INSTRUCTOR'S MANUAL TO ACCOMPANY

INTRODUCTION TO REAL ANALYSIS

Fourth Edition

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PREFACE

This manual is offered as an aid in using the fourth edition of *Introduction to Real Analysis* as a text. Both of us have frequently taught courses from the earlier editions of the text and we share here our experience and thoughts as to how to use the book. We hope our comments will be useful.

We also provide partial solutions for almost all of the exercises in the book. Complete solutions are *almost never* presented here, but we hope that enough is given so that a complete solution is within reach. Of course, there is more than one correct way to attack a problem, and you may find better proofs for some of these exercises.

We also repeat the graphs that were given in the manual for the previous editions, which were prepared for us by Professor Horacio Porta, whom we wish to thank again.

Robert G. Bartle November 20, 2010
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CONTENTS

CHAPTER 1
PRELIMINARIES

We suggest that this chapter be treated as review and covered quickly, without detailed classroom discussion. For one reason, many of these ideas will be already familiar to the students — at least informally. Further, we believe that, in practice, those notions of importance are best learned in the arena of real analysis, where their use and significance are more apparent. Dwelling on the formal aspect of sets and functions does not contribute very greatly to the students' understanding of real analysis.

If the students have already studied abstract algebra, number theory or combinatorics, they should be familiar with the use of mathematical induction. If not, then some time should be spent on mathematical induction.

The third section deals with finite, infinite and countable sets. These notions are important and should be briefly introduced. However, we believe that it is not necessary to go into the proofs of these results at this time.

Section 1.1

Students are usually familiar with the notations and operations of set algebra, so that a brief review is quite adequate. One item that should be mentioned is that two sets A and B are often proved to be equal by showing that: (i) if $x \in A$, then $x \in B$, and (ii) if $x \in B$, then $x \in A$. This type of element-wise argument is very common in real analysis, since manipulations with set identities is often not suitable when the sets are complicated.

Students are often not familiar with the notions of functions that are injective (= one-one) or surjective (= onto).

Sample Assignment: Exercises 1, 3, 9, 14, 15, 20.

Partial Solutions:

1. (a) $B \cap C = \{5, 11, 17, 23, \ldots\} = \{6k - 1 : k \in \mathbb{N}\}$, $A \cap (B \cap C) = \{5, 11, 17\}$
   (b) $(A \cap B) \setminus C = \{2, 8, 14, 20\}$
   (c) $(A \cap C) \setminus B = \{3, 7, 9, 13, 15, 19\}$
2. The sets are equal to (a) A, (b) $A \cap B$, (c) the empty set.
3. If $A \subseteq B$, then $x \in A$ implies $x \in B$, whence $x \in A \cap B$, so that $A \subseteq A \cap B \subseteq A$. Thus, if $A \subseteq B$, then $A = A \cap B$.
   Conversely, if $A = A \cap B$, then $x \in A$ implies $x \in A \cap B$, whence $x \in B$. Thus if $A = A \cap B$, then $A \subseteq B$.
4. If x is in $A \setminus (B \cap C)$, then x is in A but $x \notin B \cap C$, so that $x \in A$ and x is either not in B or not in C. Therefore either $x \in A \setminus B$ or $x \in A \setminus C$, which implies that $x \in (A \setminus B) \cup (A \setminus C)$. Thus $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Conversely, if x is in $(A \setminus B) \cup (A \setminus C)$, then $x \in A \setminus B$ or $x \in A \setminus C$. Thus $x \in A$ and either $x \notin B$ or $x \notin C$, which implies that $x \in A$ but $x \notin B \cap C$, so that $x \in A \setminus (B \cap C)$. Thus $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.

Since the sets $A \setminus (B \cap C)$ and $(A \setminus B) \cup (A \setminus C)$ contain the same elements, they are equal.

5. (a) If $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. Hence we either have (i) $x \in A$ and $x \in B$, or we have (ii) $x \in A$ and $x \in C$. Therefore, either $x \in A \cap B$ or $x \in A \cap C$, so that $x \in (A \cap B) \cup (A \cap C)$. This shows that $A \cap (B \cup C)$ is a subset of $(A \cap B) \cup (A \cap C)$.

   Conversely, let y be an element of $(A \cap B) \cup (A \cap C)$. Then either (j) $y \in A \cap B$, or (jj) $y \in A \cap C$. It follows that $y \in A$ and either $y \in B$ or $y \in C$. Therefore, $y \in A$ and $y \in B \cup C$, so that $y \in A \cap (B \cup C)$. Hence $(A \cap B) \cup (A \cap C)$ is a subset of $A \cap (B \cup C)$.

   In view of Definition 1.1.1, we conclude that the sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are equal.

   (b) Similar to (a).
6. The set D is the union of $\{x : x \in A \text{ and } x \notin B\}$ and $\{x : x \notin A \text{ and } x \in B\}$.
7. Here $A_n = \{n+1, 2(n+1), \ldots\}$.

   (a) $A_1 = \{2, 4, 6, 8, \ldots\}, A_2 = \{3, 6, 9, 12, \ldots\}, A_1 \cap A_2 = \{6, 12, 18, 24, \ldots\} = \{6k : k \in \mathbb{N}\} = A_5$.

   (b) $\bigcup A_n = \mathbb{N} \setminus \{1\}$, because if $n > 1$, then $n \in A_{n-1}$; moreover $1 \notin A_n$. Also $\bigcap A_n = \emptyset$, because $n \notin A_n$ for any $n \in \mathbb{N}$.
8. (a) The graph consists of four horizontal line segments.

   (b) The graph consists of three vertical line segments.
9. No. For example, both $(0, 1)$ and $(0, -1)$ belong to C.
10. (a) $f(E) = \{1/x^2 : 1 \le x \le 2\} = \{y : \frac{1}{4} \le y \le 1\} = [\frac{1}{4}, 1]$.

    (b) $f^{-1}(G) = \{x : 1 \le 1/x^2 \le 4\} = \{x : \frac{1}{4} \le x^2 \le 1\} = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$.
11. (a) $f(E) = \{x + 2 : 0 \le x \le 1\} = [2, 3]$, so $h(E) = g(f(E)) = g([2, 3]) = \{y^2 : 2 \le y \le 3\} = [4, 9]$.

    (b) $g^{-1}(G) = \{y : 0 \le y^2 \le 4\} = [-2, 2]$, so $h^{-1}(G) = f^{-1}(g^{-1}(G)) = f^{-1}([-2, 2]) = \{x : -2 \le x + 2 \le 2\} = [-4, 0]$.
12. If 0 is removed from E and F, then their intersection is empty, but the intersection of the images under f is $\{y : 0 < y \le 1\}$.
13. $E \setminus F = \{x : -1 \le x < 0\}$, $f(E) \setminus f(F)$ is empty, and $f(E \setminus F) = \{y : 0 < y \le 1\}$.
14. If $y \in f(E \cap F)$, then there exists $x \in E \cap F$ such that $y = f(x)$. Since $x \in E$ implies $y \in f(E)$, and $x \in F$ implies $y \in f(F)$, we have $y \in f(E) \cap f(F)$. This proves $f(E \cap F) \subseteq f(E) \cap f(F)$.
15. If $x \in f^{-1}(G) \cap f^{-1}(H)$, then $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$, so that $f(x) \in G$ and $f(x) \in H$. Then $f(x) \in G \cap H$, and hence $x \in f^{-1}(G \cap H)$. This shows

that $f^{-1}(G) \cap f^{-1}(H) \subseteq f^{-1}(G \cap H)$. The opposite inclusion is shown in Example 1.1.8(b). The proof for unions is similar.

16. If $f(a) = f(b)$, then $a/\sqrt{a^2+1} = b/\sqrt{b^2+1}$, from which it follows that $a^2 = b^2$. Since a and b must have the same sign, we get $a = b$, and hence f is injective. If $-1 < y < 1$, then $x := y/\sqrt{1-y^2}$ satisfies $f(x) = y$ (why?), so that f takes $\mathbb{R}$ onto the set $\{y : -1 < y < 1\}$. If $x > 0$, then $x = \sqrt{x^2} < \sqrt{x^2+1}$, so it follows that $f(x) \in \{y : 0 < y < 1\}$.
17. One bijection is the familiar linear function that maps a to 0 and b to 1, namely, $f(x) := (x-a)/(b-a)$. Show that this function works.
18. (a) Let $f(x) = 2x$, $g(x) = 3x$.
    (b) Let $f(x) = x^2$, $g(x) = x$, $h(x) = 1$. (Many examples are possible.)
19. (a) If $x \in f^{-1}(f(E))$, then $f(x) \in f(E)$, so that there exists $x_1 \in E$ such that $f(x_1) = f(x)$. If f is injective, then $x_1 = x$, whence $x \in E$. Therefore, $f^{-1}(f(E)) \subseteq E$. Since $E \subseteq f^{-1}(f(E))$ holds for any f, we have set equality when f is injective. See Example 1.1.8(a) for an example.
    (b) If $y \in H$ and f is surjective, then there exists $x \in A$ such that $f(x) = y$. Then $x \in f^{-1}(H)$ so that $y \in f(f^{-1}(H))$. Therefore $H \subseteq f(f^{-1}(H))$. Since $f(f^{-1}(H)) \subseteq H$ for any f, we have set equality when f is surjective. See Example 1.1.8(a) for an example.
20. (a) Since $y = f(x)$ if and only if $x = f^{-1}(y)$, it follows that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.
    (b) Since f is injective, then f^{-1} is injective on $R(f)$. And since f is surjective, then f^{-1} is defined on $R(f) = B$.
21. If $g(f(x_1)) = g(f(x_2))$, then $f(x_1) = f(x_2)$, so that $x_1 = x_2$, which implies that $g \circ f$ is injective. If $w \in C$, there exists $y \in B$ such that $g(y) = w$, and there exists $x \in A$ such that $f(x) = y$. Then $g(f(x)) = w$, so that $g \circ f$ is surjective. Thus $g \circ f$ is a bijection.
22. (a) If $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$, which implies $x_1 = x_2$, since $g \circ f$ is injective. Thus f is injective.
    (b) Given $w \in C$, since $g \circ f$ is surjective, there exists $x \in A$ such that $g(f(x)) = w$. If $y := f(x)$, then $y \in B$ and $g(y) = w$. Thus g is surjective.
23. We have $x \in f^{-1}(g^{-1}(H)) \iff f(x) \in g^{-1}(H) \iff g(f(x)) \in H \iff x \in (g \circ f)^{-1}(H)$.
24. If $g(f(x)) = x$ for all $x \in D(f)$, then $g \circ f$ is injective, and Exercise 22(a) implies that f is injective on $D(f)$. If $f(g(y)) = y$ for all $y \in D(g)$, then Exercise 22(b) implies that f maps $D(f)$ onto $D(g)$. Thus f is a bijection of $D(f)$ onto $D(g)$, and $g = f^{-1}$.

Section 1.2

The method of proof known as Mathematical Induction is used frequently in real analysis, but in many situations the details follow a routine patterns and are left to the reader by means of a phrase such as: "The proof is by Mathematical Induction". Since may students have only a hazy idea of what is involved, it may be a good idea to spend some time explaining and illustrating what constitutes a proof by induction.

Pains should be taken to emphasize that the induction hypothesis does *not* entail "assuming what is to be proved". The inductive step concerns the validity of going from the assertion for $k \in \mathbb{N}$ to that for $k+1$. The truth of falsity of the individual assertion is not an issue here.

Sample Assignment: Exercises 1, 2, 6, 11, 13, 14, 20.

Partial Solutions:

1. The assertion is true for $n=1$ because $1/(1\cdot 2)=1/(1+1)$. If it is true for $n=k$, then it follows for $k+1$ because $k/(k+1)+1/[(k+1)(k+2)]=(k+1)/(k+2)$.
2. The statement is true for $n=1$ because $[\frac{1}{2}\cdot 1\cdot 2]^2=1=1^3$. For the inductive step, use the fact that
$$\left[\tfrac{1}{2}k(k+1)\right]^2+(k+1)^3=\left[\tfrac{1}{2}(k+1)(k+2)\right]^2.$$
3. It is true for $n=1$ since $3=4-1$. If the equality holds for $n=k$, then add $8(k+1)-5=8k+3$ to both sides and show that $(4k^2-k)+(8k+3)=4(k+1)^2-(k+1)$ to deduce equality for the case $n=k+1$.
4. It is true for $n=1$ since $1=(4-1)/3$. If it is true for $n=k$, then add $(2k+1)^2$ to both sides and use some algebra to show that
$$\tfrac{1}{3}(4k^3-k)+(2k+1)^2=\tfrac{1}{3}[4k^3+12k^2+11k+3]=\tfrac{1}{3}[4(k+1)^3-(k+1)],$$
which establishes the case $n=k+1$.
5. Equality holds for $n=1$ since $1^2=(-1)^2(1\cdot 2)/2$. The proof is completed by showing $(-1)^{k+1}[k(k+1)]/2+(-1)^{k+2}(k+1)^2=(-1)^{k+2}[(k+1)(k+2)]/2$.
6. If $n=1$, then $1^3+5\cdot 1=6$ is divisible by 6. If k^3+5k is divisible by 6, then $(k+1)^3+5(k+1)=(k^3+5k)+3k(k+1)+6$ is also, because $k(k+1)$ is always even (why?) so that $3k(k+1)$ is divisible by 6, and hence the sum is divisible by 6.
7. If $5^{2k}-1$ is divisible by 8, then it follows that $5^{2(k+1)}-1=(5^{2k}-1)+24\cdot 5^{2k}$ is also divisible by 8.
8. $5^{k+1}-4(k+1)-1=5\cdot 5^k-4k-5=(5^k-4k-1)+4(5^k-1)$. Now show that 5^k-1 is always divisible by 4.
9. If $k^3+(k+1)^3+(k+2)^3$ is divisible by 9, then $(k+1)^3+(k+2)^3+(k+3)^3=k^3+(k+1)^3+(k+2)^3+9(k^2+3k+3)$ is also divisible by 9.
10. The sum is equal to $n/(2n+1)$.

11. The sum is $1+3+\cdots+(2n-1)=n^2$. Note that $k^2+(2k+1)=(k+1)^2$.
12. If $n_0>1$, let $S_1 := \{n\in\mathbb{N} : n-n_0+1\in S\}$ Apply 1.2.2 to the set S_1.
13. If $k<2^k$, then $k+1<2^k+1<2^k+2^k=2(2^k)=2^{k+1}$.
14. If $n=4$, then $2^4=16<24=4!$. If $2^k<k!$ and if $k\geq 4$, then $2^{k+1}=2\cdot 2^k< 2\cdot k!<(k+1)\cdot k!=(k+1)!$. [Note that the inductive step is valid whenever $2<k+1$, including $k=2,3$, even though the statement is false for these values.]
15. For $n=5$ we have $7\leq 2^3$. If $k\geq 5$ and $2k-3\leq 2^{k-2}$, then $2(k+1)-3= (2k-3)+2\leq 2^{k-2}+2^{k-2}=2^{(k+1)-2}$.
16. It is true for $n=1$ and $n\geq 5$, but false for $n=2,3,4$. The inequality $2k+1<2^k$, wich holds for $k\geq 3$, is needed in the induction argument. [The inductive step is valid for $n=3,4$ even though the inequality $n^2<2^n$ is false for these values.]
17. $m=6$ trivially divides n^3-n for $n=1$, and it is the largest integer to divide $2^3-2=6$. If k^3-k is divisible by 6, then since k^2+k is even (why?), it follows that $(k+1)^3-(k+1)=(k^3-k)+3(k^2+k)$ is also divisible by 6.
18. $\sqrt{k}+1/\sqrt{k+1}=(\sqrt{k}\sqrt{k+1}+1)/\sqrt{k+1}>(k+1)/\sqrt{k+1}=\sqrt{k+1}$.
19. First note that since $2\in S$, then the number $1=2-1$ belongs to S. If $m\notin S$, then $m<2^m\in S$, so $2^m-1\in S$, etc.
20. If $1\leq x_{k-1}\leq 2$ and $1\leq x_k\leq 2$, then $2\leq x_{k-1}+x_k\leq 4$, so that $1\leq x_{k+1}= (x_{k-1}+x_k)/2\leq 2$.

Section 1.3

Every student of advanced mathematics needs to know the meaning of the words "finite", "infinite", "countable" and "uncountable". For most students at this level it is quite enough to learn the definitions and read the statements of the theorems in this section, but to skip the proofs. Probably every instructor will want to show that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable (see Section 2.5).

Some students will not be able to comprehend that proofs are necessary for "obvious" statements about finite sets. Others will find the material absolutely fascinating and want to prolong the discussion forever. The teacher must avoid getting bogged down in a protracted discussion of cardinal numbers.

Sample Assignment: Exercises 1, 5, 7, 9, 11.

Partial Solutions:

1. If $T_1\neq\emptyset$ is finite, then the definition of a finite set applies to $T_2=\mathbb{N}_n$ for some n. If f is a bijection of T_1 onto T_2, and if g is a bijection of T_2 onto $\mathbb{N}_n$, then (by Exercise 1.1.21) the composite $g\circ f$ is a bijection of T_1 onto $\mathbb{N}_n$, so that T_1 is finite.

2. Part (b) Let f be a bijection of $\mathbb{N}_m$ onto A and let $C = \{f(k)\}$ for some $k \in \mathbb{N}_m$. Define g on $\mathbb{N}_{m-1}$ by $g(i) := f(i)$ for $i = 1, \ldots, k-1$, and $g(i) := f(i+1)$ for $i = k, \ldots, m-1$. Then g is a bijection of $\mathbb{N}_{m-1}$ onto $A \backslash C$. (Why?)
Part (c) First note that the union of two finite sets is a finite set. Now note that if C/B were finite, then $C = B \cup (C \setminus B)$ would also be finite.
3. (a) The element 1 can be mapped into any of the three elements of T, and 2 can then be mapped into any of the two remaining elements of T, after which the element 3 can be mapped into only one element of T. Hence there are $6 = 3 \cdot 2 \cdot 1$ different injections of S into T.
(b) Suppose a maps into 1. If b also maps into 1, then c must map into 2; if b maps into 2, then c can map into either 1 or 2. Thus there are 3 surjections that map a into 1, and there are 3 other surjections that map a into 2.
4. $f(n) := 2n + 13, n \in \mathbb{N}$.
5. $f(1) := 0$, $f(2n) := n, f(2n+1) := -n$ for $n \in \mathbb{N}$.
6. The bijection of Example 1.3.7(a) is one example. Another is the shift defined by $f(n) := n + 1$ that maps $\mathbb{N}$ onto $\mathbb{N} \setminus \{1\}$.
7. If T_1 is denumerable, take $T_2 = \mathbb{N}$. If f is a bijection of T_1 onto T_2, and if g is a bijection of T_2 onto $\mathbb{N}$, then (by Exercise 1.1.21) $g \circ f$ is a bijection of T_1 onto $\mathbb{N}$, so that T_1 is denumerable.
8. Let $A_n := \{n\}$ for $n \in \mathbb{N}$, so $\bigcup A_n = \mathbb{N}$.
9. If $S \cap T = \emptyset$ and $f : \mathbb{N} \to S, g\colon \mathbb{N} \to T$ are bijections onto S and T, respectively, let $h(n) := f((n+1)/2)$ if n is odd and $h(n) := g(n/2)$ if n is even. It is readily seen that h is a bijection of $\mathbb{N}$ onto $S \cup T$; hence $S \cup T$ is denumerable. What if $S \cap T \neq \emptyset$?
10. (a) $m + n - 1 = 9$ and $m = 6$ imply $n = 4$. Then $h(6,4) = \frac{1}{2} \cdot 8 \cdot 9 + 6 = 42$.
(b) $h(m,3) = \frac{1}{2}(m+1)(m+2) + m = 19$, so that $m^2 + 5m - 36 = 0$. Thus $m = 4$.
11. (a) $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ has $2^2 = 4$ elements.
(b) $\mathcal{P}(\{1,2,3\})$ has $2^3 = 8$ elements.
(c) $\mathcal{P}(\{1,2,3,4\})$ has $2^4 = 16$ elements.
12. Let $S_{n+1} := \{x_1, \ldots, x_n, x_{n+1}\} = S_n \cup \{x_{n+1}\}$ have $n+1$ elements. Then a subset of S_{n+1} either (i) contains x_{n+1}, or (ii) does not contain x_{n+1}. The induction hypothesis implies that there are 2^n subsets of type (i), since each such subset is the union of $\{x_{n+1}\}$ and a subset of S_n. There are also 2^n subsets of type (ii). Thus there is a total of $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ subsets of S_{n+1}.
13. For each $m \in \mathbb{N}$, the collection of all subsets of $\mathbb{N}_m$ is finite. (See Exercise 12.) Every finite subset of $\mathbb{N}$ is a subset of $\mathbb{N}_m$ for a sufficiently large m. Therefore Theorem 1.3.12 implies that $\mathcal{F}(\mathbb{N}) = \bigcup_{m=1}^{\infty} \mathcal{P}(\mathbb{N}_m)$ is countable.

CHAPTER 2

THE REAL NUMBERS

Students will be familiar with much of the factual content of the first few sections, but the process of deducing these facts from a basic list of axioms will be new to most of them. The ability to construct proofs usually improves gradually during the course, and there are much more significant topics forthcoming. A few selected theorems should be proved in detail, since some experience in writing formal proofs is important to students at this stage. However, one should not spend too much time on this material.

Sections 2.3 and 2.4 on the Completeness Property form the heart of this chapter. *These sections should be covered thoroughly.* Also the Nested Intervals Property in Section 2.5 should be treated carefully.

Section 2.1

One goal of Section 2.1 is to acquaint students with the idea of deducing consequences from a list of basic axioms. Students who have not encountered this type of formal reasoning may be somewhat uncomfortable at first, since they often regard these results as "obvious". Since there is much more to come, a sampling of results will suffice at this stage, making it clear that it is only a sampling. The classic proof of the irrationality of $\sqrt{2}$ should certainly be included in the discussion, and students should be asked to modify this argument for $\sqrt{3}$, etc.

Sample Assignment: Exercises 1(a,b), 2(a,b), 3(a,b), 6, 13, 16(a,b), 20, 23.

Partial Solutions:

1. (a) Apply appropriate algebraic properties to get $b = 0 + b = (-a + a) + b = -a + (a + b) = -a + 0 = -a$.
(b) Apply (a) to $(-a) + a = 0$ with $b = a$ to conclude that $a = -(-a)$.
(c) Apply (a) to the equation $a + (-1)a = a(1 + (-1)) = a \cdot 0 = 0$ to conclude that $(-1)a = -a$.
(d) Apply (c) with $a = -1$ to get $(-1)(-1) = -(-1)$. Then apply (b) with $a = 1$ to get $(-1)(-1) = 1$.
2. (a) $-(a + b) = (-1)(a + b) = (-1)a + (-1)b = (-a) + (-b)$.
(b) $(-a) \cdot (-b) = ((-1)a) \cdot ((-1)b) = (-1)(-1)(ab) = ab$.
(c) Note that $(-a)(-(1/a)) = a(1/a) = 1$.
(d) $-(a/b) = (-1)(a(1/b)) = ((-1)a)(1/b) = (-a)/b$.
3. (a) Add -5 to both sides of $2x + 5 = 8$ and use (A2),(A4),(A3) to get $2x = 3$. Then multiply both sides by $1/2$ to get $x = 3/2$.
(b) Write $x^2 - 2x = x(x - 2) = 0$ and apply Theorem 2.1.3(b). Alternatively, note that $x = 0$ satisfies the equation, and if $x \neq 0$, then multiplication by $1/x$ gives $x = 2$.

(c) Add -3 to both sides and factor to get $x^2-4=(x-2)(x+2)=0$. Now apply 2.1.3(b) to get $x=2$ or $x=-2$.
(d) Apply 2.1.3(b) to show that $(x-1)(x+2)=0$ if and only if $x=1$ or $x=-2$.

4. Clearly $a=0$ satisfies $a\cdot a=a$. If $a\neq 0$ and $a\cdot a=a$, then $(a\cdot a)(1/a)=a(1/a)$, so that $a=a(a(1/a))=a(1/a)=1$.
5. If $(1/a)(1/b)$ is multiplied by ab, the result is 1. Therefore, Theorem 2.1.3(a) implies that $1/(ab)=(1/a)(1/b)$.
6. Note that if $q\in\mathbb{Z}$ and if $3q^2$ is even, then q^2 is even, so that q is even. Hence, if $(p/q)^2=6$, then it follows that p is even, say $p=2m$, whence $2m^2=3q^2$, so that q is also even.
7. If $p\in\mathbb{N}$, there are three possibilities: for some $m\in\mathbb{N}\cup\{0\}$, (i) $p=3m$, (ii) $p=3m+1$, or (iii) $p=3m+2$. In either case (ii) or (iii), we have $p^2=3h+1$ for some $h\in\mathbb{N}\cup\{0\}$.
8. (a) Let $x=m/n, y=p/q$, where $m,n\neq 0, p,q\neq 0$ are integers. Then $x+y=(mq+np)/nq$ and $xy=mp/nq$ are rational.
(b) If $s:=x+y\in\mathbb{Q}$, then $y=s-x\in\mathbb{Q}$, a contradiction. If $t:=xy\in\mathbb{Q}$ and $x\neq 0$, then $y=t/x\in\mathbb{Q}$, a contradiction.
9. (a) If $x_1=s_1+t_1\sqrt{2}$ and $x_2=s_2+t_2\sqrt{2}$ are in K, then $x_1+x_2=(s_1+s_2)+(t_1+t_2)\sqrt{2}$ and $x_1x_2=(s_1s_2+2t_1t_2)+(s_1t_2+s_2t_1)\sqrt{2}$ are also in K.
(b) If $x=s+t\sqrt{2}\neq 0$ is in K, then $s-t\sqrt{2}\neq 0$ (why?) and

$$\frac{1}{x}=\frac{s-t\sqrt{2}}{(s+t\sqrt{2})(s-t\sqrt{2})}=\left(\frac{s}{s^2-2t^2}\right)-\left(\frac{t}{s^2-2t^2}\right)\sqrt{2}$$

is in K. (Use Theorem 2.1.4.)

10 (a) If $c=d$, then 2.1.7(b) implies $a+c<b+d$. If $c<d$, then $a+c<b+c<b+d$.
(b) If $c=d=0$, then $ac=bd=0$. If $c>0$, then $0<ac$ by the Trichotomy Property and $ac<bc$ follows from 2.1.7(c). If also $c\leq d$, then $ac\leq ad<bd$. Thus $0\leq ac\leq bd$ holds in all cases.

11. (a) If $a>0$, then $a\neq 0$ by the Trichotomy Property, so that $1/a$ exists. If $1/a=0$, then $1=a\cdot(1/a)=a\cdot 0=0$, which contradicts (M3). If $1/a<0$, then 2.1.7(c) implies that $1=a(1/a)<0$, which contradicts 2.1.8(b). Thus $1/a>0$, and 2.1.3(a) implies that $1/(1/a)=a$.
(b) If $a<b$, then $2a=a+a<a+b$, and also $a+b<b+b=2b$. Therefore, $2a<a+b<2b$, which, since $\frac{1}{2}>0$ (by 2.1.8(c) and part (a)), implies that $a<\frac{1}{2}(a+b)<b$.
12. Let $a=1$ and $b=2$. If $c=-3$ and $d=-1$, then $ac<bd$. On the other hand, if $c=-3$ and $d=-2$, then $bd<ac$. (Many other examples are possible.)

13. If $a \neq 0$, then 2.1.8(a) implies that $a^2 > 0$; since $b^2 \geq 0$, it follows that $a^2 + b^2 > 0$.

14. If $0 \leq a < b$, then 2.1.7(c) implies $ab < b^2$. If $a = 0$, then $0 = a^2 = ab < b^2$. If $a > 0$, then $a^2 < ab$ by 2.1.7(c). Thus $a^2 \leq ab < b^2$. If $a = 0, b = 1$, then $0 = a^2 = ab < b = 1$.

15. (a) If $0 < a < b$, then 2.1.7(c) implies that $0 < a^2 < ab < b^2$. Then by Example 2.1.13(a), we infer that $a = \sqrt{a^2} < \sqrt{ab} < \sqrt{b^2} = b$.
(b) If $0 < a < b$, then $ab > 0$ so that $1/ab > 0$, and thus $1/a - 1/b = (1/ab)(b - a) > 0$.

16. (a) To solve $(x-4)(x+1) > 0$, look at two cases. Case 1: $x - 4 > 0$ and $x + 1 > 0$, which gives $x > 4$. Case 2: $x - 4 < 0$ and $x + 1 < 0$, which gives $x < -1$. Thus we have $\{x : x > 4 \text{ or } x < -1\}$.
(b) $1 < x^2 < 4$ has the solution set $\{x : 1 < x < 2 \text{ or } -2 < x < -1\}$.
(c) The inequality is $1/x - x = (1-x)(1+x)/x < 0$. If $x > 0$, this is equivalent to $(1-x)(1+x) < 0$, which is satisfied if $x > 1$. If $x < 0$, then we solve $(1-x)(1+x) > 0$, and get $-1 < x < 0$. Thus we get $\{x : -1 < x < 0 \text{ or } x > 1\}$
(d) the solution set is $\{x : x < 0 \text{ or } x > 1\}$.

17. If $a > 0$, we can take $\varepsilon_0 := a > 0$ and obtain $0 < \varepsilon_0 \leq a$, a contradiction.

18. If $b < a$ and if $\varepsilon_0 := (a-b)/2$, then $\varepsilon_0 > 0$ and $a = b + 2\varepsilon_0 > b + \varepsilon_0$.

19. The inequality is equivalent to $0 \leq a^2 - 2ab + b^2 = (a-b)^2$.

20. (a) If $0 < c < 1$, then 2.1.7(c) implies that $0 < c^2 < c$, whence $0 < c^2 < c < 1$.
(b) Since $c > 0$, then 2.1.7(c) implies that $c < c^2$, whence $1 < c < c^2$.

21. (a) Let $S := \{n \in \mathbb{N} : 0 < n < 1\}$. If S is not empty, the Well-Ordering Property of $\mathbb{N}$ implies there is a least element m in S. However, $0 < m < 1$ implies that $0 < m^2 < m$, and since m^2 is also in S, this is a contradiction to the fact that m is the least element of S.
(b) If $n = 2p = 2q - 1$ for some p, q in $\mathbb{N}$, then $2(q-p) = 1$, so that $0 < q - p < 1$. This contradicts (a).

22. (a) Let $x := c - 1 > 0$ and apply Bernoulli's Inequality 2.1.13(c) to get $c^n = (1+x)^n \geq 1 + nx \geq 1 + x = c$ for all $n \in \mathbb{N}$, and $c^n > 1 + x = c$ for $n > 1$.
(b) Let $b := 1/c$ and use part (a).

23. If $0 < a < b$ and $a^k < b^k$, then 2.1.7(c) implies that $a^{k+1} < ab^k < b^{k+1}$ so Induction applies. If $a^m < b^m$ for some $m \in \mathbb{N}$, the hypothesis that $0 < b \leq a$ leads to a contradiction.

24. (a) If $m > n$, then $k := m - n \in \mathbb{N}$, so Exercise 22(a) implies that $c^k \geq c > 1$. But since $c^k = c^{m-n}$, this implies that $c^m > c^n$. Conversely, the hypothesis that $c^m > c^n$ and $m \leq n$ lead to a contradiction.
(b) Let $b := 1/c$ and use part (a).

25. Let $b := c^{1/mn}$. We claim that $b > 1$; for if $b \le 1$, then Exercise 22(b) implies that $1 < c = b^{mn} \le b \le 1$, a contradiction. Therefore Exercise 24(a) implies that $c^{1/n} = b^m > b^n = c^{1/m}$ if and only if $m > n$.

26. Fix $m \in \mathbb{N}$ and use Mathematical Induction to prove that $a^{m+n} = a^m a^n$ and $(a^m)^n = a^{mn}$ for all $n \in \mathbb{N}$. Then, for a given $n \in \mathbb{N}$, prove that the equalities are valid for all $m \in \mathbb{N}$.

Section 2.2

The notion of absolute value of a real number is defined in terms of the basic order properties of $\mathbb{R}$. We have put it in a separate section to give it emphasis. Many students need extra work to become comfortable with manipulations involving absolute values, especially when inequalities are involved.

We have also used this section to give students an early introduction to the notion of the ε-neighborhood of a point. As a preview of the role of ε-neighborhoods, we have recast Theorem 2.1.9 in terms of ε-neighborhhoods in Theorem 2.2.8.

Sample Assignment: Exercises 1, 4, 5, 6(a,b), 8(a,b), 9, 12(a,b), 15.

Partial Solutions:

1. (a) If $a \ge 0$, then $|a| = a = \sqrt{a^2}$; if $a < 0$, then $|a| = -a = \sqrt{a^2}$.
   (b) It suffices to show that $|1/b| = 1/|b|$ for $b \ne 0$ (why?). If $b > 0$, then $1/b > 0$ (why?), so that $|1/b| = 1/b = 1/|b|$. If $b < 0$, then $1/b < 0$, so that $|1/b| = -(1/b) = 1/(-b) = 1/|b|$.
2. First show that $ab \ge 0$ if an only if $|ab| = ab$. Then show that $(|a| + |b|)^2 = (a + b)^2$ if and only if $|ab| = ab$.
3. If $x \le y \le z$, then $|x - y| + |y - z| = (y - x) + (z - y) = z - x = |z - x|$. To establish the converse, show that $y < x$ and $y > z$ are impossible. For example, if $y < x \le z$, it follows from what we have shown and the given relationship that $|x - y| = 0$, so that $y = x$, a contradiction.
4. $|x - a| < \varepsilon \iff -\varepsilon < x - a < \varepsilon \iff a - \varepsilon < x < a + \varepsilon$.
5. If $a < x < b$ and $-b < -y < -a$, it follows that $a - b < x - y < b - a$. Since $a - b = -(b - a)$, the argument in 2.2.2(c) gives the conclusion $|x - y| < b - a$. The distance between x and y is less than or equal to $b - a$.
6. (a) $|4x - 5| \le 13 \iff -13 \le 4x - 5 \le 13 \iff -8 \le 4x \le 18 \iff -2 \le x \le 9/2$.
   (b) $|x^2 - 1| \le 3 \iff -3 \le x^2 - 1 \le 3 \iff -2 \le x^2 \le 4 \iff 0 \le x^2 \le 4 \iff -2 \le x \le 2$.
7. Case 1: $x \ge 2 \Rightarrow (x + 1) + (x - 2) = 2x - 1 = 7$, so $x = 4$.
   Case 2: $-1 < x < 2 \Rightarrow (x + 1) + (2 - x) = 3 \ne 7$, so no solution.
   Case 3: $x \le -1 \Rightarrow (-x - 1) + (2 - x) = -2x + 1 = 7$, so $x = -3$.
   Combining these cases, we get $x = 4$ or $x = -3$.

8. (a) If $x > 1/2$, then $x+1 = 2x-1$, so that $x = 2$. If $x \le 1/2$, then $x+1 = -2x+1$, so that $x = 0$. There are two solutions $\{0, 2\}$.
(b) If $x \ge 5$, the equation implies $x = -4$, so no solutions. If $x < 5$, then $x = 2$.

9. (a) If $x \ge 2$, the inequality becomes $-2 \le 1$. If $x \le 2$, the inequality is $x \ge 1/2$, so this case contributes $1/2 \le x \le 2$. Combining the cases gives us all $x \ge 1/2$.
(b) $x \ge 0$ yields $x \le 1/2$, so that we get $0 \le x \le 1/2$. $x \le 0$ yields $x \ge -1$, so that $-1 \le x \le 0$. Combining cases, we get $-1 \le x \le 1/2$.

10. (a) Either consider the three cases: $x < -1$, $-1 \le x \le 1$ and $1 < x$; or, square both sides to get $-2x > 2x$. Either approach gives $x < 0$.
(b) Consider the three cases $x \ge 0$, $-1 \le x < 0$ and $x < -1$ to get $-3/2 < x < 1/2$.

11. $y = f(x)$ where $f(x) := -1$ for $x < 0$, $f(x) := 2x - 1$ for $0 \le x \le 1$, and $f(x) := 1$ for $x > 1$.

12. Case 1: $x \ge 1 \Rightarrow 4 < (x+2) + (x-1) < 5$, so $3/2 < x < 2$.
Case 2: $-2 < x < 1 \Rightarrow 4 < (x+2) + (1-x) < 5$, so there is no solution.
Case 3: $x < -2 \Rightarrow 4 < (-x-2) + (1-x) < 5$, so $-3 < x < -5/2$.
Thus the solution set is $\{x : -3 < x < -5/2 \text{ or } 3/2 < x < 2\}$.

13. $|2x-3| < 5 \iff -1 < x < 4$, and $|x+1| > 2 \iff x < -3$ or $x > 1$. The two inequalities are satisfied simultaneously by points in the intersection $\{x : 1 < x < 4\}$.

14. (a) $|x| = |y| \iff x^2 = y^2 \iff (x-y)(x+y) = 0 \iff y = x$ or $y = -x$. Thus $\{(x, y) : y = x \text{ or } y = -x\}$.
(b) Consider four cases. If $x \ge 0$, $y \ge 0$, we get the line segment joining the points (0, 1) and (1, 0). If $x \le 0, y \ge 0$, we get the line segment joining $(-1, 0)$ and $(0, 1)$, and so on.
(c) The hyperbolas $y = 2/x$ and $y = -2/x$.
(d) Consider four cases corresponding to the four quadrants. The graph consists of a portion of a line segment in each quadrant. For example, if $x \ge 0, y \ge 0$, we obtain the portion of the line $y = x - 2$ in this quadrant.

15. (a) If $y \ge 0$, then $-y \le x \le y$ and we get the region in the upper half-plane on or between the lines $y = x$ and $y = -x$. If $y \le 0$, then we get the region in the lower half-plane on or between the lines $y = x$ and $y = -x$.
(b) This is the region on and inside the diamond with vertices (1, 0), (0, 1), $(-1, 0)$ and $(0, -1)$.

16. For the intersection, let γ be the smaller of ε and δ. For the union, let γ be the larger of ε and δ.

17. Choose any $\varepsilon > 0$ such that $\varepsilon < |a - b|$.

18. (a) If $a \le b$, then $\max\{a, b\} = b = \frac{1}{2}[a + b + (b - a)]$ and $\min\{a, b\} = a = \frac{1}{2}[a + b - (b - a)]$.
(b) If $a = \min\,\{a, b, c\}$, then $\min\{\min\{a, b\}, c\} = a = \min\{a, b, c\}$. Similarly, if b or c is $\min\{a, b, c\}$.

19. If $a \le b \le c$, then $\text{mid}\{a,b,c\} = b = \min\{b,c,c\} = \min\{\max\{a,b\},\ \max\{b,c\},\ \max\{c,a\}\}$. The other cases are similar.

Section 2.3

This section completes the description of the real number system by introducing the fundamental completeness property in the form of the Supremum Property. This property is vital to real analysis and students should attain a working understanding of it. Effort expended in this section and the one following will be richly rewarded later.

Sample Assignment: Exercises 1, 2, 5, 6, 9, 10, 12, 14.

Partial Solutions:

1. Any negative number or 0 is a lower bound. For any $x \ge 0$, the larger number $x+1$ is in S_1, so that x is not an upper bound of S_1. Since $0 \le x$ for all $x \in S_1$, then $u = 0$ is a lower bound of S_1. If $v > 0$, then v is not a lower bound of S_1 because $v/2 \in S_1$ and $v/2 < v$. Therefore $\inf S_1 = 0$.
2. S_2 has lower bounds, so that $\inf S_2$ exists. The argument used for S_1 also shows that $\inf S_2 = 0$, but that $\inf S_2$ does not belong to S_2. S_2 does not have upper bounds, so that $\sup S_2$ does not exists.
3. Since $1/n \le 1$ for all $n \in \mathbb{N}$, then 1 is an upper bound for S_3. But 1 is a member of S_3, so that $1 = \sup S_3$. (See Exercise 7 below.)
4. $\sup S_4 = 2$ and $\inf S_4 = 1/2$. (Note that both are members of S_4.)
5. It is interesting to compare algebraic and geometric approaches to these problems.
(a) $\inf A = -5/2$, $\sup A$ does not exist,
(b) $\sup B = 2$, $\inf B = -1$,
(c) $\sup C = 1$, $\inf B$ does not exist,
(d) $\sup D = 1 + \sqrt{6}$, $\inf D = 1 - \sqrt{6}$.
6. If S is bounded below, then $S' := \{-s : s \in S\}$ is bounded above, so that $u := \sup S'$ exists. If $v \le s$ for all $s \in S$, then $-v \ge -s$ for all $s \in S$, so that $-v \ge u$, and hence $v \le -u$. Thus $\inf S = -u$.
7. Let $u \in S$ be an upper bound of S. If v is another upper bound of S, then $u \le v$. Hence $u = \sup S$.
8. If $t > u$ and $t \in S$, then u is not an upper bound of S.
9. Let $u :=\ \sup S$. Since u is an upper bound of S, so is $u + 1/n$ for all $n \in \mathbb{N}$. Since u is the supremum of S and $u - 1/n < u$, then there exists $s_0 \in S$ with $u - 1/n < s_0$, whence $u - 1/n$ is not an upper bound of S.
10. Let $u := \sup A$, $v := \sup B$ and $w := \sup\{u, v\}$. Then w is an upper bound of $A \cup B$, because if $x \in A$, then $x \le u \le w$, and if $x \in B$, then $x \le v \le w$. If z is

any upper bound of $A \cup B$, then z is an upper bound of A and of B, so that $u \le z$ and $v \le z$. Hence $w \le z$. Therefore, $w = \sup(A \cup B)$.

11. Since sup S is an upper bound of S, it is an upper bound of S_0, and hence sup $S_0 \le$ sup S.
12. Consider two cases. If $u \ge s^*$, then $u = \sup(S \cup \{u\})$. If $u < s^*$, then there exists $s \in S$ such that $u < s \le s^*$, so that $s^* = \sup(S \cup \{u\})$.
13. If $S_1 := \{x_1\}$, show that $x_1 = \sup S_1$. If $S_k := \{x_1, \ldots, x_k\}$ is such that sup $S_k \in S_k$, then preceding exercise implies that $\sup\{x_1, \ldots, x_k, x_{k+1}\}$ is the larger of sup S_k and x_{k+1} and so is in S_{k+1}.
14. If $w = \inf S$ and $\varepsilon > 0$, then $w + \varepsilon$ is not a lower bound so that there exists t in S such that $t < w + \varepsilon$. If w is a lower bound of S that satisfies the stated condition, and if $z > w$, let $\varepsilon = z - w > 0$. Then there is t in S such that $t < w + \varepsilon = z$, so that z is not a lower bound of S. Thus, $w = \inf S$.

Section 2.4

This section exhibits how the supremum is used in practice, and contains some important properties of $\mathbb{R}$ that will often be used later. The Archimedean Properties 2.4.3–2.4.6 and the Density Properties 2.4.8 and 2.4.9 are the most significant. The exercises also contain some results that will be used later.

Sample Assignment: Exercises 1, 2, 4(b), 5, 7, 10, 12, 13, 14.

Partial Solutions:

1. Since $1 - 1/n < 1$ for all $n \in \mathbb{N}$, the number 1 is an upper bound. To show that 1 is the supremum, it must be shown that for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $1 - 1/n > 1 - \varepsilon$, which is equivalent to $1/n < \varepsilon$. Apply the Archimedean Property 2.4.3 or 2.4.5.
2. inf $S = -1$ and sup $S = 1$. To see the latter note that $1/n - 1/m \le 1$ for all $m, n \in \mathbb{N}$. On the other hand if $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $1/m < \varepsilon$, so that $1/1 - 1/m > 1 - \varepsilon$.
3. Suppose that $u \in \mathbb{R}$ is not the supremum of S. Then either (i) u is not an upper bound of S (so that there exists $s_1 \in S$ with $u < s_1$, whence we take $n \in \mathbb{N}$ with $1/n < s_1 - u$ to show that $u + 1/n$ is not an upper bound of S), or (ii) there exists an upper bound u_1 of S with $u_1 < u$ (in which case we take $1/n < u - u_1$ to show that $u - 1/n$ is not an upper bound of S).
4. (a) Let $u := \sup S$ and $a > 0$. Then $x \le u$ for all $x \in S$, whence $ax \le au$ for all $x \in S$, whence it follows that au is an upper bound of aS. If v is another upper bound of aS, then $ax \le v$ for all $x \in S$, whence $x \le v/a$ for all $x \in S$, showing that v/a is an upper bound for S so that $u \le v/a$, from which we conclude that $au \le v$. Therefore $au = \sup(aS)$. The statement about the infimum is proved similarly.

(b) Let $u := \sup S$ and $b < 0$. If $x \in S$, then (since $b < 0$) $bu \le bx$ so that bu is a lower bound of bS. If $v \le bx$ for all $x \in S$, then $x \le v/b$ (since $b < 0$), so that v/b is an upper bound for S. Hence $u \le v/b$ whence $v \le bu$. Therefore $bu = \inf(bS)$.

5. If $x \in S$, then $0 \le x \le u$, so that $x^2 \le u^2$ which implies $\sup T \le u^2$. If t is any upper bound of T, then $x \in S$ implies $x^2 \le t$ so that $x \le \sqrt{t}$. It follows that $u \le \sqrt{t}$, so that $u^2 \le t$. Thus $u^2 \le \sup T$.

6. Let $u := \sup f(X)$. Then $f(x) \le u$ for all $x \in X$, so that $a + f(x) \le a + u$ for all $x \in X$, whence $\sup\{a + f(x) : x \in X\} \le a + u$. If $w < a + u$, then $w - a < u$, so that there exists $x_w \in X$ with $w - a < f(x_w)$, whence $w < a + f(x_w)$, and thus w is not an upper bound for $\{a + f(x) : x \in X\}$.

7. Let $u := \sup S$, $v := \sup B$, $w := \sup(A + B)$. If $x \in A$ and $y \in B$, then $x + y \le u + v$, so that $w \le u + v$. Now, fix $y \in B$ and note that $x \le w - y$ for all $x \in A$; thus $w - y$ is an upper bound for A so that $u \le w - y$. Then $y \le w - u$ for all $y \in B$, so $v \le w - u$ and hence $u + v \le w$. Combining these inequalities, we have $w = u + v$.

8. If $u := \sup f(X)$ and $v := \sup g(X)$, then $f(x) \le u$ and $g(x) \le v$ for all $x \in X$, whence $f(x) + g(x) \le u + v$ for all $x \in X$. Thus $u + v$ is an upper bound for the set $\{f(x) + g(x) : x \in X\}$. Therefore $\sup\{f(x) + g(x) : x \in X\} \le u + v$.

9. (a) $f(x) = 2x + 1$, $\inf\{f(x) : x \in X\} = 1$.
(b) $g(y) = y$, $\sup\{g(y) : y \in Y\} = 1$.

10. (a) $f(x) = 1$ for $x \in X$. (b) $g(y) = 0$ for $y \in Y$.

11. If $x \in X$, $y \in Y$, then $g(y) \le h(x, y) \le f(x)$. If we fix $y \in Y$ and take the infimum over $x \in X$, then we get $g(y) \le \inf\{f(x) : x \in X\}$ for each $y \in Y$. Now take the supremum over $y \in Y$.

12. Let $S := \{h(x, y) : x \in X, y \in Y\}$. We have $h(x, y) \le F(x)$ for all $x \in X, y \in Y$ so that $\sup S \le \sup\{F(x) : x \in X\}$. If $w < \sup\{F(x) : x \in X\}$, then there exists $x_0 \in X$ with $w < F(x_0) = \sup \{h(x_0, y) : y \in Y\}$, whence there exists $y_0 \in Y$ with $w < h(x_0, y_0)$. Thus w is not an upper bound of S, and so $w < \sup S$. Since this is true for any w such that $w < \sup\{F(x) : x \in X\}$, we conclude that $\sup\{F(x) : x \in X\} \le \sup S$.

13. If $x \in \mathbb{Z}$, take $n := x + 1$. If $x \notin \mathbb{Z}$, we have two cases: (i) $x > 0$ (which is covered by Cor. 2.4.6), and (ii) $x < 0$. In case (ii), let $z := -x$ and use 2.4.6. If $n_1 < n_2$ are integers, then $n_1 \le n_2 - 1$ so the sets $\{y : n_1 - 1 \le y < n_1\}$ and $\{y : n_2 - 1 \le y < n_2\}$ are disjoint; thus the integer n such that $n - 1 \le x < n$ is unique.

14. Note that $n < 2^n$ (whence $1/2^n < 1/n$) for any $n \in \mathbb{N}$.

15. Let $S_3 := \{s \in \mathbb{R} : 0 \le s, s^2 < 3\}$. Show that S_3 is nonempty and bounded by 3 and let $y := \sup S_3$. If $y^2 < 3$ and $1/n < (3 - y^2)/(2y + 1)$ show that

$y+1/n \in S_3$. If $y^2 > 3$ and $1/m < (y^2-3)/2y$ show that $y-1/m \in S_3$. Therefore $y^2 = 3$.

16. Case 1: If $a > 1$, let $S_a := \{s \in \mathbb{R} : 0 \leq s, s^2 < a\}$. Show that S_a is nonempty and bounded above by a and let $z := \sup S_a$. Now show that $z^2 = a$.
 Case 2: If $0 < a < 1$, there exists $k \in \mathbb{N}$ such that $k^2 a > 1$ (why?). If $z^2 = k^2 a$, then $(z/k)^2 = a$.
17. Consider $T := \{t \in \mathbb{R} : 0 \leq t, t^3 < 2\}$. If $t > 2$, then $t^3 > 2$ so that $t \notin T$. Hence $y := \sup T$ exists. If $y^3 < 2$, choose $1/n < (2-y^3)/(3y^2+3y+1)$ and show that $(y+1/n)^3 < 2$, a contradiction, and so on.
18. If $x < 0 < y$, then we can take $r = 0$. If $x < y < 0$, we apply 2.4.8 to obtain a rational number between $-y$ and $-x$.
19. There exists $r \in \mathbb{Q}$ such that $x/u < r < y/u$.

Section 2.5

Another important consequence of the Supremum Property of $\mathbb{R}$ is the Nested Intervals Property 2.5.2. It is an interesting fact that if we assume the validity of *both* the Archimedean Property 2.4.3 and the Nested Intervals Property, then we can prove the Supremum Property. Hence these two properties could be taken as the completeness axiom for $\mathbb{R}$. However, establishing this logical equivalence would consume valuable time and not significantly advance the study of real analysis, so we will not do so. (There are other properties that could be taken as the completeness axiom.)

The discussion of binary and decimal representations is included to give the student a concrete illustration of the rather abstract ideas developed to this point. However, this material is not vital for what follows and can be omitted or treated lightly. We have kept this discussion informal to avoid getting buried in technical details that are not central to the course.

Sample Assignment: Exercises 3, 4, 5, 6, 7, 8, 10, 11.

Partial Solutions:

1. Note that $[a,b] \subseteq [a',b']$ if and only if $a' \leq a \leq b \leq b'$.
2. S has an upper bound b and a lower bound a if and only if S is contained in the interval $[a,b]$.
3. Since $\inf S$ is a lower bound for S and $\sup S$ is an upper bound for S, then $S \subseteq I_S$. Moreover, if $S \subseteq [a,b]$, then a is a lower bound for S and b is an upper bound for S, so that $[a,b] \supseteq I_S$.
4. Because z is neither a lower bound or an upper bound of S.
5. If $z \in \mathbb{R}$, then z is not a lower bound of S so there exists $x_z \in S$ such that $x_z \leq z$. Also z is not an upper bound of S so there exists $y_z \in S$ such that $z \leq y_z$. Since z belongs to $[x_z, y_z]$, it follows from the property (1) that $z \in S$.

But since $z \in \mathbb{R}$ is arbitrary, we conclude that $\mathbb{R} \subseteq S$, whence it follows that $S = \mathbb{R} = (-\infty, \infty)$.

6. Since $[a_n, b_n] = I_n \supseteq I_{n+1} = [a_{n+1}, b_{n+1}]$, it follows as in Exercise 1 that $a_n \le a_{n+1} \le b_{n+1} \le b_n$. Therefore we have $a_1 \le a_2 \le \cdots \le a_n \le \cdots$ and $b_1 \ge b_2 \ge \cdots \ge b_n \ge \cdots$.
7. Since $0 \in I_n$ for all $n \in \mathbb{N}$, it follows that $0 \in \bigcap_{n=1}^{\infty} I_n$. On the other hand if $u > 0$, then Corollary 2.4.5 implies that there exists $n \in \mathbb{N}$ with $1/n < u$, whence $u \notin [0, 1/n] = I_n$. Therefore, such a u does not belong to this intersection.
8. If $x > 0$, then there exists $n \in \mathbb{N}$ with $1/n < x$, so that $x \notin J_n$. If $y \le 0$, then $y \notin J_1$.
9. If $z \le 0$, then $z \notin K_1$. If $w > 0$, then it follows from the Archimedean Property that there exists $n_w \in \mathbb{N}$ with $w \notin (n_w, \infty) = K_{n_w}$.
10. Let $\eta := \inf\{b_n : n \in \mathbb{N}\}$; we claim that $a_n \le \eta$ for all n. Fix $n \in \mathbb{N}$; we will show that a_n is a lower bound for the set $\{b_k : k \in \mathbb{N}\}$. We consider two cases. (j) If $n \le k$, then since $I_n \supseteq I_k$, we have $a_n \le a_k \le b_k$. (jj) If $k < n$, then since $I_k \supseteq I_n$, we have $a_n \le b_n \le b_k$. Therefore $a_n \le b_k$ for all $k \in \mathbb{N}$, so that a_n is a lower bound for $\{b_k : k \in \mathbb{N}\}$ and so $a_n \le \eta$. In particular, this shows that $\eta \in [a_n, b_n]$ for all n, so that $\eta \in \bigcap I_n$.

    In view of 2.5.2, we have $[\xi, \eta] \subset I_n$ for all n, so that $[\xi, \eta] \subseteq \bigcap I_n$. Conversely, if $z \in I_n$ for all n, then $a_n \le z \le b_n$ for all n, whence it follows that $\xi = \sup\{a_n\} \le z \le \inf\{b_n\} = \eta$. Therefore $\bigcap I_n \subseteq [\xi, \eta]$ and so equality holds.
11. If $n \in \mathbb{N}$, let $c_n := a_1/2 + a_2/2^2 + \cdots + a_n/2^n$ and $d_n := a_1/2 + a_2/2^2 + \cdots + (a_n + 1)/2^n$, and let $J_n := [c_n, d_n]$. Since $c_n \le c_{n+1} \le d_{n+1} \le d_n$ for $n \in \mathbb{N}$, the intervals J_n form a nested sequence.
12. $\frac{3}{8} = (.011000\cdots)_2 = (.010111\cdots)_2$. $\frac{7}{16} = (.0111000\cdots)_2 = (.0110111\cdots)_2$.
13. (a) $\frac{1}{3} \approx (.0101)_2$ (b) $\frac{1}{3} = (.010101\cdots)_2$, the block 01 repeats.
14. We may assume that $a_n \ne 0$. If $n > m$ we multiply by 10^n to get $10p + a_n = 10q$, where $p, q \in \mathbb{N}$, so that $a_n = 10(q - p)$. Since $q - p \in \mathbb{Z}$ while a_n is one of the digits $0, 1, \ldots, 9$, it follows that $a_n = 0$, a contradiction. Therefore $n \le m$, and a similar argument shows that $m \le n$; therefore $n = m$.

    Repeating the above argument with $n = m$, we obtain $10p + a_n = 10q + b_n$, so that $a_n - b_n = 10(q - p)$, whence it follows that $a_n = b_n$. If this argument is repeated, we conclude that $a_k = b_k$ for $k = 1, \ldots, n$.
15. The problem here is that $-2/7$ is a negative number, so we write it as $-1 + 5/7$. Since $5/7 = .714285\cdots$ with the block repeating, we write $-2/7 = -1 + .714285$
16. $1/7 = .142857\cdots$, the block repeats. $2/19 = .105263157894736842\cdots$, the block repeats.
17. $1.25137\cdots137\cdots = 31253/24975$, $35.14653\cdots653\cdots = 3511139/99900$.

CHAPTER 3

SEQUENCES

Most students will find this chapter easier to understand than the preceding one for two reasons: (i) they have a partial familiarity with the notions of a sequence and its limit, and (ii) it is a bit clearer what one can use in proofs than it was for the results in Chapter 2. However, since it is essential that the students develop some technique, one should not try to go too fast.

Section 3.1

The main difficulty students have is mastering the notion of *limit* of a sequence, given in terms of ε and $K(\varepsilon)$. Students should memorize the definition accurately. The different quantifiers in statements of the form "given any ..., and there exists ..." can be confusing initially. We often use the $K(\varepsilon)$ game as a device to emphasize exactly how the quantities are related in proving statements about limits. The facts that the $\varepsilon > 0$ comes first and is arbitrary, and that the index $K(\varepsilon)$ depends on it (but is not unique) must be stressed.

The idea of deriving estimates is important and Theorem 3.1.10 is often used as a means of establishing convergence of a sequence by squeezing $|x_n - x|$ between 0 and a fixed multiple of $|a_n|$.

A careful and detailed examination of the examples in 3.1.11 is very instructive. Although some of the arguments may seem a bit artificial, the particular limits established there are useful for later work, so the results should be noted and remembered.

Sample Assignment: Exercises 1, 2(a,c), 3(b,d), 5(b,d), 6(a,c), 8, 10, 14, 15, 16.

Partial Solutions:

1. (a) 0, 2, 0, 2, 0, (b) $-1, 1/2, -1/3, 1/4, -1/5$,
   (c) 1/2, 1/6, 1/12, 1/20, 1/30, (d) 1/3, 1/6, 1/11, 1/18, 1/27.
2. (a) $2n+3$, (b) $(-1)^{n+1}/2^n$, (c) $n/(n+1)$, (d) n^2.
3. (a) 1, 4, 13, 40, 121, (b) 2, 3/2, 17/12, 577/408, 665, 857/470, 832,
   (c) 1, 2, 3, 5, 4, (d) 3, 5, 8, 13, 21.
4. Given $\varepsilon > 0$, take $K(\varepsilon) \geq |b|/\varepsilon$ if $b \neq 0$.
5. (a) We have $0 < n/(n^2+1) < n/n^2 = 1/n$. Given $\varepsilon > 0$, let $K(\varepsilon) \geq 1/\varepsilon$.
   (b) We have $|2n/(n+1) - 2| = 2/(n+1) < 2/n$. Given $\varepsilon > 0$, let $K(\varepsilon) \geq 2/\varepsilon$.
   (c) We have $|(3n+1)/(2n+5) - 3/2| = 13/(4n+10) < 13/4n$. Given $\varepsilon > 0$, let $K(\varepsilon) \geq 13/4\varepsilon$.
   (d) We have $|(n^2-1)/(2n^2+3) - 1/2| = 5/(4n^2+6) < 5/4n^2 \leq 5/4n$. Given $\varepsilon > 0$, let $K(\varepsilon) \geq 5/4\varepsilon$.

6. (a) $1/\sqrt{n+7} < 1/\sqrt{n}$, (b) $|2n/(n+2) - 2| = 4/(n+2) < 4/n$,
(c) $\sqrt{n}/(n+1) < 1/\sqrt{n}$, (d) $|(-1)^n\, n/(n^2+1)| \le 1/n$.
7. (a) $[1/\ln(n+1) < \varepsilon] \iff [\ln(n+1) > 1/\varepsilon] \iff [n+1 > e^{1/\varepsilon}]$. Given $\varepsilon > 0$, let $K \ge e^{1/\varepsilon} - 1$.
(b) If $\varepsilon = 1/2$, then $e^2 - 1 \approx 6.389$, so we choose $K = 7$. If $\varepsilon = 1/10$, then $e^{10} - 1 \approx 22{,}025.466$, so we choose $K = 22{,}026$.
8. Note that $||x_n| - 0| = |x_n - 0|$. Consider $((-1)^n)$.
9. $0 < \sqrt{x_n} < \varepsilon \iff 0 < x_n < \varepsilon^2$.
10. Let $\varepsilon := x/2$. If $M := K(\varepsilon)$, then $n \ge M$ implies that $|x - x_n| < \varepsilon = x/2$, which implies that $x_n > x - x/2 = x/2 > 0$.
11. $|1/n - 1/(n+1)| = 1/n(n+1) < 1/n^2 \le 1/n$.
12. Multiply and divide by $\sqrt{n^2+1}+n$ to obtain $\sqrt{n^2+1}-n = 1/(\sqrt{n^2+1}+n) < 1/n$.
13. Note that $n < 3^n$ so that $0 < 1/3^n < 1/n$.
14. Let $b := 1/(1+a)$ where $a > 0$. Since $(1+a)^n > \frac{1}{2}n(n-1)a^2$, we have $0 < nb^n \le n/[\frac{1}{2}n(n-1)a^2] \le 2/[(n-1)a^2]$. Thus $\lim(nb^n) = 0$.
15. Use the argument in 3.1.11(d). If $(2n)^{1/n} = 1 + k_n$, then show that $k_n^2 \le 2(2n-1)/n(n-1) < 4/(n-1)$.
16. If $n > 3$, then $0 < n^2/n! < n/(n-2)(n-1) < 1/(n-3)$.
17. $\dfrac{2^n}{n!} = \dfrac{2\cdot 2\cdot 2\cdot 2\cdots 2}{1\cdot 2\cdot 3\cdot 4\cdots n} = 2\cdot 1\cdot \dfrac{2}{3}\cdot\dfrac{2}{4}\cdots\dfrac{2}{n} \le 2\cdot\dfrac{2}{3}\cdot\dfrac{2}{3}\cdots\dfrac{2}{3} = 2\left(\dfrac{2}{3}\right)^{n-2}$.
18. If $\varepsilon := x/2$, then $n > K(\varepsilon)$ implies that $|x - x_n| < x/2$, which is equivalent to $x/2 < x_n < 3x/2 < 2x$.

Section 3.2

The results in this section, at least beginning with Theorem 3.2.3, are clearly useful in calculating limits of sequences. They are also easy to remember. The proofs of the basic theorems use techniques that will recur in later work, and so are worth attention (but not memorization). It may be pointed out to the students that the Ratio Test in 3.2.11 has the same hypothesis as the Ratio Test for the convergence of series that they encountered in their calculus course. There are additional results of this nature in the exercises.

Sample Assignment: Exercises 1, 3, 5, 7, 9, 10, 12, 13, 14.

Partial Solutions:

1. (a) $\lim(x_n) = 1$. (b) Divergence.
(c) $x_n \ge n/2$, so the sequence diverges.
(d) $\lim(x_n) = \lim(2 + 1/(n^2+1)) = 2$.
2. (a) $X := (n)$, $Y := (-n)$ or $X := ((-1)^n)$, $Y := ((-1)^{n+1})$. Many other examples are possible. (b) $X = Y := ((-1)^n)$.

3. $Y = (X+Y) - X$.
4. If $z_n := x_n y_n$ and $\lim(x_n) = x \neq 0$, then ultimately $x_n \neq 0$ so that $y_n = z_n/x_n$.
5. (a) (2^n) is not bounded since $2^n > n$ by Exercise 1.2.13.
(b) The sequence is not bounded.
6. (a) $(\lim(2+1/n))^2 = 2^2 = 4$, (b) 0, since $|(-1)^n/(n+2))| \le 1/n$,
(c) $\lim\left(\dfrac{1-1/\sqrt{n}}{1+1/\sqrt{n}}\right) = \dfrac{1}{1} = 1$, (d) $\lim(1/n^{1/2} + 1/n^{3/2}) = 0+0 = 0$.
7. If $|b_n| \le B$, $B > 0$, and $\varepsilon > 0$, let K be such that $|a_n| < \varepsilon/B$ for $n > K$. To apply Theorem 3.2.3, it is necessary that both (a_n) and (b_n) converge, but a bounded sequence may not be convergent.
8. In (3) the exponent k is fixed, but in $(1+1/n)^n$ the exponent varies.
9. Since $y_n = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$, we have $\lim(y_n) = 0$. Also we have $\sqrt{n}y_n = \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{1+1/n}+1}$, so that $\lim(\sqrt{n}y_n) = \frac{1}{2}$.
10. (a) Multiply and divide by $\sqrt{4n^2+n}+2n$ to obtain $1/(\sqrt{4+1/n}+2)$ which has limit $1/4$.
(b) Multiply and divide by $\sqrt{n^2+5n}+n$ to obtain $5/(\sqrt{1+5/n}+1)$ which has limit $5/2$.
11. (a) $(\sqrt{3})^{1/n}(n^{1/n})^{1/4}$ converges to $1 \cdot 1^{1/4} = 1$.
(b) Show that $(n+1)^{1/\ln(n+1)} = e$ for all $n \in \mathbb{N}$.
12. $\dfrac{a(a/b)^n + b}{(a/b)^n + 1}$ has limit $\dfrac{0+b}{0+1} = b$ since $0 < a/b < 1$.
13. $$\frac{(n+a)(n+b)-n^2}{\sqrt{(n+a)(n+b)}+n} = \frac{(a+b)n+ab}{\sqrt{(n+a)(n+b)}+n} \cdot \frac{1/n}{1/n} = \frac{a+b+ab/n}{\sqrt{(1+a/n)(1+b/n)}+1} \to \frac{a+b}{2}.$$
14. (a) Since $1 \le n^{1/n^2} \le n^{1/n}$, the limit is 1.
(b) Since $1 \le n! \le n^n$ implies $1 \le (n!)^{1/n^2} \le n^{1/n}$, the limit is 1.
15. Show that $b \le z_n \le 2^{1/n} b$.
16. (a) $L = a$, (b) $L = b/2$, (c) $L = 1/b$, (d) $L = 8/9$.
17. (a) $(1/n)$, (b) (n).
18. If $1 < r < L$, let $\varepsilon := L - r$. Then there exists K such that $|x_{n+1}/x_n - L| < \varepsilon$ for $n > K$. From this one gets $x_{n+1}/x_n > r$ for $n > K$. If $n > K$, then $x_n \ge r^{n-K} x_K$. Since $r > 1$, it follows that (x_n) is not bounded.
19. (a) Converges to 0, (b) Diverges,
(c) Converges to 0, (d) $n!/n^n \le 1/n$.

20. If $L < r < 1$ and $\varepsilon := r - L$, then there exists K such that $|x_n^{1/n} - L| < \varepsilon = r - L$ for $n > K$, which implies that $x_n^{1/n} < r$ for $n > K$. Then $0 < x_n < r^n$ for $n > K$, and since $0 < r < 1$, we have $\lim(r^n) = 0$. Hence $\lim(x_n) = 0$.
21. (a) (l), (b) (n).
22. Yes. The hypothesis implies that $\lim(y_n - x_n) = 0$. Since $y_n = (y_n - x_n) + x_n$, it follows that $\lim(y_n) = \lim(x_n)$.
23. It follows from Exercise 2.2.18 that $u_n = \frac{1}{2}(x_n + y_n + |x_n - y_n|)$. Theorems 3.2.3 and 3.2.9 imply that $\lim(u_n) = \frac{1}{2}[\lim(x_n) + \lim(y_n) + |\lim(x_n) - \lim(y_n)|] = \max\{\lim(x_n), \lim(y_n)\}$. Similarly for $\lim(v_n)$.
24. Since it follows from Exercises 2.2.18(b) and 2.2.19 that $\mathrm{mid}\{a, b, c\} = \min\{\max\{a, b\}, \max\{b, c\}, \max\{c, a\}\}$, this result follows from the preceding exercise.

Section 3.3

The Monotone Convergence Theorem 3.3.2 is a very important (and natural) result. It implies the *existence* of the limit of a bounded monotone sequence. Although it does not give an easy way of calculating the limit, it does give some estimates about its value. For example, if (x_n) is an increasing sequence with upper bound b, then limit x^* must satisfy $x_n \le x^* \le b$ for any $n \in \mathbb{N}$. If this is not sufficiently exact, take x_m for $m > n$ and look for a smaller bound b' for the sequence.

Sample Assignment: Exercises 1, 2, 4, 5, 7, 9, 10.

Partial Solutions:

1. Note that $x_2 = 6 < x_1$. Also, if $x_{k+1} < x_k$, then $x_{k+2} = \frac{1}{2}x_{k+1} + 2 < \frac{1}{2}x_k + 2 = x_{k+1}$. By Induction, (x_n) is a decreasing sequence. Also $0 \le x_n \le 8$ for all $n \in \mathbb{N}$. The limit $x := \lim(x_n)$ satisfies $x = \frac{1}{2}x + 2$, so that $x = 4$.
2. Show, by Induction, that $1 < x_n \le 2$ for $n \ge 2$ and that (x_n) is monotone. In fact, (x_n) is decreasing, for if $x_1 < x_2$, then we would have $(x_1 - 1)^2 < x_1^2 - 2x_1 + 1 = 0$. Since $x := \lim(x_n)$ must satisfy $x = 2 - 1/x$, we have $x = 1$.
3. If $x_k \ge 2$, then $x_{k+1} := 1 + \sqrt{x_k - 1} \ge 1 + \sqrt{2 - 1} = 2$, so $x_n \ge 2$ for all $n \in \mathbb{N}$, by Induction. If $x_{k+1} \le x_k$, then $x_{k+2} = 1 + \sqrt{x_{k+1} - 1} \le 1 + \sqrt{x_k - 1} = x_{k+1}$, so (x_n) is decreasing. The limit $x := \lim(x_n)$ satisfies $x = 1 + \sqrt{x - 1}$ so that $x = 1$ or $x = 2$. Since $x = 1$ is impossible (why?), we have $x = 2$.
4. Note that $y_1 = 1 < \sqrt{3} = y_2$, and if $y_{n+1} - y_n > 0$, then $y_{n+2} - y_{n+1} = (y_{n+1} - y_n)/(\sqrt{2 + y_{n+1}} + \sqrt{2 + y_n}) > 0$, so (y_n) is increasing by Induction. Also $y_1 < 2$ and if $y_n < 2$, then $y_{n+1} = \sqrt{2 + y_n} < \sqrt{2 + 2} = 2$, so (y_n) is bounded above. Therefore (y_n) converges to a number y which must satisfy $y = \sqrt{2 + y}$, whence $y = 2$.

5. We have $y_2 = \sqrt{p + \sqrt{p}} > \sqrt{p} = y_1$. Also $y_n > y_{n-1}$ implies that $y_{n+1} = \sqrt{p + y_n} > \sqrt{p + y_{n-1}} = y_n$, so (y_n) is increasing. An upper bound for (y_n) is $B := 1 + 2\sqrt{p}$ because $y_1 \le B$ and if $y_n \le B$ then $y_{n+1} < \sqrt{p + B} = 1 + \sqrt{p} < B$. If $y := \lim(y_n)$, then $y = \sqrt{p + y}$ so that $y = \frac{1}{2}(1 + \sqrt{1 + 4p})$.
6. Show that the sequence is monotone. The positive root of the equation $z^2 - z - a = 0$ is $z^* := \frac{1}{2}(1 + \sqrt{1 + 4a})$. Show that if $0 < z_1 < z^*$, then $z_1^2 - z_1 - a < 0$ and the sequence increase to z^*. If $z^* < z_1$, then the sequence decreases to z^*.
7. Since $x_n > 0$ for all $n \in \mathbb{N}$, we have $x_{n+1} = x_n + 1/x_n > x_n$, so that (x_n) is increasing. If $x_n \le b$ for all $n \in \mathbb{N}$, then $x_{n+1} - x_n = 1/x_n \ge 1/b > 0$ for all n. But if $\lim(x_n)$ exists, then $\lim(x_{n+1} - x_n) = 0$, a contradiction. Therefore (x_n) diverges.
8. The sequence (a_n) is increasing and is bounded above by b_1, so $\xi := \lim(a_n)$ exists. Also (b_n) is decreasing and bounded below by a_1 so $\eta := \lim(b_n)$ exists. Since $b_n - a_n \ge 0$ for all n, we have $\eta - \xi \ge 0$. Thus $a_n \le \xi \le \eta \le b_n$ for all $n \in \mathbb{N}$.
9. Show that if $x_1, x_2, \ldots, x_{n-1}$ have been chosen, then there exists $x_n \in A$ such that $x_n > u - 1/n$ and $x_n \ge x_k$ for $k = 1, 2, \ldots, n-1$.
10. Since $y_{n+1} - y_n = 1/(2n+1) + 1/(2n+2) - 1/(n+1) = 1(2n+1)(2n+2) > 0$, it follows that (y_n) is increasing. Also $y_n = 1/(n+1) + 1/(n+2) + \cdots + 1/2n < 1/(n+1) + 1/(n+1) + \cdots + 1/(n+1) = n/(n+1) < 1$, so that (y_n) is bounded above. Thus (y_n) is convergent. (It can be show that its limit is $\ln 2$).
11. The sequence (x_n) is increasing. Also $x_n < 1 + 1/1 \cdot 2 + 1/2 \cdot 3 + \cdots + 1/(n-1)n = 1 + (1 - 1/2) + (1/2 - 1/3) + \cdots + (1/(n-1) - 1/n) = 2 - 1/n < 2$, so (x_n) is bounded above and (x_n) is convergent. (It can be shown that its limit is $\pi^2/6$).
12. (a) $(1 + 1/n)^n(1 + 1/n) \to e \cdot 1 = e$, (b) $[(1 + 1/n)^n]^2 \to e^2$,
(c) $[1 + 1/(n+1)]^{n+1}/[1 + 1/(n+1)] \to e/1 = e$,
(d) $(1 - 1/n)^n = [1 + 1/(n-1)]^{-n} \to e^{-1} = 1/e$.
13. Note that if $n \ge 2$, then $0 \le s_n - \sqrt{2} \le s_n^2 - 2$.
14. Note that $0 \le s_n - \sqrt{5} \le (s_n^2 - 5)/\sqrt{5} \le (s_n^2 - 5)/2$.
15. $e_2 = 2.25$, $e_4 = 2.441\,406$, $e_8 = 2.565\,785$, $e_{16} = 2.637\,928$.
16. $e_{50} = 2.691\,588$, $e_{100} = 2.704\,814$, $e_{1000} = 2.716\,924$.

Section 3.4

The notion of a subsequence is extremely important and will be used often. It must be emphasized to students that a subsequence is not simply a collection of terms, but an ordered selection that is a sequence in its own right. Moreover, the order is inherited from the order of the given sequence. The distinction between a sequence and a set is crucial here.

The Bolzano-Weierstrass Theorem 3.4.8 is a fundamental result whose importance cannot be over-emphasized. It will be used as a crucial tool in establishing the basic properties of continuous functions in Chapter 5.

Sample Assignment: Exercises 1, 2, 3, 5, 6, 9, 12.

Partial Solutions:

1. Let $x_{2n-1} := 2n-1$, $x_{2n} := 1/2n$; that is $(x_n) = (1, 1/2, 3, 1/4, 5, 1/6, \ldots)$.
2. If $x_n := c^{1/n}$, where $0 < c < 1$, then (x_n) is increasing and bounded, so it has a limit x. Since $x_{2n} = \sqrt{x_n}$, the limit satisfies $x = \sqrt{x}$, so $x = 0$ or $x = 1$. Since $x = 0$ is impossible (why?), we have $x = 1$.
3. Since $x_n \geq 1$ for all $n \in \mathbb{N}$, $L > 0$. Further, we have $x_n = 1/x_{n-1} + 1 \Rightarrow L = 1/L + 1 \Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1}{2}(1 + \sqrt{5})$.
4. (a) $x_{2n} \to 0$ and $x_{2n+1} \to 2$.
   (b) $x_{8n} = 0$ and $x_{8n+1} = \sin(\pi/4) = 1/\sqrt{2}$ for all $n \in \mathbb{N}$.
5. If $|x_n - z| < \varepsilon$ for $n \geq K_1$ and $|y_n - z| < \varepsilon$ for $n \geq K_2$, let $K := \sup\{2K_1 - 1, 2K_2\}$. Then $|z_n - z| < \varepsilon$ for $n \geq K$.
6. (a) $x_{n+1} < x_n \iff (n+1)^{1/(n+1)} < n^{1/n} \iff (n+1)^n < n^{n+1} = n^n \cdot n \iff (1 + 1/n)^n < n$.
   (b) If $x := \lim(x_n)$, then
   $$x = \lim(x_{2n}) = \lim((2n)^{1/2n}) = \lim((2^{1/n} n^{1/n})^{1/2}) = x^{1/2},$$
   so that $x = 0$ or $x = 1$. Since $x_n \geq 1$ for all n, we have $x = 1$.
7. (a) $(1 + 1/n^2)^{n^2} \to e$,
   (b) $(1 + 1/2n)^n = ((1 + 1/2n)^{2n})^{1/2} \to e^{1/2}$,
   (c) $(1 + 1/n^2)^{2n^2} \to e^2$.
   (d) $(1 + 2/n)^n = (1 + 1/(n+1))^n \cdot (1 + 1/n)^n \to e \cdot e = e^2$.
8. (a) $(3n)^{1/2n} = ((3n)^{1/3n})^{3/2} \to 1^{3/2} = 1$,
   (b) $(1 + 1/2n)^{3n} = ((1 + 1/2n)^{2n})^{3/2} \to e^{3/2}$.
9. If (x_n) does not converge to 0, then there exists $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) with $|x_{n_k}| > \varepsilon_0$ for all $k \in \mathbb{N}$.
10. Choose m_1 such that $S \leq s_{m_1} < S + 1$. Now choose k_1 such that $k_1 \geq m_1$ and $s_{m_1} - 1 < x_{k_1} \leq s_{m_1}$. If $m_1 < m_2 < \cdots < m_{n-1}$ and $k_1 < k_2 < \cdots < k_{n-1}$ have been selected, choose $m_n > m_{n-1}$ such that $S \leq s_{m_n} < S + 1/n$. Now choose $k_n \geq m_n$ and $k_n > k_{n-1}$ such that $s_{m_n} - 1/n < x_{k_n} \leq s_{m_n}$. Then (x_{k_n}) is a subsequence of (x_n) and $|x_{k_n} - S| \leq 1/n$.

11. Show that $\lim((-1)^n x_n) = 0$.

12. Choose $n_1 \geq 1$ so that $|x_{n_1}| > 1$, then choose $n_2 > n_1$ so that $|x_{n_2}| > 2$, and, in general, choose $n_k > n_{k-1}$ so that $|x_{n_k}| > k$.

13. $(x_{2n-1}) = (-1, -1/3, -1/5, \ldots)$.

14. Choose $n_1 \geq 1$ so that $x_{n_1} \geq s - 1$, then choose $n_2 > n_1$ so that $x_{n_2} > s - 1/2$, and, in general, choose $n_k > n_{k-1}$ so that $x_{n_k} > s - 1/k$.

15. Suppose that the subsequence (x_{n_k}) converges to x. Given $n \in \mathbb{N}$ there exists K such that if $k \geq K$ then $n_k \geq n$, so that $x_{n_k} \in I_{n_k} \subseteq I_n = [a_n, b_n]$ for all $k \geq K$. By 3.2.6 we conclude that $x = \lim(x_{n_k})$ belongs to I_n for arbitrary $n \in \mathbb{N}$.

16. For example, $X = (1, 1/2, 3, 1/4, 5, 1/6, \ldots)$.

17. $\limsup(x_n) = 1$, $\sup\{x_n\} = 2$, $\liminf(x_n) = 0$, $\inf\{x_n\} = -1$.

18. If $x = \lim(x_n)$ and $\varepsilon > 0$ is given, then there exists N such that $x - \varepsilon < x_n < x + \varepsilon$ for $n \geq N$. The second inequality implies $\limsup(x_n) \leq x + \varepsilon$ and the first inequality implies $\liminf(x_n) \geq x - \varepsilon$. Then $0 \leq \limsup(x_n) - \liminf(x_n) \leq 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, equality follows. Conversely, if $x = \liminf(x_n) = \limsup(x_n)$, then there exists N_1 such that $x_n < x + \varepsilon$ for $n > N_1$, and N_2 such that $x - \varepsilon < x_n$ for $n \geq N_2$. Now take N to be the larger of N_1 and N_2.

19. If $v > \limsup(x_n)$ and $u > \limsup(y_n)$, then there are at most finitely many n such that $x_n > v$ and at most finitely many n such that $y_n > v$. Therefore, there are at most finitely many n such that $x_n + y_n > v + u$, which implies $\limsup(x_n + y_n) \leq v + u$. This proves the stated inequality. For an example of strict inequality, one can take $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$.

Section 3.5

At first, students may encounter a little difficulty in working with Cauchy sequences. It should be emphasized that in proving that a sequence (x_n) is a Cauchy sequence, the indices n, m in Definition 3.5.1 are completely independent of one another (however, one can always assume that $m > n$). On the other hand, to prove that a sequence is *not* a Cauchy sequence, a particular relationship between n and m can be assumed in the process of showing that the definition is violated.

The significance of Cauchy criteria for convergence is not immediately apparent to students. Its true role in analysis will be slowly revealed by its use in subsequent chapters.

We have included the discussion of contractive sequences to illustrate just one way in which Cauchy sequences can arise.

Sample Assignment: Exercises 1, 2, 3, 5, 7, 9, 10.

Partial Solutions:

1. For example, $((-1)^n)$.
2. (a) If $m > n$, then $|(1+1/m) - (1+1/n)| < 2/n$.
   (b) $0 < 1/(n+1)! + \cdots + 1/m! < 1/2^n$, since $2^k < k!$ for $k \geq 4$.
3. (a) Note that $|(-1)^n - (-1)^{n+1}| = 2$ for all $n \in \mathbb{N}$.
   (b) Take $m = 2n$, so $x_m - x_n = x_{2n} - x_n \geq 1$ for all n.
   (c) Take $m = 2n$, so $x_m - x_n = x_{2n} - x_n = \ln 2n - \ln n = \ln 2$ for all n.
4. Use $|x_m y_m - x_n y_n| \leq |y_m||x_m - x_n| + |x_n||y_m - y_n|$ and the fact that Cauchy sequences are bounded.
5. $\lim(\sqrt{n+1} - \sqrt{n}) = \lim\left(\dfrac{1}{\sqrt{n+1} + \sqrt{n}}\right) = 0$. However, if $m = 4n$, then $\sqrt{4n} - \sqrt{n} = \sqrt{n}$ for all n.
6. Let $x_n := 1 + 1/2 + \cdots + 1/n$, which is not a Cauchy sequence. (Why?) However, for any $p \in \mathbb{N}$, then $0 < x_{n+p} - x_n = 1/(n+1) + \cdots + 1/(n+p) \leq p/(n+1)$, which has limit 0.
7. If x_n, x_m are integers and $|x_m - x_n| < 1$, then $x_n = x_m$.
8. Let $u := \sup\{x_n : n \in \mathbb{N}\}$. If $\varepsilon > 0$, let H be such that $u - \varepsilon < x_H \leq u$. If $m \geq n \geq H$, then $u - \varepsilon < x_n \leq x_m \leq u$ so that $|x_m - x_n| < \varepsilon$.
9. If $m > n$, then $|x_m - x_n| < r^n + r^{n+1} + \cdots + r^{m-1} \leq r^n/(1-r)$, which converges to 0 since $0 < r < 1$.
10. If $L := x_2 - x_1$, then $|x_{n+1} - x_n| = L/2^{n-1}$, whence it follows that (x_n) is a Cauchy sequence. To find the limit, show that $x_{2n+1} = x_1 + L/2 + L/2^3 + L/2^5 + \cdots + L/2^{2n-1}$, whence $\lim(x_n) = x_1 + (2/3)L = (1/3)x_1 + (2/3)x_2$.
11. Note that $|y_n - y_{n+1}| = (2/3)|y_n - y_{n-1}|$. Since $y_2 > y_1$, the limit is $y = y_1 + (3/5)(y_2 - y_1) = (2/5)y_1 + (3/5)y_2$.
12. Show that $|x_{n+1} - x_n| < \frac{1}{4}|x_n - x_{n-1}|$. The limit is $\sqrt{2} - 1$.
13. Note that $x_n \geq 2$ for all n, so that $|x_{n+1} - x_n| = |1/x_n - 1/x_{n-1}| = |x_n - x_{n-1}|/x_n x_{n-1} \leq \frac{1}{4}|x_n - x_{n-1}|$. The limit is $1 + \sqrt{2}$.
14. Let $x_{n+1} = (x_n^3 + 1)/5$ and $x_1 := 0$. Four iterations give $r = 0.201\,64$ to 5 decimal places.

Section 3.6

This section can be omitted on a first reading. However, it is short, relatively easy, and prepares the way for Section 4.3. One must frequently emphasize that ∞ and $-\infty$ are *not* real numbers, but merely convenient abbreviations. While there is no reason to expect that one can manipulate with properly divergent sequences as one does in Theorem 3.2.3, there are some results in this direction.

Sample Assignment: Exercises 1, 2, 3, 5, 8, 9.

Partial Solutions:

1. If the set $\{x_n : n \in \mathbb{N}\}$ is not bounded above, choose $n_{k+1} > n_k$ such that $x_{n_k} \geq k$ for $k \in \mathbb{N}$.
2. (a) $x_n := \sqrt{n}, y_n := n$, (b) $x_n := n, y_n := \sqrt{n}$.
3. Note that $|x_n - 0| < \varepsilon$ if and only if $1/x_n > 1/\varepsilon$.
4. (a) $[\sqrt{n} > a] \iff [n > a^2]$, (b) $\sqrt{n+1} > \sqrt{n}$,
   (c) $\sqrt{n-1} \geq \sqrt{n/2}$ when $n \geq 2$, (d) $n/\sqrt{n+1} \geq \sqrt{n}/2$.
5. No. As in Example 3.4.6(c), there is a subsequence (n_k) with $n_k \sin(n_k) > \frac{1}{2} n_k$, and there is a subsequence (m_k) with $m_k \sin(m_k) < -\frac{1}{2} m_k$.
6. If (y_n) does not converge to 0, there exists $c > 0$ and a subsequence (y_{n_k}) with $|y_{n_k}| \geq c$. Hence $|x_{n_k}| = |x_{n_k} y_{n_k} / y_{n_k}|$ is bounded, contradicting the fact that (x_n) is properly divergent.
7. (a) There exists N_1 such that if $n > N_1$, then $0 < x_n < y_n$. If $\lim(x_n) = \infty$ then $\lim(y_n) = \infty$.
   (b) Suppose that $|y_n| \leq M$ for some $M > 0$. Given $\varepsilon > 0$ there exists N_ε such that if $n \geq N_\varepsilon$ then $0 < x_n/y_n \leq \varepsilon/M$. Therefore $|x_n| \leq (\varepsilon/M) y_n \leq \varepsilon$ for $n \geq N_\varepsilon$.
8. (a) $n < (n^2 + 2)^{1/2}$.
   (b) Since $\sqrt{n} \leq n$, then $\sqrt{n}/(n^2+1) \leq n/(n^2+1) < 1/n$.
   (c) Since $n < (n^2+1)^{1/2}$, then $n^{1/2} < (n^2+1)^{1/2}/n^{1/2}$.
   (d) If the sequence were convergent, the subsequence corresponding to $r_k = k^2$ would converge, contrary to Example 3.4.6(c).
9. (a) Since $x_n/y_n \to \infty$, there exists K_1 such that if $n \geq K_1$, then $x_n \geq y_n$. Now apply Theorem 3.6.4(a).
   (b) Let $0 < x_n < M$. If (y_n) does not converge to 0, there exist $\varepsilon_0 > 0$ and a subsequence (y_{n_k}) such that $\varepsilon_0 < y_{n_k}$. Since $\lim(x_n/y_n) = \infty$, there exists K such that if $k > K$, then $M/\varepsilon_0 < x_{n_k}/y_{n_k}$, which is a contradiction.
10. Apply Theorem 3.6.5.

Section 3.7

This section gives a brief introduction to infinite series, a topic that will be discussed further in Chapter 9. However, since the basic results are merely a reformulation of the material in Sections 3.1–3.6, it is useful to treat this section here — especially, if there is a possibility that one might not be able to cover Chapter 9 in class.

It must be made clear to the students that there is a significant difference between a "sequence" of numbers and a "series" of numbers. Indeed, a series is a special kind of sequence, where the terms are obtained by adding terms in a given sequence. For a series to be convergent, the given terms must approach 0 "sufficiently fast". Unfortunately there is no clear demarcation line between the convergent and the divergent series. Thus it is especially important for the students to acquire a collection of series that are known to be convergent (or divergent), so that these known series can be used for the purpose of comparison. The specific series that are discussed in this section are particularly useful in this connection.

Although much of the material in this section will be somewhat familiar to the students, most of them will not have heard of the Cauchy Condensation Criterion (Exercise 15), which is a very powerful test *when it applies.*

Sample Assignment: Exercises 1, 2, 3(a,b), 4, 8, 12, 15, 16, 17.

Partial Solutions:

1. Show that the partial sums of $\sum b_n$ form a subsequence of the partial sums of $\sum a_n$.
2. If $a_n = b_n$ for $n > K$, show that the partial sums s_n of $\sum a_n$ and t_n of $\sum b_n$ satisfy $s_n - t_n = s_K - t_K$ for all $n > K$.
3. (a) Note that $1/(n+1)(n+2) = 1/(n+1) - 1/(n+2)$, so the series is telescoping and converges to 1.
   (b) $1/(\alpha+n)(\alpha+n+1) = 1/(\alpha+n) - 1/(\alpha+n+1)$.
   (c) $1/n(n+1)(n+2) = 1/2n - 1/(n+1) + 1/2(n+2)$, so that $\sum_1^N = 1/4 - 1/2(N+1) + 1/2(N+2)$.
4. If $s_n := \sum_1^n x_k$ and $t_n := \sum_1^n y_k$, then $s_n + t_n = \sum_1^n (x_k + y_k)$.
5. No. Let $z_k := x_k + y_k$ so that $y_k = z_k + (-1)x_k$. If $\sum(-1)x_k$ and $\sum z_k$ are convergent, then $\sum y_k$ is convergent.
6. (a) $(2/7)^2[1/(1-2/7)] = 4/35$. (b) $(1/9)[1/(1-1/9)] = 1/8$.
7. $r^2(1 + r^2 + (r^2)^2 + \cdots) = r^2/(1-r^2)$
8. $S = \varepsilon + \varepsilon^2 + \varepsilon^3 + \cdots = \varepsilon/(1-\varepsilon)$.
   $S = 1/9 = 0.111\cdots$ if $\varepsilon = 0.1$, and $S = 1/99 = 0.0101\cdots$ if $\varepsilon = 0.01$.
9. (a) The sequence $(\cos n)$ does not converge to 0.
   (b) Since $|(\cos n)/n^2| \le 1/n^2$, the convergence of $\sum(\cos n)/n^2$ follows from Example 3.7.6(c) and Theorem 3.7.7.
10. Note that the "even" sequence (s_{2n}) is decreasing, and the "odd" sequence (s_{2n+1}) is increasing and $-1 \le s_n \le 0$. Moreover $0 \le s_{2n} - s_{2n+1} = 1/\sqrt{2n+1}$.
11. If convergent, then $a_n \to 0$, so there exists $M > 0$ such that $0 < a_n \le M$, whence $0 < a_n^2 \le M a_n$, and the Comparison Test 3.7.7 applies.
12. $\sum 1/n^2$ is convergent, but $\sum 1/n$ is not.
13. Recall that if $a, b \ge 0$ then $2\sqrt{ab} \le a + b$, so $\sqrt{a_n a_{n+1}} \le (a_n + a_{n+1})/2$. Now apply the Comparison Test 3.7.7.

14. Show that $b_k \geq a_1/k$ for $k \in \mathbb{N}$, whence $b_1 + \cdots + b_n \geq a_1(1 + \cdots + 1/n)$.

15. Evidently $2a(4) \leq a(3) + a(4) \leq 2a(2)$ and $2^2a(8) \leq a(5) + \cdots + a(8) \leq 2^2a(4)$, etc. The stated inequality follows by addition. Now apply the Comparison Test 3.7.7.

16. Clearly $n^p < (n+1)^p$ if $p > 0$, so that the terms in the series are decreasing. Since $2^n \cdot (1/2^{pn}) = (1/2^{p-1})^n$, the Cauchy Condensation Test asserts that the convergence of the p-series is the same as that of the geometric series with ratio $1/2^{p-1}$, which is < 1 when $p > 1$ and is ≥ 1 when $p \leq 1$.

17. (a) The terms are decreasing and $2^n/2^n \ln(2^n) = 1/(n \ln 2)$. Since the harmonic series $\sum 1/n$ diverges, so does $\sum 1/(n \ln n)$.
(b) $2^n/2^n(\ln 2^n)(\ln\ln 2^n) = 1/(n \ln 2)(\ln n(\ln 2))$. Now use the Limit Comparison Test 3.7.8 and part (a).

18. (a) The terms are decreasing and $2^n/2^n(\ln 2^n)^c = (1/n^c) \cdot (1/\ln 2)^c$. Now use the fact that $\sum(1/n^c)$ converges when $c > 1$.
(b) Since $\ln(n/2) < \ln(n \ln 2)$, we have $1/(\ln(n \ln n))^c < 1/(\ln(n/2))^c$. Now apply (a).

CHAPTER 4

LIMITS

In this chapter we begin the study of functions of a real variable. This and the next chapter are the most important ones in the book, since all subsequent material depends on the results in them. In Section 4.1 the concept of a limit of a function at a point is introduced, and in Section 4.2 the basic properties of limits are established. Both of these sections are necessary preparation for Chapter 5. However, Section 4.3 can be omitted on a first reading, if time is short.

Examples are a vital part of real analysis. Although certain examples do not need to be discussed in detail, we advise that the students be urged to study them carefully. One way of encouraging this is to ask for examples of various phenomena on examinations.

Section 4.1

Attention should be called to the close parallel between Section 3.1 and this section. It should be noted that here $\delta(\varepsilon)$ plays the same role as $K(\varepsilon)$ did in Section 3.1. The proof of the Sequential Convergence Theorem 4.1.8 is instructive and the result is important. As a rule of thumb, the ε-δ formulation of the limit is used to establish a limit, while sequences are more often used to (i) evaluate a limit, or (ii) prove that a limit fails to exist.

Sample Assignment: Exercises 1, 3, 6, 8, 9, 10(b,d), 11(a), 12(a,c), 15.

Partial Solutions:

1. (a–c) If $|x-1| \le 1$, then $|x+1| \le 3$ so that $|x^2-1| \le 3|x-1|$. Thus, $|x-1| < 1/6$ assures that $|x^2-1| < 1/2$, etc.
   (d) Since $x^3-1=(x-1)(x^2+x+1)$, if $|x-1|<1$, then $0<x<2$ and so $|x^3-1| \le 7|x-1|$.
2. (a) Since $|\sqrt{x}-2| = |x-4|/(\sqrt{x}+2) \le \frac{1}{2}|x-4|$, then $|x-4| < 1$ implies that $|\sqrt{x}-2| < \frac{1}{2}$.
   (b) If $|x-4| < 2 \times 10^{-2} = .02$, then $|\sqrt{x}-2| < .01$.
3. Apply the definition of the limit.
4. If $\lim\limits_{y \to c} f(y) = L$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |y-c| < \delta$, then $|f(y)-L| < \varepsilon$. Now let $x := y - c$ so that $y = x + c$, to conclude that $\lim\limits_{x \to 0} f(x+c) = L$.
5. If $0 < x < a$, then $0 < x + c < a + c < 2a$, so that $|x^2 - c^2| = |x+c||x-c| \le 2a|x-c|$. Given $\varepsilon > 0$, take $\delta := \varepsilon/2a$.
6. Take $\delta := \varepsilon/K$.

7. Let $b := |c| + 1$. If $|x| < b$, then $|x^2 + cx + c^2| \le 3b^2$. Hence $|x^3 - c^3| \le (3b^2)|x - c|$ for $|x| < b$.

8. Note that $\sqrt{x} - \sqrt{c} = (\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})/(\sqrt{x} + \sqrt{c}) = (x - c)/(\sqrt{x} + \sqrt{c})$. Hence, if $c \neq 0$, we have $|\sqrt{x} - \sqrt{c}| \le (1/\sqrt{c})|x - c|$, so that we can take $\delta := \varepsilon\sqrt{c}$. If $c = 0$, we can take $\delta := \varepsilon^2$, so that if $0 < x < \delta$, then $|\sqrt{x} - 0| < \varepsilon$.

9. (a) If $|x - 2| < 1/2$, then $x > 3/2$, so $x - 1 > 1/2$ whence $0 < 1/(x - 1) < 2$ and so $|1/(1 - x) + 1| = |(x - 2)/(x - 1)| \le 2|x - 2|$. Thus we can take $\delta := \inf\{1/2, \varepsilon/2\}$.
(b) If $|x - 1| < 1$, then $x + 1 > 1$, so $1/(x + 1) < 1$ whence $|x/(1 + x) - 1/2| = |x - 1|/(2|x + 1|) \le |x - 1|/2 \le |x - 1|$. Thus we may take $\delta := \inf\{1, \varepsilon\}$.
(c) If $x \neq 0$, then $|x^2/|x| - 0| = |x|$. Take $\delta := \varepsilon$.
(d) If $|x - 1| < 1$, then $|2x - 1| < 3$ and $1/|x + 1| < 1$, so that $|(x^2 - x + 1)/(x + 1) - 1/2| = |2x - 1||x - 1|/(2|x + 1|) \le (3/2)|x - 1|$, so we may take $\delta := \inf\{1, 2\varepsilon/3\}$.

10. (a) If $|x - 2| < 1$, then $|x^2 + 4x - 12| = |x + 6||x - 2| < 9|x - 2|$. We may take $\delta := \inf\{1, \varepsilon/9\}$.
(b) If $|x + 1| < 1/4$, then $-5/4 < x < -3/4$ so that $1/2 < 2x + 3 < 3/2$, and thus $0 < 1/(2x + 3) < 2$. Then $|(x + 5)/(2x + 3) - 4| = 7|x + 1|/|2x + 3| < 14|x + 1|$, so that we may take $\delta := \inf\{1/4, \varepsilon/14\}$.

11. (a) If $|x - 3| < 1/2$, then $x > 5/2$, so $4x - 9 > 1$ and then $1/|4x - 9| < 1$. Then $\left|\frac{2x+3}{4x-9} - 3\right| = \left|\frac{10x-30}{4x-9}\right| \le 10|x - 3|$. Thus we take $\delta = \inf\{1/2, \varepsilon/10\}$.
(b) If $|x - 6| < 1$, then $x + 1 < 8$, and $x + 3 > 8$, so that $(x + 1)/(x + 3) < 1$. Then $\left|\frac{x^2-3x}{x+3} - 2\right| = \left|\frac{x+1}{x+3}\right| |x - 6| \le |x - 6|$. Thus we take $\delta = \inf\{1, \varepsilon\}$.

12. (a) Let $x_n := 1/n$. (b) Let $x_n := 1/n^2$.
(c) Let $x_n := 1/n$ and $y_n := -1/n$.
(d) Let $x_n := 1/\sqrt{n\pi}$ and $y_n := 1/\sqrt{\pi/2 + 2\pi n}$.

13. If $|f(y) - L| < \varepsilon$ for $|y| < \delta$, then $|g(x) - L| < \varepsilon$ for $0 < |x| < \delta/a$.

14. (a) Given $\varepsilon > 0$, choose $\delta > 0$ such that $0 < |x - c| < \delta$ implies $(f(x))^2 < \varepsilon^2$.
(b) If $f(x) := \operatorname{sgn}(x)$, then $\lim_{x\to 0}(f(x))^2 = 1$, but $\lim_{x\to 0} f(x)$ does not exist.

15. (a) Since $|f(x) - 0| \le |x|$, we can take $\delta := \varepsilon$ to show that $\lim_{x\to 0} f(x) = 0$.
(b) If $c \neq 0$ is rational, let (x_n) be a sequence of irrational numbers that converges to c; then $f(c) = c \neq 0 = \lim(f(x_n))$. If c is irrational, let (r_n) be a sequence of rational numbers that converges to c; then $f(c) = 0 \neq c = \lim(f(r_n))$.

16. Since I is an open interval containing c, there exists $a > 0$ such that the a-neighborhood $V_a(c) \subseteq I$. For $\varepsilon > 0$, if $\delta > 0$ is chosen so that $\delta \le a$, then it will apply to both f and f_1.

17. The restriction of sgn to [0, 1] has a limit at 0.

Section 4.2

Note the close parallel between this section and Section 3.2. While the proofs should be read carefully, the main interest here is in the application of the theorems to the calculation of limits.

Sample Assignment: Exercises 1, 2, 4, 5, 9, 11, 12.

Partial Solutions:

1. (a) 15, (b) -3, (c) $1/12$, (d) $1/2$.
2. (a) The limit is 1.
(b) Since $(x^2-4)/(x-2)=x+2$ for $x\neq 2$, the limit is 4. Note that Theorem 4.2.4(b) cannot be applied here.
(c) The quotient equals $x+2$ for $x\neq 0$. Hence the limit is 2.
(d) The quotient equals $1/(\sqrt{x}+1)$ for $x\neq 1$. The limit is $1/2$.
3. Multiply the numerator and denominator by $\sqrt{1+2x}+\sqrt{1+3x}$. The limit is $-1/2$.
4. If $x_n := 1/2\pi n$ for $n\in\mathbb{N}$, then $\cos(1/x_n)=1$. Also, if $y_n := 1/(2\pi n+\pi/2)$ for $n\in\mathbb{N}$, then $\cos(1/y_n)=0$. Hence $\cos(1/x)$ does not have a limit as $x\to 0$. Since $|x\cos(1/x)|\leq |x|$, the Squeeze Theorem 4.2.7 applies.
5. If $|f(x)|\leq M$ for $x\in V_\delta(c)$, then $|f(x)g(x)-0|\leq M|g(x)-0|$ for $x\in V_\delta(c)$.
6. Given $\varepsilon>0$, choose $\delta_1>0$ so that if $0<|x-c|<\delta_1, x\in A$, then $|f(x)-L|<\varepsilon/2$. Choose $\delta_2>0$ so that if $0<|x-c|<\delta_2, x\in A$, then $|g(x)-M|<\varepsilon/2$. Take $\delta := \inf\{\delta_1,\delta_2\}$. If $x\in A$ satisfies $0<|x-c|<\delta$, then $|(f(x)+g(x))-(L+M)|\leq |f(x)-L|+|g(x)-M|<\varepsilon/2+\varepsilon/2=\varepsilon$.
7. Let (x_n) be any sequence in $A\setminus\{c\}$ that converges to c. Then $(f(x_n))$ converges to L and $(h(x_n))$ converges to H. By 3.23(b), $(f(x_n)/h(x_n))$ converges to L/H. Since (x_n) is an arbitrary sequence in $A\setminus\{c\}$, it follows from 4.1.8 that $\lim\limits_{x\to c} f/h = L/H$.
8. If $|x|\leq 1, k\in\mathbb{N}$, then $|x^k|=|x|^k\leq 1$, whence $-x^2\leq x^{k+2}\leq x^2$. Thus, if $n\geq 2$, we have $|x^n-0|\leq |x^2-0|$ for $|x|\leq 1$. Consequently $\lim\limits_{x\to 0} x^n=0$.
9. (a) Note that $g(x)=(f+g)(x)-f(x)$.
(b) No; for example, take $f(x)=x^2$ and $g(x):=1/x$ for $x>0$.
10. Let $f(x):=1$ if x is rational and $f(x):=0$ if x is irrational, and let $g(x):=1-f(x)$. Then $f(x)+g(x)=1$ for all $x\in\mathbb{R}$, so that $\lim\limits_{x\to 0}(f+g)=1$, and $f(x)g(x)=0$ for all $x\in\mathbb{R}$, so that $\lim\limits_{x\to 0} fg=0$.
11. (a) No limit, (b) 0, (c) No limit, (d) 0.
12. Since $f((k+1)y)=f(ky+y)=f(ky)+f(y)$, an induction argument shows that $f(ny)=nf(y)$ for all $n\in\mathbb{N}$, $y\in\mathbb{R}$. If we substitute $y:=1/n$, we get $f(1/n)=f(1)/n$, whence $L=\lim\limits_{x\to 0} f(x)=\lim(f(1/n))=0$. Since $f(x)-f(c)=$

$f(x-c)$, we infer that $\lim_{x \to c}(f(x) - f(c)) = \lim_{x \to c} f(x-c) = \lim_{z \to 0} f(z) = 0$, so that $\lim_{x \to c} f(x) = f(c)$.

13. (a) $g(f(x)) = g(x+1) = 2$ if $x \neq 0$, so that $\lim_{x \to 0} g(f(x)) = 2$, but $g(\lim_{x \to 0} f(x)) = g(f(0)) = g(1) = 0$. Not equal.
(b) $f(g(x)) = g(x) + 1 = 3$ if $x \neq 1$, so that $\lim_{x \to 1} f(g(x)) = 3$, and $f(\lim_{x \to 1} g(x)) = f(2) = 3$. Equal.
14. If $\lim_{x \to c} f(x) = L$, then $||f(x)| - |L|| \leq |f(x) - L|$ implies that $\lim_{x \to c} |f(x)| = |L|$.
15. This follows from Theorem 3.2.10 and the Sequential Criterion 4.1.8. Alternatively, an ε-δ proof can be given.

Section 4.3

This section can play the role of reinforcing the notion of the limit, since it provides several extensions of this concept. However, the results obtained here are used in only a few places later, so that it is easy to omit this section on a first reading. In fact, one-sided limits are used only once or twice in subsequent chapters.

In any case, we advise that the discussion of this section be quite brief. Indeed, it is quite reasonable to give a short introduction to it in a class, and leave it to the students to return to it later, when needed.

Sample Assignment: Exercises 2, 3, 4, 5(a,c,e,g), 8, 9.

Partial Solutions:

1. Modify the proof of Theorem 4.1.8 appropriately. Note that $0 < |x - c| < \delta$ is replaced by $0 < x - c < \delta$ since $x > c$.
2. Let $f(x) := \sin(1/x)$ for $x < 0$ and $f(x) := 0$ for $x > 0$.
3. Given $\alpha > 0$, if $0 < x < 1/\alpha^2$, then $\sqrt{x} < 1/\alpha$, and so $f(x) > \alpha$. Since α is arbitrary, $\lim_{x \to 0+} x/(x-1) = \infty$.
4. If $\alpha > 0$, then $f(x) > \alpha$ if and only if $|1/f(x) - 0| < 1/\alpha$.
5. (a) If $\alpha > 1$ and $1 < x < \alpha/(\alpha - 1)$, then $\alpha < x/(x-1)$, hence we have $\lim_{x \to 1+} x/(x-1) = \infty$.
(b) The right-hand limit is ∞; the left-hand limit is $-\infty$.
(c) Since $(x+2)/\sqrt{x} > 2/\sqrt{x}$, the limit is ∞.
(d) Since $(x+2)/\sqrt{x} > \sqrt{x}$, the limit is ∞.
(e) If $x > 0$, then $1/\sqrt{x} < (\sqrt{x+1})/x$, so the right-hand limit is ∞. What is the left-hand limit?
(f) 0. (g) 1. (h) -1.
6. Modify the proof of Theorem 4.3.2 (using Definition 4.3.10). Note that $0 < x - c < \delta(\varepsilon)$ is replaced by $x > K(\varepsilon)$.
7. Use Theorem 4.3.11.

8. Note that $|f(x)-L|<\varepsilon$ for $x>K$ if and only if $|f(1/z)-L|<\varepsilon$ for $0<z<1/K$.
9. There exists $\alpha>0$ such that $|xf(x)-L|<1$ whenever $x>\alpha$. Hence $|f(x)|<(|L|+1)/x$ for $x>\alpha$.
10. Modify the proof of Theorem 4.3.11 (using Definition 4.3.13). Note that $|f(x)-L|<\varepsilon$ is replaced by $f(x)>\alpha$ [respectively, $f(x)<\alpha$].
11. Let $\alpha>0$ be arbitrary and let $\beta>(2/L)\alpha$. There exists $\delta_1>0$ such that if $0<|x-c|<\delta_1$ then $f(x)>L/2$, and there exists $\delta_2>0$ such that if $0<|x-c|<\delta_2$, then $g(x)>\beta$. If $\delta_3:=\inf\{\delta_1,\delta_2\}$, and if $0<|x-c|<\delta_3$ then $f(x)g(x)>(L/2)\beta>\alpha$. Since α is arbitrary, then $\lim\limits_{x\to c} fg=\infty$. Let $c=0$ and let $f(x):=|x|$ and $g(x):=1/|x|$ for $x\neq 0$.
12. Let $f(x)=g(x):=x$ (or let $f(x):=x$ and $g(x):=x+1/x$).
    No. If $h(x):=f(x)-g(x)$, then $f(x)/g(x)=1+h(x)/g(x)\to 1$.
13. Suppose that $|f(x)-L|<\varepsilon$ for $x>K$, and that $g(y)>K$ for $y>H$. Then $|f\circ g(y)-L|<\varepsilon$ for $y>H$.

CHAPTER 5

CONTINUOUS FUNCTIONS

This chapter can be considered to be the heart of the course. We now use all the machinery that has been developed to this point in order to study the most important class of functions in analysis, namely, *continuous functions*. In Section 5.3, the fundamental properties of continuous functions are proved, and this section is the most important of this chapter. Sufficient time should be spent on it to allow adequate study of the proofs and examples. Section 5.4 on uniform continuity is also an important section. Section 5.5 contains a different approach to the basic theorems in Sections 5.3 and 5.4, using the idea of a "gauge".

The results on monotone functions in Section 5.6 are interesting, but they are not central to this course and these results will not be used often in later parts of this book.

Section 5.1

This important section is absolutely basic to everything that will follow. Every effort should be made to have the students master the notions presented here. They should memorize the definition of continuity and its various equivalents, and they should study the examples very carefully.

Sample Assignment: Exercises 1, 3, 4(a,b), 5, 7, 11, 12, 13.

Partial Solutions:

3. We will establish the continuity of h at b. Since f is continuous at b, given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if $b - \delta_1 < x < b$, then $|f(x) - f(b)| < \varepsilon$. Similarly, there exists $\delta_2 > 0$ such that if $b < x < b + \delta_2$, then $|g(x) - g(b)| < \varepsilon$. Let $\delta := \inf\{\delta_1, \delta_2\}$ so that $|h(x) - h(b)| < \varepsilon$ for $|x - b| < \delta$, whence h is continuous at b.
4. (a) Continuous if $x \neq 0, \pm 1, \pm 2, \ldots,$
   (b) Continuous if $x \neq \pm 1, \pm 2, \ldots,$
   (c) Continuous if $\sin x \neq 0, 1,$
   (d) Continuous if $x \neq 0, \pm 1, \pm 1/2, \ldots.$
5. Yes. Define $f(2) := \lim_{x \to 2} f(x) = 5$.
6. Given $\varepsilon > 0$, choose $\delta > 0$ such that if $x \in V_\delta(c) \cap A$, then $|f(x) - f(c)| < \varepsilon/2$. Then if $y \in V_\delta(c) \cap A$, we have $|f(y) - f(x)| \leq |f(x) - f(c)| + |f(c) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
7. Let $\varepsilon := f(c)/2$, and let $\delta > 0$ be such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$, which implies that $f(x) > f(c) - \varepsilon = f(c)/2 > 0$.
8. Since f is continuous at x, we have $f(x) = \lim(f(x_n)) = 0$. Thus $x \in S$.

9. (a) If $|f(x)-f(c)|<\varepsilon$ for $x\in V_\delta\cap B$, then $|g(x)-g(c)|=|f(x)-f(c)|<\varepsilon$ for $x\in V_\delta(c)\cap A$.
   (b) Let $f=\operatorname{sgn}$ on $B:=[0,1], g=\operatorname{sgn}$ on $A:=(0,1]$ and $c=0$.
10. Note that $\big||x|-|c|\big|\le|x-c|$.
11. Let $c\in\mathbb{R}$ be given and let $\varepsilon>0$. If $|x-c|<\varepsilon/K$, then $|f(x)-f(c)|\le K|x-c|<K(\varepsilon/K)=\varepsilon$.
12. If x is irrational, then by the Density Theorem 2.4.8 there exists a sequence (r_n) of rational numbers that converges to x. Then $f(x)=\lim(f(r_n))=0$.
13. Since $|g(x)-6|\le\sup\{|2x-6|,|x-3|\}=2|x-3|$, then g is continuous at $x=3$. If $c\ne 3$, let (x_n) be a sequence of rational numbers converging to c and let (y_n) be a sequence of irrational numbers converging to c. Then $\lim(g(x_n))=2c\ne c+3=\lim(g(y_n))$, so g is not continuous at c.
14. Let $c\in A$. If k is continuous at c, it follows from 4.2.2 that k is bounded on some neighborhood $(c-\delta,c+\delta)$. Let $m\in\mathbb{N}$ be given; then there exists a prime number p such that $1/p<\delta$ and $p\ge m$. (Why?) There must be at least one rational number q/p with $c-\delta<q/p<c+\delta$; otherwise there exists an integer q_0 such that $q_0/p\le c-\delta$ and $c+\delta\le(q_0+1)/p$, which implies that $2\delta\le 1/p$, a contradiction. We conclude that $k(x)=p\ge m$ for at least one point $x\in(c-\delta,c+\delta)$. But this is a contradiction.
15. Let $I_n:=(0,1/n]$ for $n\in\mathbb{N}$. Show that $(\sup f(I_n))$ is a decreasing sequence and $(\inf f(I_n))$ is an increasing sequence. If $\lim(\sup f(I_n))=\lim(\inf f(I_n))$, then $\lim\limits_{x\to 0} f$ exists. Let $x_n,y_n\in I_n$ be such that $f(x_n)>\sup f(I_n)-1/n$ and $f(y_n)<\inf f(I_n)+1/n$.

Section 5.2

Note the similarity of this section with Sections 4.2 and 3.2. However, Theorem 5.2.6 concerning composite functions is a new result, and an *important* one. Its importance may be suggested by the fact, noted in 5.2.8, that it implies several of the earlier results.

The significance of this section should be clear: it enables us to establish the continuity of many functions.

Sample Assignment: Exercises 1, 3, 5, 6, 10, 12, 13.

Partial Solutions:

1. (a) Continuous on $\mathbb{R}$, (b) Continuous for $x\ge 0$,
   (c) Continuous for $x\ne 0$, (d) Continuous on $\mathbb{R}$.
2. Use 5.2.1(a) and Induction; or, use 5.2.8 with $g(x):=x^n$.
3. Let f be the Dirichlet discontinuous function (Example 5.1.6(g)) and let $g(x):=1-f(x)$.
4. Continuous at every noninteger.

5. The function g is not continuous at $1 = f(0)$.
6. Given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $|y-b| < \delta_1$, then $|g(y)-g(b)| < \varepsilon$. Further, there exists $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x)-b| < \delta_1$. Hence, if $0 < |x-c| < \delta$, then we have $|(g \circ f)(x) - g(b)| < \varepsilon$, so that $\lim\limits_{x \to c}(g \circ f)(x) = g(b)$.
7. Let $f(x) := 1$ if x is rational, and $f(x) := -1$ if x is irrational.
8. Yes. Given $x \in \mathbb{R}$, let (r_n) be a sequence of rational numbers with $r_n \to x$.
9. Show that an arbitrary real number is the limit of a sequence of numbers of the form $m/2^n$, where $m \in \mathbb{Z}, n \in \mathbb{N}$.
10. If $c \in P$, then $f(c) > 0$. Now apply Theorem 4.2.9.
11. If $h(x) := f(x) - g(x)$, then h is continuous and $S = \{x \in \mathbb{R} : h(x) \geqslant 0\}$.
12. First show that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$; then note that $f(x - x_0) = f(x) - f(x_0)$. Consequently f is continuous at the point x_0 if and only if it is continuous at 0. Thus, if f is continuous at x_0, then it is continuous at 0, and hence everywhere.
13. First show that $f(0) = 0$ and (by Induction) that $f(x) = cx$ for $x \in \mathbb{N}$, and hence also for $x \in \mathbb{Z}$. Next show that $f(x) = cx$ for $x \in \mathbb{Q}$. Finally, if $x \notin \mathbb{Q}$, let $x = \lim(r_n)$ for some sequence in $\mathbb{Q}$.
14. First show that either $g(0) = 0$ or $g(0) = 1$. Next, if $g(\alpha) = 0$ for some $\alpha \in \mathbb{R}$ and if $x \in \mathbb{R}$, let $y := x - \alpha$ so that $x = \alpha + y$; hence $g(x) = g(\alpha + y) = g(\alpha)g(y) = 0$. Thus, if $g(\alpha) = 0$ for some α, then it follows that $g(x) = 0$ for all $x \in \mathbb{R}$.

    Now suppose that $g(0) = 1$ so that $g(c) \neq 0$ for any $c \in \mathbb{R}$. If g is continuous at 0, then given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|h| < \delta$, then $|g(h) - 1| < \varepsilon/|g(c)|$. Since $g(c+h) - g(c) = g(c)(g(h)-1)$, it follows that $|g(c+h) - g(c)| = |g(c)||g(h) - 1| < \varepsilon$, provided $|h| < \delta$. Therefore g is continuous at c.
15. If $f(x) \geq g(x)$, then both expressions given $h(x) = f(x)$; and if $f(x) \leq g(x)$, then $h(x) = g(x)$ in both cases.

Section 5.3

In this section, we establish some very important properties of continuous functions. Unfortunately, students often regard these properties as being "obvious", so that one must convince them that if the hypotheses of the theorems are dropped, then the conclusions may not hold. Thus, for example, if any *one* of the three hypotheses [(i) I is closed, (ii) I is bounded, (iii) f is continuous at every point of I] of Theorem 5.3.2 is dropped, then the conclusion that f is bounded may not hold, even though the other two hypotheses are retained. Similarly for Theorems 5.3.4 and 5.3.9. Thus, each theorem must be accompanied by examples. In 5.3.7, we do not assume that I is a closed bounded interval, but we work within a closed bounded subinterval of I.

The proofs of Theorems 5.3.2 and 5.3.4 presented here are based on the Bolzano-Weierstrass Theorem. In Section 5.5 different proofs are presented based on the concept of a "gauge". In Chapter 11 these theorems are extended to general "compact" sets in $\mathbb{R}$ by using the Heine-Borel Theorem.

Students often misunderstand Theorems 5.3.9 and 5.3.10, believing that the image of an interval with endpoints $f(a), f(b)$. Consequently, Figure 5.3.3 should be stressed in an attempt to dispell this misconception. Also, examples can be given to show that the continuous image of an interval (a, b) can be any type of interval, and not necessarily an open interval or a bounded interval.

Sample Assignment: Exercises 1, 3, 5, 6, 7, 8, 10, 13, 15.

Partial Solutions:

1. Apply either the Boundedness Theorem 5.3.2 to $1/f$, or the Maximum-Minimum Theorem 5.3.4 to conclude that $\inf f(I) > 0$.

   Alternatively, if $x_n \in I$ such that $0 < f(x_n) < 1/n$, then there is a subsequence (x_{n_k}) that converges to a point $x_0 \in I$. Since $f(x_0) = \lim(f(x_{n_k})) = 0$, we have a contradiction.
2. If $f(x_n) = g(x_n)$ and $\lim(x_n) = x_0$, then $f(x_0) = \lim(f(x_n)) = \lim(g(x_n)) = g(x_0)$.
3. Let x_1 be arbitrary and let $x_2 \in I$ be such that $|f(x_2)| \leq \frac{1}{2}|f(x_1)|$. By Induction, choose x_{n+1} such that $|f(x_{n+1})| \leq \frac{1}{2}|f(x_n)| \leq \left(\frac{1}{2}\right)^n |f(x_1)|$. Apply the Bolzano-Weierstrass Theorem to obtain a subsequence that converges to some $c \in I$. Now show that $f(c) = 0$.

   Alternatively, show that if the minimum value of $|f|$ on I is not 0, then a contradiction arises.
4. Suppose that p has odd degree n and that the coefficient a_n of x^n is positive. By 4.3.16, we have $\lim\limits_{x\to\infty} p(x) = \infty$ and $\lim\limits_{x\to-\infty} p(x) = -\infty$. Hence $p(\alpha) < 0$ for some $\alpha < 0$ and $p(\beta) > 0$ for some $\beta > 0$. Therefore there is a zero of p in $[\alpha, \beta]$.
5. In the intervals [1.035, 1.040] and [−7.026, −7.025].
6. Note that $g(0) = f(0) - f(1/2)$ and $g(1/2) = f(1/2) - f(1) = -g(0)$. Hence there is a zero of g at some $c \in [0, 1/2]$. But if $0 = g(c) = f(c) - f(c + 1/2)$, then we have $f(c) = f(c + 1/2)$.
7. In the interval [0.7390, 0.7391].
8. In the interval [1.4687, 1.4765].
9. (a) 1, (b) 6.
10. $1/2^n < 10^{-5}$ implies that $n > (5 \ln 10)/\ln 2 \approx 16.61$. Take $n = 17$.
11. If $f(w) < 0$, then it follows from Theorem 4.2.9 that there exists a δ-neighborhood $V_\delta(w)$ such that $f(x) < 0$ for all $x \in V_\delta(w)$. But since $w < b$,

this contradicts the fact that $w = \sup W$. There is a similar contradiction if we assume that $f(w) > 0$. Therefore $f(w) = 0$.

12. Since $f(\pi/4) < 1$ while $f(0) = 1$ and $f(\pi/2) > 1$, it follows that $x_0 \in (0, \pi/2)$. If $\cos x_0 > x_0^2$, then there exists a δ-neighborhood $V_\delta(x_0) \subseteq I$ on which $f(x) = \cos x$, so that x_0 is not an absolute minimum point for f.
13. If $f(x) = 0$ for all $x \in \mathbb{R}$, then all is trivial; hence, assume that f takes on some nonzero values. To be specific, suppose $f(c) > 0$ and let $\varepsilon := \frac{1}{2} f(c)$, and let $M > 0$ be such that $|f(x)| < \varepsilon$ provided $|x| > M$. By Theorem 5.3.4, there exists $c^* \in [-M, M]$ such that $f(c^*) \geq f(x)$ for all $x \in [-M, M]$ and we deduce that $f(c^*) \geq f(x)$ for all $x \in \mathbb{R}$. To see that a minimum value need not be attained, consider $f(x) := 1/(x^2 + 1)$.
14. Apply Theorem 4.2.9 to $\beta - f(x)$.
15. If $0 < a < b \leq \infty$, then $f((a, b)) = (a^2, b^2)$; if $-\infty \leq a < b < 0$, then $f((a, b)) = (b^2, a^2)$. If $a < 0 < b$, then $f((a, b))$ is not an open interval, but equals $[0, c)$ where $c := \sup\{a^2, b^2\}$. Images of closed intervals are treated similarly.
16. For example, if $a < 0 < b$ and $c := \inf\{1/(a^2 + 1), 1/(b^2 + 1)\}$, then $g((a, b)) = (c, 1]$. If $0 < a < b$, then $g((a, b)) = (1/(b^2 + 1), 1/(a^2 + 1))$. Also $g([-1, 1]) = [1/2, 1]$. If $a < b$, then $h((a, b)) = (a^3, b^3)$ and $h((a, b]) = (a^3, b^3]$.
17. Yes. Use the Density Theorem 2.4.8.
18. If f is not bounded on I, then for each $n \in \mathbb{N}$ there exists $x_n \in I$ such that $|f(x_n)| \geq n$. Then a subsequence of (x_n) converges to $x_0 \in I$. The assumption that f is bounded on a neighborhood of x_0 leads to a contradiction.
19. Consider $g(x) := 1/x$ for $x \in J := (0, 1)$.

Section 5.4

The idea of uniform continuity is a subtle one that often causes difficulties for students. The point, of course, is that for a uniformly continuous function $f : A \to \mathbb{R}$, the δ can be chosen to depend *only* on ε and *not* on the points in A. The Uniform Continuity Theorem 5.4.3 guarantees that every continuous function on a closed bounded interval is uniformly continuous; however, a continuous function defined on an interval may be uniformly continuous even when the interval is not closed and bounded. For example, every Lipschitz function is uniformly continuous, no matter what the nature of its domain is. A condition for a function to be uniformly continuous on a bounded open interval is given in 5.4.8. The extension of the Uniform Continuity Theorem to compact sets in given in Chapter 11.

One interesting application of uniform continuity is the approximation of continuous functions by "simpler" functions. Consequently we have included a brief discussion of this topic here. The Weierstrass Approximation Theorem 5.4.14 is a fundamental result in this area and we have stated it without proof.

Sample Assignment: Exercises 1, 2, 3, 6, 7, 8, 11, 12, 15.

Partial Solutions:

1. Since $1/x - 1/u = (u-x)/xu$, it follows that $|1/x - 1/u| \le (1/a^2)|x-u|$ for $x, u \in [a, \infty)$.
2. If $x, u \ge 1$, then $|1/x^2 - 1/u^2| = (1/x^2u + 1/xu^2)|x-u| \le 2|x-u|$, and it follows that f is uniformly continuous on $[1, \infty)$. If $x_n := 1/n, u_n := 1/(n+1)$, then $|x_n - u_n| \to 0$ but $|f(x_n) - f(u_n)| = 2n+1 \ge 1$ for all n, so f is not uniformly continuous on $(0, \infty)$.
3. (a) Let $x_n := n + 1/n, u_n := n$. Then $|x_n - u_n| \to 0$, but $f(x_n) - f(u_n) = 2 + 1/n^2 \ge 2$ for all n.
(b) Let $x_n := 1/2n\pi, u_n := 1/(2n\pi + \pi/2)$. Note that $|g(x_n) - g(u_n)| = 1$ for all n.
4. Show that $|f(x) - f(u)| \le [(|x|+|u|)/(1+x^2)(1+u^2)]|x-u| \le (1/2+1/2)|x-u| = |x-u|$. (Note that $x \mapsto x/(1+x^2)$ attains a maximum of $1/2$ at $x = 1$.)
5. Note that $|(f(x)+g(x)) - (f(u)+g(u))| \le |f(x) - f(u)| + |g(x) - g(u)| < \varepsilon$ provided that $|x-u| < \inf\{\delta_f(\varepsilon/2), \delta_g(\varepsilon/2)\}$.
6. If M is a bound for both f and g on A, show that $|f(x)g(x) - f(u)g(u)| \le M|f(x) - f(u)| + M|g(x) - g(u)|$ for all $x, u \in A$.
7. Since $\lim\limits_{x\to 0}(\sin x)/x = 1$, there exists $\delta > 0$ such that $\sin x \ge x/2$ for $0 \le x < \delta$. Let $x_n := 2n\pi$ and $u_n := 2n\pi + 1/n$, so that $\sin x_n = 0$ and $\sin u_n = \sin(1/n)$. If $h(x) := x \sin x$, then $|h(x_n) - h(u_n)| = u_n \sin(1/n) \ge (2n\pi + 1/n)/2n > \pi > 0$ for sufficiently large n.
8. Given $\varepsilon > 0$ there exists $\delta_f > 0$ such that $|y - v| < \delta_f$ implies $|f(y) - f(v)| < \varepsilon$. Now choose $\delta_g > 0$ so that $|x - u| < \delta_g$ implies $|g(x) - g(u)| < \delta_f$.
9. Note that $|1/f(x) - 1/f(u)| \le (1/k^2)|f(x) - f(u)|$.
10. There exists $\delta > 0$ such that if $|x-u| < \delta, x, u \in A$, then $|f(x) - f(u)| < 1$. If A is bounded, it is contained in the finite union of intervals of length δ.
11. If $|g(x) - g(0)| \le K|x - 0|$ for all $x \in [0,1]$, then $\sqrt{x} \le Kx$ for $x \in [0,1]$. But if $x_n := 1/n^2$, then K must satisfy $n \le K$ for all $n \in \mathbb{N}$, which is impossible.
12. Given $\varepsilon > 0$, choose $0 < \delta_1 < 1$ so that $|f(x) - f(u)| < \varepsilon$ whenever $|x-u| < \delta_1$ and $x, u \in [0, a+1]$. Also choose $0 < \delta_2 < 1$ so that $|f(x) - f(u)| < \varepsilon$ whenever $|x-u| < \delta_2$ and $x, u \in [a, \infty)$. Now let $\delta := \inf\{\delta_1, \delta_2\}$. If $|x-u| < \delta$, then since $\delta < 1$, either $x, u \in [0, a+1]$ or $x, u, \in [a, \infty)$, so that $|f(x) - f(u)| < \varepsilon$.
13. Note that $|f(x) - f(u)| \le |f(x) - g_\varepsilon(x)| + |g_\varepsilon(x) - g_\varepsilon(u)| + |g_\varepsilon(u) - f(u)|$.
14. Since f is bounded on $[0, p]$, it follows that it is bounded on $\mathbb{R}$. Since f is continuous on $J := [-1, p+1]$, it is uniformly continuous on J. Now show that this implies that f is uniformly continuous on $\mathbb{R}$.

15. Assume $|f(x)-f(y)| \le K_f|x-y|$ and $|g(x)-g(y)| \le K_g|x-y|$ for all x, y in A. (a) $|(f(x)+g(x))-(f(y)+g(y))| \le |f(x)-f(y)|+|g(x)-g(y)| \le (K_f+K_g)|x+y|$.
(b) If $|f(x)| \le B_f$ and $|g(x)| \le B_g$ for all x in A, then

$$\begin{aligned}|f(x)g(x)-f(y)g(y)| &= |f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y)| \\ &\le B_f|g(x)-g(y)|+B_g|f(x)-f(y)| \\ &\le (B_fK_g+B_gK_f)|x-y|.\end{aligned}$$

(c) Consider $f(x)=x$.

16. If $|f(x)-f(y)| \le K|x-y|$ for all x, y in I, then has Lipschitz constant K on I. Then for disjoint subintervals $[x_k, y_k], n=1,2,\ldots,n$, we have $\Sigma|f(x_k)-f(y_k)| \le \Sigma K|x_k-y_k|$, so that if $\varepsilon>0$ is given and $\delta=\varepsilon/nK$, then $\Sigma|f(x_k)-f(y_k)| \le \varepsilon$. Thus f is absolutely continuous on I.

Section 5.5

In this section we introduce the notion of a "gauge" which will be used in the development of the generalized Riemann integral in Chapter 10. We will also use gauges to give alternate proofs of the main theorems in Section 5.3 and 5.4, Dini's Theorem 8.2.6, and the Lebesgue Integrability Criterion in Appendix C.

Sample Assignment: Exercises 1, 2, 4, 6, 7, 9.

Partial Solutions:

1. (a) The δ-intervals are $[0-\frac{1}{4}, 0+\frac{1}{4}]=[-\frac{1}{4}, \frac{1}{4}], [\frac{1}{2}-\frac{1}{4}, \frac{1}{2}+\frac{1}{4}]=[\frac{1}{4}, \frac{3}{4}]$ and $[\frac{3}{4}-\frac{3}{8}, \frac{3}{4}+\frac{3}{8}]=[\frac{3}{8}, \frac{9}{8}]$.
(b) The third δ-interval is $[\frac{3}{10}, \frac{9}{10}]$ which does not contain $[\frac{1}{2}, 1]$.
2. (a) Yes. Since $\delta(t) \le \delta_1(t)$, every δ-fine partition is δ_1-fine.
(b) Yes. The third δ_1-interval is $[\frac{3}{20}, \frac{21}{20}]$ which contains $[\frac{1}{2}, 1]$.
3. No. The first δ_2-interval is $[-\frac{1}{10}, \frac{1}{10}]$ and does not contain $[0, \frac{1}{4}]$.
4. (b) If $t \in (\frac{1}{2}, 1)$ then $[t-\delta(t), t+\delta(t)]=[-\frac{1}{2}+\frac{3}{2}t, \frac{1}{2}+\frac{1}{2}t] \subset (\frac{1}{4}, 1)$.
5. Routine verification.
6. We could have two subintervals having c as a tag with one of them not contained in the δ-interval around c. Consider constant gauges $\delta' := 1$ on [0, 1] and $\delta'' := \frac{1}{2}$ on [1, 2], so that $\delta(1)=\frac{1}{2}$. If $\dot{\mathcal{P}}'$ consists of the single pair $([0,1], 1)$, it is δ'-fine. However, $\dot{\mathcal{P}}'$ is not δ-fine.
7. Clearly $\delta^*(t)>0$ so that δ^* is a gauge on $[a, b]$. If

$$\dot{\mathcal{P}} := \{([a, x_1], t_1), \ldots ([x_{k-1}, c], t_k), ([c, x_{k+1}], t_{k+1}), \ldots, ([x_n, b], t_n)\}$$

is δ^*-fine, then it is clear that $\dot{\mathcal{P}}' := \{([a, x_1], t_1), \ldots, ([x_{k-1}, c], t_k)\}$ is a δ'-fine partition of $[a, c]$ and $\dot{\mathcal{P}}'' := \{([c, x_{k+1}], t_{k+1}), \ldots, ([x_n, b], t_n)\}$ is a δ''-fine partition of $[c, b]$. Evidently $\dot{\mathcal{P}} = \dot{\mathcal{P}}' \cup \dot{\mathcal{P}}''$.

8. (a) If $[a, b]$ a δ-fine partition $\dot{\mathcal{P}}'$ and $[c, b]$ has a δ-fine partition $\dot{\mathcal{P}}''$, then $\dot{\mathcal{P}}' \cup \dot{\mathcal{P}}''$ is a δ-fine partition of $[a, b]$, contrary to hypothesis.
   (b) Let I_1 be $[a, c]$ if it does not have a δ-fine partition; otherwise, let I_1 be $[c, b]$, so the length of I_1 is $(b-a)/2$. Now bisect I_1 and let I_2, which has length $(b-a)/2^2$, be an interval that has no δ-fine partition. Continue this process by Induction.
   (c) By the Nested Intervals Theorem there exists a common point ξ. By the Archimedean Property there exists $p \in \mathbb{N}$ such that $(b-a)/2^p < \delta(\xi)$. Since $\xi \in I_p$ and the length of I_p is $(b-a)/2^p$, it follows that $I_p \subset [\xi - \delta(\xi), \xi + \delta(\xi)]$.
9. The hypothesis that f is locally bounded presents us with a gauge δ. If $\{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a δ-fine partition of $[a, b]$ and M_i is a bound for $|f|$ on $[x_{i-1}, x_i]$, let $M := \sup\{M_i : i = 1, \ldots, n\}$.
10. The hypothesis that f is locally increasing presents us with a gauge δ. If $\{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a δ-fine partition of $[a, b]$, then f is increasing on each interval $[x_{i-1}, x_i]$. By Induction it follows that $f(x_i) \le f(x_j)$ for $i < j$. If $x < y$ belong to $[a, b]$, then $x \in [x_{i-1}, x_i]$ and $y \in [x_{j-1}, x_j]$ where $i \le j$. If $i = j$, the fact that f is increasing on $[x_{i-1}, x_i]$ implies that $f(x) \le f(y)$. If $i < j$, then $f(x) \le f(x_i) \le f(x_{j-1}) \le f(y)$.

Section 5.6

The collection of monotone functions is a special, but very useful class of functions. This is particularly the case since most functions that arise in elementary analysis are either monotone, or their domains can be written as a union of intervals on which their restrictions are monotone. Theorem 5.6.4 shows that a monotone function is automatically continuous except (at most) at a countable set of points.

It will also be seen in Theorem 5.6.5 that continuous strictly monotone functions have continuous strictly monotone inverse functions.

Sample Assignment: Exercises 1, 2, 4, 5, 7, 10, 12.

Partial Solutions:

1. If $x \in [a, b]$, then $f(a) \le f(x)$.
2. If $x_1 \le x_2$, then $f(x_1) \le f(x_2)$ and $g(x_1) \le g(x_2)$, whence $f(x_1) + g(x_1) \le f(x_2) + g(x_2)$.
3. Note that $(fg)(0) = 0 > (fg)(1/2) = -1/4$.
4. If $0 \le f(x_1) \le f(x_2)$ and $0 \le g(x_1) \le g(x_2)$, then $f(x_1)g(x_1) \le f(x_2)g(x_1) \le f(x_2)g(x_2)$.

5. If $L := \inf\{f(x) : x \in (a,b]\}$ and $\varepsilon > 0$, then there exists $x_\varepsilon \in (a,b]$ with $L \leq f(x_\varepsilon) < L + \varepsilon$. Since f is increasing, then $L \leq f(x) < L + \varepsilon$ for $x \in (a, x_\varepsilon]$; hence $\lim\limits_{x \to a+} f$ exists and equals L.

   Conversely, if $K := \lim\limits_{x \to a+} f$, then given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in (a, a+\delta)$, then $K - \varepsilon < f(x) < K + \varepsilon$. It follows from this that $K - \varepsilon \leq L < K + \varepsilon$; since $\varepsilon > 0$ is arbitrary, we have $K = L$.

6. If f is continuous at c, then $\lim(f(x_n)) = f(c)$, since $c = \lim(x_n)$. Conversely, since $0 \leq j_f(c) \leq f(x_{2n}) - f(x_{2n+1})$, it follows that $j_f(c) = 0$, so f is continuous at c.

7. It follows from Exercises 2.4.4, 2.4.6 and the Principle of the Iterated Infima, (analogous to the result in Exercise 2.4.12), that

$$\begin{aligned} j_f(c) &= \inf\{f(y) : y \in I, c < y\} - \sup\{f(x) : x \in I, x < c\} \\ &= \inf\{f(y) : y \in I, c < y\} + \inf\{-f(x) : x \in I, x < c\} \\ &= \inf\{f(y) - f(x) : x, y \in I, x < c < y\} \end{aligned}$$

8. Let $x_1 \in I$ be such that $y = f(x_1)$ and $x_2 \in I$ be such that $y = g(x_2)$. If $x_2 \leq x_1$, then $y = g(y_2) < f(x_2) \leq f(x_1) = y$, a contradiction.

9. If $x \in I$ is rational, then $f(x) = x$ is also rational so $f(f(x)) = f(x) = x$; if $y \in I$ is irrational, then $f(y) = 1 - y$ is irrational so $f(f(y)) = f(1-y) = 1 - (1-y) = y$. Suppose that $x_1 \neq x_2, x_j \in I$; if $x_1 \in \mathbb{Q}$ and $x_2 \notin \mathbb{Q}$, then $f(x_1) \in \mathbb{Q}$ and $f(x_2) \notin \mathbb{Q}$, which implies that $f(x_1) \neq f(x_2)$. The other cases are similar. Since $|f(x) - 1/2| = |x - 1/2|$, then f is continuous at $1/2$. If $x \neq 1/2, x \in \mathbb{Q}$, take a sequence (y_n) of irrationals converging to x, so that $f(y_n) = 1 - y_n \to 1 - x \neq x$. Similarly for the case $x \neq 1/2, x \notin \mathbb{Q}$.

10. If f has an absolute maximum at $c \in (a,b)$, and if f is injective, we have $f(a) < f(c)$ and $f(b) < f(c)$. Either $f(a) \leq f(b)$ or $f(b) < f(a)$. In the first case, either $f(a) = f(b)$ or $f(a) < f(b) < f(c)$, whence there exists $b' \in (a,c)$ such that $f(b') = f(b)$. Either possibility contradicts the assumption that f is injective. The case $f(b) < f(a)$ is similar.

11. Note that f^{-1} is continuous at every point of its domain $[0,1] \cup (2,3]$. The function f is not continuous at $x = 1$.

12. Let $a \in (0,1)$ be arbitrary. If $f(a) < f(0)$, then there exists $a' \in (a,1)$ with $f(a') = f(0)$, a contradiction. Also $f(a) = f(0)$ is excluded by hypothesis. Therefore we must have $f(0) < f(a)$, and a similar argument yields $f(a) < f(1)$. If $b \in (a,1)$ is given, then $f(b) < f(a)$ implies that there exists $a'' \in (b,1)$ with $f(a) = f(a'')$, a contradiction. Since $f(b) = f(a)$ is excluded, we must have $f(b) > f(a)$.

13. Assume that h is continuous on $[0,1]$ and let $c_1 < c_2$ be the two points in $[0,1]$ where h attains its supremum. If $0 < c_1$, choose a_1, a_2 such that $0 < a_1 < c_1 < a_2 < c_2$. Let k satisfy $\sup\{h(a_1), h(a_2)\} < k < h(c_1) = h(c_2)$; then there exist

three numbers b_j such that $a_1 < b_1 < c_1 < b_2 < a_2 < b_3 < c_2$ where $k = h(b_j)$, a contradiction. Now consider the points where h attains its infimum.

14. Let $x > 0$ and consider the case $m, p, n, q \in \mathbb{N}$. Let $y := x^{1/n}$ and $z := x^{1/q}$ so that $y^n = x = z^q$, whence (by Exercise 2.1.26) $y^{np} = x^p = z^{qp}$. Since $np = mq$, we have $(y^m)^q = y^{mq} = z^{pq} = (z^p)^q$, from which it follows that $y^m = z^p$, or $(x^{1/n})^m = (x^{1/q})^p$, or $x^{m/n} = x^{p/q}$. Now consider the case where $m, p \in \mathbb{Z}$.
15. Let $x > 0$ and consider the case where $r = m/n$ and $s = p/q$, where $m, n, p, q \in \mathbb{N}$. Since $r = mq/nq$ and $s = pn/qn$, it follows from the preceding exercise that $x^r = (x^{1/nq})^{mq}$ and $x^s = (x^{1/nq})^{pn}$ so that (by Exercise 2.1.26) $x^r x^s = (x^{1/nq})^{mq+pn} = x^{(mq+pn)/nq} = x^{r+s}$. Similarly, $x^r = (x^{1/n})^m > 0$ and if $y > 0$, then (by 5.6.7) $y^s = (y^p)^{1/q}$ so that $(x^r)^s = (((x^{1/n})^m)^p)^{1/q}$. This implies that $((x^r)^s)^q = (x^{1/n})^{mp} = (x^{mp})^{1/n}$ so that $((x^r)^s)^{qn} = x^{mp}$, whence $(x^r)^s = x^{mp/qn} = x^{rs}$. Now consider the case where $m, p \in \mathbb{Z}$.

CHAPTER 6

DIFFERENTIATION

The basic properties and applications of the derivative are given in the first two sections of this chapter. Section 6.1 is a survey of the techniques of differentiation from a rigorous viewpoint. Since the students will be familiar with most of the results (though not the proofs), the section can be covered reasonably quickly. Section 6.2 contains material that is new to students, since in introductory calculus courses the Mean Value Theorem is not usually given the emphasis it deserves. Sections 6.3 and 6.4 are optional and can be discussed in either order and to whatever depth that time permits.

Section 6.1

This section contains the calculational rules of differentiation that students learn and use in introductory calculus courses. However, the emphasis here is on the rigorous establishment of these results rather than on the development of calculational skills.

The topic that students will find troublesome is the differentiation of composite and inverse functions. We feel that the use of Carathéodory's Theorem 6.1.5 is a considerable simplification of the proofs of these results.

Sample Assignment: Exercises 1(a,b), 2, 4, 5, 9, 11, 13, 15.

Partial Solutions:

1. (a) $f'(x) = \lim_{h\to 0}[(x+h)^3 - x^3]/h = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$,

   (b) $g'(x) = \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{1}{x+h} - \dfrac{1}{x}\right) = \lim_{h\to 0} \dfrac{-1}{x(x+h)} = \dfrac{-1}{x^2}$,

   (c) $h'(x) = \lim_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$,

   (d) $$k'(x) = \lim_{h\to 0} \frac{1/\sqrt{x+h} - 1/\sqrt{x}}{h} = \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x+h} + \sqrt{x})} = \frac{-1}{2x\sqrt{x}}.$$

2. $\lim_{x\to 0}(f(x) - f(0))/(x-0) = \lim_{x\to 0} x^{1/3}/x = \lim_{x\to 0} 1/x^{2/3}$ does not exist.

3. (a) $(\alpha f)'(c) = \lim_{x\to c} \dfrac{\alpha f(x) - \alpha f(c)}{x-c} = \alpha \lim_{x\to c} \dfrac{f(x) - f(c)}{x-c} = \alpha f'(c)$,

   (b) $$(f+g)'(c) = \lim_{x\to c} \frac{(f(x)+g(x)) - (f(c)+g(c))}{x-c} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c} = f'(c) + g'(c).$$

4. Note that $|f(x)/x| \le |x|$ for $x \in \mathbb{R}$.
5. (a) $f'(x) = (1-x^2)/(1+x^2)^2$,
   (b) $g'(x) = (x-1)/\sqrt{5-2x+x^2}$,
   (c) $h'(x) = m(\sin x^k)^{m-1}(\cos x^k)(kx^{k-1})$,
   (d) $k'(x) = 2x\sec^2(x^2)$.
6. The function f' is continuous for $n \ge 2$ and is differentiable for $n \ge 3$.
7. By definition $g'(c) = \lim\limits_{h\to 0} |f(c+h)|/h$, if this limit exists. If $0 = |f'(c)| = \lim\limits_{h\to 0} |f(c+h)/h|$, it follows that $g'(c) = 0$. If $f'(c) = L \ne 0$, then we have $\lim(f(c \pm 1/n)/(\pm 1/n)) = L$, while $\lim(|f(c \pm 1/n)|/(\pm 1/n))] = \pm L$, so that $|f|'(c)$ does not exist.
8. (a) $f'(x) = 2$ for $x > 0$; $f'(x) = 0$ for $-1 < x < 0$; and $f'(x) = -2$ for $x < -1$,
   (b) $g'(x) = 3$ if $x > 0$; $g'(x) = 1$ if $x < 0$; $g'(0)$ does not exist,
   (c) $h'(x) = 2|x|$ for all $x \in \mathbb{R}$,
   (d) $k'(x) = (-1)^n \cos x$ for $n\pi < x < (n+1)\pi, n \in \mathbb{Z}; k'(n\pi)$ does not exist,
   (e) $p'(0) = 0$; if $x \ne 0$, then $p'(x)$ does not exist.
9. If f is an even function, then $f'(-x) = \lim\limits_{h\to 0}[f(-x+h) - f(-x)]/h = -\lim\limits_{h\to 0}[f(x-h) - f(x)]/(-h) = -f'(x)$.
10. If $x \ne 0$, then $g'(x) = 2x\sin(1/x^2) - (2/x)\cos(1/x^2)$. Moreover, $g'(0) = \lim\limits_{h\to 0} h\ \sin(1/h^2) = 0$. If $x_n := 1/\sqrt{2n\pi}$, then $x_n \to 0$ and $|g'(x_n)| = 2\sqrt{2n\pi}$, so g' is unbounded in every neighborhood of 0.
11. (a) $f'(x) = 2/(2x+3)$, (b) $g'(x) = 6(L(x^2))^2/x$,
    (c) $h'(x) = 1/x$, (d) $k'(x) = 1/(xL(x))$.
12. $r > 1$.
13. Many examples are possible. For example, let $f(x) := x$ for x rational and $f(x) := 0$ for x irrational.
14. $1/h'(0) = 1/2$, $1/h'(1) = 1/5$ and $1/h'(-1) = 1/5$.
15. $D[\text{Arccos } y] = 1/D[\cos x] = -1/\sin x = -1/\sqrt{1-y^2}$.
16. $D[\text{Arctan } y] = 1/D[\tan x] = 1/\sec^2 x = 1/(1+y^2)$.
17. Given $\varepsilon > 0$, let $\delta(\varepsilon) > 0$ be such that if $0 < |w-c| < \delta(\varepsilon), w \in I$, then $|f(w) - f(c) - (w-c)f'(c)| < \varepsilon|w-c|$. Now take $w = u$ and $w = v$ as described and subtract and add the term $f(c) - f'(c)c$ and use the Triangle Inequality to get

$$\begin{aligned}|f(v) - f(u) - f'(c)(v-u)| &\le |f(v) - f(c) - f'(c)(v-c)| \\ &+ |f(c) - f(u) - f'(c)(c-u)| \le \varepsilon|v-c| + \varepsilon|c-u|.\end{aligned}$$

Since $v-c \geq 0$ and $c-u \geq 0$, then $|v-c| = v-c$ and $|c-u| = c-u$, so that the final term equals $\varepsilon(v-c+c-u) = \varepsilon(v-u)$.

Section 6.2

The Mean Value Theorem is stated for a function f on an interval $[a,b]$. However, many of its applications use intervals of the form $[a,x]$ or $[x_1,x_2]$ where x or x_1, x_2 are points in $[a,b]$. The shift from a "fixed interval" to what seems to be a "variable interval" can cause confusion for some students. A word of explanation when this first occurs will help to alleviate this confusion.

WARNING: Exercises 16 and 18 are rather difficult.

Sample Assignment: Exercises 2(a, b), 3(a, b), 6, 7, 9, 10, 12, 13, 17.

Partial Solutions:

1. (a) Increasing on $[3/2,\infty)$, decreasing on $(-\infty,3/2]$,
(b) Increasing on $(-\infty,3/8]$, decreasing on $[3/8,\infty)$,
(c) Increasing on $(-\infty,-1]$ and $[1,\infty)$,
(d) Increasing on $[0,\infty)$.
2. (a) $f'(x) = 1-1/x^2$. Relative minimum at $x=1$; relative maximum at $x=-1$,
(b) $g'(x) = (1+x)(1-x)/(1+x^2)^2$. Relative minimum at $x=-1$; relative maximum at $x=1$,
(c) $h'(x) = 1/2\sqrt{x} - 1/\sqrt{x+2}$. Relative maximum at $x = 2/3$,
(d) $k'(x) = 2(x^3-1)/x^3$. Relative minimum at $x=1$.
3. (a) Relative minima at $x = \pm 1$; relative maxima at $x = 0, \pm 4$,
(b) Relative maximum at $x=1$; relative minima at $x = 0, 2$,
(c) Relative minima at $x = -2, 3$; relative maximum at $x = 2$,
(d) $k'(x) = 4(x-6)/3(x-8)^{2/3}$. Relative minimum at $x=6$; relative maxima at $x = 0, 9$.
4. $x = (1/n)(a_1 + \cdots + a_n)$.
5. Show that $f'(x) < 0$ for $x > 1$. Then f is strictly decreasing on $[1,\infty)$ so that $f(a/b) < f(1)$ for $a > b > 0$.
6. If $x < y$, there exists c in (x,y) such that $|\sin x - \sin y| = |\cos c||y-x|$.
7. There exists c with $1 < c < x$ such that $\ln x = (x-1)/c$. Now use the inequality $1/x < 1/c < 1$.
8. If $h > 0$ and $a+h < b$, there exists $c_h \in (a, a+h)$ such that $f(a+h) - f(a) = hf'(c_h)$. Since $c_h \to a$ as $h \to 0+$, it follows that $f'(a) = \lim\limits_{h\to 0+} [f(a+h) - f(a)]/h = \lim\limits_{h \to 0+} f'(c_h) = A$. Now consider $h < 0$.
9. $f(x) = x^4(2 + \sin(1/x)) > 0$ for all $x \neq 0$, so f has an absolute minimum at $x=0$. We have $f'(x) = 8x^3 + 4x^3 \sin(1/x) - x^2\cos(1/x)$ for $x \neq 0$. Now verify that $f'(1/2n\pi) < 0$ for $n \geq 2$ and $f'(2/(4n+1)\pi) > 0$ for $n \geq 1$.

10. $g'(0) = \lim_{x\to 0}(1 + 2x\sin(1/x)) = 1 + 0 = 1$, and if $x \neq 0$, then $g'(x) = 1 + 4x\sin(1/x) - 2\cos(1/x)$. Now show that $g'(1/2n\pi) < 0$ and that we have $g'(2/(4n+1)\pi) > 0$ for $n \in \mathbb{N}$.

11. For example, $f(x) := \sqrt{x}$.

12. Apply Darboux's Theorem 6.2.12. If $g(x) := a$ for $x < 0, g(x) := x + b$ for $x \geq 0$, where a, b are any constants, then $g'(x) = h(x)$ for $x \neq 0$.

13. If $x_1 < x_2$, then there exists $c \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(c) > 0$.

14. Apply Darboux's Theorem 6.2.12.

15. Suppose that $|f'(x)| \leq K$ for $x \in I$. For $x, y \in I$, apply the Mean Value Theorem to get $|f(x) - f(y)| = |(x-y)f'(c)| \leq K|x-y|$.

16. (a) Given $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that if $x \geq n_\varepsilon$, then $|f'(x) - b| < \varepsilon$. Hence if $x \geq n_\varepsilon$ and $h > 0$, there exists $y_x \in (x, x+h)$ such that

$$\left| \frac{f(x+h) - f(x)}{h} - b \right| = |f'(y_x) - b| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $\lim_{x\to\infty}(f(x+h) - f(x))/h = b$.
(b) Assume that $b \neq 0$ and let $\varepsilon < |b|/2$. Let n_ε be as in part (a). Since $\lim_{x\to\infty} f$ exists, we may also assume that if $x, y \geq n_\varepsilon$, then $|f(x) - f(y)| < \varepsilon$. Hence there exists x_ε in $(n_\varepsilon, n_\varepsilon + 1)$ such that $\varepsilon > |f(n_\varepsilon + 1) - f(n_\varepsilon)| = |f'(x_\varepsilon)| \geq |b|/2$. Since $\varepsilon > 0$ is arbitrary, the hypothesis that $b \neq 0$ is contradicted.
(c) If $x \geq n_\varepsilon$, then there exists $y_\varepsilon \in (n_\varepsilon, x)$ such that $f(x) - f(n_\varepsilon) = (x - n_\varepsilon)f'(y_\varepsilon)$, so that we have

$$f(x)/x - b = f'(y_\varepsilon) - b + f(n_\varepsilon)/x - n_\varepsilon f'(y_\varepsilon)/x.$$

Since $y_\varepsilon > n_\varepsilon$, we have $|f'(y_\varepsilon) - b| < \varepsilon$. Moreover $|f(n_\varepsilon)/x| < \varepsilon$ if x is sufficiently large; since f' is bounded on $[n_\varepsilon, \infty)$, then $|n_\varepsilon f'(y_\varepsilon)/x| < \varepsilon$ if x is sufficiently large. Therefore, $\lim_{x\to\infty} f(x)/x = b$.

17. Apply the Mean Value Theorem to the function $g - f$ on $[0, x]$.

18. Given $\varepsilon > 0$, let $\delta = \delta(\varepsilon)$ be as in Definition 6.1.1, and let $x < c < y$ be such that $0 < |x - y| < \delta$. Since $f(x) - f(y) = f(x) - f(c) + f(c) - f(y)$, a simple calculation shows that

$$\frac{f(x) - f(y)}{x - y} = \frac{x - c}{x - y} \cdot \frac{f(x) - f(c)}{x - c} + \frac{c - y}{x - y} \cdot \frac{f(y) - f(c)}{y - c}.$$

Since both $(x-c)/(x-y)$ and $(c-y)/(x-y)$ are positive and have sum 1, it follows that

$$\begin{aligned}&\left|\frac{f(x)-f(y)}{x-y}-f'(c)\right|\\ &\quad\le\left(\frac{x-c}{x-y}\right)\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|+\left(\frac{c-y}{x-y}\right)\left|\frac{f(y)-f(c)}{y-c}-f'(c)\right|\\ &\quad<\left(\frac{x-c}{x-y}+\frac{c-y}{x-y}\right)\varepsilon=\varepsilon.\end{aligned}$$

Note that if one (but not both) of x and y equal c, the conclusion still holds.

19. Let $x,y\in I, x\neq y$; then

$$f'(x)-f'(y)=f'(x)-\frac{f(x)-f(y)}{x-y}+\frac{f(x)-f(y)}{x-y}-f'(y),$$

so that

$$\left|f'(x)-f'(y)\right|\le\left|f'(x)-\frac{f(x)-f(y)}{x-y}\right|+\left|\frac{f(x)-f(y)}{x-y}-f'(y)\right|.$$

If f is uniformly differentiable on I, given $\varepsilon>0$ there exists $\delta>0$ such that if $0<|x-y|<\delta, x,y\in I$, then both terms on the right side are less than ε. Hence we have $|f'(x)-f'(y)|<2\varepsilon$ for $|x-y|<\delta,\ x,y\in I$, whence we conclude that f' is (uniformly) continuous on I.

20. (a,b) Apply the Mean Value Theorem.
(c) Apply Darboux's Theorem to the results of (a) and (b).

Section 6.3

The proofs of the various cases of L'Hospital's Rules range from fairly trivial to rather complicated. The only really difficult argument in this section is the proof of Theorem 6.3.5, which deals with the case ∞/∞. This requires a more subtle analysis than the other cases.

This section may be regarded as optional. Students are already familiar with the mechanics of L'Hospital's Rules.

Sample Assignment: Exercises 1, 2, 4, 6, 7(a,b), 8(a,b), 9(a,b), 13, 14.

Partial Solutions:

1. $A=B(\lim_{x\to c}f(x)/g(x))=0$.
2. If $A>0$, then f is positive on a neighborhood of c and $\lim_{x\to c}(g(x)/f(x))=0$. Since $g(x)/f(x)>0$ on a neighborhood of c, we use the fact that $f(x)/g(x)=1/[g(x)/f(x)]$ to get a limit of ∞.

3. In fact, if $x \in (0,1]$ then $f(x)/g(x) = \sin(1/x)$, which does not have a limit at 0. [Note that 6.3.1 cannot applied be applied since $g'(0)=0$, and 6.3.3 cannot be applied since $f'(x)/g'(x)$ does not have a limit at 0.]
4. Note that $f'(0)=0$, but that $f'(x)$ does not exist if $x \neq 0$.
5. Recall that $\lim_{x\to 0}(\sin x)/x = 1$, but that $\lim_{x\to 0}\cos(1/x)$ does not exist.
6. (a) $\displaystyle\lim_{x\to 0}\frac{e^x - e^{-x} - 2}{1-\cos x} = \lim_{x\to 0}\frac{e^x + e^{-x}}{\sin x} = \lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x} = 2.$

   (b) $\displaystyle\lim_{x\to 0}\frac{x^2 - \sin^2 x}{x^4} = \lim_{x\to 0}\frac{2x - 2\sin x\cos x}{4x^3} = \lim_{x\to 0}\frac{2x - \sin 2x}{4x^3}$

   $$= \lim_{x\to 0}\frac{1-\cos 2x}{6x^2} = \lim_{x\to 0}\frac{2\sin 2x}{12x} = \lim_{x\to 0}\frac{4\cos 2x}{12} = \frac{1}{3}.$$
7. (a) 1, (b) 1, (c) 0, (d) 1/3.
8. (a) 1, (b) ∞, (c) 0, (d) 0.
9. (a) 0, (b) 0, (c) 0, (d) 0.
10. (a) 1, (b) 1, (c) e^3, (d) 0.
11. (a) 1, (b) 1, (c) 1, (d) 0.
12. Let $h(x) := e^x f(x)$. Then $h'(x) = e^x(f(x)+f'(x))$, so that $\lim_{x\to\infty} h'(x)/e^x = \lim_{x\to\infty}(f(x)+f'(x)) = L$, by hypothesis. Give $\varepsilon > 0$, there exists $\alpha > 0$ such that $L - \varepsilon/2 < h'(x)/e^x < L + \varepsilon/2$ for all $x > \alpha$. If $\alpha < y < x$, then by 6.3.2 there exists $c > \alpha$ with

$$\frac{h(x)-h(y)}{e^x - e^y} = \frac{h'(c)}{e^c},$$

and therefore $L-\varepsilon/2 < (h(x)-h(y))/(e^x-e^y) < L+\varepsilon/2$. But since $e^x - e^y > 0$, this implies that

$$\frac{e^x - e^y}{e^x}\cdot(L-\varepsilon/2) < \frac{h(x)-h(y)}{e^x} < \frac{e^x-e^y}{e^x}\cdot(L+\varepsilon/2).$$

Add $h(y)/e^x$ to all sides and rearrange terms to get

$$(L-\varepsilon/2) + \frac{h(y) - e^y(L-\varepsilon/2)}{e^x} < \frac{h(x)}{e^x} < (L+\varepsilon/2) + \frac{h(y)-e^y(L+\varepsilon/2)}{e^x}.$$

For fixed y, we note that $\lim_{x\to\infty}[h(y) - e^y(L\pm\varepsilon/2)]/e^x = 0$. Since $h(x)/e^x = f(x)$, it follows that for sufficiently large x we have $L - \varepsilon < f(x) < L + \varepsilon$. Therefore $\lim_{x\to\infty} f(x) = L$, which implies that $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}(f(x)+f'(x)) - \lim_{x\to\infty} f(x) = L - L = 0$.

[Note. If e^x is replaced by a function $g(x)$ such that $g'(x) > 0$ for large values of x, then the above argument can be modified slightly to prove the following version of L'Hospital's Rule: If h and g are differentiable functions on $(0,\infty)$ that satisfy $\lim\limits_{x\to\infty} h'(x)/g'(x)=L$ and $\lim\limits_{x\to\infty} g'(x)=\infty$, then $\lim\limits_{x\to\infty} h(x)=L$.]

13. The limit is 1.
14. $\displaystyle\lim_{x\to c}\frac{x^c-c^x}{x^x-c^c}=\lim_{x\to c}\frac{cx^{c-1}-(\ln c)c^x}{(1+\ln x)x^x}=\frac{1-\ln c}{1+\ln c}.$

Section 6.4

The applications of Taylor's Theorem are similar in spirit to those of the Mean Value Theorem, but the technical details can be more complicated since higher order derivatives are involved. Instead of estimating f', the use of Taylor's Theorem usually requires the estimation of the remainder R_n.

The applications that are presented here are independent of one another and they need not all be covered to illustrate the use and importance of Taylor's Theorem. If Newton's Method is discussed, students should be encouraged to program the algorithm on a computer or programmable calculator; comparison of the rate of convergence with the bisection method of locating roots is instructive.

Sample Assignment: Exercises 1, 2, 4, 5, 7, 8, 12, 14(a,b), 19, 20, 23.

Partial Solutions:

1. $f^{(2n-1)}(x)=(-1)^n a^{2n-1}\sin ax$ and $f^{(2n)}(x)=(-1)^n a^{2n}\cos ax$ for $n\in\mathbb{N}$.
2. $g'(x)=3x^2$ for $x\ge 0$, $g'(x)=-3x^2$ for $x<0$, and $g''(x)=6|x|$ for $x\in\mathbb{R}$.
3. Use the relation: $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$ for $0\le k\le n$, where $k,n\in\mathbb{N}$.
4. Apply Taylor's Theorem to $f(x):=\sqrt{1+x}$ at $x_0:=0$ and note that $R_1(x)<0$ and $R_2(x)>0$ for $x>0$.
5. $1.095<\sqrt{1.2}<1.1$ and $1.375<\sqrt{2}<1.5$.
6. $R_2(0.2)<0.0005$ and $R_2(1)<0.0625$.
7. $R_2(x)=(1/6)(10/27)(1+c)^{-8/3}x^3<(5/81)x^3$, where $0<c<x$.
8. $R_n(x)=e^c(x-x_0)^{n+1}/(n+1)!\to 0$ as $n\to\infty$.
9. $|R_n(x)|\le |x-x_0|^n/n!\to 0$ an $n\to\infty$.
10. Use Induction to show $h^{(n)}(0)=0$ for $n\in\mathbb{N}$. If $x\ne 0$, then $h^{(n)}(x)$ is the sum of terms of the form $e^{-1/x^2}/x^k$; therefore, if $h^{(n)}(0)=0$, then $h^{(n+1)}(0)=\lim\limits_{x\to 0}h^{(n)}(x)/x=0$. Since $P_n(x)=0$ for all x and all n, while $h(x)\ne 0$ for $x\ne 0$, the remainder $R_n(x)$ cannot converge to 0 for $x\ne 0$.
11. With $n=4$, $\ln 1.5=0.40$; with $n=7$, $\ln 1.5=0.405$.
12. Use $P_6(x)$ and note that $7!=5040$.
13. For $f(x)=e^x$ at $x_0=0$, the remainder at $x=1$ satisfies the inequality $R_n(1)\le 3/(n+1)!<10^{-7}$ if $n\ge 10$. $P_{10}(1)=2.718\,281\,8$ to seven places.

14. (a) No. (b) No. (c) No. (d) Relative minimum.
15. Apply the Mean Value Theorem to f no $[a, x_0]$ and on $[b, x_0]$ get c_1 and c_2 such that $f'(c_1) = f'(c_2)$. Now apply the Mean Value Theorem to f' on $[c_1, c_2]$.
16. To obtain the formula, apply 6.3.3 and 6.3.1.
17. Apply Taylor's Theorem to f at $x_0 = c$ to get $f(x) \geq f(c) + f'(c)(x - c)$.
18. Apply Taylor's Theorem to f and then to g at $x_0 = c$. Then form the quotient and use the continuity of the nth derivatives.
19. Since $f(2) < 0$ and $f(2.2) > 0$, there is a zero of f in [2.0, 2.2]. The value of x_4 is approximately 2.094 551 5.
20. $r_1 \approx 1.452\,626\,88$ and $r_2 \approx -1.164\,035\,14$.
21. $r \approx 1.324\,717\,96$.
22. $r_1 \approx 0.158\,594\,34$ and $r_2 \approx 3.146\,193\,22$.
23. $r_1 \approx 0.5$ and $r_2 \approx 0.809\,016\,99$.
24. $r \approx 0.739\,085\,13$.

CHAPTER 7

THE RIEMANN INTEGRAL

Students will, of course, have met with the Riemann integral in calculus, although few of them would be able to define it with any precision. The approach used here is almost certain to be the same as that used in their calculus course, although it will probably come as news to the students that the subintervals in the partitions do not need to have equal length.

If the students have a strong background, it is possible to go quickly through this chapter and then discuss part of Chapter 10, dealing with the generalized Riemann integral. However, for most classes in a one semester course, all of the time available may be needed to cover this chapter. In that case, Chapter 10 can be assigned as a "extra topic" to special students.

Since the most important results in this chapter are the Fundamental Theorems given in Section 7.3, discussion should be focussed to lead to these results.

Section 7.4 is an optional section on the Darboux approach to the integral using upper and lower integrals. The relative merits of the two approaches are discussed in the introduction to the section. Section 7.5 deals with methods of approximating integrals.

Section 7.1

Most of the results will be familiar to students. Discuss the examples carefully and sample some of the proofs. Leave time for a discussion of some of the exercises.

Sample Assignment: Exercises 1(a,c), 2(a,c), 6, 9, 10, 12, 14.

Partial Solutions:

1. (a) $\|\mathcal{P}_1\| = 2$, (b) $\|\mathcal{P}_2\| = 2$ (c) $\|\mathcal{P}_3\| = 1.4$, (d) $\|\mathcal{P}_4\| = 2$.
2. (a) $0^2 \cdot 1 + 1^2 \cdot 1 + 2^2 \cdot 2 = 0 + 1 + 8 = 9$,
(b) $1^2 \cdot 1 + 2^2 \cdot 1 + 4^2 \cdot 2 = 1 + 4 + 32 = 37$,
(c) $0^2 \cdot 2 + 2^2 \cdot 1 + 3^2 \cdot 1 = 0 + 4 + 9 = 13$,
(d) $2^2 \cdot 2 + 3^2 \cdot 1 + 4^2 \cdot 1 = 8 + 9 + 16 = 33$.
3. Definition 7.1.1 requires that if $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then $|S(f;\dot{\mathcal{P}}) - L| < \varepsilon$. Therefore, if $\|\dot{\mathcal{P}}\| \le \delta_\varepsilon/2$, then $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$ so that $|S(f;\dot{\mathcal{P}}) - L| \le \varepsilon$. Hence we take $\delta'_\varepsilon := \delta_\varepsilon/2$. On the other hand, if $\|\dot{\mathcal{P}}\| \le \eta_\varepsilon$ implies that $|S(f;\dot{\mathcal{P}}) - L| \le \varepsilon$, we set $\delta_\varepsilon := (1/2)\eta_{\varepsilon/2}$. Then if $\|\dot{\mathcal{P}}\| \le \delta_\varepsilon$ then $\|\dot{\mathcal{P}}\| < \eta_{\varepsilon/2}$ so that $|S(f;\dot{\mathcal{P}}) - L| \le \varepsilon/2 < \varepsilon$.
4. (b) If $u \in U_2$, then $u \in [x_{i-1}, x_i]$ with tag $t_i \in [1, 2]$, so that (i) $x_{i-1} \le t_i \le 2$ which implies that $u \le x_i \le x_{i-1} + \|\dot{\mathcal{P}}\| \le 2 + \|\dot{\mathcal{P}}\|$ and (ii) $1 \le t_i \le x_i$ which implies that $1 - \|\dot{\mathcal{P}}\| \le x_i - \|\dot{\mathcal{P}}\| \le x_{i-1} \le u$. Therefore u belongs to $[1 - \|\dot{\mathcal{P}}\|, 2 + \|\dot{\mathcal{P}}\|]$.

On the other hand, if $1+\|\dot{\mathcal{P}}\| \le v \le 2-\|\dot{\mathcal{P}}\|$ and $v \in [x_{i-1}, x_i]$, then (i) $1+\|\dot{\mathcal{P}}\| \le x_i$ which implies that $1 \le x_i - \|\dot{\mathcal{P}}\| \le x_{i-1} \le t_i$ and (ii) $x_{i-1} \le 2-\|\dot{\mathcal{P}}\|$ which implies that $t_i \le x_i \le x_{i-1} + \|\dot{\mathcal{P}}\| \le 2$. Therefore we get $t_i \in [1,2]$.

5. (a) If $u \in [x_{i-1}, x_i]$, then $x_{i-1} \le u$ so that $c_1 \le t_i \le x_i \le x_{i-1} + \|\dot{\mathcal{P}}\|$ whence $c_1 - \|\dot{\mathcal{P}}\| \le x_{i-1} \le u$. Also $u \le x_i$ so that $x_i - \|\dot{\mathcal{P}}\| \le x_{i-1} \le t_i \le c_2$, whence $u \le x_i \le c_2 + \|\dot{\mathcal{P}}\|$.

   (b) If $c_1 + \|\dot{\mathcal{P}}\| \le v \le x_i$ then $c_1 \le x_i - \|\dot{\mathcal{P}}\| \le x_{i-1}$ and if $v \le c_2 - \|\dot{\mathcal{P}}\|$, then $x_i \le x_{i-1} + \|\dot{\mathcal{P}}\| \le c_2$. Therefore $c_1 \le x_{i-1} \le t_i \le x_i \le c_2$.

6. (a) If $\dot{\mathcal{P}}$ is a tagged partition of [0, 2], let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ having tags in [0, 1], and let $\dot{\mathcal{P}}_2$ be the subset of $\dot{\mathcal{P}}$ having tags in [1, 2]. The union of the subintervals in $\dot{\mathcal{P}}_1$ contains the interval $[0, 1-\|\dot{\mathcal{P}}\|]$ and is contained in $[0, 1+\|\dot{\mathcal{P}}\|]$, so that $2(1-\|\dot{\mathcal{P}}\|) \le S(f;\dot{\mathcal{P}}_1) \le 2(1+\|\dot{\mathcal{P}}\|)$. Similarly, the union of the subintervals in $\dot{\mathcal{P}}_2$ contains $[1+\|\dot{\mathcal{P}}\|, 2]$ and is contained in $[1-\|\dot{\mathcal{P}}\|, 2]$, so that $1-\|\dot{\mathcal{P}}\| \le S(f;\dot{\mathcal{P}}_2) \le 1+\|\dot{\mathcal{P}}\|$. Therefore $3-3\|\dot{\mathcal{P}}\| \le S(f;\dot{\mathcal{P}}) = S(f;\dot{\mathcal{P}}_1) + S(f;\dot{\mathcal{P}}_2) \le 3+3\|\dot{\mathcal{P}}\|$, whence $|S(f;\dot{\mathcal{P}}) - 3| \le 3\|\dot{\mathcal{P}}\|$, and we should take $\|\dot{\mathcal{P}}\| < \varepsilon/3$.

   (b) If $\dot{\mathcal{P}}_0$ is the subset of $\dot{\mathcal{P}}$ having tags at 1, then the union of the (at most two) subintervals in $\dot{\mathcal{P}}_0$ is contained in $[1-\|\dot{\mathcal{P}}\|, 1+\|\dot{\mathcal{P}}\|]$, so that $|S(h;\dot{\mathcal{P}}_0)| \le 3 \cdot 2\|\dot{\mathcal{P}}\|$, and $|S(h;\dot{\mathcal{P}}) - 3| \le 9\|\dot{\mathcal{P}}\|$.

7. Use the fact that $\sum_{i=1}^{n+1} k_i f_i = (\sum_{i=1}^{n} k_i f_i) + k_{n+1} f_{n+1}$.

8. Since $-M \le f(x) \le M$ for $x \in [a,b]$, Theorem 7.1.5(c) implies that we have $-M(b-a) \le \int_a^b f \le M(b-a)$ whence the inequality follows.

9. Given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$ then $|S(f;\dot{\mathcal{P}}) - \int_a^b f| < \varepsilon$. Since $\|\dot{\mathcal{P}}_n\| \to 0$, there exists K_ε such that if $n > K_\varepsilon$ then $\|\dot{\mathcal{P}}_n\| < \delta_\varepsilon$, whence $|S(f;\dot{\mathcal{P}}_n) - \int_a^b f| < \varepsilon$. Therefore, $\int_a^b f = \lim_n S(f;\dot{\mathcal{P}}_n)$.

10. Since g is not bounded, it is not Riemann integrable. Let $\dot{\mathcal{P}}_n$ be the partition of [0, 1] into n equal subintervals with tags at the left endpoints, which are rational numbers.

11. If $f \in \mathcal{R}[a,b]$, then Exercise 9 implies that both sequences of Riemann sums converge to $\int_a^b f$.

12. Let $\mathcal{P}_n$ be the partition of [0, 1] into n equal parts. If $\dot{\mathcal{P}}_n$ is this partition with rational tags, then $S(f;\dot{\mathcal{P}}_n) = 1$, while if $\dot{\mathcal{Q}}_n$ is this partition with irrational tags, then $S(f;\dot{\mathcal{Q}}_n) = 0$.

13. If $\|\dot{\mathcal{P}}\| < \delta_\varepsilon := \varepsilon/4\alpha$, then the union of the subintervals in $\dot{\mathcal{P}}$ with tags in $[c,d]$ contains the interval $[c+\delta_\varepsilon, d-\delta_\varepsilon]$ and is contained in $[c-\delta_\varepsilon, d+\delta_\varepsilon]$. Therefore $\alpha(d-c-2\delta_\varepsilon) \le S(\varphi;\dot{\mathcal{P}}) \le \alpha(d-c+2\delta_\varepsilon)$, whence $|S(\varphi;\dot{\mathcal{P}}) - \alpha(d-c)| \le 2\alpha\delta_\varepsilon < \varepsilon$.

14. (a) Since $0 \le x_{i-1} < x_i$, we have $0 \le x_{i-1}^2 < x_i x_{i-1} < x_i^2$, so that $3x_{i-1}^2 < x_i^2 + x_i x_{i-1} + x_i^2 = 3q_i^2 < 3x_i^2$. Therefore $0 \le x_{i-1} \le q_i \le x_i$.

(b) In fact, $(x_i^2 + x_i x_{i-1} + x_{i-1}^2) \cdot (x_i - x_{i-1}) = x_i^3 - x_{i-1}^3$.
(c) The terms in $S(Q; \dot{\mathcal{Q}})$ telescope.
(d) If $\dot{\mathcal{P}}$ has the tags t_i and $\|\dot{\mathcal{P}}\| < \delta$, then $|t_i - q_i| < \delta$ so that we have $|S(Q;\dot{\mathcal{P}}) - S(Q;\dot{\mathcal{Q}})| < \delta(b-a)$.

15. Let $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a tagged partition of $[a,b]$ and let $\dot{\mathcal{Q}} := \{([x_{i-1}+c, x_i+c], t_i+c)\}_{i=1}^n$ so that $\dot{\mathcal{Q}}$ is a tagged partition of $[a+c, b+c]$ and $\|\dot{\mathcal{Q}}\| = \|\dot{\mathcal{P}}\|$. Moreover, $S(g;\dot{\mathcal{Q}}) = S(f;\dot{\mathcal{P}})$ so that $|S(g;\dot{\mathcal{Q}}) - \int_a^b f| = |S(f;\dot{\mathcal{P}}) - \int_a^b f| < \varepsilon$ when $\|\dot{\mathcal{Q}}\| < \delta_\varepsilon$.

Section 7.2

The Cauchy Criterion follows the standard pattern. It is used to obtain the Squeeze Theorem 7.2.3, which is the tool used in proving the important integrability theorems 7.2.5, 7.2.7 and 7.2.8. The only "tricky" proof is that of the Additivity Theorem 7.2.9, but that proof can be soft-pedaled since the validity of the theorem will seem obvious to most students.

Sample Assignment: Exercises 1, 2, 7, 8, 11, 12, 15, 18.

Partial Solutions:

1. If the conditions in 7.2.2(b) is satisfied, we can taken $\eta = 1/n$ and obtain the condition in the statement. Conversely, if the statement holds and $\eta > 0$ is given, we can take $n \in \mathbb{N}$ such that $1/n < \eta$ to get the desired $\dot{\mathcal{P}}_n$, $\dot{\mathcal{Q}}_n$.
2. If the tags are all rational, then $S(h;\dot{\mathcal{P}}) \geq 1$, while if the tags are all irrational, then $S(h;\dot{\mathcal{P}}) = 0$.
3. Let $\dot{\mathcal{P}}_n$ be the partition of [0, 1] into n equal subintervals with $t_1 = 1/n$ and $\dot{\mathcal{Q}}_n$ be the same subintervals tagged by irrational points.
4. No. Let $f(x) := x$ if x is rational and $f(x) := 0$ if x is irrational in [0, 1]. There is no squeeze; that is, $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon)$ is not small.
5. If $c_1, \ldots, c_n$ are the distinct values taken by φ, then $\varphi^{-1}(c_j)$ is the union of a finite collection $\{J_{j1}, \ldots, J_{jr_j}\}$ of disjoint subintervals of $[a,b]$. We can write $\varphi = \sum_{j=1}^n \sum_{k=1}^{r_j} c_j \varphi J_{jk}$.
6. Not necessarily. The Dirichlet function takes on only two values, but $\mathbb{Q} \cap [0,1]$ and $[0,1] \setminus \mathbb{Q}$ are not intervals.
7. If $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$, take $\varphi(x) := f(t_i)$ for $x \in [x_{i-1}, x_i)$ and $\varphi(b) := 0$, so that φ is a simple function. By the formula in Theorem 7.2.5, we have $\int_a^b \varphi = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = S(f;\dot{\mathcal{P}})$.
8. If $f(c) > 0$ for some $c \in (a,b)$, there exists $\delta > 0$ such that $f(x) > \frac{1}{2} f(c)$ for $|x - c| \leq \delta$. Then $\int_a^b f \geq \int_{c-\delta}^{c+\delta} f \geq \frac{1}{2} f(c)(2\delta) > 0$. If c is an endpoint, a similar argument applies.
9. The function $f(0) := 1$ and $f(x) := 0$ elsewhere on $[0,1]$ has integral 0. More dramatically, consider Thomae's function in Example 7.1.7.

10. Let $h := f - g$ so that h is continuous. By Bolzano's Theorem 5.3.7, if h is never 0, then either $h(x) > 0$, or $h(x) < 0$ for all $x \in [a, b]$. In the first case there exists $\gamma > 0$ such that $h(x) \ge \gamma$, whence $\int_a^b h \ge \gamma(b-a) > 0$.

11. Since $\alpha_c(x) = f(x)$ for $x \in [c, b]$, then $\alpha_c \in \mathcal{R}[c, b]$; similarly $\omega_c \in \mathcal{R}[c, b]$. The Additivity Theorem 7.2.9 implies that α_c and ω_c are in $\mathcal{R}[a, b]$. Moreover, $\int_a^b (\omega_c - \alpha_c) = 2M(c-a) < \varepsilon$ when $c - a < \varepsilon/2M$. The Squeeze Theorem 7.2.3 implies that $f \in \mathcal{R}[a, b]$. Further, $|\int_a^b f - \int_c^b f| = |\int_a^c f| \le M(c-a)$.

12. Indeed, $|g(x)| \le 1$ for all $x \in [0, 1]$. Since g is continuous on every interval $[c, 1]$ where $0 < c < 1$, it belongs to $\mathcal{R}[c, 1]$ and the preceding exercise applies.

13. Let $f(x) := 1/x$ for $x \in (0, 1]$ and $f(0) := 0$. Then $f \in \mathcal{R}[c, 1]$ for every $c \in (0, 1)$, but $f \notin \mathcal{R}[0, 1]$ since f is not bounded.

14. Use Mathematical Induction.

15. Suppose $E = \{a = c_0 < c_1 < \cdots < c_m = b\}$. Since f if continuous on the interval (c_{i-1}, c_i), a two-sided version of Exercise 11 implies that its restriction is in $\mathcal{R}[c_{i-1}, c_i]$. The preceding exercise implies that $f \in \mathcal{R}[a, b]$. The case where a or b is not in E is similar.

16. Let $m := \inf f(x)$ and $M := \sup f(x)$. By Theorem 7.1.5(c), we have $m(b-a) \le \int_a^b f \le M(b-a)$. By Bolzano's Theorem 5.3.7, there exists $c \in [a, b]$ such that $f(c) = (\int_a^b f)/(b-a)$.

17. Since $g(x) > 0$, we have $mg(x) \le f(x)g(x) \le Mg(x)$ for all $x \in [a, b]$, whence $m \int_a^b g \le \int_a^b fg \le M \int_a^b g$. Since $\int_a^b g > 0$ (why?), Bolzano's Theorem 5.3.7 implies that there exists $c \in [a, b]$ such that $f(c) = (\int_a^b fg)/(\int_a^b g)$.

18. Let $M := \sup f$ and let $p \in [a, b]$ be such that $f(p) = M$. Given $\varepsilon > 0$ there exists an interval $[c, d]$ containing p with $d - c > 0$ such that $M - \varepsilon \le f(x) \le M$ for $x \in [c, d]$. Therefore

$$(M-\varepsilon)^n (d-c) \le \int_c^d f^n \le \int_a^b f^n \le M^n (b-a).$$

If we take the nth root, we have $(M-\varepsilon)(d-c)^{1/n} \le M_n \le M(b-a)^{1/n}$. Now use the fact that $\alpha^{1/n} \to 1$ for $\alpha > 0$ to complete the details.

19. The Additivity Theorem implies that the restrictions of f to $[-a, 0]$ and $[0, a]$ are Riemann integrable. Let $\dot{\mathcal{P}}_n$ be a sequence of tagged partitions of $[0, a]$ with $\|\dot{\mathcal{P}}_n\| \to 0$ and let $\dot{\mathcal{P}}_n^*$ be the corresponding "symmetric" partition of $[-a, a]$.
(a) Show that $S(f; \dot{\mathcal{P}}_n^*) = 2S(f; \dot{\mathcal{P}}_n) \to 2\int_0^a f$.
(b) Show that $S(f; \dot{\mathcal{P}}_n^*) = 0$.

20. Note that $x \mapsto f(x^2)$ is an even continuous function.

Section 7.3

The main results are the Fundamental Theorems, given in 7.3.1 and 7.3.5, and the Lebesgue Integrability Criterion, stated in 7.3.12. The First Form 7.3.1 allows for a finite set E where the function F may not be differentiable. It is useful to point out that if $E = \emptyset$, then one does not need to assume that F is continuous at every point of $[a, b]$. However, one often encounters functions where F is not differentiable at every point (for example $F(x) = \sqrt{x}$). It is also worth stressing that the hypothesis 7.3.1(c) is *essential*. The Second Form 7.3.5 is complementary to the First Form, but is nowhere nearly as important in most situations.

The notion of a null set is an important one. No doubt most students will think of countable sets, but it is worth pointing out that there are uncountable null sets; however, it may be best to wait until the students encounter the Cantor set in Section 11.2 before too much is made of this fact. Similarly, the proof of the Lebesgue Criterion is given in Appendix C, but it is probably too complicated for the average student at this level.

Sample Assignment: Exercises 2, 3, 5, 7, 9, 13, 18(a,c).

Partial Solutions:

1. Suppose that $E := \{a = c_0 < c_1 < \cdots < c_m = b\}$ contains the points in $[a, b]$ where the derivative $F'(x)$ either does not exist, or does not equal $f(x)$. If we apply the proof of the 7.3.1 to $[c_{i-1}, c_i]$, we have that $f \in \mathcal{R}[c_{i-1}, c_i]$ and $\int_{c_{i-1}}^{c_i} f = F(c_i) - F(c_{i-1})$. Exercise 7.2.14 and Corollary 7.2.10 imply that $f \in \mathcal{R}[a, b]$ and that $\int_a^b f = \sum_{i=1}^m (F(c_i) - F(c_{i-1})) = F(b) - F(a)$.
2. We note that H_n is continuous on $[a, b]$ and $H_n'(x) = x^n$ for all $x \in [a, b]$, so $\int_a^b x^n dx = H_n(b) - H_n(a)$. Here $E = \emptyset$.
3. Let $E := \{-1, 1\}$. If $x \notin E$, the Chain Rule 6.1.6 implies that $G'(x) = \frac{1}{2}\operatorname{sgn}(x^2 - 1) \cdot 2x = x \operatorname{sgn}(x^2 - 1) = g(x)$. Also $g \in \mathcal{R}[-2, 3]$.
4. Indeed, $B'(x) = |x|$ for all x.
5. (a) We have $\Phi_C'(x) = \Phi'(x) = f(x)$ for all $x \in [a, b]$, so Φ_C is also an antiderivative of f on $[a, b]$.
6. By Theorem 7.2.13, we have $F_a(z) = \int_a^c f + \int_c^z f$, so that $F_c = F_a - \int_a^c f$.
7. Let h be Thomae's function. There is no function $H : [0, 1] \to \mathbb{R}$ such that $H'(x) = h(x)$ for x in some nondegenerate open interval; otherwise Darboux's Theorem 6.2.12 would be contradicted on this interval. So a finite set E will not suffice for this function.
8. Note that $F(0) = 0 = \lim_{x \to 0+} F(x)$ and that if $n \in \mathbb{N}$, then

$$\lim_{x \to n-} F(x) = (n-1)n/2 = F(n) = \lim_{x \to n+} F(x).$$

Therefore, F is continuous for $x \geq 0$. Also $F'(x) = n-1 = [\![x]\!]$ for x in $(n-1, n), n \in \mathbb{N}$. However, F does not have (a two-sided) derivative at $n = 0, 1, 2, \ldots$. Since there are only a finite number of these points in $[a, b]$, the Fundamental Theorem 7.3.1 implies that $\int_a^b [\![x]\!] dx = F(b) - F(a)$.

9. (a) $G(x) = F(x) - F(c)$, (b) $H(x) = F(b) - F(x)$, (c) $S(x) = F(\sin x) - F(x)$.
10. If $F(x) := \int_a^x f$, then since f is continuous on $[a, b]$, Theorem 7.3.6 implies that $F'(x) = f(x)$ for all $x \in [a, b]$. Since $G(x) = F(v(x))$, the statement follows from the Chain Rule 6.1.6.
11. (a) $F'(x) = 2x(1+x^6)^{-1}$, (b) $F'(x) = (1+x^2)^{1/2} - 2x(1+x^4)^{1/2}$.
12. $F(x) := x^2/2$ for $0 \leq x < 1$, $F(x) := x - 1/2$ for $1 \leq x < 2$, and $F(x) := (x^2-1)/2$ for $2 \leq x \leq 3$. If $x \neq 2$ then $F'(x) = f(x)$, but $F'(2)$ does not exist.
13. For $0 \leq x \leq 2$, we have $G(x) = \int_0^x (-1) dt = -x$; and for $2 \leq x \leq 3$, we have $G(x) = \int_0^2 (-1) dt + \int_2^x 1 dt = -2 + (x-2) = x - 4$. $G(x)$ is not differentiable at $x = 2$.
14. The Fundamental Theorem implies that if $f'(x) \leq 2$ for $0 \leq x \leq 2$, then $f(x) - f(0) = \int_0^x f'(x)\, dx \leq \int_0^x 2\, dx = 2x$, so that $f(x) \leq 2x + f(0) = 2x - 1$ for $0 \leq x \leq 2$. Then $f(2) \leq 3$ so that $f(2) = 4$ is impossible.
15. Since $g(x) = \int_0^{x+c} f - \int_0^{x-c} f$ and f is continuous, then $g'(x) = f(x+c) - f(x-c)$.
16. If $F(x) := \int_0^x f = -\int_1^x f$, then $F'(x) = f(x) = -f(x)$, so that $2f(x) = 0$ and hence $f(x) = 0$ for all $x \in [0, 1]$.
18. (a) Take $\varphi(t) = 1 + t^2$ to get $\frac{1}{2}\int_{t=0}^{t=1} (\varphi(t))^{1/2} \cdot \varphi'(t) dt = \frac{1}{2}\int_{x=1}^{x=2} x^{1/2} dx = \frac{1}{3}x^{3/2}\big|_1^2 = \frac{1}{3}(2^{3/2} - 1)$.

    (b) Take $\varphi(t) = 1 + t^3$ to get $\frac{1}{3}\int_{t=0}^{t=2} (\varphi(t))^{-1/2} \cdot \varphi'(t) dt = \frac{1}{3}\int_{x=1}^{x=9} x^{-1/2} dx = \frac{2}{3}x^{1/2}\big|_1^9 = \frac{2}{3}(9^{1/2} - 1) = \frac{4}{3}$.

    (c) Take $\varphi(t) = 1 + \sqrt{t}$ to get $2\int_{t=1}^{t=4} (\varphi(t))^{1/2} \cdot \varphi'(t) dt = 2\int_{x=2}^{x=3} x^{1/2} dx = \frac{4}{3}x^{3/2}\big|_2^3 = \frac{4}{3}(3^{3/2} - 2^{3/2})$.

    (d) Take $\varphi(t) = t^{1/2}$ to get $2\int_{t=1}^{t=4} \cos(\varphi(t)) \cdot \varphi'(t) dt = 2\int_{x=1}^{x=2} \cos x\, dx = 2(\sin 2 - \sin 1)$.
19. In (a)–(c) $\varphi'(0)$ does not exist. For (a), one can integrate over $[c, 4]$ and let $c \to 0+$. For (b) the integrand is not bounded near 0, so the integral does not exist. For (c), note that the integrand is even, so the integral equals $2\int_0^1 (1+t)^{1/2}\, dt$. For (d), $\varphi'(1)$ does not exist, so integrate over $[0, c]$ and let $c \to 1-$.
20. (b) It is clear that $\bigcup_n Z_n$ is contained in $\bigcup_{n,k} J_k^n$ and that the sum of the lengths of these intervals is $\leq \sum_n \varepsilon/2^n = \varepsilon$.
21. (a) The Product Theorem 7.3.16 implies that $(tf \pm g)^2 \geq 0$ is integrable.

    (b) We have $\mp 2t\int_a^b fg \leq t^2\int_a^b f^2 + \int_a^b g^2$. Now divide by t to obtain (b).

(c) Let $t \to \infty$ in (b).
(d) If $\int_a^b f^2 \neq 0$, let $t = (\int_a^b g^2 / \int_a^b f^2)^{1/2}$ in (b). Now replace f and g by $|f|$ and $|g|$.

22. Note that the composite function $\operatorname{sgn} \circ h$ is Dirichlet's function, which is not Riemann integrable.

Section 7.4

This optional section presents an alternative approach to the integral developed by Gaston Darboux. Instead of Riemann sums using tags, this approach employs upper and lower sums using suprema and infima. The material in this section is independent of the earlier sections of the chapter until the equivalence of the two approaches is discussed. Because of time pressure, instructors will need to make decisions about selection of material and this section provides an option.

Sample Assignment: Exercises 1, 4, 7, 9, 10, 12, 13, 14.

Partial Solutions:

1. (a) $L(f;P_1) = (0+0+1)\cdot 1 = 1, \quad U(f;P_1) = (1+1+2)\cdot 1 = 4.$
   (b) $L(f;P_2) = (1/2+0+0+1/2+1+3/2)\cdot \dfrac{1}{2} = 7/4,$
   $U(f;P_2) = (1+1/2+1/2+1+3/2+2)\cdot(1/2) = 13/4.$

2. If $P = (a, x_{2.}, x_3, \ldots, x_{n-1}, b)$ is any partition of $[a,b]$ and $f(x) = c$ for all x, then $L(f;P) = U(f;P) = c((x_2 - a) + (x_3 - x_2) + (x_4 - x_3) + \cdots + (b - x_{n-1})) = c(b-a)$.

3. If P is a partition, then $\inf\{f(x) : x \in I_k\} \leq \inf\{g(x) : x \in I_k\}$ for each k, so that $L(f;P) \leq L(g;P)$. Since P is an arbitrary partition, we have $L(f) \leq L(g)$.

4. If $k > 0$, then $\inf\{kf(x) : x \in I_j\} = k\inf\{f(x) : x \in I_j\}$, whence $L(kf;P) = k\,L(f;P)$. It follows that $L(kf) = k\,L(f)$.

5. It follows from Exercise 3 that $L(f) \leq L(g) \leq L(h)$ and $U(f) \leq U(g) \leq U(h)$. But if $L(f) = U(f) = A$ and $L(h) = U(h) = A$, it follows that $L(g) = A = U(g)$, whence g is Darboux integrable with integral A.

6. Given $\varepsilon > 0$, consider the partition $P_\varepsilon = (0, 1-\varepsilon/2, 1+\varepsilon/2, 2)$. Then $U(f;P_\varepsilon) = 2$ and $L(f;P_\varepsilon) = 2-\varepsilon$. It follows that the integral is equal to 2.

7. (a) If $P_\varepsilon = (0, 1/2-\varepsilon, 1/2+\varepsilon, 1)$, then $L(g;P_\varepsilon) = 1/2-\varepsilon$ and $U(g;P_\varepsilon) = 1/2+\varepsilon$.
   (b) Here $L(g;P_\varepsilon) = 1/2-\varepsilon$ and $U(g;P_\varepsilon) = 1/2+13\varepsilon$.

8. If for some $c \in I$ we have $f(c) > 0$, then (by Theorem 4.2.9) there exists $\delta > 0$ such that $f(x) > f(c)/2 > 0$ for $|x - c| < \delta, x \in I$. Thus for some partition P_c, we have $L(f; P_c) > 0$, and therefore $L(f) > 0$.

9. Given $\varepsilon > 0$, let P_1 and P_2 be partitions of I such that $L(f_j) - \varepsilon/2 < L(f_j; P_j)$, and let $P_\varepsilon = P_1 \cup P_2$ so that $L(f_j) - \varepsilon/2 < L(f_j; P_\varepsilon)$ for $j = 1, 2$. If $I_1, \ldots, I_m$ are the subintervals of I arising from P_ε, then it follows that
$$\inf\{f_1(x) : x \in I_k\} + \inf\{f_2(x) : x \in I_k\} \le \inf\{f_1(x) + f_2(x) : x \in I_k\}.$$
Thus we obtain $L(f_1; P_\varepsilon) + L(f_2; P_\varepsilon) \le L(f_1 + f_2; P_\varepsilon) \le L(f_1 + f_2)$. Hence $L(f_1) + L(f_2) - \varepsilon \le L(f_1 + f_2)$, where $\varepsilon > 0$ is arbitrary.

10. Let f_1 be the Dirichlet function (see Example 7.4.7(d)) and let $f_2 = 1 - f_1$. Then $L(f_1) = L(f_2) = 0$, but $L(f_1 + f_2) = 1$.

11. If $|f(x)| \le M$ for $x \in [a, b]$ and $\varepsilon > 0$, let P_ε be a partition such that the total length of the subintervals that contain any of the points $c_1, c_2, \ldots, c_n$ is less than ε/M. Then $U(f; P_n) - L(f; P_n) < \varepsilon$, so the Integrability Criterion 7.4.8 applies. Also $0 \le U(f; P_n) \le \varepsilon$, so that $U(f) = 0$.

12. $$\begin{aligned} L(f; P_n) &= (0^2 + 1^2 + \cdots + (n-1)^2)/n^3 = (n-1)n(2n-1)/6n^3 \\ &= \frac{1}{3}\Big(1 - \frac{3}{2n} + \frac{1}{2n^2}\Big) \\ U(f; P_n) &= (1^2 + 2^2 + \cdots + n^2)/n^3 = n(n+1)(2n+1)/6n^3 \\ &= \frac{1}{3}\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right) \end{aligned}$$
Therefore, $1/3 = \sup\{L(f; P_n) : n \in N\} \le L(f) \le U(f) \le \inf\{U(f; P_n) : n \in N\} = 1/3$, and we conclude that $L(f) = U(f) = 1/3$.

13. It follows from Lemma 7.4.2 that if P is a refinement of P_ε, then $L(f; P_\varepsilon) \le L(f; P)$ and $U(f; P) \le U(f; P_\varepsilon)$, so that $U(f; P) - L(f; P) \le U(f; P_\varepsilon) - L(f; P_\varepsilon)$.

14. (a) By the Uniform Continuity Theorem 5.4.3, f is uniformly continuous on I. Therefore if $\varepsilon > 0$ is given, there exists $\delta > 0$ such that if u, v in I and $|u - v| < \delta$, then $|f(u) - f(v)| < \varepsilon/(b - a)$. Let n be such that $n > (b - a)/\delta$ and $P_n = (x_0, x_1, \ldots, x_n)$ be the partition of I into n equal parts so that $x_k - x_{k-1} = (b - a)/n < \delta$. Applying the Maximum-Minimum Theorem 5.3.4 to each subinterval, we get u_k, v_k in I_k so that $f(u_k) = M_k$ and $f(v_k) = m_k$. Then $M_k - m_k = f(u_k) - f(v_k) < \varepsilon/(b - a)$. Then $0 \le U(f; P_n) - L(f; P_n) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows from Corollary 7.4.9 that f is integrable on I.

(b) If f is increasing on I, let P_n be the partition as in (a). Then $f(x_k) = M_k$ and $f(x_{k-1}) = m_k$. Then we have the "telescoping" sum

$$\begin{aligned}\sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) &= \frac{b-a}{n}\sum_{k=1}^{n}(f(x_k) - f(x_{k-1}))\\ &= \frac{b-a}{n}(f(x_1) - f(x_0) + f(x_2) - f(x_1)\\ &\quad + \cdots + f(x_n) - f(x_{n-1}))\\ &= \frac{b-a}{n}(f(b) - f(a)).\end{aligned}$$

Now given $\varepsilon > 0$, choose $n > (b-a)(f(b) - f(a))/\varepsilon$. Then for partition P_n, we get $U(f; P_n) - L(f; P_n) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) \leq \varepsilon$. Corollary 7.4.9 implies that f is integrable on I.

15. We have $0 \leq U(f; P_n) - L(f; P_n) \leq K(b-a)^2/n$, and therefore

$$0 \leq U(f; P_n) - \int_a^b f \leq K(b-a)^2/n.$$

Section 7.5

The proofs of the error estimates for the Trapezoidal, Midpoint and Simpson formulas involve application of the Mean Value Theorem and the Bolzano Intermediate Value Theorem. Since they are not particularly instructive, they are given in Appendix D and the instructor may not wish to discuss them. Consequently, it should be possible to cover this section in a single lesson.

Attention should be paid to the fact that, in the presence of convexity (or concavity) of the integrand, one has bounds for the error in the Trapezoidal and Midpoint Rules without examining the second derivative of the integrand.

Sample Assignment: Exercises 1, 2, 7, 8, 9, 17.

Partial Solutions:

1. Use (4) with $n = 4, a = 1, b = 2, h = 1/4$. Here $1/4 \leq f''(c) \leq 2$, so $T_4 \approx 0.697\,02$.
2. Use (10) with $n = 4, a = 1, b = 2, h = 1/4$. Since $f^{(4)}(x) = 24/x^5$, we have $3/4 \leq f^{(4)}(c) \leq 2$. Here $S_4 \approx 0.693\,25$.
3. $T_4 \approx 0.782\,79$.
4. The index n must satisfy $2/12n^2 < 10^{-6}$; hence $n > 1000/\sqrt{6} \approx 408.25$.
5. $S_4 \approx 0785\,39$.
6. The index n must satisfy $96/180n^4 < 10^{-6}$; hence $n \geq 28$.

7. Note that $p^{(4)}(x) = 0$ for all x.
8. Use the fact that $f''(x) \geq 0$ in (4) and (7). Geometrically the inequality is reasonable because, if the function is convex, then the chord of the trapezoid lies above the tangent to the graph. If $f''(x) \leq 0$, then the graph is concave and the inequality is reversed.
9. A direct calculation.
10. A direct calculation.
11. Use Exercise 10.
12. The integral is equal to the area of one quarter of the unit circle. The error estimates cannot be used because the derivatives of h are unbounded on $[0, 1]$. Since $h''(x) \leq 0$, the inequality is $T_n(h) < \pi/4 < M_n(h)$. See Exercise 8.
13. Interpret K as an area. Show that $h''(x) = -(1-x^2)^{3/2}$ and that $h^{(4)}(x) = -3(1+4x^2)(1+x^2)^{-7/2}$. To eight decimal places, $\pi = 3.141\,592\,65$.
14. Approximately $3.653\,484\,49$.
15. Approximately $4.821\,159\,32$.
16. Approximately $0.835\,648\,85$.
17. Approximately $1.851\,937\,05$.
18. 1.
19. Approximately $1.198\,140\,23$.
20. Approximately $0.904\,524\,24$.

CHAPTER 8

SEQUENCES OF FUNCTIONS

In this chapter we study the pointwise and uniform convergence of sequences of functions, so it draws freely from results in Chapter 3. After introducing these concepts in Section 8.1, we show in Section 8.2 that one can interchange certain important limiting operations (e.g., differentiation and integration) when the convergence is uniform. Both of these sections are important and should be discussed in detail.

Section 8.3 and 8.4 and more special. In Section 8.3 we use the results in Section 8.2 to establish the exponential function on a firm foundation, after which the logarithm is treated. In Section 8.4 we do the same for the sine and cosine functions. Most of these properties will be familiar to the students, although the approach will surely be new to them. A detailed discussion of Section 8.3 and 8.4 can be omitted if time is short.

Section 8.1

The distinction between ordinary (= pointwise) convergence and uniform convergence of a sequence of functions on a set A is a subtle one. It centers on whether the index $K(\varepsilon, x)$ can be chosen to be independent of the point $x \in A$; that is, whether the set $\{K(\varepsilon, x) : x \in A\}$ is bounded in $\mathbb{R}$. If so, we can take $K(\varepsilon)$ to be the supremum of this set. However, it is not always easy to determine whether this set is bounded for each $\varepsilon > 0$. Often it is easier to obtain estimates for the uniform norms introduced in Definition 8.1.7.

Sample Assignment: Exercises 1, 2, 3, 4, 7, 11, 12, 13, 14, 17.

Partial Solutions:

1. Note that $0 \le f_n(x) \le x/n \to 0$ as $n \to \infty$.
2. Note that $f_n(0) = 0$ for all n. If $x > 0$, we have $|f_n(x)| \le 1/(nx) \to 0$ as $n \to \infty$.
3. Note that $f_n(0) = 0$ for all $n \in \mathbb{N}$. If $x > 0$, then $|f_n(x) - 1| < 1/(nx) \to 0$ as $n \to \infty$.
4. If $x \in [0, 1)$, then $|f_n(x)| \le x^n \to 0$. If $x = 1$, then $f_n(1) = 1/2$ for all $n \in \mathbb{N}$. If $x > 1$, then $|f_n(x) - 1/ = 1/(1 + x^n) \le (1/x)^n \to 0$.
5. Note that $f_n(0) = 0$ for all n. If $x > 0$, then $|f_n(x)| \le 1/(nx) \to 0$.
6. Note that $f_n(0) = 0$ for all n. If $0 < \varepsilon < \pi/2$, let $M_\varepsilon := \tan(\pi/2 - \varepsilon) > 0$ so that if $y > M_\varepsilon$, then $\pi/2 - \varepsilon < \operatorname{Arctan} y < \pi/2$. Therefore if $n > M_\varepsilon/x$, then $\pi/2 - \varepsilon < \operatorname{Arctan} nx < \pi/2$. Similarly if $x < 0$.

7. Note that $f_n(0)=1$ for all n. If $x>0$, then $0<e^{-x}<1$ so that $0 \le e^{-nx} = (e^{-x})^n \to 0$.
8. Note that $f_n(0)=0$ for all n. If $x>0$, it follows from Exercise 7 that $0 \le x(e^{-x})^n \to x \cdot 0 = 0$.
9. For both functions $f_n(0)=0$ for all n. If $x>0$, then $0 \le x^2 e^{-nx} = x^2(e^{-x})^n \to 0$, since $0<e^{-x}<1$. For the second function, use Theorem 3.2.11 and $[(n+1)^2\ x^2 e^{-(n+1)x}]/[n^2 x^2\ e^{-nx}] = (1+1/n)^2 e^{-x} \to e^{-x} < 1$.
10. If $x \in \mathbb{Z}$, then $\cos \pi x = \pm 1$, so that $(\cos \pi x)^2 = 1$ and the limit equals 1. If $x \notin \mathbb{Z}$, then $0 \le (\cos \pi x)^2 < 1$ and the limit equals 0.
11. If $x \in [0,a]$, then $|f_n(x)| \le a/n$. However, $f_n(n) = 1/2$.
12. If $x \in [a,\infty)$, then $|f_n(x)| \le 1/(na)$. However, $f_n(1/n) = 1/2$.
13. If $a>0$, then $|f_n(x)-1| \le 1/(na)$ on $[a,\infty)$. However, $f_n(1/n) = 1/2$.
14. If $x \in [0,b]$, then $|f_n(x)| \le b^n$. However, $f_n(2^{-1/n}) = 1/3$.
15. If $x \in [a,\infty)$, then $|f_n(x)| \le 1/(na)$. However, $f_n(1/n) = \frac{1}{2}\sin 1 > 0$.
16. If $0<\varepsilon<\pi/2$, let $M_\varepsilon := \tan(\pi/2-\varepsilon) > 0$, so that if $na \ge M_\varepsilon$, then $\pi/2 - \varepsilon \le \operatorname{Arctan} na < \pi/2$. Hence, if $x \ge a$ and $n \ge M_\varepsilon/a$, then $nx \ge M_\varepsilon$ and $\pi/2-\varepsilon \le \operatorname{Arctan} nx < \pi/2$. However, $f_n(1/n) = \operatorname{Arctan} 1 = \pi/4 > 0$.
17. If $x \in [a,\infty)$, then $|f_n(x)| \le (e^{-a})^n$. However, $f_n(1/n) = 1/e$.
18. The maximum of f_n on $[0,\infty)$ is at $x=1/n$, so $\|f_n\|_{[0,\infty)} = 1/(ne)$.
19. The maximum of f_n on $[0,\infty)$ is at $x=2/n$, so $\|f_n\|_{[0,\infty)} = 4/(ne)^2$.
20. If n is sufficiently large, the maximum of f_n on $[a,\infty)$ is at $x=a>0$, so that $\|f_n\|_{[a,\infty)} = n^2a^2/e^{na} \to 0$. However, $\|f_n\|_{[0,\infty)} = f_n(2/n) = 4/e^2$.
21. Given $\varepsilon > 0$, let $K_1(\varepsilon/2)$ be such that if $n \ge K_1(\varepsilon/2)$ and $x \in A$, then $|f_n(x) - f(x)| < \varepsilon/2$; also let $K_2(\varepsilon/2)$ be such that if $n \ge K_2(\varepsilon/2)$ and $x \in A$, then $|g_n(x)-g(x)| < \varepsilon/2$. Let $K_3 := \sup\{K_1(\varepsilon/2), K_2(\varepsilon/2)\}$ so that if $n \ge K_3$ and $x \in A$, then $|(f_n+g_n)(x) - (f+g)(x)| \le |f_n(x)-f(x)| + |g_n(x)-g(x)| < \varepsilon$.
22. We have $|f_n(x)-f(x)| = 1/n$ for all $x \in \mathbb{R}$. Hence (f_n) converges uniformly on $\mathbb{R}$ to f. However, $|f_n^2(n) - f^2(n)| \ge 2$ so that (f_n^2) does not converge uniformly on $\mathbb{R}$ to f.
23. Let M be a bound for $(f_n(x))$ and $(g_n(x))$ on A, whence also $|f(x)| \le M$. The Triangle Inequality gives $|f_n(x)g_n(x) - f(x)g(x)| \le M[|f_n(x)-f(x)| + |g_n(x)-g(x)|]$ for $x \in A$.
24. Since g is uniformly continuous on $[-M,M]$, given $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that if $|u-v|<\delta_\varepsilon$ and $u,v \in A$, then $|g(u)-g(v)|<\varepsilon$. If (f_n) converges uniformly to f on A, given $\delta>0$ there exists $K(\delta)$ such that if $n \ge K(\delta)$ and $x \in A$, then $|f_n(x)-f(x)|<\delta$. Therefore, if $n \ge K(\delta_\varepsilon)$ and $x \in A$, then $|g(f_n(x)) - g(f(x))| < \varepsilon$.

Section 8.2

The proof of Theorem 8.2.2 is short and understandable; it should be discussed in detail. That of Theorem 8.2.3 is more delicate; observe that it depends on the Mean Value Theorem 6.2.4 in two places. Note especially that the hypothesis in Theorem 8.2.3 is that the *sequence of derivatives* is uniformly convergent (and that the uniform convergence of the sequence of functions is a conclusion, rather than a hypothesis). The only delicate part of the proof of Theorem 8.2.4 is to show that the limit function is integrable. The Bounded Convergence Theorem 8.2.5 will be considerably strengthened in Section 10.4. Dini's Theorem 8.2.6 is interesting in that it shows that monotone convergence for continuous functions to a continuous limit implies the uniformity of the convergence on $[a, b]$.

Sample Assignment: Exercises 1, 2, 4, 5, 7, 8, 14, 16, 19.

Partial Solutions:

1. The limit function is $f(x) := 0$ for $0 \le x < 1, f(1) := 1/2$ and $f(x) := 1$ for $1 < x \le 2$. Since it is discontinuous, while the f_n are all continuous, the convergence cannot be uniform.
2. The convergence is not uniform, because $f_n(1/n) = n$, while $f(x) = 0$ for all $x \in [0, 1]$.
3. Let $f_n(x) := 1/n$ if x is rational and $f_n(x) := 0$ if x is irrational.
4. If $\varepsilon > 0$ is given, let K be such that if $n \ge K$, then $\|f_n - f\|_I < \varepsilon/2$. Then $|f_n(x_n) - f(x_0)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| \le \varepsilon/2 + |f(x_n) - f(x_0)|$. Since f is continuous (by Theorem 8.2.2) and $x_n \to x_0$, then $|f(x_n) - f(x_0)| < \varepsilon/2$ for $n \ge K'$, so that $|f_n(x_n) - f(x_0)| < \varepsilon$ for $n \ge \max\{K, K'\}$.
5. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, u \in \mathbb{R}$ and $|x - u| < \delta$, then $|f(x) - f(u)| < \varepsilon$. Now require that $1/n < \delta$.
6. Here $f(0) = 1$ and $f(x) = 0$ for $x \in (0, 1]$. Since the f_n are continuous on [0, 1] but f is not, the convergence cannot be uniform on [0, 1]. Alternatively, note that $f_n(1/n) \to 1/e$.
7. Given $\varepsilon := 1$, there exists $K > 0$ such that if $n \ge K$ and $x \in A$, then $|f_n(x) - f(x)| < 1$, so that $|f_n(x)| \le |f_K(x)| + 1$ for all $x \in A$. If $M := \max\{\|f_1\|_A, \ldots, \|f_{K-1}\|_A, \|f_K\|_A + 1\}$, then $|f_n(x)| \le M$ for all $n \in \mathbb{N}, x \in A$. Therefore $|f(x)| \le M$ for all $x \in A$.
8. Since $0 \le f_n(x) \le nx \le n$ on [0, 1] and $0 \le f_n(x) \le 1/x \le 1$ on $[1, \infty)$, we have $0 \le f_n(x) \le n$ on $[0, \infty)$. Moreover, $f(x) := \lim(f_n(x)) = 0$ for $x = 0$ and $f(x) = 1/x$ for $x > 0$, which is not bounded on $[0, \infty)$. Since f is not continuous on $[0, \infty)$, the convergence is not uniform. Alternatively, $f_n(1/\sqrt{n}) = \sqrt{n}/2$.

9. Since $f(x)=0$ for all $x\in[0,1]$, we have $f'(1)=0$. Also $\|f_n-f\|_{[0,1]}\le 1/n$, so the convergence of (f_n) is uniform. Also $g(x)=\lim(x^{n-1})$ so that $g(1)=1$. The convergence of the sequence of derivatives is not uniform on [0,1].

10. Here $\|g_n\|_{[0,\infty)}\le 1/n$ so (g_n) converges uniformly to the zero function. However, $\lim(g_n'(x))=-1$ for $x=0$, and $=0$ for $x>0$. Hence $(\lim g_n)'(0)=0\neq 1=\lim(g_n'(0))$. The sequence (g_n') does not converge uniformly.

11. The Fundamental Theorem 7.3.1 implies that $\int_a^x f_n' = f_n(x)-f_n(a)$. Now apply Theorem 8.2.4.

12. The function $f_n(x):=e^{-nx^2}$ is decreasing on [1, 2] and $\|f_n\|_{[1,2]}=e^{-n}$. Hence Theorem 8.2.4 can be applied.

13. If $a>0$, then $\|f_n\|_{[a,\pi]}\le 1/(na)$ and Theorem 8.2.4 applies. On the interval $[0,\pi]$ the limit function is $f(0):=1$ and $f(x):=0$ for $x\in(0,\pi]$. Moreover $\|f_n\|_{[0,\pi]}=1$. Hence it follows from Theorem 8.2.5 that $\int_0^\pi f=0$. This can also be proved directly by changing the variable $v=nx$ and estimating the integrals.

14. The limit function is $f(0):=0, f(x):=1$ for $x\in(0,1]$ and the convergence is not uniform on $[0,1]$. Her $\lim\int_0^1 f_n=\lim(1-(1/n)\ln(n+1))=1$ and $\int_0^1 f=1$.

15. Here $g_n(0)=0$ for all n, and $g_n(x)\to 0$ for $x\in(0,1]$ by Theorem 3.2.11. The function g_n is maximum on [0, 1] at $x=1/(n+1)$, whence $\|g_n\|_{[0,1]}\le 1$ for all n. Now apply Theorem 8.2.5. Or evaluate the integrals directly.

16. Each f_n is Riemann integrable since it has only a finite number of discontinuities. (See Exercise 7.1.13 or the Lebesgue Integrability Criterion 7.3.12.)

17. Here $f(x):=0$ for $x\in[0,1]$ and we have $\|f_n-f\|_{[0,1]}=1$.

18. Here $f(x):=0$ for $x\in[0,1)$ and $f(1):=1$ and we have $\|f_n-f\|_{[0,1]}=1$.

19. Here $f(x):=0$ for all $x\in[0,\infty)$ and $|f_n(n)-f(n)|=1$.

20. Let $f_n(x):=x^n$ for $x\in[0,1]$.

Section 8.3

There are many different approaches to the exponential and logarithmic functions: see R.G. Bartle's *Elements of Real Analysis*. The approach here is based on obtaining the exponential function as a limit of polynomials (which, in fact, are the partial sums of the Maclaurin series for the exponential function). The uniqueness and basic properties of the exponential function are based on the differential equation and initial conditions it satisfies.

The logarithm is obtained as the function that is inverse to the exponential function. The power functions $x\mapsto a^\alpha$ and the functions $x\mapsto \log_a x$ are often useful, but can largely be left to the student.

Sample Assignment: Exercises 1, 2, 3, 5, 6, 8, 10, 13.

Partial Solutions:

1. To establish the inequality, let $A := x > 0$ and let $m \to \infty$ in (5). For the estimate on e, take $x = 1$ and $n = 3$ to obtain $|e - 2\frac{2}{3}| < 1/12$, so $e < 2\frac{3}{4}$. Since $(E_n(1))$ is increasing, we also have $2\frac{2}{3} < e$.
2. Note that if $n \geq 9$, then $2/(n+1)! < 6 \times 10^{-7} < 5 \times 10^{-6}$. Hence $e \approx 2.71828$.
3. Evidently $E_n(x) \leq e^x$ for all $n \in \mathbb{N}$, $x \geq 0$. To obtain the other inequality, apply Taylor's Theorem 6.4.1 to $[0, a]$ and note that if $c \in [0, a]$, then $1 \leq e^c \leq e^a$.
4. To obtain the inequality, replace n by $n+1$ and take $a = 1$ in Exercise 3. Since $2 < e < 3$, we have $e/(n+1)$ for $n \geq 2$. If $e = m/n$, then $en! - (1 + 1 + \cdots + 1/n!)n!$ is an integer in (0, 1), which is impossible.
5. Note that $0 \leq t^n/(1+t) \leq t^n$ for $t \in [0, x]$.
6. $\ln 1.1 \approx 0.0953$ and $\ln 1.4 \approx 0.3365$. Take $n > 19{,}999$.
7. $\ln 2 \approx 0.6931$. Note that $e/2 - 1 < 0.36$ and $(0.36)^8/8 < 0.000\,04$.
8. If $f(0) = 0$, the argument in 8.3.4 show that $f(x) = 0$ for all x, so we take $K = 0$. If $f(0) \neq 0$, then $g(x) := f(x)/f(0)$ is such that $g'(x) = g(x)$ for all x and $g(0) = 1$. It follows from 8.3.4 that $g(x) = E(x)$, whence $f(x) = f(1)e^x$.
9. Note that if the means are equal, then we must have $1 + x_k = E(x_k)$ for all k. It follows that $x_k = 0$, whence $a_k = A$ for all k.
10. $L'(1) = \lim[L(1 + 1/n) - L(1)]/(1/n) = \lim L((1 + 1/n)^n) = L(\lim(1 + 1/n)^n) = L(e) = 1$.
11. (a) Since $L(1) = 0$, we have $1^\alpha = E(\alpha L(1)) = E(0) = 1$.
    (b) This follows from the fact that $E(z) > 0$ for all $z \in \mathbb{R}$.
    (c) $(xy)^a = E(\alpha L(xy)) = E(\alpha L(x) + \alpha L(y)) = E(\alpha L(x)) \cdot E(\alpha L(y)) = x^\alpha \cdot y^\alpha$.
    (d) Since $(1/y)^\alpha = E(\alpha L(1/y)) = E(-\alpha L(y)) = (E(\alpha L(y))^{-1} = (y^\alpha)^{-1}$, the statement follows from (c).
12. (a) $x^{\alpha+\beta} = E((\alpha+\beta)L(x)) = E(\alpha L(x) + \beta L(x)) = E(\alpha L(x)) \cdot E(\beta L(x)) = x^\alpha \cdot x^\beta$.
    (b) $(x^\alpha)^\beta = E(\beta L(x^\alpha)) = E(\beta\alpha L(x)) = x^{\alpha\beta}$, and similarly for $(x^\beta)^\alpha$.
    (c) $x^{-\alpha} = E(-\alpha L(x)) = (E(\alpha L(x))^{-1} = (x^\alpha)^{-1} = 1/x^\alpha$.
    (d) If $x > 1$, then $L(x) > 0$, so that if $\alpha < \beta$, then $\alpha L(x) < \beta L(x)$. Since E is strictly increasing, we deduce that $x^\alpha = E(\alpha L(x)) < E(\beta L(x)) = x^\beta$.
13. (a) If $\alpha > 0$, it follows from 8.3.13 that $x \mapsto x^\alpha$ is strictly increasing. Since $\lim\limits_{x\to 0+} L(x) = -\infty$, use 8.3.7 to show that $\lim\limits_{x\to 0+} x^\alpha = 0$.
    (b) If $\alpha < 0$, then αL is strictly decreasing, whence $x \mapsto x^\alpha$ is strictly decreasing. Here $\lim\limits_{x\to 0+} \alpha L(x) = \infty$, so $\lim\limits_{x\to 0+} x^\alpha = \lim\limits_{y\to\infty} E(y) = \infty$, and $\lim\limits_{x\to\infty} \alpha L(x) = -\infty$, so $\lim\limits_{x\to\infty} x^\alpha = \lim\limits_{y\to-\infty} E(y) = 0$.

14. By 8.3.14, if $x>0$ and $a>0$, $a \neq 1$, then $(\log_a x)(\ln a) = \ln x$, whence $a^{\log_a x} = E((\log_a x)(\ln a)) = E(\ln x) = x$ for $x>0$. Similarly, since $\ln(a^y) = \ln(E(y \ln a)) = y \ln a$, we have $\log_a(a^y) = (\ln a^y)/(\ln a) = y$ for all $y \in \mathbb{R}$.
15. Use 8.3.14 and 8.3.9(vii).
16. Use 8.3.14 and 8.3.9(viii).
17. Indeed, we have $\log_a x = (\ln x)/(\ln a) = [(\ln x)/(\ln b)] \cdot [(\ln b)/(\ln a)]$ if $a \neq 1$, $b \neq 1$. Now take $a = 10, b = e$.

Section 8.4

Although the characterization of the sine and cosine functions given here is not the traditional approach to these functions, it has several advantages. Indeed, the most important properties of these functions follow quickly from the fact that they satisfy the differential equation $f''(x) = -f(x)$ for all $x \in \mathbb{R}$, and that any function satisfying this differential equation is a linear combination of sin and cos. [Other approaches to the trigonometric functions are sketched in R. G. Bartle's *Elements of Real Analysis*.]

Sample Assignment: Exercises 1, 2, 4, 6, 7, 8.

Partial Solutions:

1. If $n > 2|x|$, then $|\cos x - C_n(x)| \le (16/15)|x|^{2n}/(2n)!$. Hence $\cos(0.2) \approx 0.980\,067$ and $\cos 1 \approx 0.549\,302$. As for the sine function, if $n > 2|x|$, then $|\sin x - S_n(x)| \le (16/15)|x|^{2n}/(2n)!$. Hence $\sin(0.2) \approx 0.198\,669$ and $\sin 1 \approx 0.841\,471$.
2. It follows from Corollary 8.4.3 that $|\sin x| \le 1$ and $|\cos x| \le 1$.
3. If $x<0$, then $-x>0$ so property (vii) never holds. However, if $x<0$, we have $-(-x) \le S(-x) = -S(x) \le -x$, whence $-|x| = x \le S(x) \le -x = |x|$. It follows from 8.4.8(ix) that $-x^3/6 \le -(\sin x - x) \le 0$ if $x \ge 0$. Hence, if $x<0$, we have $x^3/6 \le -(\sin x - x) \le 0$, whence $|\sin x - x| \le |x|^3/6$.
4. We integrate 8.4.8(x) twice on $[0, x]$. Note that the polynomial on the left has a zero in the interval $[1.56, 1.57]$, so $1.56 \le \pi/2$.
5. Exercise 8.4.4 shows that $C_4(x) \le \cos x \le C_3(x)$ for all $x \in \mathbb{R}$. Integrating several times, we get $S_4(x) \le \sin x \le S_5(x)$ for all $x>0$. Show that $S_4(3.05) > 0$ and $S_5(3.15) < 0$. (This procedure can be sharpened.)
6. Clearly $s_n(0) = 0$ and $c_n(0) = 1$; also $s_n'(x) = c_n(x)$ and $c_{n+1}'(x) = s_n(x)$ for $x \in \mathbb{R}, n \in \mathbb{N}$. Show that if $|x| \le A$ and $m > n > 2A$, then $|c_m(x) - c_n(x)| < (16/15)A^{2n}/(2n)!$, whence it follows that $|c(x) - c_n(x)| < (16/15)A^{2n}/(2n)!$ and the convergence of (c_n) to c is uniform on each interval $[-A, A]$. Similarly for (s_n). Since $c_{n+1}'' = c_n$ and $s_{n+1}'' = s_n$ for $n \in \mathbb{N}$, property (j) holds. Property (jj) is evident, and it follows from $s_n' = c_n$ and $c_{n+1}' = s_n$ that $s' = c$ and $c' = s$.

7. Note that the derivative $D[(c(x))^2 - (s(x))^2] = 0$ for all $x \in \mathbb{R}$. To establish uniqueness, argue as in 8.4.4.

8. Let $g(x) := f(0)c(x) + f'(0)s(x)$ for $x \in \mathbb{R}$, so that $g''(x) = g(x), g(0) = f(0)$ and $g'(0) = f'(0)$. Therefore the function $h(x) := f(x) - g(x)$ has the property that $h''(x) = h(x)$ for all $x \in \mathbb{R}$ and $h(0) = 0, h'(0) = 0$. Thus it follows as in the proof of 8.4.4 that $g(x) = f(x)$ for all $x \in \mathbb{R}$, so that $f(x) = f(0)c(x) + f'(0)s(x)$. Now note that $f_1(x) := e^x$ and $f_2(x) := e^{-x}$ satisfy $f''(x) = f(x)$ for $x \in \mathbb{R}$. Hence $f_1(x) = c(x) + s(x)$ and $f_2(x) = c(x) - s(x)$, whence it follows that $c(x) = \frac{1}{2}(e^x + e^{-x})$ and $s(x) = \frac{1}{2}(e^x - e^{-x})$.

9. If $\varphi(x) := c(-x)$, show that $\varphi''(x) = \varphi(x)$ and $\varphi(0) = 1, \varphi'(0) = 0$, so that $\varphi(x) = c(x)$ for all $x \in \mathbb{R}$. Therefore c is even.

10. It follows from Exercise 8 that $c(x) > e^x/2 > 0$ for all $x \in \mathbb{R}$. Therefore s is strictly increasing on $\mathbb{R}$ and, since $s(0) = 0$, it follows that c is strictly increasing on $(0, \infty)$. Thus $1 = c(0) < c(x)$ for all $x \in (0, \infty)$; since c is even, we deduce that $c(x) \geq 1$ for all $x \in \mathbb{R}$. Since $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to \infty} e^{-x} = 0$, it follows from Exercise 8 that $\lim_{x \to \infty} c(x) = \lim_{x \to \infty} s(x) = \infty$.

CHAPTER 9

INFINITE SERIES

Students have been exposed to much of the material in this chapter in their introductory calculus course; however, their recollection of this material probably will not go much beyond the mechanical application of some of the "Tests". At this point, they should be prepared to approach the subject on a more sophisticated level.

Instructors will recall that an introduction to series was given in Section 3.7 and it would be well to review that section very briefly. Since Section 9.1 is quite short it is possible to do that in one lesson. Although much of Section 9.2 will be familiar, the short Section 9.3 will probably be new. Section 9.4 is an interesting one, and ties this discussion together with Chapter 8.

Section 9.1

The notion of absolute convergence is rather subtle, and should be stressed. The discussion about rearrangements will help the student to realize the importance of absolute convergence. If the Cauchy Condensation Test in Exercise 3.7.15 has not been discussed before, it should be covered now.

Sample Assignment: Exercises 1, 2, 4, 7, 8, 11, 12.

Partial Solutions:

1. Let s_n be the nth partial sum of $\sum_1^\infty a_n$, let t_n be the nth partial sum of $\sum_1^\infty |a_n|$, and suppose that $a_n \geq 0$ for $n > P$. If $m > n > P$, show that $t_m - t_n = s_m - s_n$. Now apply the Cauchy Criterion.
2. Replace each strictly negative term by 0 to obtain $\sum p_n$, and replace each strictly positive term by 0 to obtain $\sum q_n$. If $\sum p_n$ is convergent, then $\sum q_n$ is also convergent since $q_n = a_n - p_n$ and $\sum a_n$ is convergent (see Exercise 3.7.4). In this case, $|a_n| = p_n - q_n$, so that $\sum a_n$ is absolutely convergent, a contradiction.
3. Take positive terms until the partial sum exceeds 1, then take negative terms until the partial sum is less than 1, then take positive terms until the partial sum exceeds 2, etc.
4. It was used together with the Cauchy Criterion to assure that given $\varepsilon > 0$ there exists $N < q$ such that $\sum_{N+1}^{q} |x_k| < \varepsilon$.
5. Yes. Let $M > 0$ be such that every partial sum t_n of $\sum |a_n|$ satisfies $0 \leq t_n \leq M$. If $\sum b_k$ is rearrangement of $\sum a_n$ and if $u_k := |b_1| + \cdots + |b_k|$, then there exists $n \in \mathbb{N}$ such that every term $|b_i|$ in u_k is contained in t_n and hence $0 \leq u_k \leq t_n \leq M$. Therefore $\sum b_k$ is absolutely convergent.

6. Use Mathematical Induction to show that if $n \geq 2$, then $s_n = -\ln 2 - \ln n + \ln(n+1)$. Yes, since $a_n < 0$ for all $n \geq 2$.
7. (a) If $|b_k| \leq M$ for all $k \in \mathbb{N}$, then $|a_n b_n + \cdots + a_m b_m| \leq M(|a_n| + \cdots + |a_m|)$. Now apply the Cauchy Criterion 3.7.4.
   (b) Let $b_k := +1$ if $a_k \geq 0$ and $b_k := -1$ if $a_k < 0$. Then $\sum a_k b_k = \sum |a_k|$.
8. Let $a_k := (-1)^k/\sqrt{k}$, so $a_k^2 = 1/k$.
9. Since $s_{2n} - s_n = a_{n+1} + \cdots + a_{2n} \geq n a_{2n} = \frac{1}{2}(2na_{2n})$, then $\lim(2na_{2n}) = 0$. Similarly $s_{2n+1} - s_n \geq (n+1)a_{2n+1} \geq \frac{1}{2}(2n+1)a_{2n+1}$, so that we have $\lim(2n+1)a_{2n+1} = 0$. Consequently $\lim(na_n) = 0$.
10. Consider $\sum_2^\infty 1/(n \ln n)$, which diverges by Exercise 3.7.17(a).
11. Indeed, if $|n^2 a_n| \leq M$ for all n, then $|a_n| \leq M/n^2$ so Example 3.7.6(c) and the Comparison Test 3.7.7(a) apply.
12. If $0 < a < 1$, then $a^n \to 0$, so $1/(1+a^n) \to 1$ and the series diverges by the nth Term Test 3.7.3. Similarly, if $a = 1$, then $1/(1+a^n) = \frac{1}{2}$. If $a > 1$, then $1/(1+a^n) < (1/a)^n$ and the series converges by comparison with a geometric series with ratio $1/a < 1$.
13. (a) Rationalize to obtain $\sum x_n$ where $x_n := [\sqrt{n}(\sqrt{n+1} + \sqrt{n})]^{-1}$ and note that $x_n \approx y_n := 1/(2n)$. Now apply the Limit Comparison Test 3.7.8 to show the series diverges.
    (b) Rationalize and compare with $\sum 1/n^{3/2}$ to show the series converges.
14. If $\sum a_n$ is absolutely convergent, then the partial sums (t_n) of $\sum |a_n|$ are bounded, say by M. It is evident that the absolute value of the partial sums of any subseries of a_n are also bounded by M, so these subseries are also (absolutely) convergent.

    Conversely, if every subseries of $\sum a_n$ is convergent, then the subseries consisting of the strictly positive (and strictly negative) terms are absolutely convergent, whence it follows that $\sum a_n$ is absolutely convergent.
15. If (i) exists, let $s_n := \sum_{k=1}^\infty c_k$. For fixed $n \in \mathbb{N}$, choose i_0, j_0 such that $\{c_1, \ldots, c_n\} \subseteq \{a_{ij} : i \leq i_0, j \leq j_0\}$. Then $s_n \leq \sum_{i=1}^{i_0} \sum_{j=1}^{j_0} a_{ij} \leq \sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} = B$. Since $n \in \mathbb{N}$ is arbitrary, it follows that C exists and $C \leq B$.

    If (ii) holds, given $n \in \mathbb{N}$, choose $m \in \mathbb{N}$ such that $\{a_{i1}, \ldots, a_{in}\} \subseteq \{c_1, \ldots, c_m\}$. Then $\sum_{j=1}^n a_{ij} \leq \sum_{k=1}^m c_k \leq C$ so that $\sum_{j=1}^\infty a_{ij} \leq C$ for all $j \in \mathbb{N}$. Now choose N such that $\{a_{ij} : i \leq m, j \leq n\} \subseteq \{c_k : k \leq N\}$. Then $\sum_{i=1}^m \sum_{j=1}^n a_{ij} \leq \sum_{k=1}^N c_k \leq C$. First let $n \to \infty$, then let $m \to \infty$ to get $B \leq C$.
16. Note that $\sum_{j=1}^\infty a_{ij} = -1$ if $i = 1$ and $= 0$ if $i > 1$, whence $\sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} = -1$. On the other hand, $\sum_{i=1}^\infty a_{ij} = 1$ if $j = 1$ and $= 0$ if $j > 1$, whence $\sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij} = 1$.

Section 9.2

The results presented in this section are primarily designed to test for absolute convergence. All of these tests are very useful, but they are not definitive in the sense that there are some series that do not yield to them, but require more delicate tests (such as Kummer's and Gauss's Tests that are presented in more advanced treatises).

Sample Assignment: Exercises 1, 2(a,b), 3(a,c,e), 5, 7(a,c), 9, 12(a,d), 16.

Partial Solutions:

1. (a) Convergent; compare with $\sum 1/n^2$.
   (b) Divergent; apply 9.2.1 with $b_n := 1/n$.
   (c) Divergent; note that $2^{1/n} \to 1$.
   (d) Convergent; apply 9.2.3 or 9.2.5.
2. (a) Divergent; apply 9.2.1 with $b_n := 1/n$.
   (b) Convergent; apply 3.7.7 or 9.2.1 with $y_n := n^{-3/2}$.
   (c) Convergent; use 9.2.4 and note that $(n/(n+1))^n \to 1/e < 1$.
   (d) Divergent; the nth term does not tend to 0.
3. (a) $(\ln n)^p < n$ for large n, by L'Hospital's Rule.
   (b) Convergent; apply 9.2.3.
   (c) Convergent; note that $(\ln n)^{\ln n} > n^2$ for large n. Now apply 3.7.7 or 9.2.1.
   (d) Divergent; note that $(\ln n)^{\ln \ln n} = \exp((\ln \ln n)^2) < \exp(\ln n) = n$ for large n. Now apply 3.7.7 or 9.2.1.
   (e) Divergent; apply 9.2.6 or Exercise 3.7.15.
   (f) Convergent; apply 9.2.6 or Exercise 3.7.15.
4. (a) Convergent; apply 9.2.2 or 9.2.4.
   (b) Divergent; apply 9.2.4.
   (c) Divergent; note that $e^{\ln n} = n$.
   (d) Convergent; note that $(\ln n)\exp(-n^{1/2}) < n\exp(-n^{1/2}) < 1/n^2$ for large n, by L'Hospital's Rule.
   (e) Divergent; apply 9.2.4.
   (f) Divergent; apply 9.2.4.
5. Compare with $\sum 1/n^2$.
6. Apply the Integral Test 9.2.6.
7. (a,b) Convergent; apply 9.2.5.
   (c) Divergent; note that $x_n \geq (2/4)(4/6)\cdots(2n/(2n+2)) = 1/(n+1)$. Or, apply 9.2.9.
   (d) Convergent; apply 9.2.9.
8. Here $\lim(x_n^{1/n}) = a < 1$.
9. If $m > n \geq K$, then $|s_m - s_n| \leq |x_{n+1}| + \cdots + |x_m| < r^{n+1}/(1-r)$. Now let $m \to \infty$.

10. Relation (5) implies that $|x_{n+k}| \leq r^k |x_n|$ when $n \geq K$. Therefore, if $m > n \geq K$, we have $|s_m - s_n| \leq (r + r^2 + \cdots + r^{m-n})|x_n| < |x_n|(r/(1-r))$. Now take the limit as $m \to \infty$.
11. Let $m > n \geq K$. Use (12) to get $(a-1)(|x_{n+1}| + \cdots + |x_m|) \leq n|x_{n+1}| - m|x_{m+1}|$, whence $|s_m - s_n| \leq |x_{n+1}| + \cdots + |x_m| \leq |x_{n+1}|n/(a-1)$. Now take the limit as $m \to \infty$.
12. (a) A crude estimate of the remainder is given by $s - s_4 = 1/6 \cdot 7 + 1/7 \cdot 8 + \cdots < \int_5^\infty x^{-2}\,dx = 1/5$. Similarly $s - s_{10} < 1/11$ and $s - s_n < 1/(n+1)$, so that 999 terms suffice to get $s - s_{999} < 1/1000$. [In this case the series telescopes and we have $s_n = 1/2 - 1/(n+2)$.]
(d) If $n \geq 4$, then $x_{n+1}/x_n \leq 5/8$ so (by Exercise 10) $|s - s_4| \leq 5/12$. If $n \geq 10$, then $x_{n+1}/x_n \leq 11/20$ so that $|s - s_{10}| \leq (10/2^{10})(11/9) < 0.012$. If $n = 14$, then $|s - s_{14}| < 0.000\,99$. Alternatively, if $n \geq 4$, then $x_n^{1/n} \leq 1/\sqrt{2}$ so (by Exercise 9) $|s - s_4| \leq 1/4(\sqrt{2} - 1) < 0.61$. If $n \geq 10$, then $x_n^{1/n} \leq (1/2)(10)^{1/10}$ so that $|s - s_{10}| < 0.017$. If $n = 15$, then $|s - s_{15}| < 0.000\,69$.
13. (b) Here $1/5\sqrt{6} + 1/6\sqrt{7} + \cdots < \int_4^\infty x^{-3/2}\,dx = 1$. Therefore we have $\sum_{n+1}^\infty < \int_n^\infty x^{-3/2}dx = 2/\sqrt{n}$, so $|s - s_{10}| < 0.633$ and $|s - s_n| < 0.001$ when $n > 4 \times 10^6$.
(c) If $n \geq 4$, then $|s - s_n| \leq (0.694)x_n$ so that $|s - s_4| < 0.065$. If $n \geq 10$, then $|s - s_n| \leq (0.628)x_n$ so that $|s - s_{10}| < 0.000\,023$.
14. Note that $s_{3n} > 1 + 1/4 + 1/7 + \cdots + 1/(3n+1)$, which is not bounded.
15. Since $\ln n = \int_1^n t^{-1}\,dt < 1/1 + 1/2 + \cdots + 1/(n-1)$, it follows that $1/n < c_n$. Since $c_n - c_{n+1} = \ln(n+1) - \ln n - 1/(n+1) = 1/\theta_n - 1/(n+1)$ by the Mean Value Theorem, where $\theta_n \in (n, n+1)$, we have $c_n - c_{n+1} > 0$. Therefore the decreasing sequence (c_n) converges, say to C. An elementary calculation shows that $b_n = c_{2n} - c_n + \ln 2$, so that $b_n \to \ln 2$.
16. Note that, for an integer with n digits, there are 9 ways of picking the first digit and 10 ways of picking each of the other $n-1$ digits. Thus there are 8 "sixless" values n_k from 1 to 9, there are $8 \cdot 9$ such values from 10 to 99, there are $8 \cdot 9^2$ values between 100 and 999, and so on. Hence $\sum 1/n_k$ is dominated by $8/1 + 8 \cdot 9/10 + 8 \cdot 9^2/10^2 + \cdots = 80$.

    There is one value of m_k from 1 to 9, there is one value from 10 to 19, one from 20 to 29, etc. Hence the (grouped) terms of $\sum 1/m_k$ dominate $1/10 + 1/20 + \cdots = (1/10)\sum 1/k$, which is divergent.

    There are 9 values of p_k from 1 to 9, there are 9 such values from 10 to 19, and so on. Hence the (grouped) terms of $\sum 1/p_k$ dominate $9(1/10) + 9(1/20) + \cdots = (9/10)\sum 1/k$, which is divergent.
17. The terms are positive and $\lim(n(1 - x_{n+1}/x_n)) = q - p$; therefore, it follows from 9.2.9 that the series is convergent if $q > p + 1$ and is divergent if $q < p + 1$. If $q = p + 1$, use 9.2.1 with $y_n := 1/n$ to establish divergence.
18. Here $\lim(n(1 - x_{n+1}/x_n)) = (c - a - b) + 1$, so the series is convergent if $c > a + b$ and is divergent if $c < a + b$. [If $c = a + b$ and $ab > 0$, one can show

that $x_{n+1}/x_n \geq n/(n+1)$ so that (nx_n) is an increasing sequence, whence the series is divergent. The restriction that $ab \geq 0$ can be removed by using a stronger test, such as Kummer's or Gauss's test.]

19. Here $b_1 + b_2 + \cdots + b_n = A^{1/2} - (A - A_n)^{1/2} \to A^{1/2}$, so that $\sum b_n$ converges to $A^{1/2}$. Also $b_n > 0$ and $a_n/b_n = (A - A_{n-1})^{1/2} + (A - A_n)^{1/2} \to 0$.
20. Here (b_n) is a decreasing sequence converging to 0 and $b_1 + b_2 + \cdots + b_n > (a_1 + a_2 + \cdots + a_n)/\sqrt{A_n} = \sqrt{A_n}$, so the series $\sum b_n$ diverges. Also $b_n/a_n = 1/\sqrt{A_n} \to 0$.

Section 9.3

In this short section, we present some results that often enable one to handle series that are conditionally convergent. The easiest and most useful one is the Alternating Series Test 9.3.2, since alternating series often arise (e.g., from power series with positive coefficients). In addition, the estimate for the rapidity of convergence (in Exercise 2) is particularly easy to apply. The tests due to Dirichlet and Abel are more complicated, but apply to more general series.

Sample Assignment: Exercises 1, 2, 5, 7, 9, 10. (Warning: Exercises 11 and 15(c,f) are rather difficult.)

Partial Solutions:

1. (a) Absolutely convergent. (b) Conditionally convergent.
   (c) Divergent. (d) Conditionally convergent.
2. Show by induction that $s_2 < s_4 < s_6 < \cdots < s_5 < s_3 < s_1$. Hence the limit lies between s_n and s_{n+1} so that $|s - s_n| < |s_{n+1} - s_n| = z_{n+1}$.
3. Let $z_{2n-1} := 1/n$ and $z_{2n} := 0$. Or, if it is desired to have $z_n > 0$ for all n, take $z_{2n-1} := 1/n$ and $z_{2n} := 1/n^2$.
4. Let $(y_n) := (+1, -1, +1, -1, \ldots)$.
5. One can use Dirichlet's Test with $(y_n) := (+1, -1, -1, +1, +1, -1, -1, \ldots)$ to establish the convergence. Or group the terms in pairs (after the first), use the Alternating Series Test to establish the convergence of the grouped series, and note that $|s_{2n} - s_{2n-1}| = 1/2n \to 0$ so that $|s_k - s| \to 0$.
6. If $q > p$, then $X := (1/n^{q-p})$ is a convergent monotone sequence. Now apply Abel's Test with $y_n := a_n/n^p$.
7. If $f(x) := (\ln x)^p/x^q$, then $f'(x) < 0$ for x sufficiently large. L'Hospital's Rule shows that the terms in the alternating series approach 0.
8. (a) Convergent by 9.3.2.
   (b) Divergent; use 9.2.1 with $y_n := 1/(n+1)$.
   (c) Divergent; the terms do not approach 0.
   (d) Divergent; use 9.2.1 with $y_n := 1/n$.

9. If $t > 0$, the sequence (e^{-nt}) decreases to 0, so Dirichlet's Test applies.

10. The convergence of $\sum (a_n/n)$ follows from Dirichlet's Test; the convergence of $\sum (s_n/n(n+1))$ follows by comparison with $\sum (1/n^2)$. To obtain the equality, use Abel's Lemma with $x_k := 1/k, y_k := a_k$ and $n = 0$. Then let $m \to \infty$.

11. Dirichlet's Test does not apply (directly, at least), since the partial sums of the series generated by $(1, -1, -1, 1, 1, 1, \ldots)$ are not bounded. To establish the convergence, one can group the terms $1 - (1/2 + 1/3) + (1/4 + 1/5 + 1/6) - \cdots$ to get an alternating series. The block consisting of k terms ends with $n_k := 1 + 2 + \cdots + k = k(k+1)/2$, and starts with $n_{k-1} + 1$. The sum of this block of k terms is greater than the integral of $1/x$ over the interval $[n_{k-1} + 1, n_k + 1]$ and less than the integral of $1/x$ over the interval $[n_{k-1}, n_k]$; hence this sum is greater than $\ln[(n_k + 1)/(n_{k-1} + 1)]$ and less than $\ln[n_k/n_{k-1}]$. Since it is seen that $n_{k+1}/n_k < (n_k + 1)/(n_{k-1} + 1)$ when $k > 2$, it follows that the terms in the grouped series are decreasing. Moreover, since $\ln(n_k/n_{k-1}) \to 0$, it follows that the terms of the grouped series approach 0; consequently, the grouped series converges. This means that the subsequence (s_{n_k}) of the partial sums of the original series converges. But it is readily seen that if $n_{k-1} \le n \le n_k$, then s_n lies between the partial sums $s_{n_{k-1}}$ and s_{n_k}. Hence $\lim(s_n) = \lim(s_{n_k})$, and the series converges.

12. Let $|s_n| \le B$ for all n. If $m > n$, then it follows from Abel's Lemma that $|\sum_{k=n+1}^{m} x_k y_k| \le B[|x_m| + |x_{n+1}| + \sum_{k=n+1}^{m} |x_k - x_{k+1}|]$. Since $x_n \to 0$ and $\sum |x_k - x_{k+1}|$ is convergent, the dominant term approaches 0, and the Cauchy Criterion applies.

13. Since (a_n) and (b_n) are bounded monotone sequences, they are convergent. Hence, if $\varepsilon > 0$ is given, there exists $M(\varepsilon)$ such that if $m > n \ge M(\varepsilon)$, then $0 \le a_{n+1} - a_{m+1} < \varepsilon$ and $0 \le b_{m+1} - b_{n+1} < \varepsilon$. Since $x_k - x_{k+1} = (a_k - a_{k+1}) + (b_{k+1} - b_k)$, one has $\sum_{k=n+1}^{m} |x_k - x_{k+1}| = (a_{n+1} - a_{m+1}) + (b_{m+1} - b_{n+1}) < 2\varepsilon$.

14. By Abel's Lemma, $\sum_{k=n+1}^{m} a_k/k = s_m/m - s_n/(n+1) + \sum_{n+1}^{m-1} s_k/k(k+1)$. Thus $|\sum_{n+1}^{m} a_k/k| \le M[(1/m)^{1-r} + (1/(n+1))^{1-r} + \sum_{n+1}^{m-1} 1/k^{2-r}]$. Since $1 - r > 0$, the first two terms approach 0; since $p := 2 - r > 1$, the series $\sum 1/k^p$ is convergent by 9.2.7(d), so the final terms tends to 0.

15. (a) Use Abel's Test with $x_n := 1/n$.
(b) Use the Cauchy Inequality with $x_n := \sqrt{a_n}, y_n := 1/n$, to get $\sum \sqrt{a_n}/n \le (\sum a_n)^{1/2}(\sum 1/n^2)^{1/2}$, establishing convergence.
(c) Let $\sum (-1)^{k-1} c_k, c_k > 0$, be conditionally convergent. Since $\pi/2 > 3/2$, each interval $I_k := [(k-1)\pi + \pi/4, (k-1)\pi + 3\pi/4]$ contains at least one integer point; we let $n_k \in I_k$ be the integer nearest $(k - 1/2)\pi$ so that $|\sin n_k| > 1/2$. Let $a_{n_k} := (-1)^{k-1} c_k$ and $a_n := 0$ if $n \ne n_k$ so that $\sum a_n$ is convergent. However, $(-1)^{k-1} \sin n_k > 1/2$ so that $b_{n_k} := a_{n_k} \sin n_k > c_k/2$; hence $\sum b_n$ is divergent.

(d) Let $a_n := [n(\ln n)^2]^{-1}$, which converges by the Integral Test. However, $b_n := [\sqrt{n}\ln n]^{-1}$, which diverges.
(e) The sequence $(n^{1/n})$ decreases to 1 (see Example 3.1.11(d)); hence Abel's Test 9.3.5 applies to give convergence.
(f) If $\sum a_n$ is absolutely convergent, so is $\sum b_n$; otherwise $\sum b_n$ may diverge. Indeed, $(1/k)/(1+1/k) = 1/(k+1)$; hence, if the block of terms $1/p$, $-1/2p$, $-1/2p$ appears in $\sum a_n$, then the sum of the corresponding block of terms in $\sum b_n$ is $1/(p+1) - 2/(2p+1) = -1/(p+1)(2p+1)$. Consequently, if this block of three terms is repeated $2p+1$ times in $\sum a_n$, the sum of the corresponding terms in $\sum b_n$ is $-1/(n+1)$. Now let (a_n) consist of the block $1/1$, $-1/2$, $-1/2$ repeated 3 times, followed by the block $1/2$, $-1/4$, $-1/4$ repeated 5 times, followed by the block $1/3$, $-1/6$, $-1/6$ repeated 7 times, and so on. Then $\sum a_n$ converges to 0, but $\sum b_n = -\sum 1/(p+1)$ is divergent.

Section 9.4

The notion of convergence [respectively, uniform convergence] of a series of functions is nothing more than the convergence [resp., uniform convergence] of the sequence of partial sums of the functions. The importance of the uniform convergence is that it enables one to interchange limit operations (as in Theorems 9.4.2–9.4.4). While the Weierstrass M-Test 9.4.6 is only a sufficient condition for uniform convergence, it is often very useful.

The use of the Ratio Test to determine the radius of convergence will be familiar to most students, but since the limit of $|a_{n+1}/a_n|$ does not always exist, the Cauchy-Hadamard Theorem 9.4.9 (which always applies) is very important.

It is stressed that the results in 9.4.11 and 9.4.12 are *for power series only*. For general series of functions, one may not be able to integrate or differentiate the series term-by-term.

Sample Assignment: Exercises 1(a,c,e), 2, 5, 6(a,c,e), 7, 11, 15, 16, 17.

Partial Solutions:

1. (a) Take $M_n := 1/n^2$ in the Weierstrass M-Test.
(b) If $a > 0$, take $M_n := (1/a^2)/n^2$ to show uniform convergence for $|x| \geq a$. The series is convergent for all $x \neq 0$, but it is not uniformly convergent on $\mathbb{R} \setminus \{0\}$, since if $x_n := 1/n$, then $f_n(x_n) = 1$.
(c) Since $|\sin y| \leq |y|$, the series converges for all x. But since $f_n(n^2) = \sin 1 > 0$, the series is not uniformly convergent on $\mathbb{R}$. However, if $a > 0$, the series is uniformly convergent for $|x| \leq a$ since then $|f_n(x)| \leq a/n^2$.
(d) If $0 \leq x \leq 1$, the nth term does not go to 0, so the series is divergent. If $1 < x < \infty$, the series is convergent, since $(x^n+1)^{-1} \leq (1/x)^n$. It is uniformly convergent on $[a, \infty)$ for $a > 1$. However, it is not uniformly convergent on $(1, \infty)$; take $x_n := (1+1/n)^{1/n}$.

(e) Since $0 \le f_n(x) \le x^n$, the series is convergent on [0,1) and uniformly convergent on $[0,a]$ for any $a \in (0,1)$. It is not uniformly convergent on [0,1); take $x_n := 1 - 1/n$. The series is divergent on $[1,\infty)$ since the terms do not approach 0.
(f) If $x \ge 0$, the series is alternating and is convergent with $|s(x) - s_n(x)| \le 1/(n+1)$. Hence it is uniformly convergent.

2. Since $|\sin nx| \le 1$, we can take $M_n := |a_n|$.

3. If $\varepsilon > 0$, there exists M such that if $n \ge M$, then $|c_n \sin nx + \cdots + c_{2n} \sin 2nx| < \varepsilon$ for all x. If $x \in [\pi/6n, 5\pi/12n]$, then $\sin kx \ge 1/2$ for $k = n, \ldots, 2n$, so that $(n+1)c_{2n} < 2\varepsilon$. It follows that $2nc_{2n} < 4\varepsilon$ and $(2n+1)c_{2n+1} < 4\varepsilon$ for $n \ge M$.

4. If $\rho = \infty$, then the sequence $(|a_n|^{1/n})$ is not bounded. Hence if $|x_0| > 0$, then there are infinitely many $k \in \mathbb{N}$ with $|a_k| > 1/|x_0|$ so that $|a_k x_0^k| > 1$. Thus the series is not convergent when $x_0 \ne 0$.
   If $\rho = 0$ and $x_0 \ne 0$, then since $|a_n|^{1/n} < 1/2|x_0|$ for all $n \ge n_0$, it follows that $|a_n x_0^n| < 1/2^n$ for $n \ge n_0$, whence $\sum a_n x_0^n$ is convergent.

5. Suppose that $L := \lim(|a_n|/|a_{n+1}|)$ exists and that $0 < L < \infty$. If follows from the Ratio Test that $\sum a_n x^n$ converges for $|x| < L$ and diverges for $|x| > L$. Therefore it follows from the Cauchy-Hadamard Theorem that $L = R$. [Alternatively, if $0 < \varepsilon < L$, it can be shown by Induction that there exists $m \in \mathbb{N}$ such that $|a_m|(L+\varepsilon)^{-k} < |a_{m+k}| < |a_m|(L-\varepsilon)^{-k}$. Hence there exists $A > 0, B > 0$ such that $A(L+\varepsilon)^{-r} < |a_r| < B(L-\varepsilon)^{-r}$ for $r \ge m$, whence $A^{1/r}/(L+\varepsilon) < |a_r|^{1/r} < B^{1/r}/(L-\varepsilon)$ for $r \ge m$. We conclude that $\rho = 1/L$, so $L = R$.]
   If $L = 0$ and $\varepsilon > 0$ is given, then we have $|a_n| < \varepsilon |a_{n+1}|$ for $n \ge n_\varepsilon$, whence $|a_n x^n| < |a_{n+1} x^{n+1}|$ for $|x| \ge \varepsilon$ so that the terms do not go to 0 for $|x| \ge \varepsilon$ and the series diverges for these values. Since $\varepsilon > 0$ is arbitrary, we have $L = R = 0$.
   If $L = \infty$, given $M > 0$, there exists n_M such that if $n \ge n_M$ then $|a_{n+1}| < (1/M)|a_n|$. Hence if $|x| < M/2$, we have $|a_{n+1} x^{n+1}| \le \frac{1}{2}|a_n x^n|$ for all $n \ge n_M$ and so the series converges for $|x| < M/2$. But since $M > 0$ is arbitrary, we deduce that $L = R = \infty$.
   For example, take $a_n := \frac{1}{2}$ for n even and $a_n := 2$ for n odd. Here L does not exist, but $R = 1$.

6. (a) $\rho = \lim(1/n) = 0$ so $R = \infty$.
(b) $|a_n/a_{n+1}| = (n+1)/(1+1/n)^\alpha$, so $R = \infty$.
(c) $\lim |a_n/a_{n+1}| = 1/e$, so $R = 1/e$.
(d) $\lim |a_n/a_{n+1}| = 1$. Alternatively, since $1/n \le \ln n \le n$, we have $1/n^{1/n} \le [1/\ln n]^{1/n} \le n^{1/n}$, so $\rho = 1$ and $R = 1$.
(e) $\lim |a_n/a_{n+1}| = 4$, so $R = 4$.
(f) Since $\lim(n^{1/\sqrt{n}}) = 1$, we have $R = 1$.

7. Since $0 \le a_n \le 1$ for all n, but $a_n = 1$ for infinitely many n, it follows that $\rho = \limsup |a_n|^{1/n} = 1$, whence $R = 1$.
8. Note that $|a_n|^{1/n} \le |na_n|^{1/n} = n^{1/n}|a_n|^{1/n}$ and use that $\lim(n^{1/n}) = 1$.
9. By 3.1.11(c) we have $p^{1/n} \to 1$.
10. By the Uniqueness Theorem 9.4.13, $a_n = (-1)^n a_n$ for all n, so that $a_n = 0$ for n odd.
11. It follows from Taylor's Theorem 6.4.1 that if $|x| < r$, then $|R_n(x)| \le r^{n+1}B/(n+1)! \to 0$ as $n \to \infty$.
12. If $n \in \mathbb{N}$, there exists a polynomial P_n such that $f^{(n)}(x) = e^{-1/x^2}P_n(1/x)$ for $x \neq 0$.
13. Let $g(x) := 0$ for $x \ge 0$ and $g(x) := e^{-1/x^2}$ for $x < 0$. Show that $g^{(n)}(0) = 0$ for all n.
14. If $m \in \mathbb{N}$, the series reduces to a finite sum and holds for all $x \in \mathbb{R}$, so we consider the case where m is not an integer. If $x \in [0,1)$ and $n > m, n \in \mathbb{N}$, then by Taylor's Theorem there exists $c_x \in (0,x)$ such that $0 \le R_n(x) \le \binom{m}{n+1} x^{m+1}$ $(1+c_x)^{n+1-m} \le \binom{m}{n+1} x^{n+1}$. Use Theorem 3.2.11 to show that $R_n(x) \to 0$ as $n \to \infty$.
15. Here $s_n(x) = (1 - x^{n+1})/(1-x)$.
16. Substitute $-y$ for x in Exercise 15 and integrate from $y = 0$ to $y = x$ for $|x| < 1$, which is justified by Theorem 9.4.11.
17. If $|x| < 1$, it follows from Exercise 15 that $(1+x^2)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. If we apply Theorem 9.4.11 and integrate from 0 to x, we get the given expansion for $\operatorname{Arctan} x$, valid for $|x| < 1$.
18. If $|x| < 1$, it follows from Exercise 14 that $(1-x^2)^{1/2} = \sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n x^{2n}$. Now integrate from 0 to x and evaluate the binomial coefficient.
19. Integrate $e^{-t^2} = \sum_{n=0}^{\infty}(-1)^n t^{2n}/n!$ to get

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \quad \text{for } x \in \mathbb{R}.$$

20. Apply Exercise 14 and the fact that

$$\int_0^{\pi/2} (\sin x)^{2n}\,dx = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}.$$

CHAPTER 10

THE GENERALIZED RIEMANN INTEGRAL

This chapter will certainly be new for the students, and it is also likely that it contains material that will not be familiar to most instructors. However, the close parallel between Section 10.1 and Sections 7.1–7.3 should make it easier to absorb the material. Indeed, the only difference between the generalized Riemann integral and the ordinary Riemann integral is that slightly different orderings are used for the collection of tagged partitions. It is quite surprising that such a "slight difference" in the ordering of the partitions makes such a big difference in the resulting classes of integrable functions.

As we have noted the material in the first part of Section 10.1 is very similar to that in Chapter 7. In Section 10.2 we learn that there is no such thing as an "improper integral", and that the generalized Riemann integral is not an "absolute" integral. Section 10.3 shows how to extend the integral to functions whose domain is not bounded; while this procedure seems a bit unnatural, it is quite simple. (Section 10.3 can be omitted if time is short.)

The final Section 10.4 contains some important results; especially the Monotone and Dominated Convergence Theorems. Most treatments of the Lebesgue integral *start* with the notion of a measurable function, but using our approach it almost seems to be an afterthought. That is not the case, but just a reflection of the fact that all of the functions we have been dealing with are measurable.

Section 10.1

In order to use the definition to show that a function is in $\mathcal{R}^*[a,b]$, we need to construct a set of gauges δ_ε. This is done for the specific functions in Example 10.1.4. (Usually that is rather difficult, except for ordinary Riemann integrable functions where a constant gauge suffices.) In most of the other results in this section, these gauges are constructed from other gauges; thus a gauge for $f+g$ is constructed from gauges for f and g. In the Fundamental Theorem 10.1.9, the gauge for $f = F'$ is constructed using the differentiability of F. The Fundamental Theorems are the highpoint of this section; the later material can be treated more lightly.

Sample Assignment: Exercises 1, 4, 7(a,c,e), 11, 13, 15, 16, 20.

Partial Solutions:

1. (a) Since $t_i - \delta(t_i) \le x_{i-1}$ and $x_i \le t_i + \delta(t_i)$, then $0 \le x_i - x_{i-1} \le 2\delta(t_i)$.
(b) Apply (a) to each subinterval.
(c) If $\dot{\mathcal{Q}} = \{([y_{j-1}, y_j], s_j)\}_{j=1}^m$ satisfies $\|\dot{\mathcal{Q}}\| \le \delta_*$, then $s_j - \delta(s_j) \le s_j - \delta_* \le y_{j-1}$ and $y_j \le s_j + \delta_* \le s_j + \delta(s_j)$, so that $s_j \in [y_{j-1}, y_j] \subseteq [s_j - \delta_*, s_j + \delta_*] \subseteq [s_j - \delta(s_j), s_j + \delta(s_j)]$. Thus $\dot{\mathcal{Q}}$ is δ-fine.
(d) $\inf\{1/2^{k+2}\} = 0$.

2. (a) If t is a tag for two subintervals, it belongs to both of them, so it is the right endpoint of one and the left endpoint of the other subinterval.
(b) Consider the tagged partition $\{([0,1],1),([1,2],1),([2,3],3),([3,4],3)\}$.

3. (a) If $\dot{\mathcal{P}} = \{([x_{i-1},x_i],t_i)\}_{i=1}^n$ and if t_k is a tag for both subintervals $[x_{k-1},x_k]$ and $[x_k,x_{k+1}]$, we must have $t_k = x_k$. We replace these two subintervals by the subinterval $[x_{k-1},x_{k+1}]$ with the tag t_k, keeping the δ-fineness property. Since $f(t_k)(x_k - x_{k-1}) + f(t_k)(x_{k+1} - x_k) = f(t_k)(x_{k+1} - x_{k-1})$, this consolidation of the subintervals does not change the value of the Riemann sums. A finite number of such consolidations will result in the desired partition $\dot{\mathcal{Q}}_1$.
(b) No. The tagged partition $\{([0,1],0)([1,2],2)\}$ of $[0,2]$ has the property that every tag belongs to exactly one subinterval.
(c) If t_k is the tag for the subinterval $[x_{k-1},x_k]$ and is an endpoint of this subinterval, we make no change. However, if $t_k \in (x_{k-1},x_k)$, then we replace $[x_{k-1},x_k]$ by the two intervals $[x_{k-1},t_k]$ and $[t_k,x_k]$ both tagged by t_k, keeping the δ-fineness property. Since $f(t_k)(x_k - x_{k-1}) = f(t_k)(t_k - x_{k-1}) + f(t_k)(x_k - t_k)$, this splitting of a subinterval into two subintervals does not change the value of the Riemann sums.

4. If $x_{k-1} \le 1 \le x_k$ and if t_k is the tag for $[x_{k-1},x_k]$, then we cannot have $t_k > 1$, since then $t_k - \delta(t_k) = \frac{1}{2}(t_k+1) > 1$. Similarly, we cannot have $t_k < 1$, since then $t_k + \delta(t_k) = \frac{1}{2}(t_k+1) < 1$. Therefore we must have $t_k = 1$.
   If the subintervals $[x_{k-1},x_k]$ and $[x_k,x_{k+1}]$ both have the number 1 as tag, then $1 - .01 = 1 - \delta(1) \le x_{k-1} < x_{k+1} \le 1 + \delta(1) = 1 + .01$ so that $x_{k+1} - x_{k-1} \le 0.02$.

5. (a) Let $\delta(t) := \frac{1}{2}\min\{|t-1|,|t-2|,|t-3|\}$ if $t \ne 1,2,3$ and $\delta(t) := 1$ for $t = 1,2,3$.
(b) Let $\delta_2(t) := \min\{\delta(t),\delta_1(t)\}$, where δ is as in part (a).

6. If $f \in \mathcal{R}^*[a,b]$ and $\varepsilon > 0$ is given, then there exists δ_ε as in Definition 10.1.1, and we let $\gamma_\varepsilon := \delta_\varepsilon$. If $\dot{\mathcal{P}}$ satisfies the stated condition, then $\dot{\mathcal{P}}$ is δ_ε-fine and so $|S(f;\dot{\mathcal{P}}) - L| < \varepsilon$.
   Conversely, suppose the stated condition is satisfied for some gauge γ_ε, and let $\delta_\varepsilon := \frac{1}{2}\gamma_\varepsilon$. If $\dot{\mathcal{P}}$ is δ_ε-fine, then $0 < x_i - x_{i-1} \le 2\delta_\varepsilon(t_i) = \gamma_\varepsilon(t_i)$, so the hypothesis implies that $|S(f;\dot{\mathcal{P}}) - L| < \varepsilon$. Therefore $f \in \mathcal{R}^*[a,b]$ in the sense of Definition 10.1.1.

7. (a) $F_1(x) := (2/3)x^{3/2} + 2x^{1/2}$,
(b) $F_2(x) := (2/3)(1-x)^{3/2} - 2(1-x)^{1/2}$,
(c) $F_3(x) := (2/3)x^{3/2}(\ln x - 2/3)$ for $x \in (0,1]$ and $F_3(0) := 0$,
(d) $F_4(x) := 2x^{1/2}(\ln x - 2)$ for $x \in (0,1]$ and $F_4(0) := 0$,
(e) $F_5(x) := -\sqrt{1-x^2} + \operatorname{Arcsin} x$.
(f) $F_6(x) := \operatorname{Arcsin}(x-1)$.

8. Although the partition $\dot{\mathcal{P}}_0$ in the proof of 7.1.5 may be δ_ε-fine for some gauge δ_ε, the tagged partition $\dot{\mathcal{P}}_z$ need not be δ_ε-fine, since the value $\delta_\varepsilon(z)$ may be

much smaller than $\delta_\varepsilon(x_j)$. For the ordinary Riemann integral, we were only concerned with the norms $\|\dot{\mathcal{P}}_0\|$, $\|\dot{\mathcal{P}}_z\|$, which are equal.

9. If f were integrable, then $\int_0^1 f \geq \int_0^1 s_n = 1/2 + 1/3 + \cdots + 1/(n+1)$.
10. We enumerate the nonzero rational numbers as $r_k = m_k/n_k$ and define $\delta_\varepsilon(m_k/n_k) := \varepsilon/(n_k 2^{k+1})$ and $\delta_\varepsilon(x) := 1$ otherwise.
11. The function F is continuous on $[a,b]$, and $F'(x) = f(x)$ for $x \in [0,1] \setminus \mathbb{Q}$. Since $\mathbb{Q}$ is countable, the Fundamental Theorem 10.1.9 applies.
12. The function M is not continuous on $[-2,2]$, so Theorem 10.1.9 does not apply. In fact, by Exercise 9 the function $x \mapsto 1/x$ is not in $\mathcal{R}^*[0,2]$ no matter how we define it at 0.
13. In fact, L_1 is continuous and $L_1'(x) = l_1(x)$ for $x \neq 0$, so Theorem 10.1.9 applies.
14. (a) This is possible since F is continuous at c_k.
    (b) Since $f(c_k) = 0$, then we have $|F(x_i) - F(x_{i-1}) - f(c_k)(x_i - x_{i-1})| \leq |F(x_i) - F(c_k)| + |F(x_{i-1}) - F(c_k)| \leq \varepsilon/2^{k+1}$.
    (c) The point c_k can be the tag for at most two subintervals. The sum of such terms with tags in E is $< \varepsilon$, and the sum of the terms with tags in $I \setminus E$ is $< \varepsilon(b-a)$.
15. Since $C_1'(x) = (3/2)x^{1/2}\cos(1/x) + x^{-1/2}\sin(1/x)$ for $x > 0$, this function is in $\mathcal{R}^*[0,1]$. Since the first term in C_1' has a continuous extension on $[0,1]$, it is integrable; therefore the second term in also integrable.
16. We have $C_2'(x) = \cos(1/x) + (1/x)\sin(1/x)$ for $x > 0$. By the analogue of Exercise 7.2.12, the first term belongs to $\mathcal{R}[0,1]$ and therefore to $\mathcal{R}^*[0,1]$.
17. (a) Take $x = \varphi(t) := t^2 + t - 2$ so $\varphi'(t) = 2t + 1$ and $E_\varphi = \emptyset$ to get $\int_{x=4}^{x=10} \operatorname{sgn} x \, dx = |10| - |4| = 6$.
    (b) Take $x = \varphi(t) := \sqrt{t}$ so $x^2 = t$, $\varphi'(t) = 1/(2\sqrt{t})$ and $E_\varphi = \{0\}$. We get $\int_{x=0}^{x=2} 2x^2(1+x)^{-1} dx = 2\int_0^2 (x - 1 + (1+x)^{-1}) dx = 2(2 + \ln 3)$.
    (c) Take $x = \varphi(t) := \sqrt{t-1}$ so that $t = x^2 + 1$, $\varphi'(t) = 1/(2\sqrt{t-1})$ and $E_\varphi = \{1\}$. We get $\int_{x=0}^{x=2} 2(x^2+1)^{-1} dx = 2 \text{ Arctan } 2$.
    (d) Take $x = \varphi(t) := \operatorname{Arcsin} t$ so $t = \sin x$, $\varphi'(t) = (1 - x^2)^{-1/2}$ and $E_\varphi = \{1\}$. We get $\int_{x=0}^{x=\pi/2} \cos^2 x \, dx = \frac{1}{2}\int_0^{\pi/2}(1 + \cos 2x) dx = (\frac{1}{2}x + \frac{1}{4}\sin 2x)|_0^{\pi/2} = \frac{1}{4}\pi$.
18. Let $f(x) := 1/\sqrt{x}$ for $x \in (0,1]$ and $f(0) := 0$ and use Exercise 9.
19. (a) In fact $f(x) := F'(x) = \cos(\pi/x) + (\pi/x)\sin(\pi/x)$ for $x > 0$; we set $f(0) := 0, F'(0) = 0$. Then f and $|f|$ are continuous at every point in $(0,1]$. It follows as in Exercise 16 that $f \in \mathcal{R}^*[0,1]$.
    (b) Since $F(a_k) = 0$ and $F(b_k) = (-1)^k/k$, Theorem 10.1.9 implies that $1/k = |F(b_k) - F(a_k)| = |\int_{a_k}^{b_k} f| \leq \int_{a_k}^{b_k} |f|$.
    (c) If $|f| \in \mathcal{R}^*[0,1]$, then $\sum_{k=1}^n 1/k \leq \sum_{k=1}^n \int_{a_k}^{b_k} |f| \leq \int_0^1 |f|$ for all $n \in \mathbb{N}$, which is a contradiction.

20. Indeed, $\operatorname{sgn}(f(x)) = (-1)^k = m(x)$ on $[a_k, b_k]$ so $m(x) \cdot f(x) = |m(x)f(x)|$ for $x \in [0,1]$. Since the restrictions of m and $|m|$ to every interval $[c,1]$ for $0 < c < 1$ are step functions, they belong to $\mathcal{R}[c,1]$. By Exercise 7.2.11, m and $|m|$ belong to $\mathcal{R}[0,1]$ and $\int_0^1 m = \sum_{k=1}^{\infty}(-1)^k/k(2k+1)$ and $\int_0^1 |m| = \sum_{k=1}^{\infty} 1/k(2k+1)$.
21. Indeed, $\varphi(x) = \Phi'(x) = |\cos(\pi/x)| + (\pi/x)\sin(\pi/x) \cdot \operatorname{sgn}(\cos(\pi/x))$ for $x \notin E$ by Example 6.1.7(c). Evidently φ is not bounded near 0. It is seen that if $x \in [a_k, b_k]$, then $\varphi(x) = |\cos(\pi/x)| + (\pi/x)|\sin(\pi/x)| = |\varphi(x)|$ so that $\int_{a_k}^{b_k} |\varphi| = \Phi(b_k) - \Phi(a_k) = 1/k$, from which it follows that $|\varphi| \notin \mathcal{R}^*[0,1]$.
22. Here $\psi(x) = \Psi'(x) = 2x|\cos(\pi/x)| + \pi\sin(\pi/x) \cdot \operatorname{sgn}(\cos(\pi/x))$ for $x \notin \{0\} \cup E_1$ by Example 6.1.7(c). Since ψ is bounded, Exercise 7.2.11 applies. We cannot apply Theorem 7.3.1 to evaluate $\int_0^b \psi$ since E is not finite, but Theorem 10.1.9 applies and $\psi \in \mathcal{R}[0,1]$. Corollary 7.3.15 implies that $|\psi| \in \mathcal{R}[0,1]$.
23. If $p \geq 0$, then $mp \leq fp \leq Mp$, where m and M denote the infimum and the supremum of f on $[a,b]$, so that $m\int_a^b p \leq \int_a^b fp \leq M\int_a^b p$. If $\int_a^b p = 0$, the result is trivial; otherwise, the conclusion follows from Bolzano's Intermediate Value Theorem 5.3.7.
24. By the Multiplication Theorem 10.1.14, $fg \in \mathcal{R}^*[a,b]$. If g is increasing, then $g(a)f \leq fg \leq g(b)f$ so that $g(a)\int_a^b f \leq \int_a^b fg \leq g(b)\int_a^b f$. Let $K(x) := g(a)\int_a^x f + g(b)\int_x^b f$, so that K is continuous and takes all values between $K(b)$ and $K(a)$.

Section 10.2

The proof of Hake's Theorem (which is omitted) is another instance where one has to construct a set of gauges for the function; here one uses the gauges of the restrictions of the function to a sequence of intervals $[a, \gamma_n]$, where $\gamma_n \to b$.

It is not possible to overestimate the importance of the Lebesgue integral. Usually this integral is obtained in a *very different way*.

Sample Assignment: Exercises 1, 2, 5, 6(a,b), 7(a,c,e), 9, 11.

Partial Solutions:

1. Indeed $\int_a^c f \to A$ as $c \to b-$ if and only if the sequential condition holds.
2. (a) If $G(x) := 3x^{1/3}$ for $x \in [0,1]$ then $\int_c^1 g = G(1) - G(c) \to G(1) = 3$.
   (b) We have $\int_c^1 (1/x)dx = \ln c$, which does not have a limit in $\mathbb{R}$ as $c \to 0$.
3. Here $\int_0^c (1-x)^{-1/2}dx = 2 - 2(1-c)^{1/2} \to 2$ as $c \to 1-$.
4. Since $\int_\gamma^b \omega = \lim_{c\to b}\int_\gamma^c \omega$, given $\varepsilon > 0$ there exists $\gamma_\varepsilon \geq \gamma$ such that if $\gamma_\varepsilon \leq c_1 < c_2 < b$, then $|\int_a^{c_2} f - \int_a^{c_1} f| \leq \int_{c_1}^{c_2} \omega < \varepsilon$. By the Cauchy Criterion, the limit $\lim_{c\to b}\int_a^c f$ exists. Now apply Hake's Theorem.

5. Because of continuity, $g_1 \in \mathcal{R}^*[c,1]$ for all $c \in (0,1)$. If $\omega(x) := x^{-1/2}$, then $|g_1(x)| \le \omega(x)$ for all $x \in [0,1]$. The "left version" of the preceding exercise implies that $g_1 \in \mathcal{R}^*[0,1]$ and the above inequality and the Comparison Test 10.2.4 implies that $g_1 \in \mathcal{L}[0,1]$.

6. (a,b) Both functions are bounded on [0, 1] (use L'Hospital) and continuous in (0, 1).
(c) If $x \in (0, \frac{1}{2}]$ the integrand is dominated by $|(\ln \frac{1}{2}) \ln x|$. If $x \in [\frac{1}{2}, 1)$ the integrand is dominated by $|(\ln \frac{1}{2}) \ln(1-x)|$.
(d) If $x \in (0, \frac{1}{2}]$ the integrand is dominated by $(2/\sqrt{3})|\ln x|$. If $x \in [\frac{1}{2}, 1)$, the integrand is bounded and continuous.

7. (a) Convergent, since $|f_1(x)| \le 1/\sqrt{x}$.
(b) Divergent, since $f_2(x) \ge 1/(2x^{3/2})$ for $x \in (0,1]$.
(c) Divergent, since $-f_3(x) \ge \ln 2/x$ for $x \in (0, \frac{1}{2}]$.
(d) Convergent, since $|f_4(x)| \le 2|\ln x|$ on $(0, \frac{1}{2}]$ and is bounded on $[\frac{1}{2}, 1)$.
(e) Convergent, since $|f_5(x)| \le |\ln x|$ for $x \in (0,1]$.
(f) Divergent, since $f_6(x) \ge 1/(x-1)$ for $x \in [\frac{1}{2}, 1)$.

8. If $f \in \mathcal{R}[a,b]$, then f is bounded and is in $\mathcal{R}^*[a,b]$. Thus the Comparison Test 10.2.4 applies.

9. Let $f(x) := 1/\sqrt{x}$ for $x \in (0,1]$ and $f(0) := 0$.

10. By the Multiplication Theorem 10.1.4, the product $fg \in \mathcal{R}^*[a,b]$. Since $|f(x)g(x)| \le B|f(x)|$, then $fg \in \mathcal{L}[a,b]$ and $\|fg\| \le B\|f\|$.

11. (a) Let $f(x) := (-1)^k 2^k/k$ for $x \in [c_{k-1}, c_k)$ and $f(1) := 0$, where the c_k are as in Example 10.2.2(a). Then $f^+ := \max\{f, 0\}$ equals $2^k/k$ on $[c_{k-1}, c_k)$ when k is even and equals 0 elsewhere. Hence $\int_0^{c_{2k}} f^+ = 1/2 + 1/4 + \cdots + 1/2n$, so $f^+ \notin \mathcal{R}^*[0,1]$.
(b) From the first formula in the proof of Theorem 10.2.7, we have $f^+ = \max\{f, 0\} = \frac{1}{2}(f + |f|)$. Thus, if $f \in \mathcal{L}[a,b]$, then both $f, |f| \in \mathcal{R}^*[a,b]$ and so f^+ belongs to $\mathcal{R}^*[a,b]$. Since $f^+ \ge 0$, it belongs to $\mathcal{L}[a,b]$.

12. If $\alpha \le f$ and $\alpha \le g$, then $\alpha \le \min\{f,g\}$. The second equality in the proof of Theorem 10.2.7 implies that $0 \le |f-g| = f + g - 2\min\{f,g\} \le f + g - 2\alpha$. Therefore $f + g - 2\alpha \in \mathcal{L}[a,b]$ and the Comparison Theorem 10.2.4 implies that $f + g - 2\min\{f,g\} \in \mathcal{L}[a,b]$, whence $\min\{f,g\} \in \mathcal{R}^*[a,b]$.

13. (j) Evidently, $\mathrm{dist}(f,g) = \int_a^b |f-g| \ge 0$.
(jj) If $f(x) = g(x)$ for all $x \in [a,b]$, then $\mathrm{dist}(f,g) = \int_a^b |f-g| = \int_a^b 0 = 0$.
(jjj) $\mathrm{dist}(f,g) = \int_a^b |f-g| = \int_a^b |g-f| = \mathrm{dist}(g,f)$.
(jv) Since $|f-h| \le |f-g| + |g-h|$, we have $\mathrm{dist}(f,h) = \int_a^b |f-h| \le \int_a^b |f-g| + \int_a^b |g-h| = \mathrm{dist}(f,g) + \mathrm{dist}(g,h)$.

14. Consider the Dirichlet function.

15. By 10.2.10(iv), $\|f\| = \|f \pm g \mp g\| \le \|f \pm g\| + \|g\|$, whence $\|f\| - \|g\| \le \|f \pm g\|$. Similarly, $\|g\| \le \|g \pm f \mp f\| \le \|g \pm f\| + \|f\|$, whence $\|g\| - \|f\| \le \|f \pm g\|$. Now combine.

16. If (f_n) converges to f in $\mathcal{L}[a,b]$, given $\varepsilon > 0$ there exists $K(\varepsilon/2)$ such that if $m, n \ge K(\varepsilon/2)$ then $\|f_m - f\| < \varepsilon/2$ and $\|f_n - f\| < \varepsilon/2$. Therefore $\|f_m - f_n\| \le \|f_m - f\| + \|f - f_n\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus we may take $H(\varepsilon) := K(\varepsilon/2)$.

17. Indeed, $\|f_n\| = \int_0^1 x^n\,dx = 1/n$ and $\|f_n - \theta\| = 1/n$.

18. If $m > n$, then $\|g_m - g_n\| \le 1/n + 1/m \to 0$. One can take $g := \operatorname{sgn}$.

19. Since $\|h_{2n} - h_n\| = 1$, there is no such $h \in \mathcal{L}[0,1]$.

20. Here $\|k_n\| = 1/n$ and we can take $k = \theta$ the 0-function, or any other function in $\mathcal{L}[0,1]$ with $\int_0^1 |k| = 0$.

Section 10.3

Although it is important to extend the integral to functions defined on unbounded intervals, this section can be omitted if time is short.

Sample Assignment: Exercises 1, 3, 5, 7, 13, 15, 17(a,b), 18(a,b).

Partial Solutions:

1. Let $b \ge \max\{a, 1/\delta(\infty)\}$. If $\dot{\mathcal{P}}$ is a δ-fine partition of $[a,b]$, show that $\dot{\mathcal{P}}$ is a δ-fine subpartition of $[a,\infty)$.
2. The Cauchy Criterion for the existence of $\lim_\gamma \int_a^\gamma f$ is: given $\varepsilon > 0$ there exists $K(\varepsilon) \ge a$ such that if $q > p \ge K(\varepsilon)$, then $|\int_p^q f| = |\int_a^q f - \int_a^p f| < \varepsilon$.
3. If $f \in \mathcal{L}[a,\infty)$, apply the preceding exercise to $|f|$. Conversely, if $\int_p^q |f| < \varepsilon$ for $q > p \ge K(\varepsilon)$, then $|\int_a^q f - \int_a^p f| \le \int_p^q |f| < \varepsilon$ so that $\lim_\gamma \int_a^\gamma f$ and $\lim_\gamma \int_a^\gamma |f|$ exist; therefore $f, |f| \in \mathcal{R}^*[a,\infty)$ and so $f \in \mathcal{L}[a,\infty)$.
4. If $f \in \mathcal{L}[a,\infty)$, the existence of $\lim_\gamma \int_a^\gamma |f|$ implies that V is a bounded set. Conversely, if V is bounded, let $v := \sup V$. If $\varepsilon > 0$, there exists K such that $v - \varepsilon < \int_a^K |f| \le v$. If $K \le p < q$, we have $\int_p^q |f| < \varepsilon$, so the preceding exercise applies.
5. If $f, g \in \mathcal{L}[a,\infty)$, then $f, |f|, g$ and $|g|$ belong to $\mathcal{R}^*[a,\infty)$, so Example 10.3.3(a) implies that $f+g$ and $|f|+|g|$ belong to $\mathcal{R}^*[a,\infty)$ and $\int_a^\infty (|f|+|g|) = \int_a^\infty |f| + \int_a^\infty |g|$. Since $|f+g| \le |f| + |g|$, it follows that $\int_a^\gamma |f+g| \le \int_a^\gamma |f| + \int_a^\gamma |g| \le \int_a^\infty |f| + \int_a^\infty |g|$, whence $\|f+g\| \le \|f\| + \|g\|$.
6. Indeed, $\int_1^\gamma (1/x)dx = \ln\gamma$, which does not have limit as $\gamma \to \infty$. Or, use Exercise 2 and the fact that $\int_p^{2p} (1/x)dx = \ln 2 > 0$ for all $p \ge 1$.

7. Since f is continuous on $[1,\infty)$, both $f, |f| \in \mathcal{R}^*[1,\gamma]$ for $1 \le \gamma$. If $\gamma \le p < q$, then $|\int_p^q f| \le \int_p^q |f| \le K\int_p^q (1/x^2)dx \le K(1/p - 1/q)$. Therefore, by Exercise 2, both $f, |f|$ belong to $\mathcal{R}^*[1,\infty)$.

8. If $\gamma > 0$, then $\int_0^\gamma \cos x\, dx = \sin\gamma$, which does not have a limit as $\gamma \to \infty$.

9. (a) We have $\int_0^\gamma e^{-sx}dx = (1/s)(1 - e^{-s\gamma}) \to 1/s$.
(b) Let $G(x) := -(1/s)e^{-sx}$ for $x \in [0,\infty)$ and $G(\infty) := 0$, so G is continuous on $[0,\infty)$ and $G(x) \to G(\infty)$. By the Fundamental Theorem 10.3.5, we have $\int_0^\infty g = G(\infty) - G(0) = 1/s$.

10. (a) Integrate by parts to get $1/s^2$ plus a term that $\to 0$ as $\gamma \to \infty$.
(b) Let $G_1(x) := -(x/s)e^{-sx} - (1/s^2)e^{-sx}$ for $x \in [0,\infty)$ and $G_1(\infty) := 0$, so that $G_1'(x) = xe^{-sx}$ and $G_1(x) \to 0$ as $x \to \infty$. The Fundamental Theorem implies that $\int_0^\infty xe^{-sx}dx = G_1(\infty) - G_1(0) = 1/s^2$.

11. Use Mathematical Induction. The case $n = 1$ is Exercise 10. Assuming the formula holds for $k \in \mathbb{N}$, we integrate by parts.

12. (a) If $x \ge e$, then $(\ln x)/x \ge 1/x$. Since $\int_1^\infty (1/x)dx$ is not convergent, neither is the given one.
(b) Integrate by parts on $[1,\gamma]$ and then let $\gamma \to \infty$.

13. (a) Since $|\sin x| \ge 1/\sqrt{2} > 1/2$ and $1/x > 1/(n+1)\pi$ for x in the interval $(n\pi + \pi/4, n\pi + 3\pi/4)$, then $|(1/x)\sin x| \ge 1/(2\pi(n+1))$ on this interval, which has length $\pi/2$. Therefore $\int_{n\pi}^{(n+1)\pi} |(1/x)\sin x| dx \ge 1/(4(n+1))$.
(b) If $\gamma > (n+1)\pi$, then $\int_0^\gamma |D| \ge (1/4)(1/1 + 1/2 + \cdots + 1/(n+1))$.

14. The integrand is bounded, so is in $\mathcal{R}^*[0,\gamma]$. Integrating by parts, we get $\int_p^q x^{-1/2}\sin x\, dx = -x^{-1/2}\cos\big|_p^q - (1/2)\int_p^q x^{-3/2}\cos x\, dx$. Since $|\cos x| \le 1$, we have $|\int_p^q x^{-1/2}\sin x\, dx| \le q^{-1/2} + p^{-1/2} + (1/2)\int_p^q x^{-3/2}\, dx$ which is $\le (5/4)(q^{-1/2} + p^{-1/2}) \to 0$ as $p \to \infty$.

15. Let $u = \varphi(x) = x^2$ so that $\int_0^\gamma \sin(x^2)dx = (1/2)\int_0^{\gamma^2} u^{-1/2}\sin u\, du$. Now apply Exercise 14.

16. (a) Convergent. Since $\ln x \in \mathcal{R}^*[0,\gamma]$ the given integrand is in $\mathcal{R}^*[0,\gamma]$. Since $(\ln x)/\sqrt{x} \to 0$ as $x \to \infty$, then $|(\ln x)/(x^2+1)| \le K/x^{3/2}$ for x sufficiently large.
(b) Divergent. As in (a), the integrand is in $\mathcal{R}^*[0,1]$. Since $4x^2 > x^2 + 1$ for $x \ge 1$, then $(\ln x)/\sqrt{x^2+1} > (\ln x)/2x \ge 1/2x$ for $x \ge e$, so that the integrand is not in $\mathcal{R}^*[e,\infty]$.
(c) Divergent. If $x \in [0,1]$, then $2 > x+1$ so that $1/x(x+1) > 1/2x$. Thus the restriction of the integrand is not in $\mathcal{R}^*[0,1]$, so the integrand is not in $\mathcal{R}^*[0,\infty]$.
(d) Convergent. The integrand is dominated by $1/x^2$ on $[0,\infty]$.
(e) Divergent. $x/\sqrt[3]{1+x^3} \to 1$ as $x \to \infty$, so that $1/2 < x/\sqrt[3]{1+x^3}$ for x sufficiently large, so $1/(2x) < 1/\sqrt[3]{1+x^3}$ for x sufficiently large.

(f) Convergent. Since $0 \leq \operatorname{Arctan} x \leq \pi/2$ for $x \geq 0$ so the integrand is dominated by $1/(x^{3/2}+1) < 1/x^{3/2}$ for $x \geq 1$.

17. (a) If $f_1(x) := \sin x$, then $f_1 \notin \mathcal{R}^*[0,\infty)$. By Exercise 14, if $f_2(x) := x^{-1/2}\sin x$, then $f_2 \in \mathcal{R}^*[0,\infty)$ and $\varphi_2(x) := 1/\sqrt{x}$ is bounded and decreasing on $[1,\infty)$.
(b) If $f(x) := \sin(x^2)$, then Exercise 15 implies that $f \in \mathcal{R}^*[0,\infty)$. Here $\varphi(x) := x/(x+1)$ is bounded and increasing on $[0,\infty)$.
(c) Take $f(x) := x^{-1/2}\sin x \in \mathcal{R}^*[0,\infty)$ by Exercise 14 and $\varphi(x) := (x+1)/x$ so that φ is bounded and decreasing on $[0,\infty)$.
(d) Take $\varphi(x) := \operatorname{Arctan} x$, so φ is bounded and increasing on $[0,\infty)$, while $f(x) := 1/(x^{3/2}+1) < x^{-3/2}$ for $x \geq 1$.

18. (a) $f(x) := \sin x$ is continuous so is in $\mathcal{R}^*[0,\gamma]$. Also $F(x) := \int_0^x \sin t\,dt = 1-\cos x$ is bounded by 2 on $[0,\infty)$ and $\varphi(x) := 1/x$ decreases to 0.
(b) Take $\varphi(x) := 1/\ln x$ so φ decreases monotonely to 0.
(c) $F(x) := \int_0^x \cos t\,dt = \sin x$ is bounded by 1 on $[0,\infty)$ and $\varphi(x) := x^{-1/2}$ decreases monotonely to 0.
(d) $\varphi(x) := x/(x+1)$ increases to 1 (not 0).

19. Let $u = \varphi(x) := x^2$ so that $\int_0^\gamma x^{1/2}\sin(x^2)dx = (1/2)\int_0^{\gamma^2} u^{-1/4}\sin u\,du$. By the Chartier-Dirichlet Test, this integral converges and Hake's Theorem applies.

20. (a) If $\gamma > 0$, then $\int_0^\gamma e^{-x}dx = 1 - e^{-\gamma} \to 1$ so $e^{-x} \in \mathcal{R}^*[0,\infty)$. Similarly $e^{-|x|} = e^x \in \mathcal{R}^*(-\infty,0]$.
(b) $|x-2|/e^{-x/2} \to 0$ as $x \to \infty$, so $|x-2| \leq e^{-x/2}$ for x sufficiently large, so the integrand is dominated by $e^{-x/2}$ for x large. Therefore the integrand is in $\mathcal{R}^*[0,\infty)$ and similarly on $(-\infty,0]$.
(c) We have $0 \leq e^{-x^2} \leq e^{-x}$ for $|x| \geq 1$, so $e^{-x^2} \in \mathcal{R}^*[0,\infty)$. Similarly on $(-\infty,0]$.
(d) The integrand approaches 1 as $x \to 0$. Since $e^x/(e^x - e^{-x}) \to 1$ as $x \to \infty$, we have $2x/(e^x - e^{-x}) \leq 4xe^{-x}$ for x sufficiently large. Therefore the integrand is in $\mathcal{R}^*[0,\infty)$. Similarly on $(-\infty,0]$.

Section 10.4

This section contains some very important results.

Sample Assignment: Exercises 1, 3(a,c,e), 5, 6, 9, 11, 14.

Partial Solutions:

1. (a) Converges to 0 at $x = 0$, to 1 on (0, 1]. Not uniform. Bounded by 1. Increasing. Limit = 1.
(b) Converges to 0 on [0, 1), to $\frac{1}{2}$ at $x = 1$, to 1 on (1,2]. Not uniform. Bounded by 1. Not monotone (although decreasing on [0,1] and increasing on [1,2]). Limit = 1.

(c) Converges to 1 on [0, 1), to $\frac{1}{2}$ at $x = 1$. Not uniform. Bounded by 1. Increasing. Limit$=1$.
(d) Converges to 1 on [0, 1), converges to $\frac{1}{2}$ at $x = 1$, to 0 on (1, 2]. Not uniform. Bounded by 1. Not monotone (although increasing on [0, 1] and decreasing on (1, 2]). Limit$=1$.

2. (a) Converges to $\sqrt{x}$ on [0, 1]. Uniform. Bounded by 1. Increasing. Limit$=$ 2/3.
(b) Define to equal 0 at $x = 0$, converges to $1/\sqrt{x}$ on (0, 1), to $\frac{1}{2}$ at $x = 1$. Not uniform. Not bounded. Dominated by $1\sqrt{x}$. Increasing. Limit$=2$.
(c) Converges to $\frac{1}{2}$ at $x = 1$, to 0 on (1, 2]. Not uniform. Bounded by 1. Decreasing. Limit$=0$.
(d) Define to equal 0 at $x = 0$, converges to $1/2\sqrt{x}$ on (0, 1), to 1 at $x = 1$. Not uniform. Not bounded. Dominated by $1/2\sqrt{x}$. Decreasing. Limit$=1$.

3. (a) Converges to 1 at $x = 0$, to 0 on (0,1]. Not uniform. Bounded by 1. Decreasing. Limit$=0$.
(b) Define to be 0 at $x = 0$. The functions do not have (a finite) integral. Converges to 0. Not uniform. Not bounded. Decreasing. Integral of limit$=0$.
(c) Converges to 0. Not uniform. Bounded by $1/e$. Not monotone. Limit$=0$.
(d) Converges to 0. Not uniform. Not bounded. Not monotone. Limit$=$ $\int_0^\infty ye^{-y}dy = 1$.
(e) Converges to 0. Not uniform. Bounded by $1/\sqrt{2e}$. Not monotone. Limit$=0$.
(f) Converges to 0. Not uniform. Not bounded. Not monotone. Not dominated. Limit$=\frac{1}{2}\int_0^\infty e^{-y}dy = \frac{1}{2}$.

4. (a) Since $f_k(x) \to 0$ for $x \in [0,1)$ and $|f_k(x)| \le 1$, the Dominated Convergence Theorem applies.
(b) $f_k(x) \to 0$ for $x \in [0,1)$, but $(f_k(1))$ is not bounded. No obvious dominating function. Integrate by parts and use (a). The result shows that the Dominated Convergence Theorem does not apply.

5. Note that f_k is a step function and $\int_0^2 f_k = k(1/k) = 1$. If $x \in (0,2]$, there exists k_x such that $2/k < x$ for $k \ge k_x$; therefore $f_k(x) \to 0$.

6. Suppose that $(f_k(c))$ converges for some $c \in [a,b]$. By the Fundamental Theorem, we have $f_k(x) - f_k(c) = \int_c^x f_k'$. By the Dominated Convergence Theorem, $\int_c^x f_k' \to \int_c^x g$, whence $(f_k(x))$ converges for all $x \in [a,b]$. Note that if $f_k(x) := (-1)^k$, then $(f_k(x))$ does not converge for any $x \in [a,b]$.

7. Indeed, $g(x) := \sup\{f_k(x) : k \in \mathbb{N}\}$ equals $1/k$ on $(k-1,k]$, so that $\int_0^n g = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Hence $g \notin \mathcal{R}^*[0,\infty)$.

8. Indeed, $\int_0^\infty e^{-tx}\,dx = (-1/t)e^{-tx}\big|_{x=0}^{x=\infty} = 1/t$. If we integrate by parts, then we get $\int_0^\infty xe^{-tx}\,dx = (-x/t - 1/t^2)e^{-tx}\big|_{x=0}^{x=\infty} = 1/t^2$.

9. Indeed, $\int_0^\infty e^{-tx}\sin x\,dx = -[e^{-tx}(t\sin x + \cos x)/(t^2+1)]\big|_{x=0}^{x=\infty} = 1/(t^2+1)$.

10. (a) If $a > 0$, then $|(e^{-tx}\sin x)/x| \le e^{-ax}$ for $t \in J_a := (a, \infty)$. If $t_k \in J_a$ and $t_k \to t_0 \in J_a$, then the argument in 10.4.6(d) shows that E is continuous at t_0. Also, if $t_k \ge 1$, then $|(e^{-t_k x}\sin x)/x| \le e^{-x}$ and the Dominated Convergence Theorem implies that $E(t_k) \to 0$. Thus $E(t) \to 0$ as $t \to \infty$.
(b) It follows as 10.4.6(e) that $E'(t_0) = -\int_0^\infty e^{-t_0 x}\sin x\,dx = -1/(t_0^2 + 1)$.
(c) By 10.1.9, $E(s) - E(t) = \int_t^s E'(t)dt = -\int_t^s (t^2+1)^{-1}dt = \operatorname{Arctan} t - \operatorname{Arctan} s$ for $s, t > 0$. But $E(s) \to 0$ and Arctan $s \to \pi/2$ as $s \to \infty$.
(d) We do not know that E is continuous as $t \to 0+$.

11. (a) Note that $e^{-t^2(x^2+1)} \le 1$ for $t \ge 0$ and that $e^{-t^2(x^2+1)} \to 0$ as $t \to \infty$ for all $x \ge 0$. Thus the Dominated Convergence Theorem can be applied to sequences to give the continuity of G and the fact that $G(t) \to 0$ as $t \to \infty$.
(b) The partial derivative equals $-2te^{-t^2}e^{-t^2x^2}$, which is bounded by 2 for $t \ge 0, x \in [0,1]$. An argument as in 10.4.6(e) gives the formula for $G'(t)$.
(c) Indeed, $F'(t) = 2e^{-t^2}\int_0^t e^{-x^2}\,dx$ for $t \ge 0$, so $F'(t) = -G'(t)$ for $t \ge 0$.
(d) Since $\lim_{t\to\infty} F(t) = \frac{1}{4}\pi$, we have $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$.

12. Fix $x \in I$. As in 10.4.6(e), if $t, t_0 \in [a,b]$, there exists t_x between t, t_0 such that $f(t,x) - f(t_0,x) = (t-t_0)\frac{\partial f}{\partial t}(t_x, x)$. Therefore $\alpha(x) \le [f(t,x) - f(t_0,x)]/(t-t_0) \le \omega(x)$ when $t \ne t_0$. Now argue as before and use the Dominated Convergence Theorem 10.4.5.

13. (a) If (s_k) is a sequence of step functions converging to f a.e., and (t_k) is a sequence of step functions converging to g a.e., then it follows from Theorem 10.4.9(a) and Exercise 2.2.18 that $(\max\{s_k, t_k\})$ is a sequence of step functions that converges to $\max\{f, g\}$ a.e. Similarly, for $\min\{f, g\}$.
(b) By part (a), the functions $\max\{f, g\}$, $\max\{g, h\}$ and $\max\{h, f\}$ are measurable. Now apply Exercises 2.2.18 and 2.2.19.

14. (a) Since $f_k \in \mathcal{M}[a,b]$ is bounded, it belongs to $\mathcal{R}^*[a,b]$. The Dominated Convergence Theorem implies that $f \in \mathcal{R}^*[a,b]$. The Measurability Theorem 10.4.11 now implies that $f \in \mathcal{M}[a,b]$.
(b) Since $t \mapsto \operatorname{Arctan} t$ is continuous, Theorem 10.4.9(b) implies that $f_k := \operatorname{Arctan} \circ g_k \in \mathcal{M}[a,b]$. Further, $|f_k(x)| \le \frac{1}{2}\pi$ for $x \in [a,b]$, so (f_k), is also a bounded sequence in $\mathcal{M}[a,b]$.
(c) If $g_k \to g$ a.e., from the continuity of Arctan, it follows that $f_k \to f$ a.e. Part (a) implies that $f \in \mathcal{M}[a,b]$ and Theorem 10.4.9(b) applied to $\varphi = \tan$ implies that $g = \tan \circ f \in \mathcal{M}[a,b]$.

15. (a) Since $\mathbf{1}_E$ is bounded, it is in $\mathcal{R}^*[a,b]$ if and only if it is in $\mathcal{M}[a,b]$.
(b) Indeed, $\mathbf{1}_\emptyset$ is the 0-function, and if J is any subinterval of $[a,b]$, then $\mathbf{1}_J$ is a step function.
(c) This follows from the fact that $\mathbf{1}_{E'} = 1 - \mathbf{1}_E$.
(d) We have $x \in E \cup F$ if and only if $x \in E$ *or* $x \in F$. Thus

$$\mathbf{1}_{E\cup F}(x) = 1 \iff \mathbf{1}_E(x) = 1 \text{ or } \mathbf{1}_F(x) = 1 \iff \max\{\mathbf{1}_E(x), \mathbf{1}_F(x)\} = 1.$$

Similarly, $x \in E \cap F$ if and only if $x \in E$ *and* $x \in F$. Thus

$$\mathbf{1}_{E\cap F}(x) = 1 \Longleftrightarrow \mathbf{1}_E(x) = 1 \text{ and } \mathbf{1}_F(x) = 1 \Longleftrightarrow \min\{\mathbf{1}_E(x), \mathbf{1}_F(x)\} = 1.$$

Further, $E \setminus F = E \cap F'$.
(e) If (E_k) is an increasing sequence in $\mathbb{M}[a,b]$, then $(\mathbf{1}_{E_k})$ is an increasing sequence in $\mathcal{M}[a,b]$. Moreover, $\mathbf{1}_E(x) = \lim_k \mathbf{1}_{E_k}(x)$, and we can apply Theorem 10.4.9(c). Similarly, $(\mathbf{1}_{F_k})$ is a decreasing sequence in $\mathcal{M}[a,b]$ and $\mathbf{1}_F(x) = \lim_k \mathbf{1}_{F_k}(x)$.
(f) Let $A_n := \bigcup_{k=1}^n E_k$, so that (A_n) is an increasing sequence in $\mathbb{M}[a,b]$ with $\bigcup_{n=1}^\infty A_n = E$, so (e) applies. Similarly, if $B_n := \bigcap_{k=1}^n F_k$, then (B_n) is a decreasing sequence in $\mathbb{M}[a,b]$ with $\bigcap_{n=1}^\infty B_n = F$.

16. (a) $m(\emptyset) = \int_a^b 0 = 0$ and $0 \le \mathbf{1}_E \le 1$ implies $0 \le m(E) = \int_a^b \mathbf{1}_E \le \int_a^b 1 = b - a$.
(b) Since $\mathbf{1}_{[c,d]}$ is a step function, then $m([c,d]) = d - c$. The other characteristic functions are a.e. to $\mathbf{1}_{[c,d]}$, so have the same integral.
(c) Since $\mathbf{1}_{E'} = 1 - \mathbf{1}_E$, we have $m(E') = \int_a^b (1 - \mathbf{1}_E) = (b-a) - m(E)$.
(d) Note that $\mathbf{1}_{E\cup F} + \mathbf{1}_{E\cap F} = \mathbf{1}_E + \mathbf{1}_F$. Therefore, $m(E \cup F) + m(E\cap F) = \int_a^b (\mathbf{1}_{E\cup F} + \mathbf{1}_{E\cap F}) = \int_a^b (\mathbf{1}_E + \mathbf{1}_F) = m(E) + m(F)$.
(e) If $E \cap F = \emptyset$, then (d) and (a) imply that $m(E\cup F) + 0 = m(E) + m(F)$.
(f) If (E_k) is increasing in $\mathbb{M}[a,b]$ to E, then $(\mathbf{1}_{E_k})$ is increasing in $\mathcal{M}[a,b]$ to $\mathbf{1}_E$. The Monotone Convergence Theorem 10.4.4 implies that $\mathbf{1}_E \in \mathcal{M}[a,b]$ and that $m(E_k) = \int_a^b \mathbf{1}_{E_k} \to \int_a^b \mathbf{1}_E = m(E)$.
(g) If (C_k) is pairwise disjoint and $E_n := \bigcup_{k=1}^n C_k$ for $n \in \mathbb{N}$, then, by Induction in part (e), we have $m(E_n) = m(C_1) + \cdots + m(C_n)$. But, since $\bigcup_{k=1}^\infty C_k = \bigcup_{n=1}^\infty E_n$ and (E_n) is increasing, (f) implies that

$$m\left(\bigcup_{k=1}^\infty C_k\right) = \lim_n m(E_n) = \lim_n \sum_{k=1}^n m(C_k) = \sum_{n=1}^\infty m(C_k).$$

CHAPTER 11

A GLIMPSE INTO TOPOLOGY

We present in this chapter an introduction into the subject of topology. In the first edition of this book, most of the ideas presented here were discussed as the notions naturally arose. However, our experience in teaching from that edition was that some of the students were confused by ideas that they felt were very abstract and difficult. Consequently, in later editions, we have dealt only with open and closed *intervals* in Chapters 1 through 10, even though some of the results that were established held for general open and closed (or at least compact) subsets of $\mathbb{R}$.

Some instructors may wish to blend part of this material into their presentation of the earlier material. Others may decide to omit the entire chapter, or to assign it only to the better students as a unifying "special project".

In the final section, we give the definitions of a metric function and a metric space. They are very important for further developments in analysis as well as in the field of topology, and we feel that they are quite natural ideas. This section will serve as a springboard to students who continue their study of analysis beyond this course.

Section 11.1

Here the notions of open and closed subsets of $\mathbb{R}$ are introduced and such sets are characterized. The final topic of this section is the Cantor set $\mathbb{F}$, which should expand the imagination of the students.

Sample Assignment: Exercises 1, 2, 4, 9, 10, 13, 18, 23.

Partial Solutions:

1. If $|x-u| < \inf\{x, 1-x\}$, then $u < x+(1-x) = 1$ and $u > x-x = 0$, so that $0 < u < 1$.
2. If $x \in (a, \infty)$, then take $\varepsilon_x := x-a$. The complement of $[b, \infty)$ is the open set $(-\infty, b)$.
3. Suppose that $G_1, \ldots, G_k, G_{k+1}$ are open sets and that $G_1 \cup \cdots \cup G_k$ is open. It then follows from the fact that the union of two open sets is open that $G_1 \cup \cdots \cup G_k \cup G_{k+1} = (G_1 \cup \cdots \cup G_k) \cup G_{k+1}$ is open.
4. If $x \in (0,1]$, then $x \in (0, 1+1/n)$ for all $n \in \mathbb{N}$. Also, if $x > 1$, then $x-1 > 0$ so there exists $n_x \in \mathbb{N}$ such that $x-1 > 1/n_x$, whence $x \notin (0, 1+1/n_x)$.
5. The complement of $\mathbb{N}$ is the union $(-\infty, 1) \cup (1, 2) \cup \cdots$ of open intervals.
6. The sequence $(1/n)$ belongs to A and converges to $0 \notin A$, so A is not closed, by 11.1.7. Alternatively, use 11.1.8 and the fact that 0 is a cluster point of A.
7. Corollary 2.4.9 of the Density Theorem implies that every neighborhood of a point x in $\mathbb{Q}$ contains a point not in $\mathbb{Q}$. Hence $\mathbb{Q}$ is not an open set.

8. If F is a closed set, its complement $\mathcal{C}(F)$ is open and $G \setminus F = G \cap \mathcal{C}(F)$.
9. This is a rephrasing of Definition 11.1.2.
10. Note that x is a boundary point of $A \iff$ every neighborhood V of x contains points in A and points in $\mathcal{C}(A) \iff x$ is a boundary point of $\mathcal{C}(A)$.
11. Note that if $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then precisely one of the following statements is true: (i) x is an interior point of A, (ii) x is a boundary point of A, and (iii) x is an interior point of $\mathcal{C}(A)$. Hence, if A is open, then it does not contain any boundary points (since it contains only interior points) of A. Conversely, if A does not contain any boundary points of A, then all of its points are interior points of A.
12. Let F be closed and let x be a boundary point of F. If $x \notin F$, then $x \in \mathcal{C}(F)$. Since $\mathcal{C}(F)$ is an open set, there exists a neighborhood V of x such that $V \subseteq \mathcal{C}(F)$, contradicting the hypothesis that x is a boundary point of F. Conversely, if F contains all of its boundary points and if $y \notin F$, then y is not a boundary point of F, so there exists a neighborhood V of y such that $V \subseteq \mathcal{C}(F)$. This implies that $\mathcal{C}(F)$ is open, so that F is closed.

    (Alternative proof.) The sets F and $\mathcal{C}(F)$ have the same boundary points. Therefore F contains all of its boundary points $\iff \mathcal{C}(F)$ does not contain any of its boundary points $\iff \mathcal{C}(F)$ is open.
13. Since A° is the union of open sets, it is open (by 11.1.4(a)). If G is an open set with $G \subseteq A$, then $G \subseteq A^\circ$ (by its definition). Also $x \in A^\circ \iff x$ belongs to an open set $V \subseteq A \iff x$ is an interior point of A.
14. Since A° is the union of subsets of A, we have $A^\circ \subseteq A$. It follows that $(A^\circ)^\circ \subseteq A^\circ$. Since A° is an open subset of A° and $(A^\circ)^\circ$ is the union of all open sets contains in A°, then $A^\circ \subseteq (A^\circ)^\circ$. Therefore $A^\circ = (A^\circ)^\circ$

    Since A° is an open set in A and B° is an open set in B, then $A^\circ \cap B^\circ$ is an open set in $A \cap B$, whence $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. And since $A^\circ \cap B^\circ$ is an open set in A, then $(A \cap B)^\circ \subseteq A^\circ$; similarly $(A \cap B)^\circ \subseteq B^\circ$, so that $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. Therefore $(A \cap B)^\circ = A^\circ \cap B^\circ$.

    If $A := \mathbb{Q}$ and $B := \mathbb{R} \setminus \mathbb{Q}$, then $A^\circ = B^\circ = \emptyset$, while $A \cup B = \mathbb{R}$, whence $(A \cup B)^\circ = \mathbb{R}$.
15. Since A^- is the intersection of all closed sets containing A, then by 11.1.5(a) it is a closed set containing A. Since $\mathcal{C}(A^-)$ is open, then $z \in \mathcal{C}(A^-) \iff z$ has a neighborhood $V_\varepsilon(z)$ in $\mathcal{C}(A^-) \iff z$ is neither an interior point nor a boundary point of A.
16. For any set B, since B^- is a closed set containing B, then $B \subseteq B^-$. If we take $B = A^-$, we get $A^- \subseteq (A^-)^-$. Since A^- is a closed set containing A^-, we have $(A^-)^- \subseteq A^-$. Therefore $(A^-)^- = A^-$.

    Since $(A \cup B)^-$ is a closed set containing A, then $A^- \subseteq (A \cup B)^-$. Similarly $B^- \subseteq (A \cup B)^-$, so we conclude that $A^- \cup B^- \subseteq (A \cup B)^-$. Conversely $A \subseteq A^-$ and $B \subseteq B^-$, and since A^- and B^- are closed, it follows from 11.1.5(b) that $A^- \cup B^-$ is closed; hence $(A \cup B)^- \subseteq A^- \cup B^-$.

If $A := \mathbb{Q}$, then $A^- = \mathbb{R}$; if $B := \mathbb{R} \setminus \mathbb{Q}$, then $B^- = \mathbb{R}$. Therefore $A \cap B = \emptyset$ while $A^- = B^- = A^- \cup B^- = \mathbb{R}$.

17. Take $A = \mathbb{Q}$.
18. Either $u := \sup F$ belongs to F, or u is a cluster point of F. If F is closed, then 11.1.8 implies that any cluster point of F belongs to F.
19. If $G \neq \emptyset$ is open and $x \in G$, then there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq G$, whence it follows that $a := x - \varepsilon$ is in A_x.
20. If $a_x \in G$, then since G is open, there exists $\varepsilon > 0$ with $(a_x - \varepsilon, a_x + \varepsilon) \subseteq G$. This contradicts the definition of a_x.
21. If $a_x < y < x$ then since $a_x := \inf A_x$ there exists $a' \in A_x$ such that $a_x < a' \leq y$. Therefore $(y, x] \subseteq (a', x] \subseteq G$ and $y \in G$.
22. If $b_x \neq b_y$, then either (i) $b_x < b_y$ or (ii) $b_y < b_x$. In case (i), then $b_x \in I_y = (a_y, b_y) \subseteq G$, contrary to $b_x \notin G$. In case (ii), then $b_y \in I_x = (a_x, b_x) \subseteq G$, contrary to $b_y \notin G$.
23. If $x \in \mathbb{F}$ and $n \in \mathbb{N}$, the interval I_n in F_n containing x has length $1/3^n$. Let y_n be an endpoint of I_n with $y_n \neq x$. Then $0 < |y_n - x| \leq 1/3^n$. Since y_n is an endpoint of I_n, it also belongs to $\mathbb{F}$. Consequently $x = \lim(y_n)$ is a cluster point of $\mathbb{F}$.
24. If $x \in \mathbb{F}$ and $n \in \mathbb{N}$, the interval I_n in F_n containing x has length $1/3^n$. Let z_n be the midpoint of I_n, so that $0 < |z_n - x| \leq 1/3^n$. Since z_n does not belong to F_{n+1}, it follows that $z_n \in \mathcal{C}(\mathbb{F})$. Consequently $x = \lim(z_n)$ is a cluster point of $\mathcal{C}(\mathbb{F})$.

Section 11.2

Most students will find the notion of compactness to be difficult, especially when they learn that they must be prepared to consider *every* open cover of the set. They also find the Heine-Borel Theorem 11.2.5 a bit disappointing, since compact sets in $\mathbb{R}$ turn out to be of a very easily described nature. But this is exactly the reason why the Heine-Borel Theorem is important: it makes the determination of a compact set in $\mathbb{R}$ a relatively simple matter. Students need to be told that the situation is different in more complicated topological spaces; unfortunately, they will have to take that fact on faith until a later course.

While the sequential characterizations of compact sets are somewhat special, they are easier to grasp than the covering aspects.

Sample Assignment: Exercises 1, 3, 4, 5, 6, 9, 11.

Partial Solutions:

1. Let $G_n := (1 + 1/n, 3)$ for $n \in \mathbb{N}$.
2. Let $G_n := (n - 1/2, n + 1/2)$ for $n \in \mathbb{N}$.

3. Let $G_n := (1/2n, 2)$ for $n \in \mathbb{N}$.
4. If $\mathcal{G}$ is an open cover of F, then $\mathcal{G} \cup \{\mathcal{C}(F)\}$ is an open cover of K.
5. If $\mathcal{G}_1$ is an open cover of K_1 and $\mathcal{G}_2$ is an open cover of K_2, then $\mathcal{G}_1 \cup G_2$ is an open cover of $K_1 \cup K_2$.
6. Let K be a bounded infinite subset of $\mathbb{R}$; we want to show that K has a cluster point. If not, then it follows from Theorem 11.1.8 that K must be closed. Since K is bounded, it follows from the Heine-Borel Theorem 11.2.5 that K is compact. If $k \in K$ is arbitrary, then since $k \in K$ is not a cluster point of K, we conclude that there exists an open neighborhood J_k of k that contains no point of $K \setminus \{k\}$. But since $\{J_k : k \in K\}$ is an open cover of K, it follows that there exists a finite number of points $k_1, \ldots, k_n$ such that $\{J_{k_i} : i = 1, \ldots, n\}$ covers K. But this implies that K is a finite set.
7. Let $K_n := [0, n]$ for $n \in \mathbb{N}$.
8. If $\{K_\alpha\}$ is a collection of compact subsets of $\mathbb{R}$, it follows from the Heine-Borel Theorem 11.2.5 that each set K_α is closed and bounded. Hence, from 11.1.5(a) the set $K_0 := \bigcap K_\alpha$ is also closed. Since K_0 is also bounded (since it is a subset of a bounded set), it follows from the Heine-Borel Theorem that K_0 is compact, as asserted.
9. For each $n \in \mathbb{N}$, let $x_n \in K_n$. Since the set $\{x_n\} \subseteq K_1$, we infer that the sequence (x_n) is a bounded sequence. By the Bolzano-Weierstrass Theorem, (x_n) has a subsequence (x_{m_r}) that converges to a point x_0. Since $x_{m_r} \in K_n$ for all $r \geq n$, it follows that $x_0 = \lim(x_{m_r})$ belongs to $\bigcap K_n$.
10. Since $K \neq \emptyset$ is bounded, it follows that $\inf K$ exists in $\mathbb{R}$. If $K_n := \{k \in K : k \leq (\inf K) + 1/n\}$, then K_n is closed and bounded, hence compact. By the preceding exercise $\bigcap K_n \neq \emptyset$, but if $x_0 \in \bigcap K_n$, then $x_0 \in K$ and it is readily seen that $x_0 = \inf K$. [Alternatively, use Theorem 11.2.6.]
11. For $n \in \mathbb{N}$, let $x_n \in K$ be such that $|c - x_n| \leq \inf\{|c - x| : x \in K\} + 1/n$. Now apply Theorem 11.2.6.
12. Let $K \subseteq \mathbb{R}$ be compact and let $c \in \mathbb{R}$. If $n \in \mathbb{N}$, there exists $x_n \in K$ such that $\sup\{|c - x| : x \in K\} - 1/n < |c - x_n|$. It follows from the Bolzano-Weierstrass Theorem that there exists a subsequence (x_{n_k}) that converges to a point b, which also belongs to the compact set K. Moreover, we have
$$|c - b| = \lim |c - x_{n_k}| \geq \sup\{|c - x| : x \in K\}.$$
But since $b \in k$, it also follows that $|c - b| \leq \sup\{|c - x| : x \in K\}$.
13. The family $\{V_{\delta_x}(x) : x \in K\}$ forms an open cover of the compact set $[a, b]$. Therefore it can be replaced by a finite subcover, say $\{V_{x_1}, \ldots, V_{x_n}\}$. If b_j is a bound for f on V_{x_j}, then $\sup\{b_1, \ldots, b_n\}$ is a bound for f on $[a, b]$.
14. Suppose K_1 and K_2 are disjoint compact sets and assume that $\inf\{|x - y| : x \in K_1, y \in K_2\} = 0$. Then there exist sequences (x_n) in K_1 and (y_n) in K_2

such that $|x_n - y_n| < 1/n$ for $n \in \mathbb{N}$. Let (x'_k) be a subsequence of (x_n) that converges to a point $x_0 \in K_1$, and let (y'_k) be the corresponding subsequence of (y_n). Then (y'_k) has a subsequence (y''_k) that converges to a point $y_0 \in K_2$. If (x''_k) is the corresponding subsequence of (x'_k), then we conclude that $|x_0 - y_0| = \lim |x''_k - y''_k| = 0$, from which it follows that $x_0 = y_0$, so that K_1 and K_2 are not disjoint, contrary to the hypothesis.

15. Let $F_1 := \{n : n \in \mathbb{N}\}$ and $F_2 := \{n + 1/n : n \in \mathbb{N}, n \geq 2\}$.

Section 11.3

The relationship between continuous functions and open sets is very important and the interplay between continuous functions and compact sets is further clarified here. Students who go on to more advanced courses in topology should find this to be a very useful introduction.

Sample Assignment: Exercises 1, 2, 4, 5, 6, 9.

Partial Solutions:

1. (a) If $a < b \leq 0$, then $f^{-1}(I) = \emptyset$. If $a < 0 < b$, then $f^{-1}(I) = (-\sqrt{b}, \sqrt{b})$. If $0 \leq a < b$, then $f^{-1}(I) = (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$.
(b) If $I := (a, b)$ where $a < 0 < b$, then $f(I) = [0, c)$, where $c := \sup\{a^2, b^2\}$.
2. (a) f maps the interval $(-1, 1)$ to $(1/2, 1]$. (b) f maps $[0, \infty)$ to $(0, 1]$.
3. $f^{-1}(G) = f^{-1}([0, \varepsilon)) = [1, 1 + \varepsilon^2)) = (0, 1 + \varepsilon^2) \cap I$.
4. Let $G := (1/2, 3/2)$. Let $F := [-1/2, 1/2]$.
5. $f^{-1}((-\infty, \alpha))$ is the inverse image of the open set $(-\infty, \alpha)$.
6. The set $\{x \in \mathbb{R} : f(x) \leq \alpha\} = f^{-1}((-\infty, \alpha])$ and $(-\infty, \alpha]$ is closed in $\mathbb{R}$.
7. If $x_n \in \mathbb{R}$ is such that $f(x_n) = k$ for all $n \in \mathbb{N}$ and if $x = \lim(x_n)$, then $f(x) = \lim(f(x_n)) = k$. Alternatively, $\{x \in \mathbb{R} : f(x) = k\} = \{x \in \mathbb{R} : f(x) \leq k\} \cap \{x \in \mathbb{R} : f(x) \geq k\}$.
8. Let f be the Dirichlet Discontinuous Function.
9. First note that if $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then we have $x \in f^{-1}(\mathbb{R} \setminus A) \iff f(x) \in \mathbb{R} \setminus A \iff f(x) \notin A \iff x \notin f^{-1}(A) \iff x \in \mathbb{R} \setminus f^{-1}(A)$; therefore, $f^{-1}(\mathbb{R} \setminus A) = \mathbb{R} \setminus f^{-1}(A)$. Now use the fact a set $F \subseteq \mathbb{R}$ is closed if and only if $\mathbb{R} \setminus F$ is open, together with Corollary 11.3.3.
10. If (x_n) is a sequence in I such that $f(x_n) = g(x_n)$ for all n and if $x_n \to x_0$, then it follows from the fact that $I = [a, b]$ is closed that $x_0 \in I$. Moreover, since f and g are continuous at x_0, then $f(x_0) = \lim f(x_n) = \lim g(x_n) = g(x_0)$.

Section 11.4

We have merely introduced the notion of a metric space, but we think it may be useful and stimulating for students to see that a great deal of what has been done can be extended to much wider generality. Some instructors may wish to use this brief section as a springboard for further discussion; other may decide to omit it completely.

Sample Assignment: Exercises 1, 3, 4, 7, 9, 10.

Partial Solutions:

1. If $P_1 := (x_1, y_1), P_2 := (x_2, y_2), P_3 := (x_3, y_3)$, then

$$\begin{aligned} d_1(P_1, P_2) &\le (|x_1 - x_3| + |x_3 - x_2|) + (|y_1 - y_3| + |y_3 - y_2|) \\ &= d_1(P_1, P_3) + d_1(P_3, P_2). \end{aligned}$$

   Thus d_1 satisfies the Triangle Inequality.
   To see that d_∞ satisfies the Triangle Inequality, note that $|x_1 - x_3| \le d_\infty(P_1, P_3)$ and $|y_1 - y_3| \le d_\infty(P_1, P_3)$, and also that $|x_3 - x_2| \le d_\infty(P_3, P_2)$ and $|y_3 - y_2| \le d_\infty(P_3, P_2)$. Therefore, we have $|x_1 - x_2| \le |x_1 - x_3| + |x_3 - x_2| \le d_\infty(P_1, P_3) + d_\infty(P_3, P_2)$ and $|y_1 - y_2| \le |y_1 - y_3| + |y_3 - y_2| \le d_\infty(P_1, P_3) + d_\infty(P_3, P_2)$, whence it follows that $d_\infty(P_1, P_2) = \sup\{|x_1 - x_2|, |y_1 - y_2|\} \le d_\infty(P_1, P_3) + d_\infty(P_3, P_2)$.
2. Since $|f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)| \le d_\infty(f, h) + d_\infty(h, g)$ for all $x \in [0, 1]$, it follows that $d_\infty(f, g) \le d_\infty(f, h) + d_\infty(h, g)$ and d_∞ satisfies the Triangle Inequality.
   We also have $d_1(f, g) = \int_0^1 |f - g| \le \int_0^1 \{|f - h| + |h - g|\} = \int_0^1 |f - h| + \int_0^1 |h - g| = d_1(f, h) + d_1(h, g)$.
3. We have $s \ne t$ if and only if $d(s, t) = 1$. If $s \ne t$, the value of $d(s, u) + d(u, t)$ is either 1 or 2 depending on whether u equals s or t, or neither.
4. Since $d_\infty(P_n, P) = \sup\{|x_n - x|, |y_n - y|\}$, if $d_\infty(P_n, P) \to 0$ then it follows that both $|x_n - x| \to 0$ and $|y_n - y| \to 0$, whence $x_n \to x$ and $y_n \to y$. Conversely, if $x_n \to x$ and $y_n \to y$, then $|x_n - x| \to 0$ and $|y_n - y| \to 0$, whence $d_\infty(P_n, P) \to 0$.
5. If $(x_n), (y_n)$ converge to x, y, respectively, then $d(P_n, P) = |x_n - x| + |y_n - y| \to 0$, so that (P_n) converges to P. Conversely, since $|x_n - x| \le d(P_n, P)$, if $d(P_n, P) \to 0$, then $\lim(x_n) = x$, and similarly for (y_n).
6. If a sequence (x_n) in S converges to x relative to the discrete metric d, then $d(x_n, x) \to 0$ which implies that $x_n = x$ for all sufficiently large n. The converse is trivial.
7. Show that a set consisting of a single point is open. Then it follows that every set is an open set, so that every set is also a closed set.
8. (a) $\{(x, y) : |x| + |y| \le 1\}$ is the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$, including its interior.
   (b) $\{(x, y) : |x| \le 1, |y| \le 1\}$ is the square with vertices $(1, 1), (-1, 1), (-1, -1)$ and $(1, -1)$, including its interior.

9. For a given $y \in V_\varepsilon(x)$, let $\delta := \varepsilon - d(x,y)$; then $\delta > 0$. Show that $V_\delta(y) \subseteq V_\varepsilon(x)$. Since y is arbitrary, it follows that $V_\varepsilon(x)$ is an open set.

10. Let $G \subseteq S_2$ be open in (S_2, d_2) and let $x \in f^{-1}(G)$ so that $f(x) \in G$. Then there exists an ε-neighborhood $V_\varepsilon(f(x)) \subseteq G$. Since f is continuous at x, there exists a δ-neighborhood $V_\delta(x)$ such that $f(V_\delta(x)) \subseteq V_\varepsilon(f(x))$. Since $x \in f^{-1}(G)$ is arbitrary, we conclude that $f^{-1}(G)$ is open in (S_1, d_1).

    If $f^{-1}(G)$ is open in S_1 for every open set $G \subseteq S_2$, and if $x_0 \in S_1$, we let $y_0 := f(x_0) \in S_2$. If $V_\varepsilon(y_0)$ is any (open) ε-neighborhood of y_0, the hypothesis is that $f^{-1}(V_\varepsilon(y_0))$ is an open set in S_1. Since it contains x_0, there is a δ-neighborhood $V_\delta(x_0) \subseteq f^{-1}(V_\varepsilon(y_0))$, whence $f(V_\delta(x_0)) \subseteq V_\varepsilon(y_0)$. Therefore f is continuous at the arbitrary point $x_0 \in S_1$.

11. Let $\mathcal{G} = \{G_\alpha\}$ be a cover of $f(S) \subseteq \mathbb{R}$ by open sets in $\mathbb{R}$. It follows from 11.4.11 that each set $f^{-1}(G_\alpha)$ is open in (S,d). Therefore, the collection $\{f^{-1}(G_\alpha)\}$ is an open cover of S. Since (S,d) is compact, a finite subcollection $\{f^{-1}(G_\alpha), \ldots, f^{-1}(G_{\alpha_N})\}$ covers S, whence it follows that the sets $\{G_{\alpha_1}, \ldots, G_{\alpha_N}\}$ must form a finite subcover of $\mathcal{G}$ for $f(S)$. Since $\mathcal{G}$ was an arbitrary open cover of $f(S)$, we conclude that $f(S)$ is compact.

12. Modify the proof of Theorem 11.2.4.

SELECTED GRAPHS

We include in this manual the graphs of a few selected functions that have a special place in real analysis. They can be photocopied onto transparencies and used with an overhead projector, if desired. They are just an indication of how computer graphics can be used to exhibit the properties of some of the important examples in real analysis.

The graphs were constructed using the facilities of the CALCULUS AND *MATHEMATICA* project that was developed at the University of Illinois at Urbana-Champaign by Professors H. Porta and J. J. Uhl, Jr. This is an exciting and innovative computer laboratory in which the students use the graphic and calculational power of *Mathematica* to learn calculus. The teaching of calculus is thus brought into the modern world by having students actively interact with modern technology.

(A) Figures 1, 2, 3. Here $f(x) := x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) := 0$.

Three graphs are shown using three different plot ranges, which gives a mild zoom effect. Figure 1 gives a global perspective, and shows that for "large" values of x, the graph approaches the line $y = x$. Note that

$$f(x) = x \cdot \left[\frac{\sin(1/x)}{1/x}\right] \qquad \text{and} \qquad \lim_{x\to\infty}\left[\frac{\sin(1/x)}{1/x}\right] = 1.$$

As the plot range becomes smaller, the oscillations become increasingly dominant. The guiding parabolas $y = x^2$ and $y = -x^2$ are shown as dashed curves. It is important to note that the scales on the coordinate axes are different in the three graphs; in Figure 1, the scale is 1 to 10, while in Figure 3 the scale is 10 to 1. With these scales, the parabolic curves have the same shape in all three figures.

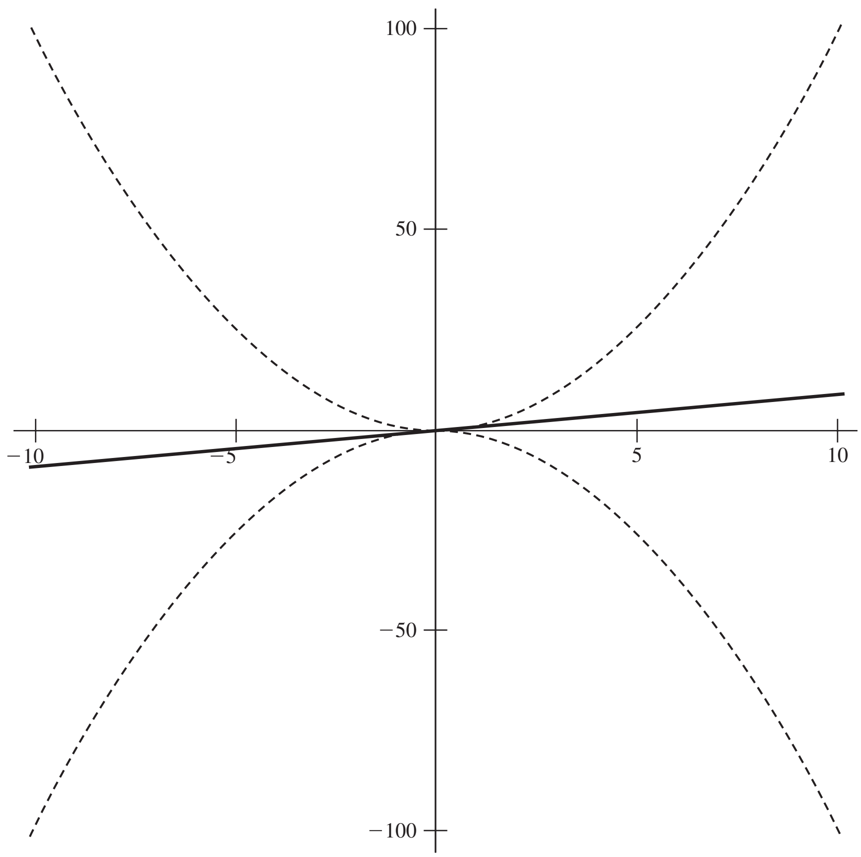

Figure 1. $f(x) = x^2 \sin(1/x)$, plot range $-10 < x < 10$

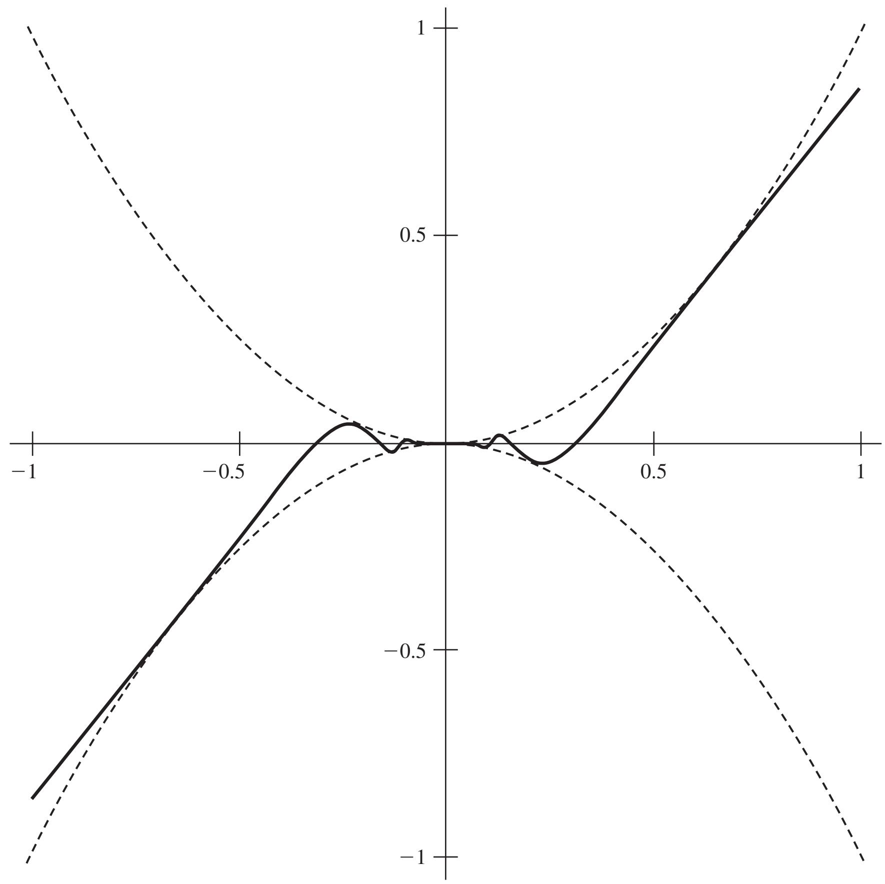

Figure 2. $f(x) = x^2 \sin(1/x)$, plot range $-1 < x < 1$

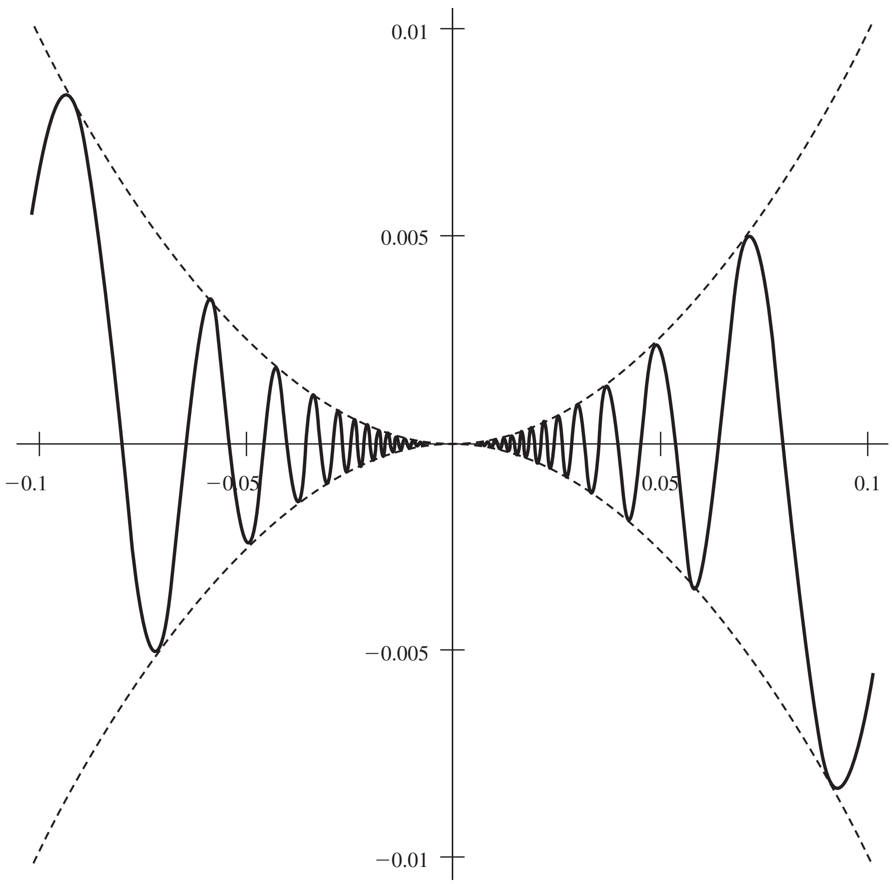

Figure 3. $f(x) = x^2 \sin(1/x)$, plot range $-0.1 < x < 0.1$

(B) Figures 4,5,6. Here $g(x) := x^4[2 + \sin(1/x)]$ for $x \neq 0$ and $g(0) := 0$.

There are three graphs of the function g for three different plot ranges, with the effect of enlarging the graph near the origin. As the plot range decreases, the oscillations become more apparent.

The function g has an absolute minumum at $x = 0$, but the derivative g' changes sign infinitely often in every neighborhood of $x = 0$; thus the first derivative test does not apply. The guiding curves $y = x^4$ and $y = 3x^4$ are shown as dashed curves. Note that the scale on the coordinate axes is different in each figure.

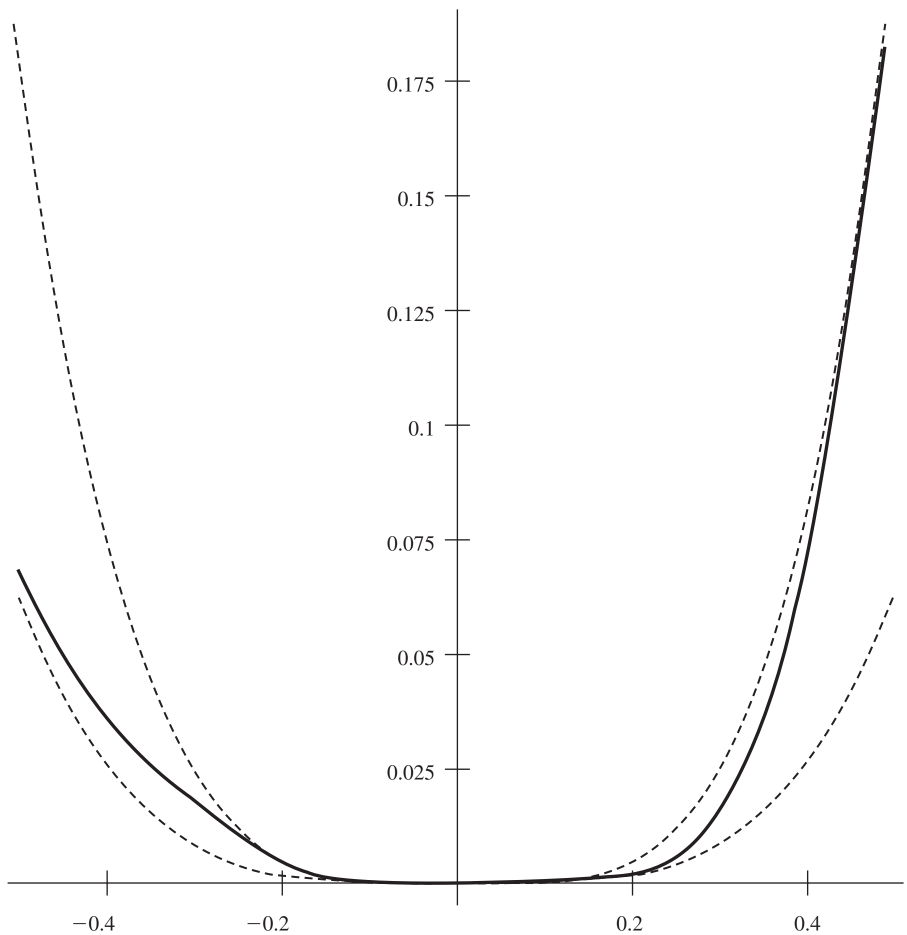

Figure 4. $g(x) = x^4[2 + \sin(1/x)]$, plot range $-0.4 < x < 0.4$

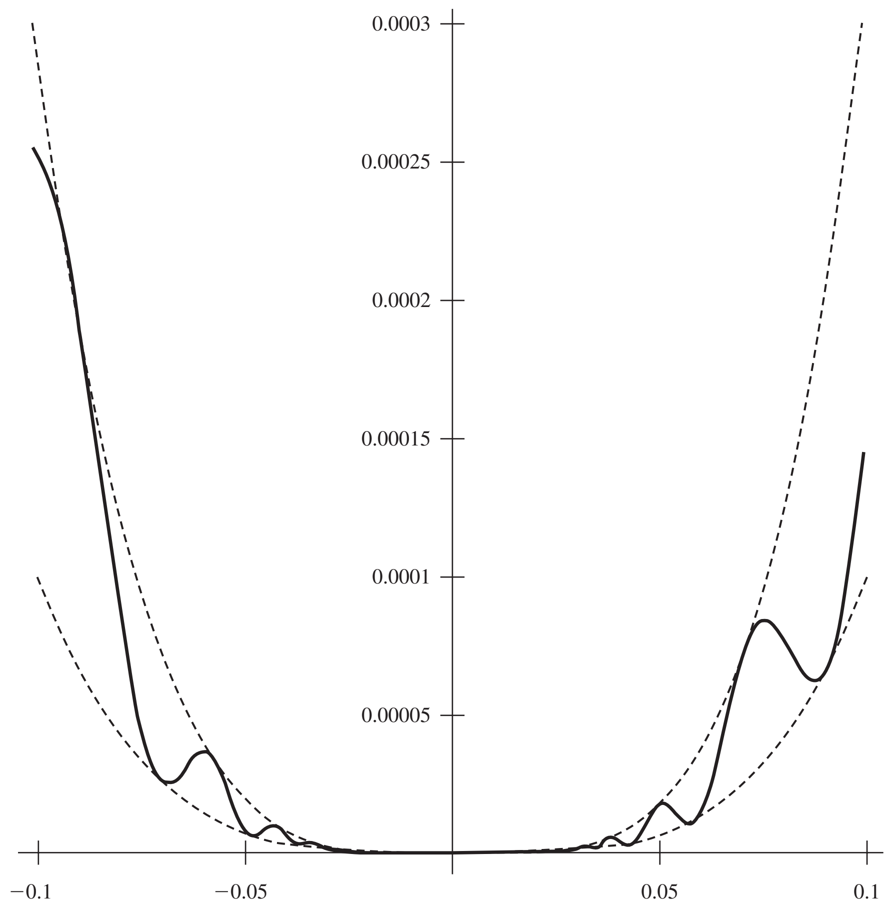

Figure 5. $g(x) = x^4[2 + \sin(1/x)]$, plot range $-0.1 < x < 0.1$

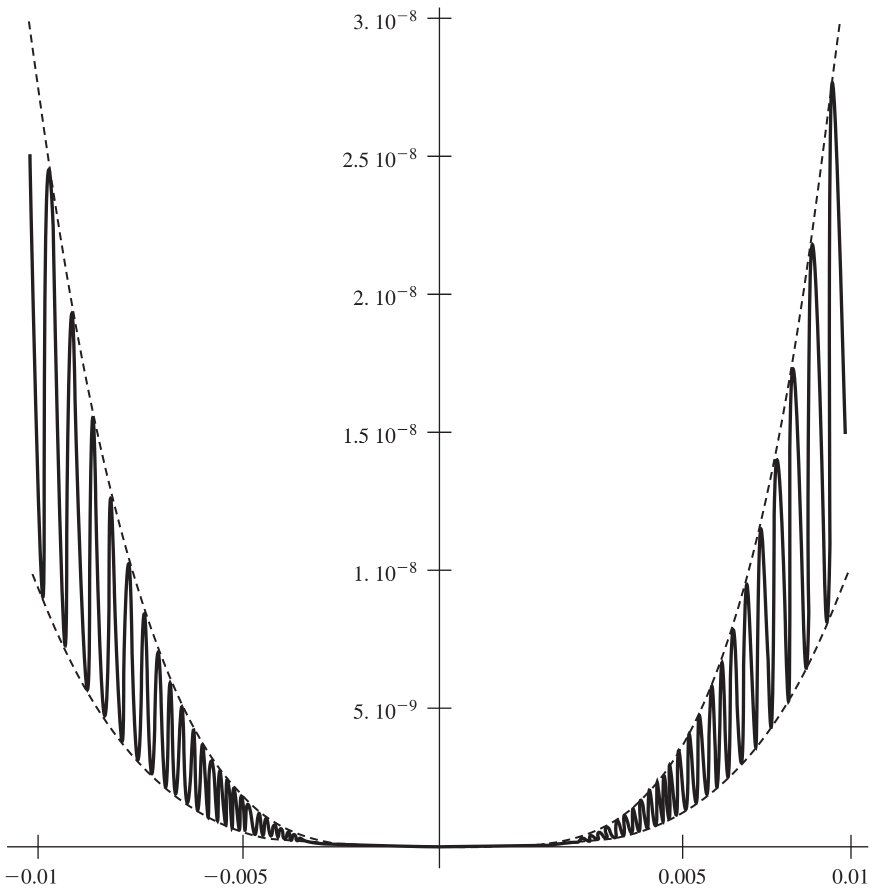

Figure 6. $g(x) = x^4[2 + \sin(1/x)]$, plot range $-0.01 < x < 0.01$

(C) Figure 7. Here $h(x) := 1/n$ if $x = m/n$, where m, n are relatively prime natural numbers with $m < n$, and $h(x) := 0$ if $0 < x < 1$ and x is irrational.

The graph of this function (Thomae's function on (0, 1)) is shown. This graph has been plotted for values of n from 1 to 70, and we have decreased the level of shading near the x-axis to keep the graph from becoming a blotch of ink for larger values of n. There is a gap of white just above the x-axis which must be filled in by the imagination. No attempt is made to represent the points on the x-axis.

This function is interesting because it is discontinuous at *every* rational number, and so has an infinite number of discontinuities. However, it is Riemann integrable.

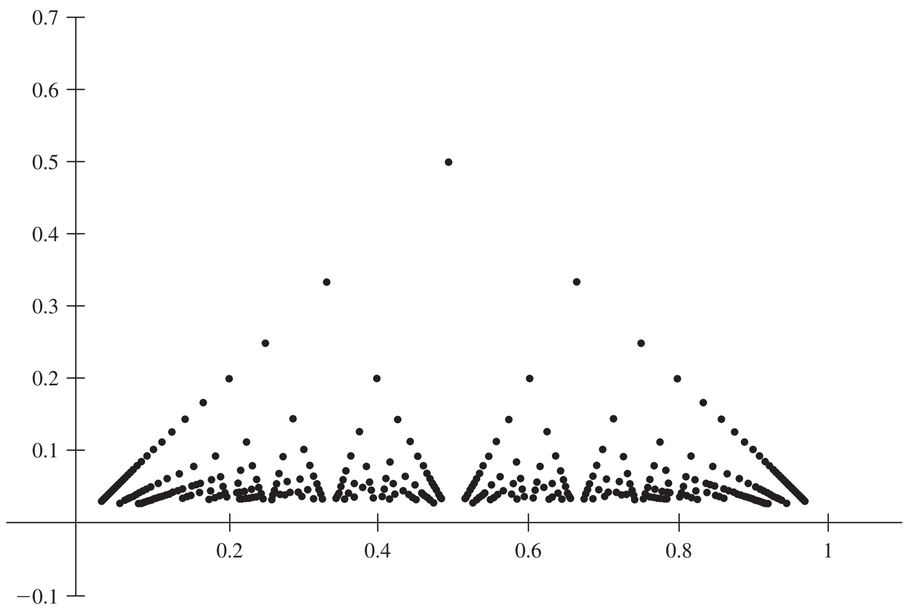

Figure 7. Thomae's function