that have some kind of analogue of the Division Algorithm. To see how to describe such an analogue, note that the degree of a polynomial in $F[x]$ can be thought of as defining a function from the nonzero polynomials in $F[x]$ to the nonnegative integers. By identifying the key properties of this function we obtain this

Definition An integral domain R is a **Euclidean domain** if there is a function δ from the nonzero elements of R to the nonnegative integers with these properties:

(i) If a and b are nonzero elements of R, then $\delta(a) \le \delta(ab)$.

(ii) If $a, b \in R$ and $b \neq 0_R$, then there exist $q, r \in R$ such that $a = bq + r$ and either $r = 0_R$ or $\delta(r) < \delta(b)$.

EXAMPLE 5

If F is a field, then the polynomial domain $F[x]$ is a Euclidean domain with the function δ given by $\delta(f(x))$ = degree of $f(x)$. Property (i) follows from Theorem 4.2 because

$$\begin{aligned}\delta(f(x)g(x)) = \deg f(x)g(x) &= \deg f(x) + \deg g(x)\\ &\ge \deg f(x) = \delta(f(x)),\end{aligned}$$

and property (ii) is just the Division Algorithm (Theorem 4.6).

EXAMPLE 6

$\mathbb{Z}$ is a Euclidean domain with the function δ given by $\delta(a) = |a|$. Property (i) holds because $|ab| = |a||b| \ge |a|$ for all nonzero a and b. If $a, b \in \mathbb{Z}$, with $b > 0$, then by the Division Algorithm (Theorem 1.1) there are integers q and r such that $a = bq + r$ and $0 \le r < b$. Either $r = 0$, or r and b are both positive, in which case, $\delta(r) = |r| = r < b = |b| = \delta(b)$. Therefore, property (ii) holds when $b > 0$. For the case when $b < 0$, see Exercise 9.

EXAMPLE 7

We shall prove that the ring of Gaussian integers $\mathbb{Z}[i] = \{s + ti \mid s, t \in \mathbb{Z}\}$ is a Euclidean domain with the function δ given by $\delta(s + ti) = s^2 + t^2$. Since $s + ti = 0$ if and only if both s and t are 0, we see that $\delta(s + ti) \ge 1$ when $s + ti \neq 0$. Verify that for any $a = s + ti$ and $b = u + vi$ in $\mathbb{Z}[i]$, $\delta(ab) = \delta(a)\,\delta(b)$ (Exercise 17). Then when $b \neq 0$ we have

$$\delta(a) = \delta(a) \cdot 1 \le \delta(a)\delta(b) = \delta(ab),$$

so that property (i) holds. If $b \neq 0$, verify that a/b is a complex number that can be written in the form $c + di$, where $c, d \in \mathbb{Q}$ (Exercise 11). Since $c \in \mathbb{Q}$, it lies between two consecutive integers; and similarly for d. Hence, there are integers m and n such that $|m - c| \le 1/2$ and $|n - d| \le 1/2$. Since $a/b = c + di$,

$$\begin{aligned} a &= b[c + di] = b[(c - m + m) + (d - n + n)i]\\ &= b[(m + ni) + ((c - m) + (d - n)i)]\\ &= b[m + ni] + b[(c - m) + (d - n)i]\\ &= bq + r,\end{aligned}$$

where $q = m + ni \in \mathbb{Z}[i]$ and $r = b[(c - m) + (d - n)i]$. Since $r = a - bq$ and a, b, $q \in \mathbb{Z}[i]$, we see that $r \in \mathbb{Z}[i]$. Property (ii) holds because

$$\begin{aligned}\delta(r) &= \delta(b)\delta[(c - m) + (d - n)i] = \delta(b)[(c - m)^2 + (d - n)^2] \\ &\le \delta(b)[(1/2)^2 + (1/2)^2] = (1/2) \cdot \delta(b) < \delta(b).\end{aligned}$$

NOTE: The remainder of this section is optional. The development here is elementary and assumes only the basic facts about rings in Section 3.1. A more sophisticated approach is presented in Section 10.2, where ideals are used to develop the key facts about a wider class of domains that includes Euclidean domains as a special case. Thus this section develops some remarkably strong results with a minimum of mathematical tools, whereas Section 10.2 obtains the same results more efficiently in a wider setting.

It is possible that a given integral domain may be made into a Euclidean domain in more than one way by defining the function δ differently (see Exercises 12 and 13). Whenever the Euclidean domains in the preceding examples are mentioned, however, you may assume that the function δ is the one defined above.

In $F[x]$, the units are the polynomials of degree 0 (Corollary 4.5), that is, the polynomials that have the same degree as the identity polynomial 1_F. Furthermore, if k is a constant (unit in $F[x]$), then $f(x)$ and $kf(x)$ have the same degree. Analogous facts hold in any Euclidean domain.

Theorem 10.2

Let R be a Euclidean domain and u a nonzero element of R. Then the following conditions are equivalent:

(1) u is a unit.

(2) $\delta(u) = \delta(1_R)$.

(3) $\delta(c) = \delta(uc)$ for some nonzero $c \in R$.

Proof ▸ (1) ⇒ (2) Exercise 15.

(2) ⇒ (3) Statement (3) holds with $c = 1_R$ because $\delta(1_R) = \delta(u) = \delta(u \cdot 1_R)$.

(3) ⇒ (1) According to (ii) in the definition of a Euclidean domain (with c and uc in place of a and b), there exist $q, r \in R$ such that

$$c = (uc)q + r \quad \text{and either} \quad r = 0_R \qquad \text{or} \qquad \delta(r) < \delta(uc).$$

If $\delta(c) \le \delta(uc)$, then by part (i) of the definition (with c and $1_R - uq$ in place of a and b) and statement (3),

$$\delta(c) \le \delta(c(1_R - uq)) = \delta(c - ucq) = \delta(r) < \delta(uc) = \delta(c),$$

so that $\delta(c) < \delta(c)$, a contradiction. Hence, we must have $r = 0_R$. Thus $c = (uc)q$, which implies that $1_R = uq$. Therefore, u is a unit. ∎

In the remainder of this section we shall develop the basic facts about greatest common divisors, irreducibles, and unique factorization in Euclidean domains. The development here parallels the ones given in Chapter 1 for $\mathbb{Z}$ and in Chapter 4 for $F[x]$ and most of the arguments are the same ones used there, with appropriate modifications. Alternatively, the major results in Sections 1.2–1.3 and 4.2–4.3 may be considered as special cases of the theorems proved here.

Greatest Common Divisors

The integers are ordered by $\leq$ and polynomials in $F[x]$ are partially ordered by their degrees. This made it natural to define greatest common divisors in these domains in terms of size or degree. The same idea carries over to Euclidean domains, where "size" is measured by the function δ.

Definition Let R be a Euclidean domain and $a, b \in R$ (not both zero). A **greatest common divisor** of a and b is an element d such that

(i) $d \mid a$ and $d \mid b$;
(ii) if $c \mid a$ and $c \mid b$, then $\delta(c) \leq \delta(d)$.

Any two elements of a Euclidean domain R have at least one common divisor, namely 1_R. If $c \mid a$, say $a = ct$, then $\delta(c) \leq \delta(ct) = \delta(a)$. Consequently, every common divisor c of a and b satisfies $\delta(c) \leq max\{\delta(a), \delta(b)\}$, which implies that there is a common divisor of largest possible δ value. In other words, greatest common divisors always exist.

When gcd's were defined in $\mathbb{Z}$ and $F[x]$, an extra condition was included in each case: The gcd of two integers is the *positive* common divisor of largest absolute value and the gcd of two polynomials is the *monic* common divisor of highest degree. These extra conditions guarantee that greatest common divisors in $\mathbb{Z}$ and $F[x]$ are unique. In arbitrary Euclidean domains there are no such extra conditions and greatest common divisors are not unique. Thus the preceding definition is consistent with, but not identical to, what was done in $\mathbb{Z}$ and $F[x]$.

EXAMPLE 8

$\mathbb{Z}$ is a Euclidean domain with $\delta(a) = |a|$. Under the preceding definition, 2 is the gcd of 10 and 18 just as before. However, -2 also satisfies this definition because -2 divides both 10 and 18 and any common divisor of 10 and 18 has absolute value $\leq |-2|$. Note that the greatest common divisors 2 and -2 are associates in $\mathbb{Z}$.

Theorem 10.3

Let R be a Euclidean domain and $a, b \in R$ (not both zero).

(1) If d is a greatest common divisor of a and b, then every associate of d is also a greatest common divisor of a and b.

(2) Any two greatest common divisors of a and b are associates.

(3) If d is a greatest common divisor of a and b, then there exit $u, v \in R$ such that $d = au + bv$.

Proof ▸ (1) Exercise 16.

We now find a particular greatest common divisor of a and b that will then be used to prove statements (2) and (3). Let

$$S = \{\delta(w) \mid 0_R \neq w \in R \text{ and } w = as + bt \text{ for some } s, t \in R\}.$$

Since at least one of $a = a1_R + b0_R$ and $b = a0_R + b1_R$ is nonzero by hypothesis, S is a nonempty set of nonnegative integers. By the Well-Ordering Axiom, S contains a smallest element, that is, there are elements d^*, u^*, v^* of R such that $d^* = au^* + bv^*$ and

(A) for every nonzero w of the form $as + bt$ (with $s, t \in R$), $\delta(d^*) \leq \delta(w)$.

We claim that d^* is a greatest common divisor of a and b. To prove this we first show that $d^* \mid a$. By the definition of Euclidean domain, there are elements q, r such that $a = d^*q + r$ and either $r = 0_R$ or $\delta(r) < \delta(d^*)$. Note that

$$\begin{aligned} r &= a - d^*q = a - (au^* + bv^*)q \\ &= a - aqu^* - bv^*q = a(1_R - qu^*) + b(-v^*q). \end{aligned}$$

Thus r is a linear combination of a and b, and, hence, we cannot have $\delta(r) < \delta(d^*)$ by (A). Therefore, $r = 0_R$, so that $a = d^*q$ and $d^* \mid a$. A similar argument shows that $d^* \mid b$ and, hence, d^* is a common divisor of a and b.

Let c be any other common divisor of a and b. Then $a = cs$ and $b = ct$ for some $s, t \in R$ and hence

(B) $$d^* = au^* + bv^* = (cs)u^* + (ct)v^* = c(su^* + tv^*).$$

Thus by part (i) of the definition of Euclidean domain $\delta(c) \leq \delta(c(su^* + tv^*)) = \delta(d^*)$. Therefore, d^* is a greatest common divisor of a and b. Note that (B) also shows that

(C) every common divisor c of a and b divides d^*.

This completes the preliminaries. We now prove the rest of the theorem.

(2) Let d be any greatest common divisor of a and b. Since d divides both a and b and d^* is a greatest common divisor, we must have $\delta(d) \leq \delta(d^*)$ by part (ii) of the definition. The same definition with the roles of d and

d^* reversed shows that $\delta(d^*) \le \delta(d)$. Hence, $\delta(d) = \delta(d^*)$. By (C) we know that $d \mid d^*$, say $d^* = dk$. Therefore, $\delta(d) = \delta(d^*) = \delta(dk)$. Hence, k is a unit by Theorem 10.2 and d is an associate of d^*. Since every gcd is an associate of d^*, any two of them must be associates of each other by Exercise 6.

(3) If d is a greatest common divisor of a and b, then as we saw in the previous paragraph $d^* = dk$, with k a unit. Since $d^* = au^* + bv^*$, we have

$$d = d^*k^{-1} = (au^* + bv^*)k^{-1} = a(u^*k^{-1}) + b(v^*k^{-1}).$$

Hence, $d = au + bv$, with $u = u^*k^{-1}$and $v = v^*k^{-1}$. ■

Corollary 10.4

Let R be a Euclidean domain and $a, b \in R$ (not both zero). Then d is a greatest common divisor of a and b if and only if d satisfies these conditions:

(i) $d \mid a$ and $d \mid b$;

(ii) if $c \mid a$ and $c \mid b$, then $c \mid d$.

Proof ▸ If d is a greatest common divisor of a and b, then d satisfies (i) by definition. Suppose c is a common divisor of a and b. Let d^* be as in (***) in the proof of Theorem 10.3. Then $c \mid d^*$, say $d^* = ct$. Furthermore, d^* is an associate of d by Theorem 10.3 so that $d^* = dk$, with k a unit. Hence, $d = d^*k^{-1} = (ct)k^{-1} = c(tk^{-1})$, so that $c \mid d$. Therefore, condition (ii) holds. The proof of the converse is Exercise 18. ■

The Euclidean Algorithm (Exercise 15 of Section 1.2) provides the most efficient way of calculating the greatest common divisor of two integers. With minor modification its proof carries over to Euclidean domains and provides a constructive method of finding both greatest common divisors and the coefficients needed to write the gcd of a and b as a linear combination of a and b. See Exercise 31.

Unique Factorization

Elements a and b of a Euclidean domain are said to be **relatively prime** if one of their greatest common divisors is 1_R. In any domain the units are the associates of 1_R. Thus by Theorem 10.3, a and b are relatively prime if and only if one of their greatest common divisors is a unit.

Theorem 10.5

Let R be a Euclidean domain and $a, b, c \in R$. If $a \mid bc$ and a and b are relatively prime, then $a \mid c$.

Proof ▸ Copy the proof of Theorem 1.4, using Theorem 10.3 in place of Theorem 1.2. ■

Corollary 10.6

Let p be an irreducible element in a Euclidean domain R.

(1) If $p \mid bc$, then $p \mid b$ or $p \mid c$.

(2) If $p \mid a_1a_2 \cdots a_n$, then p divides at least one of the a_i.

Proof ▸ (1) Let d be a greatest common divisor of p and b. Since d divides p, we know that d is either an associate of p or a unit. If d is an associate of p, then p is also a greatest common divisor of p and b by Theorem 10.3; in particular, $p \mid b$. If d is a unit, then p and b are relatively prime and, hence, $p \mid c$ by Theorem 10.5.

(2) Copy the proof of Corollary 1.6, using (1) in place of Theorem 1.5. ■

Theorem 10.7

Let R be a Euclidean domain. Every nonzero, nonunit element of R is the product of irreducible elements,* and this factorization is unique up to associates; that is, if

$$p_1p_2 \cdots p_r = q_1q_2 \cdots q_s$$

with each p_i and q_j irreducible, then $r = s$ and, after reordering and relabeling if necessary,

$$p_i \text{ is an associate of } q_i \text{ for } i = 1, 2, \ldots, r.$$

Proof ▸ Let S be the set of all nonzero nonunit elements of R that are *not* the product of irreducibles. We shall show that S is empty, which proves that every nonzero nonunit element has at least one factorization as a product of irreducibles. Suppose, on the contrary, that S is nonempty. Then the set $\{\delta(s) \mid s \in S\}$ is a nonempty set of nonnegative integers, which contains a smallest element by the Well-Ordering Axiom. That is, there exists $a \in S$ such that

$$(*) \qquad \delta(a) \leq \delta(s) \quad \text{for every} \quad s \in S.$$

Since $a \in S$, a is not itself irreducible. By the definition of irreducibility, $a = bc$ with both b and c nonunits. Now $\delta(b) \leq \delta(bc)$ by the definition of Euclidean domain. If $\delta(b) = \delta(bc)$, then b would be a unit by Theorem 10.2, which is a contradiction. Hence, $\delta(b) < \delta(bc) = \delta(a)$, so that $b \notin S$ by (*). A similar argument shows that $c \notin S$. By the definition of S, both b and c are the product of irreducibles and, hence, so is $a = bc$. This contradicts the fact

*We allow the possibility of a product with just one factor in case the original element is itself irreducible.

that $a \in S$. Therefore, S is empty, and every nonzero nonunit element of R is the product of irreducibles. To show that this factorization is unique up to associates, copy the proof of Theorem 4.14, replacing *constant* by *unit* and Corollary 4.13 by Corollary 10.6. ■

Exercises

NOTE: *Unless stated otherwise, R is an integral domain.*

A. **1.** Show that $\mathbb{Z}[\sqrt{d}]$ is a subring of $\mathbb{C}$. If $d \geq 0$, show that $\mathbb{Z}[\sqrt{d}]$ is a subring of $\mathbb{R}$.

2. Let $d \neq \pm 1$ be a square-free integer (that is, d has no integer divisors of the form c^2 except $(\pm 1)^2$). Prove that in $\mathbb{Z}[\sqrt{d}]$, $r + s\sqrt{d} = r_1 + s_1\sqrt{d}$ if and only if $r = r_1$ and $s = s_1$. Give an example to show that this result may be false if d is not square-free.

3. If the statement is true, prove it; if it is false, give a counterexample:

(a) If $a \mid b$ and $c \mid d$ in R, then $ac \mid bd$.

(b) If $a \mid b$ and $c \mid d$ in R, then $(a + c) \mid (b + d)$.

4. Prove that c and d are associates in R if and only if $c \mid d$ and $d \mid c$.

5. If $a = bc$ with $a \neq 0$ and b and c nonunits, show that a is not an associate of b.

6. Denote the statement "a is an associate of b" by $a \sim b$. Prove that $\sim$ is an equivalence relation; that is, for all $r, s, t \in R$: (i) $r \sim r$. (ii) If $r \sim s$, then $s \sim r$. (iii) If $r \sim s$ and $s \sim t$, then $r \sim t$.

7. Prove that every associate of an irreducible element is irreducible.

8. If u and v are units, prove that u and v are associates.

9. Show that the function δ in Example 6 has property (ii) in the definition of a Euclidean domain in the case when $b < 0$. [*Hint:* Apply the Division Algorithm with a as dividend and $|b|$ as divisor. Then modify the result.]

10. Is $2x + 2$ irreducible in $\mathbb{Z}[x]$? Why not?

11. If $a = s + ti$ and $b = u + vi$ are in $\mathbb{Z}[i]$ and $b \neq 0$, show that $a/b = c + di$, where

$$c = \frac{su + tv}{u^2 + v^2} \text{ and } d = \frac{tu - sv}{u^2 + v^2}.$$

12. **(a)** Show that $\mathbb{Z}$ is a Euclidean domain with the function δ given by $\delta(n) = n^2$.

(b) Is $\mathbb{Q}$ a Euclidean domain when δ is defined by $\delta(r) = r^2$?

13. Let R be a Euclidean domain with function δ and let k be a positive integer.

(a) Show that R is also a Euclidean domain under the function θ given by $\theta(r) = \delta(r) + k$.

(b) Show that R is also a Euclidean domain under the function β given by $\beta(r) = k\delta(r)$.

14. Let F be a field. Prove that F is a Euclidean domain with the function δ given by $\delta(a) = 0$ for each nonzero $a \in F$.

15. Let R be a Euclidean domain and $u \in R$. Prove that u is a unit if and only if $\delta(u) = \delta(1_R)$.

16. If d is the greatest common divisor of a and b in a Euclidean domain, prove that every associate of d is also a greatest common divisor of a and b.

17. **(a)** If $a = s + ti$ and $b = u + vi$ are nonzero elements of $\mathbb{Z}[i]$, show that $\delta(ab) = \delta(a)\delta(b)$, where $\delta(r + si) = r^2 + s^2$.

(b) If R is a Euclidean domain, is it true that $\delta(ab) = \delta(a)\delta(b)$ for all nonzero $a, b \in R$?

18. Complete the proof of Corollary 10.4 by showing that an element d satisfying conditions (i) and (ii) is a greatest common divisor of a and b.

19. Show that the elements q and r in the definition of a Euclidean domain are not necessarily unique. [*Hint:* In $\mathbb{Z}[i]$, let $a = -4 + i$ and $b = 5 + 3i$; consider $q = -1$ and $q = -1 + i$.]

B. 20. If any two nonzero elements of R are associates, prove that R is a field.

21. If every nonzero element of R is either irreducible or a unit, prove that R is a field.

22. **(a)** Show that $1 + i$ is not a unit in $\mathbb{Z}[i]$. [*Hint:* What is the inverse of $1 + i$ in $\mathbb{C}$?]

(b) Show that 2 is not irreducible in $\mathbb{Z}[i]$.

23. Let p be a nonzero, nonunit element of R such that whenever $p \mid cd$, then $p \mid c$ or $p \mid d$. Prove that p is irreducible.

24. If $f: R \to S$ is a surjective homomorphism of integral domains, p is irreducible in R, and $f(p) \neq 0_S$, is $f(p)$ irreducible in S?

25. Let R be a Euclidean domain. Prove that

(a) $\delta(1_R) \leq \delta(a)$ for all nonzero $a \in R$.

(b) If a and b are associates, then $\delta(a) = \delta(b)$.

(c) If $a \mid b$ and $\delta(a) = \delta(b)$, then a and b are associates.

26. Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain with $\delta(r + s\sqrt{-2}) = r^2 + 2s^2$.

27. Let $\omega = (-1 + \sqrt{-3})/2$ and $\mathbb{Z}[\omega] = \{r + s\omega \mid r, s \in \mathbb{Z}\}$. Prove that $\mathbb{Z}[\omega]$ is a Euclidean domain with $\delta(r + s\omega) = (r + s\omega)(r + s\omega^2) = r^2 - rs + s^2$. [*Hint:* Note that $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$ (Why?).]

28. Prove or disprove: Let R be a Euclidean domain; then $I = \{a \in R \mid \delta(a) > \delta(1_R)\}$ is an ideal in R.

29. Let R be a Euclidean domain. If the function δ is a constant function, prove that R is a field.

30. **(a)** Prove that $1 - i$ is irreducible in $\mathbb{Z}[i]$. [*Hint:* If $a \mid (1 - i)$, then $1 - i = ab$; see Exercises 17(a) and 25.]

(b) Write 2 as a product of irreducibles in $\mathbb{Z}[i]$. [*Hint:* Try $1 - i$ as a factor.]

C. 31. State and prove the Euclidean Algorithm for finding the gcd of two elements of a Euclidean domain.

32. Let R be a Euclidean domain such that $\delta(a + b) \leq \max\{\delta(a), \delta(b)\}$ for all nonzero $a, b \in R$. Prove that q and r in the definition of Euclidean domain are unique.

10.2 Principal Ideal Domains and Unique Factorization Domains

A Euclidean domain is, in effect, a domain that has an analogue of the Division Algorithm. Consequently, all the proofs used for the integers and polynomial rings, most of which ultimately depended on the Division Algorithm, can be readily carried over to Euclidean domains. We now consider domains that may not have an analogue of the Division Algorithm but do have the other important arithmetic properties of $\mathbb{Z}$, such as unique factorization and greatest common divisors.

Definition

A **principal ideal domain (PID)** is an integral domain in which every ideal is principal.

The next theorem shows, for example, that $\mathbb{Z}$, $\mathbb{Q}[x]$, and $\mathbb{Z}[i]$ are all principal ideal domains because all of them are Euclidean domains (see Examples 5–7 of Section 10.1). Example 8 of Section 6.1 shows that the polynomial ring $\mathbb{Z}[x]$ is not a PID.

Theorem 10.8

Every Euclidean domain is a principal ideal domain.

Proof ▸ Suppose I is a nonzero ideal in a Euclidean domain R. Then the set $\{\delta(i) \mid i \in I\}$ is a nonempty set of nonnegative integers, which contains a smallest element by the Well-Ordering Axiom. That is, there exists $b \in I$ such that

$$(*) \qquad \delta(b) \leq \delta(i) \qquad \text{for every} \qquad i \in I.$$

We claim that I is the principal ideal $(b) = \{rb \mid r \in R\}$. Since $b \in I$ and I is an ideal, $rb \in I$ for every $r \in R$; hence, $(b) \subseteq I$. Conversely, suppose $c \in I$. Then there exist $q, r \in R$ such that

$$c = bq + r \qquad \text{and} \qquad r = 0_R \qquad \text{or} \qquad \delta(r) < \delta(b).$$

Since $r = c - bq$ and both c and b are in I, we must have $r \in I$. Hence, it is impossible to have $\delta(r) < \delta(b)$ by $(*)$. Consequently, $r = 0_R$ and $c = bq + r = bq \in (b)$. Thus $I \subseteq (b)$ and, hence, $I = (b)$. Therefore, R is a PID. ■

The converse of Theorem 10.8 is false: There are principal ideal domains that are not Euclidean domains (see Wilson and Williams [21]). Thus the class of Euclidean domains is strictly contained in the class of principal ideal domains.

In our development of the integers, polynomial rings, and Euclidean domains we first considered greatest common divisors and used them to prove unique factorization. Although this approach could also be used with principal ideal domains, it is just as easy to proceed directly to unique factorization.* We begin by developing the connection between divisibility and principal ideals in any integral domain.

Lemma 10.9

Let a and b be elements of an integral domain R. Then

(1) $(a) \subseteq (b)$ if and only if $b \mid a$.

(2) $(a) = (b)$ if and only if $b \mid a$ and $a \mid b$.

(3) $(a) \subsetneqq (b)$ if and only if $b \mid a$ and b is not an associate of a.

Proof ▸ (1) Note first that the principal ideal (b) consists of all multiples of b, that is, all elements divisible by b. Hence,

$$a \in (b) \qquad \text{if and only if} \qquad b \mid a.$$

Now if $(a) \subseteq (b)$, then a is in the ideal (b), so that $b \mid a$. Conversely, if $b \mid a$, then $a \in (b)$, which implies that every multiple of a is also in the ideal (b). Hence, $(a) \subseteq (b)$.

(2) $(a) = (b)$ if and only if $(a) \subseteq (b)$ and $(b) \subseteq (a)$. By (1), $(a) \subseteq (b)$ and $(b) \subseteq (a)$ if and only if $b \mid a$ and $a \mid b$.

(3) To prove this, use (1), (2), and Exercise 4 in Section 10.1, which shows that $a \mid b$ and $b \mid a$ if and only if b is an associate of a. ■

To understand the origin of the next definition, it may help to recall the typical process for factoring an integer a_1 as a product of primes. Find a prime divisor p_1 of a_1 and factor: $a_1 = p_1a_2$. Next find a prime divisor p_2 of a_2 and factor: $a_2 = p_2a_3$, so that $a_1 = p_1p_2a_3$. Now find a prime divisor p_3 of a_3 and factor again: $a_3 = p_3a_4$ and $a_1 = p_1p_2p_3a_4$. Continue in this manner. Since a_1 has only a finite number of prime divisors, we must eventually have some a_k prime so that $a_k = p_k \cdot 1$ and $a_1 = p_1p_2p_k \cdots p_k \cdot 1$. The only way to continue factoring (with positive factors and without changing the p's) is to use the fact that $1 = 1 \cdot 1$ repeatedly to write a_1 as

$$a_1 = p_1p_2p_3 \cdots p_k \cdot 1 \cdot 1 \cdot 1 \cdots 1.$$

Now look at the same procedure from the point of view of ideals. We have $a_2 \mid a_1$, $a_3 \mid a_2$, $a_4 \mid a_3, \ldots, 1 \mid a_k$, $1 \mid 1$, $1 \mid 1$, and so on. Consequently, by Lemma 10.9 this factorization process leads to a chain of ideals

$$(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots \subseteq (a_k) \subseteq (1) \subseteq (1) \subseteq (1) \subseteq \cdots$$

*Greatest common divisors are discussed at the end of this section; also see Exercises 20–22.

in which all the ideals are equal after some point. This suggests that factorization as a product of irreducibles is somehow related to chains of principal ideals in which all the ideals are equal after some point and motivates the following definition.

Definition An integral domain R satisfies the **ascending chain condition (ACC) on principal ideals** provided that whenever $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots$, then there exists a positive integer n such that $(a_i) = (a_n)$ for all $i \geq n$.

Note that in this definition the identical ideals beginning with (a_n) may not be the ideal (1_R). Nevertheless, the preceding discussion suggests the possibility that $\mathbb{Z}$ has the ACC on principal ideals. This is indeed the case as we now prove.

Lemma 10.10

Every principal ideal domain R satisfies the ascending chain condition on principal ideals.

Proof ▸ If $(a_1) \subseteq (a_2) \subseteq \cdots$ is an ascending chain of ideals in R, let A be the set-theoretic union $\bigcup_{i \geq 1} (a_i)$. We claim that A is an ideal. Suppose $a, b \in A$; then $a \in (a_j)$ and $b \in (a_k)$ for some $j, k \geq 1$. Either $j \leq k$ or $k \leq j$, say $j \leq k$. Then $(a_j) \subseteq (a_k)$, so that $a, b \in (a_k)$. Since (a_k) is an ideal, we know that $a - b \in (a_k) \subseteq A$ and $ra \in (a_k) \subseteq A$ for any $r \in R$. Therefore, A is an ideal by Theorem 6.1. Since R is a PID, $A = (c)$ for some $c \in R$. Since $A = \bigcup_{i \geq 1} (a_i)$, we know that $c \in (a_n)$ for some n. Consequently, $(c) \subseteq (a_n)$ and for each $i \geq n$

$$(a_n) \subseteq (a_i) \subseteq \bigcup_{i \geq 1} (a_i) = A = (c) \subseteq (a_n).$$

Therefore, $(a_i) = (a_n)$ for each $i \geq n$. ■

As we shall see, Lemma 10.10 is the key to showing that every nonzero nonunit element in a PID can be factored as a product of irreducibles. The fact that this factorization is essentially unique is a consequence of the next lemma.

Lemma 10.11

Let R be a principal ideal domain. If p is irreducible in R and $p \mid bc$, then $p \mid b$ or $p \mid c$.

Proof* ▸ If $p \mid bc$, then bc is in the ideal (p). If (p) were known to be a prime ideal, we could conclude that $b \in (p)$ or $c \in (p)$, that is, that $p \mid b$ or $p \mid c$. Since every maximal ideal is prime by Corollary 6.16, we need only show

*For an alternate proof using greatest common divisors in place of Corollary 6.16, see Exercise 23.

that (p) is a maximal ideal. Suppose I is any ideal with $(p) \subseteq I \subseteq R$. Since R is a PID, $I = (d)$ for some $d \in R$. Then $(p) \subseteq (d) = I$ implies that $d \mid p$. Since p is irreducible, d must be either a unit or an associate of p. If d is a unit, then $I = (d) = R$ by Exercise 9 of Section 6.1. If d is an associate of p, say $d = pu$, then $p \mid d$ and, hence, $(d) \subseteq (p)$. In this case, $(p) \subseteq (d) \subseteq (p)$, so that $(p) = (d) = I$. Therefore, (p) is maximal, and the proof is complete. ■

Theorem 10.12

Let R be a principal ideal domain. Every nonzero, nonunit element of R is the product of irreducible elements,* and this factorization is unique up to associates; that is, if

$$p_1p_2 \cdots p_r = q_1q_2 \cdots q_s$$

with each p_i and q_j irreducible, then $r = s$ and, after reordering and relabeling if necessary,

$$p_i \text{ is an associate of } q_i \text{ for } i = 1, 2, \ldots, r.$$

Proof ▸ Let a be a nonzero, nonunit element in R. We must show that a has at least one factorization. Suppose, on the contrary, that a is *not* a product of irreducibles. Then a is not itself irreducible. So $a = a_1b_1$ for some nonunits a_1 and b_1 (otherwise every factorization of a would include a unit and a would be irreducible by Theorem 10.1). If both a_1 and b_1 are products of irreducibles, then so is a. Thus at least one of them, say a_1, is not a product of irreducibles. Since b_1 is not a unit, a_1 is not an associate of a (Exercise 5 in Section 10.1). Consequently, $(a) \subsetneqq (a_1)$ by part (3) of Lemma 10.9.

Now repeat the preceding argument with a_1 in place of a. This leads to a nonzero nonunit a_2 such that $(a_1) \subsetneqq (a_2)$ and a_2 is not a product of irreducibles. Continuing this process indefinitely would lead to a strictly ascending chain of principal ideals $(a_1) \subsetneqq (a_2) \subsetneqq (a_3) \subsetneqq \cdots$, contradicting Lemma 10.10. Therefore, a must have at least one factorization as a product of irreducibles.

Now we must show that this factorization is unique up to associates. To do this, adapt the proof of Theorem 4.14 (the case when $R = F[x]$) to the general situation by replacing the word *constant* by *unit* and using Lemma 10.11 and Exercise 2 in place of Corollary 4.13. ■

To appreciate the importance of Theorem 10.12, it may be beneficial to examine a domain in which unique factorization fails.

*We allow the possibility of a product with just one factor in case the original element is itself irreducible.

EXAMPLE 1

Let $\mathbb{Q}_{\mathbb{Z}}[x]$ denote the set of polynomials with rational coefficients and integer constant terms. For instance, x, $\frac{1}{2}x$, and 2 are in $\mathbb{Q}_{\mathbb{Z}}[x]$, but $x^2 + \frac{1}{2}$ and $\frac{1}{4}$ are not. Verify that $\mathbb{Q}_{\mathbb{Z}}[x]$ is an integral domain and that the constant polynomial 2 is irreducible in $\mathbb{Q}_{\mathbb{Z}}[x]$ (Exercise 16). The irreducible element 2 is a factor of $x \in \mathbb{Q}_{\mathbb{Z}}[x]$ because $x = 2 \cdot \left(\frac{1}{2}x\right)$. Similarly, 2 is an irreducible factor of $\frac{1}{2}x$ because $\frac{1}{2}x = 2 \cdot \left(\frac{1}{4}x\right)$. Hence, $x = 2 \cdot 2 \cdot \left(\frac{1}{4}x\right)$. In fact, the process of factoring out irreducible 2's *never ends* because

$$(*)\quad x = 2\cdot\left(\frac{1}{2}x\right) = 2\cdot 2\cdot\left(\frac{1}{4}x\right) = 2\cdot 2\cdot 2\cdot\left(\frac{1}{8}x\right) = \cdots$$
$$= 2\cdot 2\cdots 2\cdot\left(\frac{1}{2^n}x\right) = \cdots.$$

In view of this, it should not be surprising that x cannot be factored as a product of irreducibles of $\mathbb{Q}_{\mathbb{Z}}[x]$ (Exercise 17).

Compare this situation with the prime factorization of a_1 in $\mathbb{Z}$ as described on page 333. In $\mathbb{Z}$ the factorization becomes trivial after a finite number of steps (the only remaining factors are 1's), and all the ideals in the corresponding chain are equal after that point. In the factorization $(*)$ in $\mathbb{Q}_{\mathbb{Z}}[x]$, however, things are different. The remaining factors each time a 2 is factored from x are the elements

$$x, \frac{1}{2}x, \frac{1}{4}x, \frac{1}{8}x \ldots, \frac{1}{2^n}x \ldots.$$

No two of these elements are associates (Exercise 3) and each element is 2 times the following one, that is, each element is divisible by the following one. Therefore, by part (3) of Lemma 10.9

$$(x) \subsetneq \left(\frac{1}{2}x\right) \subsetneq \left(\frac{1}{4}x\right) \subsetneq \left(\frac{1}{8}x\right) \subsetneq \cdots.$$

Hence, the ACC for principal ideals does not hold in $\mathbb{Q}_{\mathbb{Z}}[x]$.

Unique Factorization Domains

In our study of Euclidean domains and principal ideal domains, the main result was that unique factorization held. Now we reverse the process and consider domains in which unique factorization always holds to see what other properties from ordinary arithmetic they may have.

Definition

An integral domain R is a **unique factorization domain (UFD)** provided that every nonzero, nonunit element of R is the product of irreducible elements,* and this factorization is unique up to associates; that is, if

$$p_1p_2 \cdots p_r = q_1q_2 \cdots q_s$$

with each p_i and q_j irreducible, then $r = s$ and, after reordering and relabeling if necessary,

$$p_i \text{ is an associate of } q_i \text{ for } i = 1, 2, \ldots, r.$$

EXAMPLE 2

Theorem 10.12 shows that every PID is a unique factorization domain. In particular, the ring $\mathbb{Z}[i]$ of Gaussian integers is a UFD.

EXAMPLE 3

As noted in Example 1, $\mathbb{Q}_{\mathbb{Z}}[x]$ is not a unique factorization domain because the element x has no factorization as a product of a finite number of irreducibles. In Section 10.3 we shall see that $\mathbb{Z}[\sqrt{-5}]$ fails to be a UFD for a different reason: Every element is a product of irreducibles, but this factorization is not unique.

EXAMPLE 4

A proof that the polynomial ring $\mathbb{Z}[x]$ is a UFD is given in Section 10.5. Since $\mathbb{Z}[x]$ is not a principal ideal domain (see Example 8 of Section 6.1), we see that the class of all unique factorization domains is strictly larger than the class of all principal ideal domains.

NOTE: The remainder of this section is optional and is not needed for the sequel.

When working with two integers, you can always arrange things so that the same primes appear in the factorizations of both elements. For instance, consider the prime factorizations $-18 = 2 \cdot 3 \cdot (-3)$ and $40 = 2 \cdot (-2) \cdot (-2) \cdot 5$. The list of all primes that appear in both factorizations is 2, 3, -3, 2, -2, -2, 5, but several of these primes are associates of each other. By eliminating any prime on the list that is an associate of an earlier number on the list we obtain the list 2, 3, 5 in which no two numbers are associates. We can write both 18 and 40 as products of these three primes and the units ± 1:

$$-18 = 2 \cdot 3 \cdot (-3) = (-1) \cdot 2 \cdot 3 \cdot 3 = (-1) \cdot 2^0 \cdot 3^2 \cdot 5^0$$
$$40 = 2 \cdot (-2) \cdot (-2) \cdot 5 = (-1)(-1) \cdot 2 \cdot 2 \cdot 2 \cdot 5 = (1) \cdot 2^3 \cdot 3^0 \cdot 5^1$$

Essentially the same procedure works in any UFD.

*We allow the possibility of a product with just one factor in case the original element is itself irreducible.

Theorem 10.13

If c and d are nonzero elements in a unique factorization domain R, then there exist units u and v and irreducibles $p_1, p_2, \ldots, p_k$, no two of which are associates, such that

$$c = up_1^{m_1}p_2^{m_2}\cdots p_k^{m_k} \quad \text{and} \quad d = vp_1^{n_1}p_2^{n_2}\cdots p_k^{n_k},$$

where each m_i and n_i is a nonnegative integer. Furthermore,

$$c \mid d \qquad \text{if and only if} \qquad m_i \leq n_i \qquad \text{for each} \qquad i = 1, 2, \ldots, k.$$

In the example preceding the theorem, with $c = -18$ and $d = 40$, we had $u = -1, v = 1$, $p_1 = 2, p_2 = 3$, and $p_3 = 5$.

Proof of Theorem 10.13 ▸ Since R is a UFD, both c and d can be factored, say $c = q_1q_2 \cdots q_s$ and $d = r_1r_2 \cdots r_t$ with each q_i and r_j irreducible. In the list $q_1, q_2, \ldots, q_s, r_1, r_2, \ldots, r_t$ delete any element that has an associate appearing earlier on the list and denote the remaining elements by $p_1, p_2, \ldots, p_k$. Then each p_i is irreducible, no two of them are associates of each other, and each one of the q's and r's is an associate of some p_i. Consequently, in the factorization $c = q_1q_2 \cdots q_s$ each q_j is of the form wp_i with w a unit. By rearranging terms, c can be written (product of units) (product of p's). The product of these units is itself a unit, call it u. By rearranging the p's in this product and inserting other p's with zero exponents if necessary, we can write $c = up_1^{m_1}p_2^{m_2} \cdots p_k^{m_k}$, with each $m_i \geq 0$. A similar procedure works for d and proves the first part of the theorem.

To prove the first half of the last statement of the theorem, suppose $c \mid d$. Then $d = cb$ for some $b \in R$. Since the irreducible p_i appears exactly n_i times in the factorization of d, it must also appear exactly n_i times in the factorization of cb. But p_i already appears m_i times in the factorization of c and may possibly appear in the factorization of b, so we must have $m_i \leq n_i$. Conversely, suppose that $m_i \leq n_i$ for every i. Verify that $d = ca$, where

$$a = (u^{-1}v)(p_1^{n_1-m_1}p_2^{n_2-m_2}\cdots p_k^{n_k-m_k}).$$

Therefore, $c \mid d$. ■

Corollary 10.14

Every unique factorization domain satisfies the ascending chain condition on principal ideals.

Proof ▸ First, suppose (c) and (d) are principal ideals in a UFD R such that $(d) \subsetneqq (c)$. Then $c \mid d$ and c is not an associate of d by Lemma 10.9. If c and d are written in the form given by Theorem 10.13, then each $m_i \leq n_i$. If $m_i = n_i$ for every i, then $c = uv^{-1}d$, which means that c is an associate of d, a contradiction. Hence, there must be some index j for which $m_j < n_j$.

Suppose $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots$ is a chain of principal ideals in R. Lemma 10.9 shows that each a_i divides a_1. By Theorem 10.13 we may assume that $a_1 = vp_1^{n_1}p_2^{n_2} \cdots p_k^{n_k}$ and that each a_i is of the form $a_i = up_1^{m_1}p_2^{m_2} \cdots p_k^{m_k}$, where the p_j are nonassociate irreducibles. If there are just a finite number of strict inclusions ($\subsetneqq$) in the chain of ideals, then there are only equalities after a certain point and the ACC holds. There cannot be an infinite number of strict inclusions because the first paragraph shows that each time a strict inclusion occurs, one of the exponents on one of the p's must decrease. Consequently, after a finite number of strict inclusions, there would be an a_n of the form $a_n = up_1^0 \cdots = p_k^0 = u$. Thus a_n is a unit, which implies that $(a_n) = R$ by Exercise 9 of Section 6.1. For each $i \geq n$ we have $(a_n) \subseteq (a_i) \subseteq R = (a_n)$, so that $(a_n) = (a_i)$. Therefore, R satisfies the ACC on principal ideals. ∎

Irreducibles in a unique factorization domain have a property that we have used frequently in the special cases of Euclidean domains and principal ideal domains.

Theorem 10.15

Let p be an irreducible element in a unique factorization domain R. If $p \mid bc$, then $p \mid b$ or $p \mid c$.

Proof ▸ If b or c is 0_R, then there is nothing to prove because $p \mid 0_R$. If c is a unit and $p \mid bc$, then $pt = bc$ for some $t \in R$ and $ptc^{-1} = b$. Hence, $p \mid b$; similarly, if b is a unit, then $p \mid c$. If both b and c are nonzero nonunits, then $b = q_1 \cdots q_k$ and $c = q_{k+1} \cdots q_s$ with the q_i (not necessarily distinct) irreducibles. Since $p \mid bc$, we have $pr = bc = q_1 \cdots q_s$ for some $r \in R$. The irreducible p must be an associate of some q_t by unique factorization. Therefore, p divides q_i and, hence, divides b or c. ∎

We are now in a position to characterize unique factorization domains.

Theorem 10.16

An integral domain R is a unique factorization domain if and only if

(1) R has the ascending chain condition on principal ideals; and

(2) whenever p is irreducible in R and $p \mid cd$, then $p \mid c$ or $p \mid d$.

As the proof of the theorem shows, condition (1) corresponds to the existence of an irreducible factorization for each nonzero nonunit element and condition (2), to the uniqueness of this factorization. The two conditions are independent: (1) fails and (2) holds in $\mathbb{Q}_{\mathbb{Z}}[x]$ (see Example 1 and Exercise 33), whereas (1) holds and (2) fails in $\mathbb{Z}[\sqrt{-5}]$ (as we shall see in Example 4 and Exercise 21 of Section 10.3).

Proof of Theorem 10.16 ▸ If R is a UFD, then R satisfies (1) and (2) by Corollary 10.14 and Theorem 10.15. Conversely, assume R satisfies (1) and (2) and let a be a nonzero nonunit element of R. The argument used in the proof of Theorem 10.12, which depends only on the ACC, is valid here and shows that a can be factored as a product of irreducibles. To show that this factorization is unique, adapt the proof of Theorem 4.14 (the case when $R = F[x]$) to the general situation by replacing the word *constant* by *unit* and using (2) and Exercise 2 in place of Corollary 4.13. ■

Greatest Common Divisors

Greatest common divisors were a useful tool in our study of $\mathbb{Z}$, $F[x]$, and other Euclidean domains. In each case the gcd of two elements was defined to be a common divisor of "largest size," where size was measured by absolute value in $\mathbb{Z}$, by polynomial degree in $F[x]$, and by the function δ in an arbitrary Euclidean domain. Unfortunately, there may be no similar way to measure "size" in an arbitrary integral domain, so greatest common divisors must be defined in terms of divisibility properties alone:

Definition

Let $a_1, a_2, \ldots, a_n$ be elements (not all zero) of an integral domain R. A **greatest common divisor** of $a_1, a_2 \ldots, a_n$ is an element d of R such that

(i) d divides each of the a_i;

(ii) if $c \in R$ and c divides each of the a_i, then $c \mid d$.

Corollaries 1.3, 4.9, and 10.4 show that this definition is equivalent to the definitions used previously in $\mathbb{Z}$, $F[x]$, and other Euclidean domains. The only difference is that greatest common divisors in $\mathbb{Z}$ and $F[x]$, are no longer unique (see the discussion on page 326).

Theorem 10.17

Let d be a greatest common divisor of $a_1, a_2, \ldots, a_n$ in an integral domain R. Then

(1) Every associate of d is also a gcd of $a_1, \ldots, a_n$.

(2) Any two greatest common divisors of $a_1, \ldots, a_n$ are associates.

Proof ▸ (1) Exercise 7.

(2) Suppose both d and t are gcd's of $a_1, \ldots, a_n$. Then t divides each a_i, and, therefore, $t \mid d$ by (ii) in the definition of the greatest common divisor d. But d also divides each a_i, and, hence, $d \mid t$ by (ii) in the definition of the gcd t. Since $t \mid d$ and $d \mid t$, we know that d and t are associates by Exercise 4 of Section 10.1. ■

WARNING: In some integral domains a finite set of elements may not have a greatest common divisor (see Exercise 13 in Section 10.3).

Theorem 10.18

Let $a_1, a_2, \ldots, a_n$ (not all zero) be elements in a unique factorization domain R. Then $a_1, \ldots, a_n$ have a greatest common divisor in R.

Proof ▸ The gcd of any set of elements is the gcd of the nonzero members of the set, so we may assume that each a_i is nonzero. By Theorem 10.13 there are irreducibles $p_1, \ldots, p_t$ (no two of which are associates), units $u_1, \ldots, u_n$, and nonnegative integers m_{ij} such that

$$\begin{aligned} a_1 &= u_1 p_1^{m_{11}} p_2^{m_{12}} p_3^{m_{13}} \cdots p_t^{m_{1t}} \\ a_2 &= u_2 p_1^{m_{21}} p_2^{m_{22}} p_3^{m_{23}} \cdots p_t^{m_{2t}} \\ &\vdots \\ a_n &= u_n p_1^{m_{n1}} p_2^{m_{n2}} p_3^{m_{n3}} \cdots p_t^{m_{nt}}. \end{aligned}$$

Let k_1 be the smallest exponent that appears on p_1; that is, k_1 is the minimum of $m_{11}, m_{21}, m_{31}, \ldots, m_{n1}$. Similarly, let k_2 be the smallest exponent that appears on p_2, and so on. Use Theorem 10.13 to verify that $d = p_1^{k_1} p_2^{k_2} \ldots p_t^{k_t}$ is a gcd of $a_1, \ldots, a_n$. ■

In an arbitrary unique factorization domain, it may not be possible to write the gcd of elements a and b as a linear combination of a and b as it was in $\mathbb{Z}$ and $F[x]$. In Section 10.5, for example, we shall see that 1 is a gcd of the polynomials x and 2 in the UFD $\mathbb{Z}[x]$, but 1 is not a linear combination of x and 2 in $\mathbb{Z}[x]$ (Exercise 6). In a principal ideal domain, however, the gcd of a and b can always be written as a linear combination of a and b (Exercise 20).

Exercises

A. 1. If a, b are nonzero elements of an integral domain and a is a nonunit, prove that $(ab) \subsetneq (b)$.

2. Suppose p is an irreducible element in an integral domain R such that whenever $p \mid bc$, then $p \mid b$ or $p \mid c$. If $p \mid a_1 a_2 \cdots a_n$, prove that p divides at least one a_i.

3. (a) Prove that the only units in $\mathbb{Q}_{\mathbb{Z}}[x]$ are 1 and -1. [*Hint:* Theorem 4.2.]

   (b) If $f(x) \in \mathbb{Q}_{\mathbb{Z}}[x]$, show that its only associates are $f(x)$ and $-f(x)$.

4. Is a field a UFD?

5. Give an example to show that a subdomain of a unique factorization domain need not be a UFD.

6. Prove that 1 is not a linear combination of the polynomials 2 and x in $\mathbb{Z}[x]$, that is, prove it is impossible to find $f(x), g(x) \in \mathbb{Z}[x]$ such that $2f(x) + xg(x) = 1$.

7. Let d be a gcd of $a_1, \ldots, a_k$ in an integral domain. Prove that every associate of d is also a gcd of $a_1, \ldots, a_k$.

8. Let p be an irreducible element in an integral domain. Prove that 1_R is a gcd of p and a if and only if $p \nmid a$.

B. 9. Let R be a PID. If (c) is a nonzero ideal in R, then show that there are only finitely many ideals in R that contain (c). [*Hint:* Consider the divisors of c.]

10. Prove that an ideal (p) in a PID is maximal if and only if p is irreducible.

11. Prove that every ideal in a principal ideal domain R (except R itself) is contained in a maximal ideal. [*Hint:* Exercise 10.]

12. Prove that an ideal in a PID is prime if and only if it is maximal. [*Hint:* Exercise 10.]

13. Let $f: R \to S$ be a surjective homomorphism of rings with identity.
 (a) If R is a PID, prove that every ideal in S is principal.
 (b) Show by example that S need not be an integral domain.

14. Let p be a fixed prime integer and let R be the set of all rational numbers that can be written in the form a/b with b not divisible by p. Prove that
 (a) R is an integral domain containing $\mathbb{Z}$. [Note $n = n/1$].
 (b) If $a/b \in R$ and $p \nmid a$, then a/b is a unit in R.
 (c) If I is a nonzero ideal in R and $I \neq R$, then I contains p^t for some $t > 0$.
 (d) R is a PID. (If I is an ideal, show that $I = (p^k)$, where p^k is the smallest power of p in I.)

15. Let I be a nonzero ideal in $\mathbb{Z}[i]$. Show that the quotient ring $\mathbb{Z}[i]/I$ is finite.

16. (a) If p is prime in $\mathbb{Z}$, prove that the constant polynomial p is irreducible in $\mathbb{Q}_{\mathbb{Z}}[x]$. [*Hint:* Theorem 4.2 and Exercise 3.]
 (b) If p and q are positive primes in $\mathbb{Z}$ with $p \neq q$, prove that p and q are not associates in $\mathbb{Q}_{\mathbb{Z}}[x]$.

17. (a) Show that the only divisors of x in $\mathbb{Q}_{\mathbb{Z}}[x]$ are the integers (constant polynomials) and first-degree polynomials of the form $\frac{1}{n}x$ with $0 \neq n \in \mathbb{Z}$.
 (b) For each nonzero $n \in \mathbb{Z}$, show that the polynomial $\frac{1}{n}x$ is not irreducible in $\mathbb{Q}_{\mathbb{Z}}[x]$. [*Hint:* Theorem 10.1.]
 (c) Show that x cannot be written as a finite product of irreducible elements in $\mathbb{Q}_{\mathbb{Z}}[x]$.

18. A ring R is said to satisfy the **ascending chain condition (ACC) on ideals** if whenever $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is a chain of ideals in R (not necessarily principal ideals), then there is an integer n such that $I_j = I_n$ for all $j \geq n$. Prove that if every ideal in a commutative ring R is finitely generated, then R satisfies the ACC. [*Hint:* See Theorem 6.3 and adapt the proof of Lemma 10.10.]

19. A ring R is said to satisfy the **descending chain condition (DCC) on ideals** if whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a chain of ideals in R, then there is an integer n such that $I_j = I_n$ for all $j \geq n$.

(a) Show that $\mathbb{Z}$ does not satisfy the DCC.

(b) Show that an integral domain R is a field if and only if R satisfies the DCC. [*Hint:* If $0 \neq a \in R$ is not a unit, what can be said about the chain of ideals $(a) \supseteq (a^2) \supseteq (a^3) \supseteq \cdots$?]

20. Let R be a PID and $a, b \in R$, not both zero. Prove that a, b have a greatest common divisor that can be written as a linear combination of a and b. [*Hint:* Let I be the ideal generated by a and b (see Theorem 6.3); then $I = (d)$ for some $d \in R$. Show that d is a gcd of a and b.]

21. Let R be a PID and S an integral domain that contains R. Let $a, b, d \in R$. If d is a gcd of a and b in R, prove that d is a gcd of a and b in S. [*Hint:* See Exercise 20.]

22. Extend Exercise 20 to any finite number of elements.

23. Give an alternative proof of Lemma 10.11 as follows. If $p \mid b$, there is nothing to prove. If $p \nmid b$, then 1_R is a gcd of p and b by Exercise 8. Now show that $p \mid c$ by copying the proof of Theorem 1.4 with p in place of a and Exercise 20 in place of Theorem 1.2.

24. Let R be an integral domain. Prove that R is a PID if and only if (i) every ideal of R is finitely generated (Theorem 6.3) and (ii) whenever $a, b \in R$, the sum ideal $(a) + (b)$ is principal. [Sum is defined in Exercise 20 of Section 6.1.]

25. Let R be an integral domain in which any two elements (not both 0_R) have a gcd. Let (r, s) denote any gcd of r and s. Use $\sim$ to denote associates as in Exercise 6 of Section 10.1. Prove that for all $r, s, t \in R$:

(a) If $s \sim t$, then $rs \sim rt$.

(b) If $s \sim t$, then $(r, s) \sim (r, t)$.

(c) $r(s, t) \sim (rs, rt)$.

(d) $(r, (s, t)) \sim ((r, s), t)$. [*Hint:* Show that both are gcd's of r, s, t.]

26. Let R be an integral domain in which any two elements (not both 0_R) have a gcd. With the notation of Exercise 25, prove that if $(b, c) \sim 1_R$ and $(b, d) \sim 1_R$, then $(b, cd) \sim 1_R$. [*Hint:* By Exercise 25(a) and (c), $d \sim (bd, cd)$, so that $1_R \sim (b, d) \sim (b, (bd, cd))$. Apply parts (d), (c), and (a) of Exercise 25 to show that $(b, (bd, cd)) \sim (b, cd)$.

27. Let R be an integral domain in which any two elements (not both zero) have a gcd. Let p be an irreducible element of R. Prove that whenever $p \mid cd$, then $p \mid c$ or $p \mid d$. [*Hint:* Exercises 8 and 26.]

28. If R is a UFD, if a, b, and c are elements such that $a \mid c$ and $b \mid c$, and if 1_R is a gcd of a and b, prove that $ab \mid c$.

29. Let R be a UFD. If $a \mid bc$ and if 1_R is a gcd of a and b, prove that $a \mid c$.

30. A **least common multiple (lcm)** of the nonzero elements $a_1, \ldots, a_k$ is an element b such that (i) each a_i divides b and (ii) if each a_i divides an element c, then $b \mid c$. Prove that any finite set of nonzero elements in a UFD has a least common multiple.

31. Prove that nonzero elements a and b in R have a least common multiple if and only if the intersection of the principal ideals (a) and (b) is also a principal ideal.

C. 32. Prove that every ideal I in $\mathbb{Z}[\sqrt{d}]$ is finitely generated (Theorem 6.3) as follows. Let $I_0 = I \cap \mathbb{Z}$ and let $I_1 = \{b \in \mathbb{Z} \mid a + b\sqrt{d} \in I \text{ for some } a \in \mathbb{Z}\}$.

(a) Prove that I_0 and I_1 are ideals in $\mathbb{Z}$. Therefore, $I_0 = (r_0)$ and $I_1 = (r_1)$ for some $r_i \in \mathbb{Z}$.

(b) Prove that $I_0 \subseteq I_1$.

(c) By the definition of I_1 there exists $a_1 \in \mathbb{Z}$ such that $a_1 + r_1\sqrt{d}$ is in I. Prove that I is the ideal generated by r_0 and $a_1 + r_1\sqrt{d}$. [*Hint:* If $r + s\sqrt{d} \in I$, then $s \in I_1$ so that $s = r_1 s_1$. Show that $(r + s\sqrt{d}) - s_1(a_1 + r_1\sqrt{d}) \in I_0$; use this to write $r + s\sqrt{d}$ as a linear combination of r_0 and $a_1 + r_1\sqrt{d}$.]

33. Prove that $p(x)$ is irreducible in $\mathbb{Q}_{\mathbb{Z}}[x]$ if and only if $p(x)$ is either a prime integer or an irreducible polynomial in $\mathbb{Q}[x]$ with constant term ± 1. Conclude that every irreducible $p(x)$ in $\mathbb{Q}_{\mathbb{Z}}[x]$ has the property that whenever $p(x) \mid c(x)d(x)$, then $p(x) \mid c(x)$ or $p(x) \mid d(x)$.

34. Show that every nonzero $f(x)$ in $\mathbb{Q}_{\mathbb{Z}}[x]$ can be written in the form $cx^n p_1(x) \cdots p_k(x)$, with $c \in \mathbb{Q}$, $n \geq 0$, and each $p_i(x)$ nonconstant irreducible in $\mathbb{Q}_{\mathbb{Z}}[x]$ and that this factorization is unique in the following sense: If $f(x) = dx^m q_1(x) \cdots q_t(x)$ with $d \in \mathbb{Q}$, $m \geq 0$, and each $q_i(x)$ nonconstant irreducible in $\mathbb{Q}_{\mathbb{Z}}[x]$, then $c = \pm d$, $m = n$, $k = t$, and, after relabeling if necessary, each $p_i(x) = \pm q_i(x)$.

35. Prove that any two nonzero polynomials in $\mathbb{Q}_{\mathbb{Z}}[x]$ have a gcd.

36. **(a)** Prove that $f(x)$ is irreducible in $\mathbb{Z}[x]$ if and only if $f(x)$ is either a prime integer or an irreducible polynomial in $\mathbb{Q}[x]$ such that the gcd in $\mathbb{Z}$ of the coefficients of $f(x)$ is 1.

(b) Prove that $\mathbb{Z}[x]$ is a UFD. [*Hint:* See Theorems 4.14 and 4.23.]

10.3 Factorization of Quadratic Integers*

In this section we take a closer look at the domains $\mathbb{Z}[\sqrt{d}]$. Because unique factorization frequently fails in these domains, they provide a simplified model of the kinds of difficulties that played a crucial role in the historical origin of the concept of an ideal. These domains also illustrate how ideals can be used to "restore" unique factorization in some domains that lack it. We begin with a brief sketch of the relevant history.

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*The prerequisites for this section are pages 322–324 of Section 10.1 and the definition of unique factorization domain (page 337).

Early in the last century, Gauss proved the "Law of Biquadratic Reciprocity," which provides a fast way of determining whether or not a congruence of the form $x^4 \equiv c \pmod{n}$ has a solution. Although the statement of this theorem involves only integers, Gauss's proof was set in the larger domain $\mathbb{Z}[i]$. He proved and used the fact that $\mathbb{Z}[i]$ is a unique factorization domain.

Since Gauss's proof involved $\mathbb{Z}[i]$ and i is a complex fourth root of 1, the German mathematician E. Kummer thought that analogous theorems for congruences of degree p might involve unique factorization in the domain.

$$\mathbb{Z}[\omega] = \{a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-1}\omega^{p-1} \mid a_i \in \mathbb{Z}\},$$

where $\omega = \cos(2\pi/p) + i \sin(2\pi/p)$ is a complex pth root of 1. He was unable to develop higher-order reciprocity theorems because he discovered that $\mathbb{Z}[\omega]$ may not be a UFD.*

Later in the century questions about unique factorization arose in connection with the following problem. It is easy to find many nonzero integer solutions of the equation $x^2 + y^2 = z^2$, such as 3, 4, 5, or 5, 12, 13. But no one has ever found nonzero integer solutions for $x^3 + y^3 = z^3$ or $x^4 + y^4 = z^4$, which suggests that

$$x^n + y^n = z^n \text{ has no nonzero integer solutions when } n > 2.$$

This statement is known as **Fermat's Last Theorem** because in the late 1630s Fermat wrote it in the margin of his copy of Diophantus' *Arithmetica* and added "I have discovered a truly remarkable proof, but the margin is too small to contain it." Fermat's "proof" has never been found. Most mathematicians today doubt that he actually had a valid one.

In 1847 the French mathematician G. Lame thought he had found a proof of Fermat's Last Theorem in the case when n is prime.† His proof used the fact that for any odd positive prime p, $x^p + y^p$ can be factored in the domain $\mathbb{Z}[\omega]$ described above:

$$x^p + y^p = (x + y)(x + \omega y)(x + \omega^2 y) \cdots (x + \omega^{p-1} y).$$

Lame's purported proof depended on the assumption that $\mathbb{Z}[\omega]$ is a unique factorization domain. When he became aware of Kummer's work, he realized that his proof could not be carried through.

Kummer had already found a way to avoid the difficulty. He invented what he called "ideal numbers" and proved that unique factorization *does* hold for these ideal numbers. This work eventually led to a proof that Fermat's Theorem is true for a large class of primes, including almost all the primes less than 100. This was a remarkable breakthrough and deeply influenced later work on the problem.§ But it had even greater significance in the development of modern algebra. For Kummer's "ideal numbers" were what we now call ideals.

We shall return to ideals at the end of the section. Now we consider factorization in the domains $\mathbb{Z}[\sqrt{d}]$. These domains are similar to the ones that Kummer used and

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*The domain $\mathbb{Z}[\omega]$ is a UFD for every prime p less than 23 and fails to be a UFD for every larger prime.

†If the theorem is true for prime exponents, then it is true for all exponents; see Exercise 1.

§Fermat's Last Theorem was finally proved in 1994 by Andrew Wiles. His proof uses results and techniques not available until relatively recently.

illustrate in simplified form the problems he faced and his method of solution. We shall assume that the integer d is **square-free**, meaning that $d \neq 1$ and d has no integer factors of the form c^2 except $(\pm 1)^2$. The following function is the key to factorization in $\mathbb{Z}[\sqrt{d}]$.

Definition

The function $N: \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}$ given by

$$N(s + t\sqrt{d}) = (s + t\sqrt{d})(s - t\sqrt{d}) = s^2 - dt^2$$

is called the **norm.**

For example, in $\mathbb{Z}[\sqrt{3}]$,

$$N(5 + 2\sqrt{3}) = 5^2 - 3 \cdot 2^2 = 13 \quad \text{and} \quad N(2 - 4\sqrt{3}) = 2^2 - 3(-4)^2 = -44.$$

Note that

when $d < 0$, the norm of every element is nonnegative.

For instance, in $\mathbb{Z}[\sqrt{-5}]$,

$$N(s + t\sqrt{-5}) = s^2 - (-5)t^2 = s^2 + 5t^2 \geq 0.$$

In Example 7 of Section 10.1, we saw that the norm makes $\mathbb{Z}[i] = \mathbb{Z}[\sqrt{-1}]$ into a Euclidean domain. This is not true in general, but we do have

Theorem 10.19

If d is a square-free integer, then for all $a, b \in \mathbb{Z}[\sqrt{d}]$

(1) $N(a) = 0$ if and only if $a = 0$.

(2) $N(ab) = N(a)N(b)$.

Proof ▸ (1) If $a = s + t\sqrt{d}$, then $N(a) = s^2 - dt^2$ so that $N(a) = 0$ if and only if $s^2 = dt^2$. If $d = -1$, then $s^2 = -t^2$ can occur in $\mathbb{Z}$ if and only if $s = 0 = t$, that is, if and only if $a = 0$. So suppose $d\ -1$. Every prime in the factorization of s^2 and t^2 must occur an even number of times. But the prime factors of d do not repeat because d is square-free. So if p is a prime factor of d, it must occur an odd number of times in the factorization of dt^2. By unique factorization in $\mathbb{Z}$, the equation $s^2 = dt^2$ is impossible unless $s = 0 = t$, that is, unless $a = 0$.

(2) Let $a = r + s\sqrt{d}$ and $b = m + n\sqrt{d}$. The proof is a straightforward computation (Exercise 3). ■

Theorem 10.20

Let d be a square-free integer. Then $u \in \mathbb{Z}[\sqrt{d}]$ is a unit if and only if $N(u) = \pm 1$.

Proof ▶ If u is a unit, then $uv = 1$ for some $v \in \mathbb{Z}[\sqrt{d}]$. By Theorem 10.19, $N(u)N(v) = N(uv) = N(1) = 1^2 - d \cdot 0^2 = 1$. Since $N(u)$ and $N(v)$ are integers, the only possibilities are $N(u) = \pm 1$ and $N(v) = \pm 1$. Conversely, if $u = s + t\sqrt{d}$ and $N(u) = \pm 1$, let $\overline{u} = s - t\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$. Then by the definition of the norm, $u\overline{u} = N(u) = \pm 1$. Hence, $u(\pm\overline{u}) = 1$ and u is a unit. ■

EXAMPLE 1

In $\mathbb{Z}[\sqrt{2}]$ the element $3 + 2\sqrt{2}$ is a unit because $N(3 + 2\sqrt{2}) = 3^2 - 2 \cdot 2^2 = 1$. Verify that the inverse of $3 + 2\sqrt{2}$ is $3 - 2\sqrt{2}$. Every power of a unit is also a unit, so $\mathbb{Z}[\sqrt{2}]$ has infinitely many units, including $(3 + 2\sqrt{2}), (3 + 2\sqrt{2})^2, (3 + 2\sqrt{2})^3, \ldots$

According to Theorem 10.20 we can determine every unit $s + t\sqrt{d}$ in $\mathbb{Z}[\sqrt{d}]$ by finding all the integer solutions (for s and t) of the equations $s^2 - dt^2 = \pm 1$. When $d > 1$, these equations have infinitely many solutions (see the preceding example and Burton [12]). When $d = -1$, the equations reduce to $s^2 + t^2 = 1$.* The only integer solutions are $s = \pm 1, t = 0$, and $s = 0, t = \pm 1$. So the only units in $\mathbb{Z}[i] = \mathbb{Z}[\sqrt{-1}]$ are ± 1 and $\pm i$. If $d < -1$, say $d = -k$ with $k > 1$, then the equations reduce to $s^2 + kt^2 = 1$.* Since $k > 1$, the only integer solutions are $s = \pm 1, t = 0$. Thus we have

Corollary 10.21

Let d be a square-free integer. If $d > 1$, then $\mathbb{Z}[\sqrt{d}]$ has infinitely many units. The units in $\mathbb{Z}[\sqrt{-1}]$ are ± 1 and $\pm i$. If $d < -1$, then the units in $\mathbb{Z}[\sqrt{d}]$ are ± 1.

Corollary 10.22

Let d be a square-free integer. If $p \in \mathbb{Z}[\sqrt{d}]$ and $N(p)$ is a prime integer in $\mathbb{Z}$, then p is irreducible in $\mathbb{Z}[\sqrt{d}]$.

Proof ▶ Since $N(p)$ is prime, $N(p) \neq \pm 1$, so p is not a unit in $\mathbb{Z}[\sqrt{d}]$ by Theorem 10.20. If $p = ab$ in $\mathbb{Z}[\sqrt{d}]$, then by Theorem 10.19, $N(p) = N(a)N(b)$ in $\mathbb{Z}$. Since $N(a)$, $N(b)$, $N(p)$ are integers and $N(p)$ is prime, we must have $N(a) = \pm 1$ or $N(b) = \pm 1$. So a or b is a unit by Theorem 10.20. Therefore, p is irreducible by Theorem 10.1. ■

*Since the left side of the equation is always nonnegative, -1 cannot be on the right side.

EXAMPLE 2

The element $1 - i$ is irreducible in $\mathbb{Z}[i]$ because $N(1 - \sqrt{-1}) = 2$. Similarly, $1 + i$ is also irreducible. Therefore, a factorization of 2 as a product of irreducibles in $\mathbb{Z}[i]$ is given by $2 = (1 + i)(1 - i)$.

The converse of Corollary 10.22 is false. For instance, in $\mathbb{Z}[\sqrt{-5}]$ the norm of $1 + \sqrt{-5}$ is 6, which is not prime in $\mathbb{Z}$. But the next example shows that $1 + \sqrt{-5}$ *is* irreducible in $\mathbb{Z}[\sqrt{-5}]$.

EXAMPLE 3

To show that $1 + \sqrt{-5}$ is irreducible in $\mathbb{Z}[\sqrt{-5}]$, suppose $1 + \sqrt{-5} = ab$. By Theorem 10.1 we need only show that a or b is a unit. By Theorem 10.19, $N(a)N(b) = N(ab) = N(1 + \sqrt{-5}) = 6$. Since $N(a)$ and $N(b)$ are nonnegative integers, the only possibilities are $N(a) = 1, 2, 3$, or 6. If $a = s + t\sqrt{-5}$ and $N(a) = 2$, then $s^2 + 5t^2 = 2$. It is easy to see that this equation has no integer solutions for s and t; so $N(a) = 2$ is impossible. A similar argument shows that $N(a) = 3$ is impossible. If $N(a) = 1$, then a is a unit by Theorem 10.20. If $N(a) = 6$, then $N(b) = 1$ and b is a unit. Therefore, $1 + \sqrt{-5}$ is irreducible.

We have seen an example of an integral domain in which a nonzero, nonunit element could not be factored as a product of irreducibles (Exercise 17 in Section 10.2). We shall now see that $\mathbb{Z}[\sqrt{d}]$ may fail to be a UFD for a different reason: Although factorization as a product of irreducibles is always possible in $\mathbb{Z}[\sqrt{d}]$, it may not be unique.

Theorem 10.23

Let d be a square-free integer. Then every nonzero, nonunit element in $\mathbb{Z}[\sqrt{d}]$ is a product of irreducible elements.*

Proof ▸ Let S be the set of all nonzero, nonunits in $\mathbb{Z}[\sqrt{d}]$ that are *not* the product of irreducibles. We must show that S is empty. So suppose, on the contrary, that S is nonempty. Then the set $W = \{|N(t)| \mid t \in S\}$ is a nonempty set of positive integers. By the Well-Ordering Axiom, W contains a smallest integer. Thus there is an element $a \in S$ such that $|N(a)| \leq |N(t)|$ for every $t \in S$. Since $a \in S$ we know that a is not itself irreducible. So there exist nonunits $b, c \in \mathbb{Z}[\sqrt{d}]$ such that $a = bc$. At least one of b, c must be in S (otherwise a would be a product of irreducibles and, hence, not in S), say $b \in S$. Since b and c are nonunits, $|N(b)| > 1$ and $|N(c)| > 1$ by Theorem 10.20. But $|N(a)| = |N(b)||N(c)|$ by Theorem 10.19, so we must have $1 < |N(b)| < |N(a)|$. But $b \in S$, so $|N(a)| \leq |N(b)|$ by the choice of a. This is a contraction. Therefore, S is empty, and the theorem is proved. ■

*As usual, we allow a "product" with just one factor.

EXAMPLE 4

The domain $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain. The element 6 in $\mathbb{Z}[\sqrt{-5}]$ has two factorizations:

$$6 = 2 \cdot 3 \qquad \text{and} \qquad 6 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

The proof that $1 + \sqrt{-5}$ is irreducible was given in Example 3. The proofs that 2, 3, and $1 - \sqrt{-5}$ are irreducible are similar. For instance, if $2 = ab$, then $N(a)N(b) = N(ab) = N(2) = 4$ so that $N(a) = 1, 2$, or 4. But $N(a) = 2$ is impossible because the equation $s^2 + 5t^2 = 2$ has no integer solutions. So either $N(a) = 1$ and a is a unit, or $N(a) = 4$. In the latter case $N(b) = 1$ and b is a unit. Therefore, 2 is irreducible by Theorem 10.1. Since the only units in $\mathbb{Z}[\sqrt{-5}]$ are ± 1, it is clear that neither 2 nor 3 is an associate of $1 + \sqrt{-5}$ or $1 - \sqrt{-5}$. Thus the factorization of 6 as a product of irreducibles is not unique up to associates and $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

The preceding example demonstrates that the irreducible 2 divides the product $(1 + \sqrt{-5})(1 - \sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$ but does not divide either $1 + \sqrt{-5}$ or $1 - \sqrt{-5}$. So when unique factorization fails, an irreducible element p may not have the property that when $p \mid cd$, then $p \mid c$ or $p \mid d$.* Another consequence of the failure of unique factorization is the possible absence of greatest common divisors (Exercise 13).

Unique Factorization of Ideals

We are now in the position that Kummer was in a century and a half ago and the question is: How can some kind of unique factorization be restored in domains such as $\mathbb{Z}[\sqrt{-5}]$? Kummer's answer was to change the focus from elements to ideals.† The product IJ of ideals I and J is defined to be the set of all sums of elements of the form ab, with $a \in I$ and $b \in J$; that is,

$$IJ = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid n \geq 1, a_k \in I, b_k \in J\}.$$

Exercise 36 in Section 6.1 shows that IJ is an ideal. Instead of factoring an element a as a product of irreducibles, Kummer factored the principal ideal (a) as a product of prime ideals.

EXAMPLE 5

We shall express the principal ideal (6) in $\mathbb{Z}[\sqrt{-5}]$ as a product of prime ideals. The irreducible factorization of elements $6 = 2 \cdot 3$ seems a natural place to start, and it is easy to prove that the ideal (6) is the product ideal (2)(3) (Exercise 16). But (2) is not a prime ideal (for instance, the product $(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6$ is in (2) but neither of the factors is in (2)). So we must look elsewhere. Let P be the ideal in $\mathbb{Z}[\sqrt{-5}]$ generated by 2 and $1 + \sqrt{-5}$, that is,

$$P = \{2a + (1 + \sqrt{-5})b \mid a, b \in \mathbb{Z}[\sqrt{-5}]\}.$$

*This is not particularly surprising in view of Theorem 10.16.

†Kummer used different terminology, but the ideas here are essentially his. We use the modern terminology of ideals that was introduced by R. Dedekind, who generalized Kummer's theory.

Then P is an ideal by Theorem 6.3. Exercise 17 shows that $r + s\sqrt{-5} \in P$ if and only if r and s are both even or both odd. This implies that the only distinct cosets in $\mathbb{Z}[\sqrt{-5}]/P$ are $0 + P$ and $1 + P$, as we now see: If $m + n\sqrt{-5}$ has m odd and n even, then $(m + n\sqrt{-5}) - 1 = (m - 1) + n\sqrt{-5} \in P$ because $m - 1$ and n are even. Hence, $(m + n\sqrt{-5}) + P = 1 + P$. Similarly, if m is even and n is odd, then $(m - 1) + n\sqrt{-5} \in P$ because $m - 1$ and n are odd. It follows that the quotient ring $\mathbb{Z}[\sqrt{-5}]/P$ is isomorphic to $\mathbb{Z}_2$. Therefore, P is a prime ideal in $\mathbb{Z}[\sqrt{-5}]$ by Theorem 6.14. A similar argument (Exercise 19) shows that Q_1 and Q_2 are prime ideals, where

$$Q_1 = \{3a + (1 + \sqrt{-5})b \mid a, b \in \mathbb{Z}[\sqrt{-5}]\},$$
$$Q_2 = \{3a + (1 - \sqrt{-5})b \mid a, b \in \mathbb{Z}[\sqrt{-5}]\}.$$

Exercises 18 and 19 show that the product ideal $P^2 = PP$ is precisely the ideal (2) and that $Q_1Q_2 = (3)$. Therefore, the ideal (6) is a product of four prime ideals: $(6) = (2)(3) = P^2Q_1Q_2$.

Kummer went on to show that in the domains he was considering, the factorization of an ideal as a product of prime ideals is unique except for the order of the factors. This result was later generalized by R. Dedekind. In order to state this generalization precisely, we need to fill in some background.

An **algebraic number** is a complex number that is the root of some monic polynomial with rational coefficients. If t is an algebraic number and t is the root of a polynomial degree n in $\mathbb{Q}[x]$, then

$$\mathbb{Q}(t) = \{a_0 + a_1t + a_2t^2 + \cdots + a_{n-1}t^{n-1} \mid a_i \in \mathbb{Q}\}$$

is a subfield of $\mathbb{C}$ and every element in $\mathbb{Q}(t)$ is an algebraic number.* An **algebraic integer** is a complex number that is the root of some monic polynomial with *integer* coefficients. It can be shown that the set of all algebraic integers in $\mathbb{Q}(t)$ is an integral domain. If ω is a complex root of $x^p - 1$, then the domain $\mathbb{Z}[\omega]$ that Kummer used is in fact the domain of all algebraic integers in $\mathbb{Q}(\omega)$ (see Ireland and Rosen [13; page 199]). So Kummer's results are a special case of

Theorem 10.24

Let t be an algebraic number and R the domain of all algebraic integers in $\mathbb{Q}(t)$. Then every ideal in R (except 0 and R) is the product of prime ideals and this factorization is unique up to the order of the factors.

For a proof see Ireland and Rosen [13; page 174].

Most of the rings $\mathbb{Z}[\sqrt{d}]$ are also special cases of Theorem 10.24. For if d is a squarefree integer, then $t = \sqrt{d}$ is an algebraic number (because it is a root of $x^2 - d$) and $\mathbb{Q}(\sqrt{d}) = \{a_0 + a_1\sqrt{d} \mid a_i \in \mathbb{Q}\}$. The algebraic integers in the field $\mathbb{Q}(\sqrt{d})$ are called

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*For a proof see Theorems 11.7 and 11.9.

quadratic integers. Every element $r + s\sqrt{d}$ of $\mathbb{Z}[\sqrt{d}]$ is a quadratic integer in $\mathbb{Q}(\sqrt{d})$ because it is a root of this monic polynomial in $\mathbb{Z}[x]$:

$$x^2 - 2rx + (r^2 - ds^2) = (x - (r + s\sqrt{d}))(x - (r - s\sqrt{d})).$$

When $d \equiv 2$ or $3 \pmod 4$, then $\mathbb{Z}[\sqrt{d}]$ is the domain R of *all* quadratic integers in $\mathbb{Q}(\sqrt{d})$, but when $d \equiv 1 \pmod 4$, there are quadratic integers in R that are not in $\mathbb{Z}[\sqrt{d}]$ (see Exercise 22).*

Theorem 10.24 has proved very useful in algebraic number theory. But it does not answer many questions about unique factorization of elements, such as: If R is the domain of all quadratic integers in $\mathbb{Q}(\sqrt{d})$, for what values of d is R a UFD? When $d < 0$, R is a UFD if and only if $d = -1, -2, -3, -7, -11, -19, -43, -67$, or -163 (see Stark [19]). When $d > 0$, R is known to be a UFD for $d = 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 22, 23, 29$, and many other values. But there is no complete list as there is when d is negative. It is conjectured that R is a UFD for infinitely many values of d.

Exercises

A. 1. If $x^k + y^k = z^k$ has no nonzero integer solutions and $k \mid n$, then show that $x^n + y^n = z^n$ has no nonzero integer solutions.

2. Let ω be a complex number such that $\omega^p = 1$. Show that

$$\mathbb{Z}[\omega] = \{a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{p-1}\omega^{p-1} \mid a_i \in \mathbb{Z}\}$$

is an integral domain. [*Hint:* $\omega^p = 1$ implies $\omega^{p+1} = \omega$, $\omega^{p+2} = \omega^2$, etc.]

3. If $a = r + s\sqrt{d}$ and $b = m + n\sqrt{d}$ in $\mathbb{Z}[\sqrt{d}]$, show that $N(ab) = N(a)N(b)$.

4. Explain why $\mathbb{Z}[\sqrt{-5}]$ is not a Euclidean domain for any function δ.

5. If $a \in \mathbb{Q}$ is an algebraic integer, as defined on page 350, show that $a \in \mathbb{Z}$. [*Hint:* Theorem 4.21.]

B. 6. In which of these domains is 5 an irreducible element?

(a) $\mathbb{Z}$ (b) $\mathbb{Z}[i]$ (c) $\mathbb{Z}[\sqrt{-2}]$

7. In $\mathbb{Z}[\sqrt{-7}]$, factor 8 as a product of two irreducible elements and as a product of three irreducible elements. [*Hint:* Consider $(1 + \sqrt{-7})(1 - \sqrt{-7})$.]

8. Factor each of the elements below as a product of irreducibles in $\mathbb{Z}[i]$, [*Hint:* Any factor of a must have norm dividing $N(a)$.]

(a) 3 (b) 7 (c) $4 + 3i$ (d) $11 + 7i$

9. (a) Verify that each of $5 + \sqrt{2}$, $2 - \sqrt{2}$, $11 - 7\sqrt{2}$, and $2 + \sqrt{2}$ is irreducible in $\mathbb{Z}[\sqrt{2}]$.

*Since d is square-free, $d \not\equiv 0 \pmod 4$.

(b) Explain why the fact that

$$(5 + \sqrt{2})(2 - \sqrt{2}) = (11 - 7\sqrt{2})(2 + \sqrt{2})$$

does *not* contradict unique factorization in $\mathbb{Z}[\sqrt{2}]$.

10. Find two different factorizations of 9 as a product of irreducibles in $\mathbb{Z}[\sqrt{-5}]$.

11. Show that $\mathbb{Z}[\sqrt{-6}]$ is not a UFD. [*Hint:* Factor 10 in two ways.]

12. Show that $\mathbb{Z}[\sqrt{10}]$ is not a UFD. [*Hint:* Factor 6 in two ways.]

13. Show that 6 and $2 + 2\sqrt{-5}$ have no greatest common divisor in $\mathbb{Z}[\sqrt{-5}]$. [*Hint:* A common divisor a of 6 and $2 + 2\sqrt{-5}$ must have norm dividing both $N(6) = 36$ and $N(2 + 2\sqrt{-5}) = 24$; hence, $a = r + s\sqrt{-5}$ with $r^2 + 5s^2 = N(a) = 1, 2, 3, 4, 6$, or 12. Use this to find the common divisors. Verify that none of them is divisible by all the others, as required of a gcd. Also see Example 4.]

14. Show that 1 is a gcd of 2 and $1 + \sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$, but 1 cannot be written in the form $2a + (1 + \sqrt{-5})b$ with $a, b \in \mathbb{Z}[\sqrt{-5}]$.

15. Prove that every principal ideal in a UFD is a product of prime ideals uniquely except for the order of the factors.

16. Show that $(6) = (2)(3)$ in $\mathbb{Z}[\sqrt{-5}]$. (The product of ideals is defined on page 349.)

17. Let P be the ideal $\{2a + (1 + \sqrt{-5})b \mid a, b \in \mathbb{Z}[\sqrt{-5}]\}$ in $\mathbb{Z}[\sqrt{-5}]$. Prove that $r + s\sqrt{-5} \in P$ if and only if $r \equiv s \pmod 2$ (that is, r and s are both even or both odd).

18. Let P be as in Exercise 17. Prove that P^2 is the principal ideal (2).

19. Let Q_1 be the ideal $\{3a + (1 + \sqrt{-5})b \mid a, b \in \mathbb{Z}[\sqrt{-5}]\}$ and Q_2 the ideal $\{3a + (1 - \sqrt{-5})b \mid a, b \in \mathbb{Z}[\sqrt{-5}]\}$ in $\mathbb{Z}[\sqrt{-5}]$.

(a) Prove that $r + s\sqrt{-5} \in Q_1$ if and only if $r \equiv s \pmod 3$.

(b) Show that $\mathbb{Z}[\sqrt{-5}]/Q_1$ has exactly three distinct cosets.

(c) Prove that $\mathbb{Z}[\sqrt{-5}]/Q_1$ is isomorphic to $\mathbb{Z}_3$; conclude that Q_1 is a prime ideal.

(d) Prove that Q_2 is a prime ideal. [*Hint:* Adapt (a)–(c).]

(e) Prove that $Q_1Q_2 = (3)$.

20. If $r + s\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ with $s \neq 0$, then prove that 2 is not in the principal ideal $(r + s\sqrt{-5})$.

21. If d is a square-free integer, prove that $\mathbb{Z}[\sqrt{d}]$ satisfies the ascending chain condition on principal ideals.

C. 22. Let d be a square-free integer and let $\mathbb{Q}(\sqrt{d})$ be as defined on page 350. We know that $\mathbb{Z}[\sqrt{d}] \subseteq \mathbb{Q}(\sqrt{d})$ and every element of $\mathbb{Z}[\sqrt{d}]$ is a quadratic integer. Determine all the quadratic integers in $\mathbb{Q}(\sqrt{d})$ as follows.

(a) Show that every element of $\mathbb{Q}(\sqrt{d})$ is of the form $(r + s\sqrt{d})/t$, where $r, s, t \in \mathbb{Z}$ and the gcd (r, s, t) of r, s, t is 1. Hereafter, let $a = (r + s\sqrt{d})/t$ denote such an arbitrary element of $\mathbb{Q}(\sqrt{d})$.

(b) Show that a is a root of

$$p(x) = x^2 - \left(\frac{2r}{t}\right)x + \left(\frac{r^2 - ds^2}{t^2}\right) \in Q[x].$$

[*Hint:* Show that $p(x) = (x - a)(x - \bar{a})$, where $\bar{a} = (r - s\sqrt{d})/t$.]

(c) If $s \neq 0$, show that $p(x)$ is irreducible in $\mathbb{Q}[x]$.

(d) Prove that a is a quadratic integer if and only if $p(x)$ has integer coefficients. [*Hint:* If $s \neq 0$, use Exercise 5; if $s \neq 0$ and a is a root of a monic polynomial $f(x) \in \mathbb{Z}[x]$, use Theorem 4.23 to show that a is a root of some monic $g(x) \in \mathbb{Z}[x]$, with $g(x)$ irreducible in $\mathbb{Q}[x]$. Apply (c) and Theorem 4.14 to show $g(x) = p(x)$.]

(e) If a is a quadratic integer, show that $t \mid 2r$ and $t^2 \mid 4ds^2$. Use this fact to prove that t must be 1 or 2. [*Hint:* d is square-free, $(r, s, t) = 1$; use (b) and (d).]

(f) If $d \equiv 2$ or 3 (mod 4), show that a is a quadratic integer if and only if $t = 1$. [*Hint:* If $t = 2$, then $r^2 \equiv ds^2$ (mod 4) by (b) and (d). If s is even, reach a contradiction to the fact that $(r, s, t) = 1$; if s is odd, use Exercise 7 of Section 2.1 to get a contradiction.]

(g) If $d \equiv 1$ (mod 4) and $a \in \mathbb{Q}(\sqrt{d})$, show that a is a quadratic integer if and only if $t = 1$, or $t = 2$ and both r and s are odd. [*Hint:* Use (d).]

(h) Use (f) and (g) to show that the set of all quadratic integers in $\mathbb{Q}(\sqrt{d})$ is $\mathbb{Z}[\sqrt{d}]$ if $d \equiv 2$ or 3 (mod 4) and $\left\{\frac{m + n\sqrt{d}}{2} \mid m, n, \in Z \text{ and } m \equiv n (\text{mod } 2)\right\}$ if $d \equiv 1$ (mod 4).

10.4 The Field of Quotients of an Integral Domain*

For any integral domain R we shall construct a field F that contains R and consists of "quotients" of elements of R. When the domain R is $\mathbb{Z}$, then F will be the field $\mathbb{Q}$ of rational numbers. So you may view these proceedings either as a rigorous formalization of the construction of $\mathbb{Q}$ from $\mathbb{Z}$ or as a generalization of this construction to arbitrary integral domains. The field F will be the essential tool for studying factorization in $R[x]$ in Section 10.5.

Our past experience with rational numbers will serve as a guide for the formal development. But all the proofs will be independent of any prior knowledge of the rationals.

A rational number a/b is determined by the pair of integers a, b (with $b \neq 0$). But different pairs may determine the same rational number; for instance, $\frac{1}{2} = \frac{3}{6} = \frac{4}{8}$, and in general

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc.$$

*This section is independent of the rest of Chapter 10. Its prerequisites are Chapter 3 and Appendix D.

This suggests that the rationals come from some kind of equivalence relation on pairs of integers (equivalent pairs determine the same rational number). We now formalize this idea.

Let R be an integral domain and let S be this set of pairs:

$$S = \{(a, b) \mid a, b \in R \text{ and } b \neq 0_R\}.$$

Define a relation $\sim$ on the set S by

$$(a, b) \sim (c, d) \qquad \text{means} \qquad ad = bc \text{ in } R.$$

Theorem 10.25

The relation $\sim$ is an equivalence relation on S.

Proof ▸ *Reflexive:* Since r is commutative $ab = ba$, so that $(a, b) \sim (a, b)$ for every pair (a, b) in S. *Symmetric:* If $(a, b) \sim (c, d)$, then $ad = bc$. By commutativity $cb = da$, so that $(c, d) \sim (a, b)$. *Transitive:* Suppose that $(a, b) \sim (c, d)$ and $(c, d) \sim (r, s)$. Then $ad = bc$ and $cs = dr$. Multiplying $ad = bc$ by s and using $cs = dr$ we have $ads = (bc)s = b(cs) = bdr$. Since $d \quad 0_R$ by the definition of S and R is an integral domain we can cancel d from $ads = bdr$ and conclude that $as = br$. Therefore, $(a, b) \sim (r, s)$. ■

The equivalence relation $\sim$ partitions S into disjoint equivalence classes by Corollary D.2 in Appendix D. For convenience we shall denote the equivalence class of (a, b) by $[a, b]$ rather than the more cumbersome $[(a, b)]$. Let F denote the set of all equivalence classes under $\sim$. Note that by Theorem D.l,

$$[a, b] = [c, d] \text{ in } F \qquad \text{if and only if} \qquad (a, b) \sim (c, d) \text{ in } S.$$

Therefore, by the definition of $\sim$,

$$\boldsymbol{[a, b] = [c, d] \text{ in } F \qquad \text{if and only if} \qquad ad = bc \text{ in } R.}$$

We want to make the set F into a field. Addition and multiplication of equivalence classes are defined by

$$[a, b] + [c, d] = [ad + bc, bd]$$
$$[a, b][c, d] = [ac, bd].*$$

In order for this definition to make sense, we must first show that the quantities on the right side of the equal sign are actually elements of the set F. Now $[a, b]$ is the

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*These definitions are motivated by the arithmetical rules for rational numbers (just replace the fraction r/s by the equivalence class $[r, s]$):

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

equivalence class of the pair (a, b) in S. By the definition of S we have $b \neq 0_R$; similarly, $d \neq 0_R$. Since R is an integral domain, $bd \neq 0_R$. Thus $(ad + bc, bd)$ and (ac, bd) are in the set S, so that the equivalence classes $[ad + bc, bd]$ and $[ac, bd]$ *are* elements of F. But more is required in order to guarantee that addition and multiplication in F are well defined.

In ordinary arithmetic, $\frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$ and replacing $\frac{1}{2}$ by $\frac{4}{8}$ produces the same answer because $\frac{4}{8} \cdot \frac{3}{5} = \frac{12}{40} = \frac{3}{10}$. The answer doesn't depend on how the fractions are represented. Similarly, in F we must show that arithmetic does not depend on the way the equivalence classes are written:

Lemma 10.26

Addition and multiplication in F are independent of the choice of equivalence class representatives. In other words, if $[a, b] = [a', b']$ and $[c, d] = [c', d']$, then

$$[ad + bc, bd] = [a'd' + b'c', b'd']$$

and

$$[ac, bd] = [a'c', b'd'].$$

Proof ▸ As noted above $[ad + bc, bd] = [a'd' + b'c', b'd']$ in F if and only if $(ad + bc)b'd' = bd(a'd' + b'c')$ in R. So we shall prove this last statement. Since $[a, b] = [a', b']$ and $[c, d] = [c', d']$ we know that

$$(*) \qquad ab' = ba' \quad \text{and} \quad cd' = dc'.$$

Multiplying the first equation by dd' and the second by bb' and adding the results show that

$$\begin{aligned} ab'dd' &= ba'dd' \\ cd'bb' &= dc'bb' \\ \hline ab'dd' + cd'bb' &= ba'dd' + dc'bb' \\ (ad + bc)b'd' &= bd(a'd' + b'c'). \end{aligned}$$

Therefore, $[ad + bc, bd] = [a'd' + b'c', b'd']$.

For the second part of the proof multiply the first equation in $(*)$ by cd' and the second by ba' so that

$$ab'cd' = ba'cd' \qquad \text{and} \qquad cd'ba' = dc'ba'.$$

By commutativity the right side of the first equation is the same as the left side of the second equation so that the other sides of the two equations are equal: $ab'cd' = dc'ba'$. Consequently,

$$(ac)(b'd') = ab'cd' = dc'ba' = (bd)(a'c').$$

The two ends of this equation show that $[ac, bd] = [a'c', b'd']$. ■

Lemma 10.27

If R is an integral domain and F is as above, then for all nonzero $a, b, c, d, k \in R$:

(1) $[0_R, b] = [0_R, d]$;

(2) $[a, b] = [ak, bk]$;

(3) $[a, a] = [c, c]$.

Proof ▸ Exercise 1. ■

Lemma 10.28

With the addition and multiplication defined above, F is a field.

Proof ▸ Closure of addition and multiplication follows from Lemma 10.26 and the remarks preceding it. Addition is commutative in F because addition and multiplication in R are commutative:

$$[a, b] + [c, d] = [ad + bc, bd] = [cb + da, db] = [c, d] + [a, b].$$

Let 0_F be the equivalence class $[0_R, b]$ for *any* nonzero $b \in R$ (by (1) in Lemma 10.27 *all* pairs of the form $(0_R, b)$ with $b \neq 0_R$ are in the same equivalence class). If $[a, b] \in F$, then by (2) in Lemma 10.27 (with $k = b$):

$$[a, b] + 0_F = [a, b] + [0_R, b] = [ab + b0_R, bb] = [ab, bb] = [a, b].$$

Therefore, 0_F is the zero element of F. The negative of $[a, b]$ in F is $[-a, b]$ because

$$[a, b] + [-a, b] = (ab - ba, b^2] = [0_R, b^2] = 0_F.$$

The proofs that addition is associative and that multiplication is associative and commutative are left to the reader (Exercise 2), as is the verification that $[1_R, 1_R]$ is the multiplicative identity element in F. If $[a, b]$ is a nonzero element of F, then $a \neq 0_R$. Hence, $[b, a]$ is a well-defined element of F and by (3) in Lemma 10.27

$$[a, b][b, a] = [ab, ba] = [1_R ab, 1_R ab] = [1_R, 1_R].$$

Therefore, $[b, a]$ is the multiplicative inverse of $[a, b]$. To see that the distributive law holds in F, note that

$$\begin{aligned} [a, b]([c, d] + [r, s]) &= [a, b][cs + dr, ds] \\ &= [a(cs + dr), b(ds)] \\ &= [acs + adr, bds]. \end{aligned}$$

On the other hand, by (2) in Lemma 10.27 (with $k = b$)

$$\begin{aligned}[a, b][c, d] + [a, b][r, s] &= [ac, bd] + [ar, bs] \\ &= [(ac)(bs) + (bd)(ar), (bd)(bs)] \\ &= [(acs + adr)b, (bds)b] \\ &= [acs + adr, bds].\end{aligned}$$

Therefore, $[a, b]([c, d] + [r, s]) = [a, b][c, d] + [a, b][r, s]$. ■

We usually identify the integers with rational numbers of the form $a/1$. The same idea works in the general case:

Lemma 10.29

Let R be an integral domain and F the field of Lemma 10.28. Then the subset $R^* = \{[a, 1_R] \mid a \in R\}$ of F is an integral domain that is isomorphic to R.

Proof ▸ Verify that R^* is a subring of F (Exercise 3). Clearly $[1_R, 1_R]$, the identity element of F, is in R^*, so R^* is an integral domain. Define a map $f: R \to R^*$ by $f(a) = [a, 1_R]$. Then f is a homomorphism:

$$\begin{aligned}f(a) + f(c) &= [a, 1_R] + [c, 1_R] = [a1_R + 1_Rc, 1_R1_R] \\ &= [a + c, 1_R] = f(a + c) \\ f(a)f(c) &= [a, 1_R][c, 1_R] = [ac, 1_R] = f(ac).\end{aligned}$$

If $f(a) = f(c)$, then $[a, 1_R] = [c, 1_R]$, which implies that $a1_R = 1_Rc$ by the boldface statement following Theorem 10.25. Thus $a = c$ and f is injective. Since f is obviously surjective, f is an isomorphism. ■

The equivalence class notation for elements of F is awkward and doesn't convey the promised idea of "quotients". This is easily remedied by a change of notation. Instead of denoting the equivalence class of (a, b) by $[a, b]$,

denote the equivalence class of (a, b) by a/b.

If we translate various statements above from the brackets notation to the new quotient notation, things begin to look quite familiar:

Theorem 10.30

Let R be an integral domain. Then there exists a field F whose elements are of the form a/b with $a, b \in R$ and $b \neq 0_R$, subject to the equality condition

$$\frac{a}{b} = \frac{c}{d} \text{ in } F \qquad \text{if and only if} \qquad ad = bc \text{ in } R.$$

Addition and multiplication in F are given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

The set of elements in F of the form $a/1_R$ $(a \in R)$ is an integral domain isomorphic to R.

Proof ▸ Lemmas 10.28 and 10.29 and the notation change preceding the theorem.* ■

It is now clear that if $R = \mathbb{Z}$, then the field F is precisely $\mathbb{Q}$. So Theorem 10.30 may be taken as a formal construction of $\mathbb{Q}$ from $\mathbb{Z}$. In the general case, we shall follow the same custom we use with $\mathbb{Q}$: The ring R will be *identified* with its isomorphic copy in F. Then we can say that R is the subset of F consisting of elements of the form $a/1_R$. The field F is called the **field of quotients** of R.

EXAMPLE 1

Let F be a field. The field of quotients of the polynomial domain $F[x]$ is denoted by $F(x)$ and consists of all $f(x)/g(x)$, where $f(x), g(x) \in F[x]$ and $g(x) \neq 0_K$. The field $F(x)$ is called the **field of rational functions** over F.

The field of quotients of an integral domain R is the smallest field that contains R in the following sense.†

Theorem 10.31

Let R be an integral domain and F its field of quotients. If K is a field containing R, then K contains a subfield E such that $R \subseteq E \subseteq K$ and E is isomorphic to F.

Proof ▸ If $a/b \in F$, then $a, b \in R$ and b is nonzero. Since $R \subseteq K$, b^{-1} exists. Define a map $f: F \to K$ by $f(a/b) = ab^{-1}$. Exercise 9 shows that f is well defined, that is, $a/b = c/d$ in F implies $f(a/b) = f(c/d)$ in K. Exercise 10 shows that f is an injective homomorphism. If E is the image of F under f, then $F \cong E$. For each $a \in R$, $a = a1_R{}^{-1} = f(a/1_R) \in E$, so $R \subseteq E \subseteq K$. ■

Exercises

NOTE: *Unless noted otherwise, R is an integral domain and F its field of quotients.*

A. 1. Prove Lemma 10.27.

2. Complete the proof of Lemma 10.28 by showing that

   (a) Addition of equivalence classes is associative.

   (b) Multiplication of equivalence classes is associative.

   (c) Multiplication of equivalence classes is commutative.

3. Show that $R^* = \{[a, 1_R] \mid a \in R\}$ is a subring of F.

*At this point you may well ask, "Why didn't we adopt the quotient notation sooner?" The reason is psychological rather than mathematical. The quotient notation makes things look so much like the familiar rationals that there is a tendency to *assume* everything works like it always did, instead of actually carrying out the formal (and tiresome) details of the rigorous development.

†Theorem 10.31 is not used in the sequel.

B. 4. If R is itself a field, show that $R = F$.

5. If $R = \mathbb{Z}[i]$, then show that $F \cong \{r + si \mid r, s \in \mathbb{Q}\}$.

6. If $R = \mathbb{Z}[\sqrt{d}]$, then show that $F \cong \{r + s\sqrt{d} \mid r, s \in \mathbb{Q})$.

7. Show that there are infinitely many integral domains R such that $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$, each of which has $\mathbb{Q}$ as its field of quotients. [*Hint:* Exercise 28 in Section 3.1.]

8. Let $f: R \to R_1$ be an isomorphism of integral domains. Let F be the field of quotients of R and F_1 the field of quotients of R_1. Prove that the map $f^*: F \to F_1$ given by $f^*(a/b) = f(a)/f(b)$ is an isomorphism.

9. If R is contained in a field K and $a/b = c/d$ in F, show that $ab^{-1} = cd^{-1}$ in K. [*Hint:* $a/b = c/d$ implies $ad = bc$ in K.]

10. **(a)** Prove that the map f in the proof of Theorem 10.31 is injective. [*Hint:* $f(a/b) = f(c/d)$ implies $ab^{-1} = cd^{-1}$; show that $ad = bc$.]

    (b) Use a straightforward calculation to show that f is a homomorphism.

11. Let $a, b \in R$. Assume there are positive integers m, n such that $a^m = b^m$, $a^n = b^n$, and $(m, n) = 1$. Prove that $a = b$. [Remember that negative powers of a and b are not necessarily defined in R, but they do make sense in the field F; for instance, $a^{-2} = 1_R/a^2$.]

12. Let R be an integral domain of characteristic 0 (see Exercises 41–43 in Section 3.2).

    (a) Prove that R has a subring isomorphic to $\mathbb{Z}$ [*Hint:* Consider $\{n1_R \mid n \in \mathbb{Z}\}$.]

    (b) Prove that a field of characteristic 0 contains a subfield isomorphic to $\mathbb{Q}$. [*Hint:* Theorem 10.31.]

13. Prove that Theorem 10.30 is valid when R is a commutative ring with no zero divisors (not necessarily an integral domain). [*Hint:* Show that for any nonzero $a \in R$, the class $[a, a]$ acts as a multiplicative identity for F and the set $\{[ra, a] \mid r \in R\}$ is a subring of F that is isomorphic to R. The even integers are a good model of this situation.]

10.5 Unique Factorization in Polynomial Domains*

Throughout this section R is a unique factorization domain. We shall prove that the polynomial ring $R[x]$ is also a UFD. The basic idea of the proof is quite simple: Given a polynomial $f(x)$, factor it repeatedly as a product of polynomials of lower degree until $f(x)$ is written as a product of irreducibles. To prove uniqueness, consider $f(x)$ as

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*The prerequisites for this section are pages 322–324 of Section 10.1, the definition of unique factorization domain (together with Theorems 10.13, 10.15, and 10.18), and Section 10.4. Theorems 10.13, 10.15, and 10.18 depend only on the definition of UFD and may be read independently of the rest of Section 10.2.

a polynomial in $F[x]$, where F is the field of quotients of R. Use the fact that $F[x]$ is a UFD (Theorem 4.14) to show that factorization in $R[x]$ is unique. There are some difficulties, however, in carrying out this program.

EXAMPLE 1

The polynomial $3x^2 + 6$ cannot be factored as a product of two polynomials of lower degree in $\mathbb{Z}[x]$ and is irreducible in $\mathbb{Q}[x]$. But $3x^2 + 6$ is *reducible* in $\mathbb{Z}[x]$ because $3x^2 + 6 = 3(x^2 + 2)$ and neither 3 nor $x^2 + 2$ is a unit in $\mathbb{Z}[x]$.

So the first step is to examine the role of constant polynomials in $R[x]$. By Corollary 4.5 and Exercise 1

the units in $R[x]$ are the units in R

and

the irreducible constant polynomials in $R[x]$ are the irreducible elements of R.

For example, the units of $\mathbb{Z}[x]$ are ± 1. The constant polynomial 3 is irreducible in $\mathbb{Z}[x]$ even though it is a unit in $\mathbb{Q}[x]$.

The constant irreducible factors of a polynomial in $R[x]$ may be found by factoring out any constants and expressing them as products of irreducible elements in R.

EXAMPLE 2

In $\mathbb{Z}[x]$,

$$6x^2 + 18x + 12 = 6(x^2 + 3x + 2) = 2 \cdot 3(x^2 + 3x + 2).$$

Note that $x^2 + 3x + 2$ is a polynomial whose only *constant* divisors in $\mathbb{Z}[x]$ are the units ± 1. This example suggests a strategy for the general case.

Let R be a unique factorization domain. A nonzero polynomial in $R[x]$ is said to be **primitive** if the only constants that divide it are the units in R. For instance, $x^2 + 3x + 2$ and $3x^4 - 5x^3 + 2x$ are primitive in $\mathbb{Z}[x]$. Primitive polynomials of degree 0 are units. Every primitive polynomial of degree 1 must be irreducible by Theorem 10.1 (because every factorization includes a constant (Theorem 4.2) and every such constant must be a unit). However, primitive polynomials of higher degree need not be irreducible (such as $x^2 + 3x + 2 = (x + 1)(x + 2)$ in $\mathbb{Z}[x]$). On the other hand, an irreducible polynomial of positive degree has no constant divisors except units by Theorems 4.2 and 10.1. So

an irreducible polynomial of positive degree is primitive.

Furthermore, as the example illustrates,

every nonzero polynomial $f(x) \in R[x]$ factors as $f(x) = cg(x)$ with $g(x)$ primitive.

To prove this claim, let c be a greatest common divisor of the coefficients of $f(x)$.* Then $f(x) = cg(x)$ for some $g(x)$. Now we show that $g(x)$ is primitive. If $d \in R$ divides $g(x)$, then $g(x) = dh(x)$ so that $f(x) = cdh(x)$. Since cd is a constant divisor of $f(x)$, it must divide the coefficients of $f(x)$ and, hence, must divide the gcd c. Thus $cdu = c$ for some $u \in R$. Since $c \neq 0_R$ we see that $du = 1_R$ and d is a unit. Therefore, $g(x)$ is primitive.

Using these facts about primitive polynomials, we can now modify the argument given at the beginning of the section and prove the first of the two conditions necessary for $R[x]$ to be a UFD.

Theorem 10.32

Let R be a unique factorization domain. Then every nonzero, nonunit $f(x)$ in $R[x]$ is a product of irreducible polynomials.†

Proof▸ Let $f(x) = cg(x)$ with $g(x)$ primitive. Since R is a UFD c is either a unit or a product of irreducible elements in R (and, hence, in $R[x]$). So we need to prove only that $g(x)$ is either a unit or a product of irreducibles in $R[x]$. If $g(x)$ is a unit or is itself irreducible, there is nothing to prove. If not, then by Theorem 10.1 $g(x) = h(x)k(x)$ with neither $h(x)$ or $k(x)$ a unit. Since $g(x)$ is primitive, its only divisors of degree 0 are units, so we must have $0 < \deg h(x) < \deg g(x)$ and $0 < \deg k(x) < \deg g(x)$. Furthermore, $h(x)$ and $k(x)$ are primitive (any constant that divides one of them must divide $g(x)$ and hence be a unit). If they are irreducible, we're done. If not, we can repeat the preceding argument and factor them as products of primitive polynomials of lower degree, and so on. This process must stop after a finite number of steps because the degrees of the factors get smaller at each stage and every primitive polynomial of degree 1 is irreducible. So $g(x)$ is a product of irreducibles in $R[x]$. ■

The proof that factorization in $R[x]$ is unique depends on several technical facts that will be developed next. But to get an idea of how all the pieces fit together, you may want to read the proof of Theorem 10.38 now, referring to the intermediate results as needed and accepting them without proof. Then you can return to this point and read the proofs, knowing where the argument is headed.

Lemma 10.33

Let R be a unique factorization domain and $g(x), h(x) \in R[x]$. If p is an irreducible element of R that divides $g(x)h(x)$, then p divides $g(x)$ or p divides $h(x)$.

Proof▸ Copy the proof of Lemma 4.22, which is the special case $R = \mathbb{Z}$. Just replace $\mathbb{Z}$ by R and *prime* by *irreducible* and use Theorem 10.15 in place of Theorem 1.5. ■

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*The gcd c exists by Theorem 10.18.

†As usual we allow a "product" with just one factor.

Corollary 10.34 Gauss's Lemma

Let R be a unique factorization domain. Then the product of primitive polynomials in $R[x]$ is primitive.

Proof ▸ If $g(x)$ and $h(x)$ are primitive and $g(x)h(x)$ is not, then $g(x)h(x)$ is divisible by some nonunit $c \in R$. Consequently, each irreducible factor p of c divides $g(x)h(x)$. By Lemma 10.33, p divides $g(x)$ or $h(x)$, contradicting the fact that they are primitive. Therefore, $g(x)h(x)$ is primitive. ■

Theorem 10.35

Let R be a unique factorization domain and r, s nonzero elements of R. Let $f(x)$ and $g(x)$ be primitive polynomials in $R[x]$ such that $rf(x) = sg(x)$. Then r and s are associates in R and $f(x)$ and $g(x)$ are associates in $R[x]$.

Proof ▸ If r is a unit, then $f(x) = r^{-1}sg(x)$. Since $r^{-1}s$ divides the primitive polynomial $f(x)$, it must be a unit, say $(r^{-1}s)u = 1_R$. Hence, $f(x)$ and $g(x)$ are associates in $R[x]$. Furthermore, u is a unit in R and $su = r$ so that r and s are associates in R.

If r is a nonunit, then $r = p_1p_2 \cdots p_k$ with each p_i irreducible. Then $p_1p_2 \cdots p_kf(x) = sg(x)$, so p_1 divides $sg(x)$. By Lemma 10.33 p_1 divides s or $g(x)$. Since p_1 is a nonunit and $g(x)$ is primitive, p_1 must divide s, say $s = p_1t$. Then $p_1p_2 \cdots p_kf(x) = sg(x) = p_1tg(x)$. Canceling p_1 shows that $p_2 \cdots p_kf(x) = tg(x)$. Repeating the argument with p_2 shows that $p_3 \cdots p_kf(x) = zg(x)$, where $p_2z = t$ and, hence, $p_1p_2z = p_1t = s$. After k such steps we have $f(x) = wg(x)$ and $s = p_1p_2 \cdots p_kw$ for some $w \in R$. Since w divides the primitive polynomial $f(x)$, w is a unit. Therefore, $f(x)$ and $g(x)$ are associates in $R[x]$. Since $s = p_1 \cdots p_kw = rw$, r and s are associates in R. ■

Corollary 10.36

Let R be a unique factorization domain and F its field of quotients. Let $f(x)$, $g(x)$ be primitive polynomials in $R[x]$. If $f(x)$ and $g(x)$ are associates in $F[x]$, then they are associates in $R[x]$.

Proof ▸ If $f(x)$ and $g(x)$ are associates in $F[x]$, then $g(x) = \frac{r}{s}f(x)$ for some nonzero $\frac{r}{s} \in F$ by Corollary 4.5. Consequently, $sg(x) = rf(x)$ in $R[x]$. Therefore, $f(x)$ and $g(x)$ are associates in $R[x]$ by Theorem 10.35. ■

Corollary 10.37

Let R be a unique factorization domain and F its field of quotients. If $f(x) \in R[x]$ has positive degree and is irreducible in $R[x]$, then $f(x)$ is irreducible in $F[x]$.

Proof ▸ If $f(x)$ is not irreducible in $F[x]$, then $f(x) = g(x)h(x)$ for some $g(x)$, $h(x) \in F[x]$ with positive degree. Let b be a least common denominator of the coefficients of $g(x)$. Then $bg(x)$ has coefficients in R. So $bg(x) = ag_1(x)$ with $a \in R$ and $g_1(x)$ primitive of positive degree in $R[x]$. Hence, $g(x) = \frac{a}{b}g_1(x)$. Similarly $h(x) = \frac{c}{d}h_1(x)$ with c, $d \in R$ and $h_1(x)$ primitive of positive degree in $R[x]$. Therefore, $f(x) = g(x)h(x) = \frac{a}{b}g_1(x)\frac{c}{d}h_1(x) = \frac{ac}{bd}g_1(x)h_1(x)$, so that $bdf(x) = acg_1(x)h_1(x)$ in $R[x]$. Now $f(x)$ is primitive because it is irreducible and $g_1(x)h_1(x)$ is primitive by Corollary 10.34. So bd is an associate of ac by Theorem 10.35, say $bdu = ac$ for some unit $u \in R$. Therefore, $f(x) = \frac{ac}{bd}g_1(x)h_1(x) = ug_1(x)h_1(x)$. Since $ug_1(x)$ and $h_1(x)$ are polynomials of positive degree in $R[x]$, this contradicts the irreducibility of $f(x)$. Therefore, $f(x)$ must be irreducible in $F[x]$. ■

Theorem 10.38

If R is a unique factorization domain, then so is $R[x]$.

Proof ▸ Every nonzero nonunit $f(x)$ in $R[x]$ is a product of irreducibles by Theorem 10.32. Any such factorization consists of irreducible constants (that is, irreducibles in R) and irreducible polynomials of positive degree. Suppose

$$c_1 \cdots c_m p_1(x) \cdots p_k(x) = d_1 \cdots d_n q_1(x) \cdots q_t(x)$$

with each c_i, d_j irreducible in R and each $p_i(x)$, $q_j(x)$ irreducible of positive degree in $R[x]$ (and, hence, primitive).* Then $p_1(x) \cdots p_k(x)$ and $q_1(x) \cdots q_t(x)$ are primitive by Corollary10.34. So Theorem 10.35 shows that $c_1 \cdots c_m$ is an associate of $d_1 \cdots d_n$ in R and $p_1(x) \cdots p_k(x)$ is an associate of $q_1(x) \cdots q_t(x)$ in $R[x]$. Hence, $c_1 \cdots c_m = ud_1d_2 \cdots d_n$ for some unit $u \in R$. Associates of irreducibles are irreducible (Exercise 7 of Section 10.1), so ud_1 is irreducible. Since R is a UFD, we must have $m = n$ and (after relabeling if necessary) c_1 is an associate of ud_1 (and hence of d_1), and c_i is an associate of d_i for $i \geq 2$. Let F be the field of quotients of R. Each of the $p_i(x)$, $q_j(x)$ is irreducible in $F[x]$ by Corollary 10.37. Unique factorization in $F[x]$ (Theorem 4.14) and an argument similar to the one just given for R show that $k = t$ and (after relabeling if necessary) each $p_i(x)$ is an associate of $q_i(x)$ in $F[x]$. Consequently, $p_i(x)$ and $q_i(x)$ are associates in $R[x]$ by Corollary 10.36. Therefore, $R[x]$ is a UFD. ■

*It may be that neither factorization contains constants, but this doesn't affect the argument. It is not possible to have irreducible constants in one factorization but not in the other (Exercise 5).

An immediate consequence of Theorems 1.8 and 10.38 and Example 8 of Section 6.1 is

Corollary 10.39

$\mathbb{Z}[x]$ is a unique factorization domain that is not a principal ideal domain.

As illustrated in the preceding discussion, theorems about $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are quite likely to carry over to an arbitrary UFD and its field of quotients. Among such results are the Rational Root Test and Eisenstein's Criterion (Exercises 9–11).

Exercises

NOTE: *Unless stated otherwise R is a UFD and F its field of quotients.*

A. **1.** Let R be any integral domain and $p \in R$. Prove that p is irreducible in R if and only if the constant polynomial p is irreducible in $R[x]$. [*Hint:* Corollary 4.5 may be helpful.]

2. Give an example of polynomials $f(x), g(x) \in R[x]$ such that $f(x)$ and $g(x)$ are associates in $F[x]$ but not in $R[x]$. Does this contradict Corollary10.36?

3. If $c_1 \cdots c_m f(x) = g(x)$ with $c_i \in R$ and $g(x)$ primitive in $R[x]$, prove that each c_i is a unit.

4. If $g(x)$ is primitive in $R[x]$, prove that every nonconstant polynomial in $R[x]$ that divides $g(x)$ is also primitive.

B. **5.** Prove that a polynomial is primitive if and only if 1_R is a greatest common divisor of its coefficients. This property is often taken as the definition of primitive.

6. If $f(x)$ is primitive in $R[x]$ and irreducible in $F[x]$, prove that $f(x)$ is irreducible in $R[x]$.

7. If R is a ring such that $R[x]$ is a UFD, prove that R is a UFD.

8. If R is a ring such that $R[x]$ is a principal ideal domain, prove that R is a field.

9. Verify that the Rational Root Test (Theorem 4.21) is valid with $\mathbb{Z}$ and $\mathbb{Q}$ replaced by R and F.

10. Verify that Theorem 4.23 is valid with $\mathbb{Z}$ and $\mathbb{Q}$ replaced by R and F.

11. Verify that Eisenstein's Criterion (Theorem 4.24) is valid with $\mathbb{Z}$ and $\mathbb{Q}$ replaced by R and F and *prime* replaced by *irreducible*.

12. Show that $x^3 - 6x^2 + 4ix + 1 + 3i$ is irreducible in $(\mathbb{Z}[i])[x]$. [*Hint:* Exercise 11.]

CHAPTER 11

Field Extensions

High-school algebra deals primarily with the three fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ and plane geometry, with the set $\mathbb{R} \times \mathbb{R}$. Calculus is concerned with functions from $\mathbb{R}$ to $\mathbb{R}$. Indeed, most classical mathematics is set in the field $\mathbb{C}$ and its subfields. Other fields play an equally important role in more recent mathematics. They are used in analysis, algebraic geometry, and parts of number theory, for example, and have numerous applications, including coding theory and algebraic cryptography.

In this chapter we develop the basic facts about fields that are needed to prove some famous results in the theory of equations (Chapter 12) and to study some of the topics listed above. The principal theme is the relationship of a field with its various subfields.

11.1 Vector Spaces

An essential tool for the study of fields is the concept of a vector space, which is introduced in this section. Vector spaces are treated in detail in books and courses on linear algebra. Here we present only those topics that are needed for our study of fields. If you have had a course in linear algebra, you can probably skip most of this section. Nevertheless, it would be a good idea to review the main results, particularly Theorems 11.4 and 11.5.

Consider the additive abelian group* $M(\mathbb{R})$ of all 2×2 matrices over the field $\mathbb{R}$ of real numbers. If r is a real number and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $M(\mathbb{R})$, then the

*Except for the last two results in the chapter, group theory is not a prerequisite for this chapter. In this section you need only know that an additive abelian group is a set with an addition operation that satisfies Axioms 1–5 in the definition of a ring (page 44).

product of the number r and the matrix A is defined to be the matrix $rA = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$. This operation, which is called *scalar multiplication*, takes a real number (field element) and a matrix (group element) and produces another matrix (group element). This is an example of a more general concept. Let F be a field and G an additive abelian group.* Then a **scalar multiplication** is an operation such that for each $a \in F$ and each $v \in G$ there is a unique element $av \in G$.

Definition

Let F be a field. A **vector space over F** is an additive abelian group* V equipped with a scalar multiplication such that for all $a, a_1, a_2 \in F$ and $v, v_1, v_2 \in V$:

(i) $a(v_1 + v_2) = av_1 + av_2$;
(ii) $(a_1 + a_2)v = a_1v + a_2v$;
(iii) $a_1(a_2v) = (a_1a_2)v$;
(iv) $1_Fv = v$.

EXAMPLE 1

Scalar multiplication in $M(\mathbb{R})$, as defined above, makes $M(\mathbb{R})$ into a vector space over $\mathbb{R}$ (Exercise 1).

EXAMPLE 2

Consider the set $\mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. Then $\mathbb{Q}^2$ is a group under addition (Theorem 3.1 or 7.4); its zero element is $(0, 0)$ and the negative of (s, t) is $(-s, -t)$. For $a \in \mathbb{Q}$ and $(s, t) \in \mathbb{Q}^2$, scalar multiplication is defined by $a(s, t) = (as, at)$. Under these operations $\mathbb{Q}^2$ is a vector space over $\mathbb{Q}$ (Exercise 2).

EXAMPLE 3

The preceding example can be generalized as follows. If F is any field and $n \geq 1$ an integer, let $F^n = F \times F \times \cdots \times F$ (n summands). Then F^n is a vector space over F, with addition defined coordinatewise:

$$(s_1, s_2, \ldots, s_n) + (t_1, t_2, \ldots, t_n) = (s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$$

and scalar multiplication defined by:

$$a(s_1, s_2, \ldots, s_n) = (as_1, as_2, \ldots, as_n) \qquad a \in F$$

(see Exercise 5).

*See the preceding footnote.

EXAMPLE 4

The complex numbers $\mathbb{C}$ form a vector space over the real numbers $\mathbb{R}$, with addition of complex numbers (vectors) defined as usual and with scalar multiplication being ordinary multiplication (the product of a real number and a complex number is a complex number).

Special terminology is used in situations like the preceding example. If F and K are fields with $F \subseteq K$, we say that K is an **extension field** of F. For instance, the complex numbers $\mathbb{C}$ are an extension field of the field $\mathbb{R}$ of real numbers. As the preceding example shows, the extension field $\mathbb{C}$ can be considered as a vector space over $\mathbb{R}$. The same thing is true in the general case.

If K is an extension field of F, then K is a vector space over F, with addition of vectors being ordinary addition in K and scalar multiplication being ordinary multiplication in K

(the product of an element the subfield F and an element of K is an element of K). For the purposes of this chapter, extension fields are the most important examples of vector spaces.

If V is a vector space over a field F, then the following properties hold for any $v \in V$ and $a \in F$ (Exercise 21):

$$0_F v = 0_V, \; a0_V = 0_V, \qquad -(av) = (-a)v = a(-v).$$

Spanning Sets

Suppose V is a vector space over a field F and that w and $v_1, v_2, \ldots, v_n$ are elements of V. We say that w is a **linear combination** of $v_1, v_2, \ldots, v_n$ if w can be written in the form

$$w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

for some $a_i \in F$.

Definition

If every element of a vector space V over a field F is a linear combination of $v_1, v_2, \ldots, v_n$, we say that the set $\{v_1, v_2, \ldots, v_n\}$ **spans** V over F.

EXAMPLE 5

The set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans the vector space $\mathbb{Q}^3$ over $\mathbb{Q}$ because every element (a, b, c) of $\mathbb{Q}^3$ is a linear combination of these three vectors:

$$(a, b, c) = a\,(1, 0, 0) + b\,(0, 1, 0) + c\,(0, 0, 1).$$

EXAMPLE 6

Every element of $\mathbb{C}$ (considered as a vector space over $\mathbb{R}$) is a linear combination of 1 and i because every element can be written in the form $a1 + bi$, with

$a, b \in \mathbb{R}$. Thus the set $\{1, i\}$ spans $\mathbb{C}$ over $\mathbb{R}$. The set $\{1 + i, 5i, 2 + 3i\}$ also spans $\mathbb{C}$ because any $a + bi \in \mathbb{C}$ is a linear combination of these three elements with coefficients in $\mathbb{R}$:

$$a + bi = 3a(1 + i) + \frac{b}{5}(5i) + (-a)(2 + 3i).$$

Linear Independence and Bases

The set $\{1, i\}$ not only spans the extension field $\mathbb{C}$ of $\mathbb{R}$, but it also has this property: If $a1 + bi = 0$, then $a = 0$ and $b = 0$. In other words, when a linear combination of 1 and i is 0, then all the coefficients are 0. On the other hand, the set $\{1 + i, 5i, 2 + 3i\}$ does not have this property because some linear combinations of these elements are 0 even though the coefficients are not; for instance,

$$2(1 + i) + \frac{1}{5}(5i) - 1(2 + 3i) = 0.$$

The distinction between these two situations will be crucial in our study of field extensions.

Definition

A subset $\{v_1, v_2, \ldots, v_n\}$ of a vector space V over a field F is said to be **linearly independent** over F provided that whenever

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0_V$$

with each $c_i \in F$, then $c_i = 0_F$ for every i. A set that is not linearly independent is said to be **linearly dependent**.

Thus, a set $\{u_1, u_2, \ldots, u_m\}$ is linearly dependent over F if there exist elements $b_1, b_2, \ldots, b_m$ of F, at least one of which is nonzero, such that $b_1u_1 + b_2u_2 + \cdots + b_mu_m = 0_V$.

EXAMPLE 7

The remarks preceding the definition show that the subset $\{1, i\}$ of $\mathbb{C}$ is linearly independent over $\mathbb{R}$ and that the set $\{1 + i, 5i, 2 + 3i\}$ is linearly dependent. Note, however, that both of these sets span $\mathbb{C}$.

EXAMPLE 8

Consider the subset $\{(3, 0, 0), (0, 0, 4)\}$ of the vector space $\mathbb{Q}^3$ over $\mathbb{Q}$ and suppose $c_1, c_2 \in \mathbb{Q}$ are such that $c_1(3, 0, 0) + c_2(0, 0, 4) = (0, 0, 0)$. Then

$$(0, 0, 0) = c_1(3, 0, 0) + c_2(0, 0, 4) = (3c_1, 0, 4c_2),$$

which implies that $c_1 = 0 = c_2$. Hence, $\{(3, 0, 0), (0, 0, 4)\}$ is linearly independent over $\mathbb{Q}$. However, the set $\{(3, 0, 0), (0, 0, 4)\}$ does not span $\mathbb{Q}^3$ because there is no way to write the vector (0, 5, 0), for example, in the form $a_1(3, 0, 0) + a_2(0, 0, 4) = (3a_1, 0, 4a_2)$ with $a_i \in \mathbb{Q}$.

Let V be a vector space over a field F. The preceding examples show that linear independence and spanning do not imply each other; a subset of V may have one, both, or neither of these properties. A subset that has both properties is given a special name.

Definition A subset $\{v_1, v_2, \ldots, v_n\}$ of a vector space V over a field F is said to be a **basis** of V if it spans V and is linearly independent over F.

EXAMPLE 9

Example 5 shows that the subset $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans the vector space $\mathbb{Q}^3$ over $\mathbb{Q}$. This set is also linearly independent over $\mathbb{Q}$ (Exercise 8) and, hence, is a basis.

EXAMPLE 10

Examples 6 and 7 show that the set $\{1, i\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$. We claim that the set $\{1 + i, 2i\}$ is also a basis of $\mathbb{C}$ over $\mathbb{R}$. If $c_1(1 + i) + c_2(2i) = 0$, with $c_1, c_2 \in \mathbb{R}$, then $c_1 1 + (c_1 + 2c_2)i = 0$. This can happen only if $c_1 = 0$ and $c_1 + 2c_2 = 0$. But this implies that $2c_2 = 0$ and, hence, $c_2 = 0$. Therefore, $\{1 + i, 2i\}$ is linearly independent. In order to see that $\{1 + i, 2i\}$ spans $\mathbb{C}$, note that the element $a + bi \in \mathbb{C}$ can be written as $a(1 + i) + \left(\frac{b - a}{2}\right)2i$.

One situation always leads to linear dependence. Let V be a vector space over a field F and S a subset of V. Suppose that $v, u_1, u_2, \ldots, u_t$ are some of the elements of S and that v is a linear combination of $u_1, u_2, \ldots, u_t$, say $v = a_1u_1 + \cdots + a_tu_t$, with each $a_i \in F$. If $w_1, \ldots, w_r$ are the rest of the elements of S, then

$$v = a_1u_1 + \cdots + a_tu_t + 0_Fw_1 + \cdots + 0_Fw_r$$

and, hence,

$$-1_Fv + a_1u_1 + \cdots + a_tu_t + 0_Fw_1 + \cdots + 0_Fw_r = 0_V.$$

Since at least one of these coefficients is nonzero (namely -1_F), S is linearly dependent. We have proved this useful fact:

If $v \in V$ is a linear combination of $u_1, u_2, \ldots, u_t \in V$, then any set containing v and all the u_i is linearly dependent.

In fact, somewhat more is true.

Lemma 11.1

Let V be a vector space over a field F. The subset $\{u_1, u_2, \ldots, u_n\}$ of V is linearly dependent over F if and only if some u_k is a linear combination of the preceding ones, $u_1, u_2, \ldots, u_{k-1}$.

Proof ▸ If some u_k is a linear combination of the preceding ones, then the set is linearly dependent by the remarks preceding the lemma. Conversely, suppose $\{u_1, \ldots, u_n\}$ is linearly dependent. Then there must exist elements $c_1, \ldots, c_n \in F$, *not all zero*, such that $c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0_V$. Let k be the largest index such that c_k is nonzero. Then $c_i = 0_F$ for $i > k$ and

$$c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0_V$$

$$c_ku_k = -c_1u_1 - c_2u_2 - \cdots - c_{k-1}u_{k-1}.$$

Since F is a field and $c_k \neq 0$, c_k^{-1} exists; multiplying the preceding equation by c_k^{-1} shows that u_k is a linear combination of the preceding u's:

$$u_k = (-c_1c_k^{-1})u_1 + (-c_2c_k^{-1})u_2 + \cdots + (-c_{k-1}c_k^{-1})u_{k-1}. \quad ■$$

The next lemma gives an upper limit on the size of a linearly independent set. It says, in effect, that if V can be spanned by n elements over F, then every linearly independent subset of V contains at most n elements.

Lemma 11.2

Let V be a vector space over the field F that is spanned by the set $\{v_1, v_2, \ldots, v_n\}$. If $\{u_1, u_2, \ldots, u_m\}$ is any linearly independent subset of V, then $m \leq n$.

Proof ▸ By the definition of spanning, every element of V (in particular u_1) is a linear combination of $v_1, \ldots, v_n$. So the set $\{u_1, v_1, v_2, \ldots, v_n\}$ is linearly dependent. Therefore, one of its elements is a linear combination of the preceding ones by Lemma 11.1, say $v_i = a_1u_1 + b_1v_1 + \cdots + b_{i-1}v_{i-1}$. If v_i is deleted, then the remaining set

$$(*) \qquad \{u_1, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$$

still spans V since every element of V is a linear combination of the v's and any appearance of v_i can be replaced by $a_1u_1 + b_1v_1 + \cdots + b_{i-1}v_{i-1}$. In particular, u_2 is a linear combination of the elements of the set $(*)$. Consequently, the set

$$\{u_1, u_2, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$$

is linearly dependent. By Lemma 11.1 one of its elements is a linear combination of the preceding ones. This element can't be one of the u's because this would imply that the u's were linearly dependent. So some

v_j is a linear combination of u_1, u_2, and the v's that precede it. Deleting v_j produces the set

$$\{u_1, u_2, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}.$$

This set still spans V since every element of V is a linear combination of the v's and v_i, v_j can be replaced by linear combinations of u_1, u_2, and the other v's. In particular, u_3 is a linear combination of the elements in this new set. We can continue this process, at each stage adding a u, deleting a v, and producing a set that spans V. If $m > n$, we will run out of v's before all the u's are inserted, resulting in a set of the form $\{u_1, u_2, \ldots, u_n\}$ that spans V. But this would mean that u_m would be a linear combination of $u_1, \ldots, u_n$, contradicting the linear independence of $\{u_1, \ldots, u_m\}$. Therefore, $m \leq n$. ■

Theorem 11.3

Let V be a vector space over a field F. Then any two finite bases of V over F have the same number of elements.

Proof ▸ Suppose $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n)$ are bases of V over F. Then the v's span V and the u's are linearly independent, so $m \leq n$ by Lemma 11.2. Now reverse the roles: The u's span V and the v's are linearly independent, so $n \leq m$ by Lemma 11.2 again. Therefore, $m = n$. ■

According to Theorem 11.3, the *number* of elements in a basis of V over F does not depend on which basis is chosen. So this number is a property of V.

Definition

If a vector space V over a field F has a finite basis, then V is said to be **finite dimensional** over F. The **dimension of V over F** is the number of elements in any basis of V and is denoted $[V{:}F]$. If V does not have a finite basis, then V is said to be **infinite dimensional** over F.

EXAMPLE 11

The dimension of $\mathbb{Q}^3$ over $\mathbb{Q}$ is 3 because $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis. More generally, if F is a field, then F^n is an n-dimensional vector space over F (Exercise 27).

EXAMPLE 12

$[\mathbb{C}{:}\mathbb{R}] = 2$ since $\{1, i\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$. On the other hand, the extension field $\mathbb{R}$ of $\mathbb{Q}$ is an infinite-dimensional vector space over $\mathbb{Q}$. The proof of this fact is omitted here because it requires some nontrivial facts about the cardinality of infinite sets.

Applications to Extension Fields

In the remainder of this section, K is an extension field of a field F. We say that K is a **finite-dimensional extension** of F if K, considered as a vector space over F, is finite dimensional over F.

Remark If $[K:F] = 1$ and $\{u\}$ is a basis, then every element of K is of the form cu for some $c \in F$. In particular, $1_F = cu$, and, hence, $u = c^{-1}$ is in F. Thus, $K = F$. On the other hand, if $K = F$, it is easy to see that $\{1_F\}$ is a basis and, hence, $[K:F] = 1$. Therefore,

$$[K:F] = 1 \quad \textbf{if and only if} \quad K = F.$$

If F, K, and L are fields with $F \subseteq K \subseteq L$, then both K and L can be considered as vector spaces over F, and L can be considered as a vector space over K. It is reasonable to ask how the dimensions $[K:F]$, $[L:K]$, and $[L:F]$ are related. Here is the answer.

Theorem 11.4

Let F, K, and L be fields with $F \subseteq K \subseteq L$. If $[K:F]$ and $[L:K]$ are finite, then L is a finite-dimensional extension of F and $[L:F] = [L:K][K:F]$.

Proof ▶ Suppose $[K:F] = m$ and $[L:K] = n$. Then there is a basis $\{u_1, \ldots, u_m\}$ of K over F and a basis $\{v_1, \ldots, v_n\}$ of L over K. Each u_i and v_j is nonzero by Exercise 19; hence, all the products u_iv_j are nonzero. The set of all products $\{u_iv_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ has exactly mn elements (no two of them can be equal because $u_iv_j = u_kv_t$ implies that $u_iv_j - u_kv_t = 0_K$ with $u_i, u_k \in K$, contradicting the linear independence of the v's over K). We need to show only that this set of mn elements is a basis of L over F because in that case $[L:K][K:F] = nm = [L:F]$.

If w is any element of L, then w is a linear combination of the basis elements $v_1, \ldots, v_n$, say

$$(*) \qquad w = b_1v_1 + b_2v_2 + \cdots + b_nv_n, \qquad \text{with each } b_j \in K.$$

Each $b_j \in K$ is a linear combination of the basis elements $u_1, \ldots, u_m$ so there are $a_{ij} \in F$ such that

$$\begin{aligned}
b_1 &= a_{11}u_1 + a_{21}u_2 + \cdots + a_{m1}u_m \\
b_2 &= a_{12}u_1 + a_{22}u_2 + \cdots + a_{m2}u_m \\
&\;\;\vdots \\
b_n &= a_{1n}u_1 + a_{2n}u_2 + \cdots + a_{mn}u_m.
\end{aligned}$$

Substituting the right side of each of these expressions in $(*)$ shows that w is a sum of terms of the form $a_{ij}u_iv_j$ with $a_{ij} \in F$. Therefore, the set of all products u_iv_j spans L over F.

To show linear independence, suppose $c_{ij} \in F$ and

$$(**) \qquad \sum_{i,j} c_{ij}u_iv_j = c_{11}u_1v_1 + c_{12}u_1v_2 + \cdots + c_{mn}u_mv_n = 0_F.$$

By collecting all the terms involving v_1, then all those involving v_2, and so on, we can rewrite (**) as

$$\begin{aligned}(c_{11}u_1 + c_{21}u_2 + \cdots + c_{m1}u_m)v_1 \\ + (c_{12}u_1 + c_{22}u_2 + \cdots + c_{m2}u_m)v_2 \\ + \cdots + (c_{1n}u_1 + c_{2n}u_2 + \cdots + c_{mn}u_m)\, v_n = 0_F.\end{aligned}$$

The coefficients of the v's are elements of K, so the linear independence of the v's implies that for each $j = 1, 2, \ldots, n$

$$c_{1j}u_1 + c_{2j}u_2 + \cdots + c_{mj}u_m = 0_F.$$

Since each $c_{ij} \in F$ and the u's are linearly independent over F, we must have $c_{ij} = 0_F$ for all i, j. This completes the proof of linear independence, and the theorem is proved. ■

The following result will be needed for the proof of Theorem 11.15 in Section 11.4.

Theorem 11.5

Let K and L be finite dimensional extension fields of F and let $f\colon K \to L$ be an isomorphism such that $f(c) = c$ for every $c \in F$. Then $[K:F] = [L:F]$.

Proof▸ Suppose $[K:F] = n$ and $\{u_1, \ldots, u_n\}$ is a basis of K over F. In order to prove that $[L:F] = n$ also, we need only show that $\{f(u_1), \ldots, f(u_n)\}$ is a basis of L over F. Let $v \in L$; then since f is an isomorphism, $v = f(u)$ for some $u \in K$. By the definition of basis, $u = c_1u_1 + \cdots + c_nu_n$ with each $c_i \in F$. Hence, $v = f(u) = f(c_1u_1 + \cdots + c_nu_n) = f(c_1)f(u_1) + \cdots + f(c_n)f(u_n)$. But $f(c_i) = c_i$ for every i, so that $v = c_1f(u_1) + \cdots + c_nf(u_n)$. Therefore, $\{f(u_1), \ldots, f(u_n)\}$ spans L. To show linear independence, suppose that

$$d_1f(u_1) + \cdots + d_nf(u_n) = 0_F$$

with each $d_i \in F$. Then since $f(d_i) = d_i$ we have

$$\begin{aligned} f(d_1u_1 + \cdots + d_nu_n) &= f(d_1)f(u_1) + \cdots + f(d_n)f(u_n) \\ &= d_1f(u_1) + \cdots + d_nf(u_n) = 0_F.\end{aligned}$$

Since the isomorphism f is injective, $d_1u_1 + \cdots + d_nu_n = 0_F$ by Theorem 6.11. But the u's are linearly independent in K, and, hence, every $d_i = 0_F$. Thus $\{f(u_1), \ldots, f(u_n)\}$ is linearly independent and, therefore, a basis. ■

Exercises

NOTE: *V denotes a vector space over a field F, and K denotes an extension field of F.*

A. 1. Show that $M(\mathbb{R})$ is a vector space over $\mathbb{R}$.

2. Show that $\mathbb{Q}^2$ is a vector space over $\mathbb{Q}$.

3. Show that the polynomial ring $\mathbb{R}[x]$ (with the usual addition of polynomials and product of a constant and a polynomial) is a vector space over $\mathbb{R}$.

4. If $n \geq 1$ is an integer, let $\mathbb{R}_n[x]$ denote the set consisting of the constant polynomial 0 and all polynomials in $\mathbb{R}[x]$ of degree $\leq n$. Show that $\mathbb{R}_n[x]$ (with the usual addition of polynomials and product of a constant and a polynomial) is a vector space over $\mathbb{R}$.

5. If $n \geq 1$ is an integer, show that F^n is a vector space over F.

6. If $\{v_1, v_2, \ldots, v_n\}$ spans K over F and w is any element of K, show that $\{w, v_1, v_2, \ldots, v_n\}$ also spans K.

7. Show that $\{i, 1 + 2i, 1 + 3i\}$ spans $\mathbb{C}$ over $\mathbb{R}$.

8. Show that the subset $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $\mathbb{Q}^3$ is linearly independent over $\mathbb{Q}$.

9. Show that $\{\sqrt{2}, \sqrt{2} + i, \sqrt{3} - i\}$ is linearly dependent over $\mathbb{R}$.

10. If v is a nonzero element of V, prove that $\{v\}$ is linearly independent over F.

11. Prove that any subset of V that contains 0_V is linearly dependent over F.

12. If the subset $\{u, v, w\}$ of V is linearly independent over F, prove that $\{u, u + v, u + v + w\}$ is linearly independent.

13. If $S = \{v_1, \ldots, v_k\}$ is a linearly dependent subset of V, then prove that any subset of V that contains S is also linearly dependent over F.

14. If the subset $T = \{u_1, \ldots, u_t\}$ of V is linearly independent over F, then prove that any nonempty subset of T is also linearly independent.

15. Let b and d be distinct nonzero real numbers and c any real number. Prove that $\{b, c + di\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$.

16. If K is an n-dimensional extension field of $\mathbb{Z}_p$, what is the maximum possible number of elements in K?

17. Let $\{v_1, \ldots, v_n\}$ be a basis of V over F and let $c_1, \ldots, c_n$ be nonzero elements of F. Prove that $\{c_1v_1, c_2v_2, \ldots, c_nv_n\}$ is also a basis of V over F.

18. Show that $\{1, [x]\}$ is a basis of $\mathbb{Z}_2[x]/(x^2 + x + 1)$ over $\mathbb{Z}_2$.

19. If $\{v_1, v_2 \ldots, v_n\}$ is a basis of v, prove that $v_i \neq 0_V$ for every i.

20. Let F, K, and L be fields such that $F \subseteq K \subseteq L$. If $S = \{v_1, v_2, \ldots, v_n\}$ spans L over F, explain why S also spans L over K.

B. 21. For any vector $v \in V$ and any element $a \in F$, prove that

(a) $0_F v = 0_V$. [*Hint:* Adapt the proof of Theorem 3.5.]

(b) $a0_V = 0_V$.

(c) $-(av) = (-a)\,v = a(-v)$.

22. (a) Prove that the subset $\{1, \sqrt{2}\}$ of $\mathbb{R}$ is linearly independent over $\mathbb{Q}$.
    (b) Prove that $\sqrt{3}$ is not a linear combination of 1 and $\sqrt{2}$ with coefficients in $\mathbb{Q}$. Conclude that $\{1, \sqrt{2}\}$ does not span $\mathbb{R}$ over $\mathbb{Q}$.
23. (a) Show that $\{1, \sqrt{2}, \sqrt{3}\}$ is linearly independent over $\mathbb{Q}$.
    (b) Show that $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is linearly independent over $\mathbb{Q}$.
24. Let v be a nonzero real number. Prove that $\{1, v\}$ is linearly independent over $\mathbb{Q}$ if and only if v is irrational.
25. (a) Let $k \geq 1$ be an integer. Show that the subset $\{1, x, x^2, x^3, \ldots, x^k\}$ of $\mathbb{R}[x]$ is linearly independent over $\mathbb{R}$ (see Exercise 3).
    (b) Show that $\mathbb{R}[x]$ is infinite dimensional over $\mathbb{R}$.
26. Show that the vector space $\mathbb{R}_n[x]$ of Exercise 4 has dimension $n + 1$ over $\mathbb{R}$.
27. If F is a field, show that the vector space F^n has dimension n over F.
28. Prove that K has exactly one basis over F if and only if $K = F \cong \mathbb{Z}_2$.
29. Assume $1_F + 1_F \neq 0_F$. If $\{u, v, w\}$ is a basis of V over F, prove that the set $\{u + v, v + w, u + w\}$ is also a basis.
30. Prove that $\{v_1, \ldots, v_n\}$ is a basis of V over F if and only if every element of V can be written in a unique way as a linear combination of $v_1, \ldots, v_n$ ("unique" means that if $w = c_1v_1 + \cdots + c_nv_n$ and $w = d_1v_1 + \cdots + d_nv_n$, then $c_i = d_i$ for every i).
31. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be irreducible in $F[x]$ and let L be the extension field $F[x]/(p(x))$ of F. Prove that L has dimension n over F. [*Hint:* Corollary 5.5, Theorems 5.8 and 5.10, and Exercise 30 may be helpful.]
32. If $S = \{v_1, \ldots, v_t\}$ spans V over F, prove that some subset of S is a basis of K over F. [*Hint:* Use Lemma 11.1 repeatedly to eliminate v's until you reduce to a set that still spans V and is linearly independent.]
33. If the subset $\{u_1, \ldots, u_t\}$ of V is linearly independent over F and $w \in V$ is *not* a linear combination of the u's, prove that $\{u_1, \ldots, u_t, w\}$ is linearly independent.
34. If V is infinite-dimensional over F, then prove that for any positive integer k, V contains a set of k vectors that is linearly independent over F. [*Hint:* Use induction; Exercise 10 is the case $k = 1$, and Exercise 33 can be used to prove the inductive step.]
35. Assume that the subset $\{v_1, \ldots, v_n\}$ of V is linearly independent over F and that $w = c_1v_1 + \cdots + c_nv_n$, with $c_i \in F$. Prove that the set $\{w - v_1, w - v_2, \ldots, w - v_n\}$ is linearly independent over F if and only if $c_1 + \cdots + c_n \neq 1_F$.
36. Assume that V is finite-dimensional over F and S is a linearly independent subset of V. Prove that S is contained in a basis of V. [*Hint:* Let $[V:F] = n$ and $S = \{u_1, \ldots, u_m\}$; then $m \leq n$ by Lemma 11.2. If S does not span V, then there must be some w that is not a linear combination of the u's. Apply Exercise 33 to obtain a larger independent set; if it doesn't span, repeat the argument. Use Lemma 11.2 to show that the process must end with a basis that contains S.]

37. Assume that $[V:F] = n$ and prove that the following conditions are equivalent:
    (i) $\{v_1, \ldots, v_n\}$ spans V over F.
    (ii) $\{v_1, \ldots, v_n\}$ is linearly independent over F.
    (iii) $\{v_1, \ldots, v_n\}$ is a basis of V over F.
38. Let F, K, and L be fields such that $F \subseteq K \subseteq L$. If $[L:F]$ is finite, then prove that $[L:K]$ and $[K:F]$ are also finite and both are $\leq [L:F]$. [*Hint:* Use Exercises 20 and 32 to show that $[L:K]$ is finite. To show that $[K:F]$ is finite, suppose $[L:F] = n$. The set $\{1_K\}$ is linearly independent by Exercise 10; if it doesn't span K, proceed as in the hint to Exercise 36 to build larger and larger linearly independent subsets of K. Use Lemma 11.2 and the fact that $[L:F] = n$ to show that the process must end with a basis of K containing at most n elements.]
39. If $[K:F] = p$, with p prime, prove that there is no field E such that $F \subsetneq E \subsetneq K$. [*Hint:* Exercise 38 and Theorem 11.4.]

11.2 Simple Extensions

Field extensions can be considered from two points of view. You can look upward from a field to its extensions or downward to its subfields. Chapter 5 provided an example of the upward point of view. We took a field F and an irreducible polynomial $p(x)$ in $F[x]$ and formed the field of congruence classes (that is, the quotient field) $F[x]/(p(x))$. Theorem 5.11 shows that $F[x]/(p(x))$ is an extension field of F that contains a root of $p(x)$.

In this section we take the downward view, starting with a field K and a subfield F. If $u \in K$, what can be said about the subfields of K that contain both u and F? Is there a smallest such subfield? If u is the root of some irreducible $p(x)$ in $F[x]$, how is this smallest subfield related to the extension field $F[x]/(p(x))$, which also contains a root of $p(x)$?

The theoretical answer to the first two questions is quite easy. Let K be an extension field of F and $u \in K$. Let $F(u)$ denote the intersection of all subfields of K that contain both F and u (this family of subfields is nonempty since K at least is in it). Since the intersection of any family of subfields of K is itself a field (Exercise 1), $F(u)$ is a field. By its definition, $F(u)$ is contained in every subfield of K that contains F and u, and, hence, $F(u)$ is the smallest subfield of K containing F and u. $F(u)$ is said to be a **simple extension** of F.

As a practical matter, this answer is not entirely satisfactory. A more explicit description of the simple extension field $F(u)$ is needed. It turns out that the structure of $F(u)$ depends on whether or not u is the root of some polynomial in $F[x]$. So we pause to introduce some terminology.

Definition

An *element* u of an extension field K of F is said to be **algebraic** over F if u is the root of some nonzero polynomial in $F[x]$. An element of K that is not the root of any nonzero polynomial in $F[x]$ is said to be **transcendental** over F.

EXAMPLE 1

In the extension field $\mathbb{C}$ of $\mathbb{R}$, i is algebraic over $\mathbb{R}$ because i is the root of $x^2 + 1 \in \mathbb{R}[x]$. You can easily verify that element $2 + i$ of $\mathbb{C}$ is a root of $x^3 - x^2 - 7x + 15 \in \mathbb{Q}[x]$. Thus $2 + i$ is algebraic over $\mathbb{Q}$. Similarly, $\sqrt[5]{3}$ is algebraic over $\mathbb{Q}$ since it is a root of $x^5 - 3$.

EXAMPLE 2

Every element c in a field F is algebraic over F because c is the root of $x - c \in F[x]$.

EXAMPLE 3

The real numbers π and e are transcendental over $\mathbb{Q}$ (proof omitted). Hereafter we shall concentrate on algebraic elements. For more information on transcendental elements, see Exercises 10 and 24–26.

If u is an algebraic element of an extension field K of F, then there may be many polynomials in $F[x]$ that have u as a root. The next theorem shows that all of them are multiples of a single polynomial; this polynomial will enable us to give a precise description of the simple extension field $F(u)$.

Theorem 11.6

Let K be an extension field of F and $u \in K$ an algebraic element over F. Then there exists a unique monic irreducible polynomial $p(x)$ in $F[x]$ that has u as a root. Furthermore, if u is a root of $g(x) \in F[x]$, then $p(x)$ divides $g(x)$.

Proof ▸ Let S be the set of all nonzero polynomials in $F[x]$ that have u as a root. Then S is nonempty because u is algebraic over F. The degrees of polynomials in S form a nonempty set of nonnegative integers, which must contain a smallest element by the Well-Ordering Axiom. Let $p(x)$ be a polynomial of smallest degree in S. Every nonzero constant multiple of $p(x)$ is a polynomial of the same degree with u as a root. So we can choose $p(x)$ to be monic (if it isn't, multiply by the inverse of its leading coefficient).

If $p(x)$ were not irreducible in $F[x]$, there would be polynomials $k(x)$ and $t(x)$ such that $p(x) = k(x)t(x)$, with $\deg k(x) < \deg p(x)$ and $\deg t(x) < \deg p(x)$. Consequently, $k(u)t(u) = p(u) = 0_F$ in K. Since K is a field either $k(u) = 0_F$ or $t(u) = 0_F$, that is, either $k(x)$ or $t(x)$ is in S. This is impossible since $p(x)$ is a polynomial of smallest degree in S. Hence, $p(x)$ is irreducible.

Next we show that $p(x)$ divides every $g(x)$ in S. By the Division Algorithm, $g(x) = p(x)q(x) + r(x)$, where $r(x) = 0_F$ or $\deg r(x) < \deg p(x)$. Since u is a root of both $g(x)$ and $p(x)$,

$$r(u) = g(u) - p(u)q(u) = 0_F + 0_F q(u) = 0_F.$$

So u is a root of $r(x)$. If $r(x)$ were nonzero, then $r(x)$ would be in S, contradicting the fact that $p(x)$ is a polynomial of smallest degree in S. Therefore, $r(x) = 0_F$, so that $g(x) = p(x)q(x)$. Hence, $p(x)$ divides every polynomial in S.

To show that $p(x)$ is unique, suppose $t(x)$ is a monic irreducible polynomial in S. Then $p(x) \mid t(x)$. Since $p(x)$ is irreducible (and, hence, nonconstant) and $t(x)$ is irreducible, we must have $t(x) = cp(x)$ for some $c \in F$. But $p(x)$ is monic, so c is the leading coefficient of $cp(x)$ and, hence, of $t(x)$. Since $t(x)$ is monic, we must have $c = 1_F$. Therefore, $p(x) = t(x)$ and $p(x)$ is unique. ■

If K is an extension field of F and $u \in K$ is algebraic over F, then the monic, irreducible polynomial $p(x)$ in Theorem 11.6 is called the **minimal polynomial** of u over F. The uniqueness statement in Theorem 11.6 means that once we have found any monic, irreducible polynomial in $F[x]$ that has u as a root, it must be the minimal polynomial of u over F.

EXAMPLE 4

$x^2 - 3$ is a monic, irreducible polynomial in $\mathbb{Q}[x]$ that has $\sqrt{3} \in \mathbb{R}$ as a root. Therefore, $x^2 - 3$ is the minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}$. Note that $x^2 - 3$ is reducible over $\mathbb{R}$ since it factors as $(x - \sqrt{3})(x + \sqrt{3})$ in $\mathbb{R}[x]$. So the minimal polynomial of $\sqrt{3}$ over $\mathbb{R}$ is $x - \sqrt{3}$, which is monic and irreducible in $\mathbb{R}[x]$.

EXAMPLE 5

Let $u = \sqrt{3} + \sqrt{5} \in \mathbb{R}$. Then $u^2 = 3 + 2\sqrt{3}\sqrt{5} + 5 = 8 + 2\sqrt{15}$. Hence, $u^2 - 8 = 2\sqrt{15}$ so that $(u^2 - 8)^2 = 60$, or, equivalently, $(u^2 - 8)^2 - 60 = 0$. Therefore, $u = \sqrt{3} + \sqrt{5}$ is a root of $(x^2 - 8)^2 - 60 = x^4 - 16x^2 + 4 \in \mathbb{Q}[x]$. Verify that this polynomial is irreducible in $\mathbb{Q}[x]$ (Exercise 14). Hence, it must be the minimal polynomial of $\sqrt{3} + \sqrt{5}$ over $\mathbb{Q}$.

The minimal polynomial of u provides the connection between the upward and downward views of simple field extensions and allows us to give a useful description of $F(u)$.

Theorem 11.7

Let K be an extension field of F and $u \in K$ an algebraic element over F with minimal polynomial $p(x)$ of degree n. Then

(1) $F(u) \cong F[x]/(p(x))$.

(2) $\{1_F, u, u^2, \ldots, u^{n-1}\}$ is a basis of the vector space $F(u)$ over F.

(3) $[F(u) : F] = n$.

Theorem 11.7 shows that when u is algebraic over F, then $F(u)$ does not depend on K but is completely determined by $F[x]$ and the minimal polynomial $p(x)$. Consequently, we sometimes say that $F(u)$ is the **field obtained by adjoining u to F.**

Proof of Theorem 11.7 ▸ (1) Since $F(u)$ is a field containing u, it must contain every positive power of u. Since $F(u)$ also contains F, $F(u)$ must contain every element of the form $b_0 + b_1u + b_2u^2 + \cdots + b_tu^t$ with $b_i \in F$, that is, $F(u)$ contains the element $f(u)$ for every $f(x) \in F[x]$. Verify that the map $\varphi: F[x] \to F(u)$ given by $\varphi(f(x)) = f(u)$ is a homomorphism of rings. A polynomial in $F[x]$ is in the kernel of φ precisely when it has u as a root. By Theorem 11.6 the kernel of φ is the principal ideal $(p(x))$. The First Isomorphism Theorem 6.13 shows that $F[x]/(p(x))$ is isomorphic to Im φ under the map that sends congruence class (coset) $[f(x)]$ to $f(u)$. Furthermore, since $p(x)$ is irreducible, the quotient ring $F[x]/(p(x))$, and, hence, Im φ, are fields by Theorem 5.10. Every constant polynomial is mapped to itself by φ and $\varphi(x) = u$. So Im φ is a subfield of $F(u)$ that contains both F and u. Since $F(u)$ is the smallest subfield of K containing F and u, we must have $F(u) = \text{Im } \varphi \cong F[x]/(p(x))$.

(2) and (3) Since $F(u) = \text{Im } \varphi$, every nonzero element of $F(u)$ is of the form $f(u)$ for some $f(x) \in F[x]$. If $\deg p(x) = n$, then by the Division Algorithm $f(x) = p(x)q(x) + r(x)$, where $r(x) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1} \in F[x]$. Consequently, $f(u) = p(u)q(u) + r(u) = 0_Fq(u) + r(u) = r(u) = b_01_F + b_1u + \cdots + b_{n-1}u^{n-1}$. Therefore, the set $\{1_F, u, u^2, \ldots, u^{n-1}\}$ spans $F(u)$. To show that this set is linearly independent, suppose $c_0 + c_1u + \cdots + c_{n-1}u^{n-1} = 0_F$ with each $c_i \in F$. Then u is a root of $c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$, so this polynomial (which has degree $\leq n-1$) must be divisible by $p(x)$ (which has degree n). This can happen only when $c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ is the zero polynomial; that is, each $c_i = 0_F$. Thus $\{1_F, u, u^2, \ldots, u^{n-1}\}$ is linearly independent over F and, therefore, a basis of $F(u)$. Hence, $[F(u) : F] = n$. ■

EXAMPLE 6

The minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}$ is $x^2 - 3$. Applying Theorem 11.7 with $n = 2$ we see that $\{1, \sqrt{3}\}$ is a basis of $\mathbb{Q}(\sqrt{3})$ over $\mathbb{Q}$, whence $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$. Similarly, Example 5 shows that $\sqrt{3} + \sqrt{5}$ has minimal polynomial $x^4 - 16x^2 + 4$ over $\mathbb{Q}$ so that $[\mathbb{Q}(\sqrt{3} + \sqrt{5}) : \mathbb{Q}] = 4$ and $\{1, \sqrt{3} + \sqrt{5}, (\sqrt{3} + \sqrt{5})^2, (\sqrt{3} + \sqrt{5})^3\}$ is a basis.

An immediate consequence of Theorem 11.7 is that

if u and v have the same minimal polynomial $p(x)$ in $F[x]$, then $F(u)$ is isomorphic to $F(v)$.

The reason is that both $F(u)$ and $F(v)$ are isomorphic to $F[x]/(p(x))$ and, hence, to each other. Note that this result holds even when u and v are not in the same extension field of F. The remainder of this section, which is not needed until Section 11.4, deals with generalizations of this idea. We shall consider not only simple extensions of the same field, but also simple extensions of two different, but isomorphic, fields.

Suppose F and E are fields and that $\sigma: F \to E$ is an isomorphism. Verify that the map from $F[x]$ to $E[x]$ that maps $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ to the polynomial $\sigma f(x) = \sigma(a_0) + \sigma(a_1)x + \sigma(a_2)x^2 + \cdots + \sigma(a_n)x^n$ is an isomorphism of rings (Exercise 21 in Section 4.1). Note that if $f(x) = c$ is a constant polynomial in $F[x]$ (that is, an element of F), then this isomorphism maps it onto $\sigma(c) \in E$. Consequently, we say that the isomorphism $F[x] \to E[x]$ **extends** the isomorphism $\sigma: F \to E$, and we denote the extended isomorphism by σ as well.

Corollary 11.8

Let $\sigma: F \to E$ be an isomorphism of fields. Let u be an algebraic element in some extension field of F with minimal polynomial $p(x) \in F[x]$. Let v be an algebraic element in some extension field of E, with minimal polynomial $\sigma p(x) \in E[x]$. Then σ extends to an isomorphism of fields $\overline{\sigma}: F(u) \to E(v)$ such that $\overline{\sigma}(u) = v$ and $\overline{\sigma}(c) = \sigma(c)$ for every $c \in F$.

The special case when σ is the identity map $F \to F$ states whenever u and v have the same minimal polynomial, then $F(u) \cong F(v)$ under a function that maps u to v and every element of F to itself.

Proof of Corollary 11.8 ▸ The isomorphism σ extends to an isomorphism (also denoted σ) $F[x] \to E[x]$ by the remarks preceding the corollary. The proof of Theorem 11.7 shows that there is an isomorphism $\overline{\tau}: E[x]/(\sigma p(x)) \to E(v)$ given by $\overline{\tau}([g(x)]) = g(v)$. Let π be the surjective homomorphism

$$E[x] \to E[x]/(\sigma p(x))$$

that maps $g(x)$ to $[g(x)]$ and consider the composition

$$\begin{array}{ccccccc} F[x] & \xrightarrow{\sigma} & E[x] & \xrightarrow{\pi} & E[x]/(\sigma p(x)) & \xrightarrow{\overline{\tau}} & E(v) \\ f(x) & \longrightarrow & \sigma f(x) & \longrightarrow & [\sigma f(x)] & \longrightarrow & \sigma f(v). \end{array}$$

Since all three maps are surjective, so is the composite function. The kernel of the composite function consists of all $h(x) \in F[x]$ such that $\sigma h(v) = 0_E$. Since $\overline{\tau}$ is an isomorphism, $\sigma h(v) = 0_E$ if and only if $[\sigma h(x)]$ is the zero class in $E[x]/(\sigma p(x))$, that is, if and only if $\sigma h(x)$ is a multiple of $\sigma p(x)$. But if $\sigma h(x) = k(x) \cdot \sigma p(x)$, then applying the inverse of the isomorphism σ shows that $h(x) = \sigma^{-1}(k(x))p(x)$. Thus the kernel of the composite function is the principal ideal $(p(x))$ in $F[x]$. Therefore, $F[x]/(p(x)) \cong E(v)$ by the First Isomorphism Theorem 6.13; the proof of that theorem shows that this isomorphism (call it θ) is given by $\theta([f(x)]) = \sigma f(v)$. Note that $\theta([x]) = v$ and that for each $c \in F$, $\theta([c]) = \sigma(c)$. So we have the following situation, where $\overline{\varphi}$ is the isomorphism of Theorem 11.7:

$$\begin{array}{ccccc} F[u] & \xleftarrow{\overline{\varphi}} & F[x]/(p(x)) & \xrightarrow{\theta} & E(v) \\ f[u] & \longleftarrow & [f(x)] & \longrightarrow & \sigma f(v) \\ c & \longleftarrow & [c] & \longrightarrow & \sigma(c) \qquad\qquad c \in F. \end{array}$$

The composite function $\theta \circ \overline{\varphi}^{-1} : F(u) \to E(v)$ is an isomorphism that extends σ and maps u to v. ■

EXAMPLE 7

The polynomial $x^3 - 2$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion. It has a root in $\mathbb{R}$, namely $\sqrt[3]{2}$. Verify that $\sqrt[3]{2}\omega$ is also a root of $x^3 - 2$ in $\mathbb{C}$, where $\omega = \dfrac{-1 + \sqrt{3}i}{2}$ is a complex cube root of 1. Applying Corollary 11.8 to the identity map $\mathbb{Q} \to \mathbb{Q}$ we see that the real subfield $\mathbb{Q}(\sqrt[3]{2})$ is isomorphic to the complex subfield $\mathbb{Q}(\sqrt[3]{2}\omega)$ under a map that sends $\sqrt[3]{2}$ to $\sqrt[3]{2}\omega$ and each element of $\mathbb{Q}$ to itself.

Exercises

NOTE: *Unless stated otherwise, K is an extension field of the field F.*

A. 1. Let $\{E_i \mid i \in I\}$ be a family of subfields of K. Prove that $\bigcap_{i \in I} E_i$ is a subfield of K.

2. If $u \in K$, prove that $F(u^2) \subseteq F(u)$.

3. If $u \in K$ and $c \in F$, prove that $F(u + c) = F(u) = F(cu)$.

4. Prove that $\mathbb{Q}(3 + i) = \mathbb{Q}(1 - i)$.

5. Prove that the given element is algebraic over $\mathbb{Q}$:

   (a) $3 + 5i$ **(b)** $\sqrt{i - \sqrt{2}}$ **(c)** $1 + \sqrt[3]{2}$

6. If $u \in K$ and u^2 is algebraic over F, prove that u is algebraic over F.

7. If L is a field such that $F \subseteq K \subseteq L$ and $u \in L$ is algebraic over F, show that u is algebraic over K.

8. If $u, v \in K$ and $u + v$ is algebraic over F, prove that u is algebraic over $F(v)$.

9. Prove that $\sqrt{\pi}$ is algebraic over $\mathbb{Q}(\pi)$.

10. If $u \in K$ is transcendental over F and $0_F \neq c \in F$, prove that each of $u + 1_F$, cu, and u^2 is transcendental over F.

11. Find $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}]$.

12. If $a + bi \in \mathbb{C}$ and $b \neq 0$, prove that $\mathbb{C} = \mathbb{R}(a + bi)$.

13. If $[K:F]$ is prime and $u \in K$ is algebraic over F, show that either $F(u) = K$ or $F(u) = F$.

14. Prove that $x^4 - 16x^2 + 4$ is irreducible in $\mathbb{Q}[x]$.

B. 15. Show that every element of $\mathbb{C}$ is algebraic over $\mathbb{R}$ [*Hint:* See Lemma 4.29.]

16. If $u \in K$ is algebraic over F and $c \in F$, prove that $u + 1_F$ and cu are algebraic over F.

17. Find the minimal polynomial of the given element over $\mathbb{Q}$:

    (a) $\sqrt{1 + \sqrt{5}}$ **(b)** $\sqrt{3}i + \sqrt{2}$

18. Find the minimal polynomial of $\sqrt{2} + i$ over $\mathbb{Q}$ and over $\mathbb{R}$.

19. Let u be an algebraic element of K whose minimal polynomial in $F[x]$ has prime degree. If E is a field such that $F \subseteq E \subseteq F(u)$, show that $E = F$ or $E = F(u)$.

20. Let u be an algebraic element of K whose minimal polynomial in $F[x]$ has odd degree. Prove that $F(u) = F(u^2)$.

21. Let $F = \mathbb{Q}(\pi^4)$ and $K = \mathbb{Q}(\pi)$. Show that π is algebraic over F and find a basis of K over F.

22. If r and s are nonzero, prove that $\mathbb{Q}(\sqrt{r}) = \mathbb{Q}(\sqrt{s})$ if and only if $r = t^2 s$ for some $t \in \mathbb{Q}$.

23. If K is an extension field of $\mathbb{Q}$ such that $[K:\mathbb{Q}] = 2$, prove that $K = \mathbb{Q}(\sqrt{d})$ for some square-free integer d. [Square-free means d is not divisible by p^2 for any prime p.]

24. If $u \in K$ is transcendental over F, prove that $F(u) \cong F(x)$, where $F(x)$ is the field of quotients of $F[x]$, as in Example 1 of Section 10.4. [*Hint:* Consider the map from $F(x)$ to $F(u)$ that sends $f(x)/g(x)$ to $f(u)g(u)^{-1}$.]

25. If $u \in K$ is transcendental over F, prove that all elements of $F(u)$, except those in F, are transcendental over F.

26. Let $F(x)$ be as in Exercise 24. Show that $\dfrac{x^3}{x+1} \in F(x)$ is transcendental over F.

11.3 Algebraic Extensions

The emphasis in the last section was on a single algebraic element. Now we consider extensions that consist entirely of algebraic elements.

Definition An extension field K of a field F is said to be an **algebraic extension** of F if every element of K is algebraic over F.

EXAMPLE 1

If $a + bi \in \mathbb{C}$, then $a + bi$ is a root of

$$(x - (a + bi))(x - (a - bi)) = x^2 - 2ax + (a^2 + b^2) \in \mathbb{R}[x].$$

Therefore, $a + bi$ is algebraic over $\mathbb{R}$, and, hence, $\mathbb{C}$ is an algebraic extension of $\mathbb{R}$. On the other hand, neither $\mathbb{C}$ nor $\mathbb{R}$ is an algebraic extension of $\mathbb{Q}$ since there are real numbers (such as π and e) that are not algebraic over $\mathbb{Q}$.

Every algebraic element u over F lies in some finite-dimensional extension field of F, namely $F(u)$, by Theorem 11.7. On the other hand, if we begin with a finite-dimensional extension of F we have

Theorem 11.9

If K is a finite-dimensional extension field of F, then K is an algebraic extension of F.

Proof ▸ By hypothesis, K has a finite basis over F, say $\{v_1, v_2, \ldots, v_n\}$. Since these n elements span K, Lemma 11.2 implies that every linearly independent set in K must have n or fewer elements.

If $u \in K$, there are two possibilities: (1) $u^i = u^j$ with $0 \le i < j$; and (2) all nonnegative powers of u are distinct. In Case (1), u is a root of the polynomial $x^i - x^j \in F[x]$ and hence, is algebraic over F. In Case (2), $\{1_F, u, u^2, \ldots, u^n\}$ is a set of $n + 1$ elements in K and must, therefore, be linearly dependent over F. Consequently, there are elements c_i in F, not all zero, such that $c_0 1_F + c_1 u + c_2 u^2 + \cdots + c_n u^n = 0_F$. Therefore, u is the root of the nonzero polynomial $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ in $F[x]$ and, hence, algebraic over F. ■

If an extension field K of F contains a transcendental element u, then K must be infinite dimensional over F (otherwise u would be algebraic by Theorem 11.9). Nevertheless, the converse of Theorem 11.9 is false since there do exist infinite-dimensional algebraic extensions (Exercise 16).

Simple extensions have a nice property. You need only verify that the single element u is algebraic over F to conclude that the entire field $F(u)$ is an algebraic extension (because $F(u)$ is finite dimensional by Theorem 11.7 and, hence, algebraic by Theorem 11.9). This suggests that generalizing the notion of simple extension might lead to fields whose algebraicity could be determined by checking just a finite number of elements.

If $u_1, \ldots, u_n$ are elements of an extension field K of F, let

$$F(u_1, u_2, \ldots, u_n)$$

denote the intersection of all the subfields of K that contain F and every u_i. As in the case of simple extensions, $F(u_1, \ldots, u_n)$ is the smallest subfield of K that contains F and all the u_i. $F(u_1, \ldots, u_n)$ is said to be a **finitely generated extension** of F, generated by $u_1, \ldots, u_n$.

EXAMPLE 2

The field $\mathbb{Q}(\sqrt{3}, i)$ is the smallest subfield of $\mathbb{C}$ that contains both the field $\mathbb{Q}$ and the elements $\sqrt{3}$ and i.

EXAMPLE 3

A finitely generated extension may actually be a simple extension. For instance, the field $\mathbb{Q}(i)$ contains both i and $-i$, so $\mathbb{Q}(i, -i) = \mathbb{Q}(i)$.

EXAMPLE 4

Every finite-dimensional extension is also finitely generated. If $\{u_1, \ldots, u_n]$ is a basis of K over F, then all linear combinations of the u_i (coefficients in F) are in $F(u_1, \ldots, u_n)$. Therefore, $K = F(u_1, \ldots, u_n)$.

The key to dealing with finitely generated extensions is to note that they can be obtained by taking successive simple extensions. For instance, if K is an extension field of F and $u, v \in K$, then $F(u, v)$ is a subfield of K that contains both F and u and, hence, must contain $F(u)$. Since v is in $F(u, v)$, this latter field must contain $F(u)(v)$, the smallest subfield containing both $F(u)$ and v. But $F(u)(v)$ is a field containing F, u, and v and, hence, must contain $F(u, v)$. Therefore, $F(u, v) = F(u)(v)$. Thus the finitely generated extension $F(u, v)$ can be obtained from a chain of simple extensions:

$$F \subseteq F(u) \subseteq F(u)(v) = F(u, v).$$

EXAMPLE 5

The extension field $\mathbb{Q}(\sqrt{3}, i)$ can be obtained by this sequence of simple extensions:

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{3})(i) = \mathbb{Q}(\sqrt{3}, i).$$

As we saw in Example 4 of Section 11.2, $x^2 - 3$ is the minimal polynomial of $\sqrt{3}$ over $\mathbb{Q}$, so that $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$ by Theorem 11.7. Similarly, $x^2 + 1$ [whose coefficients are in $\mathbb{Q}(\sqrt{3})$] is the minimal polynomial of i over $\mathbb{Q}(\sqrt{3})$ because its roots $\pm i$ are not in $\mathbb{Q}(\sqrt{3})$, so $x^2 + 1$ is irreducible over $\mathbb{Q}(\sqrt{3})$ by Corollary 4.19. By Theorem 11.7 again, $[\mathbb{Q}(\sqrt{3})(i) : \mathbb{Q}(\sqrt{3})] = 2$. Consequently, by Theorem 11.4,

$$[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3})(i) : \mathbb{Q}(\sqrt{3})][\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2 \cdot 2 = 4.$$

Thus, the finitely generated extension $\mathbb{Q}(\sqrt{3}, i)$ is finite dimensional and, hence, algebraic over $\mathbb{Q}$ by Theorem 11.9.

Essentially the same argument works in the general case and provides a useful way to determine that an extension is algebraic:

Theorem 11.10

If $K = F(u_1, \ldots, u_n)$ is a finitely generated extension field of F and each u_i is algebraic over F, then K is a finite-dimensional algebraic extension of F.

Proof ▸ The field K can be obtained from this chain of extensions:

$$F \subseteq F(u_1) \subseteq F(u_1, u_2) \subseteq F(u_1, u_2, u_3) \subseteq \cdots$$
$$\subseteq F(u_1, \ldots, u_{n-1}) \subseteq F(u_1, \ldots, u_n) = K.$$

Furthermore, $F(u_1, u_2) = F(u_1)(u_2)$, $F(u_1, u_2, u_3) = F(u_1, u_2)(u_3)$, and in general $F(u_1, \ldots, u_i)$ is the simple extension $F(u_1, \ldots, u_{i-1})(u_i)$. Each u_i is algebraic over F and, hence, algebraic over $F(u_1, \ldots, u_{i-1})$ by Exercise 7 of Section 11.2. But every simple extension by an algebraic element is finite dimensional by Theorem 11.7. Therefore,

$$[F(u_1, \ldots, u_i):F(u_1, \ldots, u_{i-1})]$$

is finite for each $i = 2, \ldots, n$. Consequently, by repeated application of Theorem 11.4, we see that $[K:F]$ is the product

$$[K:F(u_1, \ldots, u_{n-1})] \cdots [F(u_1, u_2, u_3):F(u_1, u_2)][F(u_1, u_2):F(u_1)][F(u_1):F].$$

Thus $[K:F]$ is finite, and, hence, K is algebraic over F by Theorem 11.9. ■

EXAMPLE 6

Both $\sqrt{3}$ and $\sqrt{5}$ are algebraic over $\mathbb{Q}$, so $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ is a finite-dimensional algebraic extension field of $\mathbb{Q}$ by Theorem 11.10. We can calculate the dimension of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$ by considering this chain of simple extensions:

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{3})(\sqrt{5}) = \mathbb{Q}(\sqrt{3}, \sqrt{5}).$$

We know that $[\mathbb{Q}(\sqrt{3}):\mathbb{Q}] = 2$. To determine $[\mathbb{Q}(\sqrt{3})(\sqrt{5}):\mathbb{Q}(\sqrt{3})]$ we shall find the minimal polynomial of $\sqrt{5}$ over $\mathbb{Q}(\sqrt{3})$. The obvious candidate is $x^2 - 5$; it is irreducible in $\mathbb{Q}[x]$, but we must show that it is irreducible over $\mathbb{Q}(\sqrt{3})$, in order to conclude that it is the minimal polynomial. If $\sqrt{5}$ or $-\sqrt{5}$ is in $\mathbb{Q}(\sqrt{3})$, then $\pm\sqrt{5} = a + b\sqrt{3}$, with $a, b \in \mathbb{Q}$. Squaring both sides shows that $5 = a^2 + 2ab\sqrt{3} + 3b^2$, whence $\sqrt{3} = \dfrac{5 - a^2 - 3b^2}{2ab}$, contradicting the fact that $\sqrt{3}$ is irrational; a similar contradiction results if $a = 0$ or $b = 0$. Therefore, $\pm\sqrt{5}$ are not in $\mathbb{Q}(\sqrt{3})$, and, hence, $x^2 - 5$ is irreducible over $\mathbb{Q}(\sqrt{3})$ by Corollary 4.19. So $x^2 - 5$ is the minimal polynomial of $\sqrt{5}$ over $\mathbb{Q}(\sqrt{3})$, and $[\mathbb{Q}(\sqrt{3})(\sqrt{5}) : \mathbb{Q}(\sqrt{3})] = 2$ by Theorem 11.7. Consequently, by Theorem 11.4

$$[\mathbb{Q}(\sqrt{3}, \sqrt{5}):\mathbb{Q}] = [\mathbb{Q}(\sqrt{3})(\sqrt{5}):\mathbb{Q}(\sqrt{3})][\mathbb{Q}(\sqrt{3}):\mathbb{Q}] = 2 \cdot 2 = 4.$$

The remainder of this section is not used in the sequel. Theorem 11.4 tells us that the top field in a chain of finite-dimensional extensions is finite dimensional over the ground field. Here is an analogous result for algebraic extensions that may not be finite dimensional.

Corollary 11.11

If L is an algebraic extension field of K and K is an algebraic extension field of F, then L is an algebraic extension of F.

Proof ▸ Let $u \in L$. Since u is algebraic over K, there exist $a_i \in K$ such that $a_0 + a_1u + a_2u^2 + \cdots + a_mu^m = 0_K$. Since each of the a_i is in the field $F(a_1, \ldots, a_m)$, u is actually algebraic over $F(a_1, \ldots, a_m)$. Consequently, in the extension chain

$$F \subseteq F(a_1, \ldots, a_m) \subseteq F(a_1, \ldots, a_m)(u) = F(a_1, \ldots, a_m, u)$$

$F(a, \ldots, a_m)(u)$ is finite dimensional over $F(a_1, \ldots, a_m)$ by Theorem 11.7. Furthermore, $[F(a_1, \ldots, a_m):F]$ is finite by Theorem 11.10 since each a_i is algebraic over F. Therefore, $F(a_1, \ldots, a_m, u)$ is finite dimensional over F by Theorem 11.4 and, hence, is algebraic over F by Theorem 11.9. Thus u is algebraic over F. Since u was an arbitrary element of L, L is an algebraic extension of F. ■

Corollary 11.12

Let K be an extension field of F and let E be the set of all elements of K that are algebraic over F. Then E is a subfield of K and an algebraic extension field of F.

Proof ▸ Every element of F is algebraic over F, so $F \subseteq E$. If $u, v \in E$, then u and v are algebraic over F by definition. The subfield $F(u, v)$ is an algebraic extension of F by Theorem 11.10, and, hence, $F(u, v) \subseteq E$. Since $F(u, v)$ is a field, $u + v, uv, -u, -v \in F(u, v) \subseteq E$. Similarly, if u is nonzero, then $u^{-1} \in F(u, v) \subseteq E$. Therefore, E is closed under addition and multiplication; negatives and inverses of elements of E are also in E. Hence, E is a field. ■

EXAMPLE 7

If $K = \mathbb{C}$ and $F = \mathbb{Q}$ in Corollary 11.12, then the field E is called the field of **algebraic numbers.** The field E is an infinite-dimensional algebraic extension of $\mathbb{Q}$ (Exercise 16). Algebraic numbers were discussed in a somewhat different context on page 350.

Exercises

NOTE: *Unless stated otherwise, K is an extension field of the field F.*

A. 1. If $u, v \in K$, verify that $F(u)(v) = F(v)(u)$.

2. If K is a finite field, show that K is an algebraic extension of F.

3. Find a basis of the given extension field of $\mathbb{Q}$.

(a) $\mathbb{Q}(\sqrt{5}, i)$ (b) $\mathbb{Q}(\sqrt{5}, \sqrt{7})$ (c) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ (d) $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$

4. Find a basis of $\mathbb{Q}(\sqrt{2}, + \sqrt{3})$ over $\mathbb{Q}(\sqrt{3})$.

5. Show that $[\mathbb{Q}(\sqrt{3}, i):\mathbb{Q}] = 4$.

6. Verify that $[\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{10}):\mathbb{Q}] = 4$.

7. If $[K:F]$ is finite and u is algebraic over K, prove that $[K(u):K] \leq [F(u):F]$.

8. If $[K:F]$ is finite and u is algebraic over K, prove that $[K(u):F(u)] \leq [K:F]$. [*Hint:* Show that any basis of K over F spans $K(u)$ over $F(u)$.]

9. If $[K:F]$ is finite and u is algebraic over K, prove that $[F(u):F]$ divides $[K(u):F]$.

B. 10. Prove that $[K:F]$ is finite if and only if $K = F(u_1, \ldots, u_n)$, with each u_i algebraic over F. [This is a stronger version of Theorem 11.10.]

11. Assume that $u, v \in K$ are algebraic over F, with minimal polynomials $p(x)$ and $q(x)$, respectively.

(a) If $\deg p(x) = m$ and $\deg q(x) = n$ and $(m, n) = 1$, prove that $[F(u, v):F] = mn$.

(b) Show by example that the conclusion of part (a) may be false if m and n are not relatively prime.

(c) What is $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}):\mathbb{Q}]$?

12. Let D be a ring such that $F \subseteq D \subseteq K$. If K is algebraic over F, prove that D is a field. [*Hint:* To find the inverse of a nonzero $u \in D$, use Theorem 11.7 to show that $F(u) \subseteq D$.]

13. Let $p(x)$ and $q(x)$ be irreducible in $F[x]$ and assume that $\deg p(x)$ is relatively prime to $\deg q(x)$. Let u be a root of $p(x)$ and v a root of $q(x)$ in some extension field of F. Prove that $q(x)$ is irreducible over $F(u)$.

14. (a) Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a chain of fields. Prove that the union of all the F_i is also a field.

(b) If each F_i is algebraic over F_1, show that the union of the F_i is an algebraic extension of F_1.

15. Let E be the field of all elements of K that are algebraic over F, as in Corollary 11.12. Prove that every element of the set $K - E$ is transcendental over E.

16. Let E be the field of algebraic numbers (see Example 7). Prove that E is an infinite dimensional algebraic extension of $\mathbb{Q}$. [*Hint:* It suffices to show that $[E:\mathbb{Q}] \geq n$ for every positive integer n. Consider roots of the polynomial $x^n - 2$ and Eisenstein's Criterion.]

17. Assume that $1_F + 1_F \neq 0_F$. If $u \in F$, let $\sqrt{u}$ denote a root of $x^2 - u$ in K. Prove that $F(\sqrt{u} + \sqrt{v}) = F(\sqrt{u}, \sqrt{v})$. [*Hint:* $1, (\sqrt{u} + \sqrt{v})$, $(\sqrt{u} + \sqrt{v})^2, (\sqrt{u} + \sqrt{v})^3$, etc., must span $F(\sqrt{u} + \sqrt{v})$ by Theorem 11.7. Use this to show that $\sqrt{u}$ and $\sqrt{v}$ are in $F(\sqrt{u} + \sqrt{v})$.]

18. If $n_1, \ldots, n_t$ are distinct positive integers, show that

$$[\mathbb{Q}(\sqrt{n_1}, \ldots, \sqrt{n_t}) : \mathbb{Q}] \leq 2^t.$$

C. 19. If each n_i is prime in Exercise 18, show that $\leq$ may be replaced by $=$.

11.4 Splitting Fields

Let F be a field and $f(x)$ a polynomial in $F[x]$. Previously we considered extension fields of F that contained a root of $f(x)$. Now we investigate extension fields that contain *all* the roots of $f(x)$.

The word "all" in this context needs some clarification. Suppose $f(x)$ has degree n. Then by Corollary 4.17, $f(x)$ has at most n roots in any field. So if an extension field K of F contains n distinct roots of $f(x)$, one can reasonably say that K contains "all" the roots of $f(x)$, even though there may be another extension of F that also contains n roots of $f(x)$. On the other hand, suppose that K contains fewer than n roots of $f(x)$. It might be possible to find an extension field of K that contains additional roots of $f(x)$. But if no such extension of K exists, it is reasonable to say that K contains "all" the roots. We can express this condition in a usable form as follows.

Let K be an extension field of F and $f(x)$ a nonconstant polynomial of degree n in $F[x]$. If $f(x)$ factors in $K[x]$ as

$$f(x) = c(x - u_1)(x - u_2) \cdots (x - u_n)$$

then we say that **$f(x)$ splits over the field K.** In this case, the (not necessarily distinct) elements $u_1, \ldots, u_n$ are the only roots of $f(x)$ in K or in any extension field of K. For if v is in some extension of K and $f(v) = 0_F$, then $c(v - u_1)(v - u_2) \cdots (v - u_n) = 0_F$. Now c is nonzero since $f(x)$ is nonconstant. Hence one of the $v - u_i$ must be zero, that is, $v = u_i$. So if $f(x)$ splits over K, we can reasonably say that K contains all the roots of $f(x)$. The next step is to consider the *smallest* extension field that contains all the roots of $f(x)$.

Definition

If F is a field and $f(x) \in F[x]$, then an extension field K of F is said to be a **splitting field** (or **root field**) of $f(x)$ over F provided that

(i) $f(x)$ splits over K, say $f(x) = c(x - u_1)(x - u_2) \cdots (x - u_n)$;
(ii) $K = F(u_1, u_2, \ldots, u_n)$.

EXAMPLE 1

If $x^2 + 1$ is considered as a polynomial in $\mathbb{R}[x]$, then $\mathbb{C}$ is a splitting field since $x^2 + 1 = (x + i)(x - i)$ in $\mathbb{C}[x]$ and $\mathbb{C} = \mathbb{R}(i) = \mathbb{R}(i, -i)$. Similarly, $\mathbb{Q}(\sqrt{2})$ is a splitting field of the polynomial $x^2 - 2$ in $\mathbb{Q}[x]$ since $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ and $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, -\sqrt{2})$.

EXAMPLE 2

The polynomial $f(x) = x^4 - x^2 - 2$ in $\mathbb{Q}[x]$ factors as $(x^2 - 2)(x^2 + 1)$, so its roots in $\mathbb{C}$ are $\pm\sqrt{2}$ and $\pm i$. Therefore, $\mathbb{Q}(\sqrt{2}, i)$ is a splitting field of $f(x)$ over $\mathbb{Q}$.

EXAMPLE 3

Every first-degree polynomial $cx + d$ in $F[x]$ splits over F since $cx + d = c(x - (-c^{-1}d))$ with $-c^{-1}d \in F$. Obviously, F is the smallest field containing both F and $c^{-1}d$, that is, $F = F(c^{-1}d)$. So F itself is the splitting field of $cx + d$ over F.

EXAMPLE 4

The concept of splitting field depends on the polynomial *and* the base field. For instance, $\mathbb{C}$ is a splitting field of $x^2 + 1$ over $\mathbb{R}$ but not over $\mathbb{Q}$ because $\mathbb{C}$ is not the extension $\mathbb{Q}(i, -i) = \mathbb{Q}(i)$. See Exercise 1 for a proof.

At this point we need to answer two major questions about splitting fields: Does every polynomial in $F[x]$ have a splitting field over F? If it has more than one splitting field over F, how are they related?

The informal answer to the first question is easy. Given $f(x) \in F[x]$, we can find an extension $F(u)$ that contains a root u of $f(x)$ by Corollary 5.12. By the Factor Theorem in $F(u)[x]$, we know that $f(x) = (x - u)g(x)$. By Corollary 5.12 again there is an extension $F(u)(v)$ of $F(u)$ that contains a root v of $g(x)$. Continuing this, we eventually get a splitting field of $f(x)$. We can formalize this argument via induction and prove slightly more:

Theorem 11.13

Let F be a field and $f(x)$ a nonconstant polynomial of degree n in $F(x)$. Then there exists a splitting field K of $f(x)$ over F such that $[K{:}F] \leq n!$.

Proof ▸ The proof is by induction on the degree of $f(x)$. If $f(x)$ has degree 1, then F itself is a splitting field of $f(x)$ and $[F{:}F] = 1 \leq 1!$. Suppose the theorem is true for all polynomials of degree $n - 1$ and that $f(x)$ has degree n. By Theorem 4.14 $f(x)$ has an irreducible factor in $F[x]$ Multiplying this polynomial by the inverse of its leading coefficient produces a monic irreducible factor $p(x)$ of $f(x)$. By Theorem 5.11 there is an extension field that contains a root u of $p(x)$ (and, hence, of $f(x)$). Furthermore, $p(x)$ is necessarily the minimal polynomial of u. Consequently, by Theorem 11.7 $[F(u) : F] = \deg p(x) \leq \deg f(x) = n$. The Factor Theorem 4.16 shows that $f(x) = (x - u)g(x)$ for some $g(x) \in F(u)[x]$. Since $g(x)$ has degree $n - 1$, the induction hypothesis guarantees the existence of a splitting field K of $g(x)$ over $F(u)$ such that $[K{:}F(u)] \leq (n - 1)!$. In $K[x]$,

$$g(x) = c(x - u_1)(x - u_2) \cdots (x - u_{n-1})$$

and, hence, $f(x) = c(x - u)(x - u_1) \cdots (x - u_{n-1})$. Since

$$K = F(u)(u_1, \ldots, u_{n-1}) = F(u, u_1, \ldots, u_{n-1})$$

we see that K is a splitting field of $f(x)$ over F such that $[K{:}F] = [K{:}F(u)][F(u){:}F] \leq ((n - 1)!)n = n!$. This completes the inductive step and the proof of the theorem. ■

The relationship between two splitting fields of the same polynomial is quite easy to state:

Any two splitting fields of a polynomial in $F[x]$ are isomorphic.

Surprisingly, the easiest way to prove this fact is to prove a stronger result of which this is a special case.

Theorem 11.14

Let $\sigma{:}F \to E$ be an isomorphism of fields, $f(x)$ a nonconstant polynomial in $F[x]$, and $\sigma f(x)$ the corresponding polynomial in $E[x]$. If K is a splitting field of $f(x)$ over F and L is a splitting field of $\sigma f(x)$ over E, then σ extends to an isomorphism $K \cong L$.

If $F = E$ and σ is the identity map $F \to F$, then the theorem states that any two splitting fields of $f(x)$ are isomorphic.

Proof of Theorem 11.14 ▸ The proof is by induction on the degree of $f(x)$. If $\deg f(x) = 1$, then by the definition of splitting field $f(x) = c(x - u)$ in $K[x]$ and $K = F(u)$. But $f(x) = cx - cu$ is in $F[x]$, so we must have c and cu in F. Hence, $u = c^{-1}cu$ is also in F. Therefore, $K = F(u) = F$. On page 380 we saw that σ extends to an isomorphism $F[x] \cong E[x]$; hence, $\sigma f(x)$ also has degree 1, and a similar argument shows that $E = L$. In this case, σ itself is an isomorphism with the required properties.

Suppose the theorem is true for polynomials of degree $n - 1$ and that $f(x)$ has degree n. As in the proof of Theorem 11.13, $f(x)$ has a monic irreducible factor $p(x)$ in $F[x]$ by Theorem 4.14. Since σ extends to an isomorphism $F[x] \cong E[x]$, (page 380), $\sigma p(x)$ is a monic irreducible factor of $\sigma f(x)$ in $E[x]$. Every root of $p(x)$ is also a root of $f(x)$, so K contains all the roots of $p(x)$, and similarly L contains all the roots of $\sigma p(x)$. Let u be a root of $p(x)$ in K and v a root of $\sigma p(x)$ in L. Then σ extends to an isomorphism $F(u) \to E(v)$ that maps u to v by Corollary 11.8, and the situation looks like this:

$$\begin{array}{ccc} K & & L \\ \cup\!| & & \cup\!| \\ F(u) & \xrightarrow{\cong} & E(v) \\ \cup\!| & & \cup\!| \\ F & \xrightarrow{\sigma} & E. \end{array}$$

The Factor Theorem 4.16 shows that $f(x) = (x - u)g(x)$ in $F(u)[x]$ and, hence, in $E(v)[x]$

$$\sigma f(x) = \sigma(x - u)\sigma g(x) = (x - \sigma u)\sigma g(x) = (x - v)\sigma g(x).$$

Now $f(x)$ splits over K, say $f(x) = c(x - u)(x - u_2) \cdots (x - u_n)$. Since $f(x) = (x - u)g(x)$, we have $g(x) = c(x - u_2) \cdots (x - u_n)$. The smallest subfield containing all the roots of $g(x)$ and the field $F(u)$ is $F(u, u_2, \ldots, u_n) = K$, so K is a splitting field of $g(x)$ over $F(u)$. Similarly, L is a splitting field of $\sigma g(x)$ over $E(v)$. Since $g(x)$ has degree $n - 1$, the induction hypothesis implies that the isomorphism $F(u) \cong E(v)$ can be extended to an isomorphism $K \cong L$. This completes the inductive step and the proof of the theorem. ■

A splitting field of some polynomial over F contains all the roots of that polynomial by definition. Surprisingly, however, splitting fields have a much stronger property, which we now define.

Definition

An algebraic extension field K of F is **normal** provided that whenever an irreducible polynomial in $F[x]$ has one root in K, then it splits over K (that is, has all its roots in K).

Theorem 11.15

The field K is a splitting field over the field F of some polynomial in $F[x]$ if and only if K is a finite-dimensional, normal extension of F.

Proof ▸ If K is a splitting field of $f(x) \in F[x]$, then $K = F(u_1, \ldots, u_n)$, where the u_i are all the roots of $f(x)$. Consequently, $[K:F]$ is finite by Theorem 11.10. Let $p(x)$ be an irreducible polynomial in $F[x]$ that has a root v in K. Consider $p(x)$ as a polynomial in $K[x]$ and let L be a splitting field of $p(x)$ over K, so that $F \subseteq K \subseteq L$. To prove that $p(x)$ splits over K, we need only show that every root of $p(x)$ in L is actually in K.

Let $w \in L$ be any root of $p(x)$ other than v. By Corollary 11.8 (with $E = F$ and σ the identity map), there is an isomorphism $F(v) \cong F(w)$ that maps v to w and maps every element of F to itself. Consider the subfield $K(w)$ of L; the situation looks like this:

$$\begin{array}{ccc} K & & K(w) \\ \cup\!\mid & & \cup\!\mid \\ F(v) & \cong & F(w) \\ \cup\!\mid & & \cup\!\mid \\ F & = & F. \end{array}$$

Since

$$K(w) = F(u_1, \ldots, u_n)(w) = F(u_1, \ldots, u_n, w) = F(w)(u_1, \ldots, u_n)$$

we see that $K(w)$ is a splitting field of $f(x)$ over $F(w)$. Furthermore, since $v \in K$ and K is a splitting field of $f(x)$ over F, K is also a splitting field of $f(x)$ over the subfield $F(v)$. Consequently, by Theorem 11.14 the isomorphism $F(v) \cong F(w)$ extends to an isomorphism $K \to K(w)$ that maps v to w and every element of F to itself. Therefore, $[K:F] = [K(w):F]$ by Theorem 11.5. In the extension chain $F \subseteq K \subseteq K(w)$, $[K(w):K]$ is finite by Theorem 11.7 and $[K:F]$ is finite by the remarks in the first paragraph of the proof. So Theorem 11.4 implies that

$$[K:F] = [K(w):F] = [K(w):K][K:F].$$

Canceling $[K:F]$ on each end shows that $[K(w):K] = 1$, and, therefore, $K(w) = K$. But this means that w is in K. Thus every root of $p(x)$ in L is in K, and $p(x)$ splits over K. Therefore, K is normal over F.

Conversely, assume K is a finite-dimensional, normal extension of F with basis $\{u_1, \ldots, u_n\}$. Then $K = F(u_1, \ldots, u_n)$. Each u_i is algebraic over F by Theorem 11.9 with minimal polynomial $p_i(x)$. Since each $p_i(x)$ splits over K by normality, $f(x) = p_1(x) \cdots p_n(x)$ also splits over K. Therefore, K is the splitting field of $f(x)$. ■

EXAMPLE 5

The field $\mathbb{Q}(\sqrt[3]{2})$ contains the real root $\sqrt[3]{2}$ of the irreducible polynomial $x^3 - 2 \in \mathbb{Q}[x]$ but does not contain the complex root $\sqrt[3]{2}\omega$ (as described in Example 7 of Section 11.2). Therefore, $\mathbb{Q}(\sqrt[3]{2})$ is not a normal extension of $\mathbb{Q}$ and, hence, cannot be the splitting field of *any* polynomial in $\mathbb{Q}[x]$.

At this point it is natural to ask if a field F has an extension field over which *every* polynomial in $F[x]$ splits. In other words, is there an extension field that contains all the roots of all the polynomials in $F[x]$? The answer is "yes," but the proof is beyond the scope of this book. A field over which every nonconstant polynomial splits is said to be **algebraically closed.** For example, the Fundamental Theorem of Algebra and Corollary 4.28 show that the field $\mathbb{C}$ of complex numbers is algebraically closed.

If K is an algebraic extension of F and K is algebraically closed, then K is called the **algebraic closure** of F. The word "the" is justified by a theorem analogous to Theorem 11.14 that says any two algebraic closures of F are isomorphic. For example, $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$ since $\mathbb{C} = \mathbb{R}(i)$ is an algebraic extension of $\mathbb{R}$ that is algebraically closed. The field $\mathbb{C}$ is not the algebraic closure of $\mathbb{Q}$, however, since $\mathbb{C}$ is not algebraic over $\mathbb{Q}$. The subfield E of algebraic numbers (see Example 7 of Section 11.3) is the algebraic closure of $\mathbb{Q}$ (Exercise 20).

Exercises

NOTE: *F is a field.*

A. **1.** Show that $\sqrt{2}$ is not in $\mathbb{Q}(i)$ and, hence, $\mathbb{C} \neq \mathbb{Q}(i)$. [*Hint:* Show that $\sqrt{2} = a + bi$, with $a, b \in \mathbb{Q}$, leads to a contradiction.]

2. Show that $x^2 - 3$ and $x^2 - 2x - 2$ are irreducible in $\mathbb{Q}[x]$ and have the same splitting field, namely $\mathbb{Q}(\sqrt{3})$.

3. Find a splitting field of $x^4 - 4x^2 - 5$ over $\mathbb{Q}$ and show that it has dimension 4 over $\mathbb{Q}$.

4. If $f(x) \in \mathbb{R}[x]$, prove that $\mathbb{R}$ or $\mathbb{C}$ is a splitting field of $f(x)$ over $\mathbb{R}$.

5. Let K be a splitting field of $f(x)$ over F. If E is a field such that $F \subseteq E \subseteq K$, show that K is a splitting field of $f(x)$ over E.

6. Let K be a splitting field of $f(x)$ over F. If $[K:F]$ is prime, $u \in K$ is a root of $f(x)$, and $u \notin F$, show that $K = F(u)$.

7. If u is algebraic over F and $K = F(u)$ is a normal extension of F, prove that K is a splitting field over F of the minimal polynomial of u.

8. Which of the following are normal extensions of $\mathbb{Q}$?

(a) $\mathbb{Q}(\sqrt{3})$ (b) $\mathbb{Q}(\sqrt[3]{3})$ (c) $\mathbb{Q}(\sqrt{5}, i)$

9. Prove that no finite field is algebraically closed. [*Hint:* If the elements of the field F are $a_1, \ldots, a_n$, with a_1 nonzero, consider $a_1 + (x - a_1)(x - a_2) \cdots (x - a_n) \in F[x]$.]

B. **10.** By finding quadratic factors, show that $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a splitting field of $x^4 + 2x^3 - 8x^2 - 6x - 1$ over $\mathbb{Q}$.

11. Find and describe a splitting field of $x^4 + 1$ over $\mathbb{Q}$.

12. Find a splitting field of $x^4 - 2$

(a) over $\mathbb{Q}$. (b) over $\mathbb{R}$.

13. Find a splitting field of $x^6 + x^3 + 1$ over $\mathbb{Q}$.

14. Show that $\mathbb{Q}(\sqrt{2}, i)$ is a splitting field of $x^2 - 2\sqrt{2}x + 3$ over $\mathbb{Q}(\sqrt{2})$.

15. Find a splitting field of $x^2 + 1$ over $\mathbb{Z}_3$.

16. Find a splitting field of $x^3 + x + 1$ over $\mathbb{Z}_2$.

17. If K is an extension field of F such that $[K:F] = 2$, prove that K is normal.

18. Let F, E, K be fields such that $F \subseteq E \subseteq K$ and $E = F(u_1, \ldots, u_t)$, where the u_i are some of the roots of $f(x) \in F[x]$. Prove that K is a splitting field of $f(x)$ over F if and only if K is a splitting field of $f(x)$ over E.

19. Prove that the following conditions on a field K are equivalent:

(i) Every nonconstant polynomial in $K[x]$ has a root in K.

(ii) Every nonconstant polynomial in $K[x]$ splits over K (that is, K is algebraically closed).

(iii) Every irreducible polynomial in $K[x]$ has degree 1.

(iv) There is no algebraic extension field of K except K itself.

20. Let K be an extension field of F and E the subfield of all elements of K that are algebraic over F, as in Corollary 11.12. If K is algebraically closed, prove that E is an algebraic closure of F. [The special case when $F = \mathbb{Q}$ and $K = \mathbb{C}$ shows that the field E of algebraic numbers is an algebraic closure of $\mathbb{Q}$.]

21. Let K be an algebraic extension field of F such that every polynomial in $F(x)$ splits over K. Prove that K is an algebraic closure of F.

C. 22. If K is a finite-dimensional extension field of F and $\sigma: F \to K$ is a homomorphism of fields, prove that there exists an extension field L of K and a homomorphism $\tau: K \to L$ such that $\tau(a) = \sigma(a)$ for every $a \in F$.

23. Prove that a finite-dimensional extension field K of F is normal if and only if it has this property: Whenever L is an extension field of K and $\sigma: K \to L$ an injective homomorphism such that $\sigma(c) = c$ for every $c \in F$, then $\sigma(K) \subseteq K$.

11.5 Separability

Every polynomial has a splitting field that contains all its roots. These roots may all be distinct, or there may be repeated roots.* In this section we consider the case when the roots are distinct and use the information obtained to prove a very useful fact about finite-dimensional extensions.

Let F be a field. A polynomial $f(x) \in F[x]$ of degree n is said to be **separable** if it has n distinct roots in some splitting field.† Equivalently, $f(x)$ is separable if it has no repeated roots in any splitting field. If K is an extension field of F, then an element $u \in K$ is said to be **separable** over F if u is algebraic over F and its minimal polynomial $p(x) \in F[x]$ is separable. The extension field K is said to be a **separable extension** (or to be **separable over** F) if every element of K is separable over F. Thus a separable extension is necessarily algebraic.

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*A repeated root occurs when $f(x) = (x - u_1) \cdots (x - u_n)$ in the splitting field and some $u_i = u_j$ with $i \neq j$.

†Since any two splitting fields are isomorphic, this means that $f(x)$ has n distinct roots in every splitting field.

EXAMPLE 1

The polynomial $x^2 + 1 \in \mathbb{Q}[x]$ is separable since it has distinct roots i and $-i$ in $\mathbb{C}$. But $f(x) = x^4 - x^3 - x + 1$ is not separable because it factors as $(x - 1)^2(x^2 + x + 1)$. Hence, $f(x)$ has one repeated root and a total of three distinct roots in $\mathbb{C}$.

There are several tests for separability that make use of the following concept. The **derivative** of

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n \in F[x]$$

is defined to be the polynomial

$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} \in F[x]^*.$$

You should use Exercises 4 and 5 to verify that derivatives defined in this algebraic fashion have these familiar properties.

$$(f + g)'(x) = f'(x) + g'(x)$$
$$(fg)'(x) = f(x)g'(x) + f'(x)g(x).$$

Lemma 11.16

Let F be a field and $f(x) \in F[x]$. If $f(x)$ and $f'(x)$ are relatively prime in $F[x]$, then $f(x)$ is separable.

Note that the lemma operates entirely in $F[x]$ and does not require any knowledge of the splitting field to determine separability. For other separability criteria, see Exercises 8–10.

Proof of Lemma 11.16 ▸ We shall prove the contrapositive: If $f(x)$ is not separable, then $f(x)$ and $f'(x)$ are not relatively prime (which is logically equivalent to the statement of the theorem).† Let K be a splitting field of $f(x)$ and suppose that $f(x)$ is not separable. Then $f(x)$ must have a repeated root u in K. Hence, $f(x) = (x - u)^2g(x)$ for some $g(x) \in K[x]$ and

$$f'(x) = (x - u)^2g'(x) + 2(x - u)g(x).$$

Therefore, $f'(u) = 0_Fg'(u) + 0_Fg(u) = 0_F$ and u is also a root of $f'(x)$. If $p(x) \in F[x]$ is the minimal polynomial of u, then $p(x)$ is nonconstant and divides both $f(x)$ and $f'(x)$. Therefore, $f(x)$ and $f'(x)$ are not relatively prime. ■

*When $F = \mathbb{R}$, this is the usual derivative of elementary calculus. But our definition is purely algebraic and applies to polynomials over any field, whereas the limits used in calculus may not be defined in some fields.

†See Appendix A (pages 503, 504 and 506) for the definition and use of the contrapositive in proofs.

Recall that for a positive integer n and $c \in F$,

$$nc \text{ is the element } c + c + \cdots + c \ (n \text{ summands}).$$

A field F is said to have **characteristic 0** if $n1_F \neq 0_F$ for every positive n. For example, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ all have characteristic 0, but $\mathbb{Z}_3$ does not (since $3 \cdot 1 = 0$ in $\mathbb{Z}_3$). Every field of characteristic 0 is infinite (Exercise 3). If F has characteristic 0, then for every positive n and $c \in F$,

$$nc = c + \cdots + c = (1_F + \cdots + 1_F)c = (n1_F)c \qquad \text{with } n1_F \neq 0_F.$$

So $nc = 0_F$ if and only if $c = 0_F$. This fact is the key to separability in fields of characteristic 0:

Theorem 11.17

Let F be a field of characteristic 0. Then every irreducible polynomial in $F[x]$ is separable, and every algebraic extension field K of F is a separable extension.

The theorem may be false if F does not have characteristic 0 (Exercise 15).

Proof of Theorem 11.17 ▸ An irreducible $p(x) \in F[x]$ is nonconstant and, hence,

$$p(x) = cx^n + (\text{lower-degree terms}), \qquad \text{with } c \neq 0_F \text{ and } n \geq 1.$$

Then

$$p'(x) = (nc)x^{n-1} + (\text{lower-degree terms}), \qquad \text{with } nc \neq 0_F.$$

Therefore, $p'(x)$ is a nonzero polynomial of lower degree than the irreducible $p(x)$. So $p(x)$ and $p'(x)$ must be relatively prime. Hence, $p(x)$ is separable by Lemma 11.16. In particular, the minimal polynomial of each $u \in K$ is separable. So K is a separable extension. ■

Separable extensions are particularly nice because every finitely generated (in particular, every finite-dimensional) separable extension is actually simple:

Theorem 11.18*

If K is a finitely generated separable extension field of F, then $K = F(u)$ for some $u \in K$.

Proof ▸ By hypothesis $K = F(u_1, \ldots, u_n)$. The proof is by induction on n. There is nothing to prove when $n = 1$ and $K = F(u_1)$. In the next paragraph we shall show that the theorem is true for $n = 2$. Assume inductively that it is true for $n = k - 1$ and suppose $n = k$. By induction and the case $n = 2$, there exist $t, u \in K$ such that

$$K = F(u_1, \ldots, u_k) = F(u_1, \ldots, u_{k-1})(u_k) = F(t)(u_k) = F(t, u_k) = F(u).$$

*This theorem will be used only in Section 12.2.

To complete the proof, we assume $K = F(v, w)$ and show that K is a simple extension of F. Assume first that F is infinite (which is always the case in characteristic 0 by Exercise 3). Let $p(x) \in F[x]$ be the minimal polynomial of v and $q(x) \in F[x]$ the minimal polynomial of w. Let L be a splitting field of $p(x)q(x)$ over F. Let $w = w_1, w_2, \ldots, w_n$ be the roots of $q(x)$ in L. By the definition of separability, all the w_i are distinct. Let $v = v_1, v_2, \ldots, v_m$ be the roots of $p(x)$ in L. Since F is infinite, there exists $c \in F$ such that

$$(*) \qquad c \neq \frac{v_i - v}{w - w_j} \qquad \text{for all } 1 \leq i \leq m,\ 1 < j \leq n.$$

Let $u = v + cw$. We claim that $K = F(u)$. To show that $w \in F(u)$, let $h(x) = p(u - cx) \in F(u)[x]$ and note that w is a root of $h(x)$:

$$h(w) = p(u - cw) = p(v) = 0_F.$$

Suppose some w_j (with $j \neq 1$) is also a root of $h(x)$. Then $p(u - cw_j) = 0_F$, so that $u - cw_j$ is one of the roots of $p(x)$, say $u - cw_j = v_i$. Since $u = v + cw$, we would have

$$v + cw - cw_j = v_i \qquad \text{or, equivalently,} \qquad c = \frac{v_i - v}{w - w_j}.$$

This contradicts $(*)$. Therefore, w is the only common root of $q(x)$ and $h(x)$.

Let $r(x)$ be the minimal polynomial of w over $F(u)$. Then $r(x)$ divides $q(x)$, so that every root of $r(x)$ is a root of $q(x)$. But $r(x)$ also divides $h(x)$, so all its roots are roots of $h(x)$. By the preceding paragraph, $r(x)$ has a single root w in L. Therefore, $r(x) \in F(u)[x]$ must have degree 1, and, hence, its root w is in $F(u)$. Since $v = u - cw$, with u, $w \in F(u)$, we see that $v \in F(u)$ and, hence, $K = F(v, w) \subseteq F(u)$. But $u = v + cw \in K$, so $F(u) \subseteq K$, whence $K = F(u)$. This completes the proof when F is infinite. For the case of finite F, see Theorem 11.28 in the next section. ■

EXAMPLE 2

Applying the proof of the theorem to $\mathbb{Q}(\sqrt{3}, \sqrt{5})$, we have $v = \sqrt{3}$, $v_2 = -\sqrt{3}$, $w = \sqrt{5}$, $w_2 = -\sqrt{5}$, so we can choose $c = 1$. Then $u = \sqrt{3} + \sqrt{5}$ and $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ is the simple extension $\mathbb{Q}(\sqrt{3} + \sqrt{5})$.

Exercises

NOTE: *K is an extension field of the field F.*

A. 1. If K is separable over F and E is a field with $F \subseteq E \subseteq K$, show that K is separable over E.

2. If F has characteristic 0, show that K has characteristic 0.

3. Prove that every field of characteristic 0 is infinite. [*Hint:* Consider the elements $n1_F$ with $n \in \mathbb{Z}$, $n > 0$.]

B. 4. If $f(x), g(x) \in F[x]$, prove

(a) $(f+g)'(x) = f'(x) + g'(x)$.

(b) If $c \in F$, then $(cf)'(x) = cf'(x)$.

5. (a) If $f(x) = cx^n \in F[x]$ and $g(x) = b_0 + b_1x + \cdots + b_kx^k \in F[x]$, prove that $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$.

(b) If $f(x), g(x)$ are any polynomials in $F[x]$, prove that $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$. [*Hint:* If $f(x) = a_0 + a_1x + \cdots + a_nx^n$, then $(fg)(x) = a_0g(x) + a_1xg(x) + \cdots + a_nx^ng(x)$; use part (a) and Exercise 4.]

6. If $f(x) \in F[x]$ and n is a positive integer, prove that the derivative of $f(x)^n$ is $nf(x)^{n-1}f'(x)$. [*Hint:* Use induction on n and Exercise 5.]

7. (a) If F has characteristic 0, $f(x) \in F[x]$, and $f'(x) = 0_F$, prove that $f(x) = c$ for some $c \in F$.

(b) Give an example in $\mathbb{Z}_2[x]$ to show that part (a) may be false if F does not have characteristic 0.

8. Prove that $u \in K$ is a repeated root of $f(x) \in F[x]$ if and only if u is a root of both $f(x)$ and $f'(x)$. [*Hint:* $f(x) = (x-u)^mg(x)$ with $m \geq 1$, $g(x) \in K[x]$, and $g(u) \neq 0_F$, u is a repeated root of $f(x)$ if and only if $m > 1$. Use Exercises 5 and 6 to compute $f'(x)$.]

9. Prove that $f(x) \in F[x]$ is separable if and only if $f(x)$ and $f'(x)$ are relatively prime. [*Hint:* See Lemma 11.16 and Exercise 8.]

10. Let $p(x)$ be irreducible in $F[x]$. Prove that $p(x)$ is separable if and only if $p'(x) \neq 0_F$.

11. Assume F has characteristic 0 and K is a splitting field of $f(x) \in F[x]$. If $d(x)$ is the greatest common divisor of $f(x)$ and $f'(x)$ and $h(x) = f(x)/d(x) \in F[x]$, prove

(a) $f(x)$ and $h(x)$ have the same roots in K.

(b) $h(x)$ is separable.

12. Use the proof of Theorem 11.18 to express each of these as simple extensions of $\mathbb{Q}$:

(a) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ (b) $\mathbb{Q}(\sqrt{3}, i)$ (c) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$

13. If p and q are distinct primes, prove that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$.

14. Assume that F is infinite, that $v, w \in K$ are algebraic over F, and that w is the root of a separable polynomial in $F[x]$. Prove that $F(v, w)$ is a simple extension of F. [*Hint:* Adapt the proof of Theorem 11.18.]

15. Here is an example of an irreducible polynomial that is not separable. Let $F = \mathbb{Z}_2(t)$ be the quotient field of $\mathbb{Z}_2[t]$ (the ring of polynomials in

the indeterminate t with coefficients in $\mathbb{Z}_2$), as in Example 1 of Section 10.4.

(a) Prove that $x^2 - t$ is an irreducible polynomial in $F[x]$. [*Hint:* If $x^2 - t$ has a root in F, then there are polynomials $g(t)$, $h(t)$ in $\mathbb{Z}_2[t]$ such that $[g(t)/h(t)]^2 = t$; this leads to a contradiction; apply Corollary 4.19.]

(b) Prove that $x^2 - t \in F[x]$ is *not* separable. [*Hint:* Show that its derivative is zero and use Exercise 10.]

11.6 Finite Fields

Finite fields have applications in many areas, including projective geometry, combinatories, experimental design, and cryptography. In this section, finite fields are characterized in terms of field extensions and splitting fields, and their structure is completely determined up to isomorphism.

We begin with some definitions and results that apply to rings that need *not* be fields or even finite. But our primary interest will be in their implications for finite fields.

Let R be a ring with identity. Recall that for a positive integer m and $c \in R$, mc is the element $c + c + \cdots + c$ (m summands). The ring R is said to have **characteristic 0** if $m1_R \neq 0_R$ for every positive m. On the other hand, if $m1_R = 0_R$ for some positive m, then there is a smallest such m by the Well-Ordering Axiom. Then R is said to have **characteristic n** if n is the smallest positive integer such that $n1_R = 0_R$.* For example, $\mathbb{Q}$ has characteristic 0 and $\mathbb{Z}_3$ has characteristic 3.

Lemma 11.19

If R is an integral domain, then the characteristic of R is either 0 or a positive prime.

Proof ▸ If R has characteristic 0, there is nothing to prove. So assume R has characteristic $n > 0$. If n were not prime, then there would exist positive integers k, t such that $n = kt$, with $k < n$ and $t < n$. The distributive laws show that

$$(k1_R)(t1_R) = \underbrace{(1_R + \cdots + 1_R)}_{k \text{ summands}}\underbrace{(1_R + \cdots + 1_R)}_{t \text{ summands}}$$

$$= 1_R1_R + \cdots + 1_R1_R = 1_R + \cdots + 1_R \qquad [kt \text{ summands}]$$

$$= (kt)1_R = n1_R = 0_R.$$

*If you have read Chapter 7, you will recognize that when the characteristic of R is positive, it is simply the order of the element 1_R in the additive group of R.

Since R is an integral domain either $k1_R = 0_R$ or $t1_R = 0_R$, contradicting the fact that n is the smallest positive integer such that $n1_R = 0_R$. Therefore, n is prime. ■

Lemma 11.20

Let R be a ring with identity of characteristic $n > 0$. Then $k1_R = 0_R$ if and only if $n \mid k$.*

Proof ▸ If $n \mid k$, say $k = nd$, then $k1_R = nd1_R = (n1_R)(d1_R) = 0_R\,(d1_R) = 0_R$. Conversely, suppose $k1_R = 0_R$. By the Division Algorithm, $k = nq + r$ with $0 \le r < n$. Now $n1_R = 0_R$, so that

$$r1_R = r1_R + 0_R = r1_R + nq1_R = (r + nq)1_R = k1_R = 0_R.$$

Since $r < n$ and n is the smallest positive integer such that $n1_R = 0_R$ by the definition of characteristic, we must have $r = 0$. Therefore, $k = nq$ and $n \mid k$. ■

Theorem 11.21

Let R be a ring with identity. Then

(1) The set $P = \{k1_R \mid k \in \mathbb{Z}\}$ is a subring of R.

(2) If R has characteristic 0, then $P \cong \mathbb{Z}$.

(3) If R has characteristic $n > 0$, then $P \cong \mathbb{Z}_n$.

Proof ▸ Define $f:\mathbb{Z} \to R$ by $f(k) = k1_R$. Then

$$f(k + t) = (k + t)1_R = k1_R + t1_R = f(k) + f(t).$$

The distributive laws (as in the proof of Lemma 11.19) show that

$$f(kt) = (kt)1_R = (k1_R)(t1_R) = f(k)f(t).$$

Therefore, f is a homomorphism. The image of f is precisely the set P, and, therefore, P is a ring by Corollary 3.11. Consequently, f can be considered as a surjective homomorphism from $\mathbb{Z}$ onto P. Then $P \cong \mathbb{Z}/(\mathrm{Ker}\, f)$ by the First Isomorphism Theorem 6.13. If R has characteristic 0, then the only integer k such that $k1_R = 0_R$ is $k = 0$. So the kernel of f is the ideal (0) in $\mathbb{Z}$, and $P \cong \mathbb{Z}/(0) \cong \mathbb{Z}$. If R has characteristic $n > 0$, then Lemma 11.20 shows that the kernel of f is the principal ideal (n) consisting of all multiples of n. Hence, $P \cong \mathbb{Z}/(n) = \mathbb{Z}_n$. ■

---

*This lemma is just a special case (in additive notation) of part (1) of Theorem 7.9, with $a = 1_R$ and $e = 0_R$.

According to Theorem 11.21 a field of characteristic 0 contains a copy of $\mathbb{Z}$ and, hence, must be infinite. Therefore, by Lemma 11.19 we have

Corollary 11.22

Every finite field has characteristic p for some prime p.

The converse of Corollary 11.22 is false, however, since there are infinite fields of characteristic p (Exercise 8).

If K is a field of prime characteristic p (in particular, if K is finite), then Theorem 11.21 shows that K contains a subfield P isomorphic to $\mathbb{Z}_p$. This field P is called the **prime subfield** of K and is contained in every subfield of K (because every subfield contains 1_K and, hence, contains $t1_K$ for every integer t).* See Exercise 4 for another description of P. We shall identify the prime subfield P with its isomorphic copy $\mathbb{Z}_p$; then

every field of characteristic p contains $\mathbb{Z}_p$.

The number of elements in a finite field K is called the **order** of K. To determine the order of a finite field K of characteristic p, we consider K as an extension field of its prime subfield $\mathbb{Z}_p$:

Theorem 11.23

A finite field K has order p^n, where p is the characteristic of K and $n = [K: \mathbb{Z}_p]$.

Proof ▸ There is certainly a finite set of elements that spans K over $\mathbb{Z}_p$ (the set K itself, for example). Consequently, by Exercise 32 of Section 11.1, K has a finite basis $\{u_1, u_2, \ldots, u_n\}$ over $\mathbb{Z}_p$. Every element of K can be written uniquely in the form

$$(*) \qquad c_1u_1 + c_2u_2 + \cdots + c_nu_n$$

with each $c_i \in \mathbb{Z}_p$ by Exercise 30 of Section 11.1. Since there are exactly p possibilities for each c_i, there are precisely p^n distinct linear combinations of the form $(*)$. So K has order p^n, with n = number of elements in the basis $= [K:\mathbb{Z}_p]$. ■

Theorem 11.23 limits the possible size of a finite field. For instance, there cannot be a field of order 6 since 6 is not a power of any prime. It also suggests several questions: Is there a field of order p^n for every prime p and every positive integer n?

*If K has characteristic 0, then K contains an isomorphic copy P of $\mathbb{Z}$. Since K contains the multiplicative inverse of every nonzero element of P, it follows that K contains a copy of the field $\mathbb{Q}$. As in the case of characteristic p, this field (called the **prime subfield**) is contained in every subfield of K. See Theorem 10.31 (with $R = P \cong \mathbb{Z}$ and $F \cong \mathbb{Q}$) for a more precise statement and proof.

How are two fields of order p^n related? The answers to these questions are given in Theorem 11.25 and its corollaries. In order to prove that theorem, we need a technical lemma.

Lemma 11.24 The Freshman's Dream*

Let p be a prime and R a commutative ring with identity of characteristic p. Then for every $a, b \in R$ and every positive integer n,

$$(a + b)^{p^n} = a^{p^n} + b^{p^n}.$$

Proof ▸ The proof is by induction on n. If $n = 1$, then the Binomial Theorem in Appendix E shows that

$$(a + b)^p = a^p + \binom{p}{1}a^{p-1}b + \cdots + \binom{p}{r}a^{p-r}b^r$$
$$+ \cdots + \binom{p}{p-1}ab^{p-1} + b^p.$$

Each of the middle coefficients $\binom{p}{r} = \dfrac{p!}{r!(p-r)!}$ is an integer by Exercise 6 in Appendix E. Since every term in the denominator is strictly less than the prime p, the factor of p in the numerator does not cancel, and, therefore, $\binom{p}{r}$ is divisible by p, say $\binom{p}{r} = tp$. Since R has characteristic p,

$$\binom{p}{r}a^{p-r}b^r = tp1_Ra^{p-r}b^r = t(p1_R)a^{p-r}b^r = t0_Ra^{p-r}b^r = 0_R.$$

Thus all the middle terms are zero and $(a + b)^p = a^p + b^p$. So the theorem is true when $n = 1$. Assume the theorem is true when $n = k$. Using this assumption and the case when $n = 1$ shows that

$$(a + b)^{p^{k+1}} = ((a + b)^{p^k})^p$$
$$= (a^{p^k} + b^{p^k})^p = (a^{p^k})^p + (a^{p^k})^p = a^{p^{k+1}} + b^{p^{k+1}}.$$

Therefore, the theorem is true when $n = k + 1$ and, hence, for all n by induction. ■

*Terminology due to Vincent O. McBrien.

Theorem 11.25

Let K be an extension field of $\mathbb{Z}_p$ and n a positive integer. Then K has order p^n if and only if K is a splitting field of $x^{p^n} - x$ over $\mathbb{Z}_p$.

Proof ▸ Assume K is a splitting field of $f(x) = x^{p^n} - x \in \mathbb{Z}_p(x)$. Since $f'(x) = p^n x^{p^n - 1} - 1 = 0x^{p^n-1} - 1 = -1, f(x)$ is separable by Lemma 11.16. Let E be the subset of K consisting of the p^n distinct roots of $x^{p^n} - x$. Note that $c \in E$ if and only if $c^{p^n} = c$. We shall show that the set E is actually a subfield of K. If $a, b \in E$, then by Lemma 11.24.

$$(a + b)^{p^n} = a^{p^n} + b^{p^n} = a + b.$$

Therefore, $a + b \in E$, and E is closed under addition. The set E is closed under multiplication since $(ab)^{p^n} = a^{p^n} b^{p^n} = ab$. Obviously, 0_K and 1_K are in E. If a is a nonzero element of E, then $-a$ and a^{-1} are in E because, for example,

$$(a^{-1})^{p^n} = a^{-p^n} = (a^{p^n})^{-1} = a^{-1}.$$

The argument for $-a$ is similar (Exercise 7). Therefore, E is a subfield of K. Since the splitting field K is the smallest subfield containing the set E of roots, we must have $K = E$. Therefore, K has order p^n.

Conversely, suppose K has order p^n. We need only show that every element of K is a root of $x^{p^n} - x$, for in that case, the p^n distinct elements of K are all the possible roots and K is a splitting field of $x^{p^n} - x$.* Clearly 0_K is a root, so let c be any nonzero element of K. Let $c_1, c_2, \ldots, c_k$ be all the nonzero elements of K (where $k = p^n - 1$ and c is one of the c_i) and let u be the product $u = c_1 c_2 c_3 \cdots c_k$. The k elements $cc_1, cc_2, \ldots, cc_k$ are all distinct (since $cc_i = cc_j$ implies $c_i = c_j$), so they are just the nonzero elements of K in some other order, and their product is the element u. Therefore,

$$u = (cc_1)(cc_2) \cdots (cc_K) = c^k(c_1 c_2 c_3 \cdots c_k) = c^k u.$$

Canceling u shows that $c^k = 1_K$ and, hence, $c^{k+1} = c$, or equivalent $c^{k+1} - c = 0_K$. Since $k + 1 = p^n$, c is a root of $x^{p^n} - x$. ■

Theorem 11.25 has several important consequences; together with the theorem they provide a complete characterization of all finite fields.

Corollary 11.26

For each positive prime p and positive integer n, there exists a field of order p^n.

Proof ▸ A splitting field of $x^{p^n} - x$ over $\mathbb{Z}_p$ exists by Theorem 11.13; it has order p^n by Theorem 11.25 ■

*A short proof, using group theory, is given in Exercise 22.

Corollary 11.27

Two finite fields of the same order are isomorphic.

Proof ▸ If K and L are fields of order p^n, then both are splitting fields of $x^{p^n} - x$ over $\mathbb{Z}_p$ by Theorem 11.25 and, hence, are isomorphic by Theorem 11.14 (with σ the identity map on $\mathbb{Z}_p$). ■

According to Corollary 11.27, there is (up to isomorphism) a unique field of order p^n. This field is called the **Galois field of order** p^n. We complete our study of finite fields with two results whose proofs depend on group theory.

Theorem 11.28

Let K be a finite field and F a subfield. Then K is a simple extension of F.

Proof ▸ By Theorem 7.16 the multiplicative group of nonzero elements of K is cyclic. If u is a generator of this group, then the subfield $F(u)$ contains 0_F and all powers of u and, hence, contains every element of K. Therefore, $K = F(u)$. ■

Corollary 11.29

Let p be a positive prime. For each positive integer n, there exists an irreducible polynomial of degree n in $\mathbb{Z}_p[x]$.

Proof ▸ There is an extension field K of $\mathbb{Z}_p$ of order p^n by Corollary 11.26. By Theorem 11.28, $K = \mathbb{Z}_p(u)$ for some $u \in K$. The minimal polynomial of u in $\mathbb{Z}_p[x]$ is irreducible of degree $[K:\mathbb{Z}_p]$ by Theorem 11.7. Theorem 11.23 shows that $[K:\mathbb{Z}_p] = n$. ■

Exercises

A. 1. If R is a ring with identity and $m, n \in \mathbb{Z}$, prove that $(m1_R)(n1_R) = (mn)1_R$. [The case of positive m, n was done in the proof of Lemma 11.19.]

2. What is the characteristic of

(a) $\mathbb{Q}$ (b) $\mathbb{Z}_2 \times \mathbb{Z}_6$ (c) $\mathbb{Z}_3[x]$

(d) $M(\mathbb{R})$ (e) $M(\mathbb{Z}_3)$

3. Let R be a ring with identity of characteristic $n \geq 0$. Prove that $na = 0_R$ for every $a \in R$.

4. If K is a field of prime characteristic p, prove that its prime subfield is the intersection of all the subfields of K.

5. Let F be a subfield of a finite field K. If F has order q, show that K has order q^n, where $n = [K:F]$.

6. Show that a field K of order p^n contains all kth roots of 1_K, where $k = p^n - 1$.

7. Let E be the set of roots of $x^{p^n} - x \in \mathbb{Z}_p[x]$ in some splitting field. If $a \in E$, prove that $-a \in E$.

B. 8. Let p be prime and let $\mathbb{Z}_p(x)$ be the field of quotients of the polynomial ring $\mathbb{Z}_p[x]$ (as in Example 1 of Section 10.4). Show that $\mathbb{Z}_p(x)$ is an infinite field of characteristic p.

9. Let R be a commutative ring with identity of prime characteristic p. If a, $b \in R$ and $n \geq 1$, prove that $(a - b)^{p^n} = a^{p^n} - b^{p^n}$.

10. Let K be a finite field of characteristic p. Prove that the map $f:K \to K$ given by $f(a) = a^p$ is an isomorphism. Conclude that every element of K has a pth root in K.

11. Show that the Freshman's Dream (Lemma 11.24) may be false if the characteristic p is not prime or if R is noncommutative. [*Hint:* Consider $\mathbb{Z}_6$ and $M(\mathbb{Z}_2)$.]

12. If c is a root of $f(x) \in \mathbb{Z}_p[x]$, prove that c^p is also a root.

13. Prove **Fermat's Little Theorem:** If p is a prime and $a \in \mathbb{Z}$, then $a^p \equiv a \pmod{p}$. If a is relatively prime to p, then $a^{p-1} \equiv 1 \pmod{p}$. [*Hint:* Translate congruence statements in $\mathbb{Z}$ into equality statements in $\mathbb{Z}_p$ and use Theorem 11.25.]

14. Let F be a field and $f(x)$ a monic polynomial in $F[x]$, whose roots are all distinct in any splitting field K. Let E be the set of roots of $f(x)$ in K. If the set E is actually a subfield of K, prove that F has characteristic p for some prime p and that $f(x) = x^{p^n} - x$ for some $n \geq 1$.

15. (a) Show that $x^3 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$ and construct a field of order 8.

(b) Show that $x^3 - x + 1$ is irreducible in $\mathbb{Z}_3[x]$ and construct a field of order 27.

(c) Show that $x^4 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$ and construct a field of order 16.

16. Let K be a finite field of characteristic p, F a subfield of K, and m a positive integer. If $L = \{a \in K \mid a^{p^m} \in F\}$, prove that

(a) L is a subfield of K that contains F.

(b) $L = F$. [*Hint:* Use Exercise 10 to show that the map $g:K \to K$ given by $g(a) = a^{p^m}$ is an isomorphism such that $g(F) = F$. What is $g^{-1}(F)$?]

17. If E and F are subfields of a finite field K and E is isomorphic to F, prove that $E = F$.

18. Let K be a field and k, n positive integers.

(a) Prove that $x^k - 1_K$ divides $x^n - 1_K$ in $K[x]$ if and only if $k \mid n$ in $\mathbb{Z}$. [*Hint:* $n = kq + r$ by the Division Algorithm; show that $x^n - 1_K = (x^k - 1_K)h(x) + (x^r - 1_K)$, where $h(x) = x^{n-k} + x^{n-2k} + \cdots + x^{n-qk}$.]

(b) If $p \geq 2$ is an integer, prove that $(p^k - 1) \mid (p^n - 1)$ if and only if $k \mid n$. [*Hint:* Copy the proof of part (a) with p in place of x.]

19. Let K be a finite field of order p^n.

(a) If F is a subfield of K, prove that F has order p^d for some d such that $d \mid n$. [*Hint:* Exercise 18 may be helpful.]

(b) If $d \mid n$, prove that K has a unique subfield of order p^d. [*Hint:* See Exercise 17 and Corollary 11.27 for the uniqueness part.]

20. Let p be prime and $f(x)$ an irreducible polynomial of degree 2 in $\mathbb{Z}_p[x]$. If K is an extension field of $\mathbb{Z}_p$ of order p^3, prove that $f(x)$ is irreducible in $K[x]$.

21. Prove that every element in a finite field can be written as the sum of two squares.

22. Use part (2) of Corollary 8.6 to prove that every nonzero element c of a finite field K of order p^n satisfies $c^{p^n - 1} = 1_K$. Conclude that c is a root of $x^{p^n} - x$ and use this fact to prove Theorem 11.25.

Application

BCH codes (Section 16.3) may be covered at this point if desired.

CHAPTER 12

Galois Theory

A major question in classical algebra was whether or not there were formulas for the solution of higher-degree polynomial equations (analogous to the quadratic formula for second-degree equations). Although formulas for third- and fourth-degree equations were found in the sixteenth century, no further progress was made for almost 300 years. Then Ruffini and Abel provided the surprising answer: There is no formula for the solution of *all* polynomial equations of degree n when $n \geq 5$. This result did not rule out the possibility that the solutions of special types of equations might be obtainable from a formula. Nor did it give any clue as to which equations might be solvable by formula.

It was the amazingly original work of Galois that provided the full explanation, including a criterion for determining which polynomial equations *can* be solved by a formula. Galois' ideas had a profound influence on the development of later mathematics, far beyond the scope of the original solvability problem.

The solutions of the equation $f(x) = 0$ lie in some extension of the coefficient field of $f(x)$. Galois' remarkable discovery was the close connection between such field extensions and groups (Section 12.1). A detailed description of the connection is given by the Fundamental Theorem of Galois Theory in Section 12.2. This theorem is the principal tool for proving Galois' Criterion for the solvability of equations by formula (Section 12.3).

12.1 The Galois Group

The key to studying field extensions is to associate with each extension a certain group, called its *Galois group*. The properties of the Galois group and theorems of group theory can then be used to establish important facts about the field extension. In this section we define the Galois group and develop its basic properties. *Throughout this section F is a field.*

Definition

Let K be an extension field of F. An **F-automorphism of K** is an isomorphism $\sigma:K \to K$ that fixes F elementwise (that is, $\sigma(c) = c$ for every $c \in F$). The set of all F-automorphisms of K is denoted $\mathrm{Gal}_F K$ and is called the **Galois group** of K over F.

The use of the word "group" in the definition is justified by:

Theorem 12.1

If K is an extension field of F, then $\mathrm{Gal}_F K$ is a group under the operation of composition of functions.

Proof ▹ $\mathrm{Gal}_F K$ is nonempty since the identity map $\iota:K \to K$ is an automorphism.* If $\sigma, \tau \in \mathrm{Gal}_F K$ then $\sigma \circ \tau$ is an isomorphism from K to K by Exercise 27 of Section 3.3. For each $c \in F$, $(\sigma \circ \tau)(c) = \sigma(\tau(c)) = \sigma(c) = c$. Hence, $\sigma \circ \tau \in \mathrm{Gal}_F K$, and $\mathrm{Gal}_F K$ is closed. Composition of functions is associative, and the identity map ι is the identity element of $\mathrm{Gal}_F K$. Every bijective function has an inverse function by Theorem B.1 in Appendix B. If $\sigma \in \mathrm{Gal}_F K$, then σ^{-1} is an isomorphism from K to K by Exercise 29 of Section 3.3. Verify that $\sigma^{-1}(c) = c$ for every $c \in F$ (Exercise 1). Therefore, $\sigma^{-1} \in \mathrm{Gal}_F K$, and $\mathrm{Gal}_F K$ is a group. ■

EXAMPLE 1.A†

The complex conjugation map $\sigma:\mathbb{C} \to \mathbb{C}$ given by $\sigma(a + bi) = a - bi$ is an automorphism of $\mathbb{C}$, as shown in Example 3 of Section 3.3. For every real number a,

$$\sigma(a) = \sigma(a + 0i) = a - 0i = a.$$

So σ is in $\mathrm{Gal}_{\mathbb{R}}\mathbb{C}$. Note that i and $-i$ are the roots of $x^2 + 1 \in \mathbb{R}$ and that σ maps these roots onto each other: $\sigma(i) = -i$ and $\sigma(-i) = i$. This is an example of the next Theorem.

Theorem 12.2

Let K be an extension field of F and $f(x) \in F[x]$. If $u \in K$ is a root of $f(x)$ and $\sigma \in \mathrm{Gal}_F K$, then $\sigma(u)$ is also a root of $f(x)$.

*Throughout this chapter, ι denotes the identity map on the field under discussion.

†Throughout this section and the next, three basic examples appear repeatedly. The first appearance of Example 1 is labeled 1.A, its second appearance 1.B, etc.; the first appearance of Example 2 is labeled 2.A, and so on.

Proof ▸ If $f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$, then

$$c_0 + c_1u + c_2u^2 + \cdots + c_nu^n = 0_F.$$

Since σ is a homomorphism and $\sigma(c_i) = c_i$ for each $c_i \in F$,

$$\begin{aligned} 0_F = \sigma(0_F) &= \sigma(c_0 + c_1u + c_2u^2 + \cdots + c_nu^n) \\ &= \sigma(c_0) + \sigma(c_1)\sigma(u) + \sigma(c_2)\sigma(u)^2 + \cdots + \sigma(c_n)\sigma(u)^n \\ &= c_0 + c_1\sigma(u) + c_2\sigma(u)^2 + \cdots + c_n\sigma(u)^n = f(\sigma(u)). \end{aligned}$$

Therefore, $\sigma(u)$ is a root of $f(x)$. ■

Let $u \in K$ be algebraic over F with minimal polynomial $p(x) \in F[x]$. Theorem 12.2 states that every image of u under an automorphism of the Galois group must also be a root of $p(x)$. Conversely, is *every* root of $p(x)$ in K the image of u under some automorphism of $\mathrm{Gal}_F K$? Here is one case where the answer is yes.

Theorem 12.3

Let K be the splitting field of some polynomial over F and let $u, v \in K$. Then there exists $\sigma \in \mathrm{Gal}_F K$ such that $\sigma(u) = v$ if and only if u and v have the same minimal polynomial in $F[x]$.

Proof ▸ If u and v have the same minimal polynomial, then by Corollary 11.8 there is an isomorphism $\sigma: F(u) \cong F(v)$ such that $\sigma(u) = v$, and σ fixes F elementwise. Since K is a splitting field of some polynomial over F, it is a splitting field of the same polynomial over both $F(u)$ and $F(v)$. Therefore, σ extends to an F-automorphism of K (also denoted σ) by Theorem 11.14. In other words, $\sigma \in \mathrm{Gal}_F K$ and $\sigma(u) = v$. The converse is an immediate consequence of Theorem 12.2. ■

EXAMPLE 1.B

Example 1.A shows that $\mathrm{Gal}_{\mathbb{R}}\mathbb{C}$ has at least two elements, the identity map ι and the complex conjugation map σ. We now prove that these are the only elements in $\mathrm{Gal}_{\mathbb{R}}\mathbb{C}$. Let τ be any automorphism in $\mathrm{Gal}_{\mathbb{R}}\mathbb{C}$. Since i is a root of $x^2 + 1$, $\tau(i) = \pm i$ by Theorem 12.2. If $\tau(i) = i$, then since τ fixes every element of $\mathbb{R}$,

$$\tau(a + bi) = \tau(a) + \tau(b)\tau(i) = a + bi,$$

and, hence, $\tau = \iota$. Similarly, if $\tau(i) = -i$, then

$$\tau(a + bi) = \tau(a) + \tau(b)\tau(i) = a + b(-i) = a - bi,$$

and, therefore, $\tau = \sigma$. Thus $\mathrm{Gal}_{\mathbb{R}}\mathbb{C} = \{\iota, \sigma\}$ is a group of order 2 and, hence, isomorphic to $\mathbb{Z}_2$ by Theorem 8.7.

The preceding example shows that an $\mathbb{R}$-automorphism of $\mathbb{C} = \mathbb{R}(i)$ is completely determined by its action on i. The same thing is true in the general case:

Theorem 12.4

Let $K = F(u_1, \ldots, u_n)$ be an algebraic extension field of F. If $\sigma, \tau \in \mathrm{Gal}_F K$ and $\sigma(u_i) = \tau(u_i)$ for each $i = 1, 2, \ldots, n$, then $\sigma = \tau$. In other words, an automorphism in $\mathrm{Gal}_F K$ is completely determined by its action on $u_1, \ldots, u_n$.

Proof ▸ Let $\beta = \tau^{-1} \circ \sigma \in \mathrm{Gal}_F K$. We shall show that β is the identity map ι. Since $\sigma(u_i) = \tau(u_i)$ for every i,

$$\beta(u_i) = (\tau^{-1} \circ \sigma)(u_i) = \tau^{-1}(\sigma(u_i)) = \tau^{-1}(\tau(u_i)) = (\tau^{-1} \circ \tau)(u_i) = \iota(u_i) = u_i.$$

Let $v \in F(u_1)$. By Theorem 11.7 there exist $c_i \in F$ such that $v = c_0 + c_1u_1 + c_2u_1^2 + \cdots + c_{m-1}u_1^{m-1}$, where m is the degree of the minimal polynomial of u_1. Since β is a homomorphism that fixes u_1 and every element of F,

$$\begin{aligned}\beta(v) &= \beta(c_0 + c_1u_1 + c_2u_1^2 + \cdots + c_{m-1}u_1^{m-1}) \\ &= \beta(c_0) + \beta(c_1)\beta(u_1) + \beta(c_2)\beta(u_1^2) + \cdots + \beta(c_{m-1})\beta(u_1^{m-1}) \\ &= c_0 + c_1u_1 + c_2u_1^2 + \cdots + c_{m-1}u_1^{m-1} = v\end{aligned}$$

Therefore, $\beta(v) = v$ for every $v \in F(u_1)$. Repeating this argument with $F(u_1)$ in place of F and u_2 in place of u_1 shows that $\beta(v) = v$ for every $v \in F(u_1)(u_2) = F(u_1, u_2)$. Another repetition, with $F(u_1, u_2)$ in place of F and u_3 in place of u_1, shows that $\beta(v) = v$ for every $v \in F(u_1, u_2, u_3)$. After a finite number of repetitions we have $\beta(v) = v$ for every $v \in F(u_1, u_2, \ldots, u_n) = K$, that is, $\iota = \beta = \tau^{-1} \circ \sigma$. Therefore,

$$\tau = \tau \circ \iota = \tau \circ (\tau^{-1} \circ \sigma) = (\tau \circ \tau^{-1}) \circ \sigma = \iota \circ \sigma = \sigma. \quad ■$$

EXAMPLE 2.A

By Theorem 12.2 any automorphism in the Galois group of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$ takes $\sqrt{3}$ to $\sqrt{3}$ or $-\sqrt{3}$, the roots of $x^2 - 3$. Similarly, it must take $\sqrt{5}$ to $\pm\sqrt{5}$, the roots of $x^2 - 5$. Since an automorphism is completely determined by its action on $\sqrt{3}$ and $\sqrt{5}$ by Theorem 12.4, there are *at most* four automorphisms in $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3}, \sqrt{5})$, corresponding to the four possible actions on $\sqrt{3}$ and $\sqrt{5}$:

$$\begin{array}{cccc} \sqrt{3} \xrightarrow{\iota} \sqrt{3} & \sqrt{3} \xrightarrow{\tau} -\sqrt{3} & \sqrt{3} \xrightarrow{\alpha} \sqrt{3} & \sqrt{3} \xrightarrow{\beta} -\sqrt{3} \\ \sqrt{5} \longrightarrow \sqrt{5} & \sqrt{5} \longrightarrow \sqrt{5} & \sqrt{5} \longrightarrow -\sqrt{5} & \sqrt{5} \longrightarrow -\sqrt{5}. \end{array}$$

We now show that $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3}, \sqrt{5})$ is a group of order 4 by constructing nonidentity automorphisms τ, α, β with these actions. To construct τ, note that $x^2 - 3$ is the minimal polynomial of both $\sqrt{3}$ and $-\sqrt{3}$ over $\mathbb{Q}$. By Corollary 11.8, there is an isomorphism $\sigma: \mathbb{Q}(\sqrt{3}) \cong \mathbb{Q}(-\sqrt{3})$ such that $\sigma(\sqrt{3}) = -\sqrt{3}$, and σ fixes $\mathbb{Q}$ elementwise. Example 6 of Section 11.3 shows that $x^2 - 5$ is the minimal polynomial of $\sqrt{5}$ over $\mathbb{Q}(\sqrt{3})$. By Corollary 11.8 again, σ extends to a $\mathbb{Q}$-automorphism τ of $\mathbb{Q}(\sqrt{3})(\sqrt{5}) = \mathbb{Q}(\sqrt{3},\sqrt{5})$ such that $\tau(\sqrt{5}) = \sqrt{5}$. Therefore, $\tau \in \mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3},\sqrt{5})$ and $\tau(\sqrt{3}) = \sigma(\sqrt{3}) = -\sqrt{3}$ and $\tau(\sqrt{5}) = \sqrt{5}$. A similar two-step argument produces automorphisms α and β with the actions listed above. Furthermore, each of τ, α, β has order 2 in $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3},\sqrt{5})$; for instance,

$$(\tau \circ \tau)(\sqrt{3}) = \tau(\tau(\sqrt{3})) = \tau(-\sqrt{3}) = -\tau(\sqrt{3}) = -(-\sqrt{3}) = \sqrt{3} = \iota(\sqrt{3})$$

and $(\tau \circ \tau)(\sqrt{5}) = \sqrt{5} = \iota(\sqrt{5})$. Therefore, $\tau \circ \tau = \iota$ by Theorem 12.4. Use Theorem 8.8 to conclude that $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3},\sqrt{5}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or compute the operation table directly (Exercise 4). For instance, you can readily verify that $(\tau \circ \alpha)(\sqrt{3}) = \beta(\sqrt{3})$ and $(\tau \circ \alpha)(\sqrt{5}) = \beta(\sqrt{5})$ and, hence, $\tau \circ \alpha = \beta$ by Theorem 12.4.

In the preceding example, $\mathbb{Q}(\sqrt{3},\sqrt{5})$ is the splitting field of $f(x) = (x^2 - 3)(x^2 - 5)$, and every automorphism in the Galois group permutes the four roots $\sqrt{3}, -\sqrt{3}, \sqrt{5}, -\sqrt{5}$ of $f(x)$. This is an illustration of

Corollary 12.5

If K is the splitting field of a separable polynomial $f(x)$ of degree n in $F[x]$, then $\mathrm{Gal}_F K$ is isomorphic to a subgroup of S_n.

Proof ▶ By separability $f(x)$ has n distinct roots in K, say $u_1, \ldots, u_n$. Consider S_n to be the group of permutations of the set $R = \{u_1, \ldots, u_n\}$. If $\sigma \in \mathrm{Gal}_F K$, then $\sigma(u_1), \sigma(u_2), \ldots, \sigma(u_n)$ are roots of $f(x)$ by Theorem 12.2. Furthermore, since σ is injective, they are all distinct and, hence, must be $u_1, u_2, \ldots, u_n$ in some order. In other words, the restriction of σ to the set R (denoted $\sigma \mid R$) is a permutation of R. Define a map $\theta: \mathrm{Gal}_F K \to S_n$ by $\theta(\sigma) = \sigma \mid R$. Since the operation in both groups is composition of functions, it is easy to verify that θ is a homomorphism of groups. $K = F(u_1, \ldots, u_n)$ by the definition of splitting field. If $\sigma \mid R = \tau \mid R$, then $\sigma(u_i) = \tau(u_i)$ for every i, and, hence, $\sigma = \tau$ by Theorem 12.4. Therefore, θ is an injective homomorphism, and thus $\mathrm{Gal}_F k$ is isomorphic to $\mathrm{Im}\,\theta$, a subgroup of S_n, by Theorem 7.20. ■

If K is the splitting field of $f(x)$, we shall usually

identify $\mathrm{Gal}_F K$ with its isomorphic subgroup in S_n

by identifying each automorphism with the permutation it induces on the roots of $f(x)$.

EXAMPLE 3.A

Let K be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Verify that the roots of $x^3 - 2$ are $\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2$, where $w = (-1 + \sqrt{3}i)/2$ is a complex cube root of 1. Then $\mathrm{Gal}_{\mathbb{Q}} K$ is a subgroup of S_3. By Theorem 12.3, there is at least one automorphism

σ that maps the first root $\sqrt[3]{2}$ to the second $\sqrt[3]{2}\omega$; it must take the third root $\sqrt[3]{2}\omega^2$ to itself *or* to the first root $\sqrt[3]{2}$ by Theorem 12.2. So σ is either the permutation (12) or (123) in S_3.

CAUTION: When K is the splitting field of a polynomial $f(x) \in F[x]$, then by Corollary 12.5 every element of $\text{Gal}_F K$ produces a permutation of the roots of $f(x)$, but not vice versa: A permutation of the roots need not come from an F-automorphism of K. For example, $\mathbb{Q}(\sqrt{3},\sqrt{5})$ is a splitting field of $f(x) = (x^2-3)(x^2-5)$, but by Example 2.A there is no $\mathbb{Q}$-automorphism of $\mathbb{Q}(\sqrt{3},\sqrt{5})$ that gives this permutation of the roots

$$\begin{array}{cccc} \sqrt{3} & -\sqrt{3} & \sqrt{5} & -\sqrt{5} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \sqrt{5} & -\sqrt{5} & \sqrt{3} & -\sqrt{3} \end{array}$$

Let K be an extension field of F. A field E such that $F \subseteq E \subseteq K$ is called an **intermediate field** of the extension. In this case, we can consider K as an extension of E. The Galois group $\text{Gal}_E K$ consists of all automorphisms of K that fix E elementwise. Every such automorphism automatically fixes each element of F since $F \subseteq E$. Hence, every automorphism in $\text{Gal}_E K$ is in $\text{Gal}_F K$, that is,

if E is an intermediate field, $\text{Gal}_E K$ is a subgroup of $\text{Gal}_F K$.

EXAMPLE 2.B

$\mathbb{Q}(\sqrt{3})$ is an intermediate field of the extension $\mathbb{Q}(\sqrt{3},\sqrt{5})$ of $\mathbb{Q}$. Example 2.A shows that $\text{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3},\sqrt{5}) = \{\iota, \tau, \alpha, \beta\}$. The automorphisms that fix every element of $\mathbb{Q}(\sqrt{3})$ are exactly the ones that map $\sqrt{3}$ to itself by Theorem 12.4. Therefore,

$$\text{Gal}_{\mathbb{Q}(\sqrt{3})}\mathbb{Q}(\sqrt{3},\sqrt{5})$$

is the subgroup $\{\iota, \alpha\}$ of $\{\iota, \tau, \alpha, \beta\}$.

We now have a natural way of associating a subgroup of the Galois group with each intermediate field of the extension. Conversely, if H is a subgroup of the Galois group, we can associate an intermediate field with H by using

Theorem 12.6

Let K be an extension field of F. If H is a subgroup of $\text{Gal}_F K$, let

$$E_H = \{k \in K \mid \sigma(k) = k \text{ for every } \sigma \in H\}.$$

Then E_H is an intermediate field of the extension.

The field E_H is called the **fixed field** of the subgroup H.

Proof of Theorem 12.6 ► If $c, d \in E_H$ and $\sigma \in H$, then

$$\sigma(c + d) = \sigma(c) + \sigma(d) = c + d \quad \text{and} \quad \sigma(cd) = \sigma(c)\sigma(d) = cd.$$

Therefore, E_H is closed under addition and multiplication. Since $\sigma(0_F) = 0_F$ and $\sigma(1_F) = 1_F$ for every automorphism, 0_F and 1_F are in E_H. Theorem 3.10 shows that for any nonzero c in E_H and any σ in H,

$$\sigma(-c) = -\sigma(c) = -c \quad \text{and} \quad \sigma(c^{-1}) = \sigma(c)^{-1} = c^{-1}.$$

Therefore, $-c \in E_H$ and $c^{-1} \in E_H$. Hence, E_H is a subfield of K. Since H is a subgroup of $\mathrm{Gal}_F K$, $\sigma(c) = c$ for every $c \in F$ and every $\sigma \in H$. Therefore, $F \subseteq E_H$. ■

EXAMPLE 2.C

Consider the subgroup $H = \{\iota, \alpha\}$ of the Galois group $\{\iota, \tau, \alpha, \beta\}$ of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$. Since $\alpha(\sqrt{3}) = \sqrt{3}$, the subfield $\mathbb{Q}(\sqrt{3})$ is contained in the fixed field E_H of H. To prove that $E_H = \mathbb{Q}(\sqrt{3})$, you must show that the elements of $\mathbb{Q}(\sqrt{3})$ are the *only* ones that are fixed by ι and α; see Exercise 14.

EXAMPLE 1.C

As we saw in Example 1.B, $\mathrm{Gal}_{\mathbb{R}}\mathbb{C} = \{\iota, \sigma\}$, where σ is the complex conjugation map. Obviously, the fixed field of the identity subgroup is the entire field $\mathbb{C}$. Since σ fixes every real number and moves every nonreal one, the fixed field of $\mathrm{Gal}_{\mathbb{R}}\mathbb{C}$ is the field $\mathbb{R}$.

Unlike the situation in the preceding example, the ground field F need not always be the fixed field of the group $\mathrm{Gal}_F K$.

EXAMPLE 3.B

Every automorphism in the Galois group of $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$ must map $\sqrt[3]{2}$ to a root of $x^3 - 2$ by Theorem 12.2. Example 3.A shows that $\sqrt[3]{2}$ is the only real root of this polynomial. Since $\mathbb{Q}(\sqrt[3]{2})$ consists entirely of real numbers by Theorem 11.7, every automorphism in $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt[3]{2})$ must map $\sqrt[3]{2}$ to itself. Therefore, $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt[3]{2})$ consists of the identity automorphism alone by Theorem 12.4. So the fixed field of $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt[3]{2})$ is the entire field $\mathbb{Q}(\sqrt[3]{2})$.

Exercises

NOTE: *Unless stated otherwise, K is an extension field of the field F.*

A. 1. If σ is an F-automorphism of K, show that σ^{-1} is also an F-automorphism of K.

2. Assume $[K:F]$ is finite. Is it true that every F-automorphism of K is completely determined by its action on a basis of K over F?

3. If $[K:F]$ is finite, $\sigma \in \mathrm{Gal}_F K$, and $u \in K$ is such that $\sigma(u) = u$, show that $\sigma \in \mathrm{Gal}_{F(u)} K$.

4. Write out the operation table for the group

$$\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3}, \sqrt{5}) = \{\iota, \tau, \alpha, \beta\}.$$

[See Example 2.A.]

5. Let $f(x) \in F[x]$ be separable of degree n and K a splitting field of $f(x)$. Show that the order of $\mathrm{Gal}_F K$ divides $n!$.

6. If K is an extension field of $\mathbb{Q}$ and σ is an automorphism of K, prove that σ is a $\mathbb{Q}$-automorphism. [*Hint:* $\sigma(1) = 1$ implies that $\sigma(n) = n$ for all $n \in \mathbb{Z}$.]

B. 7. **(a)** Show that $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{2})$ has order 2 and, hence, is isomorphic to $\mathbb{Z}_2$. [*Hint:* The minimal polynomial is $x^2 - 2$; see Theorem 11.7.]

(b) If $d \in \mathbb{Q}$ and $\sqrt{d} \notin \mathbb{Q}$, show that $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{d})$ is isomorphic to $\mathbb{Z}_2$.

8. Show that $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}\,|\,(\sqrt[4]{2}) \neq \langle\iota\rangle$.

9. **(a)** Let $\omega = (-1 + \sqrt{3}i)/2$ be a complex cube root of 1. Find the minimal polynomial $p(x)$ of ω over $\mathbb{Q}$ and show that ω^2 is also a root of $p(x)$. [*Hint:* w is a root of $x^3 - 1$.]

(b) What is $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\omega)$?

10. **(a)** Find $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{2}, \sqrt{3})$. [*Hint:* See Example 2.A.]

(b) If p, q are distinct positive primes, find $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{p}, \sqrt{q})$.

11. Find $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{2}, i)$. [*Hint:* Consider $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, i)$ and proceed as in Example 2.A.]

12. Show that $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

13. If F has characteristic 0 and K is the splitting field of $f(x) \in F[x]$, prove that the order of $\mathrm{Gal}_F K$ is $[K:F]$. [*Hint:* $K = F(u)$ by Theorems 11.17 and 11.18.]

14. Let H be the subgroup $\{\iota, \alpha\}$ of $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3}, \sqrt{5}) = \{\iota, \tau, \alpha, \beta\}$. Show that the fixed field of H is $\mathbb{Q}(\sqrt{3})$. [*Hint:* Verify that $\mathbb{Q}(\sqrt{3}) \subseteq E_H \subseteq \mathbb{Q}(\sqrt{3}, \sqrt{5})$; what is $[\mathbb{Q}(\sqrt{3}, \sqrt{5}):\mathbb{Q}(\sqrt{3})]$?]

15. **(a)** Show that every automorphism of $\mathbb{R}$ maps positive elements to positive elements. [*Hint:* Every positive element of $\mathbb{R}$ is a square.]

(b) If $a, b \in \mathbb{R}$, $a < b$, and $\sigma \in \mathrm{Gal}_{\mathbb{Q}}\mathbb{R}$, prove that $\sigma(a) < \sigma(b)$. [*Hint:* $a < b$ if and only if $b - a > 0$.]

(c) Prove that $\mathrm{Gal}_{\mathbb{Q}}\mathbb{R} = \langle\iota\rangle$. [*Hint:* If $c < r < d$, with $c, d \in \mathbb{Q}$, then $c < \sigma(r) < d$; show that this implies $\sigma(r) = r$.]

C. 16. Suppose $\zeta, \zeta^2, \ldots, \zeta^n = 1$ are n distinct roots of $x^n - 1$ in some extension field of $\mathbb{Q}$. Prove that $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\zeta)$ is abelian.

17. Let E be an intermediate field that is normal over F and $\sigma \in \mathrm{Gal}_F K$. Prove that $\sigma(E) = E$.

12.2 The Fundamental Theorem of Galois Theory

The essential idea of Galois theory is to relate properties of an extension field with properties of its Galois group. The key to doing this is the Fundamental Theorem of Galois Theory, which will be proved in this section.

Throughout this section, *K is a finite-dimensional extension field of F.* Let S be the set of all intermediate fields and T the set of all subgroups of the Galois group $\mathrm{Gal}_F K$. Define a function $\varphi: S \to T$ by this rule:

$$\text{For each intermediate field } E, \qquad \varphi(E) = \mathrm{Gal}_E K.$$

The function φ is called the **Galois correspondence.** Note that K (considered as a subfield of itself) corresponds to the identity subgroup of $\mathrm{Gal}_F K$, and the subfield F corresponds to the entire group $\mathrm{Gal}_F K$ (considered as a subgroup of itself).

EXAMPLE 2.D*

Consider the Galois correspondence for the extension $\mathbb{Q}(\sqrt{3},\sqrt{5})$ of $\mathbb{Q}$ and the intermediate field $\mathbb{Q}(\sqrt{3})$. By the preceding remarks and Example 2.B on page 412, we have

$$\begin{aligned}
\mathbb{Q}(\sqrt{3},\sqrt{5}) &\longrightarrow \mathrm{Gal}_{\mathbb{Q}(\sqrt{3},\sqrt{5})}\mathbb{Q}(\sqrt{3},\sqrt{5}) = \{\iota\}.\\
\mathbb{Q}(\sqrt{3}) &\longrightarrow \mathrm{Gal}_{\mathbb{Q}(\sqrt{3})}\mathbb{Q}(\sqrt{3},\sqrt{5}) = \{\iota, \alpha\}.\\
\mathbb{Q} &\longrightarrow \mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3},\sqrt{5}) = \{\iota, \tau, \alpha, \beta\}.
\end{aligned}$$

Example 2.C shows that $E = \mathbb{Q}(\sqrt{3})$ is the fixed field of the subgroup $H = (\iota, \alpha\} = \mathrm{Gal}_{\mathbb{Q}(\sqrt{3})}\mathbb{Q}(\sqrt{3},\sqrt{5})$. Furthermore, $K = \mathbb{Q}(\sqrt{3},\sqrt{5}) = \mathbb{Q}(\sqrt{3})(\sqrt{5})$ is a normal, separable extension of the fixed field $E = \mathbb{Q}(\sqrt{3})$ because it's the splitting field of $x^2 - 5$ (Theorem 11.15) and has characteristic 0 (Theorem 11.17).

We now construct the tools necessary to show that, under appropriate assumptions, the Galois correspondence is a bijective map from the set of intermediate fields to the set of subgroups of $\mathrm{Gal}_F K$.

Lemma 12.7

Let K be a finite-dimensional extension field of F. If H is a subgroup of the Galois group $\mathrm{Gal}_F K$ and E is the fixed field of H, then K is a simple, normal, separable extension of E.

Example 2.D above (with $K = \mathbb{Q}(\sqrt{3},\sqrt{5})$, $E = \mathbb{Q}(\sqrt{3})$, and $H = \{\iota, \alpha\}$) is an illustration of Lemma 12.7.

*The numbering scheme for examples in Sections 12.1 and 12.2 is explained on page 408.

Proof of Lemma 12.7 ▸ Each $u \in K$ is algebraic over F by Theorem 11.9 and, hence, algebraic over E by Exercise 7 in Section 11.2. Every automorphism in H must map u to some root of its minimal polynomial $p(x) \in E[x]$ by Theorem 12.2. Therefore, *u has a finite number of distinct images under automorphisms in H*, say $u = u_1, u_2, \ldots, u_t \in K$.

If $\sigma \in H$ and $u_i = \tau(u)$ (with $\tau \in H$), then $\sigma(u_i) = \sigma(\tau(u))$. Since $\sigma \circ \tau \in H$, we see that $\sigma(u_i)$ is also an image of u and, hence, must be in the set $\{u_1, u_2, \ldots, u_t\}$. Since σ is injective, the elements $\sigma(u_1), \ldots, \sigma(u_t)$ are t distinct images of u and, hence, must be the elements $u_1, u_2, \ldots, u_t$ in some order. In other words, *every automorphism in H permutes* $u_1, u_2, \ldots, u_t$. Let

$$f(x) = (x - u_1)(x - u_2) \cdots (x - u_t).$$

Since the u_i are distinct, $f(x)$ is separable. We claim that $f(x)$ is actually in $E[x]$. To prove this, let $\sigma \in H$ and recall that σ induces an isomorphism $K[x] \cong K[x]$ (also denoted σ), as described on page 380. Then

$$\sigma f(x) = (x - \sigma(u_1))(x - \sigma(u_2)) \cdots (x - \sigma(u_t)).$$

Since σ permutes the u_i, it simply rearranges the factors of $f(x)$, and, hence, $\sigma f(x) = f(x)$. Therefore, every automorphism of H maps the coefficients of the separable polynomial $f(x)$ to themselves, and, hence, these coefficients are in E, the fixed field of H. Since $u = u_1$ is a root of $f(x) \in E[x]$, u is separable over E. Hence, K is a separable extension of E.

The field K is finitely generated over F (since $[K:F]$ is finite; see Example 4 in Section 11.3). Consequently, K is finitely generated over E, and, hence, $K = E(u)$ for some $u \in K$ by Theorem 11.18. Let $f(x)$ be as in the preceding paragraph. Then $f(x)$ splits in $K[x]$, and, hence, $K = E(u)$ is the splitting field of $f(x)$ over E. Therefore, K is normal over E by Theorem 11.15. ■

Theorem 12.8

Let K be a finite-dimensional extension field of F. If H is a subgroup of the Galois group $Gal_F K$ and E is the fixed field of H, then $H = Gal_E K$ and $|H| = [K:E]$. Therefore, the Galois correspondence is surjective.

Proof ▸ Lemma 12.7 shows that $K = E(u)$ for some $u \in K$. If $p(x)$, the minimal polynomial of u over E, has degree n, then $[K:E] = n$ by Theorem 11.7. Distinct automorphisms of $\mathrm{Gal}_E K$ map u onto distinct roots of $p(x)$ by Theorems 12.2 and 12.4. So the number of distinct automorphisms in $\mathrm{Gal}_E K$ is at most n, the number of roots of $p(x)$. Now $H \subseteq \mathrm{Gal}_E K$ by the definition of the fixed field E. Consequently,

$$|H| \leq |\mathrm{Gal}_E K| \leq n = [K:E].$$

Let $f(x)$ be as in the proof of Lemma 12.7. Then H contains at least t automorphisms (the number of distinct images of u under H). Since $u = u_1$ is a root of $f(x)$, $p(x)$ divides $f(x)$. Hence,

$$|H| \geq t = \deg f(x) \geq \deg p(x) = n = [K:E].$$

Combining these inequalities, we have

$$|H| \leq |\mathrm{Gal}_E K| \leq [K:E] \leq |H|.$$

Therefore, $|H| = |\mathrm{Gal}_E K| = [K:E]$, and, hence, $H = \mathrm{Gal}_E K$. ■

EXAMPLE 3.C

The Galois group $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt[3]{2}) = \langle\iota\rangle$ by Example 3.B, so both of the intermediate fields $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}$ are associated with $\langle\iota\rangle$ under the Galois correspondence. Note that $\mathbb{Q}(\sqrt[3]{2})$ is *not* a normal extension of $\mathbb{Q}$ [it doesn't contain the complex roots of $x^3 - 2$, so this polynomial has a root but doesn't split in $\mathbb{Q}(\sqrt[3]{2})$].

Galois Extensions

Although the Galois correspondence is surjective by Theorem 12.8, the preceding example shows that it may not be injective. In order to guarantee injectivity, additional hypotheses on the extension are necessary. The preceding proofs and example suggest that normality and separability are likely candidates.

Definition

If K is a finite-dimensional, normal, separable extension field of the field F, we say that K is a **Galois extension** of F or that K is **Galois over** F.

A Galois extension of characteristic 0 *is simply a splitting field* by Theorems 11.15 and 11.17.

Theorem 12.9

Let K be a Galois extension of F and E an intermediate field. Then E is the fixed field of the subgroup $\mathrm{Gal}_E K$.

If E and L are intermediate fields with $\mathrm{Gal}_E K = \mathrm{Gal}_L K$, then Theorem 12.9 shows that both E and L are the fixed field of the *same* group, and, hence, $E = L$. Therefore, *the Galois correspondence is injective for Galois extensions.*

Proof of Theorem 12.9 ▸ The fixed field E_0 of $\mathrm{Gal}_E K$ contains E by definition. To show that $E_0 \subseteq E$, we prove the contrapositive: If $u \notin E$, then u is moved by some automorphism in $\mathrm{Gal}_E K$, and, hence, $u \notin E_0$. Since K is a Galois extension of the intermediate field E (normal by Theorem 11.15 and Exercise 5 of Section 11.4; separable by Exercise 1 of Section 11.5), it is an algebraic extension of E. Consequently, u is algebraic over E with minimal polynomial $p(x) \in E[x]$ of degree ≥ 2 (if $\deg p(x) = 1$, then u would be in E). The roots of $p(x)$ are distinct by separability, and all of them are in K by normality. Let v be a root of $p(x)$ other than u. Then there exists $\sigma \in \mathrm{Gal}_E K$ such that $\sigma(u) = v$ by Theorem 12.3. Therefore, $u \notin E_0$, and, hence, $E_0 = E$. ■

Corollary 12.10

Let K be a finite-dimensional extension field of F. Then K is Galois over F if and only if F is the fixed field of the Galois group $\mathrm{Gal}_F K$.

Proof ▸ If K is Galois over F, then Theorem 12.9 (with $E = F$) shows that F is the fixed field of $\mathrm{Gal}_F K$. Conversely, if F is the fixed field of $\mathrm{Gal}_F K$, then Lemma 12.7 (with $E = F$) shows that K is Galois over F. ∎

In view of Corollary 12.10, a Galois extension is often *defined* to be a finite-dimensional one in which F is the fixed field of $\mathrm{Gal}_F K$. When reading other books on Galois theory, it's a good idea to check which definition is being used so that you don't make unwarranted assumptions.

EXAMPLE 2.E

The field $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ is a Galois extension of $\mathbb{Q}$ because it is the splitting field of $f(x) = (x^2 - 3)(x^2 - 5)$. So the Galois correspondence is bijective by Theorem 12.8 and the remarks after Theorem 12.9. The Galois group $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{3}, \sqrt{5}) = \{\iota, \tau, \alpha, \beta\}$ by Example 2.A. Verify the accuracy of the chart below, in which subfields and subgroups in the same relative position correspond to each other under the Galois correspondence. For instance, $\mathbb{Q}(\sqrt{3})$ corresponds to $\{\iota, \alpha\}$ by Example 2.B.

Intermediate Fields

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{3}, \sqrt{5}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{3}) \arrow[dr, dash] & \mathbb{Q}(\sqrt{5}) \arrow[d, dash] & \mathbb{Q}(\sqrt{15}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Subgroups

$$\begin{tikzcd}
 & (\iota) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
(\iota, \alpha) \arrow[dr, dash] & (\iota, \tau) \arrow[d, dash] & (\iota, \beta) \arrow[dl, dash] \\
 & (\iota, \tau, \alpha, \beta) &
\end{tikzcd}$$

Note that *all* the intermediate fields are themselves Galois extensions of $\mathbb{Q}$ (for instance, $\mathbb{Q}(\sqrt{5})$ is the splitting field of $x^2 - 5$). Furthermore, the corresponding subgroups of the Galois group are normal. A similar situation holds in the general case, as we now see.

Theorem 12.11 The Fundamental Theorem of Galois Theory

If K is a Galois extension field of F, then

(1) There is bijection between the set S of all intermediate fields of the extension and the set T of all subgroups of the Galois group $\mathrm{Gal}_F K$, given by assigning each intermediate field E to the subgroup $\mathrm{Gal}_E K$. Furthermore,

$$[K:E] = |\mathrm{Gal}_E K| \quad \text{and} \quad [E:F] = [\mathrm{Gal}_F K : \mathrm{Gal}_E K].$$

(2) An intermediate field E is a normal extension of F if and only if the corresponding group $\mathrm{Gal}_E K$ is a normal subgroup of $\mathrm{Gal}_F K$, and in this case $\mathrm{Gal}_F E \cong \mathrm{Gal}_F K/\mathrm{Gal}_E K$.

Proof ▹ Theorem 12.8 and the remarks after Theorem 12.9 prove the first statement in part (1). Each intermediate field E is the fixed field of $\mathrm{Gal}_E K$ by Theorem 12.9. Consequently, $[K:E] = |\mathrm{Gal}_E K|$ by Theorem 12.8. In particular, if $F = E$, then $[K:F] = |\mathrm{Gal}_F K|$. Therefore, by Lagrange's Theorem 8.5 and Theorem 11.4,

$$[K:E][E:F] = [K:F] = |\mathrm{Gal}_F K| = |\mathrm{Gal}_E K|\,[\mathrm{Gal}_F K:\mathrm{Gal}_E K].$$

Dividing the first and last terms of this equation by $[K:E] = |\mathrm{Gal}_E K|$ shows that

$$[E:F] = [\mathrm{Gal}_F K:\mathrm{Gal}_E K].$$

To prove part (2), assume first that $\mathrm{Gal}_E K$ is a normal subgroup of $\mathrm{Gal}_F K$. If $p(x)$ is an irreducible polynomial in $F[x]$ with a root u in E, we must show that $p(x)$ splits in $E[x]$. Since K is normal over F, we know that $p(x)$ splits in $K[x]$. So we need to show only that each root v of $p(x)$ in K is actually in E. There is an automorphism σ in $\mathrm{Gal}_F K$ such that $\sigma(u) = v$ by Theorem 12.3. If τ is any element of $\mathrm{Gal}_E K$, then normality implies $\tau \circ \sigma = \sigma \circ \tau_1$ for some $\tau_1 \in \mathrm{Gal}_E K$. Since $u \in E$, we have $\tau(v) = \tau(\sigma(u)) = \sigma(\tau_1(u)) = \sigma(u) = v$. Hence, v is fixed by every element τ in $\mathrm{Gal}_E K$ and, therefore, must be in the fixed field of $\mathrm{Gal}_E K$, namely E (see Theorem 12.9).

Conversely, assume that E is a normal extension of F. Then E is finite dimensional over F by part (1). By Lemma 12.12, which is proved below, there is a surjective homomorphism of groups $\theta:\mathrm{Gal}_F K \to \mathrm{Gal}_F E$ whose kernel is $\mathrm{Gal}_E K$. Then $\mathrm{Gal}_E K$ is a normal subgroup of $\mathrm{Gal}_F K$ by Theorem 8.16, and $\mathrm{Gal}_F K/\mathrm{Gal}_E K \cong \mathrm{Gal}_F E$ by the First Isomorphism Theorem 8.20. ■

EXAMPLE 3.D

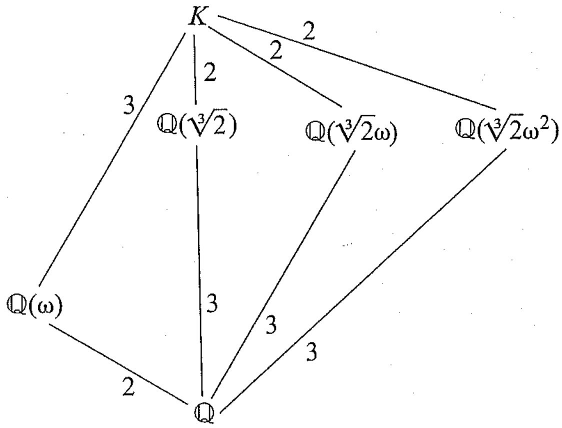
The splitting field K of $x^3 - 2$ is a Galois extension of $\mathbb{Q}$ whose Galois group is a subgroup of S_3 by Example 3.A.* Note that $\mathbb{Q} \subseteq \mathbb{Q}\sqrt[3]{2}) \subseteq K$. Since $x^3 - 2$ is the minimal polynomial of $\sqrt[3]{2}$, $[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}] = 3$ by Theorem 11.7. Neither of the other roots $(\sqrt[3]{2}\omega$ and $\sqrt[3]{2}\omega^2)$ is a real number, and, hence, neither is in $\mathbb{Q}(\sqrt[3]{2})$. So $[K:\mathbb{Q}] > 3$. Since $[K:\mathbb{Q}] \le 6$ (Theorems 11.13, 11.14) and $[K:\mathbb{Q}]$ is divisible by 3 (Theorem 11.4), we must have $[K:\mathbb{Q}] = 6$. Thus $\mathrm{Gal}_{\mathbb{Q}} K$ has order 6 by Theorem 12.11 and is S_3.

The only proper subgroups of S_3 are the cyclic group $\langle(123)\rangle$ of order 3 and three cyclic groups of order 2: $\langle(12)\rangle$, $\langle(13)\rangle$, $\langle(23)\rangle$. Verify that the Galois correspondence is as follows, where subgroups and subfields in the same relative position correspond to each other. The integer by the line connecting two

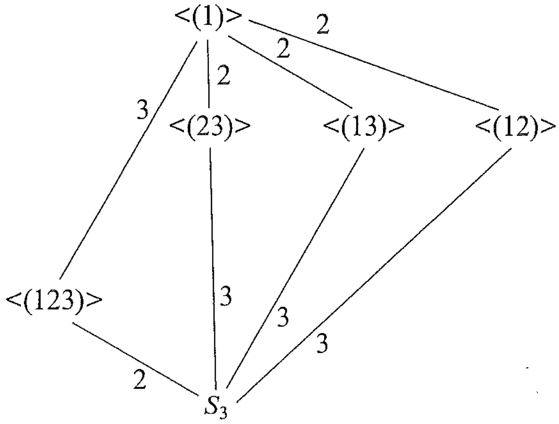
*We consider S_3 as the group of permutations of the roots $\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2$ in this order. For instance, (12) interchanges $\sqrt[3]{2}$ and $\sqrt[3]{2}\omega$ and fixes $\sqrt[3]{2}\omega^2$.

subfields is the dimension of the larger over the smaller. The integer by the line connecting two subgroups is the index of the smaller in the larger.

Intermediate Fields



Subgroups



The field $\mathbb{Q}(\omega)$ is an intermediate field because $\omega = \left(\frac{1}{2}\right)(\sqrt[3]{2})^2(\sqrt[3]{2}\omega) \in K$. $\mathbb{Q}(\omega)$ is the splitting field of $x^2 + x + 1$ (Exercise 3) and, hence, Galois over $\mathbb{Q}$. The corresponding subgroup is the normal subgroup $\langle(123)\rangle$. On the other hand, Example 3.C shows that $\mathbb{Q}(\sqrt[3]{2})$ is *not* Galois over $\mathbb{Q}$; the corresponding subgroup $\langle(23)\rangle$ is not normal in S_3.

The preceding example illustrates an important fact:

The Galois correspondence is inclusion-reversing.

For instance, $\mathbb{Q} \subseteq \mathbb{Q}(\omega)$, but the corresponding subgroups satisfy the reverse inclusion: $S_3 \supseteq \langle(123)\rangle$.

Finally, we complete the proof of the Fundamental Theorem by proving

Lemma 12.12

Let K be a finite-dimensional normal extension field of F and E an intermediate field, which is normal over F. Then there is a surjective homomorphism of groups $\theta\colon \mathrm{Gal}_F K \to \mathrm{Gal}_F E$ whose kernel is $\mathrm{Gal}_E K$.

Proof▸ Let $\sigma \in \mathrm{Gal}_F K$ and $u \in E$. Then u is algebraic over F with minimal polynomial $p(x)$. Since E is a normal extension of F, $p(x)$ splits in $E[x]$, that is, all the roots of $p(x)$ are in E. Since $\sigma(u)$ must be some root of $p(x)$ by Theorem 12.2, we see that $\sigma(u) \in E$. Therefore, $\sigma(E) \subseteq E$ for every $\sigma \in \mathrm{Gal}_F K$. Thus the restriction of σ to E (denoted $\sigma \mid E$) is an F-isomorphism $E \cong \sigma(E)$. Hence, $[E:F] = [\sigma(E):F]$ by Theorem 11.5. Since $F \subseteq \sigma(E) \subseteq E$, we have $[E:F] = [E:\sigma(E)][\sigma(E):F]$ by Theorem 11.4, which forces $[E:\sigma(E)] = 1$. Therefore, $E = \sigma(E)$, and $\sigma \mid E$ is actually an automorphism in $\mathrm{Gal}_F E$.

Define a function $\theta\colon \mathrm{Gal}_F K \to \mathrm{Gal}_F E$ by $\theta(\sigma) = \sigma \mid E$. It is easy to verify that θ is a homomorphism of groups. Its kernel consists of the automorphisms of K whose restriction to E is the identity map, that is, the subgroup $\mathrm{Gal}_E K$.

To show that θ is surjective, note that K is a splitting field over F by Theorem 11.15, and, hence, K is a splitting field of the same polynomial over E. Consequently, every $\tau \in \mathrm{Gal}_F E$ can be extended to an F-automorphism σ in $\mathrm{Gal}_F K$ by Theorem 11.14. This means that $\sigma \mid E = \tau$, that is, $\theta(\sigma) = \tau$. Therefore, θ is surjective. ■

In the preceding proof, the normality of K was not used until the last paragraph. So the first paragraph proves this useful fact:

Corollary 12.13

Let K be an extension field of F and E an intermediate field that is normal over F. If $\sigma \in \mathrm{Gal}_F K$, then $\sigma \mid E \in \mathrm{Gal}_F E$.

Exercises

NOTE: *K is an extension field of the field F.*

A. 1. If K is Galois over F, show that there are only finitely many intermediate fields.

2. If K is a normal extension of $\mathbb{Q}$ and $[K:\mathbb{Q}] = p$, with p prime, show that $\mathrm{Gal}_{\mathbb{Q}} K \cong \mathbb{Z}_p$.

3. **(a)** Show that $\omega = \left(-1 + \sqrt{3}i\right)/2$ is a root of $x^3 - 1$.

(b) Show that ω and ω^2 are roots of $x^2 + x + 1$. Hence, $\mathbb{Q}(\omega)$ is the splitting field of $x^2 + x + 1$.

4. Exhibit the Galois correspondence of intermediate fields and subgroups for the given extension of $\mathbb{Q}$:

   (a) $\mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Q}$, but $\sqrt{d} \notin \mathbb{Q}$.

   (b) $\mathbb{Q}(\omega)$, where ω is as in Exercise 3.

5. If K is Galois over F and $\mathrm{Gal}_F K$ is an abelian group of order 10, how many intermediate fields does the extension have and what are their dimensions over F?

6. Give an example of extension fields K and L of F such that both K and L are Galois over F, $K \neq L$, and $\mathrm{Gal}_F K \cong \mathrm{Gal}_F L$.

B. 7. Exhibit the Galois correspondence for the given extension of $\mathbb{Q}$:

   (a) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ (b) $\mathbb{Q}(i, \sqrt{2})$

8. If K is Galois over F, $\mathrm{Gal}_F K$ is abelian, and E is an intermediate field that is normal over F, prove that $\mathrm{Gal}_E K$ and $\mathrm{Gal}_F E$ are abelian.

9. Let K be Galois over F and assume $\mathrm{Gal}_F K \cong \mathbb{Z}_n$.

   (a) If E is an intermediate field that is normal over F, prove that $\mathrm{Gal}_E K$ and $\mathrm{Gal}_F E$ are cyclic.

   (b) Show that there is exactly one intermediate field for each positive divisor of n and that these are the only intermediate fields.

10. Two intermediate fields E and L are said to be **conjugate** if there exists $\sigma \in \mathrm{Gal}_F K$ such that $\sigma(E) = L$. Prove that E and L are conjugate if and only if $\mathrm{Gal}_E K$ and $\mathrm{Gal}_L K$ are conjugate subgroups of $\mathrm{Gal}_F K$ (as defined on page 308).

11. (a) Show that $K = \mathbb{Q}(\sqrt[4]{2}, i)$ is a splitting field of $x^4 - 2$ over $\mathbb{Q}$.

    (b) Prove that $[K:\mathbb{Q}] = 8$ and conclude from Theorem 12.11 that $\mathrm{Gal}_{\mathbb{Q}} K$ has order 8. [*Hint:* $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[4]{2}) \subseteq \mathbb{Q}(\sqrt[4]{2}, i)$.]

    (c) Prove that there exists $\sigma \in \mathrm{Gal}_{\mathbb{Q}} K$ such that $\sigma(\sqrt[4]{2}) = (\sqrt[4]{2})i$ and $\sigma(i) = i$ and that σ has order 4.

    (d) By Corollary 12.13 restriction of the complex conjugation map to K is an element τ of $\mathrm{Gal}_{\mathbb{Q}} K$. Show that

    $$\mathrm{Gal}_{\mathbb{Q}} K = \{\sigma, \sigma^2, \sigma^3, \sigma^4 = \iota, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}.$$

    [*Hint:* Use Theorem 12.4 to show these elements are distinct.]

    (e) Prove that $\mathrm{Gal}_{\mathbb{Q}} K \cong D_4$. [*Hint:* Map σ to r_1 to τ to v.]

12. Let K be as in Exercise 11. Prove that $\mathrm{Gal}_{\mathbb{Q}(i)} K \cong \mathbb{Z}_4$.

C. 13. Let K be as in Exercise 11. Exhibit the Galois correspondence for this extension. [Among the intermediate fields are $\mathbb{Q}((1 + i)\sqrt[4]{2})$ and $\mathbb{Q}((1 - i)\sqrt[4]{2})$.]

14. Exhibit the Galois correspondence for the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ of $\mathbb{Q}$. [The Galois group has seven subgroups of order 2 and seven of order 4.]

12.3 Solvability by Radicals

The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by the well-known formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This fact was known in ancient times. In the sixteenth century, formulas for the solution of cubic and quartic equations were discovered. For instance, the solutions of $x^3 + bx + c = 0$ are given by

$$x = \sqrt[3]{(-c/2) + \sqrt{d}} + \sqrt[3]{(-c/2) - \sqrt{d}}$$
$$x = \omega\left(\sqrt[3]{(-c/2) + \sqrt{d}}\right) + \omega^2\left(\sqrt[3]{(-c/2) - \sqrt{d}}\right)$$
$$x = \omega^2\left(\sqrt[3]{(-c/2) + \sqrt{d}}\right) + \omega\left(\sqrt[3]{(-c/2) - \sqrt{d}}\right),$$

where $d = (b^3/27) + (c^2/4)$, $\omega = (-1 + \sqrt{3}i)/2$ is a complex cube root of 1, and the other cube roots are chosen so that

$$\left(\sqrt[3]{(-c/2) + \sqrt{d}}\right)\left(\sqrt[3]{(-c/2) - \sqrt{d}}\right) = -b/3.^*$$

In the early 1800s Ruffini and Abel independently proved that, for $n \geq 5$, there is no formula for solving *all* equations of degree n. But the complete analysis of the problem is due to Galois, who provided a criterion for determining which polynomial equations *are* solvable by formula. This criterion, which is presented here, will enable us to exhibit a fifth-degree polynomial equation that cannot be solved by a formula. To simplify the discussion, we shall assume that *all fields have characteristic* 0.

As illustrated above, a "formula" is a specific procedure that starts with the coefficients of the polynomial $f(x) \in F[x]$ and arrives at the solutions of the equation $f(x) = 0_F$ by using only the field operations (addition, subtraction, multiplication, division) *and* the extraction of roots (square roots, cube roots, fourth roots, etc.). In this context, an **nth root** of an element c in F is any root of the polynomial $x^n - c$ in some extension field of F.

If $f(x) \in F[x]$, then performing field operations does not get you out of the coefficient field F (closure!). But taking an nth root may land you in an extension field. Taking an mth root after that may move you up to still another extension field. Thus the existence of a formula for the solutions of $f(x) = 0_F$ implies that these solutions lie in a special kind of extension field of F.

EXAMPLE 1

Applying the cubic formula above to the polynomial $x^3 + 3x + 2$ shows that the solutions of $x^3 + 3x + 2 = 0$ are

$$\sqrt[3]{-1 + \sqrt{2}} + \sqrt[3]{-1 - \sqrt{2}},$$
$$\omega\sqrt[3]{-1 + \sqrt{2}} + (\omega^2)\sqrt[4]{-1 - \sqrt{2}},$$
$$(\omega^2)\sqrt[3]{-1 + \sqrt{2}} + \omega\sqrt[4]{-1 - \sqrt{2}}.$$

*The formulas for the general cubic and the quartic are similar but more complicated.

All these solutions lie in the extension chain:

$$\begin{array}{ccccccccc} \mathbb{Q} \subseteq \mathbb{Q}(\omega) & \subseteq & \mathbb{Q}(\omega, \sqrt{2}) & \subseteq & \mathbb{Q}(\omega, \sqrt{2}, \sqrt[3]{-1+\sqrt{2}}) & \subseteq & \mathbb{Q}(\omega, \sqrt{2}, \sqrt[3]{-1+\sqrt{2}}, \sqrt[3]{-1-\sqrt{2}}) \\ \| \quad\ \ \| & & \| & & \| & & \| \\ F_0 \subseteq F_1 & \subseteq & F_2 & \subseteq & F_3 & \subseteq & F_4. \end{array}$$

Each field in this chain is a simple extension of the preceding one and is of the form $F_j(u)$, where $u^n \in F_j$ for some n (that is, u is an nth root of some element of F_j):

$F_1 = F_0(\omega)$, where $\omega^3 = 1 \in F_0$.

$F_2 = F_1(\sqrt{2})$, where $(\sqrt{2})^2 = 2 \in F_0 \subseteq F_1$.

$F_3 = F_2(\sqrt[3]{-1+\sqrt{2}})$, where $(\sqrt[3]{-1+\sqrt{2}})^3 = -1+\sqrt{2} \in F_2$.

$F_4 = F_3(\sqrt[3]{-1-\sqrt{2}})$, where $(\sqrt[3]{-1-\sqrt{2}})^3 = -1-\sqrt{2} \in F_2 \subseteq F_3$.

Since F_4 contains all the solutions of $x^3 + 3x + 2 = 0$, it also contains a splitting field of $x^3 + 3x + 2$.

The preceding example is an illustration of the next definition.

Definition

A field K is said to be a **radical extension** of a field F if there is a chain of fields

$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_t = K$$

such that for each $i = 1, 2, \ldots, t$,

$$F_i = F_{i-1}(u_i) \text{ and some power of } u_i \text{ is in } F_{i-1}.$$

Let $f(x) \in F[x]$. The equation $f(x) = 0_F$ is said to be **solvable by radicals** if there is a radical extension of F that contains a splitting field of $f(x)$. The example above shows that $x^3 + 3x + 2 = 0$ is solvable by radicals.

The preceding discussion shows that if there is a formula for its solutions, then the equation $f(x) = 0_F$ is solvable by radicals. Contrapositively, if $f(x) = 0_F$ is *not* solvable by radical, then there cannot be a formula (in the sense discussed above) for finding its solutions.

Solvable Groups

Before stating Galois' Criterion for an equation to be solvable by radicals, we need to introduce a new class of groups. A group G is said to be **solvable** if it has a chain of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{n-1} \supseteq G_n = \langle e \rangle$$

such that each G_i is a normal subgroup of the preceding group G_{i-1} and the quotient group G_{i-1}/G_i is abelian.

EXAMPLE 2

Every abelian group G is solvable because every quotient group of G is abelian, so the sequence $G \supseteq \langle e \rangle$ fulfills the conditions in the definition.

EXAMPLE 3

Let $\langle(123)\rangle$ be the cyclic subgroup of order 3 in S_3. The chain $S_3 \supseteq \langle(123)\rangle \supseteq \langle(1)\rangle$ shows that S_3 is solvable. But for other symmetric groups we have

Theorem 12.14

For $n \geq 5$ the group S_n is not solvable.

Proof ▸ Suppose, on the contrary, that S_n is solvable and that

$$S_n = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_t = \langle(1)\rangle$$

is the chain of subgroups required by the definition. Let (rst) be any 3-cycle in S_n and let u, v be any elements of $\{1, 2, \ldots, n\}$ other than r, s, t (u and v exist because $n \geq 5$). Since S_n/G_1 is abelian, Theorem 8.14 (with $a = (tus)$, $b = (srv)$) shows that G_1 must contain

$$(tus)(srv)(tus)^{-1}(srv)^{-1} = (tus)(srv)(tsu)(svr) = (rst).$$

Therefore, G_1 contains all the 3-cycles. Since G_1/G_2 is abelian, we can repeat the argument with G_1 in place of S_n and G_2 in place of G_1 and conclude that G_2 contains all the 3-cycles. The fact that each G_{i-1}/G_i is abelian and continued repetition lead to the conclusion that the identity subgroup G_t contains all the 3-cycles, which is a contradiction. Therefore, S_n is not solvable. ■

Theorem 12.15

Every homomorphic image of a solvable group G is solvable.

Proof ▸ Suppose that $f: G \to H$ is a surjective homomorphism and that $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_t = \langle e_G \rangle$ is the chain of subgroups in the definition of solvability. For each i, let $H_i = f(G_i)$ and consider this chain of subgroups:

$$H = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_t = f(\langle e_G \rangle) = \langle e_H \rangle.$$

Exercise 22 of Section 8.2 shows that H_i is a normal subgroup of H_{i-1} for each $i = 1, 2, \ldots, t$. Let $a, b \in H_{i-1}$. Then there exist $c, d \in G_{i-1}$ such that $f(c) = a$ and $f(d) = b$. Since G_{i-1}/G_i is abelian by solvability, $cdc^{-1}d^{-1} \in G_i$ by Theorem 8.14. Consequently,

$$aba^{-1}b^{-1} = f(c)f(d)f(c^{-1})f(d^{-1}) = f(cdc^{-1}d^{-1}) \in f(G_i) = H_i.$$

Therefore, H_{i-1}/H_i is abelian by Theorem 8.14, and H is solvable. ■

Galois' Criterion

If $f(x) \in F[x]$, then the **Galois group of the polynomial $f(x)$** is $\mathrm{Gal}_F K$, where K is a splitting field of $f(x)$ over F.* **Galois' Criterion** states that

$f(x) = 0_F$ is solvable by radicals if and only if the Galois group of $f(x)$ is a solvable group.

In order to prove Galois' solvability criterion, we need more information about radical extensions and nth roots. If F is a field and ζ is a root of $x^n - 1_F$ in some extension field of F (so that $\zeta^n = 1_F$), then ζ is called an **nth root of unity.** The derivative nx^{n-1} of $x^n - 1_F$ is nonzero (since F has characteristic 0) and relatively prime to $x^n - 1_F$. Therefore, $x^n - 1_F$ is separable by Lemma 11.16. So there are exactly n distinct nth roots of unity in any splitting field K of $x^n - 1_F$. If ζ and τ are nth roots of unity in K, then

$$(\zeta\tau)^n = \zeta^n\tau^n = 1_F\,1_F = 1_F,$$

so that $\zeta\tau$ is also an nth root of unity. Since the set of nth roots of unity is closed under multiplication, it is a subgroup of order n of the multiplicative group of the field K (Theorem 7.12) and is, therefore, cyclic by Theorem 7.16 or Corollary 9.11. A generator of this cyclic group of nth roots of unity in K is called a **primitive nth root of unity.** Thus ζ is a primitive nth root of unity if and only if $\zeta, \zeta^2, \zeta^3, \ldots, \zeta^n = 1_F$ are the n distinct nth roots of unity.

EXAMPLE 4

The fourth roots of unity in $\mathbb{C}$ are $1, -1, i, -i$. Since $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$, i is a primitive fourth root of unity. Similarly, $-i$ is also a primitive fourth root of unity. DeMoivre's Theorem shows that for any positive n,

$$\cos(2\pi/n) + i\sin(2\pi/n) \text{ is a primitive } n\text{th root of unity in } \mathbb{C}.$$

When $n = 3$, this states that

$$\omega = \cos(2\pi/3) + i\sin(2\pi/3) = (-1/2) + (\sqrt{3}/2)i$$

is a primitive cube root of unity.

Lemma 12.16

Let F be a field and ζ a primitive nth root of unity in F. Then F contains a primitive dth root of unity for every positive divisor d of n.

Proof ▸ By hypothesis ζ has order n in the multiplicative group of F. If $n = dt$, then ζ^t has order d by Theorem 7.9. So ζ^t generates a subgroup of order d, each of whose elements must have order dividing d by Corollary 8.6. In other words, $((\zeta^t)^k)^d = 1_F$ for every k. Thus the d distinct powers ζ^t,

*Since any two splitting fields of $f(x)$ are isomorphic by Theorem 11.14, it follows that the corresponding Galois groups are isomorphic. So the Galois group of $f(x)$ is independent of the choice of K.

$(\zeta^t)^2, \ldots, (\zeta^t)^{d-1}, (\zeta^t)^d = 1_F$ are roots of $x^d - 1_F$. Since $x^d - 1_F$ has at most d roots and every dth root of unity is a root of $x^d - 1_F$, ζ^t is a primitive dth root of unity. ∎

We can now tie together the preceding themes and prove two theorems that are special cases of Galois' Criterion as well as essential tools for proving the general case.

Theorem 12.17

Let F be a field of characteristic 0 and ζ a primitive nth root of unity in some extension field of F. Then $K = F(\zeta)$ is a normal extension of F, and $\mathrm{Gal}_F K$ is abelian.

Proof ▸ The field $K = F(\zeta)$ contains all the powers of ζ and is, therefore, a splitting field of $x^n - 1_F$.* Hence, K is normal over F by Theorem 11.15. Every automorphism in the Galois group must map ζ onto a root of $x^n - 1_F$ by Theorem 12.2. So if $\sigma, \tau \in \mathrm{Gal}_F K$, then $\sigma(\zeta) = \zeta^k$ and $\tau(\zeta) = \zeta^t$ for some positive integers k, t. Consequently,

$$(\sigma \circ \tau)(\zeta) = \sigma(\tau(\zeta)) = \sigma(\zeta^t) = \sigma(\zeta)^t = (\zeta^k)^t = \zeta^{kt}.$$
$$(\tau \circ \sigma)(\zeta) = \tau(\sigma(\zeta)) = \tau(\zeta^k) = \tau(\zeta)^k = (\zeta^t)^k = \zeta^{kt}.$$

Therefore, $\sigma \circ \tau = \tau \circ \sigma$ by Theorem 12.4, and $\mathrm{Gal}_F K$ is abelian. ∎

Theorem 12.18

Let F be a field of characteristic 0 that contains a primitive nth root of unity. If u is a root of $x^n - c \in F[x]$ in some extension field of F, then $K = F(u)$ is a normal extension of F, and $\mathrm{Gal}_F K$ is abelian.

Proof† ▸ By hypothesis, $u^n = c$. If ζ is a primitive nth root of unity in F, then for any k,

$$(\zeta^k u)^n = (\zeta^k)^n u^n = (\zeta^n)^k u^n = 1_F c = c.$$

Consequently, since $\zeta, \zeta^2, \ldots, \zeta^n = 1_F$ are distinct elements of F, the elements $\zeta u, \zeta^2 u, \zeta^3 u, \ldots, \zeta^n u = u$ are the n distinct roots of $x^n - c$. Hence, $K = F(u)$ is a splitting field of $x^n - c$ over F and is, therefore, normal over F by Theorem 11.15.§ If $\sigma, \tau, \in \mathrm{Gal}_F K$, then $\sigma(u) = \zeta^k u$ and $\tau(u) = \zeta^t u$ for some k, t by Theorem 12.2. Consequently, since ζ^k and ζ^t are in F,

---

*The field $K = F(\zeta)$ is a radical extension of F since $\zeta^n = 1_F$. Thus $x^n - 1_F = 0_F$ is solvable by radicals. So the theorem, which says that $\mathrm{Gal}_F K$ (the Galois group of $x^n - 1_F$), is abelian (and hence, solvable), is a special case of Galois' Criterion.

†For an alternate proof showing that $\mathrm{Gal}_F K$ is actually cyclic, see Exercise 22.

§The field $K = F(u)$ is also a radical extension of F since $u^n = c \in F$, so $x^n - c = 0_F$ is solvable by radicals. Hence, the theorem is another special case of Galois' Criterion.

$$(\sigma \circ \tau)(u) = \sigma(\tau(u)) = \sigma(\zeta^t u) = \sigma(\zeta^t)\sigma(u) = \zeta^t(\zeta^k u) = \zeta^{t+k}u.$$
$$(\tau \circ \sigma)(u) = \tau(\sigma(u)) = \tau(\zeta^k u) = \tau(\zeta^k)\tau(u) = \zeta^k(\zeta^t u) = \zeta^{t+k}u.$$

Therefore, $\sigma \circ \tau = \tau \circ \sigma$ by Theorem 12.4, and $\mathrm{Gal}_F K$ is abelian. ■

Theorem 12.19 Galois' Criterion

Let F be a field of characteristic 0 and $f(x) \in F[x]$. Then $f(x) = 0_F$ is solvable by radicals if and only if the Galois group of $f(x)$ is solvable.

We shall prove only the half of the theorem that is needed below; see Section V.9 of Hungerford [5] for the other half.

Proof of Theorem 12.19 ▸ Assume that $f(x) = 0_F$ is solvable by radicals. The proof, whose details are on pages 429–431, is in three steps:

1. Theorem 12.21: There is a *normal* radical extension K of F that contains a splitting field E of $f(x)$.*
2. The field E is normal over F by Theorem 11.15.
3. Theorem 12.22: Any intermediate field of K that is normal over F has a solvable Galois group; in particular, $\mathrm{Gal}_F E$ (the Galois group of $f(x)$) is solvable. ■

Before completing the proof of Theorem 12.19, we use it to demonstrate the insolvability of the quintic.

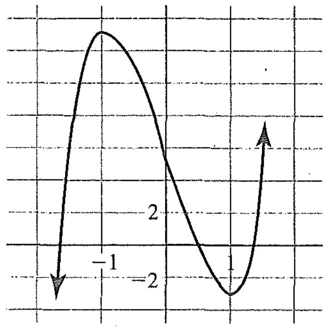
EXAMPLE 5

We claim that the Galois group of the polynomial $f(x) = 2x^5 - 10x + 5 \in \mathbb{Q}[x]$ is S_5, which is not solvable by Theorem 12.14. Consequently, the equation $2x^5 - 10x + 5 = 0$ is *not* solvable by radicals by Theorem 12.19. So, as explained on page 424,

there is no formula (involving only field operations and extraction of roots) for the solution of all fifth-degree polynomial equations.

To prove our claim, note that the derivative of $f(x)$ is $10x^4 - 10$, whose only real roots are ± 1 (the others being $\pm i$). Then $f''(x) = 40x^3$, and the second-derivative test of elementary calculus shows that $f(x)$ has exactly one relative maximum at $x = -1$, one relative minimum at $x = 1$, and one point of inflection at $x = 0$. So its graph must have the general shape shown on the next page. In particular, $f(x)$ has exactly three real roots.

*This is a crucial technical detail. The definition of solvability by radicals guarantees only a radical extension of F containing E. But a radical extension need not be normal over F (Exercise 19), and if it is not, the Fundamental Theorem 12.11 can't be used.



Note that $f(x)$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion (with $p = 5$). If K is a splitting field of $f(x)$ in $\mathbb{C}$, then $\mathrm{Gal}_{\mathbb{Q}}K$ has order $[K:\mathbb{Q}]$ by the Fundamental Theorem. If r is any root of $f(x)$, then $[K:\mathbb{Q}] = [K:\mathbb{Q}(r)]\,[\mathbb{Q}(r):\mathbb{Q}]$ by Theorem 11.4 and $[\mathbb{Q}(r):\mathbb{Q}] = 5$ by Theorem 11.7. So the order of $\mathrm{Gal}_{\mathbb{Q}}K$ is divisible by 5. It follows that $\mathrm{Gal}_{\mathbb{Q}}K$ contains an element of order 5.*

The group $\mathrm{Gal}_{\mathbb{Q}}K$, considered as a group of permutations of the roots of $f(x)$, is a subgroup of S_5 (Corollary 12.5). But the only elements of order 5 in S_5 are the 5-cycles (see Exercise 19 in Section 7.5). So $\mathrm{Gal}_{\mathbb{Q}}K$ contains a 5-cycle. Complex conjugation induces an automorphism on K (Corollary 12.13). This automorphism interchanges the two nonreal roots of $f(x)$ and fixes the three real ones. Thus $\mathrm{Gal}_{\mathbb{Q}}K$ contains a transposition. Exercise 8 shows that the only subgroup of S_5 that contains both a 5-cycle and a transposition is S_5 itself. Therefore, $\mathrm{Gal}_{\mathbb{Q}}K = S_5$ as claimed.

We now complete the proof of Galois' Criterion, beginning with a technical lemma whose import will become clear in the next theorem.

Lemma 12.20

Let F, E, L be fields of characteristic 0 with

$$F \subseteq E \subseteq L = E(v) \qquad \text{and} \qquad v^k \in E.$$

If L is finite dimensional over F and E is normal over F, then there exists an extension field M of L, which is a radical extension of E and a normal extension of F.

Proof ▸ By Theorem 11.15, E is the splitting field over F of some $g(x) \in F[x]$. Let $p(x) \in F[x]$ be the minimal polynomial of v over F and let M be a splitting field of $g(x)p(x)$ over F. Then M is normal over F by Theorem 11.15. Furthermore, $F \subseteq E \subseteq L \subseteq M$ (since $L = E(v)$ and E is generated over F by the roots of $g(x)$). Let $v = v_1, v_2, \ldots, v_r$ be all the roots of $p(x)$ in M. For each i there exists $\sigma_i \in \mathrm{Gal}_F M$ such that $\sigma_i(v) = v_i$ by

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*If you have read Chapter 9 use Corollary 9.14; otherwise, use Exercise 9 in this section.

Theorem 12.3. Corollary 12.13 shows that $\sigma_i(E) \subseteq E$. By hypothesis, $v^k = b \in E$; so for each i,

$$(v_i)^k = \sigma_i(v)^k = \sigma_i(v^k) = \sigma_i(b) \in E \subseteq E(v_1, \ldots, v_{i-1}).$$

Consequently,

$$E \subseteq L = E(v_1) \subseteq E(v_1, v_2) \subseteq E(v_1, v_2, v_3) \subseteq \cdots \subseteq E(v_1, v_2, \ldots, v_r) = M$$

is a radical extension of E. ■

Theorem 12.21

Let F be a field of characteristic 0 and $f(x) \in F[x]$. If $f(x) = 0_F$ is solvable by radicals, then there is a normal radical extension field of F that contains a splitting field of $f(x)$.

Proof ▸ By definition some splitting field K of $f(x)$ is contained in a radical extension

$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq F_t,$$

where $F_i = F_{i-1}(u_i)$ and $(u_i)^{n_i}$ is in F_{i-1} for each $i = 1, 2, \ldots, t$. Applying Lemma 12.20 with $E = F$, $L = F_1$, and $v = u_1$ produces a normal radical extension field M_1 of F that contains F_1. By hypothesis $(u_2)^{n_2} \in F_1 \subseteq M_1$. Applying Lemma 12.20 with $E = M_1$, $v = u_2$, and $L = M_1(u_2)$ produces a normal extension field M_2 of F that is a radical extension of M_1 and, hence, a radical extension of F. Furthermore, M_2 contains $F_2 = F_1(u_2)$. Continued repetition of this argument leads to a normal radical extension field M_t of F that contains F_t and, hence, contains K. ■

Theorem 12.22

Let K be a normal radical extension field of F and E an intermediate field, all of characteristic 0. If E is normal over F, then $Gal_F E$ is a solvable group.

Proof ▸ By hypothesis there is a chain of subfields

$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq F_t = K,$$

where $F_i = F_{i-1}(u_i)$ and $(u_i)^{n_i}$ is in F_{i-1} for each $i = 1, 2, \ldots, t$. Let n be the least common multiple of $n_1, n_2, \ldots, n_t$ and let ζ be a primitive nth root of unity. For each $i \geq 0$, let $E_i = F_i(\zeta)$. Then for each $i \geq 1$

$$E_i = F_i(\zeta) = F_{i-1}(u_i)(\zeta) = F_{i-1}(u_i, \zeta) = F_{i-1}(\zeta)(u_i) = E_{i-1}(u_i).$$

Since $(u_i)^{n_i} \in F_{i-1} \subseteq E_{i-1}$ for $i \geq 1$ and $\zeta^n \in F$,

$$F \subseteq E_0 \subseteq E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \subseteq E_t = L$$

is a radical extension of F that contains K (and, hence, E).* The normal extension $K = F_t$ is the splitting field of some polynomial $p(x) \in F[x]$ by Theorem 11.15, and, hence, $L = E_t = F_t(\zeta)$ is the splitting field of $p(x)(x^n - 1_F)$ over F. Therefore, L is Galois over F by Theorems 11.15 and 11.17.

Consider the following chain of subgroups of $\text{Gal}_F L$:

$$\text{Gal}_F L \supseteq \text{Gal}_{E_0} L \supseteq \text{Gal}_{E_1} L \supseteq \text{Gal}_{E_2} L \supseteq \cdots \supseteq \text{Gal}_{E_{t-1}} L \supseteq \text{Gal}_L L = \langle \iota \rangle.$$

We shall show that each subgroup is normal in the preceding one and that each quotient is abelian. Since each n_i divides n, E_0 contains a primitive n_ith root of unity by Lemma 12.16. Consequently, by Theorem 12.18 each E_i (with $i \geq 1$) is a normal extension of E_{i-1}, and the Galois group $\text{Gal}_{E_{i-1}} E_i$ is abelian. Since L is Galois over F, it is Galois over every E_j. Applying the Fundamental Theorem 12.11 to the extension L of E_{i-1}, we see that $\text{Gal}_{E_i} L$ is a normal subgroup of $\text{Gal}_{E_{i-1}} L$ and that the quotient group $\text{Gal}_{E_{i-1}} L/\text{Gal}_{E_i} L$ is isomorphic to the abelian group $\text{Gal}_{E_{i-1}} E_i$. Similarly by Theorems 12.11 and 12.17, E_0 is normal over F, $\text{Gal}_{E_0} L$ is normal in $\text{Gal}_F L$, and $\text{Gal}_F L/\text{Gal}_{E_0} L$ is isomorphic to the abelian group $\text{Gal}_F E_0$. Therefore, $\text{Gal}_F L$ is a solvable group.

Since E is normal over F, the Fundamental Theorem shows that $\text{Gal}_E L$ is normal in $\text{Gal}_F L$ and $\text{Gal}_F L/\text{Gal}_E L$ is isomorphic to $\text{Gal}_F E$. So $\text{Gal}_F E$ is the homomorphic image of the solvable group $\text{Gal}_F L$ (see Theorem 8.18) and is, therefore, solvable by Theorem 12.15. ∎

Exercises

NOTE: *F denotes a field, and all fields have characteristic* 0.

A. 1. Find a radical extension of $\mathbb{Q}$ containing the given number:

(a) $\sqrt[4]{1 + \sqrt{7}} - \sqrt[5]{2 + \sqrt{5}}$

(b) $(\sqrt[5]{\sqrt{2} + i})/(\sqrt[3]{5})$

(c) $(\sqrt[3]{3 - \sqrt{2}})/(4 + \sqrt{2})$

2. Show that $x^2 - 3$ and $x^2 - 2x - 2 \in \mathbb{Q}[x]$ have the same Galois group. [*Hint:* What is the splitting field of each?]

3. If K is a radical extension of F, prove that $[K:F]$ is finite. [*Hint:* Theorems 11.7 and 11.4.]

*The construction of L does not use the hypothesis that K is normal over F, and, as we shall see below, every field in the chain is a normal extension of the immediately preceding one. But this is *not* enough to guarantee that L is normal (hence Galois) over F (Exercise 19). We need the hypothesis that K is normal over F to guarantee this, so that we can use the Fundamental Theorem on L.

4. Prove that for $n \geq 5$, A_n is not solvable. [*Hint:* Adapt the proof of Theorem 12.14.]

5. (a) Show that S_4 is a solvable group. [*Hint:* Consider the subgroup $H = \{(12)(34), (13)(24), (14)(23), (1)\}$ of A_4.]

   (b) Show that D_4 is a solvable group.

6. If G is a simple nonabelian group, prove that G is not solvable. [This fact and Theorem 8.26 provide another proof that A_n is not solvable for $n \geq 5$.]

7. List all the nth roots of unity in $\mathbb{C}$ when $n =$

   (a) 2 (b) 3 (c) 4 (d) 5 (e) 6

B. 8. Let G be a subgroup of S_5 that contains a transposition $\sigma = (rs)$ and a 5-cycle α. Prove that $G = S_5$ as follows.

   (a) Show that for some k, α^k is of the form $(rsxyz)$. Let $\tau = \alpha^k \in G$; by relabeling we may assume that $\sigma = (12)$ and $\tau = (12345)$.

   (b) Show that (12), (23), (34), (45) $\in G$. [*Hint:* Consider $\tau^k \sigma \tau^{-k}$ for $k \geq 1$].

   (c) Show that (13), (14), (15) $\in G$. [*Hint:* (12)(23)(12) = ?]

   (d) Show that every transposition is in G. Therefore, $G = S_5$ by Theorem 7.26.

9. Let G be a group of order n. If $5|n$, prove that G contains an element of order 5 as follows. Let S be the set of all ordered 5-tuples (r, s, t, u, v) with $r, s, t, u, v \in G$ and $rstuv = e$.

   (a) Show that S contains exactly n^4 5-tuples. [*Hint:* If $r, s, t, u, \in G$ and $v = (rstu)^{-1}$, then $(r, s, t, u, v) \in S$.]

   (b) Two 5-tuples in S are said to be *equivalent* if one is a cyclic permutation of the other.* Prove that this relation is an equivalence relation on S.

   (c) Prove that an equivalence class in S either has exactly five 5-tuples in it or consists of a single 5-tuple of the form (r, r, r, r, r).

   (d) Prove that there are at least two equivalence classes in S that contain a single 5-tuple. [*Hint:* One is $\{(e, e, e, e, e)\}$. If this is the only one, show that $n^4 \equiv 1 \pmod 5$. But $5 \mid n$, so $n^4 \equiv 0 \pmod 5$, which is a contradiction.]

   (e) If $\{(c, c, c, c, c)\}$, with $c \neq e$, is a single-element equivalence class, prove that c has order 5.

10. If N is a normal subgroup of G, N is solvable, and G/N is solvable, prove that G is solvable.

11. Prove that a subgroup H of a solvable group G is solvable. [*Hint:* If $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \langle e \rangle$ is the solvable series for G, consider the groups $H_i = H \cap G_i$. To show that H_{i-1}/H_i is abelian, verify that the map $H_{i-1}/H_i \to G_{i-1}/G_i$ given by $H_i x \to G_i x$ is a well-defined injective homomorphism.]

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*For instance, (r, s, t, u, v) is equivalent to each of (s, t, u, v, r), (t, u, v, r, s), (u, v, r, s, t), (v, r, s, t, u), (r, s, t, u, v) and to no other 5-tuples in S.

12. Prove that the Galois group of an irreducible quadratic polynomial is isomorphic to $\mathbb{Z}_2$.

13. Prove that the Galois group of an irreducible cubic polynomial is isomorphic to $\mathbb{Z}_3$ or S_3.

14. Prove that the Galois group of an irreducible quartic polynomial is solvable. [*Hint:* Corollary 12.5 and Exercises 5 and 11.]

15. Let $p(x)$, $q(x)$ be irreducible quadratics. Prove that the Galois group of $f(x) = p(x)q(x)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2$. [*Hint:* If u is a root of $p(x)$ and v a root of $q(x)$, then there are two cases: $v \notin F(u)$ and $v \in F(u)$.]

16. Use Galois' Criterion to prove that every polynomial of degree ≤ 4 is solvable by radicals. [*Hint:* Exercises 12–15.]

17. Find the Galois group G of the given polynomial in $\mathbb{Q}[x]$:

(a) $x^6 - 4x^3 + 4$ [*Hint:* Factor.]

(b) $x^4 - 5x^2 + 6$

(c) $x^5 + 6x^3 + 9x$

(d) $x^4 + 3x^3 - 2x - 6$

(e) $x^5 - 10x - 5$ [*Hint:* See Example 5.]

18. Determine whether the given equation over $\mathbb{Q}$ is solvable by radicals:

(a) $x^6 + 2x^3 + 1 = 0$ **(b)** $3x^5 - 15x + 5 = 0$

(c) $2x^5 - 5x^4 + 5 = 0$ **(d)** $x^5 - x^4 - 16x + 16 = 0$

19. **(a)** Prove that $\mathbb{Q}(\sqrt{2}i)$ is normal over $\mathbb{Q}$ by showing it is the splitting field of $x^2 + 2$.

(b) Prove that $\mathbb{Q}(\sqrt[4]{2}(1 - i))$ is normal over $\mathbb{Q}(\sqrt{2}i)$ by showing that it is the splitting field of $x^2 + 2\sqrt{2}i$.

(c) Show that $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}i) \subseteq \mathbb{Q}(\sqrt[4]{2}(1 - i))$ is a radical extension of $\mathbb{Q}$ with $[\mathbb{Q}(\sqrt[4]{2}(1 - i)):\mathbb{Q}] = 4$ and note that $\mathbb{Q}$ contains all second roots of unity (namely $\pm$ 1).

(d) Let $L = \mathbb{Q}(\sqrt[4]{2}(1 - i))$. Show that $v = \sqrt[4]{2}(1 + i)$ is *not* in L. [*Hint:* If $v \in L$ and $u = \sqrt[4]{2}(1 - i) \in L$, show that $v/u = i$ and $(v - u)/2i = \sqrt[4]{2} \in L$, which implies that $[L:\mathbb{Q}] \geq \mathbb{Q}(\sqrt[4]{2}, i):\mathbb{Q}]$, contradicting (c) and Exercise 12(b) in Section 12.2.]

(e) Prove that $L = \mathbb{Q}(\sqrt[4]{2}(1 - i))$ is *not* normal over $\mathbb{Q}$ [*Hint:* u and v (as in (d)) are roots of the irreducible polynomial $x^4 + 8$.]

20. Let ζ be a primitive fifth root of unity. Assume Exercise 21 in Section 4.5 and prove that $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\zeta)$, the Galois group of $x^5 - 1$, is cyclic of order 4.

21. What is the Galois group of $x^5 + 32$ over $\mathbb{Q}$? [*Hint:* Show that $\mathbb{Q}(\zeta)$ is a splitting field, where ζ is a primitive fifth root of unity; see Exercise 20.]

22. Prove that the group $\mathrm{Gal}_F K$ in Theorem 12.18 is cyclic. [*Hint:* Define a map f from $\mathrm{Gal}_F K$ to the additive group $\mathbb{Z}_n$ by $f(\sigma) = k$, where $\sigma(u) = \zeta^k u$. Show that f is a well-defined injective homomorphism and use Theorem 7.17.]

C. 23. If p is prime and G is a subgroup of S_p that contains a transposition and a p-cycle, prove that $G = S_p$. [Exercise 8 is the case $p = 5$.]

24. If $f(x) \in \mathbb{Q}[x]$ is irreducible of prime degree p and $f(x)$ has exactly two nonreal roots, prove that the Galois group of $f(x)$ is S_p. [Example 5 is essentially the case $p = 5$.]

25. Construct a polynomial in $\mathbb{Q}[x]$ of degree 7 whose Galois group is S_7.

PART 3

EXCURSIONS AND APPLICATIONS

CHAPTER 13

Public-Key Cryptography

Prerequisites: Section 2.3

Codes have been used for centuries by merchants, spies, armies, and diplomats to transmit secret messages. In recent times, the large volume of sensitive material in government and corporate computerized data banks (much of which is transmitted by satellite or over telephone lines) has increased the need for efficient, high-security codes.

It is easy to construct unbreakable codes for one-time use. Consider this "code pad":

Actual Word:	morning	evening	Monday	Tuesday	attack
Code Word:	bat	glxt	king	button	figle

If I send you the message FIGLE BUTTON BAT, there is no way an enemy can know for certain that it means "attack on Tuesday morning" unless he or she has a copy of the pad. Of course, if the same code is used again, the enemy might well be able to break it by analyzing the events that occur after each message.

Although one-time code pads are unbreakable, they are cumbersome and inefficient when many long messages must be routinely sent. Even if the encoding and decoding are done by a computer, it is still necessary to design and supply a new pad (at least as long as the message) to each participant for every message and to make all copies of these pads secure from unauthorized persons. This is expensive and impractical when hundreds of thousands of words must be encoded and decoded every day.

For frequent computer-based communication among several parties, the ideal code system would be one in which

1. Each person has efficient, reusable, computer algorithms for encoding and decoding messages.
2. Each person's decoding algorithm is *not* obtainable from his or her encoding algorithm in any reasonable amount of time.

A code system with these properties is called a **public-key system**. Although it may not be clear how condition 2 could be satisfied, it is easy to see the advantages of a public-key system.

The *encoding* algorithm of each participant could be publicly announced—perhaps published in a book (like a telephone directory)—thus eliminating the need for couriers and the security problems associated with the distribution of code pads. This would not compromise secrecy because of condition 2: Knowing a person's *encoding* algorithm would not enable you to determine his or her *decoding* algorithm. So you would have no way of decoding messages sent to another person in his or her code, even though you could send coded messages to that person.

Since the encoding algorithms for a public-key system are available to everyone, forgery appears to be a possibility. Suppose, for example, that a bank receives a coded message claiming to be from Anne and requesting the bank to transfer money from Anne's account into Tom's account. How can the bank be sure the message was actually sent by Anne?

The answer is as simple as it is foolproof. Coding and decoding algorithms are inverses of each other: Applying one after the other (in either order) produces the word you started with. So Anne first uses her secret *decoding* algorithm to write her name; say it becomes Gybx. She then applies the bank's public encoding algorithm to Gybx and sends the result (her "signature") along with her message. The bank uses its secret decoding algorithm on this "signature" and obtains Gybx. It then applies Anne's public *encoding* algorithm to Gybx, which turns it into Anne. The bank can then be sure the message is from Anne, because no one else could use her *decoding* algorithm to produce the word Gybx that is encoded as Anne.

One public-key system was developed by R. Rivest, A. Shamir, and L. Adleman in 1977. Their system, now called the RSA system, is based on elementary number theory. Its security depends on the difficulty of factoring large integers. Here are the mathematical preliminaries needed to understand the RSA system.

Lemma 13.1

Let $p, r, s, c \in \mathbb{Z}$ with p prime. If $p \nmid c$ and $rc \equiv sc \pmod{p}$, then $r \equiv s \pmod{p}$.

Proof ▶ Since $rc \equiv sc \pmod{p}$, p divides $rc - sc = (r - s)c$. By Theorem 1.5 $p \mid (r - s)$ or $p \mid c$. Since $p \nmid c$, we have $p \mid (r - s)$, and, hence, $r \equiv s \pmod{p}$. ■

Lemma 13.2 Fermat's Little Theorem

If p is prime, $a \in \mathbb{Z}$, and $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

Proof* ▶ None of the numbers $a, 2a, 3a, \ldots, (p - 1)a$ is congruent to 0 modulo p by Exercise 1. Consequently, each of them must be congruent to one of $1, 2, 3, \ldots, p - 1$ by Corollary 2.5 and Theorem 2.3. If two of them were congruent to the same one, say $ra \equiv i \equiv sa \pmod{p}$ with

$$1 \leq i, r, s \leq p - 1,$$

*A proof based on group theory is outlined in Exercise 38 of Section 7.3, and one based on field theory is in Exercise 13 of Section 11.6.

then we would have $r \equiv s \pmod{p}$ by Lemma 13.1 (with $c = a$). This is impossible because no two of the numbers $1, 2, 3, \ldots, p-1$ are congruent modulo p (the difference of any two is less than p and, hence, not divisible by p). Therefore, in some order $a, 2a, 3a, \ldots, (p-1)a$ are congruent to $1, 2, 3, \ldots, p-1$. By repeated use of Theorem 2.2,

$$a \cdot 2a \cdot 3a \cdots (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdots (p-1) \pmod{p}.$$

Rearranging the left side shows that

$$\begin{aligned} a \cdot a \cdot a \cdots a \cdot 1 \cdot 2 \cdot 3 \cdots (p-1) &\equiv 1 \cdot 2 \cdot 3 \cdots (p-1) \pmod{p} \\ a^{p-1}(1 \cdot 2 \cdot 3 \cdots (p-1)) &\equiv 1(1 \cdot 2 \cdot 3 \cdots (p-1)) \pmod{p}. \end{aligned}$$

Now $p \nmid (1 \cdot 2 \cdot 3 \cdots (p-1))$ (if it did, p would divide one of the factors by Corollary 1.6. Therefore, $a^{p-1} \equiv 1 \pmod{p}$ by Lemma 13.1 (with $c = 1 \cdot 2 \cdot 3 \cdots (p-1)$). ∎

Throughout the rest of this discussion p and q are distinct positive primes. Let $n = pq$ and $k = (p-1)(q-1)$. Choose d such that $(d, k) = 1$. Then the equation $dx = 1$ has a solution in $\mathbb{Z}_k$ by Theorem 2.9 (with $n = k$). Therefore, the congruence $dx \equiv 1 \pmod{k}$ has a solution in $\mathbb{Z}$; call it e.

Theorem 13.3

Let p, q, n, k, e, d be as in the preceding paragraph. Then $b^{ed} \equiv b \pmod{n}$ for every $b \in \mathbb{Z}$.

Proof ▸ Since e is a solution of $dx \equiv 1 \pmod{k}$, $de - 1 = kt$ for some t. Hence, $ed = kt + 1$, so that

$$b^{ed} = b^{kt+1} = b^{kt}b^1 = b^{(p-1)(q-1)t}b = (b^{p-1})^{(q-1)t}b.$$

If $p \nmid b$, then by Lemma 13.2,

$$b^{ed} = (b^{p-1})^{(q-1)t}b \equiv (1)^{(q-1)t}\, b \equiv b \pmod{p}.$$

If $p \mid b$, then b and every one of its powers are congruent to 0 modulo p. Therefore, in every case, $b^{ed} \equiv b \pmod{p}$. A similar argument shows that $b^{ed} \equiv b \pmod{q}$. By the definition of congruence,

$$p \mid (b^{ed} - b) \qquad \text{and} \qquad q \mid (b^{ed} - b).$$

Therefore, $pq \mid (b^{ed} - b)$ by Exercise 2. Since $pq = n$, this means that n divides $(b^{ed} - b)$, and, hence, $b^{ed} \equiv b \pmod{n}$. ∎

The **least residue modulo n** of an integer c is the remainder r when c is divided by n. By the Division Algorithm, $c = nq + r$, so that $c - r = nq$, and, hence, $c \equiv r \pmod{n}$. Since two numbers strictly between 0 and n cannot be congruent modulo n, the least residue of c is the only integer between 0 and n that is congruent to c modulo n.

We can now describe the mechanics of the RSA system, after which we shall show how it satisfies the conditions for a public-key system. The message to be sent is first converted to numerical form by replacing each letter or space by a two-digit number:*

$$\text{space} = 00, \text{A} = 01, \text{B} = 02, \ldots, \text{Y} = 25, \text{Z} = 26.$$

For instance, the word GO is written as the number 0715 and WEST is written 23051920, so that the message "GO WEST" becomes the number 07150023051920, which we shall denote by B.

Let p, q, n, k, d, e, be as in Theorem 13.3, with p and q chosen so that $B < pq = n$. To encode message B, compute the least residue of B^e modulo n; denote it by C. Then C is the coded form of B. Send C in any convenient way.

The person who receives C decodes it by computing the least residue of C^d modulo n. This produces the original message for the following reasons. Since B^e, is congruent modulo n to its least residue C, Theorem 13.3 shows that

$$C^d \equiv (B^e)^d = B^{ed} \equiv B \pmod{n}.$$

The least residue of C^d is the only number between 0 and n that is congruent to C^d modulo n and $0 < B < n$. So the original message B is the least residue of C^d.

Before presenting a numerical example, we show that the RSA system satisfies the conditions for a public-key system:

1. When the RSA system is used in practice, p and q are large primes (several hundred digits each). Such primes can be quickly identified by a computer. Even though B, e, C, d are large numbers, there are fast algorithms for finding the least residues of B^e and C^d modulo n. They are based on binary representation of the exponent and do *not* require direct computation of B^e or C^d (which would be gigantic numbers). See Knuth [31] for details. So the encoding and decoding algorithms of the RSA system *are* computationally efficient.
2. To use the RSA system, each person in the network uses a computer to choose appropriate p, q, d and then determines n, k, e. The numbers e and n for the encoding algorithm are publicly announced, but the prime factors p, q of n and the numbers d and k are kept secret. Anyone with a computer can encode messages by using e and n. But there is no practical way for outsiders to determine d (and, hence, the decoding algorithm) without first finding p and q by factoring n.† With present technology this would take thousands of years! So the RSA system appears secure, as long as new and very fast methods of factoring are not developed.

Even when n is chosen as above, there may be some messages that in numerical form are larger than n. In such cases the original message is broken into several blocks, each of which is less than n. Here is an example, due to Rivest-Shamir-Adleman.

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*More numbers could be used for punctuation marks, numerals, special symbols, etc. But this will be sufficient for illustrating the basic concepts.

†Alternatively, one might try to find k and then solve the congruence $ex \equiv 1 \pmod{k}$ to get d. But this can be shown to be computationally equivalent to factoring n, so no time is saved.

EXAMPLE 1

Let $p = 47$ and $q = 59$. Then $n = pq = 47 \cdot 59 = 2773$ and $k = (p - 1)(q - 1) = 46 \cdot 58 = 2668$.* Let $d = 157$. A graphing calculator or computer quickly verifies that $(157, 2668) = 1$ and that the solution of $157x \equiv 1 \pmod{2668}$ is $e = 17$.† We shall encode the message "IT'S ALL GREEK TO ME." We can encode only numbers less than $n = 2773$. So we write the message in two-letter blocks (and denote spaces by #):

I T	S #	A L	L #	G R
0920	1900	0112	1200	0718
E E	K #	T O	# M	E #
0505	1100	2015	0013	0500.

Then each block is a number less than 2773. The first block, 0920, is encoded by using $e = 17$ and a computer to calculate the least residue of 920^{17} modulo 2773:

$$920^{17} \equiv 948 \pmod{2773}.$$

The other blocks are encoded similarly, so the coded form of the message is

0948	2342	1084	1444	2663
2390	0778	0774	0219	1655.

A person receiving this message would use $d = 157$ to decode each block. For instance, to decode 0948, the computer calculates

$$948^{157} \equiv 920 \pmod{2773}.$$

This is the original first block $0920 = \text{IT}$.

For more information on cryptography and the RSA system, see Hoffstein, Pipher, and Silveman [33], Rivest-Shamir-Adleman [34], Simmons [35], and Trappe and Washington [36].

Exercises

A. **1.** Let p be a prime and $k, a \in \mathbb{Z}$ such that $p \nmid a$ and $0 < k < p$. Prove that $ka \not\equiv 0 \pmod{p}$. [*Hint:* Theorem 1.5.]

2. If p and q are distinct primes such that $p \mid c$ and $q \mid c$, prove that $pq \mid c$. [*Hint:* If $c = pk$, then $q \mid pk$; use Theorem 1.5.]

*These numbers will illustrate the concepts. But they are too small to provide a secure code since 2773 can be factored by hand.

†To solve the congruence on a calculator, use the Technology Tip on page 12 to find u and v such that $157u + 2668v = 1$. Then $157u - 1 = 2668v$, which means that $157u \equiv 1 \pmod{2668}$.

3. Use a calculator and the RSA encoding algorithm with $e = 3$, $n = 2773$ to encode these messages:

(a) GO HOME **(b)** COME BACK **(c)** DROP DEAD

[*Hint:* Use 2-letter blocks and don't omit spaces.]

4. Prove this version of Fermat's Little Theorem: If p is a prime and $a \in \mathbb{Z}$, then $a^p \equiv a \pmod{p}$. [*Hint:* Consider two cases, $p \mid a$ and $p \nmid a$; use Lemma 13.2 in the second case.]

B. 5. Find the decoding algorithm for the code in Exercise 3.

6. Let C be the coded form of a message that was encoded by using the RSA algorithm. Suppose that you discover that C and the encoding modulus n are *not* relatively prime. Explain how you could factor n and thus find the decoding algorithm. [The probability of such a C occurring is less than 10^{-99} when the prime factors p, q, of n have more than 100 digits.]

CHAPTER 14

The Chinese Remainder Theorem

Prerequisites: Section 2.1 and Appendix C for Section 14.1; Section 3.1 for Section 14.2; Section 6.2 for Section 14.3.

The Chinese Remainder Theorem (Section 14.1) is a famous result in number theory that was known to Chinese mathematicians in the first century. It also has practical applications in computer arithmetic (Section 14.2). An extension of the theorem to rings other than $\mathbb{Z}$ has interesting consequences in ring theory (Section 14.3). Although obviously motivated by Section 14.1, Section 14.3 is independent of the rest of the chapter and may be read at any time after you have read Section 6.2.

14.1 Proof of the Chinese Remainder Theorem

A **congruence** is an equation with integer coefficients in which "=" is replaced by "$\equiv \pmod{n}$." The same equation can lead to different congruences, such as

$$6x + 5 \equiv 7 \pmod 3 \qquad \text{or} \qquad 6x + 5 \equiv 7 \pmod 5.$$

Only integers make sense as solutions of congruences, so the techniques of solving equations are not always applicable to congruences. For instance, the equation $6x + 5 = 7$ has $x = 1/3$ as a solution, but the congruence $6x + 5 \equiv 7 \pmod 3$ has no solutions (Exercise 3), and $6x + 5 \equiv 7 \pmod 5$ has infinitely many solutions (Exercise 4).

A number of theoretical problems and practical applications require the solving of a **system of linear congruences,** such as

$$\begin{aligned} x &\equiv 2 \pmod 4 \\ x &\equiv 5 \pmod 7 \\ x &\equiv 0 \pmod{11} \\ x &\equiv 8 \pmod{15} \end{aligned}$$

A **solution of the system** is an integer that is a solution of *every* congruence in the system. We shall examine some cases in which a system of linear congruences must have a solution.

Lemma 14.1

If m and n are relatively prime positive integers and $a, b \in \mathbb{Z}$, then the system

$$x \equiv a \pmod{m}$$
$$x \equiv b \pmod{n}$$

has a solution.

Proof ▸ Since $(m, n) = 1$, there exist integers u and v such that $mu + nv = 1$ by Theorem 1.2. This equation and the definition of congruence lead to four conclusions:

(i) $mu \equiv 0 \pmod{m}$ (ii) $nv \equiv 1 \pmod{m}$ [*Because* $1 - nv = mu$.]
(iii) $nv \equiv 0 \pmod{n}$ (iv) $mu \equiv 1 \pmod{n}$ [*Because* $1 - mu = nv$.]

Let $t = bmu + anv$. Then by (i), (ii), and Theorem 2.2,

$$t = bmu + anv \equiv b \cdot 0 + a \cdot 1 = a \pmod{m},$$

so that $t \equiv a \pmod{m}$. Similarly, by (iii), (iv), and Theorem 2.2,

$$t = bmu + anv \equiv b \cdot 1 + a \cdot 0 = b \pmod{n},$$

so that $t \equiv b \pmod{n}$. Therefore, t is a solution of the system. ■

The proof of Lemma 14.1, provides the

Solution Algorithm for the System in Lemma 14.1

1. Find u and v such that $mu + nv = 1$.*
2. Then $t = bmu + anv$ is a solution of the system

EXAMPLE 1

To solve the system

$$x \equiv 2 \pmod{4}$$
$$x \equiv 5 \pmod{7},$$

apply the algorithm with $m = 4, n = 7, a = 2, b = 5$:

1. It is easy to see that $u = 2, v = -1$ satisfy $4u + 7v = 1$.
2. Therefore, a solution of the system is

$$t = bmu + anv = 5 \cdot 4 \cdot 2 + 2 \cdot 7 \cdot (-1) = 26.$$

*This can be done by hand by using the Euclidean Algorithm; see Exercise 15 in Section 1.2. It can also be done on a computer or graphing calculator; see the Technology Tip on page 12.

Theorem 14.2 The Chinese Remainder Theorem*

Let $m_1, m_2, \ldots, m_r$ be pairwise relatively prime positive integers (meaning that $(m_i, m_j) = 1$ whenever $i \neq j$). Let $a_1, a_2, \ldots, a_r$ be any integers.

(1) The system

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ x &\equiv a_3 \pmod{m_3} \\ &\ \ \vdots \\ x &\equiv a_r \pmod{m_r} \end{aligned}$$

has a solution.

(2) If t is one solution of the system, then an integer z is also a solution if and only if $z \equiv t \pmod{m_1 m_2 m_3 \cdots m_r}$.

For reasons that will become apparent below, we shall use induction to prove the first part of the theorem. For a proof that does not use induction, see Exercise 21.

Proof of Theorem 14.2 ▸ (1) The proof is by induction on the number r of congruences in the system. If $r = 2$, then there is a solution by Lemma 14.1 (with $m = m_1$, $n = m_2$, $a = a_1$, $b = a_2$). So suppose inductively that there is a solution when $r = k$ and consider the system

$$(*) \qquad \begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ x &\equiv a_3 \pmod{m_3} \\ &\ \ \vdots \\ x &\equiv a_k \pmod{m_k} \\ x &\equiv a_{k+1} \pmod{m_{k+1}} \end{aligned}$$

By the induction hypothesis, the system consisting of the first k congruences in $(*)$ *has* a solution s. Furthermore, $m_1 m_2 m_3 \cdots m_k$ and m_{k+1} are relatively prime (Exercise 5). Consequently, by Lemma 14.1, the system

$$(**) \qquad \begin{aligned} x &\equiv s \quad \pmod{m_1 m_2 m_3 \cdots m_k} \\ x &\equiv a_{k+1} \pmod{m_{k+1}} \end{aligned}$$

*So named because it was known to Chinese mathematicians in the first century.

has a solution t. The number t necessarily satisfies

$$t \equiv s \pmod{m_1 m_2 m_3 \cdots m_k}.$$

Consequently, for each $i = 1, 2, 3, \ldots, k$,

$$t \equiv s \pmod{m_i}.$$

(Reason: If $t - s$ is divisible by $m_1 m_2 m_3 \cdots m_k$, then it is divisible by each m_i). Now s is a solution of the first k congruences in $(**)$, so for each $i \leq k$

$$t \equiv s \pmod{m_i} \qquad \text{and} \qquad s \equiv a_i \pmod{m_i}.$$

By transitivity (Theorem 2.1),

$$t \equiv a_i \pmod{m_i} \qquad \text{for } i = 1, 2, \ldots, k.$$

Since t is a solution of $(**)$, it must also satisfy $t \equiv a_{k+1} \pmod{m_{k+1}}$. Hence, t is a solution of the system $(*)$, so that there is a solution when $r = k + 1$. Therefore, by induction, every such system has a solution.

(2) If z is any other solution of the system, then for each $i = 1, 2, \ldots, r$,

$$z \equiv a_i \pmod{m_i} \qquad \text{and} \qquad t \equiv a_i \pmod{m_i}.$$

By transitivity (Theorem 2.1), $z \equiv t \pmod{m_i}$. Thus

$$m_1 \mid (z - t), m_2 \mid (z - t), m_3 \mid (z - t), \ldots, m_r \mid (z - t).$$

Therefore, $m_1 m_2 m_3 \cdots m_r \mid (z - t)$ by Exercise 7. Hence,

$$z \equiv t \pmod{m_1 m_2 m_3 \cdots m_r}.$$

Conversely, if $z \equiv t \pmod{m_1 m_2 m_3 \cdots m_r}$, then, as above, $z \equiv t \pmod{m_i}$ for each $i = 1, 2, \ldots, r$. Since $t \equiv a_i \pmod{m_i}$, transitivity shows that $z \equiv a_i \pmod{m_i}$ for each i. Therefore, z is a solution of the system. ■

The proof of Theorem 14.2 actually provides an effective computational algorithm for solving large systems: Solve the first two by Lemma 14.1, then repeat the inductive step as often as needed to determine a solution of the entire system.

EXAMPLE 2

We shall solve the system

$$\begin{aligned} x &\equiv 2 \pmod{4} \\ x &\equiv 5 \pmod{7} \\ x &\equiv 0 \pmod{11} \\ x &\equiv 8 \pmod{15}. \end{aligned}$$

Example 1 shows that $x = 26$ is a solution of the system consisting of the first two congruences:

$$x \equiv 2 \pmod{4}$$
$$x \equiv 5 \pmod{7}.$$

Next we solve the system

$$x \equiv 26 \pmod{4 \cdot 7}$$
$$x \equiv 0 \pmod{11}.$$

First, note that $u = 2$ and $v = -5$ satisfy $28u + 11v = 1$.* Then the Solution Algorithm preceding Example 1 (with $a = 26$, $m = 4 \cdot 7 = 28$, $b = 0$, $n = 11$) shows that a solution is

$$bmu + anv = 0 \cdot 28 \cdot 2 + 26 \cdot 11 \cdot (-5) = -1430.$$

You can readily verify that $x = -1430$ is also a solution of the system consisting of the first three congruences:

$$x \equiv 2 \pmod{4}$$
$$x \equiv 5 \pmod{7}$$
$$x \equiv 0 \pmod{11}.$$

Finally, we solve this system:

$$x \equiv -1430 \pmod{4 \cdot 7 \cdot 11}$$
$$x \equiv 8 \qquad \pmod{15}.$$

Note that $u = 2$ and $v = -41$ satisfy $308u + 15v = 1$.* So by the Solution Algorithm (with $a = -1430$, $m = 4 \cdot 7 \cdot 11 = 308$, $b = 8$, $n = 15$), a solution is

$$bmu + anv = 8 \cdot 308 \cdot 2 + (-1430) \cdot 15 \cdot (-41) = 884{,}378.$$

You can verify that $x = 884{,}378$ is a solution of the entire system

$$x \equiv 2 \pmod{4}$$
$$x \equiv 5 \pmod{7}$$
$$x \equiv 0 \pmod{11}$$
$$x \equiv 8 \pmod{15}.$$

Since $4 \cdot 7 \cdot 11 \cdot 15 = 4620$ and $884{,}378 \equiv 1958 \pmod{4620}$, as you can easily verify, $x = 1958$ is also a solution of the system by Theorem 14.2. When working by hand, the smaller solution is easier to use. So we say that the solutions of the system are all numbers that are congruent to 1958 modulo 4620.

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*The values for u and v were found with a graphing calculator program; see the Technology Tip on page 12.

Technology Tip: Systems such as the one in Example 2 can be solved by the Chinese Remainder Theorem program for TI graphing calculators that can be downloaded from our website (ADDRESS TBA). In Example 2, when asked, you enter the list of constants {2, 5, 0, 8} and the corresponding list of moduli {4, 7, 11, 15}. The program then produces the solution, as shown in Figure 1.

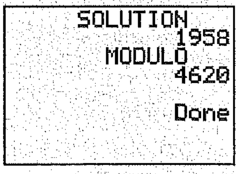

FIGURE 1

To solve the same system with Maple, use the command
chrem ([2, 5, 0, 8], [4, 7, 11, 15]); .

Exercises

A. 1. If $u \equiv v \pmod{n}$ and u is a solution of $6x + 5 \equiv 7 \pmod{n}$, then show that v is also a solution. [*Hint:* Theorem 2.2.]

2. If $6x + 5 \equiv 7 \pmod{n}$ has a solution, show that one of the numbers $1, 2, 3, \ldots, n - 1$ is also a solution. [*Hint:* Exercise 1 and Corollary 2.5.]

3. Show that $6x + 5 \equiv 7 \pmod{3}$ has no solutions. [*Hint:* Exercise 2.]

4. Show that $6x + 5 \equiv 7 \pmod{5}$ has infinitely many solutions. [*Hint:* Exercises 1 and 2.]

5. If $m_1, m_2, \ldots, m_k, m_{k+1}$ are pairwise relatively prime positive integers (that is, $(m_i, m_j) = 1$ when $i \neq j$), prove that $m_1 m_2 \cdots m_k$ and m_{k+1} are relatively prime. [*Hint:* If they aren't, then some prime p divides both of them (Why?). Use Corollary 1.6 to reach a contradiction.]

6. If $(m, n) = 1$ and $m \mid d$ and $n \mid d$, prove that $mn \mid d$. [*Hint:* If $d = mk$, then $n \mid mk$; use Theorem 1.4.]

7. Let $m_1, m_2, \ldots, m_r$ be pairwise relatively prime positive integers (that is, $(m_i m_j) = 1$ when $i \neq j$). Assume that $m_i \mid d$ for each i. Prove that $m_1 m_2 m_3 \cdots m_r \mid d$. [*Hint:* Use Exercises 5 and 6 repeatedly.]

In Exercises 8–13, solve the system of congruences.

8. $x \equiv 5 \pmod{6}$
$x \equiv 7 \pmod{11}$

9. $x \equiv 3 \pmod{11}$
$x \equiv 4 \pmod{17}$

10. $x \equiv 1 \pmod{2}$
$x \equiv 2 \pmod{3}$
$x \equiv 3 \pmod{5}$

11. $x \equiv 2 \pmod{5}$
$x \equiv 0 \pmod{6}$
$x \equiv 3 \pmod{7}$

12. $x \equiv 1 \pmod{5}$
$x \equiv 3 \pmod{6}$
$x \equiv 5 \pmod{11}$
$x \equiv 10 \pmod{13}$

13. $x \equiv 1 \pmod{7}$
$x \equiv 6 \pmod{11}$
$x \equiv 0 \pmod{12}$
$x \equiv 9 \pmod{13}$
$x \equiv 0 \pmod{17}$

B. 14. (Ancient Chinese Problem) A gang of 17 bandits stole a chest of gold coins. When they tried to divide the coins equally among themselves, there were three left over. This caused a fight in which one bandit was killed. When the remaining bandits tried to divide the coins again, there were ten left over. Another fight started, and five of the bandits were killed. When the survivors divided the coins, there were four left over. Another fight ensued in which four bandits were killed. The survivors then divided the coins equally among themselves, with none left over. What is the smallest possible number of coins in the chest?

15. If $(a, n) = d$ and $d \mid b$, show that $ax \equiv b \pmod{n}$ has a solution. [*Hint:* $b = dc$ for some c, and $au + nv = d$ for some u, v (Why?). Multiply the last equation by c; what is auc congruent to modulo n?]

16. If $(a, n) = d$ and $d \nmid b$, show that $ax \equiv b \pmod{n}$ has no solutions.

17. If $(a, n) = 1$ and s, t are solutions of $ax \equiv b \pmod{n}$, prove that $s \equiv t \pmod{n}$. [*Hint:* Show that $n \mid (as - at)$ and use Theorem 1.4.]

18. If $(a, n) = d$ and s, t are solutions of $ax \equiv b \pmod{n}$, prove that $s \equiv t \pmod{n/d}$.

19. If $(m, n) \equiv d$, prove that the system

$$x \equiv a \pmod{m}$$
$$x \equiv b \pmod{n}$$

has a solution if and only if $a \equiv b \pmod{d}$.

20. If s, t are solutions of the system in Exercise 19, prove that $s \equiv t \pmod{r}$, where r is the least common multiple of m and n.

21. (Alternate Proof of part (1) of the Chinese Remainder Theorem) For each $i = 1, 2, \ldots, r$, let N_i be the product of all the m_j *except* m_i, that is,

$$N_i = m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_r.$$

(a) For each i, show that $(N_i, m_i) = 1$, and that there are integers u_i and v_i such that $N_i u_i + m_i v_i = 1$.

(b) For each i and j such that $i \neq j$, show that $N_i u_i \equiv 0 \pmod{m_j}$.

(c) For each i, show that $N_i u_i \equiv 1 \pmod{m_i}$.

(d) Show that $t = a_i N_1 u_1 + a_2 N_2 u_2 + a_1 N_1 u_3 + \cdots + a_r N_r u_r$ is a solution of the system.

14.2 Applications of the Chinese Remainder Theorem

Every computer has a limit on the size of integers that can be used in machine arithmetic, called the **word size.** In a large computer this might be 2^{35}. Computer arithmetic with integers larger than the word size requires time-consuming multiprecision techniques. In such cases an alternate method of addition and multiplication, based on the Chinese Remainder Theorem, is often faster.

For any numbers r, s, t, n less than the word size, a large computer can quickly calculate

$r + s$ and $r \cdot s$ (even when the answer is larger than the word size);

the least residue of t modulo n* (including the case when t exceeds the word size—see Exercise 2);

sums and products in $\mathbb{Z}_n$.

Finally, a computer can use a slight variation of the Chinese Remainder Theorem solution algorithm (Theorem 14.2) to solve systems of congruences. But this may involve numbers larger than the word size and, hence, require slower multiprecision techniques.

To get an idea of how the alternate method works, imagine that the word size of our computer is 100, so that multiprecision techniques must be used for larger numbers. The following example shows how to multiply two four-digit numbers on such a computer, with minimal use of multiprecision techniques.

EXAMPLE 1

We shall multiply 3456 by 7982 by considering various systems of congruences and using the Chinese Remainder Theorem. We begin by choosing several numbers as moduli and finding the least residues of 3456 and 7982 for each modulus:†

$$(*)\qquad \begin{array}{ll} 3456 \equiv 74 \pmod{89} & 7982 \equiv 61 \pmod{89} \\ 3456 \equiv 36 \pmod{95} & 7982 \equiv 2 \pmod{95} \\ 3456 \equiv 61 \pmod{97} & 7982 \equiv 28 \pmod{97} \\ 3456 \equiv 26 \pmod{98} & 7982 \equiv 44 \pmod{98} \\ 3456 \equiv 90 \pmod{99} & 7982 \equiv 62 \pmod{99}. \end{array}$$

Then by Theorem 2.2 we know that $3456 \cdot 7982 \equiv 74 \cdot 61 \pmod{89}$. Taking the least residue of $74 \cdot 61$ modulo 89 and proceeding in similar fashion for the other congruences, we have

*The **least-residue modulo** n of a number t is the remainder r when t is divided by n. By the Division Algorithm, $t = nq + r$ so that $t - r = nq$ and $t \equiv r \pmod{n}$.

†The reason why 89, 95, 97, 98, and 99 were chosen as moduli will be explained below.

$$(**)\qquad \begin{aligned} 3456 \cdot 7982 &\equiv 74 \cdot 61 \equiv 64 \pmod{89} \\ 3456 \cdot 7982 &\equiv 36 \cdot 2 \ \ \equiv 72 \pmod{95} \\ 3456 \cdot 7982 &\equiv 61 \cdot 28 \equiv 59 \pmod{97} \\ 3456 \cdot 7982 &\equiv 26 \cdot 44 \equiv 66 \pmod{98} \\ 3456 \cdot 7982 &\equiv 90 \cdot 62 \equiv 36 \pmod{99}. \end{aligned}$$

Therefore, $3456 \cdot 7982$ is a solution of this system:

$$(***)\qquad \begin{aligned} x &\equiv 64 \pmod{89} \\ x &\equiv 72 \pmod{95} \\ x &\equiv 59 \pmod{97} \\ x &\equiv 66 \pmod{98} \\ x &\equiv 36 \pmod{99}. \end{aligned}$$

The Chinese Remainder Theorem* shows that one solution of $(***)$ is 27,585,792 and that every solution (including $3456 \cdot 7982$) is congruent to this one modulo $89 \cdot 95 \cdot 97 \cdot 98 \cdot 99 = 7{,}956{,}949{,}770$ (which we denote hereafter by M). Since no two numbers between 0 and M can be congruent modulo M, 27,585,792 is the *only* solution between 0 and M. We know that $0 < 3456 \cdot 7982 < 10^4 \cdot 10^4 = 10^8 < M$. Since $3456 \cdot 7982$ is a solution, we must have $3456 \cdot 7982 = 27{,}585{,}792$.

Now look at this example from a different perspective. If you think of the least residue of a number modulo n as an element of $\mathbb{Z}_n$, then the congruences in $(*)$ say that the integer 3456 may be represented by the element (74, 36, 61, 26, 90) in the ring $\mathbb{Z}_{89} \times \mathbb{Z}_{95} \times \mathbb{Z}_{97} \times \mathbb{Z}_{98} \times \mathbb{Z}_{99}$. Similarly, 7982 is represented by (61, 2, 28, 44, 62). Saying that $74 \cdot 61 \equiv 64 \pmod{89}$ in $(**)$ is the same as saying $74 \cdot 61 = 64$ in $\mathbb{Z}_{89}$. So the congruences in $(**)$ are equivalent to multiplication in $\mathbb{Z}_{89} \times \mathbb{Z}_{95} \times \mathbb{Z}_{97} \times \mathbb{Z}_{98} \times \mathbb{Z}_{99}$:

$$\begin{aligned} (74, 36, 61, 26, 90) \cdot (61, 2, 28, 44, 62) &= (74 \cdot 61, 36 \cdot 2, 61 \cdot 28, 26 \cdot 44, 90 \cdot 62) \\ &= (64, 72, 59, 66, 36). \end{aligned}$$

The solution of $(***)$ shows that the element (64, 72, 59, 66, 36) of the ring $\mathbb{Z}_{89} \times \mathbb{Z}_{95} \times \mathbb{Z}_{97} \times \mathbb{Z}_{98} \times \mathbb{Z}_{99}$ represents the integer 27,585,792.

The procedure in the case of a realistic word size is now clear. Let $m_1, \ldots, m_r$ be pairwise relatively prime positive integers:

1. Represent each integer t as an element of $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r}$ by taking the congruence class of t modulo each m_i.
2. Do the arithmetic in $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r}$.
3. Use the Chinese Remainder Theorem to convert the answer into integer form.

The m_i must be chosen so that their product M is larger than any number that will result from the computations. Otherwise, the conversion process in Step 3 may fail (Exercises 3–5). This is sometimes done, as in the example, by taking the m_i to be as

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*Up to this point, all computations have been quickly performed by our imaginary computer. This is the first place where slower multiprecision calculations may be needed because of numbers that exceed the word size.

large as possible without exceeding the word size of the computer. If smaller moduli are chosen, more of them may be necessary to ensure that M is large enough.

The conversion process from integer to modular representation and back (Steps 1 and 3) requires time that is not needed in conventional integer multiplication (especially Step 3, which may involve multiprecision techniques). But this need be done only once for each number, at input and output. The modular representation may be used for all intermediate calculations. It is much faster than direct computation with large integers, especially in a computer with parallel processing capability, which can work simultaneously in each $\mathbb{Z}_{m_i}$. Under appropriate conditions the speed advantage in Step 2 outweighs the disadvantage of the extra time required for Steps 1 and 3. For more details, see Knuth [31].

It is sometimes necessary to find an exact solution (not a decimal approximation) of a system of linear equations. When there are hundreds of equations or unknowns in the system and the coefficients are large integers, the usual computer methods will produce only approximate solutions because they round off very large numbers during the intermediate calculations. The Chinese Remainder Theorem is the basis of a method of finding exact solutions of such systems.

Very roughly, the idea is this. Let $m_1, \ldots, m_r$ be distinct primes (and, hence, pairwise relatively prime).* For each m_i, translate the given system of equations into a system over $\mathbb{Z}_{m_i}$ by replacing the integer coefficients by their congruence classes modulo m_i. Then solve each of these new systems by the usual methods (Gauss-Jordan elimination works equally well over the field $\mathbb{Z}_{m_i}$ as over $\mathbb{R}$, and round-off is not a problem with the smaller numbers in $\mathbb{Z}_{m_i}$). Finally, use the Chinese Remainder Theorem and matrix algebra to convert these solutions modulo m_i into a solution of the original system.†

Exercises

A. **1.** Assume that your computer has word size 100. Use the method outlined in the text to find the sum 123,684 + 413,456, using $m_1 = 95$, $m_2 = 97$, $m_3 = 98$, $m_4 = 99$.

2. **(a)** Find the least residue of 64,397 modulo 12, using only arithmetic in $\mathbb{Z}_{12}$. [*Hint:* Use Theorems 2.2 and 2.3 and the fact that $64{,}397 = (((6 \cdot 10 + 4)10 + 3)10 + 9)10 + 7$.]

(b) Let n be a positive integer less than the word size of your computer and t any integer (possibly larger than the word size). Explain how you might find the least residue of t modulo n, using only arithmetic in $\mathbb{Z}_n$ (and thus avoiding the need for multiprecision methods).

*Considerations of size similar to those discussed above play a role in the selection of the m_i.

†This conversion is a bit trickier than may first appear. For instance, the system

$$\begin{aligned} 8x + 5y &= 12 \\ 4x + 5y &= 10 \end{aligned} \quad \text{becomes} \quad \begin{aligned} x + 5y &= 5 \\ 4x + 5y &= 3 \end{aligned} \quad \text{over } \mathbb{Z}_7.$$

You can verify that $x = 4$, $y = 3$ is a solution of the $\mathbb{Z}_7$ system. It is not immediately clear how to get from this to the solution of the original system, which is $x = 1/2$, $y = 8/5$.

3. Use the method outlined in the text to represent 7 and 8 as elements of $\mathbb{Z}_3 \times \mathbb{Z}_5$. Show that the product of these representatives in $\mathbb{Z}_3 \times \mathbb{Z}_5$ is (2, 1). If you use the Chinese Remainder Theorem as in the text to convert (2, 1) to integer form, do you get 56? Why not? This example shows why the method won't work when the product of the m_i is less than the answer to the arithmetic problem in question. Also see Exercise 5.

B. 4. Let $f:\mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5$ be given by $f(t) = ([t]_3, [t]_4, [t]_5)$, where $[t]_n$ is the congruence class of t in $\mathbb{Z}_n$. The function f may be thought of as representing t as an element of $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5$ by taking its least residues.

(a) If $0 \leq r, s < 60$, prove that $f(r) = f(s)$ if and only if $r = s$. [*Hint:* Theorem 14.2.]

(b) Give an example to show that if r or s is greater than 60, then part (a) may be false.

5. Let $m_1, m_2, \ldots, m_r$ be pairwise relatively prime positive integers and $f:\mathbb{Z} \to \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$, the function given by

$$f(t) = ([t]_{m_1}, [t]_{m_2}, \ldots, [t]_{m_r}),$$

where $[t]_{m_i}$ is the congruence class of t in $\mathbb{Z}_{m_i}$. Let $M = m_1 m_2 \cdots m_r$. If $0 \leq r, s < M$, prove that $f(r) = f(s)$ if and only if $r = s$. [Exercise 4 is a special case.]

6. Assume Exercise 7(c). If your computer has word size 2^{35}, what m_i might you choose in order to do arithmetic with integers as large as 2^{184} (approximately 2.45×10^{55})?

C. 7. (a) If a and b are positive integers, prove that the least residue of $2^a - 1$ modulo $2^b - 1$ is $2^r - 1$, where r is the least residue of a modulo b.

(b) If a and b are positive integers, prove that the greatest common divisor of $2^a - 1$ and $2^b - 1$ is $2^t - 1$, where t is the gcd of a and b. [*Hint:* Use the Euclidean Algorithm and part (a).]

(c) Let a and b be positive integers. Prove that $2^a - 1$ and $2^b - 1$ are relatively prime if and only if a and b are relatively prime.

14.3 The Chinese Remainder Theorem for Rings

The Chinese Remainder Theorem for two congruences can be extended from $\mathbb{Z}$ to other rings by expressing it in terms of ideals. The key to doing this is the definition of congruence modulo an ideal (Section 6.1) and the following fact: When A and B are ideals in a ring R, the set of sums $\{a + b \mid a \in A, b \in B\}$ is denoted $A + B$ and is itself an ideal (Exercise 20 of Section 6.1).

Let m and n be integers. Let I be the ideal of all multiples of m in $\mathbb{Z}$ and J the ideal of all multiples of n. Then *congruence modulo m is the same as congruence modulo the ideal I.* If $(m, n) = 1$, then $mu + nv = 1$ for some $u, v \in \mathbb{Z}$. Multiplying this equation by any integer r shows that $m(ur) + n(vr) = r$. Thus every integer is the sum of a multiple of m and a multiple of n, that is, the sum of an element of the ideal I and an element of the ideal J. Therefore, $I + J$ is the entire ring $\mathbb{Z}$. *So the condition $(m, n) = 1$ amounts to saying $I + J = \mathbb{Z}$.*

When $(m, n) = 1$, the intersection of the ideals I and J is the ideal consisting of all multiples of mn (Exercise 6 of Section 14.1). So *two integers are congruent modulo mn precisely when they are congruent modulo the ideal* $I \cap J$.

The italicized statements in the preceding paragraphs tell us how to translate the Chinese Remainder Theorem for two congruences into the language of ideals. By replacing the ideals in that discussion by ideals in *any* ring R, we obtain

Theorem 14.3 Chinese Remainder Theorem for Rings

Let I and J be ideals in a ring R such that $I + J = R$. Then for any $a, b \in R$, the system

$$x \equiv a \pmod{I}$$
$$x \equiv b \pmod{J}$$

has a solution. Any two solutions of the system are congruent modulo $I \cap J$.

When R has an identity, the theorem can be extended to the case of r ideals $I_1, I_2, \ldots, I_r$ and congruences $x \equiv a_k \pmod{I_k}$, under the hypotheses that $I_i + I_j = R$ whenever $i \neq j$ (see Exercise 6 and Hungerford [5; p. 131]).

Proof of Theorem 14.3 ▸ Since $I + J = R$ and $b - a \in R$, there exist $i \in I, j \in J$ such that $i + j = b - a$. Hence, $a + i = b - j$. Let $t = a + i$; then

$$t - a = (a + i) - a = i \in I,$$

so that $t \equiv a \pmod{I}$. Similarly, since $a + i = b - j$

$$t - b = (a + i) - b = (b - j) - b = -j \in J.$$

Hence, $t \equiv b \pmod{J}$, and t is a solution of the system. If z is also a solution, then

$$z \equiv a \pmod{I} \qquad \text{and} \qquad t \equiv a \pmod{I} \qquad \text{imply that} \qquad z \equiv t \pmod{I}$$

by Theorem 6.4. Similarly, $z \equiv t \pmod{J}$. This means that $z - t \in I$ and $z - t \in J$. Therefore, $z - t \in I \cap J$ and $z \equiv t \pmod{I \cap J}$. ∎

One consequence of the Chinese Remainder Theorem is a useful isomorphism of rings.

Theorem 14.4

If I and J are ideals in a ring R and $I + J = R$, then there is an isomorphism of rings

$$R/(I \cap J) \cong R/I \times R/J.$$

Proof ▸ Define a map $f: R \to R/I \times R/J$ by $f(r) = (r + I, r + J)$. Then f is a homomorphism because

$$\begin{aligned} f(r) + f(s) &= (r + I, r + J) + (s + I, s + J) \\ &= ((r + s) + I, (r + s) + J) = f(r + s) \end{aligned}$$

and

$$\begin{aligned} f(r)f(s) &= (r + I, r + J)(s + I, s + J) \\ &= (rs + I, rs + J) = f(rs). \end{aligned}$$

To show that f is surjective, let $(a + I, b + J) \in R/I \times R/J$. We must find an element of R whose image under f is $(a + I, b + J)$. By Theorem 14.3 there is a solution $t \in R$ for this system:

$$\begin{aligned} x &\equiv a \pmod{I} \\ x &\equiv b \pmod{J}. \end{aligned}$$

But $t \equiv a \pmod{I}$ implies that $t + I = a + I$ by Theorem 6.6. Similarly, $t \equiv b \pmod{J}$ implies $t + J = b + J$, so that

$$f(t) = (t + I, t + J) = (a + I, b + J).$$

Therefore, f is surjective.

Let K be the kernel of f. By the First Isomorphism Theorem 6.13, R/K is isomorphic to $R/I \times R/J$. Now K consists of all elements $r \in R$ such that $f(r)$ is the zero element in $R/I \times R/J$, that is, all r such that

$$(r + I, r + J) = (0_R + I, 0_R + J),$$

or equivalently,

$$r + I = 0_R + I \qquad \text{and} \qquad r + J = 0_R + J.$$

But $r + I = 0_R + I$ means that $r \equiv 0_R \pmod{I}$, and, hence, $r \in I$. Similarly, $r + J = 0_R + J$ implies $r \in J$. Therefore, $r \in I \cap J$. So $I \cap J$ is the kernel of f, and $R/(I \cap J) = R/\mathrm{Ker}\, f \cong R/I \times R/J$. ■

Corollary 14.5

If $(m, n) = 1$, then there is an isomorphism of rings $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

Proof ▸ In the ring $\mathbb{Z}$, the ideal (m) consists of all multiples of m and the ideal (n) of all multiples of n. The first three paragraphs of this section show that $(m) + (n) = \mathbb{Z}$ and that $(m) \cap (n)$ is the ideal (mn) of all multiples of mn. Furthermore, the quotient rings $\mathbb{Z}/(mn)$, $\mathbb{Z}/(m)$, and $\mathbb{Z}/(n)$ are, respectively, $\mathbb{Z}_{mn}$, $\mathbb{Z}_m$, and $\mathbb{Z}_n$. Therefore, by Theorem 14.4 (with $R = \mathbb{Z}$, $I = (m)$, $J = (n)$) there is an isomorphism

$$\mathbb{Z}_{mn} = \mathbb{Z}/(mn) = \mathbb{Z}/((m) \cap (n)) \cong \mathbb{Z}/(m) \times \mathbb{Z}/(n) = \mathbb{Z}_m \times \mathbb{Z}_n. \quad \blacksquare$$

Corollary 14.6

If $n = p_1^{n_1}p_2^{n_2}p_3^{n_3}\cdots p_t^{n_t}$, where the p_i are distinct positive primes and each $n_i > 0$, then there is an isomorphism of rings

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \mathbb{Z}_{p_3^{n_3}} \times \cdots \times \mathbb{Z}_{p_t^{n_t}}.$$

Proof ▸ Since the p_j are distinct primes, $p_i^{n_i}$ and the product $p_{i+1}^{n+1}\cdots p_t^{n_t}$ are relatively prime for each i. So repeated use of Corollary 14.5 shows that

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}p_3^{n_3}\cdots p_t^{n_t}} \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \mathbb{Z}_{p_3^{n_3}\cdots p_t^{n_t}} \cong \cdots$$
$$\cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \mathbb{Z}_{p_3^{n_3}} \times \cdots \times \mathbb{Z}_{p_t^{n_t}}. \quad \blacksquare$$

Exercises

A. 1. (a) Show that $\mathbb{Z}_5 \times \mathbb{Z}_{12}$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{20}$.

(b) Is $\mathbb{Z}_4 \times \mathbb{Z}_{35}$ isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_{28}$?

2. If I and J are ideals in a ring R and $a \in I$, $b \in J$, show that $ab \in I \cap J$.

B. 3. If $(m, n) \neq 1$, show that $\mathbb{Z}_{mn}$ is *not* isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$. [*Hint:* If $(m, n) = d$, then $\dfrac{mn}{d}$ is an integer (Why?). If there were an isomorphism, then $1 \in \mathbb{Z}_{mn}$ would be mapped to $(1, 1) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Reach a contradiction by showing that $\dfrac{mn}{d} \cdot 1 \neq 0$ in $\mathbb{Z}_{mn}$, but $\dfrac{mn}{d} \cdot (1, 1) = (0, 0)$ in $\mathbb{Z}_m \times \mathbb{Z}_n$.]

4. Which of the following rings are isomorphic: $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_7$, $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_7$, $\mathbb{Z}_{84}$, $\mathbb{Z}_7 \times \mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{14}$, $\mathbb{Z}_4 \times \mathbb{Z}_{21}$?

5. If I_1, I_2, I_3 are ideals in a ring R with identity such that $I_1 + I_3 = R$ and $I_2 + I_3 = R$, prove that $(I_1 \cap I_2) + I_3 = R$. [*Hint:* If $r \in R$, then $r = i_1 + i_3$ and $1_R = t_2 + t_3$ for some $i_1 \in I_1$, $t_2 \in I_2$, and $i_3, t_3 \in I_3$. Then $r = (i_1 + i_3)(t_2 + t_3)$; multiply this out to show that r is in $(I_1 \cap I_2) + I_3$. Exercise 2 may be helpful.]

6. Let I_1, I_2, I_3 be ideals in a ring R with identity such that $I_i + I_j = R$ whenever $i \neq j$. If $a_i \in R$, prove that the system

$$x \equiv a_1 \pmod{I_1}$$
$$x \equiv a_2 \pmod{I_2}$$
$$x \equiv a_3 \pmod{I_3}$$

has a solution and that any two solutions are congruent modulo $I_1 \cap I_2 \cap I_3$. [*Hint:* If s is a solution of the first two congruences, use Exercise 5 and Theorem 14.3 to show that the system

$$\begin{aligned} x &\equiv s \pmod{I_1 \cap I_2} \\ x &\equiv a_3 \pmod{I_3} \end{aligned}$$

has a solution, and it is a solution of the original system.]

CHAPTER 15

Geometric Constructions

Prerequisites: Sections 4.1, 4.4, and 4.5.

Since the sixth century B.C., mathematicians have studied geometric constructions with straightedge (unmarked ruler) and compass. Despite their prowess in geometry, the ancient Greeks were never able to perform certain constructions using only straightedge and compass, such as

> ***Duplication of the Cube:*** Construct the edge of a cube having twice the volume of a given cube.*
>
> ***Trisection of the Angle:*** Construct an angle one third the size of a given angle.
>
> ***Squaring the Circle:*** Construct a square whose area is equal to the area of a given circle.

Finally in the last century it was proved that each of these constructions is impossible. This chapter presents an elementary proof of the impossibility of the first two constructions listed above (the third is discussed in Exercise 21).

Many people remain fascinated by these problems, particularly angle trisection, and continue to publish what they say are "solutions," even though it has been proved that there are none (see, for example Dudley [37]). Consequently, it is important to understand just what we claim is impossible here and what constitutes a proof.

The ancient Greeks knew that all the constructions listed above could readily be carried out provided that additional tools were permitted. For instance, any angle can be trisected using a compass and straightedge with just one mark on it. The Greeks also

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*This problem supposedly had its origin in an ancient legend: Athens was afflicted by a plague and its people were told by the oracle at Delos that the plague would end when they built a new altar to Apollo in the shape of a cube that had twice the volume of the old altar, which was also a cube.

knew that some angles, such as 90°, *can* be trisected by straightedge and compass alone (Exercise 3). So the issue is not whether these constructions can ever be performed, but whether they can be performed in *every* possible case using *only* an (unmarked) straightedge and a compass. Furthermore, physical measurement alone is not sufficient to justify such constructions because no measuring device is absolutely accurate. Justification requires a valid mathematical proof based on accepted principles and the rules of logic.

The key to the impossibility proofs presented here (and to every other known proof of these facts) is to translate the geometric problem into an equivalent algebraic one. Under this translation process, as we shall see, constructions with a straightedge correspond to solving linear equations and constructions with a compass to solving quadratic equations. Before we can begin this translation process, we present a typical straightedge-and-compass construction to give you a feel for what we are dealing with.

EXAMPLE 1

Given points O and P, construct a line perpendicular to line OP through O as follows. Construct the circle with center O and radius OP; it intersects line OP at points R and P, as shown on the left side of Figure 1. Segments OR and OP are radii of the circle and thus have the same length. Now construct the circle with center R and radius RP and the circle with center P and radius RP. These circles intersect in points A and B as shown in the center of Figure 1. Segments RP, RA, and PA have the same length. (Why?)

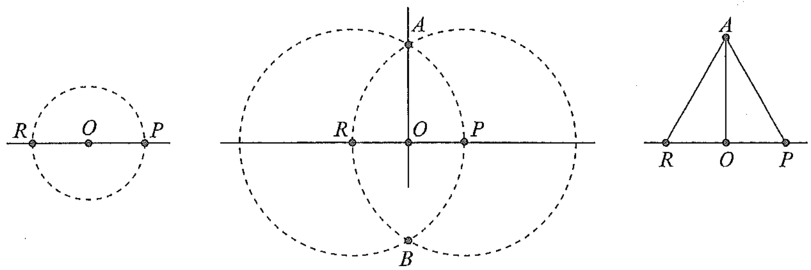

FIGURE 1

Draw the line AO. In triangle RAP, shown on the right of Figure 1, the sides RA and PA are congruent, as are the sides OR and OP. Side OA is congruent to itself. Therefore, triangles ORA and OPA are congruent by side-side-side. Since angles ROA and POA are congruent and supplementary, each of them must be a right angle. Therefore, line AO is perpendicular to line OP at O.

Outline of the Argument

Now we begin the translation from geometry to algebra. The following outline should help you to see where we're headed and to keep things straight as we go along. The capitalized headings here correspond to the headings on the subsections below.

CONSTRUCTIBLE POINTS We begin with any two points and determine what additional points can be constructed from them by straightedge-and-compass constructions; these are the *constructible points.* Next we use the distance between the original two points as the unit length and coordinatize the plane.

CONSTRUCTIBLE NUMBERS A number r is said to be *constructible* if the point $(r, 0)$ is a constructible point. We then examine the equations of lines and circles determined by constructible points and the coordinates of their intersection points. This leads to a characterization of constructible numbers in terms of certain subfields of $\mathbb{R}$ and square roots of positive elements of $\mathbb{R}$.

ROOTS OF POLYNOMIALS The characterization of constructible numbers is then used to show that certain cubic polynomials have no constructible numbers as roots.

IMPOSSIBILITY PROOFS Finally, we demonstrate the impossibility of the constructions in question by using proof by contradiction: If the construction were possible, then one of the cubic polynomials mentioned in the preceding paragraph would have a constructible number as a root, which is a contradiction.

Constructible Points

We first give a formal mathematical description of straightedge-and-compass constructions, such as those in Example 1, that begin with two points O and P. Let S be the set $\{O, P\}$. Form the line determined by the two points of S. Form the two circles with centers O and P and radius OP. Let S_1 be the set of all points of intersection of this line and these circles, together with the points O, P in the original set S. Repeat this process with S_1. Form every line determined by pairs of points in S_1. Form every circle whose radius is the distance between some pair of points in S_1 and whose center is a point in S_1. Let S_2 be the set of all points of intersection of these lines and circles, together with the points in S_1. Repeat the process with S_2. Continuing in this way produces a sequence of sets

$$S \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$$

A **constructible point** is any point that lies in some S_i. A **constructible line** is a line that contains at least two constructible points. A **constructible circle** is one whose center is a constructible point and whose radius has length equal to the distance between some pair of constructible points. For example, all the labeled points and all the lines and circles in Figure 1 are constructible. Note that points of intersection of constructible lines and circles are constructible points.

Now we coordinatize the plane by taking O as the origin, the distance from O to P as the unit length, and the line OP as the x-axis, and P having coordinates $(1, 0)$. Figure 1 shows that the y-axis (the line AO) is a constructible line. The point $(0, 1)$ is constructible since it is the intersection of the y-axis and the constructible circle with center O and radius OP. A similar argument shows that

$(r, 0)$ is constructible if and only if $(0, r)$ is constructible.

Constructible Numbers

A real number r is said to be a **constructible number** if the point $(r, 0)$ is a constructible point. Every integer is a constructible number (Exercise 4). If r is the distance between two constructible points A and B, then r is a constructible number because $(r, 0)$ is the intersection of the constructible x-axis and the constructible circle with center O and radius r. Exercise 18 shows that

a point is constructible if and only if its coordinates are constructible numbers.

Theorem 15.1

Let a, b, c, d be constructible numbers with $c \neq 0$ and $d > 0$. Then each of $a + b$, $a - b$, ab, a/c, and $\sqrt{d}$ is a constructible number.

Proof ▸ We first assume a and c are positive and show that a/c is a constructible number. Since a and c are constructible numbers, the points $(a, 0)$ and $(0, c)$ are constructible and so is the line L they determine. The line through the constructible point $(0, 1)$ parallel to L is constructible (Exercise 19). It intersects the x-axis at the constructible point $(x, 0)$, as shown on the left side of Figure 2. Hence, x is a constructible number. Use similar triangles to show that $\frac{1}{c} = \frac{x}{a}$, which implies that $x = a/c$. When $a = 0$ or when a or c is negative, Exercise 13 shows that a/c is a constructible.

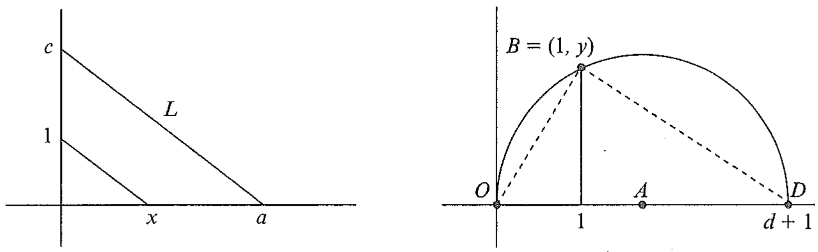

FIGURE 2

If $b = 0$, then $ab = 0$ is certainly constructible. If $b \neq 0$, then $1/b$ is constructible by the previous paragraph, and hence $a/(1/b) = ab$ is also constructible. Exercise 2 shows that $a + b$ and $a - b$ are constructible.

The number $d + 1$ is constructible by Exercise 2. So the midpoint A of the line segment joining the constructible points $(0, 0)$ and $(d + 1, 0)$ is constructible (Exercise 20). Hence, the circle with center A and radius $(d + 1)/2$ is constructible. The constructible line that is perpendicular to the x-axis at the point $(1, 0)$ intersects this circle at the constructible point $B = (1, y)$, as shown on the right of Figure 2. A theorem in plane geometry states that an angle that is inscribed in a semi-circle (such as OBD) is a right angle. Use the three right triangles on the right side of Figure 2 and the Pythagorean Theorem to show that $y^2 = d$ and, therefore, $y = \sqrt{d}$. It follows that $y = \sqrt{d}$ is a constructible number. ■

Corollary 15.2

Every rational number is constructible.

Proof ▸ Every integer is constructible (Exercise 4). Therefore, every quotient of a pair of integers (rational number) is constructible by Theorem 15.1. ■

In order to determine exactly which real numbers are constructible, we must examine the equations of constructible lines and circles.

Lemma 15.3

Let F be a subfield of the field $\mathbb{R}$ of real numbers.

(1) If a line contains two points whose coordinates are in F, then the line has an equation of the form

$$ax + by + c = 0, \qquad \text{where } a, b, c \in F.$$

(2) If the center of a circle is a point whose coordinates are in F and the radius of the circle is a number whose square is in F, then the circle has an equation of the form

$$x^2 + y^2 + rx + sy + t = 0, \qquad \text{where } r, s, t \in F.$$

Proof ▸ (1) Suppose (x_1, y_1) and (x_2, y_2) are points on the line with $x_i, y_i \in F$. If $x_1 \neq x_2$, the two-point formula for the equation of a line shows that the line has equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\underbrace{\left(\frac{y_2 - y_1}{x_2 - x_1}\right)}_{a} x - \underbrace{1}_{b}y + \underbrace{\left[-x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1\right]}_{c} = 0$$

$$ax + by + c = 0$$

Since F is a field and $x_i, y_i \in F$, each of a, b, c is in F. The case when $x_1 = x_2$ is left to the reader.

(2) If (x_1, y_1) is the center and k the radius, with $x_1, y_1, k^2 \in F$, then the equation of the circle is

$$(x - x_1)^2 + (y - y_1)^2 = k^2$$

$$x^2 + y^2 + (-2x_1)x + (-2y_1)y + [x_1^2 + y_1^2 - k^2] = 0.$$

The coefficients are in F. ■

Lemma 15.4

Let F be a subfield of $\mathbb{R}$ and k a positive element of F such that $\sqrt{k} \notin F$. Let $F(\sqrt{k})$ be the set $\{a + b\sqrt{k} \mid a, b \in F\}$. Then

(1) $F(\sqrt{k})$ is a subfield of $\mathbb{R}$ that contains F.

(2) Every element of $F(\sqrt{k})$ can be written uniquely in the form $a + b\sqrt{k}$, with $a, b \in F$.

Proof▸ (1) Exercise 15.

(2) If $a + b\sqrt{k} = a_1 + b_1\sqrt{k}$, with $a, b, a_1, b_1 \in F$, then $a - a_1 = (b_1 - b)\sqrt{k}$. If $b - b_1 \neq 0$, then $\sqrt{k} = (a - a_1)(b_1 - b)^{-1}$, which is an element of F. This contradicts the fact that $\sqrt{k} \notin F$. Hence, $b_1 - b_1 = 0$, and, therefore, $a - a_1 = (0)\sqrt{k} = 0$. Thus $a = a_1$ and $b = b_1$. ∎

The field $F(\sqrt{k})$ is called a **quadratic extension field** of F. Quadratic extension fields play a crucial role in determining which numbers are constructible.

Lemma 15.5

Let F be a subfield of $\mathbb{R}$. Let L_1 and L_2 be lines whose equations have coefficients in F. Let C_1 and C_2 be circles whose equations have coefficients in F. Then

(1) If L_1 intersects L_2, then the point of intersection has coordinates in F.

(2) If C_1 intersects C_2, then the points of intersection have coordinates in F or in some quadratic extension field $F(\sqrt{k})$.

(3) If L_1 intersects C_1, then the points of intersection have coordinates in F or in some quadratic extension field $F(\sqrt{k})$.

Proof▸ (1) Suppose L_1 and L_2 have equations

$$L_1 : a_1x + b_1y = c_1$$
$$L_2 : a_2x + b_2y = c_2$$

with $a_i, b_i, c_i \in F$. Since L_1 intersects L_2, these equations have a simultaneous solution. By using elimination or determinants, we see that this solution is

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Since $a_i, b_i, c_i \in F$, the point of intersection (x, y) has coordinates in the field F.

(2) Suppose C_1 and C_2 have equations

$$C_1 : x^2 + y^2 + r_1x + s_1y + t_1 = 0$$
$$C_2 : x^2 + y^2 + r_2x + s_2y + t_2 = 0$$

with $r_i, s_i, t_i \in F$. The coordinates of the intersection points satisfy both equations and, hence, must satisfy the equation obtained by subtracting the second equation from the first:

$$(r_1 - r_2)x + (s_1 - s_2)y + (t_1 - t_2) = 0.$$

This is the equation of a line, and its coefficients are in F. Since the intersection points of C_1 and C_2 lie on this line and on the circle C_1, we need only prove (3) to complete the proof of the theorem.

(3) Let L_1 and C_1 have the equations given above. At least one of a_1, b_1 must be nonzero, say $b_1 \neq 0$. Solve the equation of L_1 for y and substitute this result in the equation for C_1. Verify that this leads to an equation of the form $ax^2 + bx + c = 0$, with $a, b, c \in F$. The solutions of this equation are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = A \pm B\sqrt{k},$$

where $A = -b/2a$, $B = 1/2a$, and $k = b^2 - 4ac$ are elements of F. Since L_1 and C_1 intersect, we know that $k \geq 0$. Using the equation for L_1, we see that the coordinates of the points of intersection of L_1 and C_1 are

$$x = A + B\sqrt{k} \quad \text{and} \quad y = \frac{c_1 - a_1A}{b_1} - \frac{a_1B}{b_1}\sqrt{k}$$

$$x = A - B\sqrt{k} \quad \text{and} \quad y = \frac{c_1 - a_1A}{b_1} - \frac{a_1B}{b_1}\sqrt{k}.$$

If $k = 0$, these reduce to a single point of intersection. Since $b_1 \neq 0$, all these coordinates lie either in F (if $\sqrt{k} \in F$) or in the quadratic extension $F(\sqrt{k})$ (if $\sqrt{k} \notin F$). ■

Theorem 15.6

If a real number r is constructible, then there is a finite chain of fields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \mathbb{R}$ such that $r \in F_n$ and each F_i is a quadratic extension of the preceding field, that is,

$$F_1 = \mathbb{Q}(\sqrt{c_0}), \qquad F_2 = F_1(\sqrt{c_1}) \qquad F_3 = F_2(\sqrt{c_2}), \ldots, F_n = F_{n-1}(\sqrt{c_{n-1}}),$$

where $c_i \in F_i$ but $\sqrt{c_i} \notin F_i$ for $i = 0, 1, 2, \ldots, n - 1$.

A finite chain of fields as in the theorem is called a **quadratic extension chain.**

Proof of Theorem 15.6 ▸ Let r be a constructible number. Then the point $(r, 0)$ can be constructed from the points $O = (0, 0)$ and $P = (1, 0)$ by a finite sequence of operations of the following types:

(i) Form the line determined by A and B, where A, B are previously constructed points or elements of $\{O, P\}$;
(ii) Form the circle with center A and radius the distance from B to C, where A, B, C are previously constructed points or elements of $\{O, P\}$;
(iii) Determine the points of intersection of lines and circles formed in (i) and (ii).

This process begins with the points O and P whose coordinates are in $\mathbb{Q}$. Lines or circles determined by them will have equations with rational coefficients by Lemma 15.3. The intersections of such lines and circles will be points whose coordinates are either in $\mathbb{Q}$ or in some quadratic extension $\mathbb{Q}(\sqrt{c_0})$ by Lemma 15.5. The lines and circles determined by these points will have equations with coefficients in the field $F_1 = \mathbb{Q}(\sqrt{c_0})$ by Lemma 15.3. The intersections of such lines and circles will have coefficients either in F_1 or in some quadratic extension $F_1(\sqrt{c_1})$ by Lemma 15.5. Continuing in this fashion, we see that at each stage of the construction of $(r, 0)$ the points in question have coordinates in some field F_i and at the next stage the newly created points have coordinates in F_i or in a quadratic extension $F_i(\sqrt{c_i})$. After a finite number of such steps we reach the point $(r, 0)$, which necessarily has coordinates in the last field of the quadratic extension chain $\mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n$. ■

Roots of Polynomials

There are two ways to show that some real numbers are *not* constructible. The method presented here is elementary and depends only on Chapter 4. But if you've covered Sections 11.1 and 11.2, skip to Theorem 15.9 and use the footnote below in place of the proof given there.*

Lemma 15.7

Let F be a subfield of $\mathbb{R}$ and $f(x) \in F[x]$. Suppose that $k \in F$ but $\sqrt{k} \notin F$. If $a + b\sqrt{k}$ is a root of $f(x)$, then $a - b\sqrt{k}$ is also a root of $f(x)$.

Proof ▸ If $u = r + s\sqrt{k} \in F(\sqrt{k})$, let $\bar{u}$ denote $r - s\sqrt{k}$. This operation is well defined because every element of $F(\sqrt{k})$ can be written uniquely in the form $r + s\sqrt{k}\,(r, s \in F)$ by Lemma 15.4. Verify that for any $u, v \in F(\sqrt{k})$, $\overline{(u + v)} = \bar{u} + \bar{v}$ and $\overline{uv} = \bar{u} \cdot \bar{v}$. Also note that $u = \bar{u}$ if and only if $s = 0$, that is, if and only if $u \in F$. The rest of the proof is identical to the proof of Lemma 4.29, which is the special case when $F = \mathbb{R}$, $k = -1$, and $\sqrt{k} = i$. ■

Lemma 15.8

Let F be a subfield of a field K. Let $f(x)$, $g(x) \in F[x]$ and $h(x) \in K[x]$. If $f(x) = g(x)h(x)$, then $h(x)$ is actually in $F[x]$.

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*If $k \in F$ and $\sqrt{k} \notin F$, then $x^2 - k \in F[x]$ is the minimal polynomial of $\sqrt{k}$ over F, and, hence, $[F(\sqrt{k}):F] = 2$ by Theorem 11.7. If $\mathbb{Q} \subseteq \cdots \subseteq F_n$ is a quadratic extension chain, then $[F_n:\mathbb{Q}]$ must be a power of 2 by Theorem 11.4. Therefore, the minimal polynomial of a constructible number u has degree 2^k for some k (since this degree is the dimension $[\mathbb{Q}(u):\mathbb{Q}]$, which must divide $[F_n:\mathbb{Q}]$). Consequently, no constructible number can be the root of an irreducible cubic in $\mathbb{Q}[x]$. Since a cubic polynomial in $\mathbb{Q}[x]$ with no rational roots is irreducible by Corollary 4.19, no such polynomial can have a constructible number as a root.

Proof ▸ By the Division Algorithm in $F[x]$, there are polynomials $k(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)k(x) + r(x)$, with $r(x) = 0$ or $\deg r(x) < \deg g(x)$. Since $F \subseteq K$, all these polynomials are in $K[x]$. Now consider the Division Algorithm in $K[x]$, which says that there is a *unique* quotient and remainder. We have $f(x) = g(x)k(x) + r(x)$, and by hypothesis we also have $f(x) = g(x)h(x) + 0$. By uniqueness, we must have $r(x) = 0$ and $h(x) = k(x)$. Since $k(x) \in F[x]$, the lemma is proved. ■

Theorem 15.9

Let $f(x)$ be a cubic polynomial in $\mathbb{Q}[x]$. If $f(x)$ has no roots in $\mathbb{Q}$, then $f(x)$ has no constructible numbers as roots.

The theorem implies, for example, that $\sqrt[3]{2}$ is *not* a constructible number because it is a root of $x^3 - 2$, which has no rational roots by the Rational Root Test (Theorem 4.21).

Proof of Theorem 15.9 ▸ Suppose on the contrary that $f(x)$ has real roots that are constructible. Each such root lies in a quadratic extension chain of $\mathbb{Q}$ by Theorem 15.6. Among all the quadratic extension chains containing a root of $f(x)$, choose one of the smallest possible length, say $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n$. This means that $f(x)$ has a root r in F_n and that no quadratic extension chain of length $n - 1$ or less contains any root of $f(x)$. Note that $F_n \neq \mathbb{Q}$ since $f(x)$ has no rational roots. By the Factor Theorem 4.16 $f(x) = (x - r)t(x)$ for some $t(x) \in F_n[x]$. Now $r \in F_n$, and by the definition of a quadratic extension chain $F_n = F_{n-1}(\sqrt{k})$ for some $k \in F_{n-1}$ with $\sqrt{k} \notin F_{n-1}$. Therefore $r = a + b\sqrt{k}$ with $a, b \in F_{n-1}$. We must have $b \neq 0$; otherwise, r would be in the chain $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1}$, contradicting the fact that $f(x)$ has no roots in a chain of length $n - 1$. By Lemma 15.7 $\bar{r} = a - b\sqrt{k}$ is also a root of $f(x) = (x - r)t(x)$. Since $\bar{r} \neq r$ (because $b \neq 0$) $\bar{r}$ must be a root of $t(x)$. By the Factor Theorem

$$f(x) = (x - r)(x - \bar{r})h(x) \text{ for some } h(x) \in F_n[x].$$

Let $g(x) = (x - r)(x - \bar{r})$ and observe that the coefficients of $g(x)$ are in F_{n-1}:

$$g(x) = \left(x - \left(a + b\sqrt{k}\right)\right)\left(x - \left(a - b\sqrt{k}\right)\right) = x^2 - 2ax + (a^2 - kb^2).$$

Therefore, $f(x) = g(x)h(x)$ with $f(x), g(x) \in F_{n-1}[x]$. Consequently, $h(x) \in F_{n-1}[x]$ by Lemma 15.8. Now $f(x)$ has degree 3 and $g(x)$ has degree 2, so $h(x)$ must have degree 1 by Theorem 4.2. Since every first degree polynomial over a field has a root in that field, $h(x)$—and, hence, $f(x)$—has a root in F_{n-1}. This contradicts the choice of $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n$ as a quadratic extension chain of minimal length containing a root of $f(x)$. Therefore, $f(x)$ has no constructible numbers as roots. ■

Impossibility Proofs

Finally, we are in a position to prove the impossibility of the constructions discussed at the beginning of the chapter. In what follows, it is assumed that whenever a point, line radius, etc., may be chosen arbitrarily, a *constructible* point, line, radius, etc., will be chosen. This guarantees that all points, lines, etc., produced by the construction process will be constructible ones.

DUPLICATION OF THE CUBE Label the endpoints of one edge of the given cube as O and P and use this edge OP as the unit segment for coordinatizing the plane. Since the given cube has side length 1, its volume is also 1. If there were some way to construct with straightedge and compass the side of a cube of volume 2, then the length c of this side would be a constructible number such that $c^3 = 2$. Thus c would be a root of $x^3 - 2$. But this polynomial has no rational roots by the Rational Root Test and, hence, no constructible ones by Theorem 15.9. This contradiction shows that duplication of the cube by straightedge and compass is impossible.

TRISECTION OF THE ANGLE It suffices to prove that an angle of 60° cannot be trisected by straightedge and compass. Choose two points O, P and coordinatize the plane with O as origin and $P = (1, 0)$. The point $Q = (1/2, \sqrt{3}/2)$ is constructible since its coordinates are constructible numbers by Theorem 15.1 and Corollary 15.2. Furthermore, Q lies on the unit circle $x^2 + y^2 = 1$. Therefore, angle POQ has cosine $1/2$ (the first coordinate of Q) and, hence, has measure 60°. If it were possible to trisect this angle with straightedge and compass, there would be a finite sequence of constructions that would result in a constructible point R such that the angle ROP has measure 20°, as shown in Figure 3.

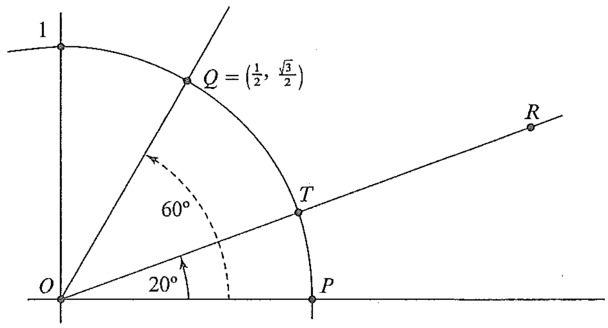

FIGURE 3

The point T where the constructible line OR meets the constructible unit circle is a constructible point. Hence, its first coordinate, which is cos 20°, is a constructible number. Therefore, 2 cos 20° is a constructible number by Theorem 15.1. But for any angle of t degrees, elementary trigonometry (Exercise 5) shows that

$$\cos 3t = 4\cos^3 t - 3\cos t.$$

If $t = 20°$, then this identity becomes

$$\cos 60° = 4\cos^3 20° - 3\cos 20°$$
$$\frac{1}{2} = 4\cos^3 20° - 3\cos 20°.$$

Multiplying by 2 and rearranging, we have

$$(2 \cos 20°)^3 - 3(2 \cos 20°) - 1 = 0.$$

Thus the supposedly constructible number 2 cos 20° is a root of $x^3 - 3x - 1$. The Rational Root Test shows that his polynomial has no rational roots and, hence, no constructible ones by Theorem 15.9. This is a contradiction. Therefore, an angle of 60° cannot be trisected by straightedge and compass.

Exercises

A. **1.** Prove that r is a constructible number if and only if $-r$ is constructible.

2. Let a, b be constructible numbers. Prove that $a + b$ and $a - b$ are constructible.

3. Use straightedge and compass to construct an angle of

(a) 30° (b) 45°

(c) Show that angles of 90° and 45° can be trisected with straightedge and compass.

4. Prove that every integer is a constructible number. [*Hint:* 1 is constructible (Why?); construct a circle with center (1, 0) and radius 1 to show 2 is constructible.]

5. Prove that $\cos 3t = 4 \cos^3 t - 3 \cos t$. [*Hint:* These identities may be helpful: (1) $\cos(t_1 + t_2) = \cos t_1 \cos t_2 - \sin t_1 \sin t_2$; (2) $\cos 2t = 2 \cos^2 t - 1$ and $\sin 2t = 2 \sin t \cos t$; (3) $\sin^2 t + \cos^2 t = 1$.]

6. Is it possible to trisect an angle of $3t$ degrees if $\cos 3t = 1/3$? What if $\cos 3t = 11/16$?

B. **7.** Consider a rectangular box with a square bottom of edge x and height y. Assume the volume of the box is 3 cubic units and its surface area is 7 square units. Can the edges of such a box be constructed with straightedge and compass?

8. Use straightedge and compass to construct a line segment of length $1 + \sqrt{3}$, beginning with the unit segment.

9. Is it possible to construct with straightedge and compass an isosceles triangle of perimeter 8 and area 1?

10. (a) Prove that the sum of two constructible angles is constructible. [A constructible angle is an angle whose sides are constructible lines.]

(b) Prove that it is impossible to construct an angle of 1° with straightedge and compass, starting with the unit segment. [*Hint:* If it were possible, what could be said about an angle of 20°?]

11. Prove that an angle of t degrees is constructible if and only if $\cos t$ is a constructible number.

12. Prove that r is a constructible number if and only if a line segment of length $|r|$ can be constructed by straightedge and compass, beginning with a segment of length 1.

13. Let a, c be constructible numbers with $c \neq 0$. Prove that a/c is constructible. [*Hint:* The case when $a > 0$, $c > 0$ was done in the proof of Theorem 15.1.]

14. Prove that the set of all constructible numbers is a field.

15. Let F be a subfield of $\mathbb{R}$ and $k \in F$. Prove that $F(\sqrt{k}) = \{a + b\sqrt{k} | a, b \in F\}$ is a subfield of $\mathbb{C}$ that contains F. If $k > 0$, show that F is a subfield of $\mathbb{R}$. [*Hint:* Adapt the hint for Exercise 39 in Section 3.1.]

16. Prove the converse of Theorem 15.6: If r is in some quadratic extension chain, then r is a constructible number. [*Hint:* Theorem 15.1 and Corollary 15.2.]

17. Let C be a constructible point and L a constructible line. Prove that the line through C perpendicular to L is constructible. [*Hint:* The case when C is on L was done in Example 1. If C is not on L and D is a constructible point on L, the circle with center C and radius CD is constructible and meets L at the constructible points D and E. The circles with center D, radius CD and center E, radius CE intersect at constructive points C and Q. Show that line CQ is perpendicular to L.]

18. Prove that (r, s) is a constructible point if and only if r and s are constructible numbers. [*Hint:* The lines through (r, s) perpendicular to the axes are constructible by Exercise 17.]

19. Let A be a constructible point not on the constructible line L. Prove that the line through A parallel to L is constructible [*Hint:* Use Exercise 17 to find a constructible line M through A, perpendicular to L. Then construct a line through A perpendicular to M.]

20. Prove that the midpoint of the line segment between two constructible points is a constructible point. [*Hint:* Adapt the hint to Exercise 17.]

C. 21. *Squaring the Circle* Given a circle of radius r, show that it is impossible to construct by straightedge and compass the side of a square whose area is the same as that of the given circle. You may assume the nontrivial fact that π is not the root of any polynomial in $\mathbb{Q}[x]$.

CHAPTER 16

Algebraic Coding Theory

Prerequisites: Section 7.4 and Appendix F for Section 16.1; Section 8.4 for Section 16.2; Section 11.6 for Section 16.3.

Coding theory deals with the fast and accurate transmission of messages over an electronic "channel" (telephone, telegraph, radio, TV, satellite, computer relay, etc.) that is subject to "noise" (atmospheric conditions, interference from nearby electronic devices, equipment failures, etc.). The noise may cause errors so that the message received is not the same as the one that was sent. The aim of coding theory is to enable the receiver to detect such errors and, if possible, to correct them.*

The use of abstract algebra to solve coding problems was pioneered by Richard W. Hamming, whose name appears several times in this chapter. In 1950 he developed a large class of error-correcting codes, some of which are presented here.

16.1 Linear Codes

Verbal messages are normally converted to numerical form for electronic transmission. When computers are involved, this is usually done by means of a binary code, in which messages are expressed as strings of 0's and 1's. Such messages are easily

*Thus coding theory has virtually no connection with the secret codes discussed in Chapter 13. The purpose of the latter was to conceal the message, whereas the purpose here is to guarantee its clarity.

handled because the internal processing units on most computers represent letters, numerals, and symbols in this way. The discussion here deals only with such binary codes.*

Throughout this chapter we assume that we have a **binary symmetric channel,** meaning that:

1. The probability of a 0 being incorrectly received as a 1 is the same as the probability of a 1 being incorrectly received as a 0;
2. The probability of a transmission error in a single digit is less than .5; and
3. Multiple transmission errors occur independently.†

Here is a simple example that gives a flavor of the subject.

EXAMPLE 1

Suppose that the message to be sent is a single digit, either 1 or 0. The message might be, for example, a signal to tell a satellite whether or not to orbit a distant planet. With a single-digit message, the receiver has no way to tell if an error has occurred. But suppose instead that a four-digit message is sent: 1111 for 1 or 0000 for 0. Then this code can correct single errors. For instance, if 1101 is received, then it seems likely that a single error has been made and that 1111 is the correct message. It's possible, of course, that three errors were made and the correct message is 0000. But this is much less likely than a single error.§ The code can *detect* double errors, but not correct them. For instance, if 1100 is received, then two errors probably have been made, but the intended message isn't clear.

Example 1 illustrates in simplified form the basic components of coding theory. The numerical *message words* (0 and 1) are translated into *codewords* (0000 and 1111). Only codewords are transmitted, but in the example any four-digit string of 0's and 1's is a possible *received word.* By comparing received words with codewords and deciding the most likely error, a *decoder* detects errors and, when possible, corrects them.** Finally, the corrected codewords are translated back to message words, or an error is signaled for received words that can't be corrected.

Now consider Example 1 from a different viewpoint. Think of the message words 0 and 1 as elements of $\mathbb{Z}_2$, and the received words as the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (with its elements written as 4-digit strings of 0's and 1's). Using Theorem 7.12, you

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*"Binary" refers to the fact that these codes are based on $\mathbb{Z}_2$. Although binary codes are the most common, other codes can be constructed by using any finite field in place of $\mathbb{Z}_2$.

†The accuracy rate of message transmission depends on these probabilities. Since elementary probability is not a prerequisite for this book, our discussion of such questions will be minimal; see Exercises 27–31.

§If the probability of receiving a wrong digit is .01, then three or four errors occur in a message word less than .0004% of the time (once in 250,000 transmissions); see Exercise 27.

This is sometimes called **maximum-likelihood decoding.

can easily verify that the set of codewords {0000, 1111} is a subgroup of order 2 of the received words, as shown schematically here:

Message Words	**Codewords**	**Received Words**
$\mathbb{Z}_2$		$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
0	0000	
1	1111	

Next, we extend these ideas to the general case. For each positive integer n,

$$\mathbf{B(n)} \quad \textbf{denotes} \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \ (\boldsymbol{n} \textbf{ copies}).$$

With coördinatewise addition, $B(n)$ is an additive group of order 2^n (Exercise 10). The elements of $B(n)$ will be written as strings of 0's and 1's of length n.

Definition

If $0 < k < n$, then an **(n, k) binary linear code** consists of a subgroup C of $B(n)$ of order 2^k.

For convenience, C is often called an (n, k) code, a linear code, or just a code.* The elements of C are called **codewords.** Only codewords are transmitted, but any element of $B(n)$ can be a **received word.**

The code in Example 1 is $C = \{0000, 1111\}$, a subgroup of order 2^1 of the group $B(4) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of order 2^4. So this is a (4, 1) code, in which the set of message words is $B(1) = \mathbb{Z}_2$. Similarly, in the general case of an (n, k) code, we shall consider $B(k) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (k copies of $\mathbb{Z}_2$), which has order 2^k to be the set of message words.

Although any method of assigning each message word to a unique code word can be used, the assignment made in Example 1 is convenient because the first digit in each code word is the corresponding message word: $0 \to 0000$ and $1 \to 1111$. The (n, k) codes discussed below have the same feature: The first k digits of an n-digit codeword form the corresponding message word.

EXAMPLE 2

We shall construct the (6, 5) **parity-check code.** The message words are the elements of $B(5)$, that is, all five-digit strings of 0's and 1's. A message word is converted to a codeword (element of $B(6)$) by adding a sixth digit to the string; the extra digit is the *sum* (in $\mathbb{Z}_2$) of the digits in the message word. For instance, if the message word is 11011, then $1 + 1 + 0 + 1 + 1 = 0$, so the corresponding codeword in $B(6)$ is 110110. Similarly, the message word $10101 \in B(5)$ has $1 + 0 + 1 + 0 + 1 = 1$, so the corresponding codeword is $101011 \in B(6)$.

An element of $B(6)$ is a codeword if and only if the sum of its digits is 0. [Reason: If the sum of the message-word digits is 0, a 0 is added to make the codeword; if the sum of the message-word digits is 1, a 1 is added for the

*Linear codes are also called **block codes** or **group codes.**

codeword and $1 + 1 = 0$; see Exercise 12 for the converse.] Using this property, it is easy to show that the set C of codewords *is* a subgroup of $B(6)$ (Exercise 13).

This code can detect single transmission errors (1 is received as 0 or 0 as 1) because the sum of the digits in the received word is 1 instead of 0. The same is true for any *odd* number of errors. But it cannot detect an even number of errors, nor can it correct any errors. For each $n \geq 2$, an $(n, n - 1)$ parity-check code can be constructed in the same way.

When retransmission of messages is easy, a parity-check code can be very useful. Such codes are frequently used in banking and in the internal arithmetic of computers. But when retransmission is expensive, difficult, or impossible, an error-correcting code is more desirable. We now develop the mathematical tools for determining the number of errors a code can detect or correct.

Definition

The **Hamming weight** of an element u of $B(n)$ is the number of nonzero coordinates in u; it is denoted $\mathrm{Wt}(u)$.

EXAMPLE 3

If $u = 11011$ in $B(5)$, then $\mathrm{Wt}(u) = 4$. Similarly, $v = 1010010 \in B(7)$ has weight 3, and 0000000 has weight 0.

Definition

Let $u, v \in B(n)$. The **Hamming distance** between u and v, denoted $d(u, v)$, is the number of coordinates in which u and v differ.*

EXAMPLE 4

If $u = 00101$ and $v = 10111$ in $B(5)$, then $d(u, v) = 2$ because u and v differ in the first and fourth coordinates. In $B(4)$ the distance between 0000 and 1111 is 4.

Lemma 16.1

If $u, v, w \in B(n)$, then

(1) $d(u, v) = \mathrm{Wt}(u - v)$;

(2) $d(u, v) \leq d(u, w) + d(w, v)$.

Proof ▸ (1) A coordinate of $u - v$ is nonzero if and only if u and v differ in that coordinate. So the number of nonzero coordinates in $u - v$, namely $\mathrm{Wt}(u - v)$, is the same as the number of coordinates in which u and v differ, namely $d(u, v)$.

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*In other words, if $u = u_1u_2 \cdots u_n$ and $v = v_1v_2 \cdots v_n$ (with each u_i, v_i either 1 or 0), then $d(u, v)$ is the *number* of indices i such that $u_i \neq v_i$.

(2) It suffices by (1) to prove that $\text{Wt}(u - v) \leq \text{Wt}(u - w) + \text{Wt}(w - v)$. The left side of this inequality is the number of nonzero coordinates of $u - v$, and the right side is the total number of nonzero coordinates in $u - w$ and $w - v$. So we need to verify only that whenever $u - v$ has nonzero ith coordinate, at least one of $u - w$ and $w - v$ also has nonzero ith coordinate. Using the subscript i to denote ith coordinates, suppose the ith coordinate $u_i - v_i$ of $u - v$ is nonzero. If the ith coordinate $u_i - w_i$ of $u - w$ is nonzero, then there is nothing to prove. If $u_i - w_i = 0$, then $u_i = w_i$, and, hence, $w_i - v_i = u_i - v_i \neq 0$. Therefore, the ith coordinate $w_i - v_i$ of $w - v$ is nonzero. ■

If a codeword u is transmitted and the word w is received, then the number of errors in the transmission is the number of coordinates in which u and w differ, that is, the Hamming distance from u to w. Since a large number of transmission errors is less likely than a small number (Exercise 27), the nearest codeword to a received word is most likely to be the codeword that was transmitted. Therefore, a *received word is decoded as the codeword that is nearest to it in Hamming distance*. If there is more than one codeword nearest to it, the decoder signals an error.* This process is called **nearest-neighbor decoding.**†

Definition A linear code is said to **correct t errors** if every codeword that is transmitted with t or fewer errors is correctly decoded by nearest-neighbor decoding.

Theorem 16.2

A linear code corrects t errors if and only if the Hamming distance between any two codewords is at least $2t + 1$.

Proof ▶ Assume that the distance between any two codewords is at least $2t + 1$. If the codeword u is transmitted with t or fewer errors and received as w, then $d(u, w) \leq t$. If v is any other codeword, then $d(u, v) \geq 2t + 1$ hypothesis. Hence, by Lemma 16.1,

$$2t + 1 \leq d(u, v) \leq d(u, w) + d(w, v) \leq t + d(w, v).$$

Subtracting t from both sides of $2t + 1 \leq t + d(w, v)$ shows that $d(w, v) \geq t + 1$. Since $d(u, w) \leq t$, u is the closest codeword to w, so nearest-neighbor decoding correctly decodes w as u. Hence, the code corrects t errors. The proof of the converse is Exercise 15. ■

*Alternatively, the decoder can be programmed to choose one of the nearest codewords arbitrarily. This is usually done when retransmission is difficult or impossible.

†Under our assumptions in this chapter, nearest-neighbor decoding coincides with maximum-likelihood decoding.

Since only codewords are transmitted, errors are detected whenever a received word is not a codeword.

Definition

A linear code is said to **detect t errors** if the received word in any transmission with at least one, but no more than t errors, is not a codeword.

Theorem 16.3

A linear code detects t errors if and only if the Hamming distance between any two codewords is at least $t + 1$.

Proof ▸ Assume that the distance between any two codewords is at least $t + 1$. If the codeword u is transmitted with at least one, but not more than t errors, and received as w, then

$$0 < d(u, w) \leq t, \qquad \text{and hence} \qquad d(u, w) < t + 1.$$

So w cannot be a codeword. Therefore, the code detects t errors. The proof of the converse is Exercise 16. ■

If u and v are distinct codewords, then $d(u, v)$ is the weight of the nonzero codeword $u - v$ by Lemma 16.1. Conversely, the weight of any nonzero codeword w is the distance between the distinct codewords w and $\mathbf{0} = 000 \cdots 0 \in B(n)$ because $\mathrm{Wt}(w) = \mathrm{Wt}(w - \mathbf{0}) = d(w, \mathbf{0})$. Therefore, *the minimum Hamming distance between any two codewords is the same as the smallest Hamming weight of all the nonzero codewords.* Combining this fact with Theorems 16.2 and 16.3 yields.

Corollary 16.4

A linear code detects $2t$ errors and corrects t errors if and only if the Hamming weight of every nonzero codeword is at least $2t + 1$.

EXAMPLE 5

Let the message words be 00, 10, 01, 11 $\in B(2)$ and construct a (10, 2) code by assigning to each message word the codeword (element of $B(10)$) obtained by repeating the message word five times:

$$0000000000, 1010101010, 0101010101, 1111111111.$$

The set C of codewords is closed under addition and, hence, a subgroup of order 2^2 (Theorem 7.12). So C is a (10, 2) code. Every nonzero codeword has Hamming weight at least $5 = 2 \cdot 2 + 1$. By Corollary 16.4 (with $t = 2$), the code C corrects two errors and detects four errors.

Systematic Codes

By constructing codes that repeat the message words a large number of times (five in the last example), you can always guarantee a high degree of error detection and correction. The disadvantage to such repetition codes is their inefficiency when long messages must be sent. It is time consuming and expensive to transmit a large number of digits for each message word. So the goal is to construct codes that achieve an acceptable accuracy rate without unnecessarily reducing the transmission rate.

One efficient technique for constructing linear codes is based on matrix multiplication. Codes constructed in this way are automatically equipped with an encoding algorithm that assigns each message word to a unique codeword.

EXAMPLE 6

We shall construct a (7, 4) code. The message words will be the elements of $B(4)$, and the codewords elements of $B(7)$. Message words are considered as row vectors and converted to codewords by right multiplying by the following matrix, whose entries are in $\mathbb{Z}_2$:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

For instance, the message word 1101 is converted to the codeword 1101001 because

$$(1 \ 1 \ 0 \ 1)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} = (1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1).$$

The complete set C of codewords may be found similarly:

Message Word	Codeword	Message Word	Codeword
0000	0000000	1000	1000011
0001	0001111	1001	1001100
0010	0010110	1010	1010101
0011	0011001	1011	1011010
0100	0100101	1100	1100110
0101	0101010	1101	1101001
0110	0110011	1110	1110000
0111	0111100	1111	1111111

Theorem 16.6 below shows that C is actually a subgroup of $B(7)$. So C is a (7, 4) code, called the **(7, 4) Hamming code.** The preceding table shows that every nonzero codeword has Hamming weight at least $3 = 2 \cdot 1 + 1$. Hence, by Corollary 16.4 (with $t = 1$) this code corrects single errors and detects double errors.

The table in Example 6 shows that codewords in the Hamming (7, 4) code have a special form: The first four digits of each codeword form the corresponding message word. For instance, *1101*001 is the codeword for 1101.* An (n, k) code in which the first k digits of each codeword form the corresponding message word is called a **systematic code.** All the examples above are systematic codes. Systematic codes are convenient because codewords are easily translated back to message words: Just take the first k digits.

We can construct other systematic codes by following a procedure similar to that in the last example. A $\boldsymbol{k \times n}$ **standard generator matrix** is a $k \times n$ matrix G with entries in $\mathbb{Z}_2$ of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & a_{11} & \dots & a_{1n-k} \\ 0 & 1 & 0 & \dots & 0 & 0 & a_{21} & \dots & a_{2n-k} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & a_{(k-1)1} & \dots & a_{k-1\,n-k} \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{k1} & \dots & a_{k\,n-k} \end{pmatrix} = (I_k \mid A),$$

where I_k is the $k \times k$ identity matrix and A is a $k \times (n-k)$ matrix. For instance, the matrix G in Example 6 is a 4×7 standard generator matrix. It has the form $(I_4 \mid A)$, where A is a 4×3 matrix.

A standard generator matrix can be used as an encoding algorithm to convert elements of $B(k)$ into codewords (elements of $B(n)$) by right multiplication. Each $u \in B(k)$ is considered as a row vector of length k. The matrix product uG is then a row vector of length n, that is, an element of $B(n)$. Because the first k columns of G form the identity matrix I_k, *the first k coordinates of the codeword uG form the corresponding message word $u \in B(k)$* (Exercise 23). In order to justify calling uG a "codeword," we must show that the set of all such elements is a subgroup of $B(n)$.

Lemma 16.5

If $f{:}B(k) \to B(n)$ is an injective homomorphism of groups, then the image of f is an (n, k) code.

Proof ▸ Im f is a subgroup of $B(n)$ that is isomorphic to $B(k)$ by Theorem 7.20. Therefore, Im f has order 2^k and, hence, is an (n, k) code. ■

Theorem 16.6

If G is a $k \times n$ standard generator matrix, then $\{uG \mid u \in B(k)\}$ is a systematic (n, k) code.

Proof ▸ Define a function $f{:}B(k) \to B(n)$ by $f(u) = uG$. The image of f is $\{f(u) \mid u \in B(k)\} = \{uG \mid u \in B(k)\}$. By Lemma 16.5 and the italicized

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*The last three digits of each codeword are **check digits** that can be used to determine if a received word is a codeword; see Exercise 22.

remarks preceding it, we need to show only that f is an injective homomorphism of groups. Since matrix multiplication is distributive,

$$f(u + v) = (u + v)G = uG + vG = f(u) + f(v).$$

Hence, f is a homomorphism of groups.

If $u = u_1u_2 \cdots u_k \in B(k)$, then the first k coordinates of uG are $u_1u_2 \cdots u_k$ because G is a standard generator matrix, and similarly for $v = v_1v_2 \cdots v_k \in B(k)$. We use this fact to show that f is injective. If $f(u) = f(v)$, then in $B(n)$

$$u_1u_2 \cdots u_k{*}{*}{*}{*}{*} = uG = f(u) = f(v) = vG = v_1v_2 \cdots v_k{*}{*}{*}{*}{*},$$

where the $*$'s indicate the remaining coordinates of uG and vG. Since these elements of $B(n)$ are equal, they must be equal in every coordinate. In particular, $u_1 = v_1, u_2 = v_2, \ldots, u_k = v_k$. Therefore, $u = v$ in $B(k)$, and f is injective. ■

EXAMPLE 7

By Theorem 16.6, the standard generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

generates the (6, 3) code $\{uG \mid u \in B(3)\}$. Verify that the encoding algorithm $u \to uG$ produces these codewords:

Message Word	Codeword	Message Word	Codeword
000	000000	100	100011
001	001110	101	101101
010	010101	110	110110
011	011011	111	111000

Since the Hamming weight of every nonzero codeword is at least 3, this code corrects single errors and detects double errors by Corollary 16.4 (with $t = 1$).

Describing a large code by means by a standard generator matrix is much more efficient than listing all the codewords. For instance, in a (50, 30) code there are only 1500 entries in the 30×50 generator matrix, but more than a billion codewords.

Linear algebra can be used to show that every systematic linear code is given by a standard generator matrix. The standard generator matrices for the codes in the examples above are in Exercises 7–9.

Exercises

A. 1. Show that $C = \{0000, 0101, 1010, 1111\}$ is a (4, 2) code.

2. Find the Hamming weight of

(a) $0110110 \in B(7)$ (b) $11110011 \in B(8)$

(c) $000001 \in B(6)$ (d) $101101101101 \in B(12)$

3. Find the Hamming distance between

(a) 0010101 and 1010101

(b) 110010101 and 100110010

(c) 111111 and 000011

(d) 00001000 and 10001000

4. Use nearest-neighbor decoding in the Hamming (7, 4) code to detect errors and, if possible, decode these received words:

(a) 0111000 (b) 1101001

(c) 1011100 (d) 0010010

5. List all codewords generated by the standard generator matrix:

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$

6. Determine the number of errors that each of the codes in Exercise 5 will detect and the number of errors each will correct.

7. Show that the standard generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

generates the (6, 5) parity-check code in Example 2. [*Hint:* List all the codewords generated by G; then list all the codewords in the parity-check code; compare the two lists.]

8. Show that the standard generator matrix

$$G = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

generates the (10, 2) repetition code in Example 5. [*Hint:* See the hint for Exercise 7.]

9. Show that 1×4 standard generator matrix $(1\ \ 1\ \ 1\ \ 1)$ generates the code in Example 1.

10. Prove that $B(n) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors) with coordinatewise addition is an abelian group of order 2^n.

B. 11. Prove that for any $u, v, w \in B(n)$,

(a) $d(u, v) = d(v, u)$.

(b) $d(u, v) = 0$ if and only if $u = v$.

(c) $d(u, v) = d(u + w, v + w)$.

12. Prove that an element of $B(6)$ is a codeword in the (6, 5) parity-check code (Example 2) if the sum of its digits is 0. [*Hint:* Compare the sum of the first five digits with the sixth digit.]

13. Prove that the set of all codewords in the (6, 5) parity-check code (Example 2) is a subgroup of $B(6)$. [*Hint:* Use Exercise 12.]

14. If u and v are distinct codewords of a code that corrects t errors, explain why $d(u, v) \geq t$.

15. Complete the proof of Theorem 16.2 by showing that if a code corrects t errors, then the Hamming distance between any two codewords is at least $2t + 1$. [*Hint:* If u, v are codewords with $d(u, v) \leq 2t$, obtain a contradiction by constructing a word w that differs from u in exactly t coordinates and from v in t or fewer coordinates; see Exercise 14.]

16. Complete the proof of Theorem 16.3 by showing that if a code detects t errors, then the Hamming distance between any two codewords is at least $t + 1$.

17. Construct a (5, 2) code that corrects single errors.

18. Show that no (6, 3) code corrects double errors.

19. Construct a (7, 3) code in which every nonzero codeword has Hamming weight at least 4.

20. Is there a (6, 2) code in which every nonzero codeword has Hamming weight at least 4?

21. Suppose only three messages are needed (for instance, "go," "slow down," "stop"). Find the smallest possible n so that these messages may be transmitted in an (n, k) code that corrects single errors.

22. Let G be the standard generator matrix for the (7, 4) Hamming code in Example 6.

(a) If $u = (u_1, u_2, u_3, u_4)$ is a message word, show that the corresponding codeword uG is

$$(u_1, u_2, u_3, u_4, u_2 + u_3 + u_4, u_1 + u_3 + u_4, u_1 + u_2 + u_4).$$

(b) If $v = (v_1, v_2, v_3, v_4, v_5, v_6, v_7) \in B(7)$, show that v is a codeword if and only if its last three coordinates (the check digits) satisfy these equations:

$$v_5 = v_2 + v_3 + v_4$$
$$v_6 = v_1 + v_3 + v_4$$
$$v_7 = v_1 + v_2 + v_4$$

23. If G is a $k \times n$ standard generating matrix and $u = u_1u_2u_3 \cdots u_k$ is a message word, show that the first k digits of the codeword uG are $u_1, u_2, \ldots, u_k$.

24. If C is a linear code, prove that either every codeword has even Hamming weight or exactly half of the codewords have even Hamming weight.

25. Prove that the elements of even Hamming weight in $B(n)$ form an $(n, n-1)$ code.

26. If $k < n$ and $f{:}B(k) \to B(n)$ is a homomorphism of groups, is $\text{Im} f$ a linear code? Is $\text{Im} f$ an (n, k) linear code?

NOTE: *A knowledge of elementary probability and a calculator are needed for Exercises 27–31.*

27. Assume that the probability of transmitting a single digit incorrectly is .01 and that a four-digit codeword is transmitted. Construct a suitable probability tree and compute the probability that the codeword is transmitted with

(a) no errors; **(b)** one error;

(c) two errors; **(d)** three errors;

(e) four errors; **(f)** at least three errors.

28. Do Exercise 27 for a five-digit codeword.

29. Suppose the probability of transmitting a single digit incorrectly is greater than .5. Explain why "inverse decoding" (decoding 1 as 0 and 0 as 1) should be employed.

30. Assume that the probability of transmitting a single digit incorrectly is .01 and that M is a 500-digit message.

(a) What is the probability that M will be transmitted with no errors?

(b) Suppose each digit is transmitted three times (111 for each 1, 000 for each 0) and that each received digit is decoded by "majority rule" (111, 110, 101, 011 are decoded as 1 and 000, 001, 010, 100 as 0). What is the probability that the message received when M is transmitted will be correctly decoded? [*Hint:* Find the probability that a single digit will be correctly decoded after transmission.]

31. **(a)** Show that the number of ways that k errors can occur in an n-digit message is $\binom{n}{k}$, where $\binom{n}{k}$ is the binomial coefficient.

(b) If p is the probability that a single digit is transmitted incorrectly and q is the probability that it is transmitted correctly, show that the probability that k errors occur in an n-digit message is $\binom{n}{k} p^k q^{n-k}$.

16.2 Decoding Techniques

Nearest-neighbor decoding for an (n, k) code was implemented in Section 16.1 by comparing each received word with all 2^k codewords in order to decode it. But when k is very large, this brute-force technique may be impractical or impossible. So we now develop decoding techniques that are sometimes more efficient. One of them is based on groups and cosets.

EXAMPLE 1

Let C be the (5, 2) code {00000, 10110, 01101, 11011}. From the elements of $B(5)$ *not* in C, choose one of smallest weight (which in this case is weight 1), say $e_1 = 10000$. Form its coset $e_1 + C$ by adding e_1 successively to the elements of C and list the coset elements, with $e_1 + c$ directly below c for each $c \in C$:

C:	00000	10110	01101	11011
$e_1 + C$:	10000	00110	11101	01011

Thus, for example, 11101 is directly below $01101 \in C$ because $e_1 + 01101 = 10000 + 01101 = 11101$. Among the elements not listed above, choose one of smallest weight, say $e_2 = 01000$, and list its coset in the same way (with $e_2 + c$ below $c \in C$):

C:	00000	10110	01101	11011
$e_1 + C$:	10000	00110	11101	01011
$e_2 + C$:	01000	11110	00101	10011

Among the elements not yet listed, choose one of smallest weight and list its coset, and continue in this way until every element of $B(5)$ is on the table. Verify that this is a complete table:

00000	10110	01101	11011	*Codewords*
10000	00110	11101	01011	
01000	11110	00101	10011	
00100	10010	01001	11111	*Received Words*
00010	10100	01111	11001	
00001	10111	01100	11010	
11000	01110	10101	00011	
10001	00111	11100	01010	

The decoding rule (which will be justified below) is: *Decode a received word w as the codeword at the top of the column in which w appears.* For instance, 01001 (fourth row) is decoded as 01101; and 01010 (last row) is decoded as 11011. Similarly, 11000 (seventh row) is decoded as 00000.

The decoding table in the example is called a **standard array,** and the decoding rule **standard-array decoding** or **coset decoding.** The same procedure can be used to construct a standard array for any code C. Its rows are the cosets of C, with C itself as the first row. Each is of the form $e + C$, where e is the **coset leader** (an element of smallest weight in the coset and listed first in the row). The element $e + c$ (with $c \in C$) is listed in the column below c and is decoded as c.

Theorem 16.7

Let C be an (n, k) code. Standard-array decoding for C is nearest-neighbor decoding.

Proof ▸ If $w \in B(n)$, then $w = e + v \in e + C$, where e is a coset leader and v is the codeword at the top of the column containing w. Standard-array decoding decodes w as v. We must show that v is a nearest codeword to w. If $u \in C$ is any other codeword, then $w - u$ is an element of $w + C$. But $w + C$ is the coset of e (because $e = w - v \in w + C$). By construction, the coset leader e has smallest weight in its coset, so $\mathrm{Wt}(w - u) \geq \mathrm{Wt}(e)$. Therefore, by Lemma 16.1,

$$d(w, u) = \mathrm{Wt}(w - u) \geq \mathrm{Wt}(e) = \mathrm{Wt}(w - v) = d(w, v).$$

Thus v is a nearest codeword to w. ∎

When nearest-neighbor decoding is implemented by a standard array, a codeword is automatically chosen whenever there is more than one codeword that is nearest to a received word w (rather than an error being signaled). So incorrect decoding may occur in such cases. The code in the last example corrects single errors (every codeword has weight at least 3; see Corollary 16.4). Since two or more errors are much less likely than a single one, standard-array decoding for this code has a high rate of accuracy (Exercise 18).

Once a standard array has been constructed, it's much more efficient for decoding than brute-force comparison with all codewords. Unfortunately, constructing a standard array for a large code may require as much computer time and memory as brute force. But when a code is given by a generator matrix, a much shorter decoding array is possible, as we now see.

Consider an (n, k) code with $k \times n$ standard generator matrix $G = (I_k \mid A)$. The **parity-check matrix** of the code is the $n \times (n - k)$ matrix $H = \left(\dfrac{A}{I_{n-k}}\right)$.*

EXAMPLE 2

Verify that the standard generator matrix for the (5, 2) code {00000, 10110, 01101, 11011} of Example 1 is

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} = (I_2 \mid A).$$

*Since the generator matrix can always be obtained from the parity-check matrix, many books on coding theory define a code in terms of its parity-check matrix rather than its generator matrix. In most books, the parity-check matrix is defined to be the transpose of our matrix H, that is, the $(k - n) \times n$ matrix whose ith row is the same as the ith column of H. The matrix H is more convenient here, and, in any case, all the results are easily translated from one notation to the other.

Here $k = 2$, $n = 5$, $n - k = 3$, and A is 2×3. So the parity-check matrix is the 5×3 matrix

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(\frac{A}{I_3}\right).$$

Verify that the product matrix GH is the 2×3 zero matrix. The phenomenon occurs in the general case as well.

Lemma 16.8

If $G = (I_k | A)$ is the standard generator matrix for a linear code and $H = \left(\frac{A}{I_{n-k}}\right)$ is its parity-check matrix, then GH is the zero matrix.

Proof ▸ The entry in row i and column j of GH is the product of the ith row of G (see page 478) and the jth column of H:*

$$(\delta_{i1}\delta_{i2} \cdots \delta_{ij} \cdots \delta_{ik} a_{i1} a_{i2} \cdots a_{ij} \cdots a_{i(n-k)}) \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{kj} \\ \delta_{1j} \\ \delta_{2j} \\ \vdots \\ \delta_{ij} \\ \vdots \\ \delta_{(n-k)j} \end{pmatrix}$$

$$\begin{aligned} &= \delta_{i1}a_{1j} + \delta_{i2}a_{2j} + \cdots + \delta_{ii}a_{ij} + \cdots + \delta_{ik}a_{kj} \\ &\quad + a_{i1}\delta_{1j} + a_{i2}\delta_{2j} + \cdots + a_{ij}\delta_{jj} + \cdots + a_{i(n-k)}\delta_{(n-k)j}. \end{aligned}$$

Since $\delta_{rs} = 0$ whenever $r \neq s$ and since addition is in $\mathbb{Z}_2$, this sum reduces to

$$\delta_{ii}a_{ij} + a_{ij}\delta_{jj} = 1a_{ij} + a_{ij}1 = a_{ij} + a_{ij} = 0. \quad \blacksquare$$

In an (n, k) code with $k \times n$ standard generator matrix G, every received word $w \in B(n)$ is a row vector of length n. Since the parity-check matrix H is $n \times (n - k)$,

*The Kronecker delta symbol δ_{rs} is defined as follows: when $r = s$, $\delta_{rs} = 1$ and when $r \neq s$, $\delta_{rs} = 0$.

the product wH is a row vector of length $n - k$, that is, an element of $B(n - k)$. Let $\mathbf{0}$ denote $000 \cdots 0 \in B(n - k)$.

EXAMPLE 3

Let H be the 5×3 parity-check matrix for the (5, 2) code in Example 2. Then $11000H = 011$ and $10110H = \mathbf{0}$:

$$(1\ 1\ 0\ 0\ 0)\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0\ 1\ 1) \quad \text{and}$$

$$(1\ 0\ 1\ 1\ 0)\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0\ 0\ 0).$$

The fact that 10110 is a codeword in this code and $10110H = 0$ is an example of the following Theorem.

Theorem 16.9

Let C be an (n, k) code with standard generator matrix G and parity-check matrix H. Then an element w in $B(n)$ is a codeword if and only if $wH = 0$.

Proof ▸ Define a function $f: B(n) \to B(n-k)$ by $f(w) = wH$. Then f is a homomorphism of groups (same argument as in the proof of Theorem 16.6). Now w is a codeword if and only if $w \in C$. Also, $w \in K$ (the kernel of f) if and only if $wH = \mathbf{0}$. So we must prove that $w \in C$ if and only if $w \in K$, that is, that $C = K$. By the definition of generator matrix, every element of C is of the form uG for some $u \in B(k)$. But $(uG)H = u(GH) = \mathbf{0}$ because GH is the zero matrix (Lemma 16.8). Therefore, $C \subseteq K$. Since C is a subgroup of order 2^k, we need to show only that K has order 2^k in order to conclude that $C = K$.

Exercise 14 shows that f is surjective. By the First Isomorphism Theorem 8.20, $B(n - k) \cong B(n)/K$, and, hence, by Lagrange's Theorem 8.5,

$$\begin{aligned} 2^n = |B(n)| &= |K|\,[B(n):K] \\ &= |K| \cdot |B(n)/K| = |K| \cdot |B(n-k)| = |K| \cdot 2^{n-k}. \end{aligned}$$

Dividing the first and last terms of this equation by 2^{n-k} shows that $|K| = 2^k$. ■

Corollary 16.10

Let C be a linear code with parity-check matrix H and let $u, v \in B(n)$. Then u and v are in the same coset of C if and only if $uH = vH$.

Proof ▸ To say that u and v are in the same coset means $u + C = v + C$. Theorem 8.2 in additive notation shows that

$$u + C = v + C \qquad \text{if and only if} \qquad u - v \in C.$$

By Theorem 16.9,

$$u - v \in C \qquad \text{if and only if} \qquad (u - v)H = \mathbf{0}.$$

Since matrix multiplication is distributive, $(u - v)H = uH - vH$. Also, $uH - vH = \mathbf{0}$ is equivalent to $uH = vH$. Hence,

$$(u - v)H = \mathbf{0} \qquad \text{if and only if} \qquad uH = vH.$$

Combining the three centered statements above proves the theorem. ■

If $w \in B(n)$ and H is the parity-check matrix, then wH is called the **syndrome** of w. By Corollary 16.10, w and its coset leader e have the same syndrome. If $w = e + v$ with $v \in C$, the standard array decodes w, as $v = w - e$. Therefore, standard-array (nearest-neighbor) decoding can be implemented as follows:

1. If w is a received word, compute the syndrome of w (that is, wH).
2. Find the coset leader e with the same syndrome (that is, $eH = wH$).
3. Decode w as $w - e$.

Since this procedure (called **syndrome decoding**) requires only that you know the syndromes of the coset leaders, the standard array can be replaced by a much shorter table.

EXAMPLE 4

The coset leaders for the (5, 2) code {00000, 10110, 01101, 11011}, as shown in Example 1, are

00000, 10000, 01000, 00100, 00010, 00001, 11000, 10001.

Multiplying each of them by the parity-check matrix H given in Example 2 produces its syndrome:

Syndrome	000	110	101	100	010	001	011	111
Coset Leader	00000	10000	01000	00100	00010	000001	11000	10001

To decode $w = 01001$, for example, we compute $01001H = 100$. The table shows that the coset leader with this syndrome is $e = 00100$. So we decode w as $w - e = 01001 - 00100 = 01101$.

Depending on the size of the code and whether or not coset leaders can be determined without constructing the entire standard array, syndrome decoding may be more efficient than brute-force nearest-neighbor decoding. For example, a (56, 48) code has 2^{48} (approximately 2.8×10^{14}) codewords but only $2^8 = 256$ cosets.

Standard-array and syndrome decoding are complete decoding schemes, meaning that they *always* find a nearest codeword for each received word. When retransmission of the message is impractical, complete decoding is a necessity. But when retransmission is feasible, it may be better to use an incomplete decoding scheme that corrects t errors and requests retransmission when more than t errors are detected. We now describe one such scheme.

Let $e_i \in B(n)$ denote the row vector with 1 in coordinate i and 0 in every other coordinate. In $B(3)$, for instance, $e_1 = 100$, $e_2 = 010$, and $e_3 = 001$. Each e_i has weight 1; in fact

$e_1, e_2, \ldots, e_n$ are the only elements of weight 1 in $B(n)$.

Consider the product of $e_2 \in B(3)$ and this matrix H:

$$e_2H = (0\ \ 1\ \ 0)\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = (0\ \ 1\ \ 1) = \text{row 2 of } H.$$

Exercise 10 shows that the same thing happens in the general case. If $e_i \in B(n)$ and H is a matrix with n rows, then

e_iH is the ith row of the matrix H.

Now assume that C is a linear code with parity-check matrix H and that *the rows of H are nonzero and no two of them are the same*. Then e_iH = ith row of $H \neq \mathbf{0}$ by hypothesis; hence, by Theorem 16.9,

e_i is not a codeword.

Furthermore, if $i \neq j$, then e_i and e_j cannot be in the same coset of C (otherwise row i of $H = e_iH = e_jH$ = row j of H by Corollary 16.10). Thus

e_i is the only element of weight 1 in its coset.

So every other element in the coset of e_i has weight at least 2.* Consequently,

e_i is *always* the coset leader in its coset.

Finally, if the syndrome of a received word w is the ith row of H, then $wH = e_iH$, so w and e_i are in the same coset by Corollary 16.10.

---

*The only element of weight 0 is $000 \cdots 0$, whose coset is C. C is not the coset of e_i because e_i is not a codeword.

The preceding paragraph suggests a convenient way to implement (possibly incomplete) syndrome decoding when the rows of H are nonzero and distinct:

1. If w is received, compute its syndrome wH.
2. If $wH = \mathbf{0}$, decode w as w (because w is a codeword by Theorem 16.9).
3. If $wH \neq \mathbf{0}$ and wH is the ith row of H, decode w by changing its ith coordinate (that is, decode w as $w - e_i$ because e_i is w's coset leader).
4. If $wH \neq \mathbf{0}$ and wH is not a row of H, do not decode and request a retransmission.

This scheme (called **parity-check matrix decoding**) can be easily implemented with large codes because there is no need to compute cosets or find coset leaders. Furthermore,

Theorem 16.11

Let C be a linear code with parity-check matrix H. If every row of H is nonzero and no two are the same, then parity-check matrix decoding corrects all single errors.

Proof ▸ When a codeword u is transmitted with exactly one error in coordinate i and received as w, then $w - u = e_i$. By Theorem 16.9, $wH = (e_i + u)H = e_iH + uH = e_iH + \mathbf{0} = e_iH$, which is the ith row of H. Therefore, w is correctly decoded as $w - e_i = u$. ■

EXAMPLE 5

Let C be the (5, 2) code whose parity-check matrix H is give in Example 2. If 10011 is received, its syndrome is

$$(1\ 0\ 0\ 1\ 1)H = (1\ 0\ 0\ 1\ 1)\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (1\quad 0\quad 1) = \text{row 2 of } H.$$

Therefore, 10011 is decoded as $10011 - e_2 = 10011 - 01000 = 11011$. If 11000 is received, verify that its syndrome is 011, which is not a row of H. Therefore, 11000 is not decoded, and a retransmission is requested.

In one important class of codes, parity-check matrix decoding is actually *complete* syndrome (nearest-neighbor) decoding.

EXAMPLE 6

The standard generator matrix G for the Hamming (7, 4) code was given in Example 6 of Section 16.1. Its parity-check matrix H has distinct, nonzero rows:

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The possible syndromes of a received word w in this code are 000 and the seven nonzero elements of $B(3)$. But all the nonzero elements of $B(3)$ appear as rows of H. So every syndrome either is 000 (decode w as itself) or is the ith row of H for some i (decode w by changing its ith coordinate). Therefore, every received word is decoded.

Example 6 is one of an infinite class of codes that can be described by using the fact that a linear code is completely determined by its parity-check matrix (from which a standard generator matrix is easily found). Let $r \geq 2$ be an integer and let $n = 2^r - 1$ and $k = 2^r - 1 - r$. Then $n - k = r$. The preceding example is the case $r = 3$. Let H be the $n \times (n - k)$ matrix whose last r rows are the identity matrix I_r and whose n rows consist of *all* the nonzero elements of $B(r)$. Since the number of nonzero elements in $B(r)$ is $2^r - 1 = n$, each nonzero element appears exactly once as a row of H. So the rows of H are distinct and nonzero. The code with this parity-check matrix is called a **Hamming code.**

In every Hamming code, all possible syndromes are rows of H. So parity-check matrix decoding is complete syndrome decoding that corrects all single errors.

Exercises

A. 1. Find the parity-check matrix of each standard generator matrix in Exercise 5 of Section 16.1.

2. Find the parity-check matrix for the code in Example 7 of Section 16.1.

3. Find the parity-check matrix for the parity-check code in Example 2 of Section 16.1. [See Exercise 7 in Section 16.1.]

4. Find the parity-check matrix for the (10, 2) repetition code in Example 5 of Section 16.1. [See Exercise 8 in Section 16.1.]

5. Find a parity-check matrix for the (15, 11) Hamming code.

6. Show that the linear code C with parity-check matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ cannot correct every single error.

7. Let C be the (4, 2) code with standard generator matrix $G = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Construct a standard array for C and find the syndrome of each coset leader.

8. Construct a standard array for the (6, 3) code in Example in 7 of Section 16.1 and find the syndrome of each coset leader.

9. Choose new coset leaders (when possible) for the (5, 2) code in Example 1 and use them to construct a standard array. How does this array compare with the one in Example 1?

10. Let $e_i = 00 \cdots 010 \cdots 00 \in B(n)$ have 1 in coordinate i and 0 elsewhere. If H is a matrix with n rows, show that e_iH is the ith row of H.

B. 11. Suppose a codeword u is transmitted and w is received. Show that standard-array decoding will decode w as u if and only if $w - u$ is a coset leader.

12. If every element of weight $\leq t$ is a coset leader in a standard array for a code C, show that C corrects t errors.

13. If a codeword u is transmitted and w is received, then $e = w - u$ is called an **error pattern.** Prove that an error will be detected if and only if the corresponding error pattern is not a codeword.

14. Prove that the function $f: B(n) \to B(n-k)$ in the proof of Theorem 16.9 is surjective. [*Hint:* If $v = v_1v_2 \cdots v_{n-k} \in B(n-k)$, show that $v = f(u)$, where $u = 000 \cdots 0v_1v_2 \cdots v_{n-k} \in B(n)$.]

15. Let C be a linear code with parity-check matrix H. Prove that C corrects single errors if and only if the rows of H are distinct and nonzero.

16. Show by example that parity-check matrix decoding with the Hamming (7, 4) code cannot detect two or more errors.

17. Show that in any Hamming code, every nonzero codeword has weight at least 3.

18. [Probability required.] In the (5, 2) code in Example 1, suppose that the probability of a transmission error in a single digit is .01.
   (a) Show that the probability of a single codeword being transmitted without error is .95099.
   (b) Show that the probability of a 100-word message being transmitted without error is less than .01.
   (c) Show that the probability of a single codeword being transmitted with exactly one error is .04803.
   (d) Show that the probability that a single codeword is correctly decoded by the standard array in Example 1 is at least .99921.
   (e) Show that the probability of a 100-word message being correctly decoded by the standard array is at least .92. [*Hint*: Compare with part (b).]

16.3 BCH Codes

The Hamming codes in the last section have efficient decoding algorithms that correct all single errors. The same is true of the BCH codes* presented here. But these codes are even more useful because they correct multiple errors.

The construction of a BCH code uses a finite *ring* whose additive group is (isomorphic to) some $B(n)$. Each ideal in such a ring is a linear code because its additive group is (isomorphic to) a subgroup of $B(n)$. The additional algebraic structure of the ring provides efficient error-correcting decoding algorithms for the code.

The finite rings in question are constructed as follows. Let n be a positive integer and $(x^n - 1)$ the principal ideal in $\mathbb{Z}_2[x]$ consisting of all multiples of $x^n - 1$. The elements of the quotient ring $\mathbb{Z}_2[x]/(x^n - 1)$ are the congruence classes (cosets) modulo $x^n - 1$. By Corollary 5.5, the distinct congruence classes in $\mathbb{Z}_2[x]/(x^n - 1)$ are in one-to-one correspondence with the polynomials of the form

$$(*) \qquad a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}, \qquad \text{with } a_i \in \mathbb{Z}_2.$$

Each such polynomial has n coefficients, and there are two possibilities for each coefficient. Hence, $\mathbb{Z}_2[x]/(x^n - 1)$ is a ring with 2^n elements. Furthermore, the n coefficients $(a_0, a_1, a_2, \ldots, a_{n-1})$ of the polynomial $(*)$ may be considered as an element of the group $B(n) = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

Theorem 16.12

The function $f: \mathbb{Z}_2[x]/(x^n - 1) \to B(n)$ given by

$$f([a_0 + a_1x + a_2x^2 + \cdots a_{n-1}x^{n-1}]) = (a_0, a_1, a_2, \ldots, a_{n-1})$$

is an isomorphism of additive groups.

Proof ▸ Exercise 7. ■

Theorem 16.12 shows that every ideal of $\mathbb{Z}_2[x]/(x^n - 1)$ can be considered as a linear code since it is (up to isomorphism) a subgroup of $B(n)$. In particular, if $g(x) \in \mathbb{Z}_2[x]$, then the congruence class (coset) of $g(x)$ generates a principal ideal I in $\mathbb{Z}_2[x]/(x^n - 1)$. The ideal I consists of all congruence classes of the form $[h(x)g(x)]$ with $h(x) \in \mathbb{Z}_2[x]$. BCH codes are of this type.

In order to define a BCH code that corrects t errors, choose a positive integer r such that $t < 2^{r-1}$. Let $n = 2^r - 1$. Then $g(x)$ is determined by considering a finite field of order 2^r, as explained below.

EXAMPLE 1

We let $t = 2$ and $r = 4$, so that $n = 2^4 - 1 = 15$. We shall construct a code in $\mathbb{Z}_2[x]/(x^{15} - 1)$ that corrects all double errors by finding an appropriate $g(x)$. To do this, we need a field of order $2^4 = 16$.

*The initials BCH stand for Bose, Chaudhuri, and Hocquenghem, who invented these codes in 1959–1960.

The polynomial $1 + x + x^4$ is irreducible in $\mathbb{Z}_2[x]$ (Exercise 3). Hence, $K = \mathbb{Z}_2[x]/(1 + x + x^4)$ is a field of order 16 by Theorem 5.10 (and the remarks after it). By Theorem 5.11, K contains a root α of $1 + x + x^4$. Using the fact that

$$1 + \alpha + \alpha^4 = 0 \qquad \text{and, hence,} \qquad \alpha^4 = 1 + \alpha^*$$

we can compute the powers of α. For example, $\alpha^6 = \alpha^2\alpha^4 = \alpha^2(1 + \alpha) = \alpha^2 + \alpha^3$. Similarly, we obtain

$$\begin{array}{lll}
\alpha^1 = \alpha & \alpha^6 = \alpha^2 + \alpha^3 & \alpha^{11} = \alpha + \alpha^2 + \alpha^3 \\
\alpha^2 = \alpha^2 & \alpha^7 = 1 + \alpha + \alpha^3 & \alpha^{12} = 1 + \alpha + \alpha^2 + \alpha^3 \\
\alpha^3 = \alpha^3 & \alpha^8 = 1 + \alpha^2 & \alpha^{13} = 1 + \alpha^2 + \alpha^3 \\
\alpha^4 = 1 + \alpha & \alpha^9 = \alpha + \alpha^3 & \alpha^{14} = 1 + \alpha^3 \\
\alpha^5 = \alpha + \alpha^2 & \alpha^{10} = 1 + \alpha + \alpha^2 & \alpha^{15} = 1
\end{array}$$

These elements are distinct and nonzero by statements (1) and (2) of Theorem 11.7 (with $u = \alpha$ and $p(x) = 1 + x + x^4$). Therefore, they are all the nonzero elements of K, and α is a generator of the multiplicative group of K.

To construct the polynomial $g(x)$, we first find the minimum polynomials of $\alpha, \alpha^2, \alpha^3, \alpha^4$ over $\mathbb{Z}_2$. By the construction of K, the minimal polynomial of α is $m_1(x) = 1 + x + x^4$. This polynomial $m_1(x)$ is also the minimal polynomial of α^2 and α^4, for instance, by the Freshman's Dream (Lemma 11.24),

$$\begin{aligned} m_1(\alpha^2) &= 1 + (\alpha^2) + (\alpha^2)^4 \\ &= 1^2 + (\alpha)^2 + (\alpha^4)^2 = (1 + \alpha + \alpha^4)^2 = 0^2 = 0. \end{aligned}$$

Verify that the minimum polynomial of α^3 is $m_3(x) = 1 + x + x^2 + x^3 + x^4$ (Exercise 5). The polynomial $g(x)$ is defined as the product $m_1(x)m_3(x)$, so that

$$\begin{aligned} g(x) &= (1 + x + x^4)(1 + x + x^2 + x^3 + x^4) \\ &= 1 + x^4 + x^6 + x^7 + x^8 \in \mathbb{Z}_2[x]. \end{aligned}$$

Let C be the ideal generated by $[g(x)]$ in $\mathbb{Z}_2[x]/(x^{15} - 1)$. Then C is a code by Theorem 16.12. We shall see that C is a (15, 7) code that corrects all single and double errors.

Just what do the codewords of C look like? By Corollary 5.5, each congruence class in $\mathbb{Z}_2[x]/(x^{15} - 1)$ is the class of a unique polynomial of the form

$$(**) \qquad a_0 + a_1x + a_2x^2 + \cdots + a_{13}x^{13} + a_{14}x^{14}, \qquad \text{with } a_i \in \mathbb{Z}_2.$$

So we shall denote the class by this polynomial.† When convenient, this polynomial will be identified (as in Theorem 16.12) with the element $a_0 a_1 a_2 \cdots a_{14} = (a_0, a_1, a_2, \ldots, a_{14})$ of $B(15)$. The codewords consist of the classes of polynomial multiples of $g(x)$. For example,

*Remember, $1 = -1$ in $\mathbb{Z}_2$.

†This is analogous to what was done in Section 2.3, when we began writing elements (classes) in $\mathbb{Z}_n$ in the form k rather than $[k]$.

Codeword in Polynomial Form	In $B(15)$ Form
$g(x) = 1 + x^4 + x^6 + x^7 + x^8$	100010111000000
$xg(x) = x(1 + x^4 + x^6 + x^7 + x^8)$	
$= x + x^5 + x^7 + x^8 + x^9$	010001011100000
$(1 + x^6)g(x) = (1 + x^6)(1 + x^4 + x^6 + x^7 + x^8)$	
$= 1 + x^4 + x^7 + x^8 + x^{10} + x^{12} + x^{13} + x^{14}$	100010011010111

If $g(x)$ is multiplied by a polynomial $h(x)$ of degree ≥ 7, then the codeword $h(x)g(x)$ has degree ≥ 15 and is not of the form (**). For example, if $h(x) = x^8$, then

$$\begin{aligned} h(x)g(x) = x^8g(x) &= x^8(1 + x^4 + x^6 + x^7 + x^8) \\ &= x^8 + x^{12} + x^{14} + x^{15} + x^{16}. \end{aligned}$$

The polynomial of the form (**) that is in the same class as $h(x)g(x)$ is the remainder when $h(x)g(x)$ is divided by $x^{15} - 1$ (see Corollary 5.5). Verify that

$$h(x)q(x) = (1 + x)(x^{15} - 1) + (1 + x + x^8 + x^{12} + x^{14}).$$

Hence, $[f(x)g(x)]$ is the codeword $1 + x + x^8 + x^{12} + x^{14}$ or, equivalently, 110000001000101.

The procedure in Example 1 is readily generalized. If t is the number of errors the code should correct, let $n = 2^r - 1$, where r is chosen so that $t < 2^{r-1}$ (in the example, $t = 2, r = 4$). By Corollary 11.26, there is a finite field K of order 2^r. By Theorem 11.28, $K = \mathbb{Z}_2(\alpha)$, where α is a generator of the multiplicative group of nonzero elements of K (and so has multiplicative order $2^r - 1 = n$). Let

$$m_1(x), m_2(x), m_3(x), \ldots, m_{2t}(x) \in \mathbb{Z}_2[x]$$

be the minimal polynomials of the elements

$$\alpha, \alpha^2, \alpha^3, \ldots, \alpha^{2t} \in K.$$

Let $g(x)$ be the product in $\mathbb{Z}_2[x]$ of the distinct polynomials on the list $m_1(x)$, $m_2(x), \ldots, m_{2t}(x)$.

The ideal C generated by $[g(x)]$ in $\mathbb{Z}_2[x]/(x^n - 1)$ is called the (primitive narrow-sense) **BCH code of length n and designed distance $2t + 1$** with **generator polynomial** $g(x)$. So the code in Example 1 is a BCH code of length 15 and designed distance 5 ($= 2 \cdot 2 + 1$). If $g(x)$ has degree m, then Exercise 14 shows that the code C is an (n, k) code, where $k = n - m$.

Theorem 16.13

A BCH code of length n and designed distance $2t + 1$ corrects t errors.

Proof ▸ The proof requires a knowledge of determinants; see Lidl-Pilz [32; page 230]. ■

Theorem 16.13 shows that there are BCH codes that will correct any desired number of errors. More importantly, from a practical viewpoint, there are efficient algorithms for decoding large BCH codes.* A complete description of them would take us too far afield. But here, in simplified form, is the underlying idea of the error-correcting procedure.

Let C be a BCH code of designed distance $2t + 1$ and generator polynomial $g(x)$. By the definition of $g(x)$, each minimal polynomial $m_1(x)$ divides $g(x)$. Hence, $g(\alpha^i) = 0$ for each $i = 1, 2, \ldots, 2t$. If $[f(x)]$ is a codeword in C, then $f(x) = h(x)g(x)$ for some $h(x)$, and, therefore,

$$f(\alpha^i) = h(\alpha^i)g(\alpha^i) = h(\alpha^i) \cdot 0 = 0.$$

Conversely, if $f(x) \in \mathbb{Z}_2[x]$ has every α^i as a root, then every $m_i(x)$ divides $f(x)$ by Theorem 11.6. This implies that $g(x) \mid f(x)$ (Exercise 8). Therefore,

$$\mathbf{[f(x)] \text{ is a codeword if and only if } f(\alpha^i) = 0 \text{ for } 1 \leq i \leq 2t}$$

The decoder receives the word $a_0a_1 \cdots a_k$, which represents the (class of) the polynomial

$$r(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k.$$

The decoder computes these elements of the field $K = \mathbb{Z}_2(\alpha)$:

$$r(\alpha), r(\alpha^2), r(\alpha^3), \ldots, r(\alpha^{2t}).$$

If all of them are 0, then $r(x)$ is a codeword by the remarks above. If certain ones are nonzero, the decoder uses them (according to a specified procedure) to construct a polynomial $D(x) \in K[x]$, called the **error-locator polynomial.** Since K is finite, the nonzero roots of $D(x)$ in K can be found by substituting each $\alpha^i \in K$ in $D(x)$].

If no more than t errors have been made, the nonzero roots of $D(x)$ give the location of the transmission errors. For instance, if α^7 is a root, then a_7 is incorrect in the received word $r(x)$; similarly if $\alpha^0 = 1$ is a root, then an error occured in transmitting a_0.

If $D(x)$ has no roots in K or if certain of the $r(\alpha^i)$ are 0, so that $D(x)$ cannot be constructed, then more than t errors have been made. So the decoder follows set procedures (omitted here) to choose arbitrarily a nearest codeword to $r(x)$.

EXAMPLE 2

In the (15, 7) BCH code of Example 1, suppose this word is received:

$$r(x) = x + x^7 + x^8 = 010000011000000.$$

*This is one reason BCH codes are widely used. For example, the European and trans-Atlantic communication system used a BCH code with $t = 6$ and $r = 8$. It is a (255, 231) code that corrects six errors with a failure probability of only 1 in 16 million.

Using the table at the beginning of Example 1 and the fact that $u + u = 0$ for every element u in K (Exercise 1), we have

$$r(\alpha) = \alpha + \alpha^7 + \alpha^8 = \alpha + (1 + \alpha + \alpha^3) + (1 + \alpha^2) = \alpha^2 + \alpha^3 = \alpha^6.$$

$$\begin{aligned} r(\alpha^3) &= \alpha^3 + (\alpha^3)^7 + (\alpha^3)^8 \\ &= \alpha^3 + \alpha^{21} + \alpha^{24} = \alpha^3 + \alpha^6 + \alpha^9 \\ &= \alpha^3 + (\alpha^2 + \alpha^3) + (\alpha + \alpha^3) = \alpha + \alpha^2 + \alpha^3 = \alpha^{11}. \end{aligned}$$

Exercise 6 shows that

$$\begin{aligned} r(\alpha^2) &= r(\alpha)^2 = (\alpha^6)^2 = \alpha^{12}; \\ r(\alpha^4) &= r(\alpha)^4 = (\alpha^6)^4 = \alpha^{24} = \alpha^9. \end{aligned}$$

The error-locator polynomial is given by this formula (which is justified in Exercise 15):

$$D(x) = x^2 + r(\alpha)x + \left(r(\alpha^2) + \frac{r(\alpha^3)}{r(\alpha)}\right).$$

Using the table at the beginning of Example 1, we see that

$$\begin{aligned} D(x) &= x^2 + \alpha^6 x + \left(\alpha^{12} + \frac{\alpha^{11}}{\alpha^6}\right) = x^2 + \alpha^6 x + (\alpha^{12} + \alpha^5) \\ &= x^2 + \alpha^6 x + \alpha^{14}. \end{aligned}$$

By substituting each of the nonzero elements of K in $D(x)$, we discover that

$$\begin{aligned} D(\alpha^5) &= (\alpha^5)^2 + \alpha^6\alpha^5 + \alpha^{14} = \alpha^{10} + \alpha^{11} + \alpha^{14} \\ &= (1 + \alpha + \alpha^2) + (\alpha + \alpha^2 + \alpha^3) + (1 + \alpha^3) = 0; \\ D(\alpha^9) &= (\alpha^9)^2 + \alpha^6\alpha^9 + \alpha^{14} = \alpha^{18} + \alpha^{15} + \alpha^{14} = \alpha^3 + 1 + \alpha^{14} \\ &= \alpha^3 + 1 + (1 + \alpha^3) = 0. \end{aligned}$$

Therefore, α^5 and α^9 are the roots of $D(x)$, so errors occurred in the coefficients of x^5 and x^9. The received word

$$r(x) = x + x^7 + x^8 = 01000\underline{0}011\underline{0}00000$$

is corrected as

$$c(x) = x + x^5 + x^7 + x^8 + x^9 = 01000\underline{1}011\underline{1}00000,$$

which *is* a codeword (see page 494).

Similarly, if $r(x) = x^2 + x^6 + x^9 + x^{10} = 001000100110000$ is received, then

$$r(\alpha) = \alpha^8, \qquad r(\alpha^2) = \alpha, \qquad r(\alpha^3) = \alpha^9, \qquad \text{and}$$

$$\begin{aligned} D(x) &= x^2 + r(\alpha)x + \left[r(\alpha^2) + \frac{r(\alpha^3)}{r(\alpha)}\right] = x^2 + \alpha^8 x + \left(\alpha + \frac{\alpha^9}{\alpha^8}\right) \\ &= x^2 + \alpha^8 x + (\alpha + \alpha) = x^2 + \alpha^8 x = x(x + \alpha^8). \end{aligned}$$

The only nonzero root of $D(x)$ is α^8, so a single error occurred in the coefficient of x^8, and the correct word is

$$c(x) = x^2 + x^6 + x^8 + x^9 + x^{10} = 001000101110000.$$

Finally, if $1 + x + x^4$ is received, then

$$r(\alpha) = 1 + \alpha + \alpha^4 = 0 \qquad \text{and} \qquad r(\alpha^3) = 1 + \alpha^3 + \alpha^{12} = \alpha^5.$$

So $D(x)$ cannot be constructed, and we conclude that more than two errors have occurred. Similarly, if $1 + x + x^3$ is received, then verify that $D(x) = x^2 + \alpha^7 x + \alpha^5$ and that $D(x)$ has no roots in K. Once again, more than two errors have occurred.

Exercises

NOTE: *Unless stated otherwise, K is the field $\mathbb{Z}_2[x]/(1 + x + x^4)$ of order 16 and α is a root of $1 + x + x^4$, as in Example 1.*

A. 1. (a) Prove that $f(x) + f(x) = 0$ for every $f(x) \in \mathbb{Z}_2[x]$.

(b) Prove that $u + u = 0$ for every u in the field K.

2. Show that the only irreducible quadratic in $\mathbb{Z}_2[x]$ is $x^2 + x + 1$. [*Hint:* List all the quadratics and use Corollary 4.19.]

3. Prove that $1 + x + x^4$ is irreducible in $\mathbb{Z}_2[x]$. [*Hint:* Exercise 2 and Theorem 4.16.]

4. Prove that the minimal polynomial of α^5 over $\mathbb{Z}_2$ is $1 + x + x^2$. [*Hint:* Use the table in Example 1.]

5. (a) Prove that the minimal polynomial of α^3 over $\mathbb{Z}_2$ is $1 + x + x^2 + x^3 + x^4$. [*Hint:* Exercise 2, Theorem 4.16, and the table in Example 1.]

(b) Show that α^4 is also a root of $1 + x + x^4$.

B. 6. If $f(x) \in \mathbb{Z}_2[x]$ and α is an element in some extension field of $\mathbb{Z}_2$, prove that for every $k \geq 1, f(\alpha^{2k}) = f(\alpha^k)^2$. [*Hint:* Lemma 11.24.]

7. (a) Show that the function f: $\mathbb{Z}_2[x]/(x^n - 1) \to B(n)$ given by

$$f([a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}]) = (a_0, a_1, a_2, \ldots, a_{n-1})$$

is surjective.

(b) Prove that f is a homomorphism of additive groups.

(c) Prove that f is injective. [*Hint:* Theorem 8.17 in additive notation.]

8. (a) Let F be a field and $f(x) \in F[x]$. If $p(x)$ and $q(x)$ are distinct monic irreducibles in $F[x]$ such that $p(x) \mid f(x)$ and $q(x) \mid f(x)$, prove that $p(x)q(x) \mid f(x)$. [*Hint:* If $f(x) = q(x)h(x)$, then $p(x) \mid q(x)h(x)$; use part (2) of Theorem 4.12.]

(b) If $m_1(x), m_2(x), \ldots, m_k(x)$ are distinct monic irreducibles in $F[x]$ such that each $m_i(x)$ divides $f(x)$, prove that $g(x) = m_1(x)m_2(x) \cdots m_k(x)$ divides $f(x)$.

9. Let C be the (15, 7) BCH code of Examples 1 and 2. Use the error-correction technique presented there to correct these received words or to determine that three or more errors have been made.

(a) $1 + x = 110000000000000$.

(b) $1 + x^3 + x^4 + x^5 = 100111000000000$.

(c) $1 + x^2 + x^4 + x^7 = 101010010000000$.

(d) $1 + x^6 + x^7 + x^8 + x^9 = 100000111100000$.

10. Show that the generator polynomial for the BCH code with $t = 3$, $r = 4$, $n = 15$ is $g(x) = 1 + x + x^2 + x^4 + x^5 + x^8 + x^{10}$. [*Hint:* Exercises 3–5 may be helpful.]

11. Let $K = \mathbb{Z}_2(\alpha)$ be a finite field of order 2^r, whose multiplicative group is generated by α. For each i, let $m_i(x)$ be the minimal polynomial of α^i over $\mathbb{Z}_2$. If $n = 2^r - 1$, prove that each $m_i(x)$ divides $x^n - 1$. [*Hint:* $\alpha^n = 1$ (Why?); use Theorem 11.6.]

12. If $g(x)$ is the generator polynomial of a BCH code in $\mathbb{Z}_2[x]/(x^n - 1)$, prove that $g(x)$ divides $x^n - 1$. [*Hint:* Exercises 11 and 8(b).]

13. Let $g(x) \in \mathbb{Z}_2[x]$ be a divisor of $x^n - 1$ and let C be the principal ideal generated by $[g(x)]$ in $\mathbb{Z}_2[x]/(x^n - 1)$. Then C is a code. Prove that C is **cyclic,** meaning that C (with codewords written as elements of $B(n)$) has this property: If $(c_0, c_1, \ldots, c_{n-1}) \in C$, then $(c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in C$. [*Hint:* $c_{n-1} + c_0x + \cdots + c_{n-2}x^{n-1} = x(c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) - c_{n-1}(x^n - 1)$.]

C. 14. Let C be the code in Exercise 13. Assume $g(x)$ has degree m and let $k = n - m$. Let J be the set of all polynomials in $\mathbb{Z}_2[x]$ of the form $a_0 + a_1x + a_2x^2 + \cdots + a_{k-1}x^{k-1}$.

(a) Prove that every element in C is of the form $[s(x)g(x)]$ with $s(x) \in J$. [*Hint:* Let $[h(x)g(x) \in C$. By the Division Algorithm, $h(x)g(x) = e(x)(x^n - 1) + r(x)$, with $\deg r(x) < n$ and $[h(x)g(x)] = [r(x)]$. Show that $r(x) = s(x)g(x)$, where $s(x) = h(x) - e(x)f(x)$ and $g(x)f(x) = x^n - 1$. Use Theorem 4.2 to show $s(x) \in J$.]

(b) Prove that C has order 2^k, and, hence, C is an (n, k) code. [*Hint:* Use Corollary 5.5 to show that if $s(x) \neq t(x)$ in J, then $[s(x)g(x)] \neq [t(x)g(x)]$ in C. How many elements are in J?]

15. Let C be the (15, 7) BCH code of Examples 1 and 2, with codewords written as polynomials of degree ≤ 14. Suppose the codeword $c(x)$ is transmitted with errors in the coefficients of x^i and x^j and $r(x)$ is received. Then $D(x) = (x + \alpha^i)(x + \alpha^j) \in K[x]$, whose roots are α^i and α^j, is the error-locator polynomial. Express the coefficients of $D(x)$ in terms of $r(\alpha)$, $r(\alpha^2)$, $r(\alpha^3)$ as follows.

(a) Show that $r(x) - c(x) = x^i + x^j$.

(b) Show that $r(\alpha^k) = \alpha^{ki} + \alpha^{kj}$ for $k = 1, 2, 3$. [See the boldface statement on page 495.]

(c) Show that $D(x) = x^2 + (\alpha^i + \alpha^j)x + \alpha^{i+j} = x^2 + r(\alpha)x + \alpha^{i+j}$.

(d) Show that $\alpha^{i+j} = r(\alpha^2) + \dfrac{r(\alpha^3)}{r(\alpha)}$. [*Hint:* Show that $r(\alpha)^3 = (\alpha^i + \alpha^j)^3 = \alpha^{3i} + \alpha^{3j} + \alpha^{i+j}(\alpha^i + \alpha^j) = r(\alpha^3) + r(\alpha)\alpha^{i+j}$ and solve for α^{i+j}; note that $r(\alpha)^2 = r(\alpha^2)$.]

16. Show that a BCH code with $t = 1$ is actually a Hamming code (see page 490).

PART 4

APPENDICES

APPENDIX A

Logic and Proof

This Appendix summarizes the basic facts about logic and proof that are needed to read this book. For a complete discussion of these topics see Galovich [7], Smith-Eggen-St. Andre [10], or Solow [11].

Logic

A **statement** is a declarative sentence that is either true or false. For instance, each of these sentences is a statement:

π is a real number.

Every triangle is isosceles.

103 bald eagles were born in the United States last year.

Note that the last sentence *is* a statement even though we may not be able to verify its truth or falsity. Neither of the following sentences is a statement:

What time is it? Wow!

Compound Statements

We frequently deal with **compound statements** that are formed from other statements by using the connectives "and" and "or". The truth of the compound statement will depend on the truth of its components. If P and Q are statements, then

"*P* and *Q*" is a true statement when *both* *P* and *Q* are true, and false otherwise.

For example,

$$\pi \text{ is a real number and } 9 < 10$$

is a true statement because both of its components are true. But

$$\pi \text{ is a real number and } 7 - 5 = 18$$

is a false statement since one of its components is false.

In ordinary English the word "or" is most often used in exclusive sense, meaning "one or the other but not both," as in

He is at least 21 years old or he is younger than 21.

But "or" can also be used in an inclusive sense, meaning "one or the other, or possibly both," as in the sentence

They will win the first game or they will win the second.

Thus the inclusive "or" has the same meaning as "and/or" in everyday language. In mathematics, *"or" is always used in the inclusive sense,* which allows the possibility that both components might be true but does not require it. Consequently, if P and Q are statements, then

"P or Q" is a true statement when at least one of P or Q is true and false when both P and Q are false.

For example, both

$$7 > 5 \qquad \text{or} \qquad 3 + 8 = 11$$

and

$$7 > 5 \qquad \text{or} \qquad 3 + 8 = 23$$

are true statements because at least one component is true in each case, but

$$4 < 2 \qquad \text{or} \qquad 5 + 3 = 12$$

is false since both components are false.

Negation

The **negation** of a statement P is the statement "it is not the case that P", which we can conveniently abbreviate as "not-P". Thus the negation of

7 is a positive integer

is the statement "it is not the case that 7 is a positive integer", which we would normally write in the less awkward form "7 is not a positive integer". If P is a statement, then

The negation of P is true exactly when P is false, and the negation of P is false exactly when P is true.

The negation of the statement "P and Q" is the statement "it is not the case that P and Q". Now "P and Q" is true exactly when both P and Q are true, so to say that this is not the case means that at least one of P or Q is false. But this occurs exactly when at least one of not-P or not-Q is true. Thus

The negation of the statement "P and Q" is the statement "not-P *or* not-Q".

For example, the negation of

f is continuous and f is differentiable at $x = 5$

is the statement

f is not continuous or f is not differentiable at $x = 5$.

The negation of the statement "P or Q" is the statement "it is not the case that P or Q". Now "P or Q" is true exactly when at least one of P or Q is true. To say that this is not the case means that both P and Q are false. But P and Q are both false exactly when not-P and not-Q are both true. Hence,

The negation of the statement "P or Q" is the statement "not-P *and* not-Q".

For instance, the negation of

119 is prime or $\sqrt{3}$ is a rational number

is the statement

119 is not prime and $\sqrt{3}$ is not a rational number.

Quantifiers

Many mathematical statements involve quantifiers. The **universal quantifier** states that a property is true *for all* the items under discussion. There are several grammatical variations of the universal quantifier, such as

For all real numbers c, $c^2 > -1$.
Every integer is a real number.
All integers are rational numbers.
For each real number a, the number $a^2 + 1$ is positive.

The **existential quantifier** asserts that *there exists* at least one object with certain properties. For example,

There exist positive rational numbers.
There exists a number x such that $x^2 - 5x + 6 = 0$.
There is an even prime number.

In mathematics, the word "some" means "at least one" and is, in effect, an existential quantifier. For instance,

Some integers are prime

is equivalent to saying "at least one integer is prime", that is,

There exists a prime integer.

Care must be used when forming the negation of statements involving quantifiers. For example, the negation of

All real numbers are rational

is "it is not the case that all real numbers are rational", which means that there is at least one real number that is irrational (= not rational). So the negation is

There exists an irrational real number.

In particular, the statements "all real numbers are not rational" and "all real numbers are irrational" are *not* negations of "all real numbers are rational". This example illustrates the general principle:

The negation of a statement with a universal quantifier is a statement with an existential quantifier.

The negation of the statement

There exists a positive integer

is "it is not the case that there is a positive integer", which means that "every integer is nonpositive" or, equivalently, "no integer is positive". Thus

The negation of a statement with an existential quantifier is a statement with a universal quantifier.

Conditional and Biconditional Statements

In mathematical proofs we deal primarily with **conditional statements** of the form

If P, then Q

which is written symbolically as $P \Rightarrow Q$. The statement P is called the **hypothesis** or **premise,** and Q is called the **conclusion.** Here are some examples:

If c and d are integers, then cd is an integer.
If f is continuous at $x = 3$, then f is differentiable there.
$a \neq 0 \Rightarrow a^2 > 0$.

There are several grammatical variations, all of which mean the same thing as "if P, then Q":

P implies Q.
P is sufficient for Q.
Q provided that P.
Q whenever P.

In ordinary usage the statement "if P, then Q" means that the truth of P guarantees the truth of Q. Consequently,

"$P \Rightarrow Q$" is a true statement when both P and Q are true and false when P is true and Q is false.

Although the situation rarely occurs, we must sometimes deal with the statement "$P \Rightarrow Q$" when P is false. For example, consider this campaign promise: "If I am elected, then taxes will be reduced". If the candidate is elected (P is true), the truth or falsity of this statement depends on whether or not taxes are reduced. But what if the candidate is *not* elected (P is false)? Regardless of what happens to taxes, you can't fairly call the campaign promise a lie. Consequently, it is customary in symbolic logic to adopt this rule:

When P is false, the statement "$P \Rightarrow Q$" is true.

The **contrapositive** of the conditional statement "$P \Rightarrow Q$" is the statement "not-Q $\Rightarrow$ not-P". For instance, the contrapositive of this statement about integers

If c is a multiple of 6, then c is even

is the statement

If c is not even, then c is not a multiple of 6.

Notice that both the original statement and its contrapositive are true. Two statements are said to be **equivalent** if one is true exactly when the other is. We claim that

The conditional statement "$P \Rightarrow Q$" is equivalent to its contrapositive "not-$Q \Rightarrow$ not-P".

To prove this equivalence, suppose $P \Rightarrow Q$ is true and consider the statement not-$Q \Rightarrow$ not-P. Suppose not-Q is true. Then Q is false. Now if P were true, then Q would necessarily be true, which is not the case. So P must be false, and, hence, not-P is true. Thus not-$Q \Rightarrow$ not-P is true. A similar argument shows that when not-$Q \Rightarrow$ not-P is true, then $P \Rightarrow Q$ is also true.

The **converse** of the conditional statement "$P \Rightarrow Q$" is the statement "$Q \Rightarrow P$". For example, the converse of the statement

If b is a positive real number, then b^2 is positive

is the statement

If b^2 is positive, then b is a positive real number.

This last statement is false since, for example, $(-3)^2$ is the positive number 9, but -3 is not positive. Thus

The converse of a true statement may be false.

There are some situations in which a conditional statement and its converse are both true. For example,

If the integer k is odd, then the integer $k + 1$ is even

is true, as is its converse

If the integer $k + 1$ is even, then the integer k is odd.

We can state this fact in succinct form by saying that "k is odd *if and only if* $k + 1$ is even". More generally, the statement

P if and only if Q,

which is abbreviated as "P iff Q" or "$P \Leftrightarrow Q$", means

$$P \Rightarrow Q \quad \textit{and} \quad Q \Rightarrow P.$$

"P if and only if Q" is called a **biconditional statement.** The rules for compound statements show that "P if and only if Q" is true exactly when both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true. In this case, the truth of P implies the truth of Q and vice versa, so that P is true exactly when Q is true. In other words, "P if and only if Q" means that P and Q are equivalent statements.

Theorems and Proof

The formal development of a mathematical topic begins with certain undefined terms and **axioms** (statements about the undefined terms that are assumed to be true). These undefined terms and axioms are used to define new terms and to construct **theorems** (true statements about these objects). The **proof** of a theorem is a complete justification of the truth of the statement.

Most theorems are conditional statements. A theorem that is not stated in conditional form is often equivalent to a conditional statement. For instance, the statement

Every integer greater than 1 is a product of primes

is equivalent to

If n is an integer and $n > 1$, then n is a product of primes.

The first step in proving a theorem that can be phrased in conditional form is to identify the hypothesis P and the conclusion Q. In order to prove the theorem "$P \Rightarrow Q$", one assumes that the hypothesis P is true and then uses it, together with axioms, definitions, and previously proved theorems, to argue that the conclusion Q is necessarily true.

Methods of Proof

Some common proof techniques are described below. While such summaries are helpful, there are no hard and fast rules that give a precise procedure for proving every possible mathematical statement. The methods of proof to be discussed here are in the nature of maps to guide you in analyzing and constructing proofs. A map may not reveal all the difficulties of the terrain, but it usually makes the route clearer and the journey easier.

DIRECT METHOD This method of proof depends on the basic rule of logic called ***modus ponens***: If R is a true statement and "$R \Rightarrow S$" is a true conditional statement, then S is a true statement. To prove the theorem "$P \Rightarrow Q$" by the direct method, you find a series of statements $P_1, P_2, \ldots, P_n$ and then verify that each of the implications $P \Rightarrow P_1$, $P_1 \Rightarrow P_2$, $P_2 \Rightarrow P_3, \ldots, P_{n-1} \Rightarrow P_n$, and $P_n \Rightarrow Q$ is true. Then the assumption that P is true and repeated use of *modus ponens* show that Q is true.

The direct method is the most widely used method of proof. In actual practice, it may be quite difficult to figure out the various intermediate statements that allow you to proceed from P to Q. In order to find them, most mathematicians use a thought process that is sometimes called the **forward-backward technique.** You begin by working forward and asking yourself, What do I know about the hypothesis P? What facts does it imply? What statements follow from these facts? And so on. At this point you may have a list of statements implied by P whose connection with the conclusion Q, if any, is not yet clear.

Now work backward from Q by asking, What facts would guarantee that Q is true? What statements would imply these facts? And so on. You now have a list of statements that imply Q. Compare it with the first list. If you are fortunate some statement will be on both lists, or more likely, there will be a statement S on the first list and a statement T on the second, and you may be able to show that $S \Rightarrow T$. Then you have $P \Rightarrow S$ and $S \Rightarrow T$ and $T \Rightarrow Q$, so that $P \Rightarrow Q$.

When you have used the forward-backward technique successfully to find a proof that $P \Rightarrow Q$, you should write the proof in finished form. This finished form may look quite different from the thought processes that led you to the proof. Your thought process jumped forward and backward, but the finished proof normally should begin with P and proceed in step-by-step logical order from P to S to T to Q. The finished proof should contain only those facts that are needed in the proof. Many statements that arise in the forward- backward process turn out to be irrelevant to the final argument, and they should *not* be included in the finished proof. As illustrated in most of the proofs in this book, the finished proof is usually written as a narrative rather than a series of conditional statements.

CONTRAPOSITIVE METHOD Since every conditional statement is equivalent to its contrapositive, you may prove "not-$Q \Rightarrow$ not-P" in order to conclude that "$P \Rightarrow Q$" is true. For example, instead of proving that for a certain function f,

$$\text{If } a \neq b, \text{ then } f(a) \neq f(b)$$

you can prove the contrapositive

$$\text{If } f(a) = f(b), \text{ then } a = b.$$

PROOF BY CONTRADICTION Suppose that you assume the truth of a statement R and that you make a valid argument that $R \Rightarrow S$ (that is, $R \Rightarrow S$ is a true statement). If the statement S is in fact a *false* statement, there is only one possible conclusion: The original statement R must have been false, because a true premise R and a true statement $R \Rightarrow S$ lead to the truth of S by *modus ponens.*

In order to use this fact to prove the theorem "$P \Rightarrow Q$", assume as usual that P is a true statement. Then apply the argument in the preceding paragraph with $R =$ not-Q. In other words, assume that *not-Q* is true and find an argument (presumably using P and previously proved results) that shows not-$Q \Rightarrow S$, where S is a statement known to be false. Conclude that not-Q must be false. But not-Q is false exactly when Q is true. Therefore, Q is true, and we have proved that $P \Rightarrow Q$. Once again, the hard part will usually be finding the statement S and proving that not-Q implies S.

EXAMPLE 1

Recall that an integer is even if it is a multiple of 2 and that an integer that is not even is said to be odd. We shall use proof by contradiction to prove this statement

$$\text{If } m^2 \text{ is even, then } m \text{ is even.}$$

Here P is the statement "m^2 is even" and Q is the statement "m is even". We assume "m is not even" or equivalently "m is odd" (statement not-Q). But every odd integer is 1 more than some even integer. Since every even integer is a multiple of 2, we must have $m = 2k + 1$ for some integer k. Then the basic laws of arithmetic show that

$$m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

This last statement says that m^2 is 1 more than a multiple of 2, that is, m^2 is odd. But we are given that m^2 is even (statement P), and, hence, "m^2 is both odd and even" (statement S). This statement is false since no integer is both odd and even. Therefore, our original assumption (not-Q) has led to a contradiction (the false statement S). Consequently, not-Q must be false, and, hence, the statement "m is even" (statement Q) is true.

In Example 1 various statements were labeled by letters so that you could easily relate the example to the general discussion. This is not usually done in proofs by contradiction, and such proofs may not be given in as much detail as in this example.

The choice of a method of proof is partly a matter of taste and partly a question of efficiency. Although any of those listed above may be used, one method may lead to a much shorter or easier-to-follow proof than another, depending on the circumstances. In addition there are methods of proof that can be applied only to certain types of statements.

PROOF BY INDUCTION This method is discussed in detail in Appendix C.

CONSTRUCTION METHOD This method is appropriate for theorems that include a statement of the type "There exists a such-and-such with property so-and-so". For instance,

There is an integer d such that $d^2 - 4d - 5 = 0$.

If r and s are distinct rational numbers, then there is a rational number between r and s.

If r is a positive real number, then there is a positive integer m such that $\frac{1}{m} < r$.

To prove such a statement, you must construct (find, build, guess, etc.) an object with the desired property. When you are reading the proof of such a statement, you need only verify that the object presented in the proof does in fact have the stated property. An existence proof may amount to nothing more than presenting an example (for instance, the integer 2 provides a proof of "there exists a positive integer"). But more often a nontrivial argument will be needed to produce the required object.

Caution Although an example is sufficient to prove an existence statement, examples can never *prove* a statement that directly or indirectly involves a universal quantifier. For instance, even if you have a million examples for which this statement is true:

If c is an integer, then $c^2 - c + 11$ is prime,

you will not have proved it. For the statement says, in effect, that for *every* integer c, a certain other integer is prime. This is *not* the case when $c = 12$ since $12^2 - 12 + 11 = 143 = 13 \cdot 11$. So the statement is false. This example demonstrates that

A counterexample is sufficient to *disprove* a statement.

The moral of the story is that when you are uncertain if a statement is true, try to find some examples where it holds or fails. If you find just one example where it fails, you have disproved the statement. If you can find only examples where the statement holds, you haven't proved it, but you do have encouraging evidence that it may be true.

Proofs of Multiconditional Statements

In order to prove the biconditional statement "P if and only if Q", you must prove *both* "$P \Rightarrow Q$" *and* "$Q \Rightarrow P$". Proving one of these statements and failing to prove the other is a common student mistake. For example, the proof of

A triangle with sides a, b, c is a right triangle with hypotenuse c if and only if $c^2 = a^2 + b^2$

consists of two separate parts. *First* you must assume that you have a right triangle with sides a, b and hypotenuse c and prove that $c^2 = a^2 + b^2$. Then you must give a *second* argument: Assume that the sides of a triangle satisfy $c^2 = a^2 + b^2$ and prove that this is a right triangle with hypotenuse c.

A statement of the form

The following conditions are equivalent: P, Q, R, S, T

is called a **multiconditional statement** and means that any one of the statements P, Q, R, S, or T implies every other one. Thus a multiconditional statement is just shorthand for a list of biconditional statements; $P \Leftrightarrow Q$ and $P \Leftrightarrow R$ and $P \Leftrightarrow S$ and $P \Leftrightarrow T$ and $Q \Leftrightarrow R$ and $Q \Leftrightarrow S$, etc. To prove this multiconditional statement you need only prove

$$P \Rightarrow Q \text{ and } Q \Rightarrow R \text{ and } R \Rightarrow S \text{ and } S \Rightarrow T \text{ and } T \Rightarrow P.$$

All the other required implications then follow immediately; for instance, from $T \Rightarrow P$ and $P \Rightarrow Q$, we know that $T \Rightarrow Q$, and similarly in the other cases.

EXAMPLE 2

In order to prove this theorem about integers:

The following conditions on a positive integer p are equivalent:

(1) p is prime.

(2) If p is a factor of ab, then p is a factor of a or p is a factor of b.

(3) If $p = rs$, *then* $r = \pm 1$ *or* $s = \pm 1$.

you must make *three* separate arguments. First, assume (1) and prove (2), so that (1) $\Rightarrow$ (2) is true. Second, you assume (2) and prove (3), so that (2) $\Rightarrow$ (3) is true. Finally, you must assume (3) and prove (1), so that (3) $\Rightarrow$ (1) is true. *Be careful:* At each stage you assume only one of the three statements and use it to prove another; the third statement does not play a role in that part of the argument.

APPENDIX B

Sets and Functions

For our purposes, a **set** is any collection of objects; for example,

The set $\mathbb{Z}$ of integers.

The set of right triangles with area 24.

The set of positive irrational numbers.

The objects in a set are called **elements** or **members** of the set. If B is a set, the statement "b is an element of B" is abbreviated as "$b \in B$". Similarly, "$b \notin B$" means "b is not an element of B". For example, if $\mathbb{Z}$ is the set of integers, then

$$2 \in \mathbb{Z} \quad \text{and} \quad \pi \notin \mathbb{Z}.$$

There are several methods of describing sets. A set may be defined by **verbal description** as in the examples above. A small finite set can be described by **listing** all its elements. Such a list is customarily placed between curly brackets; for instance,

$$\{3, 7, -4, 9\} \quad \text{or} \quad \{a, b, c, r, s, t\}.$$

Listing notation is sometimes used for infinite sets as well. For example, $\{2, 4, 6, 8, \ldots\}$ indicates the set of positive even integers. Strictly speaking, this notation is ambiguous in the infinite case since it relies on everyone's seeing the same pattern and understanding that it is to continue forever. But when the context is clear, no confusion will result.

Finally, a set can be described in terms of properties that are satisfied by its elements, and by these elements only. This is usually done with **set-builder notation.** For example,

$$\{x \mid x \text{ is an integer and } x > 9\}$$

denotes the set of all elements x *such that* x is an integer greater than 9. In general, the vertical line is shorthand for "such that" and "$\{y \mid P\}$" is read "the set of all elements y such that P". Thus each of the following is the set of even integers:

$\{x \mid x \text{ is an even integer}\}$.

$\{t \mid t \in \mathbb{Z} \text{ and } t \text{ is even}\}$.

$\{r \mid r \in \mathbb{Z} \text{ and } r \text{ is a multiple of } 2\}$.

$\{y \mid y \in \mathbb{Z} \text{ and } y = 2k \text{ for some integer } k\}$.

The Empty Set

Some special cases of set-builder notation lead to an unusual set. For instance, the set

$$\{x \mid x \text{ is an integer and } 0 < x < 1\}$$

has no elements since there is no integer between 0 and 1. The set with no elements is called the **empty set** or **null set** and is denoted $\varnothing$. For every element c,

$$c \in \varnothing \text{ is false} \quad \text{and} \quad c \notin \varnothing \text{ is true.}$$

The empty set is a very convenient concept to have around, but some care must be taken when dealing with theorems that are true only for **nonempty sets** (that is, sets that have at least one element).

Subsets

A set B is said to be a **subset** of a set C (written $B \subseteq C$) provided that every element of B is also an element of C. In other words, $B \subseteq C$ exactly when this statement is true:

$$x \in B \Rightarrow x \in C.$$

For example, the set of even integers is a subset of the set $\mathbb{Z}$ of all integers, and the set of rational numbers is a subset of the set of real numbers.

The definition of "$B \subseteq C$" allows the possibility that $B = C$ (since it is certainly true in this case that every element of B is also an element of C). In other words,

$$\mathbf{B \subseteq B \textbf{ for every set } B.}$$

If B is a subset of C and $B \neq C$ we say that B is a **proper subset** of C and write $B \subsetneqq C$.

The subset relation is easily seen to be *transitive*, that is,

$$\textbf{If } B \subseteq C \textbf{ and } C \subseteq D\textbf{, then } B \subseteq D.$$

Two sets B and C are **equal** when they have exactly the same elements. In this case every element of B is an element of C *and* every element of C is an element of B. Thus,

$$B = C \qquad \textbf{if and only if} \qquad B \subseteq C \textbf{ and } C \subseteq B.$$

This fact is the most commonly used method of proving that two sets are equal: Prove that each is a subset of the other.

Basic logic leads to a surprising fact about the empty set. Since the statement $x \in \varnothing$ is always false, the implication

$$x \in \varnothing \Rightarrow x \in C$$

is always true (see Appendix A). But this is precisely the definition of "$\varnothing$ is a subset of C". So

$$\textbf{the empty set } \varnothing \textbf{ is a subset of every set.}$$

Operations on Sets

We now review the standard ways of constructing new sets from given ones. If B and C are sets, then the **relative complement** of C in B is denoted $B - C$ and consists of the elements of B that are not in C. Thus

$$B - C = \{x \mid x \in B \text{ and } x \notin C\}.$$

For example, if E is the set of even integers, then $\mathbb{Z} - E$ is the set of odd integers.

The **intersection** of sets B and C consists of all the elements that are in *both* B and C and is denoted $B \cap C$. Thus

$$B \cap C = \{x \mid x \in B \text{ and } x \in C\}.$$

For example, if $B = \{-2, 1, \sqrt{2}, 5, \pi\}$ and C is the set of positive rational numbers, then $B \cap C = \{1, 5\}$ since 1 and 5 are the only elements in both sets. If B is the set of positive integers and C the set of negative integers, then $B \cap C = \varnothing$ since there are no elements in both sets. When B and C are sets such that $B \cap C = \varnothing$, we say that B and C are **disjoint.**

The **union** of sets B and C consists of all elements that are in at least one of B or C and is denoted $B \cup C$. Thus,

$$B \cup C = \{x \mid x \in B \text{ or } x \in C\}.$$

For example, the union of $B = \{1, 3, 5, 7\}$ and $C = \{-1, 1, 4, 9\}$ is $B \cup C = \{-1, 1, 3, 4, 5, 7, 9\}$. If B is the set of rational numbers and C is the set of irrational numbers, then $B \cup C$ is the set of all real numbers.

You should verify that union and intersection have the following properties. For any sets B, C, and D,

$$\begin{array}{rcl}
B \cup B = B & & B \cap B = B \\
B \cup \varnothing = B & & B \cap \varnothing = \varnothing \\
B \cup C = C \cup B & & B \cap C = C \cap B \\
B \subseteq B \cup C & & B \cap C \subseteq B \\
B \subseteq C & \text{if and only if} & B \cup C = C \\
B \subseteq C & \text{if and only if} & B \cap C = B \\
B \cup (C \cup D) = (B \cup C) \cup D & & B \cap (C \cap D) = (B \cap C) \cap D
\end{array}$$

$$B \cap (C \cup D) = (B \cap C) \cup (B \cap D)$$

$$B \cup (C \cap D) = (B \cup C) \cap (B \cup D).$$

The concepts of union and intersection extend readily to large, possibly infinite, collections of sets. Suppose that I is some nonempty set (called an **index set**) and that for each $i \in I$, we are given a set A_i. Then the intersection of this family of sets (denoted $\bigcap_{i \in I} A_i$) is the set of elements that are in *all* the sets A_i, that is,

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for every } i \in I\}.$$

Similarly, the union of this family of sets (denoted $\bigcup_{i \in I} A_i$) is the set of elements that are in at least one of the sets A_i, that is,

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_j \text{ for some } j \in I\}.$$

The **Cartesian product** of sets B and C is denoted $B \times C$ and consists of all ordered pairs (x, y) with $x \in B$ and $y \in C$. Equality of ordered pairs is defined by this rule:

$$(x, y) = (u, v) \qquad \text{if and only if} \qquad x = u \text{ in } B \text{ and } y = v \text{ in } C.$$

For example, if $B = \{r, s, t\}$ and $C = \{5, 7\}$, then $B \times C$ is the set

$$\{(r, 5), (r, 7), (s, 5), (s, 7), (t, 5), (t, 7)\}.$$

The set $\mathbb{R}$ of real numbers is sometimes identified with the number line. When this is done, the Cartesian product $\mathbb{R} \times \mathbb{R}$ is just the ordinary coordinate plane, the set of all points with coordinates (x, y) where $x, y \in \mathbb{R}$.

The Cartesian product of any finite number of sets $B_1, B_2, \ldots, B_n$ is defined in a similar fashion. $B_1 \times B_2 \times \cdots \times B_n$ is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \in B_i$ for each $i = 1, 2, \ldots, n$. For example, if $B = \{0, 1\}$, $\mathbb{Z}$ is the set of integers, and $\mathbb{R}$ the set of real numbers, then $B \times \mathbb{Z} \times \mathbb{R}$ is the set of all ordered triples of the form $(0, k, r)$ and $(1, k, r)$ with $k \in \mathbb{Z}$ and $r \in \mathbb{R}$. The product $B \times \mathbb{Z} \times \mathbb{R}$ is an infinite set; among its elements are $(0, -5, 3)$, $(1, 24, \pi)$, and $(1, 1, -\sqrt{3})$.

Functions

A **function** (or **map** or **mapping**) f from a set B to a set C (denoted $f{:}B \to C$) is a rule that assigns to each element b of B exactly one element c of C; c is called the **image** of b or the **value** of the function f at b and is usually denoted $f(b)$. The set B is called the **domain** and the set C the **range** of the function f.

Your previous mathematics courses dealt with a wide variety of functions. For instance, if $\mathbb{R}$ is the set of real numbers, then each of the following rules defines a function from $\mathbb{R}$ to $\mathbb{R}$:

$$f(x) = \cos x, \qquad g(x) = x^2 + 1, \qquad h(x) = x^3 - 5x + 2.$$

The rule of a function need not be given by an algebraic formula. For instance, consider the function $f{:}\mathbb{Z} \to \{0, 1\}$, whose rule is

$$f(x) = 0 \text{ if } x \text{ is even and } f(x) = 1 \text{ if } x \text{ is odd.}$$

If B is a set, then the function from B to B defined by the rule "map every element to itself" is called the **identity map** on B and is denoted ι_B. Thus $\iota_B{:}B \to B$ is defined by

$$\iota_B(x) = x \text{ for every } x \in B.$$

Composition of Functions

Let f and g be functions such that the range of f is the same as the domain of g, say $f{:}B \to C$ and $g{:}C \to D$. Then the **composite** of f and g is the function $h{:}B \to D$ whose rule is

$$h(x) = g(f(x)).$$

In other words, the composite function is obtained by first applying f and then applying g:

$$\begin{array}{ccccc} B & \xrightarrow{f} & C & \xrightarrow{g} & D \\ x & \longrightarrow & f(x) & \longrightarrow & g(f(x)). \end{array}$$

Instead of h, the usual notation for the composite function of f and g is $g \circ f$ (note the order). Thus, $g \circ f: B \to D$ is defined by $(g \circ f)(x) = g(f(x))$.

EXAMPLE 1

Let E be the set of even integers and $\mathbb{N}$ the set of nonnegative integers. Let $f: E \to \mathbb{Z}$ be defined by $f(x) = x/2$ (since x is even, $x/2$ *is* an integer). Let $g: \mathbb{Z} \to \mathbb{N}$ be given by $g(n) = n^2$. Then the composite function $g \circ f: E \to \mathbb{N}$ has this rule:

$$(g \circ f)(x) = g(f(x)) = g(x/2) = (x/2)^2 = x^2/4.$$

The composite function in the opposite order, $f \circ g$ (first apply g, then f), is *not defined* since the range of g is not the same as the domain of f. For instance, $g(3) = 9$, but the domain of f is the set of even integers; even though the rule of f makes sense for odd integers, $f(g(3)) = f(9) = 9/2$, which is not in $\mathbb{Z}$.

EXAMPLE 2

Let $f: \mathbb{Z} \to \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = x - 1$ and $g(x) = x^2$. Then the composite function $f \circ g: \mathbb{Z} \to \mathbb{Z}$ is given by the rule

$$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 - 1.$$

In this case the composite function in the opposite order $g \circ f$ is also defined; its rule is

$$(g \circ f)(x) = g(f(x)) = g(x - 1) = (x - 1)^2 = x^2 - 2x + 1.$$

Thus we have, for instance,

$$(f \circ g)(3) = 9 - 1 = 8 \qquad \text{but} \qquad (g \circ f)(3) = 9 - 6 + 1 = 4.$$

So even though both are defined, *$f \circ g$ is not the same function as $g \circ f$.*

Two functions $h: B \to C$ and $k: B \to C$ are said to be **equal** provided that $h(b) = k(b)$ for every $b \in B$.

EXAMPLE 3

Let $f: B \to C$ be any function and $\iota_C: C \to C$ the identity map on C. Then $\iota_C \circ f: B \to C$, and for every $b \in B$

$$(\iota_C \circ f)(b) = \iota_C(f(b)) = f(b).$$

Therefore $\iota_C \circ f = f$. Similarly, if ι_B is the identity map on B, then $f \circ \iota_B : B \to C$, and for every $b \in B$

$$(f \circ \iota_B)(b) = f(\iota_B(b)) = f(b).$$

Consequently,

$$\textbf{If } f: B \longrightarrow C, \textbf{ then} \quad \iota_C \circ f = f \quad \textbf{and} \quad f \circ \iota_B = f.$$

If $f: B \to C$, $g: C \to D$, and $h: D \to E$ are functions, then each of the composite functions $(f \circ g) \circ h$ and $f \circ (g \circ h)$ is a map from B to E. We claim that

$$(f \circ g) \circ h = f \circ (g \circ h).$$

The proof of this statement is simply an exercise in using the definition of composite function. For each $b \in B$

$$[(f \circ g) \circ h](b) = (f \circ g)(h(b)) = f[g(h(b))]$$

and

$$[f \circ (g \circ h)](b) = f[(g \circ h)(b)] = f[g(h(b))].$$

Since the right sides of the two equalities are identical, the composite functions $(f \circ g) \circ h$ and $f \circ (g \circ h)$ have the same effect on each $b \in B$, which proves the claim.

Binary Operations

Informally we can think of a binary operation on the integers, for example, as a rule for producing a new integer from two given ones. Ordinary addition and multiplication are operations in this sense: Given a and b we get $a + b$ and ab. Producing a new integer from a pair of given ones also suggests the idea of a function. Addition of integers may be thought of as the function f from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ whose rule is

$$f(a, b) = a + b.$$

Similarly, multiplication can be thought of as the function $g: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $g(a, b) = ab$.

With the preceding examples in mind we make this formal definition. A **binary operation** on a nonempty set B (usually called simply an **operation** on B) is a function $f: B \times B \to B$. The familiar examples suggest a new notation for the general case. We use some symbol, say $*$, to denote the operation and write $a * b$ instead of $f(a, b)$.

EXAMPLE 4

As we saw above, ordinary addition and multiplication are operations on $\mathbb{Z}$. Another operation on $\mathbb{Z}$ is defined by the function $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ whose rule is $f(a, b) = ab - 1$. If we denote this operation by $*$, then $3 * 5 = 15 - 1 = 14$, and, similarly,

$$12 * 4 = 47 \qquad -7 * 4 = -29 \qquad 0 * 8 = -1.$$

Note that $a * b = ab - 1 = ba - 1 = b * a$, so that the order of the elements doesn't matter when applying $*$, as is the case with ordinary addition and multiplication (the technical term for this property is *commutativity*). On the other hand,

$$(1 * 2) * 3 = 1 * 3 = 2 \qquad \text{but} \qquad 1 * (2 * 3) = 1 * 5 = 4,$$

so that $(a * b) * c \neq a * (b * c)$ in general. Thus $*$ is not *associative* as are addition and multiplication (meaning that $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ always).

EXAMPLE 5

Let S be a nonempty set. If $f{:}S \to S$ and $g{:}S \to S$ are functions, then their composite $f \circ g$ is also a function from S to S. So if B is the set of all functions from S to S, then composition of functions is an operation on the set B. In other words, the map that sends (f, g) to $f \circ g$ is a function from $B \times B$ to B. The discussion of composite functions above shows that the operation $\circ$ on B is associative (that is, $(f \circ g) \circ h = f \circ (g \circ h)$ always) but not commutative ($f \circ g$ need not equal $g \circ f$).

Let $*$ be an operation on a set B and $C \subseteq B$. The subset C is said to be **closed** under the operation $*$ provided that

$$\text{Whenever } a, b \in C, \text{ then } a * b \in C.$$

Consider, for example, the operation of ordinary multiplication on the set B of positive real numbers. Let C be the subset of positive integers. Then C is closed under the operation since ab is a positive integer whenever a and b are. But when the operation on B is ordinary division, then C is not closed: If a and b are integers, $a \div b$ need not be an integer (for instance, $3 \div 7 = 3/7 \notin C$).

If $*$ is an operation on a set B, then B (considered as a subset of itself) is closed under $*$ by the definition of an operation. Nevertheless many texts, including this one, routinely list the **closure** of B under $*$ as one of the properties of the operation. Although this isn't logically necessary, it calls your attention to the importance of closure and reminds you that closure cannot be taken for granted for subsets other than B.

Injective and Surjective Functions

A function $f{:}B \to C$ is said to be **injective** (or **one-to-one**) provided f maps distinct elements of B to distinct elements of C, or in functional notation: If $a \neq b$ in B, then $f(a) \neq f(b)$ in C. This rather awkward statement is equivalent to its contrapositive, so that we have this useful description:

$f{:}B \longrightarrow C$ is injective provided that
whenever $f(a) = f(b)$ in C, then $a = b$ in B.

EXAMPLE 6

Let $\mathbb{R}$ be the set of real numbers. In order to show that the function $f:\mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 3$ is injective, we assume that $f(a) = f(b)$, that is,

$$2a + 3 = 2b + 3.$$

Subtracting 3 from each side shows that $2a = 2b$; dividing both sides by 2 we conclude that $a = b$. Therefore, f is injective.

EXAMPLE 7

The map $f:\mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is *not* injective because we have $f(-3) = 9 = f(3)$, but $-3 \neq 3$. Alternatively, the distinct elements 3 and -3 have the same image.

A function $f:B \to C$ is said to be **surjective** (or **onto**) provided that every element of C is the image under f of at least one element of B, that is,

$$\text{For each } c \in C \text{ there exists } b \in B \text{ such that } f(b) = c.$$

EXAMPLE 8

Let $\mathbb{N}$ be the set of nonnegative integers and $f:\mathbb{Z} \to \mathbb{N}$ the function given by $f(x) = |x|$. Then f is surjective since every element of $\mathbb{N}$ is the image under f of at least one element of $\mathbb{Z}$ (namely itself). Note, however, that f is not injective since, for example, $f(1) = f(-1)$.

EXAMPLE 9

Let E be the set of even integers and consider the map $g:\mathbb{Z} \to E$ given by $g(x) = 4x$. We claim that the element 2 in E is *not* the image under g of any element of $\mathbb{Z}$. If $2 = g(b)$ for some $b \in \mathbb{Z}$, then $2 = 4b$, so that $1 = 2b$. This is impossible since 1 is not an integer multiple of 2. Therefore, g is *not* surjective. Note, however, that g *is* injective since $4a = 4b$ (that is, $g(a) = g(b)$) implies that $a = b$.

EXAMPLE 10

Let $\mathbb{R}$ be the set of real numbers and $f: \mathbb{R} \to \mathbb{R}$ the function given by $f(x) = 2x + 3$. To prove that f is surjective, let $c \in \mathbb{R}$; we must find $b \in \mathbb{R}$ such that $f(b) = c$. In other words, we must find a number b such that $2b + 3 = c$. To do so, we solve this last equation for b and find $b = \dfrac{c-3}{2}$. Then $f(b) = 2\left(\dfrac{c-3}{2}\right) + 3 = c - 3 + 3 = c$. Therefore, f is surjective. The map f is also injective (see Example 6).

The preceding examples demonstrate that *injectivity and surjectivity are independent concepts*. One does not imply the other, and a particular map might have one, both, or neither of these properties.

If $f:B \to C$ is a function, then the **image of** f is this subset of C:

$$\operatorname{Im} f = \{c \mid c = f(b) \text{ for some } b \in B\} = \{f(b) \mid b \in B\}.$$

For example, if $f:\mathbb{Z} \to \mathbb{Z}$ is given by $f(x) = 2x$, then $\operatorname{Im} f$ is the set of even integers since $\operatorname{Im} f = \{f(x) \mid x \in \mathbb{Z}\} = \{2x \mid x \in \mathbb{Z}\}$. Similarly, if $g:\mathbb{Z} \to \mathbb{Z}$ is given by $g(x) = |x|$, then $\operatorname{Im} g$ is the set of nonnegative integers. A map $f:B \to C$ is surjective exactly when every element of C is the image of an element of B. Thus

$$\boldsymbol{f:B \to C \text{ is surjective if and only if } \operatorname{Im} f = C.}$$

If $f:B \to C$ is a function and S is a subset of B, then the **image of the subset** S is the set

$$f(S) = \{c \mid c = f(b) \text{ for some } b \in S\} = \{f(b) \mid b \in S\}.$$

If $f:\mathbb{Z} \to \mathbb{Z}$ is given by $f(x) = 2x$, for example, and S is the set of odd integers, then $f(S) = \{2x \mid x \text{ is odd}\}$ is the set of even integers that are not multiples of 4. If the subset S is the entire set B, then $f(B)$ is precisely $\operatorname{Im} f$.

Bijective Functions

A function $f:B \to C$ is **bijective** (or a **bijection** or **one-to-one correspondence**) provided that f is both injective and surjective.

EXAMPLE 11

Examples 6 and 10 show that the map $f:\mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 3$ is bijective.

EXAMPLE 12

The map f from the set $\{1, 2, 3, 4, 5\}$ to the set $\{v, w, x, y, z\}$ given by

$$f(1) = v \qquad f(2) = w \qquad f(3) = x \qquad f(4) = y \qquad f(5) = z$$

is easily seen to be bijective.

The last example illustrates the fact that for any *finite* sets B and C, there is a bijection from B to C if and only if B and C have the same number of elements. In particular, if B is finite and $C \subsetneq B$, then there cannot be a bijection from B to C. But the situation is quite different with infinite sets.

EXAMPLE 13

Let E be the set of even integers and consider the map $f:\mathbb{Z} \to E$ given by $f(x) = 2x$. By definition every even integer is 2 times some integer, so f is surjective. Furthermore, $2a = 2b$ implies that $a = b$, so f is injective. Therefore, f is a bijection. In this case, a bit more is true. Define a map $g:E \to \mathbb{Z}$ by $g(u) = u/2$;

this makes sense since $u/2$ is an integer when u is even. Consider the composite function $g \circ f: \mathbb{Z} \to \mathbb{Z}$:

$$(g \circ f) = g(f(x)) = g(2x) = 2x/2 = x.$$

Thus $(g \circ f)(x) = x = \iota_{\mathbb{Z}}(x)$ for every x, and the composite map $g \circ f$ is just the identity map $\iota_{\mathbb{Z}}$ on $\mathbb{Z}$. Now look at the other composite, $f \circ g: E \to E$:

$$(f \circ g)(u) = f(g(u)) = f(u/2) = 2(u/2) = u.$$

Therefore, the composite map $f \circ g$ is the identity map ι_E.

Example 13 illustrates a property that all bijective functions have, as we now prove.

Theorem B.1

A function $f: B \to C$ is bijective if and only if there exists a function $g: C \to B$ such that

$$g \circ f = \iota_B \qquad \text{and} \qquad f \circ g = \iota_C.$$

Proof ▸ Assume first that f is bijective. Define $g: C \to B$ as follows. If $c \in C$, then there exists $b \in B$ such that $f(b) = c$ because f is surjective. Furthermore, since f is also injective, there is only one element b such that $f(b) = c$ (for if $f(b') = c$, then $f(b) = f(b')$ implies $b = b'$). So we can define a function $g: C \to B$ by this rule:

$$g(c) = b, \text{ where } b \text{ is the unique element of } B \text{ such that } f(b) = c.$$

Then $g(c) = b$ exactly when $f(b) = c$. Thus for any $c \in C$

$$(f \circ g)(c) = f(g(c)) = f(b) = c,$$

from which we conclude that $f \circ g = \iota_C$. Similarly, for each $u \in B$, $f(u)$ is an element of C, say $f(u) = v$, and, hence, by the definition of g, we have $g(v) = u$. Therefore,

$$(g \circ f)(u) = g(f(u)) = g(v) = u$$

and $g \circ f = \iota_B$. This proves the first half of our biconditional theorem.

To prove the other half, we assume that a map $g: C \to B$ with the stated properties is given. We must show that f is bijective. Suppose $f(a) = f(b)$. Then

$$\begin{aligned} g(f(a)) &= g(f(b)) \\ (g \circ f)(a) &= (g \circ f)(b) \\ \iota_B(a) &= \iota_B(b) \\ a &= b. \end{aligned}$$

Therefore, $f(a) = f(b)$ implies $a = b$, and f is injective. To show that f is surjective, let c be any element of C. Then $g(c) \in B$ and $f(g(c)) = (f \circ g)(c) = \iota_C(c) = c$. So we have found an element of B that f maps onto c (namely $g(c)$); hence, f is surjective. Therefore, f is bijective, and the theorem is proved. ■

If $f:B \to C$ is a bijection, then the map g in Theorem B.1 is called the **inverse** of f and is sometimes denoted by f^{-1}. Reversing the roles of f and g in Theorem B.1 shows that the inverse map g of a bijection f is itself a bijection.

Exercises

NOTE: *$\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ the set of rational numbers, and $\mathbb{R}$ the set of real numbers.*

A. **1.** Describe each set by listing:

(a) The integers strictly between -3 and 9.

(b) The negative integers greater than -10.

(c) The positive integers whose square roots are less than or equal to 4.

2. Describe each set in set-builder notation:

(a) All positive real numbers.

(b) All negative irrational numbers.

(c) All points in the coordinate plane with rational first coordinate.

(d) All negative even integers greater than -50.

3. Which of the following sets are nonempty?

(a) $\{r \in \mathbb{Q} \mid r^2 = 2\}$

(b) $\{r \in \mathbb{R} \mid r^2 + 5r - 7 = 0\}$

(c) $\{t \in \mathbb{Z} \mid 6t^2 - t - 1 = 0\}$

4. Is B a subset of C when

(a) $B = \mathbb{Z}$ and $C = \mathbb{Q}$?

(b) $B =$ all solutions of $x^2 + 2x - 5 = 0$ and $C = \mathbb{Z}$?

(c) $B = \{a, b, 7, 9, 11, -6\}$ and $C = \mathbb{Q}$?

5. If $A \subseteq B$ and $B \subseteq C$, prove that $A \subseteq C$.

6. In each part find $B - C$, $B \cap C$, and $B \cup C$:

(a) $B = \mathbb{Z}$, $C = \mathbb{Q}$. **(b)** $B = \mathbb{R}$, $C = \mathbb{Q}$.

(c) $B = \{a, b, c, 1, 2, 3, 4, 5\}$, $C = \{a, c, e, 2, 4, 6, 8\}$.

7. List the elements of $B \times C$ when $B = \{a, b, c\}$ and $C = \{0, 1, c\}$.

8. List the elements of $A \times B \times C$ when $A = \{0, 1\}$ and B, C are as in Exercise 7.

9. Let $A = \{1, 2, 3, 4\}$. Exhibit functions f and g from A to A such that $f \circ g \neq g \circ f$.

10. Do Exercise 9 when $A = \mathbb{Z}$.

11. Is the subset B closed under the given operation?

(a) B = even integers; operation: multiplication in $\mathbb{Z}$.

(b) B = odd integers; operation: addition in $\mathbb{Z}$.

(c) B = nonzero rational numbers; operation: division in the set of nonzero real numbers.

(d) B = odd integers; operation $*$ on $\mathbb{Z}$, where $a * b$ is defined to be the number $ab - (a + b) + 2$.

12. Find the image of the function f when

(a) $f{:}\mathbb{R} \to \mathbb{R}; f(x) = x^2$.

(b) $f{:}\mathbb{Z} \to \mathbb{Q}; f(x) = x - 1$.

(c) $f{:}\mathbb{R} \to \mathbb{R}; f(x) = -x^2 + 1$.

13. Let $B = \{1, 2, 3, 4\}$ and $C = \{a, b, c\}$.

(a) List four different surjective functions from B to C.

(b) List four different injective functions from C to B.

(c) List all bijective functions from C to C.

14. **(a)** Give an example of a function f that is injective but not surjective.

(b) Give an example of a function g that is surjective but not injective.

15. Let B and C be nonempty sets. Prove that the function

$$f{:}B \times C \longrightarrow C \times B$$

given by $f(x, y) = (y, x)$ is a bijection.

B. 16. List all the subsets of $\{1, 2\}$. Do the same for $\{1, 2, 3\}$ and $\{1, 2, 3, 4\}$. Make a conjecture as to the number of subsets of an n-element set. [Don't forget the empty set.]

17. Verify each of the properties of sets listed on page 511.

18. If $a, b \in \mathbb{R}$ with $a < b$, then the set $\{r \in \mathbb{R} \mid a \leq r < b\}$ is denoted $[a, b)$. Let N denote the nonnegative integers and P the positive integers. Find these unions and intersections:

(a) $\displaystyle\bigcup_{n \in N} [n, n + 1)$

(b) $\displaystyle\bigcup_{n \in P} \left[\frac{1}{n}, 2 + \frac{1}{n}\right)$

(c) $\displaystyle\bigcap_{n \in P} \left[-\frac{1}{n}, 0\right)$

(d) $\displaystyle\bigcap_{n \in P} \left[\frac{1}{n}, 2 + \frac{1}{n}\right)$

19. Prove that for any sets A, B, C:

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

20. Let A, B be subsets of U. Prove **De Morgan's laws:**

   (a) $U - (A \cap B) = (U - A) \cup (U - B)$

   (b) $U - (A \cup B) = (U - A) \cap (U - B)$

21. Prove that for any sets A, B, C:

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

22. If C is a finite set, then $|C|$ denotes the number of elements in C. If A and B are finite sets, is it true that $|A \cup B| = |A| + |B|$?

23. Let $\mathbb{R}^{**}$ denote the positive real numbers. Does the following rule define a function from $\mathbb{R}^{**}$ to $\mathbb{R}$: assign to each positive real number c the real number whose square is c?

24. Determine whether the given operation on $\mathbb{R}$ is commutative (that is, $a * b = b * a$ for all a, b) or associative (that is, $a * (b * c) = (a * b) * c$ for all a, b, c).

   (a) $a * b = 2^{ab}$ **(b)** $a * b = ab^2$

   (c) $a * b = 0$ **(d)** $a * b = (a + b)/2$

   (e) $a * b = 1$ **(f)** $a * b = b$

   (g) $a * b = a^2 + b^2$

25. Prove that the given function is injective.

   (a) $f{:}\mathbb{Z} \to \mathbb{Z}; f(x) = 2x$

   (b) $f{:}\mathbb{R} \to \mathbb{R}; f(x) = x^3$

   (c) $f{:}\mathbb{Z} \to \mathbb{Q}; f(x) = x/7$

   (d) $f{:}\mathbb{R} \to \mathbb{R}; f(x) = -3x + 5$

26. Prove that the given function is surjective.

   (a) $f{:}\mathbb{R} \to \mathbb{R}; f(x) = x^3$

   (b) $f{:}\mathbb{Z} \to \mathbb{Z}; f(x) = x - 4$

   (c) $f{:}\mathbb{R} \to \mathbb{R}; f(x) = -3x + 5$

   (d) $f{:}\mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}; f(a, b) = a/b$ when $b \neq 0$ and 0 when $b = 0$.

27. Let $f{:}B \to C$ and $g{:}C \to D$ be functions. Prove:

   (a) If f and g are injective, then $g \circ f{:}B \to D$ is injective.

   (b) If f and g are surjective, then $g \circ f$ is surjective.

28. **(a)** Let $f{:}B \to C$ and $g{:}C \to D$ be functions such that $g \circ f$ is injective. Prove that f is injective.

   (b) Give an example of the situation in part (a) in which g is not injective.

29. **(a)** Let $f{:}B \to C$ and $g{:}C \to D$ be functions such that $g \circ f$ is surjective. Prove that g is surjective.

   (b) Give an example of the situation in part (a) in which f is not surjective.

30. Let $g:B \times C \to C$ (with $B \neq \varnothing$) be the function given by $g(x, y) = y$.

(a) Prove that g is surjective.

(b) Under what conditions, if any, is g injective?

31. If $f:B \to C$ is a function, then f can be considered as a map from B to $\operatorname{Im} f$ since $f(b) \in \operatorname{Im} f$ for every $b \in B$. Show that the map $f:B \to \operatorname{Im} f$ is surjective.

32. Let B be a finite set and $f:B \to B$ is a function. Prove that f is injective if and only if f is surjective.

33. Let $f:B \to C$ be a function and let S, T be subsets of B.

(a) Prove that $f(S \cup T) = f(S) \cup f(T)$.

(b) Prove that $f(S \cap T) \subseteq f(S) \cap f(T)$.

(c) Give an example where $f(S \cap T) \neq f(S) \cap f(T)$.

34. Prove that $f:B \to C$ is injective if and only if $f(S \cap T) = f(S) \cap f(T)$ for every pair of subsets S, T of B.

35. Let $f:B \to C$ and $g:C \to D$ be bijective functions. Then the composite function $g \circ f:B \to D$ is bijective by Exercise 27. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

APPENDIX C

Well Ordering and Induction

We assume that you are familiar with ordinary arithmetic in the set $\mathbb{Z}$ of integers and with the usual order relation ($<$) on $\mathbb{Z}$. The subset of nonnegative integers will be denoted by $\mathbb{N}$. Thus

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Finally, we assume this fundamental axiom:

WELL-ORDERING AXIOM *Every nonempty subset of* $\mathbb{N}$ *contains a smallest element.*

Most people find this axiom quite plausible, but it is important to note that it may not hold if $\mathbb{N}$ is replaced by some other set of numbers; see page 3 of the text for examples.

An important consequence of the Well-Ordering Axiom is the method of proof known as mathematical induction. It can be used to prove statements such as

A set of n elements has 2^n subsets.

Denote this statement by the symbol $P(n)$ and observe that there are really infinitely many statements, one for each possible value of n:

$P(0)$: A set of 0 elements has $2^0 = 1$ subset.

$P(1)$: A set of 1 element has $2^1 = 2$ subsets.

$P(2)$: A set of 2 elements has $2^2 = 4$ subsets.

$P(3)$: A set of 3 elements has $2^3 = 8$ subsets.

And so on. To prove the original proposition we must prove that

$P(n)$ is a true statement for every $n \in \mathbb{N}$.

Here's how it can be done.

Theorem C.1 The Principle of Mathematical Induction

Assume that for each nonnegative integer n, a statement $P(n)$ is given. If

(i) $P(0)$ is a true statement; and

(ii) Whenever $P(k)$ is a true statement, then $P(k + 1)$ is also true,

then $P(n)$ is a true statement for every $n \in \mathbb{N}$.

The example of the number of subsets of a set of n elements is continued after the proof of the theorem. You may want to read that example now to see how Theorem C.1 is *applied*, which is quite different from the manner in which it is proved.

Proof of Theorem C.1 ▸ Let S be the subset of $\mathbb{N}$ consisting of those integers j for which $P(j)$ is *false*. To prove the theorem we need only show that S is empty; we shall use proof by contradiction to do this. Suppose S is nonempty. Then by the Well-Ordering Axiom, S contains a smallest element, say d. Since $P(d)$ is false by the definition of S and $P(0)$ is true by property (i), we must have $d \neq 0$. Consequently, $d \geq 1$ (because d is a nonnegative integer), and, hence, $d - 1 \geq 0$, that is, $d - 1 \in \mathbb{N}$. Since $d - 1 < d$ and d is the smallest element in S, $d - 1$ cannot be in S. Therefore, $P(d - 1)$ must be true (otherwise $d - 1$ would be in S). Property (ii) (with $k = d - 1$) implies that $P((d - 1) + 1) = P(d)$ is also a true statement. This is a contradiction since $d \in S$. Therefore, S is the empty set, and the theorem is proved. ■

In order to apply the Principle of Mathematical Induction to a series of statements, you must verify that these statements satisfy *both* properties (i) and (ii). Note that property (ii) does *not* assert that any particular $P(k)$ is actually true, but only that a conditional relationship holds: *If* $P(k)$ is true, *then* $P(k + 1)$ must also be true. So to verify property (ii), you assume the truth of $P(k)$ and use this assumption to prove that $P(k + 1)$ is true. As we shall see in the examples below, it is often possible to prove this conditional statement even though you may not be able to prove directly that a particular $P(j)$ is true. The assumption that $P(k)$ is true is called the **induction assumption** or the **induction hypothesis.**

You may have seen induction used to prove statements such as "the sum of the first n nonnegative integers is $\frac{n(n + 1)}{2}$"; here $P(n)$ is the statement: "$0 + 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$". Although such examples make nice exercises for beginners, they are not typical of the way induction is used in advanced mathematics. The examples below will give you a more comprehensive picture of inductive proof. They are a bit more complicated than the usual elementary examples but are well within your reach.

EXAMPLE 1

We shall use the Principle of Mathematical Induction to prove that for each $n \geq 0$,

A set of n elements has 2^n subsets.

If $n = 0$, then the set must be the empty set (the only set with no elements). Its one and only subset is itself (since $\varnothing$ is a subset of every set). So the statement

$$P(0)\text{: A set of 0 elements has } 2^0 = 1 \text{ subset}$$

is true (property (i) holds).

In order to verify property (ii) of Theorem C. 1, we assume the truth of

$$P(k)\text{: A set of } k \text{ elements has } 2^k \text{ subsets}$$

and use this induction hypothesis to prove

$$P(k+1)\text{: A set of } k+1 \text{ elements has } 2^{k+1} \text{ subsets.}$$

To do this, let T be any set of $k + 1$ elements and choose some element c of T. Every subset of T either contains c or does not contain c. The subsets of T that do not contain c are precisely the subsets of $T - \{c\}$. Since the set $T - \{c\}$ has one fewer element than T, it is a set of k elements and, therefore, has exactly 2^k subsets (because the induction hypothesis $P(k)$ is assumed true). Now every subset of T that contains c must be of the form $\{c\} \cup D$, where D is a subset of $T - \{c\}$. There are 2^k possible choices for D and, hence, 2^k subsets of T that contain c. Consequently, the total number of subsets of T is

$$\begin{aligned}\begin{pmatrix}\text{Number of subsets} \\ \text{that contain } c\end{pmatrix} + \begin{pmatrix}\text{Number of subsets that} \\ \text{do not contain } c\end{pmatrix} &= 2^k + 2^k \\ &= 2(2^k) \\ &= 2^{k+1}.\end{aligned}$$

Thus any set T of $k + 1$ elements has 2^{k+1} subsets, that is, $P(k + 1)$ is a true statement. We have now verified property (ii) and can, therefore, apply Theorem C.1 to conclude that $P(n)$ is true for every $n \in \mathbb{N}$; that is, every set of n elements has 2^n subsets.

The Principle of Mathematical Induction cannot be conveniently used on certain propositions, even though they appear to be suitable for inductive proof. In such cases a variation on the procedure is needed:

Theorem C.2 The Principle of Complete Induction

Assume that for each nonnegative integer n, a statement $P(n)$ is given. If

(i) $P(0)$ is a true statement; and

(ii) Whenever $P(j)$ is a true statement for all j such that $0 \leq j < t$, then $P(t)$ is also true,

then $P(n)$ is a true statement for every $n \in \mathbb{N}$.

Although commonly used, the title "complete induction" is a bit of a misnomer since, as we shall see, this form of induction is equivalent to the previous one.

Proof of Theorem C.2 ▸ For each $n \in \mathbb{N}$, let $Q(n)$ be the statement

$$P(j) \text{ is true for all } j \text{ such that } 0 \leq j \leq n.$$

Note carefully that the last inequality sign in this statement is $\leq$ and not $<$. We shall use the Principle of Mathematical Induction (Theorem C. 1) to show that $Q(n)$ is true for every $n \in \mathbb{N}$. This will mean, in particular, that $P(n)$ is true for every $n \in \mathbb{N}$. Now $Q(0)$ is the statement

$$P(j) \text{ is true for all } j \text{ such that } 0 \leq j \leq 0.$$

In other words, $Q(0)$ is just the statement "$P(0)$ is true". But we know that this is the case by hypothesis (i) in the theorem. Suppose that $Q(k)$ is true, that is,

$$P(j) \text{ is true for all } j \text{ such that } 0 \leq j \leq k.$$

By hypothesis (ii) (with $t = k + 1$), we conclude the $P(k + 1)$ is also true. Therefore, $P(j)$ is true for all j such that $0 \leq j \leq k + 1$, that is, $Q(k + 1)$ is a true statement. Thus we have shown that whenever $Q(k)$ is true, then $Q(k + 1)$ is also true. By the Principle of Mathematical Induction, $Q(n)$ is true for every $n \in \mathbb{N}$, and the proof is complete. ■

In the formal description of induction (either principle), the notation $P(n)$ is quite convenient. But it is rarely used in actual proofs by induction. The next example is more typical of the way inductive proofs are usually phrased. But even here we include more detail than is customary in such proofs.

EXAMPLE 2

We shall use the Principle of Complete Induction to prove:

If $n, b \in \mathbb{N}$ and $b > 0$, then there exist $q, r \in \mathbb{N}$ such that

$$(*) \qquad n = bq + r \quad \text{and} \quad 0 \leq r < b.$$

This statement (called the **Division Algorithm** for nonnegative integers) is just a formalization of grade-school long division: When n is divided by b, there is a quotient q and remainder r (smaller than the divisor b) such that $n = bq + r$; see the discussion on page 4 of the text.

Statement $(*)$ is true for $n = 0$ and any positive b (let $q = 0$ and $r = 0$). So property (i) of Theorem C.2 holds. Suppose that $(*)$ is true for all n such that $0 \leq n < t$ (this is the induction hypothesis). We must show that $(*)$ is true for $n = t$. If $t < b$, then $t = b0 + t$, so $(*)$ is true with $q = 0$ and $r = t$. If $b \leq t$, then $0 \leq t - b < t$, and by the induction hypothesis, $(*)$ is true for $n = t - b$. Therefore, there exist integers q_1 and r_1 such that

$$t - b = q_1 b + r_1 \qquad \text{and} \qquad 0 \leq r_1 < b.$$

Consequently,

$$t = b + q_1 b + r_1 = (1 + q_1)b + r_1 \qquad \text{and} \qquad 0 \le r_1 < b.$$

Therefore, $(*)$ is true for $n = t$ (with $q = 1 + q_1$ and $r = r_1$). Hence, property (ii) of Theorem C.2 is satisfied. By the Principle of Complete Induction, $(*)$ is true for every $n \in \mathbb{N}$.

Some mathematical statements are false (or undefined) for $n = 0$ or other small values of n but are true for $n = r$ and all subsequent integers. For instance, it can be shown that

$$3n > n + 1 \text{ for every integer } n \ge 1.$$
$$2^n > n^2 + 2 \text{ for every integer } n \ge 5.$$

Such statements can often be proved by using a variation of mathematical induction (either principle):

In order to prove that statement $P(n)$ is true for each integer $n \ge r$, follow the same basic procedure as before, starting with $P(r)$ instead of $P(0)$.

The validity of this procedure is a consequence of

Theorem C.3

Let r be a positive integer and assume that for each $n \ge r$ a statement $P(n)$ is given. If

(i) $P(r)$ is a true statement;

and either

(ii) Whenever $k \ge r$ and $P(k)$ is true, then $P(k + 1)$ is true;

or

(ii') Whenever $P(j)$ is true for all j such that $r \le j < t$, then $P(t)$ is true,

then $P(n)$ is true for every $n \ge r$.

Proof ▸ Conditions (i) and (ii) are the analogue of Theorem C. 1. Verify that the proof of Theorem C.1. carries over to the present case verbatim if 0 is replaced by r, 1 by $r + 1$, and $\mathbb{N}$ by the set $\mathbb{N}_r = \{n \mid n \in \mathbb{N} \text{ and } n \ge r\}$. Conditions (i) and (ii') are the analogue of Theorem C.2; its proof carries over similarly. ■

The final theorem to be proved here is not necessary in order to read the rest of the book. But it is a result that every serious mathematics student ought to know. It is also a good illustration of the fact that intuition can sometimes be misleading. Most people feel that the Well-Ordering Axiom is obvious, whereas the Principle of Complete Induction seems deeper and in need of some proof. But as we shall now see, these two statements are actually equivalent. Among other things, this suggests that the Well-Ordering Axiom is a good deal deeper than it first appears.

Theorem C.4

The following statements are equivalent:

(1) The Well-Ordering Axiom.

(2) The Principle of Mathematical Induction.

(3) The Principle of Complete Induction.

Proof ▸ The proof of Theorem C.1 shows that (1) $\Rightarrow$ (2), and the proof of Theorem C.2 shows that (2) $\Rightarrow$ (3). To prove (3) $\Rightarrow$ (1), we assume the Principle of Complete Induction and let S be any subset of $\mathbb{N}$. To prove that the Well-Ordering Axiom holds, we must show

If S is nonempty, then S has a smallest element.

To do so, we shall prove the equivalent contrapositive statement

If S has no smallest element, then S is empty.

Assume S has no smallest element; to prove that S is empty we need only show that the following statement is true for every $n \in \mathbb{N}$:

(**) $\qquad$ n is not an element of S.

Since 0 is the smallest element of $\mathbb{N}$, it is also the smallest element of any subset of $\mathbb{N}$ containing 0. Since S has no smallest element, 0 cannot be in S, and, hence, (**) is true when $n = 0$ (property (i) of Theorem C.2 holds). Suppose (**) is true for all j such that $0 \leq j < t$. Then none of the integers $0, 1, 2, \ldots, t - 1$ is in S, or equivalently, every element in S must be greater than or equal to t. If t were in S, then t would be the smallest element in S since $s \geq t$ for all $s \in S$. Since S has no smallest element, t is not in S. In other words, (**) is true when $n = t$. Thus the truth of (**) when $j < t$ implies its truth for t (property (ii) of Theorem C.2 holds). By the Principle of Complete Induction, (**) is true for all $n \in \mathbb{N}$. Therefore, S is empty, and the proof is complete. ■

Exercises

A. 1. Prove that the sum of the first n nonnegative integers is $n(n + 1)/2$. [*Hint:* Let $P(k)$ be the statement:

$$0 + 1 + 2 + \cdots + k = k(k + 1)/2.]$$

2. Prove that for each nonnegative integer n, $2^n > n$.

3. Prove that $2^{n-1} \leq n!$ for every nonnegative integer n. [Recall that $0! = 1$ and for $n > 0$, $n! = 1 \cdot 2 \cdot 3 \cdots (n - 1)n$.]

4. Let r be a real number, $r \neq 1$. Prove that for every integer $n \geq 1$,

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

B. **5.** Prove that 4 is a factor of $7^n - 3^n$ for every positive integer n.
[*Hint:* $7^{k+1} - 3^{k+1} = 7^{k+1} - 7 \cdot 3^k + 7 \cdot 3^k - 3^{k+1} = 7(7^k - 3^k) + (7 - 3)3^k$.]

6. Prove that 3 is a factor of $4^n - 1$ for every positive integer n.

7. Prove that 3 is a factor of $2^{2n+1} + 1$ for every positive integer n.

8. Prove that 5 is a factor of $2^{4n-2} + 1$ for every positive integer n.

9. Prove that 64 is a factor of $9^n - 8n - 1$ for every nonnegative integer n.

10. Use the Principle of Complete Induction to show that every integer greater than 1 is a product of primes. [Recall that a positive integer p is prime provided that $p > 1$ and that the only positive integer factors of p are 1 and p.]

11. Let B be a set of n elements. Prove that the number of different injective functions from B to B is $n!$. [$n!$ was defined in Exercise 3.]

12. True or false: $n^2 - n + 11$ is prime for every nonnegative integer n. Justify your answer. [Primes were defined in Exercise 10.]

13. Let B be a set of n elements.

(a) If $n \geq 2$, prove that the number of two-element subsets of B is $n(n - 1)/2$.

(b) If $n \geq 3$, prove that the number of three-element subsets of B is $n(n - 1)(n - 2)/3!$.

(c) Make a conjecture as to the number of k-element subsets of B when $n \geq k$. Prove your conjecture.

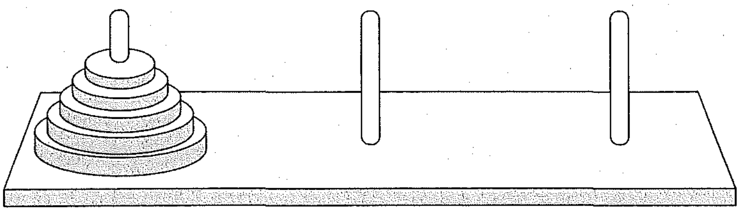
14. At a social bridge party every couple plays every other couple exactly once. Assume there are no ties.

(a) If n couples participate, prove that there is a "best couple" in the following sense: A couple u is "best" provided that for every couple v, u beats v *or* u beats a couple that beats v.

(b) Show by example that there may be more than one best couple.

15. What is wrong with the following "proof" that all roses are the same color. It suffices to prove the statement: In every set of n roses, all the roses in the set are the same color. If $n = 1$, the statement is certainly true. Assume the statement is true for $n = k$. Let S be a set of $k + 1$ roses. Remove one rose (call it rose A) from S; there are k roses remaining, and they must all be the same color by the induction hypothesis. Replace rose A and remove a different rose (call it rose B). Once again there are k roses remaining that must all be the same color by the induction hypothesis. Since the remaining roses include rose A, all the roses in S have the same color. This proves that the statement is true when $n = k + 1$. Therefore, the statement is true for all n by induction.

16. Let n be a positive integer. Suppose that there are three pegs and on one of them n rings are stacked, with each ring being smaller in diameter than the one below it, as shown here for $n = 5$:



The game is to transfer all the rings to another peg according to these rules: (i) only one ring may be moved at a time; (ii) a ring may be moved to any peg but may never be placed on top of a smaller ring; (iii) the final order of the rings on the new peg must be the same as their original order on the first peg. Prove that the game can be completed in $2^n - 1$ moves and cannot be completed in fewer moves.

17. Let x be a real number greater than -1. Prove that for every positive integer n, $(1 + x)^n \geq 1 + nx$.

C. 18. Consider maps in the plane formed by drawing a finite number of straight lines (entire lines, not line segments). Use induction to prove that every such map may be colored with just two colors in such a way that any two regions with the same line segment as a common border have different colors. Two regions that have only a single point on their common border may have the same color. [This problem is a special case of the so-called Four-Color Theorem, which states that every map in the plane (with any continuous curves or segments of curves as boundaries) can be colored with at most four colors in such a way that any two regions that share a common border have different colors.]

APPENDIX D

Equivalence Relations

This appendix may be read anytime after you've finished Appendix B, but it is not needed in the text until Section 10.4. If you read it before that point, you should have no trouble with Examples 1–3 but may have to skip some of the later examples. Chapter 2 is a prerequisite for the examples labeled "integers", Chapter 6 for those labeled "rings", and Section 8.1 for those labeled "groups".

If A is a set, then any subset of $A \times A$ is called a **relation** on A. A relation T on A is called an **equivalence relation** provided that the subset T is

(i) **Reflexive:** $(a, a) \in T$ for every $a \in A$.
(ii) **Symmetric:** If $(a, b) \in T$, then $(b, a) \in T$.
(iii) **Transitive:** If $(a, b) \in T$ and $(b, c) \in T$, then $(a, c) \in T$.

If T is an equivalence relation on A and $(a, b) \in T$, we say that ***a* is equivalent to *b*** and write $\boldsymbol{a \sim b}$ instead of $(a, b) \in T$. In this notation, the conditions defining an equivalence relation become

(i) **Reflexive:** $a \sim a$ for every $a \in A$.
(ii) **Symmetric:** If $a \sim b$, then $b \sim a$.
(iii) **Transitive:** If $a \sim b$ and $b \sim c$, then $a \sim c$.

When this notation is used, the relation is usually defined without explicit reference to a subset of $A \times A$.

EXAMPLE 1

Let A be a set and define $a \sim b$ to mean $a = b$. In other words, the equivalence relation on A is the subset $T = \{(a, b) \mid a = b\}$ of $A \times A$. Then it is easy to see that $\sim$ is an equivalence relation.

EXAMPLE 2

The relation on the set $\mathbb{R}$ of real numbers defined by

$$r \sim s \text{ means } |r| = |s|$$

is an equivalence relation, as you can readily verify.

EXAMPLE 3*

Define a relation on the set $\mathbb{Z}$ of integers by

$$a \sim b \text{ means } a - b \text{ is a multiple of 3.}$$

For example, $17 \sim 5$ since $17 - 5 = 12$, a multiple of 3. Clearly $a \sim a$ for every a since $a - a = 0 = 3 \cdot 0$. To prove property (ii), suppose $a \sim b$. Then $a - b$ is a multiple of 3. Hence, $-(a - b)$ is also a multiple of 3. But $-(a - b) = b - a$. Therefore, $b \sim a$. To prove property (iii), suppose $a \sim b$ and $b \sim c$. Then $a - b$ and $b - c$ are multiples of 3 and so is their difference $(a - b) - (b - c) = a - c$, so that $a \sim c$. Thus $\sim$ is an equivalence relation (usually called congruence modulo 3 and denoted $a \equiv b \pmod 3$).

EXAMPLE 4 (INTEGERS)

If n is a fixed positive integer, the relation of congruence modulo n on the set $\mathbb{Z}$, defined by

$$a \equiv b \pmod n \text{ if and only if } a - b \text{ is a multiple of } n,$$

is an equivalence relation by Theorem 2.1.

EXAMPLE 5 (RINGS)

If I is an ideal in the ring R, then the relation of congruence modulo I, defined by

$$a \equiv b \pmod I \text{ if and only if } a - b \in I,$$

is an equivalence relation on R by Theorem 6.4.

EXAMPLE 6 (GROUPS)

If K is a subgroup of a group G, then the relation defined by

$$a \equiv b \text{ if and only if } ab^{-1} \in K$$

is an equivalence relation on G by Theorem 8.1.

Caution It is quite possible to have a relation on a set that satisfies one or two, but not all three, of the properties that define an equivalence relation. For instance, the order relation $\leq$ on the set $\mathbb{R}$ of real numbers is reflexive and transitive but not symmetric; for other examples, see Exercises 8 and 9. Therefore, you must verify all three properties in order to prove that a particular relation is actually an equivalence relation.

*If you've already read Section 2.1, skip Examples 3 and 8; it's just congruence modulo n when $n = 3$.

Let $\sim$ be an equivalence relation on a set A. If $a \in A$, then the **equivalence class** of a (denoted $[a]$) is the set of all elements in A that are equivalent to a, that is,

$$[a] = \{b \mid b \in A \text{ and } b \sim a\}.$$

In Example 2, for instance, the equivalence class [9] of the number 9 consists of all real numbers b such that $b \sim 9$, that is, all numbers b such that $|b| = |9|$. Thus $[9] = \{9, -9\}$.

EXAMPLE 7 (RINGS, GROUPS)

If I is an ideal in a ring R, then an equivalence class under the relation of congruence modulo I is a coset $a + I = \{a + i \mid i \in I\}$. Similarly, if K is a subgroup of a group G, then an equivalence class of the relation congruence modulo K is a right coset $Ka = \{ka \mid k \in K\}$.

EXAMPLE 8

In Example 3, the equivalence class of the integer 2 consists of all integers b such that $b \sim 2$, that is, all b such that $b - 2$ is a multiple of 3. But $b - 2$ is a multiple of 3 exactly when b is of the form $b = 2 + 3k$ for some integer k. Therefore,

$$\begin{aligned}[2] &= \{2 + 3k \mid k \in \mathbb{Z}\} = \{2 + 0, 2 \pm 3, 2 \pm 6, 2 \pm 9, \ldots\} \\ &= \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}.\end{aligned}$$

A similar argument shows that the equivalence class [8] consists of all integers of the form $8 + 3k$ $(k \in \mathbb{Z})$; consequently,

$$[8] = \{\ldots, -7, -4, -1, 2, 5, 8, 11, 14, 17, \ldots\}.$$

Thus [2] and [8] are the same set. Note that $2 \sim 8$. This is an example of

Theorem D.1

Let $\sim$ be an equivalence relation on a set A and $a, b \in A$. Then

$$a \sim c \text{ if and only if } [a] = [c].$$

Proof* ▸ Assume $a \sim c$. To prove that $[a] = [c]$, we first show that $[a] \subseteq [c]$. To do this, let $b \in [a]$. Then $b \sim a$ by definition. Since $a \sim c$, we have $b \sim c$ by transitivity. Therefore, $b \in [c]$ and $[a] \subseteq [c]$. Reversing the roles of a and c in this argument and using the fact that $c \sim a$ by symmetry, show that $[c] \subseteq [a]$. Therefore, $[a] = [c]$. Conversely, assume that $[a] = [c]$. Since $a \sim a$ by reflexivity, we have $a \in [a]$, and, hence, $a \in [c]$. The definition of $[c]$ shows that $a \sim c$. ■

*If you've read Section 2.1, note that this proof and the proof of Corollary D.2 are virtually identical to the proofs of Theorem 2.3 and Corollary 2.4: just replace $\equiv$ by $\sim$.

Generally when one has two sets, there are three possibilities: The sets are equal, the sets are disjoint, or the sets have some (but not all) elements in common. With equivalence classes, the third possibility cannot occur:

Corollary D.2

Let $\sim$ be an equivalence relation on a set A. Then any two equivalence classes are either disjoint or identical.

Proof ▸ Let $[a]$ and $[c]$ be equivalence classes. If they are disjoint, then there is nothing to prove. If they are not disjoint, then $[a] \cap [c]$ is nonempty, and by definition there is an element b such that $b \in [a]$ and $b \in [c]$. By the definition of equivalence class, $b \sim a$ and $b \sim c$. Consequently, by transitivity and symmetry, $a \sim c$. Therefore, $[a] = [c]$ by Theorem D.1. ■

A **partition** of a set A is a collection of nonempty, mutually disjoint* subsets of A whose union is A. Every equivalence relation $\sim$ on A leads to a partition as follows. Since $a \in [a]$ for each $a \in A$, every equivalence class is nonempty, and every element of A is in one. Distinct equivalence classes are disjoint by Corollary D.2. Therefore,

The distinct equivalence classes of an equivalence relation on a set A form a partition of A.

Conversely, every partition of A leads to an equivalence relation whose equivalence classes are precisely the subsets of the partition (Exercise 21).

Exercises

A. 1. Let P be a plane. If p, q are points in P, then $p \sim q$ means p and q are the same distance from the origin. Prove that $\sim$ is an equivalence relation on P.

2. Define a relation on the set $\mathbb{Q}$ of rational numbers by: $r \sim s$ if and only if $r - s \in \mathbb{Z}$. Prove that $\sim$ is an equivalence relation.

3. **(a)** Prove that the following relation on the set $\mathbb{R}$ of real numbers is an equivalence relation: $a \sim b$ if and only if $\cos a = \cos b$.

   (b) Describe the equivalence class of 0 and the equivalence class of $\pi/2$.

4. If m and n are lines in a plane P, define $m \sim n$ to mean that m and n are parallel. Is $\sim$ an equivalence relation on P?

5. **(a)** Let $\sim$ be the relation on the ordinary coordinate plane defined by $(x, y) \sim (u, v)$ if and only if $x = u$. Prove that $\sim$ is an equivalence relation.

   (b) Describe the equivalence classes of this relation.

*That is, any two of the subsets are disjoint.

6. Prove that the following relation on the coordinate plane is an equivalence relation: $(x, y) \sim (u, v)$ if and only if $x - u$ is an integer.

7. Let $f: A \to B$ be a function. Prove that the following relation is an equivalence relation of A: $u \sim v$ if and only if $f(u) = f(v)$.

8. Let $A = \{1, 2, 3\}$. Use the ordered-pair definition of a relation to exhibit a relation on A with the stated properties.

(a) Reflexive, not symmetric, not transitive.

(b) Symmetric, not reflexive, not transitive.

(c) Transitive, not reflexive, not symmetric.

(d) Reflexive and symmetric, not transitive.

(e) Reflexive and transitive, not symmetric.

(f) Symmetric and transitive, not reflexive.

9. Which of the properties (reflexive, symmetric, transitive) does the given relation have?

(a) $a < b$ on the set $\mathbb{R}$ of real numbers.

(b) $A \subseteq B$ on the set of all subsets of a set S.

(c) $a \neq b$ on the set $\mathbb{R}$ of real numbers.

(d) $(-1)^a = (-1)^b$ on the set $\mathbb{Z}$ of integers.

B. 10. If r is a real number, then $[\![r]\!]$ denotes the largest integer that is $\leq r$; for instance $[\![\pi]\!] = 3$, $[\![7]\!] = 7$ and $[\![-1.5]\!] = -2$. Prove that the following relation is an equivalence relation on $\mathbb{R}$: $r \sim s$ if and only if $[\![r]\!] = [\![s]\!]$.

11. Let $\sim$ be defined on the set $\mathbb{R}^*$ of nonzero real numbers by: $a \sim b$ if and only if $a/b \in \mathbb{Q}$. Prove that $\sim$ is an equivalence relation.

12. Is the following relation an equivalence relation on $\mathbb{R}$: $a \sim b$ if and only if there exists $k \in \mathbb{Z}$ such that $a = 10^k b$.

13. In the set $\mathbb{R}[x]$ of all polynomials with real coefficients, define $f(x) \sim g(x)$ if and only if $f'(x) = g'(x)$, where $'$ denotes the derivative. Prove that $\sim$ is an equivalence relation on $\mathbb{R}[x]$.

14. Let T be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and define $f \sim g$ if and only if $f(2) = g(2)$. Prove that $\sim$ is an equivalence relation.

15. Prove that the relation on $\mathbb{Z}$ defined by $a \sim b$ if and only if $a^2 \equiv b^2 \pmod{6}$ is an equivalence relation.

16. Let $S = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$ and define $(a, b) \sim (c, d)$ if and only if $ad = bc$. Prove that $\sim$ is an equivalence relation on S.

17. Let $\sim$ be a symmetric and transitive relation on a set A. What is wrong with the following "proof" that $\sim$ is reflexive: $a \sim b$ implies $b \sim a$ by symmetry; then $a \sim b$ and $b \sim a$ imply $a \sim a$ by transitivity. [Also see Exercise 8(f).]

18.* Let G be a group and define $a \sim b$ if and only if there exists $c \in G$ such that $b = c^{-1}ac$. Prove that $\sim$ is an equivalence relation on G.

19.* **(a)** Let K be a subgroup of a group G and define $a \sim b$ if and only if $a^{-1}b \in K$. Prove that $\sim$ is an equivalence relation on G.

(b) Give an example to show that the equivalence relation in part (a) need not be the same as the relation in Example 6.

20.* Let G be a subgroup of S_n. Define a relation on the set $\{1, 2, \dots, n\}$ by $a \sim b$ if and only if $a = \sigma(b)$ for some σ in G. Prove that $\sim$ is an equivalence relation.

21. Let A be a set and $\{A_i \mid i \in I\}$ a partition of A. Define a relation on A by: $a \sim b$ if and only if a and b are in the same subset of the partition (that is, there exists $k \in I$ such that $a \in A_k$ and $b \in A_k$).

(a) Prove that $\sim$ is an equivalence relation on A.

(b) Prove that the equivalence classes of $\sim$ are precisely the subsets A_i of the partition.

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*Sections 7.2 and 7.3 are prerequisites for Exercises 18–20.

APPENDIX E

The Binomial Theorem

Appendix C and Section 3.2 are the prerequisites for this appendix. The material presented here is used in Section 11.6 and in occasional exercises elsewhere.

As we saw in Example 3 of Section 3.2,

$$(a + b)^2 = a^2 + 2ab + b^2$$

for any elements a, b in a *commutative* ring R. Similar calculations using distributivity and commutative multiplication show that

$$\begin{aligned}(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\ (a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.\end{aligned}$$

There is a pattern emerging here, but it may not be obvious unless certain facts are pointed out first.

Recall that $0!$ is defined to be 1 and that for each positive integer n, the symbol $n!$ denotes the number $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$. For each k, with $0 \leq k \leq n$, the **binomial coefficient** $\binom{n}{k}$ is defined to be the number $\dfrac{n!}{k!(n-k)!}$. This number may appear to be a fraction, but every binomial coefficient is actually an integer (Exercise 6). For instance, $\binom{4}{1} = \dfrac{4!}{1!(4-1)!} = \dfrac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 3 \cdot 2 \cdot 1} = 4$, and similarly, $\binom{4}{2} = \dfrac{4!}{2!2!} = 6$. Note that these numbers appear as coefficients in the preceding expansion of $(a + b)^4$; in fact, you can readily verify that

$$(a + b)^4 = a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + b^4.$$

This is an example of

Theorem E.1 The Binomial Theorem

Let R be a commutative ring and $a, b \in R$. Then for each positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n.$$

Proof ▸ The proof is by induction on n. If $n = 1$, the theorem states that $(a+b)^1 = a^1 + b^1$, which is certainly true. Assume that the theorem is true when $n = k$, that is, that

$$(a+b)^k = a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k-1}ab^{k-1} + b^k.$$

We must use this assumption to prove that the theorem is true when $n = k + 1$. By the definition of exponents $(a+b)^{k+1} = (a+b)(a+b)^k$. Applying the induction hypothesis to $(a+b)^k$ and using distributivity and commutative multiplication, we have

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b)\left[a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k-1}ab^{k-1} + b^k\right] \\
&= a\left[a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k-1}ab^{k-1} + b^k\right] \\
&\quad + b\left[a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k-1}ab^{k-1} + b^k\right] \\
&= \left[a^{k+1} + \binom{k}{1}a^k b + \cdots + \binom{k}{r}a^{k-r+1}b^r + \cdots + \binom{k}{k-1}a^2b^{k-1} + ab^k\right] \\
&\quad + \left[a^k b + \binom{k}{1}a^{k-1}b^2 + \cdots + \binom{k}{r}a^{k-r}b^{r+1} + \cdots + \binom{k}{k-1}ab^k + b^{k+1}\right] \\
&= a^{k+1} + \left[\binom{k}{1} + 1\right]a^k b + \left[\binom{k}{2} + \binom{k}{1}\right]a^{k-1}b^2 + \cdots \\
&\quad + \left[\binom{k}{r+1} + \binom{k}{r}\right]a^{k-r}b^{r+1} + \cdots + \left[1 + \binom{k}{k-1}\right]ab^k + b^{k+1}.
\end{aligned}$$

Exercise 5 (which you should do) shows that for $r = 0, 1, \ldots, k$

$$\binom{k}{r+1} + \binom{k}{r} = \binom{k+1}{r+1}.$$

Apply this fact to each of the coefficients in the last part of the equation above. For instance, $\binom{k}{1} + 1 = \binom{k}{1} + \binom{k}{0} = \binom{k+1}{1}$, and $\binom{k}{2} + \binom{k}{1} = \binom{k+1}{2}$, and so on. Then, from the first and last parts of the equation above we have

$$\begin{aligned}
(a+b)^{k+1} = a^{k+1} &+ \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots \\
&+ \binom{k+1}{r+1}a^{k-r}b^{r+1} + \cdots + \binom{k+1}{k}ab^k + b^{k+1}.
\end{aligned}$$

Therefore, the theorem is true when $n = k + 1$, and, hence, by induction it is true for every positive integer n. ■

Exercises

A. 1. Let x and y be real numbers. Find the coefficient of x^5y^8 in the expansion of $(2x - y^2)^9$. [*Hint:* Apply Theorem E.1 with $a = 2x$, $b = y^2$.]

2. If x and y are real numbers, what is the coefficient of $x^{12}y^6$ in the expansion of $(x^3 - 3y)^{10}$?

B. 3. Let r and n be integers with $0 < r < n$. Prove that $\binom{n}{r} = \binom{n}{n-r}$.

4. Prove that for any positive integer n, $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$. [*Hint:* $2^n = (1 + 1)^n$.]

5. Let r and k be integers such that $0 \le r \le k - 1$. Prove that $\binom{k}{r+1} + \binom{k}{r} = \binom{k+1}{r+1}$. [*Hint:* Use the fact that

$$(k - r)(k - (r + 1))! = (k - r)! = ((k + 1) - (r + 1))!]$$

to express each term on the left as a fraction with denominator $(k + 1)!(k - r)!$. Add the fractions, simplify the numerator, and compare the result with $\binom{k+1}{r+1}$.]

6. Let n be a positive integer. Use mathematical induction to prove this statement: For each integer r such that $0 \le r \le n$, $\binom{n}{r}$ is an integer [*Hint:* For $n = 1$ it is easy to calculate $\binom{1}{0} = 1 = \binom{1}{1}$; assume the statement is true for $n = k$ and use Exercise 5 to show that the statement is true for $n = k + 1$.]

7. Here are the first five rows of **Pascal's triangle:**

$$\begin{array}{lccccccccc}
\text{Row 0:} & & & & & 1 & & & & \\
\text{Row 1:} & & & & 1 & & 1 & & & \\
\text{Row 2:} & & & 1 & & 2 & & 1 & & \\
\text{Row 3:} & & 1 & & 3 & & 3 & & 1 & \\
\text{Row 4:} & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Note that each entry in a given row (except the 1's on the end) is the sum of the two numbers above it in the preceding row. For instance, the first 4 in row 4 is the sum of 1 and 3 in row 3; similarly, 6 in row 4 is the sum of the two 3's in row 3.

(a) Write out the next three rows of Pascal's triangle.

(b) Prove that the entries in row n of Pascal's triangle are precisely the coefficients in the expansion of $(a + b)^n$, that is, $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$. [*Hint:* Exercise 5 may be helpful.]

APPENDIX F

Matrix Algebra

This appendix may be read at any time after Section 3.1 but is needed only in Chapter 16. Throughout this appendix, *R is a ring with identity.*

Rings of 2×2 matrices with entries in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ were introduced in Section 3.1. These matrices are special cases of this definition: An $\boldsymbol{n \times m}$ **matrix** over R is an array of n horizontal rows and m vertical columns

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ r_{21} & r_{22} & r_{23} & \cdots & r_{2m} \\ r_{31} & r_{32} & r_{33} & \cdots & r_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & r_{nm} \end{pmatrix}$$

with each $r_{ij} \in R$. For example,

$$A = \begin{pmatrix} 7 & -6 & 4 & 10 & 0 \\ 1 & 0 & 5 & -2 & 1 \\ 3 & 3 & 4 & 12 & 9 \\ 0 & 5 & 2 & 0 & -8 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 1 & 3 \\ 3 & 2 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

4×5 over $\mathbb{Z}$ $\qquad$ 3×3 over $\mathbb{Z}_5$ $\qquad$ 2×4 over $\mathbb{Z}_2$

Matrices are usually denoted by capital letters and their entries by lowercase letters with double subscripts indicating the row and column the entry appears in. For instance, in the matrix $A = (a_{ij})$ above, the entry in row 4 and column 2 is $a_{42} = 5$. In matrix C, $c_{12} = 0$ and $c_{23} = 1$. Thus, for example, row i of an $n \times m$ matrix (r_{ij}) is

$$r_{i1} \; r_{i2} \; r_{i3} \; r_{i4} \cdots r_{im}.$$

The $n \times m$ **zero matrix** is the $n \times m$ matrix with 0_R in every entry. The **identity matrix** $\boldsymbol{I_n}$ is the $n \times n$ matrix with 1_R in positions 1-1, 2-2, 3-3, $\ldots$, n-n, and 0_R in all other positions. For example, over the ring $\mathbb{R}$,

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad I_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The identity matrix I_n can be succinctly described by $I_n = (\delta_{ij})$, where δ_{ij} is the **Kronecker delta symbol,** defined by

$$\delta_{ij} = \begin{cases} 1_R \text{ if } i = j. \\ 0_R \text{ if } i \neq j. \end{cases}$$

It is sometimes convenient to think of a large matrix as being made up of two smaller ones. For example, if A is the 3×2 matrix

$$\begin{pmatrix} 4 & 2 \\ 1 & 0 \\ 3 & 5 \end{pmatrix}$$

over $\mathbb{Z}$, then $(I_3 \mid A)$ denotes the 3×5 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 5 \end{pmatrix}.$$

Similarly, $\left(\dfrac{A}{I_2}\right)$ denotes the matrix $\begin{pmatrix} 2 & 3 \\ 4 & 6 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$.

If $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times m$ matrices, then their **matrix sum** $A + B$ is the $n \times m$ matrix with $a_{ij} + b_{ij}$ in position i-j. In other words, just add the entries in corresponding positions, as in this example over $\mathbb{Z}_5$:

$$\begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 4 \\ 1 & 1 & 3 \end{pmatrix}.$$

If A and B are of different sizes, their sum is not defined. But if A, B, C are $n \times m$ matrices, then Exercise 3 shows that *matrix addition is commutative* $[A + B = B + A]$ *and associative* $[A + (B + C) = (A + B) + C]$. The $n \times m$ zero matrix acts as an identity for addition (Exercise 4).

For reasons that are made clear in a linear algebra course, the product of matrices A and B is defined only when the number of *columns* of A is the same as the number of *rows* of B. The simplest case is the product of a $1 \times m$ matrix A consisting of a single row $(a_1\ a_2\ a_3 \cdots a_m)$ and an $m \times 1$ matrix B consisting of a single column $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.*

*A matrix with only one row is called a **row vector** and a matrix with only one column a **column vector.** Single subscripts are adequate to describe the entries of row and column vectors.

The product is defined to be the 1×1 matrix whose single entry is the element

$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + \cdots + a_mb_m.$$

For example, over $\mathbb{Z}$

$$(*) \qquad (2 \quad 3 \quad 1)\begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = 2 \cdot 4 + 3 \cdot 0 + 1 \cdot 2 = 10.$$

If A is an $n \times m$ matrix and B is an $m \times k$ matrix, then the **matrix product** AB is the $n \times k$ matrix (c_{ij}), where the entry in position i-j is the product of the ith row of A and the jth column of B:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + a_{i4}b_{4j} + \cdots + a_{im}b_{mj} = \sum_{r=1}^{m} a_{ir}b_{rj}.$$

EXAMPLE 1

The product of

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 5 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 2 & 6 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & 6 & 0 & 2 \end{pmatrix}$$

is a 2×4 matrix whose entry in position 1-1 is 10 (the product of row 1 of A and column 1 of B as shown in $(*)$ above). In position 2-3 the entry in AB is the product of row 2 of A and column 3 of B:

$$1 \cdot 6 + 5 \cdot 2 + 0 \cdot 0 = 16.$$

Similar calculations show that

$$AB = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 5 & 0 \end{pmatrix}\begin{pmatrix} 4 & 2 & 6 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & 6 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 13 & 18 & 11 \\ 4 & 7 & 16 & 8 \end{pmatrix}.$$

The product BA is *not defined* because B has four columns, but A has only two rows.

If A, B, C are matrices of appropriate sizes so that each of the products AB and BC is defined, then *matrix multiplication is associative:* $A(BC) = (AB)C$ (Exercise 7). Similarly, if E, F, G are matrices such that the products EG and FG are defined, then *the distributive law holds:* $(E + F)G = EG + FG$ (Exercise 5). The identity matrices act as *identity elements for multiplication* in this sense: If A is an $n \times m$ matrix, then $I_n \cdot A = A$ and $A \cdot I_m = A$ (Exercise 6). Even when both products AB and BA are defined, *matrix multiplication may not be commutative* (see Example 6 in Section 3.1).

Let $M_n(R)$ denote the set of all $n \times n$ matrices over the ring R. Since all the matrices in $M_n(R)$ have the same number of columns and rows, both $A + B$ and AB and BA are defined for all $A, B \in M_n(R)$. The properties of matrix addition and multiplication listed above provide the proof of

Theorem F.1

If R is a ring with identity, then the set $M_n(R)$ of all $n \times n$ matrices over R is a noncommutative ring with identity I_n.

Exercises

NOTE: *Unless stated otherwise, all matrices are over a ring R with identity.*

A. 1. Assume A and B are matrices over $\mathbb{Z}$. Find $A + B$.

(a) $A = \begin{pmatrix} 1 & 2 & -2 & 0 \\ 3 & 5 & 7 & 11 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -8 & 2 & 4 \\ 6 & 0 & 4 & 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & 0 & 2 \\ 4 & 1 & 6 \\ 0 & 1 & 0 \\ 2 & -5 & 7 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 0 & 4 \\ 0 & 7 & -6 \\ 1 & 6 & 0 \end{pmatrix}$

2. Assume A and B are matrices over $\mathbb{Z}_6$. Find AB and BA whenever the products are defined.

(a) $A = \begin{pmatrix} 2 & 4 \\ 1 & 5 \\ 3 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 5 & 3 \\ 1 & 0 & 2 \end{pmatrix}$

(c) $A = \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

B. 3. Let $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ be $n \times m$ matrices. Prove that

(a) $A + B = B + A$ **(b)** $A + (B + C) = (A + B) + C$

4. If $A = (a_{ij})$ is an $n \times m$ matrix and Z is the $n \times m$ zero matrix, prove that $A + Z = A$.

5. (a) Let E and F be $1 \times m$ row vectors and $G = (g_{ij})$ an $m \times k$ matrix. Prove that $(E + F)G = EG + FG$.

(b) Let $E = (e_{ij})$ and $F = (f_{ij})$ be $n \times m$ matrices and $G = (g_{ij})$ an $m \times k$ matrix. Prove that $(E + F)G = EG + FG$.

6. If A is an $n \times m$ matrix, prove that $I_n \cdot A = A$ and $A \cdot I_m = A$.

C. 7. Let $A = (a_{ij})$ be an $n \times m$ matrix, $B = (b_{ij})$ an $m \times k$ matrix, and $C = (c_{ij})$ a $k \times p$ matrix. Prove that $A(BC) = (AB)C$. [*Hint:* $BC = (d_{tj})$, where $d_{tj} = \sum_{r=1}^{k} b_{tr}c_{rj}$, and $AB = (e_{ir})$, where $e_{ir} = \sum_{t=1}^{m} a_{it}b_{tr}$. The i-j entry of $A(BC)$ is $\sum_{t=1}^{m} a_{it}d_{tj} = \sum_{t=1}^{m} a_{it}\left(\sum_{r=1}^{k} b_{tr}c_{rj}\right) = \sum_{t=1}^{m}\sum_{r=1}^{k} a_{it}b_{tr}c_{rj}$. Show that the i-j entry of $(AB)C$ is this same double sum.]

APPENDIX G

Polynomials

In high school there is some ambiguity about the "x" in polynomials. Sometimes x stands for a specific number (as in the equation $5x - 6 = 17$). Other times x doesn't seem to stand for any number—it's just a symbol that is algebraically manipulated (as in exercises such as $(x + 3)(x - 5) = x^2 - 2x - 15$).* Our goal here is to develop a rigorous definition of "polynomial" that removes this ambiguity. The prerequisites for this discussion are high-school algebra and Chapter 3.

As a prelude to the formal development, note that the polynomials from high school can be described without ever mentioning x. For instance, $5 + 6x - 2x^3$ is completely determined by its coefficients $(5, 6, 0, -2)$.† But $5 + 6x - 2x^3$ can also be written $5 + 6x - 2x^3 + 0x^4 + 0x^5 + 0x^6$. To allow for such additional "zero terms", we list the coefficients as an infinite sequence $(5, 6, 0, -2, 0, 0, 0, 0, \ldots)$ that ends in zeros.

Adding polynomials in this new notation is pretty much the same as before: Add the coefficients of corresponding powers of x, that is, add sequences coordinatewise:

$$\begin{array}{lcl}
5 + 6x \qquad\quad - 2x^3 & \qquad & (5,\ \ 6, 0, -2, 0, 0, 0, \ldots) \\
\underline{3 - 2x + 5x^2 - 4x^3} & & \underline{(3, -2, 5, -4, 0, 0, 0, \ldots)} \\
8 + 4x + 5x^2 - 6x^3 & & (8,\ \ 4, 5, -6, 0, 0, 0, \ldots).
\end{array}$$

Multiplication can also be described in terms of sequences, as we shall see. If you keep this model in mind, you will see clearly where the formal definitions and theorems come from.

Except in Theorem 4.1 at the end of this appendix, R is a ring with identity (not necessarily commutative). A **polynomial with coefficients in the ring R** is defined to be an infinite sequence

$$(a_0, a_1, a_2, a_3, \ldots)$$

such that each $a_i \in R$ and only finitely many of the a_i are nonzero; that is, for some index k, $a_i = 0_R$ for all $i > k$. The elements $a_i \in R$ are called the **coefficients** of the polynomial.

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*Sometimes x is also used as a variable that can take infinitely many values (as in the function $f(x) = x^3 - x$). This usage is discussed in Section 4.4.

†0 is the coefficient of x^2.

The polynomials $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ are **equal** if they are equal as sequences, that is, if $a_0 = b_0$, $a_1 = b_1$, and in general, $a_i = b_i$ for every $i \geq 0$. **Addition of polynomials** is denoted by $\oplus$ and defined by the rule

$$(a_0, a_1, a_2, \ldots) \oplus (b_0, b_1, b_2, \ldots) = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots, a_i + b_i, \ldots).$$

You should verify that the sequence on the right is actually a polynomial, that is, that after some point all its coordinates are zero (Exercise 2).

Multiplication of polynomials is denoted $\odot$ and defined by the rule*

$$\begin{aligned}
&(a_0, a_1, a_2, \ldots) \odot (b_0, b_1, b_2, \ldots.) = (c_0, c_1, c_2, \ldots), \quad \text{where}\\
&\quad c_0 = a_0b_0\\
&\quad c_1 = a_0b_1 + a_1b_0\\
&\quad c_2 = a_0b_2 + a_1b_1 + a_2b_0\\
&\qquad \vdots\\
&\quad c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + a_3b_{n-3} + \cdots + a_{n-1}b_1 + a_nb_0\\
&\qquad = \sum_{i=0}^{n} a_ib_{n-i}.
\end{aligned}$$

To show that the product defined here is actually a polynomial you must verify that after some point all the coordinates of $(c_0, c_1, \ldots)$ are zero (Exercise 2).

Theorem G.1

Let R be a ring with identity and P the set of polynomials with coefficients in R. Then P is a ring with identity. If R is commutative, then so is P.

Proof ▸ Exercise 2 shows that P is closed under addition and multiplication. To show that addition in P is commutative, we note that $a_i + b_i = b_i + a_i$ for all $a_i, b_i \in R$ because R is a ring; therefore, in P

$$\begin{aligned}
(a_0, a_1, a_2, \ldots) &\oplus (b_0, b_1, b_2, \ldots)\\
&= (a_0 + b_0, a_1 + b_1, \ldots) = (b_0 + a_0, b_1 + a_1, \ldots)\\
&= (b_0, b_1, b_2, \ldots) \oplus (a_0, a_1, a_2, \ldots).
\end{aligned}$$

Associativity of addition and the distributive laws are proved similarly. You can readily check that the multiplicative identity in P is the polynomial $(1_R, 0_R, 0_R, 0_R, \ldots)$, the zero element is the polynomial $(0_R, 0_R, 0_R, \ldots)$, and the solution of the equation $(a_0, a_1, a_2, \ldots) + X = (0_R, 0_R, 0_R, \ldots)$ is $X = (-a_0, -a_1, -a_2 \ldots)$.

To complete the proof that P is a ring with identity, we must show that multiplication is associative. Let $A, B, C \in P$, where

$$A = (a_0, a_1, a_2, \ldots) \qquad B = (b_0, b_1, b_2, \ldots) \qquad C = (c_0, c_1, c_2, \ldots).$$

*To understand the formal definition, do the following multiplication problem and look at the coefficients of each power of x in the answer: $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2)$.

Then the nth coordinate of $(A \odot B) \odot C$ is

$$(*) \qquad \sum_{i=0}^{n} (ab)_i c_{n-i} = \sum_{i=0}^{n} \left[\sum_{j=0}^{i} a_j b_{i-j} \right] c_{n-i} = \sum_{i=0}^{n} \sum_{j=0}^{i} a_j b_{i-j} c_{n-i}.$$

Exercise 6 shows that the last sum on the right is the same as

$$(**) \qquad \sum a_u b_v c_w,$$

where the sum is taken over all integers u, v, w such that $u + v + w = n$ and $u \geq 0$, $v \geq 0$, $w \geq 0$. On the other hand, the nth coordinate of $A \odot (B \odot C)$ is

$$(***) \qquad \sum_{r=0}^{n} a_r (bc)_{n-r} = \sum_{r=0}^{n} a_r \left[\sum_{s=0}^{n-r} b_s c_{n-r-s} \right] = \sum_{r=0}^{n} \sum_{s=0}^{n-r} a_r b_s c_{n-r-s}.$$

Exercise 6 shows that the last sum on the right is also equal to (**). Since the nth coordinates of $(A \odot B) \odot C$ and $A \odot (B \odot C)$ are equal for each $n \geq 0$, $(A \odot B) \odot C = A \odot (B \odot C)$. The proof of the final statement of the theorem is left to the reader (Exercise 3). ■

In the old notation, constant polynomials behave like ordinary numbers. In the new notation, constant polynomials are of the form $(r, 0, 0, 0, \ldots.)$, and essentially the same thing is true:

Theorem G.2

Let P be the ring of polynomials with coefficients in the ring R. Let R^* be the set of all polynomials in P of the form $(r, 0_R, 0_R, 0_R, \ldots)$, with $r \in R$. Then R^* is a subring of P and is isomorphic to R.

Proof ▸ Consider the function $f: R \to R^*$ given by

$$f(r) = (r, 0_R, 0_R, 0_R, \ldots).$$

You can readily verify that f is bijective. Furthermore,

$$\begin{aligned} f(r + s) &= (r + s, 0_R, 0_R, 0_R, \ldots) \\ &= (r, 0_R, 0_R, 0_R, \ldots) \oplus (s, 0_R, 0_R, 0_R) = f(r) + f(s) \end{aligned}$$

and

$$\begin{aligned} f(rs) &= (rs, 0_R, 0_R, 0_R, \ldots) \\ &= (r, 0_R, 0_R, 0_R, \ldots) \odot (s, 0_R, 0_R, 0_R, \ldots) = f(r) \odot f(s). \end{aligned}$$

Therefore, f is an isomorphism, and, hence, R^* is a subring. ■

Now that the basic facts have been established, it's time to recover the "old" notation for polynomials. First, we want polynomials in R^* to look more like "constants" (elements of R), so

$(a, 0_R, 0_R, 0_R, \ldots)$ *will be denoted by the boldface letter* $\mathbf{a}$.

Next, reverting to the original source of our sequence notation,

$$(0_R, 1_R, 0_R, 0_R, 0_R, \ldots) \textit{ will be denoted by } x.$$

There is no ambiguity about what x is here—it is a specific sequence in P; it is *not* an element of R or R^*, and it does *not* "stand for" any element of R or R^*.

This notation makes things look a bit more familiar. For instance,

$$(a, 0_R, 0_R, 0_R, \ldots) + (b, 0_R, 0_R, \ldots)(0_R, 1_R, 0_R, 0_R, \ldots)$$

becomes $\boldsymbol{a} + \boldsymbol{b}x$. Similarly, we would expect cx^3 (the "constant" c times x^3) to be the sequence $(0_R, 0_R, 0_R, c, 0_R, 0_R, \ldots)$ with c in position 3.* But we can't just *assume* that everything works as it did in the old notation. The required proof is given in the next two results.

Lemma G.3

Let P be the ring of polynomials with coefficients in the ring R and x the polynomial $(0_R, 1_R, 0_R, 0_R, \ldots)$. Then for each element $a = (a, 0_R, 0_R, \ldots)$ of R^* and each integer $n \geq 1$:

(1) $x^n = (0_R, 0_R, \ldots, 0_R, 1_R, 0_R, \ldots)$, where 1_R is in position n.

(2) $ax^n = (0_R, 0_R, \ldots, 0_R, a, 0_R, \ldots)$, where a is in position n.

Proof ▸ The polynomial x can be described like this:

$$x = (e_0, e_1, e_2, \ldots), \qquad \text{where } e_i = 0_R \text{ for all } i \neq 1, \text{ and } e_1 = 1_R.$$

Statement (1) will be proved by induction on n.† It is true for $n = 1$ by the definition of $x^1 = x$. Suppose that it is true for $n = k$, that is, suppose that

$$x^k = (d_0, d_1, d_2, \ldots), \qquad \text{where } d_i = 0_R \text{ for } i \neq k, \text{ and } d_k = 1_R.$$

Then

$$x^{k+1} = x^k x = (d_0, d_1, d_2, \ldots)(e_0, e_1, e_2, \ldots) = (r_0, r_1, r_2, \ldots),$$

where for each $j \geq 0$,

$$r_j = \sum_{i=0}^{j} d_i e_{j-i}.$$

Since $e_i = 0_R$ for $i \neq 1$ and $d_i = 0_R$ for $i \neq k$, we have

$$r_{k+1} = \underbrace{d_0e_{k+1} + \cdots + d_{k-1}e_2}_{0} + d_ke_1 + \underbrace{d_{k+1}e_0}_{0} = d_ke_1 = 1_R1_R = 1_R$$

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*Remember that in the polynomial $(r, s, t, \ldots)$ the element r is in position 0, s is in position 1, t is in position 2, etc.

†See Appendix C.

and, for $j \neq k + 1$,

$$r_j = \underbrace{d_0e_j + d_1e_{j-1} + \cdots + d_{j-2}e_2}_{0} + d_{j-1}e_1 + \underbrace{d_je_0}_{0}$$
$$= d_{j-1}e_1 = d_{j-1}1_R = d_{j-1}.$$

But $j - 1 \neq k$ since $j \neq k + 1$. Therefore, $r_j = d_{j-1} = 0_R$ for all $j \neq k + 1$. Hence, $x^{k+1} = (r_0, r_1, r_2, \ldots) = (0_R, 0_R, \ldots, 0_R, 1_R, 0_R, \ldots)$, with 1_R in position $k + 1$. So (1) is true for $n = k + 1$ and, therefore, true for all n by induction.

A similar inductive argument proves (2); see Exercise 7. ■

Theorem G.4

Let P be the ring of polynomials with coefficients in the ring R. Then P contains an isomorphic copy R^* of R and an element x such that

(1) $ax = xa$ for every $a \in R^*$.

(2) Every element of P can be written in the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

(3) If $a_0 + a_1x + \cdots + a_nx^n = b_0 + b_1x + \cdots + b_mx^m$ with $n \leq m$, then $a_i = b_i$ for $i \leq n$ and $b_i = 0_R$ for $i > n$; in particular,

(4) $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0_R$ if and only if $a_i = 0_R$ for every $i \geq 0$.

Proof ▸ Let x be as in Lemma G.3. The proof of (1) is left to the reader (Exercise 5).

(2) If $(a_0, a_1, a_2, \ldots) \in P$, then there is an index n such that $a_i = 0_R$ for all $i > n$. By Lemma G.3

$$\begin{aligned}(a_0, a_1, a_2, \ldots, a_n, 0_R, 0_R, \ldots)\\ &= (a_0, 0_R, 0_R, \ldots) + (0_R, a_1, 0_R, \ldots) + (0_R, 0_R, a_2, 0_R, \ldots)\\ &\qquad + \cdots + (0_R, \ldots, 0_R, a_n, 0_R, \ldots)\\ &= \mathbf{a}_0 + \mathbf{a}_1x + \mathbf{a}_2x^2 + \cdots + \mathbf{a}_nx^n.\end{aligned}$$

(3) Reversing the argument in (2) shows that $\mathbf{a}_0 + \mathbf{a}_1x + \cdots + \mathbf{a}_nx^n$ is the sequence $(a_0, a_1, a_2, \ldots, a_n, 0_R, 0_R, \ldots)$ and that $\mathbf{b}_0 + \mathbf{b}_1x + \cdots + \mathbf{b}_mx^m = (b_0, b_1, b_2, \ldots, b_m, 0_R, 0_R, \ldots)$. If these two sequences are equal, then we must have $a_i = b_i$ for $i \leq n$ and $0_R = b_i$ for $n < i \leq m$.

(4) is a special case of (3): Just let $\mathbf{b}_i = \mathbf{0}_R$. ■

When polynomials are written in the form $\mathbf{a}_0 + \mathbf{a}_1x + \cdots + \mathbf{a}_nx^n$, addition and multiplication look as they did in high school, except for the use of boldface print in certain symbols.

EXAMPLE 1

In the ring of polynomials with real-number coefficients, the distributive laws and Theorems G.2 and G.4 show that

$$\begin{aligned}(3x+1)(2x+5) &= (3x+1)2x + (3x+1)5\\ &= 3x2x + 1\cdot 2x + 3x5 + 1\cdot 5\\ &= 3\cdot 2xx + 1\cdot 2x + 3\cdot 5x + 1\cdot 5\\ &= 6x^2 + 17x + 5.\end{aligned}$$

In terms of elements, the distinction between boldface and regular print is important because $\boldsymbol{a}$ is a sequence, while a is an element of R. But in terms of algebraic structure, there is no need for distinction because R^* (consisting of all the boldface $\boldsymbol{a}$'s) is isomorphic to R (consisting of all the a's). Consequently, there is no harm in *identifying* R with its isomorphic copy R^* and writing the elements of $R \cong R^*$ in ordinary print.* Then polynomials look and behave as they did before. For this reason, the standard notation for the polynomial ring is $R[x]$, which we shall use hereafter instead of P.

We have now come full circle in terms of notation, with the added benefits of a rigorous justification of our past work with polynomials, a generalization of these concepts to rings, and a new viewpoint on polynomials. Beginning with a ring R with identity we have constructed an **extension ring** $R[x]$ of R (that is, a ring in which R is a subring). This extension ring contains an element x that commutes with every element of R. The element x is *not* in R and does *not* stand for an element of R. Every element of the extension ring can be written in an essentially unique way in terms of elements of R and powers of x. Because x has the property that $a_0 + a_1x + \cdots + a_nx^n = 0_R$ if and only if every $a_i = 0_R$, x is said to be **transcendental** over R or an **indeterminate** over R.†

We are now in position to prove Theorem 4.1, in which the ring R need not have an identity.

Theorem 4.1

If R is a ring, then there exists a ring T containing an element x that is not in R and has these properties:

(i) R is a subring of T.

(ii) $xa = ax$ for every $a \in R$.

*You've been making this identification for years when, for example, you treat the constant polynomial 4 as if it were the real number 4. The identification question can be avoided by rewriting the definition of polynomial to say that a polynomial is either an element of R *or* a sequence $(a_1, a_2, \ldots)$ with at least one $a_i \neq 0_R$ for $i \geq 1$ and all a_i eventually zero. Then the polynomials actually contain R as a subset. The definitions of addition and multiplication, as well as the proofs of the theorems, then have to deal with several cases. Proceed in the obvious (but tiring) way until you have proved Theorem G.4 again.

†The latter terminology is a bit misleading since x is a well-defined element of $R[x]$.

(iii) The set $R[x]$ of all elements of T of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \qquad \text{(where } n \geq 0 \text{ and } a_i \in R\text{)}$$

is a subring of T that contains R.

(iv) The representation of elements of $R[x]$ is unique: If $n \leq m$ and

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = b_0 + b_1x + b_2x^2 + \cdots + b_nx^m,$$

then $a_i = b_i$ for $i = 1, 2, \ldots, n$ and $b_i = 0_R$ for each $i > n$.

(v) $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0_R$ if and only if $a_i = 0_R$ for every i.

Proof ▸ There are two cases: (1) R has an identity; and (2) R does not have an identity.

Case 1: Use Theorems G.1 and G.4, with $T = P = R[x]$ and R^* identified with R.

Case 2: Let S be a ring with identity that contains R as a subring. With many familiar rings, an S is easy to find. For example, ring of even integers has no identity, but is a subring of $\mathbb{Z}$, which does have an identity. For the general case, use Exercise 39 of Section 3.3.

Apply Case 1 with S in place of R, to construct $S[x] = T$. The polynomials in $S[x]$ whose coefficients are actually in R form a subring of $S[x] = T$ that contains R, as you can readily verify (Exercise 10); this subring is $R[x]$. Hence, property (i) of the theorem is satisfied. Since properties (ii)–(v) hold for all elements of $S[x]$, they necessarily hold for all elements of $R[x]$. ■

Finally, note that

When R does not have an identity, the polynomial x is *not* itself in $R[x]$.

For instance, the ring of polynomials over the ring R of even integers consists of all polynomials with even coefficients. So it does not contain $x = 1x$ or any polynomial kx with k odd.

Exercises

A. **1.** Express each polynomial as a sequence and express each sequence as a polynomial.

(a) $(0, 1, 0, 1, 0, 1, 0, 0, 0, \ldots)$

(b) $(0, 1, 2, 3, 4, 5, 6, 6, 8, 9, 0, 0, 0, \ldots)$

(c) $3x^6 - 5x^4 + 12x^3 - 3x^2 + 7.5x - 11$

(d) $(x - 1)(x^3 - x^2 + 1)$

2. **(a)** If $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ are polynomials, show that their sum is a polynomial (that is, after some point all coordinates of the sum are zero).

(b) Show that $(a_1, a_2, \ldots) \odot (b_1, b_2, \ldots)$ is a polynomial. [*Hint:* If $a_i = 0_R$ for $i > k$ and $b_i = 0_R$ for $i > t$, examine the ith coordinate of the product for $i > k + t$.]

3. Prove these parts of Theorem G.1:

(a) addition in P is associative;

(b) both distributive laws hold in P;

(c) P is commutative if R is.

4. Complete the proof of Theorem G.2 by proving that

(a) f is injective; (b) f is surjective

5. Prove (1) in Theorem G.4.

B. 6. (a) In the proof of Theorem G.1 (associative multiplication in P) show that $\sum_{i=0}^{n}\sum_{j=0}^{i} a_j b_{i-j} c_{n-i} = \sum a_u b_v c_w$ where the last sum is taken over all nonnegative integers u, v, w such that $u + v + w = n$. [*Hint:* Compare the two sums term by term; the sum of the subscripts of $a_j b_{i-j} c_{n-i}$ is n; to show that $a_u b_v c_w$ is in the other sum, let $j = u$ and $i = u + v$ and verify that $n - i = w$.]

(b) Show that $\sum_{r=0}^{n}\sum_{s=0}^{n-r} a_r b_s c_{n-r-s} = \sum a_u b_v c_w$ [last sum as in part (a)].

7. Prove (2) in Lemma G.3. [*Hint:* $\boldsymbol{a} = (a_0, a_1, a_2, \ldots)$, where $a_i = 0_R$ for $i > 1$, and by (1), $x^n = (d_0, d_1, d_2, \ldots)$, where $d_n = 1_R$ and $d_i = 0_R$ for $i \neq n$; use induction on n.]

8. Let R be an integral domain. Using sequence notation, prove that the polynomial ring $R[x]$ is also an integral domain.

9. Let R be a field. Using sequence notation, prove that the polynomial ring $R[x]$ is *not* a field. [*Hint:* Is $(0_R, 1_R, 0_R, 0_R, \ldots)$ a unit?]

10. In the proof of Case (2) of Theorem 4.1, show that $R[x]$ is a subring of $S[x]$ that contains R.

C. 11. (a) Let $\mathbb{Q}[\pi]$ be the set of all real numbers of the form $r_0 + r_1\pi + r_2\pi^2 + \cdots + r_n\pi^n$, where $n \geq 0$ and each $r_i \in \mathbb{Q}$. Show that $\mathbb{Q}[\pi]$ is a subring of $\mathbb{R}$.

(b) Assume that $r_0 + r_1\pi + \cdots + r_n\pi^n = 0$ if and only if each $r_i = 0$. (This fact was first proved in 1882; the proof is beyond the scope of this book.) Prove that $\mathbb{Q}[\pi]$ is isomorphic to the polynomial ring $\mathbb{Q}[x]$.

BIBLIOGRAPHY

This list contains all the books and articles referred to in the text, as well as a number of other books suitable for collateral reading, reference, and deeper study of particular topics. The list is far from complete. For the most part readability by students has been the chief selection criterion.

Abstract Algebra in General (Undergraduate Level)

These books contain approximately the same material as Chapters 1–12 of this text, but each of them provides a slightly different viewpoint and emphasis. Only [3] has a significant overlap with Chapters 13–16.

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2. Fraleigh, J., *A First Course in Abstract Algebra*, 7th edition. Boston: Pearson, 2003.
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Abstract Algebra in General (Graduate Level)

These books have much deeper and more detailed coverage of the material in Chapters 1–12, as well as a large number of topics not discussed in the text.

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ANSWERS AND SUGGESTIONS FOR SELECTED ODD-NUMBERED EXERCISES

For exercises that ask for proofs, there may be a sketch of the full proof (you fill in minor details), a key part of the proof (you fill in the rest), or a comment that should enable you to find a proof.

Chapter 1

Section 1.1 *(page 8)*

1. (a) $q = 4; r = 1$ (b) $q = 0; r = 0$ (c) $q = -5; r = 3$
3. (a) $q = 6; r = 19$ (b) $q = -9; r = 54$ (c) $q = 62{,}720; r = 92$
5. Multiply the equation and the inequality by c. Apply the Division Algorithm appropriately.
7. If $a = 3q + 1$, then $a^2 = (3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$, which is of the form $3k + 1$ with $k = 3q^2 + 2q$. Use similar arguments when $a = 3q$ or $a = 3q + 2$.
9. By the Division Algorithm, every integer a is of the form $3q$ or $3q + 1$ or $3q + 2$. Compute a^3 in each case and proceed as in Exercise 7.

Section 1.2 *(page 14)*

1. (a) 8 (c) 1 (e) 9 (g) 592.
3. $a \mid b$ means $b = au$ for some integer u. Similarly, $b \mid c$ means $c = bv$ for some integer v. Combine these two equations to show that $c = a \cdot$ (something), which proves that $a \mid c$.
5. $a \mid b$ means $b = au$ for some integer u, and $b \mid a$ means $a = bv$ for some integer v. Combine the equations to show that $a = auv$, which implies that $1 = uv$. Since u and v are integers, what are the only possibilities?
7. $|a|$—Why?
9. Advice: Before trying to prove a simple statement, check to see if there are any obvious counterexamples.
11. (a) 1 or 2
13. (c) By parts (a) and (b), the set of common divisors of a and b is identical to the set of common divisors of b and r. What is the largest integer in this set?

19. Suppose $d \mid a$ and $d \mid b$, so that $a = du$ and $b = dv$. Since $a \mid (b + c)$, $b + c = aw$. Hence, $c = aw - b = duw - dv = d(uw - v)$, so that $d \mid c$. Since $(b, c) = 1$, what can you conclude about d and (a, b)?

21. Every common divisor of a and (b, c) is also a common divisor of (a, b) and c. [*Proof:* If $d \mid (b, c)$, then $d \mid b$ and $d \mid c$ by the definition of (b, c). If $d \mid a$ also, then d is a common divisor of a and b, and, hence, $d \mid (a, b)$ by Corollary 1.3.] A similar argument shows that the common divisors of (a, b) and c are also common divisors of a and (b, c).

25. **(a)** (a, b) divides both a and b by definition. What does this say about (a, b) and 1?

27. $d = cu + av$ for some u, v (Why?). Hence, $db = cbu + abv$. Use the fact that $ab = cw$ for some w (Why?) to show that $c \mid db$.

29. First show that every integer n is the sum of a multiple of 9 and the sum of its digits. [*Example:* $7842 = 7 \cdot 1000 + 8 \cdot 100 + 4 \cdot 10 + 2 = 7(999 + 1) + 8(99 + 1) + 4(9 + 1) + 2 = (7 \cdot 999 + 8 \cdot 99 + 4 \cdot 9) + (7 + 8 + 4 + 2) = 9(7 \cdot 111 + 8 \cdot 11 + 4) + (7 + 8 + 4 + 2)$.] Thus, every n is of the form $9k + r$, where r is the sum of the digits of n. Hence, n is divisible by 9 if and only if 9 divides r.

31. **(a)** 30; 60; 420; 72

33. Let $d = (a, b)$. Then $a = du$ and $b = dv$ for some integers u and v. Let $m = ab/d$. Show that m is a common multiple of a and b. If c is any other common multiple of a and b, use Exercise 26 to show that $m \leq c$. What does this tell you?

Section 1.3 *(page 22)*

1. **(a)** $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$ **(c)** $45{,}670 = 2 \cdot 5 \cdot 4567$

3. All of them.

5. **(a)** $3, 3^2, 3^3, \ldots, 3^s; 3 \cdot 5, 3^2 \cdot 5, 3^3 \cdot 5, \ldots, 3^s \cdot 5; 3 \cdot 5^2, 3^2 \cdot 5^2, 3^3 \cdot 5^2, \ldots, 3^s \cdot 5^2; 3 \cdot 5^3, \ldots; 3 \cdot 5^t, 3^2 \cdot 5^t, 3^3 \cdot 5^t, \ldots, 3^s \cdot 5^t; 5, 5^2, \ldots, 5^t$.

7. Because p divides a, there is an integer k such that $a = pk$. Similarly, $a + bc = pd$ for some integer d. Hence $bc = pd - a = pd - pk = p(d - k)$. Apply Theorem 1.5.

9. ($\Leftarrow$) Suppose p has the given property and let d be a divisor of p, say $p = dt$. By the property, $d = \pm 1$ (in which case $t = \pm p$) or $t = \pm 1$ (in which case $d = \pm p$). Thus the only divisors of p are ± 1 and $\pm p$, and p is prime.

11. $a - b = pv$ and $c - d = pw$ for some v, w (Why?). Add the two equations and rewrite each side of the sum equation to obtain the fact that p divides $(a + c) - (b + d)$.

17. Every prime divisor of a^2 is also a divisor of a by Theorem 1.5, and similarly for b^2.

19. $\dfrac{b}{a} = \dfrac{p_1^{s_1} \cdots p_k^{s_k}}{p_1^{r_1} \cdots p_k^{r_k}} = p_1^{s_1 - r_1} \cdots p_k^{s_k - r_k}$. Since $a \mid b$, we know that $\dfrac{b}{a}$ is an integer. Since the p_i are distinct primes, each of the exponents on the right side of the preceding equation must be nonnegative (Why?)—that is, $s_1 - r_1 \geq 0, s_2 - r_2 \geq 0, \ldots, s_k - r_k \geq 0$.

21. If c has prime decomposition $p_1 p_2 \cdots p_k$, then $ab = c^2 = p_1 p_1 p_2 p_2 \cdots p_k p_k$. Now p_1 must divide a or b by Theorem 1.5, say a. Since $(a, b) = 1$, p_1 cannot divide b. Hence, $(p_1)^2 \mid a$. By relabeling and reindexing if necessary, show that $a = p_1 p_1 p_2 p_2 \cdots p_j p_j = (p_1 p_2 \cdots p_j)^2$ and $b = p_{j+1} p_{j+1} \cdots p_k p_k = (p_{j+1} p_{j+2} \cdots p_k)^2$.

23. Suppose a and b are positive and $a^2 \mid b^2$. Suppose that $a = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ and $b = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, where $p_1, p_2, \ldots, p_k$ are distinct positive primes with each $r_i, s_i \geq 0$ (see Exercise 13). Then $a^2 = p_1^{2r_1}p_2^{2r_2}\cdots p_k^{2r_k}$ and $b^2 = p_1^{2s_1}p_2^{2s_2}\cdots p_k^{2s_k}$ and because $a^2 \mid b^2$ we have $2r_i \leq 2s_i$, and hence $r_i \leq s_i$, for each $i = 1, 2, \ldots, k$ by Exercise 19. Thus, there are nonnegative integers $u_1, \ldots, u_k$ such that $s_i = r_i + u_i$ for each i. Use this fact and the prime decompositions of a and b to show that $a \mid b$. The converse is easy.

25. Exercise 6 in Appendix E shows that $\binom{p}{k}$ is an integer. $\binom{p}{1} = p$, and for $k > 1$, the denominator of $\binom{p}{k}$ is the product of integers that are each strictly less than p.

27. If $p > 3$ is prime, then $p = 6k + 1$ or $6k + 5$ (Why can the other cases be eliminated?). If $p = 6k + 1$, then $p^2 + 2 = (6k + 1)^2 + 2 = 36k^2 + 12k + 3 = 3(12k^2 + 4k + 1)$. The other case is handled similarly.

29. Let k be the highest power of 2 that divides n. Then $n = 2^k m$ for some integer m, which must be odd because otherwise 2^{k+1} would divide n, contradicting the fact that k is the highest power of 2 that divides n. Uniqueness follows from the Fundamental Theorem of Arithmetic.

33. Verify that $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1)$. Conclude that $y^{mn} - 1 = (y^m)^n - 1$ has $y^m - 1$ as a factor. Apply this fact with $y = 2$ and $p = mn$ to show that $2^p - 1$ is composite whenever p is.

Chapter 2

Section 2.1 *(page 30)*

1. **(a)** $2^4 = 16 \equiv 1 \pmod 5$

3. **(a)** and **(c)**

5. **(a)** $5 \equiv 1 \pmod 4$, so $5^{2000} \equiv 1^{2000} \equiv 1 \pmod 4$ by Theorem 2.2. Apply Theorem 2.3.
   (b) First, find a negative number that's congruent to 4 (mod 5).

7. By Corollary 2.5, $a \equiv 0$ or $a \equiv 1$ or $a \equiv 2$ or $a \equiv 3 \pmod 4$. Hence, a^2 is congruent to 0^2 or 1^2 or 2^2 or 3^2 (mod 4) by Theorem 2.2.

9. **(a)** $(n - a)^2 = n^2 - 2na + a^2$. Hence, $(n - a)^2 - a^2$ is divisible by n.

13. ($\Rightarrow$) By the Division Algorithm, $a = qn + r$ and $b = pn + s$ with the remainders r and s satisfying $0 \leq r < n$ and $0 \leq s < n$. If $a \equiv b \pmod n$, then $a - b = kn$ (Why?), and, hence, $kn = (qn + r) - (pn + s)$, which implies that $r - s = (k - q + p)n$, that is, $n \mid (r - s)$. Since r and s are strictly less than n, this is impossible unless $r - s = 0$. To prove the converse, assume $r = s$ and show that $n \mid (a - b)$.

15. Use Theorem 1.2 and the definition of congruence.

17. Note that $10 \equiv -1 \pmod{11}$ and use Theorem 2.2.

19. $a - b = nk$ for some k (Why?). Show that any common divisor of a and n also divides b, and that any common divisor of b and n also divides a. What does this say about (a, n) and (b, n)?

21. $10 \equiv 1 \pmod 9$; hence $10^n \equiv 1^n \equiv 1 \pmod 9$ by Theorem 2.2.

Section 2.2 (page 36)

1. (a)

+	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

·	[0]	[1]
[0]	[0]	[0]
[1]	[0]	[1]

(c)

+	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[0]	[1]	[2]	[3]	[4]	[5]

·	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[0]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[0]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[0]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[0]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[0]	[6]	[5]	[4]	[3]	[2]	[1]

3. $x = [1], [3], [5]$, or $[7]$
5. $x = [1], [2], [4]$, or $[5]$
7. $x = [3]$ or $[7]$
9. (a) $[a] = [3]$ or $[5]$ (c) No
11. (a) $x = [0], [1]$, or $[2]$ (c) $x = [0], [1], [2], [3]$, or $[4]$
13. Look in $\mathbb{Z}_4$ or $\mathbb{Z}_6$.
15. (a) $[a]^2 + [b]^2$ (c) $[a]^5 + [b]^5$

Section 2.3 (page 41)

1. (a) $a = 1, 2, 3, 4, 5$, and 6 (c) $a = 1, 2, 4, 5, 7$, and 8.
3. Several possibilities, including Exercise 10.
5. Since b is a zero divisor, $bc = 0$ with $b \neq 0$ and $c \neq 0$. Hence, $(ab)c = 0$. Use the fact that a is a unit to show that $ab \neq 0$. What do you conclude?
7. $ab = 0$ in $\mathbb{Z}_p$ means $p \mid ab$ in $\mathbb{Z}$. Apply Theorem 1.5 and translate the result into $\mathbb{Z}_p$.
9. (a) Since a is a unit, $ab = 1$ for some b. If a were also a zero divisor, then we would have $ac = 0$ for some $c \neq 0$. Consider the product abc and reach a contradiction.

11. *Existence of a solution*: $au = 1$ for some u (Why?). Multiply both sides of $ax = b$ by u. *Uniqueness:* Assume that r and s are solutions of $ax = b$ and use the fact that a is a unit to show that $r = s$.

15. **(a)** 3, 9, 15.

17. If a and c are units, then $ab = 1$ and $cd = 1$ for some b, d. Use this to show that ac is a unit.

Chapter 3

Section 3.1 *(page 53)*

1. **(a)** Closure for addition.

5. **(a)** Subring without identity (every product is the zero matrix) **(c)** Not a subring

(e) Commutative subring with identity.

7. Axioms 1–5 are easy to verify. Is K closed under multiplication?

11. **(a)** *Partial proof:* Closure under addition holds since $\begin{pmatrix} a & a \\ b & b \end{pmatrix} + \begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} a+c & a+c \\ b+d & b+d \end{pmatrix} \in S$. The zero matrix is in S. Use Theorem 3.2.

(c) J fails to be a left identity for any $B \in S$ whose bottom row is nonzero—check it out.

13. Use Theorem 3.2. Closure under addition: $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in \mathbb{Z}(\sqrt{2})$ since $a + c \in \mathbb{Z}$ and $b + d \in \mathbb{Z}$. Closure under multiplication: See Example 20. Also, $0 = 0 + 0\sqrt{2} \in \mathbb{Z}(\sqrt{2})$. You do the rest.

15. **(a)**

+	(0,0)	(1,1)	(0,2)	(1,0)	(0,1)	(1,2)
(0,0)	(0,0)	(1,1)	(0,2)	(1,0)	(0,1)	(1,2)
(1,1)	(1,1)	(0,2)	(1,0)	(0,1)	(1,2)	(0,0)
(0,2)	(0,2)	(1,0)	(0,1)	(1,2)	(0,0)	(1,1)
(1,0)	(1,0)	(0,1)	(1,2)	(0,0)	(1,1)	(0,2)
(0,1)	(0,1)	(1,2)	(0,0)	(1,1)	(0,2)	(1,0)
(1,2)	(1,2)	(0,0)	(1,1)	(0,2)	(1,0)	(0,1)

·	(0,0)	(1,1)	(0,2)	(1,0)	(0,1)	(1,2)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(1,1)	(0,0)	(1,1)	(0,2)	(1,0)	(0,1)	(1,2)
(0,2)	(0,0)	(0,2)	(0,1)	(0,0)	(0,2)	(0,1)
(1,0)	(0,0)	(1,0)	(0,0)	(1,0)	(0,0)	(1,0)
(0,1)	(0,0)	(0,1)	(0,2)	(0,0)	(0,1)	(0,2)
(1,2)	(0,0)	(1,2)	(0,1)	(1,0)	(0,2)	(1,1)

19.

$+$	0	S	A	B	C	D	E	F
0	0	S	A	B	C	D	E	F
S	S	0	F	E	D	C	B	A
A	A	F	0	D	E	B	C	S
B	B	E	D	0	F	A	S	C
C	C	D	E	F	0	S	A	B
D	D	C	B	A	S	0	F	E
E	E	B	C	S	A	F	0	D
F	F	A	S	C	B	E	D	0

$\cdot$	0	S	A	B	C	D	E	F
0	0	0	0	0	0	0	0	0
S	0	S	A	B	C	D	E	F
A	0	A	A	0	0	A	A	0
B	0	B	0	B	0	B	0	B
C	0	C	0	0	C	0	C	C
D	0	D	A	B	0	D	A	B
E	0	E	A	0	C	A	E	C
F	0	F	0	B	C	B	C	F

21. The multiplicative identity is 6.

23. To prove that E is closed under $*$, you must verify that when a and b are even integers, so is $a * b = ab/2$. To prove that $*$ is associative, verify that $(a * b) * c = a * (b * c)$ as follows. By definition, $(a * b) * c = (ab/2) * c = \dfrac{(ab/2)c}{2}$. Express $a * (b * c)$ in terms of multiplication in $\mathbb{Z}$ and verify that the two expressions are equal. Commutativity of $*$ is proved similarly. To prove the distributive law, you must verify that $a * (b + c) = a * b + a * c$, that is, that $a(b + c)/2 = ab/2 + ac/2$. If there is a multiplicative identity e, then it must satisfy $e * a = a$ for every $a \in E$, which is equivalent to $ea/2 = a$ in $\mathbb{Z}$. But $ea/2 = a$ implies that $e = 2$.

25. *Partial proof:* Axiom 4: The zero element is -1 because $r \oplus (-1) = r + (-1) + 1 = r$. Axiom 5: Since -1 is the zero element, we must show that the equation $a \oplus x = -1$ has a solution. The solution is $x = -2 - a$ because $a \oplus (-2 - a) = a + (-2 - a) + 1 = -1$. To prove that this ring is an integral domain, you must assume that $a \odot b = -1$ and show that $a = -1$ or $b = -1$. Now $a \odot b = -1$ means that $ab + a + b = -1$ in $\mathbb{Q}$, that is, that $ab + a + b + 1 = 0$. Factor the left side and use the fact that $\mathbb{Q}$ is an integral domain.

27. *Partial proof:* If c and d are odd, then so is cd. Hence, $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd} \in S$, and S is closed under addition. $0 \in S$ since, for example, $0 = 0/5$. Use Theorem 3.2. As to S being a field, what is the solution of $(2/7)x = 1$?

31. **(b)** If $K = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$KA = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} = \begin{pmatrix} ak & bk \\ ck & dk \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = AK.$$

35. Consider $R = \mathbb{Z}_2$, $S = \mathbb{Z}_3$ and examine the table in the answer to Exercise 15(a).

37. **(a)** Copy the proof used for $M(\mathbb{R})$ in Example 6.

39. The proof that $\mathbb{Q}(\sqrt{2})$ is a ring is essentially the same as in Exercise 13. The hint shows how to verify that the solution of $(r + s\sqrt{2})x = 1$ is actually in $\mathbb{Q}(\sqrt{2})$.

41. **(b)** *Partial proof:* If $\begin{pmatrix} x & x \\ y & y \end{pmatrix}$ is a right identity, then

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix}\begin{pmatrix} x & x \\ y & y \end{pmatrix} = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$$

$$\begin{pmatrix} ax + ay & ax + ay \\ bx + by & bx + by \end{pmatrix} = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$$

$$\begin{pmatrix} a(x + y) & a(x + y) \\ b(x + y) & b(x + y) \end{pmatrix} = \begin{pmatrix} a & a \\ b & b \end{pmatrix}.$$

This last equation holds only when $x + y = 1$.

43. **(b)** Since H is contained in the ring $M(\mathbb{C})$, its addition is commutative and associative, its multiplication is associative, and the distributive law holds. So you need to verify only that H is closed under addition and multiplication, that the zero and identity matrices are in H, and that the negative of every matrix in H is also in H.

Section 3.2 *(page 66)*

1. **(a)** $a^2 - ab + ba - b^2$.

3. **(b)** 0, 1, 4, 9

5. **(c)** No. Suppose u is a unit in R with inverse u^{-1} and v is another inverse of u. Then $uv = 1_R$, so that $u^{-1}uv = u^{-1}1_R$, which implies that $v = u^{-1}$. Hence, there is only one inverse.

9. Closure under multiplication: $\begin{pmatrix} a & 4b \\ b & a \end{pmatrix}\begin{pmatrix} c & 4d \\ d & c \end{pmatrix} = \begin{pmatrix} ac + 4bd & 4ad + 4bc \\ bc + ad & 4bd + ac \end{pmatrix} = \begin{pmatrix} ac + 4bd & 4(ad + bc) \\ ad + bc & ac + 4bd \end{pmatrix} \in S$. Verify that S is closed under subtraction and apply Theorem 3.6.

11. S is nonempty since $0_R \in S$ (Why?). If $r, s \in S$, then by definition $mr = 0_R$ and $ms = 0_R$. Hence, $m(r - s) = mr - ms = 0_R - 0_R = 0_R$. So $r - s \in S$. Similarly, by Exercise 23, $m(rs) = (mr)s = 0_Rs = 0_R$. So $rs \in S$. Therefore, S is a subring by Theorem 3.6.

15. **(b)** Many possible examples. Almost any pair of invertible matrices in $M(\mathbb{R})$ will provide an example.

17. If $ub = 0_R$ and u is a unit with inverse v, left multiply both sides of $ub = 0_R$ by v to conclude that $b = 0_R$. If $cu = 0_R$, a similar argument (with right multiplication by v) shows that $c = 0_R$. Thus, there is no nonzero element whose product with u is 0_R and, hence, u is not a zero divisor.

19. If $(a, b)(c, d) = (1_R, 1_S)$, what can be said about ac and bd?

21. $ab = ac$ is equivalent to $a(b - c) = 0_R$.

25. **(a)** See Exercise 21 of Section 3.1 (to which the answer is "yes").

(b) Consider $1_S 1_R$ and $1_S 1_S$ and use Exercise 21.

27. No. For a counterexample, let b be almost any matrix in $M(\mathbb{R})$.

31. **(a)** $(a + a)^2 = a + a$ because $x^2 = x$ for *every* x. But $(a + a)^2 = (a + a)(a + a) = a^2 + a^2 + a^2 + a^2 = a + a + a + a$.

39. **(b)** No. You should be able to find a counterexample.

41. **(b)** 12

Section 3.3 *(page 80)*

1. The tables for $\mathbb{Z}_2 \times \mathbb{Z}_3$ are in the answer to Exercise 15 **(a)** of Section 3.1.

3. If $f(a) = f(b)$, then $(a, a) = (b, b)$, and, hence, $a = b$ by the equality rules for ordered pairs. Therefore, f is injective. $f(a + b) = (a + b, a + b) = (a, a) + (b, b) = f(a) + f(b)$. Complete the proof by showing that $f(ab) = f(a)f(b)$ and that f is surjective.

11. Many correct answers, including the following.

(a) f does not preserve addition; for example $f(4 + 9) = \sqrt{4 + 9} = \sqrt{13} \approx 3.6$, but $f(4) + f(9) = \sqrt{4} + \sqrt{9} = 2 + 3 = 5$. So $f(4 + 9) \neq f(4) + f(9)$.

(b) f does not preserve multiplication; for example $f(2 \cdot 5) = f(10) = 30$, but $f(2) \cdot f(5) = (6)(15) = 90$. So $f(2 \cdot 5) \neq f(2) \cdot f(5)$.

13. *Partial proofs:* **(a)** To prove f is surjective, let $r \in R$. Then $(r, 0_S) \in R \times S$ and $f((r, 0_s)) = r$. Hence, f is surjective.

(c) If a is a nonzero element of S, then $f((0_R, a)) = 0_R = f((0_R, 0_S))$, but $(0_R, a) \neq (0_R, 0_S)$. Hence, f is not injective.

17. *Surjective:* If $a + bi$ is a complex number, then $f(a - bi) = a - (-bi) = a + bi$. *Injective:* If $f(a + bi) = f(c + di)$, use the definition of f and the definition of equality for complex numbers (Example 11 of Section 3.1) to show that $a + bi = c + di$.

21. The multiplicative identity in $\mathbb{Z}^*$ is 0. If there is an isomorphism $f: \mathbb{Z} \to \mathbb{Z}^*$, Theorem 3.10 shows that f must satisfy $f(1) = 0$. Hence, $f(2) = f(1 + 1) = f(1) \oplus f(1) = 0 \oplus 0 = 0 + 0 - 1 = -1$. Similarly, $f(3) = f(1 + 2) = f(1) \oplus f(2) = 0 \oplus (-1) = 0 + (-1) - 1 = -2$. What is $f(4)$? $f(5)$? $f(-1)$? Find a formula for f. Then use this formula to show that f is injective, surjective, and a homomorphism.

25. f is not an isomorphism because it is not injective. For instance,

$$f\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = 1 = f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

27. **(a)** Because f and g are homomorphisms, $(f \circ g)(a + b) = f(g(a + b)) = f(g(a) + g(b)) = f(g(a)) + f(g(b)) = (f \circ g)(a) + (f \circ g)(b)$. A similar argument shows that $(f \circ g)(ab) = (f \circ g)(a) \cdot (f \circ g)(b)$. *(continues on next page)*

(b) You must show two things: (1) If f and g are injective, so is $f \circ g$; and (2) if f and g are surjective, so is $f \circ g$. To prove (1), assume $(f \circ g)(a) = (f \circ g)(b)$, that is, $f(g(a)) = f(g(b))$. Then use the injectivity of f and g to show $a = b$.

31. Since $f(0_R) = 0_S \in T$, we see that $0_R \in P$; so P is nonempty. Let $a, b \in P$; then $f(a) \in T$ and $f(b) \in T$. Hence, $f(a - b) = f(a) - f(b) \in T$. Thus, $a - b \in P$. A similar argument shows that $ab \in P$. Therefore, P is a subring by Theorem 3.6.

35. **(a)** $\mathbb{Z}$ has an identity and E doesn't. **(c)** The rings have different numbers of elements, and so no injective function is possible from $\mathbb{Z}_4 \times \mathbb{Z}_{14}$ to $\mathbb{Z}_{16}$. **(e)** The equation $x + x = 0_R$ has a *nonzero* solution in $\mathbb{Z} \times \mathbb{Z}_2$ (What is it?) but not in $\mathbb{Z}$.

37. **(b)** Since f is nonzero, there exists $a \in S$ such that $f(a) \neq 0_T$. Hence, $f(1_S)f(a) = f(1_S a) = f(a) \neq 0_T$, which implies that $f(1_S) \neq 0_T$. Show that $f(1_S)$ is an idempotent and apply part (a).

Chapter 4

Section 4.1 *(page 93)*

1. **(a)** $3x^4 + x^3 + 2x^2 + 2$ **(c)** $x^5 - 1$.

3. **(a)** $x^3; x^3 + x^2; x^3 + x; x^3 + x^2 + x; x^3 + 1; x^3 + x^2 + 1; x^3 + x + 1; x^3 + x^2 + x + 1$.

5. **(a)** $q(x) = 3x^2 - 5x + 8$; $r(x) = -4x - 6$.

(c) $q(x) = x^3 + 3x^2 + 2x + 3$; $r(x) = 4$.

9. Yes (read the definition of zero divisor and remember that R is a subset of $R[x]$).

11. The fact that $(r + s)(r - s) = r^2 - s^2$ may be helpful.

13. There exists $g(x) \in R[x]$ such that $f(x)g(x) = 0_R$ (Why?). Suppose $g(x) = b_0 + b_1x + \cdots + b_kx^k$ (with $b_k \neq 0_R$). Multiply out $f(x)g(x)$ and look at the coefficient of x^{n+k}. What must this coefficient be? And what does that say about a_n.

15. **(b)** Add one term to the polynomial in the hint for part (a).

17. If $0 \neq b \in R$, then $b \in R[x]$ and $1_R = bq(x) + r(x)$. Use the fact that $\deg b = 0$ to show that $r(x) = 0$ and $q(x) \in R$. Hence, every nonzero element of R has an inverse.

Section 4.2 *(page 99)*

1. If $0_F \neq c \in F$, then c has an inverse; hence, $f(x) = c(c^{-1}f(x))$.

5. **(a)** $x - 1$ **(c)** $x^2 - 1$ **(e)** $x - i$.

7. Since $f(x) \mid (x + 1)$ and $f(x) \mid x$, $f(x)$ must divide $(x + 1) - x = 1$. Hence, $\deg f(x) = 0$; so $f(x)$ is a constant.

9. 1_F is a linear combination of $f(x)$ and 0_F (Why?). What does this imply?

15. Every divisor of $h(x)$ is also a divisor of $f(x)$.

Section 4.3 *(page 103)*

1. **(a)** $x^3 + \dfrac{2}{3}x^2 + \dfrac{1}{3}x + \dfrac{5}{3}$ **(c)** $x^3 - ix + i$.

3. **(a)** $x^2 + x + 1; 2x^2 + 2x + 2; 3x^2 + 3x + 3; 4x^2 + 4x + 4$.

7. ($\Rightarrow$) Suppose $f(x)$ is irreducible and $g(x) = cf(x)$, with $0_F \neq c \in F$. If $g(x) = r(x)s(x)$, then $f(x) = (c^{-1}r(x))s(x)$, and, hence, either $c^{-1}r(x)$ or $s(x)$ is a nonzero constant by Theorem 4.12. If $c^{-1}r(x)$ is a constant, show that $r(x)$ is also a constant. Hence, $g(x)$ is irreducible by Theorem 4.12.

9. **(a)** $x^2 + x + 1$ **(c)** $x^2 + 1; x^2 + x + 2; x^2 + 2x + 2; 2x^2 + 2; 2x^2 + x + 1;$ $2x^2 + 2x + 1$.

11. If it were reducible, it would have a monic factor of degree 1 (Why?), that is, a factor of the form $x + a$ with $a \in \mathbb{Z}_7$. Verify that none of the seven possibilities is a factor.

13. $(x - 3)(x - 4)^3$.

15. **(a)** If $f(x) \in \mathbb{Z}_p[x]$ is a monic reducible quadratic, then it must factor as $f(x) = (cx + d)(c^{-1}x + e)$ for some $c, d, e \in \mathbb{Z}_p$ (Why?). Hence, $f(x) = c(x + dc^{-1})c^{-1}(x + ec) = (x + a)(x + b)$ with $a = dc^{-1}$ and $b = ec$. When counting the possible pairs of factors, remember that, for example, $(x + 2)(x + 3)$ is the same factorization as $(x + 3)(x + 2)$. Also consider factorizations such as $(x + 2)(x + 2)$.

23. **(a)** Proceed as in the answer to Exercise 11, with $\mathbb{Z}_5$ in place of $\mathbb{Z}_7$.

Section 4.4 *(page 109)*

1. **(a)** Many correct answers, including $f(x) = x^2 + x$.

3. **(a)** No; $f(-2) \neq 0$. **(c)** Yes.

5. The Factor Theorem may be helpful.

7. Show that every element of $\mathbb{Z}_7$ is a root of $x^7 - x$.

9. In $\mathbb{Z}_3[x]$: $x^2 + 1; x^2 + x + 2; x^2 + 2x + 2$.

13. **(a)** If $f(x) = cg(x)$ with $c \neq 0_F$, then $g(x) = c^{-1}f(x)$. Hence, $g(u) = 0_F$ implies $f(u) = 0_F$ and vice versa.

15. If $x^2 + 1$ is reducible, then $x^2 + 1 = (x + a)(x + b)$ for some $a, b \in \mathbb{Z}_p$ (see the answer to Exercise 21(a) of Section 4.3). Expand the right side.

19. **(a)** If $f(x) = (x - a)^k g(x)$ with $g(a) \neq 0$, then $f'(x) = k(x - a)^{k-1}g(x) + (x - a)^k g'(x)$. If a is a multiple root of $f(x)$, then $k \geq 2$ and $k - 1 \geq 1$. If a is a root of both $f(x)$ and $f'(x)$, show that $k \geq 2$.

23. **(a)** Let n be the maximum of the degrees of $f(x)$, $g(x)$, and $h(x)$. Using zero coefficients as necessary, we have $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_nx^n$, and $h(x) = c_0 + c_1x + \cdots + c_nx^n$. Then in $F[x]$, $g(x) + h(x) = (b_0 + b_1x + \cdots + b_nx^n) + (c_0 + c_1x + \cdots + c_nx^n) = (b_0 + c_0) + (b_1 + c_1)x + \cdots + (b_n + c_n)x^n$. Since $f(x) = g(x) + h(x)$ in $F[x]$, we must have $a_0 = b_0 + c_0$, $a_1 = b_1 + c_1$, $a_n = b_n + c_n$. Therefore, in F, $g(r) + h(r) = (b_0 + c_0) + (b_1 + c_1)r + \cdots + (b_n + c_n)r^n = a_0 + a_1r + \cdots + a_nr^n = f(r)$.

29. The proof is by induction on the degree n of $f(x)$. If $n = 0$, then $f(x)$ is a nonzero constant polynomial and therefore has no roots. So the corollary is true for $n = 0$. Now assume that the corollary is true for all polynomials of degree $k - 1$ and suppose that $\deg f(x) = k$. Prove that the corollary is true for $f(x)$ (that is, when $n = k$). [*You supply the work here.*] Conclude that the corollary is true for every degree n.

Section 4.5 *(page 119)*

1. **(a)** $(-1)(x + 1)(x - 2)(x^2 + 1)$ **(c)** $xx(x + 2)(x - 1)(3x - 1)$
   (e) $(x + 3)(2x + 1)(x^2 + 1)$.
3. Use the Rational Root Test.
5. **(a)** Let $p = 2$. **(c)** Let $p = 2$ or $p = 3$.
7. **(a)** Let $p = 5$ and use Corollary 4.19.
11. Apply Eisenstein's Criterion and Corollary 4.18.
17. A polynomial of degree k has $k + 1$ coefficients. There are n choices for each coefficient except the coefficient a_k of x^k. How many choices are there for a_k?
19. **(a)** $(x + 2)(x - 2)(x^3 + 2x^2 + 4x + 2)$

Section 4.6 *(page 123)*

1. **(a)** $1 - 2i$; $1 + 2i$; 3; -2 **(c)** $3 + 2i$; $3 - 2i$; $-1 + i$; $-1 - i$.
3. **(a)** $x^4 - 2$ in $\mathbb{Q}[x]$; $(x^2 + \sqrt{2})(x + \sqrt[4]{2})(x - \sqrt[4]{2})$ in $\mathbb{R}[x]$; $(x - \sqrt[4]{2}i)(x + \sqrt[4]{2}i)(x + \sqrt[4]{2})(x - \sqrt[4]{2})$ in $\mathbb{C}[x]$. **(c)** $(x - 1)(x^2 - 5)$ in $\mathbb{Q}[x]$; $(x - 1)(x + \sqrt{5})(x - \sqrt{5})$ in $\mathbb{R}[x]$ and $\mathbb{C}[x]$.
5. Nonreal roots of $f(x)$ occur in pairs by Lemma 4.29.

Chapter 5

Section 5.1 *(page 129)*

1. **(a)** $f(x) \equiv g(x) \pmod{p(x)}$ **(b)** $f(x) \equiv g(x) \pmod{p(x)}$
   (c) $f(x) \not\equiv g(x) \pmod{p(x)}$
3. There are eight congruence classes.
5. Use Corollary 5.5.
7. Each congruence class can be written in the form $[a]$, with $a \in F$.
9. See the answer to Exercise 13 of Section 2.1 with $f(x)$ and $g(x)$ in place of a and b.

Section 5.2 *(page 134)*

1.

+	[0]	[1]	[x]	[x + 1]	[x²]	[x² + 1]	[x² + x]	[x² + x + 1]
$[0]$	$[0]$	$[1]$	$[x]$	$[x+1]$	$[x^2]$	$[x^2+1]$	$[x^2+x]$	$[x^2+x+1]$
$[1]$	$[1]$	$[0]$	$[x+1]$	$[x]$	$[x^2+1]$	$[x^2]$	$[x^2+x+1]$	$[x^2+x]$
$[x]$	$[x]$	$[x+1]$	$[0]$	$[1]$	$[x^2+x]$	$[x^2+x+1]$	$[x^2]$	$[x^2+1]$
$[x+1]$	$[x+1]$	$[x]$	$[1]$	$[0]$	$[x^2+x+1]$	$[x^2+x]$	$[x^2+1]$	$[x^2]$
$[x^2]$	$[x^2]$	$[x^2+1]$	$[x^2+x]$	$[x^2+x+1]$	$[0]$	$[1]$	$[x]$	$[x+1]$
$[x^2+1]$	$[x^2+1]$	$[x^2]$	$[x^2+x+1]$	$[x^2+x]$	$[1]$	$[0]$	$[x+1]$	$[x]$
$[x^2+x]$	$[x^2+x]$	$[x^2+x+1]$	$[x^2]$	$[x^2+1]$	$[x]$	$[x+1]$	$[0]$	$[1]$
$[x^2+x+1]$	$[x^2+x+1]$	$[x^2+x]$	$[x^2+1]$	$[x^2]$	$[x+1]$	$[x]$	$[1]$	$[0]$

$\cdot$	$[0]$	$[1]$	$[x]$	$[x+1]$	$[x^2]$	$[x^2+1]$	$[x^2+x]$	$[x^2+x+1]$
$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$	$[x]$	$[x+1]$	$[x^2]$	$[x^2+1]$	$[x^2+x]$	$[x^2+x+1]$
$[x]$	$[0]$	$[x]$	$[x^2]$	$[x^2+x]$	$[x+1]$	$[1]$	$[x^2+x+1]$	$[x^2+1]$
$[x+1]$	$[0]$	$[x+1]$	$[x^2+x]$	$[x^2+1]$	$[x^2+x+1]$	$[x^2]$	$[1]$	$[x]$
$[x^2]$	$[0]$	$[x^2]$	$[x+1]$	$[x^2+x+1]$	$[x^2+x]$	$[x]$	$[x^2+1]$	$[1]$
$[x^2+1]$	$[0]$	$[x^2+1]$	$[1]$	$[x^2]$	$[x]$	$[x^2+x+1]$	$[x+1]$	$[x^2+x]$
$[x^2+x]$	$[0]$	$[x^2+x]$	$[x^2+x+1]$	$[1]$	$[x^2+1]$	$[x+1]$	$[x]$	$[x^2]$
$[x^2+x+1]$	$[0]$	$[x^2+x+1]$	$[x^2+1]$	$[x]$	$[1]$	$[x^2+x]$	$[x^2]$	$[x+1]$

3.

$+$	$[0]$	$[1]$	$[x]$	$[x+1]$
$[0]$	$[0]$	$[1]$	$[x]$	$[x+1]$
$[1]$	$[1]$	$[0]$	$[x+1]$	$[x]$
$[x]$	$[x]$	$[x+1]$	$[0]$	$[1]$
$[x+1]$	$[x+1]$	$[x]$	$[1]$	$[0]$

$\cdot$	$[0]$	$[1]$	$[x]$	$[x+1]$
$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$	$[x]$	$[x+1]$
$[x]$	$[0]$	$[x]$	$[1]$	$[x+1]$
$[x+1]$	$[0]$	$[x+1]$	$[x+1]$	$[0]$

7. $[ax+b]+[cx+d]=[(a+c)x+(b+d)]$;
$[ax+b][cx+d]=[(ad+bc)x+(3ac+bd)]$.

11. Consider the product of $[x]$ with itself.

Section 5.3 *(page 138)*

1. (a) Field (Use Corollary 4.19 and Theorem 5.10.)
(c) Not a field. (Show that x^4+x^2+1 is reducible.)

3. By Corollary 5.5, the distinct elements of $F[x]/(x-a)$ are the classes of the form $[c]$ with $c \in F$. Use this to show that $F[x]/(x-a)$ is isomorphic to F.

5. (a) Verify that the multiplicative inverse of $r+s\sqrt{3}$ is $\frac{r}{t}-\frac{s}{t}\sqrt{3}$, where $t=r^2-3s^2$.

7. By Corollary 5.12, there is an extension field K of F that contains a root c_1 of $f(x)$. Hence, $f(x)=(x-c_1)g(x)$ in $K[x]$. Use Corollary 5.12 again to find an extension field L of K that contains a root c_2 of $g(x)$. Continue.

9. (a) Use Corollary 4.19 and Theorem 5.10.

Chapter 6

Section 6.1 (page 148)

1. To see that K is not an ideal, consider what happens when you multiply a constant polynomial by a polynomial of positive degree.

9. **(a)** If $r \in R$ and $1_R \in I$, then $r = r \cdot 1_R \in I$. Hence, $R \subseteq I$ and thus $R = I$.

11. **(a)** $(0) = \{0\}$ and $(1) = (2) = (3) = (4) = \mathbb{Z}_5$ **(c)** $(0) = \{0\}$; $(1) = (5) = (7) = (11) = \mathbb{Z}_{12}$; $(2) = (6) = (10) = \{0, 2, 4, 6, 8, 10\}$; $(4) = (8) = \{0, 4, 8\}$; $(3) = (9) = \{0, 3, 6, 9\}$; $(6) = \{0, 6\}$.

13. No; see the answer for Exercise 11.

17. **(a)** $I \cap J$ contains 0_R (Why?) and hence is nonempty. If $a, b \in I \cap J$, then $a, b \in I$, so that $a - b$ is in I by Theorem 6.1. Similarly $a - b \in J$. Hence, $a - b \in I \cap J$. Now show that if $r \in R$, then $ra \in I \cap J$ and $ra \in I \cap J$. Apply Theorem 6.1.

27. Use Theorem 6.1. K is nonempty because $f(0_R) = 0_S$ by Theorem 3.10, and, hence, $0_R \in K$. If $a, b \in K$, then $f(a) = 0_S$ and $f(b) = 0_S$ by the definition of K. To show that $a - b \in K$, you must prove that $f(a - b) = 0_S$. If $r \in R$, you must prove that $f(ra) = 0_S$ in order to show that $ra \in K$.

29. An element of $(m) \cap (n)$ is divisible by both m and n; hence, it is in (mn) (see Exercise 17 of Section 1.2).

31. ($\Rightarrow$) If $(a) = (b) = (0_R)$, show that $a = 0_R = b$ and, hence, $a = bu$ with $u = 1_R$. If $(a) = (b) \neq (0_R)$, then both a and b are nonzero and $a = a \cdot 1_R \in (a)$. Therefore, $a \in (b)$, so that $a = bu$ for some $u \in R$. Similarly, $b = av$ for some $v \in R$. Hence, $a = bu = avu$, which implies that $uv = 1_R$ (Theorem 3.7), so that u is a unit.

35. If $I \neq (3)$, show that I contains an element b such that $(3, b) = 1$. Use Theorem 1.3 to show that $1 \in I$ and, hence, by Exercise 9(a), $I = \mathbb{Z}$.

41. **(a)** See Exercise 27 in Section 3.1.

43. **(b)** If $f(x) \in \mathbb{Z}[x]$ has constant term c, then x divides $f(x) - c$, so that $f(x) \equiv c \pmod{J}$ by part (a). Hence, $f(x) + J = c + J$ by Theorem 6.6. If b, c are distinct integers, then $b - c$ cannot be divisible by x (Why?). Hence, $b - c \notin J$ and $b \not\equiv c \pmod{J}$. Therefore, $b + J \neq c + J$ by Theorem 6.6.

47. *Half proof:* Suppose that $u \in S$. If $u^2 = u$ and $S = (u)$, then S is a subring since it is an ideal. If $s \in S$, then $s = ru$ for some $r \in \mathbb{Z}_n$. Hence, $su = (ru)u = ru^2 = ru = s$. So u is the identity element in S.

Section 6.2 (page 159)

3. By Exercise 10 in Section 6.1, the kernel of f is either (0_F) or F. Explain why it cannot be F. Hence, f is injective by Theorem 6.11 and, therefore, an isomorphism.

5. Consider the case when $R = \mathbb{Z}$ and I is the principal ideal (n). Then $\mathbb{Z}/I$ is just $\mathbb{Z}_n$. Is $\mathbb{Z}_n$ always an integral domain?

7. Apply the First Isomorphism Theorem to the identity map from R to R.

9. **(b)** The ideal consisting of all matrices in R of the form $\begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix}$, with b, c integers.

13. *Half proof:* Let $a + I \in R/I$. If there is an element $b \in R$ such that $a - b^2 \in I$, then $a \equiv b^2 \pmod{I}$. So $a + I = b^2 + I = (b + I)(b + I)$ by Theorem 6.6. Hence, $b + I$ is a square root of $a + I$ in R/I.

17. **(a)** $f(a + b) = ((a + b) + I, (a + b) + J) = ((a + I) + (b + I), (a + J) + (b + J)) = (a + I, a + J) + (b + I, b + J) = f(a) + f(b)$. A similar argument shows that $f(ab) = f(a)f(b)$. **(c)** $I \cap J$

21. Let $f: \mathbb{Z}_{20} \to \mathbb{Z}_5$ be given by $f([a]_{20}) = [a]_5$, where $[a]_n$ denotes an element of $\mathbb{Z}_n$. First, show that f is a well-defined function (independent of the choice of representative in the congruence class). Then show that f is a surjective homomorphism of rings with kernel (5). Apply the First Isomorphism Theorem.

25. If $r + J$ is a nilpotent element of R/J, then for some n, we have $0_R + J = (r + J)^n = r^n + J$. Hence, $r^n \in J$ (Why?), which means that r^n is nilpotent in R. Hence, $(r^n)^m = 0_R$ for some m. But this says $r \in J$, and, hence, $r + J$ is the zero coset $0_R + J$.

29. Define a function $f: S \to \mathbb{R} \times \mathbb{R}$ by $f\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (a, c)$. Show that f is a surjective homomorphism of rings with kernel I. Apply the First Isomorphism Theorem.

Section 6.3 *(page 166)*

1. By the definition of composite, $n = cd$ with $1 < |c| < |n|$ and $1 < |d| < |n|$. Hence, c and d cannot be multiples of n. Thus $cd = n \in (n)$, but $c \notin (n)$ and $d \notin (n)$. Therefore, (n) is not a prime ideal.

3. **(a)** Use Theorem 2.8 to show that p is prime if and only if $\mathbb{Z}_p$ is a field. But $\mathbb{Z}_p = \mathbb{Z}/(p)$; apply Theorem 6.15.

5. The maximal ideals in $\mathbb{Z}_6$ are $\{0, 3\}$ and $\{0, 2, 4\}$.

7. If R is a field, use Exercise 10 of Section 6.1. If (0_R) is a maximal ideal, use Theorem 6.15 and Exercise 7 of Section 6.2.

9. If $p = cd$, then $cd \in (p)$. Since (p) is prime, either $c \in (p)$ or $d \in (p)$, say $c \in (p)$. Hence, $c = pv$ for some $v \in R$. Use this and the fact that $p = cd$ to show that d is a unit.

15. **(b)** M is not prime because, for example, $3 \cdot 7 = 0 \in M$, but $3 \notin M$ and $7 \notin M$.

17. I is an ideal by Exercise 22 of Section 6.2. Use the fact that $J \neq S$ (Why?) and surjectivity to show that $I \neq R$. If $rs \in I$, then $f(rs) \in J$. Hence, $f(r)f(s) \in J$ (Why?), so that $f(r) \in J$ or $f(s) \in J$ by primality. Therefore, $r \in I$ or $s \in I$, and, hence, I is prime.

19. ($\Rightarrow$) Suppose R has a unique maximal ideal M. Then $M \neq R$ by definition, and so M is contained in the set of nonunits by Exercise 9 of Section 6.1. If c is a nonunit, then the ideal $(c) \neq R$ (Why?). So (c) is contained in a maximal ideal by hypothesis. But M is the only maximal ideal. So $c \in (c) \subseteq M$. Since every nonunit is in M, the set of nonunits is the ideal M.

Chapter 7

Section 7.1 *(page 180)*

1. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Each of the other permutations is its own inverse.

3. (a) 18 (c) 24 (e) 6.

5. (a) $\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 6 \\ 2 & 2 \end{pmatrix}$.

9.

$\circ$	r_0	r_1	r_2	s	t	u
r_0	r_0	r_1	r_2	s	t	u
r_1	r_1	r_2	r_0	u	s	t
r_2	r_2	r_0	r_1	t	u	s
s	s	t	u	r_0	r_1	r_2
t	t	u	s	r_2	r_0	r_1
u	u	s	t	r_1	r_2	r_0

13. $S_3 \times \mathbb{Z}_2$ is nonabelian of order 12 and $D_4 \times \mathbb{Z}_2$ is nonabelian of order 16.

17. (a) G is a group. *Closure:* If $a, b \in \mathbb{Q}$, then $a * b = a + b + 3 \in \mathbb{Q}$. *Associativity:* $(a * b) * c = (a + b + 3) * c = (a + b + 3) + c + 3 = a + b + c + 6 = a + (b + c + 3) + 3 = a * (b + c + 3) = a * (b * c)$. Verify that -3 is the identity element and that the inverse of a is $-6 - a$ because $a * (-6 - a) = a + (-6 - a) + 3 = -3$ and, similarly, $(-6 - a) * a = -3$. (c) G is a group with identity 0. The inverse of a is $-a/(1 + a)$.

19. No; there is no identity e satisfying *both* $a * e = a$ and $e * a = a$ for *every* a.

23. Most of the argument in Example 15 of Section 7.1.A can be carried over to this situation by replacing "$\neq 0$" by "$= 1$" throughout. To show that the inverse of a matrix in $SL(2, \mathbb{R})$ is also in $SL(2, \mathbb{R})$, use the formula for the inverse of a matrix (in Example 7 of Section 3.2 and in Example 15 of Section 7.1.A).

27. If $ab = ac$, then $b = eb = (a^{-1}a)b = a^{-1}(ab) = a^{-1}(ac) = (a^{-1}a)c = ec = c$.

31. Let a, b, c be distinct elements of T. Let $\sigma \in A(T)$ be given by $\sigma(a) = b$, $\sigma(b) = a$, and $\sigma(t) = t$ for every other element of T. Let $\tau \in A(T)$ be given by $\tau(a) = b$, $\tau(b) = c$, $\tau(c) = a$, and $\tau(t) = t$ for every other element of T. Verify that $(\sigma \circ \tau)(a) = a$ and $(\tau \circ \sigma)(a) = c$; hence, $\sigma \circ \tau \neq \tau \circ \sigma$.

Section 7.2 *(page 201)*

1. $e = c^{-1}c = c^{-1}c^2 = (c^{-1}c)c = ec = c$.

5. If $f(a) = f(b)$, then $a^{-1} = b^{-1}$. Hence, $(a^{-1})^{-1} = (b^{-1})^{-1}$. Therefore, by Corollary 7.6, $a = (a^{-1})^{-1} = (b^{-1})^{-1} = b$. Thus f is injective. Corollary 7.6 can also be used to prove that f is surjective.

7. (a) 2 (c) 6.

9. (a) U_{10} has order 4; U_{24} has order 8.

13. If G is a finite group of order n and $a \in G$, then the $n + 1$ elements $a^0, a, a^2, a^3, \ldots, a^n$ cannot all be distinct. Hence, $a^i = a^j$ for some i and j with $n \geq i > j$, which implies that $a^{i-j} = e$ with $0 \leq i - j \leq n$ (Why?). What does this say about $|a|$?

17. (a) $x = a^{-1}b$ is a solution of $ax = b$ because $a(a^{-1}b) = (aa^{-1})b = eb = b$. If c is also a solution, then $ac = b = a(a^{-1}b)$. Hence, $c = a^{-1}b$ by Theorem 7.5(2).

27. If $a, b \in G$, then by hypothesis, $aa = e$, $bb = e$, and $abab = e$. Left multiply both sides of the last equation by ba and simplify.

29. Let $x = a^{-1}cb^{-1}$ and show that $axb = c$. To prove uniqueness, assume $ayb = c$ and show that $y = a^{-1}cb^{-1}$.

31. (b) In S_3, let $a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. Verify that $|a| = 2$, $|b| = 2$, $ab = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, and $(ab)^4 = ab$.

33. Let $|a| = m$ and $|b| = n$, with $(m, n) = 1$. If $(ab)^k = e$ and $ab = ba$, then $a^kb^k = (ab)^k = e$, so that $a^k = b^{-k}$. Hence, $a^{kn} = (b^{-k})^n = (b^n)^{-k} = e$. Therefore, $m \mid kn$ by Theorem 7.9 and, hence, $m \mid k$ by Theorem 1.4. Similarly, $n \mid k$. So $mn \mid k$ (see Exercise 17 of Section 1.2).

35. $ab = b^4a \Rightarrow aba^{-1} = b^4 \Rightarrow ab^3a^{-1} = (aba^{-1})(aba^{-1})(aba^{-1}) = (b^4)^3 = b^{12} = e$ (because $b^6 = e$) $\Rightarrow ab^3 = a \Rightarrow b^3 = e$. Therefore, $ab = b^4a = b^3ba = eba = ba$.

Section 7.3 *(page 211)*

1. (a) $\langle 1 \rangle = U_{15}$; $\langle 2 \rangle = \langle 8 \rangle = \{1, 2, 4, 8\}$; $\langle 4 \rangle = \{1, 4\}$; $\langle 7 \rangle = \langle 13 \rangle = \{1, 4, 7, 13\}$; $\langle 11 \rangle = \{1, 11\}$; $\langle 14 \rangle = \{1, 14\}$.

5. $\langle 2 \rangle = \{\dots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \dots\}$

7. $\langle 2 \rangle = \left\{\dots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \dots\right\}$

9. $1 = 2^4$; $2 = 2^1$; $4 = 2^2$; $7 = 13^3$; $8 = 2^3$; $11 = 2 \cdot 13$; $13 = 13^1$; $14 = 2^3 \cdot 13$.

11. Using additive notation, we see that the group is cyclic with generator $(1, 1)$: $1(1, 1) = (1, 1)$; $2(1, 1) = (0, 2)$; $3(1, 1) = (1, 0)$; $4(1, 1) = (0, 1)$; $5(1, 1) = (1, 2)$; $6(1, 1) = (0, 0)$.

13. Since e_H is the identity in H, $e_He_H = e_H$. Apply Exercise 1 of Section 7.2 with $c = e_H$.

15. (a) If $a, b \in H \cap K$, then $a, b \in H$ and $a, b \in K$. Since H is a subgroup, $ab \in H$ and $a^{-1} \in H$. Similarly, $ab \in K$ and $a^{-1} \in K$. Hence, $ab \in H \cap K$ and $a^{-1} \in H \cap K$. Therefore, $H \cap K$ is a subgroup by Theorem 7.11.

29. Since H is nonempty, there is some $c \in H$. By hypothesis, $e = cc^{-1} \in H$. If $d \in H$ then since $e \in H$, we have $d^{-1} = ed^{-1} \in H$. Use this and the fact that $d = (d^{-1})^{-1}$ to show that $c, d \in H$ implies $cd \in H$. Apply Theorem 7.11.

31. If $x^{-1}ax$ and $x^{-1}bx \in x^{-1}Hx$ with $a, b \in H$, then $ab \in H$, and, hence, $(x^{-1}ax)(x^{-1}bx) = x^{-1}(ab)x \in x^{-1}Hx$. Show that $(x^{-1}ax)^{-1} = x^{-1}a^{-1}x \in x^{-1}Hx$. Apply Theorem 7.11.

33. Theorem 1.2 may be helpful.

35. ($\Rightarrow$) If a is in the center of G, then $ag = ga$ for every $g \in G$. Hence, $C(a) = \{g \in G \mid ag = ga\} = G$.

41. If $a^n, b^n \in H$, then since G is abelian, $a^nb^n = (ab)^n \in H$. Also $(a^n)^{-1} = a^{-n} = (a^{-1})^n \in H$. Apply Theorem 7.11.

43. The subgroups of $\mathbb{Z}_{12}$ are $\{0\}$, $\{0, 6\}$, $\{0, 3, 6, 9\}$, $\{0, 4, 8\}$, $\{0, 2, 4, 6, 8, 10\}$, and $\mathbb{Z}_{12}$.

47. See Exercise 33 of Section 7.2.

49. $G = \langle a \rangle = \{na \mid n \in \mathbb{Z}\}$. Assume that $g \in G$ is a solution of $x + x = a$. Then $g = ka$ for some integer k. Hence, $ka + ka = a$, which implies that a has finite order (Why?). This is a contradiction, so $x + x = a$ has no solution in G.

53. If $(m, n) = 1$, use Exercise 47. To prove that if $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic, then $(m, n) = 1$, we prove the equivalent contrapositive statement: If $(m, n) \neq 1$, then $\mathbb{Z}_m \times \mathbb{Z}_n$ is not

cyclic. If $(m, n) = d > 1$, then $m = dr$, $n = ds$, and $drs < mn$. If $(a, b) \in \mathbb{Z}_m \times \mathbb{Z}_n$, then $drs(a, b) = (drsa, drsb) = (sma, rnb) = (0, 0)$. Therefore, the order of (a, b) is a divisor of drs (by Theorem 7.9 in additive notation) and, hence, strictly less than mn. So (a, b) does not generate $\mathbb{Z}_m \times \mathbb{Z}_n$ (a group of order mn) by Theorem 7.15.

57. **(a)** Show that $U_{18} = \{1, 5, 7, 11, 13, 17\}$ is generated by 5.

Section 7.4 *(page 223)*

1. **(a)** *Homomorphism:* $f(x + y) = 3(x + y) = 3x + 3y = f(x) + f(y)$. *Surjective*: If $t \in \mathbb{R}$, then $f(t/3) = 3(t/3) = t$. *Injective*: If $f(x) = f(y)$, then $3x = 3y$, and, hence, $x = y$.

5. g is a homomorphism since for any a, b, $g(a + b) = 2(a + b) = 2a + 2b = g(a) + g(b)$. You can easily compute $f(0), f(1), \ldots, f(8)$ to see that f is injective and surjective.

7. f is a homomorphism since for any a, b, $f(ab) = |ab| = |a||b| = f(a)f(b)$. Why is f surjective?

11. g is a homomorphism since for any a, b, $g(a)g(b) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix} = g(ab)$. If $g(a) = g(b)$, then $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$, which implies that $a = b$. Hence g is injective.

13. Show that both groups are cyclic of order 4 and use Theorem 7.19.

15. $f(a^0) = f(e_G) = e_H = f(a)^0$. For positive integers, use induction: $f(a^1) = f(a) = f(a)^1$. If $f(a^k) = f(a)^k$, then $f(a^{k+1}) = f(a^k a^1) = f(a^k)f(a) = f(a)^k f(a) = f(a)^{k+1}$. Hence, $f(a^n) = f(a)^n$ for all $n \geq 0$. What about negative n?

19. ($\Rightarrow$) If G is abelian, then f is a homomorphism because $f(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = f(a)f(b)$. In this case, f is an isomorphism by Exercise 5 of Section 7.2.

21. Because f and g are homomorphisms, $(g \circ f)(ab) = g[f(ab)] = g[f(a)f(b)] = g(f(a))g(f(b)) = (g \circ f)(a)\,(g \circ f)(b)$. Hence, $g \circ f$ is a homomorphism. If $c \in K$, then since g is surjective, there exists $b \in H$ such that $g(b) = c$. Since f is surjective, there exists $a \in G$ such that $f(a) = b$. Thus, $(g \circ f)(a) = g(f(a)) = g(b) = c$ and $g \circ f$ is surjective. To complete the proof, show that f is injective.

29. If $a^n = e_G$, then by Exercise 15 and Theorem 7.20, $f(a)^n = f(a^n) = f(e_G) = e_H$. Similarly, if $f(a)^n = e_H$ then $f(a^n) = f(a)^n = e_H = f(e_G)$. Hence, $a^n = e_G$ since f is injective. So $a^n = e_G$ if and only if $f(a)^n = e_H$.

31. If $a, b \in F$, then because f is a homomorphism, $f(ab) = f(a)f(b) = ab$. So $ab \in F$, and F is closed under the group operation. Use Theorem 7.20 to show that the inverse of every element of F is also in F. Then use Theorem 7.11.

35. $K_f = \{1, 4\}$.

37. If f, $g \in \text{Inn } G$, then $f(a) = c^{-1}ac$ and $g(a) = d^{-1}ad$ for some c, d. Show that $(f \circ g)(a) = (dc)^{-1}a(dc)$ and, hence, $f \circ g \in \text{Inn } G$. Show that the inverse function h of f is given $h(a) = cac^{-1} = (c^{-1})^{-1}ac^{-1} \in \text{Inn } G$. Use Theorem 7.11.

41. See Example 6.

43. Verify that every nonidentity element of U_8 has order 2 but that this is not true for U_{10}. Hence, there is no isomorphism f by Exercise 29.

51. **(a)** If $\theta_c(x) = \theta_c(y)$, then $xc^{-1} = yc^{-1}$. Hence, $x = y$ by Theorem 7.5. Therefore, θ_c is injective. If $x \in G$, then $xc \in G$ and $\theta_c(xc) = (xc)c^{-1} = x$. Hence, θ_c is surjective.

59. **(a)** Show that h and v both induce the same inner automorphism (that is, $h^{-1}ah = v^{-1}av$ for every $a \in D_4$). Do the same for r_0 and r_2, for r_1 and r_3, and for d and t. Then show that the inner automorphisms induced by h, r_0, r_1, and d are all distinct (that is, no two of them have the same action on *every* element of D_4).

Section 7.5 *(page 233)*

1. **(a)** (173) **(c)** (1476283).

3. **(a)** (12)(45)(679) **(c)** (13)(254)(69)(78).

5. **(a)** 2 **(c)** 4.

7. **(a)** odd **(c)** even.

9. **(a)** 3 **(c)** 60.

11. There are eight 3-cycles (list them), each of order 3. Each of (12)(34), (13)(24), and (14)(23) has order 2. The identity (1) has order 1.

15. $(a_1a_2 \cdots a_k) = (a_1a_k)(a_1a_{k-1}) \cdots (a_1a_4)(a_1a_3)(a_1a_2)$. There are $k-1$ transpositions (one for each of $a_2, a_3, \ldots, a_k$). $k-1$ is even if and only if k is odd.

19. Suppose $\tau = \sigma_1\sigma_2 \cdots \sigma_r$, where the σ_i are disjoint cycles, with σ_1 having order k_1, σ_2 having order $k_2, \ldots,$ and σ_r having order k_r. Show that $\tau^n = (1)$ if and only if $\sigma_i^n = (1)$ for every i. Use Theorem 7.9 to show that $k_i \mid n$ for every i.

23. Use Theorem 7.12.

25. Verify that $\tau\sigma = \sigma^{-1}\tau$; use this to show that any product of powers of σ and powers of τ is one of: $\sigma, \sigma^2, \sigma^3, \sigma^4 = (1), \tau, \sigma\tau, \sigma^2\tau$, or $\sigma^3\tau$.

29. There are three possible cases (where a, b, c, d are distinct symbols): $(ab)(ab)$, $(ab)(ac)$, and $(ab)(cd)$. But $(ab)(ab) = (1) = (abc)^3$; $(ab)(ac) = (acb)$; and $(ab)(cd) = (acb)(acd)$.

35. Let $\tau = (ab)$ and express σ as a product of disjoint cycles. Since disjoint cycles commute by Exercise 18, all cycles in $\sigma\tau\sigma^{-1}$ not involving a or b will cancel and $\sigma\tau\sigma^{-1}$ will reduce to the form $\kappa(ab)\kappa^{-1}$, where κ has one of the following forms (in which a, b, x, y, u, v are distinct symbols): $(\cdots xaby \cdots)$; $(\cdots xbay \cdots)$; $(\cdots xay \cdots ubv \cdots)$; $(\cdots xay \cdots)$; $(\cdots ubv \cdots)$; or $(\cdots xay \cdots)(\cdots ubv \cdots)$. Verify that $\kappa(ab)\kappa^{-1}$ is a transposition in each case.

39. **(a)** The argument used in Exercise 24(a) and (b) can be used here if S_n is replaced by G, (12) is replaced by τ, B_n is replaced by the set of odd permutations in G, and A_n is replaced by the set of even permutations in G. In the Hint for Exercise 24(b), replace (12) by τ^{-1}, which is odd (Why?).

(b) See Exercise 24(c) and replace $|S_n|$ by $|G|$.

(c) Use part (b).

45. The idea is to find an injective homomorphism $S_n \to A_{n+2}$ and then apply part (4) of Theorem 7.20. First, note that any permutation in S_n can also be considered as a permutation in S_{n+2}. Let α be the transposition $(n+1, n+2)$ in S_{n+2}. Define $f: S_n \to A_{n+2}$ as follows. If σ is odd, then $f(\sigma) = \sigma\alpha$. If σ is even, then

$f(\sigma) = \sigma$. To show that f is a homomorphism, suppose that σ and τ are in S_n. Consider four cases: (1) σ and τ are both even; (2) σ is even and τ is odd; (3) σ is odd and τ is even; (4) σ and τ are both odd. Show that $f(\sigma\tau) = f(\sigma)f(\tau)$ in each case. To show that f is injective, you must show that $f(\sigma) = f(\tau)$ implies that $\sigma = \tau$. Prove it in cases 1 and 4 and show that $f(\sigma) = f(\tau)$ cannot occur in cases 2 and 3.

Chapter 8

Section 8.1 *(page 245)*

1. ($\Rightarrow$) If $Ka = K$, then $a = ea \in Ka = K$. So $a \in K$.

3. $Kr_0 = \{r_0, r_1, r_2, r_3\}$; $Kd = \{d, h, t, v\}$

7. 4 **9.** 1 **11.** 6.

17. **(a)** 1, 2, 3, 4, 6, 8, 12, 24 **(c)** 1, 2, 4, 5, 8, 10, 16, 20, 40, 80.

19. 27, 720.

21. $H \cap K$ is a subgroup of H and of K, and so its order must divide p by Lagrange's Theorem. Hence, $|H \cap K|$ is either 1 (in which case $H \cap K = \langle e \rangle$) or p (in which case $H = H \cap K = K$).

23. If $e \neq a \in G$, then $\langle a \rangle$ is a nonidentity subgroup of G. Hence, $G = \langle a \rangle$. If $|G| = |a|$ has composite order, say $|a| = td$, then $\langle a^t \rangle$ is a subgroup of order d by Theorem 7.9. Use Theorem 8.7.

25. 2.

31. List the element of G in pairs: a, a^{-1}; b, b^{-1}; c, c^{-1}, etc. with $a \neq a^{-1}$; $b \neq b^{-1}$; $c \neq c^{-1}$; etc. for as long as possible. Use the fact that there is an odd number of nonidentity elements to show that at some point you must reach a nonidentity element k such that $k = k^{-1}$. What is the order of k?

35. A proper subgroup has order n, with $1 < n < pq$ and n a divisor of pq. Use Theorem 8.7.

41. If G contains no element of order 3, show that every nonidentity element has order 11. Apply Exercise 40, with $p = 11$. What do you conclude?

Section 8.2 *(page 252)*

5. **(b)** If $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in N$ and $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G$, then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1/a & -b/ad \\ 0 & 1/d \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} 1/a & -b/ad \\ 0 & 1/d \end{pmatrix}\begin{pmatrix} a & b + cd \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & cd/a \\ 0 & 1 \end{pmatrix} \in N.$$

7. $G^* = G \times \langle e \rangle$ is a subgroup by Exercise 16 of Section 7.3. It is normal by Theorem 8.11 since for any $(c, d) \in G \times H$ and $(a, e) \in G^*$, $(c, d)^{-1}(a, e)(c, d) = (c^{-1}, d^{-1})(a, e)(c, d) = (c^{-1}ac, d^{-1}ed) = (c^{-1}ac, e) \in G^*$.

11. If $c \in G$, let f be the inner automorphism given by $f(x) = c^{-1}xc$ (see Example 9 of Section 7.4). Since N is characteristic, $f(N) \subseteq N$, that is $c^{-1}Nc \subseteq N$. Hence, N is normal by Theorem 8.11.

13. See Example 9 of Section 7.4 and Theorem 8.11.

17. First, prove that K is a subgroup of G. To show that K is normal, we show that for any $a \in G$ and $k \in K$, $a^{-1}ka \in K$:

$$\begin{aligned} f(a^{-1}ka) &= f(a^{-1})f(k)f(a) && [f \text{ is a homomorphism.}] \\ &= f(a)^{-1}f(k)f(a) && [\text{Theorem 7.20}] \\ &= f(a)^{-1}e_H f(a) && [k \in K] \\ &= f(a)^{-1}f(a) = e_H. \end{aligned}$$

Therefore, $a^{-1}ka \in K$ and K is normal by Theorem 8.11.

19. Use Exercise 15 of Section 7.3 to show that $N \cap K$ is a subgroup of K. If $g \in K$ and $n \in N \cap K$, then $g \in G$, $n \in N$, and, hence, $g^{-1}ng \in N$ by the normality of N in G. But $n \in N \cap K$ implies that $n \in K$, and, hence, $g^{-1}ng \in K$ by closure in K. Therefore, $g^{-1}ng \in N \cap K$, so that $g^{-1}(N \cap K)g \subseteq N \cap K$. Hence, $N \cap K$ is normal in K by Theorem 8.11.

21. If $n \in N$ and $k \in K$, use normality to show that $k^{-1}(n^{-1}kn) = (k^{-1}n^{-1}k)n$ is in $K \cap N = \langle e \rangle$.

23. **(a)** If $a \notin N$, then $Ne = N$ and Na are disjoint cosets (Why?). Since $[G{:}N] = 2$, these two cosets contain all the element of G. Therefore, any element that is not in N must be in Na.

27. *Partial proof:* If N is normal and $ab = n \in N$, then $ba = babb^{-1} = bnb^{-1}$ and $bnb^{-1} \in N$ by normality.

29. Let $N = \langle a \rangle$. Then $H = \langle a^k \rangle$ for some k by Theorem 7.17. If $g \in G$, then $g^{-1}ag \in N$ by normality; hence, $g^{-1}ag = a^t$ for some t. Consequently, for any $a^{ki} \in H$, $g^{-1}a^{ki}g = (g^{-1}ag)^{ki} = (a^t)^{ki} = (a^k)^{ti} \in H$.

35. N is a subgroup by Exercises 15 and 27 of Section 7.3. Show that N is normal in G.

37. By hypothesis, the cyclic group $\langle a \rangle$ is normal. Hence, $b^{-1}ab \in \langle a \rangle$, that is, $b^{-1}ab = a^k$ for some k.

Section 8.3 *(page 260)*

3. *Partial Answer:* $(Mh)(Mr_1) = M(h \circ r_1) = Md$; $(Mr_1)(Mh) = M(r_1 \circ h) = Mt = Md$.

5. Show that $\mathbb{Z}_{18}/M$ is cyclic with generator $1 + M$; then show that $1 + M$ has order 6 in $\mathbb{Z}_{18}/M$.

7. Find the orders of the groups U_{26}, $\langle 5 \rangle$, and $U_{26}/\langle 5 \rangle$ (see Example 14 of Section 7.1 or 7.1.A). Use Theorem 8.13 and 8.7.

9. $G/N \cong \mathbb{Z}_2$.

11. Since $ab = ba$ in G, $NaNb = Nab = Nba = NbNa$ in G/N.

15. The identity element of the quotient group is the coset $(0, 0) + \langle(5, 5)\rangle = \langle(5, 5)\rangle$. $(1, 0) + \langle(5, 5)\rangle$ has infinite order since for any positive integer k, $k(1, 0) = (k, 0) \notin \langle(5, 5)\rangle$. On the other hand, $(1, 1) + \langle(5, 5)\rangle$ has order 5, as you can easily verify.

19. If $b \in G$, then Nb is a square in G/N, say $Nb = (Nc)^2 = Nc^2$. Since $b \in Nb$, $b = nc^2$ for some $n \in N$. What do you know about elements of N?

21. If Tg has finite order n, then $Tg^n = (Tg)^n = Te = T$, so $g^n \in T$. What does this tell you about the order of g^n? And what, in turn, does that tell you about the order of g?

23. $\mathbb{R}^*/\mathbb{R}^{**} \cong \mathbb{Z}_2$.

25. **(a)** 9, 5, 7 **(b)** If $m, n \in \mathbb{Z}$, then $n(m/n + \mathbb{Z}) = m + \mathbb{Z} = 0 + \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$.

31. What are the possible orders of $Z(G)$? Then, what are the possible orders of $G/Z(G)$? Use Theorems 8.7 and 8.15.

37. *Hint:* Show that the function $f: A/N \times B/N \to G/N$ given by $f(Na, Nb) = Nab$ is well defined. Then show that if $a \in A$ and $b \in B$, then $Nab = Nba$. Use this fact to prove that f is a homomorphism.

Section 8.4 *(page 270)*

1. $f((a + bi) + (c + di)) = f((a + c) + (b + d)i) = b + d = f(a + bi) + f(c + di)$; the kernel is $\mathbb{Z}$.

3. You provide the proof that h is a homomorphism. The kernel is $\langle 1 \rangle$ (so h is injective by Theorem 8.17).

5. $f((x, y) + (u, v)) = f((x + u, y + v)) = y + v = f(x, y) + f(u, v)$; so f is a homomorphism. You find the kernel.

11. If $[a]_n = [b]_n$, then $n \mid (a - b)$ by Theorem 2.3. Since $k \mid n$, it follows that $k \mid (a - b)$. Use this fact to show that $[ra]_k = [rb]_k$.

13. f is well-defined by Exercise 11. f is a homomorphism because $f([a]_{16} + [b]_{16}) = f([a + b]_{16}) = [a + b]_4 = [a]_4 + [b]_4 = f([a]_{16}) + f([b]_{16})$. Find the kernel and explain why it is isomorphic to $\mathbb{Z}_4$.

17. **(a)** $\langle 0 \rangle, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_{12}$.

19. $\langle e \rangle$, S_3, and $\mathbb{Z}_2$.

21. Kernel f is a normal subgroup of G, so what can it be? What does that imply?

25. Show that f is a homomorphism. If c is any integer, then $f(0, -c) = 0 - (-c) = c$; hence f is surjective. If (a, b) is in the kernel of f, then $a - b = 0$ and, hence, $a = b$. So $(a, b) = (a, a) = a(1, 1) \in \langle (1, 1) \rangle$. Show that any element of $\langle (1, 1) \rangle$ is in the kernel; hence the kernel is $\langle (1, 1) \rangle$. Apply the First Isomorphism Theorem 8.20.

27. Verify that $f: G \times H \to G/M \times H/N$ given by $f(a, b) = (Ma, Nb)$ is a surjective homomorphism with kernel $M \times N$. Apply Theorem 8.16 and the First Isomorphism Theorem 8.20.

31. Verify that $f: \mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}_4$, given by $f(a) = ([a]_3, [a]_4)$, is a homomorphism. Use Exercise 17 of Section 1.2 to show that the kernel is $\langle 12 \rangle$. Use brute force to show that f is surjective: Verify that $f(1), f(2), \ldots, f(12)$ are all the elements of $\mathbb{Z}_3 \times \mathbb{Z}_4$.

33. Since $H \cong G/K$ by the First Isomorphism Theorem, it suffices to construct a bijection from the set S of all subgroups of G that contain K and the set T of all subgroups of G/K. If B is a subgroup of G that contains K, then B/K is a subgroup of G/K, so define $\theta: S \to T$ by $\theta(B) = B/K$. Then θ is surjective by Theorem 8.24. Show that θ is injective.

Section 8.5 *(page 277)*

1. **(a)** (123), (132), (124), (142), (134), (143), (234), (243).

3. (1).

5. Theorem 7.23 and Example 6 of Section 7.5.

9. If $N \neq (1)$, then N contains a nonidentity element σ. If $\tau \neq (1)$ is in N, then $\sigma\sigma = (1) = \sigma\tau$ implies that $\sigma = \tau$ by Theorem 7.5. Hence, $N = \{(1), \sigma\}$; and N is cyclic of order 2.

Chapter 9

Section 9.1 (page 285)

3. (a) $\{(0,0)\}$; $\{(0,0),(1,0)\}$; $\{(0,0),(0,1)\}$; $\{(0,0),(1,1)\}$; $\mathbb{Z}_2 \times \mathbb{Z}_2$.

5. $\mathbb{Z}_2 \times \mathbb{Z}_2$.

9. No.

13. (b) If D is normal, then for any $a, b \in G$, $(a, e, e)(b, b, b)(a, e, e)^{-1} \in D$. But $(a, e, e)(b, b, b)(a, e, e)^{-1} = (aba^{-1}, b, b)$. Since this is in D, we must have $aba^{-1} = b$, which implies that $ab = ba$.

23. (a) Let $M = \langle(123)\rangle$ and $N = \langle(12)\rangle$ in S_3.

25. First, verify that $N_i \cap (N_1 \cdots N_{i-1}N_{i+1} \cdots N_k) = \langle e\rangle$ implies that when $i \neq j$, then $N_i \cap N_j = \langle e\rangle$ because $N_j \subseteq N_1 \cdots N_{i-1}N_{i+1} \cdots N_k$. Use the homomorphism f in the proof of Theorem 9.1. If $f(a_1, \ldots, a_k) = e$, then $a_i = (a_1 \cdots a_{i-1})^{-1}e(a_{i+1} \cdots a_k)^{-1}$. Use Lemma 9.2 and Corollary 7.6 repeatedly to show that $a_i \in N_i \cap N_1 \cdots N_{i-1}N_{i+1} \cdots N_k = \langle e\rangle$. Hence, f is injective by Theorem 8.17.

27. (a) What are the normal subgroups of S_3?

Section 9.2 (page 297)

1. If $p^n a = 0$ and $p^m b = 0$, then $p^n(-a) = -(p^n a) = 0$ and $p^{m+n}(a + b) = p^n p^m(a + b) = p^m(p^n a) + p^n(p^m b) = 0$. Hence, $a + b \in G(p)$ and $-a \in G(p)$. Use Theorem 7.11.

3. (a) $\mathbb{Z}_4 \oplus \mathbb{Z}_3$; $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ (c) $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ (e) $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$; $\mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$ (g) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$; $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$; $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$; $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$; $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$; $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$.

5. (a) $2, 5^3$ (c) $2, 2, 2^2, 2^3, 3, 5, 5, 5, 5$.

7. (a) 2, 2 and 2, 2 (c) $2, 2^2$ and $2, 2^2$.

9. (a) G must contain an element of order p (Why?). If a has order p, then $pa = 0$.

13. If q is a prime other than p and if q divides $|G|$, use Exercise 12 to reach a contradiction.

19. (a) Exercise 1 is the special case when every element of finite order has order a power of p. Essentially the same proof works here.

Section 9.3 (page 302)

3. $\{(12)(34), (13)(24), (14)(23), (1)\}$ is the only Sylow 2-subgroup. The four Sylow 3-subgroups are $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$, $\langle(234)\rangle$.

5. (a) 1 or 4.

7. (a) Show that G has a normal Sylow 7-subgroup. (c) Show that G has a normal Sylow-11 subgroup.

9. If $a \in G$, then $(Na)^{p^n} = N$ in G/N, so that $a^{p^n} \in N$.

13. For each prime that divides $|G|$, there is exactly one Sylow subgroup by the Second Sylow Theorem. Let $p_1, p_2, \ldots, p_k$ be the distinct primes that divide $|G|$, and let $N_1, N_2, \ldots, N_k$ be the corresponding Sylow groups. Define $f: N_1 \times N_2 \times \cdots \times N_k \to G$ by $f(a_1, a_2, \ldots, a_k) = a_1a_2 \cdots a_k$. The proof of Theorem 9.1 shows that f is a homomorphism. Then $\operatorname{Im} f = N_1N_2 \cdots N_k = \{a_1a_2 \cdots ak \mid a_i \in N_i\}$ is a subgroup of G by Theorem 7.20. The Sylow subgroups

of Im f also are $N_1, N_2, \ldots, N_k$ (Why?). By the definition of Sylow subgroups, $|\text{Im } f| = |N_1| \cdot |N_2| \cdots |N_k| = |G|$. Hence, Im $f = G$, and f is surjective. By the definition of the direct product, $|N_1 \times N_2 \times \cdots \times N_k| = |N_1| \cdot |N_2| \cdots |N_k| = |G|$. Since $N_1 \times N_2 \times \cdots \times N_k$ and G have the same number of elements the surjective map f must also be injective (Why?). Therefore, f is an isomorphism.

21. Show that there is a normal Sylow 3- or 5-subgroup. Note that if there are six Sylow 5-subgroups, G has 24 distinct elements of order 5 (Why?). Similarly, if there are ten Sylow 3-subgroups, G has 20 distinct elements of order 3.

Section 9.4 *(page 310)*

1. **(a)** $\{r_0\}, \{r_2\}, \{r_1, r_3\}, \{h, v\}, \{d, t\}$.

3. Look at $H = \{r_0, r_1, r_2, r_3\}$ in D_4.

5. $\langle(123)\rangle, \langle(124)\rangle, \langle(134)\rangle, \langle(234)\rangle$.

9. If C is the conjugacy class of $a \in G$, show that $f(C)$ is the conjugacy class of $f(a)$.

15. In the equation of Exercise 14(c), verify that each $|C_i|$ is either 1 or a positive power of p. At least one $|C_i|$ is 1 beacuse $\{e\}$ is a conjugacy class. Since $|N|$ is divisible by p, there must be more than one $|C_i| = 1$ and, hence, some nonidentity element of $Z(G)$ in N.

19. If $b \in N(N(K))$, then $b^{-1}N(K)b = N(K)$. Hence, $b^{-1}Kb \subseteq N(K)$, since $K \subseteq N(K)$. Verify that both K and $b^{-1}Kb$ are Sylow p-subgroups of $N(K)$ and, hence, conjugate in $N(K)$. But K is normal in $N(K)$, and so $b^{-1}Kb = K$. Hence, $b \in N(K)$.

21. If S is a Sylow p-subgroup containing H (Exercise 24), then every Sylow p-subgroup is of the from $a^{-1}Sa$ for some $a \in G$ and, therefore, contains $a^{-1}Ha$.

Section 9.5 *(page 318)*

1. First show that $p^2 \not\equiv 1 \pmod q$. [If $p^2 \equiv 1 \pmod q$, then q divides $p + 1$ or $p - 1$ (Why?). Use the facts that $p < q$ and $q \not\equiv 1 \pmod p$ to show that both possibilities lead to a contradiction.] Then use Theorem 9.30.

5. **(a)**

	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	e	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	e	ab	a^2b	a^3b	b
a^2	a^2	a^3	e	a	a^2b	a^3b	b	ab
a^3	a^3	e	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	a^2	a	e	a^3
ab	ab	b	a^3b	a^2b	a^3	a^2	a	e
a^2b	a^2b	ab	b	a^3b	e	a^3	a^2	a
a^3b	a^3b	a^2b	ab	b	a	e	a^3	a^2

7. Use Exercise 13 of Section 9.3 and Theorem 9.9.

13. $\{1, -1\}$.

17. How many Sylow p-subgroups does G have? Use Corollary 9.16.

Chapter 10

Section 10.1 *(page 330)*

3. (a) True. *Proof:* $a \mid b$ means $b = au$ and $c \mid d$ means $d = cv$. Hence, $bd = aucv = ac(uv)$.

5. If a is an associate of b, then $a = bu$ for some unit u. Hence, $bu = a = bc$, and, therefore, $u = c$, a contradiction.

7. Suppose $q = pu$, where p is irreducible and u is a unit. Suppose $q = rs$; then $rs = pu$, and, hence, $p = (pu)u^{-1} = (rs)u^{-1} = r(su^{-1})$. Since p is irreducible, r is a unit or su^{-1} is a unit by Theorem 10.1. But if su^{-1} is a unit, say $su^{-1}w = 1$, then s is a unit. Therefore, q is irreducible by Theorem 10.1.

17. (a) $\delta(ab) = \delta((su - tv) + (sv + tu)i) = (su - tv)^2 + (sv + tu)^2 = s^2u^2 - 2stuv + t^2v^2 + s^2v^2 + 2stuv + t^2u^2 = s^2u^2 + t^2v^2 + s^2v^2 + t^2u^2 = (s^2 + t^2)(u^2 + v^2) = \delta(a)\delta(b)$.

21. If $0_R \neq a \in R$, use Theorem 10.1 to show that a^2 can't be irreducible and, hence, must be a unit. Hence, a is a unit.

23. Suppose $p = rs$. Then $p \mid r$ or $p \mid s$. Show that r or s must be a unit and apply Theorem 10.1.

29. Assume that $\delta(a) = k$ for all nonzero $a \in R$. If $b \neq 0_R$, then there exist q, r such that $1_R = bq + r$, with $r = 0_R$ or $\delta(r) < \delta(b)$. The latter condition is impossible because $\delta(r) = k = \delta(b)$. Thus $r = 0_R$, and, hence, q is a multiplicative inverse of b.

Section 10.2 *(page 341)*

1. $(ab) \subseteq (b)$ since $b \mid ab$. If $(ab) = (b)$, then $ab \mid b$, say $abu = b$. Hence, $au = 1_R$, contradicting the fact that a is a nonunit.

5. See Example 3.

11. If (a) is an ideal other than R, then a is not a unit (Why?) and, hence, must be divisible by an irreducible element p (Theorem 10.12). Hence, $(a) \subseteq (p)$, with (p) maximal by Exercise 10.

13. (b) Verify that $f: \mathbb{Z} \to \mathbb{Z}_6$, given by $f(a) = [a]$, is a surjective homomorphism.

15. By Theorem 10.8, $I = (b)$ for some nonzero b. If $a \in \mathbb{Z}[i]$, then $a = bq + r$ with $r = 0$ or $\delta(r) < \delta(b)$, and, hence, $a \equiv r \pmod{I}$. By Theorem 6.6, the number of distinct cosets of I (congruence classes mod I) is at most the number of possible r's under division by b. Show that there are only finitely many possible r's.

21. By Exercise 20, $d = au + bv$ for some $u, v \in R$. If $e \in S$ is a common divisor of a and b, then e necessarily divides d. Hence, d is a gcd of a and b in S.

29. For some d, $bc = ad$. If $a = r_1r_2 \cdots r_k$, $d = z_1z_2 \cdots z_n$, $b = p_1p_2 \cdots p_s$, and $c = q_1q_2 \cdots q_t$ with each p_i, q_i, r_i, z_i, irreducible, then $p_1p_2 \cdots p_sq_1q_2 \cdots q_t = r_1r_2 \cdots r_kz_1z_2 \cdots z_n$. So each r_i is an associate of p_j or q_j. But r_i cannot be an associate of any p_j (otherwise r_i would divide the gcd 1_R of a and b, which implies that the irreducible r_i is a unit).

Section 10.3 *(page 351)*

1. If $x = a, y = b, z = c$ is a solution of $x^n + y^n = z^n$ and $n = kt$, show that $x = a^t$, $y = b^t$, $z = c^t$ is a solution of $x^k + y^k = z^k$, contradicting the hypothesis.

3. $N(ab) = N((rm + snd) + (rn + sm)\sqrt{d}) = (rm + snd)^2 - d(rn + sm)^2 = r^2m^2 + 2mnrsd + s^2n^2d^2 - dr^2n^2 - 2mnrsd - ds^2m^2 = r^2m^2 + s^2n^2d^2 - dr^2n^2 - ds^2m^2 = (r^2 - ds^2)(m^2 - dn^2) = N(a)N(b)$.

9. **(a)** Use Corollary 10.22.

17. ($\Rightarrow$) Let $a = u + v\sqrt{-5}$ and $b = w + z\sqrt{-5}$. If $r + s\sqrt{-5} \in P$, then $r + s\sqrt{-5} = 2a + (1 + \sqrt{-5})b = 2(u + v\sqrt{-5}) + (1 + \sqrt{-5})(w + z\sqrt{-5}) = (2u + w - 5z) + (2v + w + z)\sqrt{-5}$. Hence, $r - s = (2u + w - 5z) - (2v + w + z) = 2(u - v - 3z)$, so that $r \equiv s \pmod 2$.

Section 10.4 *(page 358)*

1. (2) $[a, b] = [ak, bk]$ because $a(bk) = b(ak)$.

3. $[a, 1_R] + [b, 1_R] = [a1_R + 1_Rb, 1_R1_R] = [a + b, 1_R] \in R^*$ and $[a, 1_R][b, 1_R] = [ab, 1_R1_R] = [ab, 1_R] \in R^*$; hence, R^* is closed under addition and multiplication. The zero element $[0_R, 1_R]$ of F is in R^*. The negative of $[a, 1_R]$ is $[-a, 1_R] \in R^*$.

5. Verify that $f{:}F \to \{r + si \mid r, s \in \mathbb{Q}\}$ given by $f([a + bi, c + di]) = \left(\dfrac{ac + bd}{c^2 + d^2}\right) + \left(\dfrac{bc + ad}{c^2 + d^2}\right)i$ is an isomorphism.

11. $mu + nv = 1$ for some integers u and v by Theorem 1.2; u and v may be negative. Negative powers of a are defined in F and, hence, in F, $a = a^1 = a^{mu+nv} = a^{mu}a^{nv} = (a^m)^u(a^n)^v = (b^m)^u(b^n)^v = b^{mu+nv} = b^1 = b$.

Section 10.5 *(page 364)*

1. ($\Rightarrow$) If $f(x)$ is a unit in $R[x]$, then $f(x)g(x) = 1_R$ for some $g(x)$. By Theorem 4.2, $\deg f(x) + \deg g(x) = \deg 1_R = 0$. Hence, $\deg f(x) = 0 = \deg g(x)$, so that $f(x)$, $g(x) \in R$. Hence, $f(x)$ is a unit in R.

3. ($\Rightarrow$) Assume p is irreducible in $R[x]$. If $p = rs$ in R, then either r or s is a unit in $R[x]$. Hence, r or s is a unit in R by Exercise 1. Therefore, p is irreducible in R by Theorem 10.1.

5. Since $c_1c_2 \cdots c_mf(x) = g(x)$, each c_i divides $g(x)$. Therefore, c_i is a unit in R because $g(x)$ is primitive.

9. First use the fact that $R[x]$ is a UFD to show that R is an integral domain. If c is a nonzero, nonunit element of R, then c is a nonzero, nonunit element of $R[x]$ by Exercise 1. Hence, $c = p_1p_2 \cdots p_k$, with each p_i irreducible in $R[x]$. Theorem 4.2 shows that each $p_i \in R$. Hence, p_i is irreducible in R by Exercise 3. Use the fact that $R[x]$ is a UFD to show that this factorization is unique up to order and associates in R.

Chapter 11

Section 11.1 *(page 374)*

7. $a + bi = (b - 2a)i + a(1 + 2i) + 0(1 + 3i)$. Also, $a + bi = (-2a)i + (a - b)(1 + 2i) + b(1 + 3i)$.

9. Verify that $((-3/\sqrt{2}) - \sqrt{3})\sqrt{2} + \sqrt{3}(\sqrt{2} + i) + \sqrt{3}(\sqrt{3} - i) = 0$.

11. If the subset is $\{0_V, u_2, u_3, \ldots, u_n\}$, then $1_F 0_V + 0_F u_2 + 0_F u_3 + \cdots + 0_F u_n = 0_V$, with the first coefficient nonzero.

13. There exist $c_i \in F$, not all zero, such that $c_1 v_1 + \cdots + c_k v_k = 0_V$ since the v_i are linearly dependent. The set $\{v_1, \ldots, v_k, w_1, \ldots, w_t\}$ is linearly dependent because $c_1 v_1 + \cdots + c_k v_k + 0_F w_1 + \cdots + 0_F w_t = 0_V$ and not all the coefficients are zero.

15. For any $r + si \in \mathbb{C}$, $r + si = \left(\frac{r}{b} - \frac{cs}{bd}\right)b + \frac{s}{d}(c + di)$. Hence, $\{b, c + di\}$ spans $\mathbb{C}$ over $\mathbb{R}$. Prove that it is also linearly independent over $\mathbb{R}$.

23. **(a)** If $a + b\sqrt{2} + c\sqrt{3} = 0$, then $a + b\sqrt{2} = -c\sqrt{3}$. Squaring both sides and rearranging, show that $2ab\sqrt{2} = 3c^2 - a^2 - 2b^2$. If $ab \neq 0$, then $\sqrt{2} = (3c^2 - a^2 - 2b^2)/2ab \in \mathbb{Q}$, which contradicts the fact that $\sqrt{2}$ is irrational. Hence, $a = 0$ or $b = 0$. If $a = 0$, then $b\sqrt{2} + c\sqrt{3} = 0$. Square both sides and make a similar argument to show that $bc = 0$. Hence, $b = 0$ or $c = 0$. But $a = 0$ and $b = 0$ imply that $c\sqrt{3} = 0$, whence, $c = 0$. Similarly, $a = 0$ and $c = 0$ imply that $b = 0$.

33. Suppose $c_1 u_1 + \cdots + c_t u_t + dw = 0_V$. If $d \neq 0_F$, then $w = -d^{-1}c_1 u_1 - d^{-1}c_2 u_2 - \cdots - d^{-1}c_t u_t$, a contradiction. Hence, $d = 0_F$. Then all the $c_i = 0_F$ because $\{u_1, \ldots, u_t\}$ is linearly independent.

37. ((i) $\Rightarrow$ (iii)) Suppose $S = \{v_1, \ldots, v_n\}$ spans V over F. Then some subset T of S is a basis of V over F by Exercise 32. Since $[V{:}F] = n$, T must have n elements, and, hence, $T = S$. Use Exercise 36 to prove (ii) $\Rightarrow$ (iii). (iii) implies (i) and (ii) by the definition of basis.

Section 11.2 *(page 381)*

3. Both $F(u + c)$ and $F(u)$ contain F by definition. Since $c \in F$ and $u \in F(u)$, $u + c \in F(u)$. Therefore, $F(u) \supseteq F(u + c)$, since $F(u + c)$ is the smallest subfield containing F and $u + c$. Conversely, $u = (u + c) - c \in F(u + c)$, so that $F(u) \subseteq F(u + c)$, since $F(u)$ is the smallest subfield containing F and u. Therefore, $F(u + c) = F(u)$.

5. **(a)** Verify that $3 + 5i$ is a root of $x^2 - 6x + 34$. **(c)** Verify that $1 + \sqrt[3]{2}$ is a root of $x^3 - 3x^2 + 3x - 3$.

7. By hypothesis, u is a root of some $p(x) \in F[x]$. But $F[x] \subseteq K[x]$, so that u is a root of $p(x) \in K[x]$.

9. $\sqrt{\pi}$ is a root of $x^2 - \pi \in \mathbb{Q}(\pi)[x]$.

11. 6.

15. By the Factor Theorem, $a + bi$ is a root of $f(x) = (x - (a + bi))(x - (a - bi))$. Verify that $f(x)$ has real coefficients.

17. **(a)** $x^4 - 2x^2 - 4$.

21. π is a root of $x^4 - \pi^4 \in \mathbb{Q}(\pi^4)[x]$ and, hence, is algebraic over $\mathbb{Q}(\pi^4)$. Therefore, $\{1, \pi, \pi^2, \pi^3\}$ is a basis by Theorem 11.7.

Section 11.3 *(page 387)*

3. Many correct answers, including **(a)** $\{1, \sqrt{5}, i, \sqrt{5}i\}$ **(c)** $\{1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{15}, \sqrt{30}\}$.

5. Use Corollary 4.19 to show that $x^2 + 1$ is irreducible over $\mathbb{Q}(\sqrt{3})$ and thus is the minimal polynomial of i over $\mathbb{Q}(\sqrt{3})$. Hence, $[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}(\sqrt{3})] = 2$ and $[\mathbb{Q}(\sqrt{3}, i):\mathbb{Q}] = [\mathbb{Q}(\sqrt{3}, i):\mathbb{Q}(\sqrt{3})]\,[\mathbb{Q}(\sqrt{3}):\mathbb{Q}] = 2 \cdot 2 = 4$.

7. $[K(u):F]$ is finite by Theorems 11.7 and 11.4. Hence, u is algebraic over F by Theorem 11.9. If $p(x) \in F[x]$ is the minimal polynomial of u over F and $q(x) \in K[x]$ is the minimal polynomial of u over K, then $q(x) \mid p(x)$ by Theorem 11.6. Hence, by Theorem 11.7, $[K(u):K] = \deg q(x) \le \deg p(x) = [F(u):F]$.

9. $[F(u):F]$ and $[K(u):F(u)]$ are finite by Theorems 11.4, 11.7, and 11.9 and Exercise 8. Apply Theorem 11.4 to $F \subseteq F(u) \subseteq K(u)$.

11. **(a)** Theorem 11.4 applied to $F \subseteq F(u) \subseteq F(u, v)$ shows that $m = \deg p(x) = [F(u):F]$ divides $[F(u, v):F]$. Similarly, $n \mid [F(u, v):F]$. Hence, $mn \mid [F(u, v):F]$ by Exercise 17 of Section 1.2. Use Theorem 11.4 and Exercise 7 to show that $[F(u, v):F] \le mn$. Therefore, $[F(u, v):F] = mn$.

13. Let $h(x) \in F(u)[x]$ be the minimal polynomial of v over $F(u)$; then $h(x) \mid q(x)$. By Exercise 11(a) and Theorems 11.4 and 11.7, $(\deg p(x))(\deg q(x)) = [F(u, v):F] = [F(u, v):F(u)]\,[F(u):F] = (\deg h(x))(\deg p(x))$. Therefore, $\deg h(x) = \deg q(x)$, and, hence, $q(x) = kh(x)$ for some $k \in K$. Since $h(x)$ in irreducible over $F(u)$, so is $q(x)$.

15. If u is algebraic over E, then it is algebraic over F by Theorem 11.10 and Corollary 11.11.

Section 11.4 *(page 393)*

3. $\mathbb{Q}(\sqrt{5}, i)$ is a splitting field; it has dimension 4 by Exercise 3 of Section 11.3.

7. The minimal polynomial $p(x)$ of u is irreducible in $F[x]$ and has a root in K. Therefore, $p(x)$ splits over $K = F(u)$.

11. The fourth roots of -1 are $(\pm\sqrt{2}/2) \pm (\sqrt{2}/2)i$, so that $\mathbb{Q}(\sqrt{2}, i)$ is a splitting field.

15. $x^2 + 1$ is irreducible in $\mathbb{Z}_3[x]$ by Corollary 4.19. Hence, by Theorem 5.11, $\mathbb{Z}_3[x]/(x^2 + 1)$ is a field of nine elements that contains the roots $[x]$ and $[2x]$ of $x^2 + 1$.

21. If $p(x) \in K[x]$ is irreducible and u is a root of $p(x)$, then $K(u)$ is algebraic over K by Theorem 11.10. Therefore, u is algebraic over F by Corollary 11.11. Its minimal polynomial $q(x)$ over F splits over K and divides the irreducible $p(x)$ in $K[x]$ by Theorem 11.6. Show that $p(x)$ has degree 1 and apply Exercise 19.

Section 11.5 *(page 397)*

1. Every polynomial in $F[x]$ is also in $E[x]$.

7. **(a)** If $f(x) = a_n x^n + \cdots + a_0$ and $f'(x) = 0_F$, then for each $k > 0$, $(k1_F)\,a_k = ka_k = 0_F$. Since F has characteristic 0, $k1_F \neq 0_F$, and hence, $a_k = 0$. Therefore, $f(x) = a_0$.

9. If $f(x)$ and $f'(x)$ are not relatively prime, then their gcd has a root u in some splitting field. Hence, u is a repeated root of $f(x)$ by Exercise 8, so that $f(x)$ is not separable.

13. Use the proof of Theorem 11.18, as in Example 2.

Section 11.6 *(page 404)*

3. $na = a + a + \cdots + a = 1_R a + 1_R a + \cdots + 1_R a = (1_R + \cdots + 1_R)a = (n1_R)a = 0_R a = 0_R$.

5. Let p = characteristic F = characteristic K. F has order p^m, where $m = [F{:}\mathbb{Z}_p]$, by Theorem 11.23, and, hence, $q = p^m$. Since $[K{:}\mathbb{Z}_p] = [K{:}F]\,[F{:}\mathbb{Z}_p] = nm$, Theorem 11.23 shows that K has order $p^{mn} = q^n$.

13. Every element a of $\mathbb{Z}_p$ is a root of $x^p - x$ by the proof of Theorem 11.25. Hence, $a^p = a$ in $\mathbb{Z}_p$, which means that $a^p \equiv a \pmod p$ in $\mathbb{Z}$. If a is relatively prime to p in $\mathbb{Z}$, then a is a nonzero element of the field $\mathbb{Z}_p$ and, hence, has an inverse.

17. Since $E \cong F$, each has order p^n for some prime p. By Theorem 11.25, $E = \mathbb{Z}_p(u_1, \ldots, u_t) = F$, where the u_i are all the roots of $x^{p^n} - x$ in K.

Chapter 12

Section 12.1 *(page 413)*

1. If $\sigma(c) = c$ for every $c \in F$, then $\sigma^{-1}(c) = \sigma^{-1}(\sigma(c)) = c$.

3. Use Theorem 11.7 to show that $\sigma(c) = c$ for all $c \in F(u)$.

5. Use Corollary 12.5 and Lagrange's Theorem 8.5.

9. **(a)** $p(x) = x^2 + x + 1$ **(b)** $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\omega) \cong \mathbb{Z}_2$.

11. $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\sqrt{2}, i) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Section 12.2 *(page 421)*

1. The number of intermediate fields is the same as the number of subgroups of $\mathrm{Gal}_F K$, which is finite by Theorem 12.11.

5. Four, of dimensions 10, 5, 2, and 1.

9. **(a)** Every subgroup of $\mathbb{Z}_n \cong \mathrm{Gal}_F K$ (in particular, $\mathrm{Gal}_E K$) is cyclic and normal by Theorem 7.17. By Theorem 12.11, $\mathrm{Gal}_F E \cong \mathrm{Gal}_F K/\mathrm{Gal}_E K$; apply Exercise 24 of Section 8.3.

11. **(b)** $[\mathbb{Q}(\sqrt[4]{2}){:}\mathbb{Q}] = 4$ since $x^4 - 2$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion. $x^2 + 1$ is the minimal polynomial of i over $\mathbb{Q}(\sqrt[4]{2})$ by Corollary 4.19.

Section 12.3 *(page 431)*

1. **(a)** Many correct answers, including $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}(\sqrt{5}, \sqrt{7}) \subseteq \mathbb{Q}(\sqrt{5}, \sqrt{7}, \sqrt[4]{2 + \sqrt{5}}) \subseteq \mathbb{Q}(\sqrt{5}, \sqrt{7}, \sqrt[4]{2 + \sqrt{5}}, \sqrt[4]{1 + \sqrt{7}})$.

5. **(a)** A_4 consists of the subgroup H and the eight 3-cycles (123), (132), (124), (142), (134), (143), (234), (243). Show that H is normal in A_4. Use the fact that all groups of order ≤ 4 are abelian to show that the series $S_4 \supseteq A_4 \supseteq H \supseteq (1)$ satisfies the definition of solvability.

7. **(a)** ± 1 **(c)** $\pm 1, \pm i$ **(e)** $\pm 1, 1/2 \pm i\sqrt{3}/2, -1/2 \pm i\sqrt{3}/2$.

13. If K is the splitting field of a cubic polynomial, then $[K{:}F]$ is divisible by 3 (Why?) and ≤ 6 by Theorem 11.13. Hence, the Galois group is a subgroup of S_3 (Corollary 12.5) of order 3 or 6.

17. (a) $x^6 - 4x^3 + 4 = (x^3 - 2)^2$. $\mathbb{Q}(\sqrt[3]{2}, \omega)$ is a splitting field, where ω is a complex cube root of 1. $G \cong S_3$. (c) $x^5 + 6x^3 + 9x = x(x^2 + 3)^2$. $\mathbb{Q}(i\sqrt{3})$ is a splitting field. $G \cong \mathbb{Z}_2$. (e) $G \cong S_5$.

Chapter 13

Chapter 13 *(page 441)*

1. If $ka \equiv 0 \pmod p$, then $p \mid ka$. But $(p, k) = 1$ (Why?). Hence, $p \mid a$ by Theorem 1.5, which is a contradiction.

3. (a) 0107 0512 2421 1479.

Chapter 14

Section 14.1 *(page 448)*

3. If there is a solution, then 0, 1, or 2 is a solution by Exercise 2. Verify that this is not the case.

9. $x \equiv -30 \pmod{187}$.

11. $x \equiv -18 \pmod{210}$.

13. $x \equiv 204 \pmod{204{,}204}$.

19. ($\Leftarrow$) If $b - a = dk$ and $mu + nv = d$, then $muk + nvk = b - a$. Proceed as in the proof of Lemma 14.1.

Section 14.2 *(page 452)*

3. 7 is (1, 2) and 8 is (2, 3) in $\mathbb{Z}_3 \times \mathbb{Z}_5$. So the product is $(1 \cdot 2, 2 \cdot 3) = (2, 1)$.

5. ($\Rightarrow$) If $f(r) = f(s)$, then *both* r and s are solutions of the system $x \equiv r \pmod{m_1}$, $x \equiv r \pmod{m_2}, \ldots, x \equiv r \pmod{m_r}$.

Section 14.3 *(page 456)*

1. (a) Repeated use of Corollary 14.6 shows that both are isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5$ and, hence, to each other.

Chapter 15

Chapter 15 *(page 469)*

3. (a) Begin as in the construction of the coordinate plane. Place the compass point on (1, 0) and make a circle whose radius is the segment from (1, 0) to (3, 0). It intersects the vertical axis at Q. The right triangle with vertices (0, 0), Q, (1, 0) has hypotenuse of length 2 and one side of length 1. Hence the angle at Q (opposite the side of length 1) is a 30° angle, because $\sin^{-1}\left(\frac{1}{2}\right) = 30°$.
(c) Part (a) shows that a 90° angle can be trisected. Since a 30° angle can be *bisected*, a 45° angle can be trisected.

5. $\cos 3t = \cos(t + 2t) = \cos t \cos 2t - \sin t \sin 2t = \cos t(2\cos^2 t - 1) - \sin t(2 \sin t \cos t) = 2\cos^3 t - \cos t - 2\sin^2 t \cos t = 2\cos^3 t - \cos t - 2(1 - \cos^2 t)\cos t = 4\cos^3 t - 3\cos t$.

7. No. To prove this, show that x must be the root of a cubic polynomial in $\mathbb{Q}[x]$ that has no rational roots.

9. No.

15. If $\sqrt{k} \in F$, then $F(\sqrt{k}) = F$. If $\sqrt{k} \notin F$, then the multiplicative inverse of a nonzero element $a + b\sqrt{k}$ of $F(\sqrt{k})$ is $c + d\sqrt{k}$, where $c = a/(a^2 - kb^2)$ and $d = -b/(a^2 - kb^2)$.

Chapter 16

Section 16.1 *(page 480)*

1. Verify that C is closed under addition and, hence, is a subgroup by Theorem 7.12.

3. (a) 1 (c) 4.

5. (a) 0000, 1000, 0111, 1111 (c) 0000, 0010, 0101, 0111, 1001, 1011, 1100, 1110.

11. (c) If the ith coordinate is denoted by a subscript, then $(u + w)_i = u_i + w_i$ and $(v + w)_i = v_i + w_i$. Hence, $(u + v)_i = (v + w)_i$ if and only if $u_i = v_i$.

17. Many correct answers, including 00000, 11100, 00111, 11011.

21. $n = 5$.

25. Verify that an element of $B(n)$ has even Hamming weight if and only if it is the sum of an even number of elements of Hamming weight 1 (for instance, 110 = 100 + 010). Use this to show that the set of elements of even Hamming weight is closed under addition.

27. (a) .96059601 (c) .00058806 (e) .00000001.

Section 16.2 *(page 490)*

1. (a) $\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

3. $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

5. Several possible answers, including $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

13. An error is detected if and only if w is not a codeword. Note that $w = u + e$ and that the set of codewords is closed under addition.

Section 16.3 *(page 497)*

1. **(a)** If $f(x) = a_n x^n + \cdots + a_i x^i + \cdots + a_0$, then $f(x) + f(x) = (a_n + a_n)x^n + \cdots + (a_i + a_i)x^i + \cdots + (a_0 + a_0) = 0x^n + \cdots + 0x^i + \cdots + 0$ because $a_i + a_i = 0$ for every $a_i \in \mathbb{Z}_2$.

3. Verify that $1 + x + x^4$ has no roots in $\mathbb{Z}_2$ and, hence, no first- or third-degree factors. If there is a quadratic factor, it is either the product of two linear factors or irreducible. Use long division to show that the only irreducible quadratic (Exercise 2) is not a factor.

5. **(a)** Use the table to show that α^3 is a root of $f(x) = 1 + x + x^2 + x^3 + x^4$. It then suffices to show that $f(x)$ is irreducible. Use the method of Exercise 3.

7. **(c)** If $f([a_0 + a_1x + \cdots + a_{n-1}x^{n-1}]) = (0, 0, \ldots, 0)$, then $[a_0 + a_1x + \cdots + a_{n-1}x^{n-1}] = [0]$, so that the kernel of f is the identity subgroup. Apply Theorem 8.17.

9. **(a)** $D(x) = x^2 + \alpha^4 x + \alpha$ has roots $1 = \alpha^0$ and $\alpha = \alpha^1$. Hence, the correct word is 000000000000000. **(c)** $D(x) = x^2 + \alpha^{13}x + \alpha^4$ has roots α^9 and α^{10}. Hence, the correct word is 101010010110000.

Appendix B

Appendix B *(page 519)*

1. **(a)** $\{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$ **(c)** $\{1, 2\}$.

3. **(a)** Empty since $\sqrt{2}$ is irrational **(c)** Empty.

7. $(a, 0), (a, 1), (a, c), (b, 0), (b, 1), (b, c), (c, 0), (c, 1), (c, c)$.

11. **(a)** yes **(c)** yes.

13. **(a)** Many correct answers, including the functions f, g, h, k given by $f(1) = a$, $f(2) = b, f(3) = c, f(4) = a$; $g(1) = c, g(2) = b, g(3) = a, g(4) = b$; $h(1) = b$, $h(2) = a, h(3) = c, h(4) = c$; $k(1) = c, k(2) = a, k(3) = a, k(4) = b$. **(c)** There are six bijections from C to C.

19. If $(a, d) \in A \times (B \cup C)$, then $a \in A$ and $d \in B$ or $d \in C$. Therefore, $(a, d) \in A \times B$ or $(a, d) \in A \times C$, and, hence, $(a, d) \in (A \times B) \cup (A \times C)$. Thus $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Conversely, suppose $(r, s) \in (A \times B) \cup (A \times C)$. Then $(r, s) \in A \times B$ or $(r, s) \in A \times C$. If $(r, s) \in A \times B$, then $r \in A$ and $s \in B$ (and, hence, $s \in B \cup C$), so that $(r, s) \in A \times (B \cup C)$. Similarly, if $(r, s) \in A \times C$, then $(r, s) \in A \times (B \cup C)$. Therefore, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$, and, hence, the two sets are equal.

23. No; why not?

25. **(a)** If $f(a) = f(b)$, then $2a = 2b$. Dividing both sides by 2 shows that $a = b$. Therefore, f is injective. **(c)** If $f(a) = f(b)$, then $a/7 = b/7$, which implies that $a = b$.

27. **(a)** If $(g \circ f)(a) = (g \circ f)(b)$, then $g(f(a)) = g(f(b))$. Since g is injective, $f(a) = f(b)$. This implies that $a = b$ because f is injective. Therefore, $g \circ f$ is injective.

29. **(a)** Let $d \in D$. Since $g \circ f$ is surjective, there exists $b \in B$ such that $(g \circ f)(b) = d$. Let $c = f(b) \in C$. Then $g(c) = g(f(b)) = (g \circ f)(b) = d$. Hence, g is surjective.

Appendix C

Appendix C *(page 528)*

1. $P(0)$ is true since $0 = 0(0 + 1)/2$. If $P(k)$ is true, then $1 + 2 + \cdots + k = k(k + 1)/2$. Add $k + 1$ to both sides and show that the right side is $(k + 1)(k + 2)/2$. This says that $P(k + 1)$ is true.

3. Let $P(n)$ be the statement $2^{n-1} \leq n!$. Verify that $P(0)$ and $P(1)$ are true. If $P(k)$ is true and $k \geq 1$, then $2^{k-1} \leq k!$ and $2 \leq k + 1$. Hence, $(2^{k-1})2 \leq k!(k + 1)$, that is, $2^k \leq (k + 1)!$. Thus $P(k + 1)$ is true.

7. Verify that the statement is true when $n = 1$. Suppose the statement is true for k, that is, that 3 is a factor of $2^{2k+1} + 1$. Then $2^{2k+1} + 1 = 3t$, and, hence, $2^{2k+1} = 3t - 1$. To show that the statement is true for $k + 1$, note that $2^{2(k+1)+1} = 2^{2k+2+1} = 2^{2k+1}2^2 = (3t - 1)4 = 12t - 4 = 3(4t - 1) - 1$, and, hence, $2^{2(k+1)+1} + 1 = 3(4t - 1)$.

11. Verify that the statement is true when $n = 1$. Let $B = \{b_1, b_2, \ldots, b_n\}$. In defining an injective function from B to B, there are n possible choices for the image of b_1, $n - 1$ choices for the image of b_2 (because b_2 can't have the same image as b_1), $n - 3$ choices for the image of b_3, and so on.

13. **(a)** Verify that the statement is true when $n = 2$. Assume that a set of k elements has $k(k - 1)/2$ two-element subsets and that B has $k + 1$ elements. Choose $b \in B$ and let $C = B - \{b\}$. Every two-element subset of B consists either of two elements of C or of b and one element of C. There are $k(k - 1)/2$ subsets of the first type by the induction hypothesis.

Appendix D

Appendix D *(page 534)*

3. **(a)** $a \sim a$ since $\cos a = \cos a$. If $a \sim b$, then $\cos a = \cos b$ and, by the symmetric property of $=$, $\cos b = \cos a$; hence, $b \sim a$. If $a \sim b$ and $b \sim c$, then $\cos a = \cos b$ and $\cos b = \cos c$. Hence, $\cos a = \cos c$, and, therefore, $a \sim c$.

5. **(b)** The equivalence class of (r, s) is the vertical line through (r, s).

9. **(a)** Transitive **(c)** Symmetric.

19. **(b)** Consider the subgroup $K = \{r_0, v\}$ of D_4.

Appendix E

Appendix E *(page 539)*

1. 4032.

3. $$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-(n-r))!(n-r)!} = \binom{n}{n-r}.$$

Appendix F

Appendix F *(page 543)*

1. **(a)** $A + B = \begin{pmatrix} 1 & -6 & 0 & 4 \\ 9 & 5 & 11 & 12 \end{pmatrix}$.

3. **(a)** The entry in position i-j of $A + B$ is $a_{ij} + b_{ij}$. But $a_{ij} + b_{ij} = b_{ij} + a_{ij}$, which is the entry in position i-j of $B + A$. Hence, $A + B = B + A$.

Appendix G

Appendix G *(page 551)*

1. **(a)** $x + x^3 + x^5$ **(c)** $(-11, 7.5, -3, 12, -5, 0, 3, 0, 0, 0, \ldots)$.

3. **(a)** $$\begin{aligned}[(a_0, a_1, \ldots) \oplus (b_0, b_1, \ldots)] \oplus (c_0, c_1, \ldots) &= (a_0 + b_0, a_1 + b_1, \ldots) \oplus (c_0, c_1, \ldots) \\ &= ((a_0 + b_0) + c_0, (a_1 + b_1) + c_1, \ldots) \\ &= (a_0 + (b_0 + c_0), a_1 + (b_1 + c_1), \ldots) \\ &= (a_0, a_1, \ldots) \oplus (b_0 + c_0, b_1 + c_1, \ldots) \\ &= (a_0, a_1, \ldots) \oplus [(b_0, b_1, \ldots) \oplus (c_0, c_1, \ldots)].\end{aligned}$$

INDEX

Groups

e	Identity element, 172, 186
$\|G\|$	Order of the group G, 172, 186
S_n	Symmetric group on n symbols, 172–173, 186–187
$A(T)$	Group of permutations of the set T, 173, 187
D_4	Dihedral group of degree 4 [symmetries of the square], 173–176, 187–190
D_n	Dihedral group of degree n, 176, 190
U_n	Multiplicative group of units in $\mathbb{Z}_n$, 179, 193
$GL(2, \mathbb{R})$	General linear group of degree 2 over $\mathbb{R}$, 179, 194
$GL(2, \mathbb{Z}_2)$	General linear group of degree 2 over $\mathbb{Z}_2$, 179, 195
Q	Quaternion group, 181
$SL(2, \mathbb{R})$	Special linear group of degree 2 over $\mathbb{R}$, 182
a^{-1}	Inverse of a, 197
$\|a\|$	Order of a, 198–199
$Z(G)$	Center of the group G, 205
$\langle a \rangle$	Cyclic (sub)group generated by a, 206
$\langle S \rangle$	(Sub)group generated by the subset S, 209–210
$C(a)$	Centralizer of a, 212, 305
$N(H)$	Normalizer of the subgroup H, 213, 308
$G \cong H$	Group G is isomorphic to group H, 216
$\iota_G: G \to G$	Identity automorphism of the group G, 218
Aut G	Group of automorphisms of the group G, 225
Inn G	Group of inner automorphisms of the group G, 225
det A	Determinant of matrix A, 225
A_n	Alternating group of degree n, 233
$a \equiv b \pmod{K}$	a is congruent to b modulo the subgroup K, 238
Ka	Right coset [congruence class] of a modulo the subgroup K, 239
$[G:H]$	Index of the subgroup H in the group G, 240
aK	Left coset of a modulo the subgroup K, 248
G/N	Quotient group [or factor group] of the group G by the normal subgroup N, 255
G'	Commutator subgroup of the group G, 262
$\prod_{i \in I} G_i$	Infinite direct product of the groups G_i with $i \in I$ and I infinite, 288
$\sum_{i \in I} G_i$	Infinite direct sum of the groups G_i with $i \in I$ and I infinite, 288
$G(p)$	Subgroup consisting of the elements in the abelian group G whose orders are powers of the prime p, 290

Fields and Galois Theory

F^n	$F \times F \times \cdots \times F$ (n copies), where F is a field, 366
$[V{:}F]$	Dimension of the vector space V over the field F; *special case:* $[K{:}F]$, where K is an extension field of F considered as a vector space over F, 371
$F(u)$	Simple extension field of the field F; smallest subfield containing F and u, where K is an extension field of F and $u \in K$, 376
$F(u_1, u_2, \ldots, u_n)$	Finitely generated extension field of the field F; smallest subfield containing F and $u_1, u_2, \ldots, u_n$, where K is an extension field of F and each $u_i \in K$, 383
$f'(x)$	Derivative of the polynomial $f(x)$, 395
$\mathrm{Gal}_F K$	Galois group of K over F, where K is an extension field of F, 408
E_H	Fixed field of the subgroup H of $\mathrm{Gal}_F K$, 412
ζ	Primitive nth root of unity in a field, 426

Algebraic Coding Theory

$B(n)$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n copies of $\mathbb{Z}_2$), 473
$\mathrm{Wt}(u)$	Hamming weight of $u \in B(n)$, 474
$d(u, v)$	Hamming distance between elements u and v of $B(n)$, 474
$e_1, e_2, \ldots, e_n$	The elements of weight 1 in $B(n)$, 488

The Greek Alphabet

Alpha	α	A	Nu	ν	N
Beta	β	B	Xi	ξ	Ξ
Gamma	γ	Γ	Omicron	o	O
Delta	δ	Δ	Pi	π	Π
Epsilon	$\varepsilon, \in$	E	Rho	ρ	P
Zeta	ζ	Z	Sigma	σ	Σ
Eta	η	H	Tau	τ	T
Theta	θ	Θ	Upsilon	υ	Υ
Iota	ι	I	Phi	φ, ϕ	Φ
Kappa	κ	K	Chi	χ	X
Lambda	λ	Λ	Psi	ψ	Ψ
Mu	μ	M	Omega	ω	Ω

BIBLIOGRAPHY

This list contains all the books and articles referred to in the text, as well as a number of other books suitable for collateral reading, reference, and deeper study of particular topics. The list is far from complete. For the most part readability by students has been the chief selection criterion.

Abstract Algebra in General (Undergraduate Level)

These books contain approximately the same material as Chapters 1–12 of this text, but each of them provides a slightly different viewpoint and emphasis. Only [3] has a significant overlap with Chapters 13–16.

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2. Fraleigh, J., *A First Course in Abstract Algebra*, 7th edition. Boston: Pearson, 2003.
3. Gallian, J., *Contemporary Abstract Algebra*, 8th edition. Belmont, CA: Cengage, 2013.
4. Herstein, I. N., *Abstract Algebra*, 3rd edition. New York: Wiley, 1996.

Abstract Algebra in General (Graduate Level)

These books have much deeper and more detailed coverage of the material in Chapters 1–12, as well as a large number of topics not discussed in the text.

5. Hungerford, T. W., *Algebra*. New York: Springer, 1980.
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7. Galovich, S., *Doing Mathematics: An Introduction to Proofs and Problem Solving*, 2nd edition. Belmont, CA: Cengage, 2007.
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