Roark's Formulas for Stress and Strain

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RICHARD G. BUDYNAS

Seventh Edition

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About the Authors

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Richard G. Budynas is Professor of Mechanical Engineering at Rochester Institute of Technology. He is author of a newly revised McGraw-Hill textbook, *Applied Strength and Applied Stress Analysis*, 2nd Edition.

List of Tables

Preface to the Seventh Edition

The tabular format used in the fifth and sixth editions is continued in this edition. This format has been particularly successful when implementing problem solutions on a programmable calculator, or especially, a personal computer. In addition, though not required in utilizing this book, user-friendly computer software designed to employ the format of the tabulations contained herein are available.

The seventh edition intermixes International System of Units (SI) and United States Customary Units (USCU) in presenting example problems. Tabulated coefficients are in dimensionless form for convenience in using either system of units. Design formulas drawn from works published in the past remain in the system of units originally published or quoted.

Much of the changes of the seventh edition are organizational, such as:

- Numbering of equations, figures and tables is linked to the particular chapter where they appear. In the case of equations, the section number is also indicated, making it convenient to locate the equation, since section numbers are indicated at the top of each odd-numbered page.
- In prior editions, tables were interspersed within the text of each chapter. This made it difficult to locate a particular table and disturbed the flow of the text presentation. In this edition, all numbered tables are listed at the end of each chapter before the references.

Other changes/additions included in the seventh addition are as follows:

- Part 1 is an introduction, where Chapter 1 provides terminology such as state properties, units and conversions, and a description of the contents of the remaining chapters and appendices. The definitions incorporated in Part 1 of the previous editions are retained in the seventh edition, and are found in Appendix B as a glossary.

- Properties of plane areas are located in Appendix A.
- Composite material coverage is expanded, where an introductory discussion is provided in Appendix C, which presents the nomenclature associated with composite materials and how available computer software can be employed in conjunction with the tables contained within this book.
- Stress concentrations are presented in Chapter 17.
- Part 2, Chapter 2, is completely revised, providing a more comprehensive and modern presentation of stress and strain transformations.
- Experimental Methods. Chapter 6, is expanded, presenting more coverage on electrical strain gages and providing tables of equations for commonly used strain gage rosettes.
- Correction terms for multielement shells of revolution were presented in the sixth edition. Additional information is provided in Chapter 13 of this edition to assist users in the application of these corrections.

The authors wish to acknowledge and convey their appreciation to those individuals, publishers, institutions, and corporations who have generously given permission to use material in this and previous editions. Special recognition goes to Barry J. Berenberg and Universal Technical Systems, Inc. who provided the presentation on composite materials in Appendix C, and Dr. Marietta Scanlon for her review of this work.

Finally, the authors would especially like to thank the many dedicated readers and users of *Roark's Formulas for Stress & Strain*. It is an honor and quite gratifying to correspond with the many individuals who call attention to errors and/or convey useful and practical suggestions to incorporate in future editions.

Warren C. Young
Richard G. Budynas

Preface to the First Edition

This book was written for the purpose of making available a compact, adequate summary of the formulas, facts, and principles pertaining to strength of materials. It is intended primarily as a reference book and represents an attempt to meet what is believed to be a present need of the designing engineer.

This need results from the necessity for more accurate methods of stress analysis imposed by the trend of engineering practice. That trend is toward greater speed and complexity of machinery, greater size and diversity of structures, and greater economy and refinement of design. In consequence of such developments, familiar problems, for which approximate solutions were formerly considered adequate, are now frequently found to require more precise treatment, and many less familiar problems, once of academic interest only, have become of great practical importance. The solutions and data desired are often to be found only in advanced treatises or scattered through an extensive literature, and the results are not always presented in such form as to be suited to the requirements of the engineer. To bring together as much of this material as is likely to prove generally useful and to present it in convenient form has been the author's aim.

The scope and management of the book are indicated by the Contents. In Part 1 are defined all terms whose exact meaning might otherwise not be clear. In Part 2 certain useful general principles are stated; analytical and experimental methods of stress analysis are briefly described, and information concerning the behavior of material under stress is given. In Part 3 the behavior of structural elements under various conditions of loading is discussed, and extensive tables of formulas for the calculation of stress, strain, and strength are given.

Because they are not believed to serve the purpose of this book, derivations of formulas and detailed explanations, such as are appropriate in a textbook, are omitted, but a sufficient number of examples are included to illustrate the application of the various formulas and methods. Numerous references to more detailed discussions are given, but for the most part these are limited to sources that are generally available and no attempt has been made to compile an exhaustive bibliography.

That such a book as this derives almost wholly from the work of others is self-evident, and it is the author's hope that due acknowledgment has been made of the immediate sources of all material here presented. To the publishers and others who have generously permitted the use of material, he wishes to express his thanks. The helpful criticisms and suggestions of his colleagues, Professors E. R. Maurer, M. O. Withey, J. B. Kommers, and K. F. Wendt, are gratefully acknowledged. A considerable number of the tables of formulas have been published from time to time in *Product Engineering*, and the opportunity thus afforded for criticism and study of arrangement has been of great advantage.

Finally, it should be said that, although every care has been taken to avoid errors, it would be oversanguine to hope that none had escaped detection; for any suggestions that readers may make concerning needed corrections the author will be grateful.

Raymond J. Roark

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Part 1 Introduction

Part 2 Facts; Principles; Methods

Part 3 Formulas and Examples

Part

1

Introduction

Chapter 1

Introduction

The widespread use of personal computers, which have the power to solve problems solvable in the past only on mainframe computers, has influenced the tabulated format of this book. Computer programs for structural analysis, employing techniques such as the finite element method, are also available for general use. These programs are very powerful; however, in many cases, elements of structural systems can be analyzed quite effectively independently without the need for an elaborate finite element model. In some instances, finite element models or programs are verified by comparing their solutions with the results given in a book such as this. Contained within this book are simple, accurate, and thorough tabulated formulations that can be applied to the stress analysis of a comprehensive range of structural components.

This chapter serves to introduce the reader to the terminology, state property units and conversions, and contents of the book.

1.1 Terminology

Definitions of terms used throughout the book can be found in the glossary in Appendix B.

1.2 State Properties, Units, and Conversions

The basic state properties associated with stress analysis include the following: geometrical properties such as length, area, volume, centroid, center of gravity, and second-area moment (area moment of inertia); material properties such as mass density, modulus of elasticity, Poisson's ratio, and thermal expansion coefficient; loading properties such as force, moment, and force distributions (e.g., force per unit length, force per unit area, and force per unit volume); other proper-

TABLE 1.1 Units appropriate to structural analysis

Property	SI unit, symbol (derived units)	USCU unit,[†] symbol (derived units)
Length	meter, m	inch, in
Area	square meter (m^2)	square inch (in^2)
Volume	cubic meter (m^3)	cubic inch (in^3)
Second-area moment	(m^4)	(in^4)
Mass	kilogram, kg	(lbf-s^2/in)
Force	Newton, N (kg-m/s^2)	pound, lbf
Stress, pressure	Pascal, Pa (N/m^2)	psi (lbf/in^2)
Work, energy	Joule, J (N-m)	(lbf-in)
Temperature	Kelvin, K	degrees Fahrenheit, °F

[†] In stress analysis, the unit of length used most often is the inch.

ties associated with loading, including energy, work, and power; and stress analysis properties such as deformation, strain, and stress.

Two basic systems of units are employed in the field of stress analysis: SI units and USCU units.[†] SI units are mass-based units using the kilogram (kg), meter (m), second (s), and Kelvin (K) or degree Celsius (°C) as the fundamental units of mass, length, time, and temperature, respectively. Other SI units, such as that used for force, the Newton (kg-m/s^2), are derived quantities. USCU units are force-based units using the pound force (lbf), inch (in) or foot (ft), second (s), and degree Fahrenheit (°F) as the fundamental units of force, length, time, and temperature, respectively. Other USCU units, such as that used for mass, the slug (lbf-s^2/ft) or the nameless lbf-s^2/in, are derived quantities. Table 1.1 gives a listing of the primary SI and USCU units used for structural analysis. Certain prefixes may be appropriate, depending on the size of the quantity. Common prefixes are given in Table 1.2. For example, the modulus of elasticity of carbon steel is approximately $207\ \mathrm{GPa} = 207 \times 10^9\ \mathrm{Pa} = 207 \times 10^9\ \mathrm{N/m^2}$. Prefixes are normally used with SI units. However, there are cases where prefixes are also used with USCU units. Some examples are the kpsi ($1\ \mathrm{kpsi} = 10^3\ \mathrm{psi} = 10^3\ \mathrm{lbf/in^2}$), kip ($1\ \mathrm{kip} = 1\ \mathrm{kilopound} = 1000\ \mathrm{lbf}$), and Mpsi ($1\ \mathrm{Mpsi} = 10^6\ \mathrm{psi}$).

Depending on the application, different units may be specified. It is important that the analyst be aware of all the implications of the units and make consistent use of them. For example, if you are building a model from a CAD file in which the design dimensional units are given in mm, it is unnecessary to change the system of units or to scale the model to units of m. However, if in this example the input forces are in

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[†] SI and USCU are abbreviations for the International System of Units (from the French Systéme International d'Unités) and the United States Customary Units, respectively.

TABLE 1.2 Common prefixes

Prefix, symbol	Multiplication factor
Giga, G	10^9
Mega, M	10^6
Kilo, k	10^3
Milli, m	10^{-3}
Micro, μ	10^{-6}
Nano, n	10^{-9}

Newtons, then the output stresses will be in N/mm^2, which is correctly expressed as MPa. If in this example applied moments are to be specified, the units should be N-mm. For deflections in this example, the modulus of elasticity E should also be specified in MPa and the output deflections will be in mm.

Table 1.3 presents the conversions from USCU units to SI units for some common state property units. For example, 10 kpsi $= (6.895 \times 10^3) \cdot (10 \times 10^3) = 68.95 \times 10^6$ Pa $= 68.95$ MPa. Obviously, the multiplication factors for conversions from SI to USCU are simply the reciprocals of the given multiplication factors.

TABLE 1.3 Multiplication factors to convert from USCU units to SI units

To convert from USCU	to SI	Multiply by
Area:		
ft^2	m^2	9.290×10^{-2}
in^2	m^2	6.452×10^{-4}
Density:		
slug/ft^3 (lbf-s^2/ft^4)	kg/m^3	515.4
lbf-s^2/in^4	kg/m^3	2.486×10^{-2}
Energy, work, or moment:		
ft-lbf or lbf-ft	J or N-m	1.356
in-lbf or lbf-in	J or N-m	0.1130
Force:		
lbf	N	4.448
Length:		
ft	m	0.3048
in	m	2.540×10^{-2}
Mass:		
slug (lbf-s^2/ft)	kg	14.59
lbf-s^2/in	kg	1.216
Pressure, stress:		
lbf/ft^2	Pa (N/m^2)	47.88
lbf/in^2 (psi)	Pa (N/m^2)	6.895×10^3
Volume:		
ft^3	m^3	2.832×10^{-2}
in^3	m^3	1.639×10^{-5}

1.3 Contents

The remaining parts of this book are as follows.

Part 2: Facts; Principles; Methods. This part describes important relationships associated with stress and strain, basic material behavior, principles and analytical methods of the mechanics of structural elements, and numerical and experimental techniques in stress analysis.

Part 3: Formulas and Examples. This part contains the many applications associated with the stress analysis of structural components. Topics include the following: direct tension, compression, shear, and combined stresses; bending of straight and curved beams; torsion; bending of flat plates; columns and other compression members; shells of revolution, pressure vessels, and pipes; direct bearing and shear stress; elastic stability; stress concentrations; and dynamic and temperature stresses. Each chapter contains many tables associated with most conditions of geometry, loading, and boundary conditions for a given element type. The definition of each term used in a table is completely described in the introduction of the table.

Appendices. The first appendix deals with the properties of a plane area. The second appendix provides a glossary of the terminology employed in the field of stress analysis.

The references given in a particular chapter are always referred to by number, and are listed at the end of each chapter.

Part

2

Facts; Principles; Methods

Chapter 2

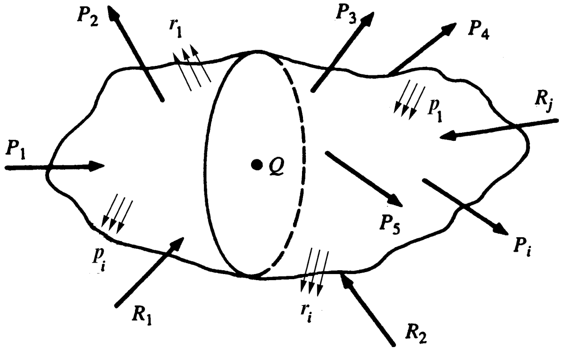
Stress and Strain: Important Relationships

Understanding the physical properties of stress and strain is a prerequisite to utilizing the many methods and results of structural analysis in design. This chapter provides the definitions and important relationships of stress and strain.

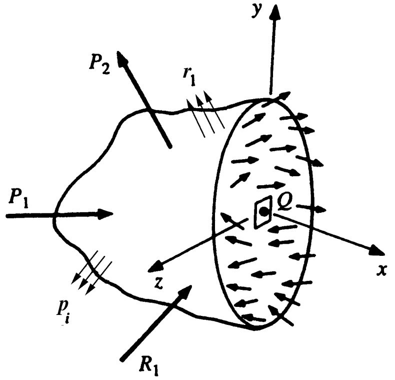
2.1 Stress

Stress is simply a distributed force on an external or internal surface of a body. To obtain a physical feeling of this idea, consider being submerged in water at a particular depth. The "force" of the water one feels at this depth is a pressure, which is a compressive *stress*, and not a finite number of "concentrated" forces. Other types of force distributions (stress) can occur in a liquid or solid. Tensile (pulling rather than pushing) and shear (rubbing or sliding) force distributions can also exist.

Consider a general solid body loaded as shown in Fig. 2.1(*a*). P_i and p_i are applied concentrated forces and applied surface force distributions, respectively; and R_i and r_i are possible support reaction force and surface force distributions, respectively. To determine the state of stress at point Q in the body, it is necessary to expose a surface containing the point Q. This is done by making a planar slice, or break, through the body intersecting the point Q. The orientation of this slice is arbitrary, but it is generally made in a convenient plane where the state of stress can be determined easily or where certain geometric relations can be utilized. The first slice, illustrated in Fig. 2.1(*b*), is arbitrarily oriented by the surface normal x. This establishes the yz plane. The external forces on the remaining body are shown, as well as the internal force (stress) distribution across the exposed internal



(*a*) Structural member

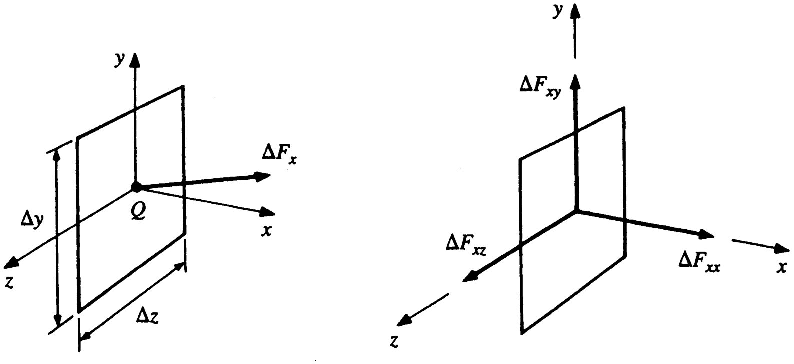


(*b*) Isolated section

Figure 2.1

surface containing Q. In the general case, this distribution will not be uniform along the surface, and will be neither normal nor tangential to the surface at Q. However, the force distribution at Q will have components in the normal and tangential directions. These components will be tensile or compressive and shear stresses, respectively.

Following a right-handed rectangular coordinate system, the y and z axes are defined perpendicular to x, and tangential to the surface. Examine an infinitesimal area $\Delta A_x = \Delta y \Delta z$ surrounding Q, as shown in Fig. 2.2(*a*). The equivalent concentrated force due to the force distribution across this area is ΔF_x, which in general is neither normal nor tangential to the surface (the subscript x is used to designate the *normal* to the area). The force ΔF_x has components in the x, y, and z directions, which are labeled ΔF_{xx}, ΔF_{xy}, and ΔF_{xz}, respectively, as shown in Fig. 2.2(*b*). Note that the first subscript



(*a*) Force on the ΔA surface

(*b*) Force components

Figure 2.2

denotes the direction normal to the surface and the second gives the actual direction of the force component. The average distributed force per unit area (average stress) in the x direction is[†]

$$\bar{\sigma}_{xx} = \frac{\Delta F_{xx}}{\Delta A_x}$$

Recalling that stress is actually a point function, we obtain the exact stress in the x direction at point Q by allowing ΔA_x to approach zero. Thus,

$$\sigma_{xx} = \lim_{\Delta A_x \to 0} \frac{\Delta F_{xx}}{\Delta A_x}$$

or,

$$\sigma_{xx} = \frac{dF_{xx}}{dA_x} \tag{2.1-1}$$

Stresses arise from the tangential forces ΔF_{xy} and ΔF_{xz} as well, and since these forces are tangential, the stresses are shear stresses. Similar to Eq. (2.1-1),

$$\tau_{xy} = \frac{dF_{xy}}{dA_x} \tag{2.1-2}$$

$$\tau_{xz} = \frac{dF_{xz}}{dA_x} \tag{2.1-3}$$

[†] Standard engineering practice is to use the Greek symbols σ and τ for normal (tensile or compressive) and shear stresses, respectively.

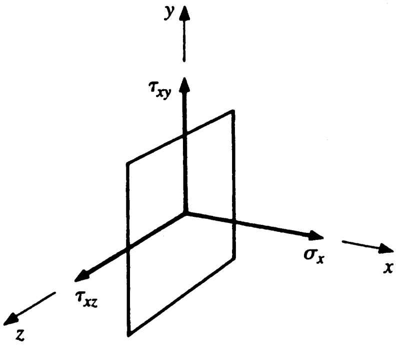


Figure 2.3 Stress components.

Since, by definition, σ represents a normal stress acting in the same direction as the corresponding surface normal, double subscripts are redundant, and standard practice is to drop one of the subscripts and write σ_{xx} as σ_x. The three stresses existing on the exposed surface at the point are illustrated together using a single arrow vector for each stress as shown in Fig. 2.3. However, it is important to realize that the stress arrow represents a force distribution (stress, force per unit area), and *not* a concentrated force. The shear stresses τ_{xy} and τ_{xz} are the components of the net shear stress acting on the surface, where the net shear stress is given by[†]

$$(\tau_x)_{\text{net}} = \sqrt{\tau_{xy}^2 + \tau_{xz}^2} \tag{2.1-4}$$

To describe the complete state of stress at point Q completely, it would be necessary to examine other surfaces by making different planar slices. Since different planar slices would necessitate different coordinates and different free-body diagrams, the stresses on each planar surface would be, in general, quite different. As a matter of fact, in general, an infinite variety of conditions of normal and shear stress exist at a given point within a stressed body. So, it would take an infinitesimal spherical surface surrounding the point Q to understand and describe the complete state of stress at the point. Fortunately, through the use of the method of *coordinate transformation*, it is only necessary to know the state of stress on *three* different surfaces to describe the state of stress on *any* surface. This method is described in Sec. 2.3.

The three surfaces are generally selected to be mutually perpendicular, and are illustrated in Fig. 2.4 using the stress subscript notation

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[†] Stresses can only be added as vectors if they exist on a common surface.

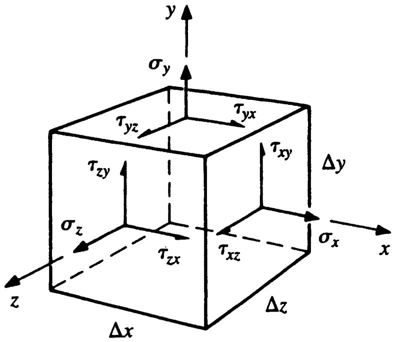


Figure 2.4 Stresses on three orthogonal surfaces.

as earlier defined. This state of stress can be written in matrix form, where the stress matrix $[\boldsymbol{\sigma}]$ is given by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{2.1-5}$$

Except for extremely rare cases, it can be shown that adjacent shear stresses are equal. That is, $\tau_{yx} = \tau_{xy}$, $\tau_{zy} = \tau_{yz}$, and $\tau_{xz} = \tau_{zx}$, and the stress matrix is symmetric and written as

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} \tag{2.1-6}$$

Plane Stress. There are many practical problems where the stresses in one direction are zero. This situation is referred to as a case of *plane stress*. Arbitrarily selecting the z direction to be stress-free with $\sigma_z = \tau_{yz} = \tau_{yz} = 0$, the last row and column of the stress matrix can be eliminated, and the stress matrix is written as

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \tag{2.1-7}$$

and the corresponding stress element, viewed three-dimensionally and down the z axis, is shown in Fig. 2.5.

2.2 Strain and the Stress–Strain Relations

As with stresses, two types of strains exist: normal and shear strains, which are denoted by ε and γ, respectively. Normal strain is the rate of change of the length of the stressed element in a particular direction.

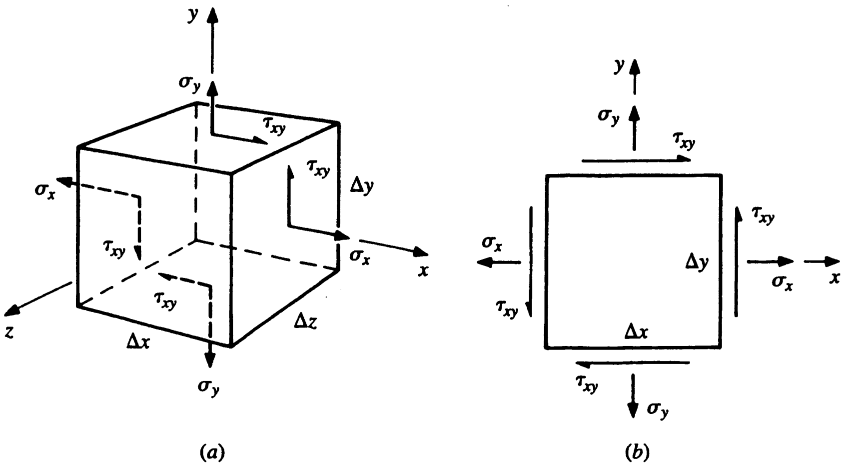


Figure 2.5 Plane stress.

Shear strain is a measure of the distortion of the stressed element, and has two definitions: the *engineering shear strain* and the *elasticity shear strain*. Here, we will use the former, more popular, definition. However, a discussion of the relation of the two definitions will be provided in Sec. 2.4. The engineering shear strain is defined as the change in the corner angle of the stress cube, in radians.

Normal Strain. Initially, consider only one normal stress σ_x applied to the element as shown in Fig. 2.6. We see that the element increases in length in the x direction and decreases in length in the y and z directions. The dimensionless rate of increase in length is defined as the *normal strain*, where ε_x, ε_y, and ε_z represent the normal strains in

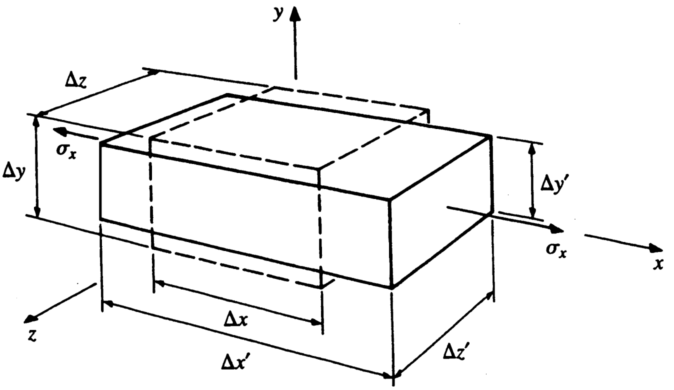


Figure 2.6 Deformation attributed to σ_x.

the x, y, and z directions respectively. Thus, the new length in any direction is equal to its original length plus the rate of increase (normal strain) times its original length. That is,

$$\Delta x' = \Delta x + \varepsilon_x \Delta x, \qquad \Delta y' = \Delta y + \varepsilon_y \Delta y, \qquad \Delta z' = \Delta z + \varepsilon_z \Delta z \tag{2.2-1}$$

There is a direct relationship between strain and stress. *Hooke's law* for a linear, homogeneous, isotropic material is simply that the normal strain is directly proportional to the normal stress, and is given by

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \tag{2.2-2a}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \tag{2.2-2b}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \tag{2.2-2c}$$

where the material constants, E and ν, are the *modulus of elasticity* (also referred to as *Young's modulus*) and *Poisson's ratio*, respectively. Typical values of E and ν for some materials are given in Table 2.1 at the end of this chapter.

If the strains in Eqs. (2.2-2) are known, the stresses can be solved for simultaneously to obtain

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)] \tag{2.2-3a}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)] \tag{2.2-3b}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)] \tag{2.2-3c}$$

For *plane stress*, with $\sigma_z = 0$, Eqs. (2.2-2) and (2.2-3) become

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \tag{2.2-4a}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{2.2-4b}$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \tag{2.2-4c}$$

and

$$\sigma_x = \frac{E}{1 - \nu^2}(\varepsilon_x + \nu\varepsilon_y) \tag{2.2-5a}$$

$$\sigma_y = \frac{E}{1 - \nu^2}(\varepsilon_y + \nu\varepsilon_x) \tag{2.2-5b}$$

Shear Strain. The change in shape of the element caused by the shear stresses can be first illustrated by examining the effect of τ_{xy} alone as shown in Fig. 2.7. The engineering shear strain γ_{xy} is a measure of the skewing of the stressed element from a rectangular parallelepiped. In Fig. 2.7(*b*), the shear strain is defined as the change in the angle *BAD*. That is,

$$\gamma_{xy} = \angle BAD - \angle B'A'D'$$

where γ_{xy} is in dimensionless radians.

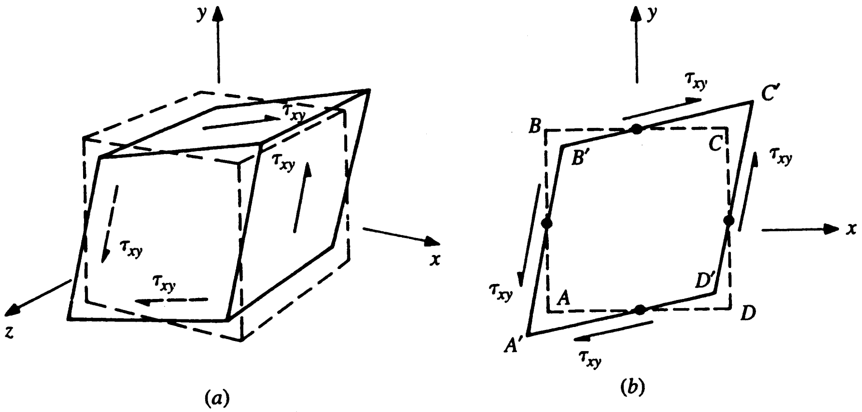
For a linear, homogeneous, isotropic material, the shear strains in the *xy*, *yz*, and *zx* planes are directly related to the shear stresses by

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{2.2-6a}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \tag{2.2-6b}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \tag{2.2-6c}$$

where the material constant, *G*, is called the *shear modulus*.



(*a*) (*b*)

Figure 2.7 Shear deformation.

It can be shown that for a linear, homogeneous, isotropic material the shear modulus is related to Poisson's ratio by (Ref. 1)

$$G = \frac{E}{2(1+\nu)} \tag{2.2-7}$$

2.3 Stress Transformations

As was stated in Sec. 2.1, knowing the state of stress on three mutually orthogonal surfaces at a point in a structure is sufficient to generate the state of stress for *any* surface at the point. This is accomplished through the use of *coordinate transformations*. The development of the transformation equations is quite lengthy and is not provided here (see Ref. 1). Consider the element shown in Fig. 2.8(*a*), where the stresses on surfaces with normals in the x, y, and z directions are known and are represented by the stress matrix

$$[\boldsymbol{\sigma}]_{xyz} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} \tag{2.3-1}$$

Now consider the element, shown in Fig. 2.8(*b*), to correspond to the state of stress *at the same point* but defined relative to a different set of surfaces with normals in the x', y', and z' directions. The stress matrix corresponding to this element is given by

$$[\boldsymbol{\sigma}]_{x'y'z'} = \begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{z'x'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{z'x'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix} \tag{2.3-2}$$

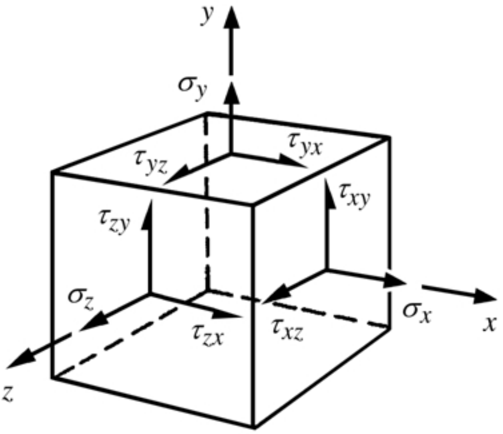
To determine $[\boldsymbol{\sigma}]_{x'y'z'}$ by coordinate transformation, we need to establish the relationship between the $x'y'z'$ and the xyz coordinate systems. This is normally done using *directional cosines*. First, let us consider the relationship between the x' axis and the xyz coordinate system. The orientation of the x' axis can be established by the angles $\theta_{x'x}$, $\theta_{x'y}$, and $\theta_{x'z}$, as shown in Fig. 2.9. The directional cosines for x' are given by

$$l_{x'} = \cos\theta_{x'x}, \qquad m_{x'} = \cos\theta_{x'y}, \qquad n_{x'} = \cos\theta_{x'z} \tag{2.3-3}$$

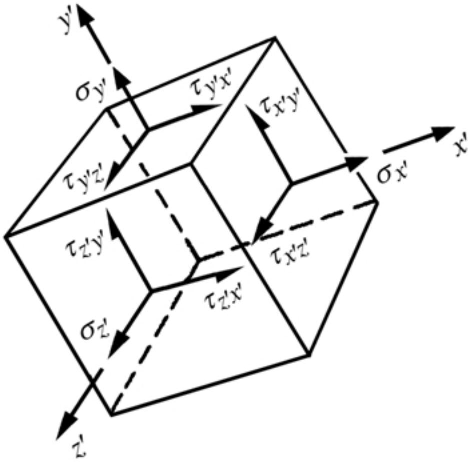
Similarly, the y' and z' axes can be defined by the angles $\theta_{y'x}$, $\theta_{y'y}$, $\theta_{y'z}$ and $\theta_{z'x}$, $\theta_{z'y}$, $\theta_{z'z}$, respectively, with corresponding directional cosines

$$l_{y'} = \cos\theta_{y'x}, \qquad m_{y'} = \cos\theta_{y'y}, \qquad n_{y'} = \cos\theta_{y'z} \tag{2.3-4}$$

$$l_{z'} = \cos\theta_{z'x}, \qquad m_{z'} = \cos\theta_{z'y}, \qquad n_{z'} = \cos\theta_{z'z} \tag{2.3-5}$$



(*a*) Stress element relative to *xyz* axes



(*b*) Stress element relative to *x′y′z′* axes

Figure 2.8 The stress at a point using different coordinate systems.

It can be shown that the transformation matrix

$$[\mathbf{T}] = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix} \tag{2.3-6}$$

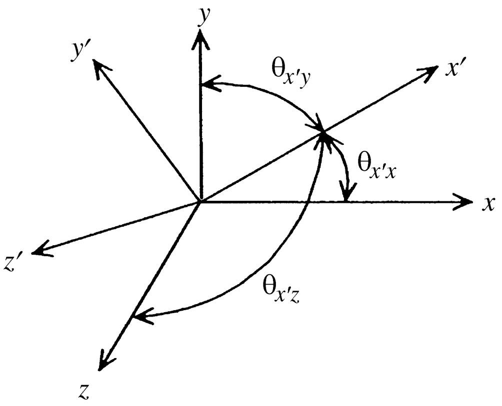


Figure 2.9 Coordinate transformation.

transforms a vector given in xyz coordinates, $\{\mathbf{V}\}_{xyz}$, to a vector in $x'y'z'$ coordinates, $\{\mathbf{V}\}_{x'y'z'}$, by the matrix multiplication

$$\{\mathbf{V}\}_{x'y'z'} = [\mathbf{T}]\{\mathbf{V}\}_{xyz} \tag{2.3-7}$$

Furthermore, it can be shown that the transformation equation for the stress matrix is given by (see Ref. 1)

$$[\boldsymbol{\sigma}]_{x'y'z'} = [\mathbf{T}][\boldsymbol{\sigma}]_{xyz}[\mathbf{T}]^T \tag{2.3-8}$$

where $[\mathbf{T}]^T$ is the *transpose* of the transformation matrix $[\mathbf{T}]$, which is simply an interchange of rows and columns. That is,

$$[\mathbf{T}]^T = \begin{bmatrix} l_{x'} & l_{y'} & l_{z'} \\ m_{x'} & m_{y'} & m_{z'} \\ n_{x'} & n_{y'} & n_{z'} \end{bmatrix} \tag{2.3-9}$$

The stress transformation by Eq. (2.3-8) can be implemented very easily using a computer spreadsheet or mathematical software. Defining the directional cosines is another matter. One method is to define the $x'y'z'$ coordinate system by a series of two-dimensional rotations from the initial xyz coordinate system. Table 2.2 at the end of this chapter gives transformation matrices for this.

EXAMPLE

The state of stress at a point relative to an *xyz* coordinate system is given by the stress matrix

$$[\boldsymbol{\sigma}]_{xyz} = \begin{bmatrix} -8 & 6 & -2 \\ 6 & 4 & 2 \\ -2 & 2 & -5 \end{bmatrix} \text{ MPa}$$

Determine the state of stress on an element that is oriented by first rotating the *xyz* axes 45° about the *z* axis, and then rotating the resulting axes 30° about the new *x* axis.

Solution. The surface normals can be found by a series of coordinate transformations for each rotation. From Fig. 2.10(*a*), the vector components for the first rotation can be represented by

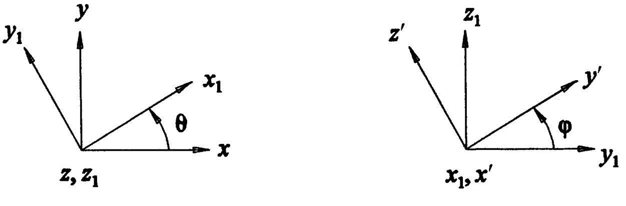
$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{a}$$

The last rotation establishes the $x'y'z'$ coordinates as shown in Fig. 2.10(*b*), and they are related to the $x_1y_1z_1$ coordinates by

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} \tag{b}$$

Substituting Eq. (*a*) in (*b*) gives

$$\begin{aligned} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta\cos\varphi & \cos\theta\cos\varphi & \sin\varphi \\ \sin\theta\sin\varphi & -\cos\theta\sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \end{aligned} \tag{c}$$



(a) First rotation

(b) Second rotation

Figure 2.10

Equation (*c*) is of the form of Eq. (2.3-7). Thus, the transformation matrix is

$$[\mathbf{T}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta\cos\varphi & \cos\theta\cos\varphi & \sin\varphi \\ \sin\theta\sin\varphi & -\cos\theta\sin\varphi & \cos\varphi \end{bmatrix} \tag{d}$$

Substituting $\theta = 45^\circ$ and $\varphi = 30^\circ$ gives

$$[\mathbf{T}] = \frac{1}{4}\begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} & 0 \\ -\sqrt{6} & \sqrt{6} & 2 \\ \sqrt{2} & -\sqrt{2} & 2\sqrt{3} \end{bmatrix} \tag{e}$$

The transpose of [**T**] is

$$[\mathbf{T}]^T = \frac{1}{4}\begin{bmatrix} 2\sqrt{2} & -\sqrt{6} & \sqrt{2} \\ 2\sqrt{2} & \sqrt{6} & -\sqrt{2} \\ 0 & 2 & 2\sqrt{3} \end{bmatrix} \tag{f}$$

From Eq. (2.3-8),

$$[\boldsymbol{\sigma}]_{x'y'z'} = \frac{1}{4}\begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} & 0 \\ -\sqrt{6} & \sqrt{6} & 2 \\ \sqrt{2} & -\sqrt{2} & 2\sqrt{3} \end{bmatrix}\begin{bmatrix} -8 & 6 & -2 \\ 6 & 4 & 2 \\ -2 & 2 & -5 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 2\sqrt{2} & -\sqrt{6} & \sqrt{2} \\ 2\sqrt{2} & \sqrt{6} & -\sqrt{2} \\ 0 & 2 & 2\sqrt{3} \end{bmatrix}$$

This matrix multiplication can be performed simply using either a computer spreadsheet or mathematical software, resulting in

$$[\boldsymbol{\sigma}]_{x'y'z'} = \begin{bmatrix} 4 & 5.196 & -3 \\ 5.196 & -4.801 & 2.714 \\ -3 & 2.714 & -8.199 \end{bmatrix} \text{ MPa}$$

Stresses on a Single Surface. If one was concerned about the state of stress on one particular surface, a complete stress transformation would be unnecessary. Let the directional cosines for the normal of the surface be given by *l*, *m*, and *n*. It can be shown that the normal stress on the surface is given by

$$\sigma = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl \tag{2.3-10}$$

and the net shear stress on the surface is

$$\begin{aligned} \tau = [&(\sigma_x l + \tau_{xy} m + \tau_{zx} n)^2 + (\tau_{xy} l + \sigma_y m + \tau_{yz} n)^2 \\ &+ (\tau_{zx} l + \tau_{yz} m + \sigma_z n)^2 - \sigma^2]^{1/2} \end{aligned} \tag{2.3-11}$$

The direction of τ is established by the directional cosines

$$\begin{aligned} l_\tau &= \frac{1}{\tau}[(\sigma_x - \sigma)l + \tau_{xy}m + \tau_{zx}n] \\ m_\tau &= \frac{1}{\tau}[\tau_{xy}l + (\sigma_y - \sigma)m + \tau_{yz}n] \\ n_\tau &= \frac{1}{\tau}[\tau_{zx}l + \tau_{yz}m + (\sigma_z - \sigma)n] \end{aligned} \tag{2.3-12}$$

EXAMPLE

The state of stress at a particular point relative to the *xyz* coordinate system is

$$[\boldsymbol{\sigma}]_{xyz} = \begin{bmatrix} 14 & 7 & -7 \\ 7 & 10 & 0 \\ -7 & 0 & 35 \end{bmatrix} \text{ kpsi}$$

Determine the normal and shear stress on a surface at the point where the surface is parallel to the plane given by the equation

$$2x - y + 3z = 9$$

Solution. The normal to the surface is established by the *directional numbers* of the plane and are simply the coefficients of x, y, and z terms of the equation of the plane. Thus, the directional numbers are 2, -1, and 3. The directional cosines of the normal to the surface are simply the normalized values of the directional numbers, which are the directional numbers divided by $\sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$. Thus

$$l = 2/\sqrt{14}, \qquad m = -1/\sqrt{14}, \qquad n = 3/\sqrt{14}$$

From the stress matrix, $\sigma_x = 14$, $\tau_{xy} = 7$, $\tau_{zx} = -7$, $\sigma_y = 10$, $\tau_{yz} = 0$, and $\sigma_z = 35$ kpsi. Substituting the stresses and directional cosines into Eq. (2.3-10) gives

$$\begin{aligned} \sigma = {} & 14(2/\sqrt{14})^2 + 10(-1/\sqrt{14})^2 + 35(3/\sqrt{14})^2 + 2(7)(2/\sqrt{14})(-1/\sqrt{14}) \\ & + 2(0)(-1/\sqrt{14})(3/\sqrt{14}) + 2(-7)(3/\sqrt{14})(2/\sqrt{14}) = 19.21 \text{ kpsi} \end{aligned}$$

The shear stress is determined from Eq. (2.3-11), and is

$$\begin{aligned} \tau = {} & \{[14(2/\sqrt{14}) + 7(-1/\sqrt{14}) + (-7)(3/\sqrt{14})]^2 \\ & + [7(2/\sqrt{14}) + 10(-1/\sqrt{14}) + (0)(3/\sqrt{14})]^2 \\ & + [(-7)(2/\sqrt{14}) + (0)(-1/\sqrt{14}) + 35(3/\sqrt{14})]^2 - (19.21)^2\}^{1/2} = 14.95 \text{ kpsi} \end{aligned}$$

From Eq. (2.3-12), the directional cosines for the direction of τ are

$$l_\tau = \frac{1}{14.95}[(14 - 19.21)(2/\sqrt{14}) + 7(-1/\sqrt{14}) + (-7)(3/\sqrt{14})] = -0.687$$

$$m_\tau = \frac{1}{14.95}[7(2/\sqrt{14}) + (10 - 19.21)(-1/\sqrt{14}) + (0)(3/\sqrt{14})] = 0.415$$

$$n_\tau = \frac{1}{14.95}[(-7)(2/\sqrt{14}) + (0)(-1/\sqrt{14}) + (35 - 19.21)(3/\sqrt{14})] = 0.596$$

Plane Stress. For the state of plane stress shown in Fig. 2.11(*a*), $\sigma_z = \tau_{yz} = \tau_{zx} = 0$. Plane stress transformations are normally performed in the *xy* plane, as shown in Fig. 2.11(*b*). The angles relating the $x'y'z'$ axes to the *xyz* axes are

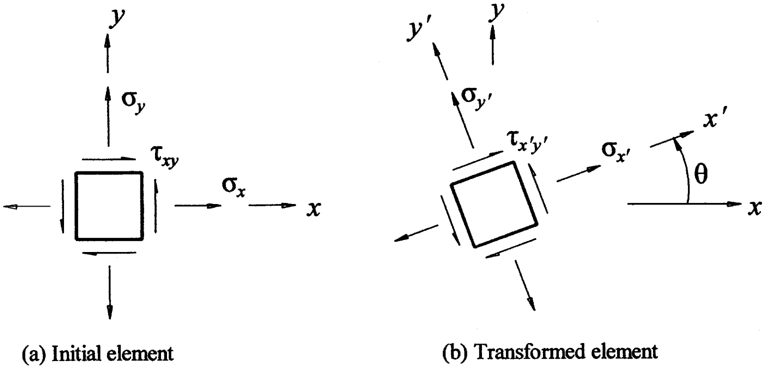
$$\begin{array}{lll} \theta_{x'x} = \theta, & \theta_{x'y} = 90^\circ - \theta, & \theta_{x'z} = 90^\circ \\ \theta_{y'x} = \theta + 90^\circ, & \theta_{y'y} = \theta, & \theta_{y'z} = 90^\circ \\ \theta_{z'x} = 90^\circ, & \theta_{z'y} = 90^\circ, & \theta_{z'z} = 0 \end{array}$$

Thus the directional cosines are

$$\begin{array}{lll} l_{x'} = \cos\theta & m_{x'} = \sin\theta & n_{x'} = 0 \\ l_{y'} = -\sin\theta & m_{y'} = \cos\theta & n_{y'} = 0 \\ l_{z'} = 0 & m_{z'} = 0 & n_{z'} = 1 \end{array}$$

The last rows and columns of the stress matrices are zero so the stress matrices can be written as

$$[\boldsymbol{\sigma}]_{xy} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \tag{2.3-13}$$



(a) Initial element (b) Transformed element

Figure 2.11 Plane stress transformations.

and

$$[\boldsymbol{\sigma}]_{x'y'} = \begin{bmatrix} \sigma_{x'} & \tau_{x'y'} \\ \tau_{x'y'} & \sigma_{y'} \end{bmatrix} \tag{2.3-14}$$

Since the plane stress matrices are 2 × 2, the transformation matrix and its transpose are written as

$$[\mathbf{T}] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad [\mathbf{T}]^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.3-15}$$

Equations (2.3-13)–(2.3-15) can then be substituted into Eq. (2.3-8) to perform the desired transformation. The results, written in long-hand form, would be

$$\begin{aligned} \sigma_{x'} &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta \\ \sigma_{y'} &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta \\ \tau_{x'y'} &= -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned} \tag{2.3-16}$$

If the state of stress is desired on a single surface with a normal rotated θ counterclockwise from the x axis, the first and third equations of Eqs. (2.3-16) can be used as given. However, using trigonometric identities, the equations can be written in slightly different form. Letting σ and τ represent the desired normal and shear stresses on the surface, the equations are

$$\begin{aligned} \sigma &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau &= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{aligned} \tag{2.3-17}$$

Equations (2.3-17) represent a set of parametric equations of a circle in the $\sigma\tau$ plane. This circle is commonly referred to as *Mohr's circle* and is generally discussed in standard mechanics of materials textbooks. This serves primarily as a teaching tool and adds little to applications, so it will not be represented here (see Ref. 1).

Principal Stresses. In general, maximum and minimum values of the normal stresses occur on surfaces where the shear stresses are zero. These stresses, which are actually the eigenvalues of the stress matrix, are called the *principal stresses*. Three principal stresses exist, σ_1, σ_2, and σ_3, where they are commonly ordered as $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$.

Considering the stress state given by the matrix of Eq. (2.3-1) to be known, the principal stresses σ_p are related to the given stresses by

$$\begin{aligned}(\sigma_x - \sigma_p)l_p + \tau_{xy}m_p + \tau_{zx}n_p &= 0\\ \tau_{xy}l_p + (\sigma_y - \sigma_p)m_p + \tau_{yz}n_p &= 0\\ \tau_{zx}l_p + \tau_{yz}m_p + (\sigma_z - \sigma_p)n_p &= 0\end{aligned} \tag{2.3-18}$$

where l_p, m_p, and n_p are the directional cosines of the normals to the surfaces containing the principal stresses. One possible solution to Eqs. (2.3-18) is $l_p = m_p = n_p = 0$. However, this cannot occur, since

$$l_p^2 + m_p^2 + n_p^2 = 1 \tag{2.3-19}$$

To avoid the zero solution of the directional cosines of Eqs. (2.3-18), the determinant of the coefficients of l_p, m_p, and n_p in the equation is set to zero. This makes the solution of the directional cosines indeterminate from Eqs. (2.3-18). Thus,

$$\begin{vmatrix}(\sigma_x - \sigma_p) & \tau_{xy} & \tau_{zx}\\ \tau_{xy} & (\sigma_y - \sigma_p) & \tau_{yz}\\ \tau_{zx} & \tau_{yz} & (\sigma_z - \sigma_p)\end{vmatrix} = 0$$

Expanding the determinant yields

$$\begin{aligned}\sigma_p^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma_p^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma_p\\ - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0\end{aligned} \tag{2.3-20}$$

where Eq. (2.3-20) is a cubic equation yielding the three principal stresses σ_1, σ_2, and σ_3.

To determine the directional cosines for a specific principal stress, the stress is substituted into Eqs. (2.3-18). The three resulting equations in the unknowns l_p, m_p, and n_p will not be independent since they were used to obtain the principal stress. Thus, only two of Eqs. (2.3-18) can be used. However, the second-order Eq. (2.3-19) can be used as the third equation for the three directional cosines. Instead of solving one second-order and two linear equations simultaneously, a simplified method is demonstrated in the following example.[†]

[†] Mathematical software packages can be used quite easily to extract the eigenvalues (σ_p) and the corresponding eigenvectors (l_p, m_p, and n_p) of a stress matrix. The reader is urged to explore software such as *Mathcad*, *Matlab*, *Maple*, and *Mathematica*, etc.

EXAMPLE

For the following stress matrix, determine the principal stresses and the directional cosines associated with the normals to the surfaces of each principal stress.

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{MPa}$$

Solution. Substituting $\sigma_x = 3$, $\tau_{xy} = 1$, $\tau_{zx} = 1$, $\sigma_y = 0$, $\tau_{yz} = 2$, and $\sigma_z = 0$ into Eq. (2.3-20) gives

$$\begin{aligned} &\sigma_p^2 - (3+0+0)\sigma_p^2 + [(3)(0) + (0)(0) + (0)(3) - 2^2 - 1^2 - 1^2]\sigma_p \\ &\quad - [(3)(0)(0) + (2)(2)(1)(1) - (3)(2^2) - (0)(1^2) - (0)(1^2)] = 0 \end{aligned}$$

which simplifies to

$$\sigma_p^2 - 3\sigma_p^2 - 6\sigma_p + 8 = 0 \tag{a}$$

The solutions to the cubic equation are $\sigma_p = 4$, 1, and -2 MPa. Following the conventional ordering,

$$\sigma_1 = 4 \text{ MPa}, \qquad \sigma_2 = 1 \text{ MPa}, \qquad \sigma_3 = -2 \text{ MPa}$$

The directional cosines associated with each principal stress are determined independently. First, consider σ_1 and substitute $\sigma_p = 4$ MPa into Eqs. (2.3-18). This results in

$$-l_1 + m_1 + n_1 = 0 \tag{b}$$

$$l_1 - 4m_1 + 2n_1 = 0 \tag{c}$$

$$l_1 + 2m_1 - 4n_1 = 0 \tag{d}$$

where the subscript agrees with that of σ_1.

Equations (*b*), (*c*), and (*d*) are no longer independent since they were used to determine the values of σ_p. Only two independent equations can be used, and in this example, any two of the above can be used. Consider Eqs. (*b*) and (*c*), which are independent. A third equation comes from Eq. (2.3-19), which is nonlinear in l_1, m_1, and n_1. Rather than solving the three equations simultaneously, consider the following approach.

Arbitrarily, let $l_1 = 1$ in Eqs. (*b*) and (*c*). Rearranging gives

$$\begin{aligned} m_1 + n_1 &= 1 \\ 4m_1 - 2n_1 &= 1 \end{aligned}$$

solving these simultaneously gives $m_1 = n_1 = \frac{1}{2}$. These values of l_1, m_1, and n_1 do not satisfy Eq. (2.3-19). However, all that remains is to normalize their values by dividing by $\sqrt{1^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{6}/2$. Thus,[†]

$$\begin{aligned} l_1 &= (1)(2/\sqrt{6}) = \sqrt{6}/3 \\ m_1 &= (1/2)(2/\sqrt{6}) = \sqrt{6}/6 \\ n_1 &= (1/2)(2/\sqrt{6}) = \sqrt{6}/6 \end{aligned}$$

Repeating the same procedure for $\sigma_2 = 1\,\text{MPa}$ results in

$$l_2 = \sqrt{3}/3, \qquad m_2 = -\sqrt{3}/3, \qquad n_2 = -\sqrt{3}/3$$

and for $\sigma_3 = -2\,\text{MPa}$

$$l_3 = 0, \qquad m_3 = \sqrt{2}/2, \qquad n_3 = -\sqrt{2}/2$$

If two of the principal stresses are equal, there will exist an infinite set of surfaces containing these principal stresses, where the normals of these surfaces are perpendicular to the direction of the third principal stress. If all three principal stresses are equal, a *hydrostatic* state of stress exists, and regardless of orientation, all surfaces contain the same principal stress with no shear stress.

Principal Stresses, Plane Stress. Considering the stress element shown in Fig. 2.11(a), the shear stresses on the surface with a normal in the z direction are zero. Thus, the normal stress $\sigma_z = 0$ is a principal stress. The directions of the remaining two principal stresses will be in the xy plane. If $\tau_{x'y'} = 0$ in Fig. 2.11(b), then $\sigma_{x'}$ would be a principal stress, σ_p with $l_p = \cos\theta$, $m_p = \sin\theta$, and $n_p = 0$. For this case, only the first two of Eqs. (2.3-18) apply, and are

$$\begin{aligned} (\sigma_x - \sigma_p)\cos\theta + \tau_{xy}\sin\theta &= 0 \\ \tau_{xy}\cos\theta + (\sigma_y - \sigma_p)\sin\theta &= 0 \end{aligned} \qquad (2.3\text{-}21)$$

As before, we eliminate the trivial solution of Eqs. (2.3-21) by setting the determinant of the coefficients of the directional cosines to zero. That is,

$$\begin{aligned} \begin{vmatrix} (\sigma_x - \sigma_p) & \tau_{xy} \\ \tau_{xy} & (\sigma_y - \sigma_p) \end{vmatrix} &= (\sigma_x - \sigma_p)(\sigma_y - \sigma_p) - \tau_{xy}^2 \\ &= \sigma_p^2 - (\sigma_x + \sigma_y)\sigma_p + (\sigma_x\sigma_y - \tau_{xy}^2) = 0 \end{aligned} \qquad (2.3\text{-}22)$$

[†] This method has one potential flaw. If l_1 is actually zero, then a solution would not result. If this happens, simply repeat the approach letting either m_1 or n_1 equal unity.

Equation (2.3-22) is a quadratic equation in σ_p for which the two solutions are

$$\sigma_p = \frac{1}{2}\left[(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right] \tag{2.3-23}$$

Since for plane stress, one of the principal stresses (σ_z) is always zero, numbering of the stresses ($\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$) cannot be performed until Eq. (2.3-23) is solved.

Each solution of Eq. (2.3-23) can then be substituted into *one* of Eqs. (2.3-21) to determine the direction of the principal stress. Note that if $\sigma_x = \sigma_y$ and $\tau_{xy} = 0$, then σ_x and σ_y are principal stresses and Eqs. (2.3-21) are satisfied for all values of θ. This means that all stresses in the plane of analysis are equal and the state of stress at the point is *isotropic* in the plane.

EXAMPLE

Determine the principal stresses for a case of plane stress given by the stress matrix

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 5 & -4 \\ -4 & 11 \end{bmatrix} \text{ kpsi}$$

Show the element containing the principal stresses properly oriented with respect to the initial *xyz* coordinate system.

Solution. From the stress matrix, $\sigma_x = 5$, $\sigma_y = 11$, and $\tau_{xy} = -4$ kpsi and Eq. (2.3-23) gives

$$\sigma_p = \tfrac{1}{2}\left[(5 + 11) \pm \sqrt{(5 - 11)^2 + 4(-4)^2}\right] = 13, \quad 3 \text{ kpsi}$$

Thus, the three principal stresses ($\sigma_1, \sigma_2, \sigma_3$), are $(13, 3, 0)$ kpsi, respectively. For directions, first substitute $\sigma_1 = 13$ kpsi into either one of Eqs. (2.3-21). Using the first equation with $\theta = \theta_1$

$$(\sigma_x - \sigma_1)\cos\theta_1 + \tau_{xy}\sin\theta_1 = (5 - 13)\cos\theta_1 + (-4)\sin\theta_1 = 0$$

or

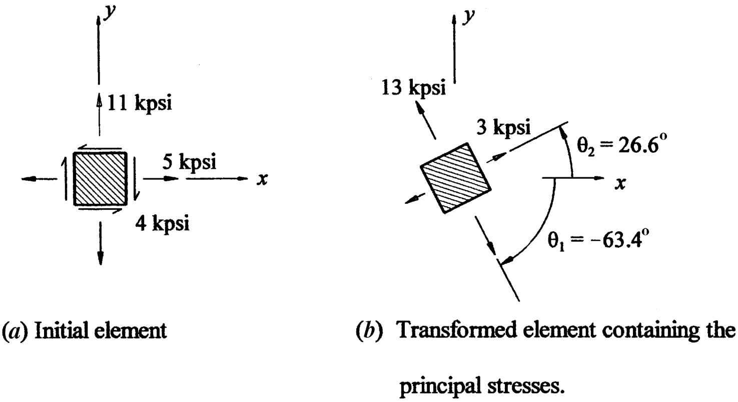
$$\theta_1 = \tan^{-1}\left(-\frac{8}{4}\right) = -63.4^\circ$$

Now for the other principal stress, $\sigma_2 = 3$ kpsi, the first of Eqs. (2.3-21) gives

$$(\sigma_x - \sigma_2)\cos\theta_2 + \tau_{xy}\sin\theta_2 = (5 - 3)\cos\theta_2 + (-4)\sin\theta_2 = 0$$

or

$$\theta_2 = \tan^{-1}\left(\frac{2}{4}\right) = 26.6^\circ$$



(*a*) Initial element

(*b*) Transformed element containing the principal stresses.

Figure 2.12 Plane stress example.

Figure 2.12(*a*) illustrates the initial state of stress, whereas the orientation of the element containing the in-plane principal stresses is shown in Fig. 2.12(*b*).

Maximum Shear Stresses. Consider that the principal stresses for a general stress state have been determined using the methods just described and are illustrated by Fig. 2.13. The 123 axes represent the normals for the principal surfaces with directional cosines determined by Eqs. (2.3-18) and (2.3-19). Viewing down a principal stress axis (e.g., the 3 axis) and performing a plane stress transformation in the plane normal to that axis (e.g., the 12 plane), one would find that the shear stress is a maximum on surfaces ±45° from the two principal stresses in that plane (e.g., σ_1, σ_2). On these surfaces, the maximum shear stress would be one-half the difference of the principal stresses [e.g., $\tau_{\max} = (\sigma_1 - \sigma_2)/2$] and will also have a normal stress equal to the average of the principal stresses [e.g., $\sigma_{\text{ave}} = (\sigma_1 + \sigma_2)/2$]. Viewing along the three principal axes would result in three shear stress

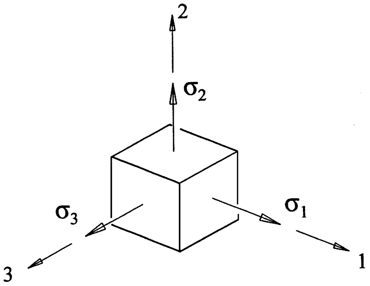


Figure 2.13 Principal stress state.

maxima, sometimes referred to as the *principal shear stresses*. These stresses together with their accompanying normal stresses are

$$\begin{array}{lll} \textbf{Plane 1, 2:} & (\tau_{\max})_{1,2} = (\sigma_1 - \sigma_2)/2, & (\sigma_{\text{ave}})_{1,2} = (\sigma_1 + \sigma_2)/2 \\ \textbf{Plane 2, 3:} & (\tau_{\max})_{2,3} = (\sigma_2 - \sigma_3)/2, & (\sigma_{\text{ave}})_{2,3} = (\sigma_2 + \sigma_3)/2 \\ \textbf{Plane 1, 3:} & (\tau_{\max})_{1,3} = (\sigma_1 - \sigma_3)/2, & (\sigma_{\text{ave}})_{1,3} = (\sigma_1 + \sigma_3)/2 \end{array} \tag{2.3-24}$$

Since conventional practice is to order the principal stresses by $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$, the largest shear stress of all is given by the third of Eqs. (2.3-24) and will be repeated here for emphasis:

$$\tau_{\max} = (\sigma_1 - \sigma_3)/2 \tag{2.3-25}$$

EXAMPLE

In the previous example, the principal stresses for the stress matrix

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 5 & -4 \\ -4 & 11 \end{bmatrix} \text{ kpsi}$$

were found to be $(\sigma_1, \sigma_2, \sigma_3) = (13, 3, 0)$ kpsi. The orientation of the element containing the principal stresses was shown in Fig. 2.12(*b*), where axis 3 was the *z* axis and normal to the page. Determine the maximum shear stress and show the orientation and complete state of stress of the element containing this stress.

Solution. The initial element and the transformed element containing the principal stresses are repeated in Fig. 2.14(*a*) and (*b*), respectively. The maximum shear stress will exist in the 1, 3 plane and is determined by substituting $\sigma_1 = 13$ and $\sigma_3 = 0$ into Eqs. (2.3-24). This results in

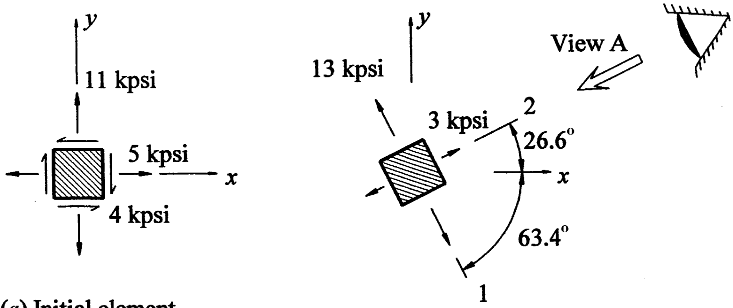
$$(\tau_{\max})_{1,3} = (13 - 0)/2 = 6.5 \text{ kpsi}, \qquad (\sigma_{\text{ave}})_{1,3} = (13 + 0)/2 = 6.5 \text{ kpsi}$$

To establish the orientation of these stresses, view the element along the axis containing $\sigma_2 = 3$ kpsi [view A, Fig. 2.14(*c*)] and rotate the surfaces ±45° as shown in Fig. 2.14(*c*).

The directional cosines associated with the surfaces are found through successive rotations. Rotating the *xyz* axes to the 123 axes yields

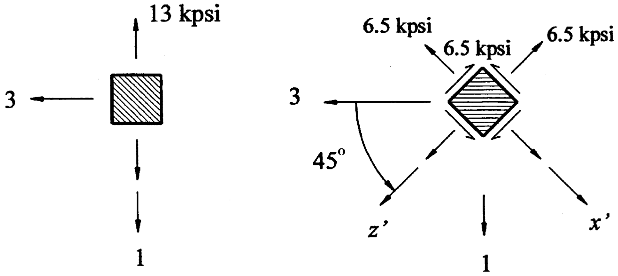
$$\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \begin{bmatrix} \cos 63.4^\circ & -\sin 63.4^\circ & 0 \\ \sin 63.4^\circ & \cos 63.4^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

$$= \begin{bmatrix} 0.4472 & -0.8944 & 0 \\ 0.8944 & 0.4472 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{a}$$



(*a*) Initial element

(*b*) Transformed element containing Principal stresses



(*c*) View A from part (*b*)

(*d*) Two-dimensional transformation of part (*c*) showing the maximum shear stress

Figure 2.14 Plane stress maximum shear stress.

A counterclockwise rotation of 45° of the normal in the 3 direction about axis 2 is represented by

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} \cos 45^\circ & 0 & -\sin 45^\circ \\ 0 & 1 & 0 \\ \sin 45^\circ & 0 & \cos 45^\circ \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$$= \begin{bmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ 0.7071 & 0 & 0.7071 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} \qquad (b)$$

Thus,

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 0.7071 & 0 & -0.7071 \\ 0 & 1 & 0 \\ 0.7071 & 0 & 0.7071 \end{bmatrix} \begin{bmatrix} 0.4472 & -0.8944 & 0 \\ 0.8944 & 0.4472 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

$$= \begin{bmatrix} 0.3162 & -0.6325 & -0.7071 \\ 0.8944 & 0.4472 & 0 \\ 0.3162 & -0.6325 & 0.7071 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

The directional cosines for Eq. (2.1-14c) are therefore

$$\begin{bmatrix} n_{x'x} & n_{x'y} & n_{x'z} \\ n_{y'x} & n_{y'y} & n_{y'z} \\ n_{z'x} & n_{z'y} & n_{z'z} \end{bmatrix} = \begin{bmatrix} 0.3162 & -0.6325 & -0.7071 \\ 0.8944 & 0.4472 & 0 \\ 0.3162 & -0.6325 & 0.7071 \end{bmatrix}$$

The other surface containing the maximum shear stress can be found similarly except for a clockwise rotation of 45° for the second rotation.

2.4 Strain Transformations

The equations for strain transformations are identical to those for stress transformations. However, the engineering strains as defined in Sec. 2.2 *will not* transform. Transformations can be performed if the shear strain is modified. All of the equations for the stress transformations can be employed simply by replacing σ and τ in the equations by ε and $\gamma/2$ (using the same subscripts), respectively. Thus, for example, the equations for plane stress, Eqs. (2.3-16), can be written for strain as

$$\begin{aligned} \varepsilon_{x'} &= \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\cos\theta\sin\theta \\ \varepsilon_{y'} &= \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \gamma_{xy}\cos\theta\sin\theta \\ \gamma_{x'y'} &= -2(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned} \tag{2.4-1}$$

2.5 Reference

1. Budynas, R. G.: “Advanced Strength and Applied Stress Analysis,” 2nd ed., McGraw-Hill, 1999.

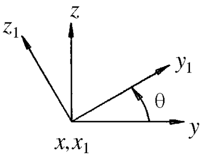
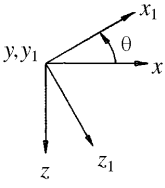
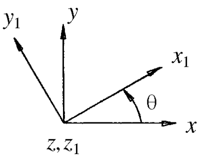
2.6 Tables

TABLE 2.1 Material properties[†]

Material	Modulus of elasticity, E		Poisson's ratio, ν	Thermal expansion coefficient, α	
	Mpsi	GPa		μ/°F	μ/°C
Aluminum alloys	10.5	72	0.33	13.1	23.5
Brass (65/35)	16	110	0.32	11.6	20.9
Concrete	4	34	0.20	5.5	9.9
Copper	17	118	0.33	9.4	16.9
Glass	10	69	0.24	5.1	9.2
Iron (gray cast)	13	90	0.26	6.7	12.1
Steel (structural)	29.5	207	0.29	6.5	11.7
Steel (stainless)	28	193	0.30	9.6	17.3
Titanium (6 A1/4 V)	16.5	115	0.34	5.2	9.5

[†] The values given in this table are to be treated as approximations of the true behavior of an actual batch of the given material.

TABLE 2.2 Transformation matrices for positive rotations about an axis[†]

Axis	Transformation matrix
x axis:  	$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$
y axis:  	$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$
z axis:  	$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$

[†] A positive rotation about a given axis is counterclockwise about the axis (as viewed from the positive axis direction).

TABLE 2.3 Transformation equations

General state of stress

$$[\boldsymbol{\sigma}]_{x'y'z'} = [\mathbf{T}][\boldsymbol{\sigma}]_{xyz}[\mathbf{T}]^T$$

where

$$[\boldsymbol{\sigma}]_{x'y'z'} = \begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{z'x'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{z'x'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix}, \qquad [\mathbf{T}] = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix}, \qquad [\boldsymbol{\sigma}]_{xyz} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Stresses on a single surface (l, m, n are directional cosines of surface normal)

$$\sigma = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl$$

$$\tau = [(\sigma_x l + \tau_{xy} m + \tau_{zx} n)^2 + (\tau_{xy} l + \sigma_y m + \tau_{yz} n)^2 + (\tau_{zx} l + \tau_{yz} m + \sigma_z n)^2 - \sigma^2]^{1/2}$$

$$l_\tau = \frac{1}{\tau}[(\sigma_x - \sigma)l + \tau_{xy} m + \tau_{zx} n]$$

$$m_\tau = \frac{1}{\tau}[\tau_{xy} l + (\sigma_y - \sigma)m + \tau_{yz} n]$$

$$n_\tau = \frac{1}{\tau}[\tau_{zx} l + \tau_{yz} m + (\sigma_z - \sigma)n]$$

l_τ, m_τ, and n_τ are directional cosines for the direction of τ.

Plane stress (θ is counterclockwise from x axis to surface normal, x')

$$\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\tau = -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Principal stresses (general case)

$$\sigma_p^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma_p^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma_p$$
$$- (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0$$

Directional cosines (l_p, m_p, n_p) are found from three of the following equations:

$$\left.\begin{aligned} (\sigma_x - \sigma_p)l_p + \tau_{xy} m_p + \tau_{zx} n_p &= 0 \\ \tau_{xy} l_p + (\sigma_y - \sigma_p)m_p + \tau_{yz} n_p &= 0 \\ \tau_{zx} l_p + \tau_{yz} m_p + (\sigma_z - \sigma_p)n_p &= 0 \end{aligned}\right\} \quad \text{select two independent equations}$$

$$l_p^2 + m_p^2 + n_p^2 = 1$$

Principal stresses (plane stress) One principal stress is zero and the remaining two are given by

$$\sigma_p = \tfrac{1}{2}\left[(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

Angle of surface normal relative to the x axis is given by

$$\theta_p = \tan^{-1}\left(\frac{\sigma_p - \sigma_x}{\tau_{xy}}\right)$$

Chapter

3

The Behavior of Bodies under Stress

This discussion pertains to the behavior of what are commonly designated as *structural materials*. That is, materials suitable for structures and members that must sustain loads without suffering damage. Included in this category are most of the metals, concrete, wood, composite materials, some plastics, etc. It is beyond the scope of this book to give more than a mere statement of a few important facts concerning the behavior of a stressed material. Extensive literature is available on every phase of the subject, and the articles contained herein will serve as an introduction only.

3.1 Methods of Loading

The mechanical properties of a material are usually determined by laboratory tests, and the commonly accepted values of ultimate strength, elastic limit, etc., are those found by testing a specimen of a certain form in a certain manner. To apply results so obtained in engineering design requires an understanding of the effects of many different variables, such as form and scale, temperature and other conditions of service, and method of loading.

The method of loading, in particular, affects the behavior of bodies under stress. There are an infinite number of ways in which stress may be applied to a body, but for most purposes it is sufficient to distinguish the types of loading now to be defined.

1. *Short-time static loading.* The load is applied so gradually that at any instant all parts are essentially in equilibrium. In testing, the load is increased progressively until failure occurs, and the total time required to produce failure is not more than a few minutes. In

service, the load is increased progressively up to its maximum value, is maintained at that maximum value for only a limited time, and is not reapplied often enough to make fatigue a consideration. The ultimate strength, elastic limit, yield point, yield strength, and modulus of elasticity of a material are usually determined by short-time static testing at room temperature.

2. *Long-time static loading.* The maximum load is applied gradually and maintained. In testing, it is maintained for a sufficient time to enable its probable final effect to be predicted; in service, it is maintained continuously or intermittently during the life of the structure. The creep, or flow characteristics, of a material and its probable permanent strength are determined by long-time static testing at the temperatures prevailing under service conditions. (See Sec. 3.6.)
3. *Repeated loading.* Typically, a load or stress is applied and wholly or partially removed or reversed repeatedly. This type of loading is important if high stresses are repeated for a few cycles or if relatively lower stresses are repeated many times; it is discussed under *Fatigue*. (See Sec. 3.8.)
4. *Dynamic loading.* The circumstances are such that the rate of change of momentum of the parts must be taken into account. One such condition may be that the parts are given definite *accelerations* corresponding to a controlled motion, such as the constant acceleration of a part of a rotating member or the repeated accelerations suffered by a portion of a connecting rod. As far as stress effects are concerned, these loadings are treated as virtually static and the *inertia forces* (Sec. 16.2) are treated exactly as though they were ordinary static loads.

A second type of *quasi-static* loading, *quick static loading*, can be typified by the rapid burning of a powder charge in a gun barrel. Neither the powder, gas, nor any part of the barrel acquires appreciable radial momentum; therefore equilibrium may be considered to exist at any instant and the maximum stress produced in the gun barrel is the same as though the powder pressure had developed gradually.

In static loading and the two types of dynamic loading just described, the loaded member is required to resist a definite *force*. It is important to distinguish this from *impact loading*, where the loaded member is usually required to absorb a definite amount of *energy*.

Impact loading can be divided into two general categories. In the first case a relatively large slow-moving mass strikes a less massive beam or bar and the *kinetic energy* of the moving mass is assumed to be converted into *strain energy* in the beam. All portions of the beam and the moving mass are assumed to stop moving simultaneously. The shape of the elastic axis of the deflected beam or bar is thus the same as in static loading. A special case of this loading, generally called *sudden loading*, occurs when a mass that is not moving is released when in contact with a beam and falls through the distance the beam deflects. This produces approximately twice the stress and deflection that would have been produced had the mass been "eased" onto the beam (see Sec. 16.4). The second case of impact loading involves the mass of the member being struck. *Stress waves* travel through the member during the impact and continue even after the impacting mass has rebounded (see Sec. 16.3).

On consideration, it is obvious that methods of loading really differ only in degree. As the time required for the load to be applied increases, short-time static loading changes imperceptibly into long-time static loading; impact may be produced by a body moving so slowly that the resulting stress conditions are practically the same as though equal deflection had been produced by static loading; the number of stress repetitions at which fatigue becomes involved is not altogether definite. Furthermore, all these methods of loading may be combined or superimposed in various ways. Nonetheless, the classification presented is convenient because most structural and machine parts function under loading that may be classified definitely as one of the types described.

3.2 Elasticity; Proportionality of Stress and Strain

In determining stress by mathematical analysis, it is customary to assume that material is elastic, isotropic, homogeneous, and infinitely divisible without change in properties and that it conforms to Hooke's law, which states that strain is proportional to stress. Actually, none of these assumptions is strictly true. A structural material is usually an aggregate of crystals, fibers, or cemented particles, the arrangement of which may be either random or systematic. When the arrangement is random the material is essentially isotropic if the part considered is large in comparison with the constituent units; when the arrangement is systematic, the elastic properties and strength are usually different in different directions and the material is anisotropic. Again, when subdivision is carried to the point where the part under consideration comprises only a portion of a single crystal, fiber, or other unit, in all probability its properties will differ from those of a larger part that is an aggregate of such units. Finally, very careful experiments show that for all materials there is probably some set and some deviation from Hooke's law for any stress, however small.

These facts impose certain limitations upon the conventional methods of stress analysis and must often be taken into account, but formulas for stress and strain, mathematically derived and based on the assumptions stated, give satisfactory results for nearly all problems of engineering design. In particular, Hooke's law may be regarded as practically true up to a proportional limit, which, though often not sharply defined, can be established for most materials with sufficient definiteness. So, too, a fairly definite elastic limit is determinable; in most cases it is so nearly equal to the proportional limit that no distinction need be made between the two.

3.3 Factors Affecting Elastic Properties

For ordinary purposes it may be assumed that the elastic properties of most metals, when stressed below a nominal proportional limit, are constant with respect to stress, unaffected by ordinary atmospheric variations of temperature, unaffected by prior applications of moderate stress, and independent of the rate of loading. When precise relations between stress and strain are important, as in the design or calibration of instruments, these assumptions cannot always be made. The fourth edition of this book (Ref. 1) discussed in detail the effects of strain rate, temperature, etc., on the elastic properties of many metals and gave references for the experiments performed. The relationships between atomic and molecular structure and the elastic properties are discussed in texts on materials science.

Wood exhibits a higher modulus of elasticity and much higher proportional limit when tested rapidly than when tested slowly. The standard impact test on a beam indicates a fiber stress at the proportional limit approximately twice as great as that found by the standard static bending test. Absorption of moisture up to the fiber saturation point greatly lowers both the modulus of elasticity and the proportional limit (Ref. 2).

Both concrete and cast iron have stress-strain curves more or less curved throughout, and neither has a definite proportional limit. For these materials it is customary to define E as the ratio of some definite stress (for example, the allowable stress or one-fourth the ultimate strength) to the corresponding unit strain; the quantity so determined is called the *secant* modulus since it represents the slope of the secant of the stress-strain diagram drawn from the origin to the point representing the stress chosen. The moduli of elasticity of cast iron are much more variable than those of steel, and the stronger grades are stiffer than the weaker ones. Cast iron suffers a distinct set from the first application of even a moderate stress; but after several repetitions of that stress, the material exhibits perfect elasticity up to, but not beyond, that stress. The modulus of elasticity is slightly less in tension than in compression.

Concrete also shows considerable variation in modulus of elasticity, and in general its stiffness increases with its strength. Like cast iron, concrete can be made to exhibit perfect elasticity up to a moderate stress by repeated loading up to that stress. Because of its tendency to yield under continuous loading, the modulus of elasticity indicated by long-time loading is much less than that obtained by progressive loading at ordinary speeds.

3.4 Load–Deformatlon Relation for a Body

If Hooke's law holds for the material of which a member or structure is composed, the member or structure will usually conform to a similar law of load-deformation proportionality and the deflection of a beam or truss, the twisting of a shaft, the dilation of a pressure container, etc., may in most instances be assumed proportional to the magnitude of the applied load or loads.

There are two important exceptions to this rule. One is to be found in any case where the stresses due to the loading are appreciably affected by the deformation. Examples of this are: a beam subjected to axial and transverse loads; a flexible wire or cable held at the ends and loaded transversely; a thin diaphragm held at the edges and loaded normal to its plane; a ball pressed against a plate or against another ball; and a helical spring under severe extension.

The second exception is represented by any case in which failure occurs through elastic instability, as in the compressive loading of a long, slender column. Here, for compression loads less than a specific *critical* (*Euler*) *load*, elastic instability plays no part and the axial deformation is linear with load. At the critical load, the type of deformation changes, and the column bends instead of merely shortening axially. For any load beyond the critical load, high bending stresses and failure occurs through excessive deflection (see Sec. 3.13).

3.5 Plasticity

Elastic deformation represents an actual change in the distance between atoms or molecules; plastic deformation represents a permanent change in their relative positions. In crystalline materials, this permanent rearrangement consists largely of group displacements of the atoms in the crystal lattice brought about by slip on planes of least resistance, parts of a crystal sliding past one another and in some instances suffering angular displacement. In amorphous materials, the rearrangement appears to take place through the individual shifting from positions of equilibrium of many atoms or molecules, the cause being thermal agitation due to external work and the result appearing as a more or less uniform flow like that of a viscous liquid. It should be noted that plastic deformation before rupture is much less for biaxial or triaxial tension than for one-way stress; for this reason metals that are ordinarily ductile may prove brittle when thus stressed.

3.6 Creep and Rupture under Long-Time Loading

More materials will creep or flow to some extent and eventually fail under a sustained stress less than the short-time ultimate strength. After a short time at load, the initial creep related to stress redistribution in the structure and strain hardening ceases and the *steady state*, or *viscous creep*, predominates. The viscous creep will continue until fracture unless the load is reduced sufficiently, but it is seldom important in materials at temperatures less than 40 to 50% of their absolute melting temperatures. Thus, creep and long-time strength at atmospheric temperatures must sometimes be taken into account in designing members of nonferrous metals and in selecting allowable stresses for wood, plastics, and concrete.

Metals. Creep is an important consideration in high-pressure steam and distillation equipment, gas turbines, nuclear reactors, supersonic vehicles, etc. Marin, Odqvist, and Finnie, in Ref. 3, give excellent surveys and list references on creep in metals and structures. Conway (Refs. 4 and 5) discusses the effectiveness of various parametric equations, and Conway and Flagella (Ref. 6) present extensive creep-rupture data for the refractory metals. Odqvist (Ref. 7) discusses the theory of creep and its application to large deformation and stability problems in plates, shells, membranes, and beams and tabulates creep constants for 15 common metals and alloys. Hult (Ref. 8) also discusses creep theory and its application to many structural problems. Penny and Marriott (Ref. 9) discuss creep theories and the design of experiments to verify them. They also discuss the development of several metals for increased resistance to creep at high temperatures as well as polymeric and composite materials at lower temperatures. Reference 10 is a series of papers with extensive references covering creep theory, material properties, and structural problems.

Plastics. The literature on the behavior of the many plastics being used for structural or machine applications is too extensive to list here.

Concrete. Under sustained compressive stress, concrete suffers considerable plastic deformation and may flow for a very long time at stresses less than the ordinary working stress. Continuous flow has been observed over a period of 10 years, though ordinarily it ceases or becomes imperceptible within 1 or 2 years. The rate of flow is greater for air than for water storage, greater for small than for large specimens, and for moderate stresses increases approximately as the applied stress. On removal of stress, some elastic recovery occurs. Concrete also shows creep under tensile stress, the early creep rate being greater than the flow rate under compression (Refs. 11 and 16).

Under very gradually applied loading concrete exhibits an ultimate strength considerably less than that found under short-time loading; in certain compression tests it was found that increasing the time of testing from 1 s to 4 h decreased the unit stress at failure about 30%, most of this decrease occurring between the extremely quick (1 or 2 s) and the conventional (several minutes) testing. This indicates that the compressive stress that concrete can sustain indefinitely may be considerably less than the ultimate strength as determined by a conventional test. On the other hand, the long-time imposition of a moderate loading appears to have no harmful effect; certain tests show that after 10 years of constant loading equal to one-fourth the ultimate strength, the compressive strength of concrete cylinders is practically the same and the modulus of elasticity is considerably greater than for similar cylinders that were not kept under load (Ref. 15).

The modulus of rupture of plain concrete also decreases with the time of loading, and some tests indicate that the long-time strength in cross-breaking may be only 55 to 75% of the short-time strength (Ref. 12).

Reference 17 is a compilation of 12 papers, each with extensive references, dealing with the effect of volumetric changes on concrete structures. Design modifications to accommodate these volumetric changes are the main thrust of the papers.

Wood. Wood also yields under sustained stress; the long-time (several years) strength is about 55% of the short-time (several minutes) strength in bending; for direct compression parallel to the grain the corresponding ratio is about 75% (Ref. 2).

3.7 Criteria of Elastic Failure and of Rupture

For the purpose of this discussion it is convenient to divide metals into two classes: (1) *ductile* metals, in which marked plastic deformation commences at a fairly definite stress (yield point, yield strength, or possibly elastic limit) and which exhibit considerable ultimate elongation; and (2) *brittle* metals, for which the beginning of plastic deformation is not clearly defined and which exhibit little ultimate elongation. Mild steel is typical of the first class, and cast iron is typical of the second; an ultimate elongation of 5% has been suggested as the arbitrary dividing line between the two classes of metals.

A ductile metal is usually considered to have failed when it has suffered *elastic failure*, i.e., when marked plastic deformation has begun. Under simple uniaxial tension this occurs when the stress reaches a value we will denote by σ_{ys}, which represents the yield strength, yield point, or elastic limit, according to which one of these is the most satisfactory indication of elastic failure for the material in question. The question arises, when does elastic failure occur under other conditions of stress, such as compression, shear, or a combination of tension, compression, and shear?

There are many theories of elastic failure that can be postulated for which the consequences can be seen in the tensile test. When the tensile specimen begins to yield at a tensile stress of σ_{ys}, the following events occur:

1. *The maximum-principal-stress theory:* the maximum principal stress reaches the tensile yield strength, σ_{ys}.
2. *The maximum-shear-stress theory* (also called the *Tresca* theory): the maximum shear stress reaches the shear yield strength, 0.5 σ_{ys}.
3. *The maximum-principal-strain theory:* the maximum principal strain reaches the yield strain, σ_{ys}/E.
4. *The maximum-strain-energy theory:* the strain energy per unit volume reaches a maximum of 0.5 σ_{ys}^2/E.
5. The *maximum-distortion-energy theory* (also called the *von Mises theory* and the *Maxwell–Huber–Hencky–von Mises theory*): the energy causing a change in shape (distortion) reaches $[(1+\nu)/(3E)]\sigma_{ys}^2$.
6. *The maximum-octahedral-shear-stress theory:* the shear stress acting on each of eight (octahedral) surfaces containing a hydrostatic normal stress, $\sigma_{ave} = (\sigma_1 + \sigma_2 + \sigma_3)/3$, reaches a value of $\sqrt{2}\sigma_{ys}/3$. It can be shown that this theory yields identical conditions as that provided by the maximum-distortion-energy theory.

Of these six theories, for ductile materials, the fifth and sixth are the ones that agree best with experimental evidence. However, the second leads to results so nearly the same and is simpler and more conservative for design applications. Thus, it is more widely used as a basis for design.

Failure theories for yield of ductile materials are based on shear or distortion. The maximum-distortion-energy theory equates the distortion energy for a general case of stress to the distortion energy when a simple tensile specimen yields. In terms of the principal stresses the distortion energy for the general case can be shown to be (see Ref. 59)

$$u_d = \frac{1+\nu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \qquad (3.7\text{-}1)$$

For the simple tensile test, yielding occurs when $\sigma_1 = \sigma_{ys}$, and $\sigma_2 = \sigma_3 = 0$. From Eq. (3.7-1), this gives a distortion energy at yield of

$$(u_d)_y = \frac{1+\nu}{3E}\sigma_{ys}^2 \qquad (3.7\text{-}2)$$

Equating the energy for the general case, Eq. (3.7-1), to that for yield, Eq. (3.7-2), gives

$$\sqrt{0.5[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} = \sigma_{ys} \qquad (3.7\text{-}3)$$

For yield under a single, uniaxial state of stress, the stress would be equated to σ_{ys}. Thus, for yield, a single, uniaxial stress *equivalent* to the general state of stress is equated to the left-hand side of Eq. (3.7-3). This equivalent stress is called the *von Mises stress*, σ_{vM}, and is given by

$$\sigma_{vM} = \sqrt{0.5[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} \qquad (3.7\text{-}4)$$

Therefore, the maximum-distortion-energy theory predicts elastic failure when the von Mises stress reaches the yield strength.

The maximum-octahedral-shear-stress theory yields identical results to that of the maximum-distortion-energy theory (see Ref. 59). Through stress transformation, a stress element can be isolated in which all normal stresses on it are equal. These normal stresses are the averages of the normal stresses of the stress matrix, which are also the averages of the principal stresses and are given by

$$\sigma_{\text{ave}} = \tfrac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \qquad (3.7\text{-}5)$$

The element with these normal stresses is an octahedron where the eight surfaces are symmetric with respect to the principal axes. The directional cosines of the normals of these surfaces, relative to the principal axes, are eight combinations of $\pm 1/\sqrt{3}$ (e.g., one set is $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$; another is $1/\sqrt{3}$, $-1\sqrt{3}$, $1/\sqrt{3}$; etc.). The octahedron is as shown in Fig. 3.1. The shear stresses on these surfaces are also equal, called the *octahedral shear stresses*, and are given by

$$\tau_{\text{oct}} = \tfrac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \qquad (3.7\text{-}6)$$

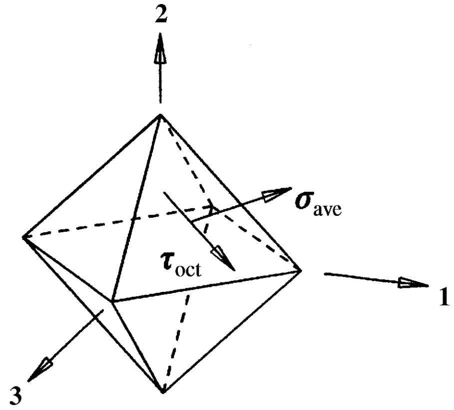


Figure 3.1 Octahedral surfaces containing octahedral shear stresses (shown relative to the principal axes, with only one set of stresses displayed).

Again, for the simple tensile test, yield occurs when $\sigma_1 = \sigma_{ys}$, and $\sigma_2 = \sigma_3 = 0$. From Eq. (3.7-6), this gives an octahedral shear stress at yield of

$$(\tau_{oct})_y = \frac{\sqrt{2}}{3}\sigma_{ys} \tag{3.7-7}$$

Equating Eqs. (3.7-6) and (3.7-7) results in Eq. (3.7-3) again, proving that the maximum-octahedral-shear-stress theory is identical to the maximum-distortion-energy theory.

The maximum-shear-stress theory equates the maximum shear stress for a general state of stress to the maximum shear stress obtained when the tensile specimen yields. If the principal stresses are ordered such that $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$, the maximum shear stress is given by $0.5(\sigma_1 - \sigma_3)$ (see Sec. 2.3, Eq. 2.3-25). The maximum shear stress obtained when the tensile specimen yields is 0.5 σ_{ys}. Thus, the condition for elastic failure for the maximum-shear-stress theory is†

$$\sigma_1 - \sigma_3 = \sigma_{ys} \tag{3.7-8}$$

The criteria just discussed concern the elastic failure of *material*. Such failure may occur locally in a *member* and may do no real damage if the volume of material affected is so small or so located as to have

---

† Plane stress problems are encountered quite often where the principal stresses are found from Eq. (2.3-23), which is

$$\sigma_p = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

This yields only two of the three principal stresses. The third principal stress for plane stress is zero. Once the *three* principal stresses are determined, they can be ordered according to $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ and then Eq. (3.7-8) can be employed.

only negligible influence on the form and strength of the member as a whole. Whether or not such local overstressing is significant depends upon the properties of the material and the conditions of service. Fatigue properties, resistance to impact, and mechanical functioning are much more likely to be affected than static strength, and a degree of local overstressing that would constitute failure in a high-speed machine part might be of no consequence whatever in a bridge member.

A brittle material cannot be considered to have definitely failed until it has broken, which can occur either through a *tensile fracture*, when the maximum tensile stress reaches the ultimate strength, or through what appears to be a *shear fracture*, when the maximum compressive stress reaches a certain value. The fracture occurs on a plane oblique to the maximum compressive stress but not, as a rule, on the plane of maximum shear stress, and so it cannot be considered to be purely a shear failure (see Ref. 14). The results of some tests on glass and Bakelite (Ref. 26) indicate that for these brittle materials either the maximum stress or the maximum strain theory affords a satisfactory criterion of rupture while neither the maximum shear stress nor the constant energy of distortion theory does. These tests also indicate that strength increases with rate of stress application and that the increase is more marked when the location of the most stressed zone changes during the loading (pressure of a sphere on a flat surface) than when this zone is fixed (axial tension).

Another failure theory that is applicable to brittle materials is the *Coulomb–Mohr theory of failure*. Brittle materials have ultimate compressive strengths σ_{uc} greater than their ultimate tensile strengths σ_{ut}, and therefore both a uniaxial tensile test and a uniaxial compressive test must be run to use the Coulomb–Mohr theory. First we draw on a single plot both Mohr's stress circle for the tensile test at the instant of failure and Mohr's stress circle for the compressive test at the instant of failure; then we complete a failure envelope simply by drawing a pair of tangent lines to the two circles, as shown in Fig. 3.2.

Failure under a complex stress situation is expected if the largest of the three Mohr circles for the given situation touches or extends outside the envelope just described. If all normal stresses are tensile, the results coincide with the maximum stress theory. For a condition where the three principal stresses are σ_A, σ_B, and σ_C, as shown in Fig. 3.2, failure is being approached but will not take place unless the dashed circle passing through σ_A and σ_C reaches the failure envelope.

The accurate prediction of the breaking strength of a member composed of brittle metal requires a knowledge of the effect of *form and scale*, and these effects are expressed by the *rupture factor* (see Sec. 3.11). In addition, what has been said here concerning brittle metals applies also to any essentially isotropic brittle material.

Thus far, our discussion of failure has been limited to *isotropic* materials. For wood, which is distinctly *anisotropic*, the possibility of

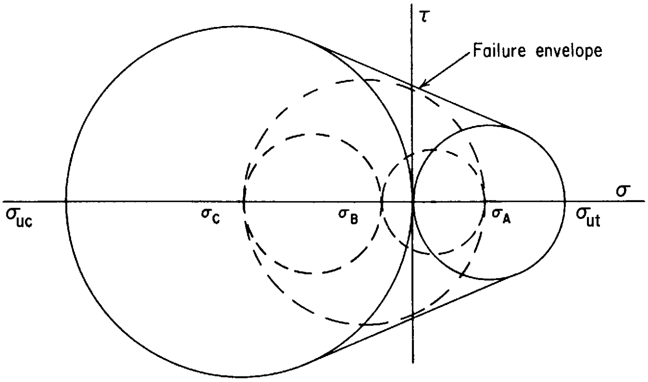


Figure 3.2

failure in each of several ways and directions must be taken into account, viz.: (1) by tension parallel to the grain, which causes fracture; (2) by tension transverse to the grain, which causes fracture; (3) by shear parallel to the grain, which causes fracture; (4) by compression parallel to the grain, which causes gradual buckling of the fibers usually accompanied by a shear displacement on an oblique plane; (5) by compression transverse to the grain, which causes sufficient deformation to make the part unfit for service. The unit stress producing each of these types of failure must be ascertained by suitable tests (Ref. 2).

Another anisotropic class of material of consequence is that of the composites. It is well known that composite members (see Secs. 7.3, 8.2, and Appendix C), such as steel reinforced concrete beams, more effectively utilize the more expensive, higher-strength materials in high-stress areas and the less expensive, lower-strength materials in the low-stress areas. Composite materials accomplish the same effect at microstructural and macrostructural levels. Composite materials come in many forms, but are generally formulated by embedding a reinforcement material in the form of fibers, flakes, particles, or laminations, in a randomly or orderly oriented fashion within a base matrix of polymeric, metallic, or ceramic material. For more detail properties of composites, see Ref. 60.

3.8 Fatigue

Practically all materials will break under numerous repetitions of a stress that is not as great as the stress required to produce immediate rupture. This phenomenon is known as *fatigue*.

Over the past 100 years the effects of surface condition, corrosion, temperature, etc., on fatigue properties have been well documented, but only in recent years has the microscopic cause of fatigue damage been attributed to *cyclic plastic flow* in the material at the source of a fatigue crack (*crack initiation*) or at the tip of an existing fatigue crack (*crack propagation*; Ref. 20). The development of extremely sensitive extensometers has permitted the separation of elastic and plastic strains when testing axially loaded specimens over short gage lengths. With this instrumentation it is possible to determine whether cyclic loading is accompanied by significant cyclic plastic strain and, if it is, whether the cyclic plastic strain continues at the same level, increases, or decreases. Sandor (Ref. 44) discusses this instrumentation and its use in detail.

It is not feasible to reproduce here even a small portion of the fatigue data available for various engineering materials. The reader should consult materials handbooks, manufacturers' literature, design manuals, and texts on fatigue. See Refs. 44 to 48. Some of the more important factors governing fatigue behavior in general will be outlined in the following material.

Number of cycles to failure. Most data concerning the number of cycles to failure are presented in the form of an S–N curve where the cyclic stress amplitude is plotted versus the number of cycles to failure. This generally leads to a straight-line log–log plot if we account for the scatter in the data. For ferrous metals a lower limit exists on the stress amplitude and is called the *fatigue limit*, or *endurance limit*. This generally occurs at a life of from 10^5 to 10^7 cycles of reversed stress, and we assume that stresses below this limit will not cause failure regardless of the number of repetitions. With the ability to separate elastic and plastic strains accurately, there are instances when a plot of plastic-strain amplitudes versus N and elastic-strain amplitudes versus N will reveal more useful information (Refs. 44 and 45).

Method of loading and size of specimen. Uniaxial stress can be produced by axial load, bending, or a combination of both. In flat-plate bending, only the upper and lower surfaces are subjected to the full range of cyclic stress. In rotating bending, all surface layers are similarly stressed, but in axial loading, the entire cross section is subjected to the same average stress. Since fatigue properties of a material depend upon the statistical distribution of defects throughout the specimen, it is apparent that the three methods of loading will produce different results.

In a similar way, the size of a bending specimen will affect the fatigue behavior while it will have little effect on an axially loaded specimen. Several empirical formulas have been proposed to represent the influence of size on a machine part or test specimen in bending. For steel, Moore (Ref. 38) suggests the equation

$$\sigma'_e\left(1-\frac{0.016}{d'}\right)=\sigma''_e\left(1-\frac{0.016}{d''}\right)$$

where σ'_e is the endurance limit for a specimen of diameter d' and σ''_e is the endurance limit for a specimen of diameter d''. This formula was based on test results obtained with specimens from 0.125 to 1.875 inches in diameter and shows good agreement within that size range. Obviously it cannot be used for predicting the endurance limit of very small specimens. The few relevant test results available indicate a considerable decrease in endurance limit for very large diameters (Refs. 22–24).

Stress concentrations. Fatigue failures occur at stress levels less than those necessary to produce the gross yielding which would blunt the sharp rise in stress at a stress concentration. It is necessary, therefore, to apply the fatigue strengths of a smooth specimen to the peak stresses expected at the stress concentrations unless the size of the stress-concentrating notch or fillet approaches the grain size or the size of an anticipated defect in the material itself (see *Factor of stress concentration in fatigue* in Sec. 3.10). References 40 and 41 discuss the effect of notches on low-cycle fatigue.

Surface conditions. Surface roughness constitutes a kind of stress raiser. Discussion of the effect of surface coatings and platings is beyond the scope of this book (see Refs. 28 and 36).

Corrosion fatigue. Under the simultaneous action of *corrosion* and repeated stress, the fatigue strength of most metals is drastically reduced, sometimes to a small fraction of the strength in air, and a true endurance limit can no longer be said to exist. Liquids and gases not ordinarily thought of as especially conducive to corrosion will often have a very deleterious effect on fatigue properties, and resistance to corrosion is more important than normal fatigue strength in determining the relative rating of different metals (Refs. 24, 25, and 31).

Range of stress. Stressing a ductile material beyond the elastic limit or yield point in tension will raise the elastic limit for subsequent cycles but lower the elastic limit for compression. The consequence of this Bauschinger effect on fatigue is apparent if one accepts the statement that fatigue damage is a result of cyclic plastic flow; i.e.,

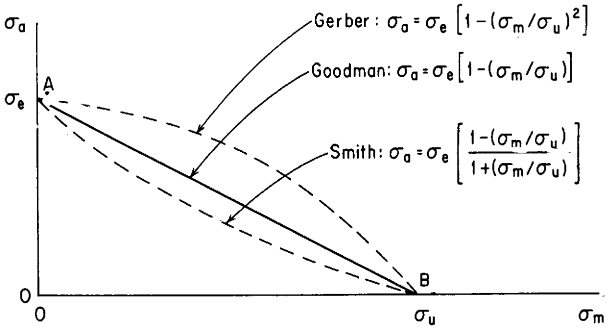


Figure 3.3

if the range of cyclic stress is reduced sufficiently, higher peak stresses can be accepted without suffering continuing damage.

Various empirical formulas for the endurance limit corresponding to any given range of stress variation have been suggested, the most generally accepted of which is expressed by the *Goodman diagram* or some modification thereof. Figure 3.3 shows one method of constructing this diagram. In each cycle, the stress varies from a maximum value $\sigma_{\max}$ to a minimum value $\sigma_{\min}$, either of which is plus or minus according to whether it is tensile or compressive. The *mean stress* is

$$\sigma_m = \tfrac{1}{2}(\sigma_{\max} + \sigma_{\min})$$

and the *alternating stress* is

$$\sigma_a = \tfrac{1}{2}(\sigma_{\max} - \sigma_{\min})$$

the addition and subtraction being algebraic. With reference to rectangular axes, σ_m is measured horizontally and σ_a vertically. Obviously when $\sigma_m = 0$, the limiting value of σ_a is the endurance limit for fully reversed stress, denoted here by σ_e. When $\sigma_a = 0$, the limiting value of σ_m is the ultimate tensile strength, denoted here by σ_u. Points *A* and *B* on the axes are thus located.

According to the Goodman theory, the ordinate to a point on the straight line *AB* represents the maximum alternating stress σ_a that can be imposed in conjunction with the corresponding mean stress σ_m. Any point above *AB* represents a stress condition that would eventually cause failure; any point below *AB* represents a stress condition with more or less margin of safety. A more conservative construction, suggested by Soderberg (Ref. 13), is to move point *B* back to σ_{ys}, the yield strength. A less conservative but sometimes preferred construction, proposed by Gerber, is to replace the straight line by the parabola.

The Goodman diagrams described can be used for steel and for aluminum and titanium alloys, but for cast iron many test results fall below the straight line AB and the lower curved line, suggested by Smith (Ref. 21), is preferred. Test results for magnesium alloys also sometimes fall below the straight line.

Figure 3.3 represents conditions where σ_m is tensile. If σ_m is compressive, σ_a is increased; and for values of σ_m less than the compression yield strength, the relationship is represented approximately by the straight line AB extended to the left with the same slope. When the mean stress and alternating stress are both torsional, σ_a is practically constant until σ_m exceeds the yield strength in shear; and for alternating bending combined with mean torsion, the same thing is true. But when σ_m is tensile and σ_a is torsional, σ_a diminishes as σ_m increases in almost the manner represented by the Goodman line. When stress concentration is to be taken into account, the accepted practice is to apply K_f (or K_t if K_f is not known) to σ_a only, not to σ_m (for K_t and K_f, see Sec. 3.10).

Residual stress. Since residual stresses, whether deliberately introduced or merely left over from manufacturing processes, will influence the *mean* stress, their effects can be accounted for. One should be careful, however, not to expect the beneficial effects of a residual stress if during the expected life of a structure it will encounter overloads sufficient to change the residual-stress distribution. Sandor (Ref. 44) discusses this in detail and points out that an occasional overload might be beneficial in some cases.

The several modified forms of the Goodman diagram are used for predicting the stress levels which will form cracks, but other more extensive plots such as the Haigh diagram (Ref. 45) can be used to predict in addition the stress levels for which cracks, once formed, will travel, fatigue lives, etc.

Combined stress. No one of the theories of failure in Sec. 3.7 can be applied to all fatigue loading conditions. The *maximum-distortion-energy theory* seems to be conservative in most cases, however. Reference 18 gives a detailed description of an acceptable procedure for designing for fatigue under conditions of combined stress. The procedure described also considers the effect of mean stress on the cyclic stress range. Three criteria for failure are discussed: *gross yielding*, *crack initiation*, and *crack propagation*. An extensive discussion of fatigue under combined stress is found in Refs. 27, 31, and 45.

Stress history. A very important question and one that has been given much attention is the influence of previous stressing on fatigue strength. One theory that has had considerable acceptance is the *linear damage law* (Miner in Ref. 27); here the assumption is made that the damage produced by repeated stressing at any level is directly proportional to the number of cycles. Thus, if the number of cycles producing failure (100% damage) at a stress range σ_1 is N_1, then the proportional damage produced by N cycles of the stress is N/N_1 and stressing at various stress levels for various numbers of cycles causes cumulative damage equal to the summation of such fractional values. Failure occurs, therefore, when $\sum N/N_1 = 1$. The formula implies that the effect of a given number of cycles is the same, whether they are applied continuously or intermittently, and does not take into account the fact that for some metals understressing (stressing below the endurance limit) raises the endurance limit. The linear damage law is not reliable for all stress conditions, and various modifications have been proposed, such as replacing 1 in the formula by a quantity x whose numerical value, either more or less than unity, must be determined experimentally. Attempts have been made to develop a better theory (e.g., Corten and Dolan, Freudenthal and Gumbel, in Ref. 32). Though all the several theories are of value when used knowledgeably, it does not appear that as yet any generally reliable method is available for predicting the life of a stressed part under variable or random loading. (See Refs. 19 and 39.) See Refs. 44 and 45 for a more detailed discussion.

A modification of the foil strain gage called an *S–N fatigue life gage* (Refs. 33 and 34) measures accumulated plastic deformation in the form of a permanent change in resistance. A given total change in resistance can be correlated with the damage necessary to cause a fatigue failure in a given material.

3.9 Brittle Fracture

Brittle fracture is a term applied to an unexpected brittle failure of a material such as low-carbon steel where large plastic strains are usually noted before actual separation of the part. Major studies of brittle fracture started when failures such as those of welded ships operating in cold seas led to a search for the effect of temperature on the mode of failure. For a brittle fracture to take place the material must be subjected to a tensile stress at a location where a crack or other very sharp notch or defect is present and the temperature must be lower than the so-called *transition temperature*. To determine a transition temperature for a given material, a series of notched specimens is tested under impact loading, each at a different temperature, and the ductility or the energy required to cause fracture is noted. There will be a limited range of temperatures over which the ductility or fracture energy will drop significantly. Careful examination of the fractured specimens will show that the material at the root of the notch has tried to contract laterally. Where the fracture energy is large, there is evidence of a large lateral contraction; and where the fracture energy is small, the lateral contraction is essentially zero. In all cases the lateral contraction is resisted by the adjacent less stressed material. The deeper and sharper cracks have relatively more material to resist lateral contraction. Thicker specimens have a greater distance over which to build up the necessary *triaxial* tensile stresses that lead to a tensile failure without producing enough shear stress to cause yielding. Thus, the term *transition temperature* is somewhat relative since it depends upon notch geometry as well as specimen size and shape. Since yielding is a flow phenomenon, it is apparent that rate of loading is also important. Static loading of sufficient intensity may start a brittle fracture, but it can continue under much lower stress levels owing to the higher rate of loading.

The ensuing research in the field of *fracture mechanics* has led to the development of both acceptable theories and experimental techniques, the discussion of which is beyond the scope of this book. Users should examine Refs. 49–58 for information and for extensive bibliographies.

3.10 Stress Concentration

The distribution of elastic stress across the section of a member may be nominally uniform or may vary in some regular manner, as illustrated by the linear distribution of stress in flexure. When the variation is abrupt so that within a very short distance the intensity of stress increases greatly, the condition is described as *stress concentration*. It is usually due to local irregularities of form such as small holes, screw threads, scratches, and similar *stress raisers*. There is obviously no hard and fast line of demarcation between the rapid variation of stress brought about by a stress raiser and the variation that occurs in such members as sharply curved beams, but in general the term *stress concentration* implies some form of irregularity not inherent in the member as such but accidental (tool marks) or introduced for some special purpose (screw thread).

The maximum intensity of elastic stress produced by many of the common kinds of stress raisers can be ascertained by mathematical analysis, photoelastic analysis, or direct strain measurement and is usually expressed by the *stress concentration factor*. This term is defined in Appendix B, but its meaning may be made clearer by an example. Consider a straight rectangular beam, originally of uniform breadth b and depth D, which has had cut across the lower face a fairly sharp transverse V-notch of uniform depth h, making the net depth of the beam section at that point $D - h$. If now the beam is subjected to a uniform bending moment M, the *nominal* fiber stress at the root of the notch may be calculated by ordinary flexure formula $\sigma = Mc/I$, which here reduces to $\sigma = 6M/[b(D-h)^2]$. But the actual stress σ' is very much greater than this because of the stress concentration that occurs at the root of the notch. The ratio σ'/σ, actual stress divided by nominal stress, is the stress concentration factor K_t for this particular case. Values of K_t for a number of common stress raisers are given in Table 17.1. The most complete single source for numerical values of stress concentration factors is Peterson (Ref. 42). It also contains an extensive bibliography.

The abrupt variation and high local intensity of stress produced by stress raisers are characteristics of *elastic behavior*. The plastic yielding that occurs on overstressing greatly mitigates stress concentration even in relatively brittle materials and causes it to have much less influence on breaking strength than might be expected from a consideration of the elastic stresses only. The practical significance of stress concentration therefore depends on circumstances. For ductile metal under static loading it is usually (though not always) of little or no importance; for example, the high stresses that occur at the edges of rivet holes in structural steel members are safely ignored, the stress due to a tensile load being assumed uniform on the net section. (In the case of eyebars and similar pin-connected members, however, a reduction of 25% in allowable stress on the net section is recommended.) For brittle material under static loading, stress concentration is often a serious consideration, but its effect varies widely and cannot be predicted either from K_t or from the brittleness of the material (see Ref. 35).

What may be termed the *stress concentration factor at rupture*, or the *strength reduction factor*, represents the significance of stress concentration for static loading. This factor, which will be denoted by K_r is the ratio of the computed stress at rupture for a plain specimen to the computed stress at rupture for the specimen containing the stress raiser. For the case just described, it would be the ratio of the modulus of rupture of the plain beam to that of the notched beam, the latter being calculated for the net section. K_r is therefore a ratio of stresses, one or both of which may be fictitious, but is nonetheless a measure of the strength-reducing effect of stress concentration. Some values of K_r are given in Table 17 of Ref. 1.

It is for conditions involving fatigue that stress concentration is most important. Even the most highly localized stresses, such as those produced by small surface scratches, may greatly lower the apparent endurance limit, but materials vary greatly in *notch sensitivity*, as susceptibility to this effect is sometimes called. Contrary to what might be expected, ductility (as ordinarily determined by axial testing) is not a measure of immunity to stress concentration in fatigue; for example, steel is much more susceptible than cast iron. What may be termed the *fatigue stress concentration factor* K_f is the practical measure of notch sensitivity. It is the ratio of the endurance limit of a plain specimen to the nominal stress at the endurance limit of a specimen containing the stress raiser.

A study of available experimental data shows that K_f is almost always less, and often significantly less, than K_t, and various methods for estimating K_f from K_t have been proposed. Neuber (Ref. 37) proposes the formula

$$K_f = 1 + \frac{K_t - 1}{1 + \pi\sqrt{\rho'/\rho}/(\pi - \omega)} \tag{3.10-1}$$

where ω is the flank angle of the notch (called θ in Table 17.1), ρ is the radius of curvature (in inches) at the root of the notch (called r in Table 17.1), and ρ' is a dimension related to the grain size, or size of some type of basic building block, of the material and may be taken as 0.0189 in for steel.

All the methods described are valuable and applicable within certain limitations, but none can be applied with confidence to all situations (Ref. 29). Probably none of them gives sufficient weight to the effect of scale in the larger size range. There is abundant evidence to show that the significance of stress concentration increases with size for both static and repeated loading, especially the latter.

An important fact concerning stress concentration is that a single isolated notch or hole has a worse effect than have a number of similar stress raisers placed close together; thus, a single V-groove reduces the strength of a part more than does a continuous screw thread of almost identical form. The deleterious effect of an unavoidable stress raiser can, therefore, be mitigated sometimes by juxtaposing additional form irregularities of like nature, but the actual superposition of stress raisers, such as the introduction of a small notch in a fillet, may result in a stress concentration factor equal to or even exceeding the product of the factors for the individual stress raisers (Refs. 30 and 43).

3.11 Effect of Form and Scale on Strength; Rupture Factor

It has been pointed out (Sec. 3.7) that a member composed of brittle material breaks in tension when the maximum tensile stress reaches the ultimate strength or in shear when the maximum compressive stress reaches a certain value. In calculating the stress at rupture in such a member it is customary to employ an *elastic-stress formula*; thus the ultimate fiber stress in a beam is usually calculated by the ordinary flexure formula. It is known that the result (modulus of rupture) is not a true stress, but it can be used to predict the strength of a similar beam of the same material. However, if another beam of the same material but of different cross section, span/depth ratio, size, or manner of loading and support is tested, the modulus of rupture will be found to be different. (The effect of the shape of the section is often taken into account by the *form factor*, and the effects of the span/depth ratio and manner of loading are recognized in the testing procedure.) Similarly, the *calculated* maximum stress at rupture in a curved beam, flat plate, or torsion member is not equal to the ultimate strength of the material, and the magnitude of the disparity will vary greatly with the material, form of the member, manner of loading, and absolute scale. In order to predict accurately the breaking load for such a member, it is necessary to take this variation into account, and the *rupture factor* (defined in Appendix B) provides a convenient means of doing so. Values of the rupture factor for a number of materials and types of members are given in Table 18 of Ref. 1.

On the basis of many experimental determinations of the rupture factor (Ref. 35) the following generalizations may be made:

1. The smaller the proportional part of the member subjected to high stress, the larger the rupture factor. This is exemplified by the facts that a beam of circular section exhibits a higher modulus of rupture than a rectangular beam and that a flat plate under a concentrated center load fails at a higher computed stress than one uniformly loaded. The extremes in this respect are, on the one hand, a uniform bar under axial tension for which the rupture factor is unity and, on the other hand, a case of severe stress concentration such as a sharply notched bar for which the rupture factor may be indefinitely large.
2. In the flexure of statically indeterminate members, the redistribution of bending moments that occurs when plastic yielding starts at the most highly stressed section increases the rupture factor. For this reason a flat plate gives a higher value than a simple beam, and a circular ring gives a higher value than a portion of it tested as a statically determinate curved beam.
3. The rupture factor seems to vary inversely with the absolute scale for conditions involving abrupt stress variation, which is consistent with the fact (already noted) that for cases of stress concentration both K_r and K_f diminish with the absolute scale.
4. As a rule, the more brittle the material, the more nearly all rupture factors approach unity. There are, however, many exceptions to this rule. It has been pointed out (Sec. 3.10) that immunity to notch effect even under static loading is not always proportional to ductility.

The practical significance of these facts is that for a given material and given factor of safety, some members may be designed with a much higher allowable stress than others. This fact is often recognized in design; for example, the allowable stress for wooden airplane spars varies according to the form factor and the proportion of the stress that is flexural.

What has been said here pertains especially to comparatively brittle materials, i.e., materials for which failure consists in fracture rather than in the beginning of plastic deformation. The effect of form on the ultimate strength of ductile members is less important, although even for steel the allowable unit stress is often chosen with regard to circumstances such as those discussed previously. For instance, in gun design the maximum stress is allowed to approach and even exceed the nominal elastic limit, the volume of material affected being very small, and in structural design extreme fiber stresses in bending are permitted to exceed the value allowed for axial loading. In testing, account must be taken of the fact that some ductile metals exhibit a higher ultimate strength when fracture occurs at a reduced section such as would be formed in a tensile specimen by a concentric groove or notch. Whatever effect of stress concentration may remain during plastic deformation is more than offset by the supporting action of the shoulders, which tends to prevent the normal "necking down."

3.12 Prestressing

Parts of an elastic system, by accident or design, may have introduced into them stresses that cause and are balanced by opposing stresses in other parts, so that the system reaches a state of stress without the imposition of any external load. Examples of such initial, or locked-up, stresses are the temperature stresses in welded members, stresses in a statically indeterminate truss due to tightening or "rigging" some of the members by turnbuckles, and stresses in the flange couplings of a pipeline caused by screwing down the nuts. The effects of such prestressing upon the rigidity and strength of a system will now be considered, the assumption being made that prestressing is not so severe as to affect the properties of the *material*.

In discussing this subject it is necessary to distinguish two types of systems, viz. one in which the component parts can sustain reversal of stress and one in which at least some of the component parts cannot sustain reversal of stress. Examples of the first type are furnished by a solid bar and by a truss, all members of which can sustain either tension or compression. Examples of the second type are furnished by the bolt-flange combination mentioned and by a truss with wire diagonals that can take tension only.

For the first type of system, prestressing has no effect on initial rigidity. Thus a plain bar with locked-up temperature stresses will exhibit the same modulus of elasticity as a similar bar from which these stresses have been removed by annealing; two prestressed helical springs arranged in parallel, the tension in one balancing the compression in the other, will deflect neither more nor less than the same two springs similarly placed without prestressing.

Prestressing will lower the elastic limit (or allowable load, or ultimate strength) provided that in the absence of prestressing all parts of the system reach their respective elastic limits (or allowable loads, or ultimate strengths) simultaneously. But if this relation between the parts does not exist, then prestressing may raise any or all of these quantities. One or two examples illustrating each condition may make this clear.

Consider first a plain bar that is to be loaded in axial tension. If there are no locked-up stresses, then (practically speaking) all parts of the bar reach their allowable stress, elastic limit, and ultimate strength simultaneously. But if there are locked-up stresses present, then the parts in which the initial tension is highest reach their elastic limit before other parts and the elastic limit of the bar as a whole is thus lowered. The load at which the allowable unit stress is first reached is similarly lowered, and the ultimate strength may also be reduced; although if the material is ductile, the equalization of stress that occurs during elongation will largely prevent this.

As an example of the second condition (all parts do not simultaneously reach the elastic limit or allowable stress) consider a thick cylinder under internal pressure. If the cylinder is not prestressed, the stress at the interior surface reaches the elastic limit first and so governs the pressure that may be applied. But if the cylinder is prestressed by shrinking on a jacket or wrapping with wire under tension, as is done in gun construction, then the walls are put into an initial state of compression. This compressive stress also is greatest at the inner surface, and the pressure required to reverse it and produce a tensile stress equal to the elastic limit is much greater than before. As another example, consider a composite member comprising two rods of equal length, one aluminum and the other steel, that are placed side by side to jointly carry a tensile load. For simplicity, it will be assumed that the allowable unit stresses for the materials are the same. Because the modulus of elasticity of the steel is about three times that of the aluminum, it will reach the allowable stress first and at a total load less than the sum of the allowable loads for the bars acting separately. But if the composite bar is properly prestressed, the steel being put into initial compression and the aluminum into initial tension (the ends being in some way rigidly connected to permit this), then on the application of a tensile load the two bars will reach the allowable stress simultaneously and the load-carrying capacity of the combination is thus greater than before. Similarly the elastic limit and sometimes the ultimate strength of a composite member may be raised by prestressing.

In a system of the second type (in which all parts *cannot* sustain stress reversal) prestressing increases the rigidity for any load less than that required to produce stress reversal. The effect of prestressing up to that point is to make the rigidity of the system the same as though all parts were effective. Thus in the case of the truss with wire diagonals it is as though the counterwires were taking compression; in the case of the flange-bolt combination it is as though the flanges were taking tension. (If the flanges are practically rigid in comparison with the bolts, there is no deformation until the applied load exceeds the bolt tension and so the system is rigid.) When the applied load becomes large enough to cause stress reversal (to make the counterwires go slack or to separate the flanges), the effect of prestressing disappears and the system is neither more nor less rigid than a similar one not prestressed provided, of course, none of the parts has been overstressed.

The elastic limit (or allowable load, or ultimate strength) of a system of this type is not affected by prestressing unless the elastic limit (or allowable load, or ultimate strength) of one or more of the parts is reached before the stress reversal occurs. In effect, a system of this type is exactly like a system of the first type until stress reversal occurs, after which all effects of prestressing vanish.

The effects of prestressing are often taken advantage of, notably in bolted joints (flanges, cylinder heads, etc.), where high initial tension in the bolts prevents stress fluctuation and consequent fatigue, and in prestressed reinforced-concrete members, where the initially compressed concrete is enabled, in effect, to act in tension without cracking up to the point of stress reversal. The example of the prestressed thick cylinder has already been mentioned.

3.13 Elastic Stability

Under certain circumstances the maximum load a member will sustain is determined not by the strength of the material but by the stiffness of the member. This condition arises when the load produces a bending or a twisting moment that is proportional to the corresponding deformation. The most familiar example is the *Euler column*. When a straight slender column is loaded axially, it remains straight and suffers only axial compressive deformation under small loads. If while thus loaded it is slightly deflected by a transverse force, it will straighten after removal of that force. But there is obviously some axial load that will just hold the column in the deflected position, and since both the bending moment due to the load and the resisting moment due to the stresses are directly proportional to the deflection, the load required thus to hold the column is independent of the amount of the deflection. If this condition of balance exists at stresses less than the elastic limit, the condition is called *elastic stability* and the load that produces this condition is called the *critical* load. Any increase of the load beyond this critical value is usually attended by immediate collapse of the member.

It is the compressive stresses within long, thin sections of a structure that can cause instabilities. The compressive stress can be elastic or inelastic and the instability can be global or local. Global instabilities can cause catastrophic failure, whereas local instabilities may cause permanent deformation but not necessarily a catastrophic failure. For the Euler column, when instability occurs, it is global since the entire cross section is involved in the deformation. Localized buckling of the edges of the flange in compression of a wide-flange I-beam in bending can occur. Likewise, the center of the web of a transversely loaded I-beam or plate girder in bending undergoes pure shear where along the diagonal (45°) compressive stresses are present and localized buckling is possible.

Other examples of elastic stability are afforded by a thin cylinder under external pressure, a thin plate under edge compression or edge shear, and a deep thin cantilever beam under a transverse end load applied at the top surface. Some such elements, unlike the simple column described previously, do not fail under the load that initiates elastic buckling but demonstrate increasing resistance as the buckling progresses. Such *postbuckling* behavior is important in many problems of shell design. Elastic stability is discussed further in Chap. 15, and formulas for the critical loads for various members and types of loadings are given in Tables 15.1 and 15.2.

3.14 References

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Chapter 4

Principles and Analytical Methods

Most of the formulas of mechanics of materials express the relations among the form and dimensions of a member, the loads applied thereto, and the resulting stress or deformation. Any such formula is valid only within certain limitations and is applicable only to certain problems. An understanding of these limitations and of the way in which formulas may be combined and extended for the solution of problems to which they do not immediately apply requires a knowledge of certain principles and methods that are stated briefly in the following articles. The significance and use of these principles and methods are illustrated in Part 3 by examples that accompany the discussion of specific problems.

4.1 Equations of Motion and of Equilibrium

The relations that exist at any instant between the motion of a body and the forces acting on it may be expressed by these two equations: (1) F_x (the component along any line x of all forces acting on a body) $= m\bar{a}_x$ (the product of the mass of the body and the x component of the acceleration of its mass center); (2) T_x (the torque about any line x of all forces acting on the body) $= dH_x/dt$ (the time rate at which its angular momentum about that line is changing). If the body in question is in equilibrium, these equations reduce to (1) $F_x = 0$ and (2) $T_x = 0$.

These equations, Hooke's law, and experimentally determined values of the elastic constants E, G, and ν constitute the basis for the mathematical analysis of most problems of mechanics of materials. The majority of the common formulas for stress are derived by considering a portion of the loaded member as a body in equilibrium under the action of forces that include the stresses sought and then solving for these stresses by applying the equations of equilibrium.

4.2 Principle of Superposition

With certain exceptions, the effect (stress, strain, or deflection) produced on an elastic system by any final state of loading is the same whether the forces that constitute that loading are applied simultaneously or in any given sequence and is the result of the effects that the several forces would produce if each acted singly.

An exception to this principle is afforded by any case in which some of the forces cause a deformation that enables other forces to produce an effect they would not have otherwise. A beam subjected to transverse and axial loading is an example; the transverse loads cause a deflection that enables the longitudinal load to produce a bending effect it would not produce if acting alone. In no case does the principle apply if the deformations are so large as to alter appreciably the geometrical relations of the parts of the system.

The principle of superposition is important and has many applications. It often makes it possible to resolve or break down a complex problem into a number of simple ones, each of which can be solved separately for like stresses, deformations, etc., which are then algebraically added to yield the solution of the original problem.

4.3 Principle of Reciprocal Deflections

Let A and B be any two points of an elastic system. Let the displacement of B in any direction U due to force P acting in any direction V at A be u; and let the displacement of A in the direction V due to a force Q acting in the direction U at B be v. Then $Pv = Qu$.

This is the general statement of the *principle of reciprocal deflections*. If P and Q are equal and parallel and u and v are parallel, the statement can be simplified greatly. Thus, for a horizontal beam with vertical loading and deflection understood, the principle expresses the following relation: A load applied at any point A produces the same deflection at any other point B as it would produce at A if applied at B.

The principle of reciprocal deflections is a corollary of the principle of superposition and so can be applied only to cases for which that principle is valid. It can be used to advantage in many problems involving deformation. Examples of the application of the principle are given in Chaps. 8 and 11.

4.4 Method of Consistent Deformations (Strain Compatibility)

Many statically indeterminate problems are easily solved by utilizing the obvious relations among the deformations of the several parts or among the deformations produced by the several loads. Thus the division of load between the parts of a composite member is readily ascertained by expressing the deformation or deflection of each part in terms of the load it carries and then equating these deformations or deflections. For example, the reaction at the supported end of a beam with one end fixed and the other supported can be found by regarding the beam as a cantilever, acted on by the actual loads and an upward end load (the reaction), and setting the resultant deflection at the support end equal to zero.

The method of consistent deformations is based on the principle of superposition; it can be applied only to cases for which that principle is valid.

4.5 Principles and Methods Involving Strain Energy

Strain energy is defined as the mechanical energy stored up in an elastically stressed system; formulas for the amount of strain energy developed in members under various conditions of loading are given in Part 3. It is the purpose of this article to state certain relations between strain energy and external forces that are useful in the analysis of stress and deformation. For convenience, external forces with points of application that do not move will here be called *reactions*, and external forces with points of application that move will be called *loads*.

External work equal to strain energy. When an *elastic* system is subjected to static loading, the external work done by the loads as they increase from zero to their maximum value is equal to the strain energy acquired by the system.

This relation may be used directly to determine the deflection of a system under a single load; for such a case, assuming a linear material, it shows that the deflection at the point of loading in the direction of the load is equal to twice the strain energy divided by the load. The relationship also furnishes a means of determining the critical load that produces elastic instability in a member. A reasonable form of curvature, compatible with the boundary conditions, is assumed, and the corresponding critical load found by equating the work of the load to the strain energy developed, both quantities being calculated for the curvature assumed. For each such assumed curvature, a corresponding approximate critical load will be found and the least load so found represents the closest approximation to the true critical load (see Refs. 3 to 5).

Method of unit loads. During the static loading of an elastic system the external work done by a *constant* force acting thereon is equal to the internal work done by the stresses caused by that constant force. This relationship is the basis of the following method for finding the deflection of any given point of an elastic system: A unit force is imagined to act at the point in question and in the direction of the deflection that is to be found. The stresses produced by such a unit force will do a certain amount of internal work during the application of the actual loads. This work, which can be readily found, is equal to the work done by the unit force; but since the unit force is constant, this work is equal to the deflection sought.

If the direction of the deflection cannot be ascertained in advance, its horizontal and vertical components can be determined separately in the way described and the resultant deflection found therefrom. Examples of application of the method are given in Sec. 7.4.

Deflection, the partial derivative of strain energy. When a linear *elastic* system is statically loaded, the partial derivative of the strain energy with respect to any one of the applied forces is equal to the movement of the point of application of that force in the direction of that force. This relationship provides a means of finding the deflection of a beam or truss under several loads (see Refs. 3, 5, and 7).

Theorem of least work.[†] When an elastic system is statically loaded, the distribution of stress is such as to make the strain energy a minimum consistent with equilibrium and the imposed boundary conditions. This principle is used extensively in the solution of statically indeterminate problems. In the simpler type of problem (beams with redundant supports or trusses with redundant members) the first step in the solution consists in arbitrarily selecting certain reactions or members to be considered redundant, the number and identity of these being such that the remaining system is just determinate. The strain energy of the entire system is then expressed in terms of the unknown redundant reactions or stresses. The partial derivative of the strain energy with respect to each of the redundant reactions or stresses is then set equal to zero and the resulting equations solved for the redundant reactions or stresses. The remaining reactions or stresses are then found by the equations of equilibrium. An example of the application of this method is given in Sec. 7.4.

[†] By *theorem of least work* is usually meant only so much of the theorem as is embodied in the first application here described, and so understood it is often referred to as *Castigliano's second theorem*. But, as originally stated by Castigliano, it had a somewhat different significance. (See his "Théorème de l'équilibre des systèmes élastiques et ses applications," Paris, 1879, or the English translation "Elastic Stresses in Structures," by E. S. Andrews, Scott, Greenwood, London. See also R. V. Southwell, Castigliano's Principle of Minimum Strain-energy, *Proc. Roy. Soc. Lond., Ser. A*, vol. 154, 1936.) The more general theory stated is called *theorem of minimum energy* by Love (Ref. 1) and *theorem of minimum resilience* by Morley (Ref. 2).

As defined by this procedure, the *theorem of least work* is implicit in Castigliano's theorem: It furnishes a method of solution identical with the method of consistent deflections, the deflection used being zero and expressed as a partial derivative of the strain energy. In a more general type of problem, it is necessary to determine which of an infinite number of possible stress distributions or configurations satisfies the condition of minimum strain energy. Since the development of software based on the finite-element method of analysis the electronic computer has made practicable the solution of many problems of this kind—shell analysis, elastic and plastic buckling, etc.—that formerly were relatively intractable.

4.6 Dimensional Analysis

Most physical quantities can be expressed in terms of mass, length, and time conveniently represented by the symbols M, L, and t, respectively. Thus velocity is Lt^{-1} acceleration is Lt^{-2}, force is MLt^{-2}, unit stress is $ML^{-1}t^{-2}$, etc. A formula in which the several quantities are thus expressed is a dimensional formula, and the various applications of this system of representation constitute *dimensional analysis*.

Dimensional analysis may be used to check formulas for homogeneity, check or change units, derive formulas, and establish the relationships between similar physical systems that differ in scale (e.g., a model and its prototype). In strength of materials, dimensional analysis is especially useful in checking formulas for homogeneity. To do this, it is not always necessary to express *all* quantities dimensionally since it may be possible to cancel some terms. Thus it is often convenient to express force by some symbol, as F, until it is ascertained whether or not all terms representing force can be canceled.

For example, consider the formula for the deflection y at the free end of a cantilever beam of length l carrying a uniform load per unit length, w. This formula (Table 8.1) is

$$y = -\frac{1}{8}\frac{wl^4}{EI}$$

To test for homogeneity, omit the negative sign and the coefficient $\frac{1}{8}$ (which is dimensionless) and write the formula

$$L = \frac{(F/L)^4}{(F/L^2)L^4}$$

It is seen that F cancels and the equation reduces at once to $L = L$, showing that the original equation was homogeneous.

Instead of the symbols M, L, t, and F, we can use the names of the *units* in which the quantities are to be expressed. Thus the above equation may be written

$$\text{inches} = \frac{(\text{pounds/inch})(\text{inches}^4)}{(\text{pounds/inches}^2)(\text{inches}^4)} = \text{inches}$$

This practice is especially convenient if it is desired to change units. Thus it might be desired to write the above formula so that y is given in inches when l is expressed in feet. It is only necessary to write

$$\text{inches} = \frac{1}{8}\frac{(\text{pounds/inch})(\text{feet} \times 12)^4}{(\text{pounds/inches}^2)\text{inches}^4}$$

and the coefficient is thus found to be 2592 instead of $\frac{1}{8}$.

By what amounts to a reversal of the checking process described, it is often possible to determine the way in which a certain term or terms should appear in a formula provided the other terms involved are known. For example, consider the formula for the critical load of the Euler column. Familiarity with the theory of flexure suggests that this load will be directly proportional to E and I. It is evident that the length l will be involved in some way as yet unknown. It is also reasonable to assume that the load is independent of the deflection since both the bending moment and the resisting moment would be expected to vary in direct proportion to the deflection. We can then write $P = kEIl^a$, where k is a dimensionless constant that must be found in some other way and the exponent a shows how l enters the expression. Writing the equation dimensionally and omitting k, we have

$$F = \frac{F}{L^2}L^4L^a \qquad \text{or} \qquad L^2 = L^{4+a}$$

Equating the exponents of L (as required for homogeneity) we find $a = -2$, showing that the original formula should be $P = kEI/l^2$. Note that the derivation of a formula in this way requires at least a partial knowledge of the relationship that is to be expressed.

A much more detailed discussion of similitude, modeling, and dimensional analysis can be found in Chaps. 15 and 8 of Refs. 6 and 7, respectively. Reference 6 includes a section where the effect of Poisson's ratio on the stresses in two- and three-dimensional problems is discussed. Since Poisson's ratio is dimensionless, it would have to be the same in model and prototype for perfect modeling and this generally is not possible. References to work on this problem are included and will be helpful.

4.7 Remarks on the Use of Formulas

No calculated value of stress, strength, or deformation can be regarded as exact. The formulas used are based on certain assumptions as to properties of materials, regularity of form, and boundary conditions that are only approximately true, and they are derived by mathematical procedures that often involve further approximations. In general, therefore, great precision in numerical work is not justified. Each individual problem requires the exercise of judgment, and it is impossible to lay down rigid rules of procedure; but the following suggestions concerning the use of formulas may be of value.

1. For most cases, calculations giving results to three significant figures are sufficiently precise. An exception is afforded by any calculation that involves the algebraic addition of quantities that are large in comparison with the final result (e.g., some of the formulas for beams under axial and transverse loading, some of the formulas for circular rings, and any case of superposition in which the effects of several loads tend to counteract each other). For such cases more significant figures should be carried throughout the calculations.
2. In view of uncertainties as to actual conditions, many of the formulas may appear to be unnecessarily elaborate and include constants given to more significant figures than is warranted. For this reason, we may often be inclined to simplify a formula by dropping unimportant terms, "rounding off" constants, etc. It is sometimes advantageous to do this, but it is usually better to use the formula as it stands, bearing in mind that the result is at best only a close approximation. The only disadvantage of using an allegedly "precise" formula is the possibility of being misled into thinking that the result it yields corresponds exactly to a real condition. So far as the time required for calculation is concerned, little is saved by simplification.
3. When using an unfamiliar formula, we may be uncertain as to the correctness of the numerical substitutions made and mistrustful of the result. It is nearly always possible to effect some sort of check by analogy, superposition, reciprocal deflections, comparison, or merely by judgment and common sense. Thus the membrane analogy (Sec. 5.4) shows that the torsional stiffness of any irregular section is greater than that of the largest inscribed circular section and less than that of the smallest circumscribed section. Superposition shows that the deflection and bending moment at the center of a beam under triangular loading (Table 8.1, case 2e) is the same as under an equal load uniformly distributed. The principle of reciprocal deflections shows that the stress and deflection at the center of a circular flat plate under eccentric concentrated load (Table 11.2, case 18) are the same as for an equal load uniformly distributed along a concentric circle with radius equal to the eccentricity (case 9a). Comparison

shows that the critical unit compressive stress is greater for a thin plate under edge loading than for a strip of that plate regarded as an Euler column. Common sense and judgment should generally serve to prevent the acceptance of grossly erroneous calculations.

4. A difficulty frequently encountered is uncertainty as to boundary conditions—whether a beam or flat plate should be calculated as freely supported or fixed, whether a load should be assumed uniformly or otherwise distributed, etc. In any such case it is a good plan to make *bracketing assumptions*, i.e., to calculate the desired quantity on the basis of each of two assumptions representing limits between which the actual conditions must lie. Thus for a beam with ends having an unknown degree of fixity, the bending moment at the center cannot be more than if the ends were freely supported and the bending moments at the ends cannot be more than if the ends were truly fixed. If so designed as to be safe for either extreme condition, the beam will be safe for any intermediate degree of fixity.

5. The stress and deflections predicted by most formulas do not account for localized effects of the loads. For example, the stresses and deflections given for a straight, simply-supported beam with a centered, concentrated lateral force only account for that due to bending. Additional compressive bearing stresses and deflections exist depending on the exact nature of the interaction of the applied and reaction forces with the beam. Normally, the state of stress and deformation at distances greater than the dimensions of the loaded regions only depend on the *net* effect of the localized applied and reaction forces and are independent of the form of these forces. This is an application of *Saint Venant's principle* (defined in Appendix B). This principle may not be reliable for thin-walled structures or for some orthotropic materials.

6. Formulas concerning the validity of which there is a reason for doubt, especially empirical formulas, should be checked dimensionally. If such a formula expresses the results of some intermediate condition, it should be checked for extreme or terminal conditions; thus an expression for the deflection of a beam carrying a uniform load over a portion of its length should agree with the corresponding expression for a fully loaded beam when the loaded portion becomes equal to the full length and should vanish when the loaded portion becomes zero.

4.8 References

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Chapter 5

Numerical Methods

The analysis of stress and deformation of the loading of simple geometric structures can usually be accomplished by closed-form techniques. As the structures become more complex, the analyst is forced to approximations of closed-form solutions, experimentation, or numerical methods. There are a great many numerical techniques used in engineering applications for which digital computers are very useful. In the field of structural analysis, the numerical techniques generally employ a method which discretizes the continuum of the structural system into a finite collection of points (or nodes) whereby mathematical relations from elasticity are formed. The most popular technique used currently is the *finite element method* (FEM). For this reason, most of this chapter is dedicated to a general description of the method. A great abundance of papers and textbooks have been presented on the finite element method, and a complete listing is beyond the scope of this book. However, some textbooks and historical papers are included for introductory purposes.

Other methods, some of which FEM is based upon, include *trial functions* via *variational methods* and *weighted residuals*, the *finite difference method* (FDM), *structural analogues*, and the *boundary element method* (BEM). FDM and BEM will be discussed briefly.

5.1 The Finite Difference Method

In the field of structural analysis, one of the earliest procedures for the numerical solutions of the governing differential equations of stressed continuous solid bodies was the *finite difference method*. In the finite difference approximation of differential equations, the derivatives in the equations are replaced by difference quotients of the values of the dependent variables at discrete mesh points of the domain. After imposing the appropriate boundary conditions on the structure, the discrete equations are solved obtaining the values of the variables at the mesh points. The technique has many disadvantages, including inaccuracies of the derivatives of the approximated solution, difficulties in imposing boundary conditions along curved boundaries, difficulties in accurately representing complex geometric domains, and the inability to utilize non-uniform and non-rectangular meshes.

5.2 The Finite Element Method

The finite element method (FEM) evolved from the use of trial functions via variational methods and weighted residuals, the finite difference method, and structural analogues (see Table 1.1 of Ref. 1). FEM overcomes the difficulties encountered by the finite-difference method in that the solution of the differential equations of the structural problem are obtained by utilizing an integral formulation to generate a system of algebraic equations with continuous piecewise-smooth (trial) functions that approximate the unknown quantities. A geometrically complex domain of the structural problem can be systematically represented by a large, but finite, collection of simpler subdomains, called *finite elements*. For structural problems, the displacement field of each element is approximated by polynomials, which are interpolated with respect to preselected points (*nodes*) on, and possibly within, the element. The polynomials are referred to as *interpolation functions*, where variational or weighted residual methods (e.g. Rayleigh–Ritz, Galerkin, etc.) are applied to determine the unknown nodal values. Boundary conditions can easily be applied along curved boundaries, complex geometric domains can be modeled, and non-uniform and non-rectangular meshes can be employed.

The modern development of FEM began in the 1940s in the field of structural mechanics with the work of Hrennikoff, McHenry, and Newmark, who used a lattice of line elements (rods and beams) for the solution of stresses in continuous solids (see Refs. 2–4). In 1943, from a 1941 lecture, Courant suggested piecewise-polynomial interpolation over triangular subregions as a method to model torsional problems (see Ref. 5).

With the advent of digital computers in the 1950s, it became practical for engineers to write and solve the stiffness equations in matrix form (see Refs. 6–8). A classic paper by Turner, Clough, Martin, and Topp published in 1956 presented the matrix stiffness equations for the truss, beam, and other elements (see Ref. 9). The expression *finite element* is first attributed to Clough (see Ref. 10).

Since these early beginnings, a great deal of effort has been expended in the development of FEM in the areas of element formulations and computer implementation of the entire solution process. The major advances in computer technology includes the rapidly expanding computer hardware capabilities, efficient and accurate matrix solver routines, and computer graphics for ease in the preprocessing stages of model building, including automatic adaptive mesh generation, and in the postprocessing stages of reviewing the solution results. A great abundance of literature has been presented on the subject, including many textbooks. A partial list of some textbooks, introductory and more comprehensive, is given at the end of this chapter. For a brief introduction to FEM and modeling techniques, see Chapters 9 and 10, respectively, of Ref. 11.

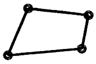
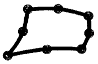
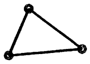
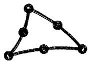
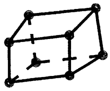
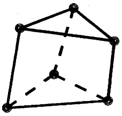
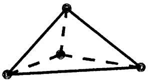
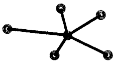
FEM is ideally suited to digital computers, in which a continuous elastic structure (continuum) is divided (discretized) into small but finite well-defined substructures (*elements*). Using matrices, the continuous elastic behavior of each element is categorized in terms of the element's material and geometric properties, the distribution of loading (static, dynamic, and thermal) within the element, and the loads and displacements at the *nodes* of the element. The element's nodes are the fundamental governing entities of the element, since it is the node where the element connects to other elements, where elastic properties of the element are established, where boundary conditions are assigned, and where forces (contact or body) are ultimately applied. A node possesses *degrees of freedom* (dof's). Degrees of freedom are the translational and rotational motion that can exist at a node. At most, a node can possess three translational and three rotational degrees of freedom. Once each element within a structure is defined locally in matrix form, the elements are then globally assembled (attached) through their common nodes (dof's) into an overall system matrix. Applied loads and boundary conditions are then specified, and through matrix operations the values of all unknown displacement degrees of freedom are determined. Once this is done, it is a simple matter to use these displacements to determine strains and stresses through the constitutive equations of elasticity.

Many geometric shapes of elements are used in finite element analysis for specific applications. The various elements used in a general-purpose commercial FEM software code constitute what is referred to as the *element library* of the code. Elements can be placed in the following categories: *line elements*, *surface elements*, *solid elements*, and *special purpose elements*. Table 5.1 provides some, but not all, of the types of elements available for finite element analysis.

Since FEM is a numerical technique that discretizes the domain of a continuous structure, errors are inevitable. These errors are:

1. *Computational errors.* These are due to round-off errors from the computer floating-point calculations and the formulations of the numerical integration schemes that are employed. Most commercial

TABLE 5.1 Sample finite element library

Element type	Name	Shape	Number of nodes	Applications
Line	Truss		2	Pin-ended bar in tension or compression
	Beam		2	Bending
	Frame		2	Axial, torsional, and bending. With or without load stiffening
Surface	4 Noded quadrilateral		4	Plane stress or strain, axisymmetry, shear panel, thin flat plate in bending
	8 Noded quadrilateral		8	Plane stress or strain, thin plate or shell in bending
	3 Noded triangular		3	Plane stress or strain, axisymmetry, shear panel, thin flat plate in bending. Prefer quad where possible. Used for transitions of quads
	6 Noded triangular		6	Plane stress or strain, axisymmetry, thin plate or shell in bending. Prefer quad where possible. Used for transitions of quads
Solid†	8 Noded hexagonal (brick)		8	Solid, thick plate (using mid-side nodes)
	6 Noded Pentagonal (wedge)		6	Solid, thick plate (using mid-side nodes). Used for transitions
	4 Noded tetrahedron (tet)		4	Solid, thick plate (using mid-side nodes). Used for transitions
Special purpose	Gap		2	Free displacement for prescribed compressive gap
	Hook		2	Free displacement for prescribed extension gap
	Rigid		Variable	Rigid constraints between nodes

† These elements are also available with mid-size nodes.

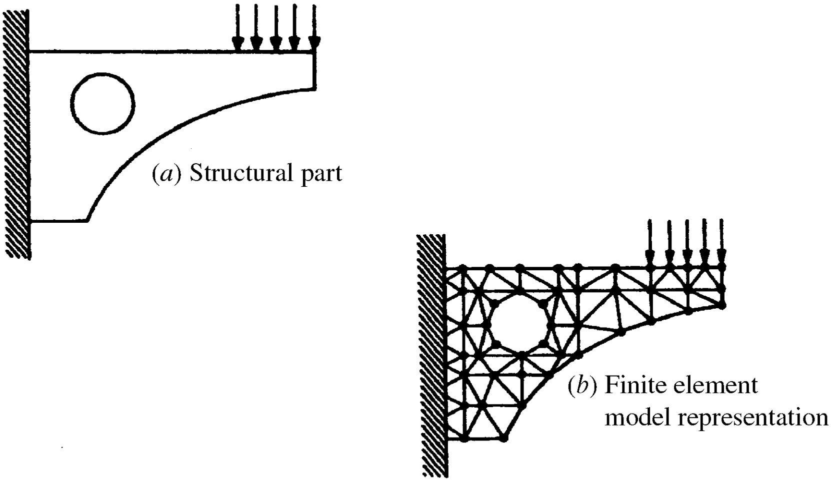
finite element codes concentrate on reducing these errors and consequently the analyst generally is concerned with discretization factors.

2. *Discretization errors.* The geometry and the displacement distribution of a true structure vary continuously. Using a finite number of elements to model the structure introduces errors in matching geometry and the displacement distribution due to the inherent limita-

tions of the elements. For example, consider the thin plate structure shown in Fig. 5.1(*a*). Figure 5.1(*b*) shows a finite element model of the structure where three-noded, plane stress, triangular elements are employed. The plane stress triangular element has a flaw, which creates two basic problems. The element has straight sides, which remain straight after deformation. The strains throughout the plane stress triangular element are constant. The first problem, a geometric one, is the modeling of curved edges. Note that the surface of the model with a large curvature appears reasonably modeled, whereas the surface of the hole is very poorly modeled. The second problem, which is much more severe, is that the strains in various regions of the actual structure are changing rapidly, and the constant strain element will only provide an approximation of the average strain at the center of the element. So, in a nutshell, the results predicted using this model will be relatively poor. The results can be improved by significantly increasing the number of elements used (increased mesh density). Alternatively, using a better element, such as an eight-noded quadrilateral, which is more suited to the application, will provide the improved results. Due to higher-order interpolation functions, the eight-noded quadrilateral element can model curved edges and provides for a higher-order function for the strain distribution.

5.3 The Boundary Element Method

The *boundary element method* (BEM), developed more recently than FEM, transforms the governing differential equations and boundary conditions into integral equations, which are converted to contain



(*a*) Structural part

(*b*) Finite element model representation

Figure 5.1 Discretization of a continuous structure.

surface integrals (see Refs. 12–16). Because only surface integrals remain, surface elements are used to perform the required integrations. This is the main advantage that BEM has over FEM, which requires three-dimensional elements throughout the entire volumetric domain. Boundary elements for a general three-dimensional solid are quadrilateral or triangular surface elements covering the surface area of the component. For two-dimensional and axisymmetric problems, only line elements tracing the outline of the component are necessary.

Although BEM offers some modeling advantages over FEM, the latter can analyze more types of engineering applications and is much more firmly entrenched in today's computer-aided-design (CAD) environment. Development of engineering applications of BEM are proceeding however, and more will be seen of the method in the future.

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Chapter 6

Experimental Methods

A structural member may be of such a form or may be loaded in such a way that the direct use of formulas for the calculation of stresses and strain produced in it is ineffective. One then must resort either to numerical techniques such as the finite element method or to experimental methods. Experimental methods can be applied to the actual member in some cases, or to a model thereof. Which choice is made depends upon the results desired, the accuracy needed, the practicality of size, and the cost associated with the experimental method. There has been a tremendous increase in the use of numerical methods over the years, but the use of experimental methods is still very effective. Many investigations make use of both numerical and experimental results to cross-feed information from one to the other for increased accuracy and cost effectiveness (see Chap. 17 in Ref. 27). Some of the more important experimental methods are described briefly in Sec. 6.1 of this chapter. Of these methods, the most popular method employs electrical resistance strain gages, and is described in more detail in Sec. 6.2. Only textbooks, reference books, handbooks, and lists of journals are referenced, since there are several organizations (see Refs. 1, 25, and 26) devoted either partially or totally to experimental methods, and a reasonable listing of papers would be excessive and soon out of date. The most useful reference for users wanting information on experimental methods is Ref. 27, the "Handbook on Experimental Mechanics," edited by A. S. Kobayashi and dedicated to the late Dr. M. Hetenyi, who edited Ref. 2. Reference 27 contains 22 chapters contributed by 27 authors under the sponsorship of the Society for Experimental Mechanics. Experimental methods applied specifically to the field of fracture mechanics are treated extensively in Refs. 13, 15, 17, 19, 22, and Chaps. 14 and 20 of Ref. 27.

6.1 Measurement Techniques

The determination of stresses produced under a given loading of a structural system by means of experimental techniques are based on the measurement of deflections. Since strain is directly related to (the rate of change of) deflection, it is common practice to say that the measurements made are that of strain. Stresses are then determined implicitly using the stress–strain relations. Deflections in a structural system can be measured through changes in resistance, capacitance, or inductance of electrical elements; optical effects of interference, diffraction, or refraction; or thermal emissions. Measurement is comparatively easy when the stress is fairly uniform over a considerable length of the part in question, but becomes more difficult when the stress is localized or varies greatly with position. Short gage lengths and great precision require stable gage elements and stable electronic amplification if used. If dynamic strains are to be measured, a suitable high-frequency response is also necessary. In an isotropic material undergoing uniaxial stress, one normal strain measurement is all that is necessary. On a free surface under biaxial stress conditions, two measured orthogonal normal strains will provide the stresses in the same directions of the measured strains. On a free surface under a general state of plane stress, three measured normal strains in different directions will allow the determination of the stresses in directions at that position (see Sec. 6.2). At a free edge in a member that is thin perpendicular to the free edge, the state of stress is uniaxial and, as stated earlier, can be determined from one normal strain tangent to the edge. Another tactic might be to measure the change in thickness or the through-thickness strain at the edge. This might be more practical, such as measuring the strain at the bottom of a groove in a thin plate. For example, assume an orthogonal xyz coordinate system where x is parallel to the edge and z is in the direction of the thickness at the edge. Considering a linear, isotropic material, from Hooke's law, $\varepsilon_z = -\nu\sigma_x/E$. Thus, $\sigma_x = -E\varepsilon_z/\nu$.

The following descriptions provide many of the successful instruments and techniques used for strain measurement. They are listed in a general order of mechanical, electrical, optical, and thermal methods. Optical and thermal techniques have been greatly enhanced by advances in digital image processing technology for computers (see Chap. 21 of Ref. 27).

1. Mechanical measurement. A direct measurement of strain can be made with an Invar tape over a gage length of several meters or with a pair of dividers over a reasonable fraction of a meter. For shorter gage lengths, mechanical amplification can be used, but friction is a problem and vibration can make them difficult to mount and to read. Optical magnification using mirrors still requires mechanical levers or rollers and is an improvement but still not satisfactory for most applications. In a laboratory setting, however, such mechanical and optical magnification can be used successfully. See Ref. 3 for more detailed descriptions. A scratch gage uses scratches on a polished target to determine strain amplitudes, and while the scratches are in general not strictly related to time, they are usually related to events in such a way as to be extremely useful in measuring some dynamic events. The scratched target is viewed with a microscope to obtain peak-to-peak strains per event, and a zero strain line can also be scratched on the target if desired (Ref. 3). The use of lasers and/or optical telescopes with electronic detectors to evaluate the motion of fiduciary marks on distant structures makes remote-displacement measurements possible, and when two such detectors are used, strains can be measured. While the technique is valuable when needed for remote measurement, generally for environmental reasons, it is an expensive technique for obtaining the strain at a single location.

2. Brittle coatings. Surface coatings formulated to crack at strain levels well within the elastic limit of most structural materials provide a means of locating points of maximum strain and the directions of principal strains. Under well-controlled environmental conditions and with suitable calibration, such coatings can yield quantitative results (Refs. 2, 3, 7, 9, 20, 21, and 27). This technique, however, is not universally applicable, since the coatings may not be readily available due to environmental problems with the coating materials.

3. Electrical strain and displacement gages. The evolution of electrical gages has led to a variety of configurations where changes in resistance, capacitance, or inductance can be related to strain and displacement with proper instrumentation (Refs. 2–5, 20, 21, 23, 24, and 27).

(*a*) *Resistance strain gage.* For the electrical resistance strain gages, the gage lengths vary from less than 0.01 in to several inches. The gage grid material can be metallic or a semiconductor. The gages can be obtained in alloys that are designed to provide minimum output due to temperature strains alone and comparatively large outputs due to stress-induced strains. Metallic bonded-foil gages are manufactured by a photoetching process that allows for a wide range of configurations of the grid(s). The semiconductor strain gages provide the largest resistance change for a given strain, but are generally very sensitive to temperature changes. They are used in transducers where proper design can provide temperature compensation The use of electrical resistance strain gages for stress analysis purposes constitute the majority of experimental applications. For this reason, Sec. 6.2 provides further information on the use of these gages.

(*b*) *Capacitance strain gage.* Capacitance strain gages are larger and more massive than bonded electric resistance strain gages and are more widely used for applications beyond the upper temperature limits of the bonded resistance strain gages.

(*c*) *Inductance strain gages.* The change in air gap in a magnetic circuit can create a large change in *inductance* depending upon the design of the rest of the magnetic circuit. The large change in inductance is accompanied by a large change in *force* across the gap, and so the very sensitive inductance strain gages can be used only on more massive structures. They have been used as overload indicators on presses with no electronic amplification necessary. The linear relationship between core motion and output voltage of a *linear differential transformer* makes possible accurate measurement of displacements over a wide range of gage lengths and under a wide variety of conditions. The use of displacement data as input for work in experimental modal analysis is discussed in Chap. 16 of Ref. 27 and in many of the technical papers in Ref. 24.

4. Interferometrlc strain gages. Whole-field interferometric techniques will be discussed later, but a simple strain gage with a short length and high sensitivity can be created by several methods. In one, a diffraction grating is deposited at the desired location and in the desired direction and the change in grating pitch under strain is measured. With a metallic grid, these strain gages can be used at elevated temperatures. Another method, also useable at high temperatures, makes use of the interference of light reflected from the inclined surfaces of two very closely spaced indentations in the surface of a metallic specimen. Both of these methods are discussed and referenced in Ref. 27.

5. Photoelastic analysis. When a beam of polarized light passes through an elastically stressed transparent isotropic material, the beam may be treated as having been decomposed into two rays polarized in the planes of the principal stresses in the material. In birefringent materials the indexes of refraction of the material encountered by these two rays will depend upon the principal stresses. Therefore, interference patterns will develop which are proportional to the differences in the principal stresses.

(*a*) *Two-dimensional analysis.* With suitable optical elements—polarizers and wave plates of specific relative retardation—both the principal stress differences and the directions of principal stresses may be determined at every point in a two-dimensional specimen (Refs. 2–6, 10, 14, 18, 27, and 28). Many suitable photoelastic plastics are available. The material properties that must be considered are transparency, sensitivity (relative index of refraction change with stress), optical and mechanical creep, modulus of elasticity, ease of machining, cost, and stability (freedom from stresses developing with time). Materials with appropriate creep properties may be used for *photoplasticity* studies (Ref. 16).

(*b*) *Three-dimensional analysis.* Several photoelastic techniques are used to determine stresses in three-dimensional specimens. If information is desired at a single point only, the optical polarizers, wave plates, and photoelastically sensitive material can be *embedded* in a transparent model (Ref. 2) and two-dimensional techniques used. A modification of this technique, *stress freezing*, is possible in some biphase materials. By heating, loading, cooling, and unloading, it is possible to lock permanently into the specimen, on a *molecular* level, strains proportional to those present under load. Since equilibrium exists at a molecular level, the specimen can be cut into two-dimensional slices and all *secondary principal stress differences* determined. The secondary principal stresses at a point are defined as the largest and smallest normal stresses in the plane of the slice; these in general will not correspond with the principal stresses at that same point in the three-dimensional structure. If desired, the specimen can be cut into cubes and the three principal stress differences determined. The individual principal stresses at a given point cannot be determined from photoelastic data taken at that point alone since the addition of a hydrostatic stress to any cube of material would not be revealed by differences in the indexes of refraction. Mathematical integration techniques, which start at a point where the hydrostatic stress component is known, can be used with photoelastic data to determine all individual principal stresses.

A third method, *scattered light photoelasticity*, uses a laser beam of intense monochromatic polarized light or a similar thin sheet of light passing through photoelastically sensitive transparent models that have the additional property of being able to scatter uniformly a small portion of the light from any point on the beam or sheet. The same general restrictions apply to this analysis as applied to the stress-frozen three-dimensional analysis except that the specimen does not have to be cut. However, the amount of light available for analysis is much less, the specimen must be immersed in a fluid with an index of refraction that very closely matches that of the specimen, and in general the data are much more difficult to analyze.

(*c*) *Photoelastic coating.* Photoelastic coatings have been sprayed, bonded in the form of thin sheets, or cast directly in place on the surface of models or structures to determine the two-dimensional surface strains. The surface is made reflective before bonding the plastics in place so the effective thickness of the photoelastic plastic is doubled and all two-dimensional techniques can be applied with suitable instrumentation.

6. Moiré techniques. All moiré techniques can be explained by optical interference, but the course-grid techniques can also be evaluated on the basis of obstructive or mechanical interference.

(*a*) *Geometric moiré.* Geometric moiré techniques use grids of alternate equally wide bands of relatively transparent or light-colored material and opaque or dark-colored material in order to observe the relative motion of two such grids. The most common technique (Refs. 2, 5, 8, and 11) uses an alternate transparent and opaque grid to produce photographically a matching grid on the flat surface of the specimen. Then the full-field relative motion is observed between the reproduction and the original when the specimen is loaded. Similarly, the original may be used with a projector to produce the photographic image on the specimen and then produce interference with the projected image after loading. These methods can use ordinary white light, and the interference is due merely to *geometric* blocking of the light as it passes through or is reflected from the grids.

Another similar technique, *shadow moiré*, produces interference patterns due to motion of the specimen at right angles to its surface between an alternately transparent and opaque grid and the shadow of the grid on the specimen.

(*b*) *Moiré interferometry.* Interferometry provides a means of producing both specimen gratings and reference gratings. Virtual reference gratings of more than 100,000 lines per inch have been utilized. Moiré interferometry provides contour maps of in-plane displacements, and, with the fine pitches attainable, differentiation to obtain strains from this experimental process is comparable to that used in the finite-element method of numerical analysis where displacement fields are generally the initial output. See Chap. 7 in Ref. 27.

7. Holographic and laser speckle interferometry. The rapid evolution of holographic and laser speckle interferometry is related to the development of high-power lasers and to the development of digital computer enhancement of the resulting images. Various techniques are used to measure the several displacement components of diffuse reflecting surfaces. Details are beyond the scope of this book and are best reviewed in Chap. 8 of Ref. 27.

8. Shadow optical method of caustics. The very simple images created by the reflection or refraction of light from the surface contours of high-gradient stress concentrations such as those at the tips of cracks make the use of the shadow optical method of caustics very useful for dynamic studies of crack growth or arrest. Chapter 9 of Ref. 27 gives a detailed discussion of this technique and a comparison to photoelastic studies for the same loadings.

9. X-ray diffraction. X-ray diffraction makes possible the determination of changes in interatomic distance and thus the measurement of elastic strain. The method has the particular advantages that it can be used at points of high stress concentration and to determine residual stresses without cutting the object of investigation.

10. Stress-pattern analysis by thermal emission. This technique uses computer enhancement of infrared detection of very small temperature changes in order to produce digital output related to stress at a point on the surface of a structure, a stress graph along a line on the surface, or a full-field isopachic stress map of the surface. Under cyclic loading, at a frequency high enough to assure that any heat transfer due to stress gradients is insignificant, the thermoelastic effect produces a temperature change proportional to the change in the sum of the principal stresses. Although calibration corrections must be made for use at widely differing ambient temperatures, the technique works over a wide range of temperatures and on a variety of structural materials including metals, wood, concrete, and plain and reinforced plastics. Tests have been made on some metals at temperatures above 700°C. Chapter 14 of Ref. 27 describes and surveys work on this technique.

6.2 Electrical Resistance Strain Gages

General. The use of electrical resistance strain gages is probably the most common method of measurement in experimental stress analysis. In addition, strain gage technology is quite important in the design of transducer instrumentation for the measurement of force, torque, pressure, etc.

Electrical resistance strain gages are based on the principal that the resistance R of a conductor changes as a function of normal strain ε. The resistance of a conductor can be expressed as

$$R = \rho \frac{L}{A} \tag{6.2-1}$$

where ρ is the resistivity of the conductor (ohms-length), and L and A are the length and cross-sectional area of the conductor respectively. It can be shown that a change in R due to changes in ρ, L and A is given by

$$\frac{\Delta R}{R} = (1 + 2\nu)\varepsilon + \frac{\Delta\rho}{\rho} \tag{6.2-2}$$

where ν is Poisson's ratio, and assuming small strain on the conductor, ε, which is given by $\Delta L/L$. If the change in the resistance of the conductor is considered to be only due to the applied strain, then Eq. (6.2-2) can be written as

$$\frac{\Delta R}{R} = S_a\varepsilon \tag{6.2-3}$$

where

$$S_a = 1 + 2\nu + \frac{\Delta\rho/\rho}{\varepsilon} \tag{6.2-4}$$

S_a is the *sensitivity* of the conductor to strain[†]. The first two terms come directly from changes in dimension of the conductor where for most metals the quantity $1 + 2\nu$ varies from 1.4 to 1.7. The last term in Eq. (6.2-4) is called the change in specific resistance relative to strain, and for some metals can account for much of the sensitivity to strain. The most commonly used material for strain gages is a copper–nickel alloy called Constantan, which has a strain sensitivity of 2.1. Other alloys used for strain gage applications are modified Karma, Nichrome V, and Isoelastic, which have sensitivities of 2.0, 2.2, and 3.6, respectively. The primary advantages of Constantan are:

1. The strain sensitivity S_a is linear over a wide range of strain and does not change significantly as the material goes plastic.
2. The thermal stability of the material is excellent and is not greatly influenced by temperature changes when used on common structural materials.
3. The metallurgical properties of Constantan are such that they can be processed to minimize the error induced due to the mismatch in the thermal expansion coefficients of the gage and the structure to which it is adhered over a wide range of temperature.

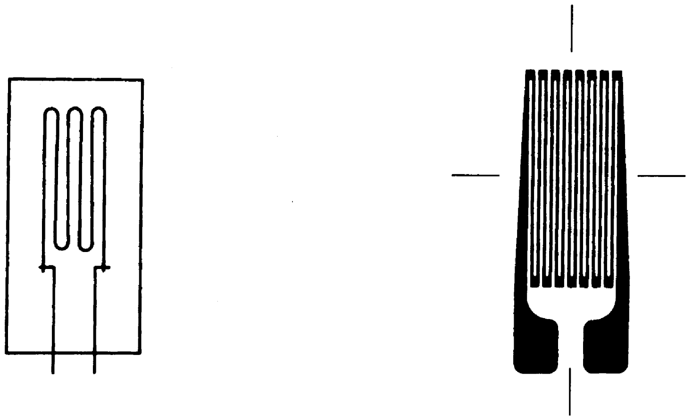
[†] When using a commercial strain indicator, one must enter the sensitivity provided by the gage manufacturer. This sensitivity is referred to the *gage factor* of the gage, S_g. This is defined slightly differently than S_a, and will be discussed shortly.

Isoelastic, with a higher sensitivity, is used for dynamic applications. Semiconductor gages are also available, and can reach sensitivities as high as 175. However, care must be exercised with respect to the poor thermal stability of these piezoresistive gages.

Most gages have a nominal resistance of 120 ohm or 350 ohm. Considering a 120-ohm Constantan gage, to obtain a measurement of strain within an accuracy of ±5 μ, it would be necessary to measure a change in resistance within ±1.2 mohm. To measure these small changes in resistance accurately, commercial versions of the Wheatstone bridge, called *strain gage indicators*, are available.

Metallic alloy electrical resistance strain gages used in experimental stress analysis come in two basic types: bonded-wire and bonded-foil (see Fig. 6.1). Today, bonded-foil gages are by far the more prevalent. The resistivity of Constantan is approximately 49 μohm · cm. Thus if a strain gage is to be fabricated using a wire 0.025 mm in diameter and is to have a resistance of 120 ohm, the gage would require a wire approximately 120 mm long. To make the gage more compact over a shorter active length, the gage is constructed with many loops as shown in Fig. 6.1. Typical commercially available bonded-foil gage lengths vary from 0.20 mm (0.008 in) to 101.6 mm (4.000 in). For normal applications, bonded-foil gages either come mounted on a very thin polyimide film carrier (backing) or are encapsulated between two thin films of polyimide. Other carrier materials are available for special cases such as high-temperature applications.

The most widely used adhesive for bonding a strain gage to a test structure is the pressure-curing methyl 2-cyanoacrylate cement. Other adhesives include epoxy, polyester, and ceramic cements.



Wire

Foil

Figure 6.1 Forms of electrical resistance strain gages.

Extreme care must be exercised when installing a gage, since a good bond and an electrically insulated gage are necessary. The installation procedures can be obtained from technical instruction bulletins supplied by the manufacturer. Once a gage is correctly mounted, wired, resistance tested for continuity and insulation from the test structure, and waterproofed (if appropriate), it is ready for instrumentation and testing.

Strain Gage Configurations. In both wire or foil gages, many configurations and sizes are available. Strain gages come in many forms for transducer or stress-analysis applications. The fundamental configurations for stress-analysis work are shown in Fig. 6.2.

A strain gage is mounted on a free surface, which in general, is in a state of plane stress where the state of stress with regards to a specific *xy* rectangular coordinate system can be unknown up to the three stresses, σ_x, σ_y, and τ_{xy}. Thus, if the state of stress is completely unknown on a free surface, it is necessary to use a three-element rectangular or delta rosette since each gage element provides only one piece of information, the indicated normal strain at the point in the direction of the gage.

To understand how the rosettes are used, consider the three-element rectangular rosette shown in Fig. 6.3(*a*), which provides normal strain components in three directions spaced at angles of 45°.

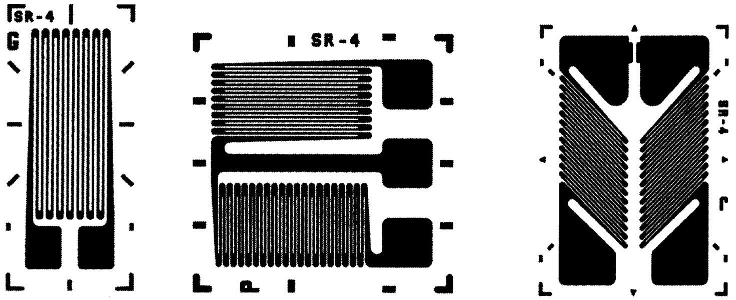
If an *xy* coordinate system is assumed to coincide with gages *A* and *C*, then $\varepsilon_x = \varepsilon_A$ and $\varepsilon_y = \varepsilon_C$. Gage *B* in conjunction with gages *A* and *C* provides information necessary to determine γ_{xy}. Recalling the first of Eqs. (2.4-1), $\varepsilon_{x'} = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta + \gamma_{xy} \cos \theta \sin \theta$, with $\theta = 45^\circ$

$$\begin{aligned}\varepsilon_B &= \varepsilon_x \cos^2 45^\circ + \varepsilon_y \sin^2 45^\circ + \gamma_{xy} \cos 45 \sin 45^\circ \\ &= \tfrac{1}{2}(\varepsilon_x + \varepsilon_y + \gamma_{xy}) = \tfrac{1}{2}(\varepsilon_A + \varepsilon_C + \gamma_{xy})\end{aligned}$$

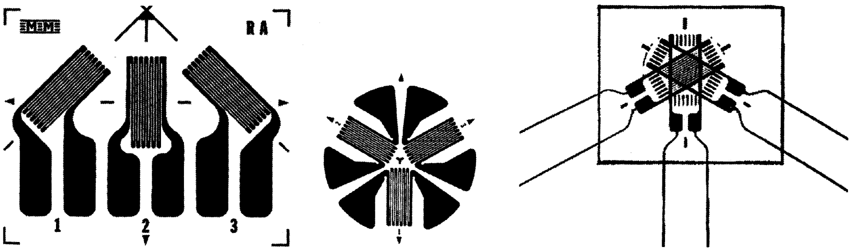
Solving for γ_{xy} yields

$$\gamma_{xy} = 2\varepsilon_B - \varepsilon_A - \varepsilon_C$$

Once ε_x, ε_y, and γ_{xy} are known, Hooke's law [Eqs. (2.2-5) and (2.2-6*a*)] can be used to determine the stresses σ_x, σ_y, and τ_{xy}.



(*a*) Uniaxial (*b*) Two-element, 90° planar rosette (*c*) Two-element, 90° planar (shear) rosette



(*d*) Three-element, 45° planar rectangular rosette (*e*) Three element, 60° delta rosette (*f*) Three-element, 60° stacked delta rosette

Figure 6.2 Examples of commonly used strain gage configurations. (*Source*: Figures *a–c* courtesy of *BLH Electronics, Inc.*, Canton, MA. Figures *d–f* courtesy of *Micro-Measurements Division of Measurements Group, Inc.*, Raleigh, NC.) *Note*: The letters *SR-4* on the *BLH* gages are in honor of E. E. Simmons and Arthur C. Ruge and their two assistants (a total of four individuals), who, in 1937–1938, independently produced the first bonded-wire resistance strain gage.

The relationship between ε_A, ε_B, and ε_C can be seen from Mohr's circle of strain corresponding to the strain state at the point under investigation [see Fig. 6.3(*b*)].

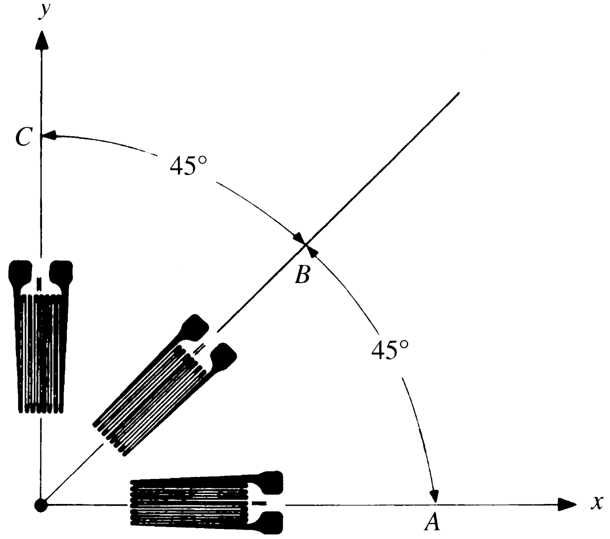
The following example shows how to use the above equations for an analysis as well as how to use the equations provided in Table 6.1.

EXAMPLE

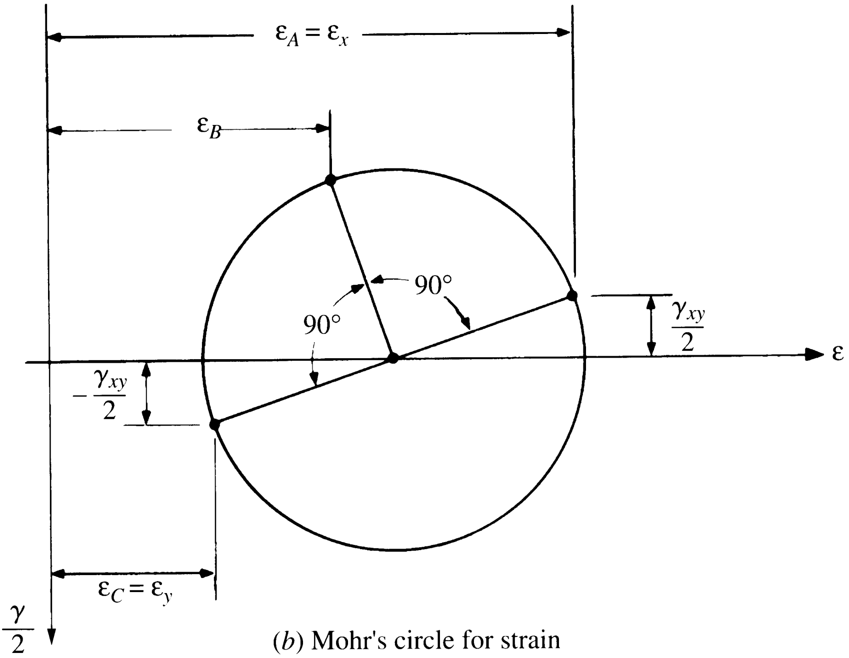
A three-element rectangular rosette strain gage is installed on a steel specimen. For a particular state of loading of the structure the strain gage readings are[†]

$$\varepsilon_A = 200\ \mu, \qquad \varepsilon_B = 900\ \mu, \qquad \varepsilon_C = 1000\ \mu$$

[†] The strain gage readings are typically corrected due to the effect of transverse strains on each gage. This will be discussed shortly.



(*a*) Rectangular rosette



(*b*) Mohr's circle for strain

Figure 6.3 Three-element strain gage rosette.

Determine the values and orientations of the principal stresses at the point. Let $E = 200\,\text{GPa}$ and $\nu = 0.285$.

Solution. From above,

$$\varepsilon_x = \varepsilon_A = 200\ \mu, \qquad \varepsilon_y = \varepsilon_C = 1000\ \mu$$

$$\gamma_{xy} = 2\varepsilon_B - \varepsilon_A - \varepsilon_C = (2)(900) - 200 - 1000 = 600\ \mu$$

The stresses can be determined using Eqs. (2.2-5) and (2.2-6*a*):

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \\
&= \frac{200(10^9)}{1-(0.285)^2}[200 + (0.285)(1000)](10^{-6}) = 105.58(10^6)\ \text{N/m}^2 = 105.58\ \text{MPa} \\
\sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \\
&= \frac{200(10^9)}{1-(0.285)^2}[1000 + (0.285)(200)](10^{-6}) = 230.09(10^6)\ \text{N/m}^2 = 230.09\ \text{MPa} \\
\tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{200(10^9)}{2(1+0.285)}600(10^{-6}) = 46.69(10^6)\ \text{N/m}^2 = 46.69\ \text{MPa}
\end{aligned}$$

Figure 6.4(*a*) shows the stresses determined in the *x* and *y* directions as related to the gage orientation shown in Fig. 6.3(*a*).

For the principal stress axes, we use Eq. (2.3-23) given by

$$\begin{aligned}
\sigma_p &= \tfrac{1}{2}\left[(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x + \sigma_y)^2 + 4\tau_{xy}^2}\right] \\
&= \tfrac{1}{2}\left[105.58 + 230.09 \pm \sqrt{(105.58 + 230.09)^2 + 4(46.67)^2}\right] \\
&= 245.65, \quad 90.01\ \text{MPa}
\end{aligned}$$

For the orientation of the principal stress axes, using the first of Eqs. (2.3-21) gives

$$\theta_p = \tan^{-1}\left(\frac{\sigma_p - \sigma_x}{\tau_{xy}}\right) \tag{a}$$

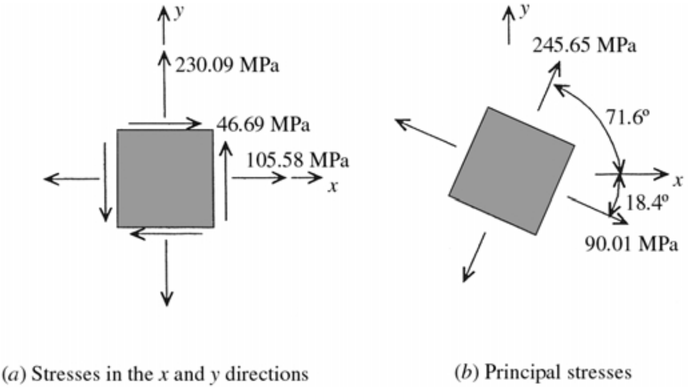
For the principal stress, $\sigma_1 = 245.65\,\text{MPa}$, Eq. (*a*) gives

$$\theta_{p1} = \tan^{-1}\left(\frac{245.65 - 105.58}{46.69}\right) = 71.6^\circ$$

For the other principal stress, $\sigma_2 = 90.01\,\text{MPa}$

$$\theta_{p2} = \tan^{-1}\left(\frac{90.01 - 105.58}{46.69}\right) = -18.4^\circ$$

Recalling that θ_p is defined positive in the counterclockwise direction, the principal stress state at the point relative to the *xy* axes of the strain gage rosette correspond to that shown in Fig. 6.4(*b*).



(a) Stresses in the x and y directions

(b) Principal stresses

Figure 6.4 (*a*) Stresses in the *x* and *y* directions. (*b*) Principal stresses.

Using the equations given in Table 6.1 at the end of the chapter,

$$\frac{\varepsilon_A+\varepsilon_C}{1-\nu}=\frac{200+1000}{1-0.285}=1678.3\ \mu$$

$$\frac{1}{1+\nu}\sqrt{(\varepsilon_A-\varepsilon_C)^2+(2\varepsilon_B-\varepsilon_A-\varepsilon_C)^2}$$

$$=\frac{1}{1+0.285}\sqrt{(200-1000)^2+[2(900)-200-1000]^2}=778.2\ \mu$$

Thus,

$$\sigma_{p1}=\frac{200(10)^9}{2}(1678.3+778.2)=245.65\ \text{MPa}$$

$$\sigma_{p2}=\frac{200(10)^9}{2}(1678.3-778.2)=90.01\ \text{MPa}$$

The principal angle is

$$\theta_p=\frac{1}{2}\tan^{-1}\left(\frac{2(900)-200-1000}{200-1000}\right)=\frac{1}{2}\tan^{-1}\left(\frac{+600}{-800}\right)=\frac{1}{2}(143.13^\circ)=71.6^\circ$$

counterclockwise from the *x* axis (*A* gage) to $\sigma_{p1}=245.65\,\text{MPa}$.[†] Note that this agrees with Fig. 6.4(*b*).

[†] When calculating θ_p, do not change the signs of the numerator and denominator in the equation. The $\tan^{-1}$ is defined from 0° to 360°. For example, $\tan^{-1}(+/+)$ is in the range 0°–90°, $\tan^{-1}(+/-)$ is in the range 90°–180°, $\tan^{-1}(-/-)$ is in the range 180°–270°, and $\tan^{-1}(-/+)$ is in the range 270°–360°. Using this definition for θ_p, the calculation will yield the counterclockwise angle from the *x* axis to σ_{p1}, the greater of the two principal stresses in the plane of the gages.

Strain Gage Corrections. There are many types of corrections that may be necessary to obtain accurate strain gage results (see Refs. 27 and 28). Two fundamental corrections that are necessary correct the indicated strain errors due to strains on the specimen perpendicular (transverse) to the longitudinal axis of the gage and changes in temperature of the gage installation. With each strain gage, the manufacturer provides much information on the performance of the gage, such as its sensitivity to longitudinal and transverse strain and how the sensitivity of the gage behaves relative to temperature changes.

(*b*) *Transverse sensitivity corrections.* The strain sensitivity of a single straight uniform length of conductor in a uniform *uniaxial strain field* ε in the longitudinal direction of the conductor is given by Eq. (6.2-3), which is $S_a = (\Delta R/R)/\varepsilon$. In a general strain field, there will be strains perpendicular to the longitudinal axis of the conductor (*transverse strains*). Due to the width of the conductor elements and the geometric configuration of the conductor in the gage pattern, the transverse strains will also effect a change in resistance in the conductor. This is not desirable, since only the effect of the strain in the direction of the gage length is being sought.

To further complicate things, the sensitivity of the strain gage provided by the gage manufacturer is *not* based on a uniaxial strain field, but that of a uniaxial *stress* field in a tensile test specimen. For a uniaxial stress field let the axial and transverse strains be ε_a and ε_t respectively. The sensitivity provided by the gage manufacturer, called the *gage factor* S_g, is defined as $S_g = (\Delta R/R)\varepsilon_a$, where under a uniaxial stress field, $\varepsilon_t = -\nu_0\varepsilon_a$. Thus

$$\frac{\Delta R}{R} = S_g\varepsilon_a, \qquad \text{with} \qquad \varepsilon_t = -\nu_0\varepsilon_a \tag{6.2-5}$$

The term ν_0 is Poisson's ratio of the material on which the manufacturer's gage factor was measured, and is normally taken to be 0.285. If the gage is used under conditions where the transverse strain is $\varepsilon_t = -\nu\varepsilon_a$, then the equation $\Delta R/R = S_g\varepsilon_a$ would yield exact results. If $\varepsilon_t \neq -\nu_0\varepsilon_a$, then some error will occur. This error depends on the sensitivity of the gage to transverse strain and the deviation of the ratio of $\varepsilon_t/\varepsilon_a$ from $-\nu_0$. The strain gage manufacturer generally supplies a transverse sensitivity coefficient, K_t, defined as S_t/S_a, where S_t is the transverse sensitivity factor. One cannot correct the indicated strain from a single strain reading. Thus it is necessary to have multiple strain readings from that of a strain gage rosette. Table 6.2 at the end of the chapter gives equations for the corrected strain values of the three most widely used strain gage rosettes. Corrected strain readings are given by ε, whereas uncorrected strains from the strain gage indicator are given by $\hat{\varepsilon}$.

EXAMPLE

In the previous example the indicated strains are $\hat{\varepsilon}_A = 200\ \mu$, $\hat{\varepsilon}_B = 900\ \mu$, and $\hat{\varepsilon}_C = 1000\ \mu$. Determine the principal stresses and directions if the transverse sensitivity coefficient of the gages are $K_{tA} = K_{tC} = 0.05$ and $K_{tB} = 0.06$.

Solution. From Table 6.2,

$$\begin{aligned}\varepsilon_A &= \frac{(1 - \nu_0 K_{tA})\hat{\varepsilon}_A - K_{tA}(1 - \nu_0 K_{tC})\hat{\varepsilon}_C}{1 - K_{tA}K_{tC}}\\ &= \frac{[1 - (0.285)(0.05)](200) - (0.05)[1 - (0.285)(0.05)](1000)}{1 - (0.05)(0.05)}\\ &= 148.23\ \mu\end{aligned}$$

$$\begin{aligned}\varepsilon_B &= \frac{(1 - \nu_0 K_{tB})\hat{\varepsilon}_B - \dfrac{K_{tB}}{1 - K_{tA}K_{tC}}[(1 - \nu_0 K_{tA})(1 - K_{tC})\hat{\varepsilon}_A + (1 - \nu_0 K_{tC})(1 - K_{tA})\hat{\varepsilon}_C]}{1 - K_{tB}}\\ &= \left([1 - (0.285)(0.06)](900) - \frac{0.06}{[1 - (0.05)(0.05)]}\right.\\ &\qquad \left.\times \{[1 - (0.285)(0.05)][1 - 0.05)](200) + [1 - (0.285)(0.05)][1 - 0.05](1000)\}\right)\\ &\qquad \times (1 - 0.06)^{-1}\\ &= 869.17\ \mu\end{aligned}$$

and

$$\begin{aligned}\varepsilon_C &= \frac{(1 - \nu_0 K_{tC})\hat{\varepsilon}_C - K_{tC}(1 - \nu_0 K_{tA})\hat{\varepsilon}_A}{1 - K_{tA}K_{tC}}\\ &= \frac{[1 - (0.285)(0.05)](1000) - (0.05)[1 - (0.285)(0.05)](200)}{1 - (0.05)(0.05)}\\ &= 978.34\ \mu\end{aligned}$$

From Table 6.1,†

$$\frac{\varepsilon_A + \varepsilon_C}{1 - \nu} = \frac{148.23 + 978.34}{1 - 0.285} = 1575.62\ \mu$$

$$\begin{aligned}&\frac{1}{1 + \nu}\sqrt{(\varepsilon_A - \varepsilon_C)^2 + (2\varepsilon_B - \varepsilon_A - \varepsilon_C)^2}\\ &\quad = \frac{1}{1 + 0.285}\sqrt{(148.23 - 978.34)^2 + [2(869.17) - 148.23 - 978.34]^2} = 802.48\ \mu\end{aligned}$$

---

† Note that if ν for the specimen was different from $\nu_0 = 0.285$, it would be used in the equations of Table 6.1 but *not* for Table 6.2.

and

$$\sigma_{p1} = \frac{200(10)^9}{2}(1575.62 + 802.48)(10^{-6}) = 237.8(10^6)\ \mathrm{N/m^2} = 237.81\ \mathrm{MPa}$$

$$\sigma_{p2} = \frac{200(10)^9}{2}(1575.62 - 802.48)(10^{-6}) = 90.01(10^6)\ \mathrm{N/m^2} = 77.31\ \mathrm{MPa}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2(869.17) - 148.23 - 978.34}{148.23 - 978.34}\right) = \frac{1}{2}\tan^{-1}\left(\frac{611.77}{-830.11}\right)$$

$$= \frac{1}{2}(143.61^\circ) = 71.8^\circ$$

The principal stress element is shown in Fig. 6.5 relative to the *xy* coordinate system of the gage rosette as shown in Fig. 6.3(*a*).

(*b*) *Corrections due to temperature changes.* Temperature changes on an installed strain gage cause a change in resistance, which is due to a mismatch in the thermal expansion coefficients of the gage and the specimen, a change in the resistivity of the gage material, and a change in the gage factor, S_g. This effect can be compensated for by two different methods. The first method of temperature compensation is achieved using an additional compensating gage on an adjacent arm of the Wheatstone bridge circuit. This compensating gage must be identical to the active gage, mounted on the same material as the active gage, and undergoing an identical temperature change as that of the active gage.

The second method involves calibration of the gage relative to temperature changes. The gage can be manufactured and calibrated for the application on a specific specimen material. The metallurgical properties of alloys such as Constantan and modified Karma can be processed to minimize the effect of temperature change over a limited range of temperatures, somewhat centered about room temperature. Gages processed in this manner are called *self-temperature-compensated* strain gages. An example of the characteristics of a

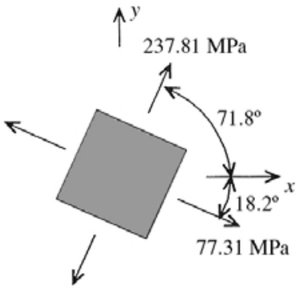
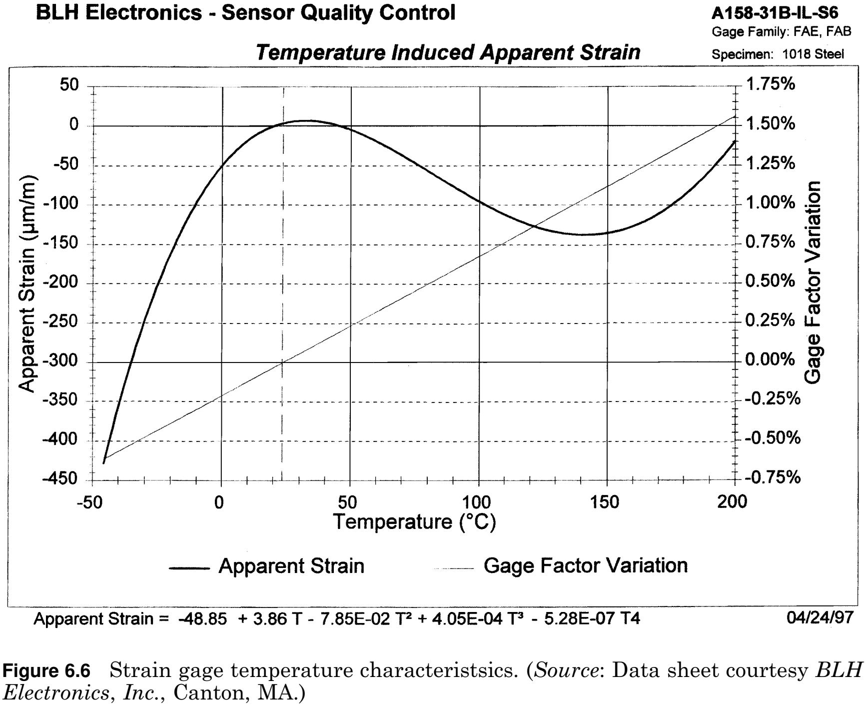


Figure 6.5 Principal stress element corrected for transverse sensitivity.

BLH self-temperature-compensated gage specifically processed for use on a low-carbon steel is shown in Fig. 6.6. Note that the apparent strain is zero at 22°C and 45°C and approximately zero in the vicinity of these temperatures. For temperatures beyond this region, compensation can be achieved by monitoring the temperature at the strain gage site. Then, using either the curve from the data sheet or the fitted polynomial equation, the strain readings can be corrected numerically. Note, however, that the curve and the polynomial equation given on the data sheet are based on a gage factor of 2.0. If corrections are anticipated, the gage factor adjustment of the strain indicator should be set to 2.0. An example that demonstrates this correction is given at the end of this section.

The gage factor variation with temperature is also presented in the data sheet of Fig. 6.6. If the strain gage indicator is initially set at $(S_g)_i$, the actual gage factor at temperature T is $(S_g)_T$, and the indicator registers a strain measurement of $\varepsilon_{\text{reading}}$, the corrected strain is

$$\varepsilon_{\text{actual}} = \frac{(S_g)_i}{(S_g)_T} \varepsilon_{\text{reading}} \tag{6.2-6}$$



BLH Electronics - Sensor Quality Control

A158-31B-IL-S6
Gage Family: FAE, FAB
Specimen: 1018 Steel

Temperature Induced Apparent Strain

Apparent Strain = -48.85 + 3.86 T - 7.85E-02 T^2 + 4.05E-04 T^3 - 5.28E-07 T4 04/24/97

Figure 6.6 Strain gage temperature characteristsics. (*Source*: Data sheet courtesy *BLH Electronics, Inc.*, Canton, MA.)

where

$$(S_g)_T = \left(1 + \frac{\Delta S_g(\%)}{100}\right)(S_g)_i \tag{6.2-7}$$

and $\Delta S_g(\%)$ being the percent variation in gage factor given in Fig. 6.6.

If a simultaneous correction for apparent strain and gage factor variation is necessary, the corrected strain is given by

$$\varepsilon_{\text{actual}} = \frac{(S_g)_i}{(S_g)_T}(\varepsilon_{\text{reading}} - \varepsilon_{\text{apparent}}) \tag{6.2-8}$$

EXAMPLE

A strain gage with the characteristics of Fig. 6.6 has a room-temperature gage factor of 2.1 and is mounted on a 1018 steel specimen. A strain measurement of $-1800\ \mu$ is recorded during the test when the temperature is 150°C. Determine the value of actual test strain if:

(a) the gage is in a half-bridge circuit with a dummy temperature compensating gage and prior to testing, the indicator is zeroed with the gage factor set at 2.1.

(b) the gage is the only gage in a quarter-bridge circuit and prior to testing, the indicator is zeroed with the gage factor set at 2.0.

Solution. From Fig. 6.6, the gage factor variation at 150°C is $\Delta S_g(\%) = 1.13\%$. Thus, from Eq. (6.2-7), the gage factor at the test temperature is

$$(S_g)_T = \left(1 + \frac{1.13}{100}\right)(2.1) = 2.124$$

(a) Since in this part, a dummy gage is present that cancels the apparent strain, the only correction that is necessary is due to the change in the gage factor. From Eq. (6.2-6),

$$\varepsilon_{\text{actual}} = \left(\frac{2.1}{2.124}\right)(-1800) = -1780\ \mu$$

which we see is a minor correction.

(b) In this part, we must use Eq. (6.2-8). Using the equation given in Fig. 6.6, the apparent strain at the test temperature is

$$\begin{aligned}\varepsilon_{\text{apparent}} &= -48.85 + (3.86)(150) - (7.85E{-}02)(150)^2 \\ &\quad + (4.05E{-}04)(150)^3 - (5.28E{-}07)(150)^4 = -136.5\ \mu\end{aligned}$$

Substituting this into Eq. (6.2-8), with $(S_g)_i = 2.0$, gives

$$\varepsilon_{\text{actual}} = \left(\frac{2.0}{2.124}\right)[-1800 - (-136.5)] = -1566\,\mu$$

which is not a minor correction.

6.3 Detection of Plastic Yielding

In parts made of ductile metal, sometimes a great deal can be learned concerning the location of the most highly stressed region and the load that produces elastic failure by noting the first signs of plastic yielding. Such yielding may be detected in the following ways.

Observation of slip lines. If yielding occurs first at some point on the surface, it can be detected by the appearance of slip lines if the surface is suitably polished.

Brittle coating. If a member is coated with some material that will flake off easily, this flaking will indicate local yielding of the member. A coating of rosin or a wash of lime or white portland cement, applied and allowed to dry, is best for this purpose, but chalk or mill scale will often suffice. By this method zones of high stress such as those that occur in pressure vessels around openings and projections can be located and the load required to produce local yielding can be determined approximately.

Photoelastic coatings. Thin photoelastic coatings show very characteristic patterns analogous to slip lines when the material beneath the coating yields.

6.4 Analogies

Certain problems in elasticity involve equations that cannot be solved but that happen to be mathematically identical with the equations that describe some other physical phenomenon which can be investigated experimentally. Among the more useful of such analogies are the following.

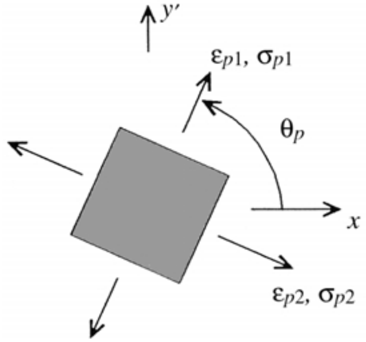
Membrane analogy. This is especially useful in determining the torsion properties of bars having noncircular sections. If in a thin flat plate holes are cut having the outlines of various sections and over each of these holes a soap film (or other membrane) is stretched and slightly distended by pressure from one side, the volumes of the bubbles thus formed are proportional to the torsional rigidities of the corresponding sections and the slope of a bubble surface at any point is proportional to the stress caused at that point of the corresponding section by a given twist per unit length of bar. By cutting in the plate one hole the shape of the section to be studied and another hole that is circular, the torsional properties of the irregular section can be determined by comparing the bubble formed on the hole of that shape with the bubble formed on the circular hole since the torsional properties of the circular section are known.

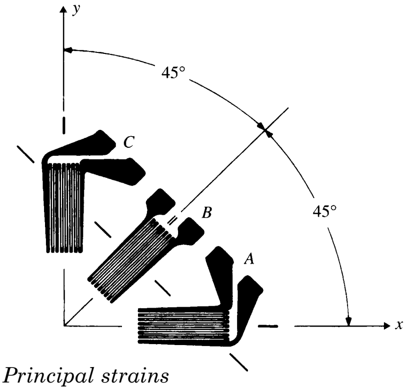
Electrical analogy for isopachic lines. *Isopachic lines* are lines along which the sums of the principal stresses are equal in a two-dimensional plane stress problem. The voltage at any point on a uniform two-dimensional conducting surface is governed by the same form of equation as is the principal stress sum. Teledeltos paper is a uniform layer of graphite particles on a paper backing and makes an excellent material from which to construct the electrical analog. The paper is cut to a geometric outline corresponding to the shape of the two-dimensional structure or part, and boundary potentials are applied by an adjustable power supply. The required boundary potentials are obtained from a photoelastic study of the part where the principal stress sums can be found from the principal stress differences on the boundaries (Refs. 2 and 3). A similar membrane analogy has the height of a nonpressurized membrane proportional to the principal stress sum (Refs. 2 and 3).

6.5 Tables

TABLE 6.1 Strain gage rosette equations applied to a specimen of a linear, isotropic material

The principal strains and stresses are given relative to the xy coordinate axes as shown.





Three-element rectangular rosette

Principal strains

$$\varepsilon_{p1} = \frac{\varepsilon_A + \varepsilon_C}{2} + \frac{1}{2}\sqrt{(\varepsilon_A - \varepsilon_C)^2 + (2\varepsilon_B - \varepsilon_A - \varepsilon_C)^2}$$

$$\varepsilon_{p2} = \frac{\varepsilon_A + \varepsilon_C}{2} - \frac{1}{2}\sqrt{(\varepsilon_A - \varepsilon_C)^2 + (2\varepsilon_B - \varepsilon_A - \varepsilon_C)^2}$$

Principal stresses

$$\sigma_{p1} = \frac{E}{2}\left[\frac{\varepsilon_A + \varepsilon_C}{1 - \nu} + \frac{1}{1 + \nu}\sqrt{(\varepsilon_A - \varepsilon_C)^2 + (2\varepsilon_B - \varepsilon_A - \varepsilon_C)^2}\right]$$

$$\sigma_{p2} = \frac{E}{2}\left[\frac{\varepsilon_A + \varepsilon_C}{1 - \nu} - \frac{1}{1 + \nu}\sqrt{(\varepsilon_A - \varepsilon_C)^2 + (2\varepsilon_B - \varepsilon_A - \varepsilon_C)^2}\right]$$

Principal angle

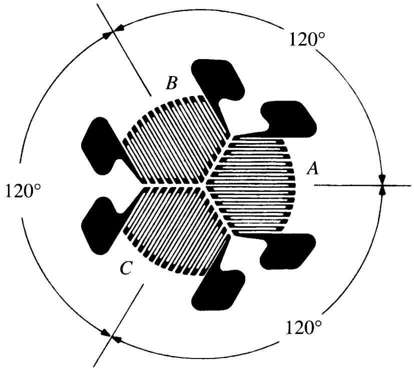
Treating the $\tan^{-1}$ as a *single-valued function*,[†] the angle *counterclockwise* from gage A to the axis containing ε_{p1} or σ_{p1} is given by

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\varepsilon_B - \varepsilon_A - \varepsilon_C}{\varepsilon_A - \varepsilon_C}\right)$$

[†] See Example in Sec. 6.2.

(*continued*)

TABLE 6.1 Strain gage rosette equations applied to a specimen of a linear, isotropic material (*Continued*)



Three-element delta rosette

Principal strains

$$\varepsilon_{p1} = \frac{\varepsilon_A + \varepsilon_B + \varepsilon_C}{3} + \frac{\sqrt{2}}{3}\sqrt{(\varepsilon_A - \varepsilon_B)^2 + (\varepsilon_B - \varepsilon_C)^2 + (\varepsilon_C - \varepsilon_A)^2}$$

$$\varepsilon_{p2} = \frac{\varepsilon_A +_B + \varepsilon_C}{3} - \frac{\sqrt{2}}{3}\sqrt{(\varepsilon_A - \varepsilon_B)^2 + (\varepsilon_B - \varepsilon_C)^2 + (\varepsilon_C - \varepsilon_A)^2}$$

Principal stresses

$$\sigma_{p1} = \frac{E}{3}\left[\frac{\varepsilon_A + \varepsilon_B + \varepsilon_C}{1 - \nu} + \frac{\sqrt{2}}{1 + \nu}\sqrt{(\varepsilon_A - \varepsilon_B)^2 + (\varepsilon_B - \varepsilon_C)^2 + (\varepsilon_C - \varepsilon_A)^2}\right]$$

$$\sigma_{p2} = \frac{E}{3}\left[\frac{\varepsilon_A + \varepsilon_B + \varepsilon_C}{1 - \nu} - \frac{\sqrt{2}}{1 + \nu}\sqrt{(\varepsilon_A - \varepsilon_B)^2 + (\varepsilon_B - \varepsilon_C)^2 + (\varepsilon_C - \varepsilon_A)^2}\right]$$

Principal angle

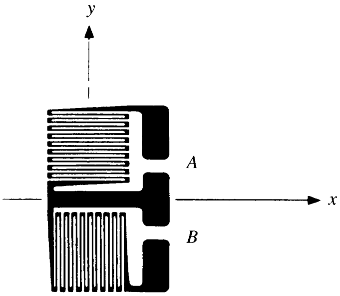
Treating the $\tan^{-1}$ as a *single-valued function*[†] the angle *counterclockwise* from gage A to the axis containing ε_{p1} or σ_{p1} is given by

$$\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{\sqrt{3}(\varepsilon_C - \varepsilon_B)}{2\varepsilon_A - \varepsilon_B - \varepsilon_C}\right]$$

[†] See Example (as applied to a rectangular rosette) in Sec. 6.2.

TABLE 6.2 Corrections for the transverse sensitivity of electrical resistance strain gages

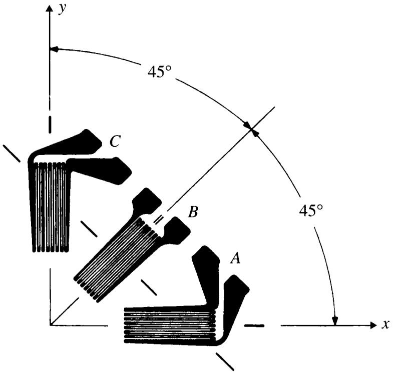
ε refers to corrected strain value, whereas $\hat{\varepsilon}$ refers to the strain read from the strain indicator. The K_t terms are the transverse sensitivity coefficients of the gages as supplied by the manufacturer. Poisson's ratio, ν_0, is normally given to be 0.285.



Two-element rectangular rosette

$$\varepsilon_x = \frac{(1-\nu_0 K_{tA})\hat{\varepsilon}_A - K_{tA}(1-\nu_0 K_{tB})\hat{\varepsilon}_B}{1-K_{tA}K_{tB}}$$

$$\varepsilon_y = \frac{(1-\nu_0 K_{tB})\hat{\varepsilon}_B - K_{tB}(1-\nu_0 K_{tA})\hat{\varepsilon}_A}{1-K_{tA}K_{tB}}$$

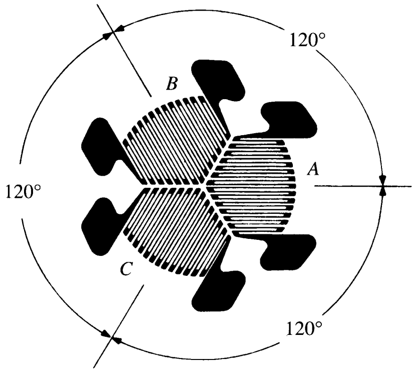


Three-element rectangular rosette

$$\varepsilon_A = \frac{(1-\nu_0 K_{tA})\hat{\varepsilon}_A - K_{tA}(1-\nu_0 K_{tC})\hat{\varepsilon}_C}{1-K_{tA}K_{tC}}$$

$$\varepsilon_B = \frac{(1-\nu_0 K_{tB})\hat{\varepsilon}_B - \dfrac{K_{tB}}{1-K_{tA}K_{tC}}[(1-\nu_0 K_{tA})(1-K_{tC})\hat{\varepsilon}_A + (1-\nu_0 K_{tC})(1-K_{tA})\hat{\varepsilon}_C]}{1-K_{tB}}$$

$$\varepsilon_C = \frac{(1-\nu_0 K_{tC})\hat{\varepsilon}_C - K_{tC}(1-\nu_0 K_{tA})\hat{\varepsilon}_A}{1-K_{tA}K_{tC}}$$



Three-element delta rosette

$$\begin{aligned}\varepsilon_A &= \kappa\{(1 - \nu_0 K_{tA})(3 - K_{tB} - K_{tC} - K_{tB}K_{tC})\hat{\varepsilon}_A - 2K_{tA}[(1 - \nu_0 K_{tB})(1 - K_{tC})\hat{\varepsilon}_B \\ &\quad + (1 - \nu_0 K_{tC})(1 - K_{tB})\hat{\varepsilon}_C]\} \\ \varepsilon_B &= \kappa\{(1 - \nu_0 K_{tB})(3 - K_{tC} - K_{tA} - K_{tC}K_{tA})\hat{\varepsilon}_B - 2K_{tB}[(1 - \nu_0 K_{tC})(1 - K_{tA})\hat{\varepsilon}_C \\ &\quad + (1 - \nu_0 K_{tA})(1 - K_{tC})\hat{\varepsilon}_A]\} \\ \varepsilon_C &= \kappa\{(1 - \nu_0 K_{tC})(3 - K_{tA} - K_{tB} - K_{tA}K_{tB})\hat{\varepsilon}_C - 2K_{tC}[(1 - \nu_0 K_{tA})(1 - K_{tB})\hat{\varepsilon}_A \\ &\quad + (1 - \nu_0 K_{tB})(1 - K_{tA})\hat{\varepsilon}_B]\}\end{aligned}$$

where

$$\kappa = (3K_{tA}K_{tB}K_{tC} - K_{tA}K_{tB} - K_{tB}K_{tC} - K_{tA}K_{tC} - K_{tA} - K_{tB} - K_{tC} + 3)^{-1}$$

6.6 References

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Part

3

Formulas and Examples

Each of the following chapters deals with a certain type of structural member or a certain condition of stress. What may be called the common, or typical, case is usually discussed first; special cases, representing peculiarities of form, proportions, or circumstances of loading, are considered subsequently. In the discussion of each case the underlying assumptions are stated, the general behavior of the loaded member is described, and formulas for the stress and deformation are given. The more important of the general equations are numbered consecutively throughout each section to facilitate reference, but, wherever possible, formulas applying to specific cases are tabulated for convenience and economy of space.

In all formulas which contain numerical constants having dimensions, the units are specified.

Most formulas contain only dimensionless constants and can be evaluated in any consistent system of units.

Chapter 7

Tension, Compression, Shear, and Combined Stress

7.1 Bar under Axial Tension (or Compression); Common Case

The bar is straight, of any uniform cross section, of homogeneous material, and (if under compression) short or constrained against lateral buckling. The loads are applied at the ends, centrally, and in such a manner as to avoid nonuniform stress distribution at any section of the part under consideration. The stress does not exceed the proportional limit.

Behavior. Parallel to the load the bar elongates (under tension) or shortens (under compression), the unit longitudinal strain being ε and the total longitudinal deflection in the length l being δ. At right angles to the load the bar contracts (under tension) or expands (under compression); the unit lateral strain ε' is the same in all transverse directions, and the total lateral deflection δ' in any direction is proportional to the lateral dimension d measured in that direction. Both longitudinal and lateral strains are proportional to the applied load. On any right section there is a uniform tensile (or compressive) stress σ; on any oblique section there is a uniform tensile (or compressive) normal stress σ_θ and a uniform shear stress τ_θ. The deformed bar under tension is represented in Fig. 7.1(*a*), and the stresses in Fig. 7.1(*b*).

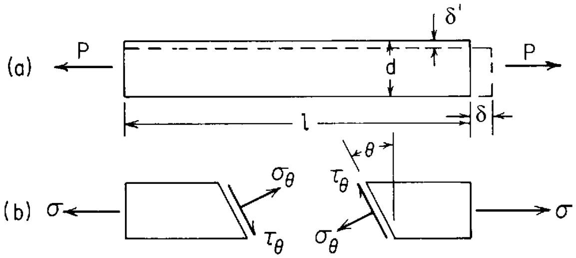

Figure 7.1

Formulas. Let

P = applied load
A = cross-sectional area (before loading)
l = length (before loading)
E = modulus of elasticity
ν = Poisson's ratio

Then

$$\sigma = \frac{P}{A} \tag{7.1-1}$$

$$\sigma_\theta = \frac{P}{A}\cos^2\theta, \qquad \max \sigma_\theta = \sigma \text{ (when } \theta = 0^\circ)$$

$$\tau = \frac{P}{2A}\sin 2\theta, \qquad \max \tau_\theta = \frac{1}{2}\sigma\text{(when } \theta = 45 \text{ or } 135^\circ)$$

$$\varepsilon = \frac{\sigma}{E} \tag{7.1-2}$$

$$\delta = l\varepsilon = \frac{Pl}{AE} \tag{7.1-3}$$

$$\varepsilon' = -\nu\varepsilon \tag{7.1-4}$$

$$\delta' = \varepsilon' d \tag{7.1-5}$$

$$\text{Strain energy per unit volume } U = \frac{1}{2}\frac{\sigma^2}{E} \tag{7.1-6}$$

$$\text{Total strain energy } U = \frac{1}{2}\frac{\sigma^2}{E}\,Al = \frac{1}{2}P\delta \tag{7.1-7}$$

For small strain, each unit area of cross section changes by $(-2\nu\varepsilon)$ under load, and each unit of volume changes by $(1 - 2\nu)\varepsilon$ under load.

In some discussions it is convenient to refer to the *stiffness* of a member, which is a measure of the resistance it offers to being deformed. The stiffness of a uniform bar under axial load is shown by Eq. (7.1-3) to be proportional to A and E directly and to l inversely, i.e., proportional AE/l.

EXAMPLE

A cylindrical rod of steel 4 in long and 1.5 in diameter has an axial compressive load of 20,000 lb applied to it. For this steel $\nu = 0.285$ and $E = 30{,}000{,}000\ \text{lb/in}^2$. Determine (*a*) the unit compressive stress σ; (*b*) the total longitudinal deformation, δ; (*c*) the total transverse deformation δ'; (*d*) the change in volume, ΔV; and (*e*) the total energy, or work done in applying the load.

Solution

$$(a)\quad \sigma = \frac{P}{A} = \frac{4P}{\pi d^2} = \frac{4(-20{,}000)}{\pi(1.5)^2} = -11{,}320\ \text{lb/in}^2$$

$$(b)\quad \varepsilon = \frac{\sigma}{E} = \frac{-11{,}320}{30{,}000{,}000} = -377(10^{-6})$$

$$\delta = \varepsilon l = (-377)(10^{-6})(4) = -1.509(10^{-3})\ \text{in}\ (''-''\ \text{means shortening})$$

$$(c)\quad \varepsilon' = -\nu\varepsilon = -0.285(-377)(10^{-6}) = 107.5(10^{-6})$$

$$\delta' = \varepsilon' d = (107.5)(10^{-6})(1.5) = 1.613(10^{-4})\ \text{in}\ (''+''\ \text{means expansion})$$

$$(d)\quad \Delta V/V = (1 - 2\nu)\varepsilon = [1 - 2(0.285)](-377)(10^{-6}) = -162.2(10^{-6})$$

$$\Delta V = -162.2(10^{-6})V = -162.2(10^{-6})\frac{\pi}{4}d^2 l = -162.2(10^{-6})\frac{\pi}{4}(1.5)^2(4)$$
$$= -1.147(10^{-3})\ \text{in}^3\ (''-''\ \text{means decrease})$$

(*e*) Increase in strain energy,

$$U = \frac{1}{2}P\delta = \frac{1}{2}(-20{,}000)(-1.509)(10^{-3}) = 15.09\ \text{in-lb}$$

7.2 Bar under Tension (or Compression); Special Cases

If the bar is not straight, it is subject to bending; formulas for this case are given in Sec. 12.4.

If the load is applied eccentrically, the bar is subject to bending; formulas for this case are given in Secs. 8.7 and 12.4. If the load is compressive and the bar is long and not laterally constrained, it must be analyzed as a column by the methods of Chapters 12 and 15.

If the stress exceeds the proportional limit, the formulas for stress given in Sec. 7.1 still hold but the deformation and work done in producing it can be determined only from experimental data relating unit strain to unit stress.

If the section is not uniform but changes *gradually*, the stress at any section can be found by dividing the load by the area of that section; the total longitudinal deformation over a length l is given by $\int_0^l \frac{P}{AE}\,dx$ and the strain energy is given by $\int_0^l \frac{1}{2}\frac{P^2}{AE}\,dx$, where dx is an infinitesimal length in the longitudinal direction. If the change in section is *abrupt* stress concentration may have to be taken into account, values of K_t being used to find elastic stresses and values of K_r being used to predict the breaking load. Stress concentration may also have to be considered if the end attachments for loading involve pinholes, screw threads, or other stress raisers (see Sec. 3.10 and Chap. 17).

If instead of being applied at the ends of a uniform bar the load is applied at an intermediate point, both ends being held, the *method of consistent deformations* shows that the load is apportioned to the two parts of the bar in inverse proportion to their respective lengths.

If a uniform bar is supported at one end in a vertical position and loaded only by its own weight, the maximum stress occurs at the supported end and is equal to the weight divided by the cross-sectional area. The total elongation is *half* as great and the total strain energy *one-third* as great as if a load equal to the weight were applied at the unsupported end. A bar supported at one end and loaded by its own weight and an axial downward load P (force) applied at the unsupported end will have the same unit stress σ (force per unit area) at all sections if it is tapered so that all sections are similar in form but vary in scale according to the formula

$$y = \frac{\sigma}{w}\log_e \frac{A\sigma}{P} \qquad (7.2\text{-}1)$$

where y is the distance from the free end of the bar to any section, A is the area of that section, and w is the density of the material (force per unit volume).

If a bar is stressed by having both ends rigidly held while a change in temperature is imposed, the resulting stress is found by calculating the longitudinal expansion (or contraction) that the change in temperature would produce if the bar were not held and then calculating the load necessary to shorten (or lengthen) it by that amount (principle of superposition). If the bar is uniform, the unit stress produced is independent of the length of the bar if restraint against buckling is provided. If a bar is stressed by being struck an axial blow at one end, the case is one of *impact* loading, discussed in Sec. 16.3.

EXAMPLES

1. Figure 7.2 represents a uniform bar rigidly held at the ends A and D and axially loaded at the intermediate points B and C. It is required to determine

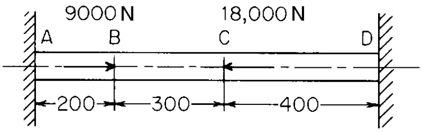

Figure 7.2

the total force in each portion of the bar *AB*, *BC*, *CD*. The loads are in newtons and the lengths in centimeters.

Solution. Each load is divided between the portions of the bar to right and left in inverse proportion to the lengths of these parts (consistent deformations), and the total force sustained by each part is the algebraic sum of the forces imposed by the individual loads (superposition). Of the 9000 N load, therefore, $\frac{7}{9}$, or 7000 N, is carried in tension by segment *AB*, and $\frac{2}{9}$, or 2000 N, is carried in compression by the segment *BD*. Of the 18,000 N load, $\frac{4}{9}$, or 8000 N, is carried in compression by segment *AC*, and $\frac{5}{9}$, or 10,000 N, is carried in tension by segment *CD*. Denoting tension by the plus sign and compression by the minus sign, and adding algebraically, the actual stresses in each segment are found to be

$$\begin{aligned} AB&: \quad 7000 - 8000 = -1000 \text{ N} \\ BC&: \quad -2000 - 8000 = -10{,}000 \text{ N} \\ CD&: \quad -2000 + 10{,}000 = +8000 \text{ N} \end{aligned}$$

The results are quite independent of the diameter of the bar and of E provided the bar is completely uniform.

If instead of being *held* at the ends, the bar is prestressed by wedging it between rigid walls under an initial compression of, say, 10,000 N and the loads at B and C are then applied, the results secured above would represent the *changes* in force the several parts would undergo. The final forces in the bar would therefore be 11,000 N compression in *AB*, 20,000 N compression in *BC*, and 2000 N compression in *CD*. But if the initial compression were less than 8000 N, the bar would break contact with the wall at D (no tension possible); there would be no force at all in *CD*, and the forces in *AB* and *BC*, now statically determinate, would be 9000 and 18,000 N compression, respectively.

2. A steel bar 24 in long has the form of a truncated cone, being circular in section with a diameter at one end of 1 in and at the other of 3 in. For this steel, $E = 30{,}000{,}000 \text{ lb/in}^2$ and the coefficient of thermal expansion is 0.0000065/°F. This bar is rigidly held at both ends and subjected to a drop in temperature of 50°F. It is required to determine the maximum tensile stress thus caused.

Solution. Using the principle of superposition, the solution is effected in three steps: (*a*) the shortening δ due to the drop in temperature is found, assuming the bar free to contract; (*b*) the force P required to produce an elongation equal to δ, that is, to stretch the bar back to its original length, is calculated; (*c*) the maximum tensile stress produced by this force P is calculated.

(*a*) $\delta = 50(0.0000065)(24) = 0.00780 \text{ in}$.

(*b*) Let d denote the diameter and A the area of any section a distance x in from the small end of the bar. Then

$$d = 1 + \frac{x}{12}, \qquad A = \frac{\pi}{4}\left(1 + \frac{x}{12}\right)^2$$

and

$$\delta = \int_0^l \frac{P}{EA}\,dx = \int_0^{24} \frac{4P}{(\pi E)(1 + x/12)^2}\,dx = \frac{4P}{\pi(30)(10^6)}\frac{(-12)}{(1 + x/12)}\bigg|_0^{24} = 3.395(10^{-7})P$$

Equating this to the thermal contraction of 0.00780 in yields

$$P = 22{,}970 \text{ lb}$$

(*c*) The maximum stress occurs at the smallest section, and is

$$\sigma = \frac{4P}{\pi d_{\min}^2} = \frac{4(22{,}970)}{\pi(1)^2} = 29{,}250 \text{ lb/in}^2$$

The result can be accepted as correct only if the proportional limit of the steel is known to be as great as or greater than the maximum stress and if the concept of a rigid support can be accepted. (See cases 8, 9, and 10 in Table 14.1.)

7.3 Composite Members

A tension or compression member may be made up of parallel elements or parts which jointly carry the applied load. The essential problem is to determine how the load is apportioned among the several parts, and this is easily done by the method of consistent deformations. If the parts are so arranged that all undergo the same total elongation or shortening, then each will carry a portion of the load proportional to its stiffness, i.e., proportional to AE/l if each is a uniform bar and proportional to AE if all these uniform bars are of equal length. It follows that if there are n bars, with section areas $A_1, A_2, \ldots, A_n$, lengths $l_1, l_2, \ldots, l_n$, and moduli E_1, $E_2, \ldots, E_n$, then the loads on the several bars P_1, $P_2, \ldots, P_n$ are given by

$$P_1 = P\frac{\dfrac{A_1E_1}{l_1}}{\dfrac{A_1E_1}{l_1} + \dfrac{A_2E_2}{l_2} + \cdots + \dfrac{A_nE_n}{l_n}} \tag{7.3-1}$$

$$P_2 = P\frac{\dfrac{A_2E_2}{l_2}}{\dfrac{A_1E_1}{l_1} + \dfrac{A_2E_2}{l_2} + \cdots + \dfrac{A_nE_n}{l_n}} \tag{7.3-2}$$

……………………………………………

A composite member of this kind can be *prestressed*. P_1, P_2, etc., then represent the *increments* of force in each member due to the applied load, and can be found by Eqs. (7.3-1) and (7.3-2), provided all bars can sustain reversal of stress, or provided the applied load is not great enough to cause such reversal in any bar which cannot sustain it. As explained in Sec. 3.12, by proper prestressing, all parts of a composite member can be made to reach their allowable loads, elastic limits, or ultimate strengths simultaneously (Example 2).

EXAMPLES

1. A ring is suspended by three vertical bars, A, B, and C of unequal lengths. The upper ends of the bars are held at different levels, so that as assembled none of the bars is stressed. A is 4 ft long, has a section area of 0.3 in^2, and is of steel for which $E = 30{,}000{,}000\ \text{lb/in}^2$; B is 3 ft long and has a section area of 0.2 in^2, and is of copper for which $E = 17{,}000{,}000\ \text{lb/in}^2$; C is 2 ft long, has a section area of 0.4 in^2, and is of aluminum for which $E = 10{,}000{,}000\ \text{lb/in}^2$. A load of 10,000 lb is hung on the ring. It is required to determine how much of this load is carried by each bar.

Solution. Denoting by P_A, P_B, and P_C the loads carried by A, B, and C, respectively, and expressing the moduli of elasticity in millions of pounds per square inch and the lengths in feet, we substitute in Eq. (7.3-1) and find

$$P_A = 10{,}000\left[\frac{\dfrac{(0.3)(30)}{4}}{\dfrac{(0.3)(30)}{4}+\dfrac{(0.2)(17)}{3}+\dfrac{(0.4)(10)}{2}}\right] = 4180\ \text{lb}$$

Similarly

$$P_B = 2100\ \text{lb} \quad \text{and} \quad P_C = 3720\ \text{lb}$$

2. A composite member is formed by passing a steel rod through an aluminum tube of the same length and fastening the two parts together at both ends. The fastening is accomplished by adjustable nuts, which make it possible to assemble the rod and tube so that one is under initial tension and the other is under an equal initial compression. For the steel rod the section area is 1.5 in^2, the modulus of elasticity 30,000,000 lb/in^2 and the allowable stress 15,000 lb/in^2. For the aluminum tube the section area is 2 in^2, the modulus of elasticity 10,000,000 lb/in^2 and the allowable stress 10,000 lb/in^2. It is desired to prestress the composite member so that under a tensile load both parts will reach their allowable stresses simultaneously.

Solution. When the allowable stresses are reached, the force in the steel rod will be $1.5(15{,}000) = 22{,}500$ lb, the force in the aluminum tube will be $2(10{,}000) = 20{,}000$ lb, and the total load on the member will be $22{,}500 + 20{,}000\ ; = 42{,}500$ lb. Let P_i denote the initial tension or compression in the members, and, as before, let tension be considered positive and compression negative. Then, since Eq. (7.3-1) gives the *increment* in force, we have for the aluminum tube

$$P_i + 42{,}500\frac{(2)(10)}{(2)(10)+(1.5)(30)} = 20{,}000$$

or

$$P_i = +6920 \text{ lb} \qquad \text{(initial tension)}$$

For the steel rod, we have

$$P_i + 42{,}500\frac{(1.5)(30)}{(2)(10)+(1.5)(30)} = 22{,}500$$

or

$$P_i = -6920 \text{ lb} \qquad \text{(initial compression)}$$

If the member were not prestressed, the unit stress in the steel would always be just three times as great as that in the aluminum because it would sustain the same unit deformation and its modulus of elasticity is three times as great. Therefore, when the steel reached its allowable stress of 15,000 lb/in^2, the aluminum would be stressed to only 5000 lb/in^2 and the allowable load on the composite member would be only 32,500 lb instead of 42,500 lb.

7.4 Trusses

A conventional truss is essentially an assemblage of straight uniform bars that are subjected to axial tension or compression when the truss is loaded at the joints. The deflection of any joint of a truss is easily found by the *method of unit loads* (Sec. 4.5). Let p_1, p_2, p_3, etc., denote the forces produced in the several members by an *assumed unit load* acting in the direction x at the joint whose deflection is to be found, and let δ_1, δ_2, δ_3, etc., denote the longitudinal deformations produced in the several members by the *actual applied loads*. The deflection Δ_x in the direction x of the joint in question is given by

$$\Delta_x = p_1\delta_1 + p_2\delta_2 + p_3\delta_3 + \cdots = \sum_{i=1}^{n} p_i\delta_i \qquad (7.4\text{-}1)$$

The deflection in the direction y, at right angles to x, can be found similarly by assuming the unit load to act in the y direction; the resultant deflection is then determined by combining the x and y deflections. Attention must be given to the *signs* of p and δ, p is positive if a member is subjected to tension and negative if under compression, and δ is positive if it represents an elongation and negative if it represents a shortening. A positive value for $\sum p\delta$ means that the deflection is in the direction of the assumed unit load, and a negative value means that it is in the opposite direction. (This procedure is illustrated in Example 1 below.)

A statically indeterminate truss can be solved by the *method of least work* (Sec. 4.5). To do this, it is necessary to write down the expression for the total strain energy in the structure, which, being simply the sum of the strain energies of the constituent bars, is given by

$$\frac{1}{2}P_1\delta_1 + \frac{1}{2}P_2\delta_2 + \frac{1}{2}P_3\delta_3 + \cdots = \sum_{i=1}^{n}\frac{1}{2}P_i\delta_i = \sum_{i=1}^{n}\frac{1}{2}\left(\frac{P^2 l}{AE}\right)_i \qquad (7.4\text{-}2)$$

Here P_1, P_2, etc., denote the forces in the individual members due to the applied loads and δ has the same meaning as above. It is necessary to express each force P_i as the sum of the two forces; one of these is the force the applied loads would produce with the redundant member removed, and the other is the force due to the unknown force (say, F) exerted by this redundant member on the rest of the structure. The total strain energy is thus expressed as a function of the known applied forces and F, the force in the redundant member. The partial derivative with respect to F of this expression for strain energy is then set equal to zero and solved for F. If there are two or more redundant members, the expression for strain energy with all the redundant forces, F_1, F_2, etc., represented is differentiated once with respect to each. The equations thus obtained are then solved simultaneously for the unknown forces. (The procedure is illustrated in Example 2.)

EXAMPLES

1. The truss shown in Fig. 7.3 is composed of tubular steel members, for which $E = 30{,}000{,}000$ lb/in^2. The section areas of the members are given in the table below. It is required to determine Δ_x and Δ_y, the horizontal and vertical components of the displacement of joint A produced by the indicated loading.

Solution. The method of unit loads is used. The force P in each member due to the applied loads is found, and the resulting elongation or shortening δ is calculated. The force p_x in each member due to a load of 1 lb acting to the right at A, and the force p_y in each member due to a load of 1 lb acting down at A are calculated. By Eq. (7.4-1), $\sum p_x\delta$, then gives the horizontal and $\sum p_y\delta$ gives the vertical displacement or deflection of A. Tensile forces and elongations are

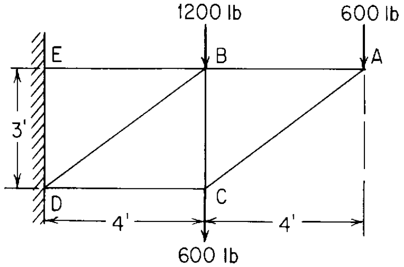

Figure 7.3

denoted by $+$, compressive forces and shortenings by $-$. The work is conveniently tabulated as follows:

Member	Area, A_i, in^2	Length, l_i, in	P_i, lb	$\delta_i = \left(\frac{Pl}{AE}\right)_i$, in	$(p_x)_i$	$(p_x\delta)_i$, in (*a*)	$(p_y)_i$	$(p_y\delta)_i$, in (*b*)
(1) *AB*	0.07862	48	800	0.01628	1.000	0.01628	1.333	0.02171
(2) *AC*	0.07862	60	−1000	−0.02544	0	0	−1.667	0.04240
(3) *BC*	0.1464	36	1200	0.00984	0	0	1.000	0.00984
(4) *BE*	0.4142	48	4000	0.01545	1.000	0.01545	2.667	0.04120
(5) *BD*	0.3318	60	−4000	−0.02411	0	0	−1.667	0.04018
(6) *CD*	0.07862	48	−800	−0.01628	0	0	−1.333	0.02171
						$\Delta_x = 0.03173$ in		$\Delta_y = 0.17704$ in

Δ_x and Δ_y are both found to be positive, which means that the displacements are in the directions of the assumed unit loads—to the right and down. Had either been found to be negative, it would have meant that the displacement was in a direction opposite to that of the corresponding unit load.

2. Assume a diagonal member, running from *A* to *D* and having a section area 0.3318 in^2 and length 8.544 ft, is to be added to the truss of Example 1; the structure is now statically indeterminate. It is required to determine the force in each member of the altered truss due to the loads shown.

Solution. We use the method of least work. The truss has one redundant member; any member except *BE* may be regarded as redundant, since if any one were removed, the remaining structure would be stable and statically determinate. We select *AD* to be regarded as redundant, denote the unknown force in *AD* by *F*, and assume *F* to be tension. We find the force in each member assuming *AD* to be removed, then find the force in each member due to a pull *F* exerted at *A* by *AD*, and then add these forces, thus getting an expression for the force in each member of the actual truss in terms of *F*. The expression for the strain energy can then be written out, differentiated with respect to *F*, equated to zero, and solved for *F*. *F* being known, the force in each member of the truss is easily found. The computations are conveniently tabulated as follows:

	Forces in members[†]			
Member	Due to applied loads without AD (*a*) (lb)	Due to pull, *F*, exerted by *AD* (*b*)	Total forces, P_i. Superposition of (*a*) and (*b*) (*c*)	Actual total values with $F = -1050$ lb in (*c*) (*d*) (lb)
(1) *AB*	800	$-0.470\,F$	$800 - 0.470\,F$	1290
(2) *AC*	−1000	$-0.584\,F$	$-1000 - 0.584\,F$	−390
(3) BC	1200	$0.351\,F$	$1200 + 0.351\,F$	830
(4) *BE*	4000	0	4000	4000
(5) *BD*	−4000	$-0.584\,F$	$-4000 - 0.584\,F$	−3390
(6) *CD*	−800	$-0.470\,F$	$-800 - 0.470\,F$	−306
(7) *AD*	0	F	F	−1050

[†] + for tension and − for compression.

$$U = \sum_{i=1}^{7} \frac{1}{2}\left(\frac{P^2 l}{AE}\right)_i = \frac{1}{2E}\left[\frac{(800 - 0.470F)^2(48)}{0.07862} + \frac{(-1000 - 0.584F)^2(60)}{0.07862}\right.$$
$$+ \frac{(1200 + 0.351F)^2(36)}{0.1464} + \frac{(4000)^2(48)}{0.4142}$$
$$+ \frac{(-4000 - 0.584F)^2(60)}{0.3318} + \frac{(-800 - 0.470F)^2(48)}{0.07862}$$
$$\left. + \frac{F^2(102.5)}{0.3318}\right]$$

Setting the partial derivative of U relative to F to zero,

$$\frac{\partial U}{\partial F} = \frac{1}{2E}\left[\frac{2(800 - 0.470F)(-0.470)(48)}{0.07862} + \frac{2(-1000 - 0.584F)(-0.584)(60)}{0.07862} + \cdots\right]$$
$$= 0$$

and solving for F gives $F = -1050$ lb.

The negative sign here simply means that AD is in compression. A positive value of F would have indicated tension. Substituting the value of F into the terms of column (*c*) yield the actual total forces in each member as tabulated in column (*d*).

7.5 Body under Pure Shear Stress

A condition of pure shear may be produced by any one of the methods of loading shown in Fig. 7.4. In Fig. 7.4(*a*), a rectangular block of length a, height b, and uniform thickness t is shown loaded by forces P_1 and P_2, uniformly distributed over the surfaces to which they are applied and satisfying the equilibrium equation $P_1 b = P_2 a$. There are equal shear stresses on all vertical and horizontal planes, so that any contained cube oriented like $ABCD$ has on each of four faces the shear stress $\tau = P_1/at = P_2/bt$ and no other stress.

In Fig. 7.4(*b*) a rectangular block is shown under equal and opposite biaxial stresses σ_t and σ_c. There are equal shear stresses on all planes inclined at 45° to the top and bottom faces, so that a contained cube

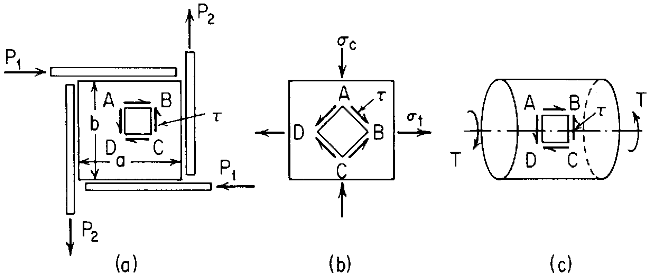

Figure 7.4

oriented like *ABCD* has on each of four faces the shear stress $\tau = \sigma_t = \sigma_c$ and no other stress.

In Fig. 7.4(*c*), a circular shaft is shown under a twisting moment T; a cube of infinitesimal dimensions, a distance z from the axis and oriented like *ABCD* has on each of four faces an essentially uniform shear stress $\tau = Tz/J$ (Sec. 10.1) and no other stress.

In whatever way the loading is accomplished, the result is to impose on an elementary cube of the loaded body the condition of stress represented in Fig. 7.5, that is, shearing stress alone on each of four faces, these stresses being equal and so directed as to satisfy the equilibrium condition $T_x = 0$ (Sec. 4.1).

The stresses, σ_θ and τ_θ on a transformed surface rotated counterclockwise through the angle θ can be determined from the transformation equations given by Eqs. (2.3-17). They are given by

$$\sigma_\theta = \tau \sin 2\theta, \qquad \tau_\theta = \tau \cos 2\theta \tag{7.5-1}$$

where $(\sigma_\theta)_{\max,\min} = \pm\tau$ at $\theta = \pm 45^\circ$.

The strains produced by pure shear are shown in Fig. 7.5(*b*), where the cube *ABCD* is deformed into a rhombohedron *A′B′C′D′*. The unit shear strain, γ, referred to as the *engineering shear strain*, is reduction of angles $\angle ABC$ and $\angle ADC$, and the increase in angles $\angle DAB$ and $\angle BCD$ in radians. Letting G denote the modulus of rigidity, the shear strain is related to the shear stress as

$$\gamma = \frac{\tau}{G} \tag{7.5-2}$$

Assuming a linear material, the strain energy per unit volume for pure shear, u_s, within the elastic range is given by

$$u_s = \frac{1}{2}\frac{\tau^2}{G} \tag{7.5-3}$$

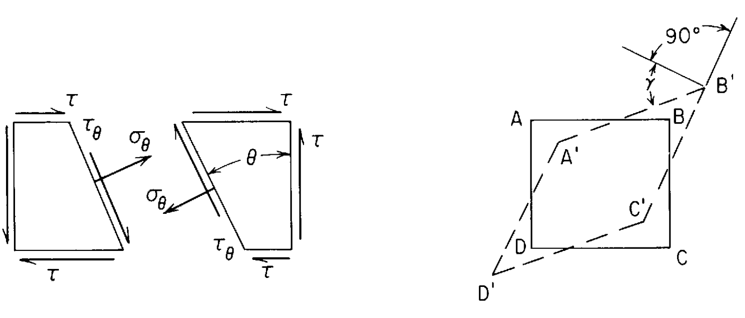

Figure 7.5 (*a*) Shear stress and transformation. (*b*) Shear strain.

The relations between τ, σ, and the strains represented in Fig. 7.5(*b*) make it possible to express G in terms of E and Poisson's ratio, ν, for a linear, homogeneous, isotropic material. The relationship is

$$G = \frac{E}{2(1+\nu)} \tag{7.5-4}$$

From known values of E (determined by a tensile test) and G (determined by a torsion test) it is thus possible to calculate ν.

7.6 Cases of Direct Shear Loading

By *direct shear loading* is meant any case in which a member is acted on by equal, parallel, and opposite forces so nearly colinear that the material between them is subjected primarily to shear stress, with negligible bending. Examples of this are provided by rivets, bolts, and pins, shaft splines and keys, screw threads, short lugs, etc. These are not really cases of pure shear; the actual stress distribution is complex and usually indeterminate because of the influence of fit and other factors. In designing such parts, however, it is usually assumed that the shear is uniformly distributed on the critical section, and since working stresses are selected with due allowance for the approximate nature of this assumption, the practice is usually permissible. In *beams* subject to transverse shear, this assumption cannot be made as a rule.

Shear and other stresses in rivets, pins, keys, etc., are discussed more fully in Chap. 14, shear stresses in beams in Chap. 8, and shear stresses in torsion members in Chap. 10.

7.7 Combined Stress

Under certain circumstances of loading, a body is subjected to a combination of tensile and compressive stresses (usually designated as *biaxial* or *triaxial stress*) or to a combination of tensile, compressive, and shear stresses (usually designated as *combined stress*). For example, the material at the inner surface of a thick cylindrical pressure vessel is subjected to triaxial stress (radial compression, longitudinal tension, and circumferential tension), and a shaft simultaneously bent and twisted is subjected to combined stress (longitudinal tension or compression, and torsional shear).

In most instances the normal and shear stresses on each of three mutually perpendicular planes are due to flexure, axial loading, torsion, beam shear, or some combination of these which separately can be calculated readily by the appropriate formulas. Normal stresses arising from different load conditions acting on the same plane can be combined simply by algebraic addition considering tensile stresses positive and compressive stresses negative. Similarly, shear stresses can be combined by algebraic addition following a consistent sign convention. Further analysis of the combined states of normal and shear stresses must be performed using the transformation techniques outlined in Sec. 2.3. The principal stresses, the maximum shear stress, and the normal and shear stresses on any given plane can be found by the equations given in Sec. 2.3.

The strains produced by any combination of stresses not exceeding the proportional limit can also be found using Hooke's law for each stress and then combined by superposition. Consideration of the strains caused by equal triaxial stresses leads to an expression for the bulk modulus of elasticity given by

$$K = \frac{E}{3(1-2\nu)} \tag{7.7-1}$$

EXAMPLES

1. A rectangular block 12 in long, 4 in high, and 2 in thick is subjected to a longitudinal tensile stress $\sigma_x = 12{,}000\ \mathrm{lb/in^2}$, a vertical compressive stress $\sigma_y = 15{,}000\ \mathrm{lb/in^2}$, and a lateral compressive stress $\sigma_z = 9000\ \mathrm{lb/in^2}$. The material is steel, for which $E = 30{,}000{,}000\ \mathrm{lb/in^2}$ and $\nu = 0.30$. It is required to find the total change in length.

Solution. The longitudinal deformation is found by superposition: The unit strain due to each stress is computed separately by Eqs. (7.1-2) and (7.1-4); these results are added to give the resultant longitudinal unit strain, which is multiplied by the length to give the total elongation. Denoting unit longitudinal strain by ε_x and total longitudinal deflection by δ_x, we have

$$\begin{aligned}\varepsilon_x &= \frac{12{,}000}{E} - \nu\frac{-15{,}000}{E} - \nu\frac{-9000}{E} \\ &= 0.000400 + 0.000150 + 0.000090 = +0.00064 \\ \delta_x &= 12(0.00064) = 0.00768\ \text{in}\end{aligned}$$

The lateral dimensions have nothing to do with the result since the lateral stresses, not the lateral loads, are given.

2. A piece of "standard extra-strong" pipe, 2 in nominal diameter, is simultaneously subjected to an internal pressure of $p = 2000\ \mathrm{lb/in^2}$ and to a twisting moment of $T = 5000$ in-lb caused by tightening a cap screwed on at one end. Determine the maximum tensile stress and the maximum shear stress thus produced in the pipe.

Solution. The calculations will be made, first, for a point at the outer surface and, second, for a point at the inner surface. The dimensions of the pipe and properties of the cross section are as follows: inner radius $r_i = 0.9695$ in, outer radius $r_o = 1.1875$ in, cross-sectional area of bore $A_b = 2.955\ \mathrm{in^2}$, cross-sectional area of pipe wall $A_w = 1.475\ \mathrm{in^2}$, and polar moment of inertial $J = 1.735\ \mathrm{in^4}$.

We take axis x along the axis of the pipe, axis y tangent to the cross section, and axis z radial in the plane of the cross section. For a point at the outer surface of the pipe, σ_x is the longitudinal tensile stress due to pressure and σ_y is the circumferential (hoop) stress due to pressure, the radial stress $\sigma_z = 0$ (since the pressure is zero on the outer surface of the pipe), and τ_{xy} is the shear stress due to torsion. Equation (7.1-1) can be used for σ_x, where $P = pA_b$ and $A = A_w$. To calculate σ_y, we use the formula for stress in thick cylinders (Table 13.5, case 1*b*). Finally, for τ_{xy}, we use the formula for torsional stress (Eq. (10.1-2). Thus,

$$\sigma_x = \frac{pA_b}{A_w} = \frac{(2000)(2.955)}{1.475} = 4007 \text{ lb/in}^2$$

$$\sigma_y = p\frac{r_i^2(r_o^2 + r_o^2)}{r_o^2(r_o^2 - r_i^2)} = 2000\frac{(0.9695^2)(1.1875^2 + 1.1875^2)}{(1.1875^2)(1.1875^2 - 0.9695^2)} = 7996 \text{ lb/in}^2$$

$$\tau_{xy} = \frac{Tr_o}{J} = \frac{(5000)(1.1875)}{1.735} = 3422 \text{ lb/in}^2$$

This is a case of plane stress where Eq. (2.3-23) applies. The principal stresses are thus

$$\begin{aligned}\sigma_p &= \tfrac{1}{2}[(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}]\\ &= \tfrac{1}{2}[(4007 + 7996) \pm \sqrt{(4007 - 7996)^2 + 4(3422^2)}] = 9962, \quad 2041 \text{ lb/in}^2\end{aligned}$$

Thus, $\sigma_{\max} = 9962 \text{ lb/in}^2$.

In order to determine the maximum shear stress, we order the *three* principal stresses such that $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. For plane stress, the out-of-plane principal stresses are zero. Thus, $\sigma_1 = 9962 \text{ lb/in}^2$, $\sigma_2 = 2041 \text{ lb/in}^2$, and $\sigma_3 = 0$. From Eq. (2.3-25), the maximum shear stress is

$$\tau_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = \tfrac{1}{2}(9962 - 0) = 4981 \text{ lb/in}^2$$

For a point on the inner surface, the stress conditions are three-dimensional since a radial stress due to the internal pressure is present. The longitudinal stress is the same; however, the circumferential stress and torsional shear stress change. For the inner surface,

$$\sigma_x = 4007 \text{ lb/in}^2$$

$$\sigma_y = p\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = 2000\frac{1.1875^2 + 0.9695^2}{1.1875^2 - 0.9695^2} = 9996 \text{ lb/in}^2$$

$$\sigma_z = -p = -2000 \text{ lb/in}^2$$

$$\tau_{xy} = \frac{Tr_i}{J} = \frac{(5000)(0.9695)}{1.735} = 2794 \text{ lb/in}^2$$

$$\tau_{yz} = \tau_{zx} = 0$$

The principal stresses are found using Eq. (2.3-20):[†]

$$\begin{aligned}\sigma_p^3 &- (4007 + 9996 - 2000)\sigma_p^2 + [(4007)(9996) + (9996)(-2000) \\ &+ (-2000)(4007) - 2794^2 - 0 - 0]\sigma_p - [(4007)(9996)(-2000) + 2(2794)(0)(0) \\ &- (4007)(0^2) - (9996)(0^2) - (-2000)(2794^2)] = 0\end{aligned}$$

or

$$\sigma_p^3 - 12.003(10^3)\sigma_p^2 + 4.2415(10^6)\sigma_p + 64.495(10^9) = 0$$

Solving this gives $\sigma_p = 11{,}100$, 2906, and -2000 lb/in^2, which are the principal stresses σ_1, σ_2, and σ_3, respectively. Obviously, the maximum tensile stress is 11,100 lb/in^2. Again, the maximum shear stress comes from Eq. (2.3-25), and is $\frac{1}{2}[11{,}100 - (-2000)] = 6550$ lb/in^2.

Note that for this problem, if the pipe is a ductile material, and one were looking at failure due to shear stress (see Sec. 3.7), the stress conditions for the pipe are more severe at the inner surface compared with the outer surface.

[†] *Note:* Since $\tau_{yz} = \tau_{zx} = 0$, σ_z is one of the principal stresses and the other two can be found from the plane stress equations. Consequently, the other two principal stresses are in the *xy* plane.

Chapter

8

Beams; Flexure of Straight Bars

8.1 Straight Beams (Common Case) Elastically Stressed

The formulas in this section are based on the following assumptions: (1) The beam is of homogeneous material that has the same modulus of elasticity in tension and compression. (2) The beam is straight or nearly so; if it is slightly curved, the curvature is in the plane of bending and the radius of curvature is at least 10 times the depth. (3) The cross section is uniform. (4) The beam has at least one longitudinal plane of symmetry. (5) All loads and reactions are perpendicular to the axis of the beam and lie in the same plane, which is a longitudinal plane of symmetry. (6) The beam is long in proportion to its depth, the span/depth ratio being 8 or more for metal beams of compact section, 15 or more for beams with relatively thin webs, and 24 or more for rectangular timber beams. (7) The beam is not disproportionately wide (see Sec. 8.11 for a discussion on the effect of beam width). (8) The maximum stress does not exceed the proportional limit.

Applied to any case for which these assumptions are not valid, the formulas given yield results that at best are approximate and that may be grossly in error; such cases are discussed in subsequent sections. The limitations stated here with respect to straightness and proportions of the beam correspond to a maximum error in calculated results of about 5%.

In the following discussion, it is assumed for convenience that the beam is horizontal and the loads and reactions vertical.

Behavior. As the beam bends, fibers on the convex side lengthen, and fibers on the concave side shorten. The neutral surface is normal to the plane of the loads and contains the centroids of all sections, hence the neutral axis of any section is the horizontal central axis. Plane sections remain plane, and hence unit fiber strains and stresses are proportional to distance from the neutral surface. Longitudinal displacements of points on the neutral surface are negligible. Vertical deflection is largely due to bending, that due to shear being usually negligible under the conditions stated.

There is at any point a longitudinal fiber stress σ, which is tensile if the point lies between the neutral and convex surfaces of the beam and compressive if the point lies between the neutral and concave surfaces of the beam. This fiber stress σ usually may be assumed uniform across the width of the beam (see Secs. 8.11 and 8.12).

There is at any point a longitudinal shear stress τ on the horizontal plane and an equal vertical shear stress on the transverse plane. These shear stresses, due to the transverse beam forces, may be assumed uniform across the width of the beam (see page 129).

Figure 8.1(a,b) represent a beam under load and show the various dimensions that appear in the formulas; Fig. 8.1(c) shows a small stress element at a point q acted on by the stresses σ and τ.

Formulas. Let I = the moment of inertia of the section of the beam with respect to the neutral axis and E = modulus of elasticity of the material.

The fiber stress σ at any point q is

$$\sigma = -\frac{My}{I} \tag{8.1-1}$$

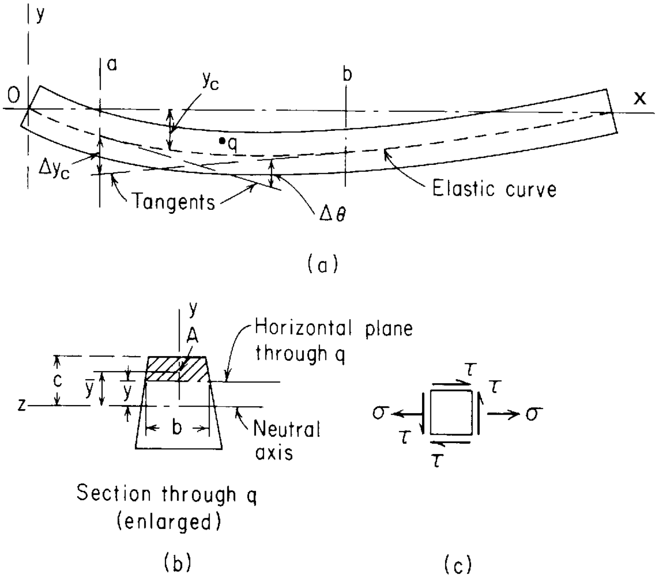

Figure 8.1

where M is the bending moment at the section containing q, and y is the vertical distance from the neutral axis to q.

The shear stress τ at any point q is

$$\tau = \frac{VA'\bar{y}}{Ib} \tag{8.1-2}$$

where V is the vertical shear at the section containing q, A' is the area of that part of the section above (or below) q, $\bar{y}$ is the distance from the neutral axis to the centroid of A', and b is the net breadth of the section measured through q.

The complementary energy of flexure U_f, is

$$U_f = \int \frac{M^2}{2EI}\,dx \tag{8.1-3}$$

where M represents the bending moment equation in terms of x, the distance from the left end of the beam to any section.

The radius of curvature ρ of the elastic curve at any section is

$$\rho = \frac{EI}{M} \tag{8.1-4}$$

where M is the bending moment at the section in question.

The general differential equation of the elastic curve is

$$EI\,\frac{d^2y_c}{dx^2} = M \tag{8.1-5}$$

where M has the same meaning as in Eq. (8.1-3) and y_c represents the vertical deflection of the centroidal axis of the beam. Solution of this equation for the vertical deflection y_c is effected by writing out the expression for M, integrating twice, and determining the constants of integration by the boundary conditions.

By the method of unit loads the vertical deflection at any point is found to be

$$y_c = \int \frac{Mm}{EI}\,dx \tag{8.1-6}$$

or by Castigliano's second theorem it is found to be

$$y_c = \frac{\partial U}{\partial P} \tag{8.1-7}$$

where M has the same meaning as in Eq. (8.1-3) and m is the equation of the bending moment due to a unit load acting vertically at the section where y_c is to be found. The integration indicated must be performed over each portion of the beam for which either M or m is expressed by a different equation. A positive result for y_c means that the deflection is in the direction of the assumed unit load; a negative result means it is in the opposite direction (see Example 2 at the end of this section).

In Eq. (8.1-7), U is given by Eq. (8.1-3) and P is a vertical load, real or imaginary, applied at the section where y_c is to be found. It is most convenient to perform the differentiation within the integral sign; as with Eq. (8.1-6), the integration must extend over the entire length of the beam, and the sign of the result is interpreted as before.

The change in slope of elastic curve $\Delta\theta$ (radians) between any two sections a and b is

$$\Delta\theta = \int_a^b \frac{M}{EI}\,dx \tag{8.1-8}$$

where M has the same meaning as in Eq. (8.1-3).

The deflection Δy_c at any section a, *measured vertically from a tangent drawn to the elastic curve at any section* b, is

$$\Delta y_c = \int_a^b \frac{M}{EI}x\,dx \tag{8.1-9}$$

where x is the distance from a to any section dx between a and b.

Important relations between the bending moment and shear equations are

$$V = \frac{dM}{dx} \tag{8.1-10}$$

$$M = \int V\,dx \tag{8.1-11}$$

These relations are useful in constructing shear and moment diagrams and locating the section or sections of maximum bending moment since Eq. (8.1-10) shows that the maximum moment occurs when V, its first derivative, passes through zero and Eq. (8.1-11) shows that the increment in bending moment that occurs between any two sections is equal to the area under the shear diagram between those sections.

Maximum fiber stress. The maximum fiber stress at any section occurs at the point or points most remote from the neutral axis and is given by Eq. (8.1-1) when $y = c$; hence

$$\sigma_{\max} = \frac{Mc}{I} = \frac{M}{I/c} = \frac{M}{S} \tag{8.1-12}$$

where S is the *section modulus*. The maximum fiber stress in the beam occurs at the section of greatest bending moment; if the section is not symmetrical about the neutral axis, the stresses should be investigated at both the section of greatest positive moment and the section of greatest negative moment.

Maximum transverse shear stress.[†] The maximum transverse shear stress in the beam occurs at the section of greatest vertical shear. The maximum transverse shear stress at any section occurs at the neutral axis, provided the net width b is as small there as anywhere else; if the section is narrower elsewhere, the maximum shear stress may not occur at the neutral axis. This maximum transverse shear stress can be expressed conveniently by the formula

$$(\tau_{\max})_V = \alpha \frac{V}{A} \tag{8.1-13}$$

where V/A is the *average* shear stress on the section and α is a factor that depends on the form of the section. For a rectangular section, $\alpha = \frac{3}{2}$ and the maximum stress is at the neutral axis; for a solid circular section, $\alpha = \frac{4}{3}$ and the maximum stress is at the neutral axis; for a triangular section, $\alpha = \frac{3}{2}$ and the maximum stress is halfway between the top and bottom of the section; for a diamond-shaped section of depth h, $\alpha = \frac{9}{8}$ and the maximum stress is at points that are a distance $h/8$ above and below the neutral axis.

In the derivation of Eq. (8.1-2) and in the preceding discussion, it is assumed that the shear stress is uniform across the width of the beam; i.e., it is the same at all points on any transverse line parallel to the neutral axis. Actually this is not the case; exact analysis (Ref. 1) shows that the shear stress varies across the width and that for a rectangle the maximum intensity occurs at the ends of the neutral axis where, for a wide beam, it is twice the average intensity. Photoelastic investigation of beams under concentrated loading shows that localized shearing stresses about four times as great as the maximum stress given by Eq. (8.1-2) occur near the points of loading and support (Ref. 2), but experience shows that this variation may be ignored and design based on the average value as determined by Eq. (8.1-2).

[†] Note that the *transverse shear stress* denoted here is the shear stress due to the vertical transverse force, V. The maximum transverse shear stress in a beam is not necessarily the maximum shear stress in the beam. One needs to look at the overall state of stress in light of stress transformations. For long beams, the maximum shear stress is normally due to the maximum fiber stress, and, using Eq. (2.3-25), the maximum shear stress is $\tau_{\max} = \frac{1}{2}\sigma_{\max} = \frac{1}{2}M/S$. For this reason, we will denote the maximum transverse shear stress in a beam as $(\tau_{\max})_V$.

For some sections the greatest horizontal shear stress at a given point occurs, not on a horizontal plane, but on an inclined longitudinal plane which cuts the section so as to make b a minimum. Thus, for a circular tube or pipe the greatest horizontal shear stress at any point occurs on a *radial* plane; the corresponding shear stress in the plane of the section is not vertical but tangential, and in computing τ by Eq. (8.1-2) b should be taken as twice the thickness of the tube instead of the net horizontal breadth of the member. (See Table 9.2, case 20 for an example of where this shear stress in a tube is of importance.)

In an I-, T-, or box section there is a horizontal shear stress on any vertical longitudinal plane through the flange, and this stress is usually a maximum at the juncture of flange and web. It may be calculated by Eq. (8.1-2), taking A' as the area outside of the vertical plane (for outstanding flanges) or between the vertical plane and the center of the beam section (for box girders), and b as the flange thickness (see the solution to Example 1*b*). The other terms have the same meanings as explained previously.

Shear stresses are not often of controlling importance except in wood or metal beams that have thin webs or a small span/depth ratio. For beams that conform to the assumptions stated previously, strength will practically always be governed by fiber stress.

Change in projected length due to bending. The apparent shortening of a beam due to bending, i.e., the difference between its original length and the horizontal projection of the elastic curve, is given by

$$\Delta l = -\frac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx \tag{8.1-14}$$

To evaluate Δl, dy/dx is expressed in terms of x [Eq. (8.1-5)] and the square of this is integrated as indicated.

The extreme fibers of the beam undergo a change in actual length due to stress given by

$$e = \int_0^l \frac{Mc}{EI} dx \tag{8.1-15}$$

By means of these equations the actual relative horizontal displacement of points on the upper or lower surface of the beam can be predicted and the necessary allowances made in the design of rocker bearings, clearance for the corners, etc.

Tabulated formulas. Table 8.1 gives formulas for the reactions, moments, slopes and deflections at each end of single-span beams supported and loaded in various ways. The table also gives formulas for the vertical shears, bending moments, slopes, and deflections at any distance x from the left end of the span.

In these formulas, the unit step function is used by itself and in combination with ordinary functions.

The unit step function is denoted by $\langle x-a\rangle^0$ where the use of the angle brackets $\langle\,\rangle$ is defined as follows: If $x < a$, $\langle x-a\rangle^0 = 0$; if $x > a$, $\langle x-a\rangle^0 = 1$. At $x = a$ the unit step function is undefined just as vertical shear is undefined directly beneath a concentrated load. The use of the angle brackets $\langle\,\rangle$ is extended to other cases involving powers of the unit step function and the ordinary function $(x-a)^n$. Thus the quantity $(x-a)^n\langle x-a\rangle^0$ is shortened to $\langle x-a\rangle^n$ and again is given a value of zero if $x < a$ and is $(x-a)^n$ if $x > a$.

In addition to the usual concentrated vertical loads, concentrated couples, and distributed loads, Table 8.1 also presents cases where the loading is essentially a development of reactions due to deformations created within the span. These cases include the concentrated angular displacement, concentrated transverse deflection, and linear temperature differential across the beam from top to bottom. *In all three cases it can be assumed that initially the beam was deformed but free of stress and then is forced to conform to the end conditions*. (In many cases no forces are created, and the formulas give only the deformed shape.)

Hetényi (Ref. 29) discusses in detail the Maclaurin series form of the equations used in this article and suggests (Ref. 53) that the deformation type of loading might be useful in solving beam problems. Thomson (Ref. 65) describes the use of the unit step function in the determination of beam deflections. By superposition, the formulas can be made to apply to almost any type of loading and support. The use of the tabulated and fundamental formulas given in this article is illustrated in the following examples.

EXAMPLES

1. For a beam supported and loaded as shown in Fig. 8.2, it is required to determine the maximum tensile stress, maximum shear stress, and maximum compressive stress, assuming, first, that the beam is made of wood with section as shown in Fig. 8.2(*a*) second, that the beam is made of wood with section as shown in Fig. 8.2(*b*); and third that the beam is a 4-in, 7.7-lb steel I-beam.

Solution. By using the equations of equilibrium (Sec. 4.1), the left and right reactions are found to be 900 and 1500 lb, respectively. The shear and moment equations are therefore

$$(x = 0 \text{ to } x = 160): \quad V = 900 - 12x$$
$$M = 900x - 12x(\tfrac{1}{2}x)$$

$$(x = 160 \text{ to } x = 200): \quad V = 900 - 12x + 1500$$
$$M = 900x - 12x(\tfrac{1}{2}x) + 1500(x - 160)$$

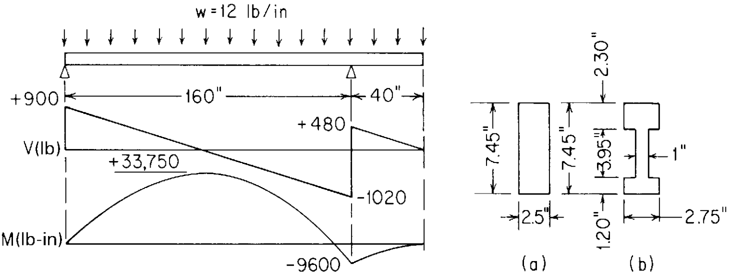

Figure 8.2

Using the step function described previously, these equations can be reduced to

$$V = 900 - 12x + 1500\langle x - 160\rangle^0$$

$$M = 900x - 6x^2 + 1500\langle x - 160\rangle$$

These equations are plotted, giving the shear and moment diagrams shown in Fig. 8.2. The maximum positive moment evidently occurs between the supports; the exact location is found by setting the first shear equation equal to zero and solving for x, which gives $x = 75$ in. Substitution of this value of x in the first moment equation gives $M = 33{,}750$ lb-in. The maximum negative moment occurs at the right support where the shear diagram again passes through zero and is 9600 lb-in.

The results obtained so far are independent of the cross section of the beam. The stresses will now be calculated for each of the sections (*a*), (*b*), and (*c*).

(*a*) For the rectangular section: $I = \frac{1}{12}bd^3 = 86.2\text{ in}^4$; $I/c = \frac{1}{6}bd^2 = 23.1\text{ in}^3$; and $A = bd = 18.60\text{ in}^2$. Therefore

$$\sigma_{\max} = \frac{M_{\max}}{I/c} = \frac{33{,}750}{23.1} = 1460\text{ lb/in}^2 \quad \text{[by Eq. (8.1-12)]}$$

This stress occurs at $x = 75$ in and is tension in the bottom fibers of the beam and compression in the top. The maximum *transverse* shear stress is[†]

$$\tau_{\max} = \frac{3}{2}\frac{V_{\max}}{A} = \frac{3}{2}\frac{1020}{18.60} = 82\text{ lb/in}^2 \qquad \text{[by Eq. (8.1-13)]}$$

which is the horizontal and vertical shear stress at the neutral axis of the section just to the left of the right support.

(*b*) For the routed section it is found (Appendix A) that the neutral axis is 4 in from the base of the section and $I = 82.6\text{ in}^4$. The maximum transverse shear stress on a horizontal plane occurs at the neutral axis since b is as small there as anywhere, and so in Eq. (8.1-2) the product $A'\bar{y}$ represents the statical

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[†] The actual maximum shear stress for this example is found from a stress transformation of the maximum bending stress. Thus, at the outer fibers of the beam at $x = 75$, $\tau_{\max} = \frac{1}{2}\sigma_{\max} = \frac{1}{2}1460 = 730\text{ lb/in}^2$.

moment about the neutral axis of all that part of the section above the neutral axis. Taking the moment of the flange and web portions separately, we find $A'\bar{y} = (2.75)(2.3)(2.30) + (1)(1.15)(0.575) = 15.2\ \text{in}^3$. Also, $b = 1.00$ in.

Since the section is not symmetrical about the neutral axis, the fiber stresses will be calculated at the section of maximum positive moment and at the section of maximum negative moment. We have

$$\text{At } x = 75 \text{ in:} \quad \sigma = \begin{cases} \dfrac{(33{,}750)(4)}{82.6} = 1630\ \text{lb/in}^2 & \text{(tension in bottom fiber)} \\ \dfrac{(33{,}750)(3.45)}{82.6} = 1410\ \text{lb/in}^2 & \text{(compression in top fiber)} \end{cases}$$

$$\text{At } x = 160 \text{ in:} \quad \sigma = \begin{cases} \dfrac{(9600)(4)}{82.6} = 456\ \text{lb/in}^2 & \text{(compression in bottom fiber)} \\ \dfrac{(9600)(3.45)}{82.6} = 400\ \text{lb/in}^2 & \text{(tension in top fibers)} \end{cases}$$

It is seen that for this beam the maximum fiber stresses in both tension and compression occur at the section of maximum positive bending moment. The maximum transverse shear stress is

$$(\tau_{\max})_V = \frac{(1020)(15.2)}{(82.6)(1)} = 188\ \text{lb/in}^2 \qquad \text{[by Eq. (8.1-2)]}$$

This is the maximum shear stress on a horizontal plane and occurs at the neutral axis of the section just to the left of the right support. The actual maximum shear stress is $\tau_{\max} = \frac{1}{2}\sigma_{\max} = \frac{1}{2} \times 1630 = 815\ \text{lb/in}^2$.

(*c*) For the steel I-beam, the structural steel handbook gives $I/c = 3.00\ \text{in}^3$ and $t = 0.190$ in. Therefore

$$\sigma_{\max} = \frac{33{,}750}{3} = 11{,}250\ \text{lb/in}^2$$

This stress occurs at $x = 75$ in, where the beam is subject to tension in the bottom fibers and compression in the top. The maximum transverse shear stress is approximated by

$$(\tau_{\max})_V \approx \frac{1020}{(4)(0.19)} = 1340\ \text{lb/in}^2$$

Although this method of calculating τ (where the shear force is assumed to be carried entirely by the web) is only approximate, it is usually sufficiently accurate to show whether or not this shear stress is important. If it indicates that this shear stress may govern, then the stress at the neutral axis may be calculated by Eq. (8.1-2). For standard I-beams, the allowable vertical shear is given by the structural-steel handbooks, making computation unnecessary.

2. The beam is shown in Fig. 8.3 has a rectangular section 2 in wide and 4 in deep and is made of spruce, where $E = 1{,}300{,}000\ \text{lb/in}^2$. It is required to determine the deflection of the left end.

Solution. The solution will be effected first by superposition, using the formulas of Table 8.1. The deflection y of the left end is the sum of the

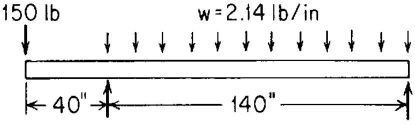


Figure 8.3

deflection y_1 produced by the distributed load and the deflection y_2 produced by the concentrated load. Each of these is computed independently of the other and added using superposition. Thus

$$y_1 = -40\theta = (-40)\left[-\frac{1}{24}\frac{(2.14)(140^3)}{EI}\right] = +\frac{9{,}800{,}000}{EI}$$

by formula for θ at A, case 2e, where $a = 0$, $l = 140$ in and $w_a = w_l = 2.14$ lb/in. y_2 is calculated as the sum of the deflection the 150-lb load would produce if the beam were *fixed* at the left support and the deflection produced by the fact that it actually assumes a slope there. The first part of the deflection is given by the formula for max y (case 1a), and the second part is found by multiplying the overhang (40 in) by the slope produced at the left end of the 140-in span by a counterclockwise couple equal to $150(40) = 6000$ lb-in applied at that point (formula for θ at A, case 3e, where $a = 0$).

$$y_2 = -\frac{1}{3}\frac{(150)(40^3)}{EI} + (-40)\left[-\frac{1}{3}\frac{(-6000)(140)}{EI}\right] = -\frac{14{,}400{,}000}{EI}$$

Adding algebraically, the deflection of the left end is

$$y = y_1 + y_2 = -\frac{4{,}600{,}000}{EI} = -0.33 \text{ in} \qquad \text{(deflection is downward)}$$

The solution of this problem can also be effected readily by using Eq. (8.1-6). The reaction at the left support due to the actual loads is 343 lb and the reaction due to a unit load acting down at the left support is 1.286. If x is measured from the extreme left end of the beam

$$M = -150x + 343\langle x-40\rangle - 2.14\frac{\langle x-40\rangle^2}{2} \quad \text{and} \quad m = -x + 1.286\langle x-40\rangle$$

Simplifying the equations, we have

$$\begin{aligned} y &= \int \frac{Mm}{EI}\,dx \\ &= \frac{1}{EI}\left[\int_0^{40}(-150x)(-x)dx + \int_{40}^{180}(-1.071x^2 + 278.8x - 15{,}430)(0.286x - 51.6)dx\right] \\ &= +0.33 \text{ in} \end{aligned}$$

(Here the plus sign means that y is in the direction of the assumed unit load, i.e., downward.)

This second solution involves much more labor than the first, and the calculations must be carried out with great accuracy to avoid the possibility of making a large error in the final result.

3. A cast iron beam is simply supported at the left end and fixed at the right end on a span of 100 cm. The cross section is 4 cm wide and 6 cm deep ($I = 72\ \text{cm}^4$). The modulus of elasticity of cast iron is $10^7\ \text{N/cm}^2$, and the coefficient of expansion is 0.000012 cm/cm/°C. It is desired to determine the locations and magnitudes of the maximum vertical deflection and the maximum bending stress in the beam. The loading consists of a uniformly increasing distributed load starting at 0 at 40 cm from the left end and increasing to 200 N/cm at the right end. In addition, the beam, which was originally 20°C, is heated to 50°C on the top and 100°C on the bottom with the temperature assumed to vary linearly from top to bottom.

Solution. Superimposing cases 2c and 6c of Table 8.1, the following reactions are determined. (*Note:* For case 2c, $w_a = 0$, $a = 40$ cm, $w_l = 200$ N/cm, and $l = 100$ cm; for case 6c, $T_1 = 50°\text{C}$, $T_2 = 100°\text{C}$, $\gamma = 0.000012$ cm/cm/°C, $t = 6$ cm, and $a = 0$.)

$$R_A = \frac{200(100-40)^3(4\cdot 100+40)}{40(100^3)} - \frac{3(10^7)(72)(0.000012)(100-50)}{2(6)(100)} = -604.8\ \text{N}$$

$$M_A = 0 \qquad y_A = 0$$

$$\theta_A = \frac{-200(100-40)^3(2\cdot 100+3\cdot 40)}{240(10^7)(72)(100)} + \frac{0.000012(100-50)(-100)}{4(6)}$$
$$= -0.0033\ \text{rad}$$

Therefore

$$y = -0.0033x - \frac{604.8x^3}{6EI} - \frac{200\langle x-40\rangle^5}{(100-40)(120)EI} + \frac{0.000012(100-50)x^2}{2(6)}$$
$$= -0.0033x - 1.4(10^{-7})x^3 - 3.86(10^{-11})\langle x-40\rangle^5 + 5.0(10^{-5})x^2$$

and

$$\frac{dy}{dx} = -0.0033 - 4.2(10^{-7})x^2 - 19.3(10^{-11})\langle x-40\rangle^4 + 10(10^{-5})x$$

The maximum deflection will occur at a position x_1 where the slope dy/dx is zero. At this time an assumption must be made as to which span, $x_1 < 40$ or $x_1 > 40$, will contain the maximum deflection. Assuming that x_1 is less than 40 cm and setting the slope equal to zero,

$$0 = -0.0033 - 4.2(10^{-7})x_1^2 + 10(10^{-5})x_1$$

Of the two solutions for x_1, 39.7 and 198 cm, only the value of 39.7 cm is valid since it is less than 40 cm. Substituting $x = 39.7$ cm into the deflection equation gives the maximum deflection of −0.061 cm. Similarly,

$$M = -604.8x - \frac{200}{6(100-40)}\langle x-40\rangle^3$$

which has a maximum negative value where x is a maximum, i.e., at the right end:

$$M_{\max} = -604.8(100) - \frac{200}{6(100-40)}(100-40)^3 = -180{,}480 \text{ N-cm}$$
$$\sigma_{\max} = \frac{Mc}{I} = \frac{180{,}480(3)}{72} = 7520 \text{ N/cm}^2$$

4. The cast iron beam in Example 3 is to be simply supported at both ends and carry a concentrated load of 10,000 N at a point 30 cm from the right end. It is desired to determine the relative displacement of the lower edges of the end section. For this example, case 1e can be used with $a = 70$ cm:

$$R_A = \frac{10{,}000(100-70)}{100} = 3000 \text{ N} \qquad M_A = 0$$
$$\theta_A = \frac{-10{,}000(70)}{6(10^7)(72)(100)}(200-70)(100-70) = -0.00632 \text{ rad}, \qquad y_A = 0$$
$$\theta_B = \frac{10{,}000(70)}{6(10^7)(72)(100)}(100^2 - 70^2) = 0.00826 \text{ rad}$$

Then

$$\begin{aligned}\frac{dy}{dx} &= -0.00632 + 2.083(10^{-6})x^2 - \frac{10{,}000}{2(10^7)(72)}\langle x-70\rangle^2 \\ &= -0.00632 + 2.083(10^{-6})x^2 - 6.94(10^{-6})\langle x-70\rangle^2\end{aligned}$$

The shortening of the neutral surface of the beam is given in Eq. (8.1-14) to be

$$\begin{aligned}\Delta l &= \frac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx \\ &= \frac{1}{2}\int_0^{70} [-0.00632 + 2.083(10^{-6})x^2]^2\, dx \\ &\quad + \frac{1}{2}\int_{70}^{100} [-0.04033 + 9.716(10^{-4})x - 4.857(10^{-6})x^2]^2\, dx\end{aligned}$$

or

$$\Delta l = 0.00135 \text{ cm} \qquad \text{(a shortening)}$$

In addition to the shortening of the neutral surface, the lower edge of the left end moves to the left by an amount $\theta_A c$ or $0.00632(3) = 0.01896$ cm. Similarly, the lower edge of the right end moves to the right by an amount $\theta_B c$ or $0.00826(3) = 0.02478$ cm. Evaluating the motion of the lower edges in this manner is equivalent to solving Eq. (8.1-15) for the total strain in the lower fibers of the beam.

The total relative motion of the lower edges of the end sections is therefore a moving apart by an amount $0.01896 + 0.02478 - 0.00135 = 0.0424$ cm.

8.2 Composite Beams and Bimetallic Strips

Beams that are constructed of more than one material can be treated by using an equivalent width technique if the maximum stresses in each of the several materials remain within the proportional limit. An equivalent cross section is developed in which the width of each component parallel to the principal axis of bending is increased in the same proportion that the modulus of elasticity of that component makes with the modulus of the assumed material of the equivalent beam.

EXAMPLE

The beam cross section shown in Fig. 8.4(*a*) is composed of three portions of equal width and depth. The top portion is made of aluminum for which $E_A = 10 \cdot 10^6$ lb/in^2; the center is made of brass for which $E_B = 15 \cdot 10^6$ lb/in^2; and the bottom is made of steel for which $E_S = 30 \cdot 10^6$ lb/in^2. Figure 8.4(*b*) shows the equivalent cross section, which is assumed to be made of aluminum. For this equivalent cross section the centroid must be located and the moment of inertia determined for the centroidal axis.

Solution

$$\bar{y} = \frac{3(2)(5) + 4.5(2)(3) + 9(2)(1)}{6 + 9 + 18} = 2.27 \text{ in}$$

$$\begin{aligned} I_x &= \frac{3(2^3)}{12} + 6(5 - 2.27)^2 + \frac{4.5(2^3)}{12} + 9(3 - 2.27)^2 + \frac{9(2^3)}{12} + 18(2.27 - 1)^2 \\ &= 89.5 \text{ in}^4 \end{aligned}$$

The equivalent stiffness *EI* of this beam is therefore $10 \cdot 10^6(89.5)$, or $895 \cdot 10^6$ lb-in^2.

A flexure stress computed by $\sigma = Mc/I_x$ will give a stress in the equivalent beam which can thereby be converted into the stress in the actual composite beam by multiplying by the modulus ratio. If a bending moment of 300,000 lb-in were applied to a beam with the cross section shown, the stress at the top surface of the equivalent beam would be $\sigma = 300{,}000(6 - 2.27)/89.5$, or 12,500 lb/in^2. Since the material at the top is the same in both the actual and equivalent beams, this is also the maximum stress in the aluminum portion of the actual beam. The stress at the bottom of the equivalent beam would be $\sigma = 300{,}000(2.27)/89.5 = 7620$ lb/in^2. Multiplying the stress by the

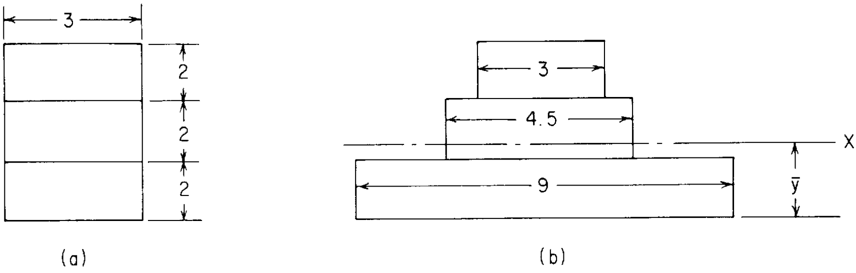


Figure 8.4

modulus ratio, the actual stress at the bottom of the steel portion of the beam would be $\sigma = 7620(30/10) = 22{,}900\ \text{lb/in}^2$.

Bimetallic strips are widely used in instruments to sense or control temperatures. The following formula gives the equivalent properties of the strip for which the cross section is shown in Fig. 8.5:

$$\text{Equivalent } EI = \frac{wt_b^3 t_a E_b E_a}{12(t_a E_a + t_b E_b)} K_1 \tag{8.2-1}$$

or

$$K_1 = 4 + 6\frac{t_a}{t_b} + 4\left(\frac{t_a}{t_b}\right)^2 + \frac{E_a}{E_b}\left(\frac{t_a}{t_b}\right)^3 + \frac{E_b}{E_a}\frac{t_b}{t_a} \tag{8.2-2}$$

All the formulas in Table 8.1, cases 1 to 5, can be applied to the bimetallic beam by using this equivalent value of EI. Since a bimetallic strip is designed to deform when its temperature differs from T_o, the temperature at which the strip is straight, Table 8.1, case 6, can be used to solve for reaction forces and moments as well as deformations of the bimetallic strip under a uniform temperature T. To do this, the term $\gamma(T_2 - T_1)/t$ is replaced by the term $6(\gamma_b - \gamma_a)(T - T_o)(t_a + t_b)/(t_b^2 K_1)$ and EI is replaced by the equivalent EI given by Eq. (8.2-1).

After the moments and deformations have been determined, the flexure stresses can be computed. The stresses due to the bending moments caused by the restraints and any applied loads are given by the following expressions:

In the top surface of material a:

$$\sigma = \frac{-6M}{wt_b^2 K_1}\left(2 + \frac{t_b}{t_a} + \frac{E_a t_a}{E_b t_b}\right) \tag{8.2-3}$$

In the bottom surface of material b:

$$\sigma = \frac{6M}{wt_b^2 K_1}\left(2 + \frac{t_a}{t_b} + \frac{E_b t_b}{E_a t_a}\right) \tag{8.2-4}$$

If there are no restraints imposed, the distortion of a bimetallic strip due to a temperature change is accompanied by flexure stresses in the two materials. This differs from a beam made of a single material which deforms free of stress when subjected to a linear temperature

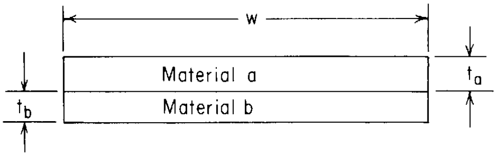


Figure 8.5

variation through the thickness if there are no restraints. Therefore the following stresses must be added algebraically to the stresses caused by the bending moments, if any:

In the top surface of material a:

$$\sigma = \frac{-(\gamma_b - \gamma_a)(T - T_o)E_a}{K_1}\left[3\frac{t_a}{t_b} + 2\left(\frac{t_a}{t_b}\right)^2 - \frac{E_b t_b}{E_a t_a}\right] \tag{8.2-5}$$

In the bottom surface of material b:

$$\sigma = \frac{(\gamma_b - \gamma_a)(T - T_o)E_b}{K_1}\left[3\frac{t_a}{t_b} + 2 - \frac{E_a}{E_b}\left(\frac{t_a}{t_b}\right)^3\right] \tag{8.2-6}$$

EXAMPLE

A bimetallic strip is made by bonding a piece of titanium alloy $\frac{1}{4}$ in wide by 0.030 in thick to a piece of stainless steel $\frac{1}{4}$ in wide by 0.060 in thick. For titanium, $E = 17 \cdot 10^6$ lb/in^2 and $\gamma = 5.7 \cdot 10^{-6}$ in/in/°F; for stainless steel, $E = 28 \cdot 10^6$ lb/in^2 and $\gamma = 9.6 \cdot 10^{-6}$ in/in/°F. It is desired to find the length of bimetal required to develop a reaction force of 5 oz at a simply supported left end when the right end is fixed and the temperature is raised 50°F; also the maximum stresses must be determined.

Solution. First find the value of K_1 from Eq. (8.2-2) and then evaluate the equivalent stiffness from Eq. (8.2-1):

$$K_1 = 4 + 6\frac{0.03}{0.06} + 4\left(\frac{0.03}{0.06}\right)^2 + \frac{17}{28}\left(\frac{0.03}{0.06}\right)^3 + \frac{28}{17}\frac{0.06}{0.03} = 11.37$$

$$\text{Equivalent } EI = \frac{0.25(0.06^3)(0.03)(28 \cdot 10^6)(17 \cdot 10^6)}{12[0.03(17 \cdot 10^6) + 0.06(28 \cdot 10^6)]}11.37 = 333 \text{ lb-in}^2$$

Under a temperature rise over the entire length, the bimetallic strip curves just as a single strip would curve under a temperature differential. To use case 6c in Table 8.1, the equivalent to $\gamma(T_2 - T_1)/t$ must be found. This equivalent value is given by

$$\frac{6(9.6 \cdot 10^{-6} - 5.7 \cdot 10^{-6})(50)(0.03 + 0.06)}{(0.06^2)(11.37)} = 0.00257 \text{ in}^{-1}$$

The expression for R_A can now be obtained from case 6c in Table 8.1 and, noting that $a = 0$, the value of the length l can be determined:

$$R_A = \frac{-3(l^2 - a^2)}{2l^3}EI\frac{\gamma}{t}(T_2 - T_1) = \frac{-3}{2l}(333)(0.00257) = \frac{-5}{16} \text{ lb}$$

Therefore $l = 4.11$ in.

The maximum bending moment is found at the fixed end and is equal to $R_A l$:

$$\max M = -\tfrac{5}{16}(4.11) = -1.285 \text{ lb-in}$$

Combining Eqs. (8.2-3) and (8.2-5), the flexure stress on the top of the titanium is

$$\begin{aligned}\sigma &= \frac{-6(-1.285)}{0.25(0.06)^2(11.37)}\left(2+\frac{0.06}{0.03}+\frac{17}{28}\frac{0.03}{0.06}\right)\\ &\quad -\frac{(9.6\cdot 10^{-6}-5.7\cdot 10^{-6})(50)(17\cdot 10^{6})}{11.37}\left[3\frac{0.03}{0.06}+2\left(\frac{0.03}{0.06}\right)^2-\frac{28}{17}\frac{0.06}{0.03}\right]\\ &= 3242+378=3620\ \text{lb/in}^2\end{aligned}$$

Likewise, the flexure stress on the bottom of the stainless steeel is

$$\begin{aligned}\sigma &= \frac{6(-1.285)}{0.25(0.06^2)(11.37)}\left[2+\frac{0.03}{0.06}+\frac{28}{17}\frac{0.06}{0.03}\right]\\ &\quad +\frac{(9.6\cdot 10^{-6}-5.7\cdot 10^{-6})(50)(28\cdot 10^{6})}{11.37}\left[3\frac{0.03}{0.06}+2-\frac{17}{28}\left(\frac{0.03}{0.06}\right)^3\right]\\ &= -4365+1644=-2720\ \text{lb/in}^2\end{aligned}$$

8.3 Three-Moment Equation

The *three-moment equation*, which expresses the relationship between the bending moments found at three *consecutive* supports in a *continuous* beam, can be readily derived for any loading shown in Table 8.1. This is accomplished by taking any two consecutive spans and evaluating the slope for each span at the end where the two spans join. These slopes, which are expressed in terms of the three moments and the loads on the spans, are then equated and the equation reduced to its usual form.

EXAMPLE

Consider two contiguous spans loaded as shown in Fig. 8.6. In addition to the loading shown, it is known that the left end of span 1 had settled an amount $y_2 - y_1$ relative to the right end of the span, and similarly that the left end of span 2 has settled an amount $y_3 - y_2$ relative to the right end. (Note that y_1, y_2, and y_3 are considered positive upward as usual.) The individual spans with their loadings are shown in Fig. 8.7(*a*,*b*). Determine the relationship between the applied loads and the moment at the intermediate support.

Solution. Using cases 2e and 3e from Table 8.1 and noting the relative deflections mentioned above, the expression for the slope at the right end of

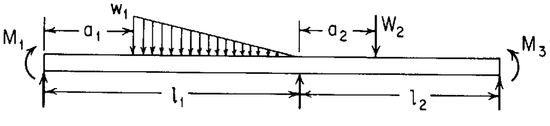


Figure 8.6

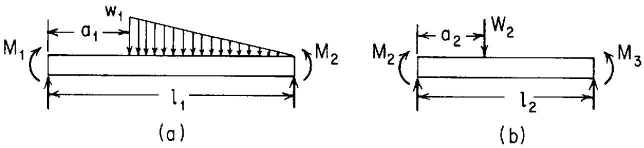


Figure 8.7

span 1 is

$$\theta_2 = \frac{w_1(l_1^2 - a_1^2)^2}{24E_1I_1l_1} - \frac{w_1(l_1 - a_1)^2}{360E_1I_1l_1}(8l_1^2 + 9a_1l_1 + 3a_1^2) + \frac{M_1l_1^2}{6E_1I_1l_1} + \frac{-M_2(l_1^2 - 3a_1^2)}{6E_1I_1l_1} + \frac{y_2 - y_1}{l_1}$$

Similarly, using cases 1e and 3e from Table 8.1, the expression for the slope at the left end of span 2 is

$$\theta_2 = \frac{-W_2a_2}{6E_2I_2l_2}(2l_2 - a_2)(l_2 - a_2) - \frac{M_2}{6E_2I_2l_2}(2l_2^2) - \frac{-M_3}{6E_2I_2l_2}(2l_2^2 - 6l_2^2 + 3l_2^2) + \frac{y_3 - y_2}{l_2}$$

Equating these slopes gives

$$\frac{M_1l_1}{6E_1I_1} + \frac{M_2l_1}{3E_1I_1} + \frac{M_2l_2}{3E_2I_2} + \frac{M_3l_2}{6E_2I_2} = \frac{-w_1(l_1 - a_1)^2}{360E_1I_1l_1}(7l_1^2 + 21a_1l_1 + 12a_1^2) - \frac{y_2 - y_1}{l_1} - \frac{W_2a_2}{6E_2I_2l_2}(2l_2 - a_2)(l_2 - a_2) + \frac{y_3 - y_2}{l_2}$$

If M_1 and M_3 are known, this expression can be solved for M_2: if not, similar expressions for the adjacent spans must be written and the set of equations solved for the moments.

The three-moment equation can also be applied to beams carrying axial tension or compression in addition to transverse loading. The procedure is exactly the same as that described above except the slope formulas to be used are those given in Tables 8.8 and 8.9.

8.4 Rigid Frames

By superposition and the matching of slopes and deflections, the formulas in Table 8.1 can be used to solve for the indeterminate reactions in rigid frames or to determine the deformations where the support conditions permit such deformations to take place. The term *rigid* in this section simply means that any deformations are small enough to have negligible effect on bending moments.

In Table 8.2 formulas are given for the indeterminate reactions and end deformations for rigid frames consisting of three members. Only in-plane deformations and only those due to bending moments have been included in the expressions found in this table. Since deformations due to transverse shear and axial loading are not included, the results are limited to those frames made up of members which are long in proportion to their depths (see assumptions 6 to 8 in Sec. 8.1). Each member must be straight and of uniform cross section having a principal axis lying in the plane of the frame. The elastic stability of the frame, either in the plane of the frame or out of this plane, has not been treated in developing Table 8.2. The effects of axial load on the bending deformations, as documented in Tables 8.7–8.9, have not been considered. A final check on a solution must verify that indeed these effects can be neglected. Very extensive compilations of formulas for rigid frames are available, notably those of Kleinlogel (Ref. 56) and Leontovich (Ref. 57).

While Table 8.2 is obviously designed for frames with three members where the vertical members both lie on the same side of the horizontal member, its use is not limited to this configuration. One can set the lengths of either of the vertical members, members 1 and 2, equal to zero and solve for reactions and deformations of two-member frames. The length of the horizontal member, member 3, should not be set to zero for two reasons: (1) It does not generally represent a real problem; and (2) the lengths of members 1 and 2 are assumed not to change, and this adds a restraint to member 3 that would force it to have zero slope if its length was very short. Another very useful application of the expressions in Table 8.2 is to apply them to frames where one of the two vertical members lies on the opposite side of the horizontal member. Instead of forming a U-shape in the side view, it forms a Z-shape. To do this one must change the signs of three variables associated with the reversed member: (1) the sign of the length of the reversed member, (2) the sign of the distance a which locates any load on the reversed member, and (3) the sign of the product EI of the reversed member. All the reactions and end-point deflections retain their directions as given in the figures in Table 8.2; that is, if member 1 is reversed and extends upward from the left end of member 3, H_A now acts at the upper end of member 1 and is positive to the left as is δ_{HA}. Example 3 illustrates this application as well as showing how the results of using Tables 8.1 and 8.2 together can be used to determine the deformations anywhere in a given frame.

When the number of members is large, as in a framed building, a relaxation method such as moment distribution might be used or a digital computer could be programmed to solve the large number of equations. In all rigid frames, corner or knee design is important; much information and experimental data relating to this problem are to be found in the reports published by the Fritz Engineering Laboratories of Lehigh University. The frames given in Table 8.2 are assumed to have rigid corners; however, corrections can be made easily once the rigidity of a given corner design is known by making use of the concentrated angular displacement loading with the displacement positioned at the corner. This application is illustrated in Example 2.

EXAMPLES

1. The frame shown in Fig. 8.8(*a*) is fixed at the lower end of the right-hand member and guided at the lower end of the left-hand member in such a way as to prevent any rotation of this end but permitting horizontal motion if any is produced by the loading. The configuration could represent the upper half of the frame shown in Fig. 8.8(*b*); for this frame the material properties and physical dimensions are given as $l_1 = 40$ in, $l_2 = 20$ in, $l_3 = 15$ in, $E_1 = E_2 = E_3 = 30 \cdot 10^6$ lb/in^2, $I_1 = 8$ in^4, $I_2 = 10$ in^4, and $I_3 = 4$ in^4. In addition to the load P of 1000 lb, the frame has been subjected to a temperature rise of 50°F since it was assembled in a stress-free condition. The coefficient of expansion of the material used in all three portions is 0.0000065 in/in/°F.

Solution. An examination of Table 8.2 shows the required end or support conditions in case 7 with the loading cases f and q listed under case 5. For cases 5 to 12 the frame constants are evaluated as follows:

$$C_{HH} = \frac{l_1^3}{3E_1I_1} + \frac{l_1^3 - (l_1 - l_2)^3}{3E_2I_2} + \frac{l_1^2 l_3}{E_3I_3} = \frac{\dfrac{40^3}{3(8)} + \dfrac{40^3 - (40 - 20)^3}{3(10)} + \dfrac{(40^2)(15)}{4}}{30(10^6)}$$

$$= \frac{2666.7 + 1866.7 + 6000}{30(10^6)} = 0.0003511 \text{ in/lb}$$

Similarly

$$C_{HV} = C_{VH} = 0.0000675 \text{ in/lb}$$
$$C_{HM} = C_{MH} = 0.00001033 \text{ lb}^{-1}$$
$$C_{VV} = 0.0000244 \text{ in/lb}$$
$$C_{VM} = C_{MV} = 0.00000194 \text{ lb}^{-1}$$
$$C_{MM} = 0.000000359 \text{ (lb-in)}^{-1}$$

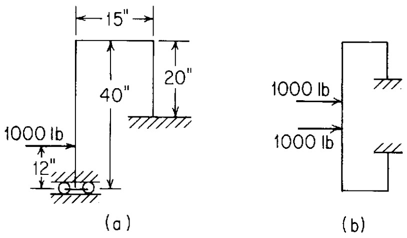


Figure 8.8

For case 7f

$$
\begin{aligned}
LF_H &= W\left(C_{HH} - aC_{HM} + \frac{a^3}{6E_1I_1}\right)\\
&= 1000\left[0.0003511 - 12(0.00001033) + \frac{12^3}{6(30 \cdot 10^6)(8)}\right]\\
&= 1000(0.0003511 - 0.000124 + 0.0000012) = 0.2282 \text{ in}
\end{aligned}
$$

Similarly

$$LF_V = 0.0442 \text{ in} \quad \text{and} \quad LF_M = 0.00632 \text{ rad}$$

For case 7q

$$LF_H = -(T - T_o)\gamma_3 l_3 = -50(0.0000065)(15) = -0.004875 \text{ in}$$

$$LF_V = 0.0065 \text{ in} \quad \text{and} \quad LF_M = 0 \text{ rad}$$

For the combined loading

$$
\begin{aligned}
LF_H &= 0.2282 - 0.004875 = 0.2233 \text{ in}\\
LF_V &= 0.0507 \text{ in}\\
LF_M &= 0.00632 \text{ in}
\end{aligned}
$$

Now the left end force, moment, and displacement can be evaluated:

$$
\begin{aligned}
V_A &= \frac{LF_V C_{MM} - LF_M C_{VM}}{C_{VV}C_{MM} - C_{VM}^2} = \frac{0.0507(0.359 \cdot 10^{-6}) - 0.00632(1.94 \cdot 10^{-6})}{(24.4 \cdot 10^{-6})(0.359 \cdot 10^{-6}) - (1.94 \cdot 10^{-6})^2}\\
&= 1189 \text{ lb}\\
M_A &= 11{,}179 \text{ lb-in}\\
\delta_{HA} &- 0.0274 \text{ in}
\end{aligned}
$$

Figure 8.9 shows the moment diagram for the entire frame.

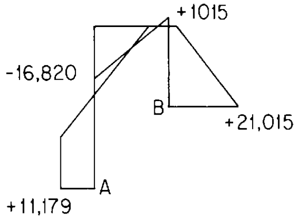


Figure 8.9 (units are in lb-in)

2. If the joint at the top of the left vertical member in Example 1 had not been rigid but instead had been found to be deformable by 10^{-7} rad for every inch-pound of bending moment applied to it, the solution can be modified as follows.

Solution. The bending moment as the corner in question would be given by $M_A - 28(1000)$, and so the corner rotation would be $10^{-7}(M_A - 28{,}000)$ rad in a direction opposite to that shown by θ_o in case $5a$. Note that the position of θ_o is at the corner, and so a would be 40 in. Therefore, the following load terms due to the corner deformation can be added to the previously determined load terms:

$$LF_H = -10^{-7}(M_A - 28{,}000)(40)$$
$$LF_V = 0$$
$$LF_M = -10^{-7}(M_A - 28{,}000)$$

Thus the resultant load terms become

$$LF_H = 0.2233 - 4 \cdot 10^{-6}M_A + 0.112 = 0.3353 - 4 \cdot 10^{-6}M_A$$
$$LF_V = 0.0507 \text{ in}$$
$$LF_M = 0.00632 - 10^{-7}M_A + 0.0028 = 0.00912 - 10^{-7}M_A$$

Again, the left end force, moment and displacement are evaluated:

$$V_A = \frac{0.0507(0.359 \cdot 10^{-6}) - (0.00912 - 10^{-7}M_A)(1.94 \cdot 10^{-6})}{4.996 \cdot 10^{-12}}$$
$$= 100 + 0.0388M_A$$
$$M_A = \frac{(0.00912 - 10^{-7}M_A)(24.4 \cdot 10^{-6}) - 0.0507(1.94 \cdot 10^{-6})}{4.996 \cdot 10^{-12}}$$
$$= 24{,}800 - 0.488M_A$$

or

$$M_A = 16{,}670 \text{ lb-in}$$
$$\delta_{HA} = -0.0460 \text{ in}$$
$$V_A = 747 \text{ lb}$$

3. Find the reactions and deformations at the four positions A to D in the pinned-end frame shown in Fig. 8.10. All lengths are given in millimeters, $M_o = 2000$ N-mm, and all members are made of aluminum for which $E = 7(10^4)\text{N/mm}^2$ with a constant cross section for which $I = 100 \text{ mm}^4$.

Solution. Case 1h of Table 8.2 covers this combination of loading and end supports, if, due to the upward reach of member 1, appropriate negative values are used. The need for negative values is described in the introduction to Sec.

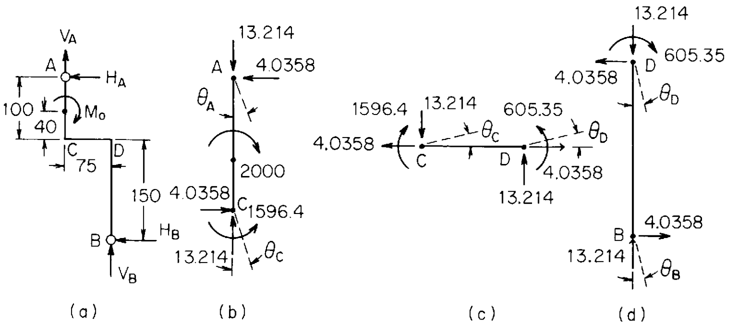


Figure 8.10

8.4. Substitute the following material properties and physical dimensions into case 1h: $l_1 = -100$ mm, $l_2 = 150$ mm, $l_3 = 75$ mm, $a = -(100 - 40) = -60$ mm, $E_1 I_1 = -7(10^6)$ N-mm^2, and $E_2 I_2 = E_3 I_3 = 7(10^6)$ N-mm^2. Using these data gives the frame and load constants as

$$\begin{aligned} &A_{HH} = 0.2708 \text{ in/lb}, \qquad A_{HM} = A_{MH} = -0.0008036 \text{ lb}^{-1} \\ &A_{MM} = 0.00001786 \text{ (lb-in)}^{-1}, \qquad LP_H = 0.0005464(2000) = 1.0928 \text{ in} \\ &LP_M = -0.000009286(2000) = -0.01857 \text{ rad} \end{aligned}$$

Using these frames and load terms in case 1, the pinned-end case, the reaction and deformations are found to be

$$\delta_{HA} = 0 \qquad M_A = 0 \qquad H_A = \frac{LP_H}{A_{HH}} = 4.0352 \text{ N} \qquad \text{(to left)}$$

and

$$\psi_A = A_{MH} H_A - LP_M = 0.01533 \text{ rad} \qquad \text{(clockwise)}$$

Applying the three independent statics equations to Fig. 8.10(*a*) results in

$$V_A = -13.214 \text{ N} \qquad H_B = -4.0358 \text{ N} \quad \text{and} \quad V_B = 13.214 \text{ N}$$

Now treat each of the three members as separate bodies in equilibrium, and find the deformations as pinned-end beams using equations from Table 8.1 as appropriate.

For member 1 as shown in Fig. 8.10(*b*): Using case 3e twice, once for each of the two moment loadings, gives

$$\begin{aligned} \theta_A &= \frac{-2000}{6(7 \cdot 10^6)(100)}[2(100)^2 - 6(60)(100) + 3(60)^2] \\ &\quad + \frac{1596.4}{6(7 \cdot 10^6)(100)}[2(100)^2 - 6(100)(100) + 3(100)^2] \\ &= -0.001325 \text{ rad} \\ \theta_C &= \frac{2000}{6(7 \cdot 10^6)(100)}[(100)^2 - 3(60)^2] - \frac{1596.4}{6(7 \cdot 10^6)(100)}[(100)^2 - 3(100)^2] \\ &= 0.007221 \text{ rad} \end{aligned}$$

To obtain the angle $\psi_A = 0.01533$ rad at position A, member 1 must be given an additional rigid-body clockwise rotation of $0.01533 - 0.001325 = 0.01401$ rad. This rigid-body motion moves position C to the left a distance of 1.401 mm and makes the slope at position C equal to $0.007221 - 0.01401 = -0.006784$ rad (clockwise).

For member 3 as shown in Fig. 8.10(*c*): Again use case 3e from Table 8.1 twice to get

$$\begin{aligned} \theta_C &= \frac{-1596.4}{6(7 \cdot 10^6)(75)}[2(75)^2 - 0 + 0] - \frac{-605.35}{6(7 \cdot 10^6)(75)}[2(75)^2 - 6(75)(75) + 3(75)^2] \\ &= -0.006782 \text{ rad} \\ \theta_D &= \frac{1596.4}{6(7 \cdot 10^6)(75)}[(75)^2 - 0] + \frac{-605.35}{6(7 \cdot 10^6)(75)}[(75)^2 - 3(75)^2] \\ &= 0.005013 \text{ rad} \end{aligned}$$

No rigid-body rotation is necessary for member 3 since the left end has the same slope as the lower end of member 1, which is as it should be.

For member 2 as shown in Fig. 8.10(*d*): Use case 3e from Table 8.1 to get

$$\theta_D = \frac{-605.35}{6(7 \cdot 10^6)(150)}[2(150)^2 - 0 + 0] = -0.004324 \text{ rad}$$

$$\theta_B = \frac{605.35}{6(7 \cdot 10^6)(150)}[(150)^2 - 0] = 0.002162 \text{ rad}$$

To match the slope at the right end of member 3, a rigid-body counterclockwise rotation of $0.005013 + 0.00432 = 0.009337$ rad must be given to member 2. This creates a slope $\psi_B = 0.009337 + 0.002162 = 0.01150$ rad counterclockwise and a horizontal deflection at the top end of $0.009337(150) = 1.401$ mm to the left. This matches the horizontal deflection of the lower end of member 1 as a final check on the calculations.

To verify that the effect of axial load on the bending deformations of the members is negligible, the Euler load on the longest member is found to be more than 100 times the actual load. Using the formulas from Table 8.8 would not produce significantly different results from those in Table 8.1.

8.5 Beams on Elastic Foundations

There are cases in which beams are supported on foundations which develop essentially continuous reactions that are proportional at each position along the beam to the deflection of the beam at that position. This is the reason for the name *elastic foundation*. Solutions are available (Refs. 41 and 42) which consider that the foundation transmits shear forces within the foundation such that the reaction force is not directly proportional to the deflection at a given location but instead is proportional to a function of the deflections near the given location; these solutions are much more difficult to use and are not justified in many cases since the linearity of most foundations is open to question anyway.

It is not necessary, in fact, that a foundation be continuous. If a discontinuous foundation, such as is encountered in the support provided a rail by the cross ties, is found to have at least three concentrated reaction forces in every half-wavelength of the deflected beam, then the solutions provided in this section are adequate.

Table 8.5 provides formulas for the reactions and deflections at the left end of a finite-length beam on an elastic foundation as well as formulas for the shear, moment, slope, and deflection at any point x along the length. The format used in presenting the formulas is designed to facilitate programming for use on a digital computer or programmable calculator.

In theory the equations in Table 8.5 are correct for any finite-length beam or for any finite foundation modulus, but for practical purposes they should not be used when βl exceeds a value of 6 because the roundoff errors that are created where two very nearly equal large numbers are subtracted will make the accuracy of the answer questionable. For this reason, Table 8.6 has been provided. Table 8.6 contains formulas for semi-infinite- and infinite-length beams on elastic foundations. These formulas are of a much simpler form since the far end of the beam is assumed to be far enough away so as to have no effect on the response of the left end to the loading. If $\beta l > 6$ and the load is nearer the left end, this is the case.

Hetényi (Ref. 53) discusses this problem of a beam supported on an elastic foundation extensively and shows how the solutions can be adapted to other elements such as hollow cylinders. Hetényi (Ref. 51) has also developed a series solution for beams supported on elastic foundations in which the stiffness parameters of the beam and foundation are not incorporated in the arguments of trigonometric or hyperbolic functions. He gives tables of coefficients derived for specific boundary conditions from which deformation, moments, or shears can be found at any specific point along the beam. Any degree of accuracy can be obtained by using enough terms in the series.

Tables of numerical values, Tables 8.3 and 8.4 are provided to assist in the solution of the formulas in Table 8.5. Interpolation is possible for values that are included but should be used with caution if it is noted that differences of large and nearly equal numbers are being encountered. A far better method of interpolation for a beam with a single load is to solve the problem twice. For the first solution move the load to the left until $\beta(l-a)$ is a value found in Table 8.3, and for the second solution move the load similarly to the right. A linear interpolation from these solutions should be very accurate.

Presenting the formulas for end reactions and displacements in Table 8.5 in terms of the constants C_i and C_{ai} is advantageous since it permits one to solve directly for loads anywhere on the span. If the loads are at the left end such that $C_i = C_{ai}$, then the formulas can be presented in a simpler form as is done in Ref. 6 of Chap. 13 for cylindrical shells. To facilitate the use of Table 8.5 when a concentrated load, moment, angular rotation, or lateral displacement is at the left end (that is, $a = 0$), the following equations are presented to simplify the numerators:

$$\begin{aligned} C_1C_2 + C_3C_4 &= C_{12}, & 2C_1^2 + C_2C_4 &= 2 + C_{11} \\ C_2C_3 - C_1C_4 &= C_{13}, & C_2^2 - 2C_1C_3 &= C_{14} \\ C_1^2 + C_3^2 &= 1 + C_{11}, & 2C_3^2 - C_2C_4 &= C_{11} \\ C_2^2 + C_4^2 &= 2C_{14}, & 2C_1C_3 + C_4^2 &= C_{14} \end{aligned}$$

EXAMPLES

1. A 6-in, 12.5-lb I-beam 20 ft long is used as a rail for an overhead crane and is in turn being supported every 2 ft of its length by being bolted to the bottom of a 5-in, 10-lb I-beam at midlength. The supporting beams are each 21.5 ft long and are considered to be simply supported at the ends. This is a case of a discontinuous foundation being analyzed as a continuous foundation. It is desired to determine the maximum bending stresses in the 6-in beam as well as in the supporting beams when a load of 1 ton is supported at one end of the crane.

Solution. The spring constant for each supporting beam is $48EI/l^3$, or $(48)(30 \cdot 10^6)(12.1)/(21.5 \cdot 12)^3 = 1013$ lb/in. If this is assumed to be distributed over a 2-ft length of the rail, the equivalent value of $b_o k_o$ is $1.013/24 = 42.2$ lb/in per inch of deflection. Therefore

$$\beta = \left(\frac{b_o k_o}{4EI}\right)^{1/4} = \left[\frac{42.2}{4(30 \cdot 10^6)(21.8)}\right]^{1/4} = 0.01127 \text{ in}^{-1}$$

and

$$\beta l = (0.01127)(240) = 2.70$$

An examination of the deflection of a beam on an elastic foundation shows that it varies cyclicly in amplitude with the sine and cosine of βx. A half-wavelength of this cyclic variation would occur over a span l_1, where $\beta l_1 = \pi$, or $l_1 = \pi/0.01127 = 279$ in. There is no question about there being at least three supporting forces over this length, and so the use of the solution for a continuous foundation is entirely adequate.

Since βl is less than 6, Table 8.5 will be used. Refer to case 1 where both ends are free. It must be pointed out that a simple support refers to a reaction force, developed by a support other than the foundation, which is large enough to prevent any vertical deflection of the end of the beam. From the table we find that $R_A = 0$ and $M_A = 0$; and since the load is at the left end, $a = 0$. When $a = 0$, the C_a terms are equal to the C terms, and so the four terms C_1, C_2, C_3, and C_4 are calculated:

$$C_1 = \cosh \beta l \cos \beta l = 7.47(-0.904) = -6.76$$

$$C_2 = \cosh \beta l \sin \beta l + \sinh \beta l \cos \beta l = 7.47(0.427) + 7.41(-0.904) = -3.50$$

Similarly $C_3 = 3.17$, $C_4 = 9.89$, and $C_{11} = 54.7$. (See Tables 8.3 and 8.4.) Therefore,

$$\theta_A = \frac{2000}{2(30 \cdot 10^6)(21.8)(0.01127^2)} \frac{(-3.50^2) - (2)(3.17)(-6.76)}{54.7} = 0.01216 \text{ rad}$$

$$y_A = \frac{2000}{2(30 \cdot 10^6)(21.8)(0.01127^3)} \frac{(9.89)(-6.76) - (3.17)(-3.50)}{54.7} = -1.092 \text{ in}$$

With the deformations at the left end known, the expression for the bending moment can be written:

$$
\begin{aligned}
M &= -y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 - \frac{W}{2\beta} F_{a2} \\
&= 1.092(2)(30 \cdot 10^6)(21.8)(0.01127^2)F_3 - 0.01216(30 \cdot 10^6)(21.8)(0.01127)F_4 \\
&\quad - \frac{2000}{2(0.01127)} F_{a2} \\
&= 181{,}400F_3 - 89{,}600F_4 - 88{,}700F_{a2}
\end{aligned}
$$

Now substituting the expressions for F_{a2}, F_3, and F_4 gives

$$
\begin{aligned}
M = {} & 181{,}400 \sinh \beta x \sin \beta x - 89{,}600(\cosh \beta x \sin \beta x - \sinh \beta x \cos \beta x) \\
& - 88{,}700(\cosh \beta x \sin \beta x + \sinh \beta x \cos \beta x)
\end{aligned}
$$

or

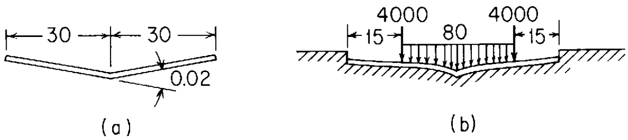
$$
M = 181{,}400 \sinh \beta x \sin \beta x - 178{,}300 \cosh \beta x \sin \beta x + 900 \sinh \beta x \cos \beta x
$$

The maximiun value of M can be found by trying values of x in the neighborhood of $x = \pi/4\beta = \pi/4(0.01127) = 69.7$ in, which would be the location of the maximum moment if the beam were infinitely long (see Table 8.6). This procedure reveals that the maximum moment occurs at $x = 66.5$ in and has a value of $-55{,}400$ lb-in.

The maximum stress in the 6-in I-beam is therefore $55{,}400(3)/21.8 = 7620$ lb/in^2. The maximum stress in the supporting 5-in I-beams is found at the midspan of the beam directly above the load. The deflection of this beam is known to be 1.092 in, and the spring constant is 1013 lb/in, so that the center load on the beam is $1.092(1013) = 1107$ lb. Therefore the maximum bending moment is $Pl/4 = 1107(21.5)(12)/4 = 71{,}400$ lb-in and the maximum stress is $71{,}400(2.5)/12.1 = 14{,}780$ lb/in^2.

2. If the 6-in I-beam in Example 1 had been much longer but supported in the same manner, Table 8.6 could have been used. Case 8 reveals that for an end load the end deflection is $-W/2EI\beta^3 = -2000/2(30 \cdot 10^6)(21.8)(0.01127^3) = -1.070$ in and the maximum moment would have equaled $-0.3225W/\beta = -0.3225(2000)/0.01127 = -57{,}200$ in-lb at 69.7 in from the left end. We should not construe from this example that increasing the length will always increase the stresses; if the load had been placed elsewhere on the span, the longer beam could have had the lower maximum stress.

3. An aluminum alloy beam 3 in wide, 2 in deep, and 60 in long is manufactured with an initial concentrated angular deformation of 0.02 rad at midlength; this initial shape is shown in Fig. 8.11(*a*). In use, the beam is placed on an elastic foundation which develops 500 lb/in^2 vertical upward pressure for every 1 in it is depressed. The beam is loaded by two concentrated loads of 4000 lb each and a uniformly distributed load of 80 lb/in over the portion between the concentrated loads. The loading is shown in Fig. 8.11(*b*). It is desired to determine the maximum bending stress in the aluminum beam.



(a) (b)

Figure 8.11

Solution. First determine the beam and foundation parameters:

$$E = 9.5 \cdot 10^6 \text{ lb/in}^2, \quad I = \tfrac{1}{12}(3)(2^3) = 2 \text{ in}^4, \quad k_0 = 500 \text{ lb/in}^2\text{/in}, \quad b_o = 3 \text{ in}$$

$$\beta = \left[\frac{3(500)}{4(9.5 \cdot 10^6)(2)}\right]^{1/4} = 0.0666, \qquad l = 60 \text{ in}, \qquad \beta l = 4.0$$

$$C_1 = -17.85, \qquad C_2 = -38.50, \qquad C_3 = -20.65, \qquad C_4 = -2.83, \qquad C_{11} = 744$$

An examination of Table 8.5 shows the loading conditions covered by the superposition of three cases in which both ends are free: case 1 used twice with $W_1 = 4000$ lb and $a_1 = 15$ in, and $W_2 = 4000$ lb and $a_2 = 45$ in; case 2 used twice with $w_3 = 80$ lb/in and $a_3 = 15$ in, and $w_4 = -80$ lb/in and $a_4 = 45$ in; case 5 used once with $\theta_o = 0.02$ and $a = 30$ in.

The loads and deformations at the left end are now evaulated by summing the values for the five different loads, which is done in the order in which the loads are mentioned. But before actually summing the end values, a set of constants involving the load positions must be determined for each case. For case 1, load 1:

$$C_{a1} = \cosh \beta(60 - 15) \cos \beta(60 - 15) = 10.068(-0.99) = -9.967$$
$$C_{a2} = -8.497$$

For case 1, load 2:

$$C_{a1} = 0.834, \qquad C_{a2} = 1.933$$

For case 2, load 3:

$$C_{a2} = -8.497, \qquad C_{a3} = 1.414$$

For case 2, load 4:

$$C_{a2} = 1.933, \qquad C_{a3} = 0.989$$

For case 5:

$$C_{a3} = 3.298, \qquad C_{a4} = 4.930$$

Therefore $R_A = 0$ and $M_A = 0$.

$$\begin{aligned}\theta_A &= \frac{4000}{2EI\beta^2}\frac{(-38.50)(-8.497)-(2)(-20.65)(-9.967)}{744}\\&\quad+\frac{4000}{2EI\beta^2}\frac{(-38.50)(1.933)-(2)(-20.65)(0.834)}{744}\\&\quad+\frac{80}{2EI\beta^3}\frac{(-38.50)(1.414)-(-20.65)(-8.497)}{744}\\&\quad+\frac{-80}{2EI\beta^3}\frac{(-38.50)(0.989)-(-20.65)(1.933)}{744}\\&\quad+0.02\frac{(-38.50)(4.93)-(2)(-20.65)(3.298)}{744}\\&=\frac{400}{EI\beta^2}(-0.0568)+\frac{4000}{EI\beta^2}(-0.02688)+\frac{80}{EI\beta^3}(-0.1545)\\&\quad-\frac{80}{EI\beta^3}(0.00125)+0.02(-0.0721)\\&=-0.007582 \text{ rad}\end{aligned}$$

Similarly,

$$y_A = -0.01172 \text{ in}$$

An examination of the equation for the transverse shear V shows that the value of the shear passes through zero at $x = 15$, 30, and 45 in. The maximum positive bending moment occurs at $x = 15$ in and is evaluated as follows, noting again that R_A and M_A are zero:

$$\begin{aligned}M_{15} &= -(-0.01172)(2)(9.5\cdot 10^6)(2)(0.06666^2)[\sinh(0.06666)(15)\sin 1]\\&\quad-(-0.007582)(9.5\cdot 10^6)(2)(0.06666)(\cosh 1 \sin 1 - \sinh 1 \cos 1)\\&= 8330 \text{ lb-in}\end{aligned}$$

Similarly, the maximum negative moment at $x = 30$ in is evaluated, making sure that the terms for the concentrated load at 15 in and the uniformly distributed load from 15 to 30 in are included:

$$M_{30} = -13{,}000 \text{ lb-in}$$

The maximum bending stress is given by $\sigma = Mc/I$ and is found to be 6500 lb/in^2.

8.6 Deformation Due to the Elasticity of Fixed Supports

The formulas in Tables 8.1, 8.2, 8.5, 8.6, and 8.8–8.10 that apply to those cases where fixed or guided end supports are specified are based on the assumption that the support is rigid and holds the fixed or guided end truly horizontal or vertical. The slight deformation that actually occurs at the support permits the beam to assume there a

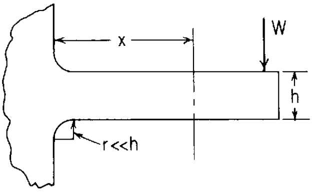


Figure 8.12

slope $\Delta\theta$, which for the conditions represented in Fig. 8.12, that is, a beam integral with a semi-infinite supporting foundation, is given by

$$\Delta\theta = \frac{16.67M}{\pi E h_1^2} + \frac{(1-\nu)V}{E h_1}$$

Here M is the bending moment per unit width and V is the shear force per unit width of the beam at the support; E is the modulus of elasticity, and ν is Poisson's ratio for the foundation material; and $h_1 = h + 1.5r$ (Ref. 54). The effect of this deformation is to increase the deflections of the beam. For a cantilever, this increase is simply $x\,\Delta\theta$, but for other support conditions the concept of the externally created angular deformation may be utilized (see Example 2 on page 144).

For the effect of many moment-loaded cantilever beams spaced closely one above the next, see Ref. 67.

8.7 Beams under Simultaneous Axial and Transverse Loading

Under certain conditions a beam may be subjected to axial tension or compression in addition to the transverse loads. Axial tension tends to straighten the beam and thus reduce the bending moments produced by the transve:rse loads, but axial compression has the opposite effect and may greatly increase the maximum bending moment and deflection and obviously must be less than the critical or buckling load. See Chap. 15. In either case a solution cannot be effected by simple superposition but must be arrived at by methods that take into account the change in deflection produced by the axial load.

For any condition of loading, the maximum normal stress in an extreme fiber is given by

$$\sigma_{\max} = \frac{P}{A} \pm \frac{Mc}{I} \qquad (8.7\text{-}1)$$

where P is the axial load (positive if tensile and negative if compressive), A is the cross-sectional area of the beam, I/c is the section modulus, and M is the maximum bending moment due to the combined effect of axial and transverse loads. (Use the plus sign if

M causes tension at the point in question and the minus sign if M causes compression.)

It is the determination of M that offers difficulty. For some cases, especially if P is small or tensile, it is permissible to ignore the small additional moment caused by P and to take M equal to M', the bending moment due to transverse loads only. Approximate formulas of the type (Ref. 33)

$$y_{\max} = \frac{y'_{\max}}{1 \pm \alpha_y P^2/EI}, \qquad \theta_{\max} = \frac{\theta'_{\max}}{1 \pm \alpha_\theta P^2/EI}$$
$$M_{\max} = \frac{M'_{\max}}{1 \pm \alpha_M P^2/EI} \tag{8.7-2}$$

have been used, but the values of α_y, α_θ, and α_M are different for each loading and each manner of supporting the beam.

Instead of tabulating the values of α, which give answers with increasing error as P increases, Tables 8.7(*a–d*) gives values of the coefficient C_P which can be used in the expressions

$$y_A = C_P y'_A, \qquad \theta_A = C_P \theta'_A, \qquad M_A = C_p M'_A, \quad \text{etc.} \tag{8.7-3}$$

where the primed values refer to the laterally loaded beam without the axial load and can be evaluated from expressions found in Table 8.1. For those cases listed where the reactions are statically indeterminate, the reaction coefficients given will enable the remaining reactions to be evaluated by applying the principles of static equilibrium. The given values of C_P are exact, based on the assumption that deflections due to transverse shear are negligible. This same assumption was used in developing the equations for transverse shear, bending moment, slope, and deflection shown in Tables 8.8 and 8.9.

Table 8.8 lists the general equations just mentioned as well as boundary values and selected maximum values for the case of axial compressive loading plus transverse loading. Since, in general, axial tension is a less critical condition, where deflections, slopes, and moments are usually reduced by the axial load, Table 8.9 is much more compact and gives only the general equations and the left-end boundary values.

Although the principle of superposition does not apply to the problem considered here, this modification of the principle can be used: The moment (or deflection) for a combination of transverse loads can be found by adding the moments (or deflections) for each transverse load combined with the entire axial load. Thus a beam supported at the ends and subjected to a uniform load, a center load, and an axial compression would have a maximum bending moment (or deflection) given by the sum of the maximum moments (or deflections) for Table

8.8, cases 1e and 2e, the end load being included once for each transverse load.

A problem closely related to the beam under combined axial and lateral loading occurs when the ends of a beam are axially restrained from motion along the axis of the beam (held) and a lateral load is applied. A solution is effected by equating the increase in length of the neutral surface of the beam Pl/AE to the decrease in length due to the curvature of the neutral surface $\frac{1}{2}\int_0^1 \theta^2 dx$ [Eq. (8.1-14)]. In general, solving the resulting equation for P is difficult owing to the presence of the hyperbolic functions and the several powers of the load P in the equation. If the beam is long, slender, and heavily loaded, this will be necessary for good accuracy; but if the deflections are small, the deflection curve can be approximated with a sine or cosine curve, obtaining the results given in Table 8.10. The following examples will illustrate the use of the formulas in Tables 8.7–8.10.

EXAMPLES

1. A 4-in, 7.7-lb steel I-beam 20 ft long is simply supported at both ends and simultaneously subjected to a transverse load of 50 lb/ft (including its own weight), a concentrated lateral load of 600 lb acting vertically downward at a position 8 ft from the left end, and an axial compression of 3000 lb. It is required to determine the maximum fiber stress and the deflection at midlength.

Solution. Here $P = 3000$ lb; $l = 240$ in; $I = 6$ in^4; $I/c = 3$ in^3; $A = 2.21$ in^2; $w_a = w_l = \frac{50}{12} = 4.17$ lb/in; and $a = 0$ for case 2e; $W = 600$ lb and $a = 96$ in for case 1e; $k = \sqrt{P/EI} = 0.00408$ in^{-1}; $kl = 0.98$. The solution will be carried out (*a*) ignoring deflection, (*b*) using coefficients from Table 8.7 and (c) using precise formulas from Table 8.8.

(*a*) $R_A = 860$ lb, and max $M_{8'} = 860(8) - 8(50)(4) = 5280$ lb-ft:

$$\text{max compressive stress} = -\frac{P}{A} - \frac{M}{I/c} = -\frac{3000}{2.21} - \frac{5280(12)}{3} = -22{,}475 \text{ lb/in}^2$$

For the uniform load (Table 8.1, case 2e):

$$y_{l/2} = \frac{-5}{384}\frac{w_a l^4}{EI} = \frac{-5(4.17)(240^4)}{384(30 \cdot 10^6)(6)} = -1.00 \text{ in}$$

For the concentrated load (Table 8.1, case 1e):

$$R_A = 360 \text{ lb}$$

$$\theta_A = \frac{-600(96)[2(240) - 96](240 - 96)}{6(30 \cdot 10^6)(6)(240)} = -0.123 \text{ rad}$$

$$y_{l/2} = -0.123(120) + \frac{360(120^3)}{6(30 \cdot 10^6)(6)} - \frac{600(120 - 96)^3}{6(30 \cdot 10^6)(6)} = -0.907 \text{ in}$$

Thus

$$\text{Total midlength deflection} = -1.907 \text{ in}$$

(*b*) From Table 8.7(*b*) (simply supported ends), coefficients are given for concentrated loads at $l/4$ and $l/2$. Plotting curves of the coefficients versus kl and using linear interpolation to correct for $a = 0.4l$ give a value of $C_P = 1.112$ for the midlength deflection and 1.083 for the moment under the load. Similarly, for a uniform load on the entire span ($a = 0$), the values of C_P are found to be 1.111 for the midlength deflection and 1.115 for the moment at midlength. If it is assumed that this last coefficient is also satisfactory for the moment at $x = 0.4l$, the following deflections and moments are calculated:

$$\text{Max } M_{8'} = 360(8)(1.083) + [500(8) - 8(50)(4)](1.115) = 3120 + 2680 = 5800 \text{ lb-ft}$$

$$\text{Max compressive stress} = -\frac{P}{A} - \frac{M}{I/c} = -\frac{3000}{2.21} - \frac{5800(12)}{3} = -24{,}560 \text{ lb/in}^2$$

$$\text{Midlength deflection} = -0.907(1.112) - 1.00(1.111) = -2.12 \text{ in}$$

(*c*) From Table 8.8 cases 1e and 2e, $R_A = 860$ lb and

$$\begin{aligned}\theta_A &= \frac{-600}{3000}\left[\frac{\sin 0.00408(240-96)}{\sin 0.98} - \frac{240-96}{240}\right] \\ &\quad + \frac{-4.17}{0.00408(3000)}\left(\tan\frac{0.98}{2} - \frac{0.98}{2}\right) \\ &= \frac{-600}{3000}\left(\frac{0.5547}{0.8305} - 0.6\right) - 0.341(0.533 - 0.49) \\ &= -0.0283 \text{ rad}\end{aligned}$$

$$\begin{aligned}\text{Max } M_{8'} &= \frac{860}{0.00408}\sin 0.00408(96) - \frac{-0.0283(3000)}{0.00408}\sin 0.392 \\ &\quad - \frac{4.17}{0.00408^2}(1 - \cos 0.392) \\ &= 80{,}500 + 7950 - 19{,}000 = 69{,}450 \text{ lb-in}\end{aligned}$$

$$\text{Max compressive stress} = -\frac{3000}{2.21} - \frac{69{,}450}{3} = -24{,}500 \text{ lb/in}^2$$

$$\begin{aligned}\text{Midlength deflection} &= \frac{-0.0283}{0.00408}\sin 0.49 + \frac{860}{0.00408(3000)}(0.49 - \sin 0.49) \\ &\quad - \frac{600}{0.00408(3000)}[0.00408(120-96) \\ &\quad - \sin 0.00408(120-96)] \\ &\quad - \frac{4.17}{0.00408^2(3000)}\left[\frac{0.00408^2(120^2)}{2}\right. \\ &\quad \left. - 1 + \cos 0.00408(120)\right] \\ &= -3.27 + 1.36 - 0.00785 - 0.192 = -2.11 \text{ in}\end{aligned}$$

The ease with which the coefficients C_P can be obtained from Tables 8.7(*a–d*) makes this a very desirable way to solve problems of axially loaded beams. Some caution must be observed, however, when interpolating for the position of the load. For example, the concentrated moment in Tables 8.7*c* and 8.7*d* shows a large variation in C_P for the end moments when the load position is changed from 0.25 to 0.50, especially under axial tension. Note that there are some cases in which C_P either changes sign or increases and then decreases when kl is increased; in these cases the loading produces both positive and negative moments and deflections in the span.

2. A solid round brass bar 1 cm in diameter and 120 cm long is rigidly fixed at both ends by supports assumed to be sufficiently rigid to preclude any relative horizontal motion of the two supports. If this bar is subjected to a transverse center load of 300 N at midlength, what is the center deflection and what is the maximum tensile stress in the bar?

Solution. Here P is an unknown tensile load; $W = 300$ N and $a = 60$ cm; $l = 120$ cm; $A = 0.785\ \text{cm}^2$, $I = 0.0491\ \text{cm}^4$; and $E = 10 \cdot 10^6\ \text{N/cm}^2$. (This situation is described in Table 8.10, case 2.) The first equation is solved for $y_{\max}$:

$$y_{\max} + \frac{0.785}{16(0.0491)} y_{\max}^3 = \frac{300(120^3)}{2(\pi^4)(10 \cdot 10^6)(0.0491)}$$
$$y_{\max} + y_{\max}^3 = 5.44$$

Therefore $y_{\max} = 1.57$ cm. The second equation is now solved for P:

$$P = \frac{\pi^2(10 \cdot 10^6)(0.785)}{4(120^2)} 1.57^2 = 3315\ \text{N}$$
$$k = \sqrt{\frac{P}{EI}} = \left[\frac{3315}{(10 \cdot 10^6)(0.0491)}\right]^{1/2} = 0.0822\ \text{cm}^{-1}$$
$$kl = 9.86$$

From Table 8.7, case 1d, the values of R_A and M_A can be calculated. (Note that θ_A and y_A are zero.) First evaluate the necessary constants:

$$C_2 = \sinh 9.86 = 9574.4$$
$$C_3 = \cosh 9.86 - 1 = 9574.4 - 1 = 9573.4$$
$$C_4 = \sinh 9.86 - 9.86 = 9564.5$$
$$C_{a3} = \cosh \frac{9.86}{2} - 1 = 69.193 - 1 = 68.193$$
$$C_{a4} = \sinh 4.93 - 4.93 = 69.186 - 4.93 = 64.256$$

$$R_A = W \frac{C_3 C_{a3} - C_2 C_{a4}}{C_3^2 - C_2 C_4} = 300 \frac{9573.4(68.193) - 9574.4(64.256)}{9573.4^2 - 9574.4(9564.5)} = 300(0.5)$$
$$= 150\ \text{N}$$
$$M_A = \frac{-W}{k} \frac{C_4 C_{a3} - C_3 C_{a4}}{C_3^2 - C_2 C_4} = \frac{-300}{0.0822} \frac{9564.5(68.193) - 9573.4(64.256)}{74{,}900}$$
$$= \frac{-300}{0.0822} 0.493 = -1800\ \text{N-cm}$$

$$\text{Max tensile stress} = \frac{P}{A} + \frac{Mc}{I} = \frac{3315}{0.785} + \frac{1800(0.5)}{0.0491} = 4220 + 18{,}330$$
$$= 22{,}550\ \text{N/cm}^2$$

$$\text{Midlength deflection} = \frac{-1800}{3315}\left(\cosh \frac{9.86}{2} - 1\right) + \frac{150}{3315(0.0822)}(\sinh 4.93 - 4.93)$$
$$= -37.0 + 35.4 = -1.6\ \text{cm}$$

This compares favorably with the value $y_{\max} = 1.57$ cm obtained from the equation which was based on the assumption of a cosine curve for the deflection.

An alternative to working with the large numerical values of the hyperbolic sines and cosines as shown in the preceding calculations would be to simplify the equations for this case where the load is at the center by using the double-angle identities for hyperbolic functions. If this is done here, the expressions simplify to

$$R_A = \frac{W}{2} \qquad M_A = \frac{-W}{k}\tanh\frac{kl}{4} \qquad y_{l/2} = \frac{-W}{kP}\left(\frac{kl}{4} - \tanh\frac{kl}{4}\right)$$

Using these expressions gives $R_A = 150$ N, $M_A = -1800$ N-cm and $y_{l/2} = -1.63$ cm. Table 8.6 for axial compression gives the formulas for these special cases, but when the lateral loads are not placed at midlength or any of the other common locations, a desk calculator or digital computer must be used. If tables of hyperbolic functions are employed, it should be kept in mind that adequate solutions can be made using values of kl close to the desired values if such values are given in the table and the desired ones are not. For example, if the values for the arguments 9.86 and 4.93 are not available but values for 10 and 5 are (note that it is necessary to maintain the correct ratio a/l), these values could be used with no noticeable change in the results. Finally, an energy approach, using Rayleigh's technique, is outlined in Chap. 6, Sec. 13, of Ref. 72. The method works well with simple, axially constrained, and unconstrained beams.

8.8 Beams of Variable Section

Stress. For a beam whose cross section changes gradually, Eqs. (8.1-1), (8.1-4), and (8.1-10)–(8.1-12) (Sec. 8.1) apply with sufficient accuracy; Eqs. (8.1-3) and (8.1-5)–(8.1-7) apply if I is treated as a variable, as in the examples that follow. All the formulas given in Table 8.1 for vertical shear and bending moments in *statically determinate* beams apply, but the formulas given for statically indeterminate beams and for deflection and slope are inapplicable to beams of nonuniform section unless the section varies in such a way that I is constant.

Accurate analysis (Ref. 3) shows that in an end-loaded cantilever beam of rectangular section which is symmetrically tapered in the plane of bending the maximum fiber stress is somewhat less than is indicated by Eq. (8.1-12) the error amounting to about 5% for a surface slope of 15° (wedge angle 30°) and about 10% for a surface slope of 20°. See also Prob. 2.35 in Ref. 66. The maximum horizontal and vertical shear stress is shown to occur at the upper and lower surfaces instead of at the neutral axis and to be approximately three times as great as the average shear stress on the section for slopes up to 20°. It is very doubtful, however, if this shear stress is often critical even in wood beams, although it may possibly start failure in short, heavily reinforced concrete beams that are deepened or "haunched" at the ends.

Such a failure, if observed, would probably be ascribed to compression since it would occur at a point of high compressive stress. It is also conceivable, of course, that this shear stress might be of importance in certain metal parts subject to repeated stress.

Abrupt changes in the section of a beam cause high local stresses, the effect of which is taken into account by using the proper factor of stress concentration (Sec. 3.10 and Table 17.1).

Deflection. Determining deflections or statically indeterminate reactions for beams of variable section can be considered in two categories: where the beam has a continuously varying cross section from one end to the other, and where the cross section varies in a stepwise fashion.

Considering the first category, where the section varies continuously, we sometimes find a variation where Eq. (8.1-5) can be integrated directly, with the moment of inertia treated as a variable. This has been accomplished in Ref. 20 for tapered beams of circular section, but using the expressions presented, one must carry more than the usual number of digits to get accurate results. In most instances, however, this is not easy, if possible, and a more productive approach is to integrate Eq. (8.1-6) numerically using small incremental lengths Δx. This has been done for a limited number of cases, and the results are tabulated in Tables 8.11(*a*)–(*d*).

These tables give coefficients by which the stated reaction forces or moments or the stated deformations for uniform beams, as given in Table 8.1, must be multiplied to obtain the comparable reactions or deformations for the tapered beams. The coefficients are dependent upon the ratio of the moment of inertia at the right end of the beam I_B to the moment of inertia at the left end I_A, assuming that the uniform beam has a moment of inertia I_A. The coefficients are also dependent upon the manner of variation between the two end values. This variation is of the form $I_x = I_A(1 + Kx/l)^n$, where x is measured from the left end and $K = (I_B/I_A)^{1/n} - 1$. Thus if the beam is uniform, $n = 0$; if the width of a rectangular cross section varies linearly, $n = 1$; if the width of a rectangular cross section varies parabolically, $n = 2$; if the depth of a rectangular cross section varies linearly, $n = 3$; and if the lateral dimensions of any cross section vary linearly and proportionately, $n = 4$. Beams having similar variations in cross section can be analysed approximately by comparing the given variations to those found in Table 8.11.

Coefficients are given for only a few values of a/l, so it is not desirable to interpolate to determine coefficients for other values of a/l. Instead it is advisable to determine the corrected deformations or reactions with the loads at the tabulated values of a/l and then interpolate. This allows the use of additional known values as shown in the second example below. For beams with symmetric end conditions, such as both ends fixed or both ends simply supported, the data given for any value of $a/l < 0.5$ can be used twice by reversing the beam end for end.

EXAMPLES

1. A tapered beam 30 in long with a depth varying linearly from 2 in at the left end to 4 in at the right end and with a constant width of 1.5 in is fixed on the right end and simply supported on the left end. A concentrated clockwise couple of 5000 lb-in is applied at midlength and it is desired to know the maximum bending stress in the beam.

Solution. First determine the left-end reaction force for a uniform cross section. From Table 8.1, case 3c, the left reaction is

$$R_A = \frac{-3M_o(l^2 - a^2)}{2l^3} = \frac{-3(5000)(30^2 - 15^2)}{2(30^3)} = -187.5 \text{ lb}$$

For the tapered beam

$$I_A = \frac{1.5(2^3)}{12} = 1 \text{ in}^4, \qquad I_B = \frac{1.5(4^3)}{12} = 8 \text{ in}^4$$

In Table 8.11(c) for $n = 3$, $I_B/I_A = 8$; and for case 3c with the loading at $l/2$, the coefficient is listed as 0.906. Therefore, the left-end reaction is $-187.5(0.906) = -170$ lb

The maximum negative moment will occur just left of midlength and will equal $-170(15) = -2550$ lb-in. The maximum positive moment will occur just right of midlength and will equal $-2550 + 5000 = 2450$ lb-in. At midlength the moment of inertia $I = 1.5(3^3)/12 = 3.37$ in^4, and so the maximum stress is given by $\sigma = Mc/I = 2550(1.5)/3.37 = 1135$ lb/in^2 just left of midlength.

2. A machine part is an 800-mm-long straight beam with a variable wide-flange cross section. The single central web has a constant thickness of 1.5 mm but a linearly varying depth from 6 mm at the left end A to 10 mm at the right end B. The web and flanges are welded together continuously over the entire length and are also welded to supporting structures at each end to provide fixed ends. A concentrated lateral load of 100 N acts normal to the central axis of the beam parallel to the web at a distance of 300 mm from the left end. The maximum bending stress and the deflection under the load are desired. The modulus of elasticity is 70 GPa, or 70,000 N/mm^2.

Solution. First determine the left-end reaction force and moment for a beam of constant cross section. From Table 8.1, case 1d,

$$R_A = \frac{W}{l^3}(l-a)^2(l+2a) = \frac{100}{800^3}(800-300)^2(800+600) = 68.36 \text{ N}$$

$$M_A = \frac{-Wa}{l^2}(l-a)^2 = \frac{-100(300)}{800^2}(800-300)^2 = -11{,}720 \text{ N-mm}$$

For the tapered beam,

$$I_A = \frac{4(10^3) - 2.5(6^3)}{12} = 288.3 \text{ mm}^4$$

$$I_B = \frac{8(14^3) - 6.5(10^3)}{12} = 1287.7 \text{ mm}^4$$

and at midlength where $x = l/2$, the moment of inertia is given by

$$I_{l/2} = \frac{6(12^3) - 4.5(8^3)}{12} = 672.0 \text{ mm}^4$$

Using the formula for the variation of I with x and these three values for the moment of inertia, approximate values for K and n can be found.

$$I_B = I_A(1+K)^n, \qquad I_{l/2} = I_A\left(1+\frac{K}{2}\right)^n$$

$$\frac{1287.7}{288.3} = 4.466 = (1+K)^n, \qquad \frac{672}{288.3} = 2.331 = \left(1+\frac{K}{2}\right)^n$$

$$4.466^{1/n} - 2(2.331)^{1/n} + 1 = 0$$

Solving this last expression gives $1/n = 0.35$, $n = 2.86$, and $K = 0.689$.

An examination of Tables 8.11(a–d) shows that for a fixed-ended beam with a concentrated load, which is case 1d in Table 8.1, values of coefficients are given only for $a/l = 0.25$ and 0.50. For this problem $a/l = 0.375$. Simple linear interpolation is not sufficiently accurate. However, if one imagines the load at $a/l = 0.25$ the values for R_A and M_A can be found. This procedure can be repeated for $a/l = 0.50$. Two other sets of data are also available. If the load were placed at the left end, $a/l = 0$, $M_A = 0$, $R_A = 100$ N, and $dM_A/da = -100$ N. If the load were placed at the right end, $a/l = 1$, $R_A = 0$, $M_A = 0$, and $dM_A/da = 0$. The variations of the tabulated coefficients with I_B/I_A and with n do not pose a comparable problem since many data are available. Plotting curves for the variation with I_B/I_A and interpolating linearly between $n = 2$ and $n = 3$ for $n = 2.86$ give the coefficients used below to find the values for R_A and M_A at $a/l = 0.25$ and 0.50:

	Untapered beam		Tapered beam where $n = 2.86$ and $I_B/I_A = 4.466$	
a/l	0.25	0.50	0.25	0.50
R_A (N)	84.38	50.00	$84.38(0.922) = 77.80$	$50(0.0805) = 40.25$
M_A (N-mm)	−11,250	−10,000	$-11{,}250(0.788) = -8865$	$-10{,}000(0.648) = -6480$

Plotting these values of R_A and M_A versus a/l for the four positions allows one to pick from the graphs at $a/l = 0.375$, $R_A = 60$ N, and $M_A = -8800$ N-mm. The use of static equilibrium now gives $M_B = -10{,}800$ N-mm and the moment at the load of 9200 N-mm. The bending stress at the left end is found to be the largest.

$$\sigma_A = \frac{M_A c_A}{I_A} = \frac{8800(5)}{288.3} = 152.6 \text{ MPa}$$

No correction coefficients for deflections are included for case ld in Table 8.11. The deflection at the load position can be found from the information in this table, however, if either end of the beam is isolated and treated as a cantilever with an end load and an end moment. The moment of inertia at the load position C is given by

$$I_C = \frac{5.5(11.5^3) - 4(7.5^3)}{12} = 556.4 \text{ mm}^4$$

Treat the left portion of the beam as a 300-mm-long cantilever, using case 1a with an end load of 60 N and case 3a with an end moment of 9200 N-mm. Determine the correction coefficients for $a/l = 0$, $n = 3$, and the moment of inertia ratio of $288.33/556.44 = 0.518$. Interpolation between data for $n = 2$ and $n = 3$ is not justified when considering the approximations already made from plotted curves. Noting that all correction coefficients in Table 8.11 are unity for $I_B/I_A = 1$ and using data points for $I_B/I_A = 0.25$ and 0.50, the correction coefficients used below were found

$$y_C = -\frac{60(300^3)(1.645)}{3(70{,}000)(556.4)} + \frac{9200(300^2)(1.560)}{2(70{,}000)(556.4)} = -22.8 + 16.6 = -6.2 \text{ mm}$$

This deflection at the load can be checked by repeating the above procedure by using the right-hand portion of the beam. The slope of the beam at the load can also be used as a check.

Alternative solution. The solution just presented was intended to illustrate appropriate methods of interpolation with the limited load positions shown in the tables. There is also an alternative solution involving superposition of cases. Remove the fixity at end A and treat the 500-mm-long right portion as a cantilever with an end load of 100 N. Use $n = 3$ as being close enough to $n = 2.86$ and $I_B/I_C = 1287.7/556.4 = 2.314$. Interpolate between $I_B/I_C = 2$ and 4 to obtain from case 1a in Table 8.11(c) the correction coefficients used below to calculate the slope and deflection at the load

$$y_C = -\frac{100(500^3)(0.555)}{3(70{,}000)(556.4)} = -59.37 \text{ mm} \qquad \theta_C = \frac{100(500^2)(0.593)}{2(70{,}000)(556.4)} = 0.1903 \text{ rad}$$

Since the left portion is unloaded and remains straight, the end deflection and slope are $y_A = -59.37 - 300(0.1903) = -116.5$ mm, and $\theta_A = 0.1903$ rad. Next treat the complete beam as a cantilever under an end load R_A and an end moment M_A. Let $I_A = 228.3 \text{ mm}^4$, $I_B/I_A = 4.466$, and again let $n = 3$. From cases 1a and 3a in Table 8.11(c),

$$y_A = \frac{R_A(800^3)(0.332)}{3(70{,}000)(288.3)} + \frac{M_A(800^2)(0.380)}{2(70{,}000)(288.3)} = 2.808R_A + 0.00602M_A$$

$$\theta_A = -\frac{R_A(800^2)(0.380)}{2(70{,}000)(288.3)} - \frac{M_A(800)(0.497)}{(70{,}000)(288.3)} = -0.006024R_A - 19.7(10^{-6})M_A$$

Adding the slopes and deflections from the load of 100 N to those above and equating each to zero to represent the fixed end gives $R_A = 60.3$ N and $M_A = -8790$ N-mm. This is a reasonable check on those values from the first solution.

The second category of determining deflections, where the cross section varies in steps from one uniform section to another, can be solved in several ways. Equation (8.1-5) can be integrated, matching slopes and deflections at the transition sections, or Eq. (8.1-6) can be integrated over the separate portions and summed to obtain the desired deflections. A third method utilizes the advantages of the step function and its application to beam deflections as given in Table 8.1. In a given portion of the span where the cross section is uniform it is apparent that the shape of the elastic curve will remain the same if the internal bending moments and the moments of inertia are increased or decreased in proportion. By this means, a modified moment diagram can be constructed which could be applied to a beam with a single constant cross section and thereby produce an elastic curve identical to the one produced by the actual moments and the several moments of inertia present in the actual span. It is also apparent that this modified moment diagram could be produced by adding appropriate loads to the beam. (See Refs. 29 and 65.) In summary, then, a new loading is constructed which will produce the required elastic curve, and the solution for this loading is carried out by using the formulas in Table 8.1. This procedure will be illustrated by the following example.

EXAMPLE

The beam shown in Fig. 8.13 has a constant depth of 4 in and a step increase in the width from 2 to 5 in at a point 5 ft from the left end. The left end is simply supported, and the right end is fixed; the loading is a uniform 200 lb/ft from $x = 3$ ft to the right end. Find the value of the reaction at the left end and the maximum stress.

Solution. For the left 5 ft, $I_1 = 2(4^3)/12 = 10.67\ \text{in}^4$. For the right 5 ft, $I_2 = 5(4^3)/12 = 26.67\ \text{in}^4$, or 2.5 times I_1.

The same M/I diagram shown in Fig. 8.13(*e*) can be produced by the loading shown in Fig. 8.14 acting upon a beam having a constant moment of inertia I_2. Note that all loads on the left portion simply have been increased by a factor of 2.5, while added loads at the 5-ft position reduce the effects of these added loads to those previously present on the right portion.

To find the left-end reaction for the beam loaded as shown in Fig. 7.14(*a*), use Table 8.1, case 1c, where $W = 1.5R_1 - 600$ and $a = 5$; case 2c, where $w_a = w_l = 500$ lb/ft and $a = 3$ ft; case 2c, again, where $w_a = w_l = -300$ lb/ft and $a = 5$ ft; and finally case 3c, where $M_o = -(7.5R_1 - 600)$ and $a = 5$ ft. Summing the expressions for R_A from these cases in the order above, we obtain

$$R_A = 2.5R_1 = \frac{(1.5R_1 - 600)(10-5)^2}{2(10^3)}[2(10)+5] + \frac{500(10-3)^3}{8(10^3)}[3(10)+3]$$

$$+\frac{(-300)(10-5)^3}{8(10^3)}[3(10)+5] - \frac{3[-(7.5R_1 - 600)]}{2(10^3)}(10^2 - 5^2)$$

which gives $R_1 = 244$ lb.

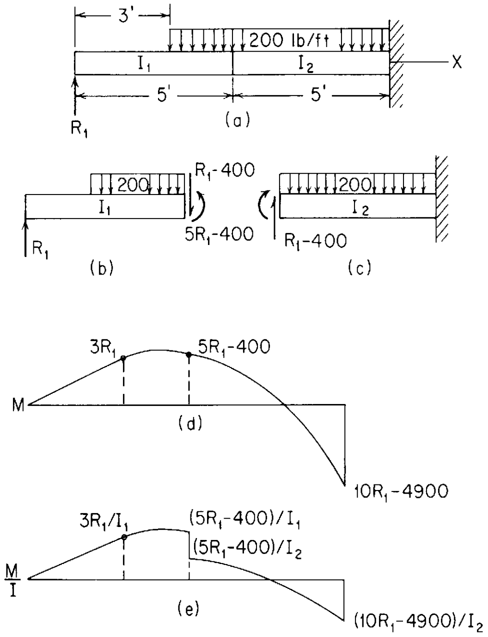


Figure 8.13

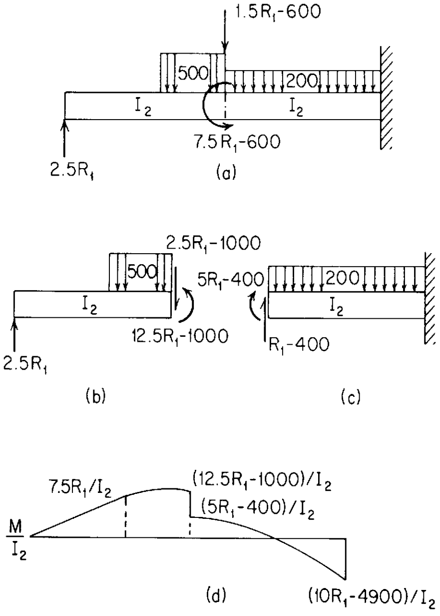


Figure 8.14

From Fig. 8.13(*a*) we can observe that the maximum positive bending moment will occur at $x = 4.22$ ft, where the transverse shear will be zero. The maximum moments are therefore

$$\text{Max} + M = 244(4.22) - \frac{200}{2}(1.22^2) = 881 \text{ lb-ft}$$

$$\text{Max} - M = 244(10) - 4900 = -2460 \text{ lb-ft} \qquad \text{at the right end}$$

The maximum stresses are $\sigma = 881(12)(2)/10.67 = 1982 \text{ lb/in}^2$ at $x = 4.22$ ft and $\sigma = 2460(12)(2)/26.67 = 2215 \text{ lb/in}^2$ at $x = 10$ ft.

8.9 Slotted Beams

If the web of a beam is pierced by a hole or slot (Fig. 8.15), the stresses in the extreme fibers *a* and *b* at any section *B* are given by

$$\sigma_a = -\frac{M_A}{I/c} - \frac{V_A x I_1/(I_1 + I_2)}{(I/c)_1} \qquad \text{(compression)}$$

$$\sigma_b = \frac{M_A}{I/c} + \frac{V_A x I_2/(I_1 + I_2)}{(I/c)_2} \qquad \text{(tension)}$$

Here M_A is the bending moment at A (midlength of the slot), V_A is the vertical shear at A, I/c is the section modulus of the net beam section at B, I_1 and I_2 are the moments of inertia, and (I/c), and $(I/c)_2$ are the section moduli of the cross sections of parts 1 and 2 about their own central axes. M and V are positive or negative according to the usual convention, and x is positive when measured to the right.

The preceding formulas are derived by replacing all forces acting on the beam to the left of A by an equivalent couple M_A and shear V_A at A. The couple produces a bending stress given by the first term of the formula. The shear divides between parts 1 and 2 in proportion to their respective I's and produces in each part an additional bending stress given by the second term of the formula. The stress at any other point in the cross section can be found similarly by adding the stresses due to M_A and those due to this secondary bending caused by the shear. (At the ends of the slot there is a stress concentration at the corners which is not taken into account here.)

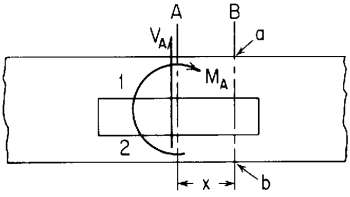


Figure 8.15

The above analysis applies also to a beam with multiple slots of equal length; all that is necessary is to modify the term $(I_1 + I_2)$. The numerator is still the I of the part in question and the denominator is the sum of the I's of all the parts 1, 2, 3, etc. The formulas can also be used for a rigid frame consisting of beams of equal length joined at their ends by rigid members; thus in Fig. 8.15 parts 1 and 2 might equally well be two separate beams joined at their ends by rigid crosspieces.

8.10 Beams of Relatively Great Depth

In beams of small span/depth ratio, the transverse shear stresses are likely to be high and the resulting deflection due to shear may not be negligible. For span/depth ratios of 3 or more, the deflection y_s due to shear is found by the method of unit loads to be

$$y_s = F \int \frac{Vv}{AG} dx \tag{8.10-1}$$

or by Castigliano's theorem to be

$$y_s = \frac{\partial U_s}{\partial P} \tag{8.10-2}$$

In Eq. (8.10-1), V is the vertical shear due to the actual loads, v is the vertical shear due to a unit load acting at the section where the deflection is desired, A is the area of the section, G is the modulus of rigidity, F is a factor depending on the form of the cross section, and the integration extends over the entire length of the beam, with due regard to the signs of V and v. For three solid sections, a rectangle, a triangle with base either up or down, and a trapezoid with parallel sides top and bottom, $F = \frac{6}{5}$; for a diamond-shaped section, $F = \frac{31}{30}$; for a solid circular section, $F = \frac{10}{9}$; for a thin-walled hollow circular section, $F = 2$; for an I- or box section with flanges and webs of uniform thickness,

$$F = \left[1 + \frac{3(D_2^2 - D_1^2)D_1}{2D_2^3}\left(\frac{t_2}{t_1} - 1\right)\right]\frac{4D_2^2}{10r^2}$$

where

D_1 = distance from neutral axis to the nearest surface of the flange

D_2 = distance from neutral axis to extreme fiber

t_1 = thickness of web (or webs in box beams)

t_2 = width of flange

r = radius of gyration of section with respect to the neutral axis

If the I- or box beam has flanges of nonuniform thickness, it may be replaced by an "equivalent" section whose flanges, of uniform thickness, have the same width and areas as those of the actual section (Ref. 19). Approximate results may be obtained for I-beams using $F = 1$ and taking for A the area of the web.

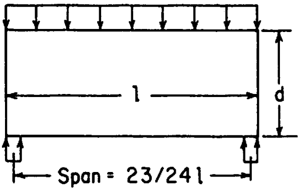
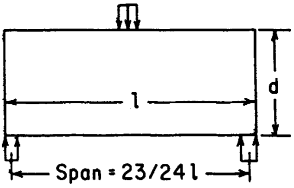
Application of Eq. (8.10-1) to several common cases of loading yields the following results:

$$\begin{aligned}
&\text{End support, center load } P && y_s = \frac{1}{4} F \frac{Pl}{AG} \\
&\text{End support, uniform load } w && y_s = \frac{1}{8} F \frac{wl^2}{AG} \\
&\text{Cantilever, end load } P && y_s = F \frac{Pl}{AG} \\
&\text{Cantilever, uniform load } w && y_s = \frac{1}{2} F \frac{wl^2}{AG}
\end{aligned}$$

In Eq. (8.10-2), $U_s = F \int (V^2/2AG) dx, P$ is a vertical load, real or imaginary, applied at the section where y_s is to be found, and the other terms have the same meaning as in Eq. (8.10-1).

The deflection due to shear will usually be negligible in metal beams unless the span/depth ratio is extremely small; in wood beams, because of the small value of G compared with E, deflection due to shear is much more important. In computing deflections it may be allowed for by using for E a value obtained from bending tests (shear deflection ignored) on beams of similar proportions or a value about 10% less than that found by testing in direct compression if the span/depth ratio is between 12 and 24. For larger ratios the effect of shear is negligible, and for lower ratios it should be calculated by the preceding method.

For extremely short deep beams, the assumption of linear stress distribution, on which the simple theory of flexure is based, is no longer valid. Equation (8.1-1) gives sufficiently accurate results for span/depth ratios down to about 3; for still smaller ratios it was believed formerly that the actual stresses were smaller than the formula indicates (Refs. 1 and 2), but more recent analyses by numerical methods (Refs. 43 and 44) indicate that the contrary is true. These analyses show that at s/d between 1.5 and 1, depending on the manner of loading and support, the stress distribution changes radically and the ratio of maximum stress to Mc/I becomes greater than 1 and increases rapidly as s/d becomes still smaller. In the following table, the influence of s/d on both maximum fiber stress and maximum horizontal shear stress is shown in accordance with the solution given in Ref. 43. Reference 44 gives comparable results, and both strain-gage measurements (Ref. 45) and photoelastic studies (Ref. 46) support the conclusions reached in these analyses.

		 Uniform load over entire l			 Uniform load over middle $\frac{1}{12}l$		
Ratio l/d	Ratio span/d	$\frac{\max \sigma_t}{Mc/I}$	$\frac{\max \sigma_c}{Mc/I}$	$\frac{\max \tau}{V/A}$	$\frac{\max \sigma_t}{Mc/I}$	$\frac{\max \sigma_c}{Mc/I}$	$\frac{\max \tau}{V/A}$
3	2.875	1.025	1.030	1.58	0.970	1.655	1.57
2.5	2.395	1.046	1.035	1.60	0.960	1.965	1.60
2.0	1.915	1.116	1.022	1.64	0.962	2.525	1.70
1.5	1.4375	1.401	0.879	1.80	1.038	3.585	1.92
1	0.958	2.725	0.600	2.43	1.513	6.140	2.39
0.5	0.479	10.95	2.365	4.53	5.460	15.73	3.78
$\frac{1}{3}$	0.3193	24.70	5.160	6.05	12.35	25.55	7.23

These established facts concerning elastic stresses in short beams seem incompatible with the contrary influence of s/d on modulus of rupture, discussed in Sec. 8.15, unless it is assumed that there is a very radical redistribution of stress as soon as plastic action sets in.

The stress produced by a concentrated load acting on a very short *cantilever* beam or projection (gear tooth, sawtooth, screw thread) can be found by the following formula, due to Heywood (Chap. 2, Ref. 28) and modified by Kelley and Pedersen (Ref. 59). As given here, the formula follows this modification, with some changes in notation. Figure 8.16 represents the profile of the beam, assumed to be of uniform thickness t. ED is the axis or center line of the beam; it bisects the angle between the sides if these are straight; otherwise it is drawn through the centers of two unequal inscribed circles. W represents the load; its line of action, or load line, intersects the beam profile at C and the beam axis at O. The inscribed parabola, with vertex at O, is tangent to the fillet on the tension side of the beam at A, which is the point of maximum tensile stress. (A can be located by making AF equal to FE by trial, F being the intersection of a perpendicular to the axis at O and a trial tangent to the fillet.) B is the corresponding point on the compression side, and D is the intersection of the beam axis with section AB. The dimensions a and e are perpendicular, respectively, to the load line and to the beam axis, r is the fillet radius, and b is the straight-line distance from A to C. The tensile stress at A is given by

$$\sigma = \frac{W}{t}\left[1 + 0.26\left(\frac{e}{r}\right)^{0.7}\right]\left[\frac{1.5a}{e^2} + \frac{\cos\beta}{2e} + \frac{0.45}{(be)^{1/2}}\right]$$

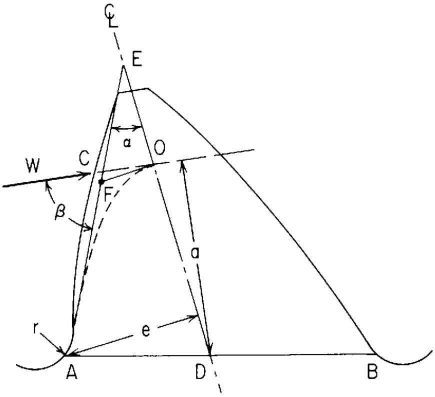


Figure 8.16

Here the quantity in the first pair of brackets is the factor of stress concentration for the fillet. In the second pair of brackets, the first term represents the bending moment divided by the section modulus, the second term represents the effect of the component of the load along the tangent line, positive when tensile, and the third term represents what Heywood calls the *proximity effect*, which may be regarded as an adjustment for the very small span/depth ratio.

Kelley and Pedersen have suggested a further refinement in locating the point of maximum stress, putting it at an angular distance equal to $25^\circ - \frac{1}{2}\alpha$, positive toward the root of the fillet. Heywood suggests locating this point at 30° from the outer end of the fillet, reducing this to 12° as the ratio of b to e increases; also, Heywood locates the moment of W about a point halfway between A and B instead of about D. For most cases the slightly different procedures seem to give comparable results and agree well with photoelastic analysis. However, more recent experimental studies (1963), including fatigue tests, indicate that actual stresses may considerably exceed those computed by the formula (Ref. 63).

8.11 Beams of Relatively Great Width

Because of prevention of the lateral deformation that would normally accompany the fiber stresses, wide beams, such as thin metallic strips, are more rigid than the formulas of Sec. 8.1 indicate. This stiffening effect is taken into account by using $E/(1 - \nu^2)$ instead of E in the formulas for deflection and curvature if the beams are very wide (Ref. 21). The anticlastic curvature that exists on narrow rectangular beams is still present at the extreme edges of very wide beams, but the central region remains flat in a transverse direction and transverse bending stresses equal to Poisson's ratio times the longitudinal bending stresses are present. For rectangular beams of moderate width, Ashwell (Ref. 10) shows that the stiffness depends not only upon the ratio of depth to width of the beam but also upon the radius of curvature to which the beam is bent. For a rectangular beam of width b and depth h bent to a radius of curvature ρ by a bending moment M, these variables are related by the expression $1/\rho = M/KEI$, where $I = bh^3/12$, and the following table of values for K is given for several values of Poisson's ratio and for the quantity $b^2/\rho h$.

	$b^2/\rho h$						
Value of ν	0.25	1.00	4.00	16.0	50.0	200.	800.
0.1000	1.0000	1.0003	1.0033	1.0073	1.0085	1.0093	1.0097
0.2000	1.0001	1.0013	1.0135	1.0300	1.0349	1.0383	1.0400
0.3000	1.0002	1.0029	1.0311	1.0710	1.0826	1.0907	1.0948
0.3333	1.0002	1.0036	1.0387	1.0895	1.1042	1.1146	1.1198
0.4000	1.0003	1.0052	1.0569	1.1357	1.1584	1.1744	1.1825
0.5000	1.0005	1.0081	1.0923	1.2351	1.2755	1.3045	1.3189

In very short wide beams, such as the concrete slabs used as highway-bridge flooring, the deflection and fiber-stress distribution cannot be regarded as uniform across the width. In calculating the strength of such a slab, it is convenient to make use of the concept of *effective width*, i.e., the width of a spanwise strip which, acting as a beam with uniform extreme fiber stress equal to the maximum stress in the slab, develops the same resisting moment as does the slab. The effective width depends on the manner of support, manner of loading, and ratio of breadth to span b/a. It has been determined by Holl (Ref. 22) for a number of assumed conditions, and the results are given in the following table for a slab that is freely supported at each of two opposite edges (Fig. 8.17). Two kinds of loading are considered, viz. uniform load over the entire slab and load uniformly distributed over a central circular area of radius c. The ratio of the effective width e to the span a is given for each of a number of ratios of c to slab thickness h and each of a number of b/a values.

	Values of e/a for				
Loading	$b/a = 1$	$b/a = 1.2$	$b/a = 1.6$	$b/a = 2$	$b/a = \infty$
Uniform	0.960	1.145	1.519	1.900	
Central, $c = 0$	0.568	0.599	0.633	0.648	0.656
Central, $c = 0.125h$	0.581	0.614	0.649	0.665	0.673
Central, $c = 0.250h$	0.599	0.634	0.672	0.689	0.697
Central, $c = 0.500h$	0.652	0.694	0.740	0.761	0.770

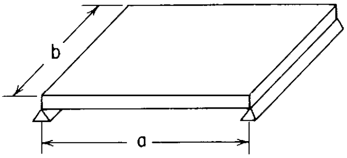


Figure 8.17

For the same case (a slab that is supported at opposite edges and loaded on a central circular area) Westergaard (Ref. 23) gives $e = 0.58a + 4c$ as an approximate expression for effective width. Morris (Ref. 24) gives $e = \frac{1}{2}e_c + d$ as an approximate expression for the effective width for midspan *off-center* loading, where e_c is the effective width for central loading and d is the distance from the load to the nearer unsupported edge.

For a slab that is *fixed* at two opposite edges and uniformly loaded, the stresses and deflections may be calculated with sufficient accuracy by the ordinary beam formulas, replacing E by $E/(1 - \nu^2)$. For a slab thus supported and loaded at the center, the maximum stresses occur under the load, except for relatively large values of c, where they occur at the midpoints of the fixed edges. The effective widths are approximately as given in the following table (values from the curves of Ref. 22). Here b/a and c have the same meaning as in the preceding table, but it should be noted that values of e/b are given instead of e/a.

Values of c	Values of e/b for				Max stress at
	$b/a = 1$	$b/a = 1.2$	$b/a = 1.6$	$b/a = 2.0$	
0	0.51	0.52	0.53	0.53	Load
$0.01a$	0.52	0.54	0.55	0.55	Load
$0.03a$	0.58	0.59	0.60	0.60	Load
$0.10a$	0.69	0.73	0.81	0.86	Fixed edges

Holl (Ref. 22) discusses the deflections of a wide beam with two edges supported and the distribution of pressure under the supported edges. The problem of determining the effective width in concrete slabs and tests made for that purpose are discussed by Kelley (Ref. 25), who also gives a brief bibliography on the subject.

The case of a very wide *cantilever* slab under a concentrated load is discussed by MacGregor (Ref. 26), Holl (Ref. 27), Jaramillo (Ref. 47), Wellauer and Seireg (Ref. 48), Little (Ref. 49), Small (Ref. 50), and others. For the conditions represented in Fig. 8.18, a cantilever plate of infinite length with a concentrated load, the bending stress σ at any point can be expressed by $\sigma = K_m(6P/t^2)$, and the deflection y at any point by $y = K_y(Pa^2/\pi D)$, where K_m and K_y are dimensionless coeffi-

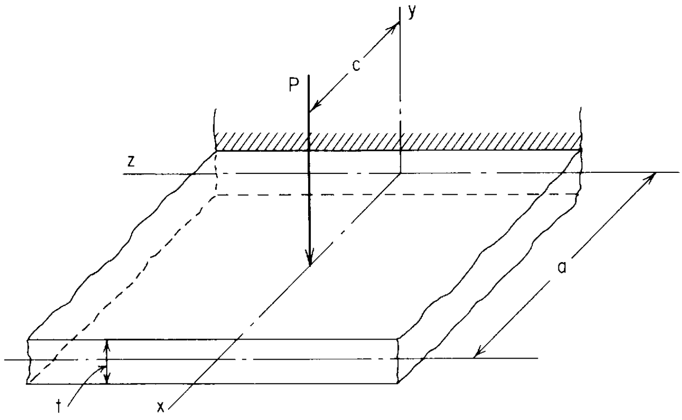


Figure 8.18

cients that depend upon the location of the load and the point, and D is as defined in Table 11.2. For the load at $x = c$, $z = 0$, the stress at any point on the fixed edge $x = 0$, $z = z$, and the deflection at any point on the free edge $x = a$, $z = z$, can be found by using the following values of K_m and K_y:

c/a \ z/a		0	0.25	0.50	1.0	1.5	2	∞
1.0	K_m	0.509	0.474	0.390	0.205	0.091	0.037	0
	K_y	0.524	0.470	0.380	0.215	0.108	0.049	0
0.75	K_m	0.428	0.387	0.284	0.140	0.059	0.023	0
	K_y	0.318	0.294	0.243	0.138	0.069	0.031	0
0.50	K_m	0.370	0.302	0.196	0.076	0.029	0.011	0
0.25	K_m	0.332	0.172	0.073	0.022	0.007	0.003	0

These values are based on the analysis of Jaramillo (Ref. 47), who assumes an infinite length for the plate, and are in good agreement, so far as comparable, with coefficients given by MacGregor (Ref. 26). They differ only slightly from results obtained by Holl (Ref. 27) for a length/span ratio of 4 and by Little (Ref. 49) for a length/span ratio of 5 and are in good agreement with available test data.

Wellauer and Seireg (Ref. 48) discuss the results of tests on beams of various proportions and explain and illustrate an empirical method by which the K_m values obtained by Jaramillo (Ref. 47) for the infinite plate under concentrated loading can be used to determine approximately the stress in a finite plate under an arbitrary transverse loading.

The stresses corresponding to the tabulated values of K_m are *spanwise* (x direction) stresses; the maximum *crosswise* (z direction) stress occurs under the load when the load is applied at the midpoint of the free edge and is approximately equal to the maximum spanwise stress for that loading.

Although the previous formulaes are based on the assumption of infinite width of a slab, tests (Ref. 26) on a plate with a width of $8\frac{1}{2}$ in and span a of $1\frac{1}{4}$ in showed close agreement between calculated and measured deflections, and Holl's analysis (Ref. 27), based on the assumption of a plate width four times the span, gives results that differ only slightly from MacGregor's (Ref. 26). The formulas given should therefore be applicable to slabs of breadth as small as four times the span.

8.12 Beams with Wide Flanges; Shear Lag

In thin metal construction, box, T-, or I-beams with very wide thin cover plates or flanges are sometimes used, and when a thin plate is stiffened by an attached member, a portion of the plate may be considered as a flange, acting integrally with the attached member which forms the web; examples are to be found in ship hulls, floors, tanks, and aircraft. In either type of construction the question arises as to what width of flange or plate would be considered effective; i.e., what width, uniformly stressed to the maximum stress that actually occurs, would provide a resisting moment equal to that of the actual stresses, which are greatest near the web and diminish as the distance from it increases.

This problem has been considered by several investigators; the results tabulated on page 174 are due to Hildebrand and Reissner (Ref. 38), Winter (Ref. 39), and Miller (Ref. 28).

Let $b =$ actual net width of the flange (or clear distance between webs in continuous plate and stiffener construction), let $l =$ span, and let $b' =$ effective width of the flange at the section of maximum bending moment. Then the approximate value of b'/b, which varies with the loading and with the ratio l/b, can be found for beams of uniform section in the table on p. 174. (In this table the case numbers refer to the manner of loading and support represented in Table 8.1.) See also Ref. 37.

Some of the more important conclusions stated in Ref. 38 can be summarized as follows.

The amount of shear lag depends not only on the method of loading and support and the ratio of span to flange width but also on the ratio of G to E and on the ratio $m = (3I_w + I_s)/(I_w + I_s)$, where I_w and I_s are

Ratio of effective width to total width b′/b for wide flanges

Case no. and load positions (from Table 8.1)	Reference no.	l/b													
		1	1.25	1.50	1.75	2	2.5	3	4	5	6	8	10	15	20
1a. $a = 0$	38	0.571	0.638	0.690	0.730	0.757	0.801	0.830	0.870	0.895	0.913	0.934	0.946		
2a. $w_a = w_l, a = 0$	38			0.550	0.600	0.632	0.685	0.724	0.780	0.815	0.842	0.876	0.899		
2a. $w_a = 0, a = 0$	38						0.609	0.650	0.710	0.751	0.784	0.826	0.858		
1e. $a = l/2$	38				0.530	0.571	0.638	0.686	0.757	0.801	0.830	0.870	0.895	0.936	0.946
1e. $a = l/2$	39							0.550	0.670	0.732	0.779	0.850	0.894	0.945	
1e. $a = l/2$	28						0.525			0.750					
1e. $a = l/4$	38				0.455	0.495	0.560	0.610	0.686	0.740	0.788	0.826	0.855	0.910	0.930
2e. $w_a = w_l, a = 0$	38				0.640	0.690	0.772	0.830	0.897	0.936	0.957	0.977	0.985	0.991	0.995

the moments of inertia about the neutral axis of the beam of the side plates and cover plates, respectively. (The values tabulated from Ref. 38 are for $G/E = 0.375$ and $m = 2$.) The value of b'/b increases with increasing m, but for values of m between 1.5 and 2.5 the variation is small enough to be disregarded. Shear lag at the critical section does not seem to be affected appreciably by the taper of the beam in width, but the taper in cover-plate thickness may have an important effect. In beams with fixed ends the effect of shear lag at the end sections is the same as for a cantilever of span equal to the distance from the point of inflection to the adjacent end.

In Ref. 39 it is stated that for a given l/b ratio the effect of shear lag is practically the same for box, I-, T-, and U-beams.

Flange in compression. The preceding discussion and tabulated factors apply to any case in which the flange is subjected to tension or to compression less than that required to produce elastic instability (see Chap. 15). When a thin flange or sheet is on the compression side, however, it may be stressed beyond the stability limit. For this condition, the effective width decreases with the actual stress. A formula for effective width used in aircraft design is

$$b' = Kt\sqrt{\frac{E}{s}}$$

where s is the maximum compressive stress (adjacent to the supporting web or webs) and K is a coefficient which may be conservatively taken as 0.85 for box beams and 0.60 for a T- or an I-beam having flanges with unsupported outer edges.

A theoretical analysis that takes into account both compressive buckling and shear lag is described in Ref. 40. Problems involving shear lag and buckling are most frequently encountered in design with thin-gage metal; good guides to such design are the books "Cold-Formed Steel Design Manual" in 5 parts including commentary, published in 1982 by the American Iron and Steel Institute, and "Aluminum Construction Manual," 4th ed., published in 1981 by the Aluminum Association. See also Ref. 68.

8.13 Beams with Very Thin Webs

In beams with extremely thin webs, such as are used in airplane construction, buckling due to shear will occur at stresses well below the elastic limit. This can be prevented if the web is made *shear-resistant* by the addition of stiffeners such as those used in plate girders, but the number of these required may be excessive. Instead of making the web shear-resistant, it may be permitted to buckle elastically without damage, the shear being carried wholly in *diagonal tension*. This tension tends to pull the upper and lower flanges together, and to prevent this, vertical struts are provided which carry the vertical component of the diagonal web tension. A girder so designed is, in effect, a Pratt truss, the web replacing the diagonal-tension members and the vertical struts constituting the compression members. In appearance, these struts resemble the stiffeners of an ordinary plate girder, but their function is obviously quite different.

A beam of this kind is called a *diagonal-tension field beam*, or *Wagner beam*, after Professor Herbert Wagner of Danzig, who is largely responsible for developing the theory. Because of its rather limited field of application, only one example of the Wagner beam will be considered here, viz. a cantilever under end load.

Let $P =$ end load, $h =$ depth of the beam, $t =$ thickness of the web, $d =$ spacing of the vertical struts, $x =$ distance from the loaded end to the section in question, H_t and $H_c =$ total stresses in the tension and compression flanges, respectively, at the given section, $C =$ total compression on a vertical strut, and $f =$ unit diagonal tensile stress in the web. Then

$$H_t = \frac{Px}{h} - \frac{1}{2}P, \qquad H_c = \frac{Px}{h} + \frac{1}{2}P, \qquad C = \frac{Pd}{h}, \qquad f = \frac{2P}{ht}$$

The vertical component of the web tension constitutes a beam loading on each individual flange between struts; the maximum value of the resulting bending moment occurs at the struts and is given by $M_f = \frac{1}{12}Pd^2/h$. The flexural stresses due to M_f must be added to the stresses due to H_t or H_c, which may be found simply by dividing H_t or H_c by the area of the corresponding flange.

The horizontal component of the web tension causes a bending moment $M = \frac{1}{8}Ph$ in the vertical strut at the end of the beam unless bending there is prevented by some system of bracing. This end strut must also distribute the load to the web, and should be designed to carry the load as a pin-ended column of length $\frac{1}{2}h$ as well as to resist the moment imposed by the web tension.

The intermediate struts are designed as pin-ended columns with lengths somewhat less than h. An adjacent portion of the web is included in the area of the column, the width of the strip considered effective being $30t$ in the case of aluminum and $60t$ in the case of steel.

Obviously the preceding formulas will apply also to a beam with end supports and center load if P is replaced by the reaction $\frac{1}{2}P$. Because of various simplifying assumptions made in the analysis, these formulas are conservative; in particular the formula for stress in the vertical struts or stiffeners gives results much larger than actual stresses that have been discovered experimentally. More accurate analyses, together with experimental data from various sources, will be found in Refs. 30, 34, 35, 62, and 69–71.

8.14 Beams Not Loaded in Plane of Symmetry; Flexural Center

The formulas for stress and deflection given in Sec. 8.1 are valid if the beam is loaded in a plane of symmetry; they are also valid if the applied loads are parallel to either principal central axis of the beam section, but unless the loads also pass through the *elastic axis*, the beam will be subjected to torsion as well as bending.

For the general case of a beam of any section loaded by a transverse load P in any plane, therefore, the solution comprises the following steps: (1) The load P is resolved into an equal and parallel force P' passing through the flexural center Q of the section, and a twisting couple T equal to the moment of P about Q; (2) P' is resolved at Q into rectangular components P'_u and P'_v, each parallel to a principal central axis of the section; (3) the flexural stresses and deflections due to P'_u and P'_v, are calculated independently by the formulas of Sec. 8.1 and superimposed to find the effect of P'; and (4) the stresses due to T are computed independently and superimposed on the stresses due to P', giving the stresses due to the actual loading. (It is to be noted that T may cause longitudinal fiber stresses as well as shear stresses. See Sec. 10.3 and the example at the end of this section.) If there are several loads, the effect of each is calculated separately and these effects added. For a distributed load the same procedure is followed as for a concentrated load.

The above procedure requires the determination of the position of the *flexural center* Q. For any section having two or more axes of symmetry (rectangle, I-beam, etc.) and for any section having a point of symmetry (equilateral triangle, Z-bar, etc.), Q is at the centroid. For any section having only one axis of symmetry, Q is on that axis but in general not at the centroid. For such sections and for unsymmetrical sections in general, the position of Q must be determined by calculation, direct experiment, or the soap-film method (Sec. 6.4).

Table 8.12 gives the position of the flexural center for each of a number of sections.

Neutral axis. When a beam is bent by one or more loads that lie in a plane not parallel to either principal central axis of the section, the neutral axis passes through the centroid but is not perpendicular to the plane of the loads. Let axes 1 and 2 be the principal central axes of the section, and let I_1 and I_2 represent the corresponding moments of inertia. Then, if the plane of the loads makes with axis 1 an angle α, the neutral axis makes with axis 2 an angle β such that $\tan\beta = (I_2/I_1)\tan\alpha$. It can be seen from this equation that the neutral axis tends to approach the principal central axis about which the moment of inertia is least.

EXAMPLE

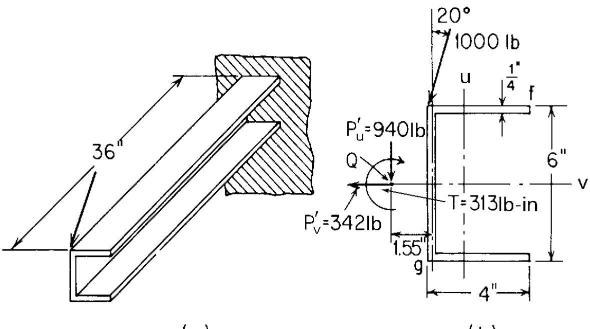
Figure 8.19(*a*) represents a cantilever beam of channel section under a diagonal end load applied at one corner. It is required to determine the maximum resulting fiber stress.

Solution. For the section (Fig. 8.19*b*): $I_u = 5.61\ \text{in}^4$, $I_v = 19.9\ \text{in}^4$; $b = 3.875$ in, $h = 5.75$ in, and $t = \frac{1}{4}$ in. By the formula from Table 8.12, $e = b^2h^2t/4I_v = 1.55$ in; therefore the flexural center is at Q, as shown. When the load is resolved into vertical and horizontal components at Q and a couple, the results are as shown in Fig. 8.19(*b*). (Vertical and horizontal components are used because the *principal central axes* u and v are vertical and horizontal.)

The maximum fiber stress will occur at the corner where the stresses due to the vertical and horizontal bending moments are of the same kind; at the upper-right corner f both stresses are tensile, and since f is farther from the u axis than the lower-left corner g where both stresses are compressive, it will sustain the greater stress. This stress will be simply the sum of the stresses due to the vertical and horizontal components of the load, or

$$\sigma = \frac{940(36)(3)}{19.9} + \frac{342(36)(2.765)}{5.61} = 5100 + 6070 = 11{,}200\ \text{lb/in}^2$$

The effect of the couple depends on the way in which the inner end of the beam is supported. If it is simply constrained against rotation in the planes of bending and twisting, the twisting moment will be resisted wholly by shear stress on the cross section, and these stresses can be found by the appropriate torsion formula of Table 10.1. If, however, the beam is built in so that the flanges are fixed in the *horizontal* plane, then part of the torque is resisted by the bending rigidity of the flanges and the corresponding moment causes a further fiber stress. This can be found by using the formulas of Sec. 10.3.



(a) (b)

Figure 8.19

For the channel section, K is given with sufficient accuracy by the formula $K = (t^3/3)(h + 2b)$ (Table 9.2, case 1), which gives $K = 0.073\ \text{in}^4$. Taking $G = 12{,}000{,}000\ \text{lb/in}^2$, and $E = 30{,}000{,}000\ \text{lb/in}^2$, and the formula for C_w as

$$C_w = \frac{h^2 b^3 t}{12}\frac{2h + 3b}{h + 6b} = 38.4\ \text{in}^6$$

the value for β can be found. From Table 10.3, the formula for β is given as

$$\beta = \left(\frac{KG}{C_w E}\right)^{1/2} = \left[\frac{0.073(12)}{38.4(30)}\right]^{1/2} = 0.0276$$

From Table 10.3, case 1b, the value of θ'' at the wall is given as

$$\theta'' = \frac{T_o}{C_w E \beta}\tanh \beta l = \frac{313}{38.4(30 \cdot 10^6)(0.0276)}\tanh 0.0276(36) = 7.47(10^{-6})\ \text{in}^{-2}$$

Therefore the longitudinal compressive stress at f can be found from the expression for σ_x in Table 10.2, case 1, as

$$\sigma_x = \frac{hb}{2}\frac{h + 3b}{h + 6b}E\theta'' = \frac{6(4)}{2}\frac{6 + 3(4)}{6 + 6(4)}(30)(10^6)(7.47)(10^{-6}) = 1610\ \text{lb/in}^2$$

The resultant fiber stress at f is $11{,}200 - 1610 = 9590\ \text{lb/in}^2$.

8.15 Straight Uniform Beams (Common Case); Ultimate Strength

When a beam is stressed beyond the elastic limit, plane sections remain plane or nearly so but unit stresses are no longer proportional to strains and hence no longer proportional to distance from the neutral surface. If the material has similar stress-strain curves in tension and compression, the stress distribution above and below the neutral surface will be similar and the neutral axis of any section which is symmetric about a horizontal axis will still pass through the centroid; if the material has different properties in tension and compression, then the neutral axis will shift away from the side on which the fibers yield the most; this shift causes an additional departure from the stress distribution assumed by the theory outlined in Sec. 8.1.

Failure In bending. The strength of a beam of ordinary proportions is determined by the maximum bending moment it can sustain. For beams of nonductile material (cast iron, concrete, or seasoned wood) this moment may be calculated by the formula $M_m = \sigma'(I/c)$ if σ', the *modulus of rupture*, is known. The modulus of rupture depends on the material and other factors (see Sec. 3.11), and attempts have been made to calculate it for a given material and section from the form of the complete stress-strain diagram. Thus for cast iron an approximate value of σ' may be found by the formula $\sigma' = K\sqrt{c/z'}\sigma_t$, where c is the distance to the extreme fiber, z' is the distance from the neutral axis to the centroid of the tensile part of the section, and K is an experimental coefficient equal to $\frac{6}{5}$ for sections that are flat at the top and bottom (rectangle, I, T, etc.) and $\frac{4}{3}$ for sections that are pointed or convex at the top and bottom (circle, diamond, etc.) (Ref. 4). Some tests indicate that this method of calculating the breaking strength of cast iron is sometimes inaccurate but generally errs on the side of safety (Ref. 5).

In general, the breaking strength of a beam can be predicted best from experimentally determined values of the rupture factor and ultimate strength or the form factor and modulus of rupture. The rupture factors are based on the ultimate tensile strength for all materials except wood, for which it is based on compressive strength. Form factors are based on a rectangular section. For structural steel, wrought aluminum, and other ductile metals, where beams do not actually break, the modulus of rupture means the computed fiber stress at the maximum bending moment (Refs. 6 to 9).

When the maximum bending moment occurs at but one section, as for a single concentrated load, the modulus of rupture is higher than when the maximum moment extends over a considerable part of the span. For instance, the modulus of rupture of short beams of brittle material is about 20 percent higher when determined by center loading than when determined by third-point loading. The disparity decreases as the span/depth ratio increases.

Beams of ductile material (structural steel or aluminum) do not ordinarily fracture under static loading but fail through excessive deflection. For such beams, if they are of relatively thick section so as to preclude local buckling, the maximum bending moment is that which corresponds to plastic yielding throughout the section. This maximum moment, or "plastic" moment, is usually denoted by M_p and can be calculated by the formula $M_p = \sigma_y Z$, where σ_y is the lower yield point of the material and Z, called the *plastic section modulus*, is the arithmetical sum of the statical moments about the neutral axis of the parts of the cross section above and below that axis. Thus, for a rectangular section of depth d and width b,

$$Z = (\tfrac{1}{2}bd)(\tfrac{1}{4}d) + (\tfrac{1}{2}bd)(\tfrac{1}{4}d) = \tfrac{1}{4}bd^2$$

This method of calculating the maximum resisting moment of a ductile-material beam is widely used in "plastic design" and is discussed further in Sec. 8.16. It is important to note that when the plastic moment has been developed, the neutral axis divides the cross-sectional area into halves and so is not always a centroidal axis. It is also important to note that the plastic moment is always greater than the moment required to just stress the extreme fiber to the lower yield point. This moment, which may be denoted by M_y, is equal to $\sigma_y I/c$, and so

$$\frac{M_p}{M_y} = \frac{Z}{I/c}$$

This ratio $Z/(I/c)$, called the *shape factor*, depends on the form of the cross section. For a solid rectangle it would be $\frac{1}{4}bd^2/\frac{1}{6}bd^2$, or 1.5; for an I-section it is usually about 1.15. Table A.1 gives formulas or numerical values for the plastic section modulus Z and for the shape factor for most of the cross sections listed.

In tubes and beams of thin open section, local buckling or crippling will sometimes occur before the full plastic resisting moment is realized, and the length of the member will have an influence. Tubes of steel or aluminum alloy generally will develop a modulus of rupture exceeding the ultimate tensile strength when the ratio of diameter to wall thickness is less than 50 for steel or 35 for aluminum. Wide-flanged steel beams will develop the full plastic resisting moment when the outstanding width/thickness ratio is less than 8.7 for $\sigma_y = 33{,}000\ \text{lb/in}^2$ or 8.3 for $\sigma_y = 36{,}000\ \text{lb/in}^2$. Charts giving the effective modulus of rupture of steel, aluminum, and magnesium tubes of various proportions may be found in Ref. 55.

Failure in shear. Failure by an actual shear fracture is likely to occur only in wood beams, where the shear strength parallel to the grain is, of course, small.

In I-beams and similar thin-webbed sections the diagonal compression that accompanies shear (Sec. 7.5) may lead to a buckling failure (see the discussion of *web buckling* that follows), and in beams of cast iron and concrete the diagonal tension that similarly accompanies shear may cause rupture. The formula for shear stress [Eq. (8.1-2)] may be considered valid as long as the *fiber* stresses do not exceed the proportional limit, and therefore it may be used to calculate the vertical shear necessary to produce failure in any case where the ultimate shearing strength of the beam is reached while the fiber stresses, at the section of maximum shear, are still within the proportional limit.

Web buckling; local failure. An I-beam or similar thin-webbed member may fail by buckling of the web owing to diagonal compression when the shear stress reaches a certain value. Ketchum and Draffin (Ref. 11) and Wendt and Withey (Ref. 12) found that in light I-beams this type of buckling occurs when the shear stress, calculated by

$\tau = 1.25\ V/\text{web area}$ (Ref. 11) or $\tau = V/\text{Web area}$ (Ref. 12), reaches a value equal to the unit load that can be carried by a vertical strip of the beam as a round-ended column. For the thin webs of the beams tested, such a thin strip would be computed as a Euler column; for heavier beams an appropriate parabolic or other formula should be used (Chap. 12).

In plate girders, web buckling may be prevented by vertical or diagonal stiffeners, usually consisting of double angles that are riveted or welded, one on each side of the web. Steel-construction specifications (Ref. 13) require that such stiffeners be provided when h/t exceeds 70 and v exceeds $64{,}000{,}000/(h/t)^2$. Such stiffeners should have a moment of inertia (figured for an axis at the center line of the web) equal to at least $0.00000016H^4$ and should be spaced so that the clear distance between successive stiffeners is not more than $11{,}000t/\sqrt{v}$ or 84 in, whichever is least. Here h is the clear depth of the web between flanges, t is the web thickness, v is the shear stress V/ht, and H is the total depth of the web. In light-metal airplane construction, the stiffeners are sometimes designed to have a moment of inertia about an axis parallel to the web given by $I = (2.29d/t)(Vh/33E)^{4/3}$, where $V =$ the (total) vertical shear and $d =$ the stiffener spacing center to center (Ref. 14).

Buckling failure may occur also as a result of vertical compression at a support or concentrated load, which is caused by either column-type buckling of the web (Refs. 11 and 12) or crippling of the web at the toe of the fillet (Ref. 15). To guard against this latter type of failure, present specifications provide that for interior loads $R/t(N + 2k) \leqslant 24{,}000$ and for end reactions $R/t(N + k) \leqslant 24{,}000$, where R is the concentrated load or end reaction, t the web thickness, N the length of bearing, and k the distance from the outer face of the flange to the web toe of the fillet. Here R is in pounds and all linear dimensions are in inches.

Wood beams will crush locally if the supports are too narrow or if a load is applied over too small a bearing area. The unit bearing stress in either case is calculated by dividing the force by the nominal bearing area, no allowance being made for the nonuniform distribution of pressure consequent upon bending (Ref. 9). Metal beams also may be subjected to high local pressure stresses; these are discussed in Chap. 14.

Lateral buckling. The compression flange of an I-beam or similar member may fail as a column as a result of lateral buckling if it is unsupported. Such buckling may be *elastic or plastic*; that is, it may occur at a maximum fiber stress below or above the elastic limit. In the first case the buckling is an example of elastic instability, for which relevant formulas are given in Table 15.1. For buckling above the elastic range analytical solutions are difficult to obtain, and empirical expressions based on experiment are used (as will be shown to be true also of the columns discussed in Chap. 12).

Moore (Ref. 16) found that standard I-beams fail by lateral buckling when

$$s' = 40{,}000 - 60\frac{ml}{r}$$

where s' is the compressive stress in the extreme fiber [computed by Eq. (8.1-1)], l is the span (in inches), r is the radius of gyration (in inches) of the beam section about a central axis parallel to the web, and m is a coefficient which depends on the manner of loading and support and has the following values:

Loading and support	Value of m
End supports, uniform load	0.667
End supports, midpoint load	0.500
End supports, single load at any point	0.500
End supports, loads at third points	0.667
End supports, loads at quarter points	0.750
End supports, loads at sixth points	0.833
Cantilever beam, uniform load	0.667
Cantilever beam, end load	1.000
Fixed-ended beam, uniform load	0.281
Fixed-ended beam, midpoint load	0.250

For very light I-beams, Ketchum and Draffin (Ref. 11) found that the lower limit of test results is given by

$$s' = 24{,}000 - 40\frac{ml}{r}$$

where the terms have the same meaning and m the same values as given previously.

The beams tested by Moore generally failed at stresses below but very close to the yield point and so probably could be regarded as representing plastic buckling. The lighter beams tested by Ketchum and Draffin, however, failed at stresses below the limit of proportionality and are examples of elastic buckling.

In Ref. 13 rules are given for the reduction in allowable compressive stress according to the unbraced length of the compression flange. A review of the literature on this subject of the lateral buckling of structural members and a bibliography through 1959 are to be found in Ref. 58.

Narrow rectangular beams may fail also as a result of buckling of the compression edge. When this buckling occurs below the elastic limit, the strength is determined by elastic stability; formulas for this case are given in Table 15.1. For buckling at stresses beyond the elastic limit, no simple formula for the critical stress can be given, but methods for calculating this critical stress are given for aluminum beams by Dumont and Hill (Ref. 17) and for wood beams by Trayer and March (Ref. 18).

8.16 Plastic, or Ultimate Strength, Design

The foregoing discussion of beams and frames is based for the most part on the assumption of purely elastic action and on the acceptance of maximum fiber stress as the primary criterion of safety. These constitute the basis of *elastic* analysis and design. An alternative and often preferred method of design, applicable to rigid frames and statically indeterminate beams made of materials capable of plastic action, is the method of *plastic*, or *ultimate strength*, design. It is based on the fact that such a frame or beam cannot deflect indefinitely or collapse until the full plastic moment M_p (see Sec. 8.15) has been developed at each of several critical sections. If it is assumed that the plastic moment—a determinable couple—does indeed act at each such section, then the problem becomes a statically determinate one and the load corresponding to the collapse condition can be readily calculated.

A simple illustration of the procedure is afforded by the beam of Fig. 8.20(*a*), corresponding to case lc of Table 8.1. Suppose it is desired to determine the maximum value of the load W that the beam can support. It is shown by elastic analysis, and is indeed apparent from inspection, that the maximum bending moments occur at the load and at the left end of the beam. The maximum possible value of each such moment is M_p. It is evident that the beam cannot collapse until the

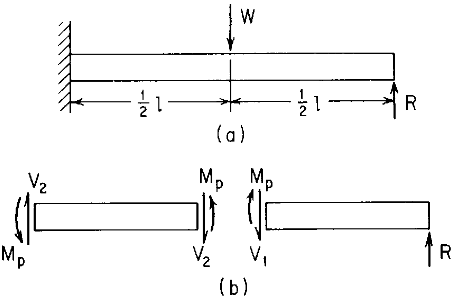


Figure 8.20

moment at each of these points reaches this value. Therefore, when W has reached its maximum value and collapse is imminent, the beam is acted on by the force system represented in Fig. 8.20(*b*); there is a *plastic hinge* and a known couple M_p at each of the critical sections and the problem is statically determinate. For equilibrium of the right half, $R = M_p/(l/2)$ and $V_1 = R$; and for equilibrium of the left half, $V_2 = W - R$ and $[W - M_p/(l/2)]l/2 = 2M_p$ or $W = 6M_p/l$.

In attempting to predict the collapse load on the basis of elastic analysis, it is easy to fall into the error of equating the maximum elastic moment $\frac{3}{16}Wl$ at the wall (Table 8.1) to M_p, thus obtaining $W = \frac{16}{3}M_p/l$. This erroneous procedure fails to take into account the fact that as W increases and yielding commences and progresses at the wall section, there is a redistribution of moments; the moment at the wall becomes less than $\frac{3}{16}Wl$, and the moment at the load becomes greater than $\frac{5}{32}Wl$ until finally each moment becomes equal to M_p. An important point to note is that although the elastic moments are affected by even a very slight departure from the assumed conditions—perfect fixity at one end and rigid support at the other—the collapse load is not thus affected. So long as the constraints are rigid enough to develop the plastic hinges as indicated, the ultimate load will be the same. Similarly, the method does not require that the beam be uniform in section, although a local reduction in section leading to the formation of a hinge at some point other than those assumed, of course, would alter the solution.

For a beam with multiple concentrated transverse loads or with distributed transverse loads, the locations of hinges are not known and must be assumed and verified. A virtual work approach to plastic collapse may permit a more rapid analysis than does the use of equilibrium equations, see Ref. 66. Verification consists of using the equilibrium conditions to construct a moment diagram and determine that no moments larger than the locally permitted values of the fully plastic moment are present.

Since nonlinear behavior does not permit superposition of results, one must consider all loads which are acting at a given time. If any of the several loads on a beam tend to cancel the maximum moments due to other loads, one must also consider the order in which the loads are applied in service to assure that the partially loaded beam has not collapsed before all loads have been applied.

Column 4 of Table A.1 contains an expression or a numerical value for the plastic section modulus Z and for the shape factor $SF = Zc/I$ for many of the cross sections. Using the plastic section modulus and the value of the yield strength of the material, one can find the full plastic moment M_p. Table 8.13 contains expressions for the loadings which will cause plastic collapse and the locations of the plastic hinges associated with each such loading.

The following example problems illustrate (1) the direct use of the tabulated material and (2) the use of the virtual work approach to a problem where two loads are applied simultaneously and where one plastic hinge location is not obvious.

EXAMPLES

1. A hollow aluminum cylinder is used as a transversely loaded beam 6 ft long with a 3-in outer diameter and a 1-in inner diameter. It is fixed at both ends and subjected to a distributed loading which increases linearly from zero at midspan to a maximum value w_l at the right end. The yield strength of this material is 27,000 psi, and the value of W_{lc} at plastic collapse is desired.

Solution. From Table A.1 case 15, the expression for the plastic section modulus is given as $Z = 1.333(R^3 - R_i^3)$, which gives

$$Z = 1.333(1.5^3 - 0.5^3) = 4.33 \text{ in}^3$$

and

$$M_p = 4.33(27{,}000) = 117{,}000 \text{ lb-in}$$

From Table 8.13 case 2d, with $w_a = 0$ for a uniformly increasing load, the locations of the fully developed plastic hinges are at the two ends and at a distance x_{h2} from the left end, where

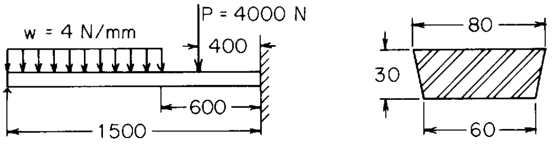
$$x_{h2} = a + \left(a^2 - al + \frac{l^2}{3} - \frac{a^3}{3l}\right)^{1/2}$$

Since $l = 72$ in and $a = 86$ in, the third hinge is found at $x_{h2} = 50.70$ in. The expression for the collapse load w_{lc} is given as

$$w_{lc} = \frac{12M_p(l-a)}{(l-x_{h2})(x_{h2}^2 - 3ax_{h2} + lx_{h2} + a^3/l)} = 1703 \text{ lb/in}$$

2. A steel beam of trapezoidal section is shown in Fig. 8.21. It is 1500 mm long, fixed at the right end and simply supported at the left end. The factor of safety of the loading shown is to be determined based on plastic collapse under a proportionately increased set of similar loads. The yield strength of the material is 200 N/mm^2 in both tension and compression. All dimensions are in millimeters.

Solution. First evaluate M_p. An examination of Table A.1 shows that the plastic section modulus for this cross section is not given. The yield strength in tension and compression is the same, so half the cross-sectional area of



(a) (b)

Figure 8.21

2100 mm^2 will be in tension and half in compression under a fully developed plastic hinge. Calculations show that a horizontal axis 16.066 mm above the base will divide this area into equal parts. The centroid of the upper portion is 7.110 mm above this axis, and the centroid of the lower portion is 7.814 mm below. Therefore

$$M_p = 200(1050)(7.110 + 7.814) = 3.134(10^6) \text{ N-mm}$$

Let the concentrated load at collapse P_c be accompanied by a uniformly distributed load w_c, which equals $P_c/1000$. During a virtual displacement of the beam when plastic rotation is taking place about the fully developed plastic hinges, the elastic deformations and any deformations due to the development of the plastic hinges remain constant and can be neglected in computing the work done by the loading. The angles shown in Fig. 8.22 are not true slopes at these locations but merely represent the virtual rotations of the fully developed plastic hinges. The location of hinge A is not known at this point in the solution, but it is either under the concentrated load or somewhere in the portion of the beam under the distributed load.

Trial 1. Assume that hinge A is under the concentrated load and the virtual displacements are represented by Fig. 8.22(*a*). The work performed by the loads during their vertical displacements is absorbed by the two plastic hinges. The hinge at A rotates through the angle $\theta + \phi$ and the hinge at B through the angle ϕ. Thus

$$M_p(\theta + \phi + \phi) = P_c 1100\theta + w_c(900)(450)\theta$$

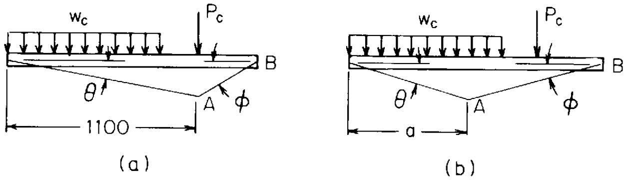
where $w_c = P_c/1000$ and from geometry $400\phi = 1100\theta$ so that $P_c = 4.319(10^{-3})M_p$, and from the equilibrium of the entire beam one obtains $R_1 = 3.206(10^{-3})M_p$. Using these two values and constructing a moment diagram, one finds that a maximum moment of $1.190M_p$ will be present at a distance of 742 mm from the left end.

Thus the assumption that hinge A was under the concentrated load was incorrect. A second trial solution will be carried out by assuming that hinge A is a distance a from the left end.

Trial 2. Figure 8.22(*b*) shows the virtual displacements for this second assumption. Again set the virtual work done by the loading equal to the energy absorbed by the two plastic hinges, or

$$M_p(\theta + 2\phi) = P_c 400\phi + \frac{w_c \theta a^2}{2} + w_c(900 - a)\left(1500 - a - \frac{900 - a}{2}\right)\phi$$

Note that $w_c = P_c/1000$ and from geometery $\phi(1500 - a) = \theta a$ so that $P_c = M_p[(1500 + a)/(1345a - 0.75a^2)]$. A minimum value of P_c is desired so



(a) (b)

Figure 8.22

this expression is differentiated with respect to a and the derivative set equal to zero. This leads to $a = 722.6$ mm and $P_c = 3.830(10^{-3})M_p$, a significantly smaller value than before and one which leads to a moment diagram with a maximum positive moment of M_p at $a = 722.6$ mm and a maximum negative moment of $-M_p$ at the right end. This then is the correct solution, and substituting the numerical value for M_p one gets $P_c = 12{,}000$ N and $w_c = 12$ N/mm. The applied loads were $P = 4000$ N and $w = 4$ N/mm, so the factor of safety is 3.0.

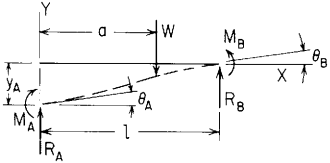
Because of the simplicity, these examples may give an exaggerated impression of the ease of plastic analysis, but they do indicate that for any indeterminate structure with strength that is determined primarily by resistance to bending, the method is well-suited to the determination of ultimate load and—through the use of a suitable factor of safety—to design. Its accuracy has been proved by good agreement between computed and experimental ultimate loads for both beams and frames. An extended discussion of plastic analysis is not appropriate here, but the interested reader will find an extensive literature on the subject (Refs. 60, 61).

8.17 Tables

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams

NOTATION: W = load (force); w = unit load (force per unit length); M_o = applied couple (force-length); θ_o externally created concentrated angular displacement (radians); Δ_o = externally created concentrated lateral displacement; T_1 and T_2 = temperatures on the top and bottom surfaces, respectively (degrees). R_A and R_B are the vertical end reactions at the left and right, respectively, and are positive upward. M_A and M_B are the reaction end moments at the left and right, respectively. All moments are positive when producing compression on the upper portion of the beam cross section. The transverse shear force V is positive when acting upward on the left end of a portion of the beam. All applied loads, couples, and displacements are positive as shown. All deflections are positive upward, and all slopes are positive when up and to the right. E is the modulus of elasticity of the beam material, and I is the area moment of inertia about the centroidal axis of the beam cross section. γ is the temperature coefficient of expansion (unit strain per degree)

1. Concentrated intermediate load



Transverse shear $= V = R_A - W\langle x-a\rangle^0$

Bending moment $= M = M_A + R_A x - W\langle x-a\rangle$

Slope $= \theta = \theta + \dfrac{M_A x}{EI} + \dfrac{R_A x^2}{2EI} - \dfrac{W}{2EI}\langle x-a\rangle^2$

Deflection $= y = y_A + \theta_A x + \dfrac{M_A x^2}{2EI} + \dfrac{R_A x^3}{6EI} - \dfrac{W}{6EI}\langle x-a\rangle^3$

(*Note*: see page 131 for a definition of the term $\langle x-a\rangle^n$.)

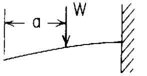
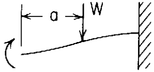
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
1a. Left end free, right end fixed (cantilever)  	$R_A = 0 \quad M_A = 0 \quad \theta_A = \dfrac{W(l-a)^2}{2EI}$ $y_A = \dfrac{-W}{6EI}(2l^3 - 3l^2a + a^3)$ $R_B = W \quad M_B = -W(l-a)$ $\theta_B = 0 \quad y_B = 0$	Max $M = M_B$; max possible value $= -Wl$ when $a = 0$ Max $\theta = \theta_A$; max possible value $= \dfrac{Wl^2}{2EI}$ when $a = 0$ Max $y = y_A$; max possible value $= \dfrac{-Wl^3}{3EI}$ when $a = 0$
1b. Left end guided, right end fixed  	$R_A = 0 \quad M_A = \dfrac{W(l-a)^2}{2l} \quad \theta_A = 0$ $y_A = \dfrac{-W}{12EI}(l-a)^2(l+2a)$ $R_B = W \quad M_B = \dfrac{-W(l^2-a^2)}{2l}$ $\theta_B = 0 \quad y_B = 0$	Max $+M = M_A$; max possible value $= \dfrac{Wl}{2}$ when $a = 0$ Max $-M = M_B$; max possible value $= \dfrac{-Wl}{2}$ when $a = 0$ Max $y = y_A$; max possible value $= \dfrac{-Wl^3}{12EI}$ when $a = 0$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

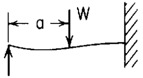
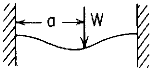
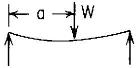
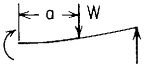
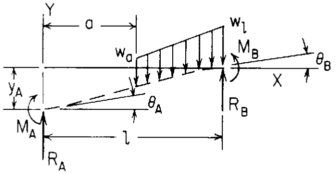
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
1c. Left end simply supported right end fixed  	$R_A = \frac{W}{2l^3}(l-a)^2(2l+a) \qquad M_A = 0$ $\theta_A = \frac{-Wa}{4EIl}(l-a)^2 \qquad y_A = 0$ $R_B = \frac{Wa}{2l^3}(3l^2-a^2) \qquad \theta_B = 0$ $M_B = \frac{-Wa}{2l^2}(l^2-a^2) \qquad y_B = 0$	$\text{Max} + M = \frac{Wa}{2l^3}(l-a)^2(2l+a)$ at $x = a$; max possible value $= 0.174Wl$ when $a = 0.366l$ $\text{Max} - M = M_B$; max possible value $= -0.1924Wl$ when $a = 0.5773l$ $\text{Max } y = \frac{-Wa}{6EI}(l-a)^2\left(\frac{a}{2l+a}\right)^{1/2}$ at $x = l\left(\frac{a}{2l+a}\right)^{1/2}$ when $a > 0.414l$ $\text{Max } y = \frac{-Wa(l^2-a^2)^3}{3EI(3l^2-a^2)^2}$ at $x = \frac{l(l^2+a^2)}{3l^2-a^2}$ when $a < 0.414l$; max possible $y = -0.0098\frac{Wl^3}{EI}$ when $x = a = 0.414l$
1d. Left end fixed, right end fixed  	$R_A = \frac{W}{l^3}(l-a)^2(l+2a)$ $M_A = \frac{-Wa}{l^2}(l-a)^2$ $\theta_A = 0 \qquad y_A = 0$ $R_B = \frac{Wa^2}{l^3}(3l-2a)$ $M_B = \frac{-Wa^2}{l^2}(l-a)$ $\theta_B = 0 \qquad y_B = 0$	$\text{Max} + M = \frac{2Wa^2}{l^3}(l-a)^2$ at $x = a$; max possible value $= \frac{Wl}{8}$ when $a = \frac{l}{2}$ $\text{Max} - M = M_A$ if $a < \frac{l}{2}$; max possible value $= -0.1481Wl$ when $a = \frac{l}{3}$ $\text{Max } y = \frac{-2W(l-a)^2a^3}{3EI(l+2a)^2}$ at $x = \frac{2al}{l+2a}$ if $a > \frac{l}{2}$; max possible value $= \frac{-Wl^3}{192EI}$ when $x = a = \frac{l}{2}$
1e. Left end simply supported, right end simply supported  	$R_A = \frac{W}{l}(l-a) \qquad M_A = 0$ $\theta_A = \frac{-Wa}{6EIl}(2l-a)(l-a) \qquad y_A = 0$ $R_B = \frac{Wa}{l} \qquad M_B = 0$ $\theta_B = \frac{Wa}{6EIl}(l^2-a^2) \qquad y_B = 0$	$\text{Max } M = R_A a$ at $x = a$; max possible value $= \frac{Wl}{4}$ when $a = \frac{l}{2}$ $\text{Max } y = \frac{-Wa}{3EIl}\left(\frac{l^2-a^2}{3}\right)^{3/2}$ at $x = l - \left(\frac{l^2-a^2}{3}\right)^{1/2}$ when $a < \frac{l}{2}$; max possible value $= \frac{-Wl^3}{48EI}$ at x $= \frac{l}{2}$ when $a = \frac{l}{2}$ $\text{Max } \theta = \theta_A$ when $a < \frac{l}{2}$; max possible value $= -0.0642\frac{Wl^2}{EI}$ when $a = 0.423l$
1f. Left end guided, right end simply supported  	$R_A = 0 \qquad M_A = W(l-a) \qquad \theta_A = 0$ $y_A = \frac{-W(l-a)}{6EI}(2l^2+2al-a^2)$ $R_B = W \qquad M_B = 0$ $\theta_B = \frac{W}{2EI}(l^2-a^2) \qquad y_B = 0$	$\text{Max } M = M_A$ for $0 < x < a$; max possible value $= Wl$ when $a = 0$ $\text{Max } \theta = \theta_B$; max possible value $= \frac{Wl^2}{2EI}$ when $a = 0$ $\text{Max } y = y_A$; max possible value $= \frac{-Wl^3}{3EI}$ when $a = 0$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

2. Partial distributed load



Transverse shear $= V = R_A - w_a\langle x-a\rangle - \dfrac{w_l - w_a}{2(l-a)}\langle x-a\rangle^2$

Bending moment $= M = M_A + R_A x - \dfrac{w_a}{2}\langle x-a\rangle^2 - \dfrac{w_l - w_a}{6(l-a)}\langle x-a\rangle^3$

Slope $= \theta = \theta_A + \dfrac{M_A x}{EI} + \dfrac{R_A x^2}{2EI} - \dfrac{w_a}{6EI}\langle x-a\rangle^3 - \dfrac{w_l - w_a}{24EI(l-a)}\langle x-a\rangle^4$

Deflection $= y = y_A + \theta_A + \theta_A x + \dfrac{M_A x^2}{2EI} + \dfrac{R_A x^3}{6EI} - \dfrac{w_a}{24EI}\langle x-a\rangle^4 - \dfrac{(w_l - w_a)}{120EI(l-a)}\langle x-a\rangle^5$

End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations

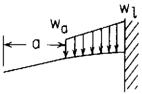
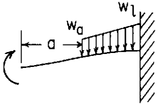
2a. Left end free, right end fixed (cantilever)

 	
	$R_A = 0 \qquad M_A = 0$
	$\theta_A = \dfrac{w_a}{6EI}(l-a)^3 + \dfrac{w_l - w_a}{24EI}(l-a)^3$
	$y_A = \dfrac{-w_a}{24EI}(l-a)^3(3l+a) - \dfrac{w_l - w_a}{120EI}(l-a)^3(4l+a)$
	$R_B = \dfrac{w_a + w_l}{2}(l-a)$
	$M_B = \dfrac{-w_a}{2}(l-a)^2 - \dfrac{w_l - w_a}{6}(l-a)^2$
	$\theta_B = 0 \qquad y_B = 0$
		If $a = 0$ and $w_l = w_a$ (uniform load on entire span), then
		Max $M = M_B = \dfrac{-w_a l^2}{2} \qquad$ Max $\theta = \theta_A = \dfrac{w_a l^3}{6EI}$
		Max $y = y_A = \dfrac{-w_a l^4}{8EI}$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		Max $M = M_B = \dfrac{-w_l l^2}{6} \qquad$ Max $\theta = \theta_A = \dfrac{w_l l^3}{24EI}$
		Max $y = y_A = \dfrac{-w_l l^4}{30EI}$
		If $a = 0$ and $w_l = 0$ (uniformly decreasing load), then
		Max $M = M_B = \dfrac{-w_a l^2}{3} \qquad$ Max $\theta = \theta_A = \dfrac{w_a l^3}{8EI}$
		Max $y = y_A = \dfrac{-11 w_a l^4}{120EI}$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations

2b. Left end guided, right end fixed

 	
	$R_A = 0 \qquad \theta_A = 0$
	$M_A = \frac{w_a}{6l}(l-a)^3 + \frac{w_l - w_a}{24l}(l-a)^3$
	$y_A = \frac{-w_a}{24EI}(l-a)^3(l+a) - \frac{w_l - w_a}{240EI}(l-a)^3(3l+2a)$
	$R_B = \frac{w_a + w_l}{2}(l-a)$
	$M_B = \frac{-w_a}{6l}(l-a)^2(2l+a) - \frac{w_l - w_a}{24l}(l-a)^2(3l+a)$
	$\theta_B = 0 \qquad y_B = 0$
		If $a = 0$ and $w_l = w_a$ (uniform load on entire span), then
		$\text{Max} - M = M_B = \frac{-w_a l^2}{3} \qquad \text{Max} + M = M_A = \frac{w_a l^2}{6}$
		$\text{Max } y = y_A = \frac{-w_a l^4}{24EI}$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$\text{Max} - M = M_B = \frac{-w_l l^2}{8} \qquad \text{Max} + M = M_A = \frac{w_l l^2}{24}$
		$\text{Max } y = y_A = \frac{-w_l l^4}{80EI}$
		If $a = 0$ and $w_l = 0$ (uniformly decreasing load), then
		$\text{Max} - M = M_B = \frac{-5w_a l^2}{24} \qquad \text{Max} + M = M_A = \frac{w_a l^2}{8}$
		$\text{Max } y = y_A = \frac{-7w_a l^4}{240EI}$

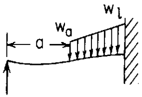
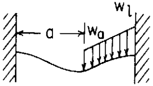
2c. Left end simply supported, right end fixed

 	
	$R_A = \frac{w_a}{8l^3}(l-a)^3(3l+a) + \frac{w_l - w_a}{40l^3}(l-a)^3(4l+a)$
	$\theta_A = \frac{-w_a}{48EIl}(l-a)^3(l+3a) - \frac{w_l - w_a}{240EIl}(l-a)^3(2l+3a)$
	$M_A = 0 \qquad y_A = 0$
	$R_B = \frac{w_a + w_l}{2}(l-a) - R_A$
	$M_B = R_A l - \frac{w_a}{2}(l-a)^2 - \frac{w_l - w_a}{6}(l-a)^2$
	$\theta_B = 0 \qquad y_B = 0$
		If $a = 0$ and $w_l = w_a$ (uniform load on entire span), then
		$R_A = \frac{3}{8}w_a l \qquad R_B = \frac{5}{8}w_a l \qquad \text{Max} - M = M_B = \frac{-w_a l^2}{8}$
		$\text{Max} + M = \frac{9w_a l^2}{128} \text{ at } x = \frac{3}{8}l \qquad \text{Max } \theta = \theta_A = \frac{-w_a l^3}{48EI}$
		$\text{Max } y = -0.0054\frac{w_a l^4}{EI} \text{ at } x = 0.4215l$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$R_A = \frac{w_l l}{10} \qquad R_B = \frac{2w_l l}{5} \qquad \text{Max} - M = M_B = \frac{-w_l l^2}{15}$
		$\text{Max} + M = 0.0298w_l l^2 \text{ at } x = 0.4472l \qquad \text{Max } \theta = \theta_A = \frac{-w_l l^3}{120EI}$
		$\text{Max } y = -0.00239\frac{w_l l^4}{EI} \text{ at } x = 0.4472l$
		If $a = 0$ and $w_l = 0$ (uniformly decreasing load), then
		$R_A = \frac{11}{40}w_a l \qquad R_B = \frac{9}{40}w_a l \qquad \text{Max} - M = M_B = \frac{-7}{120}w_a l^2$
		$\text{Max} + M = 0.0422w_a l^2 \text{ at } x = 0.329l$
		$\text{Max } \theta = \theta_A = \frac{-w_a l^3}{80EI} \qquad \text{Max } y = -0.00304\frac{w_a l^4}{EI}; \text{ at } x = 0.4025l$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

2d. Left end fixed, right end fixed

 	
	$R_A = \frac{w_a}{2l^3}(l-a)^3(l+a) + \frac{w_l - w_a}{20l^3}(l-a)^3(3l+2a)$
	$M_A = \frac{-w_a}{12l^2}(l-a)^3(l+3a) - \frac{w_l - w_a}{60l^2}(l-a)^3(2l+3a)$
	$\theta_A = 0 \qquad y_A = 0$
	$R_B = \frac{w_a + w_l}{2}(l-a) - R_A$
	$M_B = R_A l + M_A - \frac{w_a}{2}(l-a)^2 - \frac{w_l - w_a}{6}(l-a)^2$
	$\theta_B = 0 \qquad y_B = 0$
		If $a = 0$ and $w_l = w_a$ (uniform load on entire span), then
		$\text{Max} - M = M_A = M_B = \frac{-w_a l^2}{12} \qquad \text{Max} + M = \frac{w_a l^2}{24}$ at $x = \frac{l}{2}$
		$\text{Max}\ y = \frac{-w_a l^4}{384EI}$ at $x = \frac{l}{2}$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$R_A = \frac{3w_l l}{20} \qquad M_A = \frac{-w_l l^2}{30} \qquad R_B = \frac{7w_l l}{20} \qquad \text{Max} - M = M_B = \frac{-w_l l^2}{20}$
		$\text{Max} + M = 0.0215 w_l l^2$ at $x = 0.548l$
		$\text{Max}\ y = -0.001309 \frac{w_l l^4}{EI}$ at $x = 0.525l$

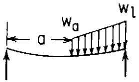
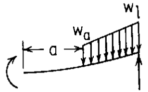
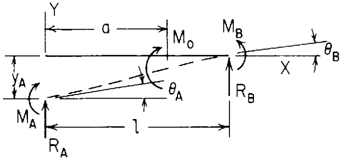
2e. Left end simply supported, right end simply supported

 	
	$R_A = \frac{w_a}{2l}(l-a)^2 + \frac{w_l - w_a}{6l}(l-a)^2$
	$M_A = 0 \qquad y_A = 0$
	$\theta_A = \frac{-w_a}{24EIl}(l-a)^2(l^2 + 2al - a^2)$
	$\quad - \frac{w_l - w_a}{360EIl}(l-a)^2(7l^2 + 6al - 3a^2)$
	$R_B = \frac{w_a + w_l}{2}(l-a) - R_A$
	$\theta_B = \frac{w_a}{24EIl}(l^2 - a^2)^2$
	$\quad + \frac{w_l - w_a}{360EIl}(l-a)^2(8l^2 + 9al + 3a^2)$
	$M_B = 0 \qquad y_B = 0$
		If $a = 0$ and $w_l = w_a$ (uniform load on entire span), then
		$R_A = R_B = \frac{w_a l}{2} \qquad \text{Max}\ M = \frac{w_a l^2}{8}$ at $x = \frac{l}{2}$
		$\text{Max}\ \theta = \theta_B = \frac{w_a l^3}{24EI} \qquad \text{Max}\ y = \frac{-5w_a l^4}{384EI}$ at $x = \frac{l}{2}$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$R_A = \frac{w_l l}{6} \qquad R_B = \frac{w_l l}{3} \qquad \text{Max}\ M = 0.0641 w_l l^2$ at $x = 0.5773l$
		$\theta_A = \frac{-7w_l l^3}{360EI} \qquad \theta_B = \frac{w_l l^3}{45EI}$
		$\text{Max}\ y = -0.00653 \frac{w_l l^4}{EI}$ at $x = 0.5195l$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations

2f. Left end guided, right end simply supported

 	
	$R_A = 0 \qquad \theta_A = 0$
	$M_A = \frac{w_a}{2}(l-a)^2 + \frac{w_l - w_a}{6}(l-a)^2$
	$y_A = \frac{-w_a}{24EI}(l-a)^2(5l^2+2al-a^2)$
	$- \frac{w_l - w_a}{120EI}(l-a)^2(9l^2+2al-a^2)$
	$R_B = \frac{w_a + w_l}{2}(l-a)$
	$\theta_B = \frac{w_a}{6EI}(l-a)^2(2l+a) + \frac{w_l - w_a}{24EI}(l-a)^2(3l+a)$
	$M_B = 0 \qquad y_B = 0$
		If $a = 0$ and $w_l = w_a$ (uniform load on entire span), then
		$\text{Max } M = M_A = \frac{w_a l^2}{2} \qquad \text{Max } \theta = \theta_B = \frac{w_a l^3}{3EI}$
		$\text{Max } y = y_A = \frac{-5w_a l^4}{24EI}$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$\text{Max } M = M_A = \frac{w_l l^2}{6} \qquad \text{Max } \theta = \theta_B = \frac{w_l l^3}{8EI}$
		$\text{Max } y = y_A = \frac{-3w_l l^4}{40EI}$
		If $a = 0$ and $w_l = 0$ (uniformly decreasing load), then
		$\text{Max } M = M_A = \frac{w_a l^2}{3} \qquad \text{Max } \theta = \theta_B = \frac{5w_a l^3}{24EI}$
		$\text{Max } y = y_A = \frac{-2w_a l^4}{15EI}$

3. Concentrated intermediate moment



Transverse shear $= V = R_A$

Bending moment $= M = M_A + R_A x + M_o \langle x-a \rangle^0$

Slope $= \theta = \theta_A + \frac{M_A x}{EI} + \frac{R_A x^2}{2EI} + \frac{M_o}{EI}\langle x-a \rangle$

Deflection $= y = y_A + \theta_A x + \frac{M_A x^2}{2EI} + \frac{R_A x^3}{6EI} + \frac{M_o}{2EI}\langle x-a \rangle^2$

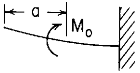
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
3a. Left end free, right end fixed (cantilever)  	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{-M_o(l-a)}{EI}$ $y_A = \frac{M_o(l^2-a^2)}{2EI}$ $R_B = 0 \qquad M_B = M_o$ $\theta_B = 0 \qquad y_B = 0$	$\text{Max } M = M_o$ $\text{Max } \theta = \theta_A$; max possible value $= \frac{-M_o l}{EI}$ when $a = 0$ $\text{Max } y = y_A$; max possible value $= \frac{M_o l^2}{2EI}$ when $a = 0$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

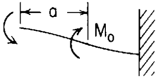
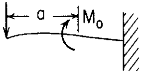
3b Left end guided, right end fixed  	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{-M_o(l-a)}{l}$ $y_A = \frac{M_o a(l-a)}{2EI}$ $R_B = 0 \qquad \theta_B = 0$ $M_B = \frac{M_o a}{l} \qquad y_B = 0$	Max $+M = M_B$; max possible value $= M_o$ when $a = l$ Max $-M = M_A$; max possible value $= -M_o$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{M_o l^2}{8EI}$ when $a = \frac{l}{2}$
3c. Left end simply supported, right end fixed  	$R_A = \frac{-3M_o}{2l^3}(l^2 - a^2)$ $\theta_A = \frac{M_o}{4EIl}(l-a)(3a-l)$ $M_A = 0 \qquad y_A = 0$ $R_B = \frac{3M_o}{2l^3}(l^2 - a^2)$ $M_B = \frac{M_o}{2l^2}(3a^2 - l^2)$ $\theta_B = 0 \qquad y_B = 0$	Max $+M = M_o + R_A a$ just right of $x = a$; max possible value $= M_o$ when $a = 0$ or $a = l$; min possible value $= 0.423M_o$ when $a = 0.577l$ Max $-M = \frac{-3M_o a}{2l^3}(l^2 - a^2)$ just left of $x = a$ if $a > 0.282l$; max possible value $= -0.577M_o$ when $a = 0.577l$ Max $-M = \frac{-M_o}{2l^2}(l^2 - 3a^2)$ at B if $a < 0.282l$; max possible value $= -0.5M_o$ when $a = 0$ Max $+y = \frac{M_o(l-a)}{6\sqrt{3(l+a)}EI}(3a-l)^{3/2}$ at $x = l\sqrt{\frac{3a-l}{3l+3a}}$; max possible value $= 0.0257\frac{M_o l^2}{EI}$ at $x = 0.474l$ when $a = 0.721l$ $\left(\text{Note: There is no positive deflection if } a < \frac{l}{3}\right)$ Max $-y$ occurs at $x = \frac{2l^3}{3(l^2 - a^2)}\left[1 - \frac{1}{2}\sqrt{1 - 6\left(\frac{a}{l}\right)^2 + 9\left(\frac{a}{l}\right)^4}\right]$; max possible value $= \frac{-M_0 l^2}{27EI}$ at $x = \frac{l}{3}$ when $a = 0$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

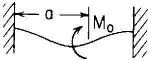
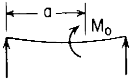
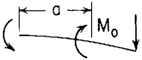
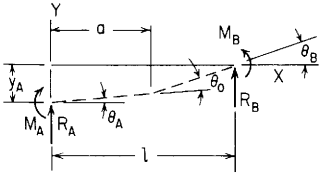
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
3d. Left end fixed, right end fixed  	$R_A = \frac{-6M_o a}{l^3}(l-a)$ $M_A = \frac{-M_o}{l^2}(l^2 - 4al + 3a^2)$ $\theta_A = 0 \qquad y_A = 0$ $R_B = -R_A$ $M_B = \frac{M_o}{l^2}(3a^2 - 2al)$ $\theta_B = 0 \qquad y_B = 0$	$\text{Max} + M = \frac{M_o}{l^3}(4al^2 - 9a^2 l + 6a^3)$ just right of $x = a$; max possible value $= M_o$ when $a = l$ $\text{Max} - M = \frac{M_o}{l^3}(4al^2 - 9a^2 l + 6a^3 - l^3)$ just left of $x = a$; max possible value $= -M_o$ when $a = 0$ $\text{Max} + y = \frac{2M_A^3}{3R_A^2 EI}$ at $x = \frac{l}{3a}(3a - l)$; max possible value $= 0.01617\frac{M_o l^2}{EI}$ at $x = 0.565l$ when $a = 0.767l$ $\left(\textit{Note}\text{: There is no positive deflection if } a < \frac{l}{3}\right)$
3e. Left end simply supported, right end simply supported  	$R_A = \frac{-M_o}{l}$ $\theta_A = \frac{-M_o}{6EIl}(2l^2 - 6al + 3a^2)$ $M_A = 0 \qquad y_A = 0$ $R_B = \frac{M_o}{l}$ $\theta_B = \frac{M_o}{6EIl}(l^2 - 3a^2)$ $M_B = 0 \qquad y_B = 0$	$\text{Max} + M = \frac{M_o}{l}(l - a)$ just right of $x = a$; max possible value $= M_o$ when $a = 0$ $\text{Max} - M = \frac{-M_o a}{l}$ just left of $x = a$; max possible value $= -M_o$ when $a = l$ $\text{Max} + y = \frac{M_o(6al - 3a^2 - 2l^2)^{3/2}}{9\sqrt{3}EIl}$ at $x = (2al - a^2 - \frac{2}{3}l^2)^{1/2}$ when $a > 0.423l$; max possible value $= 0.0642\frac{M_o l^2}{EI}$ at $x = 0.577l$ when $a = l$ (*Note*: There is no positive deflection if $a < 0.423l$)
3f. Left end guided, right end simply supported  	$R_A = 0 \qquad \theta_A = 0$ $M_A = -M_o$ $y_A = \frac{M_o a}{2EI}(2l - a)$ $R_B = 0 \qquad M_B = 0 \qquad y_B = 0$ $\theta_B = \frac{-M_o a}{EI}$	$\text{Max } M = -M_o$ for $0 < x < a$ $\text{Max } \theta = \theta_B$; max possible value $= \frac{-M_o l}{EI}$ when $a = l$ $\text{Max } y = y_A$; max possible value $= \frac{M_o l^2}{2EI}$ when $a = l$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

4. Intermediate externally created angular deformation



Transverse shear $= V = R_A$

Bending moment $= M = M_A + R_A x$

Slope $= \theta = \theta_A + \frac{M_A x}{EI} + \frac{R_A x^2}{2EI} + \theta_o \langle x - a \rangle^0$

Deflection $= y = y_A + \theta_A x + \frac{M_A x^2}{2EI} + \frac{R_A x^3}{6EI} + \theta_o \langle x - a \rangle$

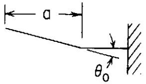
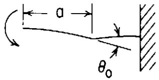
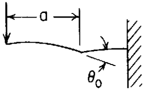
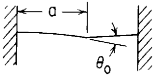
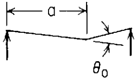
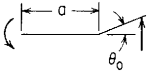
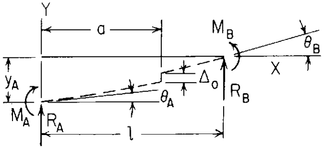
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
4a. Left end free, right end fixed 	$R_A = 0 \quad M_A = 0$ $\theta_A = -\theta_o \quad y_A = \theta_o a$ $R_B = 0 \quad M_B = 0$ $\theta_B = 0 \quad y_B = 0$	Max $y = y_A$, max possible value $= \theta_o l$ when $a = l$
4b. Left end guided, right end fixed 	$R_A = 0 \quad M_A = \frac{-EI\theta_o}{l}$ $\theta_A = 0 \quad y_A = \theta_o\left(a - \frac{l}{2}\right)$ $R_B = 0 \quad M_B = \frac{-EI\theta_o}{l}$ $\theta_B = 0 \quad y_B = 0$	Max $M = M_A$ Max $+y = y_A$ when $a > \frac{l}{2}$; max possible value $= \frac{\theta_o l}{2}$ when $a = l$ Max $-y = \frac{-\theta_o}{2l}(l - a)^2$ at $x = a$; max possible value $= \frac{-\theta_o l}{2}$ when $a = 0$
4c. Left end simply supported, right end fixed 	$M_A = 0 \quad y_A = 0$ $R_A = \frac{-3EIa\theta_o}{l^3}$ $\theta_A = -\theta_o\left(1 - \frac{3a}{2l}\right)$ $R_B = -R_A$ $M_B = \frac{-3EIa\theta_o}{l^2}$ $\theta_B = 0 \quad y_B = 0$	Max $M = M_B$; max possible value $= \frac{-3EI\theta_o}{l}$ when $a = l$ Max $+y = \theta_o a\left(1 - \frac{2l}{3a}\right)^{3/2}$ at $x = l\left(1 - \frac{2l}{3a}\right)^{1/2}$ when $a \geq \frac{2}{3}l$; max possible value $= 0.1926\theta_o l$ at $x = 0.577l$ when $a = l$ (*Note*: There is no positive deflection if $a < \frac{2}{3}l$) Max $-y = -\theta_o a\left(1 - \frac{3a}{2l} + \frac{a^3}{2l^3}\right)$ at $x = a$; max possible value $= -0.232\theta_o l$ at $x = 0.366l$ when $a = 0.366l$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
4d. Left end fixed, right end fixed 	$R_A = \frac{6EI\theta_o}{l^3}(l-2a)$ $M_A = \frac{2EI\theta}{l^2}(3a-2l)$ $\theta_A = 0 \qquad y_A = 0$ $R_B = -R_A$ $M_B = \frac{2EI\theta_o}{l^2}(l-3a)$ $\theta_B = 0 \qquad y_B = 0$	Max $+M = M_B$ when $a < \frac{l}{2}$; max possible value $= \frac{2EI\theta_o}{l}$ when $a = 0$ Max $-M = M_A$ when $a < \frac{l}{2}$; max possible value $= \frac{-4EI\theta_o}{l}$ when $a = 0$ Max $+y$ occurs at $x = \frac{l^2}{3(l-2a)}$ if $a < \frac{l}{3}$; max possible value $= \frac{4}{27} l\theta_o$ when $a = 0$ (*Note*: There is no positive deflection if $\frac{l}{3} < a < \frac{2}{3}l$) Max $-y = \frac{-2\theta_o a^2}{l^3}(l-a)^2$ at $x = a$; max possible value $= \frac{-\theta_o l}{8}$ when $a = \frac{l}{2}$
4e. Left end simply supported, right end simply supported 	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{-\theta}{l}(l-a) \qquad y_A = 0$ $R_B = 0 \qquad M_B = 0$ $y_B = 0 \qquad \theta_B = \frac{\theta_o}{l}$	Max $y = \frac{-\theta_o a}{l}(l-a)$ at $x = a$; max possible value $= \frac{-\theta_o l}{4}$ when $a = \frac{l}{2}$
4f. Left end guided, right end simply supported 	$R_A = 0 \qquad M_A = 0$ $\theta_A = 0 \qquad y_A = -\theta_o(l-a)$ $R_B = 0 \qquad M_B = 0$ $y_B = 0 \qquad \theta_B = \theta_o$	Max $y = y_A$; max possible value $= -\theta_o l$ when $a = 0$
5. Intermediate externally created lateral displacement 	Transverse shear $= V = R_A$ Bending moment $= M = M_A + R_A x$ Slope $= \theta = \theta_A + \frac{M_A x}{EI} + \frac{R_A x^2}{2EI}$ Deflection $= y = y_A + \theta_A x + \frac{M_A x^2}{2EI} + \frac{R_A x^3}{6EI} + \Delta_o \langle x-a \rangle^0$	

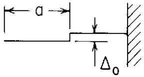
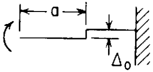
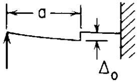
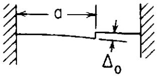
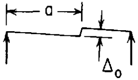
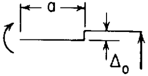
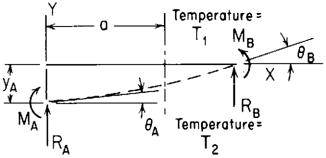
5a. Left end free, right end fixed 	$R_A = 0 \quad M_A = 0$ $\theta_A = 0 \quad y_A = -\Delta_o$ $R_B = 0 \quad M_B = 0$ $\theta_B = 0 \quad y_B = 0$	Max $y = y_A$ when $x < a$
5b. Left end guided, right end fixed 	$R_A = 0 \quad M_A = 0$ $\theta_A = 0 \quad y_A = -\Delta_o$ $R_B = 0 \quad M_B = 0$ $\theta_B = 0 \quad y_B = 0$	Max $y = y_A$ when $x < a$
5c. Left end simply supported, right end fixed 	$R_A = \frac{3EI\Delta_o}{l^3} \quad M_A = 0$ $\theta_A = \frac{-3\Delta_o}{2l} \quad y_A = 0$ $R_B = -R_A \quad M_B = \frac{3EI\Delta_o}{l^2}$ $\theta_B = 0 \quad y_B = 0$	Max $M = M_B$ Max $\theta = \theta_A$ Max $+y = \frac{\Delta_o}{2l^3}(2l^3 + a^3 - 3l^2a)$ just right of $x = a$; max possible value $= \Delta_o$ when $a = 0$ Max $-y = \frac{-\Delta_o a}{2l^3}(3l^2 - a^2)$ just left of $x = a$; max possible value $= -\Delta_o$ when $a = l$
5d. Left end fixed, right end fixed 	$R_A = \frac{12EI\Delta_o}{l^3} \quad \theta_A = 0$ $M_A = \frac{-6EI\Delta_o}{l^2} \quad y_A = 0$ $R_B = -R_A \quad M_B = -M_A$ $\theta_B = 0 \quad y_B = 0$	Max $+M = M_B$ Max $-M = M_A$ Max $\theta = \frac{-3\Delta_o}{2l}$ at $x = \frac{l}{2}$ Max $+y = \frac{\Delta_o}{l^3}(l^3 + 2a^3 - 3a^2l)$ just right of $x = a$; max possible value $= \Delta_o$ when $a = 0$ Max $-y = \frac{-\Delta_o a^2}{l^3}(3l - 2a)$ just left of $x = a$; max possible value $= -\Delta_o$ when $a = l$
5e. Left end simply supported, right end simply supported 	$R_A = 0 \quad M_A = 0$ $y_A = 0 \quad \theta_A = \frac{-\Delta_o}{l}$ $R_B = 0 \quad M_B = 0$ $y_B = 0 \quad \theta_B = \frac{-\Delta_o}{l}$	Max $+y = \frac{\Delta_o}{l}(l - a)$ just right of $x = a$; max possible value $= \Delta_o$ when $a = 0$ Max $-y = \frac{-\Delta_o a}{l}$ just left of $x = a$; max possible value $= -\Delta_o$ when $a = l$
5f. Left end guided, right end simply supported 	$R_A = 0 \quad M_A = 0$ $\theta_A = 0 \quad y_A = -\Delta_o$ $R_B = 0 \quad M_B = 0$ $\theta_B = 0 \quad y_B = 0$	Max $y = y_A$ when $x < a$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

6. Uniform temperature variation from top to bottom from a to l



Transverse shear $= V = R_A$

Bending moment $= M = M_A + R_A x$

$$\text{Slope} = \theta = \theta_A + \frac{M_A x}{EI} + \frac{R_A x^2}{2EI} + \frac{\gamma}{t}(T_2 - T_1)\langle x - a \rangle$$

$$\text{Deflection} = y = y_A + \theta_A x + \frac{M_A x^2}{2EI} + \frac{R_A x^3}{6EI} + \frac{\gamma}{2t}(T_2 - T_1)\langle x - a \rangle^2$$

$\gamma =$ temperature coefficient of expansion (unit strain/°) $t =$ depth of beam

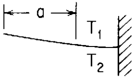
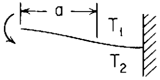
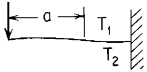
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
6a. Left end free, right end fixed  	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{-\gamma}{t}(T_2 - T_1)(l - a)$ $y_A = \frac{\gamma}{2t}(T_2 - T_1)(l^2 - a^2)$ $R_B = 0 \qquad M_B = 0$ $\theta_B = 0 \qquad y_B = 0$	$M = 0$ everywhere Max $\theta = \theta_A$; max possible value $= \frac{-\gamma l}{t}(T_2 - T_1)$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{\gamma l^2}{2t}(T_2 - T_1)$ when $a = 0$
6b. Left end guided, right end fixed  	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{-EI\gamma}{lt}(T_2 - T_1)(l - a)$ $y_A = \frac{a\gamma}{2t}(T_2 - T_1)(l - a)$ $R_B = 0 \qquad M_B = M_A$ $\theta_B = 0 \qquad y_B = 0$	$M = M_A$ everywhere; max possible value $= \frac{-EI\gamma}{t}(T_2 - T_1)$ when $a = 0$ Max $\theta = \frac{-a\gamma}{lt}(T_2 - T_1)(l - a)$ *at* $x = a$; max possible value $= \frac{-\gamma l}{4t}(T_2 - T_1)$ when $a = \frac{l}{2}$ Max $y = y_A$; max possible value $= \frac{\gamma l^2}{8t}(T_2 - T_1)$ when $a = \frac{l}{2}$
6c. Left end simply supported, right end fixed  	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{-3EI\gamma}{2tl^3}(T_2 - T_1)(l^2 - a^2)$ $\theta_A = \frac{\gamma}{4tl}(T_2 - T_1)(l - a)(3a - l)$ $R_B = -R_A \qquad M_B = R_A l$ $\theta_B = 0 \qquad y_B = 0$	Max $M = M_B$; max possible value $= \frac{-3EI\gamma}{2t}(T_2 - T_1)$ when $a = 0$ Max $+ y = \frac{\gamma(T_2 - T_1)(l - a)}{6t\sqrt{3(l + a)}}(3a - l)^{3/2}$ at $x = l\left(\frac{3a - l}{3l + 3a}\right)^{1/2}$; max possible value $= 0.0257\frac{\gamma l^2}{t}(T_2 - T_1)$ at $x = 0.474l$ when $a = 0.721l$ (*Note*: There is no positive deflection if $a < l/3$) Max $- y$ occurs at $x = \frac{2l^3}{3(l^2 - a^2)}\left[1 - \frac{1}{2}\sqrt{1 - 6\left(\frac{a}{l}\right)^2 + 9\left(\frac{a}{l}\right)^4}\right]$; max possible value $= \frac{-\gamma l^2}{27t}(T_2 - T_1)$ at $x = \frac{l}{3}$ when $a = 0$

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams (*Continued*)

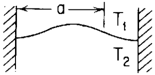
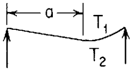
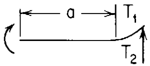
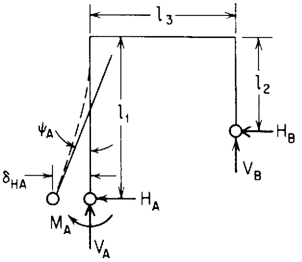
6d. Left end fixed, right end fixed  	$R_A = \frac{-6EIa\gamma}{tl^3}(T_2 - T_1)(l-a)$ $M_A = \frac{EI\gamma}{tl^2}(T_2 - T_1)(l-a)(3a-l)$ $\theta_A = 0 \qquad y_A = 0$ $R_B = -R_A$ $M_B = \frac{-EI\gamma}{tl^2}(T_2 - T_1)(l-a)(3a+l)$ $\theta_B = 0 \qquad y_B = 0$	Max $+M = M_A$; max possible value $= \frac{EI\gamma}{3t}(T_2 - T_1)$ when $a = \frac{2}{3}l$ $\left(\textit{Note}\text{: There is no positive moment if } a < \frac{l}{3}\right)$ Max $-M = M_B$; max possible value $= \frac{-4EI\gamma}{3t}(T_2 - T_1)$ when $a = \frac{l}{3}$ Max $+y = \frac{2M_A^2}{3R_A^2 EI}$ at $x = \frac{l}{3a}(3l-a)$; max possible value $= 0.01617\frac{\gamma l^2}{t}(T_2 - T_1)$ at $x = 0.565l$ when $a = 0.767l$ $\left(\textit{Note:}\text{ There is no positive deflection if } a < \frac{l}{3}\right)$
6e. Left end simply supported, right end simply supported  	$R_A = 0 \qquad M_A = 0 \qquad y_A = 0$ $\theta_A = \frac{-\gamma}{2tl}(T_2 - T_1)(l-a)^2$ $R_B = 0 \qquad M_B = 0 \qquad y_B = 0$ $\theta_B = \frac{\gamma}{2tl}(T_2 - T_l)(l^2 - a^2)$	$M = 0$ everywhere Max $+\theta = \theta_B$; max possible value $= \frac{\gamma l}{2t}(T_2 - T_1)$ when $a = 0$ Max $-\theta = \theta_A$; max possible value $= \frac{-\gamma l}{2t}(T_2 - T_1)$ when $a = 0$ Max $y = \frac{-\gamma}{8tl^2}(T_2 - T_1)(l^2 - a^2)^2$; max possible value $= \frac{-\gamma l^2}{8t}(T_2 - T_1)$ at $x = \frac{l}{2}$ when $a = 0$
6f. Left end guided, right end simply supported  	$R_A = 0 \qquad M_A = 0 \qquad \theta_A = 0$ $y_A = \frac{-\gamma}{2t}(T_2 - T_1)(l-a)^2$ $R_B = 0 \qquad M_B = 0 \qquad y_B = 0$ $\theta_B = \frac{\gamma}{l}(T_2 - T_1)(l-a)$	$M = 0$ everywhere Max $\theta = \theta_B$; max possible value $= \frac{\gamma l}{t}(T_2 - T_1)$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{-\gamma l^2}{2t}(T_2 - T_1)$ when $a = 0$

TABLE 8.2 Reaction and deflection formulas for in-plane loading of elastic frames

NOTATION: $W =$ load (force); $w =$ unit load (force per unit length); $M_o =$ applied couple (force-length); $\theta_o =$ externally created concentrated angular displacement (radians); $\Delta_o =$ externally created concentrated lateral displacement (length); $T - T_o =$ uniform temperature rise (degrees); T_1 and $T_2 =$ temperature on outside and inside, respectively (degrees). H_A and H_B are the horizontal end reactions at the left and right, respectively, and are positive to the left; V_A and V_B are the vertical end reactions at the left and right, respectively, and are positive upwards; M_A and M_B are the reaction moments at the left and right, respectively, and are positive clockwise. I_1, I_2, and I_3 are the respective area moments of inertia for bending in the plane of the frame for the three members (length to the fourth); E_1, E_2, and E_3 are the respective moduli of elasticity (force per unit area); γ_1, γ_2, and γ_3 are the respective temperature coefficients of expansions (unit strain per degree)

General reaction and deformation expressions for cases 1–4, right end pinned in all four cases



Deformation equations:

Horizontal deflection at $A = \delta_{HA} = A_{HH}H_A + A_{HM}M_A - LP_H$

Angular rotation at $A = \psi_A = A_{MH}H_A + A_{MM}M_A - LP_M$

$$\text{where } A_{HH} = \frac{l_1^3}{3E_1I_1} + \frac{I_2^3}{3E_2I_2} + \frac{l_3}{3E_3I_3}(l_1^2 + l_1l_2 + l_2^2)$$

$$A_{HM} = A_{MH} = \frac{l_1^2}{2E_1I_1} + \frac{l_3}{6E_3I_3}(2l_1 + l_2)$$

$$A_{MM} = \frac{l_1}{E_1I_1} + \frac{l_3}{3E_3I_3}$$

and where LP_H and LP_M are loading terms given below for several types of load

(*Note:* V_A, V_B, and H_B are to be evaluated from equilibrium equations after calculating H_A and M_A)

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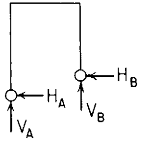
1. Left end pinned, right end pinned



Since $\delta_{HA} = 0$ and $M_A = 0$,

$$H_A = \frac{LP_H}{A_{HH}} \quad \text{and} \quad \psi_A = A_{MH}H_A - LP_M$$

The loading terms are as follows.

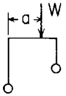
Reference no., loading	Loading terms
1a. Concentrated load on the horizontal member 	$LP_H = \frac{Wa}{6E_3I_3}\left[3l_1a - 2l_1l_3 - l_2l_3 - \frac{a^2}{l_3}(l_1 - l_2)\right]$ $LP_M = \frac{Wa}{6E_3I_3}\left(3a - 2l_3 - \frac{a^2}{l_3}\right)$

TABLE 8.2 Reaction and deflection formulas for in-plane loading of elastic frames (*Continued*)

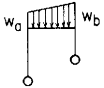
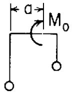
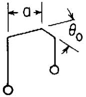
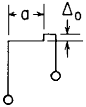
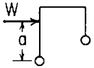
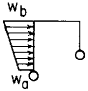
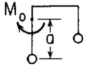
1b. Distributed load on the horizontal member		$LP_H = \frac{-w_a l_3^3}{24E_3I_3}(l_1 + l_2) - \frac{(w_b - w_a)l_3^3}{360E_3I_3}(7l_1 + 8l_2)$ $LP_M = \frac{-7w_a l_3^3}{72E_3I_3} - \frac{11(w_b - w_a)l_3^3}{180E_3I_3}$
1c. Concentrated moment on the horizontal member		$LP_H = \frac{M_o}{6E_3I_3}\left[6l_1a - 2l_1l_3 - l_2l_3 - \frac{3a^2}{l_3}(l_1 - l_2)\right]$ $LP_M = \frac{M_o}{6E_3I_3}\left(4a - 2l_3 - \frac{3a^2}{l_3}\right)$
1d. Concentrated angular displacement on the horizontal member		$LP_H = \theta_o\left[l_1 - \frac{a}{l_3}(l_1 - l_2)\right]$ $LP_M = \theta_o\left(\frac{2}{3} - \frac{a}{l_3}\right)$
1e. Concentrated lateral displacement on the horizontal member		$LP_H = \frac{-\Delta_o(l_1 - l_2)}{l_3}$ (*Note*: Δ_o could also be an increase in the length l_1 or a decrease in the length l_2) $LP_M = \frac{-\Delta_o}{l_3}$
1f. Concentrated load on the left vertical member		$LP_H = W\left(A_{HH} - aA_{HM} + \frac{a^3}{6E_1I_1}\right)$ $LP_M = W\left(A_{MH} - aA_{MM} + \frac{a^2}{2E_1I_1}\right)$
1g. Distributed load on the left vertical member		$LP_H = w_a\left(A_{HH}l_1 - A_{HM}\frac{l_1^2}{2} + \frac{l_1^4}{24E_1I_1}\right) + (w_b - w_a)\left(A_{HH}\frac{l_1}{2} - A_{HM}\frac{l_1^2}{3} + \frac{l_1^4}{30E_1I_1}\right)$ $LP_M = w_a\left[\frac{l_1^3}{6E_1I_1} + \frac{l_1l_3}{6E_3I_3}(l_1 + l_2)\right] + (w_b - w_a)\left[\frac{l_1^3}{24E_1I_1} + \frac{l_1l_3}{36E_3I_3}(2l_1 + 3l_2)\right]$
1h. Concentrated moment on the left vertical member		$LP_H = M_o\left(\frac{a^2}{2E_1I_1} - A_{HM}\right)$ $LP_M = M_o\left(\frac{a}{E_1I_1} - A_{MM}\right)$

TABLE 8.2 Reaction and deflection formulas for in-plane loading of elastic frames (*Continued*)

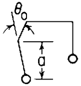
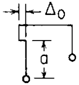
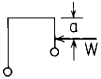
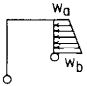
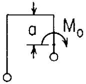
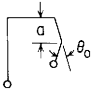
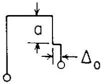
Reference no., loading	Loading terms
1i. Concentrated angular displacement on the left vertical member 	$LP_H = \theta_o(a)$ $LP_M = \theta_o(1)$
1j. Concentrated lateral displacement on the left vertical member 	$LP_H = \Delta_o(1)$ (*Note*: Δ_o could also be a decrease in the length l_3) $LP_M = 0$
1k. Concentrated load on the right vertical member 	$LP_H = W\left[\frac{1}{6E_2I_2}(3l_2a^2 - 2l_2^3 - a^3) - \frac{l_3}{6E_3I_3}(l_2 - a)(l_1 + 2l_2)\right]$ $LP_M = W\left[\frac{-l_3}{6E_3I_3}(l_2 - a)\right]$
1l. Distributed load on the right vertical member 	$LP_H = w_a\left[\frac{-5l_2^4}{24E_2I_2} - \frac{l_2^2l_3}{12E_3I_3}(l_1 + 2l_2)\right] + (w_b - w_a)\left[\frac{-3l_2^4}{40E_2I_2} - \frac{l_2^2l_3}{36E_3I_3}(l_1 + 2l_2)\right]$ $LP_M = w_a\left(\frac{-l_2^2l_3}{12E_3I_3}\right) + (w_b - w_a)\left(\frac{-l_2^2l_3}{36E_3I_3}\right)$
1m. Concentrated moment on the right vertical member 	$LP_H = M_o\left[\frac{a}{2E_2I_2}(2l_2 - a) + \frac{l_3}{6E_3I_3}(l_1 + 2l_2)\right]$ $LP_M = M_o\frac{l_3}{6E_3I_3}$
1n. Concentrated angular displacement on the right vertical member 	$LP_H = \theta_o(l_2 - a)$ $LP_M = 0$
1p. Concentrated lateral displacement on the right vertical member 	$LP_H = \Delta_o(-1)$ (*Note*: Δ_o could also be an increase in the length l_3) $LP_M = 0$

TABLE 8.2 Reaction and deflection formulas for in-plane loading of elastic frames (*Continued*)

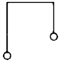
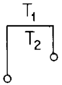
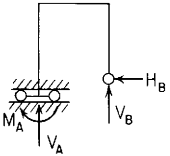
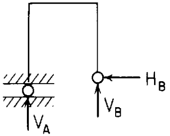
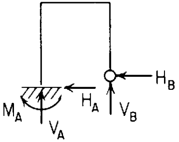
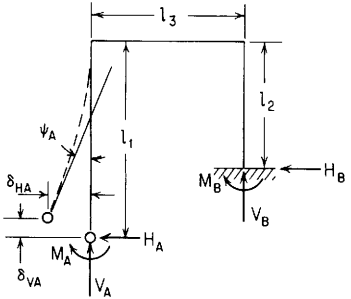
1q. Uniform temperature rise: T = uniform temperature T_o = unloaded temperature 	$LP_H = (T - T_o)\left[\gamma_3 l_3 - \frac{(l_1 - l_2)}{l_3}(\gamma_1 l_1 - \gamma_2 l_2)\right]$ γ = temperature coefficient of expansion (unit strain/degree) $LP_M = (T - T_o)\left[\frac{-1}{l_3}(\gamma_1 l_1 - \gamma_2 l_2)\right]$
1r. Uniform temperature differential from outside to inside. Average temperature is T_o 	$LP_H = \frac{(T_1 - T_2)}{2}\left[\frac{\gamma_1 l_1^2}{t_1} + \frac{\gamma_2 l_2^2}{t_2} + \frac{\gamma_3 l_3(l_1 + l_2)}{t_3}\right]$ t_1, t_2, and t_3 are beam thicknesses from inside to outside $LP_M = \frac{(T_1 - T_2)}{2}\left(\frac{4\gamma_1 l_1}{t_1} + \frac{\gamma_3 l_3}{3t_3}\right)$
2. Left and guided horizontally, right end pinned 	Since $\psi_A = 0$ and $H_A = 0$ $M_A = \frac{LP_M}{A_{MM}}$ and $\delta_{HA} = A_{HM} M_A - LP_H$ Use the loading terms for cases 1a to 1r
3. Left end roller supported along the horizontal, right end pinned 	Since H_A and M_A are both zero, this is a statically determinate case $\delta_{HA} = -LP_H$ and $\psi_A = -LP_M$ Use the loading terms for cases 1a to 1r
4. Left end fixed, right end pinned 	Since $\delta_{HA} = 0$ and $\psi_A = 0$, $H_A = \frac{A_{MM} LP_H - A_{HM} LP_M}{A_{HH} A_{MM} - (A_{HM})^2}$ and $M_A = \frac{A_{HH} LP_M - A_{HM} LP_H}{A_{HH} A_{MM} - (A_{HM})^2}$ Use the loading terms for cases 1a to 1r

TABLE 8.2 Reaction and deflection formulas for in-plane loading of elastic frames (*Continued*)

General reaction and deformation expressions for cases 5–12, right end fixed in all eight cases:



Deformation equations:

Horizontal deflection at $A = \delta_{HA} = C_{HH}H_A + C_{HV}V_A + C_{HM}M_A - LF_H$

Vertical deflection at $A = \delta_{VA} = C_{VH}H_A + C_{VV}V_A + C_{VM}M_A - LF_V$

Angular rotation at $A = \psi_A = C_{MH}H_A + C_{MV}V_A + C_{MM}M_A - LF_M$

where $C_{HH} = \dfrac{l_1^3}{3E_1I_1} + \dfrac{l_1^3 - (l_1 - l_2)^3}{3E_2I_2} + \dfrac{l_1^2 l_3}{E_3 l_3}$

$$C_{HV} = C_{VH} = \frac{l_2 l_3}{2E_2I_2}(2l_1 - l_2) + \frac{l_1 l_3^2}{2E_3I_3}$$

$$C_{HM} = C_{MH} = \frac{l_1^2}{2E_1I_1} + \frac{l_2}{2E_2I_2}(2l_1 - l_2) + \frac{l_1 l_3}{E_3I_3}$$

$$C_{VV} = \frac{l_2 l_3^2}{E_2I_2} + \frac{l_3^3}{3E_3I_3}$$

$$C_{VM} = C_{MV} = \frac{l_2 l_3}{E_2I_2} + \frac{l_3^2}{2E_3I_3}$$

$$C_{MM} = \frac{l_1}{E_1I_1} + \frac{l_2}{E_2I_2} + \frac{l_3}{E_3 l_3}$$

and where LF_H, LF_V, and LF_M are loading terms given below for several types of load

(*Note:* If desired, H_B, V_B, and M_B are to be evaluated from equilibrium equations after calculating H_A, V_A, and M_A)

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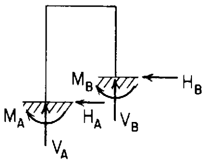
5. Left end fixed, right end fixed



Since $\delta_{HA} = 0$, $\delta_{VA} = 0$, and $\psi_A = 0$, these three equations are solved simultaneously for H_A, V_A, and M_A:

$$C_{HH}H_A + C_{HV}V_A + C_{HM}M_A = LF_H$$
$$C_{VH}H_A + C_{VV}V_A + C_{VM}M_A = LF_V$$
$$C_{MH}H_A + C_{MV}V_A + C_{MM}M_A = LF_M$$

The loading terms are given below.

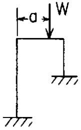
Reference no., loading	Loading terms
5a. Concentrated load on the horizontal member 	$LF_H = W\left[\dfrac{l_2}{2E_2I_2}(2l_1 - l_2)(l_3 - a) + \dfrac{l_1}{2E_3I_3}(l_3 - a)^2\right]$ $LF_V = W\left(C_{VV} - aC_{VM} + \dfrac{a^3}{6E_3I_3}\right)$ $LF_M = W\left[\dfrac{l_2}{E_2I_2}(l_3 - a) + \dfrac{1}{2E_3I_3}(l_3 - a)^2\right]$

TABLE 8.2 Reaction and deflection formulas for in-plane loading of elastic frames (*Continued*)

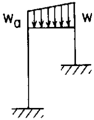
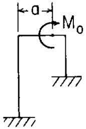
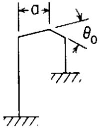
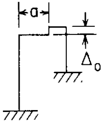
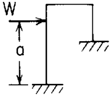
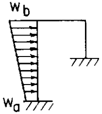
5b. Distributed load on the horizontal member		$LF_H = w_a\left[\frac{l_2 l_3^2}{4E_2I_2}(2l_1 - l_2) + \frac{l_1 l_3^3}{6E_3I_3}\right] + (w_b - w_a)\left[\frac{l_2 l_3^2}{12E_2I_2}(2l_1 - l_2) + \frac{l_1 l_3^3}{24E_3I_3}\right]$ $LF_V = w_a\left(\frac{l_2 l_3^3}{2E_2I_2} + \frac{l_3^4}{8E_3I_3}\right) + (w_b - w_a)\left(\frac{l_2 l_3^3}{6E_2I_2} + \frac{l_3^4}{30E_3I_3}\right)$ $LF_M = w_a\left(\frac{l_2 l_3^2}{2E_2I_2} + \frac{l_3^3}{6E_3I_3}\right) + (w_b - w_a)\left(\frac{l_2 l_3^2}{6E_2I_2} + \frac{l_3^3}{24E_3I_3}\right)$
5c. Concentrated moment on the horizontal member		$LF_H = M_o\left[\frac{-l_2}{2E_2I_2}(2l_1 - l_2) - \frac{l_1}{E_3I_3}(l_3 - a)\right]$ $LF_V = M_o\left(-C_{VM} + \frac{a^2}{2E_3I_3}\right)$ $LF_M = M_o\left[\frac{-l_2}{E_2I_2} - \frac{1}{E_3I_3}(l_3 - a)\right]$
5d. Concentrated angular displacement on the horizontal member		$LF_H = \theta_o(l_1)$ $LF_V = \theta_o(a)$ $LF_M = \theta_o(1)$
5e. Concentrated lateral displacement on the horizontal member		$LF_H = 0$ $LF_V = \Delta_o(1)$ $LF_M = 0$
5f. Concentrated load on the left vertical member		$LF_H = W\left(C_{HH} - aC_{HM} + \frac{a^3}{6E_1I_1}\right)$ $LF_V = W(C_{VH} - aC_{VM})$ $LF_M = W\left(C_{MH} - aC_{MM} + \frac{a^2}{2E_1I_1}\right)$
5g. Distributed load on the left vertical member		$LF_H = w_a\left(C_{HH}l_1 - C_{HM}\frac{l_1^2}{2} + \frac{l_1^4}{24E_1I_1}\right) + (w_b - w_a)\left(C_{HH}\frac{l_1}{2} - C_{HM}\frac{l_1^2}{3} + \frac{l_1^4}{30E_1I_1}\right)$ $LF_V = w_a\left(C_{VH}l_1 - C_{VM}\frac{l_1^2}{2}\right) + (w_b - w_a)\left(C_{VH}\frac{l_1}{2} - C_{VM}\frac{l_1^2}{3}\right)$ $LF_M = w_a\left(C_{MH}l_1 - C_{MM}\frac{l_1^2}{2} + \frac{l_1^3}{6E_1I_1}\right) + (w_b - w_a)\left(C_{MH}\frac{l_1}{2} - C_{MM}\frac{l_1^2}{3} + \frac{l_1^3}{8E_1I_1}\right)$

TABLE 8.2 Reaction and deflection formulas for in-plane loading of elastic frames (*Continued*)

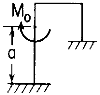
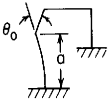
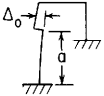
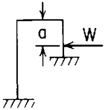
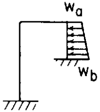
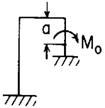
Reference no., loading	Loading terms
5h. Concentrated moment on the left vertical member 	$LF_H = M_o\left(-C_{HM} + \frac{a^2}{2E_1I_1}\right)$ $LF_V = M_o(-C_{VM})$ $LF_M = M_o\left(-C_{MM} + \frac{a}{E_1I_1}\right)$
5i. Concentrated angular displacement on the left vertical member 	$LF_H = \theta_o(a)$ $LF_V = 0$ $LF_M = \theta_o(1)$
5j. Concentrated lateral displacement on the left vertical member 	$LF_H = \Delta_o(1)$ $LF_V = 0$ $LF_M = 0$
5k. Concentrated load on the right vertical member 	$LF_H = \frac{W}{6E_2I_2}[3l_1(l_2-a)^2 - 2l_2^3 - a^3 + 3al_2^2]$ $LF_V = \frac{W}{2E_2I_2}[l_3(l_2-a)^2]$ $LF_M = \frac{W}{2E_2I_2}(l_2-a)^2$
5l. Distributed load on the right vertical member 	$LF_H = w_a\left[\frac{l_2^3}{24E_2I_2}(4l_1-3l_2)\right] + (w_b-w_a)\left[\frac{l_2^3}{120E_2I_2}(5l_1-4l_2)\right]$ $LF_V = w_a\frac{l_2^3l_3}{6E_2I_2} + (w_b-w_a)\frac{l_2^3l_3}{24E_2I_2}$ $Lf_M = w_a\frac{l_2^3}{6E_2I_2} + (w_b-w_a)\frac{l_2^3}{24E_2I_2}$
5m. Concentrated moment on the right vertical member 	$LF_H = \frac{M_o}{2E_2I_2}[-2l_1(l_2-a) - a^2 + l_2^2]$ $LF_V = \frac{M_o}{E_2I_2}[-l_3(l_2-a)]$ $LF_M = \frac{M_o}{E_2I_2}[-(l_2-a)]$

TABLE 8.2 Reaction and deflection formulas for in-plane loading of elastic frames (*Continued*)

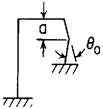
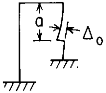
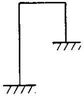
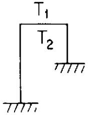
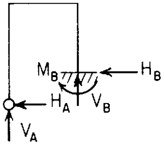
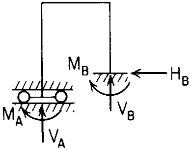
5n. Concentrated angular displacement on the right vertical member 	$LF_H = \theta_o(l_1 - a)$ $LF_V = \theta_o(l_3)$ $LF_M = \theta_o(1)$
5p. Concentrated lateral displacement on the right vertical member 	$LF_H = \Delta_o(-1)$ (*Note*: Δ_o could also be an increase in the length l_3) $LF_V = 0$ $LF_M = 0$
5q. Uniform temperature rise: $T =$ uniform temperature $T_o =$ unloaded temperature 	$LF_H = (T - T_o)(-\gamma_3 l_3)$ $\gamma =$ temperature coefficient of expansion (inches/inch/degree) $LF_V = (T - T_o)(\gamma_1 l_1 - \gamma_2 l_2)$ $LF_M = 0$
5r. Uniform temperature differential from outside to inside; average temperature is T_o 	$LF_H = (T_1 - T_2)\left[\frac{l_1^2\gamma_1}{2t_1} + \frac{l_2\gamma_2}{2t_2}(2l_1 - l_2) + \frac{l_1 l_3 \gamma_3}{t_3}\right]$ $LF_V = (T_1 - T_2)\left(\frac{l_2 l_3 \gamma_2}{t_2} + \frac{l_3^2\gamma_3}{2t_3}\right)$ t_1, t_2, and t_3 are beam thicknesses from inside to outside $LF_M = (T_1 - T_2)\left(\frac{l_1\gamma_1}{t_1} + \frac{l_2\gamma_2}{t_2} + \frac{l_3\gamma_3}{t_3}\right)$
6. Left end pinned, right end fixed 	Since $\delta_{HA} = 0, \delta_{VA} = 0$ and $M_A = 0$, $H_A = \frac{LF_H C_{VV} - LF_V C_{HV}}{C_{HH} C_{VV} - (C_{HV})^2}$ $V_A = \frac{LF_V C_{HH} - LF_H C_{HV}}{C_{HH} C_{VV} - (C_{HV})^2}$ $\psi_A = C_{MH} H_A + C_{MV} V_A - LF_M$ Use the loading terms for cases 5a to 5r
7. Left end guided horizontally, right end fixed 	Since $\delta_{VA} = 0, \psi_A = 0$ and $H_A = 0$, $V_A = \frac{LF_V C_{MM} - LF_M C_{VM}}{C_{VV} C_{MM} - (C_{VM})^2}$ $M_A = \frac{LF_M C_{VV} - LF_V C_{VM}}{C_{VV} C_{MM} - (C_{VM})^2}$ $\delta_{HA} = C_{HV} V_A + C_{HM} M_A - LF_H$ Use the loading terms for cases 5a to 5r

TABLE 8.2 Reaction and deflection formulas for in-plane loading of elastic frames (*Continued*)

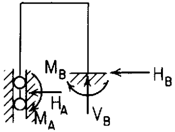
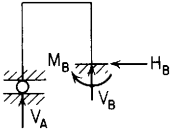
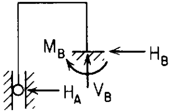
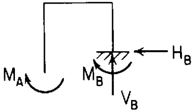
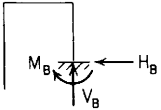
8. Left end guided vertically, right end fixed		Since $\delta_{HA} = 0, \psi_A = 0$, and $V_A = 0$, $H_A = \dfrac{LF_H C_{MM} - LF_M C_{HM}}{C_{HH} C_{MM} - (C_{HM})^2}$ $\quad M_A = \dfrac{LF_M C_{HH} - LF_H C_{HM}}{C_{HH} C_{MM} - (C_{HM})^2}$ $\delta_{VA} = C_{VH} H_A + C_{VM} M_A - LF_V$ Use the loading terms for cases 5a to 5r
9. Left end roller supported along the horizontal, right end fixed		Since $\delta_{VA} = 0, H_A = 0$, and $M_A = 0$, $V_A = \dfrac{LF_V}{C_{VV}}$ $\quad \delta_{HA} = C_{HV} V_A - LF_H$ and $\psi_A = C_{MV} V_A - LF_M$ Use the loading terms for cases 5a to 5r
10. Left end roller supported along the vertical, right end fixed		Since $\delta_{HA} = 0, V_A = 0$, and $M_A = 0$, $H_A = \dfrac{LF_H}{C_{HH}}$ $\quad \delta_{VA} = C_{VH} H_A - LF_V$ and $\psi_A = C_{MH} H_A - LF_M$ Use the loading terms for cases 5a to 5r
11. Left end guided by moment only (zero slope at the left end), right end fixed		Since $\psi_A = 0, H_A = 0$, and $V_A = 0$, $M_A = \dfrac{LF_M}{C_{MM}}$ $\quad \delta_{HA} = C_{HM} M_A - LF_H$ and $\delta_{VA} = C_{VM} M_A - LF_V$ Use the loading terms for cases 5a to 5r
12. Left end free, right end fixed		Since $H_A = 0, V_A = 0$ and $M_A = 0$, this is a statically determinate case. The deflections are given by $\delta_{HA} = -LF_H$ $\quad \delta_{VA} = -LF_V$ and $\psi_A = -LF_M$ Use the loading terms for cases 5a to 5r

TABLE 8.3 Numerical values for functions used in Table 8.5

βx	F_1	F_2	F_3	F_4
0.00	1.00000	0.00000	0.00000	0.00000
0.10	0.99998	0.20000	0.01000	0.00067
0.20	0.99973	0.39998	0.04000	0.00533
0.30	0.99865	0.59984	0.08999	0.01800
0.40	0.99573	0.79932	0.15995	0.04266
0.50	0.98958	0.99792	0.24983	0.08331
0.60	0.97841	1.19482	0.35948	0.14391
0.70	0.96001	1.38880	0.48869	0.22841
0.80	0.93180	1.57817	0.63709	0.34067
0.90	0.89082	1.76067	0.80410	0.48448
1.00	0.83373	1.93342	0.98890	0.66349
1.10	0.75683	2.09284	1.19034	0.88115
1.20	0.65611	2.23457	1.40688	1.14064
1.30	0.52722	2.35341	1.63649	1.44478
1.40	0.36558	2.44327	1.87659	1.79593
1.50	0.16640	2.49714	2.12395	2.19590
1.60	−0.07526	2.50700	2.37456	2.64573
1.70	−0.36441	2.46387	2.62358	3.14562
1.80	−0.70602	2.35774	2.86523	3.69467
1.90	−1.10492	2.17764	3.09266	4.29076
2.00	−1.56563	1.91165	3.29789	4.93026
2.10	−2.09224	1.54699	3.47170	5.60783
2.20	−2.68822	1.07013	3.60355	6.31615
2.30	−3.35618	0.46690	3.68152	7.04566
2.40	−4.09766	−0.27725	3.69224	7.78428
2.50	−4.91284	−1.17708	3.62088	8.51709
2.60	−5.80028	−2.24721	3.45114	9.22607
2.70	−6.75655	−3.50179	3.16529	9.88981
2.80	−7.77591	−4.95404	2.74420	10.48317
2.90	−8.84988	−6.61580	2.16749	10.97711
3.00	−9.96691	−8.49687	1.41372	11.33837
3.20	−12.26569	−12.94222	−0.71484	11.50778
3.40	−14.50075	−18.30128	−3.82427	10.63569
3.60	−16.42214	−24.50142	−8.09169	8.29386
3.80	−17.68744	−31.35198	−13.66854	3.98752
4.00	−17.84985	−38.50482	−20.65308	−2.82906
4.20	−16.35052	−45.41080	−29.05456	−12.72446
4.40	−12.51815	−51.27463	−38.74857	−26.24587
4.60	−5.57927	−55.01147	−49.42334	−43.85518
4.80	5.31638	−55.21063	−60.51809	−65.84195
5.00	21.05056	−50.11308	−71.15526	−92.21037
5.20	42.46583	−37.61210	−80.07047	−122.53858
5.40	70.26397	−15.28815	−85.54576	−155.81036
5.60	104.86818	19.50856	−85.35442	−190.22206
5.80	146.24469	69.51236	−76.72824	−222.97166
6.00	193.68136	137.31651	−56.36178	−250.04146

TABLE 8.4 Numerical values for denominators used in Table 8.5

βl	C_{11}	C_{12}	C_{13}	C_{14}
0.00	0.00000	0.00000	0.00000	0.00000
0.10	0.00007	0.20000	0.00133	0.02000
0.20	0.00107	0.40009	0.01067	0.08001
0.30	0.00540	0.60065	0.03601	0.18006
0.40	0.01707	0.80273	0.08538	0.32036
0.50	0.04169	1.00834	0.16687	0.50139
0.60	0.08651	1.22075	0.28871	0.72415
0.70	0.16043	1.44488	0.45943	0.99047
0.80	0.27413	1.68757	0.68800	1.30333
0.90	0.44014	1.95801	0.98416	1.66734
1.00	0.67302	2.26808	1.35878	2.08917
1.10	0.98970	2.63280	1.82430	2.57820
1.20	1.40978	3.07085	2.39538	3.14717
1.30	1.95606	3.60512	3.08962	3.81295
1.40	2.65525	4.26345	3.92847	4.59748
1.50	3.53884	5.07950	4.93838	5.52883
1.60	4.64418	6.09376	6.15213	6.64247
1.70	6.01597	7.35491	7.61045	7.98277
1.80	7.70801	8.92147	9.36399	9.60477
1.90	9.78541	10.86378	11.47563	11.57637
2.00	12.32730	13.26656	14.02336	13.98094
2.10	15.43020	16.23205	17.10362	16.92046
2.20	19.21212	19.88385	20.83545	20.51946
2.30	23.81752	24.37172	25.36541	24.92967
2.40	29.42341	29.87747	30.87363	30.33592
2.50	36.24681	36.62215	37.58107	36.96315
2.60	44.55370	44.87496	45.75841	45.08519
2.70	54.67008	54.96410	55.73686	55.03539
2.80	66.99532	67.29005	67.92132	67.21975
2.90	82.01842	82.34184	82.80645	82.13290
3.00	100.33792	100.71688	100.99630	100.37775
3.20	149.95828	150.51913	150.40258	149.96510
3.40	223.89682	224.70862	224.21451	224.02742
3.60	334.16210	335.25438	334.46072	334.55375
3.80	498.67478	500.03286	499.06494	499.42352
4.00	744.16690	745.73416	744.74480	745.31240
4.20	1110.50726	1112.19410	1111.33950	1112.02655
4.40	1657.15569	1658.85362	1658.26871	1658.96679
4.60	2472.79511	2474.39393	2474.17104	2474.76996
4.80	3689.70336	3691.10851	3691.28284	3691.68805
5.00	5505.19766	5506.34516	5506.88918	5507.03673
5.20	8213.62683	8214.49339	8215.32122	8215.18781
5.40	12254.10422	12254.71090	12255.69184	12255.29854
5.60	18281.71463	18282.12354	18283.10271	18282.51163
5.80	27273.73722	27274.04166	27274.86449	27274.16893
6.00	40688.12376	40688.43354	40688.97011	40688.27990

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations

NOTATION: W = load (force); w = unit load (force per unit length); M_o = applied couple (force-length); θ_o = externally created concentrated angular displacement (radians); Δ_o = externally created concentrated lateral displacement (length); γ = temperature coefficient of expansion (unit strain per degree); T_1 and T_2 = temperatures on top and bottom surfaces, respectively (degrees). R_A and R_B are the vertical end reactions at the left and right, respectively, and are positive upward. M_A and M_B are the reaction end moments at the left and right, respectively, and all moments are positive when producing compression on the upper portion of the beam cross section. The transverse shear force V is positive when acting upward on the left end of a portion of the beam. All applied loads, couples, and displacements are positive as shown. All slopes are in radians, and all temperatures are in degrees. All deflections are positive upward and slopes positive when up and to the right. Note that M_A and R_A are reactions, not applied loads. They exist only when necessary end restraints are provided.

The following constants and functions, involving both beam constants and foundation constants, are hereby defined in order to permit condensing the tabulated formulas which follow

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k_o = foundation modulus (unit stress per unit deflection); b_o = beam width; and $\beta = (b_o k_o / 4EI)^{1/4}$. (*Note*: See page 131 for a definition of $\langle x-a \rangle^n$.) The functions $\cosh \beta\langle x-a \rangle$, $\sinh \beta\langle x-a \rangle$, $\cos \beta\langle x-a \rangle$, and $\sin \beta\langle x-a \rangle$ are also defined as having a value of zero if $x < a$.

$F_1 = \cosh \beta x \cos \beta x$

$F_2 = \cosh \beta x \sin \beta x + \sinh \beta x \cos \beta x$

$F_3 = \sinh \beta x \sin \beta_x$

$F_4 = \cosh \beta x \sin \beta x - \sinh \beta x \cos \beta x$

$F_{a1} = \langle x-a \rangle^0 \cosh \beta\langle x-a \rangle \cos \beta\langle x-a \rangle$

$F_{a2} = \cosh \beta\langle x-a \rangle \sin \beta\langle x-a \rangle + \sinh \beta\langle x-a \rangle \cos \beta\langle x-a \rangle$

$F_{a3} = \sinh \beta\langle x-a \rangle \sin \beta\langle x-a \rangle$

$F_{a4} = \cosh \beta\langle x-a \rangle \sin \beta\langle x-a \rangle - \sinh \beta\langle x-a \rangle \cos \beta\langle x-a \rangle$

$F_{a5} = \langle x-a \rangle^0 - F_{a1}$

$F_{a6} = 2\beta(x-a)\langle x-a \rangle^0 - F_{a2}$

$C_1 = \cosh \beta l \cos \beta l$

$C_2 = \cosh \beta l \sin \beta l + \sinh \beta l \cos \beta l$

$C_3 = \sinh \beta l \sin \beta l$

$C_4 = \cosh \beta l \sin \beta l - \sinh \beta l \cos \beta l$

$C_{a1} = \cosh \beta(l-a) \cos \beta(l-a)$

$C_{a2} = \cosh \beta(l-a) \sin \beta(l-a) + \sinh \beta(l-a) \cos \beta(l-a)$

$C_{a3} = \sinh \beta(l-a) \sin \beta(l-a)$

$C_{a4} = \cosh \beta(l-a) \sin \beta(l-a) - \sinh \beta(l-a) \cos \beta(l-a)$

$C_{a5} = 1 - C_{a1}$

$C_{a6} = 2\beta(l-a) - C_{a2}$

$C_{11} = \sinh^2 \beta l - \sin^2 \beta l$

$C_{12} = \cosh \beta l \sinh \beta l + \cos \beta l \sin \beta l$

$C_{13} = \cosh \beta l \sinh \beta l - \cos \beta l \sin \beta l$

$C_{14} = \sinh^2 \beta l + \sin^2 \beta l$

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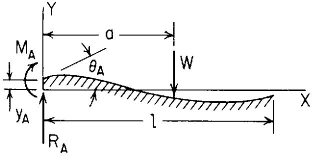
1. Concentrated intermediate load



Transverse shear $= V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 - W F_{a1}$

Bending moment $= M = M_A F_1 + \dfrac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 - \dfrac{W}{2\beta} F_{a2}$

Slope $= \theta = \theta_A F_1 + \dfrac{M_A}{2EI\beta} F_2 + \dfrac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 - \dfrac{W}{2EI\beta^2} F_{a3}$

Deflection $= y = y_A F_1 + \dfrac{\theta_A}{2\beta} F_2 + \dfrac{M_A}{2EI\beta^2} F_3 + \dfrac{R_A}{4EI\beta^3} F_4 - \dfrac{W}{4EI\beta^3} F_{a4}$

If $\beta l > 6$, see Table 8.6

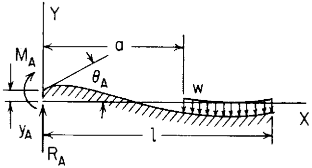
Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of end restraints

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations (*Continued*)

Right end Left end	Free	Guided	Simply supported	Fixed
Free	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_2C_{a2} - 2C_3C_{a1}}{C_{11}}$ $y_A = \frac{W}{2EI\beta^3}\frac{C_4C_{a1} - C_3C_{a2}}{C_{11}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_2C_{a3} - C_4C_{a1}}{C_{12}}$ $y_A = \frac{-W}{2EI\beta^3}\frac{C_1C_{a1} + C_3C_{a3}}{C_{12}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_1C_{a2} + C_3C_{a4}}{C_{13}}$ $y_A = \frac{-W}{4EI\beta^3}\frac{C_4C_{a4} + C_2C_{a2}}{C_{13}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{W}{2EI\beta^2}\frac{2C_1C_{a3} + C_4C_{a4}}{2 + C_{11}}$ $y_A = \frac{W}{2EI\beta^3}\frac{C_1C_{a4} - C_2C_{a3}}{2 + C_{11}}$
Guided	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{W}{2\beta}\frac{C_2C_{a2} - 2C_3C_{a1}}{C_{12}}$ $y_A = \frac{-W}{4EI\beta^3}\frac{2C_1C_{a1} + C_4C_{a2}}{C_{12}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{W}{2\beta}\frac{C_2C_{a3} - C_4C_{a1}}{C_{14}}$ $y_A = \frac{-W}{4EI\beta^3}\frac{C_2C_{a1} + C_4C_{a3}}{C_{14}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{W}{2\beta}\frac{C_1C_{a2} + C_3C_{a4}}{1 + C_{11}}$ $y_A = \frac{W}{4EI\beta^3}\frac{C_1C_{a4} - C_3C_{a2}}{1 + C_{11}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{W}{2\beta}\frac{2C_1C_{a3} + C_4C_{a4}}{C_{12}}$ $y_A = \frac{W}{4EI\beta^3}\frac{C_2C_{a4} - 2C_3C_{a3}}{C_{12}}$
Simply supported	$M_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_3C_{a2} - C_4C_{a1}}{C_{13}}$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_1C_{a2} - C_2C_{a1}}{C_{13}}$	$M_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_1C_{a1} + C_3C_{a3}}{1 + C_{11}}$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_1C_{a3} - C_3C_{a1}}{1 + C_{11}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{W}{2}\frac{C_2C_{a2} + C_4C_{a4}}{C_{14}}$ $\theta_A = \frac{W}{4EI\beta^2}\frac{C_2C_{a4} - C_4C_{a2}}{C_{14}}$	$M_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_2C_{a3} - C_1C_{a4}}{C_{13}}$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_3C_{a4} - C_4C_{a3}}{C_{13}}$
Fixed	$\theta_A = 0 \qquad y_A = 0$ $R_A = W\frac{2C_1C_{a1} + C_4C_{a2}}{2 + C_{11}}$ $M_A = \frac{W}{\beta}\frac{C_1C_{a2} - C_2C_{a1}}{2 + C_{11}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_4C_{a3} + C_2C_{a1}}{C_{12}}$ $M_A = \frac{W}{\beta}\frac{C_1C_{a3} - C_3C_{a1}}{C_{12}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_3C_{a2} - C_1C_{a4}}{C_{13}}$ $M_A = \frac{W}{2\beta}\frac{C_2C_{a4} - C_4C_{a2}}{C_{13}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = W\frac{2C_3C_{a3} - C_2C_{a4}}{C_{11}}$ $M_A = \frac{W}{\beta}\frac{C_3C_{a4} - C_4C_{a3}}{C_{11}}$

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations (*Continued*)

2. Partial uniformly distributed load



Transverse shear $= V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 - \frac{w}{2\beta} F_{a2}$

Bending moment $= M = M_A F_1 + \frac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 - \frac{w}{2\beta^2} F_{a3}$

Slope $= \theta = \theta_A F_1 + \frac{M_A}{2EI\beta} F_2 + \frac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 - \frac{w}{4EI\beta^3} F_{a4}$

Deflection $= y = y_A F_1 + \frac{\theta_A}{2\beta} F_2 + \frac{M_A}{2EI\beta^2} F_3 + \frac{R_A}{4EI\beta^3} F_4 - \frac{w}{4EI\beta^4} F_{a5}$

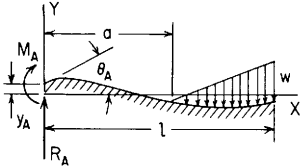
If $\beta l > 6$, see Table 8.6

Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of end restraints

Right end Left end	Free	Guided	Simply supported	Fixed
Free	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{w}{2EI\beta^3} \frac{C_2 C_{a3} - C_3 C_{a2}}{C_{11}}$ $y_A = \frac{w}{4EI\beta^4} \frac{C_4 C_{a2} - 2C_3 C_{a3}}{C_{11}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{w}{4EI\beta^3} \frac{C_2 C_{a4} - C_4 C_{a2}}{C_{12}}$ $y_A = \frac{-w}{4EI\beta^4} \frac{C_1 C_{a2} + C_3 C_{a4}}{C_{12}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{w}{2EI\beta^3} \frac{C_1 C_{a3} + C_3 C_{a5}}{C_{13}}$ $y_A = \frac{-w}{4EI\beta^4} \frac{C_4 C_{a5} + C_2 C_{a3}}{C_{13}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{w}{2EI\beta^3} \frac{C_1 C_{a4} + C_4 C_{a5}}{2 + C_{11}}$ $y_A = \frac{w}{4EI\beta^4} \frac{2C_1 C_{a5} - C_2 C_{a4}}{2 + C_{11}}$
Guided	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{w}{2\beta^2} \frac{C_2 C_{a3} - C_3 C_{a2}}{C_{12}}$ $y_A = \frac{-w}{4EI\beta^4} \frac{C_1 C_{a2} + C_4 C_{a3}}{C_{12}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{w}{4\beta^2} \frac{C_2 C_{a4} - C_4 C_{a2}}{C_{14}}$ $y_A = \frac{-w}{8EI\beta^4} \frac{C_2 C_{a2} + C_4 C_{a4}}{C_{14}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{w}{2\beta^2} \frac{C_1 C_{a3} + C_3 C_{a5}}{1 + C_{11}}$ $y_A = \frac{w}{4EI\beta^4} \frac{C_1 C_{a5} - C_3 C_{a3}}{1 + C_{11}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{w}{2\beta^2} \frac{C_1 C_{a4} + C_4 C_{a5}}{C_{12}}$ $y_A = \frac{w}{4EI\beta^4} \frac{C_2 C_{a5} - C_3 C_{a4}}{C_{12}}$
Simply supported	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{w}{2\beta} \frac{2C_2 C_{a3} - C_4 C_{a2}}{C_{13}}$ $\theta_A = \frac{w}{4EI\beta^3} \frac{2C_1 C_{a3} - C_2 C_{a2}}{C_{13}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{w}{2\beta} \frac{C_1 C_{a2} + C_3 C_{a4}}{1 + C_{11}}$ $\theta_A = \frac{w}{4EI\beta^3} \frac{C_1 C_{a4} - C_3 C_{a2}}{1 + C_{11}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{w}{2\beta} \frac{C_2 C_{a3} + C_4 C_{a5}}{C_{14}}$ $\theta_A = \frac{w}{4EI\beta^3} \frac{C_2 C_{a5} - C_4 C_{a3}}{C_{14}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{w}{2\beta} \frac{C_2 C_{a4} - 2C_1 C_{a5}}{C_{13}}$ $\theta_A = \frac{w}{4EI\beta^3} \frac{2C_3 C_{a5} - C_4 C_{a4}}{C_{13}}$
Fixed	$\theta_A = 0 \qquad y_A = 0$ $R_A = \frac{w}{\beta} \frac{C_1 C_{a2} + C_4 C_{a3}}{2 + C_{11}}$ $M_A = \frac{w}{2\beta^2} \frac{2C_1 C_{a3} - C_2 C_{a2}}{2 + C_{11}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \frac{w}{2\beta} \frac{C_4 C_{a4} + C_2 C_{a2}}{C_{12}}$ $M_A = \frac{w}{2\beta^2} \frac{C_1 C_{a4} - C_3 C_{a2}}{C_{12}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \frac{w}{\beta} \frac{C_3 C_{a3} - C_1 C_{a5}}{C_{13}}$ $M_A = \frac{w}{2\beta^2} \frac{C_2 C_{a5} - C_4 C_{a3}}{C_{13}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \frac{w}{\beta} \frac{C_3 C_{a4} - C_2 C_{a5}}{C_{11}}$ $M_A = \frac{w}{2\beta^2} \frac{2C_3 C_{a5} - C_4 C_{a4}}{C_{11}}$

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations (*Continued*)

3. Partial uniformly increasing load



$$\text{Transverse shear} = V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 - \frac{wF_{a3}}{2\beta^2(l-a)}$$

$$\text{Bending moment} = M = M_A F_1 + \frac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 - \frac{wF_{a4}}{4\beta^3(l-a)}$$

$$\text{Slope} = \theta = \theta_A F_1 + \frac{M_A}{2EI\beta} F_2 + \frac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 - \frac{wF_{a5}}{4EI\beta^4(l-a)}$$

$$\text{Deflection} = y = y_A F_1 + \frac{\theta_A}{2\beta} F_2 + \frac{M_A}{2EI\beta^2} F_3 + \frac{R_A}{4EI\beta^3} F_4 - \frac{wF_{a6}}{8EI\beta^5(l-a)}$$

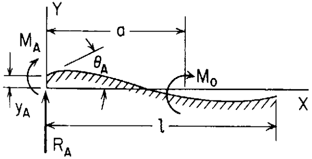
If $\beta l > 6$, see Table 8.6

Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of end restraints

Right end / Left end	Free	Guided	Simply supported	Fixed
Free	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{w(C_2 C_{a4} - 2C_3 C_{a3})}{4EI\beta^4(l-a)C_{11}}$ $y_A = \frac{w(C_4 C_{a3} - C_3 C_{a4})}{4EI\beta^5(l-a)C_{11}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{w(C_2 C_{a5} - C_4 C_{a3})}{4EI\beta^4(l-a)C_{12}}$ $y_A = \frac{-w(C_1 C_{a3} + C_3 C_{a5})}{4EI\beta^5(l-a)C_{12}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{w(C_1 C_{a4} + C_3 C_{a6})}{4EI\beta^4(l-a)C_{13}}$ $y_A = \frac{-w(C_2 C_{a4} + C_4 C_{a6})}{8EI\beta^5(l-a)C_{13}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{w(2C_1 C_{a5} + C_4 C_{a6})}{4EI\beta^4(l-a)(2+C_{11})}$ $y_A = \frac{w(C_1 C_{a6} - C_2 C_{a5})}{4EI\beta^5(l-a)(2+C_{11})}$
Guided	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{w(C_2 C_{a4} - 2C_3 C_{a3})}{4\beta^3(l-a)C_{12}}$ $y_A = \frac{-w(2C_1 C_{a3} + C_4 C_{a4})}{8EI\beta^5(l-a)C_{12}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{w(C_2 C_{a5} - C_4 C_{a3})}{4\beta^3(l-a)C_{14}}$ $y_A = \frac{-w(C_2 C_{a3} + C_4 C_{a5})}{8EI\beta^5(l-a)C_{14}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{w(C_1 C_{a4} + C_3 C_{a6})}{4\beta^3(l-a)(1+C_{11})}$ $y_A = \frac{w(C_1 C_{a6} - C_3 C_{a4})}{8EI\beta^5(l-a)(1+C_{11})}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{w(2C_1 C_{a5} + C_4 C_{a6})}{4\beta^3(l-a)C_{12}}$ $y_A = \frac{w(C_2 C_{a6} - 2C_3 C_{a5})}{8EI\beta^5(l-a)C_{12}}$
Simply supported	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{w(C_3 C_{a4} - C_4 C_{a3})}{2\beta^2(l-a)C_{13}}$ $\theta_A = \frac{w(C_1 C_{a4} - C_2 C_{a3})}{4EI\beta^4(l-a)C_{13}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{w(C_1 C_{a3} + C_3 C_{a5})}{2\beta^2(l-a)(1+C_{11})}$ $\theta_A = \frac{w(C_1 C_{a5} - C_3 C_{a3})}{4EI\beta^4(l-a)(1+C_{11})}$	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{w(C_2 C_{a4} + C_4 C_{a6})}{4\beta^2(l-a)C_{14}}$ $\theta_A = \frac{w(C_2 C_{a6} - C_4 C_{a4})}{8EI\beta^4(l-a)C_{14}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{w(C_2 C_{a5} - C_1 C_{a6})}{2\beta^2(l-a)C_{13}}$ $\theta_A = \frac{w(C_3 C_{a6} - C_4 C_{a5})}{4EI\beta^4(l-a)C_{13}}$
Fixed	$\theta_A = 0 \qquad y_A = 0$ $R_A = \frac{w(2C_1 C_{a3} + C_4 C_{a4})}{2\beta^2(l-a)(2+C_{11})}$ $M_A = \frac{w(C_1 C_{a4} - C_2 C_{a3})}{2\beta^3(l-a)(2+C_{11})}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \frac{w(C_4 C_{a5} + C_2 C_{a3})}{2\beta^2(l-a)C_{12}}$ $M_A = \frac{w(C_1 C_{a5} - C_3 C_{a3})}{2\beta^3(l-a)C_{12}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \frac{w(C_3 C_{a4} - C_1 C_{a6})}{2\beta^2(l-a)C_{13}}$ $M_A = \frac{w(C_2 C_{a6} - C_4 C_{a4})}{4\beta^3(l-a)C_{13}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \frac{w(2C_3 C_{a5} - C_2 C_{a6})}{2\beta^2(l-a)C_{11}}$ $M_A = \frac{w(C_3 C_{a6} - C_4 C_{a5})}{2\beta^3(l-a)C_{11}}$

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations (*Continued*)

4. Concentrated intermediate moment



Transverse shear $= V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 - M_o \beta F_{a4}$

Bending moment $= M = M_A F_1 + \frac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 + M_o F_{a1}$

Slope $= \theta = \theta_A F_1 + \frac{M_A}{2EI\beta} F_2 + \frac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 + \frac{M_o}{2EI\beta} F_{a2}$

Deflection $= y = y_A F_1 + \frac{\theta_A}{2\beta} F_2 + \frac{M_A}{2EI\beta^2} F_3 + \frac{R_A}{4EI\beta^3} F_4 + \frac{M_o}{2EI\beta^2} F_{a3}$

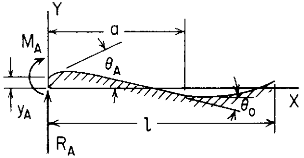
If $\beta l > 6$, see Table 8.6

Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of end restraints

Right end / Left end	Free	Guided	Simply supported	Fixed
Free	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{-M_o}{EI\beta} \frac{C_3 C_{a4} + C_2 C_{a1}}{C_{11}}$ $y_A = \frac{M_o}{2EI\beta^2} \frac{2C_3 C_{a1} + C_4 C_{a4}}{C_{11}}$	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{-M_o}{2EI\beta} \frac{C_2 C_{a2} + C_4 C_{a4}}{C_{12}}$ $y_A = \frac{M_o}{2EI\beta^2} \frac{C_3 C_{a2} - C_1 C_{a4}}{C_{12}}$	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{-M_o}{EI\beta} \frac{C_1 C_{a1} + C_3 C_{a3}}{C_{13}}$ $y_A = \frac{M_o}{2EI\beta^2} \frac{C_4 C_{a3} + C_2 C_{a1}}{C_{13}}$	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{-M_o}{EI\beta} \frac{C_1 C_{a2} + C_4 C_{a3}}{2 + C_{11}}$ $y_A = \frac{-M_o}{2EI\beta^2} \frac{2C_1 C_{a3} - C_2 C_{a2}}{2 + C_{11}}$
Guided	$R_A = 0 \quad \theta_A = 0$ $M_A = -M_o \frac{C_2 C_{a1} + C_3 C_{a4}}{C_{12}}$ $y_A = \frac{-M_o}{2EI\beta^2} \frac{C_1 C_{a4} - C_4 C_{a1}}{C_{12}}$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{-M_o}{2} \frac{C_2 C_{a2} + C_4 C_{a4}}{C_{14}}$ $y_A = \frac{M_o}{4EI\beta^2} \frac{C_4 C_{a2} - C_2 C_{a4}}{C_{14}}$	$R_A = 0 \quad \theta_A = 0$ $M_A = -M_o \frac{C_1 C_{a1} + C_3 C_{a3}}{1 + C_{11}}$ $y_A = \frac{M_o}{2EI\beta^2} \frac{C_3 C_{a1} - C_1 C_{a3}}{1 + C_{11}}$	$R_A = 0 \quad \theta_A = 0$ $M_A = -M_o \frac{C_1 C_{a2} + C_4 C_{a3}}{C_{12}}$ $y_A = \frac{M_o}{2EI\beta^2} \frac{C_3 C_{a2} - C_2 C_{a3}}{C_{12}}$
Simply supported	$M_A = 0 \quad y_A = 0$ $R_A = -M_o \beta \frac{2C_3 C_{a1} + C_4 C_{a4}}{C_{13}}$ $\theta_A = \frac{-M_o}{2EI\beta} \frac{2C_1 C_{a1} + C_2 C_{a4}}{C_{13}}$	$M_A = 0 \quad y_A = 0$ $R_A = -M_o \beta \frac{C_3 C_{a2} - C_1 C_{a4}}{1 + C_{11}}$ $\theta_A = \frac{-M_o}{2EI\beta} \frac{C_1 C_{a2} + C_3 C_{a4}}{1 + C_{11}}$	$M_A = 0 \quad y_A = 0$ $R_A = -M_o \beta \frac{C_2 C_{a1} + C_4 C_{a3}}{C_{14}}$ $\theta_A = \frac{-M_o}{2EI\beta} \frac{C_2 C_{a3} - C_4 C_{a1}}{C_{14}}$	$M_A = 0 \quad y_A = 0$ $R_A = -M_o \beta \frac{C_2 C_{a2} - 2C_1 C_{a3}}{C_{13}}$ $\theta_A = \frac{-M_o}{2EI\beta} \frac{2C_3 C_{a3} - C_4 C_{a2}}{C_{13}}$
Fixed	$\theta_A = 0 \quad y_A = 0$ $R_A = -M_o 2\beta \frac{C_4 C_{a1} - C_1 C_{a4}}{2 + C_{11}}$ $M_A = -M_o \frac{2C_1 C_{a1} + C_2 C_{a4}}{2 + C_{11}}$	$\theta_A = 0 \quad y_A = 0$ $R_A = -M_o \beta \frac{C_4 C_{a2} - C_2 C_{a4}}{C_{12}}$ $M_A = -M_o \frac{C_1 C_{a2} + C_3 C_{a4}}{C_{12}}$	$\theta_A = 0 \quad y_A = 0$ $R_A = -M_o 2\beta \frac{C_3 C_{a1} - C_1 C_{a3}}{C_{13}}$ $M_A = -M_o \frac{C_2 C_{a3} - C_4 C_{a1}}{C_{13}}$	$\theta_A = 0 \quad y_A = 0$ $R_A = -M_o 2\beta \frac{C_3 C_{a2} - C_2 C_{a3}}{C_{11}}$ $M_A = -M_o \frac{2C_3 C_{a3} - C_4 C_{a2}}{C_{11}}$

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations (*Continued*)

5. Externally created concentrated angular displacement



Transverse shear $= V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 - \theta_o 2EI\beta^2 F_{a3}$

Bending moment $= M = M_A F_1 + \dfrac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 - \theta_o EI\beta F_{a4}$

Slope $= \theta = \theta_A F_1 + \dfrac{M_A}{2EI\beta} F_2 + \dfrac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 + \theta_o F_{a1}$

Deflection $= y = y_A F_1 + \dfrac{\theta_A}{2\beta} F_2 + \dfrac{M_A}{2EI\beta^2} F_3 + \dfrac{R_A}{4EI\beta^3} F_4 + \dfrac{\theta_0}{2\beta} F_{a2}$

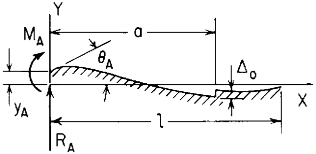
If $\beta l > 6$, see Table 8.6

Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of end restraints

Right end Left end	Free	Guided	Simply supported	Fixed
Free	$R_A = 0 \qquad M_A = 0$ $\theta_A = \theta_o \dfrac{C_2 C_{a4} - 2C_3 C_{a3}}{C_{11}}$ $y_A = \dfrac{\theta_o}{\beta} \dfrac{C_4 C_{a3} - C_3 C_{a4}}{C_{11}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = -\theta_o \dfrac{C_2 C_{a1} + C_4 C_{a3}}{C_{12}}$ $y_A = \dfrac{\theta_o}{\beta} \dfrac{C_3 C_{a1} - C_1 C_{a3}}{C_{12}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \theta_o \dfrac{C_1 C_{a4} - C_3 C_{a2}}{C_{13}}$ $y_A = \dfrac{\theta_o}{2\beta} \dfrac{C_4 C_{a2} - C_2 C_{a4}}{C_{13}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = -\theta_o \dfrac{2C_1 C_{a1} + C_4 C_{a2}}{2 + C_{11}}$ $y_A = \dfrac{-\theta_o}{\beta} \dfrac{C_1 C_{a2} - C_2 C_{a1}}{2 + C_{11}}$
Guided	$R_A = 0 \qquad \theta_A = 0$ $M_A = \theta_o EI\beta \dfrac{C_2 C_{a4} - 2C_3 C_{a3}}{C_{12}}$ $y_A = \dfrac{-\theta_o}{2\beta} \dfrac{2C_1 C_{a3} + C_4 C_{a4}}{C_{12}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = -\theta_o EI\beta \dfrac{C_2 C_{a1} + C_4 C_{a3}}{C_{14}}$ $y_A = \dfrac{\theta_o}{2\beta} \dfrac{C_4 C_{a1} - C_2 C_{a3}}{C_{14}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \theta_o EI\beta \dfrac{C_1 C_{a4} - C_3 C_{a2}}{1 + C_{11}}$ $y_A = \dfrac{-\theta_o}{2\beta} \dfrac{C_1 C_{a2} + C_3 C_{a4}}{1 + C_{11}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = -\theta_o EI\beta \dfrac{2C_1 C_{a1} + C_4 C_{a2}}{C_{12}}$ $y_A = \dfrac{\theta_o}{2\beta} \dfrac{2C_3 C_{a1} - C_2 C_{a2}}{C_{12}}$
Simply supported	$M_A = 0 \qquad y_A = 0$ $R_A = \theta_o 2EI\beta^2 \dfrac{C_3 C_{a4} - C_4 C_{a3}}{C_{13}}$ $\theta_A = \theta_o \dfrac{C_1 C_{a4} - C_2 C_{a3}}{C_{13}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \theta_o 2EI\beta^2 \dfrac{C_1 C_{a3} - C_3 C_{a1}}{1 + C_{11}}$ $\theta_A = -\theta_o \dfrac{C_1 C_{a1} + C_3 C_{a3}}{1 + C_{11}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \theta_o EI\beta^2 \dfrac{C_2 C_{a4} - C_4 C_{a2}}{C_{14}}$ $\theta_A = \dfrac{-\theta_o}{2} \dfrac{C_2 C_{a2} + C_4 C_{a4}}{C_{14}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \theta_o 2EI\beta^2 \dfrac{C_1 C_{a2} - C_2 C_{a1}}{C_{13}}$ $\theta_A = \theta_o \dfrac{C_4 C_{a1} - C_3 C_{a2}}{C_{13}}$
Fixed	$\theta_A = 0 \qquad y_A = 0$ $R_A = \theta_o 2EI\beta^2 \dfrac{2C_1 C_{a3} + C_4 C_{a4}}{2 + C_{11}}$ $M_A = \theta_o 2EI\beta \dfrac{C_1 C_{a4} - C_2 C_{a3}}{2 + C_{11}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \theta_o 2EI\beta^2 \dfrac{C_2 C_{a3} - C_4 C_{a1}}{C_{12}}$ $M_A = -\theta_o 2EI\beta \dfrac{C_1 C_{a1} + C_3 C_{a3}}{C_{12}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \theta_o 2EI\beta^2 \dfrac{C_1 C_{a2} + C_3 C_{a4}}{C_{13}}$ $M_A = -\theta_o EI\beta \dfrac{C_2 C_{a2} + C_4 C_{a4}}{C_{13}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \theta_o 2EI\beta^2 \dfrac{C_2 C_{a2} - 2C_3 C_{a1}}{C_{11}}$ $M_A = \theta_o 2EI\beta \dfrac{C_4 C_{a1} - C_3 C_{a2}}{C_{11}}$

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations (*Continued*)

6. Externally created concentrated lateral displacement



Transverse shear $= V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 - \Delta_o 2EI\beta^3 F_{a2}$

Bending moment $= M = M_A F_1 + \dfrac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 - \Delta_o 2EI\beta^2 F_{a3}$

Slope $= \theta = \theta_A F_1 + \dfrac{M_A}{2EI\beta} F_2 + \dfrac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 - \Delta_o \beta F_{a4}$

Deflection $= y = y_A F_1 + \dfrac{\theta_A}{2\beta} F_2 + \dfrac{M_A}{2EI\beta^2} F_3 + \dfrac{R_A}{4EI\beta^3} F_4 + \Delta_o F_{a1}$

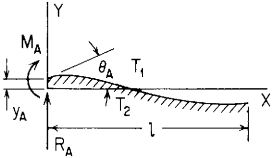
If $\beta l > 6$, see Table 8.6

Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of end restraints

Right end Left end	Free	Guided	Simply supported	Fixed
Free	$R_A = 0 \qquad M_A = 0$ $\theta_A = \Delta_o 2\beta \dfrac{C_2 C_{a3} - C_3 C_{a2}}{C_{11}}$ $y_A = \Delta_o \dfrac{C_4 C_{a2} - 2C_3 C_{a3}}{C_{11}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \Delta_o \beta \dfrac{C_2 C_{a4} - C_4 C_{a2}}{C_{12}}$ $y_A = -\Delta_o \dfrac{C_1 C_{a2} + C_3 C_{a4}}{C_{12}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \Delta_o 2\beta \dfrac{C_1 C_{a3} - C_3 C_{a1}}{C_{13}}$ $y_A = \Delta_o \dfrac{C_4 C_{a1} - C_2 C_{a3}}{C_{13}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \Delta_0 2\beta \dfrac{C_1 C_{a4} - C_4 C_{a1}}{2 + C_{11}}$ $y_A = -\Delta_o \dfrac{2C_1 C_{a1} + C_2 C_{a4}}{2 + C_{11}}$
Guided	$R_A = 0 \qquad \theta_A = 0$ $M_A = \Delta_o 2EI\beta^2 \dfrac{C_2 C_{a3} - C_3 C_{a2}}{C_{12}}$ $y_A = -\Delta_o \dfrac{C_1 C_{a2} + C_4 C_{a3}}{C_{12}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \Delta_o EI\beta^2 \dfrac{C_2 C_{a4} - C_4 C_{a2}}{C_{14}}$ $y_A = \dfrac{-\Delta_o}{2} \dfrac{C_2 C_{a2} + C_4 C_{a4}}{C_{14}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \Delta_o 2EI\beta^2 \dfrac{C_1 C_{a3} - C_3 C_{a1}}{1 + C_{11}}$ $y_A = -\Delta_o \dfrac{C_1 C_{a1} + C_3 C_{a3}}{1 + C_{11}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \Delta_o 2EI\beta^2 \dfrac{C_1 C_{a4} - C_4 C_{a1}}{C_{12}}$ $y_A = -\Delta_o \dfrac{C_2 C_{a1} + C_3 C_{a4}}{C_{12}}$
Simply supported	$M_A = 0 \qquad y_A = 0$ $R_A = \Delta_o 2EI\beta^3 \dfrac{2C_3 C_{a3} - C_4 C_{a2}}{C_{13}}$ $\theta_A = \Delta_o \beta \dfrac{2C_1 C_{a3} - C_2 C_{a2}}{C_{13}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \Delta_o 2EI\beta^3 \dfrac{C_1 C_{a2} + C_3 C_{a4}}{1 + C_{11}}$ $\theta_A = \Delta_o \beta \dfrac{C_1 C_{a4} - C_3 C_{a2}}{1 + C_{11}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \Delta_o 2EI\beta^3 \dfrac{C_2 C_{a3} - C_4 C_{a1}}{C_{14}}$ $\theta_A = -\Delta_o \beta \dfrac{C_2 C_{a1} + C_4 C_{a3}}{C_{14}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \Delta_o 2EI\beta^3 \dfrac{C_2 C_{a4} + 2C_1 C_{a1}}{C_{13}}$ $\theta_A = -\Delta_o \beta \dfrac{2C_3 C_{a1} + C_4 C_{a4}}{C_{13}}$
Fixed	$\theta_A = 0 \qquad y_A = 0$ $R_A = \Delta_o 4EI\beta^3 \dfrac{C_1 C_{a2} + C_4 C_{a3}}{2 + C_{11}}$ $M_A = \Delta_o 2EI\beta^2 \dfrac{2C_1 C_{a3} - C_2 C_{a2}}{2 + C_{11}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \Delta_o 2EI\beta^3 \dfrac{C_4 C_{a4} + C_2 C_{a2}}{C_{12}}$ $M_A = \Delta_o 2EI\beta^2 \dfrac{C_1 C_{a4} - C_3 C_{a2}}{C_{12}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \Delta_o 4EI\beta^3 \dfrac{C_3 C_{a3} + C_1 C_{a1}}{C_{13}}$ $M_A = -\Delta_o 2EI\beta^2 \dfrac{C_2 C_{a1} + C_4 C_{a3}}{C_{13}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = \Delta_o 4EI\beta^3 \dfrac{C_3 C_{a4} + C_2 C_{a1}}{C_{11}}$ $M_A = -\Delta_o 2EI\beta^2 \dfrac{2C_3 C_{a1} + C_4 C_{a4}}{C_{11}}$

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations (*Continued*)

7. Uniform temperature differential from top to bottom



$$\text{Transverse shear} = V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 + \frac{T_1 - T_2}{t}\gamma EI\beta F_4$$

$$\text{Bending moment} = M = M_A F_1 + \frac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 - \frac{T_1 - T_2}{t}\gamma EI(F_1 - 1)$$

$$\text{Slope} = \theta = \theta_A F_1 + \frac{M_A}{2EI\beta} F_2 + \frac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 - \frac{T_1 - T_2}{2t\beta}\gamma F_2$$

$$\text{Deflection} = y = y_A F_1 + \frac{\theta_A}{2\beta} F_2 + \frac{M_A}{2EI\beta^2} F_3 + \frac{R_A}{4EI\beta^3} F_4 - \frac{T_1 - T_2}{2t\beta^2}\gamma F_3$$

If $\beta l > 6$, see Table 8.6

Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of end restraints

Right end Left end	Free	Guided	Simply supported	Fixed
Free	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{(T_1 - T_2)\gamma}{\beta t}\frac{C_1 C_2 + C_3 C_4 - C_2}{C_{11}}$ $y_A = \frac{-(T_1 - T_2)\gamma}{2\beta^2 t}\frac{C_4^2 + 2C_1 C_3 - 2C_3}{C_{11}}$	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{(T_1 - T_2)\gamma}{2\beta t}\frac{C_2^2 + C_4^2}{C_{12}}$ $y_A = \frac{-(T_1 - T_2)\gamma}{2\beta^2 t}\frac{C_2 C_3 - C_1 C_4}{C_{12}}$	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{(T_1 - T_2)\gamma}{\beta t}\frac{C_1^2 + C_3 - C_4}{C_{13}}$ $y_A = \frac{-(T_1 - T_2)\gamma}{\beta t}\frac{C_1 C_2 + C_3 C_4 - C_2}{C_{13}}$	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{(T_1 - T_2)\gamma}{2\beta^2 t}\frac{C_1 C_2 + C_3 C_1}{2 + C_{11}}$ $y_A = \frac{(T_1 - T_2)\gamma}{2\beta^2 t}\frac{2C_1 C_3 - C_2^2}{2 + C_{11}}$
Guided	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{(T_1 - T_2)\gamma EI}{t}\frac{C_1 C_2 + C_3 C_4 - C_2}{C_{12}}$ $y_A = \frac{(T_1 - T_2)\gamma}{2\beta^2 t}\frac{C_4}{C_{12}}$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{(T_1 - T_2)\gamma EI}{t}$ $y_A = 0$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{(T_1 - T_2)\gamma EI}{t}\frac{C_1^2 + C_3^2 - C_1}{1 + C_{11}}$ $y_A = \frac{(T_1 - T_2)\gamma}{2\beta^2 t}\frac{C_3}{1 + C_{11}}$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{(T_1 - T_2)\gamma EI}{t}$ $y_A = 0$
Simply supported	$M_A = 0 \quad y_A = 0$ $R_A = \frac{(T_1 - T_2)\gamma\beta EI}{t}\frac{2C_1 C_3 + C_4^2 - 2C_3}{C_{13}}$ $\theta_A = \frac{(T_1 - T_2)\gamma}{2\beta t}\frac{2C_1^2 + C_2 C_4 - 2C_1}{C_{13}}$	$M_A = 0 \quad y_A = 0$ $R_A = \frac{(T_1 - T_2)\gamma\beta EI}{t}\frac{C_2 C_3 - C_1 C_4}{1 + C_{11}}$ $\theta_A = \frac{(T_1 - T_2)\gamma}{2\beta t}\frac{C_1 C_2 + C_3 C_4}{1 + C_{11}}$	$M_A = 0 \quad y_A = 0$ $R_A = \frac{(T_1 - T_2)\gamma\beta EI}{t}\frac{C_1 C_2 + C_3 C_4 - C_2}{C_{14}}$ $\theta_A = \frac{(T_1 - T_2)\gamma}{2\beta t}\frac{C_2 C_3 - C_1 C_4 + C_4}{C_{14}}$	$M_A = 0 \quad y_A = 0$ $R_A = \frac{(T_1 - T_2)\gamma\beta EI}{t}\frac{C_2^2 - 2C_1 C_3}{C_{13}}$ $\theta_A = \frac{(T_1 - T_2)\gamma}{2\beta t}\frac{2C_3^2 - C_2 C_4}{C_{13}}$
Fixed	$\theta_A = 0 \quad y_A = 0$ $R_A = \frac{(T_1 - T_2)\gamma 2\beta EI}{t}\frac{-C_4}{2 + C_{11}}$ $M_A = \frac{(T_1 - T_2)\gamma EI}{t}\frac{2C_1^2 + C_2 C_4 - 2C_1}{2 + C_{11}}$	$\theta_A = 0 \quad y_A = 0$ $R_A = 0$ $M_A = \frac{(T_1 - T_2)\gamma EI}{t}$	$\theta_A = 0 \quad y_A = 0$ $R_A = \frac{(T_1 - T_2)\gamma\beta EI}{t}\frac{-2C_3}{C_{13}}$ $M_A = \frac{(T_1 - T_2)\gamma EI}{t}\frac{C_2 C_3 - C_1 C_4 + C_4}{C_{13}}$	$\theta_A = 0 \quad y_A = 0$ $R_A = 0$ $M_A = \frac{(T_1 - T_2)\gamma EI}{t}$

TABLE 8.6 Shear, moment, slope, and deflection formulas for semi-infinite beams on elastic foundations

NOTATION: All notation is the same as that for Table 8.5. No length is defined since these beams are assumed to extend from the left end, for which restraints are defined, to a length beyond that portion affected by the loading. Note that M_A and R_A are reactions, not applied loads.

The following constants and functions, involving both beam constants and foundation constants, are hereby defined in order to permit condensing the tabulated formulas which follow

k_o = foundation modulus (unit stress per unit deflection); b_o = beam width; and $\beta = (b_o k_o / 4EI)^{1/4}$. (*Note*: See page 131 for a definition of $\langle x-a \rangle^n$.)

$F_1 = \cosh \beta x \cos \beta x$

$F_2 = \cosh \beta x \sin \beta x + \sinh \beta x \cos \beta x$

$F_3 = \sinh \beta x \sin \beta x$

$F_4 = \cosh \beta x \sin \beta x - \sinh \beta x \cos \beta x$

$F_{a1} = \langle x-a \rangle^0 \cosh \beta \langle x-a \rangle \cos \beta \langle x-a \rangle$

$F_{a2} = \cosh \beta \langle x-a \rangle \sin \beta \langle x-a \rangle + \sinh \beta \langle x-a \rangle \cos \beta \langle x-a \rangle$

$F_{a3} = \sinh \beta \langle x-a \rangle \sin \beta \langle x-a \rangle$

$F_{a4} = \cosh \beta \langle x-a \rangle \sin \beta \langle x-a \rangle - \sinh \beta \langle x-a \rangle \cos \beta \langle x-a \rangle$

$F_{a5} = \langle x-a \rangle^0 - F_{a1}$

$F_{a6} = 2\beta(x-a)\langle x-a \rangle^0 - F_{a2}$

$A_1 = 0.5e^{-\beta a} \cos \beta a$

$A_2 = 0.5e^{-\beta a}(\sin \beta a - \cos \beta a)$

$A_3 = -0.5e^{-\beta a} \sin \beta a$

$A_4 = 0.5e^{-\beta a}(\sin \beta a + \cos \beta a)$

$B_1 = 0.5e^{-\beta b} \cos \beta b$

$B_2 = 0.5e^{-\beta b}(\sin \beta b - \cos \beta b)$

$B_3 = -0.5e^{-\beta b} \sin \beta b$

$B_4 = 0.5e^{-\beta b}(\sin \beta b + \cos \beta b)$

$F_{b1} = \langle x-b \rangle^0 \cosh \beta \langle x-b \rangle \cos \beta \langle x-b \rangle$

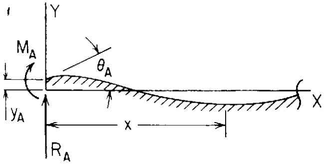
$F_{b2} = \cosh \beta \langle x-b \rangle \sin \beta \langle x-b \rangle + \sinh \langle \beta x-b \rangle \cos \beta \langle x-b \rangle$

$F_{b3} = \sinh \beta \langle x-b \rangle \sin \beta \langle x-b \rangle$

$F_{b4} = \cosh \beta \langle x-b \rangle \sin \beta - b \rangle - \sinh \beta \langle x-b \rangle \cos \beta \langle x-b \rangle$

$F_{b5} = \langle x-b \rangle^0 - F_{b1}$

$F_{b6} = 2\beta(x-b)\langle x-b \rangle^0 - F_{b2}$



Transverse shear $= V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 + LT_V$

Bending moment $= M = M_A F_1 + \dfrac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 + LT_M$

Slope $= \theta = \theta_A F_1 + \dfrac{M_A}{2EI\beta} F_2 + \dfrac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 + LT_\theta$

Deflection $= y = y_A F_1 + \dfrac{\theta_A}{2\beta} F_2 + \dfrac{M_A}{2EI\beta^2} F_3 + \dfrac{R_A}{4EI\beta^3} F_4 + LT_y$

Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of loading and left end restraints. The loading terms LT_V, LT_M, LT_θ, and LT_y are given for each loading condition.

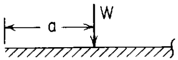
Left end restraint Loading, reference no.	Free	Guided	Simply supported	Fixed	Loading terms
1. Concentrated intermediate load (if $\beta a > 3$, see case 10) 	$R_A = 0$ $M_A = 0$ $\theta_A = \dfrac{-W}{EI\beta^2} A_2$ $y_A = \dfrac{-W}{EI\beta^3} A_1$ (if $a = 0$, see case 8)	$R_A = 0$ $\theta_A = 0$ $M_A = \dfrac{-W}{\beta} A_2$ $y_A = \dfrac{-W}{2EI\beta^3} A_4$	$M_A = 0$ $y_A = 0$ $R_A = 2WA_1$ $\theta_A = \dfrac{W}{EI\beta^2} A_3$	$\theta_A = 0$ $y_A = 0$ $R_A = 2WA_4$ $M_A = \dfrac{2W}{\beta} A_3$	$LT_V = -WF_{a1}$ $LT_M = \dfrac{-W}{2\beta} F_{a2}$ $LT_\theta = \dfrac{-W}{2EI\beta^2} F_{a3}$ $LT_y = \dfrac{-W}{4EI\beta^3} F_{a4}$

TABLE 8.6 Shear, moment, slope, and deflection formulas for semi-infinite beams on elastic foundations (*Continued*)

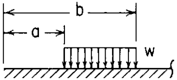
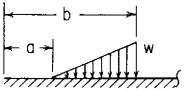
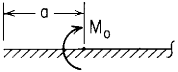
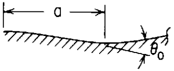
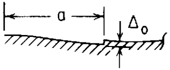
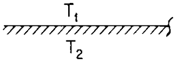
Left end restraint Loading, reference no.	Free	Guided	Simply supported	Fixed	Loading terms
2. Uniformly distributed load from a to b 	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{-w}{EI\beta^3}(B_3 - A_3)$ $y_A = \frac{-w}{2EI\beta^4}(B_2 - A_2)$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{-w}{\beta^2}(B_3 - A_3)$ $y_A = \frac{w}{2EI\beta^4}(B_1 - A_1)$	$M_A = 0 \quad y_A = 0$ $R_A = \frac{w}{\beta}(B_2 - A_2)$ $\theta_A = \frac{w}{2EI\beta^3}(B_4 - A_4)$	$\theta_A = 0 \quad y_A = 0$ $R_A = \frac{-2w}{\beta}(B_1 - A_1)$ $M_A = \frac{w}{\beta^2}(B_4 - A_4)$	$LT_V = \frac{-w}{2\beta}(F_{a2} - F_{b2})$ $LT_M = \frac{-w}{2\beta^2}(F_{a3} - F_{b3})$ $LT_\theta = \frac{-w}{4EI\beta^3}(F_{a4} - F_{b4})$ $LT_y = \frac{-w}{4EI\beta^4}(F_{a5} - F_{b5})$
3. Uniform increasing load from a to b 	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{w}{2EI\beta^4}\left(\frac{B_4 - A_4}{b - a} - 2\beta B_3\right)$ $y_A = \frac{w}{2EI\beta^5}\left(\frac{B_3 - A_3}{b - a} - \beta B_2\right)$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{w}{2\beta^3}\left(\frac{B_4 - A_4}{b - a} - 2\beta B_3\right)$ $y_A = \frac{-w}{4EI\beta^5}\left(\frac{B_2 - A_2}{b - a} - 2\beta B_1\right)$	$M_A = 0 \quad y_A = 0$ $R_A = \frac{-w}{\beta^2}\left(\frac{B_3 - A_3}{b - a} - \beta B_2\right)$ $\theta_A = \frac{w}{2EI\beta^4}\left(\frac{B_1 - A_1}{b - a} + \beta B_4\right)$	$\theta_A = 0 \quad y_A = 0$ $R_A = \frac{w}{\beta^2}\left(\frac{B_2 - A_2}{b - a} - 2\beta B_1\right)$ $M_A = \frac{w}{\beta^3}\left(\frac{B_1 - A_1}{b - a} + \beta B_4\right)$	$LT_V = \frac{-w}{2\beta^2}\left(\frac{F_{a3} - F_{b3}}{b - a} - \beta F_{b2}\right)$ $LT_M = \frac{-w}{4\beta^3}\left(\frac{F_{a4} - F_{b4}}{b - a} - 2\beta F_{b3}\right)$ $LT_\theta = \frac{-w}{4EI\beta^4}\left(\frac{F_{a5} - F_{b5}}{b - a} - \beta F_{b4}\right)$ $LT_y = \frac{-w}{8EI\beta^5}\left(\frac{F_{a6} - F_{b6}}{b - a} - 2\beta F_{b5}\right)$
4. Concentrated intermediate moment (if $\beta a > 3$, see case 11) 	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{-2M_o}{EI\beta}A_1$ $y_A = \frac{M_o}{EI\beta^2}A_4$ (if $a = 0$, see case 9)	$R_A = 0 \quad \theta_A = 0$ $M_A = -2M_oA_1$ $y_A = \frac{-M_o}{EI\beta^2}A_3$	$M_A = 0 \quad y_A = 0$ $R_A = -2M_o\beta A_4$ $\theta_A = \frac{M_o}{EI\beta}A_2$	$\theta_A = 0 \quad y_A = 0$ $R_A = 4M_o\beta A_3$ $M_A = 2M_oA_2$	$LT_V = -M_o\beta F_{a4}$ $LT_M = M_oF_{a1}$ $LT_\theta = \frac{M_o}{2EI\beta}F_{a2}$ $LT_y = \frac{M_o}{2EI\beta^2}F_{a3}$
5. Externally created concentrated angular displacement 	$R_A = 0 \quad M_A = 0$ $\theta_A = -2\theta_oA_4$ $y_A = \frac{-2\theta_o}{\beta}A_3$	$R_A = 0 \quad \theta_A = 0$ $M_A = -2\theta_oEI\beta A_4$ $y_A = \frac{\theta_o}{\beta}A_2$	$M_A = 0 \quad y_A = 0$ $R_A = 4\theta_oEI\beta^2A_3$ $\theta_A = -2\theta_oA_1$	$\theta_A = 0 \quad y_A = 0$ $R_A = -4\theta_oEI\beta^2A_2$ $M_A = -4\theta_oEI\beta A_1$	$LT_V = -2\theta_oEI\beta^2F_{a3}$ $LT_M = -\theta_oEI\beta F_{a4}$ $LT_\theta = \theta_oF_{a1}$ $LT_y = \frac{\theta_o}{2\beta}F_{a2}$

TABLE 8.6 Shear, moment, slope, and deflection formulas for semi-infinite beams on elastic foundations (*Continued*)

6. Externally created concentrated lateral displacement 	$R_A = 0 \quad M_A = 0$ $\theta_A = 4\Delta_o \beta A_3$ $y_A = 2\Delta_o A_2$	$R_A = 0 \quad \theta_A = 0$ $M_A = 4\Delta_o EI\beta^2 A_3$ $y_A = -2\Delta_o A_1$	$M_A = 0 \quad y_A = 0$ $R_A = -4\Delta_o EI\beta^3 A_2$ $\theta_A = -2\Delta_o \beta A_4$	$\theta_A = 0 \quad y_A = 0$ $R_A = 8\Delta_o EI\beta^3 A_1$ $M_A = -4\Delta_o EI\beta^2 A_4$	$LT_V = -2\Delta_o EI\beta^3 F_{a2}$ $LT_M = -2\Delta_o EI\beta^2 F_{a3}$ $LT_\theta = -\Delta_o \beta F_{a4}$ $LT_y = \Delta_o F_{a1}$
7. Uniform temperature differential from top to bottom 	$R_A = 0 \quad M_A = 0$ $\theta_A = \frac{T_1 - T_2}{t\beta}\gamma$ $y_A = -\frac{T_1 - T_2}{2t\beta^2}\gamma$	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{T_1 - T_2}{t}\gamma EI$ $y_A = 0$	$M_A = 0 \quad y_A = 0$ $R_A = \frac{T_1 - T_2}{t}\gamma EI\beta$ $\theta_A = \frac{T_1 - T_2}{2t\beta}\gamma$	$\theta_A = 0 \quad y_A = 0$ $R_A = 0$ $M_A = \frac{T_1 - T_2}{t}\gamma EI$	$LT_V = \frac{T_1 - T_2}{t}\gamma EI\beta F_4$ $LT_M = \frac{T_1 - T_2}{t}\gamma EI(1 - F_1)$ $LT_\theta = -\frac{T_1 - T_2}{2t\beta}\gamma F_2$ $LT_y = -\frac{T_1 - T_2}{2t\beta^2}\gamma F_3$

Simple loads on semi-infinite and on infinite beams on elastic foundations

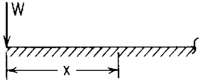
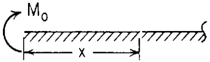
Loading, reference no.	Shear, moment, and deformation equations	Selected maximum values
8. Concentrated end load on a semi-infinite beam, left end free 	$V = -We^{-\beta x}(\cos\beta x - \sin\beta x)$ $M = -\frac{W}{\beta}e^{-\beta x}\sin\beta x$ $\theta = \frac{W}{2EI\beta^2}e^{-\beta x}(\cos\beta x + \sin\beta x)$ $y = -\frac{W}{2EI\beta^3}e^{-\beta x}\cos\beta x$	Max $V = -W$ at $x = 0$ Max $M = -0.3224\frac{W}{\beta}$ at $x = \frac{\pi}{4\beta}$ Max $\theta = \frac{W}{2EI\beta^2}$ at $x = 0$ Max $y = \frac{-W}{2EI\beta^3}$ at $x = 0$
9. Concentrated end moment on a semi-infinite beam, left end free 	$V = -2M_o\beta e^{-\beta x}\sin\beta x$ $M = M_o e^{-\beta x}(\cos\beta x + \sin\beta x)$ $\theta = -\frac{M_o}{EI\beta}e^{-\beta x}\cos\beta x$ $y = -\frac{M_o}{2EI\beta^2}e^{-\beta x}(\sin\beta x - \cos\beta x)$	Max $V = -0.6448M_o\beta$ at $x = \frac{\pi}{4\beta}$ Max $M = M_o$ at $x = 0$ Max $\theta = -\frac{M_o}{EI\beta}$ at $x = 0$ Max $y = \frac{M_o}{2EI\beta^2}$ at $x = 0$

TABLE 8.6 Shear, moment, slope, and deflection formulas for semi-infinite beams on elastic foundations (*Continued*)

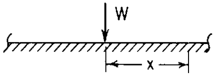
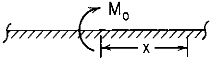
Loading, reference no.	Shear, moment, and deformation equations	Selected maximum values
10. Concentrated load on an infinite beam  	$V = -\frac{W}{2} e^{-\beta x} \cos \beta x$ $M = \frac{W}{4\beta} e^{-\beta x}(\cos \beta x - \sin \beta x)$ $\theta = \frac{W}{4EI\beta^2} e^{-\beta x} \sin \beta x$ $y = -\frac{W}{8EI\beta^3} e^{-\beta x}(\cos \beta x + \sin \beta x)$	Max $V = -\frac{W}{2}$ at $x = 0$ Max $M = \frac{W}{4\beta}$ at $x = 0$ Max $\theta = 0.0806 \frac{W}{EI\beta^2}$ at $x = \frac{\pi}{4\beta}$ Max $y = -\frac{W}{8EI\beta^3}$ at $x = 0$
11. Concentrated moment on an infinite beam  	$V = -\frac{M_o \beta}{2} e^{-\beta x}(\cos \beta x + \sin \beta x)$ $M = \frac{M_o}{2} e^{-\beta x} \cos \beta x$ $\theta = -\frac{M_o}{4EI\beta} e^{-\beta x}(\cos \beta x - \sin \beta x)$ $y = -\frac{M_o}{4EI\beta^2} e^{-\beta x} \sin \beta x$	Max $V = -\frac{M_o \beta}{2}$ at $x = 0$ Max $M = \frac{M_o}{2}$ at $x = 0$ Max $\theta = -\frac{M_o}{4EI\beta}$ at $x = 0$ Max $y = -0.0806 \frac{M_o}{EI\beta^2}$ at $x = \frac{\pi}{4\beta}$

TABLE 8.7(*a*) Reaction and deflection coefficients for beams under simultaneous axial and transverse loading: cantilver end support

Case no. in Table 8.1	Load location a/l	Coefficient listed for	Axial compressive load, $kl = \sqrt{Pl^2/EI}$					Axial tensile load, $kl = \sqrt{Pl^2/EI}$				
			0.2	0.4	0.6	0.8	1.0	0.5	1.0	2.0	4.0	8.0
1a. Conc. load	0	y_A	1.0163	1.0684	1.1686	1.3455	1.6722	0.9092	0.7152	0.3885	0.1407	0.0410
		θ_A	1.0169	1.0713	1.1757	1.3604	1.7016	0.9054	0.7039	0.3671	0.1204	0.0312
		M_B	1.0136	1.0570	1.1402	1.2870	1.5574	0.9242	0.7616	0.4820	0.2498	0.1250
	0.5	y_A	1.0153	1.0646	1.1589	1.3256	1.6328	0.9142	0.7306	0.4180	0.1700	0.0566
		θ_A	1.0195	1.0821	1.2026	1.4163	1.8126	0.8914	0.6617	0.2887	0.0506	0.0022
		M_B	1.0085	1.0355	1.0869	1.1767	1.3402	0.9524	0.8478	0.6517	0.4333	0.2454
2a. Uniform load	0	y_A	1.0158	1.0665	1.1638	1.3357	1.6527	0.9117	0.7228	0.4031	0.1552	0.0488
		θ_A	1.0183	1.0771	1.1900	1.3901	1.7604	0.8980	0.6812	0.3243	0.0800	0.0117
		M_B	1.0102	1.0427	1.1047	1.2137	1.4132	0.9430	0.8193	0.5969	0.3792	0.2188
	0.5	y_A	1.1050	1.0629	1.1548	1.3171	1.6161	0.9164	0.7373	0.4314	0.1851	0.0667
		θ_A	1.0198	1.0835	1.2062	1.4239	1.8278	0.8896	0.6562	0.2794	0.0447	0.0015
		M_B	1.0059	1.0248	1.0606	1.1229	1.2357	0.9666	0.8925	0.7484	0.5682	0.3773
2a. Uniformly increasing	0	y_A	1.0155	1.0652	1.1604	1.3287	1.6389	0.9135	0.7283	0.4137	0.1662	0.0552
		θ_A	1.0190	1.0799	1.1972	1.4051	1.7902	0.8942	0.6700	0.3039	0.0629	0.0057
		M_B	1.0081	1.0341	1.0836	1.1701	1.3278	0.9543	0.8543	0.6691	0.4682	0.2930
	0.5	y_A	1.0147	1.0619	1.1523	1.3118	1.6056	0.9178	0.7415	0.4400	0.1951	0.0740
		θ_A	1.0200	1.0843	1.2080	1.4277	1.8355	0.8887	0.6535	0.2748	0.0419	0.0012
		M_B	1.0046	1.0191	1.0467	1.0944	1.1806	0.9742	0.9166	0.8020	0.6489	0.4670
2a. Uniformly decreasing	0	y_A	1.0159	1.0670	1.1650	1.3382	1.6578	0.9110	0.7208	0.3992	0.1512	0.0465
		θ_A	1.0181	1.0761	1.1876	1.3851	1.7505	0.8992	0.6850	0.3311	0.0857	0.0136
		M_B	1.0112	1.0469	1.1153	1.2355	1.4559	0.9374	0.8018	0.5609	0.3348	0.1817
	0.5	y_A	1.0150	1.0633	1.1557	1.3189	1.6197	0.9159	0.7358	0.4284	0.1816	0.0642
		θ_A	1.0198	1.0833	1.2056	1.4226	1.8253	0.8899	0.6571	0.2809	0.0456	0.0016
		M_B	1.0066	1.0276	1.0676	1.1372	1.2632	0.9628	0.8804	0.7215	0.5279	0.3324
3a. Conc. moment	0	y_A	1.0169	1.0713	1.1757	1.3604	1.7016	0.9054	0.7039	0.3671	0.1204	0.0312
		θ_A	1.0136	1.0570	1.1402	1.2870	1.5574	0.9242	0.7616	0.4820	0.2498	0.1250
		M_B	1.0203	1.0857	1.2116	1.4353	1.8508	0.8868	0.6481	0.2658	0.0366	0.0007
	0.5	y_A	1.0161	1.0677	1.1668	1.3418	1.6646	0.9101	0.7180	0.3932	0.1437	0.0409
		θ_A	1.0186	1.0785	1.1935	1.3974	1.7747	0.8961	0.6754	0.3124	0.0664	0.0046
		M_B	1.0152	1.0641	1.1575	1.3220	1.6242	0.9147	0.7308	0.4102	0.1378	0.0183

TABLE 8.7(*b*) Reaction and deflection coefficients for beams under simultaneous axial and transverse loading: simply supported ends

			Axial compressive load, $kl = \sqrt{Pl^2/EI}$					Axial tensile load, $kl = \sqrt{Pl^2/EI}$				
Case no. in Table 8.1	Load location a/l	Coefficient listed for	0.4	0.8	1.2	1.6	2.0	1.0	2.0	4.0	8.0	12.0
1e. Conc. load	0.25	$y_{l/2}$	1.0167	1.0702	1.1729	1.3546	1.6902	0.9069	0.7082	0.3751	0.1273	0.0596
		θ_A	1.0144	1.0605	1.1485	1.3031	1.5863	0.9193	0.7447	0.4376	0.1756	0.0889
		θ_B	1.0185	1.0779	1.1923	1.3958	1.7744	0.8972	0.6805	0.3311	0.0990	0.0444
		$M_{l/4}$	1.0101	1.0425	1.1039	1.2104	1.4025	0.9427	0.8158	0.5752	0.3272	0.2217
	0.50	$y_{l/2}$	1.0163	1.0684	1.1686	1.3455	1.6722	0.9092	0.7152	0.3885	0.1407	0.0694
		θ_A	1.0169	1.0713	1.1757	1.3604	1.7016	0.9054	0.7039	0.3671	0.1204	0.0553
		$M_{l/2}$	1.0136	1.0570	1.1402	1.2870	1.5574	0.9242	0.7616	0.4820	0.2498	0.1667
2e. Uniform load	0	$y_{l/2}$	1.0165	1.0696	1.1714	1.3515	1.6839	0.9077	0.7107	0.3797	0.1319	0.0630
		θ_A	1.0163	1.0684	1.1686	1.3455	1.6722	0.9092	0.7152	0.3885	0.1407	0.0694
		$M_{l/2}$	1.0169	1.0713	1.1757	1.3604	1.7016	0.9054	0.7039	0.3671	0.1204	0.0553
	0.50	$y_{1/2}$	1.0165	1.0696	1.1714	1.3515	1.6839	0.9077	0.7107	0.3797	0.1319	0.0630
		θ_A	1.0180	1.0759	1.1873	1.3851	1.7524	0.8997	0.6875	0.3418	0.1053	0.0475
		θ_B	1.0149	1.0626	1.1540	1.3147	1.6099	0.9166	0.7368	0.4248	0.1682	0.0865
		$M_{l/2}$	1.0169	1.0713	1.1757	1.3604	1.7016	0.9054	0.7039	0.3671	0.1204	0.0553
2e. Uniformly increasing	0	$y_{l/2}$	1.0165	1.0696	1.1714	1.3515	1.6839	0.9077	0.7107	0.3797	0.1319	0.0630
		θ_A	1.0172	1.0722	1.1781	1.3656	1.7127	0.9044	0.7011	0.3643	0.1214	0.0570
		θ_B	1.0155	1.0651	1.1603	1.3280	1.6368	0.9134	0.7276	0.4097	0.1575	0.0803
		$M_{l/2}$	1.0169	1.0713	1.1757	1.3604	1.7016	0.9054	0.7039	0.3671	0.1204	0.0553
	0.50	$y_{l/2}$	1.0167	1.0702	1.1729	1.3545	1.6899	0.9069	0.7084	0.3754	0.1278	0.0601
		θ_A	1.0184	1.0776	1.1915	1.3942	1.7710	0.8976	0.6816	0.3329	0.1002	0.0450
		θ_B	1.0140	1.0588	1.1445	1.2948	1.5702	0.9215	0.7516	0.4521	0.1936	0.1048
		$M_{l/2}$	1.0183	1.0771	1.1900	1.3901	1.7604	0.8980	0.6812	0.3243	0.0800	0.0270
3e. Conc. moment	0	$y_{l/2}$	1.0169	1.0713	1.1757	1.3604	1.7016	0.9054	0.7039	0.3671	0.1204	0.0553
		θ_A	1.0108	1.0454	1.1114	1.2266	1.4365	0.9391	0.8060	0.5630	0.3281	0.2292
		θ_B	1.0190	1.0801	1.1979	1.4078	1.7993	0.8945	0.6728	0.3200	0.0932	0.0417
	0.25	$y_{l/2}$	1.0161	1.0677	1.1668	1.3418	1.6646	0.9101	0.7180	0.3932	0.1437	0.0704
		θ_A	1.0202	1.0852	1.2102	1.4318	1.8424	0.8873	0.6485	0.2595	0.0113	−0.0244
		θ_B	1.0173	1.0728	1.1795	1.3682	1.7174	0.9035	0.6982	0.3571	0.1131	0.0512

TABLE 8.7(*c*) Reaction and deflection coefficients for beams under simultaneous axial and transverse loading: left end simply supported, right end fixed

			Axial compressive load, $kl = \sqrt{Pl^2/EI}$					Axial tensile load, $kl = \sqrt{Pl^2/EI}$				
Case no. in Table 8.1	Load location a/l	Coefficient listed for	0.6	1.2	1.8	2.4	3.0	1.0	2.0	4.0	8.0	12.0
1c. Conc. load	0.25	$y_{l/2}$	1.0190	1.0804	1.2005	1.4195	1.8478	0.9507	0.8275	0.5417	0.2225	0.1108
		θ_A	1.0172	1.0726	1.1803	1.3753	1.7530	0.9553	0.8429	0.5762	0.2576	0.1338
		M_B	1.0172	1.0728	1.1818	1.3812	1.7729	0.9554	0.8443	0.5881	0.3018	0.1940
	0.50	$y_{l/2}$	1.0170	1.0719	1.1786	1.3718	1.7458	0.9557	0.8444	0.5802	0.2647	0.1416
		θ_A	1.0199	1.0842	1.2101	1.4406	1.8933	0.9485	0.8202	0.5255	0.2066	0.1005
		M_B	1.1037	1.0579	1.1432	1.2963	1.5890	0.9642	0.8733	0.6520	0.3670	0.2412
2c. Uniform load	0	$y_{l/2}$	1.0176	1.0742	1.1846	1.3848	1.7736	0.9543	0.8397	0.5694	0.2524	0.1323
		θ_A	1.0183	1.0776	1.1933	1.4042	1.8162	0.9524	0.8334	0.5561	0.2413	0.1263
		M_B	1.0122	1.0515	1.1273	1.2635	1.5243	0.9681	0.8874	0.6900	0.4287	0.3033
	0.50	$y_{l/2}$	1.0163	1.0689	1.1709	1.3549	1.7094	9.9575	0.8502	0.5932	0.2778	0.1505
		θ_A	1.0202	1.0856	1.2139	1.4496	1.9147	0.9477	0.8179	0.5224	0.2087	0.1048
		M_B	1.0091	1.0383	1.0940	1.1920	1.3744	0.9760	0.9141	0.7545	0.5126	0.3774
2c. Uniformly increasing	0	$y_{l/2}$	1.0170	1.0719	1.1785	1.3716	1.7453	0.9557	0.8444	0.5799	0.2637	0.1405
		θ_A	1.0192	1.0814	1.2030	1.4255	1.8619	0.9502	0.8259	0.5394	0.2237	0.1136
		M_B	1.0105	1.0440	1.1084	1.2230	1.4399	0.9726	0.9028	0.7277	0.4799	0.3504
	0.50	$y_{l/2}$	1.0160	1.0674	1.1669	1.3463	1.6911	0.9584	0.8533	0.6003	0.2860	0.1571
		θ_A	1.0202	1.0855	1.2138	1.4499	1.9165	0.9478	0.8183	0.5245	0.2141	0.1105
		M_B	1.0071	1.0298	1.0726	1.1473	1.2843	0.9813	0.9325	0.8029	0.5900	0.4573
2c. Uniformly decreasing	0	$y_{l/2}$	1.0180	1.0762	1.1895	1.3957	1.7968	0.9532	0.8359	0.5608	0.2431	0.1256
		θ_A	1.0177	1.0751	1.1868	1.3900	1.7857	0.9539	0.8383	0.5673	0.2531	0.1347
		M_B	1.0142	1.0600	1.1489	1.3098	1.6207	0.9630	0.8698	0.6470	0.3701	0.2495
	0.50	$y_{l/2}$	1.0165	1.0695	1.1725	1.3584	1.7169	0.9571	0.8490	0.5902	0.2743	0.1477
		θ_A	1.0202	1.0856	1.2139	1.4495	1.9140	0.9477	0.8177	0.5216	0.2066	0.1026
		M_B	1.0104	1.0439	1.1078	1.2208	1.4327	0.9726	0.9023	0.7232	0.4625	0.3257
3c. Conc. moment	0	$y_{l/2}$	1.0199	1.0842	1.2101	1.4406	1.8933	0.9485	0.8202	0.5255	0.2066	0.1005
		θ_A	1.0122	1.0515	1.1273	1.2635	1.5243	0.9681	0.8874	0.6900	0.4287	0.3030
		M_B	1.0183	1.0779	1.1949	1.4105	1.8379	0.9525	0.8348	0.5684	0.2842	0.1704
	0.50	$y_{l/2}$	1.0245	1.1041	1.2613	1.5528	2.1347	0.9368	0.7812	0.4387	0.1175	0.0390
		θ_A	1.0168	1.0707	1.1750	1.3618	1.7186	0.9562	0.8452	0.5760	0.2437	0.1176
		M_B	0.9861	0.9391	0.8392	0.6354	0.1828	1.0346	1.1098	1.1951	0.9753	0.7055

TABLE 8.7(*d*) Reaction and deflection coefficients for beams under simultaneous axial and transverse loading: fixed ends

			Axial compressive load, $kl = \sqrt{Pl^2/EI}$					Axial tensile load, $kl = \sqrt{Pl^2/EI}$				
Case no. in Table 8.1	Load location a/l	Coefficient listed for	0.8	1.6	2.4	3.2	4.0	1.0	2.0	4.0	8.0	12.0
1d. Conc. load	0.25	$y_{l/2}$	1.0163	1.0684	1.1686	1.3455	1.6722	0.9756	0.9092	0.7152	0.3885	0.2228
		R_A	1.0007	1.0027	1.0064	1.0121	1.0205	0.9990	0.9960	0.9859	0.9613	0.9423
		M_A	1.0088	1.0366	1.0885	1.1766	1.3298	0.9867	0.9499	0.8350	0.6008	0.4416
		M_B	1.0143	1.0603	1.1498	1.3117	1.6204	0.9787	0.9213	0.7583	0.4984	0.3645
	0.50	$y_{l/2}$	1.0163	1.0684	1.1686	1.3455	1.6722	0.9756	0.9092	0.7152	0.3885	0.2228
		M_A	1.0136	1.0570	1.1402	1.2870	1.5574	0.9797	0.9242	0.7616	0.4820	0.3317
		$M_{l/2}$	1.0136	1.0570	1.1402	1.2870	1.5574	0.9797	0.9242	0.7616	0.4820	0.3317
2d. Uniform load	0	$y_{l/2}$	1.0163	1.0684	1.1686	1.3455	1.6722	0.9756	0.9092	0.7152	0.3885	0.2228
		M_A	1.0108	1.0454	1.1114	1.2266	1.4365	0.9837	0.9391	0.8060	0.5630	0.4167
		$M_{l/2}$	1.0190	1.0801	1.1979	1.4078	1.7993	0.9716	0.8945	0.6728	0.3200	0.1617
	0.50	$y_{l/2}$	1.0146	1.0667	1.1667	1.3434	1.6696	0.9741	0.9077	0.7141	0.3879	0.2224
		R_A	0.9982	0.9927	0.9828	0.9677	0.9453	1.0027	1.0106	1.0375	1.1033	1.1551
		M_A	1.0141	1.0595	1.1473	1.3045	1.5999	0.9789	0.9217	0.7571	0.4868	0.3459
		M_B	1.0093	1.0390	1.0950	1.1913	1.3622	0.9859	0.9470	0.8282	0.5976	0.4488
2d. Uniformly increasing	0	$y_{l/2}$	1.0163	1.0684	1.1686	1.3455	1.6722	0.9756	0.9092	0.7152	0.3885	0.2228
		R_A	0.9995	0.9979	0.9951	0.9908	0.9845	1.0008	1.0030	1.0108	1.0303	1.0463
		M_A	1.0124	1.0521	1.1282	1.2627	1.5107	0.9814	0.9307	0.7818	0.5218	0.3750
		M_B	1.0098	1.0410	1.1001	1.2026	1.3870	0.9853	0.9447	0.8221	0.5904	0.4445
	0.50	$y_{l/2}$	1.0161	1.0679	1.1672	1.3427	1.6667	0.9758	0.9099	0.7174	0.3927	0.2274
		R_A	0.9969	0.9875	0.9707	0.9449	0.9070	1.0047	1.0182	1.0648	1.1815	1.2778
		M_A	1.0141	1.0595	1.1476	1.3063	1.6076	0.9790	0.9222	0.7602	0.4995	0.3647
		M_B	1.0075	1.0312	1.0755	1.1507	1.2819	0.9887	0.9573	0.8594	0.6582	0.5168
3d. Conc. moment	0.25	$y_{l/2}$	1.0169	1.0713	1.1757	1.3604	1.7016	0.9746	0.9054	0.7039	0.3671	0.2001
		R_A	0.9993	0.9972	0.9932	0.9867	0.9763	1.0010	1.0038	1.0122	1.0217	1.0134
		M_A	1.0291	1.1227	1.3025	1.6203	2.2055	0.9563	0.8376	0.4941	−0.0440	−0.2412
		M_B	1.0151	1.0635	1.1571	1.3244	1.6380	0.9775	0.9164	0.7404	0.4517	0.3035
	0.50	$\theta_{l/2}$	1.0054	1.0220	1.0515	1.0969	1.1641	0.9918	0.9681	0.8874	0.6900	0.5346
		R_A	1.0027	1.0110	1.0260	1.0492	1.0842	0.9959	0.9842	0.9449	0.8561	0.7960
		M_A	1.0081	1.0331	1.0779	1.1477	1.2525	0.9877	0.9525	0.8348	0.5684	0.3881

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading

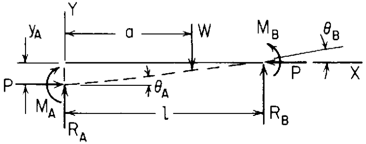
NOTATION: P = axial compressive load (force); all other notation is the same as that for Table 8.1. P must be less than P_{cr} where $P_{cr} = K_1\pi^2 EI/l^2$ and where, for cases 1a–6a and 1f-6f, $K_1 = 0.25$; for cases 1b–6b and 1e–6e; $K_1 = 1$; for cases 1c–6c, $K_1 = 2.046$; and for cases 1d–6d, $K_1 = 4$.

The following constants and functions are hereby defined in order to permit condensing the tabulated formulas which follow. $k = (P/EI)^{1/2}$. (*Note:* See page 131 for a definition of $\langle x-a\rangle^n$.) The function $\sin k\langle x-a\rangle$ is also defined as having a value of zero if $x < a$

$F_1 = \cos kx$	$F_{a1} = \langle x-a\rangle^0 \cos k(x-a)$	$C_1 = \cos kl$	$C_{a1} \cos k(l-a)$
$F_2 = \sin kx$	$F_{a2} = \sin k\langle x-a\rangle$	$C_2 = \sin kl$	$C_{a2} = \sin k(l-a)$
$F_3 = 1-\cos kx$	$F_{a3} = \langle x-a\rangle^0[1-\cos k(x-a)]$	$C_3 = 1-\cos kl$	$C_{a3} = 1-\cos k(l-a)$
$F_4 = kx-\sin kx$	$F_{a4} = k\langle x-a\rangle - \sin k\langle x-a\rangle$	$C_4 = kl-\sin kl$	$C_{a4} = k(l-a)-\sin k(l-a)$
	$F_{a5} = \frac{k^2}{2}\langle x-a\rangle^2 - F_{a3}$		$C_{a5} = \frac{k^2}{2}(l-a)^2 - C_{a3}$
	$F_{a6} = \frac{k^3}{6}\langle x-a\rangle^3 - F_{a4}$		$C_{a6} = \frac{k^3}{6}(l-a)^3 - C_{a4}$

(*Note:* M_A and R_A as well as M_B and R_B are reactions, not applied loads. They exist only when the necessary end restraints are provided.)

1. Axial compressive load plus concentrated intermediate lateral load



Transverse shear $= V = R_A F_1 - M_A k F_2 - \theta_A P F_1 - W F_{a1}$

Bending moment $= M = M_A F_1 + \frac{R_A}{k}F_2 - \frac{\theta_A P}{k}F_2 - \frac{W}{k}F_{a2}$

Slope $= \theta = \theta_A F_1 + \frac{M_A k}{P}F_2 + \frac{R_A}{P}F_3 - \frac{W}{P}F_{a3}$

Deflection $= y = y_A + \frac{\theta_A}{k}F_2 + \frac{M_A}{P}F_3 + \frac{R_A}{kP}F_4 - \frac{W}{kP}F_{a4}$

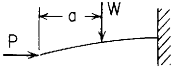
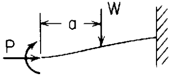
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
1a. Left end free, right end fixed (cantilever)  	$R_A = 0 \qquad M_A = 0 \qquad \theta_A = \frac{W}{P}\frac{C_{a3}}{C_1}$ $y_A = \frac{-W}{kP}\frac{C_2C_{a3} - C_1C_{a4}}{C_1}$ $R_B = W \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = \frac{-W}{k}\frac{C_2C_{a3} + C_1C_{a2}}{C_1}$	Max $M = M_B$; max possible value $= \frac{-W}{k}\tan kl$ when $a = 0$ Max $\theta = \theta_A$; max possible value $= \frac{W}{P}\frac{1-\cos kl}{\cos kl}$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{-W}{kP}(\tan kl - kl)$ when $a = 0$
1b. Left end guided, right end fixed  	$R_A = 0 \qquad M_A = \frac{W}{k}\frac{C_{a3}}{C_2} \qquad \theta_A = 0$ $y_A = \frac{-W}{kP}\frac{C_3C_{a3} - C_2C_{a4}}{C_2}$ $R_B = W \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = \frac{-W}{k}\frac{\cos ka - \cos kl}{\sin kl}$	Max $+M = M_A$; max possible value $= \frac{W}{k}\tan\frac{kl}{2}$ when $a = 0$ Max $-M = M_B$; max possible value $= \frac{-W}{k}\tan\frac{kl}{2}$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{-W}{kP}\left(2\tan\frac{kl}{2} - kl\right)$ when $a = 0$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

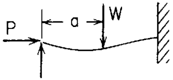
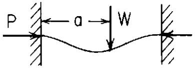
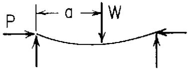
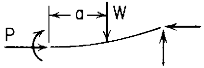
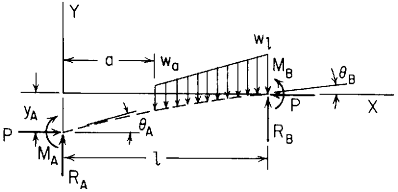
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
1c. Left end simply supported, right end fixed 	$R_A = W\frac{C_2C_{a3} - C_1C_{a4}}{C_2C_3 - C_1C_4} \qquad M_A = 0$ $\theta_A = \frac{-W}{P}\frac{C_4C_{a3} - C_3C_{a4}}{C_2C_3 - C_1C_4} \qquad y_A = 0$ $R_B = W - R_A \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = \frac{-W}{k}\frac{kl\sin ka - ka\sin kl}{\sin kl - kl\cos kl}$	Max $- M = M_B$; max possible value occurs when $a = \frac{1}{k}\cos^{-1}\frac{\sin kl}{kl}$ If $a = l/2$ (transverse center load), then $R_A = W\frac{\sin kl - \sin\frac{kl}{2} - \frac{kl}{2}\cos kl}{\sin kl - kl\cos kl}$ $M_B = -Wl\frac{\sin\frac{kl}{2}\left(1 - \cos\frac{kl}{2}\right)}{\sin kl - kl\cos kl}$
1d. Left end fixed, right end fixed 	$R_A = W\frac{C_3C_{a3} - C_2C_{a4}}{C_3^2 - C_2C_4}$ $M_A = \frac{-W}{k}\frac{C_4C_{a3} - C_3C_{a4}}{C_3^2 - C_2C_4}$ $\theta_A = 0 \qquad y_A = 0$ $R_B = W - R_A \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = M_A + R_A l - W(l - a)$	Max $- M = M_A$ if $a < \frac{l}{2}$ If $a = \frac{l}{2}$ (transverse center load), then $R_A = R_B = \frac{W}{2} \qquad M_B = M_A = \frac{-W}{2k}\tan\frac{kl}{4}$ Max $+ M = \frac{W}{2k}\tan\frac{kl}{4}$ at $x = \frac{l}{2}$ Max $y = \frac{-W}{kP}\left(\tan\frac{kl}{4} - \frac{kl}{4}\right) \qquad$ at $x = \frac{l}{2}$
1e. Left end simply supported, right end simply supported 	$R_A = \frac{W}{l}(l - a) \qquad M_A = 0 \qquad y_A = 0$ $\theta_A = \frac{-W}{P}\left[\frac{\sin k(l - a)}{\sin kl} - \frac{l - a}{l}\right]$ $R_B = W\frac{a}{l} \qquad M_B = 0 \qquad y_B = 0$ $\theta_B = \frac{W}{P}\left(\frac{\sin ka}{\sin kl} - \frac{a}{l}\right)$	Max $M = \frac{W\sin k(l - a)}{k\sin kl}\sin ka \qquad$ at $x = a$ if $\frac{l}{2} < a < \frac{\pi}{2k}$ Max $M = \frac{W\sin k(l - a)}{k\sin kl} \qquad$ at $x = \frac{\pi}{2k}$ if $a > \frac{\pi}{2k}$ and $a > \frac{l}{2}$; max possible value of $M = \frac{W}{2k}\tan\frac{kl}{2}$ at $x = a$ when $a = \frac{l}{2}$ Max $\theta = \theta_B$ if $a > \frac{l}{2}$; max possible value occurs when $a = \frac{1}{k}\cos^{-1}\frac{\sin kl}{kl}$ Max y occurs at $x = \frac{1}{k}\cos^{-1}\frac{(l - a)\sin kl}{l\sin k(l - a)}$ if $a > \frac{l}{2}$; max possible value $= \frac{-W}{2kP}\left(\tan\frac{kl}{2} - \frac{kl}{2}\right)$ at $x = \frac{l}{2}$ when $a = \frac{l}{2}$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

1f. Left end guided, right end simply supported  	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{W}{k}\frac{\sin k(l-a)}{\cos kl}$ $y_A = \frac{-W}{kp}\left[\frac{\sin k(l-a)}{\cos kl} - k(l-a)\right]$ $R_B = W \qquad M_B = 0 \qquad y_B = 0$ $\theta_B = \frac{W}{P}\left(\frac{\cos ka}{\cos kl} - 1\right)$	Max $M = M_A$; max possible value $= \frac{W}{k}\tan kl$ when $a = 0$ Max $\theta = \theta_B$; max possible value $= \frac{W}{P}\frac{1-\cos kl}{\cos kl}$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{-W}{kP}(\tan kl - kl)$ when $a = 0$
2. Axial compressive load plus distributed lateral load  	Transverse shear $= V = R_AF_1 - M_AkF_2 - \theta_APF_1 - \frac{w_a}{k}F_{a2} - \frac{w_l - w_a}{k^2(l-a)}F_{a3}$ Bending moment $= M = M_AF_1 + \frac{R_A}{k}F_2 - \frac{\theta_AP}{k}F_2 - \frac{w_a}{k^2}F_{a3} - \frac{w_l - w_a}{k^3(l-a)}F_{a4}$ Slope $= \theta = \theta_AF_1 + \frac{M_Ak}{P}F_2 + \frac{R_A}{P}F_3 - \frac{w_a}{kP}F_{a4} - \frac{w_l - w_a}{k^2P(l-a)}F_{a5}$ Deflection $= y = y_A + \frac{\theta_A}{k}F_2 + \frac{M_A}{P}F_3 + \frac{R_A}{kP}F_4 - \frac{w_a}{k^2P}F_{a5} - \frac{w_l - w_a}{k^3P(l-a)}F_{a6}$	
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations

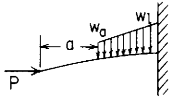
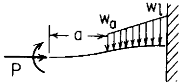
2a. Left end free, right end fixed (cantilever)

 	
	$R_A = 0 \qquad M_A = 0$
	$\theta_A = \frac{w_a}{kP}\frac{C_{a4}}{C_1} + \frac{w_l - w_a}{k^2P(l-a)}\frac{C_{a5}}{C_1}$
	$y_A = \frac{-w_a}{k^2P}\frac{C_2C_{a4} - C_1C_{a5}}{C_1} - \frac{w_l - w_a}{k^3P(l-a)}\frac{C_2C_{a5} - C_1C_{a6}}{C_1}$
	$R_B = \frac{w_a + w_l}{2}(l-a) \qquad \theta_B = 0$
	$M_B = \frac{-w_a}{k^2}\frac{C_2C_{a4} + C_1C_{a3}}{C_1} - \frac{w_l - w_a}{k^3(l-a)}\frac{C_2C_{a5} + C_1C_{a4}}{C_1}$
	$y_B = 0$
		If $a = 0$ and $w_a = w_l$ (uniform load on entire span), then
		Max $M = M_B = \frac{-w_a}{k^2}\left(1 + kl\tan kl - \frac{1}{\cos kl}\right)$
		Max $\theta = \theta_A = \frac{w_a}{kP}\left(\frac{kl}{\cos kl} - \tan kl\right)$
		Max $y = y_A = \frac{-w_a}{k^2P}\left(1 + kl\tan kl - \frac{k^2l^2}{2} - \frac{1}{\cos kl}\right)$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		Max $M = M_B = \frac{-w_l}{k^2}\left(1 + \frac{kl}{2}\tan kl - \frac{\tan kl}{kl}\right)$
		Max $\theta = \theta_A = \frac{w_l}{kP}\left(\frac{1}{kl} + \frac{kl}{2\cos kl} - \frac{1}{kl\cos kl}\right)$
		Max $y = y_A = \frac{-w_l}{k^2P}\left(1 + \frac{kl}{2}\tan kl - \frac{k^2l^2}{6} - \frac{\tan kl}{kl}\right)$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations

2b. Left end guided, right end fixed

 	
	$R_A = 0 \qquad \theta_A = 0$
	$M_A = \frac{w_a}{k^2}\frac{C_{a4}}{C_2} + \frac{w_l - w_a}{k^3(l-a)}\frac{C_{a5}}{C_2}$
	$y_A = \frac{-w_a}{k^2P}\frac{C_3C_{a4} - C_2C_{a5}}{C_2} - \frac{w_l - w_a}{k^3P(l-a)}\frac{C_3C_{a5} - C_2C_{a6}}{C_2}$
	$R_B = \frac{w_a + w_l}{2}(l-a) \qquad \theta_B = 0$
	$M_B = \frac{-w_a}{k^2}\frac{C_2C_{a3} - C_1C_{a4}}{C_2} - \frac{(w_1 - w_a)}{k^3(l-a)}\frac{C_2C_{a4} - C_1C_{a5}}{C_2}$
	$y_B = 0$
		If $a = 0$ and $w_a = w_l$ (uniform load on entire span), then
		$\text{Max} + M = M_A = \frac{w_a}{k^2}\left(\frac{kl}{\sin kl} - 1\right)$
		$\text{Max} - M = M_B = \frac{-w_a}{k^2}\left(1 - \frac{kl}{\tan kl}\right)$
		$\text{Max } y = y_A = \frac{-w_a l}{kP}\left(\tan\frac{kl}{2} - \frac{kl}{2}\right)$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$\text{Max} + M = M_A = \frac{w_l}{k^2}\left(\frac{kl}{2\sin kl} - \frac{\tan(kl/2)}{kl}\right)$
		$\text{Max} - M = M_B = \frac{-w_l}{k^2}\left(1 - \frac{kl}{2\tan kl} - \frac{1-\cos kl}{kl\sin kl}\right)$
		$\text{Max } y = y_A = \frac{-w_l}{k^2P}\left[\left(\frac{kl}{2} - \frac{2}{kl}\right)\tan\frac{kl}{2} - \frac{k^2l^2}{6} + 1\right]$

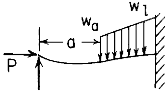
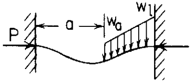
2c. Left end simply supported, right end fixed

 	
	$M_A = 0 \qquad y_A = 0$
	$R_A = \frac{w_a}{k}\frac{C_2C_{a4} - C_1C_{a5}}{C_2C_3 - C_1C_4} + \frac{w_l - w_a}{k^2(l-a)}\frac{C_2C_{a5} - C_1C_{a6}}{C_2C_3 - C_1C_4}$
	$\theta_A = \frac{-w_a}{kP}\left[\frac{C_4C_{a4} - C_3C_{a5}}{C_2C_3 - C_1C_4}\right] - \frac{w_l - w_a}{k^2P(l-a)}\frac{C_4C_{a5} - C_3C_{a6}}{C_2C_3 - C_1C_4}$
	$R_B = \frac{w_a + w_l}{2}(l-a) - R_A$
	$M_B = \frac{-w_a}{k^2}\left(\frac{C_2C_{a5} - klC_2C_{a4}}{C_2C_3 - C_1C_4} + C_{a3}\right)$
	$- \frac{(w_l - w_a)}{k^3(l-a)}\left(\frac{C_2C_{a6} - klC_2C_{a5}}{C_2C_3 - C_1C_4} + C_{a4}\right)$
	$\theta_B = 0 \qquad y_B = 0$
		If $a = 0$ and $w_a = w_l$ (uniform load on entire span), then
		$\text{Max } \theta = \theta_A = \frac{-w_a}{kP}\frac{4 - 2kl\sin kl - (2 - k^2l^2/2)(1 + \cos kl)}{\sin kl - kl\cos kl}$
		$\text{Max} - M = M_B = \frac{-w_a l}{k}\frac{\tan kl[\tan(kl/2) - kl/2]}{\tan kl - kl}$
		$R_A = \frac{w_a}{k}\frac{kl\sin kl - 1 + (1 - k^2l^2/2)\cos kl}{\sin kl - kl\cos kl}$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$\text{Max } \theta = \theta_A = \frac{-w_l l}{6P}\frac{2kl + kl\cos kl - 3\sin kl}{\sin kl - kl\cos kl}$
		$\text{Max} - M = M_B = \frac{-w_l}{k^2}\frac{(l - k^2l^2/3)\tan kl - kl}{\tan kl - kl}$
		$R_A = \frac{w_l l}{6}\left(\frac{2\tan kl}{\tan kl - kl} - \frac{6}{k^2l^2} + 1\right)$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

2d. Left end fixed, right end fixed

 	
	$\theta_A = 0 \qquad y_A = 0$
	$R_A = \frac{w_a}{k}\frac{C_3C_{a4} - C_2C_{a5}}{C_3^2 - C_2C_4} + \frac{w_l - w_a}{k^2(l-a)}\frac{C_3C_{a5} - C_2C_{a6}}{C_3^2 - C_2C_4}$
	$M_A = \frac{-w_a}{k^2}\frac{C_4C_{a4} - C_3C_{a5}}{C_3^2 - C_2C_4} - \frac{w_l - w_a}{k^3(l-a)}\frac{C_4C_{a5} - C_3C_{a6}}{C_3^2 - C_2C_4}$
	$R_B = \frac{w_a + w_l}{2}(l-a) - R_A$
	$M_B = M_A + R_Al - \frac{w_a}{2}(l-a)^2 - \frac{w_l - w_a}{6}(l-a)^2$
	$\theta_B = 0 \qquad y_B = 0$
		If $a = 0$ and $w_a = w_l$ (uniform load on entire span), then
		$\text{Max} - M = M_A = M_B = \frac{-w_a}{k^2}\left[1 - \frac{kl/2}{\tan(kl/2)}\right]$
		$\text{Max} + M = \frac{w_a}{k^2}\left[\frac{kl/2}{\sin(kl/2)} - 1\right] \qquad \text{at } x = \frac{l}{2}$
		$\text{Max } y = \frac{-w_al}{2kP}\left(\tan\frac{kl}{4} - \frac{kl}{4}\right) \qquad \text{at } x = \frac{l}{2}$
		$R_A = R_B = \frac{w_al}{2}$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$\text{Max} - M = M_B = \frac{-w_l}{k^2}\left[1 - \frac{(kl/2)\sin kl - k^2l^2/6 - (k^2l^2/3)\cos kl}{2 - 2\cos kl - kl\sin kl}\right]$
		$M_A = \frac{w_ll}{6k}\frac{3\sin kl - kl(2 + \cos kl)}{2 - 2\cos kl - kl\sin kl}$
		$R_A = \frac{w_l}{k^2l}\left(\frac{k^2l^2}{6}\frac{3 - 3\cos kl - kl\sin kl}{2 - 2\cos kl - kl\sin kl} - 1\right)$

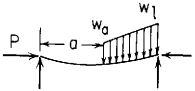
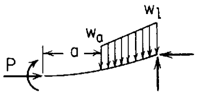
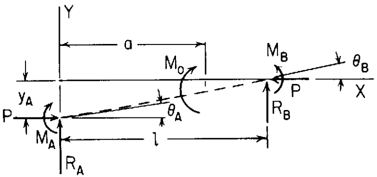
2e. Left end simply supported, right end simply supported

 	
	$M_A = 0 \qquad y_A = 0$
	$R_A = \frac{w_a}{2l}(l-a)^2 + \frac{w_l - w_a}{6l}(l-a)^2$
	$\theta_A = \frac{-w_a}{kP}\left[\frac{1 - \cos k(l-a)}{\sin kl} - \frac{k}{2l}(l-a)^2\right]$
	$\qquad - \frac{w_l - w_a}{kP}\left[\frac{k(l-a) - \sin k(l-a)}{k(l-a)\sin kl} - \frac{k}{6l}(l-a)^2\right]$
	$R_B = \frac{w_a + w_l}{2}(l-a) - R_A$
	$M_B = 0 \qquad y_B = 0$
	$\theta_B = \frac{w_a}{kP}\left[\frac{\cos ka - \cos kl}{\sin kl} - \frac{k(l^2 - a^2)}{2l}\right]$
	$\qquad + \frac{w_l - w_a}{k^2P(l-a)}\left[\frac{k^2}{6l}(3al^2 - 2l^3 - a^3) + 1\right.$
	$\qquad \left. - \frac{\sin ka + k(l-a)\cos kl}{\sin kl}\right]$
		If $a = 0$ and $w_a = w_l$ (uniform load on entire span), then
		$\text{Max} + M = \frac{w_a}{k^2}\left[\frac{1}{\cos(kl/2)} - 1\right] \qquad \text{at } x = \frac{l}{2}$
		$\text{Max } \theta = \theta_B = -\theta_A = \frac{w_a}{kP}\left(\tan\frac{kl}{2} - \frac{kl}{2}\right)$
		$\text{Max } y = \frac{-w_a}{k^2P}\left[\frac{1}{\cos(kl/2)} - \frac{k^2l^2}{8} - 1\right] \qquad \text{at } x = \frac{l}{2}$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$M = \frac{w_l}{k^2}\left(\frac{\sin kx}{\sin kl} - \frac{x}{l}\right)$; max M occurs at $x = \frac{1}{k}\cos^{-1}\frac{\sin kl}{kl}$
		$\theta_A = \frac{-w_l}{kP}\left(\frac{1}{\sin kl} - \frac{1}{kl} - \frac{kl}{6}\right)$
		$\text{Max } \theta = \theta_B = \frac{w_l}{kP}\left(\frac{1}{kl} - \frac{kl}{3} - \frac{1}{\tan kl}\right)$
		$y = \frac{-w_l}{k^2P}\left[\frac{\sin kx}{\sin kl} - \frac{x}{l} - \frac{k^2x}{6l}(l^2 - x^2)\right]$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations

2f. Left end guided, right end simply supported

 	
	$R_A = 0 \qquad \theta_A = 0$
	$M_A = \frac{w_a}{k^2}\frac{C_{a3}}{C_1} + \frac{w_l - w_a}{k^3(l-a)}\frac{C_{a4}}{C_1}$
	$y_A = \frac{-w_a}{k^2 P}\left[\frac{C_{a3}}{C_1} - \frac{k^2}{2}(l-a)^2\right] - \frac{w_l - w_a}{k^3 P(l-a)}\left[\frac{C_{a4}}{C_1} - \frac{k^3}{6}(l-a)^3\right]$
	$R_B = \frac{w_a + w_l}{2}(l-a) \qquad M_B = 0$
	$\theta_B = \frac{w_a}{kP}\left[\frac{\sin kl - \sin ka}{\cos kl} - k(l-a)\right] + \frac{w_l - w_a}{k^2 P(l-a)}\left[\frac{k(l-a)\sin kl - \cos ka}{\cos kl} - \frac{k^2(l-a)^2}{2} + 1\right]$
	$y_B = 0$
		If $a = 0$ and $w_a = w_l$ (uniform load on entire span), then
		$\text{Max } M = M_A = \frac{w_a}{k^2}\left(\frac{1}{\cos kl} - 1\right)$
		$\text{Max } \theta = \theta_B = \frac{w_a}{kP}(\tan kl - kl)$
		$\text{Max } y = y_A = \frac{-w_a}{k^2 p}\left(\frac{1}{\cos kl} - 1 - \frac{k^2 l^2}{2}\right)$
		If $a = 0$ and $w_a = 0$ (uniformly increasing load), then
		$\text{Max } M = M_A = \frac{w_l}{k^3 l}\frac{kl - \sin kl}{\cos kl}$
		$\text{Max } \theta = \theta_B = \frac{w_l}{k^2 Pl}\left(1 - \frac{k^2 l^2}{2} - \frac{1 - kl \sin kl}{\cos kl}\right)$
		$\text{Max } y = y_A = \frac{-w_l}{k^2 P}\left(\frac{kl - \sin kl}{kl \cos kl} - \frac{k^2 l^2}{6}\right)$

3. Axial compressive load plus concentrated intermediate moment



$$\text{Transverse shear} = V = R_A F_1 - M_A k F_2 - \theta_A P F_1 - M_o k F_{a2}$$

$$\text{Bending moment} = M = M_A F_1 + \frac{R_A}{k} F_2 - \frac{\theta_A P}{k} F_2 + M_o F_{a1}$$

$$\text{Slope} = \theta = \theta_A F_1 + \frac{M_A k}{P} F_2 + \frac{R_A}{P} F_3 + \frac{M_o k}{P} F_{a2}$$

$$\text{Deflection} = y = y_A + \frac{\theta_A}{k} F_2 + \frac{M_A}{P} F_3 + \frac{R_A}{kP} F_4 + \frac{M_o}{P} F_{a3}$$

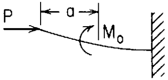
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
3a. Left end free, right end fixed (cantilever)  	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{-M_o k}{P}\frac{\sin k(l-a)}{\cos kl}$ $y_A = \frac{M_o}{P}\left(\frac{\cos ka}{\cos kl} - 1\right)$ $R_B = 0 \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = M_o \frac{\cos ka}{\cos kl}$	$\text{Max } M = M_B$; max possible value $= \frac{M_o}{\cos kl}$ when $a = 0$ $\text{Max } \theta = \theta_A$; max possible value $= \frac{-M_o k}{P}\tan kl$ when $a = 0$ $\text{Max } y = y_A$; max possible value $= \frac{M_o}{P}\left(\frac{1}{\cos kl} - 1\right)$ when $a = 0$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

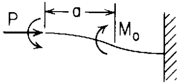
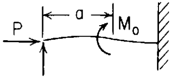
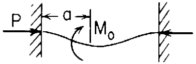
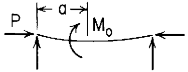
3b. Left end guided, right end fixed  	$R_A = 0 \qquad \theta_A = 0$ $M_A = -M_o \dfrac{\sin k(l-a)}{\sin kl}$ $y_A = \dfrac{M_o}{P}\left[\dfrac{\sin k(l-a) + \sin ka}{\sin kl} - 1\right]$ $R_B = 0 \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = M_o \dfrac{\sin ka}{\sin kl}$	Max $+M = M_B$; max possible value $= M_o$ when $a = l$ Max $-M = M_A$; max possible value $= -M_o$ when $a = 0$ Max $= \dfrac{-M_o k}{P}\dfrac{\sin k(l-a)}{\sin kl}$ at $x = a$; max possible value $= \dfrac{-M_o k}{2P}\tan\dfrac{kl}{2}$ when $a = \dfrac{l}{2}$ Max $y = y_A$; max possible value $= \dfrac{M_o}{P}\left[\dfrac{1}{\cos(kl/2)} - 1\right]$ when $a = \dfrac{l}{2}$
3c. Left end simply supported, right end fixed  	$M_A = 0 \qquad y_A = 0$ $R_A = -M_o k \dfrac{\cos ka - \cos kl}{\sin kl - kl\cos kl}$ $\theta_A = \dfrac{-M_o k}{P}\dfrac{C_3 C_{a3} - C_4 C_{a2}}{C_2 C_3 - C_1 C_4}$ $R_B = -R_A \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = M_o \dfrac{\sin kl - kl\cos ka}{\sin kl - kl\cos kl}$	If $a = 0$ (concentrated end moment), then $R_A = -M_o k \dfrac{1 - \cos kl}{\sin kl - kl\cos kl}$ $\theta_A = \dfrac{-M_o k}{P}\dfrac{2 - 2\cos kl - kl\sin kl}{\sin kl - kl\cos kl}$ $M_B = -M_o \dfrac{kl - \sin kl}{\sin kl - kl\cos kl}$
3d. Left end fixed, right end fixed  	$\theta_A = 0 \qquad y_A = 0$ $R_A = -M_o k \dfrac{C_3 C_{a2} - C_2 C_{a3}}{C_3^2 - C_2 C_4}$ $M_A = -M_o \dfrac{C_3 C_{a3} - C_4 C_{a2}}{C_3^2 - C_2 C_4}$ $R_B = -R_A \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = R_A l + M_A + M_o$	If $a = l/2$ (concentrated center moment), then $R_A = -M_o k \dfrac{[1/\cos(kl/2)] - 1}{2\tan(kl/2) - kl}$ $M_A = -M_o \dfrac{1 - \cos kl - kl\sin(kl/2)}{2 - 2\cos kl - kl\sin kl}$ At the center, $y = 0$ and $\theta = \left(\dfrac{-M_o k}{2P}\right)\left(\dfrac{2 - 2\cos\dfrac{kl}{2} - \dfrac{kl}{2}\sin\dfrac{kl}{2}}{\sin\dfrac{kl}{2} - \dfrac{kl}{2}\cos\dfrac{kl}{2}}\right)$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations

3e. Left end simply supported, right end simply supported

 	
	$M_A = 0 \qquad y_A = 0$
	$R_A = \frac{-M_o}{l}$
	$\theta_A = \frac{M_o}{Pl}\left[\frac{kl\cos k(l-a)}{\sin kl} - 1\right]$
	$R_B = -R_A \qquad M_B = 0 \qquad y_B = 0$
	$\theta_B = \frac{M_o}{Pl}\left(\frac{kl\cos ka}{\sin kl} - 1\right)$
		If $a = 0$ (concentrated moment at the left end), then
		$\theta_A = \frac{-M_o}{Pl}\left(1 - \frac{kl}{\tan kl}\right)$
		$\theta_B = \frac{M_o}{Pl}\left(\frac{kl}{\sin kl} - 1\right)$
		$M = M_o \cos kx\left(1 - \frac{\tan kx}{\tan kl}\right)$
		If $a = l/2$ (concentrated moment at the center), then
		$\theta_A = \theta_B = \frac{M_o}{Pl}\left[\frac{kl}{2\sin(kl/2)} - 1\right]$ and $y = 0$ at the center

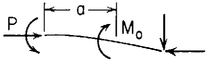
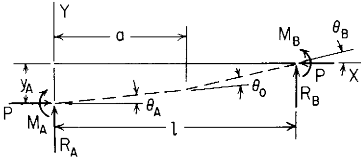
3f. Left end guided, right end simply supported  	$R_A = 0 \qquad \theta_A = 0$ $M_A = -M_o \frac{\cos k(l-a)}{\cos kl}$ $y_A = \frac{M_o}{P}\left[\frac{\cos k(l-a)}{\cos kl} - 1\right]$ $R_B = 0 \qquad M_B = 0 \qquad y_B = 0$ $\theta_B = \frac{-M_o k}{P}\frac{\sin ka}{\cos kl}$	Max $M = M_A$; max possible value $= \frac{-M_o}{\cos kl}$ when $a = l$ Max $\theta = \theta_B$; max possible value $= \frac{-M_o k}{P}\tan kl$ when $a = l$ Max $y = y_a$; max possible value $= \frac{M_o}{P}\left(\frac{1}{\cos kl} - 1\right)$ when $a = l$
4. Axial compressive load plus externally created concentrated angular displacement  	Transverse shear $= V = R_A F_1 - M_A k F_2 - \theta_A P F_1 - \theta_o P F_{a1}$ Bending moment $= M = M_A F_1 + \frac{R_A}{k} F_2 - \frac{\theta_A P}{k} F_2 - \frac{\theta_o P}{k} F_{a2}$ Slope $= \theta = \theta_A F_1 + \frac{M_A k}{P} F_2 + \frac{R_A}{P} F_3 + \theta_o F_{a1}$ Deflection $= y = y_A + \frac{\theta_A}{k} F_2 + \frac{M_A}{P} F_3 + \frac{R_A}{kP} F_4 + \frac{\theta_o}{k} F_{a2}$	

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

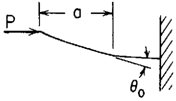
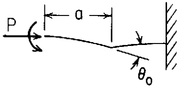
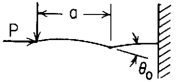
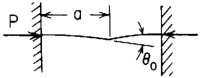
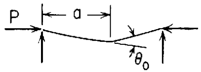
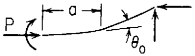
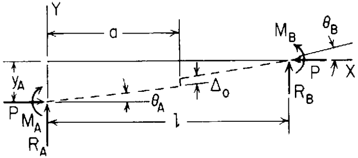
4a. Left end free, right end fixed  	$R_A = 0 \qquad M_A = 0$ $\theta_A = -\theta_o \dfrac{\cos k(l-a)}{\cos kl}$ $y_A = \dfrac{\theta_o}{k}\dfrac{\sin ka}{\cos kl}$ $R_B = 0 \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = \dfrac{\theta_o P}{k}\dfrac{\sin ka}{\cos kl}$	Max $M = M_B$; max possible value $= \dfrac{\theta_o P}{k}\tan kl$ when $a = l$ Max $\theta = \theta_A$; max possible value $= \dfrac{-\theta_o}{\cos kl}$ when $a = l$ Max $y = y_A$; max possible value $= \dfrac{\theta_o}{k}\tan kl$ when $a = l$
4b. Left end guided, right end fixed  	$R_A = 0 \qquad \theta_A = 0$ $M_A = \dfrac{-\theta_o P}{k}\dfrac{\cos k(l-a)}{\sin kl}$ $y_A = \dfrac{\theta_o}{k}\dfrac{\cos k(l-a) - \cos ka}{\sin kl}$ $R_B = 0 \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = \dfrac{-\theta_o P}{k}\dfrac{\cos ka}{\sin kl}$	Max $-M = M_B$ if $a < \dfrac{l}{2}$; max possible value $= \dfrac{-\theta_o P}{k \sin kl}$ when $a = 0$ Max $-M = M_A$ if $a > \dfrac{l}{2}$; max possible value $= \dfrac{-\theta_o P}{k \sin kl}$ when $a = l$ Max $+y = y_A$; max possible value $= \dfrac{\theta_o}{k}\tan\dfrac{kl}{2}$ when $a = l$ Max $-y$ occurs at $x = a$; max possible value $= \dfrac{-\theta_o}{k}\tan\dfrac{kl}{2}$ at $x = 0$ when $a = 0$
4c. Left end simply supported, right end fixed  	$M_A = 0 \qquad y_A = 0$ $R_A = -\theta_o P \dfrac{\sin ka}{\sin kl - kl\cos kl}$ $\theta_A = -\theta_o \dfrac{C_3 C_{a2} - C_4 C_{a1}}{C_2 C_3 - C_1 C_4}$ $R_B = -R_A \qquad \theta_B = 0 \qquad y_B = 0 \qquad M_B = R_A l$	
4d. Left end fixed, right end fixed  	$\theta_A = 0 \qquad y_A = 0$ $R_A = -\theta_o P \dfrac{C_3 C_{a1} - C_2 C_{a2}}{C_3^2 - C_2 C_4}$ $M_A = \dfrac{-\theta_o P}{k}\dfrac{C_3 C_{a2} - C_4 C_{a1}}{C_3^2 - C_2 C_4}$ $R_B = -R_A \qquad \theta_B = 0 \qquad y_B = 0 \qquad M_B = M_A + R_A l$	

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
4e. Left end simply supported, right end simply supported  	$M_A = 0 \quad y_A = 0 \quad R_A = 0$ $\theta_A = -\theta_o \frac{\sin k(l-a)}{\sin kl}$ $M_B = 0 \quad y_B = 0 \quad R_B = 0$ $\theta_B = \theta_o \frac{\sin ka}{\sin kl}$	Max $M = \frac{\theta_o P}{k} \frac{\sin k(l-a) \sin ka}{\sin kl}$ at $x = a$; max possible value $= \frac{\theta_o P}{k \cos(kl/2)}$ when $a = \frac{l}{2}$ Max $\theta = \theta_A$ if $a < l/2$; max possible value $= -\theta_o$ when $a = 0$ Max $y = \frac{-\theta_o}{k} \frac{\sin k(l-a) \sin ka}{\sin kl}$ at; $x = a$; max possible value $= \frac{-\theta_o}{k \cos(kl/2)}$ when $a = \frac{l}{2}$
4f. Left end guided, right end simply supported  	$R_A = 0 \quad \theta_A = 0$ $M_A = \frac{\theta_o P}{k} \frac{\sin k(l-a)}{\cos kl}$ $y_A = \frac{-\theta_o}{k} \frac{\sin k(l-a)}{\cos kl}$ $R_B = 0 \quad M_B = 0 \quad y_B = 0$ $\theta_B = \theta_o \frac{\cos ka}{\cos kl}$	Max $M = M_A$; max possible value $= \frac{\theta_o P}{k} \tan kl$ when $a = 0$ Max $\theta = \theta_B$; max possible value $= \frac{\theta_o}{\cos kl}$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{-\theta_o}{k} \tan kl$ when $a = 0$

5. Axial compressive load plus externally created concentrated lateral displacement



Transverse shear $= V = R_A F_1 - M_A k F_2 - \theta_A P F_1 + \Delta_o P k F_{a2}$

Bending moment $= M = M_A F_1 + \frac{R_A}{k} F_2 - \frac{\theta_A P}{k} F_2 - \Delta_o P F_{a1}$

Slope $= \theta = \theta_A F_1 + \frac{M_A k}{P} F_2 + \frac{R_A}{P} F_3 - \Delta_o k F_{a2}$

Deflection $= y = y_A + \frac{\theta_A}{k} F_2 + \frac{M_A}{P} F_3 + \frac{R_A}{kP} F_4 + \Delta_o F_{a1}$

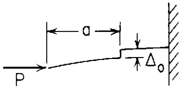
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
5a. Left end free, right end fixed (cantilever)  	$R_A = 0 \quad M_A = 0$ $\theta_A = \Delta_o k \frac{\sin k(l-a)}{\cos kl}$ $y_A = -\Delta_o \frac{\cos ka}{\cos kl}$ $R_B = 0 \quad \theta_B = 0 \quad y_B = 0$ $M_B = -\Delta_o P \frac{\cos ka}{\cos kl}$	Max $M = M_B$; max possible value $= \frac{-\Delta_o P}{\cos kl}$ when $a = 0$ Max $\theta = \theta_A$; max possible value $= \Delta_o k \tan kl$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{-\Delta_o}{\cos kl}$ when $a = 0$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

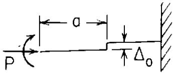
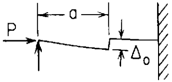
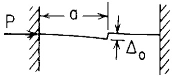
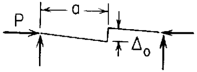
5b. Left end guided, right end fixed  	$R_A = 0 \qquad \theta_A = 0$ $M_A = \Delta_o P \dfrac{\sin k(l-a)}{\sin kl}$ $y_A = -\Delta_o \dfrac{\sin k(l-a) + \sin ka}{\sin kl}$ $R_B = 0 \qquad \theta_B = 0 \qquad y_B = 0 \qquad M_B = -\Delta_o P \dfrac{\sin ka}{\sin kl}$	Max $+M = M_A$; max possible value $= \Delta_o P$ when $a = 0$ Max $-M = M_B$; max possible value $= -\Delta_o P$ when $a = l$ Max $y = y_A$; max possible value $= \dfrac{-\Delta_o}{\cos(kl/2)}$ when $a = \dfrac{l}{2}$
5c. Left end simply supported, right end fixed  	$M_A = 0 \qquad y_A = 0$ $R_A = \Delta_o P k \dfrac{\cos ka}{\sin kl - kl \cos kl}$ $\theta_A = -\Delta_o k \dfrac{C_3 C_{a1} + C_4 C_{a2}}{C_2 C_3 - C_1 C_4}$ $R_B = -R_A \qquad \theta_B = 0 \qquad y_B = 0 \qquad M_B = R_A l$	
5d. Left end fixed, right end fixed  	$\theta_A = 0 \qquad y_A = 0$ $R_A = \Delta_o P k \dfrac{C_3 C_{a2} + C_2 C_{a1}}{C_3^2 - C_2 C_4}$ $M_A = -\Delta_o P \dfrac{C_3 C_{a1} + C_4 C_{a2}}{C_3^2 - C_2 C_4}$ $R_B = -R_A \qquad \theta_B = 0 \qquad y_B = 0 \qquad M_B = M_A + R_A l$	
5e. Left end simply supported, right end simply supported  	$R_A = 0 \qquad M_A = 0 \qquad y_A = 0$ $\theta_A = -\Delta_o k \dfrac{\cos k(l-a)}{\sin kl}$ $R_B = 0 \qquad M_B = 0 \qquad y_B = 0$ $\theta_B = -\Delta_o k \dfrac{\cos ka}{\sin kl}$	Max $+M = \Delta_o P \dfrac{\sin ka}{\sin kl} \cos k(l-a)$ at x just left of a; max possible value $= \Delta_o P$ when $a = l$ Max $-M = -\Delta_o P \dfrac{\cos ka}{\sin kl} \sin k(l-a)$ at x just right of a; max possible value $= -\Delta_o P$ when $a = 0$ Max $+y = \Delta_o \dfrac{\cos ka}{\sin kl} \sin k(l-a)$ at x just right of a; max possible value $= \Delta_o$ when $a = 0$ Max $-y = -\Delta_o \dfrac{\sin ka}{\sin kl} \cos k(l-a)$ at x just left of a; max possible value $= -\Delta_o$ when $a = l$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

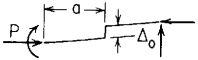
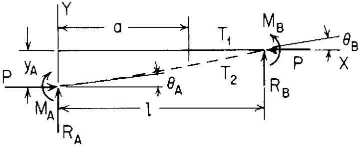
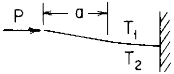
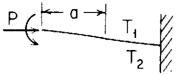
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
5f. Left end guided, right end simply supported 	$R_A = 0 \qquad \theta_A = 0$ $M_A = \Delta_o P \dfrac{\cos k(l-a)}{\cos kl}$ $y_A = -\Delta_o \dfrac{\cos k(l-a)}{\cos kl}$ $R_B = 0 \qquad M_B = 0 \qquad y_B = 0 \qquad \theta_B = \Delta_o k \dfrac{\sin ka}{\cos kl}$	Max $M = M_A$; max possible value $= \dfrac{\Delta_o P}{\cos kl}$ when $a = l$ Max $\theta = \theta_B$; max possible value $= \Delta_o k \tan kl$ when $a = l$ Max $y = y_A$; max possible value $= \dfrac{-\Delta_o}{\cos kl}$ when $a = l$
6. Axial compressive load plus a uniform temperature variation from top to bottom in the portion from a to l; t is the thickness of the beam 	Transverse shear $= V = R_A F_1 - M_A k F_2 - \theta_A P F_1 - \dfrac{\gamma(T_2 - T_1)P}{kt} F_{a2}$ Bending moment $= M = M_A F_1 + \dfrac{R_A}{k} F_2 - \dfrac{\theta_A P}{k} F_2 - \dfrac{\gamma(T_2 - T_1)EI}{t} F_{a3}$ Slope $= \theta = \theta_A F_1 + \dfrac{M_A k}{P} F_2 + \dfrac{R_A}{P} F_3 + \dfrac{\gamma(T_2 - T_1)}{kt} F_{a2}$ Deflection $= y = y_A + \dfrac{\theta_A}{k} F_2 + \dfrac{M_A}{P} F_3 + \dfrac{R_A}{kP} F_4 + \dfrac{\gamma(T_2 - T_1)}{k^2 t} F_{a3}$	
End restraints, reference no.	Boundary values	Selected maximum values of moments and deformations
6a. Left end free, right end fixed 	$R_A = 0 \qquad M_A = 0$ $\theta_A = \dfrac{-\gamma(T_2 - T_1)}{kt} \dfrac{\sin k(l-a)}{\cos kl}$ $y_A = \dfrac{\gamma(T_2 - T_1)}{k^2 t}\left(\dfrac{\cos ka}{\cos kl} - 1\right)$ $R_B = 0 \qquad \theta_B = 0 \qquad y_B = 0 \qquad M_B = P y_A$	Max $M = M_B$ max possible value $= \dfrac{\gamma(T_2 - T_1)EI}{t}\left(\dfrac{1}{\cos kl} - 1\right)$ when $a = 0$ Max $\theta = \theta_A$; max possible value $= \dfrac{-\gamma(T_2 - T_1)}{kt} \tan kl$ when $a = 0$ Max $y = y_A$; max possible value $= \dfrac{\gamma(T_2 - T_1)}{k^2 t}\left(\dfrac{1}{\cos kl} - 1\right)$ when $a = 0$
6b. Left end guided, right end fixed 	$R_A = 0 \qquad \theta_A = 0$ $M_A = \dfrac{-\gamma(T_2 - T_1)EI}{t} \dfrac{\sin k(l-a)}{\sin kl}$ $y_A = \dfrac{\gamma(T_2 - T_1)}{k^2 t} \dfrac{C_3 C_{a2} - C_2 C_{a3}}{C_2}$ $R_B = 0 \qquad \theta_B = 0 \qquad y_B = 0$ $M_B = \dfrac{\gamma(T_2 - T_1)EI}{t}\left(\dfrac{\sin ka}{\sin kl} - 1\right)$	Max $-M = M_A$; max possible value $= \dfrac{-\gamma(T_2 - T_1)EI}{t}$ when $a = l$ (*Note*: There is no positive moment in the beam) Max $\theta = \dfrac{-\gamma(T_2 - T_1)}{kt} \dfrac{\sin ka}{\sin kl} \sin k(l-a)$ at $x = a$; max possible value $= \dfrac{-\gamma(T_2 - T_1)}{2kt} \tan \dfrac{kl}{2}$ when $a = \dfrac{l}{2}$ Max $y = y_A$; max possible value $= \dfrac{\gamma(T_2 - T_1)}{k^2 t}\left[\dfrac{1}{\cos(kl/2)} - 1\right]$ when $a = \dfrac{l}{2}$

TABLE 8.8 Shear, moment, slope, and deflection formulas for beams under simultaneous axial compression and transverse loading (*Continued*)

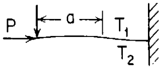
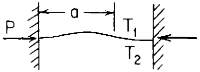
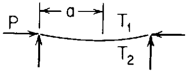
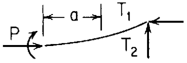
6c. Left end simply supported, right end fixed  	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{-\gamma(T_2 - T_1)P}{kt}\frac{\cos ka - \cos kl}{\sin kl - kl\cos kl}$ $\theta_A = \frac{-\gamma(T_2 - T_1)}{kt}\frac{C_3C_{a3} - C_4C_{a2}}{C_2C_3 - C_1C_4}$ $R_B = -R_A \qquad \theta_B = 0 \qquad y_B = 0 \qquad M_B = R_A l$	If $a = 0$ (temperature variation over entire span), then Max $-M = M_B = \frac{-\gamma(T_2 - T_1)Pl}{kt}\frac{1 - \cos kl}{\sin kl - kl\cos kl}$ Max $\theta = \theta_A = \frac{-\gamma(T_2 - T_1)}{kt}\frac{2 - 2\cos kl - kl\sin kl}{\sin kl - kl\cos kl}$
6d. Left end fixed, right end fixed  	$\theta_A = 0 \qquad y_A = 0$ $R_A = \frac{-\gamma(T_2 - T_1)P}{kt}\frac{C_3C_{a2} - C_2C_{a3}}{C_3^2 - C_2C_4}$ $M_A = \frac{-\gamma(T_2 - T_1)EI}{t}\frac{C_3C_{a3} - C_4C_{a2}}{C_3^2 - C_2C_4}$ $R_B = -R_A \qquad \theta_B = 0 \qquad y_B = 0 \qquad M_B = M_A + R_A l$	If $a = 0$ (temperature variation over entire span), then $R_A = R_B = 0$ $M = \frac{-\gamma(T_2 - T_1)EI}{t}$ everywhere in the span $\theta = 0$ and $y = 0$ everywhere in the span
6e. Left end simply supported, right end simply supported  	$R_A = 0 \qquad M_A = 0 \qquad y_A = 0$ $\theta_A = \frac{-\gamma(T_2 - T_1)}{kt}\frac{1 - \cos k(l - a)}{\sin kl}$ $R_B = 0 \qquad M_B = 0 \qquad y_B = 0$ $\theta_B = \frac{\gamma(T_2 - T_1)}{kt}\frac{\cos ka - \cos kl}{\sin kl}$	Max y occurs at $x = \frac{1}{k}\tan^{-1}\frac{1 - \cos kl\cos ka}{\sin kl}$; max possible value $= \frac{-\gamma(T_2 - T_1)}{k^2t}\left[\frac{1}{\cos(kl/2)} - 1\right]$ at $x = \frac{l}{2}$ when $a = 0$ Max $M = P(\max y)$ Max $\theta = \theta_B$; max possible value $= \frac{\gamma(T_2 - T_1)}{kt}\tan\frac{kl}{2}$ when $a = 0$
6f. Left end guided, right end simply supported  	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{\gamma(T_2 - T_1)EI}{t}\frac{1 - \cos k(l - a)}{\cos kl}$ $y_A = \frac{-\gamma(T_2 - T_1)}{k^2t}\frac{1 - \cos k(l - a)}{\cos kl}$ $R_B = 0 \qquad M_B = 0 \qquad y_B = 0$ $\theta_B = \frac{\gamma(T_2 - T_1)}{kt}\frac{\sin kl - \sin ka}{\cos kl}$	Max $M = M_A$; max possible value $= \frac{\gamma(T_2 - T_1)EI}{t}\left(\frac{1}{\cos kl} - 1\right)$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{-\gamma(T_2 - T_1)}{k^2t}\left(\frac{1}{\cos kl} - 1\right)$ when $a = 0$ Max $\theta = \theta_B$; max possible value $= \frac{\gamma(T_2 - T_1)}{kt}\tan kl$ when $a = 0$

TABLE 8.9 Shear, moment, slope, and deflection formulas for beams under simultaneous axial tension and transverse loading

NOTATION: P = axial tensile load (force); all other notation is the same as that for Table 8.1; see Table 8.8 for loading details.

The following constants and functions are hereby defined in order to permit condensing the tabulated formulas which follow. $k = (P/EI)^{1/2}$. (*Note:* See page 131 for a definition of $\langle x - a \rangle^n$.) The function $\sinh k\langle x - a \rangle$ is also defined as having a value of zero if $x < a$

$F_1 = \cosh kx$
$F_2 = \sinh kx$
$F_3 = \cosh kx - 1$
$F_4 = \sinh hx - kx$

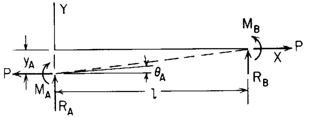
$F_{a1} = \langle x - a \rangle^0 \cosh k(x - a)$
$F_{a2} = \sinh k\langle x - a \rangle$
$F_{a3} = \langle x - a \rangle^0 [\cosh k(x - a) - 1]$
$F_{a4} = \sinh k\langle x - a \rangle - k\langle x - a \rangle$
$F_{a5} = F_{a3} - \frac{k^2}{2}\langle x - a \rangle^2$
$F_{a6} = F_{a4} - \frac{k^3}{6}\langle x - a \rangle^3$

$C_1 = \cosh kl$
$C_2 = \sinh kl$
$C_3 = \cosh kl - 1$
$C_4 = \sinh kl - kl$

$C_{a1} = \cosh k(l - a)$
$C_{a2} = \sinh k(l - a)$
$C_{a3} = \cosh k(l - a) - 1$
$C_{a4} = \sinh k(l - a) - k(l - a)$
$C_{a5} = C_{a3} - \frac{k^2}{2}(l - a)^2$
$C_{a6} = C_{a4} - \frac{k^3}{6}(l - a)^3$

(*Note:* Load terms LT_V, LT_M, LT_θ, and LT_y are found at the end of the table for each of the several loadings.)

Axial tensile load plus lateral loading



Transverse shear $= V = R_A F_1 + M_A k F_2 + \theta_A P F_1 + LT_V$

Bending moment $= M = M_A F_1 + \frac{R_A}{k} F_2 + \frac{\theta_A P}{k} F_2 + LT_M$

Slope $= \theta = \theta_A F_1 + \frac{M_A k}{P} F_2 + \frac{R_A}{P} F_3 + LT_\theta$

Deflection $= y = y_A + \frac{\theta_A}{k} F_2 + \frac{M_A}{P} F_3 + \frac{R_A}{Pk} F_4 + LT_y$

(*Note:* For each set of end restraints the two initial parameters not listed are zero. For example, with the left end free and the right end fixed, the values of R_A and M_A are zero.)

End restraints / Lateral load		Case 1, Concentrated lateral load	Case 2, Distributed lateral load	Case 3, Concentrated moment	Case 4, Concentrated angular displacement	Case 5, Concentrated lateral displacement	Case 6, Uniform temperature variation
Left end free, right end fixed (a)	θ_A	$\frac{W}{P}\frac{C_{a3}}{C_1}$	$\frac{w_a}{kP}\frac{C_{a4}}{C_1} + \frac{(w_l - w_a)C_{a5}}{k^2P(l - a)C_1}$	$\frac{-M_o k}{P}\frac{C_{a2}}{C_1}$	$-\theta_o \frac{C_{a1}}{C_1}$	$-\Delta_o k \frac{C_{a2}}{C_1}$	$\frac{-\gamma(T_2 - T_1)}{kt}\frac{C_{a2}}{C_1}$
	y_A	$\frac{-W}{kP}\left(\frac{C_2 C_{a3}}{C_1} - C_{a4}\right)$	$\frac{-w_a}{k^2P}\left(\frac{C_2 C_{a4}}{C_1} - C_{a5}\right) + \frac{-(w_l - w_a)}{k^3P(l - a)}\left(\frac{C_2 C_{a5}}{C_1} - C_{a6}\right)$	$\frac{M_o}{P}\left(\frac{C_2 C_{a2}}{C_1} - C_{a3}\right)$	$\frac{\theta_o}{k}\left(\frac{C_2 C_{a1}}{C_1} - C_{a2}\right)$	$\Delta_o\left(\frac{C_2 C_{a2}}{C_1} - C_{a1}\right)$	$\frac{\gamma(T_2 - T_1)}{k^2t}\left(\frac{C_2 C_{a2}}{C_1} - C_{a3}\right)$

TABLE 8.9 Shear, moment, slope, and deflection formulas for beams under simultaneous axial tension and transverse loading (*Continued*)

Left end guided, right end fixed (b)	M_A	$\frac{W}{k}\frac{C_{a3}}{C_2}$	$\frac{w_a}{k^2}\frac{C_{a4}}{C_2}+\frac{w_l-w_a}{k^3(l-a)}\frac{C_{a5}}{C_2}$	$-M_o\frac{C_{a2}}{C_2}$	$\frac{-\theta_o P}{k}\frac{C_{a1}}{C_2}$	$-\Delta_o P\frac{C_{a2}}{C_2}$	$\frac{-\gamma(T_2-T_1)P}{k^2t}\frac{C_{a2}}{C_2}$
	y_A	$\frac{-W}{kP}\left(\frac{C_3C_{a3}}{C_2}-C_{a4}\right)$	$\frac{-w_a}{k^2P}\left(\frac{C_3C_{a4}}{C_2}-C_{a5}\right)$ $+\frac{-(w_l-w_a)}{k^3P(l-a)}\left(\frac{C_3C_{a5}}{C_2}-C_{a6}\right)$	$\frac{M_o}{P}\left(\frac{C_3C_{a2}}{C_2}-C_{a3}\right)$	$\frac{\theta_o}{k}\left(\frac{C_3C_{a1}}{C_2}-C_{a2}\right)$	$\Delta_o\left(\frac{C_3C_{a2}}{C_2}-C_{a1}\right)$	$\frac{\gamma(T_2-T_1)}{k^2t}\left(\frac{C_3C_{a2}}{C_2}-C_{a3}\right)$
Left end simply supported, right end fixed (c)	R_A	$W\frac{C_2C_{a3}-C_1C_{a4}}{C_2C_3-C_1C_4}$	$\frac{w_a}{k}\frac{C_2C_{a4}-C_1C_{a5}}{C_2C_3-C_1C_4}$ $+\frac{w_l-w_a}{k^2(l-a)}\frac{C_2C_{a5}-C_1C_{a6}}{C_2C_3-C_1C_4}$	$-M_ok\frac{C_2C_{a2}-C_1C_{a3}}{C_2C_3-C_1C_4}$	$-\theta_oP\frac{C_2C_{a1}-C_1C_{a2}}{C_2C_3-C_1C_4}$	$\Delta_okP\frac{C_1C_{a1}-C_2C_{a2}}{C_2C_3-C_1C_4}$	$\frac{-\gamma(T_2-T_1)P}{kt}\frac{C_2C_{a2}-C_1C_{a3}}{C_2C_3-C_1C_4}$
	θ_A	$\frac{-W}{P}\frac{C_4C_{a3}-C_3C_{a4}}{C_2C_3-C_1C_4}$	$\frac{-w_a}{kP}\frac{C_4C_{a4}-C_3C_{a5}}{C_2C_3-C_1C_4}$ $+\frac{-(w_l-w_a)}{k^2P(l-a)}\frac{C_4C_{a5}-C_3C_{a6}}{C_2C_3-C_1C_4}$	$\frac{-M_ok}{P}\frac{C_3C_{a3}-C_4C_{a2}}{C_2C_3-C_1C_4}$	$-\theta_o\frac{C_3C_{a2}-C_4C_{a1}}{C_2C_3-C_1C_4}$	$\Delta_ok\frac{C_4C_{a2}-C_3C_{a1}}{C_2C_3-C_1C_4}$	$\frac{-\gamma(T_2-T_1)}{kt}\frac{C_3C_{a3}-C_4C_{a2}}{C_2C_3-C_1C_4}$
Left end fixed, right end fixed (d)	R_A	$W\frac{C_3C_{a3}-C_2C_{a4}}{C_3^2-C_2C_4}$	$\frac{w_a}{k}\frac{C_3C_{a4}-C_2C_{a5}}{C_3^2-C_2C_4}$ $+\frac{w_l-w_a}{k^2(l-a)}\frac{C_3C_{a5}-C_2C_{a6}}{C_3^2-C_2C_4}$	$-M_ok\frac{C_3C_{a2}-C_2C_{a3}}{C_3^2-C_2C_4}$	$-\theta_oP\frac{C_3C_{a1}-C_2C_{a2}}{C_3^2-C_2C_4}$	$\Delta_oPk\frac{C_2C_{a1}-C_3C_{a2}}{C_3^2-C_2C_4}$	$\frac{-\gamma(T_2-T_1)P}{kt}\frac{C_3C_{a2}-C_2C_{a3}}{C_3^2-C_2C_4}$
	M_A	$\frac{-W}{k}\frac{C_4C_{a3}-C_3C_{a4}}{C_3^2-C_2C_4}$	$\frac{-w_a}{k^2}\frac{C_4C_{a4}-C_3C_{a5}}{C_3^2-C_2C_4}$ $+\frac{-(w_l-w_a)}{k^3(l-a)}\frac{C_4C_{a5}-C_3C_{a6}}{C_3^2-C_2C_4}$	$-M_o\frac{C_3C_{a3}-C_4C_{a2}}{C_3^2-C_2C_4}$	$\frac{-\theta_oP}{k}\frac{C_3C_{a2}-C_4C_{a1}}{C_3^2-C_2C_4}$	$\Delta_oP\frac{C_4C_{a2}-C_3C_{a1}}{C_3^2-C_2C_4}$	$\frac{-\gamma(T_2-T_1)P}{k^2t}\frac{C_3C_{a3}-C_4C_{a2}}{C_3^2-C_2C_4}$
Left end simply supported, right end simply supported (e)	R_A	$\frac{W}{l}(l-a)$	$\frac{w_a}{2l}(l-a)^2+\frac{w_l-w_a}{6l}(l-a)^2$	$\frac{-M_o}{l}$	0	0	0
	θ_A	$\frac{-W}{Pkl}\left(\frac{C_4C_{a2}}{C_2}-C_{a4}\right)$	$\frac{-w_a}{Pk}\left[\frac{k(l-a)^2}{2l}-\frac{C_{a3}}{C_2}\right]$ $+\frac{-(w_l-w_a)}{Pk^2(l-a)}\left[\frac{k^2(l-a)^3}{6l}-\frac{C_{a4}}{C_2}\right]$	$\frac{M_ok}{P}\left(\frac{1}{kl}-\frac{C_{a1}}{C_2}\right)$	$-\theta_o\frac{C_{a2}}{C_2}$	$-\Delta_ok\frac{C_{a1}}{C_2}$	$\frac{-\gamma(T_2-T_1)}{kt}\frac{C_{a3}}{C_2}$

TABLE 8.9 Shear, moment, slope, and deflection formulas for beams under simultaneous axial tension and transverse loading (*Continued*)

Lateral load End restraints		Case 1, Concentrated lateral load	Case 2, Distributed lateral load	Case 3, Concentrated moment	Case 4, Concentrated angular displacement	Case 5, Concentrated lateral displacement	Case 6, Uniform temperature variation
Left end guided, right end simply supported (f)	M_A	$\frac{W}{k}\frac{C_{a2}}{C_1}$	$\frac{w_a}{k^2}\frac{C_{a3}}{C_1}+\frac{w_l-w_a}{k^3(l-a)}\frac{C_{a4}}{C_1}$	$-M_o\frac{C_{a1}}{C_1}$	$\frac{-\theta_o P}{k}\frac{C_{a2}}{C_1}$	$-\Delta_o P\frac{C_{a1}}{C_1}$	$\frac{-\gamma(T_2-T_1)P}{k^2t}\frac{C_{a3}}{C_1}$
	y_A	$\frac{-W}{Pk}\left(\frac{C_3C_{a2}}{C_1}-C_{a4}\right)$	$\frac{-w_a}{k^2P}\left[\frac{k^2(l-a)^2}{2}-\frac{C_{a3}}{C_1}\right]$ $+\frac{-(w_l-w_a)}{k^3P(l-a)}\left[\frac{k^3(l-a)^3}{6}-\frac{C_{a4}}{C_1}\right]$	$\frac{m_o}{P}\left(1-\frac{C_{a1}}{C_1}\right)$	$\frac{-\theta_o}{k}\frac{C_{a2}}{C_1}$	$-\Delta_o\frac{C_{a1}}{C_1}$	$\frac{-\gamma(T_2-T_1)}{k^2t}\frac{C_{a3}}{C_1}$
Load terms for all end restraints (a)–(f)	LT_V	$-WF_{a1}$	$\frac{-w_a}{k}F_{a2}-\frac{w_l-w_a}{k^2(l-a)}F_{a3}$	M_okF_{a2}	θ_oPF_{a1}	ΔPkF_{a2}	$\frac{\gamma(T_2-T_1)P}{kt}F_{a2}$
	LT_M	$\frac{-W}{k}F_{a2}$	$\frac{-w_a}{k^2}F_{a3}-\frac{w_l-w_a}{k^3(l-a)}F_{a4}$	M_oF_{a1}	$\frac{\theta_oP}{k}F_{a2}$	Δ_oPF_{a1}	$\frac{\gamma(T_2-T_1)P}{k^2t}F_{a3}$
	LT_θ	$\frac{-W}{P}F_{a3}$	$\frac{-w_a}{Pk}F_{a4}-\frac{(w_l-w_a)}{Pk^2(l-a)}F_{a5}$	$\frac{M_ok}{P}F_{a2}$	θ_oF_{a1}	Δ_okF_{a2}	$\frac{\gamma(T_2-T_1)}{kt}F_{a2}$
	LT_y	$\frac{-W}{Pk}F_{a4}$	$\frac{-w_a}{Pk^2}F_{a5}-\frac{w_l-w_a}{Pk^3(l-a)}F_{a6}$	$\frac{M_o}{P}F_{a3}$	$\frac{\theta_o}{k}F_{a2}$	Δ_oF_{a1}	$\frac{\gamma(T_2-T_1)}{k^2t}F_{a3}$

TABLE 8.10 Beams restrained against horizontal displacement at the ends

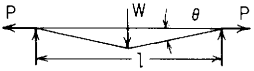
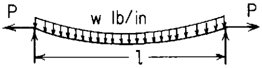
Case no., manner of loading and support	Formulas to solve for $y_{\max}$ and P
1. Ends pinned to rigid supports, concentrated center load W	$y_{\max} + \frac{A}{4I} y_{\max}^3 = \frac{2Wl^3}{\pi^4 EI}$ (Solve for $y_{\max}$) $P = \frac{\pi^2 EA}{4l^2} y_{\max}^2$ Use case 1e from Table 8.7(*b*) or Table 8.9 to determine maximum slopes and moments after solving for P
2. Ends fixed to rigid supports, concentrated center load W	$y_{\max} + \frac{A}{16I} y_{\max}^3 = \frac{Wl^3}{2\pi^4 EI}$ (Solve for $y_{\max}$) $P = \frac{\pi^2 EA}{4l^2} y_{\max}^2$ Use case 1d from Table 8.7(*d*) or Table 8.9 to determine maximum slopes and moments after solving for P
3. Ends pinned to rigid supports, uniformly distributed transverse load w on entire span	$y_{\max} + \frac{A}{4I} y_{\max}^3 = \frac{5wl^4}{4\pi^4 EI}$ (Solve for $y_{\max}$) $P = \frac{\pi^2 EA}{4l^2} y_{\max}^2$ Use case 2e from Table 8.7(*b*) or Table 8.9 to determine maximum slopes and moments after solving for P
4. Ends fixed to rigid supports, uniformly distributed transverse load w on entire span	$y_{\max} + \frac{A}{16I} y_{\max}^3 = \frac{wl^4}{4\pi^4 EI}$ (Solve for $y_{\max}$) $P = \frac{\pi^2 EA}{4l^2} y_{\max}^2$ Use case 2d from Table 8.7(*d*) or Table 8.9 to determine maximum slopes and moments after solving for P
5. Same as case 1, except beam is perfectly flexible like a cable or chain and has an unstretched length l	$\tan\theta - \sin\theta = \frac{W}{2EA}$ or if $\theta < 12^\circ$, $\theta = \left(\frac{W}{EA}\right)^{1/3}$ $P = \frac{W}{2\tan\theta}$ 
6. Same as case 3, except beam is perfectly flexible like a cable or chain and has an unstretched length l	$y_{\max} = l\left(\frac{3wl}{64EA}\right)^{1/3}$ $P = \frac{wl^2}{8y_{\max}}$ 

TABLE 8.11(*a*) Reaction and deflection coefficients for tapered beams
Moments of inertia vary as $(1 + Kx/l)^n$, where $n = 1.0$

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
1a	0	y_A θ_A	2.525 2.262	1.636 1.545	0.579 0.614	0.321 0.359	0.171 0.201
	0.25	y_A θ_A	2.663 2.498	1.682 1.631	0.563 0.578	0.303 0.317	0.159 0.168
	0.50	y_A θ_A	2.898 2.811	1.755 1.731	0.543 0.548	0.284 0.289	0.146 0.149
	0.75	y_A θ_A	3.289 3.261	1.858 1.851	0.521 0.522	0.266 0.267	0.135 0.135
1c	0.25	R_A θ_A	1.055 1.492	1.028 1.256	0.972 0.744	0.946 0.514	0.926 0.330
	0.50	R_A θ_A	1.148 1.740	1.073 1.365	0.936 0.682	0.887 0.435	0.852 0.261
1d	0.25	R_A M_A	1.046 1.137	1.026 1.077	0.968 0.905	0.932 0.797	0.895 0.686
	0.50	R_A M_A	1.163 1.326	1.085 1.171	0.915 0.829	0.837 0.674	0.771 0.542
1e	0.25	θ_A $y_{l/2}$	1.396 1.563	1.220 1.301	0.760 0.703	0.531 0.452	0.342 0.268
	0.50	θ_A $y_{l/2}$	1.524 1.665	1.282 1.349	0.718 0.674	0.476 0.416	0.293 0.239
2a. Uniform load	0	y_A θ_A	2.711 2.525	1.695 1.636	0.561 0.579	0.302 0.321	0.158 0.171
	0.25	y_A θ_A	2.864 2.745	1.742 1.708	0.547 0.556	0.289 0.296	0.149 0.154
	0.50	y_A θ_A	3.091 3.029	1.806 1.790	0.532 0.535	0.275 0.278	0.140 0.142
	0.75	y_A θ_A	3.435 3.415	1.890 1.886	0.516 0.516	0.262 0.263	0.132 0.133
2c. Uniform load	0	R_A θ_A	1.074 1.663	1.036 1.326	0.968 0.710	0.941 0.473	0.922 0.296
	0.50	R_A θ_A	1.224 1.942	1.104 1.438	0.917 0.653	0.858 0.403	0.818 0.237

TABLE 8.11(*a*) Reaction and deflection coefficients for tapered beams (*Continued*)

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
2d. Uniform load	0	R_A	1.089	1.046	0.954	0.911	0.872
		M_A	1.267	1.137	0.863	0.733	0.615
	0.50	R_A	1.267	1.130	0.886	0.791	0.717
		M_A	1.481	1.234	0.794	0.625	0.491
2e. Uniform load	0	θ_A	1.508	1.271	0.729	0.492	0.309
		$y_{l/2}$	1.678	1.352	0.676	0.420	0.243
	0.50	θ_A	1.616	1.320	0.700	0.454	0.275
		$y_{l/2}$	1.765	1.389	0.658	0.398	0.225
2a. Uniformly increasing load	0	y_A	2.851	1.737	0.549	0.291	0.150
		θ_A	2.711	1.695	0.561	0.302	0.158
	0.25	y_A	3.005	1.781	0.538	0.280	0.143
		θ_A	2.915	1.757	0.543	0.285	0.147
	0.50	y_A	3.220	1.839	0.525	0.270	0.137
		θ_A	3.172	1.827	0.527	0.272	0.138
	0.75	y_A	3.526	1.910	0.513	0.260	0.131
		θ_A	3.511	1.907	0.513	0.260	0.131
2c. Uniformly increasing load	0	R_A	1.129	1.062	0.948	0.907	0.878
		θ_A	1.775	1.372	0.686	0.442	0.269
	0.50	R_A	1.275	1.124	0.907	0.842	0.799
		θ_A	2.063	1.479	0.639	0.388	0.225
2d. Uniformly increasing load	0	R_A	1.157	1.079	0.926	0.860	0.804
		M_A	1.353	1.177	0.833	0.685	0.559
	0.50	R_A	1.334	1.157	0.870	0.767	0.690
		M_A	1.573	1.269	0.777	0.601	0.468
2e. Uniformly increasing load	0	θ_A	1.561	1.295	0.714	0.472	0.291
		$y_{l/2}$	1.722	1.370	0.667	0.409	0.234
	0.50	θ_A	1.654	1.335	0.693	0.447	0.269
		$y_{l/2}$	1.806	1.404	0.651	0.392	0.221
3a	0	y_A	2.262	1.545	0.614	0.359	0.201
		θ_A	1.848	1.386	0.693	0.462	0.297
	0.25	y_A	2.337	1.575	0.597	0.337	0.182
		θ_A	2.095	1.492	0.627	0.367	0.203
	0.50	y_A	2.566	1.658	0.566	0.305	0.159
		θ_A	2.443	1.622	0.575	0.313	0.164
	0.75	y_A	3.024	1.795	0.532	0.275	0.140
		θ_A	2.985	1.785	0.534	0.277	0.141

TABLE 8.11(*a*) Reaction and deflection coefficients for tapered beams (*Continued*)

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
3c	0	R_A θ_A	0.896 1.312	0.945 1.166	1.059 0.823	1.118 0.645	1.173 0.482
	0.50	R_A θ_A	1.016 1.148	1.014 1.125	0.977 0.794	0.952 0.565	0.929 0.365
3d	0.25	R_A M_A	0.796 1.614	0.890 1.331	1.116 0.653	1.220 0.340	1.298 0.106
	0.50	R_A M_A	0.958 0.875	0.988 0.965	0.988 0.965	0.958 0.875	0.919 0.758
3e	0	θ_A $y_{l/2}$	1.283 1.524	1.159 1.282	0.818 0.718	0.631 0.476	0.460 0.293
	0.25	θ_A $y_{l/2}$	1.628 1.651	1.338 1.345	0.666 0.671	0.393 0.408	0.208 0.229

TABLE 8.11(*b*) Reaction and deflection coefficients for tapered beams
Moments of inertia vary as $(1 + Kx/l)^n$, where $n = 2.0$

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
1a	0	y_A	2.729	1.667	0.589	0.341	0.194
		θ_A	2.455	1.577	0.626	0.386	0.235
	0.25	y_A	2.872	1.713	0.572	0.320	0.176
		θ_A	2.708	1.663	0.588	0.338	0.190
	0.50	y_A	3.105	1.783	0.549	0.296	0.157
		θ_A	3.025	1.761	0.555	0.301	0.161
	0.75	y_A	3.460	1.877	0.525	0.272	0.140
		θ_A	3.437	1.872	0.526	0.273	0.140
1c	0.25	R_A	1.052	1.028	0.970	0.938	0.905
		θ_A	1.588	1.278	0.759	0.559	0.398
	0.50	R_A	1.138	1.070	0.932	0.867	0.807
		θ_A	1.867	1.390	0.695	0.468	0.306
1d	0.25	R_A	1.049	1.027	0.969	0.934	0.895
		M_A	1.155	1.082	0.909	0.813	0.713
	0.50	R_A	1.169	1.086	0.914	0.831	0.753
		M_A	1.358	1.177	0.833	0.681	0.548
1e	0.25	θ_A	1.509	1.246	0.778	0.586	0.428
		$y_{l/2}$	1.716	1.334	0.721	0.501	0.334
	0.50	θ_A	1.668	1.313	0.737	0.525	0.363
		$y_{l/2}$	1.840	1.385	0.692	0.460	0.294
2a. Uniform load	0	y_A	2.916	1.724	0.569	0.318	0.174
		θ_A	2.729	1.667	0.589	0.341	0.194
	0.25	y_A	3.067	1.770	0.554	0.301	0.161
		θ_A	2.954	1.737	0.563	0.311	0.169
	0.50	y_A	3.282	1.830	0.537	0.283	0.148
		θ_A	3.226	1.816	0.540	0.287	0.150
	0.75	y_A	3.580	1.906	0.518	0.266	0.136
		θ_A	3.564	1.902	0.519	0.267	0.136
2c. Uniform load	0	R_A	1.068	1.035	0.965	0.932	0.899
		θ_A	1.774	1.349	0.723	0.510	0.351
	0.50	R_A	1.203	1.098	0.910	0.831	0.761
		θ_A	2.076	1.463	0.664	0.430	0.271

TABLE 8.11(*b*) Reaction and deflection coefficients for tapered beams (*Continued*)

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
2d. Uniform load	0	R_A	1.091	1.046	0.954	0.909	0.865
		M_A	1.290	1.142	0.866	0.741	0.628
	0.50	R_A	1.267	1.129	0.833	0.779	0.689
		M_A	1.509	1.239	0.795	0.625	0.486
2e. Uniform load	0	θ_A	1.645	1.301	0.747	0.542	0.382
		$y_{l/2}$	1.853	1.387	0.694	0.463	0.298
	0.50	θ_A	1.774	1.352	0.718	0.500	0.339
		$y_{l/2}$	1.955	1.426	0.675	0.438	0.274
2a. Uniformly increasing load	0	y_A	3.052	1.765	0.556	0.304	0.163
		θ_A	2.916	1.724	0.569	0.318	0.174
	0.25	y_A	3.199	1.807	0.543	0.290	0.153
		θ_A	3.116	1.784	0.550	0.297	0.158
	0.50	y_A	3.395	1.860	0.529	0.276	0.143
		θ_A	3.354	1.849	0.532	0.279	0.144
	0.75	y_A	3.653	1.923	0.515	0.263	0.134
		θ_A	3.641	1.921	0.515	0.263	0.134
2c. Uniformly increasing load	0	R_A	1.119	1.059	0.944	0.890	0.841
		θ_A	1.896	1.396	0.698	0.475	0.315
	0.50	R_A	1.244	1.116	0.898	0.810	0.736
		θ_A	2.196	1.503	0.649	0.411	0.255
2d. Uniformly increasing load	0	R_A	1.159	1.079	0.925	0.854	0.789
		M_A	1.379	1.182	0.836	0.691	0.565
	0.50	R_A	1.328	1.154	0.866	0.752	0.656
		M_A	1.596	1.272	0.777	0.598	0.457
2e. Uniformly increasing load	0	θ_A	1.708	1.326	0.732	0.521	0.360
		$y_{l/2}$	1.904	1.407	0.684	0.451	0.286
	0.50	θ_A	1.817	1.368	0.711	0.491	0.331
		$y_{l/2}$	2.001	1.442	0.668	0.430	0.268
3a	0	y_A	2.455	1.577	0.626	0.386	0.235
		θ_A	2.000	1.414	0.707	0.500	0.354
	0.25	y_A	2.539	1.608	0.609	0.363	0.211
		θ_A	2.286	1.526	0.641	0.400	0.243
	0.50	y_A	2.786	1.691	0.575	0.323	0.177
		θ_A	2.667	1.657	0.586	0.333	0.185
	0.75	y_A	3.234	1.821	0.538	0.284	0.148
		θ_A	3.200	1.812	0.540	0.286	0.149

TABLE 8.11(*b*) Reaction and deflection coefficients for tapered beams (*Continued*)

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
3c	0	R_A θ_A	0.900 1.375	0.946 1.181	1.062 0.835	1.132 0.688	1.212 0.558
	0.50	R_A θ_A	1.021 1.223	1.015 1.148	0.977 0.814	0.946 0.622	0.911 0.451
3d	0.25	R_A M_A	0.785 1.682	0.888 1.347	1.117 0.660	1.230 0.348	1.333 0.083
	0.50	R_A M_A	0.966 0.890	0.991 0.972	0.991 0.974	0.966 0.905	0.928 0.807
3e	0	θ_A $y_{l/2}$	1.364 1.668	1.179 1.313	0.833 0.737	0.682 0.525	0.549 0.363
	0.25	θ_A $y_{l/2}$	1.801 1.826	1.376 1.382	0.686 0.690	0.441 0.454	0.263 0.284

TABLE 8.11(*c*) Reaction and deflection coefficients for tapered beams
Moments of inertia vary as $(1+Kx/l)^n$, where $n=3.0$

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
1a	0	y_A θ_A	2.796 2.520	1.677 1.587	0.593 0.630	0.349 0.397	0.204 0.250
	0.25	y_A θ_A	2.939 2.777	1.722 1.674	0.575 0.592	0.327 0.346	0.184 0.200
	0.50	y_A θ_A	3.169 3.092	1.791 1.770	0.551 0.558	0.300 0.307	0.162 0.167
	0.75	y_A θ_A	3.509 3.488	1.883 1.878	0.526 0.527	0.274 0.275	0.142 0.143
1c	0.25	R_A θ_A	1.051 1.626	1.027 1.286	0.969 0.764	0.936 0.573	0.899 0.422
	0.50	R_A θ_A	1.134 1.916	1.068 1.399	0.930 0.700	0.860 0.480	0.791 0.322
1d	0.25	R_A M_A	1.050 1.161	1.027 1.084	0.969 0.911	0.934 0.818	0.895 0.724
	0.50	R_A M_A	1.171 1.378	1.086 1.179	0.914 0.834	0.829 0.684	0.748 0.553
1e	0.25	θ_A $y_{l/2}$	1.554 1.774	1.256 1.346	0.784 0.728	0.605 0.519	0.460 0.362
	0.50	θ_A $y_{l/2}$	1.723 1.907	1.324 1.397	0.743 0.699	0.543 0.477	0.391 0.318
2a. Uniform load	0	y_A θ_A	2.981 2.796	1.734 1.677	0.572 0.593	0.324 0.349	0.182 0.204
	0.25	y_A θ_A	3.130 3.020	1.779 1.747	0.556 0.566	0.306 0.317	0.167 0.176
	0.50	y_A θ_A	3.338 3.285	1.837 1.823	0.538 0.542	0.287 0.291	0.151 0.154
	0.75	y_A θ_A	3.620 3.606	1.911 1.097	0.519 0.520	0.268 0.269	0.137 0.138
2c. Uniform load	0	R_A θ_A	1.066 1.817	1.034 1.357	0.965 0.727	0.928 0.522	0.891 0.370
	0.50	R_A θ_A	1.194 2.125	1.096 1.471	0.908 0.668	0.821 0.439	0.741 0.284

TABLE 8.11(*c*) Reaction and deflection coefficients for tapered beams (*Continued*)

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
2d. Uniform load	0	R_A M_A	1.092 1.297	1.046 1.144	0.954 0.867	0.908 0.745	0.863 0.635
	0.50	R_A M_A	1.266 1.517	1.128 1.240	0.882 0.796	0.776 0.626	0.680 0.487
2e. Uniform load	0	θ_A $y_{l/2}$	1.697 1.919	1.311 1.400	0.753 0.700	0.560 0.480	0.411 0.322
	0.50	θ_A $y_{l/2}$	1.833 2.025	1.363 1.438	0.724 0.680	0.517 0.453	0.365 0.296
2a. Uniformly increasing load	0	y_A θ_A	3.115 2.981	1.773 1.734	0.559 0.572	0.309 0.324	0.169 0.182
	0.25	y_A θ_A	3.258 3.178	1.815 1.792	0.545 0.552	0.294 0.301	0.157 0.163
	0.50	y_A θ_A	3.446 3.407	1.866 1.856	0.531 0.533	0.279 0.282	0.146 0.148
	0.75	y_A θ_A	3.687 3.676	1.927 1.925	0.516 0.516	0.264 0.265	0.135 0.135
2c. Uniformly increasing load	0	R_A θ_A	1.114 1.942	1.058 1.404	0.942 0.702	0.885 0.486	0.829 0.332
	0.50	R_A θ_A	1.233 2.244	1.113 1.511	0.895 0.652	0.800 0.419	0.713 0.266
2d. Uniformly increasing load	0	R_A M_A	1.159 1.386	1.078 1.183	0.925 0.837	0.853 0.694	0.785 0.596
	0.50	R_A M_A	1.325 1.602	1.153 1.273	0.865 0.777	0.747 0.598	0.645 0.456
2e. Uniformly increasing load	0	θ_A $y_{l/2}$	1.764 1.972	1.337 1.419	0.738 0.690	0.538 0.466	0.387 0.309
	0.50	θ_A $y_{l/2}$	1.878 2.072	1.379 1.454	0.717 0.674	0.508 0.445	0.356 0.288
3a	0	y_A θ_A	2.520 2.054	1.587 1.424	0.630 0.712	0.397 0.513	0.250 0.375
	0.25	y_A θ_A	2.607 2.352	1.619 1.537	0.613 0.646	0.373 0.412	0.224 0.260
	0.50	y_A θ_A	2.858 2.741	1.702 1.668	0.579 0.590	0.330 0.342	0.185 0.194
	0.75	y_A θ_A	3.296 3.264	1.829 1.821	0.539 0.542	0.288 0.290	0.152 0.153

TABLE 8.11(*c*) Reaction and deflection coefficients for tapered beams (*Continued*)

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
3c	0	R_A	0.901	0.947	1.063	1.136	1.223
		θ_A	1.401	1.186	0.839	0.701	0.583
	0.50	R_A	1.022	1.015	0.977	0.945	0.906
		θ_A	1.257	1.157	0.820	0.642	0.483
3d	0.25	R_A	0.781	0.887	1.117	1.233	1.343
		M_A	1.705	1.352	0.663	0.355	0.088
	0.50	R_A	0.969	0.992	0.992	0.969	0.932
		M_A	0.897	0.975	0.977	0.916	0.828
3e	0	θ_A	1.397	1.186	0.838	0.699	0.579
		$y_{l/2}$	1.723	1.324	0.743	0.543	0.391
	0.25	θ_A	1.868	1.389	0.693	0.460	0.289
		$y_{l/2}$	1.892	1.394	0.697	0.471	0.308

TABLE 8.11(*d*) Reaction and deflection coefficients for tapered beams
Moments of inertia vary as $(1 + Kx/l)^n$, where $n = 4.0$

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
1a	0	y_A	2.828	1.682	0.595	0.354	0.210
		θ_A	2.552	1.593	0.632	0.402	0.258
	0.25	y_A	2.971	1.727	0.576	0.330	0.188
		θ_A	2.811	1.679	0.593	0.350	0.206
	0.50	y_A	3.200	1.796	0.553	0.303	0.165
		θ_A	3.124	1.774	0.559	0.310	0.170
	0.75	y_A	3.532	1.886	0.527	0.276	0.143
		θ_A	3.511	1.881	0.528	0.277	0.144
1c	0.25	R_A	1.051	1.027	0.969	0.935	0.896
		θ_A	1.646	1.290	0.767	0.581	0.434
	0.50	R_A	1.131	1.068	0.929	0.857	0.784
		θ_A	1.941	1.404	0.702	0.485	0.331
1d	0.25	R_A	1.051	1.027	0.969	0.935	0.896
		M_A	1.164	1.085	0.912	0.821	0.730
	0.50	R_A	1.172	1.086	0.914	0.828	0.746
		M_A	1.373	1.180	0.835	0.686	0.556
1e	0.25	θ_A	1.578	1.260	0.787	0.615	0.476
		$y_{l/2}$	1.805	1.351	0.731	0.528	0.376
	0.50	θ_A	1.752	1.329	0.746	0.552	0.406
		$y_{l/2}$	1.941	1.404	0.702	0.485	0.331
2a. Uniform load	0	y_A	3.013	1.738	0.573	0.328	0.187
		θ_A	2.828	1.682	0.595	0.354	0.210
	0.25	y_A	3.161	1.783	0.558	0.309	0.170
		θ_A	3.052	1.751	0.568	0.320	0.180
	0.50	y_A	3.365	1.841	0.539	0.289	0.154
		θ_A	3.314	1.827	0.543	0.293	0.157
	0.75	y_A	3.639	1.913	0.520	0.269	0.138
		θ_A	3.625	1.910	0.521	0.270	0.139
2c. Uniform load	0	R_A	1.065	1.034	0.964	0.927	0.888
		θ_A	1.839	1.361	0.729	0.528	0.380
	0.50	R_A	1.190	1.095	0.907	0.817	0.731
		θ_A	2.151	1.476	0.670	0.443	0.290

TABLE 8.11(*d*) Reaction and deflection coefficients for tapered beams (*Continued*)

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
2d. Uniform load	0	R_A M_A	1.092 1.301	1.046 1.145	0.954 0.867	0.908 0.747	0.862 0.639
	0.50	R_A M_A	1.266 1.521	1.128 1.241	0.882 0.796	0.774 0.627	0.676 0.488
2e. Uniform load	0	θ_A $y_{l/2}$	1.724 1.953	1.316 1.406	0.756 0.703	0.569 0.488	0.426 0.335
	0.50	θ_A $y_{l/2}$	1.864 2.061	1.396 1.445	0.727 0.683	0.526 0.461	0.379 0.307
2a. Uniformly increasing load	0	y_A θ_A	3.145 3.013	1.778 1.738	0.560 0.573	0.312 0.328	0.173 0.187
	0.25	y_A θ_A	3.287 3.207	1.819 1.796	0.546 0.553	0.297 0.304	0.160 0.166
	0.50	y_A θ_A	3.470 3.432	1.869 1.859	0.532 0.534	0.281 0.284	0.147 0.150
	0.75	y_A θ_A	3.703 3.692	1.929 1.927	0.516 0.517	0.265 0.266	0.136 0.136
2c. Uniformly increasing load	0	R_A θ_A	1.112 1.966	1.057 1.408	0.942 0.704	0.882 0.492	0.823 0.340
	0.50	R_A θ_A	1.227 2.269	1.111 1.515	0.894 0.653	0.794 0.423	0.701 0.271
2d. Uniformly increasing load	0	R_A M_A	1.159 1.390	1.078 1.184	0.924 0.837	0.852 0.695	0.783 0.572
	0.50	R_A M_A	1.323 1.605	1.153 1.274	0.864 0.777	0.744 0.598	0.639 0.456
2e. Uniformly increasing load	0	θ_A $y_{l/2}$	1.793 2.007	1.343 1.425	0.741 0.693	0.547 0.475	0.402 0.321
	0.50	θ_A $y_{l/2}$	1.909 2.108	1.385 1.461	0.719 0.677	0.516 0.453	0.369 0.299
3a	0	y_A θ_A	2.552 2.081	1.593 1.428	0.632 0.714	0.402 0.520	0.258 0.386
	0.25	y_A θ_A	2.641 2.386	1.624 1.543	0.615 0.648	0.378 0.419	0.231 0.270
	0.50	y_A θ_A	2.893 2.778	1.707 1.674	0.581 0.592	0.334 0.346	0.190 0.200
	0.75	y_A θ_A	3.326 3.295	1.833 1.825	0.540 0.543	0.290 0.292	0.154 0.155

TABLE 8.11(*d*) Reaction and deflection coefficients for tapered beams (*Continued*)

Case no. in Table 8.1	Load location a/l	Multiplier listed for	I_B/I_A				
			0.25	0.50	2.0	4.0	8.0
3c	0	R_A	0.902	0.947	1.063	1.138	1.227
		θ_A	1.414	1.189	0.841	0.707	0.595
	0.50	R_A	1.023	1.015	0.976	0.944	0.904
		θ_A	1.275	1.161	0.823	0.652	0.499
3d	0.25	R_A	0.780	0.887	1.117	1.234	1.347
		M_A	1.716	1.354	0.665	0.359	0.092
	0.50	R_A	0.971	0.993	0.993	0.971	0.935
		M_A	0.902	0.976	0.979	0.922	0.839
3e	0	θ_A	1.414	1.189	0.841	0.707	0.595
		$y_{l/2}$	1.752	1.329	0.746	0.552	0.406
	0.25	θ_A	1.903	1.396	0.697	0.470	0.304
		$y_{l/2}$	1.927	1.401	0.700	0.480	0.321

TABLE 8.12 Position of flexural center *Q* for different sections

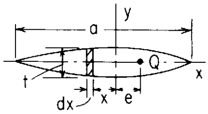
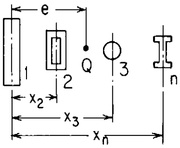
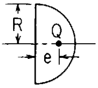
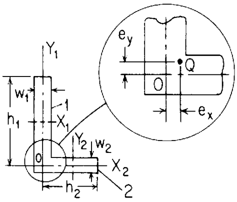
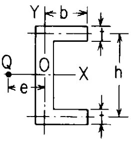
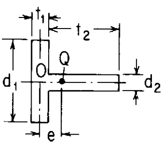
Form of section	Position of Q
1. Any narrow section symmetrical about the x axis; centroid at $x = 0$, $y = 0$  	$e = \frac{1+3\nu}{1+\nu}\frac{\int xt^3 dx}{\int t^3 dx}$ For narrow triangle (with $\nu = 0.25$), $e = 0.187a$ (Refs. 32 and 52)
2. Beam composed of n elements of any form, connected of separate, with common neutral axis (e.g., multiple-spar airplane wing)  	$e = \frac{E_2I_2x_2 + E_3I_3x_3 + \cdots + E_nI_nx_n}{E_1I_1 + E_2I_2 + E_3I_3 + \cdots + E_nI_n}$ where I_1, I_2, etc., are moments of inertia of the several elements about the X axis (that is, Q is at the centroid of the products EI for the several elements)
3. Semicircular area  	$e = \frac{8}{15\pi}\frac{3+4\nu}{1+\nu}R$ (Q is to right of centroid) (Refs. 1 and 64) For any sector of solid or hollow circular area, see Ref. 32
4. Angle  	Leg 1 = rectangle w_1h_1; leg 2 = rectangle w_2h_2 I_1 = moment of inertia of leg 1 about Y_1 (central axis) I_2 = moment of inertia of leg about Y_2 (central axis) $e_y = \frac{h_1}{2}\frac{I_1}{I_1 + I_2}$ (for e_x use X_1 and X_2 central axes) (Ref. 31) If w_1 and w_2 are small, $e_x = e_y = 0$ (practically) and Q is at 0
5. Channel  	$e = h\frac{I_{xy}}{I_x}$ where I_{xy} = product of inertia of the half section (above X) with respect to axes X and Y, and I_x = moment of inertia of whole section with respect to axis X If t is uniform, $e = (b^2 - t^2/4)h^2t/4I_x$
6. T  	$e = \frac{1}{2}(t_1 + t_2)\frac{1}{1 + d_1^3t_1/d_2^3t_2}$ For a T-beam of ordinary proportions, Q may be assumed to be at 0

TABLE 8.12 Position of flexural center *Q* for different sections (*Continued*)

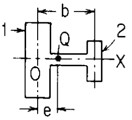
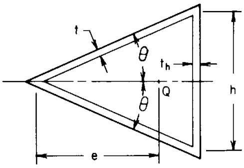
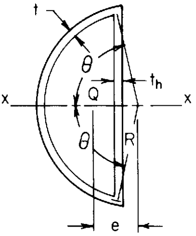
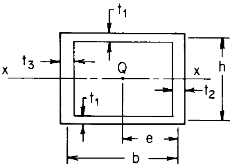
7. *I* with unequal flanges and thin web  	$e = b\frac{I_2}{I_1 + I_2}$ where I_1 and I_2, respectively, denote moments of inertia about X axis of flanges 1 and 2
8. Hollow thin-walled triangular section  	$\frac{e}{h} = \frac{1}{2\tan\theta(1 + t\sin\theta/t_h)}$
9. Hollow thin-walled section bounded by a circular arc and a straight edge  *Note:* Expressions are valid for $0 < \theta < \pi$	$I_x = tR^3\left(\theta - \sin\theta\cos\theta + \frac{2t_h}{3t}\sin^3\theta\right)$ $\frac{e}{R} = \frac{2t_hR^3\sin\theta(\cos\theta + t/t_h)}{I_x(\sin\theta + t_h\theta/t)}\left(\sin\theta - \theta\cos\theta + \frac{t_h\theta}{3t}\sin^2\theta\right)$ If $\theta = \pi/2$ $I_x = \frac{tR^3}{2}\left(\pi + \frac{4t_h}{3t}\right)$ $\frac{e}{R} = \frac{4(6 + \pi t_h/t)}{(2 + \pi t_h/t)(3\pi + 4t_h/t)}$
10. Hollow thin-walled rectangular section  	$I_x = \frac{h^3}{12}(t_2 + t_3) + \frac{t_1bh^2}{2}$ $\frac{e}{b} = \frac{bh^2}{12I_x}\left[9t_1 + \frac{t_3h}{b} - \frac{12(b + t_1h/t_3)}{2b/t_1 + h/t_3 + h/t_2}\right]$
11. For thin-walled sections, such as lipped channels, hat sections, and sectors of circular tubes, see Table 9.2. The position of the flexural centers and shear centers coincide.	

TABLE 8.13 Collapse loads with plastic hinge locations for straight beams

NOTATION: M_p = fully plastic bending moment (force-length); x_h = position of a plastic hinge (length); W_c = concentrated load necessary to produce plastic collapse of the beam (force); w_c = unit load necessary to produce plastic collapse of the beam (force per unit length); M_{oc} = applied couple necessary to produce plastic collapse (force-length). The fully plastic bending moment M_p is the product of the yield strength of the matieral σ_{ys} and the plastic section modulus Z found in Table A.1 for the given cross sections

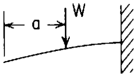
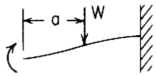
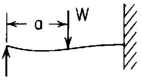
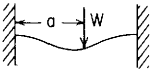
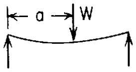
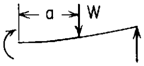
Reference no., end restraints	Collapse loads with plastic hinge locations
1a. Left end free, right end fixed (cantilever)  	$W_c = \dfrac{M_p}{l-a}$ $x_h = l$
1b. Left end guided, right end fixed  	$W_c = \dfrac{2M_p}{l-a}$ $0 \le x_{h1} \le a \qquad x_{h2} = l$
1c. Left end simply supported, right end fixed  	$W_c = \dfrac{M_p(l+a)}{a(l-a)}$ $x_{h1} = a \qquad x_{h2} = l$
1d. Left end fixed, right end fixed  	$W_c = \dfrac{2M_p l}{a(l-a)}$ $x_{h1} = 0 \qquad x_{h2} = a \qquad x_{h3} = l$
1e. Left end simply supported, right end simply supported  	$W_c = \dfrac{M_p l}{a(l-a)}$ $x_h = a$
1f. Left end guided, right end simply supported  	$W_c = \dfrac{M_p}{l-a}$ $0 < x_h < a$

TABLE 8.13 Collapse loads with plastic hinge locations for straight beams (*Continued*)

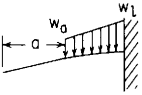
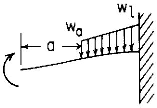
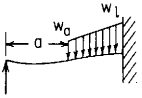
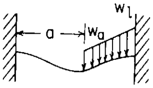
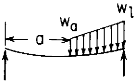
<table>
<tr><th>Reference no., end restraints</th><th>Collapse loads with plastic hinge locations</th></tr>
<tr><td>2a. Left end free, right end fixed (cantilever)
 </td><td>If $w_l = w_a$ (uniform load), then
$$w_{ac} = \frac{2M_p}{(l-a)^2} \qquad x_h = l$$
If $w_a = 0$ (uniformly increasing load), then
$$w_{lc} = \frac{6M_p}{(l-a)^2} \qquad x_h = l$$
If $w_l = 0$ (uniformly decreasing load), then
$$w_{ac} = \frac{3M_p}{(l-a)^2} \qquad x_h = l$$</td></tr>
<tr><td>2b. Left end guided, right end fixed
 </td><td>If $w_l = w_a$ (uniform load), then
$$w_{ac} = \frac{4M_p}{(l-a)^2} \qquad x_{h1} = a \qquad x_{h2} = l$$
If $w_a = 0$ (uniformly increasing load), then
$$w_{lc} = \frac{12M_p}{(l-a)^2} \qquad x_{h1} = a \qquad x_{h2} = l$$
If $w_l = 0$ (uniformly decreasing load), then
$$w_{ac} = \frac{6M_p}{(l-a)^2} \qquad x_{h1} = a \qquad x_{h2} = l$$</td></tr>
<tr><td>2c. Left end simply supported, right end fixed
 </td><td>If $w_l = w_a$ (uniform load), then
$$w_{ac} = \frac{2M_p(l + x_{h1})}{(l - x_{h1})(l x_{h_1} - a^2)}$$
where $x_{h1} = [2(l^2 + a^2)]^{1/2} - l \qquad x_{h2} = l$
If $w_a = 0$ (uniformly increasing load), then
$$w_{lc} = \frac{6K_1M_p}{(l-a)^2} \qquad x_{h1} = K_2 l \qquad x_{h2} = l$$
<table>
<tr><td>a/l</td><td>0</td><td>0.2</td><td>0.4</td><td>0.6</td><td>0.8</td></tr>
<tr><td>K_1</td><td>4.000</td><td>3.324</td><td>2.838</td><td>2.481</td><td>2.211</td></tr>
<tr><td>K_2</td><td>0.500</td><td>0.545</td><td>0.616</td><td>0.713</td><td>0.838</td></tr>
</table>
If $w_l = 0$ (uniformly decreasing load), then
$$w_{ac} = \frac{6K_3M_p}{(l-a)^2} \qquad x_{h1} = K_4 l \qquad x_{h2} = l$$
<table>
<tr><td>a/l</td><td>0</td><td>0.2</td><td>0.4</td><td>0.6</td><td>0.8</td></tr>
<tr><td>K_3</td><td>3.596</td><td>2.227</td><td>1.627</td><td>1.310</td><td>1.122</td></tr>
<tr><td>K_4</td><td>0.347</td><td>0.387</td><td>0.490</td><td>0.634</td><td>0.808</td></tr>
</table></td></tr>
</table>

TABLE 8.13 Collapse loads with plastic hinge locations for straight beams (*Continued*)

Reference no., end restraints	Collapse loads with plastic hinge locations

2d. Left end fixed, right end fixed

 	
	If $w_l = w_a$ (uniform load), then
	$w_{ac} = \frac{16M_p l^2}{(l^2 - a^2)^2} \qquad x_{h1} = 0 \qquad x_{h2} = \frac{l^2 + a^2}{2l} \qquad x_{h3} = l$
	If $w_a = 0$ (uniformly increasing load), then
	$w_{lc} = \frac{12M_p(l - a)}{(l - x_{h2})(x_{h2}^2 - 3ax_{h2} + lx_{h2} + a^3/l)}$
	$x_{h1} = 0 \qquad x_{h2} = a + \left(a^2 - al + \frac{l^2}{3} - \frac{a^3}{3l}\right)^{1/2} \qquad x_{h3} = l$
	If $w_l = 0$ (uniformly decreasing load), then
	$w_{ac} = \frac{12M_p(l - a)}{(l - x_{h2})(2lx_{h2} - 3a^2 - x_{h2}^2 + 2a^3/l)}$
	$x_{h1} = 0 \qquad x_{h2} = l - \left(\frac{l^2}{3} - a^2 + \frac{2a^3}{3l}\right)^{1/2} \qquad x_{h3} = l$

2e. Left end simply supported, right end simply supported

 	
	If $w_l = w_a$ (uniform load), then
	$w_{ac} = \frac{8M_p l^2}{(l^2 - a^2)^2} \qquad x_h = \frac{l^2 + a^2}{2l}$
	If $w_a = 0$ (uniformly increasing load), then
	$w_{lc} = \frac{6M_p(l - a)}{(l - x_h)(x_h^2 - 3ax_h + lx_h + a^3/l)}$
	$x_h = a + \left(a^2 - al + \frac{l^2}{3} - \frac{a^3}{3l}\right)^{1/2}$
	If $w_l = 0$ (uniformly decreasing load), then
	$w_{ac} = \frac{6M_p(l - a)}{(l - x_h)(2lx_h - 3a^2 - x_h^2 + 2a^3/l)}$
	$x_h = l - \left(\frac{l^2}{3} - a^2 + \frac{2a^3}{3l}\right)^{1/2}$

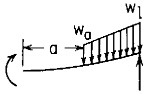
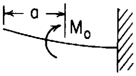
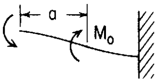
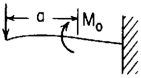
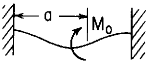
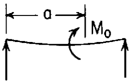
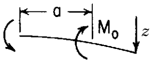
2f. Left end guided, right end simply supported

 	
	If $w_l = w_a$ (uniform load), then
	$w_{ac} = \frac{2M_p}{(l - a)^2} \qquad 0 \leqslant x_h \leqslant a$
	If $w_a = 0$ (uniformly increasing load), then
	$w_{lc} = \frac{6M_p}{(l - a)^2} \qquad 0 \leqslant x_h \leqslant a$
	If $w_l = 0$ (uniformly decreasing load), then
	$w_{ac} = \frac{3M_p}{(l - a)^2} \qquad 0 \leqslant x_h \leqslant a$

TABLE 8.13 Collapse loads with plastic hinge locations for straight beams (*Continued*)

Reference no., end restraints	Collapse loads with plastic hinge locations
3a. Left end free, right end fixed (cantilever) 	$M_{oc} = M_p$ $a < x_h < l$
3b. Left end guided, right end fixed 	$M_{oc} = 2M_p$ $0 < x_{h1} < a \qquad a < x_{h2} < l$
3c. Left end simply supported, right end fixed 	If $l/3 \leqslant a \leqslant l$, then $M_{oc} = 2M_p$ and two plastic hinges form, one on each side of and adjacent to the loading M_o If $0 \leqslant a \leqslant l/3$, then $M_{oc} = \dfrac{M_p(l+a)}{l-a}$ $x_{h1} = a$ just to the right of the loading $M_o \qquad x_{h2} = l$
3d. Left end fixed, right end fixed 	$M_{oc} = 2M_p$ and two plastic hinges form, one on each side of and adjacent to the loading M_o If $0 < a < l/2$, then a third hinge forms at the right end If $l/2 < a < l$, then the third hinge forms at the left end If $a = l/2$, two hinges form at any two locations on one side of the load and one at any location on the other side

3e. Left end simply supported, right end simply supported

	If $0 \leqslant a < l/2$, then
	$M_{oc} = \dfrac{M_p l}{l-a}$
	$x_h = a$ just to the right of the loading M_o
	If $l/2 < a \leqslant l$, then
	$M_{oc} = \dfrac{M_p l}{a}$
	$x_h = a$ just to the right of the loading M_o
	If $a = l/2$, then
	$M_{oc} = 2M_p$
	and two plastic hinges form, one on each side of and adjacent to the loading M_o

3f. Left end guided, right end simply supported 	$M_{oc} = M_p$ $0 < x_h < a$

8.18 References

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Chapter 9

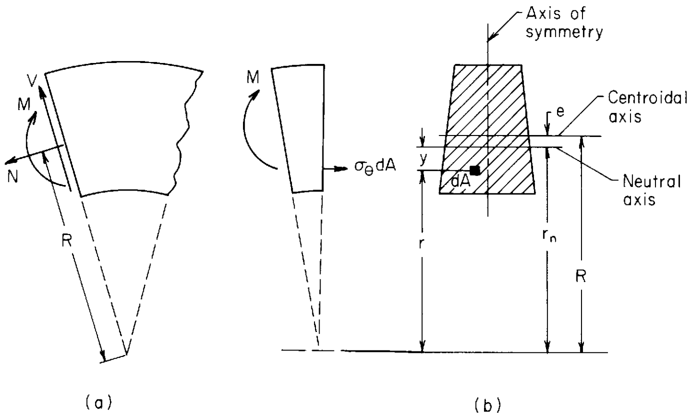
Curved Beams

9.1 Bending in the Plane of the Curve

In a straight beam having either a constant cross section or a cross section which changes gradually along the length of the beam, the neutral surface is defined as the longitudinal surface of zero fiber stress when the member is subjected to pure bending. It contains the neutral axis of every section, and these neutral axes pass through the centroids of the respective sections. In this section on bending in the plane of the curve, the use of the many formulas is restricted to those members for which that axis passing through the centroid of a given section and directed normal to the plane of bending of the member is a principal axis. The one exception to this requirement is for a condition equivalent to the beam being constrained to remain in its original plane of curvature such as by frictionless external guides.

To determine the stresses and deformations in curved beams satisfying the restrictions given above, one first identifies several cross sections and then locates the centroids of each. From these centroidal locations the curved centroidal surface can be defined. For bending in the plane of the curve there will be at each section (1) a force N normal to the cross section and taken to act through the centroid, (2) a shear force V parallel to the cross section in a radial direction, and (3) a bending couple M in the plane of the curve. In addition there will be radial stresses σ_r in the curved beam to establish equilibrium. These internal loadings are shown in Fig. 9.1(a), and the stresses and deformations due to each will be evaluated.

Circumferential normal stresses due to pure bending. When a curved beam is bent in the plane of initial curvature, plane sections remain plane, but because of the different lengths of fibers on the inner and outer portions of the beam, the distribution of unit strain, and therefore stress, is not linear. The neutral axis does not pass through the



(a) (b)

Figure 9.1

centroid of the section and Eqs. (8.1-1) and (8.1-2) do not apply. The error involved in their use is slight as long as the radius of curvature is more than about eight times the depth of the beam. At that curvature the errors in the maximum stresses are in the range of 4 to 5%. The errors created by using the straight-beam formulas become large for sharp curvatures as shown in Table 9.1, which gives formulas and selected numerical data for curved beams of several cross sections and for varying degrees of curvature. In part the formulas and tabulated coefficients are taken from the University of Illinois Circular by Wilson and Quereau (Ref. 1) with modifications suggested by Neugebauer (Ref. 28). For cross sections not included in Table 9.1 and for determining circumferential stresses at locations other than the extreme fibers, one can find formulas in texts on advanced mechanics of materials, for example, Refs. 29 and 36.

The circumferential normal stress σ_θ is given as

$$\sigma_\theta = \frac{My}{Aer} \tag{9.1-1}$$

where M is the applied bending moment, A is the area of the cross section, e is the distance from the centroidal axis to the neutral axis, and y and r locate the radial position of the desired stress from the neutral axis and the center of the curvature, respectively. See Fig. 9.1(b).

$$e = R - r_n = R - \frac{A}{\displaystyle\int_{\text{area}} dA/r} \quad \text{for } \frac{R}{d} < 8 \tag{9.1-2}$$

Equations (9.1-1) and (9.1-2) are based on derivations that neglect the contribution of radial normal stress to the circumferential strain. This assumption does not cause appreciable error for curved beams of compact cross section for which the radial normal stresses are small, and it leads to acceptable answers for beams having thin webs where, although the radial stresses are higher, they occur in regions of the cross section where the circumferential bending stresses are small. The use of the equations in Table 9.1 and of Eqs. (9.1-1) and (9.1-2) is limited to values of $R/d > 0.6$ where, for a rectangular cross section, a comparison of this mechanics-of-materials solution [Eq. (9.1-1)] to the solution using the theory of elasticity shows the mechanics of materials solution to indicate stresses approximately 10% too large.

While in theory the curved-beam formula for circumferential bending stress, Eq. (9.1-1), could be used for beams of very large radii of curvature, one should not use the expression for e from Eq. (9.1-2) for cases where R/d, the ratio of the radius of the curvature R to the depth of the cross section, exceeds 8. The calculation for e would have to be done with careful attention to precision on a computer or calculator to get an accurate answer. Instead one should use the following approximate expression for e which becomes very accurate for large values of R/d. See Ref. 29.

$$e \approx \frac{I_c}{RA} \quad \text{for } \frac{R}{d} > 8 \tag{9.1-3}$$

where I_c is the area moment of inertia of the cross section about the centroidal axis. Using this expression for e and letting R approach infinity leads to the usual straight-beam formula for bending stress.

For complex sections where the table or Eq. (9.1-3) are inappropriate, a numerical technique that provides excellent accuracy can be employed. This technique is illustrated on pp. 318–321 of Ref. 36.

In summary, use Eq. (9.1-1) with e from Eq. (9.1-2) for $0.6 < R/d < 8$. Use Eq. (9.1-1) with e from Eq. (9.1-3) for those curved beams for which $R/d > 8$ and where errors of less than 4 to 5% are desired, or use straight-beam formulas if larger errors are acceptable or if $R/d \gg 8$.

Circumferential normal stresses due to hoop tension $N(M = 0)$. The normal force N was chosen to act through the centroid of the cross section, so a constant normal stress N/A would satisfy equilibrium. Solutions carried out for rectangular cross sections using the theory of elasticity show essentially a constant normal stress with higher values on a thin layer of material on the inside of the curved section and lower values on a thin layer of material on the outside of the section. In most engineering applications the stresses due to the moment M are much larger than those due to N, so the assumption of uniform stress due to N is reasonable.

Shear stress due to the radial shear force *V*. Although Eq. (8.1-2) does not apply to curved beams, Eq. (8.1-13), used as for a straight beam, gives the *maximum* shear stress with sufficient accuracy in most instances. Again an analysis for a rectangular cross section carried out using the theory of elasticity shows that the peak shear stress in a curved beam occurs not at the centroidal axis as it does for a straight beam but toward the inside surface of the beam. For a very sharply curved beam, $R/d = 0.7$, the peak shear stress was $2.04V/A$ at a position one-third of the way from the inner surface to the centroid. For a sharply curved beam, $R/d = 1.5$, the peak shear stress was $1.56V/A$ at a position 80% of the way from the inner surface to the centroid. These values can be compared to a peak shear stress of $1.5V/A$ at the centroid for a straight beam of rectangular cross section.

If a mechanics-of-materials solution for the shear stress in a curved beam is desired, the element in Fig. 9.2(*b*) can be used and moments taken about the center of curvature. Using the normal stress distribution $\sigma_\theta = N/A + My/AeR$, one can find the shear stress expression to be

$$\tau_{r\theta} = \frac{V(R-e)}{t_r A e r^2}(RA_r - Q_r) \tag{9.1-4}$$

where t_r is the thickness of the section normal to the plane of curvature at the radial position r and

$$A_r = \int_b^r dA_1 \quad \text{and} \quad Q_r = \int_b^r r_1\, dA_1 \tag{9.1-5}$$

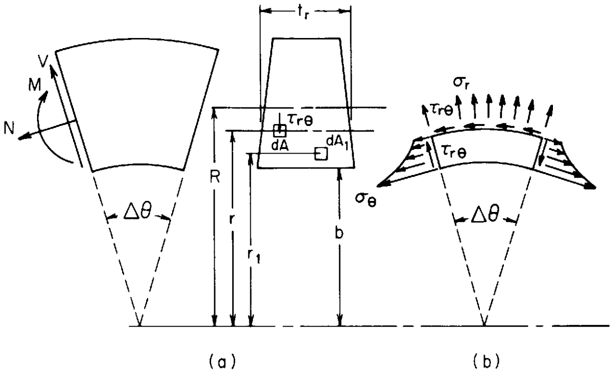


Figure 9.2

Equation (9.1-4) gives conservative answers for the peak values of shear stress in rectangular sections when compared to elasticity solutions. The locations of peak shear stress are the same in both analyses, and the error in magnitude is about 1%.

Radial stresses due to moment *M* and normal force *N*. Owing to the radial components of the fiber stresses, radial stresses are present in a curved beam; these are tensile when the bending moment tends to straighten the beam and compressive under the reverse condition. A mechanics-of-materials solution may be developed by summing radial forces and summing forces perpendicular to the radius using the element in Fig. 9.2.

$$\sigma_r = \frac{R-e}{t_r A e r}\left[(M - NR)\left(\int_b^r \frac{dA_1}{r_1} - \frac{A_r}{R-e}\right) + \frac{N}{r}(RA_r - Q_r)\right] \qquad (9.1\text{-}6)$$

Equation (9.1-6) is as accurate for radial stress as is Eq. (9.1-4) for shear stress when used for a rectangular cross section and compared to an elasticity solution. However, the complexity of Eq. (9.1-6) coupled with the fact that the stresses due to N are generally smaller than those due to M leads to the usual practice of omitting the terms involving N. This leads to the equation for radial stress found in many texts, such as Refs. 29 and 36.

$$\sigma_r = \frac{R-e}{t_r A e r} M\left(\int_b^r \frac{dA_1}{r_1} - \frac{A_r}{R-e}\right) \qquad (9.1\text{-}7)$$

Again care must be taken when using Eqs. (9.1-4), (9.1-6), and (9.1-7) to use an accurate value for e as explained above in the discussion following Eq. (9.1-3).

Radial stress is usually not a major consideration in compact sections for it is smaller than the circumferential stress and is low where the circumferential stresses are large. However, in flanged sections with thin webs the radial stress may be large at the junction of the flange and web, and the circumferential stress is also large at this position. This can lead to excessive shear stress and the possible yielding if the radial and circumferential stresses are of opposite sign. A large compressive radial stress in a thin web may also lead to a buckling of the web. Corrections for curved-beam formulas for sections having thin flanges are discussed in the next paragraph but corrections are also needed if a section has a *thin* web and *very thick* flanges. Under these conditions the individual flanges tend to rotate about their own neutral axes and larger radial and shear stresses are developed. Broughton et al. discuss this configuration in Ref. 31.

EXAMPLES

1. The sharply curved beam with an elliptical cross section shown in Fig. 9.3(*a*) has been used in a machine and has carried satisfactorily a bending moment of $2(10^6)$ N-mm. All dimensions on the figures and in the calculations are given in millimeters. A redesign does not provide as much space for this part, and a decision has been made to salvage the existing stock of this part by machining 10 mm from the inside. The question has been asked as to what maximum moment the modified part can carry without exceeding the peak stress in the original installation.

Solution. First compute the maximum stress in the original section by using case 6 of Table 9.1. $R = 100, c = 50, R/c = 2, A = \pi(50)(20) = 3142$, $e/c = 0.5[2 - (2^2 - 1)^{1/2}] = 0.1340$, $e = 6.70$, and $r_n = 100 - 6.7 = 93.3$. Using these values the stress σ_i can be found as

$$\sigma_i = \frac{My}{Aer} = \frac{2(10^6)(93.3 - 50)}{3142(6.7)(50)} = 82.3\ \text{N/mm}^2$$

Alternatively one can find σ_i from $\sigma_i = k_i Mc/I_x$, where k_i is found to be 1.616 in the table of values from case 6

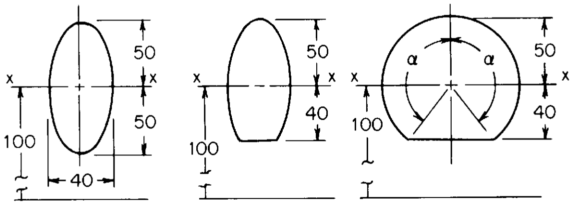
$$\sigma_i = \frac{(1.616)(2)(10^6)(50)}{\pi(20)(50)^3/4} = 82.3\ \text{N/mm}^2$$

Next consider the same section with 10 mm machined from the inner edge as shown in Fig. 9.3(*b*). Use case 9 of Table 9.1 with the initial calculations based on the equivalent modified circular section shown in Fig. 9.3(*c*). For this configuration $\alpha = \cos^{-1}(-40/50) = 2.498$ rad (143.1°), $\sin\alpha = 0.6$, $\cos\alpha = -0.8$, $R_x = 100$, $a = 50$, $a/c = 1.179$, $c = 42.418$, $R = 102.418$, and $R/c = 2.415$. In this problem $R_x > a$, so by using the appropriate expression from case 9 one obtains $e/c = 0.131$ and $e = 5.548$. R, c, and e have the same values for the machined ellipse, Fig. 9.3(*b*), and from case 18 of Table A.1 the area is found to be $A = 20(50)(\alpha - \sin\alpha\cos\alpha) = 2978$. Now the maximum stress on the inner surface can be found and set equal to $82.3\ \text{N/mm}^2$.

$$\sigma_i = 82.3 = \frac{My}{Aer} = \frac{M(102.42 - 5.548 - 60)}{2978(5.548)(60)}$$

$$82.3 = 37.19(10^6)M, \quad M = 2.21(10^6)\ \text{N-mm}$$

One might not expect this increase in M unless consideration is given to the machining away of a stress concentration. Be careful, however, to note that,



(a) (b) (c)

Figure 9.3

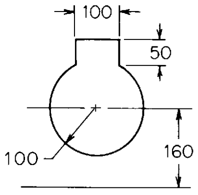


Figure 9.4

although removing the material reduced the peak stress in this case, the part will undergo greater deflections under the same moment that it carried before.

2. A curved beam with a cross section shown in Fig. 9.4 is subjected to a bending moment of 10^7 N-mm in the plane of the curve and to a normal load of 80,000 N in tension. The center of the circular portion of the cross section has a radius of curvature of 160 mm. All dimensions are given and used in the formulas in millimeters. The circumferential stresses in the inner and outer fibers are desired.

Solution. This section can be modeled as a summation of three sections: (1) a solid circular section, (2) a negative (materials removed) segment of a circle, and (3) a solid rectangular section. The section properties are evaluated in the order listed above and the results summed for the composite section.

Section 1. Use case 6 of Table 9.1. $R = 160$, $b = 200$, $c = 100$, $R/c = 1.6$, $\int dA/r = 200[1.6 - (1.6^2 - 1)^{1/2}] = 220.54$, and $A = \pi(100^2) = 31{,}416$.

Section 2. Use case 9 of Table 9.1. $\alpha = \pi/6$ (30°), $R_x = 160$, $a = 100$, $R_x/a = 1.6$, $a/c = 18.55$, $c = 5.391$, $R = 252.0$, $\int dA/r = 3.595$, and from case 20 of Table A.1, $A = 905.9$.

Section 3. Use case 1 of Table 9.1. $R = 160 + 100\cos 30^\circ + 25 = 271.6$, $b = 100$, $c = 25$, $R/c = 10.864$, $A = 5000$, $\int dA/r = 100\ \ln(11.864/9.864) = 18.462$.

For the composite section; $A = 31{,}416 - 905.9 + 5000 = 35{,}510$, $R = [31{,}416(160) - 905.9(252) + 5000(272.6)]/35{,}510 = 173.37$, $c = 113.37$, $\int dA/r = 220.54 - 3.595 + 18.462 = 235.4$, $r_n = A/(\int dA/r) = 35{,}510/235.4 = 150.85$, $e = R - r_n = 22.52$.

Using these data the stresses on the inside and outside are found to be

$$\sigma_i = \frac{My}{Aer} + \frac{N}{A} = \frac{10^7(150.85 - 60)}{35{,}510(22.52)(60)} + \frac{80{,}000}{35{,}510}$$

$$= 18.93 + 2.25 = 21.18\,\text{N/mm}^2$$

$$\sigma_o = \frac{10^7(150.85 - 296.6)}{35{,}510(22.52)(296.6)} + \frac{80{,}000}{35{,}510}$$

$$= -6.14 + 2.25 = -3.89\,\text{N/mm}^2$$

Curved beams with wide flanges. In reinforcing rings for large pipes, airplane fuselages, and ship hulls, the combination of a curved sheet and attached web or stiffener forms a curved beam with wide flanges.

Formulas for the effective width of a flange in such a curved beam are given in Ref. 9 and are as follows.

When the flange is indefinitely wide (e.g., the inner flange of a pipe-stiffener ring), the effective width is

$$b' = 1.56\sqrt{Rt}$$

where b' is the total width assumed effective, R is the mean radius of curvature of the flange, and t is the thickness of the flange.

When the flange has a definite unsupported width b (gross width less web thickness), the ratio of effective to actual width b'/b is a function of qb, where

$$q = \sqrt[4]{\frac{3(1-\nu^2)}{R^2t^2}}$$

Corresponding values of qb and b'/b are as follows:

qb	1	2	3	4	5	6	7	8	9	10	11
b'/b	0.980	0.850	0.610	0.470	0.380	0.328	0.273	0.244	0.217	0.200	0.182

For the curved beam each flange should be considered as replaced by one of corresponding effective width b', and all calculations for direct, bending, and shear stresses, including corrections for curvature, should be based on this transformed section.

Bleich (Ref. 10) has shown that under a *straightening* moment where the curvature is decreased, the radial components of the fiber stresses in the flanges bend both flanges radially away from the web, thus producing tension in the fillet between flange and web in a direction normal to both the circumferential and radial normal stresses discussed in the previous section. Similarly, a moment which increases the curvature causes both flanges to bend radially toward the web and produce compressive stresses in the fillet between flange and web. The *nominal* values of these transverse bending stresses σ' in the fillet, without any correction for the stress concentration at the fillet, are given by $|\sigma'| = |\beta\sigma_m|$, where σ_m is the circumferential bending stress at the *midthickness* of the flange. This is less than the maximum value found in Table 9.1 and can be calculated by using Eq. (9.1-1). See the first example problem. The value of the coefficient β depends upon the ratio c^2/Rt, where c is the actual unsupported projecting width of the flange to either side of the web and R and t have the same meaning they did in the expressions for b' and q. Values of β may be found from the following table; they were taken from Ref. 10, where values of b' are also tabulated.

$c^2/Rt = 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.8
$\beta = 0$	0.297	0.580	0.836	1.056	1.238	1.382	1.577
$c^2/Rt = 1$	1.2	1.4	1.5	2	3	4	5
$\beta = 1.677$	1.721	1.732	1.732	1.707	1.671	1.680	1.700

Derivations of expressions for b'/b and for β are also found in Ref. 29. Small differences in the values given in various references are due to compensations for secondary effects. The values given here are conservative.

In a similar way, the radial components of the circumferential normal stresses distort thin tubular cross sections of curved beams. This distortion affects both the stresses and deformations and is discussed in the next section.

U-shaped members. A U-shaped member having a semicircular inner boundary and a rectangular outer boundary is sometimes used as a punch or riveter frame. Such a member can usually be analyzed as a curved beam having a concentric outer boundary, but when the back thickness is large, a more accurate analysis may be necessary. In Ref. 11 are presented the results of a photoelastic stress analysis of such members in which the effects of variations in the several dimensions were determined. See case 23, Table 17.1

Deflections. If a sharply curved beam is only a small portion of a larger structure, the contribution to deflection made by the curved portion can best be calculated by using the stresses at the inner and outer surfaces to calculate strains and the strains then used to determine the rotations of the plane sections. If the structure is made up primarily of a sharply curved beam or a combination of such beams, then refer to the next section.

9.2 Deflection of Curved Beams

Deflections of curved beams can generally be found most easily by applying an energy method such as Castigliano's second theorem. One such expression is given by Eq. (8.1-7). The proper expression to use for the complementary energy depends upon the degree of curvature in the beam.

Deflection of curved beams of large radius. If for a curved beam the radius of curvature is large enough such that Eqs. (8.1-1) and (8.1-2) are acceptable, i.e., the radius of curvature is greater than 10 times the depth, then the stress distribution across the depth of the beam is very nearly linear and the complementary energy of flexure is given with sufficient accuracy by Eq. (8.1-3). If, in addition, the angular span is great enough such that deformations due to axial stress from the normal force N and the shear stresses due to transverse shear V can be neglected, deflections can be obtained by applying Eqs. (8.1-3) and (8.1-7) and rotations by Eq. (8.1-8). The following example shows how this is done.

EXAMPLE

Figure 9.5 represents a slender uniform bar curved to form the quadrant of a circle; it is fixed at the lower end and at the upper end is loaded by a vertical force V, a horizontal force H, and a couple M_0. It is desired to find the vertical deflection δ_y, the horizontal deflection δ_x, and the rotation θ of the upper end.

Solution. According to Castigliano's second theorem, $\delta_y = \partial U/\partial V$, $\delta_x = \partial U/\partial H$, and $\theta = \partial U/\partial M_0$. Denoting the angular position of any section by x, it is evident that the moment there is $M = VR\sin x + HR(1-\cos x) + M_0$. Disregarding shear and axial stress, and replacing ds by $R\,dx$, we have [Eq. (8.1-3)]

$$U = U_f = \int_0^{\pi/2} \frac{[VR\sin x + HR(1-\cos x) + M_0]^2 R\,dx}{2EI}$$

Instead of integrating this and then carrying out the partial differentiations, we will differentiate first and then integrate, and for convenience suppress the constant term EI until all computations are completed. Thus

$$\begin{aligned}
\delta_y &= \frac{\partial U}{\partial V} \\
&= \int_0^{\pi/2} [VR\sin x + HR(1-\cos x) + M_0](R\sin x)R\,dx \\
&= VR^3(\tfrac{1}{2}x - \tfrac{1}{2}\sin x\cos x) - HR^3(\cos x + \tfrac{1}{2}\sin^2 x) - M_0R^2\cos x\Big|_0^{\pi/2} \\
&= \frac{(\pi/4)VR^3 + \frac{1}{2}HR^3 + M_0R^2}{EI}
\end{aligned}$$

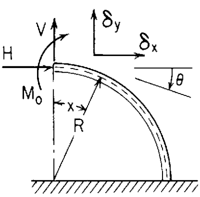


Figure 9.5

$$\begin{aligned}
\delta_x &= \frac{\partial U}{\partial H} \\
&= \int_0^{\pi/2} [VR\sin x + HR(1-\cos x) + M_0]R(1-\cos x)R\,dx \\
&= VR^3(-\cos x - \tfrac{1}{2}\sin^2 x) + HR^3(\tfrac{3}{2}x - 2\sin x + \tfrac{1}{2}\sin x\cos x) + M_0R^2(x-\sin x)\Big|_0^{\pi/2} \\
&= \frac{\frac{1}{2}VR^3 + (\frac{3}{4}\pi - 2)HR^3 + (\pi/2-1)M_0R^2}{EI} \\
\theta &= \frac{\partial U}{\partial M_0} \\
&= \int_0^{\pi/2} [VR\sin x + HR(1-\cos x) + M_0]R\,dx \\
&= -VR^2\cos x + HR^2(x-\sin x) + M_0Rx\Big|_0^{\pi/2} \\
&= \frac{VR^2 + (\pi/2-1)HR^2 + (\pi/2)M_0R}{EI}
\end{aligned}$$

The deflection produced by any one load or any combination of two loads is found by setting the other load or loads equal to zero; thus, V alone would produce $\delta_x = \frac{1}{2}VR^3/EI$, and M alone would produce $\delta_y = M_0R^2/EI$. In this example all results are positive, indicating that δ_x is in the direction of H, δ_y in the direction of V, and θ in the direction of M_0.

Distortion of tubular sections. In curved beams of thin tubular section, the distortion of the cross section produced by the radial components of the fiber stresses reduces both the strength and stiffness. If the beam curvature is not so sharp as to make Eqs. (8.1-1) and (8.1-4) inapplicable, the effect of this distortion of the section can be taken into account as follows.

In calculating deflection of curved beams of hollow circular section, replace I by KI, where

$$K = 1 - \frac{9}{10 + 12(tR/a^2)^2}$$

(Here $R =$ the radius of curvature of the beam axis, $a =$ the outer radius of tube section, and $t =$ the thickness of tube wall.) In calculating the maximum bending stress in curved beams of hollow circular section, use the formulas

$$\sigma_{\max} = \frac{Ma}{I}\frac{2}{3K\sqrt{3\beta}} \qquad \text{at } y = \frac{a}{\sqrt{3\beta}} \qquad \text{if } \frac{tR}{a^2} < 1.472$$

or

$$\sigma_{\max} = \frac{Ma}{I}\frac{1-\beta}{K} \qquad \text{at}\, y = a \qquad \text{if } \frac{tR}{a^2} > 1.472$$

where

$$\beta = \frac{6}{5 + 6(tR/a^2)^2}$$

and y is measured from the neutral axis. Torsional stresses and deflections are unchanged.

In calculating deflection or stress in curved beams of hollow square section and uniform wall thickness, replace I by

$$\frac{1 + 0.0270n}{1 + 0.0656n} I$$

where $n = b^4/R^2t^2$. (Here $R =$ the radius of curvature of the beam axis, $b =$ the length of the side of the square section, and $t =$ the thickness of the section wall.)

The preceding formulas for circular sections are from von Kármán (Ref. 4); the formulas for square sections are from Timoshenko (Ref. 5), who also gives formulas for rectangular sections.

Extensive analyses have been made for thin-walled pipe elbows with sharp curvatures for which the equations given above do not apply directly. Loadings may be *in-plane*, *out-of-plane*, or in various combinations (Ref. 8). Internal pressure increases and external pressure decreases pipe-bend stiffness. To determine ultimate load capacities of pipe bends or similar thin shells, elastic-plastic analyses, studies of the several modes of instability, and the stabilizing effects of flanges and the piping attached to the elbows are some of the many subjects presented in published works. Bushnell (Ref. 7) included an extensive list of references. Using numerical results from computer codes, graphs of stress indices and flexibility factors provide design data (Refs. 7, 19, and 34).

Deflection of curved beams of small radius. For a sharply curved beam, i.e., the radius of curvature is less than 10 times the depth, the stress distribution is not linear across the depth. The expression for the complementary energy of flexure is given by

$$U_f = \int \frac{M^2}{2AEeR} R\, dx = \int \frac{M^2}{2AEe} dx \tag{9.2-1}$$

where A is the cross-sectional area, E is the modulus of elasticity, and e is the distance from the centroidal axis to the neutral axis as given in Table 9.1. The differential change in angle dx is the same as is used in the previous example. See Fig. 9.1. Also keep clearly in mind that the bending in the plane of the curvature must be about a *principal* axis or the restraints described in the first paragraph of Sec. 9.1 must be present.

For all cross sections the value of the product AeR approaches the value of the moment of inertia I when the radius of curvature becomes greater than 10 times the depth. This is seen clearly in the following table where values of the ratio AeR/I are given for several sections and curvatures.

Case no.	Section	R/d			
		1	3	5	10
1	Solid rectangle	1.077	1.008	1.003	1.001
2	Solid circle	1.072	1.007	1.003	1.001
5	Triangle (base inward)	0.927	0.950	0.976	0.988
6	Triangle (base outward)	1.268	1.054	1.030	1.014

For curved beams of large radius the effect on deflections of the shear stresses due to V and the circumferential normal stresses due to N were small unless the length was small. For sharply curved beams the effects of these stresses must be considered. Only the effects of the radial stresses σ_r will be neglected. The expression for the complementary energy including all but the radial stresses is given by

$$U_f = \int \frac{M^2}{2AEe}\,dx + \int \frac{FV^2R}{2AG}\,dx + \int \frac{N^2R}{2AE}\,dx - \int \frac{MN}{AE}\,dx \qquad (9.2\text{-}2)$$

where all the quantities are defined in the notation at the top of Table 9.2.

The last term, hereafter referred to as the *coupling* term, involves the complementary energy developed from coupling the strains from the bending moment M and the normal force N. A positive bending moment M produces a negative strain at the position of the *centroidal* axis in a curved beam, and the resultant normal force N passes through the centroid. Reasons have been given for and against including the coupling term in attempts to improve the accuracy of calculated deformations (see Refs. 3 and 29). Ken Tepper, Ref. 30, called attention to the importance of the coupling term for sharply curved beams. The equations in Tables 9.2 and 9.3 have been modified and now include the effect of the coupling term. With this change, the formulas given in Tables 9.2 and 9.3 for the indeterminate reactions and for the deformations are no longer limited to thin rings and arches but can be used as well for thick rings and arches. As before, for thin rings and arches α and β can be set to zero with little error.

To summarize this discussion and its application to the formulas in Tables 9.2 and 9.3, one can place a given curved beam into one of three categories: a thin ring, a moderately thick ring, and a very thick or sharply curved ring. The boundaries between these categories depend upon the R/d ratio and the shape of the cross section. Reference to the preceding tabulation of the ratio AeR/I will be helpful.

For thin rings the effect of normal stress due to N and shear stress due to V can be neglected; i.e., set α and β equal to zero. For moderately thick rings and arches use the equations as they are given in Tables 9.2 and 9.3. For thick rings and arches replace the moment of inertia I with the product AeR in all equations including those for α and β. To illustrate the accuracy of this approach, the previous example problem will be repeated but for a thick ring of rectangular cross section. The rectangular cross section was chosen because a solution can be obtained by using the theory of elasticity with which to compare and evaluate the results.

EXAMPLE

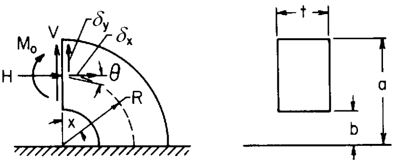
Figure 9.6 represents a thick uniform bar of rectangular cross section having a curved centroidal surface of radius R. It is fixed at the lower end, and the upper end is loaded by a vertical force V, a horizontal force H, and a couple M_o. It is desired to find the vertical deflection δ_y, the horizontal deflection δ_x, and the rotation θ of the upper end. Note that the deflections δ_y and δ_x are the values at the free end and at the radial position R at which the load H is applied.

First Solution. Again Castigliano's theorem will be used. First find the moment, shear, and axial force at the angular position x.

$$\begin{aligned} M_x &= VR\sin x + HR(1-\cos x) + M_o \\ V_x &= V\cos x + H\sin x \\ N_x &= -H\cos x + V\sin x \end{aligned}$$

Since the beam is to be treated as a thick beam the expression for complementary energy is given by

$$U + \int \frac{M_x^2}{2AEe}\,dx + \int \frac{FV_x^2R}{2AG}\,dx + \int \frac{N_x^2R}{2AE}\,dx - \int \frac{M_xN_x}{AE}\,dx$$



(a) (b)

Figure 9.6

The deflections can now be calculated

$$\delta_y = \frac{\partial U}{\partial V} = \int_0^{\pi/2} \frac{M_x}{AEe}(R\sin x)dx + \int_0^{\pi/2} \frac{FV_xR}{AG}(\cos x)dx + \int_0^{\pi/2} \frac{N_xR}{AE}(\sin x)dx$$

$$- \int_0^{\pi/2} \frac{M_x}{AE}(\sin x)dx - \int_0^{\pi/2} \frac{N_x}{AE}(R\sin x)dx$$

$$= \frac{(\pi/4)VR^3 + 0.5HR^3 + M_oR^2}{EAeR} + \frac{0.5R(\pi V/2 + H)[2F(1+\nu) - 1] - M_o}{AE}$$

$$\delta_x = \frac{\partial U}{\partial H} = \int_0^{\pi/2} \frac{M_xR}{AEe}(1 - \cos x)dx + \int_0^{\pi/2} \frac{FV_xR}{AG}(\sin x)dx + \int_0^{\pi/2} \frac{N_xR}{AE}(-\cos x)dx$$

$$- \int_0^{\pi/2} \frac{M_x}{AE}(-\cos x)dx - \int_0^{\pi/2} \frac{N_xR}{AE}(1 - \cos x)dx$$

$$= \frac{0.5VR^3 + (3\pi/4 - 2)HR^3 + (\pi/2 - 1)M_oR^2}{EAeR}$$

$$+ \frac{0.5VR[2F(1+\nu) - 1] + (\pi/4)HR[2F(1+\nu) + 8/\pi - 1] + M_o}{EA}$$

$$\theta = \frac{\partial U}{\partial M_o} = \int_0^{\pi/2} \frac{M_x}{AEe}(1)dx + \int_0^{\pi/2} \frac{FV_xR}{AG}(0)dx + \int_0^{\pi/2} \frac{N_xR}{AE}(0)dx - \int_0^{\pi/2} \frac{M_x}{AE}(0)dx$$

$$- \int_0^{\pi/2} \frac{N_x}{AE}(1)dx$$

$$= \frac{VR^2 + (\pi/2 - 1)HR^2 + (\pi/2)M_oR}{EAeR} + \frac{H - V}{AE}$$

There is no need to reduce these expressions further in order to make a numerical calculation, but it is of interest here to compare to the solutions in the previous example. Therefore, let $\alpha = e/R$ and $\beta = FEe/GR = 2F(1+\nu)/R$ as defined previously

$$\delta_y = \frac{(\pi/4)VR^3(1 - \alpha + \beta) + 0.5HR^3(1 - \alpha + \beta) + M_oR^2(1 - \alpha)}{EAeR}$$

$$\delta_x = \frac{0.5VR^3(1 - \alpha + \beta) + HR^3[(3\pi/4 - 2) + (2 - \pi/4)\alpha + (\pi/4)\beta]}{EAeR}$$

$$+ \frac{M_oR^2(\pi/2 - 1 + \alpha)}{EAeR}$$

$$\theta = \frac{VR^2(1 - \alpha) + HR^2(\pi/2 - 1 + \alpha) + (\pi/2)M_oR}{EAeR}$$

Up to this point in the derivation, the cross section has not been specified. For a rectangular cross section having an outer radius a and an inner radius b and of thickness t normal to the surface shown in Fig. 9.6(*b*), the following substitutions can be made in the deformation equations. Let $\nu = 0.3$.

$$R = \frac{a + b}{2}, \qquad A = (a - b)t, \qquad F = 1.2 \quad \text{(see Sec. 8.10)}$$

$$\alpha = \frac{e}{R} = 1 - \frac{2(a - b)}{(a + b)\ln(a/b)}, \qquad \beta = 3.12\alpha$$

In the following table the value of a/b is varied from 1.1, where $R/d = 10.5$, a thin beam, to $a/b = 5.0$, where $R/d = 0.75$, a very thick beam. Three sets of numerical values are compared. The first set consists of the three deformations δ_y, δ_x, and θ evaluated from the equations just derived and due to the vertical load V. The second set consists of the same deformations due to the same loading but evaluated by applying the equations for a thin curved beam from the first example. The third set consists of the same deformations due to the same loading but evaluated by applying the theory of elasticity. See Ref. 2. The abbreviation MM in parentheses identifies the values from the *mechanics-of-materials* solutions and the abbreviation EL similarly identifies those found from the *theory of elasticity.*

		From thick-beam theory			From thin-beam theory		
a/b	R/d	$\frac{\delta_y(\text{MM})}{\delta_y(\text{EL})}$	$\frac{\delta_x(\text{MM})}{\delta_x(\text{EL})}$	$\frac{\theta(\text{MM})}{\theta(\text{EL})}$	$\frac{\delta_y(\text{MM})}{\delta_y(\text{EL})}$	$\frac{\delta_x(\text{MM})}{\delta_x(\text{EL})}$	$\frac{\theta(\text{MM})}{\theta(\text{EL})}$
1.1	10.5	0.9996	0.9990	0.9999	0.9986	0.9980	1.0012
1.3	3.83	0.9974	0.9925	0.9991	0.9900	0.9852	1.0094
1.5	2.50	0.9944	0.9836	0.9976	0.9773	0.9967	1.0223
1.8	1.75	0.9903	0.9703	0.9944	0.9564	0.9371	1.0462
2.0	1.50	0.9884	0.9630	0.9916	0.9431	0.9189	1.0635
3.0	1.00	0.9900	0.9485	0.9729	0.8958	0.8583	1.1513
4.0	0.83	1.0083	0.9575	0.9511	0.8749	0.8345	1.2304
5.0	0.75	1.0230	0.9763	0.9298	0.8687	0.8290	1.2997

If reasonable errors can be tolerated, the strength-of-materials solutions are very acceptable when proper recognition of thick and thin beams is given.

Second Solution. Table 9.3 is designed to enable one to take any angular span 2θ and any single load or combination of loads and find the necessary indeterminate reactions and the desired deflections. To demonstrate this use of Table 9.3 in this example the deflection δ_x will be found due to a load H. Use of case 12d, with load terms from case 5d and with $\theta = \pi/4$ and $\phi = \pi/4$. Both load terms LF_H and LF_V are needed since the desired deflection δ_x is not in the direction of either of the deflections given in the table. Let $c = m = s = n = 0.7071$.

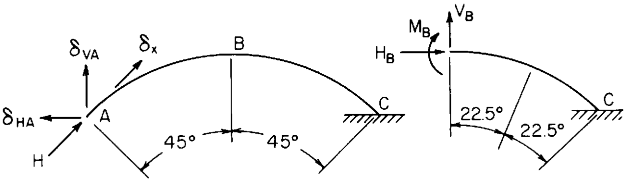
$$LF_H = H\left\{\frac{\pi}{2}0.7071 + \frac{k_1}{2}\left[\frac{\pi}{2}0.7071 - 0.7071^3(2)\right] - k_2 2(0.7071)\right\}$$

$$LF_V = H\left\{-\frac{\pi}{2}0.7071 - \frac{k_1}{2}\left[\frac{\pi}{2}0.7071 + 0.7071^3(2)\right] + k_2 4(0.7071^3)\right\}$$

$$\delta_x = (\delta_{VA} - \delta_{HA})0.7071 = \frac{-R^3}{EAeR}(LF_V - LF_H)0.7071$$

$$= \frac{-R^3 H}{EAeR}\left(-\frac{\pi}{2} - \frac{k_1}{2}\frac{\pi}{2} + 2k_2\right) = \frac{R^3 H}{EAeR}\left[\frac{3\pi}{4} - 2 + \left(2 - \frac{\pi}{4}\right)\alpha + \frac{\pi}{4}\beta\right]$$

This expression for δ_x is the same as the one derived directly from Castigliano's theorem. For angular spans of 90 or 180° the direct derivation is not difficult, but for odd-angle spans the use of the equations in Table 9.3 is recommended.



(a) (b)

Figure 9.7

The use of the equations in Table 9.3 is also recommended when deflections are desired at positions other than the load point. For example, assume the deflections of the midspan of this arch are desired when it is loaded with the end load H as shown in Fig. 9.7(a). To do this, isolate the span from B to C and find the loads H_B, V_B, and M_B which act at point B. This gives $H_B = V_B = 0.7071H$ and $M_B = HR(1 - 0.7071)$. Now, superpose cases 12c, 12d, and 12n using these loads and $\theta = \phi = \pi/8$. In a problem with neither end fixed, a rigid-body motion may have to be superposed to satisfy the boundary conditions.

Deflection of curved beams of variable cross section and/or radius. None of the tabulated formulas applies when either the cross section or the radius of curvature varies along the span. The use of Eqs. (9.2-1) and (9.2-2), or of comparable expressions for thin curved beams, with numerical integration carried out for a finite number of elements along the span provides an effective means of solving these problems. This is best shown by example.

EXAMPLE

A rectangular beam of constant thickness and a depth varying linearly along the length is bent such that the centroidal surface follows the curve $x = 0.25y^2$ as shown in Fig. 9.8. The vertical deflection at the loaded end is desired. To keep the use of specific dimensions to a minimum let the depth of the curved beam at the fixed end = 1.0, the thickness = 0.5, and the horizontal location of the load $P = 1.0$. The beam will be subdivided into eight segments, each spanning 0.25 units in the y direction. Normally a constant length along the span is used, but using constant Δy gives shorter spans where moments are larger and curvatures are sharper. The numerical calculations are also easier. Use will be made of the following expressions in order to provide the tabulated

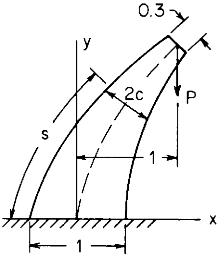


Figure 9.8

information from which the needed summation can be found. Note that y_i and x_i are used here as the y and x positions of the midlength of each segment

$$x = 0.25y^2, \qquad \frac{dx}{dy} = 0.5y, \qquad \frac{d^2x}{d^2y} = 0.5, \qquad \Delta l = \Delta y(1 + x_i)^{1/2}$$

$$R = \frac{[1 + (dx/dy)^2]^{3/2}}{d^2x/d^2y}$$

$$\frac{e}{c} = \frac{R}{c} - \frac{2}{\ln[(R/c + 1)/(R/c - 1)]} \qquad \text{for } \frac{R}{2c} < 8$$

[see Eq. (9.1-1) and case 1 of Table 9.1] or

$$\frac{e}{c} = \frac{I_c}{RAc} = \frac{t(2c)^3}{12(Rt2c^2)} = \frac{c}{3R} \qquad \text{for } \frac{R}{2c} > 8$$

[see Eq. (9.1-3)].

The desired vertical deflection of the loaded end can be determined from Castigliano's theorem, using Eq. (9.2-2) for U_f in summation form rather than integral form. This reduces to

$$\delta = \frac{\delta U}{\delta P} = \frac{P}{E}\sum\left[\frac{(M/P)^2}{eR} + F\left(\frac{V}{P}\right)^2 2(1+\nu) + \left(\frac{N}{P}\right)^2 - 2\frac{M}{P}\frac{N}{P}\right]\frac{\Delta l}{A}$$

$$= \frac{P}{E}\sum\frac{\Delta l}{A}[B]$$

where $[B]$ and $[B]\Delta l/A$ are the last two columns in the following table. The internal forces and moments can be determined from equilibrium equations as

$$\frac{M}{P} = -(1 - x_i), \quad \theta_i = \tan^{-1}\frac{dx}{dy}, \quad V = P\sin\theta_i, \quad \text{and} \quad N = -P\cos\theta_i$$

In the evaluation of the above equations for this problem, $F = 1.2$ and $\nu = 0.3$. In the table below one must fill in the first five columns in order to find the total length of the beam before the midsegment depth 2c can be found and the table completed.

Element no.	y_i	x_i	R	Δl	c	R/c
1	0.125	0.004	2.012	0.251	0.481	4.183
2	0.375	0.035	2.106	0.254	0.442	4.761
3	0.625	0.098	2.300	0.262	0.403	5.707
4	0.875	0.191	2.601	0.273	0.362	7.180
5	1.125	0.316	3.020	0.287	0.320	9.451
6	1.375	0.473	3.574	0.303	0.275	13.019
7	1.625	0.660	4.278	0.322	0.227	18.860
8	1.875	0.879	5.151	0.343	0.176	29.243
				2.295		

Element no.	e/c	M/P	V/P	N/P	$[B]$	$[B]\frac{\Delta l}{A}$
1	0.0809	−0.996	0.062	−0.998	11.695	6.092
2	0.0709	−0.965	0.184	−0.983	13.269	7.627
3	0.0589	−0.902	0.298	−0.954	14.370	9.431
4	0.0467	−0.809	0.401	−0.916	14.737	11.101
5	0.0354	−0.684	0.490	−0.872	14.007	12.569
6	0.0256	−0.527	0.567	−0.824	11.856	13.105
7	0.0177	−0.340	0.631	−0.776	8.049	11.431
8	0.0114	−0.121	0.684	−0.730	3.232	6.290
						77.555

Therefore, the deflection at the load and in the direction of the load is $77.56P/E$ in whatever units are chosen as long as the depth at the fixed end is unity. If one maintains the same length-to-depth ratio and the same shape, the deflection can be expressed as $\delta = 77.56P/(E2t_o)$, where t_o is the constant thickness of the beam.

Michael Plesha (Ref. 33) provided a finite-element solution for this configuration and obtained for the load point a vertically downward deflection of 72.4 units and a horizontal deflection of 88.3 units. The 22 elements he used were nine-node, quadratic displacement, Lagrange elements. The reader is invited to apply a horizontal dummy load and verify the horizontal deflection.

9.3 Circular Rings and Arches

In large pipelines, tanks, aircraft, and submarines the circular ring is an important structural element, and for correct design it is often necessary to calculate the stresses and deflections produced in such a ring under various conditions of loading and support. The circular arch of uniform section is often employed in buildings, bridges, and machinery.

Rings. A closed circular ring may be regarded as a *statically indeterminate beam* and analyzed as such by the use of Castigliano's second theorem. In Table 9.2 are given formulas thus derived for the bending moments, tensions, shears, horizontal and vertical deflections, and rotations of the load point in the plane of the ring for various loads and supports. By superposition, these formulas can be combined so as to cover almost any condition of loading and support likely to occur.

The ring formulas are based on the following assumptions: (1) The ring is of uniform cross section and has symmetry about the plane of curvature. An exception to this requirement of symmetry can be made if moment restraints are provided to prevent rotation of each cross section out of its plane of curvature. Lacking the plane of symmetry and any external constraints, out-of-plane deformations will accompany in-plane loading. Meck, in Ref. 21, derives expressions concerning the coupling of in-plane and out-of-plane deformations of circular rings of arbitrary compact cross section and resulting instabilities. (2) All loadings are applied at the radial position of the centroid of the cross section. For thin rings this is of little concern, but for radially thick rings a concentrated load acting in other than a radial direction and not at the centroidal radius must be replaced by a statically equivalent load at the centroidal radius and a couple. For case 15, where the loading is due to gravity or a constant linear acceleration, and for case 21, where the loading is due to rotation around an axis normal to the plane of the ring, the proper distribution of loading through the cross section is accounted for in the formulas. (3) It is nowhere stressed beyond the elastic limit. (4) It is not so severely deformed as to lose its essentially circular shape. (5) Its deflection is due primarily to bending, but for thicker rings the deflections due to deformations caused by axial tension or compression in the ring and/or by transverse shear stresses in the ring may be included. To include these effects, we can evaluate first the coefficients α and β, the *axial stress deformation factor*, and the *transverse shear deformation factor*, and then the constants k_1 and k_2. Such corrections are more often necessary when composite or sandwich construction is employed. If no axial or shear stress corrections are desired, α and β are set equal to zero and the values of k are set equal to unity. (6) In the case of pipes acting as beams between widely spaced supports, the distribution of shear stress across the section of the pipe is in accordance with Eq. (8.1-2), and the direction of the resultant shear stress at any point of the cross section is tangential.

Note carefully the deformations given regarding the point or points of loading as compared with the deformations of the horizontal and vertical diameters. For many of the cases listed, the numerical values of load and deflection coefficients have been given for several positions of the loading. These coefficients do not include the effect of axial and shear deformation.

No account has been taken in Table 9.2 of the effect of radial stresses in the vicinity of the concentrated loads. These stresses and the local deformations they create can have a significant effect on overall ring deformations and peak stresses. In case 1 a reference is made to Sec. 14.3 in which thick-walled rollers or rings are loaded on the outer ends of a diameter. The stresses and deflections given here are different from those predicted by the equations in case 1. If a concentrated load is used only for purposes of superposition, as is often the case, there is no cause for concern, but if an actual applied load is concentrated over a small region and the ring is sharply curved with thick walls, then one must be aware of the possible errors.

EXAMPLES

1. A pipe with a diameter of 13 ft and thickness of $\frac{1}{2}$ in is supported at intervals of 44 ft by rings, each ring being supported at the extremities of its horizontal diameter by vertical reactions acting at the centroids of the ring sections. It is required to determine the bending moments in a ring at the bottom, sides, and top, and the maximum bending moment when the pipe is filled with water.

Solution. We use the formulas for cases 4 and 20 of Table 9.2. Taking the weight of the water as 62.4 lb/ft^3 and the weight of the shell as 20.4 lb/ft^2, the total weight W of 44 ft of pipe carried by one ring is found to be 401,100 lb. Therefore, for case 20, $W = 401{,}100$ lb; and for case 4, $W = 250{,}550$ lb and $\theta = \pi/2$. Assume a thin ring, $\alpha = \beta = 0$.

At bottom:

$$\begin{aligned} M = M_C &= 0.2387(401{,}100)(6.5)(12) - 0.50(200{,}550)(78) \\ &= 7.468(10^6) - 7.822(10^6) = -354{,}000\ \text{lb-in} \end{aligned}$$

At top:

$$\begin{aligned} M &= M_A = 0.0796(401{,}100)(78) - 0.1366(200{,}550)(78) = 354{,}000\ \text{lb-in} \\ N &= N_A = 0.2387(401{,}100) - 0.3183(200{,}500) = 31{,}900\ \text{lb} \\ V &= V_A = 0 \end{aligned}$$

At sides:

$$M = M_A - N_A R(1-u) + V_A Rz + LT_M$$

where for $x = \pi/2$, $u = 0$, $z = 1$, and $LT_M = (WR/\pi)(1 - u - xz/2) = [401{,}100(78)/\pi]\ (1 - \pi/4) = 2.137(10^6)$ for case 20, and $LT_M = 0$ for case 4 since $z - s = 0$. Therefore

$$M = 354{,}000 - 31{,}900(78)(1-0) + 0 + 2.137(10^6) = 2800\ \text{lb-in}$$

The value of 2800 lb-in is due to the small differences in large numbers used in the superposition. An exact solution would give zero for this value. It is apparent that at least four digits must be carried.

To determine the location of maximum bending moment let $0 < x < \pi/2$ and examine the expression for M:

$$\begin{aligned} M &= M_A - N_A R(1 - \cos x) + \frac{WR}{\pi}\left(1 - \cos x - \frac{x \sin x}{2}\right) \\ \frac{dM}{dx} &= -N_A R \sin x + \frac{WR}{\pi}\sin x - \frac{WR}{2\pi}\sin x - \frac{WRx}{2\pi}\cos x \\ &= 31{,}950R \sin x - 63{,}800Rx \cos x \end{aligned}$$

At $x = x_1$, let $dM/dx = 0$ or $\sin x_1 = 2x_1 \cos x_1$, which yields $x_1 = 66.8^\circ(1.166\,\text{rad})$. At $x = x_1 = 66.8^\circ$,

$$M = 354{,}00 - 31{,}900(78)(1 - 0.394) + \frac{401{,}100(78)}{\pi}\left[1 - 0.394 - \frac{1.166(0.919)}{2}\right]$$
$$= -455{,}000\,\text{lb-in} \quad \text{(max negative moment)}$$

Similarly, at $x = 113.2^\circ$, $M = 455{,}000\,\text{lb-in}$ (max positive moment).

By applying the supporting reactions outside the center line of the ring at a distance a from the centroid of the section, side couples that are each equal to $Wa/2$ would be introduced. The effect of these, found by the formulas for case 3, would be to reduce the maximum moments, and it can be shown that the optimum condition obtains when $a = 0.04R$.

2. The pipe of Example 1 rests on soft ground, with which it is in contact over 150° of its circumference at the bottom. The supporting pressure of the soil may be assumed to be radial and uniform. It is required to determine the bending moment at the top and bottom and at the surface of the soil. Also the bending stresses at these locations and the change in the horizontal diameter must be determined.

Solution. A section of pipe 1 in long is considered. The loading may be considered as a combination of cases 12, 15, and 16. Owing to the weight of the pipe (case 15, $w = 0.1416\,\text{lb/in}$), and letting $K_T = k_1 = k_2 = 1$, and $\alpha = \beta = 0$,

$$M_A = \frac{0.1416(78)^2}{2} = 430\,\text{lb-in}$$
$$N_A = \frac{0.1416(78)}{2} = 5.52\,\text{lb}$$
$$V_A = 0$$

and at $x = 180 - \frac{150}{2} = 105^\circ = 1.833$ rad,

$$LT_M = -0.1416(78^2)[1.833(0.966) - 0.259 - 1] = -440\,\text{lb-in}$$

Therefore

$$M_{105^\circ} = 430 - 5.52(78)(1 + 0.259) - 440 = -552\,\text{lb-in}$$
$$M_C = 1.5(0.1416)(78) = 1292\,\text{lb-in}$$

Owing to the weight of contained water (case 16, $\rho = 0.0361\,\text{lb/in}^3$),

$$M_A = \frac{0.0361(78^3)}{4} = 4283\,\text{lb-in/in}$$
$$N_A = \frac{0.0361(78^2)(3)}{4} = 164.7\,\text{lb/in}$$
$$V_A = 0$$

and at $x = 105^\circ$,

$$LT_M = 0.0361(78^3)\left[1 + 0.259 - \frac{1.833(0.966)}{2}\right] = 6400\,\text{lb-in/in}$$

Therefore

$$M_{105^\circ} = 4283 - 164.7(78)(1 + 0.259) + 6400 = -5490\,\text{lb-in/in}$$

$$M_C = \frac{0.0361(78^3)(3)}{4} = 12{,}850\,\text{lb-in/in}$$

Owing to earth pressure and the reversed reaction (case 12, $\theta = 105^\circ$),

$$2wR\sin\theta = 2\pi R(0.1416) + 0.0361\pi R^2 = 759\,\text{lb} \quad (w = 5.04\,\text{lb/in})$$

$$\begin{aligned} M_A &= \frac{-5.04(78^2)}{\pi}[0.966 + (\pi - 1.833)(-0.259) - 1(\pi - 1.833 - 0.966)] \\ &= -2777\,\text{in-lb} \\ N_A &= \frac{-5.04(78)}{\pi}[0.966 + (\pi - 1.833)(-0.259)] = -78.5\,\text{lb} \\ V_A &= 0 \\ LT_M &= 0 \\ M_{105^\circ} &= -2777 + 78.5(78)(1.259) = 4930\,\text{lb-in} \\ M_C &= -5.04(78^2)\frac{1.833(1 - 0.259)}{\pi} = -13{,}260\,\text{lb-in} \end{aligned}$$

Therefore, for the 1 in section of pipe

$$\begin{aligned} M_A &= 430 + 4283 - 2777 = 1936\,\text{lb-in} \\ \sigma_A &= \frac{6M_A}{t^2} = 46{,}500\,\text{lb/in}^2 \\ M_{105^\circ} &= -552 - 5490 + 4930 = -1112\,\text{lb-in} \\ \sigma_{105^\circ} &= 26{,}700\,\text{lb/in}^2 \\ M_C &= 1292 + 12{,}850 - 13{,}260 = 882\,\text{lb-in} \\ \sigma_C &= 21{,}200\,\text{lb/in}^2 \end{aligned}$$

The change in the horizontal diameter is found similarly by superimposing the three cases. For E use $30(10^6)/(1 - 0.285^2) = 32.65(10^6)\,\text{lb/in}^2$, since a plate is being bent instead of a narrow beam (see page 169). For I use the moment of inertia of a 1-in-wide piece, 0.5 in thick:

$$I = \tfrac{1}{12}(1)(0.5^3) = 0.0104\,\text{in}^4, \qquad EI = 340{,}000\,\text{lb-in}^2$$

From case 12:

$$\Delta D_H = \frac{-5.04(78^4)}{340{,}000}\left[\frac{(\pi - 1.833)(-0.259) + 0.966}{2} - \frac{2}{\pi}(\pi - 1.833 - 0.966)\right]$$

$$= \frac{-5.04(78)^4}{340{,}000}(0.0954) = -52.37\,\text{in}$$

From case 15:

$$\Delta D_H = \frac{0.4292(0.1416)78^4}{340{,}000} = 6.616\,\text{in}$$

From case 16:

$$\Delta D_H = \frac{0.2146(0.0361)78^5}{340{,}000} = 65.79\,\text{in}$$

The total change in the horizontal diameter is 20 in. It must be understood at this point that the anwers are somewhat in error since this large a deflection does violate the assumption that the loaded ring is very nearly circular. This was expected when the stresses were found to be so large in such a thin pipe.

Arches. Table 9.3 gives formulas for end reactions and end deformations for circular arches of constant radius of curvature and constant cross section under 18 different loadings and with 14 combinations of end conditions. The corrections for axial stress and transverse shear are accomplished as they were in Table 9.2 by the use of the constants α and β. Once the indeterminate reactions are known, the bending moments, axial loads, and transverse shear forces can be found from equilibrium equations. If deformations are desired for points away from the ends, the unit-load method [Eq. (8.1-6)] can be used or the arch can be divided at the position where the deformations are desired and either portion analyzed again by the formulas in Table 9.3. Several examples illustrate this last approach. Note that in many instances the answer depends upon the difference of similar large terms, and so appropriate attention to accuracy must be given.

EXAMPLES

1. A $WT4 \times 6.5$ structural steel T-beam is formed in the plane of its web into a circular arch of 50-in radius spanning a total angle of 120°. The right end is fixed, and the left end has a pin which is constrained to follow a horizontal slot in the support. The load is applied through a vertical bar welded to the beam, as shown in Fig. 9.9. Calculate the movement of the pin at the left end, the maximum bending stress, and the rotation of the bar at the point of attachment to the arch.

Solution. The following material and cross-sectional properties may be used for this beam. $E = 30(10^6)\,\text{lb/in}^2$, $G = 12(10^6)\,\text{lb/in}^2$, $I_x = 2.90\,\text{in}^4$, $A = 1.92\,\text{in}^2$, flange thickness $= 0.254$ in, and web thickness $= 0.230$ in. The loading on the

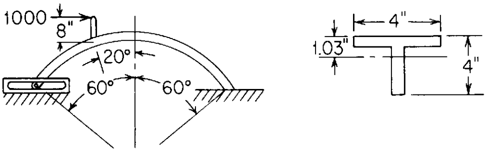


Figure 9.9

arch can be replaced by a concentrated moment of 8000 lb-in and a horizontal force of 1000 lb at a position indicated by $\phi = 20^\circ$ (0.349 rad). $R = 50$ in and $\theta = 60^\circ$ (1.047 rad). For these loads and boundary conditions, cases 9b and 9n of Table 9.3 can be used.

Since the radius of 50 in is only a little more than 10 times the depth of 4 in, corrections for axial load and shear will be considered. The axial-stress deformation factor $\alpha = I/AR^2 = 2.9/1.92(50^2) = 0.0006$. The transverse-shear deformation factor $\beta = FEI/GAR^2$, where F will be approximated here by using $F = 1$ and $A =$ web area $= 4(0.23) = 0.92$. This gives $\beta = 1(30)(10^6)(2.90)/12(10^6)(0.92)(50^2) = 0.003$. The small values of α and β indicate that bending governs the deformations, and so the effect of axial load and transverse shear will be neglected. Note that $s = \sin 60^\circ$, $c = \cos 60^\circ$, $n = \sin 20^\circ$, and $m = \cos 20^\circ$.

For case 9b,

$$\begin{aligned} LF_H &= 1000\left[\frac{1.0472 + 0.3491}{2}(1 + 2\cos 20^\circ \cos 60^\circ) - \frac{\sin 60^\circ \cos 60^\circ}{2}\right. \\ &\quad \left. - \frac{\sin 20^\circ \cos 20^\circ}{2} - \cos 20^\circ \sin 60^\circ - \sin 20^\circ \cos 60^\circ\right] \\ &= 1000(-0.00785) = -7.85\,\text{lb} \end{aligned}$$

Similarly,

$$LF_V = 1000(-0.1867) = -186.7\,\text{lb} \quad \text{and} \quad LF_M = 1000(-0.1040) = -104.0\,\text{lb}$$

For the case 9n,

$$\begin{aligned} LF_H &= \frac{8000}{50}(-0.5099) = -81.59\,\text{lb} \\ LF_V &= \frac{8000}{50}(-1.6489) = -263.8\,\text{lb} \\ LF_M &= \frac{8000}{50}(-1.396) = -223.4\,\text{lb} \end{aligned}$$

Also,

$$\begin{aligned} B_{VV} &= 1.0472 + 2(1.0472)\sin^2 60^\circ - \sin 60^\circ \cos 60^\circ = 2.1850\,\text{lb} \\ B_{HV} &= 0.5931\,\text{lb} \\ B_{MV} &= 1.8138\,\text{lb} \end{aligned}$$

Therefore,

$$V_A = -\frac{186.7}{2.1850} - \frac{263.8}{2.1850} = -85.47 - 120.74 = -206.2\,\text{lb}$$

$$\delta_{HA} = \frac{50^3}{30(10^6)(2.9)}[0.5931(-206.2) + 7.85 + 81.59] = -0.0472\,\text{in}$$

The expression for the bending moment can now be obtained by an equilibrium equation for a position located by an angle x measured from the left end:

$$M_x = V_A R[\sin\theta - \sin(\theta - x)] + 8000\langle x - (\theta - \phi)\rangle^0$$
$$- 1000R[\cos(\theta - x) - \cos\phi]\langle x - (\theta - \phi)\rangle^0$$

At $x = 40°-$ $\quad M_x = -206.2(50)[\sin 60° - \sin(60° - 40°)] = -5403\,\text{lb-in}$

At $x = 40°+$ $\quad M_x = -5403 + 8000 = 2597\,\text{lb-in}$

At $x = 60°$ $\quad M_x = -206.2(50)(0.866) + 8000 - 1000(50)(1 - 0.940)$
$\quad = -3944\,\text{lb-in}$

At $x = 120°$ $\quad M_x = 12{,}130\,\text{lb-in}$

The maximum bending stress is therefore

$$\sigma = \frac{12{,}130(4 - 1.03)}{2.9} = 12{,}420\,\text{lb/in}^2$$

To obtain the rotation of the arch at the point of attachment of the bar, we first calculate the loads on the portion to the right of the loading and then establish an equivalent symmetric arch (see Fig. 9.10). Now from cases 12a, 12b, and 12n, where $\theta = \phi = 40°(0.698\,\text{rad})$, we can determine the load terms:

For case 12a $\quad LF_M = -148[2(0.698)(0.643)] = -133\,\text{lb}$

For case 12b $\quad LF_M = 1010[0.643 + 0.643 - 2(0.698)(0.766)] = 218\,\text{lb}$

For case 12n $\quad LF_M = \frac{2597}{50}(-0.698 - 0.698) = -72.5\,\text{lb}$

Therefore, the rotation at the load is

$$\psi_A = \frac{-50^2}{30(10^6)(2.9)}(-133 + 218 - 72.5) = -0.00036\,\text{rad}$$

We would not expect the rotation to be in the opposite direction to the applied moment, but a careful examination of the problem shows that the point on the arch where the bar is fastened moves to the right 0.0128 in. Therefore, the net motion in the direction of the 1000-lb load on the end of the 8-in bar is 0.0099 in, and so the applied load does indeed do positive work on the system.

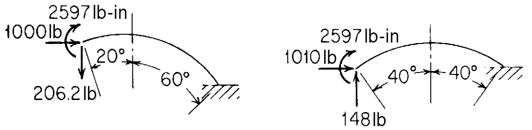


Figure 9.10

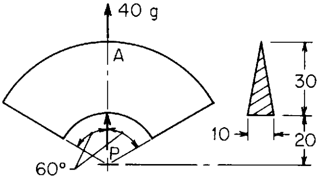


Figure 9.11

2. The deep circular arch of titanium alloy has a triangular cross section and spans 120° as shown in Fig. 9.11. It is driven by the central load P to produce an acceleration of $40g$. The tensile stress at A and the deformations at the extreme ends are required. All dimensions given and used in the formulas are in centimeters.

Solution. This is a statically determinate problem, so the use of information from Table 9.3 is needed only to obtain the deformations. Superposing the central load and the forces needed to produce the acceleration on the total span can be accomplished readily by using cases 3a and 3h. This solution, however, will provide only the horizontal and rotational deformations of the ends. Using the symmetry one can also superpose the loadings from cases 12h and 12i on the left half of the arch and obtain all three deformations. Performing both calculations provides a useful check. All dimensions are given in centimeters and used with expressions from Table 9.1, case 5, to obtain the needed factors for this section. Thus, $b = 10$, $d = 30$, $A = 150$, $c = 10$, $R = 30$, $R/c = 3$, $e/c = 0.155$, $e = 1.55$ and for the peak stresses, $k_i = 1.368$ and $k_o = 0.697$. The titanium alloy has a modulus of elasticity of 117 GPa [$11.7(10^6)\text{N/cm}^2$], a Poisson's ratio of 0.33, and a mass density of $4470\,\text{kg/m}^3$, or $0.00447\,\text{kg/cm}^3$. One g of acceleration is $9.81\,\text{m/s}^2$, and 1 cm of arc length at the centroidal radius of 30 cm will have a volume of $150\,\text{cm}^3$ and a mass of 0.6705 kg. This gives a loading parallel to the driving force P of $0.6705(40)(9.81) = 263$ N/cm of centroidal arc length. Since this is a very sharply curved beam, $R/d = 1$, one must recognize that the resultant load of 263 N/cm does not act through the centroid of the cross-sectional area but instead acts through the mass center of the differential length. The radius to this point is given as R_{cg} and is found from the expression $R_{cg}/R = 1 + I/AR^2$, where I is the area moment of inertia about the centroidal axis of the cross section. Therefore, $R_{cg}/R = 1 + (bd^3/36)/(bd/2)R^2 = 1.056$. Again due to the sharp curvature the axial- and shear-stress contributions to deformation must be considered. From the introduction to Table 9.3 we find that $\alpha = h/R = 0.0517$ and $\beta = 2F(1+\nu)h/R = 0.1650$, where $F = 1.2$ for a triangular cross section as given in Sec. 8.10. Therefore, $k_1 = 1 - \alpha + \beta = 1.1133$, and $k_2 = 1 - \alpha = 0.9483$.

For a first solution use the full span and superpose cases 3a and 3h. To obtain the load terms LP_H and LP_M use cases 1a and 1h.

For case 1a, $W = -263(30)(2\pi/3) = -16{,}525$ N, $\theta = 60°$, $\phi = 0°$, $s = 0.866$, $c = 0.500$, $n = 0$, and $m = 1.000$.

$$LP_H = -16{,}525\left[\frac{\pi}{3}(0.866)(0.5) - 0 + \frac{1.1133}{2}(0.5^2 - 1.0^2)\right.$$
$$\left. + 0.9483(0.5)(0.5 - 1.0)\right]$$
$$= -16{,}525(-0.2011) = 3323\,\text{N}$$

Similarly, $LP_M = 3575$ N.

For case 1*h*, $w = 263\,\text{N/cm}$, $R = 30$, $R_{cg}/R = 1.056$, $\theta = 60^\circ$, $s = 0.866$, and $c = 0.5000$.

$$LP_H = 263(30)(-0.2365) = -1866\,\text{N}$$

and

$$LP_M = 263(30)(-0.2634) = -2078\,\text{N}$$

Returning now to case 3 where M_A and H_A are zero, one finds that $V_A = 0$ by superposing the loadings. To obtain δ_{HA} and ψ_A we superpose cases a and h and substitute *AhR* for *I* because of the sharp curvature

$$\delta_{HA1} = -30^3 \frac{3323 - 1866}{11.7(10^6)(150)(1.55)(30)} = -482(10^{-6})\,\text{cm}$$

$$\psi_{A1} = -30^2 \frac{3575 - 2078}{8.161(10^{10})} = -16.5(10^{-6})\,\text{rad}$$

Now for the second and more complete solution, use will be made of cases 12h and 12i. The left half spans 60°, so $\theta = 30^\circ$, $s = 0.5000$, and $c = 0.8660$. In this solution the central symmetry axis of the left half being used is inclined at 30° to the gravitational loading of 263 N/cm. Therefore, for case 5h, $w = 263\cos 30^\circ = 227.8\,\text{N/cm}$

$$\begin{aligned} LF_H = 227.8(30)\Bigg\{&\frac{1.1133}{2}\left[2\left(\frac{\pi}{6}\right)(0.866^2) - \frac{\pi}{6} - 0.5(0.866)\right] \\ &+ 0.9483(1.056+1)\left[\frac{\pi}{6} - 0.5(0.866)\right] + 1.056(2)(0.866)\left(\frac{\pi}{6}0.866 - 0.5\right)\Bigg\} \\ = 227.8(30)&(-0.00383) = -26.2\,\text{N} \end{aligned}$$

Similarly

$$LF_V = 227.8(30)(0.2209) = 1510\,\text{N} \quad \text{and} \quad LF_M = 227.8(30)(0.01867) = 1276\,\text{N}$$

For case 5i, $w = -263\sin 30^\circ = -131.5\,\text{N/cm}$ and again $\theta = 30^\circ$

$$LF_H = -131.5(30)(0.0310) = -122.3\,\text{N}$$

$$LF_V = -131.5(30)(-0.05185) = 204.5\,\text{N}$$

$$LF_M = -131.5(30)(-0.05639) = 222.5\,\text{N}$$

Using case 12 and superposition of the loadings gives

$$\delta_{HA2} = -30^3 \frac{-26.2 - 122.3}{8.161(10^{10})} = 49.1(10^{-6})\,\text{cm}$$

$$\delta_{VA2} = -30^3 \frac{1510 + 204.5}{8.161(10^{10})} = -567(10^{-6})\,\text{cm}$$

$$\psi_{A2} = -30^2 \frac{1276 + 222.5}{8.161(10^{10})} = -16.5(10^{-6})\ \text{rad}$$

Although the values of ψ_A from the two solutions check, one further step is needed to check the horizontal and vertical deflections of the free ends. In the last solution the reference axes are tilted at 30°. Therefore, the horizontal and vertical deflections of the left end are given by

$$\delta_{HA} = \delta_{HA2}(0.866) + \delta_{VA2}(0.5) = -241(10^{-6})\,\text{cm}$$

$$\delta_{VA} = \delta_{HA2}(-0.5) + \delta_{VA2}(0.866) = -516(10^{-6})\,\text{cm}$$

Again the horizontal deflection of -0.000241 cm for the left half of the arch checks well with the value of -0.000482 cm for the entire arch. With the two displacements of the centroid and the rotation of the end cross section now known, one can easily find the displacements of any other point on the end cross section.

To find the tensile stress at point A we need the bending moment at the center of the arch. This can be found by integration as

$$M = \int_{\pi/6}^{\pi/2} -263R\,d\theta(R_{cg}\cos\theta) = -263RR_{cg}\sin\theta\Big|_{\pi/6}^{\pi/2} = -125{,}000\,\text{N-cm}$$

Using the data from Table 9.1, the stress in the outer fiber at the point A is given by

$$\sigma_A = \frac{k_o Mc}{I} = \frac{0.697(125{,}000)(20)}{10(30^3)/36} = 232\,\text{N/cm}^2$$

9.4 Elliptical Rings

For an elliptical ring of semiaxes a and b, under equal and opposite forces W (Fig. 9.12), the bending moment M_1 at the extremities of the major axis is given by $M_1 = K_1 Wa$, and for equal and opposite outward forces applied at the ends of the minor axis, the moment M_1 at the ends of the major axis is given by $M_1 = -K_2 Wa$, where K_1 and K_2 are coefficients which depend on the ratio a/b and have the following values:

a/b	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
K_1	0.318	0.295	0.274	0.255	0.240	0.227	0.216	0.205
K_2	0.182	0.186	0.191	0.195	0.199	0.203	0.206	0.208

a/b	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5
K_1	0.195	0.185	0.175	0.167	0.161	0.155	0.150	0.145
K_2	0.211	0.213	0.215	0.217	0.219	0.220	0.222	0.223

Burke (Ref. 6) gives charts by which the moments and tensions in elliptical rings under various conditions of concentrated loading can be found; the preceding values of K were taken from these charts.

Timoshenko (Ref. 13) gives an analysis of an elliptical ring (or other ring with two axes of symmetry) under the action of a uniform

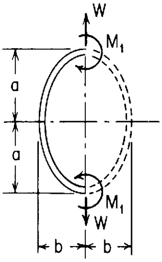


Figure 9.12

outward pressure, which would apply to a tube of elliptical section under internal pressure. For this case $M = Kpa^2$, where M is the bending moment at a section a distance x along the ring from the end of the minor axis, p is the outward normal pressure per linear inch, and K is a coefficient that depends on the ratios b/a and x/S, where S is one-quarter of the perimeter of the ring. Values of K are given in the following table; M is positive when it produces tension at the inner surface of the ring:

b/a x/S	0.3	0.5	0.6	0.7	0.8	0.9
0	−0.172	−0.156	−0.140	−0.115	−0.085	−0.045
0.1	−0.167	−0.152	−0.135	−0.112	−0.082	−0.044
0.2	−0.150	−0.136	−0.120	−0.098	−0.070	−0.038
0.4	−0.085	−0.073	−0.060	−0.046	−0.030	−0.015
0.6	0.020	0.030	0.030	0.028	0.022	0.015
0.7	0.086	0.090	0.082	0.068	0.050	0.022
0.8	0.160	0.150	0.130	0.105	0.075	0.038
0.9	0.240	0.198	0.167	0.130	0.090	0.046
1.0	0.282	0.218	0.180	0.140	0.095	0.050

Values of M calculated by the preceding coefficients are correct only for a ring of uniform moment of inertia I; if I is not uniform, then a correction ΔM must be added. This correction is given by

$$\Delta M = \frac{-\displaystyle\int_0^x \frac{M}{I}\,dx}{\displaystyle\int_0^x \frac{dx}{I}}$$

The integrals can be evaluated graphically. Reference 12 gives charts for the calculation of moments in elliptical rings under uniform radial loading; the preceding values of K were taken from these charts.

9.5 Curved Beams Loaded Normal to Plane of Curvature

This type of beam usually presents a statically indeterminate problem, the degree of indeterminacy depending upon the manner of loading and support. Both bending and twisting occur, and it is necessary to distinguish between an analysis that is applicable to compact or flangeless sections (circular, rectangular, etc.) in which torsion does not produce secondary bending and one that is applicable to flanged sections (I-beams, channels, etc.) in which torsion may be accompanied by such secondary bending (see Sec. 10.3). It is also necessary to distinguish among three types of constraints that may or may not occur at the supports, namely: (1) the beam is prevented from *sloping*, its horizontal axis held horizontal by a bending couple; (2) the beam is prevented from *rolling*, its vertical axis held vertical by a twisting couple; and (3) in the case of a flanged section, the flanges are prevented from turning about their vertical axes by horizontal secondary bending couples. These types of constraints will be designated here as (1) fixed as to slope, (2) fixed as to roll, and (3) flanges fixed.

Compact sections. Table 9.4 treats the curved beam of uniform cross section under concentrated and distributed loads normal to the plane of curvature, out-of-plane concentrated bending moments, and concentrated and distributed torques. Expressions are given for transverse shear, bending moment, twisting moment, deflection, bending slope, and roll slope for 10 combinations of end conditions. To keep the presentation to a reasonable size, use is made of the singularity functions discussed in detail previously and an extensive list of constants and functions is given. In previous tables the representative functional values have been given, but in Table 9.4 the value of β depends upon both bending and torsional properties, and so a useful set of tabular values would be too large to present. The curved beam or ring of circular cross section is so common, however, that numerical coefficients are given in the table for $\beta = 1.3$ which will apply to a solid or hollow circular cross section of material for which Poisson's ratio is 0.3.

Levy (Ref. 14) has treated the closed circular ring of arbitrary compact cross section for six loading cases. These cases have been chosen to permit apropriate superposition in order to solve a large number of problems, and both isolated and distributed out-of-plane loads are discussed. Hogan (Ref. 18) presents similar loadings and supports. In a similar way the information in Table 9.4 can be used by appropriate superposition to solve most out-of-plane loading problems on closed rings of compact cross section if strict attention is given to the symmetry and boundary conditions involved. Several simple examples of this reasoning are described in the following three cases.

1. If a closed circular ring is supported on any number of equally spaced simple supports (two or more) and if identical loading on each span is symmetrically placed relative to the center of the span, then each span can be treated by boundary condition f of Table 9.4. This boundary condition has both ends with no deflection or slope, although they are free to roll as needed.
2. If a closed circular ring is supported on any even number of equally spaced simple supports and if the loading on any span is antisymmetrically placed relative to the center line of each span and symmetrically placed relative to each support, then boundary condition f can be applied to each full span. This problem can also be solved by applying boundary condition g to each half span. Boundary condition g has one end simply supported and slope-guided and the other end simply supported and roll-guided.
3. If a closed circular ring is supported on any even number of equally spaced simple supports (four or more) and if each span is symmetrically loaded relative to the center of the span with adjacent spans similarly loaded in opposite directions, then boundary condition i can be applied to each span. This boundary condition has both ends simply supported and roll-guided.

Once any indeterminate reaction forces and moments have been found and the indeterminate internal reactions found at at least one location in the ring, all desired internal bending moment, torques, and transverse shears can be found by equilibrium equations. If a large number of such calculations need be made, one should consider using a theorem published in 1922 by Biezeno. For details of this theorem see Ref. 32. A brief illustration of this work for loads normal to the plane of the ring is given in Ref. 29.

A treatment of curved beams on elastic foundations is beyond the scope of this book. See Ref. 20.

The following examples illustrate the applications of the formulas in Table 9.4 to both curved beams and closed rings with out-of-plane loads.

EXAMPLES

1. A piece of 8-in standard pipe is used to carry water across a passageway 40 ft wide. The pipe must come out of a wall normal to the surface and enter normal to a parallel wall at a position 16.56 ft down the passageway at the same elevation. To accomplish this a decision was made to bend the pipe into two opposite arcs of 28.28-ft radius with a total angle of 45° in each arc. If it is assumed that both ends are rigidly held by the walls, determine the maximum combined stress in the pipe due to its own weight and the weight of a full pipe of water.

Solution. An 8-in standard pipe has the following properties: $A = 8.4\,\text{in}^2$, $I = 72.5\,\text{in}^4$, $w = 2.38\,\text{lb/in}$, $E = 30(10^6)\,\text{lb/in}^2$, $\nu = 0.3$, $J = 145\,\text{in}^4$, $\text{OD} = 8.625\,\text{in}$, $\text{ID} = 7.981\,\text{in}$, and $t = 0.322\,\text{in}$. The weight of water in a 1-in length of pipe is 1.81 lb. Owing to the symmetry of loading it is apparent that at the center of the span where the two arcs meet there is neither slope nor roll. An examination of Table 9.4 reveals that a curved beam that is fixed at the right end and roll- and slope-guided at the left end is not included among the 10 cases. Therefore, a solution will be carried out by considering a beam that is fixed at the right end and free at the left end with a uniformly distributed load over the entire span and both a concentrated moment and a concentrated torque on the left end. (These conditions are covered in cases 2a, 3a, and 4a.)

Since the pipe is round, $J = 2I$; and since $G = E/2(1+\nu)$, $\beta = 1.3$. Also note that for all three cases $\phi = 45^\circ$ and $\theta = 0^\circ$. For these conditions, numerical values of the coefficients are tabulated and the following expressions for the deformations and moments can be written directly from superposition of the three cases:

$$
\begin{aligned}
y_A &= 0.3058\frac{M_oR^2}{EI} - 0.0590\frac{T_oR^2}{EI} - 0.0469\frac{(2.38+1.81)R^4}{EI}\\
\Theta_A &= -0.8282\frac{M_oR}{EI} - 0.0750\frac{T_oR}{EI} + 0.0762\frac{4.19R^3}{EI}\\
\psi_A &= 0.0750\frac{M_oR}{EI} + 0.9782\frac{T_oR}{EI} + 0.0267\frac{4.19R^3}{EI}\\
V_B &= 0 + 0 - 4.19R(0.7854)\\
M_B &= 0.7071M_o - 0.7071T_o - 0.2929(4.19)R^2\\
T_B &= 0.7071M_o + 0.7071T_o - 0.0783(4.19)R^2
\end{aligned}
$$

Since both Θ_A and ψ_A are zero and $R = 28.28(12) = 339.4\,\text{in}$,

$$
\begin{aligned}
0 &= -0.8282M_o - 0.0750T_o + 36{,}780\\
0 &= 0.0750M_o + 0.9782T_o + 12{,}888
\end{aligned}
$$

Solving these two equations gives $M_o = 45{,}920\,\text{lb-in}$ and $T_o = -16{,}700\,\text{lb-in}$. Therefore,

$$
y_A = -0.40\,\text{in}, \qquad M_B = -97{,}100\,\text{lb-in}
$$

$$
T_B = -17{,}000\,\text{lb-in}, \qquad V_B = -1120\,\text{lb}
$$

The maximum combined stress would be at the top of the pipe at the wall where $\sigma = Mc/I = 97{,}100(4.3125)/72.5 = 5575\,\text{lb/in}^2$ and $\tau = Tr/J = 17{,}100(4.3125)/145 = 509\,\text{lb/in}^2$

$$
\sigma_{\max} = \frac{5775}{2} + \sqrt{\left(\frac{5775}{2}\right)^2 + 509^2} = 5819\,\text{lb/in}^2
$$

2. A hollow steel rectangular beam 4 in wide, 8 in deep, and with 0.1-in wall thickness extends over a loading dock to be used as a crane rail. It is fixed to a warehouse wall at one end and is simply supported on a post at the other. The beam is curved in a horizontal plane with a radius of 15 ft and covers a total angular span of 60°. Calculate the torsional and bending stresses at the wall when a load of 3000 lb is 20° out from the wall. Neglect the weight of the beam.

Solution. The beam has the following properties: $R = 180\,\text{in}$; $\phi = 60°(\pi/3\ \text{rad})$; $\theta = 40°$; $\phi - \theta = 20°(\pi/9\,\text{rad})$; $I = \frac{1}{12}[4(8^3) - 3.8(7.8^3)] = 20.39\,\text{in}^4$; $K = 2(0.1^2)(7.9^2)(3.9^2)/[8(0.1) + 4(0.1) - 2(0.1^2)] = 16.09\,\text{in}^4$ (see Table 10.1, case 16); $E = 30(10^6)$; $G = 12(10^6)$; and $\beta = 30(10^6)(20.39)/12(10^6)(16.09) = 3.168$. Equations for a curved beam that is fixed at one end and simply supported at the other with a concentrated load are found in Table 9.4, case 1b. To obtain the bending and twisting moments at the wall requires first the evaluation of the end reaction V_A, which, in turn, requires the following constants:

$$C_3 = -3.168\left(\frac{\pi}{3} - \sin 60°\right) - \frac{1 + 3.168}{2}\left(\frac{\pi}{3}\cos 60° - \sin 60°\right) = 0.1397$$

$$C_{a3} = -3.168\left(\frac{\pi}{9} - \sin 20°\right) - C_{a2} = 0.006867$$

Similarly,

$$C_6 = C_1 = 0.3060, \qquad C_{a6} = C_{a1} = 0.05775$$

$$C_9 = C_2 = -0.7136, \qquad C_{a9} = C_{a2} = -0.02919$$

Therefore,

$$V_A = 3000\frac{-0.02919(1 - \cos 60°) - 0.05775 \sin 60° + 0.006867}{-0.7136(1 - \cos 60°) - 0.3060 \sin 60° + 0.1397} = 359.3\,\text{lb}$$

$$M_B = 359.3(180)(\sin 60°) - 3000(180)(\sin 20°) = -128{,}700\,\text{lb-in}$$

$$T_B = 359.3(180)(1 - \cos 60°) - 3000(180)(1 - \cos 20°) = -230\,\text{lb-in}$$

At the wall,

$$\sigma = \frac{Mc}{I} = \frac{128{,}700(4)}{20.39} = 25{,}240\,\text{lb/in}^2$$

$$\tau = \begin{cases} \dfrac{VA'\bar{y}}{Ib} = \dfrac{(3000 - 359.3)[4(4)(2) - 3.9(3.8)(1.95)]}{20.39(0.2)} = 2008\,\text{lb/in}^2 \\ \qquad\qquad\qquad\qquad\qquad \text{(due to transverse shear)} \\ \dfrac{T}{2t(a - t)(b - t)} = \dfrac{230}{2(0.1)(7.9)(3.9)} = 37.3\,\text{lb/in}^2 \\ \qquad\qquad\qquad\qquad\qquad \text{(due to torsion)} \end{cases}$$

3. A solid round aluminum bar is in the form of a horizontal closed circular ring of 100-in radius resting on three equally spaced simple supports. A load of 1000 lb is placed midway between two supports, as shown in Fig. 9.13(*a*). Calculate the deflection under this load if the bar is of such diameter as to make the maximum normal stress due to combined bending and torsion equal to 20,000 lb/in^2. Let $E = 10(10^6)\,\text{lb/in}^2$ and $\nu = 0.3$.

Solution. The reactions R_B, R_C, and R_D are statically determinate, and a solution yields $R_B = -333.3\,\text{lb}$ and $R_C = R_D = 666.7\,\text{lb}$. The internal bending

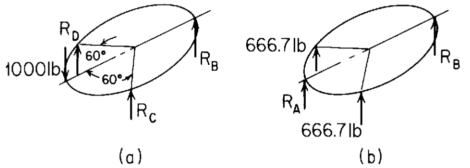


Figure 9.13

and twisting moments are statically indeterminate, and so an energy solution would be appropriate. However, there are several ways that Table 9.4 can be used by superimposing various loadings. The method to be described here is probably the most straightforward.

Consider the equivalent loading shown in Fig. 9.13(*b*), where $R_B = -333.3\,\text{lb}$ and $R_A = -1000\,\text{lb}$. The only difference is in the point of zero deflection. Owing to the symmetry of loading, one-half of the ring can be considered slope-guided at both ends, points *A* and *B*. Case 1f gives tabulated values of the necessary coefficients for $\phi = 180^\circ$ and $\theta = 60^\circ$. We can now solve for the following values:

$$V_A = -666.7(0.75) = -500\,\text{lb}$$

$$M_A = -666.7(100)(-0.5774) = 38{,}490\,\text{lb-in}$$

$$\psi_A = \frac{-666.7(100^2)}{EI}(-0.2722) = \frac{1.815(10^6)}{EI}$$

$$T_A = 0 \qquad y_A = 0 \quad \Theta_A = 0$$

$$M_B = -666.7(100)(-0.2887) = 19{,}250\,\text{lb-in}$$

$$M_{60^\circ} = -666.7(100)(0.3608) = -24{,}050\,\text{lb-in}$$

The equations for *M* and *T* can now be examined to determine the location of the maximum combined stress:

$$M_x = -50{,}000 \sin x + 38{,}490 \cos x + 66{,}667 \sin(x - 60^\circ)\langle x - 60^\circ\rangle^0$$

$$T_x = -50{,}000(1 - \cos x) + 38{,}490 \sin x + 66{,}667[1 - \cos(x - 60^\circ)]\langle x - 60^\circ\rangle^0$$

A careful examination of the expression for *M* shows no maximum values except at the ends and at the position of the load The torque, however, has a maximum value of 13,100 in-lb at $x = 37.59^\circ$ and a minimum value of -8790 in-lb at $x = 130.9^\circ$. At these same locations the bending moments are zero. At the position of the load, the torque $T = 8330$ lb-in. Nowhere is the combined stress larger than the bending stress at point *A*. Therefore,

$$\sigma_A = 20{,}000 = \frac{M_A c}{I} = \frac{38{,}490 d/2}{(\pi/64)d^4} = \frac{392{,}000}{d^3}$$

which gives

$$d = 2.70\,\text{in} \quad \text{and} \quad I = 2.609\,\text{in}^4$$

To obtain the deflection under the 1000-lb load in the original problem, first we must find the deflection at the position of the load of 666.7 lb in Fig. 9.13(*b*). At $x = 60^\circ$,

$$y_x = 0 + 0 + \frac{1.815(10^6)(100)}{10(10^6)2.609}(1 - \cos 60^\circ) + \frac{38{,}490(100^2)}{10(10^6)(2.609)}F_1 + 0 + \frac{-500(100^3)}{10(10^6)(2.609)}F_3$$

where

$$F_1 = \frac{1 + 1.3}{2}\frac{\pi}{3}\sin 60^\circ - 1.3(1 - \cos 60^\circ) = 0.3029 \quad \text{and} \quad F_3 = 0.1583$$

Therefore,

$$y_{60} = 3.478 + 5.796 - 3.033 = 6.24\,\text{in}$$

If the entire ring were now rotated as a rigid body about point B in order to lower points C and D by 6.24 in, point A would be lowered a distance of $6.24(2)/(1 + \cos 60^\circ) = 8.32$ in, which is the downward deflection of the 1000-lb load.

The use of a fictitious support, as was done in this problem at point A, is generalized for asymmeric loadings, both in-plane and out-of-plane, by Barber in Ref. 35.

Flanged sections. The formulas in Table 9.4 for flangeless or compact sections apply also to flanged sections when the ends are fixed as to slope only or when fixed as to slope and roll but not as to flange bending and if the loads are distributed or applied only at the ends. If the flanges are fixed or if concentrated loads are applied within the span, the additional torsional stiffness contributed by the bending resistance of the flanges [*warping restraint* (see Sec. 10.3)] may appreciably affect the value and distribution of twisting and bending moments. The flange stresses caused by the secondary bending or warping may exceed the primary bending stresses. References 15 to 17 and 22 show methods of solution and give some numerical solutions for simple concentrated loads on curved I-beams with both ends fixed completely. Brookhart (Ref. 22) also includes results for additional boundary conditions and uniformly distributed loads. Results are compared with cases where the warping restraint was not considered.

Dabrowski (Ref. 23) gives a thorough presentation of the theory of curved thin-walled beams and works out many examples including multispan beams and beams with open cross sections, closed cross sections, and cross sections which contain both open and closed elements; an extensive bibliography is included. Vlasov (Ref. 27) also gives a very thorough derivation and discusses, among many other topics, vibrations, stability, laterally braced beams of open cross section, and thermal stresses. He also examines the corrections necessary to account for shear deformation in flanges being warped. Verden (Ref. 24) is primarily concerned with multispan curved beams and works out many examples. Sawko and Cope (Ref. 25) and Meyer (Ref. 26) apply finite-element analysis to curved box girder bridges.

9.6 Tables

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve

NOTATION: $R =$ radius of curvature measured to centroid of section; $c =$ distance from centroidal axis to extreme fiber on concave side of beam; $A =$ area of section; $e =$ distance from centroidal axis to neutral axis measured toward center of curvature; $I =$ moment of inertia of cross section about centroidal axis perpendicular to plane of curvature; and $k_i = \sigma_i/\sigma$ and $k_o = \sigma_o/\sigma$ where $\sigma_i =$ actual stress in exteme fiber on concave side, $\sigma_o =$ actual stress in extreme fiber on convex side, and $\sigma =$ fictitious unit stress in corresponding fiber as computed by ordinary flexure formula for a straight beam

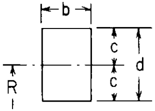
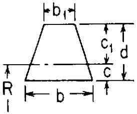
Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
1. Solid rectangular section  	$\frac{e}{c} = \frac{R}{c} - \frac{2}{\ln \frac{R/c+1}{R/c-1}}$ (*Note*: e/c, k_i, and k_o are independent of the width b) $k_i = \frac{1}{3e/c}\frac{1-e/c}{R/c-1}$ $\int_{\text{area}} \frac{dA}{r} = b\frac{R/c+1}{R/c-1}$ $k_o = \frac{1}{3e/c}\frac{1+e/c}{R/c+1}$	$\frac{R}{c} = 1.20$ 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.00 $\frac{e}{c} = 0.366$ 0.284 0.236 0.204 0.180 0.115 0.085 0.056 0.042 0.033 $k_i = 2.888$ 2.103 1.798 1.631 1.523 1.288 1.200 1.124 1.090 1.071 $k_o = 0.566$ 0.628 0.671 0.704 0.730 0.810 0.853 0.898 0.922 0.937
2. Trapezoidal section  	$\frac{d}{c} = \frac{3(1+b_1/b)}{1+2b_1/b}, \quad \frac{c_1}{c} = \frac{d}{c} - 1$ $\frac{e}{c} = \frac{R}{c} - \frac{\frac{1}{2}(1+b_1/b)(d/c)^2}{\left[\frac{R}{c}+\frac{c_1}{c}-\frac{b_1}{b}\left(\frac{R}{c}-1\right)\right]\ln\left(\frac{R/c+c_1/c}{R/c-1}\right)-\left(1-\frac{b_1}{b}\right)\frac{d}{c}}$ $k_i = \frac{1}{2e/c}\frac{1-e/c}{R/c-1}\frac{1+4b_1/b+(b_1/b)^2}{(1+2b_1/b)^2}$ $k_o = \frac{c_1/c}{2e/c}\frac{c_1/c+h/c}{R/c+c_1/c}\frac{1+4b_1/b+(b_1/b)^2}{(2+b_1/b)^2}$ (*Note*: while $e/c, k_i$, k_o depend upon the width ratio b_1/b, they are independent of the width b)	(When $b_1/b = \frac{1}{2}$) $\frac{R}{c} = 1.20$ 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.00 $\frac{e}{c} = 0.403$ 0.318 0.267 0.232 0.206 0.134 0.100 0.067 0.050 0.040 $k_i = 3.011$ 2.183 1.859 1.681 1.567 1.314 1.219 1.137 1.100 1.078 $k_o = 0.544$ 0.605 0.648 0.681 0.707 0.790 0.836 0.885 0.911 0.927

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

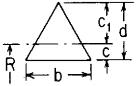
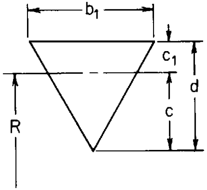
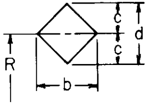
3. Triangular section, base inward  	$\frac{e}{c}=\frac{R}{c}-\frac{4.5}{\left(\frac{R}{c}+2\right)\ln\left(\frac{R/c+2}{R/c-1}\right)-3}, \quad c=\frac{d}{3}$ $k_i=\frac{1}{2e/c}\frac{1-e/c}{R/c-1}$ $\int_{\text{area}}\frac{dA}{r}=b\left[\left(\frac{R}{3c}+\frac{2}{3}\right)\ln\left(\frac{R/c+2}{R/c-1}\right)-1\right]$ $k_o=\frac{1}{4e/c}\frac{2+e/c}{R/c+2}$ (*Note:* e/c, k_i, and k_o are independent of the width b)	$\frac{R}{c}$ = 1.20 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.00 $\frac{e}{c}$ = 0.434 0.348 0.296 0.259 0.232 0.155 0.117 0.079 0.060 0.048 k_i = 3.265 2.345 1.984 1.784 1.656 1.368 1.258 1.163 1.120 1.095 k_o = 0.438 0.497 0.539 0.573 0.601 0.697 0.754 0.821 0.859 0.883
4. Triangular section, base outward  	$\frac{e}{c}=\frac{R}{c}-\frac{1.125}{1.5-\left(\frac{R}{c}-1\right)\ln\frac{R/c+0.5}{R/c-1}}, \quad c=\frac{2d}{3}$ $k_i=\frac{1}{8e/c}\frac{1-e/c}{R/c-1}$ $\int_{\text{area}}\frac{dA}{r}=b_1\left[1-\frac{2}{3}\left(\frac{R}{c}-1\right)\ln\frac{R/c+0.5}{R/c-1}\right]$ $k_o=\frac{1}{4e/c}\frac{2e/c+1}{2R/c+1}$ (*Note:* e/c, k_i, and k_o are independent of the width b_1)	$\frac{R}{c}$ = 1.20 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.00 $\frac{e}{c}$ = 0.151 0.117 0.097 0.083 0.073 0.045 0.033 0.022 0.016 0.013 k_i = 3.527 2.362 1.947 1.730 1.595 1.313 1.213 1.130 1.094 1.074 k_o = 0.636 0.695 0.735 0.765 0.788 0.857 0.892 0.927 0.945 0.956
5. Diamond  	$\frac{e}{c}=\frac{R}{c}-\frac{1}{\frac{R}{c}\ln\left[1-\left(\frac{c}{R}\right)^2\right]+\ln\frac{R/c+1}{R/c-1}}$ $k_i=\frac{1}{6e/c}\frac{1-e/c}{R/c-1}$ $k_o=\frac{1}{6e/c}\frac{1+e/c}{R/c+1}$ $\int_{\text{area}}\frac{dA}{r}=b\left[\frac{R}{c}\ln\left[1-\left(\frac{c}{R}\right)^2\right]+\ln\frac{R/c+1}{R/c-1}\right]$ (*Note:* e/c, k_i, and k_o are independent of the width b)	$\frac{R}{c}$ = 1.200 1.400 1.600 1.800 2.000 3.000 4.000 6.000 8.000 10.000 $\frac{e}{c}$ = 0.175 0.138 0.116 0.100 0.089 0.057 0.042 0.028 0.021 0.017 k_i = 3.942 2.599 2.118 1.866 1.709 1.377 1.258 1.159 1.115 1.090 k_o = 0.510 0.572 0.617 0.652 0.681 0.772 0.822 0.875 0.904 0.922

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

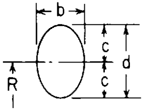
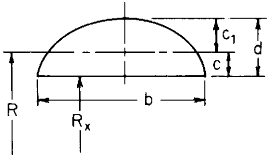
Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
6. Solid circular or elliptical section  	$\frac{e}{c}=\frac{1}{2}\left[\frac{R}{c}-\sqrt{\left(\frac{R}{c}\right)^2-1}\right]$ $k_i=\frac{1}{4e/c}\frac{1-e/c}{R/c-1}$ $k_o=\frac{1}{4e/c}\frac{1+e/c}{R/c+1}$, $\int_{\text{area}}\frac{dA}{r}=\pi b\left[\frac{R}{c}-\sqrt{\left(\frac{R}{c}\right)^2-1}\right]$ (*Note:* $e/c, k_i$, and k_o are independent of the width b)	$\frac{R}{c}$ = 1.20 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.000 $\frac{e}{c}$ = 0.268 0.210 0.176 0.152 0.134 0.086 0.064 0.042 0.031 0.025 k_i = 3.408 2.350 1.957 1.748 1.616 1.332 1.229 1.142 1.103 1.080 k_o = 0.537 0.600 0.644 0.678 0.705 0.791 0.837 0.887 0.913 0.929
7. Solid semicircle or semiellipse, base inward  (*Note:* For a semicircle, $b/2=d$)	$R=R_x+c$, $\frac{d}{c}=\frac{3\pi}{4}$ (*Note:* $e/c, k_i$ and k_o are independent of the width b) $k_i=\frac{0.3879}{e/c}\frac{1-e/c}{R/c-1}$ $k_o=\frac{0.2860}{e/c}\frac{e/c+1.3562}{R/c+1.3562}$ For $R_x \geqslant d{:}\,R/c \geqslant 3.356$ and $\int_{\text{area}}\frac{dA}{r}=\frac{\pi R_x b}{2d}-b-\frac{b}{d}\sqrt{R_x^2-d^2}\left(\frac{\pi}{2}-\sin^{-1}\frac{d}{R_x}\right)$ $\frac{e}{c}=\frac{R}{c}-\frac{(d/c)^2/2}{\frac{R}{c}-2.5-\sqrt{\left(\frac{R}{c}-1\right)^2-\left(\frac{d}{c}\right)^2}\left(1-\frac{2}{\pi}\sin^{-1}\frac{d/c}{R/c-1}\right)}$ For $R_x < d : R/c < 3.356$ and $\int_{\text{area}}\frac{dA}{r}=\frac{\pi R_x b}{2d}-b+\frac{b}{d}\sqrt{d^2-R_x^2}\ln\frac{d+\sqrt{d^2-R_x^2}}{R_x}$ $\frac{e}{c}=\frac{R}{c}-\frac{(d/c)^2/2}{\frac{R}{c}-2.5+\frac{2}{\pi}\sqrt{\left(\frac{d}{c}\right)^2-\left(\frac{R}{c}-1\right)^2}\ln\frac{d/c+\sqrt{(d/c)^2-(R/c-1)^2}}{R/c-1}}$	$\frac{R}{c}$ = 1.200 1.400 1.600 1.800 2.000 3.000 4.000 6.000 8.000 10.000 $\frac{e}{c}$ = 0.388 0.305 0.256 0.222 0.197 0.128 0.096 0.064 0.048 0.038 k_i = 3.056 2.209 1.878 1.696 1.579 1.321 1.224 1.140 1.102 1.080 k_o = 0.503 0.565 0.609 0.643 0.671 0.761 0.811 0.867 0.897 0.916

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

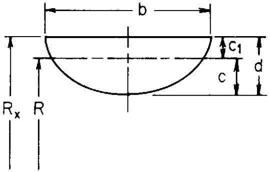
8. Solid semicircle or semiellipse, base outward  (*Note:* for a semicircle, $b/2 = d$)	$R = R_x - c_1, \quad \frac{d}{c} = \frac{3\pi}{3\pi - 4}, \quad \frac{c_1}{c} = \frac{4}{3\pi - 4}$ (*Note:* $e/c, k_i$, and k_o are independent of the width b) $\int_{\text{area}} \frac{dA}{r} = \frac{\pi R_x b}{2d} + b - \frac{b}{d}\sqrt{R_x^2 - d^2}\left(\frac{\pi}{2} + \sin^{-1}\frac{d}{R_x}\right)$ $\frac{e}{c} = \frac{R}{c} - \frac{(d/c)^2/2}{\frac{R}{c} + \frac{10}{3\pi - 4} - \sqrt{\left(\frac{R}{c} + \frac{c_1}{c}\right)^2 - \left(\frac{d}{c}\right)^2}\left(1 + \frac{2}{\pi}\sin^{-1}\frac{d/c}{R/c + c_1/c}\right)}$ $k_i = \frac{0.2109}{e/c}\frac{1 - e/c}{R/c - 1}$ $k_o = \frac{0.2860}{e/c}\frac{e/c + 0.7374}{R/c + 0.7374}$	$\frac{R}{c} =$ 1.200 1.400 1.600 1.800 2.000 3.000 4.000 6.000 8.000 10.000 $\frac{e}{c} =$ 0.244 0.189 0.157 0.135 0.118 0.075 0.055 0.036 0.027 0.021 $k_i =$ 3.264 2.262 1.892 1.695 1.571 1.306 1.210 1.130 1.094 1.073 $k_o =$ 0.593 0.656 0.698 0.730 0.755 0.832 0.871 0.912 0.933 0.946

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

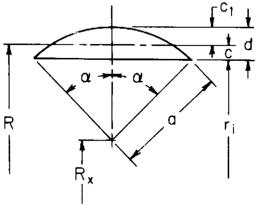
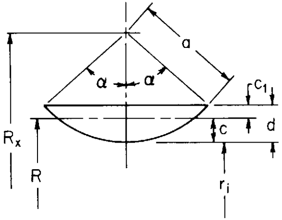
Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
9. Segment of a solid circle, base inward  *Note:* $R, c, e/c, k_i$, and k_o are all independent of the width of the segment provided all horizontal elements of the segment change width proportionately. Therefore to use these expressions for a segment of an ellipse, find the circle which has the same radial dimensions R_x, r_i, and d and evaluate $e/c, k_i$, and k_o which have the same values for both the circle and ellipse. To find dA/r for the ellipse, multiply the value of dA/r for the circle by the ratio of the horizontal to vertical semi-axes for the ellipse. See the example.	$R = R_x + c + a\cos\alpha$ $\frac{a}{c} = \frac{3\alpha - 3\sin\alpha\cos\alpha}{3\sin\alpha - 3\alpha\cos\alpha - \sin^3\alpha}$, $\frac{c_1}{c} = \frac{3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha}{3\sin\alpha - 3\alpha\cos\alpha - \sin^3\alpha}$ $k_i = \frac{I}{Ac^2}\frac{1}{e/c}\frac{1 - e/c}{R/c - 1}$ where expressions for I and A are found in Table A.1, case 19 $k_o = \frac{I}{Ac^2}\frac{1}{(e/c)(c_1/c)}\frac{e/c + c_1/c}{R/c + c_1 c}$ For $R_x \geqslant a : R/c \geqslant (a/c)(1 + \cos\alpha) + 1$ and $\int_{\text{area}} \frac{dA}{r} = 2R_x\alpha - 2a\sin\alpha - 2\sqrt{R_x^2 - a^2}\left(\frac{\pi}{2} - \sin^{-1}\frac{a + R_x\cos\alpha}{R_x + a\cos\alpha}\right)$ $\frac{e}{c} = \frac{R}{c} - \frac{(\alpha - \sin\alpha\cos\alpha)a/c}{\frac{2\alpha R_x}{a} - 2\sin\alpha - 2\sqrt{\left(\frac{R_x}{a}\right)^2 - 1}\left(\frac{\pi}{2} - \sin^{-1}\frac{1 + (R_x/a)\cos\alpha}{R_x/a + \cos\alpha}\right)}$ (*Note:* Values of $\sin^{-1}$ between $-\pi/2$ and $\pi/2$ are to be taken in above expressions.) For $R_x < a$: $R/c < (a/c)(1 + \cos\alpha) + 1$ and $\int_{\text{area}} \frac{dA}{r} = 2R_x\alpha - 2a\sin\alpha + 2\sqrt{a^2 - R_x^2}\ln\frac{\sqrt{a^2 - R_x^2}\sin\alpha + a + R_x\cos\alpha}{R_x + a\cos\alpha}$ $\frac{e}{c} = \frac{R}{c}$ $- \frac{(\alpha - \sin\alpha\cos\alpha)a/c}{\frac{2\alpha R_x}{a} - 2\sin\alpha + 2\sqrt{1 - \left(\frac{R_x}{a}\right)^2}\ln\frac{\sqrt{1 - (R_x/a)^2}\sin\alpha + 1 + (R_x/a)\cos\alpha}{R_x/a + \cos\alpha}}$	For $\alpha = 60^\circ$: $\frac{R}{c}$ = 1.200 1.400 1.600 1.800 2.000 3.000 4.000 6.000 8.000 10.000 $\frac{e}{c}$ = 0.401 0.317 0.266 0.232 0.206 0.134 0.101 0.067 0.051 0.041 k_i = 3.079 2.225 1.891 1.707 1.589 1.327 1.228 1.143 1.104 1.082 k_o = 0.498 0.560 0.603 0.638 0.665 0.755 0.806 0.862 0.893 0.913 For $\alpha = 30^\circ$: $\frac{R}{c}$ = 1.200 1.400 1.600 1.800 2.000 3.000 4.000 6.000 8.000 10.000 $\frac{e}{c}$ = 0.407 0.322 0.271 0.236 0.210 0.138 0.103 0.069 0.052 0.042 k_i = 3.096 2.237 1.900 1.715 1.596 1.331 1.231 1.145 1.106 1.083 k_o = 0.495 0.556 0.600 0.634 0.662 0.752 0.803 0.860 0.891 0.911

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

10. Segment of a solid circle, base outward



Note: $R, c, e/c, k_i$, and k_o are all independent of the width of the segment provided all horizontal elements of the segment change width proportionately. To use these expressions for a segment of an ellipse, refer to the explanation in case 9.

$$R = R_x + c - a$$

$$\frac{a}{c} = \frac{3\alpha - 3\sin\alpha\cos\alpha}{3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha}, \qquad \frac{c_1}{c} = \frac{3\sin\alpha - 3\alpha\cos\alpha - \sin^3\alpha}{3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha}$$

$$k_i = \frac{I}{Ac^2}\frac{1}{e/c}\frac{1-e/c}{R/c-1}$$

where expressions for I and A are found in Table A.1, case 19

$$k_o = \frac{I}{Ac^2}\frac{1}{(e/c)(c_1/c)}\frac{e/c + c_1/c}{R/c + c_1/c}$$

$$\int_{\text{area}} \frac{dA}{r} = 2R_x\alpha + 2a\sin\alpha - 2\sqrt{R_x^2 - a^2}\left(\frac{\pi}{2} + \sin^{-1}\frac{a - R_x\cos\alpha}{R_x - a\cos\alpha}\right)$$

$$\frac{e}{c} = \frac{R}{c} - \frac{(\alpha - \sin\alpha\cos\alpha)a/c}{\dfrac{2\alpha R_x}{a} + 2\sin\alpha - 2\sqrt{\left(\dfrac{R_x}{a}\right)^2} - 1\left(\dfrac{\pi}{2} + \sin^{-1}\dfrac{1 - (R_x/a)\cos\alpha}{R_x/a - \cos\alpha}\right)}$$

(*Note:* Values of $\sin^{-1}$ between $-\pi/2$ and $\pi/2$ are to be taken in above expressions.)

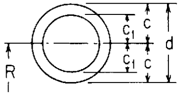
For $\alpha = 60°$:

$\frac{R}{c} =$	1.200	1.400	1.600	1.800	2.000	3.000	4.000	6.000	8.000	10.000
$\frac{e}{c} =$	0.235	0.181	0.150	0.129	0.113	0.071	0.052	0.034	0.025	0.020
$k_i =$	3.241	2.247	1.881	1.686	1.563	1.301	1.207	1.127	1.092	1.072
$k_o =$	0.598	0.661	0.703	0.735	0.760	0.836	0.874	0.914	0.935	0.948

For $\alpha = 30°$:

$\frac{R}{c} =$	1.200	1.400	1.600	1.800	2.000	3.000	4.000	6.000	8.000	10.000
$\frac{e}{c} =$	0.230	0.177	0.146	0.125	0.110	0.069	0.051	0.033	0.025	0.020
$k_i =$	3.232	2.241	1.876	1.682	1.560	1.299	1.205	1.126	1.091	1.072
$k_o =$	0.601	0.663	0.706	0.737	0.763	0.838	0.876	0.916	0.936	0.948

11. Hollow circular section



$$\frac{e}{c} = \frac{1}{2}\left[\frac{2R}{c} - \sqrt{\left(\frac{R}{c}\right)^2 - 1} - \sqrt{\left(\frac{R}{c}\right)^2 - \left(\frac{c_1}{c}\right)^2}\right]$$

$$k_i = \frac{1}{4e/c}\frac{1-e/c}{R/c-1}\left[1 + \left(\frac{c_1}{c}\right)^2\right]$$

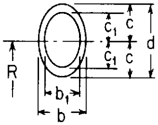
$$k_o = \frac{1}{4e/c}\frac{1+e/c}{R/c+1}\left[1 + \left(\frac{c_1}{c}\right)^2\right]$$

(*Note:* For thin-walled tubes the discussion on page 277 should be considered)

(When $c_1/c = \frac{1}{2}$)

$\frac{R}{c} =$	1.20	1.40	1.60	1.80	2.00	3.00	4.00	6.00	8.00	10.00
$k_i =$	3.276	2.267	1.895	1.697	1.573	1.307	1.211	1.130	1.094	1.074
$\frac{e}{c} =$	0.323	0.256	0.216	0.187	0.166	0.107	0.079	0.052	0.039	0.031
$k_o =$	0.582	0.638	0.678	0.708	0.733	0.810	0.852	0.897	0.921	0.936

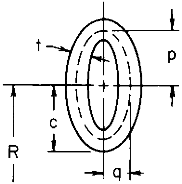
TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
12. Hollow elliptical section 12a. Inner and outer perimeters are ellipses, wall thickness is not constant 	$\frac{e}{c}=\frac{R}{c}-\frac{\frac{1}{2}[1-(b_1/b)(c_1/c)]}{\frac{R}{c}-\sqrt{\left(\frac{R}{c}\right)^2-1}-\frac{b_1/b}{c_1/c}\left[\frac{R}{c}-\sqrt{\left(\frac{R}{c}\right)^2-\left(\frac{c_1}{c}\right)^2}\right]}$ $k_i=\frac{1}{4e/c}\frac{1-e/c}{R/c-1}\frac{1-(b_1/b)(c_1/c)^3}{1-(b_1/b)(c_1/c)}$ $k_o=\frac{1}{4e/c}\frac{1+e/c}{R/c+1}\frac{1-(b_1/b)(c_1/c)^3}{1-(b_1/b)(c_1/c)}$ (*Note:* While $e/c, k_i$, and k_o depend upon the width ratio b_1/b, they are independent of the width b)	(When $b_1/b=\frac{3}{5}, c_1/c=\frac{4}{5}$)

(When $b_1/b=\frac{3}{5}, c_1/c=\frac{4}{5}$)

$\frac{R}{c}=$	1.20	1.40	1.60	1.80	2.00	3.00	4.00	6.00	8.00	10.00
$\frac{e}{c}=$	0.345	0.279	0.233	0.202	0.178	0.114	0.085	0.056	0.042	0.034
$k_i=$	3.033	2.154	1.825	1.648	1.535	1.291	1.202	1.125	1.083	1.063
$k_o=$	0.579	0.637	0.677	0.709	0.734	0.812	0.854	0.899	0.916	0.930

12b. Constant wall thickness, midthickness perimeter is an ellipse (shown dashed)



Note: There is a limit on the maximum wall thickness allowed in this case. Cusps will form on the inner perimeter at the ends of the major axis if this maximum is exceeded. If $p/q \leqslant 1$, then $t_{\max}=2p^2/q$. If $p/q \geqslant 1$, then $t_{\max}=2q^2/p$

There is no closed-form solution for this case, so numerical solutions were run for the ranges $1.2 < R/c < 5$; $0 < t < t_{\max}$; $0.2 < p/q < 5$. Results are expressed below in terms of the solutions for case 12a for which $c=p+t/2, c_1=p-t/2, b=2q+t$, and $b_1=2q-t$.

$\frac{e}{c}=K_1\left(\frac{e}{c}\text{ from case 12a}\right)$, $k_i=K_2(k_i\text{ from case 12a)}$

$k_o=K_3(k_o\text{ from case 12a)}$

where K_1, K_2, and K_3 are given in the following table and are essentially independent of t and R/c.

p/q	0.200	0.333	0.500	0.625	1.000	1.600	2.000	3.000	4.000	5.000
K_1	0.965	0.985	0.995	0.998	1.000	1.002	1.007	1.027	1.051	1.073
K_2	1.017	1.005	1.002	1.001	1.000	1.000	1.000	0.998	0.992	0.985
K_3	0.982	0.992	0.998	0.999	1.000	1.002	1.004	1.014	1.024	1.031

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

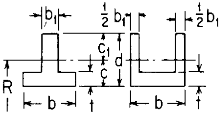
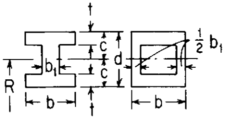
13. T-beam or channel section  	$\frac{d}{c} = \frac{2[b_1/b + (1-b_1/b)(t/d)]}{b_1/b + (1-b_1/b)(t/d)^2}, \qquad \frac{c_1}{c} = \frac{d}{c} - 1$ $\frac{e}{c} = \frac{R}{c} - \frac{(d/c)[b_1/b + (1-b_1/b)(t/d)]}{\frac{b_1}{b}\ln\frac{d/c+R/c-1}{(d/c)(t/d)+R/c-1} + \ln\frac{(d/c)(t/d)+R/c-1}{R/c-1}}$ $k_i = \frac{I_c}{Ac^2(R/c-1)}\frac{1-e/c}{e/c}$ where $\frac{I_c}{Ac^2} = \frac{1}{3}\left(\frac{d}{c}\right)^2\left[\frac{b_1/b+(1-b_1/b)(t/d)^3}{b_1/b+(1-b_1/b)(t/d)}\right] - 1$ $k_o = \frac{I_c}{Ac^2(e/c)}\frac{d/c+e/c-1}{R/c+d/c-1}\frac{1}{d/c-1}$ (*Note:* While $e/c, k_i$, and k_o depend upon the width ratio b_1/b, they are independent of the width b)	(When $b_1/b = \frac{1}{4}, t/d = \frac{1}{4}$) $\frac{R}{c} = 1.200 \quad 1.400 \quad 1.600 \quad 1.800 \quad 2.000 \quad 3.000 \quad 4.000 \quad 6.000 \quad 8.000 \quad 10.000$ $\frac{e}{c} = 0.502 \quad 0.419 \quad 0.366 \quad 0.328 \quad 0.297 \quad 0.207 \quad 0.160 \quad 0.111 \quad 0.085 \quad 0.069$ $k_i = 3.633 \quad 2.538 \quad 2.112 \quad 1.879 \quad 1.731 \quad 1.403 \quad 1.281 \quad 1.176 \quad 1.128 \quad 1.101$ $k_o = 0.583 \quad 0.634 \quad 0.670 \quad 0.697 \quad 0.719 \quad 0.791 \quad 0.832 \quad 0.879 \quad 0.905 \quad 0.922$
14. Symmetrical I-beam or hollow rectangular section  	$\frac{e}{c} = \frac{R}{c} = \frac{2[t/c+(1-t/c)(b_1/b)]}{\ln\frac{R/c^2+(R/c+1)(t/c)-1}{(R/c)^2-(R/c-1)(t/c)-1} + \frac{b_1}{b}\ln\frac{R/c-t/c+1}{R/c+t/c-1}}$ $k_i = \frac{I_c}{Ac^2(R/c-1)}\frac{1-e/c}{e/c}$ where $\frac{I_c}{Ac^2} = \frac{1}{3}\frac{1-(1-b_1/b)(1-t/c)^3}{1-(1-b_1/b)(1-t/c)}$ $k_o = \frac{I_c}{Ac^2(R/c+1)}\frac{1+e/c}{e/c}$ (*Note:* While $e/c, k_i$, and k_o depend upon the width ratio b_1/b, they are independent of the width b)	(When $b_1/b = \frac{1}{3}, t/d = \frac{1}{6}$) $\frac{R}{c} = 1.20 \quad 1.40 \quad 1.60 \quad 1.80 \quad 2.00 \quad 3.00 \quad 4.00 \quad 6.00 \quad 8.00 \quad 10.00$ $\frac{e}{c} = 0.489 \quad 0.391 \quad 0.330 \quad 0.287 \quad 0.254 \quad 0.164 \quad 0.122 \quad 0.081 \quad 0.060 \quad 0.048$ $k_i = 2.156 \quad 1.876 \quad 1.630 \quad 1.496 \quad 1.411 \quad 1.225 \quad 1.156 \quad 1.097 \quad 1.071 \quad 1.055$ $k_o = 0.666 \quad 0.714 \quad 0.747 \quad 0.771 \quad 0.791 \quad 0.853 \quad 0.886 \quad 0.921 \quad 0.940 \quad 0.951$

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

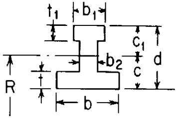
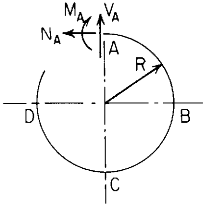
Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
15. Unsymmetrical I-beam section  	$A = bd[b_1/b + (1 - b_2/b)(t/d) - (b_1/b - b_2/b)(1 - t_1/d)]$ $\frac{d}{c} = \frac{2A/bd}{(b_1/b - b_2/b)(2 - t_1/d)(t_1/d) + (1 - b_2/b)(t/d)^2 + b_2/b}$ $\frac{e}{c} = \frac{R}{c} - \frac{(A/bd)(d/c)}{\ln\frac{R/c + t/c - 1}{R/c - 1} + \frac{b_2}{b}\ln\frac{R/c + c_1/c - t_1/c}{R/c + t/c - 1} + \frac{b_1}{b}\ln\frac{R/c + c_1/c}{R/c + c_1/c - t_1/c}}$ $k_i = \frac{I_c}{Ac^2(R/c - 1)}\frac{1 - e/c}{e/c}$ where $\frac{I_c}{Ac^2} = \frac{1}{3}\left(\frac{d}{c}\right)^2\left[\frac{b_1/b + (1 - b_2/b)(t/d)^3 - (b_1/b - b_2/b)(1 - t_1/d)^3}{b_1/b + (1 - b_2/b)(t/d) - (b_1/b - b_2/b)(1 - t_1/d)}\right] - 1$ $k_o = \frac{I_c}{Ac^2(e/c)}\frac{d/c + e/c - 1}{R/c + d/c - 1}\frac{1}{d/c - 1}$ (*Note:* While $e/c, k_i$, and k_o depend upon the width ratios b_1/b and b_2/b, they are independent of the width b)	(When $b_1/b = \frac{2}{3}, b_2/b = \frac{1}{6}, t_1/d = \frac{1}{6}, t/d = \frac{1}{3}$) $\frac{R}{c} = 1.20$ 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.00 $\frac{e}{c} = 0.491$ 0.409 0.356 0.318 0.288 0.200 0.154 0.106 0.081 0.066 $k_i = 3.589$ 2.504 2.083 1.853 1.706 1.385 1.266 1.165 1.120 1.094 $k_o = 0.671$ 0.721 0.754 0.779 0.798 0.856 0.887 0.921 0.938 0.950

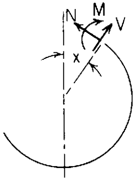
TABLE 9.2 Formulas for circular rings

NOTATION: W = load (force); w and v = unit loads (force per unit of circumferential length); ρ = unit weight of contained liquid (force per unit volume); M_o = applied couple (force-length). M_A, M_B, M_C, and M are internal moments at A, B, C, and x, respectively, positive as shown. N_A, N, V_A, and V are internal forces, positive as shown. E = modulus of elasticity (force per unit area); ν = Poisson's ratio; A = cross-sectional area (length squared); R = radius to the centroid of the cross section (length); I = area moment of inertia of ring cross section about the principal axis perpendicular to the plane of the ring (length^4). [Note that for a pipe or cylinder, a representative segment of unit axial length may be used by replacing EI by $Et^3/12(1-\nu^2)$.] e = positive distance measured radially inward from the centroidal axis of the cross section to the neutral axis of pure bending (see Sec. 9.1). θ, x, and ϕ are angles (radians) and are limited to the range zero to π for all cases except 18 and 19; $s = \sin\theta, c = \cos\theta$, $z = \sin x$, $u = \cos x$, $n = \sin\phi$, and $m = \cos\phi$.

ΔD_V and ΔD_H are changes in the vertical and horizontal diameters, respectively, and an increase is positive. ΔL is the change in the lower half of the vertical diameter or the vertical motion relative to point C of a line connecting points B and D on the ring. Similarly ΔL_W is the vertical motion relative to point C of a horizontal line connecting the load points on the ring. ΔL_{WH} is the change in length of a horizontal line connecting the load points on the ring. ψ is the angular rotation (radians) of the load point in the plane of the ring and is positive in the direction of positive θ. For the distributed loadings the load points just referred to are the points where the distributed loading starts, i.e., the position located by the angle θ. The reference to points A, B, and C and to the diameters refer to positions on a circle of radius R passing through the centroids of the several sections; i.e., diameter $= 2R$. It is important to consider this when dealing with thick rings. Similarly, all concentrated and distributed loadings are assumed to be applied at the radial position of the centroid with the exception of the cases where the ring is loaded by its own weight or by dynamic loading, cases 15 and 21. In these two cases the actual radial distribution of load is considered. If the loading is on the outer or inner surfaces of thick rings, an equivalent loading at the centroidal radius R must be used. See the examples to determine how this might be accomplished.

The hoop-stress deformation factor is $\alpha = I/AR^2$ for thin rings or $\alpha = e/R$ for thick rings. The transverse (radial) shear deformation factor is $\beta = FEI/GAR^2$ for thin rings or $\beta = 2F(1+\nu)e/R$ for thick rings, where G is the shear modulus of elasticity and F is a shape factor for the cross section (see Sec. 8.10). The following constants are defined to simplify the expressions whcih follow. Note that these constants are unity if no correction for hoop stress or shear stress is necessary or desired for use with thin rings. $k_1 = 1 - \alpha + \beta$, $k_2 = 1 - \alpha$.





General formulas for moment, hoop load, and radial shear

$$M = M_A - N_A R(1-u) + V_A Rz + LT_M$$
$$N = N_A u + V_A z + LT_N$$
$$V = -N_A z + V_A u + LT_V$$

where LT_M, LT_N, and LT_V are load terms given below for several types of load.

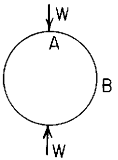
Note: Due to symmetry in most of the cases presented, the loads beyond 180° are not included in the load terms. Only for cases 16, 17, and 19 should the equations for M, N, and V be used beyond 180°.

Note: The use of the bracket $\langle x-\theta\rangle^0$ is explained on page 131 and has a value of zero unless $x > \theta$

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms | Formulas for moments, loads, and deformations and some selected numerical values

1.



$LT_M = \frac{-WRz}{2} \quad LT_N = \frac{-Wz}{2}$

$LT_v = \frac{-Wu}{2}$

$$M_A = \frac{WRk_2}{\pi}$$

$$N_A = 0$$

$$V_A = 0$$

$$\Delta D_H = \frac{WR^3}{EI}\left(\frac{k_1}{2} - k_2 + \frac{2k_2^2}{\pi}\right)$$

$$\Delta D_V = \frac{-WR^3}{EI}\left(\frac{\pi k_1}{4} - \frac{2k_2^2}{\pi}\right)$$

Max $+M = M_A = 0.3183WRk_2$

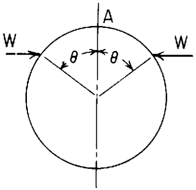
Max $-M = M_B = -(0.5 - 0.3183k_2)WR$

If $\alpha = \beta = 0$,

$$\Delta D_H = 0.1366\frac{WR^3}{EI} \quad \text{and} \quad \Delta D_V = -0.1488\frac{WR^3}{EI}$$

Note: For concentrated loads on thick-walled rings, study the material in Sec. 14.3 on hollow pins and rollers. Radial stresses under the concentrated loads have a significant effect not considered here.

2.



$LT_M = -WR(c-u)\langle x-\theta\rangle^0$
$LT_N = Wu\langle x-\theta\rangle^0$
$LT_V = -Wz\langle x-\theta\rangle^0$

$$M_A = \frac{-WR}{\pi}[(\pi-\theta)(1-c) - s(k_2-c)]$$

$$M_C = \frac{-WR}{\pi}[\theta(1+c) - s(k_2+c)]$$

$$N_A = \frac{-W}{\pi}[\pi - \theta + sc]$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \frac{-WR^3}{EI\pi}[0.5\pi k_1(\theta - sc) + 2k_2\theta c - 2k_2^2 s] & \text{if } \theta \leqslant \frac{\pi}{2} \\ \frac{-WR^3}{EI\pi}[0.5\pi k_1(\pi - \theta + sc) - 2k_2(\pi-\theta)c - 2k_2^2 s] & \text{if } \theta \geqslant \frac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{WR^3}{EI}\left[\frac{k_1 s^2}{2} - k_2\left(1 - c + \frac{2\theta c}{\pi}\right) + \frac{2k_2^2 s}{\pi}\right]$$

$$\Delta L = \begin{cases} \frac{WR^3}{EI}\left[\frac{\theta c}{2} + \frac{k_1(\theta - sc)}{2\pi} - k_2\left(\frac{\theta c}{\pi} + \frac{s}{2}\right) + \frac{k_2^2 s}{\pi}\right] & \text{if } \theta \leqslant \pi/2 \\ \frac{WR^3}{EI}\left[\frac{(\pi-\theta)c}{2} + \frac{k_1(\theta - sc - \pi c^2)}{2\pi} - k_2\left(1 + \frac{\theta c}{\pi} - \frac{s}{2}\right) + k_2^2 s/\pi\right] & \text{if } \theta \geqslant \pi/2 \end{cases}$$

$$\Delta L_W = \frac{WR^3}{EI\pi}[(\pi-\theta)\theta sc + 0.5k_1 s^2(\theta - sc) + k_2(2\theta s^2 - \pi s^2 - \theta c - \theta) + k_2^2 s(1+c)]$$

$$\Delta L_{WH} = \frac{-WR^3}{EI\pi}[(\pi-\theta)2\theta c^2 - k_1(\pi sc + s^2c^2 - 2\theta sc - \pi\theta + \theta^2) - 2k_2 sc(\pi - 2\theta) - 2k_2^2 s^2]$$

$$\Delta\psi = \frac{-WR^2}{EI\pi}[(\pi-\theta)\theta c - k_2 s(sc + \pi - 2\theta)]$$

Max $+M = \frac{WRs(k_2 - c^2)}{\pi}$ at $x = \theta$

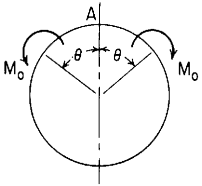
$$\text{Max} -M = \begin{cases} M_A & \text{if } \theta \leqslant \frac{\pi}{2} \\ M_C & \text{if } \theta \geqslant \frac{\pi}{2} \end{cases}$$

If $\alpha = \beta = 0, M = K_M WR, N = K_N W, \Delta D = K_{\Delta D} WR^3/EI$, $\Delta\psi = K_{\Delta\psi} WR^2/EI$, etc.

θ	30°	45°	60°
K_{M_A}	−0.0903	−0.1538	−0.1955
K_{M_θ}	0.0398	0.1125	0.2068
K_{N_A}	−0.9712	−0.9092	−0.8045
$K_{\Delta D_H}$	−0.0157	−0.0461	−0.0891
$K_{\Delta D_V}$	0.0207	0.0537	0.0930
$K_{\Delta L}$	0.0060	0.0179	0.0355
$K_{\Delta L_W}$	0.0119	0.0247	0.0391
$K_{\Delta L_{WH}}$	−0.0060	−0.0302	−0.0770
$K_{\Delta\psi}$	0.0244	0.0496	0.0590

TABLE 9.2 Formulas for circular rings (*Continued*)

3.



$LT_M = M_o\langle x-\theta\rangle^0$
$LT_N = 0$
$LT_V = 0$

$$M_A = \frac{-M_o}{\pi}\left(\pi-\theta-\frac{2sk_2}{k_1}\right)$$

$$M_C = \frac{M_o}{\pi}\left(\theta-\frac{2sk_2}{k_1}\right)$$

$$N_A = \frac{M_o}{R\pi}\left(\frac{2sk_2}{k_1}\right)$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{M_oR^2}{EI}k_2\left(\dfrac{2\theta}{\pi}-s\right) & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{M_oR^2}{EI}k_2\left(\dfrac{2\theta}{\pi}-2+s\right) & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{M_oR^2}{EI}k_2\left(\frac{2\theta}{\pi}-1+c\right)$$

$$\Delta L = \begin{cases} \dfrac{-M_oR^2}{EI}\left[\dfrac{\theta}{2}-\dfrac{k_2(\theta+s)}{\pi}\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-M_oR^2}{EI}\left[\dfrac{\pi-\theta}{2}-\dfrac{k_2(\theta+s+\pi c)}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta L_W = \frac{-M_oR^2}{EI\pi}[(\pi-\theta)\theta s - k_2(s^3+\theta+\theta c)]$$

$$\Delta L_{WH} = \frac{M_oR^2}{EI\pi}[2\theta c(\pi-\theta)+2k_2 s(2\theta-\pi-sc)]$$

$$\Delta\psi = \frac{M_oR}{EI\pi}\left[\theta(\pi-\theta)-\frac{2s^2k_2^2}{k_1}\right]$$

$\text{Max} + M = \dfrac{M_o}{\pi}\left(\theta+\dfrac{2sck_2}{k_1}\right)$ at x just greater than θ

$\text{Max} - M = \dfrac{-M_o}{\pi}\left(\pi-\theta-\dfrac{2sck_2}{k_1}\right)$ at x just less than θ

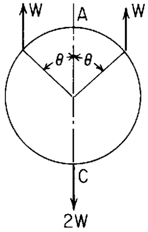
If $\alpha=\beta=0$, $M = K_M M_o$, $N = K_N M_o/R$, $\Delta D = K_{\Delta D} M_o R^2/EI$, $\Delta\psi = K_{\Delta\psi} M_o R/EI$, etc.

θ	30°	45°	60°	90°
K_{M_A}	−0.5150	−0.2998	−0.1153	0.1366
K_{N_A}	0.3183	0.4502	0.5513	0.6366
K_{M_θ}	−0.5577	−0.4317	−0.3910	−0.5000
$K_{\Delta D_H}$	−0.1667	−0.2071	−0.1994	0.0000
$K_{\Delta D_V}$	0.1994	0.2071	0.1667	0.0000
$K_{\Delta L}$	0.0640	0.0824	0.0854	0.0329
$K_{\Delta L_W}$	0.1326	0.1228	0.1022	0.0329
$K_{\Delta L_{WH}}$	−0.0488	−0.0992	−0.1180	0.0000
$K_{\Delta\psi}$	0.2772	0.2707	0.2207	0.1488

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values

4.



$LT_M = WR(z - s)\langle x - \theta \rangle^0$
$LT_N = Wz\langle x - \theta \rangle^0$
$LT_V = Wu\langle x - \theta \rangle^0$

$$M_A = \frac{-WR}{\pi}[s(s - \pi + \theta) + k_2(1 + c)]$$

$$M_C = \frac{-WR}{\pi}[s\theta - s^2 + k_2(1 + c)]$$

$$N_A = \frac{-W}{\pi}s^2$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-WR^3}{EI\pi}\left[\pi k_1\left(1 - \dfrac{s^2}{2}\right) - 2k_2(\pi - \theta s) + 2k_2^2(1 + c)\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-WR^3}{EI\pi}\left[\dfrac{\pi k_1 s^2}{2} - 2sk_2(\pi - \theta) + 2k_2^2(1 + c)\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{WR^3}{EI\pi}\left[\frac{\pi k_1(\pi - \theta - sc)}{2} + k_2 s(\pi - 2\theta) - 2k_2^2(1 + c)\right]$$

$$\Delta L = \begin{cases} \dfrac{WR^3}{2EI}\left[\theta s + k_1\left(\dfrac{\pi}{2} - \dfrac{s^2}{\pi}\right) - k_2\left(1 - c + \dfrac{2\theta s}{\pi}\right)\right. \\ \qquad\qquad \left. - \dfrac{2k_2^2(1 + c)}{\pi}\right] & \text{if} \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{WR^3}{2EI}\left[s(\pi - \theta) + k_1\left(\pi - \theta - sc - \dfrac{s^2}{\pi}\right) - k_2\left(1 + c + \dfrac{2\theta s}{\pi}\right)\right. \\ \qquad\qquad \left. - \dfrac{2k_2^2(1 + c)}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta L_W = \frac{WR^3}{EI\pi}\left[\theta s^2(\pi - \theta) + \frac{\pi k_1(\pi - \theta - sc - s^4/\pi)}{2}\right.$$
$$\left. + k_2 s(\pi c - 2\theta - 2\theta c) - k_2^2(1 + c)^2\right]$$

$$\Delta L_{WH} = \frac{-WR^3}{EI\pi}[2\theta sc(\pi - \theta) + k_1 s^2(\theta - sc)$$
$$-2k_2(\pi s^2 - \theta s^2 + \theta c + \theta c^2) + 2k_2^2 s(1 + c)]$$

$$\Delta\psi = \frac{WR^2}{EI\pi}[-\theta s(\pi - \theta) + k_2(\theta + \theta c + s^3)]$$

Max $+M$ occurs at an angular position $x_1 = \tan^{-1}\dfrac{-\pi}{s^2}$ if $\theta < 106.3°$

Max $+M$ occurs at the load if $\theta \geqslant 106.3°$

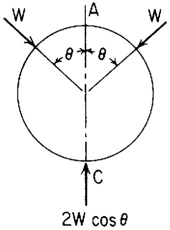
Max $-M = M_C$

If $\alpha = \beta = 0$, $M = K_M WR$, $N = K_N W$, $\Delta D = K_{\Delta D} WR^3/EI$,
$\Delta\psi = K_{\Delta\psi} WR^2/EI$, etc.

θ	30°	60°	90°	120°	150°
K_{M_A}	−0.2569	−0.1389	−0.1366	−0.1092	−0.0389
K_{N_A}	−0.0796	−0.2387	−0.3183	−0.2387	−0.0796
K_{M_C}	−0.5977	−0.5274	−0.5000	−0.4978	−0.3797
K_{M_θ}	−0.2462	−0.0195	0.1817	0.2489	0.1096
$K_{\Delta D_H}$	−0.2296	−0.1573	−0.1366	−0.1160	−0.0436
$K_{\Delta D_V}$	0.2379	0.1644	0.1488	0.1331	0.0597
$K_{\Delta L}$	0.1322	0.1033	0.0933	0.0877	0.0431
$K_{\Delta L_W}$	0.2053	0.1156	0.0933	0.0842	0.0271
$K_{\Delta L_{WH}}$	−0.0237	−0.0782	−0.1366	−0.1078	−0.0176
$K_{\Delta\psi}$	0.1326	0.1022	0.0329	−0.0645	−0.0667

TABLE 9.2 Formulas for circular rings (*Continued*)

5.



$LT_M = -WR\sin(x-\theta)\langle x-\theta\rangle^0$
$LT_N = -W\sin(x-\theta)\langle x-\theta\rangle^0$
$LT_V = -W\cos(x-\theta)\langle x-\theta\rangle^0$

$$M_A = \frac{-WR}{\pi}[s(\pi-\theta) - k_2(1+c)]$$

$$M_C = \frac{-WR}{\pi}[s\theta - k_2(1+c)]$$

$$N_A = \frac{-W}{\pi}s(\pi-\theta)$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-WR^3}{EI}\left[k_1\left(\dfrac{\theta s}{2} - c\right) + 2k_2c - \dfrac{2k_2^2(1+c)}{\pi}\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-WR^3}{EI}\left[\dfrac{k_1 s(\pi-\theta)}{2} - \dfrac{2k_2^2(1+c)}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{WR^3}{EI}\left[\frac{k_1(s-\pi c+\theta c)}{2} - k_2 s + \frac{2k_2^2(1+c)}{\pi}\right]$$

$$\Delta L = \begin{cases} \dfrac{WR^3}{2EI}\left[k_1\left(\dfrac{\theta s}{\pi} - \dfrac{\pi c}{2}\right) - k_2(1-c) + \dfrac{2k_2^2(1+c)}{\pi}\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{WR^3}{2EI}\left[k_1\left(\dfrac{\theta s}{\pi} - \pi c + \theta c\right) + k_2(1+c-2s) \right. & \\ \qquad\qquad \left. + \dfrac{2k_2^2(1+c)}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta L_W = \frac{WR^3}{EI}\left\{k_1\frac{s - s^3(1-\theta/\pi) - c(\pi-\theta)}{2} + k_2\left[\frac{\theta s(1+c)}{\pi} - s\right] + \frac{k_2^2(1+c)^2}{\pi}\right\}$$

$$\Delta L_{WH} = \frac{-WR^3}{EI\pi}[k_1 s(\pi-\theta)(\theta - sc) + 2\theta c k_2(1+c) - 2sk_2^2(1+c)]$$

$$\Delta\psi = \frac{WR^2}{EI\pi}[\pi s^2 - \theta(1+c+s^2)]k_2$$

Max $+M = M_C$ if $\theta \leqslant 60^\circ$

Max $+M$ occurs at the load if $\theta > 60^\circ$ where

$$M_\theta = \frac{WR}{\pi}[k_2(1+c) - sc(\pi-\theta)]$$

$$\text{Max } -M = \begin{cases} M_C & \text{if } \theta \geqslant 90^\circ \\ M_A & \text{if } 60^\circ \leqslant \theta \leqslant 90^\circ \end{cases}$$

Max $-M$ occurs at an angular

$$\text{position } x_1 = \tan^{-1}\frac{-\pi c}{\theta s} \text{ if } \theta \leqslant 60^\circ$$

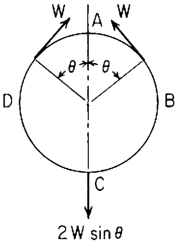
If $\alpha = \beta = 0$, $M = K_M WR$, $N = K_N W$, $\Delta D = K_{\Delta D}WR^3/EI$, $\Delta\psi = K_{\Delta\psi}WR^2/EI$, etc.

θ	30°	60°	90°	120°	150°
K_{M_A}	0.1773	–0.0999	–0.1817	–0.1295	–0.0407
K_{N_A}	–0.4167	–0.5774	–0.5000	–0.2887	–0.0833
K_{M_C}	0.5106	0.1888	–0.1817	–0.4182	–0.3740
K_{M_θ}	0.2331	0.1888	0.3183	0.3035	0.1148
$K_{\Delta D_H}$	0.1910	0.0015	–0.1488	–0.1351	–0.0456
$K_{\Delta D_V}$	–0.1957	–0.0017	0.1366	0.1471	0.0620
$K_{\Delta L}$	–0.1115	–0.0209	0.0683	0.0936	0.0447
$K_{\Delta L_W}$	–0.1718	–0.0239	0.0683	0.0888	0.0278
$K_{\Delta L_{WH}}$	0.0176	–0.0276	–0.1488	–0.1206	– 0.0182
$K_{\Delta\psi}$	–0.1027	0.0000	0.0000	0.0833	–0.0700

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values

6.



$LT_M = -WR[1-\cos(x-\theta)]\langle x-\theta\rangle^0$
$LT_N = W\cos(x-\theta)\langle x-\theta\rangle^0$
$LT_V = -W\sin(x-\theta)\langle x-\theta\rangle^0$

$$M_A = \frac{-WR}{\pi}[s(1+k_2)-(\pi-\theta)(1-c)]$$

$$M_C = \frac{-WR}{\pi}[s(k_2-1)+\theta(1+c)]$$

$$N_A = \frac{-W}{\pi}[s+(\pi-\theta)c]$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-WR^3}{EI}\left[\dfrac{k_1(s+\theta c)}{2} - 2k_2\left(s-\dfrac{\theta}{\pi}\right) + \dfrac{2k_2^2 s}{\pi}\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-WR^3}{EI}\left[\dfrac{k_1(s+\pi c-\theta c)}{2} - 2k_2\left(1-\dfrac{\theta}{\pi}\right) + \dfrac{2k_2^2 s}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{WR^3}{EI}\left[\frac{k_1 s(\pi-\theta)}{2} + k_2\left(1-c-\frac{2\theta}{\pi}\right) - \frac{2k_2^2 s}{\pi}\right]$$

$$\Delta L = \begin{cases} \dfrac{WR^3}{EI}\left[\dfrac{\theta}{2} + \dfrac{k_1(\pi^2 s+2\theta c-2s)}{4\pi} - k_2\left(\dfrac{s}{2}+\dfrac{\theta}{\pi}\right) - \dfrac{k_2^2 s}{\pi}\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{WR^3}{EI}\left[\dfrac{\pi}{2} - \dfrac{\theta}{2} + \dfrac{k_1(\pi s-\theta s+\theta c/\pi-s/\pi-c)}{2}\right. & \\ \qquad\qquad \left. - k_2\left(\dfrac{\theta}{\pi}+\dfrac{s}{2}+c\right) - \dfrac{k_2^2 s}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta L_W = \frac{WR^3}{EI\pi}\left[\theta s(\pi-\theta) + \frac{k_1 s(\theta sc - s^2 - sc\pi + \pi^2 - \theta\pi)}{2} - k_2\theta(1+s^2+c) - k_2^2 s(1+c)\right]$$

$$\Delta L_{WH} = \frac{-WR^3}{EI\pi}[2\theta c(\pi-\theta) - k_1(sc^2\pi - 2\theta sc^2 + s^2 c - \theta c\pi + \theta^2 c - \theta s^3) - 2k_2 s(\pi-\theta+\theta c) + 2k_2^2 s^2]$$

$$\Delta\psi = \frac{-WR^2}{EI\pi}[\theta(\pi-\theta) - k_2 s(\theta+s+\pi c-\theta c)]$$

$$\text{Max} + M = \frac{WR}{\pi}[\pi s\sin x_1 - (s-\theta c)\cos x_1 - k_2 s - \theta]$$
at an angular position $x_1 = \tan^{-1}\dfrac{-\pi s}{s-\theta c}$

(*Note* : $x_1 > \theta$ and $x_1 > \pi/2$)

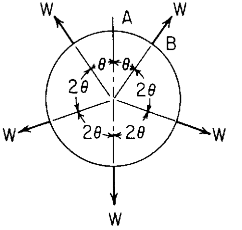
$\text{Max} - M = M_C$

If $\alpha = \beta = 0$, $M = K_M WR$, $N = K_N W$, $\Delta D = K_{\Delta D} WR^3/EI$,
$\Delta\psi = K_{\Delta\psi} WR^2/EI$, etc.

θ	30°	60°	90°	120°	150°
K_{M_A}	−0.2067	−0.2180	−0.1366	−0.0513	−0.0073
K_{N_A}	−0.8808	−0.6090	−0.3183	−0.1090	−0.0148
K_{M_C}	−0.3110	−0.5000	−0.5000	−0.3333	−0.1117
$K_{\Delta D_H}$	−0.1284	−0.1808	−0.1366	−0.0559	−0.0083
$K_{\Delta D_V}$	0.1368	0.1889	0.1488	0.0688	0.0120
$K_{\Delta L}$	0.0713	0.1073	0.0933	0.0472	0.0088
$K_{\Delta L_W}$	0.1129	0.1196	0.0933	0.0460	0.0059
$K_{\Delta L_{WH}}$	−0.0170	−0.1063	−0.1366	−0.0548	− 0.0036
$K_{\Delta\psi}$	0.0874	0.1180	0.0329	−0.0264	−0.0123

TABLE 9.2 Formulas for circular rings (*Continued*)

7. Ring under any number of equal radial forces equally spaced



For $0 < x < \theta$ $\quad M = \dfrac{WR(u/s - k_2/\theta)}{2} \qquad N = \dfrac{Wu}{2s} \qquad V = \dfrac{-Wz}{2s}$

$$\text{Max} + M = M_A = \frac{WR(1/s - k_2/\theta)}{2} \qquad \text{Max} - M = \frac{-WR}{2}\left(\frac{k_2}{\theta} - \frac{c}{s}\right) \text{ at each load position}$$

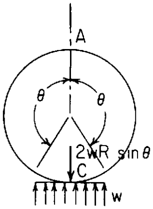
$$\text{Radial displacement at each load point} = \Delta R_B = \frac{WR^3}{EI}\left[\frac{k_1(\theta - sc)}{4s^2} + \frac{k_2 c}{2s} - \frac{k_2^2}{2\theta}\right]$$

$$\text{Radial displacement at } x = 0, 2\theta, \ldots = \Delta R_A = \frac{-WR^3}{EI}\left[\frac{k_1(s - \theta c)}{4s^2} - \frac{k_2}{2s} + \frac{k_2^2}{2\theta}\right]$$

If $\alpha = \beta = 0$, $M = K_M WR$, $\Delta R = K_{\Delta R} WR^3/EI$

θ	15°	30°	45°	60°	90°
K_{M_A}	0.02199	0.04507	0.07049	0.09989	0.18169
K_{M_B}	−0.04383	−0.08890	−0.13662	−0.18879	−0.31831
$K_{\Delta R_B}$	0.00020	0.00168	0.00608	0.01594	0.07439
$K_{\Delta R_A}$	−0.00018	−0.00148	−0.00539	−0.01426	−0.06831

8.



$\left(\text{Note: } \theta \geqslant \dfrac{\pi}{2}\right)$

$$LT_M = \frac{-wR^2}{2}(z - s)^2 \langle x - \theta \rangle^0$$
$$LT_N = -wRz(z - s)\langle x - \theta \rangle^0$$
$$LT_V = -wRu(z - s)\langle x - \theta \rangle^0$$

$$M_A = \frac{wR^2}{2\pi}\left[\pi(s^2 - 0.5) - \frac{sc - \theta}{2} - s^2\left(\theta + \frac{2s}{3}\right) - k_2(2s + sc - \pi + \theta)\right]$$

$$M_C = \frac{-wR^2}{2\pi}\left[\frac{\pi}{2} + \frac{sc}{2} - \frac{\theta}{2} + \theta s^2 - \frac{2s^3}{3} + k_2(2s + sc - \pi + \theta)\right]$$

$$N_A = \frac{-wRs^3}{3\pi}$$

$$V_A = 0$$

$$\Delta D_H = \frac{-wR^4}{2EI\pi}\left[\frac{k_1 \pi s^3}{3} + k_2(\pi - 2\pi s^2 - \theta + 2\theta s^2 + sc) + 2k_2^2(2s + sc - \pi + \theta)\right]$$

$$\Delta D_V = \frac{wR^4}{2EI\pi}\left[k_1\pi\left(\pi s - \theta s - \tfrac{2}{3} - c + \frac{c^3}{3}\right) - k_2(\pi c^2 + sc - \theta + 2\theta s^2) - 2k_2^2(2s + sc - \pi + \theta)\right]$$

$$\Delta L = \frac{wR^4}{4EI\pi}\left[\pi(\pi - \theta)\frac{2s^2 - 1}{2} - \frac{\pi sc}{2} - 2\pi k_1\left(\tfrac{2}{3} - \pi s + c + \theta s - \frac{c^3}{3} + \frac{s^3}{3\pi}\right) - k_2(sc + \pi - \theta + 2\theta s^2 + 2s\pi - \pi^2 + \pi\theta + \pi sc) - 2k_2^2(2s + sc - \pi + \theta)\right]$$

Max $+ M$ occurs at an angular position x_1 where $x_1 > \theta$, $x_1 > 123.1°$, and

$$\tan x_1 + \frac{3\pi(s - \sin x_1)}{s^3} = 0$$

Max $- M = M_C$

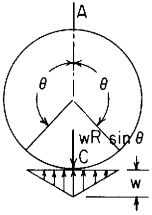
If $\alpha = \beta = 0$, $M = K_M wR^2$, $N = K_N wR$, $\Delta D = K_{\Delta D} wR^4/EI$, etc.

θ	90°	120°	135°	150°
K_{M_A}	−0.0494	−0.0329	−0.0182	−0.0065
K_{N_A}	−0.1061	−0.0689	−0.0375	−0.0133
K_{M_C}	−0.3372	−0.2700	−0.1932	−0.1050
$K_{\Delta D_H}$	−0.0533	−0.0362	−0.0204	−0.0074
$K_{\Delta D_V}$	0.0655	0.0464	0.0276	0.0108
$K_{\Delta L}$	0.0448	0.0325	0.0198	0.0080

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values

9.



$\left(Note\text{: } \theta \geqslant \frac{\pi}{2}\right)$

$$LT_M = \frac{wR^2}{6s}(z-s)^3\langle x-\theta\rangle^0$$

$$LT_N = \frac{wRz}{s^2}(z-s)^2\langle x-\theta\rangle^0$$

$$LT_V = \frac{wRu}{2s}(z-s)^2\langle x-\theta\rangle^0$$

$$M_A = \frac{wR^2}{36\pi s}\{(\pi-\theta)(6s^3-9s)-3s^4+8+8c-5s^2c-6k_2[3s(s-\pi+\theta)+s^2c+2+2c]\}$$

$$M_C = \frac{-wR^2}{36\pi s}\{9s(\pi-\theta)+6\theta s^3-3s^4-8-8c+5s^2c+6k_2[3s(s-\pi+\theta)+s^2c+2+2c]\}$$

$$N_A = \frac{-wRs^3}{12\pi}$$

$$V_A = 0$$

$$\Delta D_H = \frac{-wR^4}{18EI\pi}\left\{\frac{3k_1\pi s^3}{4}-k_2\left[(\pi-\theta)(6s^2-9)+\frac{8(1+c)}{s}-5sc\right]+6k_2^2\left[sc+\frac{2(1+c)}{s}-3(\pi-\theta-s)\right]\right\}$$

$$\Delta D_V = \frac{wR^4}{18EI\pi}\left\{18\pi k_1\left[\left(\frac{s}{4}+\frac{1}{16s}\right)(\pi-\theta)-\frac{13c}{48}-\frac{s^2c}{24}-\frac{1}{3}\right]+k_2\left[(\pi-\theta)(3s^2-9)-3s^2\theta+\frac{8(1+c)}{s}-5sc\right]-6k_2^2\left[sc+\frac{2(1+c)}{s}-3(\pi-\theta-s)\right]\right\}$$

$$\Delta L = \frac{wR^4}{EI}\left[(\pi-\theta)\left(\frac{s^2}{12}-\frac{1}{8}\right)+\frac{1+c}{9s}-\frac{5sc}{72}+k_1\frac{(\pi-\theta)(12s+3/s)-13c-2s^2c-16-2s^3/\pi}{48}\right.$$
$$\left.-k_2\frac{(1+c)(2\pi-\frac{8}{3})/s-3(\pi-\theta)(\pi-1)+2\theta s^2+3\pi s+sc(\pi+\frac{5}{3})}{12\pi}-k_2^2\frac{3(s-\pi+\theta)+2(1+c)s+sc}{6\pi}\right]$$

Max $+M$ occurs at an angular position x_1 where $x_1 > \theta, x_1 > 131.1^\circ$, and $\tan x_1 + \dfrac{6\pi(s-\sin x_1)^2}{s^4} = 0$

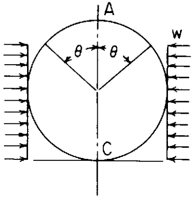
Max $-M = M_C$

If $\alpha = \beta = 0, M = K_M wR^2, N = K_N wR, \Delta D = K_{\Delta D}wR^4/EI$, etc.

θ	90°	120°	135°	150°
K_{M_A}	−0.0127	−0.0084	−0.0046	−0.0016
K_{N_A}	−0.0265	−0.0172	−0.0094	−0.0033
K_{M_C}	−0.1263	−0.0989	−0.0692	−0.0367
$K_{\Delta D_H}$	−0.0141	−0.0093	−0.0052	−0.0019
$K_{\Delta D_V}$	0.0185	0.0127	0.0074	0.0028
$K_{\Delta L}$	0.0131	0.0092	0.0054	0.0021

TABLE 9.2 Formulas for circular rings (*Continued*)

10.



$LT_M = \frac{-wR^2}{2}(c-u)^2\langle x-\theta\rangle^0$

$LT_N = wRu(c-u)\langle x-\theta\rangle^0$

$LT_V = -wRz(c-u)\langle x-\theta\rangle^0$

$$M_A = \frac{-wR^2}{4\pi}\left[(\pi-\theta)(4c+2s^2-1)+s\left(4-\frac{4s^2}{3}-c\right)-2k_2(\pi-\theta+sc)\right]$$

$$M_C = \frac{-wR^2}{4\pi}\left[3\pi+\theta+4\theta c-2\theta s^2-4s-sc+\frac{4s^3}{3}-2k_2(\pi-\theta+sc)\right]$$

$$N_A = \frac{-wR}{\pi}\left(\pi c+s-\theta c-\frac{s^3}{3}\right)$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \frac{-wR^4}{6EI\pi}[\pi k_1(s^3+3\theta c+4-3s)+3k_2(\pi-\theta+2\theta c^2-sc)-6k_2^2(\pi-\theta+sc)] & \text{for } \theta \leqslant \frac{\pi}{2} \\ \frac{-wR^4}{2EI\pi}\left\{\pi k_1\left[c(\pi-\theta)+s-\frac{s^3}{3}\right]+k_2[(\pi-\theta)(2s^2-1)-sc]-2k_2^2(\pi-\theta+sc)\right\} & \text{for } \theta \geqslant \frac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{wR^4}{3EI\pi}\{\pi k_1(2-c^3+3c)+3k_2[2\theta s^2-\theta+sc-\pi(1+2c+s^2)]+6k_2^2(\pi-\theta+sc)\}$$

$$\Delta L = \begin{cases} \frac{wR^4}{12EI\pi}[1.5\pi(\theta-2\theta s^2-sc)+2k_1(2\pi+s^3+3\theta c-3s)+3k_2(sc+\theta\pi+2\theta s^2-3\pi-\theta-\pi sc)+6k_2^2(\pi-\theta+sc)] & \text{for } \theta \leqslant \frac{\pi}{2} \\ \frac{wR^4}{12EI\pi}\{1.5\pi[(\pi-\theta)(1-2s^2)+sc]+2k_1(2\pi+s^3+3\theta c-3s-\pi c^3)+3k_2[(\pi+1)(\pi-\theta+sc)+2\theta s^2-4\pi(1+c)]+6k_2^2(\pi-\theta+sc)\} & \text{for } \theta \geqslant \frac{\pi}{2} \end{cases}$$

Max $+M$ occurs at an angular position x_1 where $x_1 > \theta, x_1 > 90^\circ$, and $x_1 = \cos^{-1}\frac{s^3/3+\theta c-s}{\pi}$

Max $-M = M_C$

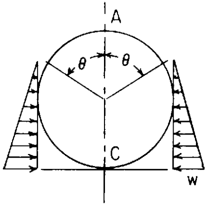
If $\alpha = \beta = 0, M = K_M wR^2, N = K_N wR, \Delta D = K_{\Delta D} wR^4/EI$, etc.

θ	0°	30°	45°	60°	90°	120°	135°	150°
K_{M_A}	−0.2500	−0.2434	−0.2235	−0.1867	−0.0872	−0.0185	−0.0052	−0.00076
K_{N_A}	−1.0000	−0.8676	−0.7179	−0.5401	−0.2122	−0.0401	−0.0108	−0.00155
K_{M_C}	−0.2500	−0.2492	−0.2448	−0.2315	−0.1628	−0.0633	−0.0265	−0.00663
$K_{\Delta D_H}$	−0.1667	−0.1658	−0.1610	−0.1470	−0.0833	−0.0197	−0.0057	−0.00086
$K_{\Delta D_V}$	0.1667	0.1655	0.1596	0.1443	0.0833	0.0224	0.0071	0.00118
$K_{\Delta L}$	0.0833	0.0830	0.0812	0.0756	0.0486	0.0147	0.0049	0.00086

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values

11.



$$LT_M = \frac{-wR^2}{6(1+c)}(c-u)^3\langle x-\theta\rangle^0$$

$$LT_N = \frac{wRu}{2(1+c)}(c-u)^2\langle x-\theta\rangle^0$$

$$LT_V = \frac{-wRz}{2(1+c)}(c-u)^2\langle x-\theta\rangle^0$$

$$M_A = \frac{-wR^2}{(1+c)\pi}\left[(\pi-\theta)\frac{3+12c^2+2c+4cs^2}{24} - \frac{3s^3c-3s-5s^3}{36} + \frac{5sc}{8} - k_2\left(\frac{\pi c}{2} - \frac{\theta c}{2} + \frac{s^3}{3} + \frac{sc^2}{2}\right)\right]$$

$$M_C = \frac{-wR^2}{(1+c)\pi}\left[(\pi-\theta)\frac{-3-12c^2+2c+4cs^2}{24} + \frac{\pi(1+c)^3}{6} + \frac{3s^3c+3s+5s^3}{36} - \frac{5sc}{8} - k_2\left(\frac{\pi c}{2} - \frac{\theta c}{2} + \frac{s^3}{3} + \frac{sc^2}{2}\right)\right]$$

$$N_A = \frac{-wR}{(1+c)\pi}\left[(\pi-\theta)\frac{1+4c^2}{8} + \frac{5sc}{8} - \frac{s^3c}{12}\right]$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-wR^4}{EI(1+c)\pi}\left\{\pi k_1\left(\dfrac{\theta+4\theta c^2-5sc}{16} + \dfrac{s^3c+16c}{24}\right) - k_2\left(\dfrac{5sc^2+3\theta c+6\theta s^2c-8s}{18} - \dfrac{\pi c}{2}\right) - k_2^2\left[c(\pi-\theta) + \dfrac{2s^3}{3} + sc^2\right]\right\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-wR^4}{EI(1+c)\pi}\left\{\pi k_1\left[(\pi-\theta)\dfrac{1+4c^2}{16} + \dfrac{5sc}{16} - \dfrac{s^3c}{24}\right] - k_2\left[\dfrac{5sc^2-8s}{18} - (\pi-\theta)\dfrac{c+2s^2c}{6}\right] - k_2^2\left[c(\pi-\theta) + s - \dfrac{s^3}{3}\right]\right\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{wR^4}{EI(1+c)}\left\{k_1\left[\frac{(1+c)^2}{6} - \frac{s^4}{24}\right] + k_2\left(\frac{5sc^2+3\theta c+6\theta s^2c-8s}{18\pi} + \frac{s^2}{2} + \frac{c^3}{6} - c - \frac{2}{3}\right) + k_2^2\frac{c(\pi-\theta)+s-s^3/3}{\pi}\right\}$$

$$\Delta L = \begin{cases} \dfrac{wR^4}{EI(1+c)}\left\{\dfrac{3s+5s^3+6\theta c^3-9\theta c-16}{72} + k_1\left(\dfrac{c}{3} + \dfrac{1}{16} + \dfrac{12\theta c^2+3\theta+2s^3c-15sc}{48\pi}\right)\right. \\ \qquad \left. + k_2\left[\dfrac{1-s^3}{6} - \dfrac{c(3+sc-\theta)}{4} + \dfrac{3\theta c+6\theta s^2c-3s-5s^3}{36\pi}\right] + k_2^2\dfrac{c(\pi-\theta)/2+sc^2/2+s^3/3}{\pi}\right\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{wR^4}{EI(1+c)}\left\{\dfrac{-(\pi-\theta)c(1+2s^2)}{24} - \dfrac{sc^2}{24} - \dfrac{s^3}{9} + k_1\left(\dfrac{c}{3} + \dfrac{1}{16} - \dfrac{c^4}{24} + \dfrac{12\theta c^2+3\theta+2s^3c-15sc}{48\pi}\right)\right. \\ \qquad \left. + k_2\left[\dfrac{2s-2+sc^2}{12} + \dfrac{c(\pi-\theta-3-2c)}{4} + \dfrac{3\theta c+6\theta s^2c-3s-5s^3}{36\pi}\right] + k_2^2\dfrac{3c(\pi-\theta)+2s+sc^2}{6\pi}\right\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

Max $+M$ occurs at an angular position x_1 where $x_1 > \theta$, $x_1 > 96.8^\circ$, and $x_1 = \arccos\left\{c - \left[(c^2+0.25)\left(1-\dfrac{\theta}{\pi}\right) + \dfrac{sc(5-2s^2/3)}{4\pi}\right]^{1/2}\right\}$

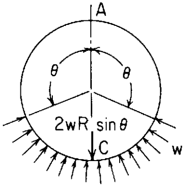
Max $-M = M_C$

If $\alpha = \beta = 0$, $M = K_M wR^2$, $N = K_N wR$, $\Delta D = K_{\Delta D} wR^4/EI$, etc.

θ	0°	30°	45°	60°	90°	120°	135°	150°
K_{M_A}	−0.1042	−0.0939	−0.0808	−0.0635	−0.0271	− 0.0055	−0.0015	−0.00022
K_{N_A}	−0.3125	−0.2679	−0.2191	−0.1628	−0.0625	− 0.0116	−0.0031	−0.00045
K_{M_C}	−0.1458	−0.1384	−0.1282	−0.1129	−0.0688	−0.0239	−0.0096	−0.00232
$K_{\Delta D_H}$	−0.8333	−0.0774	−0.0693	−0.0575	−0.0274	− 0.0059	−0.0017	−0.00025
$K_{\Delta D_V}$	0.0833	0.0774	0.0694	0.0579	0.0291	0.0071	0.0022	0.00035
$K_{\Delta L}$	0.0451	0.0424	0.0387	0.0332	0.0180	0.0048	0.0015	0.00026

TABLE 9.2 Formulas for circular rings (*Continued*)

12.



$LT_M = -wR^2[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

$LT_N = -wR[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

$LT_V = -wR\sin(x-\theta)\langle x-\theta\rangle^0$

$$M_A = \frac{-wR^2}{\pi}[s+\pi c-\theta c-k_2(\pi-\theta-s)]$$

$$M_C = \frac{-wR^2}{\pi}[\pi-s+\theta c-k_2(\pi-\theta-s)]$$

$$N_A = \frac{-wR}{\pi}(s+\pi c-\theta c)$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-wR^4}{EI}\left[\dfrac{k_1(s+\theta c)}{2}+2k_2(1-s)-\dfrac{2k_2^2(\pi-\theta-s)}{\pi}\right] & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-wR^4}{EI}\left[\dfrac{k_1(s+\pi c-\theta c)}{2}-\dfrac{2k_2^2(\pi-\theta-s)}{\pi}\right] & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{wR^4}{EI}\left[\frac{k_1 s(\pi-\theta)}{2}-k_2(1+c)+\frac{2k_2^2(\pi-\theta-s)}{\pi}\right]$$

$$\Delta L = \begin{cases} \dfrac{wR^4}{2EI\pi}\left[k_1\left(\dfrac{\pi^2 s}{2}-s+\theta c\right)+k_2\pi(\theta-s-2)+2k_2^2(\pi-\theta-s)\right] & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{wR^4}{2EI\pi}[k_1(\pi^2 s-\pi\theta s-\pi c-s+\theta c)+k_2\pi(\pi-\theta-s-2-2c)+2k_2^2(\pi-\theta-s)] & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

Max $+M$ occurs at an angular position x_1 where $x_1 > \theta, x_1 > 90^\circ$, and $x_1 = \tan^{-1}\dfrac{-\pi s}{s-\theta c}$

Max $-M = M_C$

If $\alpha = \beta = 0, M = K_M wR^2, N = K_N wR, \Delta D = K_{\Delta D} wR^4/EI$, etc.

θ	30°	60°	90°	120°	150°
K_{M_A}	−0.2067	−0.2180	−0.1366	−0.0513	−0.0073
K_{N_A}	−0.8808	−0.6090	−0.3183	−0.1090	−0.0148
K_{M_C}	−0.3110	−0.5000	−0.5000	−0.3333	−0.1117
$K_{\Delta D_H}$	−0.1284	−0.1808	−0.1366	−0.0559	−0.0083
$K_{\Delta D_V}$	0.1368	0.1889	0.1488	0.0688	0.0120
$K_{\Delta L}$	0.0713	0.1073	0.0933	0.0472	0.0088

TABLE 9.2 Formulas for circular rings (*Continued*)

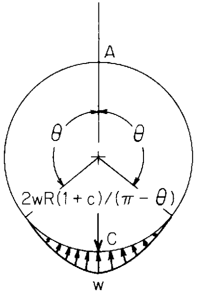
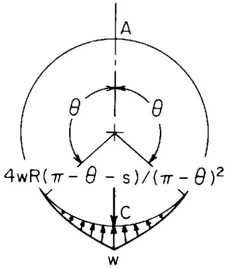
Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
13.  The radial pressure w_x varies linearly with x from 0 at $x = \theta$ to w at $x = \pi$. $LT_M = \frac{-wR^2}{\pi - \theta}(x - \theta - zc + us)\langle x - \theta \rangle^0$ $LT_N = \frac{-wR}{\pi - \theta}(x - \theta - zc + us)\langle x - \theta \rangle^0$ $LT_V = \frac{-wR}{\pi - \theta}(1 - uc - zs)\langle x - \theta \rangle^0$	$M_A = \frac{-wR^2}{\pi(\pi - \theta)}\left\{2 + 2c - s(\pi - \theta) + k_2\left[1 + c - \frac{(\pi - \theta)^2}{2}\right]\right\}$ $M_C = \frac{-wR^2}{\pi(\pi - \theta)}\left\{\pi(\pi - \theta) - 2 - 2c - s\theta + k_2\left[1 + c - \frac{(\pi - \theta)^2}{2}\right]\right\}$ $N_A = \frac{-wR}{\pi(\pi - \theta)}[2 + 2c - s(\pi - \theta)]$ $V_A = 0$ $\Delta D_H = \begin{cases} \frac{-wR^4}{EI(\pi - \theta)}\left\{k_1\left(1 - \frac{s\theta}{2}\right) + k_2(\pi - 2\theta - 2c) + k_2^2\frac{2 + 2c - (\pi - \theta)^2}{\pi}\right\} & \text{for } \theta \leqslant \frac{\pi}{2} \\ \frac{-wR^4}{EI(\pi - \theta)}\left\{k_1\left[1 + c - \frac{s(\pi - \theta)}{2}\right] + k_2^2\frac{2 + 2c - (\pi - \theta)^2}{\pi}\right\} & \text{for } \theta \geqslant \frac{\pi}{2} \end{cases}$ $\Delta D_V = \frac{wR^4}{EI(\pi - \theta)}\left\{k_1\frac{s + c(\pi - \theta)}{2} - k_2(\pi - \theta - s) - k_2^2\frac{2 + 2c - (\pi - \theta)^2}{\pi}\right\}$ $\Delta L = \begin{cases} \frac{wR^4}{2EI\pi(\pi - \theta)}\left\{k_1\left(\frac{\pi^2 c}{2} - 2c + 2\pi - 2 - \theta s\right) - k_2\pi\left[2(\pi - \theta) - 1 + c - \frac{\pi^2}{4} + \frac{\theta^2}{2}\right] - k_2^2[2 + 2c - \pi - \theta)^2\right\} & \text{for } \theta \leqslant \frac{\pi}{2} \\ \frac{wR^4}{2EI\pi(\pi - \theta)}\left\{k_1[\pi c(\pi - \theta) + 2\pi s - 2c - 2 - \theta s] - k_2\pi\left[2(\pi - \theta) + 1 + c - 2s - \frac{(\pi - \theta)^2}{2}\right] - k_2^2[2 + 2c - (\pi - \theta)^2]\right\} & \text{for } \theta \geqslant \frac{\pi}{2} \end{cases}$ Max $+M$ occurs at an angular position x_1 where $x_1 > \theta, x_1 > 103.7°$, and x_1 is found from $\left(1 + c + \frac{s\theta}{2}\right)\sin x_1 + c\cos x_1 - 1 = 0$ Max $-M = M_C$

TABLE 9.2 Formulas for circular rings (*Continued*)

14.



The radial pressure w_x varies with $(x-\theta)^2$ from 0 at $x=\theta$ to w at $x=\pi$

$$LT_M = \frac{-wR^2}{(\pi-\theta)^2}[(x-\theta)^2 - 2 + 2uc + 2zs]\langle x-\theta\rangle^0$$

$$LT_N = \frac{-wR}{(\pi-\theta)^2}[(x-\theta)^2 - 2 + 2uc + 2zs]\langle x-\theta\rangle^0$$

$$LT_V = \frac{-2wR}{(\pi-\theta)^2}(x-\theta-zc+us) \times \langle x-\theta\rangle^0$$

$$M_A = \frac{-wR^2}{\pi(\pi-\theta)^2}\left\{2(\pi-\theta)(2-c) - 6s + k_2\left[2(\pi-\theta-s) - \frac{(\pi-\theta)^3}{3}\right]\right\}$$

$$M_C = \frac{-wR^2}{\pi(\pi-\theta)^2}\left\{2\theta(2-c) + 6s - 6\pi + \pi(\pi-\theta)^2 + k_2\left[2(\pi-\theta-s) - \frac{(\pi-\theta)^3}{3}\right]\right\}$$

$$N_A = \frac{-wR}{\pi(\pi-\theta)^2}[2(\pi-\theta)(2-c) - 6s]$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-wR^4}{EI(\pi-\theta)^2}\left\{k_1\left(\dfrac{\pi^2}{4} - 6 - \theta^2 s + 3s - 3\theta c + \dfrac{3\pi}{2} + c + \theta s - 2\theta\right) + k_2\left(\dfrac{\pi^2}{2} - 4 + 4s - 2\theta\pi + 2\theta^2\right) - 2k_2^2\dfrac{(\pi-\theta)^3 - 6(\pi-\theta-s)}{3\pi}\right\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-wR^4}{EI(\pi-\theta)^2}\left\{k_1[(2-c)(\pi-\theta) - 3s] - 2k_2^2\dfrac{(\pi-\theta)^3 - 6(\pi-\theta-s)}{3\pi}\right\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{wR^4}{EI(\pi-\theta)^2}\left\{k_1[2 + 2c - s(\pi-\theta)] + k_2(2 + 2c - (\pi-\theta)^2] + 2k_2^2\frac{(\pi-\theta)^3 - 6(\pi-\theta-s)}{3\pi}\right\}$$

Max $+M$ occurs at an angular position x_1 where $x_1 > \theta, x_1 > 108.6^\circ$, and x_1 is found from

$$(x_1 - \theta + s\cos x_1) + (3s - 2\pi + 2\theta - \theta c)\sin x_1 = 0$$

Max $-M = M_C$

TABLE 9.2 Formulas for circular rings (*Continued*)

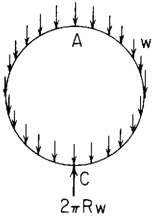
Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
14. Continued	$\Delta L = \begin{cases} \dfrac{wR^4}{EI(\pi-\theta)^2}\left\{k_1\dfrac{3s+2\theta-\theta c}{\pi}+\pi-2\theta-\dfrac{\pi s}{2}\right)+k_2\left[2+s-\theta+\dfrac{\pi^3}{8}+\dfrac{\theta^3}{6}-\dfrac{\theta\pi^2}{4}-(\pi-\theta)^2\right]+k_2^2\dfrac{(\pi-\theta)^3-6(\pi-\theta-s)}{3\pi}\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{wR^4}{EI(\pi-\theta)^2}\left\{k_1\left[\dfrac{3s+2\theta-\theta c}{\pi}-s(\pi-\theta)+3c\right]+k_2\left[\dfrac{(\pi-\theta)^3}{6}-(\pi-\theta)^2-\pi+\theta+s+2+2c\right]+\ k_2^2\dfrac{(\pi-\theta)^3-6(\pi-\theta-s)}{3\pi}\right\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$
15. Ring supported at base and loaded by own weight per unit length of circumference w  $LT_M = -wR^2[xz + K_T(u-1)]$ $LT_N = -wRxz$ $LT_V = -wRxu$	$M_A = wR^2\left[k_2 - 0.5 - \dfrac{(K_T-1)\beta}{k_1}\right]$ where $K_T = 1 + \dfrac{I}{AR^2}$ $M_C = wR^2\left[k_2 + 0.5 + \dfrac{(K_T-1)\beta}{k_1}\right]$ $N_A = wR\left[0.5 + \dfrac{(K_T-1)k_2}{k_1}\right]$ $V_A = 0$ $\Delta D_H = \dfrac{wR^3}{EAe}\left(\dfrac{k_1\pi}{2} - k_2\pi + 2k_2^2\right)$ $\Delta D_V = \dfrac{-wR^3}{EAe}\left(\dfrac{k_1\pi^2}{4} - 2k_2^2\right)$ $\Delta L = \dfrac{-wR^3}{EAe}\left[1 + \dfrac{3k_1\pi^2}{16} - \dfrac{k_2\pi}{2} - k_2^2 + (K_T-1)\alpha\right]$ *Note:* The constant K_T accounts for the radial distribution of mass in the ring. Max $+M = M_C$ Max $-M$ occurs at an angular position x_1 where $\dfrac{x_1}{\tan x_1} = -0.5 + \dfrac{(K_T-1)\beta}{k_1}$ For a thin ring where $K_T \approx 1$, Max $-M = -wR^2(1.6408 - k_2)$ at $x = 105.23^\circ$ If $\alpha = \beta = 0$, $M_A = \dfrac{wR^2}{2}$ $N_A = \dfrac{wR}{2}$ $\Delta D_H = 0.4292\dfrac{wR^4}{EI}$ $\Delta D_V = -0.4674\dfrac{wR^4}{EI}$ $\Delta L = -0.2798\dfrac{wR^4}{EI}$ Max $+M = \frac{3}{2}wR^2$ at C

TABLE 9.2 Formulas for circular rings (*Continued*)

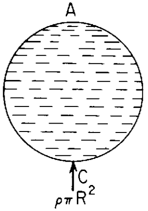
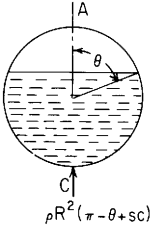
16. Unit axial segment of pipe filled with liquid of weight per unit volume ρ and supported at the base  $LT_M = \rho R^3\left(1 - u - \frac{xz}{2}\right)$ $LT_N = \rho R^2\left(1 - u - \frac{xz}{2}\right)$ $LT_V = \rho R^2\left(\frac{z}{2} - \frac{xu}{2}\right)$	$M_A = \rho R^3\left(0.75 - \frac{k_2}{2}\right)$ $M_C = \rho R^3\left(1.25 - \frac{k_2}{2}\right)$ $N_A = 0.75\rho R^2$ $V_A = 0$ $\Delta D_H = \frac{\rho R^5 12(1-\nu^2)}{Et^3}\left[\frac{k_1\pi}{4} + k_2\left(2 - \frac{\pi}{2}\right) - k_2^2\right]$ $\Delta D_V = \frac{-\rho R^5 12(1-\nu^2)}{Et^3}\left(\frac{k_1\pi^2}{8} - 2k_2 + k_2^2\right)$ $\Delta L = \frac{-\rho R^5 12(1-\nu^2)}{Et^3}\left[\frac{k_1 3\pi^2}{32} - k_2\left(0.5 + \frac{\pi}{4}\right) + \frac{k_2^2}{2}\right]$ *Note:* For this case and case 17, $\alpha = \frac{t^2}{12R^2(1-\nu^2)}$ $\beta = \frac{t^2}{6R^2(1-\nu)}$ where $t =$ pipe wall thickness	Max $+M = M_C$ Max $-M = -\rho R^3\left(\frac{k_2}{2} - 0.1796\right)$ at $x = 105.23^\circ$ If $\alpha = \beta = 0$, $\Delta D_H = 0.2146\frac{\rho R^5 12(1-\nu^2)}{Et^3}$ $\Delta D_V = -0.2337\frac{\rho R^5 12(1-\nu^2)}{Et^3}$ $\Delta L = -0.1399\frac{\rho R^5 12(1-\nu^2)}{Et^3}$
17. Unit axial segment of pipe partly filled with liquid of weight per unit volume ρ and supported at the base  	*Note:* see case 16 for expressions for α and β $M_A = \frac{\rho R^3}{4\pi}\{2\theta s^2 + 3sc - 3\theta + \pi + 2\pi c^2 + 2k_2[sc - 2s + (\pi - \theta)(1 - 2c)]\}$ $N_A = \frac{\rho R^2}{4\pi}[3sc + (\pi - \theta)(1 + 2c^2)]$ $V_A = 0$ $\Delta D_H = \begin{cases} \frac{\rho R^5 3(1-\nu^2)}{2Et^3\pi}\left\{k_1\pi(sc + 2\pi - 3\theta + 2\theta c^2) + 8k_2\pi\left(2c - sc - \frac{\pi}{2} + \theta\right) + 8k_2^2[(\pi - \theta)(1 - 2c) + sc - 2s]\right\} & \text{for } \theta \leqslant \frac{\pi}{2} \\ \frac{\rho R^5 3(1-\nu^2)}{2Et^3\pi}\{k_1\pi[(\pi - \theta)(1 + 2c^2) + 3sc] + 8k_2^2[(\pi - \theta)(1 - 2c) + sc - 2s]\} & \text{for } \theta \geqslant \frac{\pi}{2} \end{cases}$ $\Delta D_V = \frac{-\rho R^5 3(1-\nu^2)}{2Et^3\pi}\{k_1\pi[s^2 + (\pi - \theta)(\pi - \theta + 2sc)] - 4k_2\pi(1 + c)^2 - 8k_2^2[(\pi - \theta)(1 - 2c) + sc - 2s]\}$	

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
17. Continued	

$$LT_M = \frac{\rho R^3}{2}[2c - z(x - \theta + sc) - u(1 + c^2)]\langle x - \theta \rangle^0$$

$$LT_N = \frac{\rho R^2}{2}[2c - z(x - \theta + sc) - u(1 + c^2)]\langle x - \theta \rangle^0$$

$$LT_V = \frac{\rho R^2}{2}[zc^2 - u(x - \theta + sc)] \times \langle x - \theta \rangle^0$$

$$\Delta L = \begin{cases} \dfrac{-\rho R^5 3(1 - \nu^2)}{2Et^3\pi}\left\{k_1\left[2\theta c^2 + \theta - 3sc + \pi^2\left(sc - \theta + \dfrac{3\pi}{4}\right)\right] + 2k_2\pi[2 + 2\theta c - 2s - 4c - \pi + \theta - sc] - 4k_2^2[(\pi - \theta)(1 - 2c) + sc - 2s]\right\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-\rho R^5 3(1 - \nu^2)}{2Et^3\pi}\{k_1[2\theta c^2 + \theta - 3sc + \pi(\pi - \theta)(\pi - \theta + 2sc) - 3\pi c^2] + 2k_2\pi[2s - 2(1 + c)^2 - sc - (\pi - \theta)(1 - 2c)] - 4k_2^2[(\pi - \theta)(1 - 2c) + sc - 2s]\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\text{Max} + M = M_C = \frac{\rho R^3}{4\pi}\{4\pi c + \pi + 2\theta c^2 + \theta - 3sc + 2k_2[(\pi - \theta)(1 - 2c) + sc - 2s]\}$$

Max $-M$ occurs at an angular position where $x_1 > \theta, x_1 > 105.23^\circ$, and x_1 is found from

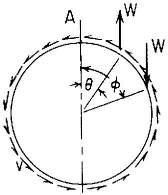
$$(\theta + 2\theta c^2 - 3sc - \pi)\tan x_1 + 2\pi(\theta - sc - x_1) = 0$$

If $\alpha = \beta = 0, M = K_M \rho R^3, N = K_N \rho R^2, \Delta D = K_{\Delta D}\rho R^5 12(1 - \nu^2)/Et^3$, etc.

θ	0°	30°	45°	60°	90°	120°	135°	150°
K_{M_A}	0.2500	0.2290	0.1935	0.1466	0.0567	0.0104	0.0027	0.00039
K_{N_A}	0.7500	0.6242	0.4944	0.3534	0.1250	0.0216	0.0056	0.00079
K_{M_C}	0.7500	0.7216	0.6619	0.5649	0.3067	0.0921	0.0344	0.00778
$K_{\Delta D_H}$	0.2146	0.2027	0.1787	0.1422	0.0597	0.0115	0.0031	0.00044
$K_{\Delta D_V}$	−0.2337	−0.2209	−0.1955	−0.1573	−0.0700	−0.0150	−0.0043	−0.00066
$K_{\Delta L}$	−0.1399	−0.1333	−0.1198	−0.0986	−0.0465	−0.0106	−0.0031	−0.00050

TABLE 9.2 Formulas for circular rings (*Continued*)

18.



$$v = \frac{W}{2\pi R}(\sin\phi - \sin\theta)$$

$$\begin{aligned} LT_M = & -\frac{WR}{2\pi}(n-s)(x-z) \\ & + WR(z-s)\langle x-\theta\rangle^0 \\ & - WR(z-n)\langle x-\phi\rangle^0 \end{aligned}$$

$$\begin{aligned} LT_N = & \frac{W}{2\pi}(n-s)z \\ & + Wz\langle x-\theta\rangle^0 \\ & - Wz\langle x-\phi\rangle^0 \end{aligned}$$

$$\begin{aligned} LT_V = & \frac{-W}{2\pi}(n-s)(1-u) \\ & + Wu\langle x-\theta\rangle^0 \\ & - Wu\langle x-\phi\rangle^0 \end{aligned}$$

$$M_A = \frac{WR}{2\pi}[n^2 - s^2 - (\pi-\phi)n + (\pi-\theta)s - k_2(c-m)]$$

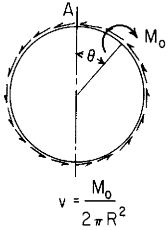
$$N_A = \frac{W}{2\pi}(n^2 - s^2)$$

$$V_A = \frac{W}{2\pi}(\theta - \phi + s - n + sc - nm)$$

If $\alpha = \beta = 0, M = K_M WR, N = K_N W, V = K_V W$

θ	$\phi - \theta$	30°	45°	60°	90°	120°	135°	150°	180°
0°	K_{M_A}	−0.1899	−0.2322	−0.2489	−0.2500	−0.2637	−0.2805	−0.2989	−0.3183
	K_{N_A}	0.0398	0.0796	0.1194	0.1592	0.1194	0.0796	0.0398	0.0000
	K_{V_A}	−0.2318	−0.3171	−0.3734	−0.4092	−0.4022	−0.4080	−0.4273	−0.5000
30°	K_{M_A}	−0.0590	−0.0613	−0.0601	−0.0738	−0.1090	−0.1231	−0.1284	−0.1090
	K_{N_A}	0.0796	0.1087	0.1194	0.0796	−0.0000	−0.0291	−0.0398	−0.0000
	K_{V_A}	−0.1416	−0.1700	−0.1773	−0.1704	−0.1955	−0.2279	−0.2682	−0.3408
45°	K_{M_A}	−0.0190	−0.0178	−0.0209	−0.0483	−0.0808	−0.0861	−0.0808	−0.0483
	K_{N_A}	0.0689	0.0796	0.0689	0.0000	−0.0689	−0.0796	−0.0689	−0.0000
	K_{V_A}	−0.0847	−0.0920	−0.0885	−0.0908	−0.1426	−0.1829	−0.2231	−0.2749
60°	K_{M_A}	−0.0011	−0.0042	−0.0148	−0.0500	−0.0694	−0.0641	−0.0500	−0.0148
	K_{N_A}	0.0398	0.0291	−0.0000	−0.0796	−0.1194	−0.1087	−0.0796	0.0000
	K_{V_A}	−0.0357	−0.0322	−0.0288	−0.0539	−0.1266	−0.1668	−0.1993	−0.2243
90°	K_{M_A}	−0.0137	−0.0305	−0.0489	−0.0683	−0.0489	−0.0305	−0.0137	0.0000
	K_{N_A}	−0.0398	−0.0796	−0.1194	−0.1592	−0.1194	−0.0796	−0.0398	0.0000
	K_{V_A}	0.0069	0.0012	−0.0182	−0.0908	−0.1635	−0.1829	−0.1886	−0.1817

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
19.  $LT_M = \frac{-M_o}{2\pi}(x-z) + M_o\langle x-\theta\rangle^0$ $LT_N = \frac{M_o z}{2\pi R}$ $LT_V = \frac{-M_o}{2\pi R}(1-u)$	$M_A = \frac{-M_o}{2\pi}\left(\pi - \theta - \frac{2k_2 s}{k_1}\right)$ $N_A = \frac{M_o}{\pi R}\left(\frac{k_2 s}{k_1}\right)$ $V_A = \frac{-M_o}{2\pi R}\left(1 + \frac{2k_2 c}{k_1}\right)$ Max $+M = \frac{M_o}{2}$ for x just greater than θ Max $-M = \frac{-M_o}{2}$ for x just less than θ At $x = \theta + 180^\circ, M = 0$ Other maxima are, for $\alpha = \beta = 0$ $M\begin{cases} -0.1090M_o & \text{at } x = \theta + 120^\circ \\ 0.1090M_0 & \text{at } x = \theta + 240^\circ \end{cases}$ If $\alpha = \beta = 0, M = k_M M_o, N = K_N M_o/R, V = K_V M_o/R$

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
K_{M_A}	−0.5000	−0.2575	−0.1499	−0.0577	0.0683	0.1090	0.1001	0.0758	0.0000
K_{N_A}	0.0000	0.1592	0.2251	0.2757	0.3183	0.2757	0.2250	0.1592	0.0000
K_{V_C}	−0.4775	−0.4348	−0.3842	−0.3183	−0.1592	0.0000	0.0659	0.1165	0.1592

TABLE 9.2 Formulas for circular rings (*Continued*)

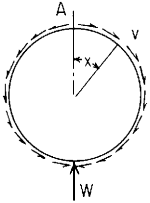
20. Bulkhead or supporting ring in pipe, supported at bottom and carrying total load W transferred by tangential shear v distributed as shown  $v = \frac{W \sin x}{\pi R}$ $LT_M = \frac{WR}{\pi}\left(1 - u - \frac{xz}{2}\right)$ $LT_N = \frac{-W}{2\pi} xz$ $LT_V = \frac{W}{2\pi}(z - xu)$ for $0 < x < 180^\circ$	$M_A = \frac{WR}{2\pi}(k_2 - 0.5)$ $M_C = \frac{WR}{2\pi}(k_2 + 0.5)$ $N_A = \frac{0.75W}{\pi}$ $V_A = 0$ $\Delta D_H = \frac{WR^3}{EI}\left(\frac{k_1}{4} - \frac{k_2}{2} + \frac{k_2^2}{\pi}\right)$ $\Delta D_V = \frac{-WR^3}{EI}\left(\frac{k_1 \pi}{8} - \frac{k_2^2}{\pi}\right)$ $\Delta L = \frac{-WR^3}{4EI\pi}\left[4 + k_1 \frac{3\pi^2}{8} - k_2(\pi + 2) - 2k_2^2\right]$	Max $+M = M_C$ Max $-M = \frac{-WR}{4\pi}(3.2815 - 2k_2)$ at $x = 105.2^\circ$ If $\alpha = \beta = 0$, $M_A = 0.0796WR$ $N_A = 0.2387W$ $V_A = 0$ $\Delta D_H = 0.0683\frac{WR^3}{EI}$ $\Delta D_V = -0.0744\frac{WR^3}{EI}$ $\Delta R = -0.0445\frac{WR^3}{EI}$

TABLE 9.2 Formulas for circular rings (*Continued*)

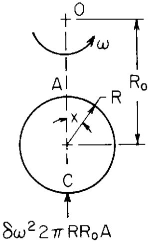
Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
21. Ring rotating at angular rate ω rad/s about an axis perpendicular to the plane of the ring. Note the requirement of symmetry of the cross section in Sec. 9.3.  $\delta =$ mass density of ring material $LT_M = \delta\omega^2 AR^3 \left\{ K_T(1-u) - \frac{R_o}{R}[xz - K_T(1-u)] \right\}$ $LT_N = \delta\omega^2 AR^2 \left[K_T(1-u) - \frac{R_o}{R} xz \right]$ $LT_V = \delta\omega^2 AR^2 \left[zK_T(2u-1) - \frac{R_o}{R} xu \right]$	$M_A = \delta\omega^2 AR^3 \left\{ K_T\alpha + \frac{R_o}{R}\left[k_2 - 0.5 - \frac{(K_T-1)\beta}{k_1} \right] \right\}$ where $K_T = 1 + \frac{I}{AR^2}$ $M_C = \delta\omega^2 AR^3 \left\{ K_T\alpha + \frac{R_o}{R}\left[k_2 + 0.5 + \frac{(K_T-1)\beta}{k_1} \right] \right\}$ $N_A = \delta\omega^2 AR^2 \left\{ K_T + \frac{R_o}{R}\left[0.5 + (K_T-1)\frac{k_2}{k_1} \right] \right\}$ $V_A = 0$ $\Delta D_H = \frac{\delta\omega^2 R^4}{Ee}\left[2K_T k_2 \alpha + \frac{R_o}{R}\left(\frac{k_1\pi}{2} - k_2\pi + 2k_2^2 \right) \right]$ $\Delta D_V = \frac{\delta\omega^2 R^4}{Ee}\left[2K_T k_2 \alpha - \frac{R_o}{R}\left(\frac{k_1\pi^2}{4} - 2k_2^2 \right) \right]$ $\Delta L = \frac{\delta\omega^2 R^4}{Ee}\left[K_T k_2 \alpha - \frac{R_o}{R}\left(\frac{k_1 3\pi^2}{16} + k_2 - \frac{k_2\pi}{2} - k_2^2 + K_T\alpha \right) \right]$ *Note:* The constant K_T accounts for the radial distribution of mass in the ring. Max $+M = M_C$ Max $-M$ occurs at an angular position x_1 where $\frac{x_1}{\tan x_1} = -0.5 + \frac{(K_T-1)\beta}{k_1}$ For a thin ring where $K_T \approx 1$, Max $-M = -\delta\omega^2 AR^3 \left[\frac{R_o}{R}(1.6408 - k_2) - \alpha \right]$ at $x = 105.23°$

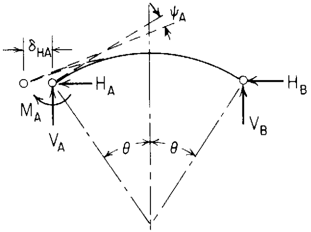
TABLE 9.3 Reaction and deformation formulas for circular arches

NOTATION: $W =$ load (force); $w =$ unit load (force per unit of circumferential length); $M_o =$ applied couple (force-length). $\theta_o =$ externally created concentrated angular displacement (radians); $\Delta_o =$ externally created concentrated radial displacement; $T - T_o =$ uniform temperature rise (degrees); T_1 and $T_2 =$ temperatures on outside and inside, respectively (degrees). H_A and H_B are the horizontal end reactions at the left and right, respectively, and are positive to the left; V_A and V_B are the vertical end reactions at the left and right ends, respectively, and are positive upward; M_A and M_B are the reaction moments at the left and right, respectively, and are positive clockwise. $E =$ modulus of elasticity (force per unit area); $\nu =$ Poisson's ratio; A is the cross-sectional area; R is the radius ot the centroid of the cross section; $I =$ area moment of inertia of arch cross section about the principal axis perpendicular to the plane of the arch. [Note that for a wide curved plate or a sector of a cylinder, a representative segment of unit axial length may be used by replacing EI by $Et^3/12(1-\nu^2)$.] e is the positive distance measured radially inward from the centroidal axis of the cross section to the neutral axis of pure bending (see Sec. 9.1). θ (radians) is one-half of the total subtended angle of the arch and is limited to the range zero to π. For an angle θ close to zero, round-off errors may cause troubles; for an angle θ close to π, the possibility of static or elastic instability must be considered. Deformations have been assumed small enough so as to not affect the expressions for the internal bending moments, radial shear, and circumferential normal forces. Answers should be examined to be sure that such is the case before accepting them. ϕ (radians) is the angle measured counterclockwise from the midspan of the arch to the position of a concentrated load or the start of a distributed load. $s = \sin\theta$, $c = \cos\theta$, $n = \sin\phi$, and $m = \cos\phi$. $\gamma =$ temperature coefficient of expansion.

The references to end points A and B refer to positions on a circle of radius R passing through the centroids of the several sections. It is important to note this carefully when dealing with thick rings. Similarly, all concentrated and distributed loadings are assumed to be applied at the radial position of the centroid with the exception of cases h and i where the ring is loaded by its own weight or by a constant linear acceleration. In these two cases the actual radial distribution of load is considered. If the loading is on the outer or inner surfaces of thick rings, a statically equivalent loading at the centroidal radius R must be used. See examples to determine how this might be accomplished.

The hoop-stress deformation factor is $\alpha = I/AR^2$ for thin rings or $\alpha = e/R$ for thick rings. The transverse- (radial-) shear deformation factor is $\beta = FEI/GAR^2$ for thin rings or $\beta = 2F(1+\nu)e/R$ for thick rings, where G is the shear modulus of elasticity and F is a shape factor for the cross section (see Sec. 8.10). The following constants are defined to simplify the expressions which follow. Note that these constants are unity if no correction for hoop stress or shear stress is necessary or desired for use with thin rings. $k_1 = 1 - \alpha + \beta$, $k_2 = 1 - \alpha$.

General reaction and expressions for cases 1–4; right end pinned in all four cases, no vertical motion at the left end



Deformation equations:

$$\text{Horizontal deflection at } A = \delta_{HA} = \frac{R^3}{EI}\left(A_{HH}H_A + A_{HM}\frac{M_A}{R} - LP_H\right)$$

$$\text{Angular rotation at } A = \psi_A = \frac{R^2}{EI}\left(A_{MH}H_A + A_{MM}\frac{M_A}{R} - LP_M\right)$$

where $A_{HH} = 2\theta c^2 + k_1(\theta - sc) - k_2 2sc$

$A_{MH} = A_{HM} = k_2 s - \theta c$

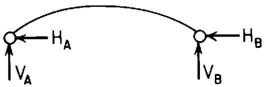
$A_{MM} = \frac{1}{4s^2}[2\theta s^2 + k_1(\theta + sc) - k_2 2sc]$

and where LP_H and LP_M are loading terms given below for several types of load.

(*Note:* If desired, V_A, V_B, and H_B can be evaluated from equilibrium equations after calculating H_A and M_A)

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

1. Left end pinned, right end pinned



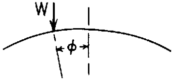
Since $\delta_{HA} = 0$ and $M_A = 0$,

$$H_A = \frac{LP_H}{A_{HH}} \quad \text{and} \quad \psi_A = \frac{R^2}{EI}(A_{MH}H_A - LP_M)$$

The loading terms are given below.

Reference no., loading	Loading terms and some selected numerical values

1a. Concentrated vertical load



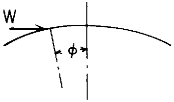
$$LP_H = W\left[\theta sc - \phi nc + \frac{k_1}{2}(c^2 - m^2) + k_2 c(c - m)\right]$$

$$LP_M = \frac{W}{2}\left[\phi n - \theta s + \frac{k_1}{2s^2}(\theta n - \phi s + snc - snm) - k_2(c - m)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{W}$	−0.0143	−0.0100	−0.1715	−0.1105	−0.5000	−0.2500
$\frac{LP_M}{W}$	−0.0639	−0.0554	−0.2034	−0.1690	−0.2854	−0.1978

1b. Concentrated horizontal load



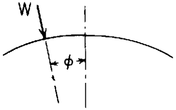
$$LP_H = W\left[\theta c^2 + \phi mc + \frac{k_1}{2}(\theta + \phi - sc - nm) - k_2 c(s + n)\right]$$

$$LP_M = \frac{W}{2}\left[-\theta c - \phi m + \frac{k_1}{2s^2}(\theta c - \theta m + sm^2 - scm) + k_2(s + n)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{W}$	0.0050	0.0057	0.1359	0.1579	0.7854	0.9281
$\frac{LP_M}{W}$	0.0201	0.0222	0.1410	0.1582	0.3573	0.4232

1c. Concentrated radial load



$$LP_H = W\left[\theta c(cn + sm) + \frac{k_1}{2}(\theta n + \phi n - scn - s^2 m) - k_2 c(1 + sn - cm)\right]$$

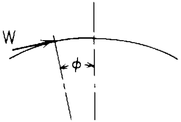
$$LP_M = \frac{W}{2}\left[-\theta(cn + sm) + \frac{k_1}{2s^2}(\theta cn - \phi sm) + k_2(1 + sn - cm)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{W}$	−0.0143	−0.0082	−0.1715	−0.0167	−0.5000	0.4795
$\frac{LP_M}{W}$	−0.0639	−0.0478	−0.2034	−0.0672	−0.2854	0.1594

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

1d. Concentrated tangential load



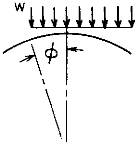
$$LP_H = W\left[\theta c(cm - sn) + \phi c + \frac{k_1}{2}(\theta m + \phi m - scm - c^2 n) - k_2 c(sm + cn)\right]$$

$$LP_M = \frac{W}{2}\left[\theta(sn - cm) - \phi + \frac{k_1}{2s^2}(\theta cm - \theta + \phi sn - sc + sm) + k_2(sm + cn)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{W}$	0.0050	0.0081	0.1359	0.1920	0.7854	0.8330
$\frac{LP_M}{W}$	0.0201	0.0358	0.1410	0.2215	0.3573	0.4391

1e. Uniform vertical load on partial span



$$LP_H = \frac{wR}{4}\left[\theta c(1 + 4sn + 2s^2) + \phi c(m^2 - n^2) - c(sc + mn)\right.$$

$$\left. + \frac{2k_1}{3}(n^3 - 3ns^2 - 2s^3) + 2k_2 c(2cn + cs - \theta - \phi - mn)\right]$$

$$LP_M = \frac{wR}{8}\left\{mn + sc - \theta(4sn + 2s^2 + 1) - \phi(m^2 - n^2)\right.$$

$$+ \frac{k_1}{s}\left[\frac{\theta}{s}(n^2 + s^2) + 2(c - m) - \tfrac{2}{3}(c^3 - m^3) + c(n^2 - s^2) - 2\phi n\right]$$

$$\left. + 2k_2(\theta + \phi + mn - sc - 2cn)\right\}$$

If $\phi = 0$ (the full span is loaded)

$$LP_H = \frac{wR}{6}[3c(2\theta s^2 + \theta - sc) - 4k_1 s^3 + 6k_2 c(sc - \theta)]$$

$$LP_M = \frac{wR}{4}[sc - \theta - 2\theta s^2 + 2k_2(\theta - sc)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{wR}$	−0.0046	−0.0079	−0.0969	−0.1724	−0.3333	−0.6280
$\frac{LP_M}{wR}$	−0.0187	−0.0350	−0.1029	−0.2031	−0.1667	−0.3595

1f. Uniform horizontal load on left side only

$$LP_H = \frac{wR}{12}[3\theta c(1 - 6c^2 + 4c) + 3sc^2 + k_1(6\theta - 6sc - 12\theta c + 12c - 8s^3)$$

$$+ 6k_2 c(3sc - 2s - \theta)]$$

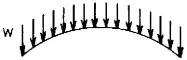
$$LP_M = \frac{wR}{8}\left\{6\theta c^2 - \theta - 4\theta c - sc + \frac{k_1}{3s^2}[s(2 - 3c + c^3) - 3\theta(1 - c)^2]\right.$$

$$\left. + 2k_2(\theta + 2s - 3sc)\right\}$$

For $\alpha = \beta = 0$

θ	30°	60°	90°
$\frac{LP_H}{wR}$	0.0010	0.0969	1.1187
$\frac{LP_M}{wR}$	0.0040	0.1060	0.5833

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

<table>
<tr><th>Reference no., loading</th><th colspan="2">Loading terms and some selected numerical values</th></tr>
<tr><td>1g. Uniform horizontal load on right side only
 </td><td>$LP_H = \frac{wR}{12}[3\theta c(1+2c^2-4c)+3sc^2+2k_1(2s^3-3\theta+3sc)+6k_2c(2s-sc-\theta)]$
$LP_M = \frac{wR}{8}\left\{4\theta c-2\theta c^2-\theta-sc-\frac{k_1}{3s^2}[s(2-3c+c^3)-3\theta(1-c)^2] + 2k_2(\theta-2s+sc)\right\}$</td><td>For $\alpha=\beta=0$<table><tr><th>θ</th><th>30°</th><th>60°</th><th>90°</th></tr><tr><td>$\frac{LP_H}{wR}$</td><td>−0.0004</td><td>−0.0389</td><td>−0.4521</td></tr><tr><td>$\frac{LP_M}{wR}$</td><td>−0.0015</td><td>−0.0381</td><td>−0.1906</td></tr></table></td></tr>
<tr><td>1h. Vertical loading uniformly distributed along the circumference (by gravity or linear acceleration)

(*Note:* The full span is loaded)</td><td>$LP_H = wR\left\{2\theta^2 sc+\left(\frac{k_1}{2}+k_2\right)(2\theta c^2-\theta-sc)+\frac{R_{cg}}{R}[k_2(\theta-sc)-2c(s-\theta c)]\right\}$
$LP_M = wR\left[\left(\frac{R_{cg}}{R}+k_2\right)(s-\theta c)-\theta^2 s\right]$
where R_{cg} is the radial distance to the center of mass for a differential length of the circumference for radially thicker arches. $R_{cg}/R = 1+I_c/(AR^2)$. I_c is the area moment of inertia about the centroidal axis of the cross section. For radially thin arches let $R_{cg}=R$. See the discussion on page 333.</td><td>For $\alpha=\beta=0$ and for $R_{cg}=R$<table><tr><th>θ</th><th>30°</th><th>60°</th><th>90°</th></tr><tr><td>$\frac{LP_H}{wR}$</td><td>−0.0094</td><td>−0.2135</td><td>−0.7854</td></tr><tr><td>$\frac{LP_M}{wR}$</td><td>−0.0440</td><td>−0.2648</td><td>−0.4674</td></tr></table></td></tr>
<tr><td>1i. Horizontal loading uniformly distributed along the circumference (by gravity or linear acceleration)

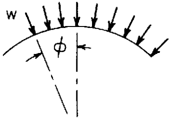
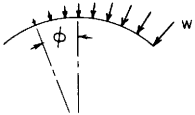
(*Note:* The full span is loaded)</td><td>$LP_H = wR\theta[2\theta c^2+k_1(\theta-sc)-2k_2 sc]$
$LP_M = \frac{wR}{2s}\left[-2\theta^2 sc+\frac{k_1}{2s}(2\theta^2 c+\theta s+s^2 c)+k_2(2\theta s^2-\theta-sc) - \frac{R_{cg}}{R}(k_1-k_2)(\theta+sc)\right]$
See case 1h for a definition of the radius R_{cg}</td><td>For $\alpha=\beta=0$ and for $R_{cg}=R$<table><tr><th>θ</th><th>30°</th><th>60°</th><th>90°</th></tr><tr><td>$\frac{LP_H}{wR}$</td><td>0.0052</td><td>0.2846</td><td>2.4674</td></tr><tr><td>$\frac{LP_M}{wR}$</td><td>0.0209</td><td>0.2968</td><td>1.1781</td></tr></table></td></tr>
<tr><td>1j. Partial uniformly distributed radial loading
 </td><td>$LP_H = wRc\left[\theta(1-cm+sn)+\frac{k_1}{2c}(scm+c^2n-\theta m-\phi m)+k_2(sm+cn-\theta-\phi)\right]$
$LP_M = \frac{wR}{2}\left[\theta(cm-1-sn)+\frac{k_1}{2s^2}[\theta-\theta cm-\phi sn+sc-sm]+k_2(\theta+\phi-sm-cn)\right]$
If $\phi=\theta$ (the full span is loaded)
$LP_H = wRc[2\theta s^2-k_1(\theta-sc)-2k_2(\theta-sc)]$
$LP_M = wR[-\theta s^2+k_2(\theta-sc)]$</td><td>For $\alpha=\beta=0$<table><tr><th>θ</th><th colspan="2">30°</th><th colspan="2">60°</th><th colspan="2">90°</th></tr><tr><td>ϕ</td><td>0°</td><td>15°</td><td>0°</td><td>30°</td><td>0°</td><td>45°</td></tr><tr><td>$\frac{LP_H}{wR}$</td><td>−0.0050</td><td>−0.0081</td><td>−0.1359</td><td>−0.1920</td><td>−0.7854</td><td>−0.8330</td></tr><tr><td>$\frac{LP_M}{wR}$</td><td>−0.0201</td><td>−0.0358</td><td>−0.1410</td><td>−0.2215</td><td>−0.3573</td><td>−0.4391</td></tr></table></td></tr>
</table>

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

1k. Partial uniformly increasing distributed radial loading



$$LP_H = \frac{wR}{\theta+\phi}\left\{\theta c(\theta+\phi-cn-sm)+\frac{k_1}{2}[n(sc-\theta-\phi)+s^2m+2c-2m]\right.$$

$$\left.+k_2\left[scn+s^2m-\frac{c}{2}(\theta+\phi)^2+c-m\right]\right\}$$

$$LP_M = \frac{wR}{\theta+\phi}\left\{\frac{\theta}{2}(cn+sm-\theta-\phi)+\frac{k_1}{4s^2}[(\theta+\phi)(\theta+sc)+\phi sm-\theta cn\right.$$

$$\left.-2s^2-2sn]+\frac{k_2}{4}[(\theta+\phi)^2+2cm-2sn-2]\right\}$$

If $\phi = \theta$ (the full span is loaded)

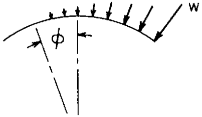
$$LP_H = \frac{wRc}{\theta}\left[\theta(\theta-cs)-\frac{k_1 s}{2c}(\theta-sc)-k_2(\theta^2-s^2)\right]$$

$$LP_M = \frac{wR}{2\theta}\left[\theta sc-\theta^2+\frac{k_1}{2s^2}(\theta^2+\theta sc-2s^2)+k_2(\theta^2-s^2)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{wR}$	−0.0018	−0.0035	−0.0518	−0.0915	−0.3183	−0.5036
$\frac{LP_M}{wR}$	−0.0072	−0.0142	−0.0516	−0.0968	−0.1366	−0.2335

1l. Partial second-order increase in distributed radial loading



$$LP_H = \frac{wRc}{(\theta+\phi)^2}\left\{\theta(\theta+\phi)^2-2\theta(1-cm+sn)+\frac{k_1}{c}[(\theta+\phi)(2c+m)\right.$$

$$\left.-c(sm+cn)-2n-2s]+\frac{k_2}{3}[6(\theta+\phi-sm-cn)-(\theta+\phi)^3]\right\}$$

$$LP_M = \frac{wR}{(\theta+\phi)^2}\left\{\theta(1-cm+sn)-\frac{\theta}{2}(\theta+\phi)^2\right.$$

$$+\frac{k_1}{4s^2}[(\theta+\phi)^2(\theta+sc)+2s(\phi n+3m-3c)-4s^2(\theta+\phi)-2\theta(1-cm)]$$

$$\left.+k_2\left[sm+cn-\theta-\phi+\frac{(\theta+\phi)^3}{6}\right]\right\}$$

If $\phi = \theta$ (the full span is loaded)

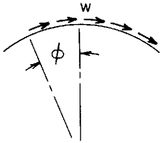
$$LP_H = \frac{wR}{6\theta^2}[6\theta^3c-6\theta s^2c+3k_1(3\theta c-3s+s^3)+2k_2(3\theta c-2\theta^3c-3sc^2)]$$

$$LP_M = \frac{wR}{2\theta^2}\left[\theta s^2-\theta^3+k_1\frac{\theta}{2s^2}(\theta^2+\theta sc-2s^2)+k_2\left(sc-\theta+\frac{2\theta^3}{3}\right)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{wR}$	−0.0010	−0.0019	−0.0276	−0.0532	−0.1736	−0.3149
$\frac{LP_M}{wR}$	−0.0037	−0.0077	−0.0269	−0.0542	−0.0726	−0.1388

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

<table>
<tr><th>Reference no., loading</th><th colspan="2">Loading terms and some selected numerical values</th></tr>
<tr>
<td>1m. Partial uniformly distributed tangential loading

</td>
<td>

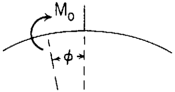
$$LP_H = \frac{wR}{2}\{2\theta c(cn + sm) - c(\theta^2 - \phi^2) + k_1[n(\theta + \phi) + c(cm - sn - 2) + e] + k_2 2c(cm - sn - 1)\}$$

$$LP_M = \frac{wR}{4}\left\{\theta^2 - \phi^2 - 2\theta(cn + sm) + \frac{k_1}{s^2}[\theta(cn - cs - \theta - \phi) - \phi s(c + m) + 2s(s + n)] + k_2 2(1 + sn - cm)\right\}$$

If $\phi = \theta$ (the full span is loaded)

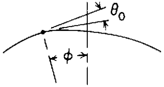
$$LP_H = wR[2\theta c^2 s + k_1 s(\theta - sc) - 2k_2 cs^2]$$

$$LP_M = wR\left[-\theta sc + \frac{k_1}{2s^2}(2s^2 - \theta sc - \theta^2) + k_2 s^2\right]$$

</td>
<td>For $\alpha = \beta = 0$
<table>
<tr><td>θ</td><td colspan="2">30°</td><td colspan="2">60°</td><td colspan="2">90°</td></tr>
<tr><td>ϕ</td><td>0°</td><td>15°</td><td>0°</td><td>30°</td><td>0°</td><td>45°</td></tr>
<tr><td>$\frac{LP_H}{wR}$</td><td>0.0010</td><td>0.0027</td><td>0.0543</td><td>0.1437</td><td>0.5000</td><td>1.1866</td></tr>
<tr><td>$\frac{LP_M}{wR}$</td><td>0.0037</td><td>0.0112</td><td>0.0540</td><td>0.1520</td><td>0.2146</td><td>0.5503</td></tr>
</table>
</td>
</tr>
<tr>
<td>1n. Concentrated couple

</td>
<td>

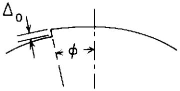
$$LP_H = \frac{M_o}{R}(\phi c - k_2 n)$$

$$LP_M = \frac{M_o}{4s^2 R}[-2s^2\phi - k_1(\theta + sc) + k_2 2sm]$$

</td>
<td>For $\alpha = \beta = 0$
<table>
<tr><td>θ</td><td colspan="2">30°</td><td colspan="2">60°</td><td colspan="2">90°</td></tr>
<tr><td>ϕ</td><td>0°</td><td>15°</td><td>0°</td><td>30°</td><td>0°</td><td>45°</td></tr>
<tr><td>$\frac{LP_H R}{M_o}$</td><td>0.0000</td><td>−0.0321</td><td>0.0000</td><td>−0.2382</td><td>0.0000</td><td>−0.7071</td></tr>
<tr><td>$\frac{LP_M R}{M_o}$</td><td>0.0434</td><td>−0.1216</td><td>0.0839</td><td>−0.2552</td><td>0.1073</td><td>−0.4318</td></tr>
</table>
</td>
</tr>
<tr>
<td>1p. Concentrated angular displacement

</td>
<td colspan="2">

$$LP_H = \frac{\theta_o EI}{R^2}(m - c)$$

$$LP_M = \frac{\theta_o EI}{R^2}\left(\frac{1}{2} + \frac{n}{2s}\right)$$

</td>
</tr>
<tr>
<td>1q. Concentrated radial displacement

</td>
<td colspan="2">

$$LP_H = \frac{\Delta_o EI}{R^3}n$$

$$LP_M = \frac{\Delta_o EI}{R^3}\left(-\frac{m}{2s}\right)$$

</td>
</tr>
</table>

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

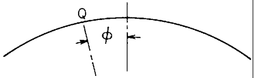
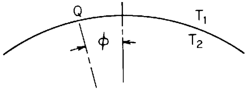
1r. Uniform temperature rise over that span to the right of point Q 	$LP_H = -(T - T_o)\frac{\gamma EI}{R^2}(s + n)$ $LP_M = (T - T_o)\frac{\gamma EI}{2R^2 s}(m - c)$ $T =$ uniform temperature $T_o =$ unloaded temperature
1s. Linear temperature differential through the thickness t for that span to the right of point Q 	$LP_H = (T_1 - T_2)\frac{\gamma EI}{Rt}(n + s - \theta c - \phi c)$ $LP_M = (T_1 - T_2)\frac{\gamma EI}{2Rts}(\theta s + \phi s - m + c)$ where t is the radial thickness and T_o, the unloaded temperature, is the temperature at the radius of the centroid
2. Left end guided horizontally, right end pinned 	Since $\psi_A = 0$ and $H_A = 0$ $M_A = \frac{LP_M}{A_{MM}}R$ and $\delta_{HA} = \frac{R^3}{EI}\left(A_{HM}\frac{M_A}{R} - LP_H\right)$ Use load terms given above for cases 1a–1s
3. Left end roller supported in vertical direction only, right end pinned 	Since both M_A and H_A are zero, this is a statically determinate case: $\delta_{HA} = \frac{-R^3}{EI}LP_H$ and $\psi_A = \frac{-R^2}{EI}LP_M$ Use load terms given above for cases 1a–1s

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

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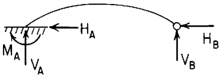
4. Left end fixed, right end pinned

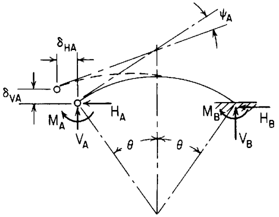


Since $\delta_{HA} = 0$ and $\psi_A = 0$,

$$H_A = \frac{A_{MM}LP_H - A_{HM}LP_M}{A_{HH}A_{MM} - A_{HM}^2} \qquad \text{and} \qquad \frac{M_A}{R} = \frac{A_{HH}LP_M - A_{HM}LP_H}{A_{HH}A_{MM} - A_{HM}^2}$$

Use load terms given above for cases 1a–1s

General reaction and deformation expressions for cases 5–14, right end fixed in all 10 cases.



Deformation equations:

$$\text{Horizontal deflection at } A = \delta_{HA} = \frac{R^3}{EI}\left(B_{HH}H_A + B_{HV}V_A + B_{HM}\frac{M_A}{R} - LF_H\right)$$

$$\text{Vertical deflection at } A = \delta_{VA} = \frac{R^3}{EI}\left(B_{VH}H_A + B_{VV}V_A + B_{VM}\frac{M_A}{R} - LF_V\right)$$

$$\text{Angular rotation at } A = \psi_A = \frac{R^2}{EI}\left(B_{MH}H_A + B_{MV}V_A + B_{MM}\frac{M_A}{R} - LF_M\right)$$

where $B_{HH} = 2\theta c^2 + k_1(\theta - sc) - k_2 2sc$
$B_{HV} = B_{VH} = -2\theta sc + k_2 2s^2$
$B_{HM} = B_{MH} = -2\theta c + k_2 2s$
$B_{VV} = 2\theta s^2 + k_1(\theta + sc) - k_2 2sc$
$B_{VM} = B_{MV} = 2\theta s$
$B_{MM} = 2\theta$

and where LF_H, LF_V, and LF_M are loading terms given below for several types of load

(*Note:* If desired, H_B, V_B, and M_B can be evaluated from equilibrium equations after calculating H_A, V_A, and M_A)

---

5. Left end fixed, right end fixed

Since $\delta_{HA} = 0, \delta_{VA} = 0, \psi_A = 0$, these equations must be solved simultaneously for H_A, V_A, and M_A/R
The loading terms are given in cases 5a–5s

$$B_{HH}H_A + B_{HV}V_A + B_{HM}M_A/R = LF_H$$

$$B_{VH}H_A + B_{VV}V_A + B_{VM}M_A/R = LF_V$$

$$B_{MH}H_A + B_{MV}V_A + B_{MM}M_A/R = LF_M$$

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

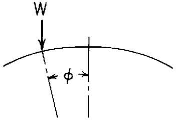
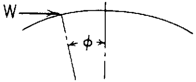
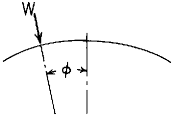
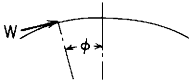
<table>
<tr><th>Reference no., loading</th><th colspan="2">Loading terms and some selected numerical values</th></tr>
<tr>
<td>5a. Concentrated vertical load
 </td>
<td>$LF_H = W\left[-(\theta+\phi)cn + \frac{k_1}{2}(c^2-m^2) + k_2(1+sn-cm)\right]$
$LF_V = W\left[(\theta+\phi)sn + \frac{k_1}{2}(\theta+\phi+sc+nm) - k_2(2sc-sm+cn)\right]$
$LF_M = W[(\theta+\phi)n + k_2(m-c)]$</td>
<td>For $\alpha = \beta = 0$
<table>
<tr><td>θ</td><td colspan="2">30°</td><td colspan="2">60°</td><td colspan="2">90°</td></tr>
<tr><td>ϕ</td><td>0°</td><td>15°</td><td>0°</td><td>30°</td><td>0°</td><td>45°</td></tr>
<tr><td>$\frac{LF_H}{W}$</td><td>0.0090</td><td>0.0253</td><td>0.1250</td><td>0.3573</td><td>0.5000</td><td>1.4571</td></tr>
<tr><td>$\frac{LF_V}{W}$</td><td>0.1123</td><td>0.2286</td><td>0.7401</td><td>1.5326</td><td>1.7854</td><td>3.8013</td></tr>
<tr><td>$\frac{LF_M}{W}$</td><td>0.1340</td><td>0.3032</td><td>0.5000</td><td>1.1514</td><td>1.0000</td><td>2.3732</td></tr>
</table></td>
</tr>
<tr>
<td>5b. Concentrated horizontal load
 </td>
<td>$LF_H = W\left[(\theta+\phi)mc + \frac{k_1}{2}(\theta+\phi-sc-nm) - k_2(sm+cn)\right]$
$LF_V = W\left[-(\theta+\phi)sm + \frac{k_1}{2}(c^2-m^2) + k_2(1-2c^2+cm+sn)\right]$
$LF_M = W[-(\theta+\phi)m + k_2(s+n)]$</td>
<td>For $\alpha = \beta = 0$
<table>
<tr><td>θ</td><td colspan="2">30°</td><td colspan="2">60°</td><td colspan="2">90°</td></tr>
<tr><td>ϕ</td><td>0°</td><td>15°</td><td>0°</td><td>30°</td><td>0°</td><td>45°</td></tr>
<tr><td>$\frac{LF_H}{W}$</td><td>−0.0013</td><td>0.0011</td><td>−0.0353</td><td>0.0326</td><td>−0.2146</td><td>0.2210</td></tr>
<tr><td>$\frac{LF_V}{W}$</td><td>−0.0208</td><td>−0.0049</td><td>−0.2819</td><td>−0.0621</td><td>−1.0708</td><td>−0.2090</td></tr>
<tr><td>$\frac{LF_M}{W}$</td><td>−0.0236</td><td>0.0002</td><td>−0.1812</td><td>0.0057</td><td>−0.5708</td><td>0.0410</td></tr>
</table></td>
</tr>
<tr>
<td>5c. Concentrated radial load
 </td>
<td>$LF_H = W\left[\frac{k_1}{2}(\theta n+\phi n-scn-s^2m) + k_2(m-c)\right]$
$LF_V = W\left[\frac{k_1}{2}(\theta m+\phi m+scm+c^2n) + k_2(s+n-2scm-2c^2n)\right]$
$LF_M = W[k_2(1+sn-cm)]$</td>
<td>For $\alpha = \beta = 0$
<table>
<tr><td>θ</td><td colspan="2">30°</td><td colspan="2">60°</td><td colspan="2">90°</td></tr>
<tr><td>ϕ</td><td>0°</td><td>15°</td><td>0°</td><td>30°</td><td>0°</td><td>45°</td></tr>
<tr><td>$\frac{LF_H}{W}$</td><td>0.0090</td><td>0.0248</td><td>0.1250</td><td>0.3257</td><td>0.5000</td><td>1.1866</td></tr>
<tr><td>$\frac{LF_V}{W}$</td><td>0.1123</td><td>0.2196</td><td>0.7401</td><td>1.2962</td><td>1.7854</td><td>2.5401</td></tr>
<tr><td>$\frac{LF_M}{W}$</td><td>0.1340</td><td>0.2929</td><td>0.5000</td><td>1.0000</td><td>1.0000</td><td>1.7071</td></tr>
</table></td>
</tr>
</table>

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

Reference no., loading	Loading terms and some selected numerical values

5d. Concentrated tangential load



$$LF_H = W\left[(\theta+\phi)c + \frac{k_1}{2}(\theta m + \phi m - scm - c^2 n) - k_2(s+n)\right]$$

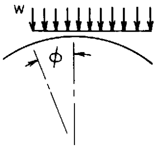
$$LF_V = W\left[-(\theta+\phi)s - \frac{k_1}{2}(\theta n + \phi n + scn + s^2 m) + k_2(2s^2 m + 2scn + c - m)\right]$$

$$LF_M = W[-\theta - \phi + k_2(sm + cn)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{W}$	−0.0013	−0.0055	−0.0353	−0.1505	−0.2146	−0.8741
$\frac{LF_V}{W}$	−0.0208	−0.0639	−0.2819	−0.8200	−1.0708	−2.8357
$\frac{LF_M}{W}$	−0.0236	−0.0783	−0.1812	−0.5708	−0.5708	−1.6491

5e. Uniform vertical load on partial span



$$LF_H = \frac{wR}{4}\left\{c[(1-2n^2)(\theta+\phi) - sc - mn] - \frac{2k_1}{3}(2s^3 + 3s^2 n - n^3) + 2k_2[s + 2n + sn^2 - c(\theta+\phi+mn)]\right\}$$

$$LF_V = \frac{wR}{4}\left\{s[(1-2m^2)(\theta+\phi) + sc + mn] + \frac{2k_1}{3}[3n(\theta+\phi+sc) + 3m - m^3 - 2c^3] + 2k_2[s(\theta+\phi-2sc+nm-4cn) - cn^2]\right\}$$

$$LF_M = \frac{wR}{4}[(1-2m^2)(\theta+\phi) + nm + sc + 2k_2(\theta+\phi+nm-sc-2cn)]$$

If $\phi = \theta$ (the full span is loaded)

$$LF_H = \frac{wR}{2}\left[\theta c(1-2s^2) - sc^2 - \frac{k_1 4s^3}{3} + k_2 2(s^3 + s - c\theta)\right]$$

$$LF_V = wR\left[\frac{s}{2}(\theta s^2 - \theta c^2 + sc) + k_1 s(\theta + sc) + k_2 s(\theta - 3sc)\right]$$

$$LF_M = wR[\tfrac{1}{2}(\theta s^2 - \theta c^2 + sc) + k_2(\theta - sc)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	0.0012	0.0055	0.0315	0.1471	0.1667	0.8291
$\frac{LF_V}{wR}$	0.0199	0.0635	0.2371	0.7987	0.7260	2.6808
$\frac{LF_M}{wR}$	0.0226	0.0778	0.1535	0.5556	0.3927	1.5531

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

5f. Uniform horizontal load on left side only

$$LF_H = \frac{wR}{4}\left[\theta c(4s^2-1)+sc^2+2k_1\left(\theta-2\theta c-sc+2sc^2+\frac{2s^2}{3}\right)+2k_2(sc^2-s^3-\theta c)\right]$$

$$LF_V = \frac{wR}{4}\left\{-\theta s(4s^2-1)-s^2c-\frac{2k_1}{3}(1-3c^2+2c^3)+2k_2[\theta s+2(1-c)(1-2c^2)]\right\}$$

$$LF_M = \frac{wR}{4}[-\theta(4s^2-1)-sc+2k_2(2s-3sc+\theta)]$$

For $\alpha = \beta = 0$

θ	30°	60°	90°
$\frac{LF_H}{wR}$	0.0005	0.0541	0.6187
$\frac{LF_V}{wR}$	0.0016	0.0729	0.4406
$\frac{LF_M}{wR}$	0.0040	0.1083	0.6073

5g. Uniform horizontal load on right side only

$$LF_H = \frac{wR}{4}\left[sc^2-\theta c+\frac{2k_1}{3}(2s^3+3sc-3\theta)+2k_2(s-\theta c)\right]$$

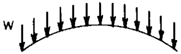
$$LF_V = \frac{wR}{4}\left[\theta s-s^2c+\frac{2k_1}{3}(1-3c^2+2c^3)-2k_2(2-4c^2+2c^3-\theta s)\right]$$

$$LF_M = \frac{wR}{4}[\theta-sc+2k_2(\theta-2s+sc)]$$

For $\alpha = \beta = 0$

θ	30°	60°	90°
$\frac{LF_H}{wR}$	0.0000	0.0039	0.0479
$\frac{LF_V}{wR}$	0.0009	0.0448	0.3448
$\frac{LF_M}{wR}$	0.0010	0.0276	0.1781

5h. Vertical loading uniformly distributed along the circumference (by gravity or linear acceleration)



(*Note:* The full span is loaded)

$$LF_H = wR\left[\frac{k_1}{2}(2\theta c^2-\theta-sc)+k_2\left(\frac{R_{cg}}{R}+1\right)(\theta-sc)+\frac{R_{cg}}{R}2c(\theta c-s)\right]$$

$$LF_V = wR\left[k_1\theta(\theta+sc)+2k_2s(s-2\theta c)-\frac{R_{cg}}{R}2s(\theta c-s)\right]$$

$$LF_M = 2wR\left(\frac{R_{cg}}{R}+k_2\right)(s-\theta c)$$

See case 1h for a definition of the radius R_{cg}

For $\alpha = \beta = 0$ and $R_{cg} = R$

θ	30°	60°	90°
$\frac{LF_H}{wR}$	0.0149	0.4076	2.3562
$\frac{LF_V}{wR}$	0.1405	1.8294	6.4674
$\frac{LF_M}{wR}$	0.1862	1.3697	4.0000

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

Reference no., loading — Loading terms and some selected numerical values

5i. Horizontal loading uniformly distributed along the circumference (by gravity or linear acceleration)

(*Note:* The full span is loaded)

$$LF_H = wR\left[k_1\theta(\theta - sc) + \frac{R_{cg}}{R}2s(\theta c - k_2 s)\right]$$

$$LF_V = wR\left[\frac{k_1}{2}(2\theta c^2 - \theta - sc) + \left(\frac{R_{cg}}{R} + 1\right)k_2(sc - \theta) + \left(2k_2 - \frac{R_{cg}}{R}\right)2\theta s^2\right]$$

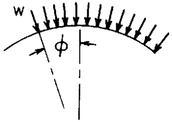
$$LF_M = wR\left(k_2 - \frac{R_{cg}}{R}\right)2\theta s$$

See case 1h for a definition of the radius R_{cg}

For $\alpha = \beta = 0$ and $R_{cg} = R$

θ	30°	60°	90°
$\frac{LF_H}{wR}$	0.0009	0.0501	0.4674
$\frac{LF_V}{wR}$	−0.0050	−0.1359	−0.7854
$\frac{LF_M}{wR}$	0.0000	0.0000	0.0000

5j. Partial uniformly distributed radial loading



$$LF_H = wR\left[\frac{k_1}{2}(scm + c^2 n - \theta m - \phi m) + k_2(s + n - \theta c - \phi c)\right]$$

$$LF_V = wR\left[\frac{k_1}{2}(\theta n + \phi n + scn + s^2 m) + k_2(\theta s + \phi s - 2scn + 2c^2 m - c - m)\right]$$

$$LF_M = wR[k_2(\theta + \phi - sm - cn)]$$

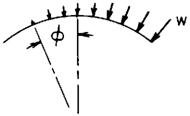
If $\phi = \theta$ (the full span is loaded)

$$LF_H = wR[k_1 c(sc - \theta) + 2k_2(s - \theta c)]$$
$$LF_V = wR[k_1 s(\theta + sc) + 2k_2 s(\theta - 2sc)]$$
$$LF_M = wR[2k_2(\theta - sc)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	0.0013	0.0055	0.0353	0.1505	0.2146	0.8741
$\frac{LF_V}{wR}$	0.0208	0.0639	0.2819	0.8200	1.0708	2.8357
$\frac{LF_M}{wR}$	0.0236	0.0783	0.1812	0.5708	0.5708	1.6491

5k. Partial uniformly increasing distributed radial loading



$$LF_H = \frac{wR}{\theta + \phi}\left\{\frac{k_1}{2}[scn - (\theta + \phi)n + 2c - m - c^2 m] + \frac{k_2}{2}[(\theta + \phi)(2s - \theta c - \phi c) + 2c - 2m]\right\}$$

$$LF_V = \frac{wR}{\theta + \phi}\left\{\frac{k_1}{2}[2s + 2n - (\theta + \phi)m - smc - c^2 n) + \frac{k_2}{2}[(\theta + \phi)(\theta s + \phi s - 2c) - 2s - 2n + 4smc + 4c^2 n]\right\}$$

$$LF_M = \frac{wR}{\theta + \phi}\left\{\frac{k_2}{2}[(\theta + \phi)^2 + 2(cm - sn - 1)]\right\}$$

If $\phi = \theta$ (the full span is loaded)

$$LF_H = \frac{wR}{2\theta}[k_1 s(sc - \theta) + 2k_2\theta(s - \theta c)]$$

$$LF_V = \frac{wR}{2\theta}[k_1(2s - sc^2 - \theta c) + 2k_2(2sc^2 + s\theta^2 - s - \theta c)]$$

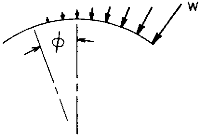
$$LF_M = \frac{wR}{2\theta}[k_2(\theta^2 - s^2)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	0.0003	0.0012	0.0074	0.0330	0.0451	0.1963
$\frac{LF_V}{wR}$	0.0054	0.0169	0.0737	0.2246	0.2854	0.8245
$\frac{LF_M}{wR}$	0.0059	0.0198	0.0461	0.1488	0.1488	0.4536

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

5l. Partial second-order increase in distributed radial loading



$$LF_H = \frac{wR}{(\theta+\phi)^2}\left\{k_1[(\theta+\phi)(2c+m) - 2s - 2n - c^2n - scm] + \frac{k_2}{3}[3(\theta+\phi)(\theta s+\phi s+2c) - 6s - 6n - c(\theta+\phi)^3]\right\}$$

$$LF_V = \frac{wR}{(\theta+\phi)^2}\left\{k_1[(\theta+\phi)(2s-n) + mc^2 - 3m - scn + 2c] - \frac{k_2}{3}[3(\theta+\phi)(\theta c+\phi c+2s) - 6c - 6m + 12c(mc - sn) - s(\theta+\phi)^3]\right\}$$

$$LF_M = \frac{wR}{(\theta+\phi)^2}\left\{\frac{k_2}{3}[6(sm + cn - \theta - \phi) + (\theta+\phi)^3]\right\}$$

If $\phi = \theta$ (the full span is loaded)

$$LF_H = \frac{wR}{2\theta^2}\left[k_1(3\theta c - 3s + s^3) + 2k_2\left(\theta c - s + s\theta^2 - \frac{2c\theta^3}{3}\right)\right]$$

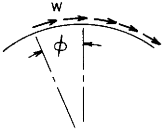
$$LF_V = \frac{wR}{2\theta^2}\left[k_1 s(\theta - sc) + 2k_2\left(2s^2c - \theta s - c\theta^2 + \frac{2s\theta^3}{3}\right)\right]$$

$$LF_M = \frac{wR}{\theta^2}\left[k_2\left(sc - \theta + \frac{2\theta^3}{3}\right)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	0.0001	0.0004	0.0025	0.0116	0.0155	0.0701
$\frac{LF_V}{wR}$	0.0022	0.0070	0.0303	0.0947	0.1183	0.3579
$\frac{LF_M}{wR}$	0.0024	0.0080	0.0186	0.0609	0.0609	0.1913

5m. Partial uniformly distributed tangential loading



$$LF_H = \frac{wR}{2}[(\theta+\phi)^2c + k_1(\theta n + \phi n - scn - s^2m + 2m - 2c) + 2k_2(m - c - \theta s - \phi s)]$$

$$LF_V = \frac{wR}{2}[-(\theta+\phi)^2s + k_1(\theta m + \phi m + c^2n + scm - 2s - 2n) + 2k_2(\theta c + \phi c + 2s^2n - n - 2scm + s)]$$

$$LF_M = \frac{wR}{2}[-(\theta+\phi)^2 + 2k_2(1 + sn - cm)]$$

If $\phi = \theta$ (the full span is loaded)

$$LF_H = wR[2\theta^2c + k_1s(\theta - sc) - k_2 2\theta s]$$

$$LF_V = wR[-2\theta^2s + k_1(\theta c - s - s^3) + 2k_2(\theta c + s - 2sc^2)]$$

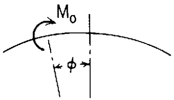
$$LF_M = wR(-2\theta^2 + k_2 2s^2)$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	−0.0001	−0.0009	−0.0077	−0.0518	−0.0708	−0.4624
$\frac{LF_V}{wR}$	−0.0028	−0.0133	−0.0772	−0.3528	−0.4483	−1.9428
$\frac{LF_M}{wR}$	−0.0031	−0.0155	−0.0483	−0.2337	−0.2337	−1.0687

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

<table>
<tr><th>Reference no., loading</th><th>Loading terms and some selected numerical values</th></tr>
<tr><td>5n. Concentrated couple



</td><td>

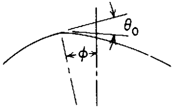
$LF_H = \frac{M_o}{R}[(\theta + \phi)c - k_2(s + n)]$

$LF_V = \frac{M_o}{R}[-(\theta + \phi)s + k_2(c - m)]$

$LF_M = \frac{M_o}{R}(-\theta - \phi)$

For $\alpha = \beta = 0$

<table>
<tr><td>θ</td><td>30°</td><td></td><td>60°</td><td></td><td>90°</td><td></td></tr>
<tr><td>ϕ</td><td>0°</td><td>15°</td><td>0°</td><td>30°</td><td>0°</td><td>45°</td></tr>
<tr><td>$\frac{LF_H R}{M_o}$</td><td>−0.0466</td><td>−0.0786</td><td>−0.3424</td><td>−0.5806</td><td>−1.0000</td><td>−1.7071</td></tr>
<tr><td>$\frac{LF_V R}{M_o}$</td><td>−0.3958</td><td>−0.4926</td><td>−1.4069</td><td>−1.7264</td><td>−2.5708</td><td>−3.0633</td></tr>
<tr><td>$\frac{LF_M R}{M_o}$</td><td>−0.5236</td><td>−0.7854</td><td>−1.0472</td><td>−1.5708</td><td>−1.5708</td><td>−2.3562</td></tr>
</table>
</td></tr>
<tr><td>5p. Concentrated angular displacement

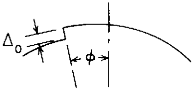


</td><td>

$LF_H = \frac{\theta_o EI}{R^2}(m - c)$

$LF_V = \frac{\theta_o EI}{R^2}(s - n)$

$LF_M = \frac{\theta_o EI}{R^2}(1)$
</td></tr>
<tr><td>5q. Concentrated radial displacement

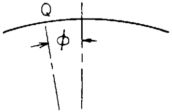


</td><td>

$LF_H = \frac{\Delta_o EI}{R^3}(n)$

$LF_V = \frac{\Delta_o EI}{R^3}(m)$

$LF_M = 0$
</td></tr>
<tr><td>5r. Uniform temperature rise over that span to the right of point Q



</td><td>

$LF_H = -(T - T_o)\frac{\gamma EI}{R^2}(n + s)$

$LF_V = (T - T_o)\frac{\gamma EI}{R^2}(c - m)$

$LF_M = 0$

T = uniform temperature
T_o = unloaded temperature
</td></tr>
</table>

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

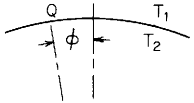
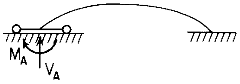
5s. Linear temperature differential through the thickness t for that span to the right of point Q 	$LF_H = (T_1 - T_2)\frac{\gamma EI}{Rt}(n + s - \theta c - \phi c)$ $LF_V = (T_1 - T_2)\frac{\gamma EI}{Rt}(m - c + \theta s + \phi s)$ $LF_M = (T_1 - T_2)\frac{\gamma EI}{Rt}(\theta + \phi)$ *Note:* The temperature at the centroidal axis is the initial unloaded temperature
6. Left end pinned, right end fixed 	Since $\delta_{HA} = 0, \delta_{VA} = 0,$ and $M_A = 0,$ $H_A = \frac{B_{VV} LF_H - B_{HV} LF_V}{B_{HH} B_{VV} - B_{HV}^2}$ $V_A = \frac{B_{HH} LF_V - B_{HV} LF_H}{B_{HH} B_{VV} - B_{HV}^2}$ $\psi_A = \frac{R^2}{EI}(B_{MH} H_A + B_{MV} V_A - LF_M)$ Use load terms given above for cases 5a–5s
7. Left end guided in horizontal direction, right end fixed 	Since $\delta_{VA} = 0, \psi_A = 0,$ and $H_A = 0,$ $V_A = \frac{B_{MM} LF_V - B_{MV} LF_M}{B_{VV} B_{MM} - B_{MV}^2}$ $\frac{M_A}{R} = \frac{B_{VV} LF_M - B_{MV} LF_V}{B_{VV} B_{MM} - B_{MV}^2}$ $\delta_{HA} = \frac{R^3}{EI}\left(B_{HV} V_A + B_{HM} \frac{M_A}{R} - LF_H\right)$ Use load terms given above for cases 5a–5s
8. Left end guided in vertical direction, right end fixed 	Since $\delta_{HA} = 0, \psi_A = 0,$ and $V_A = 0,$ $H_A = \frac{B_{MM} LF_H - B_{HM} LF_M}{B_{HH} B_{MM} - B_{HM}^2}$ $\frac{M_A}{R} = \frac{B_{HH} LF_M - B_{HM} LF_H}{B_{HH} B_{MM} - B_{HM}^2}$ $\delta_{VA} = \frac{R^3}{EI}\left(B_{VH} H_A + B_{VM} \frac{M_A}{R} - LF_V\right)$ Use load terms given above for cases 5a–5s

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

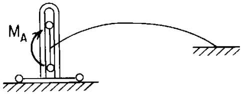
9. Left end roller supported in vertical direction only, right end fixed 	Since $\delta_{VA} = 0, H_A = 0,$ and $M_A = 0,$ $V_A = \frac{LF_V}{B_{VV}}, \qquad \delta_{HA} = \frac{R^3}{EI}(B_{HV}V_A - LF_H)$ $\psi_A = \frac{R^2}{EI}(B_{MV}V_A - LF_M)$ Use load terms given above for cases 5a–5s
10. Left end roller supported in horizontal direction only, right end fixed 	Since $\delta_{HA} = 0, V_A = 0,$ and $M_A = 0,$ $H_A = \frac{LF_H}{B_{HH}}, \qquad \delta_{VA} = \frac{R^3}{EI}(B_{VH}H_A - LF_V)$ $\psi_A = \frac{R^2}{EI}(B_{MH}H_A - LF_M)$ Use load terms given above for cases 5a–5s
11. Left end restrained against rotation only, right end fixed 	Since $\psi_A = 0, H_A = 0,$ and $V_A = 0,$ $\frac{M_A}{R} = \frac{LF_M}{B_{MM}}, \qquad \delta_{HA} = \frac{R^3}{EI}\left(B_{HM}\frac{M_A}{R} - LF_H\right)$ $\delta_{VA} = \frac{R^3}{EI}\left(B_{VM}\frac{M_A}{R} - LF_V\right)$ Use load terms given above for cases 5a–5s
12. Left end free, right end fixed 	Since $H_A = 0, V_A = 0,$ and $M_A = 0,$ this is a statically determinate problem. The deflections at the free end are given by $\delta_{HA} = \frac{-R^3}{EI}LF_H, \qquad \delta_{VA} = \frac{-R^3}{EI}LF_V$ $\psi_A = \frac{-R^2}{EI}LF_M$ Use load terms given above for cases 5a–5s

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

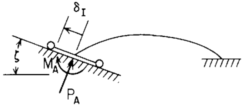
13. Left end guided along an inclined surface at angle ζ, right end fixed  	Since there is no deflection perpendicular to the incline, the following three equations must be solved for $M_A, P_A,$ and δ_I: $\delta_I \frac{EI\cos\zeta}{R^3} = P_A(B_{HV}\cos\zeta - B_{HH}\sin\zeta) + B_{HM}\frac{M_A}{R} - LF_H$ $\delta_I \frac{EI\sin\zeta}{R^3} = P_A(B_{VV}\cos\zeta - B_{VH}\sin\zeta) + B_{VM}\frac{M_A}{R} - LF_V$ $0 = P_A(B_{MV}\cos\zeta - B_{MH}\sin\zeta) + B_{MM}\frac{M_A}{R} - LF_M$ Use load terms given above for cases 5a–5s
14. Left end roller supported along an inclined surface at angle ζ, right end fixed  	Since there is no deflection perpendicular to the incline and $M_A = 0$, the following equations give $P_A, \delta_I,$ and ψ_A: $P_A = \frac{LF_V\cos\zeta - LF_H\sin\zeta}{B_{HH}\sin^2\zeta - 2B_{HV}\sin\zeta\cos\zeta + B_{VV}\cos^2\zeta}$ $\delta_I = \frac{R^3}{EI}\{P_A[B_{HV}(\cos^2\zeta - \sin^2\zeta) + (B_{VV} - B_{HH})\sin\zeta\cos\zeta] - LF_H\cos\zeta - LF_V\sin\zeta\}$ $\psi_A = \frac{R^2}{EI}[P_A(B_{MV}\cos\zeta - B_{MH}\sin\zeta) - LF_M]$ Use load terms given above for cases 5a–5s

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature

NOTATION: $W =$ applied load normal to the plane of curvature (force); $M_o =$ applied bending moment in a plane tangent to the curved axis of the beam (force-length); $T_o =$ applied twisting moment in a plane normal to the curved axis of the beam (force-length); $w =$ distributed load (force per unit length); $t_o =$ distributed twisting moment (force-length per unit length); $V_A =$ reaction force, $M_A =$ reaction bending moment, $T_A =$ reaction twisting moment, $y_A =$ deflection normal to the plane of curvature, $\Theta_A =$ slope of the beam axis in the plane of the moment M_A, and $\psi_A =$ roll of the beam cross section in the plane of the twisting moment T_A, all at the left end of the beam. Similarly, $V_B, M_B, T_B, y_B, \Theta_B$, and ψ_B are the reactions and displacements at the right end: V, M, T, y, Θ, and ψ are internal shear forces, moments, and displacements at an angular position x rad from the left end. All loads and reactions are positive as shown in the diagram; y is positive upward; Θ is positive when y increases as x increases; and ψ is positive in the direction of T.

$R =$ radius of curvature of the beam axis (length); $E =$ modulus of elasticity (force per unit area); $I =$ area moment of inertia about the bending axis (length to the fourth power) (note that this must be a principal axis of the beam cross section); $G =$ modulus of rigidity (force per unit area); $\nu =$ Poisson's ratio; $K =$ torsional stiffness constant of the cross section (length to the fourth power) (see page 383); $\theta =$ angle in radians from the left end to the position of the loading; $\phi =$ angle (radians) subtended by the entire span of the curved beam. See page 131 For a definition of the term $\langle x-\theta\rangle^n$.

The following constants and functions are hereby defined to permit condensing the tabulated formulas which follow. $\beta = EI/GK$.

$F_1 = \frac{1+\beta}{2} x\sin x - \beta(1-\cos x)$ $\qquad$ $C_1 = \frac{1+\beta}{2}\phi\sin\phi - \beta(1-\cos\phi)$

$F_2 = \frac{1+\beta}{2}(x\cos x - \sin x)$ $\qquad$ $C_2 = \frac{1+\beta}{2}(\phi\cos\phi - \sin\phi)$

$F_3 = -\beta(x-\sin x) - \frac{1+\beta}{2}(x\cos x - \sin x)$ $\qquad$ $C_3 = -\beta(\phi-\sin\phi) - \frac{1+\beta}{2}(\phi\cos\phi - \sin\phi)$

$F_4 = \frac{1+\beta}{2}x\cos x + \frac{1-\beta}{2}\sin x$ $\qquad$ $C_4 = \frac{1+\beta}{2}\phi\cos\phi + \frac{1-\beta}{2}\sin\phi$

$F_5 = -\frac{1+\beta}{2}x\sin x$ $\qquad$ $C_5 = -\frac{1+\beta}{2}\phi\sin\phi$

$F_6 = F_1$ $\qquad$ $C_6 = C_1$

$F_7 = F_5$ $\qquad$ $C_7 = C_5$

$F_8 = \frac{1-\beta}{2}\sin x - \frac{1+\beta}{2}x\cos x$ $\qquad$ $C_8 = \frac{1-\beta}{2}\sin\phi - \frac{1+\beta}{2}\phi\cos\phi$

$F_9 = F_2$ $\qquad$ $C_9 = C_2$

$F_{a1} = \left\{\frac{1+\beta}{2}(x-\theta)\sin(x-\theta) - \beta[1-\cos(x-\theta)]\right\}\langle x-\theta\rangle^0$ $\qquad$ $C_{a1} = \frac{1+\beta}{2}(\phi-\theta)\sin(\phi-\theta) - \beta[1-\cos(\phi-\theta)]$

$F_{a2} = \frac{1+\beta}{2}[(x-\theta)\cos(x-\theta) - \sin(x-\theta)]\langle x-\theta\rangle^0$ $\qquad$ $C_{a2} = \frac{1+\beta}{2}[(\phi-\theta)\cos(\phi-\theta) - \sin(\phi-\theta)]$

$F_{a3} = \{-\beta[x-\theta-\sin(x-\theta)] - F_{a2}\}\langle x-\theta\rangle^0$ $\qquad$ $C_{a3} = -\beta[\phi-\theta-\sin(\phi-\theta)] - C_{a2}$

$F_{a4} = \left[\frac{1+\beta}{2}(x-\theta)\cos(x-\theta) + \frac{1-\beta}{2}\sin(x-\theta)\right]\langle x-\theta\rangle^0$ $\qquad$ $C_{a4} = \frac{1+\beta}{2}(\phi-\theta)\cos(\phi-\theta) + \frac{1-\beta}{2}\sin(\phi-\theta)$

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

$F_{a5} = -\frac{1+\beta}{2}(x-\theta)\sin(x-\theta)\langle x-\theta\rangle^0$	$C_{a5} = -\frac{1+\beta}{2}(\phi-\theta)\sin(\phi-\theta)$
$F_{a6} = F_{a1}$	$C_{a6} = C_{a1}$
$F_{a7} = F_{a5}$	$C_{a7} = C_{a5}$
$F_{a8} = \left[\frac{1-\beta}{2}\sin(x-\theta) - \frac{1+\beta}{2}(x-\theta)\cos(x-\theta)\right]\langle x-\theta\rangle^0$	$C_{a8} = \frac{1-\beta}{2}\sin(\phi-\theta) - \frac{1+\beta}{2}(\phi-\theta)\cos(\phi-\theta)$
$F_{a9} = F_{a2}$	$C_{a9} = C_{a2}$
$F_{a12} = \frac{1+\beta}{2}[(x-\theta)\sin(x-\theta) - 2 + 2\cos(x-\theta)]\langle x-\theta\rangle^0$	$C_{a12} = \frac{1+\beta}{2}[(\phi-\theta)\sin(\phi-\theta) - 2 + 2\cos(\phi-\theta)]$
$F_{a13} = \left[\beta\left\{1-\cos(x-\theta) - \frac{(x-\theta)^2}{2}\right\} - F_{a12}\right]\langle x-\theta\rangle^0$	$C_{a13} = \beta\left[1-\cos(\phi-\theta) - \frac{(\phi-\theta)^2}{2}\right] - C_{a12}$
$F_{a15} = F_{a2}$	$C_{a15} = C_{a2}$
$F_{a16} = F_{a3}$	$C_{a16} = C_{a3}$
$F_{a18} = \left[1-\cos(x-\theta) - \frac{1+\beta}{2}(x-\theta)\sin(x-\theta)\right]\langle x-\theta\rangle^0$	$C_{a18} = 1-\cos(\phi-\theta) - \frac{1+\beta}{2}(\phi-\theta)\sin(\phi-\theta)$
$F_{a19} = F_{a12}$	$C_{a19} = C_{a12}$

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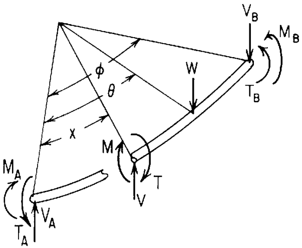
1. Concentrated intermediate lateral load



Transverse shear $= V = V_A - W\langle x-\theta\rangle^0$

Bending moment $= M = V_A R\sin x + M_A\cos x - T_A\sin x - WR\sin(x-\theta)\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A\sin x + T_A\cos x - WR[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

Deflection $= y = y_A + \Theta_A R\sin x + \psi_A R(1-\cos x) + \frac{M_A R^2}{EI}F_1 + \frac{T_A R^2}{EI}F_2 + \frac{V_A R^3}{EI}F_3 - \frac{WR^3}{EI}F_{a3}$

Bending slope $= \Theta = \Theta_A\cos x + \psi_A\sin x + \frac{M_A R}{EI}F_4 + \frac{T_A R}{EI}F_5 + \frac{V_A R^2}{EI}F_6 - \frac{WR^2}{EI}F_{a6}$

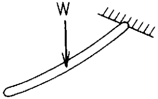
Roll slope $= \psi = \psi_A\cos x - \Theta_A\sin x + \frac{M_A R}{EI}F_7 + \frac{T_A R}{EI}F_8 + \frac{V_A R^2}{EI}F_9 - \frac{WR^2}{EI}F_{a9}$

For tabulated values: $V = K_V W,\quad M = K_M WR,\quad T = K_T WR,\quad y = K_y\frac{WR^3}{EI},\quad \Theta = K_\Theta\frac{WR^2}{EI},\quad \psi = K_\psi\frac{WR^2}{EI}$

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no. — Formulas for boundary values and selected numerical values

1a. Right end fixed, left end free



$V_A = 0 \quad M_A = 0 \quad T_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$y_A = \frac{-WR^3}{EI}[C_{a6}\sin\phi - C_{a9}(1-\cos\phi) - C_{a3}], \qquad \Theta_A = \frac{WR^2}{EI}(C_{a6}\cos\phi - C_{a9}\sin\phi)$$

$$\psi_A = \frac{WR^2}{EI}(C_{a9}\cos\phi + C_{a6}\sin\phi)$$

$$V_B = -W$$

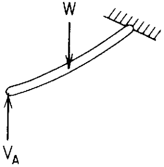
$$M_B = -WR\sin(\phi-\theta)$$

$$T_B = -WR[1-\cos(\phi-\theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	−0.1607	−1.2485	−0.6285	−0.1576	−7.6969	−3.7971	−0.6293
$K_{\Theta A}$	0.3058	1.1500	0.3938	0.0535	2.6000	−0.1359	−0.3929
$K_{\psi A}$	0.0590	0.5064	0.3929	0.1269	3.6128	2.2002	0.3938
K_{VB}	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000
K_{MB}	−0.7071	−1.0000	−0.8660	−0.5000	−0.0000	−0.8660	−0.8660
K_{TB}	−0.2929	−1.0000	−0.5000	−0.1340	−2.0000	−1.5000	−0.5000

1b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{C_{a9}(1-\cos\phi) - C_{a6}\sin\phi + C_{a3}}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\Theta_A = \frac{WR^2}{EI}\frac{(C_{a3}C_9 - C_{a9}C_3)\sin\phi + (C_{a9}C_6 - C_{a6}C_9)(1-\cos\phi) + (C_{a6}C_3 - C_{a3}C_6)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\psi_A = \frac{WR^2}{EI}\frac{[C_{a6}(C_3+C_9) - C_6(C_{a3}+C_{a9})]\sin\phi + (C_{a9}C_3 - C_{a3}C_9)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$V_B = V_A - W$$

$$M_B = V_A R\sin\phi - WR\sin(\phi-\theta)$$

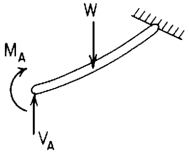
$$T_B = V_A R(1-\cos\phi) - WR[1-\cos(\phi-\theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.5136	0.1420	0.5034	0.1262	0.4933	0.0818
$K_{\Theta A}$	−0.0294	−0.0148	−0.1851	−0.0916	−1.4185	−0.6055
$K_{\psi A}$	0.0216	0.0106	0.1380	0.0630	0.4179	0.0984
K_{MB}	−0.1368	−0.1584	−0.3626	−0.3738	−0.8660	−0.8660
K_{TB}	0.0165	0.0075	0.0034	−0.0078	−0.5133	−0.3365
$K_{M\theta}$	0.1329	0.0710	0.2517	0.1093	0.4272	0.0708

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

1c. Right end fixed, left end supported and slope guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{(C_{a9}C_4 - C_{a6}C_7)(1-\cos\phi) + (C_{a6}C_1 - C_{a3}C_4)\cos\phi + (C_{a3}C_7 - C_{a9}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = WR\frac{(C_{a6}C_9 - C_{a9}C_6)(1-\cos\phi) + (C_{a3}C_6 - C_{a6}C_3)\cos\phi + (C_{a9}C_3) - C_{a3}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = \frac{WR^2}{EI}\frac{C_{a3}(C_4C_9 - C_6C_7) + C_{a6}(C_3C_7 - C_1C_9) + C_{a9}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$V_B = V_A - W$$

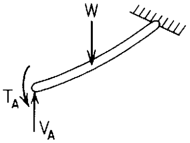
$$M_B = V_AR\sin\phi + M_A\cos\phi - WR\sin(\phi - \theta)$$

$$T_B = V_AR(1-\cos\phi) + M_A\sin\phi - WR[1 - \cos(\phi - \theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.7407	0.2561	0.7316	0.2392	0.6686	0.1566
K_{MA}	−0.1194	−0.0600	−0.2478	−0.1226	−0.5187	−0.2214
$K_{\psi A}$	−0.0008	−0.0007	−0.0147	−0.0126	−0.2152	−0.1718
K_{MB}	−0.0607	−0.1201	−0.1344	−0.2608	−0.3473	−0.6446
K_{TB}	−0.0015	−0.0015	−0.0161	−0.0174	−0.1629	−0.1869
$K_{M\theta}$	0.0764	0.0761	0.1512	0.1458	0.3196	0.2463

1d. Right end fixed, left end supported and roll guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{[(C_{a3} + C_{a9})C_5 - C_{a6}(C_2 + C_8)]\sin\phi + (C_{a3}C_8 - C_{a9}C_2)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = WR\frac{[C_{a6}(C_3 + C_9) - C_6(C_{a3} + C_{a9})]\sin\phi + (C_{a9}C_3 - C_{a3}C_9)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\frac{C_{a3}(C_5C_9 - C_6C_8) + C_{a6}(C_3C_8 - C_2C_9) + C_{a9}(C_2C_6 - C_3C_5)}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$V_B = V_A - W$$

$$M_B = V_AR\sin\phi - T_A\sin\phi - WR\sin(\phi - \theta)$$

$$T_B = V_AR(1-\cos\phi) + T_A\cos\phi - WR[1 - \cos(\phi - \theta)]$$

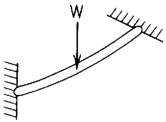
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.5053	0.1379	0.4684	0.1103	0.3910	0.0577
K_{TA}	−0.0226	−0.0111	−0.0862	−0.0393	−0.2180	−0.0513
$K_{\Theta A}$	−0.0252	−0.0127	−0.1320	−0.0674	−1.1525	−0.5429
K_{MB}	−0.1267	−0.1535	−0.3114	−0.3504	−0.8660	−0.8660
K_{TB}	−0.0019	−0.0015	−0.0316	−0.0237	−0.5000	−0.3333
$K_{M\theta}$	0.1366	0.0745	0.2773	0.1296	0.5274	0.0944

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no. — Formulas for boundary values and selected numerical values

1e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{C_{a3}(C_4C_8 - C_5C_7) + C_{a6}(C_2C_7 - C_1C_8) + C_{a9}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = WR\frac{C_{a3}(C_5C_9 - C_6C_8) + C_{a6}(C_3C_8 - C_2C_9) + C_{a9}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = WR\frac{C_{a3}(C_6C_7 - C_4C_9) + C_{a6}(C_1C_9 - C_3C_7) + C_{a9}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A - W$$

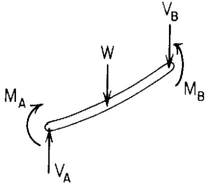
$$M_B = V_AR\sin\phi + M_A\cos\phi - T_A\sin\phi - WR\sin(\phi - \theta)$$

$$T_B = V_AR(1 - \cos\phi) + M_A\sin\phi + T_A\cos\phi - WR[1 - \cos(\phi - \theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	180°	270°	360°	
θ	15°	30°	60°	90°	90°	180°
K_{VA}	0.7424	0.7473	0.7658	0.7902	0.9092	0.5000
K_{MA}	−0.1201	−0.2589	−0.5887	−0.8488	−0.9299	−0.3598
K_{TA}	0.0009	0.0135	0.1568	0.5235	0.7500	1.0000
K_{MB}	−0.0606	−0.1322	−0.2773	−0.2667	0.0701	−0.3598
K_{TB}	−0.0008	−0.0116	−0.1252	−0.3610	−0.2500	−1.0000
$K_{M\theta}$	0.0759	0.1427	0.2331	0.2667	0.1592	0.3598

1f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = W\frac{[-C_1\sin\phi + C_4(1 - \cos\phi)][1 - \cos(\phi - \theta)] + C_{a3}\sin^2\phi - C_{a6}\sin\phi(1 - \cos\phi)}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$M_A = +WR\frac{C_{a6}(1 - \cos\phi)^2 - C_{a3}(1 - \cos\phi)\sin\phi + [C_3\sin\phi - C_6(1 - \cos\phi)][1 - \cos(\phi - \theta)]}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$\psi_A = \frac{WR^2}{EI}\frac{(C_{a3}C_4 - C_{a6}C_1)(1 - \cos\phi) - (C_{a3}C_6 - C_{a6}C_3)\sin\phi - (C_3C_4 - C_1C_6)[1 - \cos(\phi - \theta)]}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$V_B = V_A - W$$

$$M_B = V_AR\sin\phi + M_A\cos\phi - WR\sin(\phi - \theta)$$

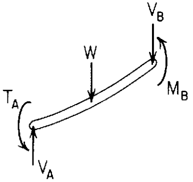
$$\psi_B = \psi_A\cos\phi + \frac{M_AR}{EI}C_7 + \frac{V_AR^2}{EI}C_9 - \frac{WR^2}{EI}C_{a9}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	180°	270°
θ	15°	30°	60°	90°
K_{VA}	0.7423	0.7457	0.7500	0.7414
K_{MA}	−0.1180	−0.2457	−0.5774	−1.2586
$K_{\psi A}$	−0.0024	−0.0215	−0.2722	−2.5702
K_{MB}	−0.0586	−0.1204	−0.2887	−0.7414
$K_{\psi B}$	−0.0023	−0.0200	−0.2372	−2.3554
$K_{M\theta}$	0.0781	0.1601	0.3608	0.7414

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

1g. Right end supported and slope-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = W\frac{(C_5\sin\phi - C_2\cos\phi)[1-\cos(\phi-\theta)] + C_{a3}\cos^2\phi - C_{a6}\sin\phi\cos\phi}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = WR\frac{(C_3\cos\phi - C_6\sin\phi)[1-\cos(\phi-\theta)] - (C_{a3}\cos\phi - C_{a6}\sin\phi)(1-\cos\phi)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\,\frac{(C_2C_6 - C_3C_5)[1-\cos(\phi-\theta)] + (C_{a3}C_5 - C_{a6}C_2)(1-\cos\phi) + (C_{a6}C_3 - C_{a3}C_6)\cos\phi}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$V_B = V_A - W$$

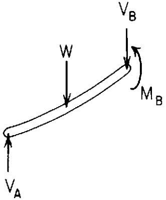
$$M_B = V_AR\sin\phi - T_A\sin\phi - WR\sin(\phi-\theta)$$

$$\psi_B = -\Theta_A\sin\phi + \frac{T_AR}{EI}C_8 + \frac{V_AR^2}{EI}C_9 - \frac{WR^2}{EI}C_{a9}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.5087	0.1405	0.5000	0.1340	0.6257	0.2141
K_{TA}	−0.0212	−0.0100	−0.0774	−0.0327	−0.2486	−0.0717
$K_{\Theta A}$	−0.0252	−0.0127	−0.1347	−0.0694	−1.7627	−0.9497
K_{MB}	−0.1253	−0.1524	−0.2887	−0.3333	−0.8660	−0.8660
$K_{\psi B}$	−0.0016	−0.0012	−0.0349	−0.0262	−0.9585	−0.6390
$K_{M\theta}$	0.1372	0.0753	0.2887	0.1443	0.7572	0.2476

1h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = W\frac{1-\cos(\phi-\theta)}{1-\cos\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\left\{\frac{C_{a3}\sin\phi + C_6[1-\cos(\phi-\theta)]}{1-\cos\phi} - \frac{C_3\sin\phi[1-\cos(\phi-\theta)]}{(1-\cos\phi)^2} - C_{a6}\right\}$$

$$\psi_A = \frac{WR^2}{EI}\left\{\frac{C_{a6}\sin\phi - C_{a3}\cos\phi}{1-\cos\phi} - (C_6\sin\phi - C_3\cos\phi)\frac{1-\cos(\phi-\theta)}{(1-\cos\phi)^2}\right\}$$

$$V_B = V_A - W$$

$$M_B = V_AR\sin\phi - WR\sin(\phi-\theta)$$

$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_AR^2}{EI}C_9 - \frac{WR^2}{EI}C_{a9}$$

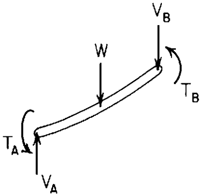
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.4574	0.1163	0.5000	0.1340	0.7500	0.2500
$K_{\Theta A}$	−0.0341	−0.0169	−0.1854	−0.0909	−2.0859	−1.0429
$K_{\psi A}$	0.0467	0.0220	0.1397	0.0591	0.4784	0.1380
K_{MB}	−0.1766	−0.1766	−0.3660	−0.3660	−0.8660	−0.8660
$K_{\psi B}$	0.0308	0.0141	0.0042	−0.0097	−0.9878	−0.6475
$K_{M\theta}$	0.1184	0.0582	0.2500	0.1160	0.6495	0.2165

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

1i. Right end supported and roll-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{(C_{a3} + C_{a9})\sin\phi + (C_2 + C_8)\sin(\phi - \theta)}{(C_2 + C_3 + C_8 + C_9)\sin\phi}$$

$$T_A = WR\frac{(C_{a3} + C_{a9})\sin\phi - (C_3 + C_9)\sin(\phi - \theta)}{(C_2 + C_3 + C_8 + C_9)\sin\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\frac{C_{a3}(C_8 + C_9) - C_{a9}(C_2 + C_3) + (C_2C_9 - C_3C_8)\sin(\phi - \theta)/\sin\phi}{(C_2 + C_3 + C_8 + C_9)\sin\phi}$$

$$V_B = V_A - W$$

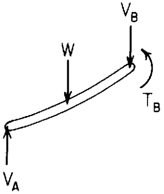
$$T_B = V_AR(1 - \cos\phi) + T_A\cos\phi - WR[1 - \cos(\phi - \theta)]$$

$$\Theta_B = \Theta_A\cos\phi + \frac{V_AR^2}{EI}C_6 + \frac{T_AR}{EI}C_5 - \frac{WR^2}{EI}C_{a6}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	270°
θ	15°	30°	90°
K_{VA}	0.6667	0.6667	0.6667
K_{TA}	−0.0404	−0.1994	0.0667
$K_{\Theta A}$	−0.0462	−0.3430	−4.4795
K_{TB}	0.0327	0.1667	−1.3333
$K_{\Theta B}$	0.0382	0.3048	1.7333
$K_{M\theta}$	0.1830	0.4330	0.0000

1j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{\sin(\phi - \theta)}{\sin\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\left\{\frac{C_{a3}\cos\phi - C_{a9}(1 - \cos\phi)}{\sin\phi} - [C_3\cos\phi - C_9(1 - \cos\phi)]\frac{\sin(\phi - \theta)}{\sin^2\phi}\right\}$$

$$\psi_A = \frac{WR^2}{EI}\left[C_{a3} + C_{a9} - (C_3 + C_9)\frac{\sin(\phi - \theta)}{\sin\phi}\right]$$

$$V_B = V_A - W$$

$$T_B = V_AR(1 - \cos\phi) - WR[1 - \cos(\phi - \theta)]$$

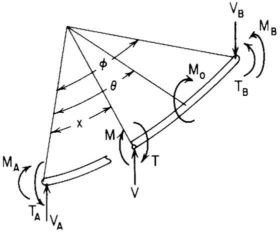
$$\Theta_B = \Theta_A\cos\phi + \psi_A\sin\phi + \frac{V_AR^2}{EI}C_6 - \frac{WR^2}{EI}C_{a6}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		270°	
θ	15°	30°	30°	60°	90°	180°
K_{VA}	0.7071	0.3660	0.8660	0.5000	0.0000	−1.0000
$K_{\Theta A}$	−0.0575	−0.0473	−0.6021	−0.5215	−3.6128	0.0000
$K_{\psi A}$	0.0413	0.0334	0.4071	0.3403	−4.0841	−8.1681
K_{TB}	0.0731	0.0731	0.3660	0.3660	−2.0000	−2.0000
$K_{\Theta B}$	0.0440	0.0509	0.4527	0.4666	6.6841	14.3810
$K_{M\theta}$	0.1830	0.1830	0.4330	0.4330	0.0000	0.0000

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

2. Concentated intermediate bending moment



Transverse shear $= V = V_A$

Bending moment $= M = V_A R \sin x + M_A \cos x - T_A \sin x + M_o \cos(x-\theta)\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A \sin x + T_A \cos x + M_o \sin(x-\theta)\langle x-\theta\rangle^0$

Vertical deflection $= y = y_A + \Theta_A R \sin x + \psi_A R(1-\cos x) + \dfrac{M_A R^2}{EI}F_1 + \dfrac{T_A R^2}{EI}F_2 + \dfrac{V_A R^3}{EI}F_3 + \dfrac{M_o R^2}{EI}F_{a1}$

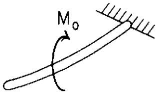
Bending slope $= \Theta = \Theta_A \cos x + \psi_A \sin x + \dfrac{M_A R}{EI}F_4 + \dfrac{T_A R}{EI}F_5 + \dfrac{V_A R^2}{EI}F_6 + \dfrac{M_o R}{EI}F_{a4}$

Roll slope $= \psi = \psi_A \cos x - \Theta_A \sin x + \dfrac{M_A R}{EI}F_7 + \dfrac{T_A R}{EI}F_8 + \dfrac{V_A R^2}{EI}F_9 + \dfrac{M_o R}{EI}F_{a7}$

For tabulated values $V = K_V \dfrac{M_o}{R}$, $\quad M = K_M M_o$, $\quad T = K_T M_o \quad y = K_y \dfrac{M_o R^2}{EI}$, $\quad \Theta = K_\Theta \dfrac{M_o R}{EI}$, $\quad \psi = K_\psi \dfrac{M_o R}{EI}$

End restraints,reference no.	Formulas for boundary values and selected numerical values

2a. Right end fixed, left end free



$V_A = 0 \quad M_A = 0 \quad T_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$y_A = \frac{M_o R^2}{EI}[C_{a4}\sin\phi - C_{a7}(1-\cos\phi) - C_{a1}]$$

$$\Theta_A = \frac{M_o R}{EI}(C_{a7}\sin\phi - C_{a4}\cos\phi)$$

$$\psi_A = -\frac{M_o R}{EI}(C_{a4}\sin\phi + C_{a7}\cos\phi)$$

$V_B = 0, \quad M_B = M_o\cos(\phi-\theta)$

$T_B = M_o\sin(\phi-\theta)$

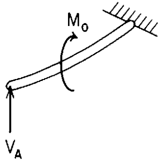
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	0.3058	1.1500	1.1222	0.6206	2.6000	4.0359	1.6929
$K_{\Theta A}$	−0.8282	−1.8064	−1.0429	−0.3011	−3.6128	−1.3342	0.4722
$K_{\psi A}$	0.0750	0.1500	−0.4722	−0.4465	0.0000	−2.0859	−1.0429
K_{MB}	0.7071	0.0000	0.5000	0.8660	−1.0000	−0.5000	0.5000
K_{TB}	0.7071	1.0000	0.8660	0.5000	0.0000	0.8660	0.8660

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no. | Formulas for boundary values and selected numerical values

2b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = \frac{-M_o}{R}\frac{C_{a7}(1-\cos\phi) - C_{a4}\sin\phi + C_{a1}}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\Theta_A = -\frac{M_o R}{EI}\frac{(C_{a1}C_9 - C_{a7}C_3)\sin\phi + (C_{a7}C_6 - C_{a4}C_9)(1-\cos\phi) + (C_{a4}C_3 - C_{a1}C_6)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\psi_A = -\frac{M_o R}{EI}\frac{[(C_{a4}(C_9+C_3) - C_6(C_{a1}+C_{a7})]\sin\phi + (C_{a7}C_3 - C_{a1}C_9)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$V_B = V_A$$

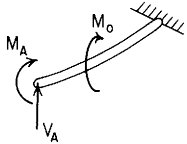
$$M_B = V_A R\sin\phi + M_o\cos(\phi-\theta)$$

$$T_B = V_A R(1-\cos\phi) + M_o\sin(\phi-\theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−1.9021	−0.9211	−0.8989	−0.4971	−0.3378	−0.5244	−0.2200
$K_{\Theta A}$	−0.2466	−0.7471	−0.0092	0.2706	−2.7346	0.0291	1.0441
$K_{\psi A}$	0.1872	0.6165	−0.0170	−0.1947	1.2204	−0.1915	−0.2483
K_{MB}	−0.6379	−0.9211	−0.3989	0.3689	−1.0000	−0.5000	0.5000
K_{TB}	0.1500	0.0789	−0.0329	0.0029	−0.6756	−0.1827	0.4261

2c. Right end fixed, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o}{R}\frac{(C_{a7}C_4 - C_{a4}C_7)(1-\cos\phi) + (C_{a4}C_1 - C_{a1}C_4)\cos\phi + (C_{a1}C_7 - C_{a7}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = -M_o\frac{(C_{a4}C_9 - C_{a7}C_6)(1-\cos\phi) + (C_{a1}C_6 - C_{a4}C_3)\cos\phi + (C_{a7}C_3 - C_{a1}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = -\frac{M_o R}{EI}\frac{C_{a1}(C_4C_9 - C_6C_7) + C_{a4}(C_3C_7 - C_1C_9) + C_{a7}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$V_B = V_A$$

$$M_B = V_A R\sin\phi + M_A\cos\phi + M_o\cos(\phi-\theta)$$

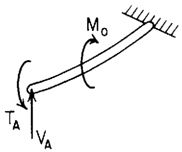
$$T_B = V_A R(1-\cos\phi) + M_A\sin\phi + M_o\sin(\phi-\theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	−1.7096	−1.6976	−0.8876	−0.8308	−0.5279	−0.3489
K_{MA}	−0.0071	0.3450	−0.0123	0.3622	0.0107	0.3818
$K_{\psi A}$	−0.0025	0.0029	−0.0246	0.0286	−0.1785	0.2177
K_{MB}	−0.3478	0.0095	−0.3876	0.0352	−0.5107	0.1182
K_{TB}	−0.0057	0.0056	−0.0338	0.0314	−0.1899	0.1682

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

2d. Right end fixed, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o}{R}\frac{[(C_{a1}+C_{a7})C_5 - C_{a4}(C_2+C_8)]\sin\phi + (C_{a1}C_8 - C_{a7}C_2)\cos\phi}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = -M_o\frac{[C_{a4}(C_3+C_9) - (C_{a1}+C_{a7})C_6]\sin\phi + (C_{a7}C_3 - C_{a1}C_9)\cos\phi}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = -\frac{M_oR}{EI}\frac{C_{a1}(C_5C_9 - C_6C_8) + C_{a4}(C_3C_8 - C_2C_9) + C_{a7}(C_2C_6 - C_3C_5)}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$V_B = V_A$$

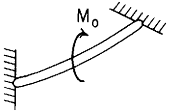
$$M_B = V_AR\sin\phi - T_A\sin\phi + M_o\cos(\phi-\theta)$$

$$T_B = V_AR(1-\cos\phi) + T_A\cos\phi + M_o\sin(\phi-\theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	30°	60°
K_{VA}	−1.9739	−1.0773	−0.8946	−0.4478	−0.6366	−0.4775	−0.1592
K_{TA}	−0.1957	−0.3851	0.0106	0.1216	−0.6366	0.0999	0.1295
$K_{\Theta A}$	−0.2100	−0.5097	−0.0158	0.1956	−1.9576	−0.0928	0.8860
K_{MB}	−0.5503	−0.6923	−0.4052	0.2966	−1.0000	−0.5000	0.5000
K_{TB}	−0.0094	−0.0773	−0.0286	0.0522	−0.6366	−0.1888	0.4182

2e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o}{R}\frac{C_{a1}(C_4C_8 - C_5C_7) + C_{a4}(C_2C_7 - C_1C_8) + C_{a7}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = -M_o\frac{C_{a1}(C_5C_9 - C_6C_8) + C_{a4}(C_3C_8 - C_2C_9 + C_{a7}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = -M_o\frac{C_{a1}(C_6C_7 - C_4C_9) + C_{a4}(C_1C_9 - C_3C_7) + C_{a7}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A$$

$$M_B = V_AR\sin\phi + M_A\cos\phi - T_A\sin\phi + M_o\cos(\phi-\theta)$$

$$T_B = V_AR(1-\cos\phi) + M_A\sin\phi + T_A\cos\phi + M_o\sin(\phi-\theta)$$

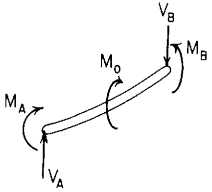
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	180°	270°	360°	
θ	15°	30°	60°	90°	90°	180°
K_{VA}	−1.7040	−0.8613	−0.4473	−0.3115	−0.1592	−0.3183
K_{MA}	−0.0094	−0.0309	−0.0474	0.0584	−0.0208	0.5000
K_{TA}	0.0031	0.0225	0.1301	0.2788	0.5908	−0.3183
K_{MB}	−0.3477	−0.3838	−0.4526	−0.4097	−0.0208	−0.5000
K_{TB}	−0.0036	−0.0262	−0.1586	−0.3699	−0.4092	−0.3183

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no. | Formulas for boundary values and selected numerical values

2f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = +\frac{M_o}{R}\frac{[C_1 \sin\phi - C_4(1-\cos\phi)]\sin(\phi-\theta) - C_{a1}\sin^2\phi + C_{a4}\sin\phi(1-\cos\phi)}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$M_A = -M_o\frac{[C_3\sin\phi - C_6(1-\cos\phi)]\sin(\phi-\theta) - C_{a1}(1-\cos\phi)\sin\phi + C_{a4}(1-\cos\phi)^2}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$\psi_A = \frac{MoR}{EI}\frac{(C_3C_4 - C_1C_6)\sin(\phi-\theta) + (C_{a1}C_6 - C_{a4}C_3)\sin\phi - (C_{a1}C_4 - C_{a4}C_1)(1-\cos\phi)}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$V_B = V_A$$

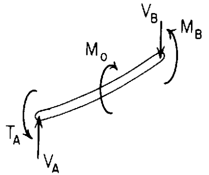
$$M_B = V_A R\sin\phi + M_A\cos\phi + M_o\cos(\phi-\theta)$$

$$\psi_B = \psi_A\cos\phi + \frac{M_A R}{EI}C_7 + \frac{V_A R^2}{EI}C_9 + \frac{M_o R}{EI}C_{a7}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	180°	270°
θ	15°	30°	60°	90°
K_{VA}	−1.7035	−0.8582	−0.4330	−0.2842
K_{MA}	−0.0015	−0.0079	−0.0577	−0.2842
$K_{\psi A}$	−0.0090	−0.0388	−0.2449	−1.7462
K_{MB}	−0.3396	−0.3581	−0.4423	−0.7159
$K_{\psi B}$	−0.0092	−0.0418	−0.2765	−1.8667

2g. Right end supported and slope-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{M_o}{R}\frac{C_{a1}\cos^2\phi - C_{a4}\sin\phi\cos\phi + (C_5\sin\phi - C_2\cos\phi)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = -M_o\frac{(C_{a4}\sin\phi - C_{a1}\cos\phi)(1-\cos\phi) + (C_3\cos\phi - C_6\sin\phi)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = \frac{-M_o R}{EI}\frac{(C_{a1}C_5 - C_{a4}C_2)(1-\cos\phi) + (C_{a4}C_3 - C_{a1}C_6)\cos\phi + (C_2C_6 - C_3C_5)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\cos\phi}$$

$$V_B = V_A$$

$$M_B = V_A R\sin\phi - T_A\sin\phi + M_o\cos(\phi-\theta)$$

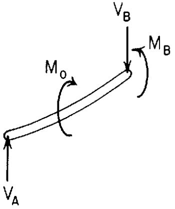
$$\psi_B = -\Theta_A\sin\phi + \frac{T_A R}{EI}C_8 + \frac{V_A R^2}{EI}C_9 + \frac{M_o R}{EI}C_{a7}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−1.9576	−1.0000	−0.8660	−0.5000	−0.3378	−0.3888	−0.3555
K_{TA}	−0.1891	−0.3634	0.0186	0.1070	−0.6756	0.0883	0.1551
$K_{\Theta A}$	−0.2101	−0.5163	−0.0182	0.2001	−2.7346	−0.3232	1.3964
K_{MB}	−0.5434	−0.6366	−0.3847	0.2590	−1.0000	−0.5000	0.5000
$K_{\psi B}$	−0.0076	−0.0856	−0.0316	0.0578	−1.2204	−0.3619	0.8017

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

2h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{M_o}{R}\frac{\sin(\phi-\theta)}{1-\cos\phi}$$

$$\Theta_A = -\frac{M_o R}{EI}\left[\frac{C_{a1}\sin\phi + C_6\sin(\phi-\theta)}{1-\cos\phi} - \frac{C_3\sin\phi\sin(\phi-\theta)}{(1-\cos\phi)^2} - C_{a4}\right]$$

$$\psi_A = -\frac{M_o R}{EI}\left[\frac{C_{a4}\sin\phi - C_{a1}\cos\phi}{1-\cos\phi} + \frac{(C_3\cos\phi - C_6\sin\phi)\sin(\phi-\theta)}{(1-\cos\phi)^2}\right]$$

$$V_B = V_A$$

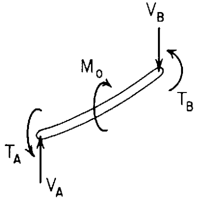
$$M_B = V_A R\sin\phi + M_o\cos(\phi-\theta)$$

$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_A R^2}{EI}C_9 + \frac{M_o R}{EI}C_{a7}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−2.4142	−1.0000	−0.8660	−0.5000	0.0000	−0.4330	−0.4330
$K_{\Theta A}$	−0.2888	−0.7549	−0.0060	0.2703	−3.6128	−0.2083	1.5981
$K_{\psi A}$	0.4161	0.6564	−0.0337	−0.1933	1.3000	−0.1700	−0.2985
K_{MB}	−1.0000	−1.0000	−0.3660	0.3660	−1.0000	−0.5000	0.5000
$K_{\psi B}$	0.2811	0.0985	−0.0410	0.0036	−1.3000	−0.3515	0.8200

2i. Right end supported and roll-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o}{R}\frac{(C_{a1}+C_{a7})\sin^2\phi + (C_2+C_8)\cos(\phi-\theta)\sin\phi}{(C_2+C_3+C_8+C_9)\sin^2\phi}$$

$$T_A = -M_o\frac{(C_{a1}+C_{a7})\sin^2\phi - (C_3+C_9)\cos(\phi-\theta)\sin\phi}{(C_2+C_3+C_8+C_9)\sin^2\phi}$$

$$\Theta_A = -\frac{M_o R}{EI}\frac{[C_{a1}(C_8+C_9) - C_{a7}(C_2+C_3)]\sin\phi + (C_2C_9 - C_3C_8)\cos(\phi-\theta)}{(C_2+C_3+C_8+C_9)\sin^2\phi}$$

$$V_B = V_A$$

$$T_B = V_A R(1-\cos\phi) + T_A\cos\phi + M_o\sin(\phi-\theta)$$

$$\Theta_B = \Theta_A\cos\phi + \frac{T_A R}{EI}C_5 + \frac{V_A R^2}{EI}C_6 + \frac{M_o R}{EI}C_{a4}$$

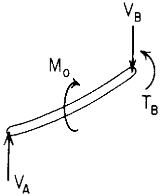
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		270°	
θ	0°	15°	0°	30°	0°	90°
K_{VA}	−1.2732	−1.2732	−0.6366	−0.6366	−0.2122	−0.2122
K_{TA}	−0.2732	−0.0485	−0.6366	−0.1366	−0.2122	0.7878
$K_{\Theta A}$	−0.3012	−0.0605	−0.9788	−0.2903	−5.1434	0.1259
K_{TB}	0.1410	0.0928	0.3634	0.2294	−1.2122	−0.2122
$K_{\Theta B}$	0.1658	0.1063	0.6776	0.3966	0.4259	2.0823

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no. | Formulas for boundary values and selected numerical values

2j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$

$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o \cos(\phi - \theta)}{R \sin\phi}$$

$$\Theta_A = -\frac{M_o R}{EI}\left\{\frac{C_{a1}\cos\phi - C_{a7}(1-\cos\phi)}{\sin\phi} - \frac{[C_3\cos\phi - C_9(1-\cos\phi)]\cos(\phi-\theta)}{\sin^2\phi}\right\}$$

$$\psi_A = -\frac{M_o R}{EI}\left[C_{a1} + C_{a7} - \frac{(C_3 + C_9)\cos(\phi-\theta)}{\sin\phi}\right]$$

$$V_B = V_A$$

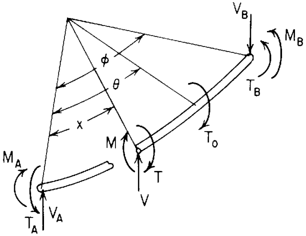
$$T_B = V_A R(1-\cos\phi) + M_o \sin(\phi - \theta)$$

$$\Theta_B = \Theta_A \cos\phi + \psi_A \sin\phi + \frac{V_A R^2}{EI} C_6 + \frac{M_o R}{EI} C_{a4}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°			90°		
θ	0°	15°	30°	0°	30°	60°
K_{VA}	−1.0000	−1.2247	−1.3660	0.0000	−0.5000	−0.8660
$K_{\Theta A}$	−0.3774	−0.0740	0.1322	−1.8064	−0.4679	0.6949
$K_{\psi A}$	0.2790	0.0495	−0.0947	1.3000	0.2790	−0.4684
K_{TB}	0.4142	0.1413	−0.1413	1.0000	0.3660	−0.3660
$K_{\Theta B}$	0.2051	0.1133	−0.0738	1.1500	0.4980	−0.4606

3. Concentrated intermediate twisting moment (torque)



Transverse shear $= V = V_A$

Bending moment $= M = V_A R \sin x + M_A \cos x - T_A \sin x - T_0 \sin(x-\theta)\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A \sin x + T_A \cos x + T_0 \cos(x-\theta)\langle x-\theta\rangle^0$

Vertical deflection $= y = y_A + \Theta_A R \sin x + \psi_A R(1-\cos x) + \frac{M_A R^2}{EI}F_1 + \frac{T_A R^2}{EI}F_2 + \frac{V_A R^3}{EI}F_3 + \frac{T_o R^2}{EI}F_{a2}$

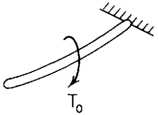
Bending slope $= \Theta = \Theta_A \cos x + \psi_A \sin x + \frac{M_A R}{EI}F_4 + \frac{T_A R}{EI}F_5 + \frac{V_A R^2}{EI}F_6 + \frac{T_o R}{EI}F_{a5}$

Roll slope $= \psi = \psi_A \cos x - \Theta_A \sin x + \frac{M_A R}{EI}F_7 + \frac{T_A R}{EI}F_8 + \frac{V_A R^2}{EI}F_9 + \frac{T_o R}{EI}F_{a8}$

For tabulated values: $V = K_V \frac{T_o}{R}, \quad M = K_M T_o, \quad T = K_T T_o, \quad y = K_y \frac{T_o R^2}{EI}, \quad \Theta = K_\Theta \frac{T_o R}{EI}, \quad \psi = K_\psi \frac{T_o R}{EI}$

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

3a. Right end fixed, left end free



$V_A = 0 \quad M_A = 0 \quad T_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$y_A = \frac{T_o R^2}{EI}[C_{a5}\sin\phi - C_{a8}(1-\cos\phi) - C_{a2}]$$

$$\Theta_A = -\frac{T_o R}{EI}(C_{a5}\cos\phi - C_{a8}\sin\phi)$$

$$\psi_A = -\frac{T_o R}{EI}(C_{a8}\cos\phi + C_{a5}\sin\phi)$$

$V_B = 0$

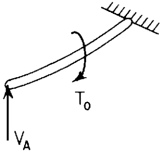
$M_B = -T_o \sin(\phi - \theta)$

$T_B = T_o \cos(\phi - \theta)$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	−0.0590	−0.5064	0.0829	0.3489	−3.6128	0.0515	1.8579
$K_{\Theta A}$	−0.0750	−0.1500	−0.7320	−0.5965	0.0000	−2.0859	−1.0429
$K_{\psi A}$	0.9782	1.8064	1.0429	0.3011	3.6128	1.0744	−0.7320
K_{MB}	−0.7071	−1.0000	−0.8660	−0.5000	0.0000	−0.8660	−0.8660
K_{TB}	0.7071	0.0000	0.5000	0.8660	−1.0000	−0.5000	0.5000

3b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{T_o}{R}\frac{C_{a8}(1-\cos\phi) - C_{a5}\sin\phi + C_{a2}}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\Theta_A = -\frac{T_o R}{EI}\frac{(C_{a2}C_9 - C_{a8}C_3)\sin\phi + (C_{a8}C_6 - C_{a5}C_9)(1-\cos\phi) + (C_{a5}C_3 - C_{a2}C_6)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\psi_A = -\frac{T_o R}{EI}\frac{[C_{a5}(C_9 + C_3) - C_6(C_{a2} + C_{a8}]\sin\phi + (C_{a8}C_3 - C_{a2}C_9)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$V_B = V_A$

$M_B = V_A R\sin\phi - T_o\sin(\phi - \theta)$

$T_B = V_A R(1-\cos\phi) + T_o\cos(\phi - \theta)$

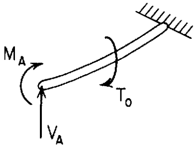
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.3668	0.4056	−0.0664	−0.2795	0.4694	−0.0067	−0.2414
$K_{\Theta A}$	−0.1872	−0.6165	−0.6557	−0.2751	−1.2204	−2.0685	−0.4153
$K_{\psi A}$	0.9566	1.6010	1.0766	0.4426	1.9170	1.0985	0.1400
K_{MB}	−0.4477	−0.5944	−0.9324	−0.7795	0.0000	−0.8660	−0.8660
K_{TB}	0.8146	0.4056	0.4336	0.5865	−0.0612	−0.5134	0.0172

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

3c. Right end fixed, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{T_o}{R}\frac{(C_{a8}C_4 - C_{a5}C_7)(1-\cos\phi) + (C_{a5}C_1 - C_{a2}C_4)\cos\phi + (C_{a2}C_7 - C_{a8}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = -T_o\frac{(C_{a5}C_9 - C_{a8}C_6)(1-\cos\phi) + (C_{a2}C_6 - C_{a5}C_3)\cos\phi + (C_{a8}C_3 - C_{a2}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = -\frac{T_oR}{EI}\frac{C_{a2}(C_4C_9 - C_6C_7) + C_{a5}(C_3C_7 - C_1C_9) + C_{a8}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$V_B = V_A$

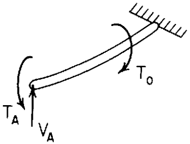
$$M_B = V_AR\sin\phi + M_A\cos\phi - T_o\sin(\phi - \theta)$$

$$T_B = V_AR(1-\cos\phi) + M_A\sin\phi + T_o\cos(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	1.8104	1.1657	0.7420	0.0596	0.6201	0.2488	−0.1901
K_{MA}	−0.7589	−0.8252	−0.8776	−0.3682	−0.4463	−0.7564	−0.1519
$K_{\psi A}$	0.8145	1.0923	0.5355	0.2156	1.3724	0.1754	−0.0453
K_{MB}	0.0364	0.1657	−0.1240	−0.4404	0.4463	−0.1096	−0.7141
K_{TB}	0.7007	0.3406	0.3644	0.5575	0.2403	−0.0023	0.1199

3d. Right end fixed, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{T_o}{R}\frac{[(C_{a2} + C_{a8})C_5 - C_{a5}(C_2 + C_8)]\sin\phi + (C_{a2}C_8 - C_{a8}C_2)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = -T_o\frac{[C_{a5}(C_3 + C_9) - C_6(C_{a2} + C_{a8})]\sin\phi + (C_{a8}C_3 - C_{a2}C_9)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = -\frac{T_oR}{EI}\frac{C_{a2}(C_5C_9 - C_6C_8) + C_{a5}(C_3C_8 - C_2C_9) + C_{a8}(C_2C_6 - C_3C_5)}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$V_B = V_A$

$$M_B = V_AR\sin\phi - T_A\sin\phi - T_o\sin(\phi - \theta)$$

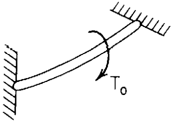
$$T_B = V_AR(1-\cos\phi) + T_A\cos\phi + T_o\cos(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	−0.3410	−0.4177	−0.3392	−0.3916	−0.2757	−0.2757
K_{TA}	−0.6694	−0.3198	−0.6724	−0.2765	−0.5730	−0.0730
$K_{\Theta A}$	−0.0544	−0.0263	−0.2411	−0.1046	−1.3691	−0.3262
K_{MB}	−0.2678	−0.3280	−0.5328	−0.6152	−0.8660	−0.8660
K_{TB}	0.2928	0.6175	0.1608	0.4744	−0.4783	0.0217

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

3e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{T_o}{R}\frac{C_{a2}(C_4C_8 - C_5C_7) + C_{a5}(C_2C_7 - C_1C_8) + C_{a8}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = -T_o\frac{C_{a2}(C_5C_9 - C_6C_8) + C_{a5}(C_3C_8 - C_2C_9) + C_{a8}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = -T_o\frac{C_{a2}(C_6C_7 - C_4C_9) + C_{a5}(C_1C_9 - C_3C_7) + C_{a8}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A$$

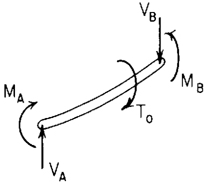
$$M_B = V_AR\sin\phi + M_A\cos\phi - T_A\sin\phi - T_o\sin(\phi - \theta)$$

$$T_B = V_AR(1 - \cos\phi) + M_A\sin\phi + T_A\cos\phi + T_o\cos(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	180°	270°	360°	
θ	15°	30°	60°	90°	90°	180°
K_{VA}	0.1704	0.1705	0.1696	0.1625	0.1592	0.0000
K_{MA}	−0.2591	−0.4731	−0.6994	−0.7073	−0.7500	0.0000
K_{TA}	−0.6187	−0.4903	−0.1278	0.2211	0.1799	0.5000
K_{MB}	−0.1252	−0.2053	−0.1666	0.0586	0.2500	0.0000
K_{TB}	0.2953	0.1974	−0.0330	−0.1302	0.1799	−0.5000

3f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = \frac{T_o}{R}\frac{[C_1\sin\phi - C_4(1 - \cos\phi)]\cos(\phi - \theta) - C_{a2}\sin^2\phi + C_{a5}(1 - \cos\phi)\sin\phi}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$M_A = -T_o\frac{[C_3\sin\phi - C_6(1 - \cos\phi)]\cos(\phi - \theta) - C_{a2}(1 - \cos\phi)\sin\phi + C_{a5}(1 - \cos\phi)^2}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$\psi_A = \frac{T_oR}{EI}\frac{(C_3C_4 - C_1C_6)\cos(\phi - \theta) + (C_{a2}C_6 - C_{a5}C_3)\sin\phi - (C_{a2}C_4 - C_{a5}C_1)(1 - \cos\phi)}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$V_B = V_A$$

$$M_B = V_AR\sin\phi + M_A\cos\phi - T_o\sin(\phi - \theta)$$

$$\psi_B = \psi_A\cos\phi + \frac{M_AR}{EI}C_7 + \frac{V_AR^2}{EI}C_9 + \frac{T_oR}{EI}C_{a8}$$

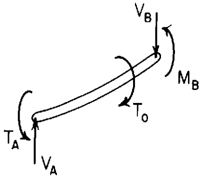
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	0°	15°	0°	30°	0°	60°
K_{VA}	1.0645	0.5147	0.8696	0.4252	0.5000	0.2500
K_{MA}	−1.4409	−1.4379	−0.8696	−0.9252	−0.3598	−0.7573
$K_{\psi A}$	1.6003	1.3211	1.2356	0.6889	1.4564	0.1746
K_{MB}	−0.9733	−1.1528	−0.1304	−0.4409	0.3598	−0.1088
$K_{\psi B}$	1.1213	1.1662	0.4208	0.4502	0.3500	−0.0034

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

3g. Right end supported and slope-guided, left end supported and roll-guided.



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{T_o}{R}\frac{C_{a2}\cos^2\phi - C_{a5}\sin\phi\cos\phi + (C_5\sin\phi - C_2\cos\phi)\cos(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = -T_o\frac{(C_{a5}\sin\phi - C_{a2}\cos\phi)(1-\cos\phi) + (C_3\cos\phi - C_6\sin\phi)\cos(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = -\frac{T_oR}{EI}\frac{(C_{a2}C_5 - C_{a5}C_2)(1-\cos\phi) + (C_{a5}C_3 - C_{a2}C_6)\cos\phi + (C_2C_6 - C_3C_5)\cos(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$V_B = V_A$$

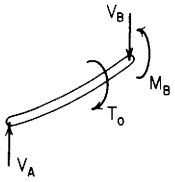
$$M_B = V_AR\sin\phi - T_A\sin\phi - T_o\sin(\phi-\theta)$$

$$\psi_B = -\Theta_A\sin\phi + \frac{T_AR}{EI}C_8 + \frac{V_AR^2}{EI}C_9 + \frac{T_oR}{EI}C_{a8}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	−0.8503	−1.4915	−0.5000	−0.8660	−0.0512	−0.2859
K_{TA}	−0.8725	−0.7482	−0.7175	−0.4095	−0.6023	−0.0717
$K_{\Theta A}$	−0.0522	−0.0216	−0.2274	−0.0640	−1.9528	−0.2997
K_{MB}	−0.4843	−0.7844	−0.6485	−0.9566	−0.8660	−0.8660
$K_{\psi B}$	0.2386	0.5031	0.1780	0.5249	−0.9169	0.0416

3h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{T_o\cos(\phi-\theta)}{R(1-\cos\phi)}$$

$$\Theta_A = -\frac{T_oR}{EI}\left[\frac{C_{a2}\sin\phi + C_6\cos(\phi-\theta)}{1-\cos\phi} - \frac{C_3\sin\phi\cos(\phi-\theta)}{(1-\cos\phi)^2} - C_{a5}\right]$$

$$\psi_A = -\frac{T_oR}{EI}\left[\frac{C_{a5}\sin\phi - C_{a2}\cos\phi}{1-\cos\phi} + (C_3\cos\phi - C_6\sin\phi)\frac{\cos(\phi-\theta)}{(1-\cos\phi)^2}\right]$$

$$V_B = V_A$$

$$M_B = V_AR\sin\phi - T_o\sin(\phi-\theta)$$

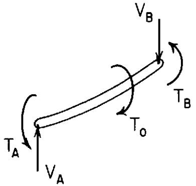
$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_AR^2}{EI}C_9 + \frac{T_oR}{EI}C_{a8}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−2.4142	0.0000	−0.5000	−0.8660	0.5000	0.2500	−0.2500
$K_{\Theta A}$	−0.4161	−0.6564	−0.6984	−0.3328	−1.3000	−2.7359	−0.3929
$K_{\psi A}$	2.1998	1.8064	1.2961	0.7396	1.9242	1.1590	0.1380
K_{MB}	−2.4142	−1.0000	−1.3660	−1.3660	0.0000	−0.8660	−0.8660
$K_{\psi B}$	1.5263	0.5064	0.5413	0.7323	−0.1178	−0.9878	0.0332

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

3i. Right end supported and roll-guided, left end supported and roll-guided.



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = 0$$

$$T_A = -T_o \frac{(C_{a2} + C_{a8}) \sin^2 \phi + (C_3 + C_9) \sin(\phi - \theta) \sin \phi}{(C_2 + C_3 + C_8 + C_9) \sin^2 \phi}$$

$$\Theta_A = -\frac{T_o R}{EI} \frac{[C_{a2}(C_8 + C_9) - C_{a8}(C_2 + C_3)] \sin \phi - (C_2 C_9 - C_3 C_8) \sin(\phi - \theta)}{(C_2 + C_3 + C_8 + C_9) \sin^2 \phi}$$

$$V_B = 0$$

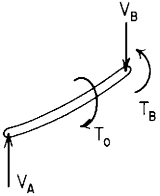
$$T_B = V_A R(1 - \cos \phi) + T_A \cos \phi + T_o \cos(\phi - \theta)$$

$$\Theta_B = \Theta_A \cos \phi + \frac{T_A R}{EI} C_5 + \frac{V_A R^2}{EI} C_6 + \frac{T_o R}{EI} C_{a5}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	270°
θ	15°	30°	90°
K_{VA}	0.0000	0.0000	0.0000
K_{TA}	−0.7071	−0.8660	0.0000
$K_{\Theta A}$	−0.0988	−0.6021	−3.6128
K_{TB}	0.3660	0.5000	−1.0000
$K_{\Theta B}$	0.0807	0.5215	0.0000

3j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = \frac{T_o \sin(\phi - \theta)}{R \sin \phi}$$

$$\Theta_A = -\frac{T_o R}{EI} \left\{ \frac{C_{a2} \cos \phi - C_{a8}(1 - \cos \phi)}{\sin \phi} + [C_3 \cos \phi - C_9(1 - \cos \phi)] \frac{\sin(\phi - \theta)}{\sin^2 \phi} \right\}$$

$$\psi_A = -\frac{T_o R}{EI} \left[C_{a2} + C_{a8} + (C_3 + C_9) \frac{\sin(\phi - \theta)}{\sin \phi} \right]$$

$$V_B = V_A$$

$$T_B = V_A R(1 - \cos \phi) + T_o \cos(\phi - \theta)$$

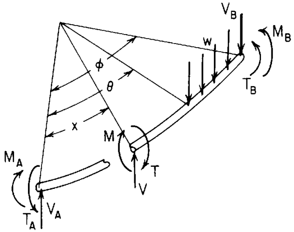
$$\Theta_B = \Theta_A \cos \phi + \psi_A \sin \phi + \frac{V_A R^2}{EI} C_6 + \frac{T_o R}{EI} C_{a5}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°			90°		
θ	0°	15°	30°	0°	30°	60°
K_{VA}	1.0000	0.7071	0.3660	1.0000	0.8660	0.5000
$K_{\Theta A}$	−0.2790	−0.2961	−0.1828	−1.3000	−1.7280	−1.1715
$K_{\psi A}$	1.0210	0.7220	0.3737	2.0420	1.7685	1.0210
K_{TB}	1.0000	1.0731	1.0731	1.0000	1.3660	1.3660
$K_{\Theta B}$	0.1439	0.1825	0.1515	0.7420	1.1641	0.9732

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

4. Uniformly distributed lateral load



Transverse shear $= V = V_A - wR\langle x-\theta\rangle^1$

Bending moment $= M = V_A R\sin x + M_A\cos x - T_A\sin x - wR^2[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A\sin x + T_A\cos x - wR^2[x-\theta-\sin(x-\theta)]\langle x-\theta\rangle^0$

Vertical deflection $= y = y_A + \Theta_A R\sin x + \psi_A R(1-\cos x) + \dfrac{M_A R^2}{EI}F_1 + \dfrac{T_A R^2}{EI}F_2 + \dfrac{V_A R^3}{EI}F_3 - \dfrac{wR^4}{EI}F_{a13}$

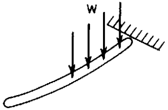
Bending slope $= \Theta = \Theta_A\cos x + \psi_A\sin x + \dfrac{M_A R}{EI}F_4 + \dfrac{T_A R}{EI}F_5 + \dfrac{V_A R^2}{EI}F_6 - \dfrac{wR^3}{EI}F_{a16}$

Roll slope $= \psi = \psi_A\cos x - \Theta_A\sin x + \dfrac{M_A R}{EI}F_7 + \dfrac{T_A R}{EI}F_8 + \dfrac{V_A R^2}{EI}F_9 - \dfrac{wR^3}{EI}F_{a19}$

For tabulated values: $V = K_V wR,\quad M = K_M wR^2,\quad T = K_T wR^2,\quad y = K_y\dfrac{wR^4}{EI},\quad \Theta = K_\Theta\dfrac{wR^3}{EI},\quad \psi = K_\psi\dfrac{wR^3}{EI}$

End restraints, reference no.	Formulas for boundary values and selected numerical values

4a. Right end fixed, left end free



$V_A = 0 \quad M_A = 0 \quad T_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$y_A = -\frac{wR^4}{EI}[C_{a16}\sin\phi - C_{a19}(1-\cos\phi) - C_{a13}]$$

$$\Theta_A = \frac{wR^3}{EI}(C_{a16}\cos\phi - C_{a19}\sin\phi)$$

$$\psi_A = \frac{wR^3}{EI}(C_{a19}\cos\phi + C_{a16}\sin\phi)$$

$$V_B = -wR(\phi-\theta)$$

$$M_B = -wR^2[1-\cos(\phi-\theta)]$$

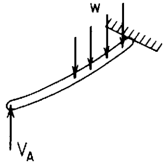
$$T_B = -wR^2[\phi-\theta-\sin(\phi-\theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	−0.0469	−0.7118	−0.2211	−0.0269	−8.4152	−2.2654	−0.1699
$K_{\Theta A}$	0.0762	0.4936	0.1071	0.0071	0.4712	−0.6033	−0.1583
$K_{\psi A}$	0.0267	0.4080	0.1583	0.0229	4.6000	1.3641	0.1071
K_{MB}	−0.2929	−1.0000	−0.5000	−0.1340	−2.0000	−1.5000	−0.5000
K_{TB}	−0.0783	−0.5708	−0.1812	−0.0236	−3.1416	−1.2284	−0.1812

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

4b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{C_{a19}(1-\cos\phi) - C_{a16}\sin\phi + C_{a13}}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\Theta_A = \frac{wR^3}{EI}\frac{(C_{a13}C_9 - C_{a19}C_3)\sin\phi + (C_{a19}C_6 - C_{a16}C_9)(1-\cos\phi) + (C_{a16}C_3 - C_{a13}C_6)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\psi_A = \frac{wR^3}{EI}\frac{[C_{a16}(C_3+C_9) - C_6(C_{a13}+C_{a19})]\sin\phi + (C_{a19}C_3 - C_{a13}C_9)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$V_B = V_A - wR(\phi - \theta)$$

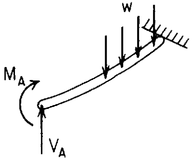
$$M_B = V_A R\sin\phi - wR^2[1-\cos(\phi-\theta)]$$

$$T_B = V_A R(1-\cos\phi) - wR^2[\phi - \theta - \sin(\phi-\theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.2916	0.5701	0.1771	0.0215	1.0933	0.2943	0.0221
$K_{\Theta A}$	−0.1300	−0.1621	−0.0966	−0.0177	−2.3714	−1.3686	−0.2156
$K_{\psi A}$	0.0095	0.1192	0.0686	0.0119	0.6500	0.3008	0.0273
K_{MB}	−0.0867	−0.4299	−0.3229	−0.1124	−2.0000	−1.5000	−0.5000
K_{TB}	0.0071	−0.0007	−0.0041	−0.0021	−0.9549	−0.6397	−0.1370

4c. Right end fixed, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{(C_{a19}C_4 - C_{a16}C_7)(1-\cos\phi) + (C_{a16}C_1 - C_{a13}C_4)\cos\phi + (C_{a13}C_7 - C_{a19}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = wR^2\frac{(C_{a16}C_9 - C_{a19}C_6)(1-\cos\phi) + (C_{a13}C_6 - C_{a16}C_3)\cos\phi + (C_{a19}C_3 - C_{a13}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = \frac{wR^3}{EI}\frac{C_{a13}(C_4C_9 - C_6C_7) + C_{a16}(C_3C_7 - C_1C_9) + C_{a19}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$V_B = V_A - wR(\phi - \theta)$$

$$M_B = V_A R\sin\phi + M_A\cos\phi - wR^2[1-\cos(\phi-\theta)]$$

$$T_B = V_A R(1-\cos\phi) + M_A\sin\phi - wR^2[\phi - \theta - \sin(\phi-\theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

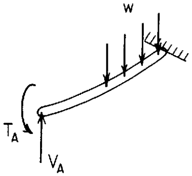
ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.3919	0.7700	0.2961	0.0434	1.3863	0.4634	0.0487
K_{MA}	−0.0527	−0.2169	−0.1293	−0.0237	−0.8672	−0.5005	−0.0789
$K_{\psi A}$	−0.0004	−0.0145	−0.0111	−0.0027	−0.4084	−0.3100	−0.0689
K_{MB}	−0.0531	−0.2301	−0.2039	−0.0906	−1.1328	−0.9995	−0.4211
K_{TB}	−0.0008	−0.0178	−0.0143	−0.0039	−0.3691	−0.3016	−0.0838

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.

Formulas for boundary values and selected numerical values

4d. Right end fixed, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$

$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{[(C_{a13}+C_{a19})C_5 - C_{a16}(C_2+C_8)]\sin\phi + (C_{a13}C_8 - C_{a19}C_2)\cos\phi}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = wR^2\frac{[C_{a16}(C_3+C_9) - C_6(C_{a13}+C_{a19})]\sin\phi + (C_{a19}C_3 - C_{a13}C_9)\cos\phi}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\frac{C_{a13}(C_5C_9 - C_6C_8) + C_{a16}(C_3C_8 - C_2C_9) + C_{a19}(C_2C_6 - C_3C_5)}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$V_B = V_A - wR(\phi - \theta)$$

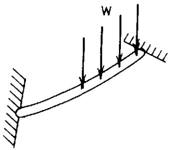
$$M_B = V_AR\sin\phi - T_A\sin\phi - wR^2[1 - \cos(\phi - \theta)]$$

$$T_B = V_AR(1 - \cos\phi) + T_A\cos\phi - wR^2[\phi - \theta - \sin(\phi - \theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.2880	0.5399	0.1597	0.0185	0.9342	0.2207	0.0154
K_{TA}	−0.0099	−0.0745	−0.0428	−0.0075	−0.3391	−0.1569	−0.0143
$K_{\Theta A}$	−0.0111	−0.1161	−0.0702	−0.0131	−1.9576	−1.1171	−0.1983
K_{MB}	−0.0822	−0.3856	−0.2975	−0.1080	−2.0000	−1.5000	−0.5000
K_{TB}	−0.0010	−0.0309	−0.0215	−0.0051	−0.9342	−0.6301	−0.1362

4e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$

$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{C_{a13}(C_4C_8 - C_5C_7) + C_{a16}(C_2C_7 - C_1C_8) + C_{a19}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = wR^2\frac{C_{a13}(C_5C_9 - C_6C_8) + C_{a16}(C_3C_8 - C_2C_9) + C_{a19}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = wR^2\frac{C_{a13}(C_6C_7 - C_4C_9) + C_{a16}(C_1C_9 - C_3C_7) + C_{a19}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A - wR(\phi - \theta)$$

$$M_B = V_AR\sin\phi + M_A\cos\phi - T_A\sin\phi - wR^2[1 - \cos(\phi - \theta)]$$

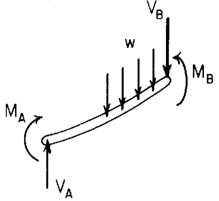
$$T_B = V_AR(1 - \cos\phi) + M_A\sin\phi + T_A\cos\phi - wR^2[\phi - \theta - \sin(\phi - \theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°		360°
θ	0°	15°	0°	30°	0°	60°	0°
K_{VA}	0.3927	0.1548	0.7854	0.3080	1.5708	0.6034	3.1416
K_{MA}	−0.0531	−0.0316	−0.2279	−0.1376	−1.0000	−0.6013	−2.1304
K_{TA}	0.0005	0.0004	0.0133	0.0102	0.2976	0.2259	3.1416
K_{MB}	−0.0531	−0.0471	−0.2279	−0.2022	−1.0000	−0.8987	−2.1304
K_{TB}	−0.0005	−0.0004	−0.0133	−0.0108	−0.2976	−0.2473	−3.1416

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

4f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = wR\frac{[C_4(1-\cos\phi) - C_1\sin\phi][\phi-\theta-\sin(\phi-\theta)] + C_{a13}\sin^2\phi - C_{a16}\sin\phi(1-\cos\phi)}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$M_A = wR^2\frac{[C_3\sin\phi - C_6(1-\cos\phi)][\phi-\theta-\sin(\phi-\theta)] + C_{a16}(1-\cos\phi)^2 - C_{a13}\sin\phi(1-\cos\phi)}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$\psi_A = \frac{wR^3}{EI}\frac{(C_{a13}C_4 - C_{a16}C_1)(1-\cos\phi) - (C_{a13}C_6 - C_{a16}C_3)\sin\phi - (C_3C_4 - C_1C_6)[\phi-\theta-\sin(\phi-\theta)]}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$V_B = V_A - wR(\phi-\theta)$$

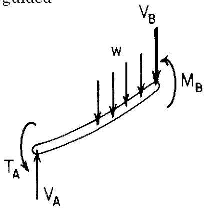
$$M_B = V_AR\sin\phi + M_A\cos\phi - wR^2[1-\cos(\phi-\theta)]$$

$$\psi_B = \psi_A\cos\phi + \frac{M_AR}{EI}C_7 + \frac{V_AR^2}{EI}C_9 - \frac{wR^3}{EI}C_{a19}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	0°	15°	0°	30°	0°	60°
K_{VA}	0.3927	0.1549	0.7854	0.3086	1.5708	0.6142
K_{MA}	−0.0519	−0.0308	−0.2146	−0.1274	−1.0000	−0.6090
$K_{\psi A}$	−0.0013	−0.0010	−0.0220	−0.0171	−0.5375	−0.4155
K_{MB}	−0.0519	−0.0462	−0.2146	−0.1914	−1.0000	−0.8910
$K_{\psi B}$	−0.0013	−0.0010	−0.0220	−0.0177	−0.5375	−0.4393

4g. Right end supported and slope-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = wR\frac{(C_5\sin\phi - C_2\cos\phi)[\phi-\theta-\sin(\phi-\theta)] + C_{a13}\cos^2\phi - C_{a16}\sin\phi\cos\phi}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = wR^2\frac{(C_3\cos\phi - C_6\sin\phi)[\phi-\theta-\sin(\phi-\theta)] - (C_{a13}\cos\phi - C_{a16}\sin\phi)(1-\cos\phi)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\frac{(C_2C_6 - C_3C_5)[\phi-\theta-\sin(\phi-\theta)] + (C_{a13}C_5 - C_{a16}C_2)(1-\cos\phi) + (C_{a16}C_3 - C_{a13}C_6)\cos\phi}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$V_B = V_A - wR(\phi-\theta)$$

$$M_B = V_AR\sin\phi - T_A\sin\phi - wR^2[1-\cos(\phi-\theta)]$$

$$\psi = -\Theta_A\sin\phi + \frac{T_AR}{EI}C_8 + \frac{V_AR^2}{EI}C_9 - \frac{wR^3}{EI}C_{a19}$$

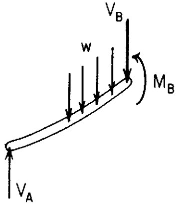
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.2896	0.5708	0.1812	0.0236	1.3727	0.5164	0.0793
K_{TA}	−0.0093	−0.0658	−0.0368	−0.0060	−0.3963	−0.1955	−0.0226
$K_{\Theta A}$	−0.0111	−0.1188	−0.0720	−0.0135	−3.0977	−1.9461	−0.3644
K_{MB}	−0.0815	−0.3634	−0.2820	−0.1043	−2.0000	−1.5000	−0.5000
$K_{\psi B}$	−0.0008	−0.0342	−0.0238	−0.0056	−1.7908	−1.2080	−0.2610

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no. — Formulas for boundary values and selected numerical values

4h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = wR\frac{\phi - \theta - \sin(\phi - \theta)}{1 - \cos\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\left\{\frac{C_{a13}\sin\phi + C_6[\phi - \theta - \sin(\phi - \theta)]}{1 - \cos\phi} - \frac{C_3\sin\phi[\phi - \theta - \sin(\phi - \theta)]}{(1 - \cos\phi)^2} - C_{a16}\right\}$$

$$\psi_A = \frac{wR^3}{EI}\left\{\frac{C_{a16}\sin\phi - C_{a13}\cos\phi}{1 - \cos\phi} - (C_6\sin\phi - C_3\cos\phi)\frac{\phi - \theta - \sin(\phi - \theta)}{(1 - \cos\phi)^2}\right\}$$

$$V_B = V_A - wR(\phi - \theta)$$

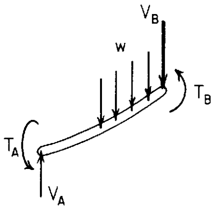
$$M_B = V_A R\sin\phi - wR^2[1 - \cos(\phi - \theta)]$$

$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_A R^2}{EI}C_9 - \frac{wR^3}{EI}C_{a19}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.2673	0.5708	0.1812	0.0236	1.5708	0.6142	0.0906
$K_{\Theta A}$	−0.0150	−0.1620	−0.0962	−0.0175	−3.6128	−2.2002	−0.3938
$K_{\psi A}$	0.0204	0.1189	0.0665	0.0109	0.7625	0.3762	0.0435
K_{MB}	−0.1039	−0.4292	−0.3188	−0.1104	−2.0000	−1.5000	−0.5000
$K_{\psi B}$	−0.0133	−0.0008	−0.0051	−0.0026	−1.8375	−1.2310	−0.2637

4i. Right end supported and roll-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{(C_{a13} + C_{a19})\sin\phi + (C_2 + C_8)[1 - \cos(\phi - \theta)]}{(C_2 + C_3 + C_8 + C_9)\sin\phi}$$

$$T_A = wR^2\frac{(C_{a13} + C_{a19})\sin\phi - (C_3 + C_9)[1 - \cos(\phi - \theta)]}{(C_2 + C_3 + C_8 + C_9)\sin\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\frac{C_{a13}(C_8 + C_9) - C_{a19}(C_2 + C_3) + (C_2C_9 - C_3C_8)[1 - \cos(\phi - \theta)]/\sin\phi}{(C_2 + C_3 + C_8 + C_9)\sin\phi}$$

$$V_B = V_A - wR(\phi - \theta)$$

$$T_B = V_A R(1 - \cos\phi) + T_A\cos\phi - wR^2[\phi - \theta - \sin(\phi - \theta)]$$

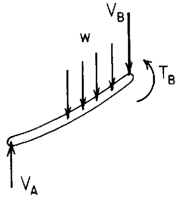
$$\Theta_B = \Theta_A\cos\phi + \frac{T_A R}{EI}C_5 + \frac{V_A R^2}{EI}C_6 - \frac{wR^3}{EI}C_{a16}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$),

ϕ	45°		90°		270°	
θ	0°	15°	0°	30°	0°	90°
K_{VA}	0.3927	0.1745	0.7854	0.3491	2.3562	1.0472
K_{TA}	−0.0215	−0.0149	−0.2146	−0.1509	3.3562	3.0472
$K_{\Theta A}$	−0.0248	−0.0173	−0.3774	−0.2717	−10.9323	−6.2614
K_{TB}	0.0215	0.0170	0.2146	0.1679	−3.3562	−2.0944
$K_{\Theta B}$	0.0248	0.0194	0.3774	0.2912	10.9323	9.9484

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

4j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$

$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{1-\cos(\phi-\theta)}{\sin\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\left[\frac{C_{a13}\cos\phi - C_{a19}(1-\cos\phi)}{\sin\phi} - [C_3\cos\phi - C_9(1-\cos\phi)]\frac{1-\cos(\phi-\theta)}{\sin^2\phi}\right]$$

$$\psi_A = \frac{wR^3}{EI}\left[C_{a13} + C_{a19} - (C_3+C_9)\frac{1-\cos(\phi-\theta)}{\sin\phi}\right]$$

$$V_B = V_A - wR(\phi-\theta)$$

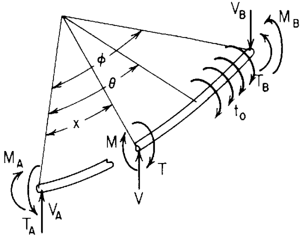
$$T_B = V_A R(1-\cos\phi) - wR^2[\phi-\theta-\sin(\phi-\theta)]$$

$$\Theta_B = \Theta_A\cos\phi + \psi_A\sin\phi + \frac{V_A R^2}{EI}C_6 - \frac{wR^3}{EI}C_{a16}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°			90°		
θ	0°	15°	30°	0°	30°	60°
K_{VA}	0.4142	0.1895	0.0482	1.0000	0.5000	0.1340
$K_{\Theta A}$	−0.0308	−0.0215	−0.0066	−0.6564	−0.4679	−0.1479
$K_{\psi A}$	0.0220	0.0153	0.0047	0.4382	0.3082	0.0954
K_{TB}	0.0430	0.0319	0.0111	0.4292	0.3188	0.1104
$K_{\Theta B}$	0.0279	0.0216	0.0081	0.5367	0.4032	0.1404

5. Uniformly distributed torque



Transverse shear $= V = V_A$

Bending moment $= M = V_A R\sin x + M_A\cos x - T_A\sin x - t_o R[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A\sin x + T_A\cos x + t_o R\sin(x-\theta)\langle x-\theta\rangle^0$

Vertical deflection $= y = y_A + \Theta_A R\sin x + \psi_A R(1-\cos x) + \dfrac{M_A R^2}{EI}F_1 + \dfrac{T_A R^2}{EI}F_2 + \dfrac{V_A R^3}{EI}F_3 + \dfrac{t_o R^3}{EI}F_{a12}$

Bending slope $= \Theta = \Theta_A\cos x + \psi_A\sin x + \dfrac{M_A R}{EI}F_4 + \dfrac{T_A R}{EI}F_5 + \dfrac{V_A R^2}{EI}F_6 + \dfrac{t_o R^2}{EI}F_{a15}$

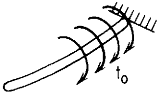
Roll slope $= \psi = \psi_A\cos x - \Theta_A\sin x + \dfrac{M_A R}{EI}F_7 + \dfrac{T_A R}{EI}F_8 + \dfrac{V_A R^2}{EI}F_9 + \dfrac{t_o R^2}{EI}F_{a18}$

For tabulated values: $V = K_V t_o, \quad M = K_M t_o R, \quad T = K_T t_o R, \quad y = K_y\dfrac{t_o R^3}{EI}, \quad \Theta = K_\Theta\dfrac{t_o R^2}{EI}, \quad \psi = K_\psi\dfrac{t_o R^2}{EI}$

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no. | Formulas for boundary values and selected numerical values

5a. Right end fixed, left end free



$V_A = 0 \quad M_A = 0 \quad T_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$y_A = \frac{t_o R^3}{EI}[C_{a15} \sin\phi - C_{a18}(1 - \cos\phi) - C_{a12}]$$

$$\Theta_A = -\frac{t_o R^2}{EI}(C_{a15} \cos\phi - C_{a18} \sin\phi)$$

$$\psi_A = -\frac{t_o R^2}{EI}(C_{a18} \cos\phi + C_{a15} \sin\phi)$$

$$V_B = 0$$

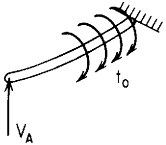
$$M_B = -t_o R[1 - \cos(\phi - \theta)]$$

$$T_B = t_o R \sin(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	0.0129	0.1500	0.2562	0.1206	0.6000	2.5359	1.1929
$K_{\Theta A}$	−0.1211	−0.8064	−0.5429	−0.1671	−3.6128	−2.2002	−0.3938
$K_{\psi A}$	0.3679	1.1500	0.3938	0.0535	2.0000	−0.5859	−0.5429
K_{MB}	−0.2929	−1.0000	−0.5000	−0.1340	−2.0000	−1.5000	−0.5000
K_{TB}	0.7071	1.0000	0.8660	0.5000	0.0000	0.8660	0.8660

5b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -t_o \frac{C_{a18}(1 - \cos\phi) - C_{a15} \sin\phi + C_{a12}}{C_9(1 - \cos\phi) - C_6 \sin\phi + C_3}$$

$$\Theta_A = -\frac{t_o R^2}{EI} \frac{(C_{a12}C_9 - C_{a18}C_3)\sin\phi + (C_{a18}C_6 - C_{a15}C_9)(1 - \cos\phi) + (C_{a15}C_3 - C_{a12}C_6)\cos\phi}{C_9(1 - \cos\phi) - C_6 \sin\phi + C_3}$$

$$\psi_A = -\frac{t_o R^2}{EI} \frac{[C_{a15}(C_9 + C_3) - C_6(C_{a12} + C_{a18})]\sin\phi + (C_{a18}C_3 - C_{a12}C_9)\cos\phi}{C_9(1 - \cos\phi) - C_6 \sin\phi + C_3}$$

$$V_B = V_A$$

$$M_B = V_A R \sin\phi - t_o R[1 - \cos(\phi - \theta)]$$

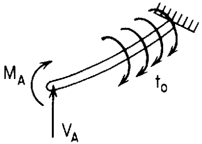
$$T_B = V_A R(1 - \cos\phi) + t_o R \sin(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−0.0801	−0.1201	−0.2052	−0.0966	−0.0780	−0.3295	−0.1550
$K_{\Theta A}$	−0.0966	−0.6682	−0.3069	−0.0560	−3.4102	−1.3436	0.0092
$K_{\psi A}$	0.3726	1.2108	0.4977	0.1025	2.2816	0.6044	0.0170
K_{MB}	−0.3495	−1.1201	−0.7052	−0.2306	−2.0000	−1.5000	−0.5000
K_{TB}	0.6837	0.8799	0.6608	0.4034	−0.1559	0.2071	0.5560

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

5c. Right end fixed, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -t_o \frac{(C_{a18}C_4 - C_{a15}C_7)(1-\cos\phi) + (C_{a15}C_1 - C_{a12}C_4)\cos\phi + (C_{a12}C_7 - C_{a18}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = -t_0 R \frac{(C_{a15}C_9 - C_{a18}C_6)(1-\cos\phi) + (C_{a12}C_6 - C_{a15}C_3)\cos\phi + (C_{a18}C_3 - C_{a12}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = -\frac{t_oR^2}{EI} \frac{C_{a12}(C_4C_9 - C_6C_7) + C_{a15}(C_3C_7 - C_1C_9) + C_{a18}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$V_B = V_A$$

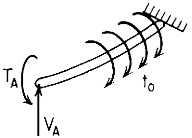
$$M_B = V_A R \sin\,\phi + M_A \cos\phi - t_o R[1 - \cos(\phi - \theta)]$$

$$T_B = V_A R(1 - \cos\,\phi) + M_A \sin\phi + t_o R \sin(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.6652	0.7038	0.1732	−0.0276	0.3433	−0.1635	−0.1561
K_{MA}	−0.3918	−0.8944	−0.4108	−0.0749	−1.2471	−0.4913	0.0034
$K_{\psi A}$	0.2993	0.6594	0.2445	0.0563	0.7597	0.0048	0.0211
K_{MB}	−0.0996	−0.2962	−0.3268	−0.1616	−0.7529	−1.0087	−0.5034
K_{TB}	0.6249	0.8093	0.6284	0.3975	0.6866	0.5390	0.5538

5d. Right end fixed, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -t_o \frac{[(C_{a12} + C_{a18})C_5 - C_{a15}(C_2 + C_8)]\sin\phi + (C_{a12}C_8 - C_{a18}C_2)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = -t_o R \frac{[C_{a15}(C_3 + C_9) - C_6(C_{a12} + C_{a18})]\sin\phi + (C_{a18}C_3 - C_{a12}C_9)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = -\frac{t_oR^2}{EI} \frac{C_{a12}(C_5C_9 - C_6C_8) + C_{a15}(C_3C_8 - C_2C_9) + C_{a18}(C_2C_6 - C_3C_5)}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$V_B = V_A$$

$$M_B = V_A R \sin\,\phi - T_A \sin\phi - t_o R[1 - \cos(\phi - \theta)]$$

$$T_B = V_A R(1 - \cos\,\phi) + T_A \cos\phi + t_o R \sin(\phi - \theta)$$

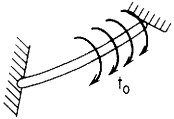
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−0.2229	−0.4269	−0.3313	−0.1226	−0.6366	−0.4775	−0.1592
K_{TA}	−0.3895	−0.7563	−0.3109	−0.0640	−1.1902	−0.3153	−0.0089
$K_{\Theta A}$	−0.0237	−0.2020	−0.1153	−0.0165	−1.9576	−0.9588	0.0200
K_{MB}	−0.1751	−0.6706	−0.5204	−0.1926	−2.0000	−1.5000	−0.5000
K_{TB}	0.3664	0.5731	0.5347	0.3774	−0.0830	0.2264	0.5566

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

5e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$

$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -t_o \frac{C_{a12}(C_4C_8 - C_5C_7) + C_{a15}(C_2C_7 - C_1C_8) + C_{a18}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = -t_oR \frac{C_{a12}(C_5C_9 - C_6C_8) + C_{a15}(C_3C_8) - C_2C_9) + C_{a18}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = -t_oR \frac{C_{a12}(C_6C_7 - C_4C_9) + C_{a15}(C_1C_9 - C_3C_7) + C_{a18}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A$$

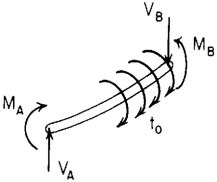
$$M_B = V_AR\sin\phi + M_A\cos\phi - T_A\sin\phi - t_oR[1 - \cos(\phi - \theta)]$$

$$T_B = V_AR(1 - \cos\phi) + M_A\sin\phi + T_A\cos\phi + t_oR\sin(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	0°	15°	0°	30°	0°	60°
K_{VA}	0.0000	−0.0444	0.0000	−0.0877	0.0000	−0.1657
K_{MA}	−0.1129	−0.0663	−0.3963	−0.2262	−1.0000	−0.4898
K_{TA}	−0.3674	−0.1571	−0.6037	−0.2238	−0.5536	−0.0035
K_{MB}	−0.1129	−0.1012	−0.3963	−0.3639	−1.0000	−1.0102
K_{TB}	0.3674	0.3290	0.6037	0.5522	0.5536	0.5382

5f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$

$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = t_o \frac{[C_1\sin\phi - C_4(1 - \cos\phi)]\sin(\phi - \theta) - C_{a12}\sin^2\phi + C_{a15}(1 - \cos\phi)\sin\phi}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$M_A = -t_oR \frac{[C_3\sin\phi - C_6(1 - \cos\phi)]\sin(\phi - \theta) - C_{a12}(1 - \cos\phi)\sin\phi + C_{a15}(1 - \cos\phi)^2}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$\psi_A = \frac{t_oR^2}{EI} \frac{(C_3C_4 - C_1C_6)\sin(\phi - \theta) + (C_{a12}C_6 - C_{a15}C_3)\sin\phi - (C_{a12}C_4 - C_{a15}C_1)(1 - \cos\phi)}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$V_B = V_A$$

$$M_B = V_AR\sin\phi + M_A\cos\phi - t_oR[1 - \cos(\phi - \theta)]$$

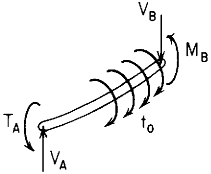
$$\psi_B = \psi_A\cos\phi + \frac{M_AR}{EI}C_7 + \frac{V_AR^2}{EI}C_9 + \frac{t_oR^2}{EI}C_{a18}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	0°	15°	0°	30°	0°	60°
K_{VA}	0.0000	−0.2275	0.0000	−0.3732	0.0000	−0.4330
K_{MA}	−1.0000	−0.6129	−1.0000	−0.4928	−1.0000	−0.2974
$K_{\psi A}$	1.0000	0.6203	1.0000	0.5089	1.0000	0.1934
K_{MB}	−1.0000	−0.7282	−1.0000	−0.8732	−1.0000	−1.2026
$K_{\psi B}$	1.0000	0.7027	1.0000	0.7765	1.0000	0.7851

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

5g. Right end supported and slope-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -t_o \frac{C_{a12}\cos^2\phi - C_{a15}\sin\phi\cos\phi + (C_5\sin\phi - C_2\cos\phi)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = -t_o R \frac{(C_{a15}\sin\phi - C_{a12}\cos\phi)(1-\cos\phi) + (C_3\cos\phi - C_6\sin\phi)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = -\frac{t_o R^2}{EI} \frac{(C_{a12}C_5 - C_{a15}C_2)(1-\cos\phi) + (C_{a15}C_3 - C_{a12}C_6)\cos\phi + (C_2C_6 - C_3C_5)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$V_B = V_A$$

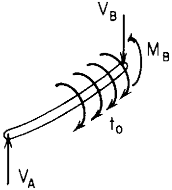
$$M_B = V_A R\sin\phi - T_A\sin\phi - t_o R[1-\cos(\phi-\theta)]$$

$$\psi_B = -\Theta_A\sin\phi + \frac{T_A R}{EI}C_8 + \frac{V_A R^2}{EI}C_9 + \frac{t_o R^2}{EI}C_{a18}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−0.8601	−1.0000	−0.8660	−0.5000	−0.5976	−0.5837	−0.4204
K_{TA}	−0.6437	−0.9170	−0.4608	−0.1698	−1.1953	−0.3014	0.0252
$K_{\Theta A}$	−0.0209	−0.1530	−0.0695	0.0158	−2.0590	−0.6825	0.6993
K_{MB}	−0.4459	−1.0830	−0.9052	−0.4642	−2.0000	−1.5000	−0.5000
$K_{\psi B}$	0.2985	0.6341	0.5916	0.4716	−0.1592	0.4340	1.0670

5h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{t_o\sin(\phi-\theta)}{1-\cos\phi}$$

$$\Theta_A = -\frac{t_o R^2}{EI}\left[\frac{C_{a12}\sin\phi + C_6\sin(\phi-\theta)}{1-\cos\phi} - \frac{C_3\sin\phi\sin(\phi-\theta)}{(1-\cos\phi)^2} - C_{a15}\right]$$

$$\psi_A = -\frac{t_o R^2}{EI}\left[\frac{C_{a15}\sin\phi - C_{a12}\cos\phi}{1-\cos\phi} + (C_3\cos\phi - C_6\sin\phi)\frac{\sin(\phi-\theta)}{(1-\cos\phi)^2}\right]$$

$$V_B = V_A$$

$$M_B = V_A R\sin\phi - t_o R[1-\cos(\phi-\theta)]$$

$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_A R^2}{EI}C_9 + \frac{t_o R^2}{EI}C_{a18}$$

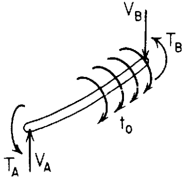
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−2.4142	−1.0000	−0.8660	−0.5000	0.0000	−0.4330	−0.4330
$K_{\Theta A}$	−0.2888	−0.7549	−0.3720	−0.0957	−3.6128	−1.0744	0.7320
$K_{\psi A}$	1.4161	1.6564	0.8324	0.3067	2.3000	0.5800	−0.0485
K_{MB}	−2.0000	−2.0000	−1.3660	−0.6340	−2.0000	−1.5000	−0.5000
$K_{\psi B}$	1.2811	1.0985	0.8250	0.5036	−0.3000	0.3985	1.0700

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no. | Formulas for boundary values and selected numerical values

5i. Right end supported and roll-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = 0$$

$$T_A = -t_o R \frac{(C_{a12} + C_{a18}) \sin^2 \phi + (C_3 + C_9) \sin \phi [1 - \cos(\phi - \theta)]}{(C_2 + C_3 + C_8 + C_9) \sin^2 \phi}$$

$$\Theta_A = -\frac{t_o R^2}{EI} \frac{[C_{a12}(C_8 + C_9) - C_{a18}(C_2 + C_3)] \sin \phi - (C_2 C_9 - C_3 C_8)[1 - \cos(\phi - \theta)]}{(C_2 + C_3 + C_8 + C_9) \sin^2 \phi}$$

$$V_B = 0$$

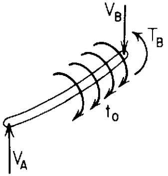
$$T_B = T_A \cos \phi + t_o R \sin(\phi - \theta)$$

$$\Theta_B = \Theta_A \cos \phi + \frac{T_A R}{EI} C_5 + \frac{t_o R^2}{EI} C_{a15}$$

If $\beta = 1.3$ (solid or hollow round cross section, v = 0.3)

ϕ	45°		90°		270°	
θ	0°	15°	0°	30°	0°	90°
K_{VA}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
K_{TA}	−0.4142	−0.1895	−1.0000	−0.5000	1.0000	2.0000
$K_{\Theta A}$	−0.0527	−0.0368	−0.6564	−0.4679	−6.5692	−2.3000
K_{TB}	0.4142	0.3660	1.0000	0.8660	−1.0000	0.0000
$K_{\Theta B}$	0.0527	0.0415	0.6564	0.5094	6.5692	7.2257

5j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = \frac{t_o[1 - \cos(\phi - \theta)]}{\sin \phi}$$

$$\Theta_A = -\frac{t_o R^2}{EI} \left\{ \frac{C_{a12} \cos \phi - C_{a18}(1 - \cos \phi)}{\sin \phi} + \frac{[C_3 \cos \phi - C_9(1 - \cos \phi)][1 - \cos(\phi - \theta)]}{\sin^2 \phi} \right\}$$

$$\psi_A = -\frac{t_o R^2}{EI} \left\{ C_{a12} + C_{a18} + \frac{(C_3 + C_9)[1 - \cos(\phi - \theta)]}{\sin \phi} \right\}$$

$$V_B = V_A$$

$$T_B = V_A R(1 - \cos \phi) + t_o R \sin(\phi - \theta)$$

$$\Theta_B = \Theta_A \cos \phi + \phi_A \sin \phi + \frac{V_A R^2}{EI} C_6 + \frac{t_o R^2}{EI} C_{a15}$$

If $\beta = 1.3$ (solid or hollow round cross section, v = 0.3)

ϕ	45°			90°		
θ	0°	15°	30°	0°	30°	60°
K_{VA}	0.4142	0.1895	0.0482	1.0000	0.5000	0.1340
$K_{\Theta A}$	−0.1683	−0.0896	−0.0247	−1.9564	−1.1179	−0.3212
$K_{\psi A}$	0.4229	0.1935	0.0492	2.0420	1.0210	0.2736
K_{TB}	0.8284	0.5555	0.2729	2.0000	1.3660	0.6340
$K_{\Theta B}$	0.1124	0.0688	0.0229	1.3985	0.8804	0.2878

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Chapter 10

Torsion

10.1 Straight Bars of Uniform Circular Section under Pure Torsion

The formulas in this section are based on the following assumptions: (1) The bar is straight, of uniform circular section (solid or concentrically hollow), and of homogeneous isotropic material; (2) the bar is loaded only by equal and opposite twisting couples, which are applied at its ends in planes normal to its axis; and (3) the bar is not stressed beyond the elastic limit.

Behavior. The bar twists, each section rotating about the longitudinal axis. Plane sections remain plane, and radii remain straight. There is at any point a shear stress τ on the plane of the section; the magnitude of this stress is proportional to the distance from the center of the section, and its direction is perpendicular to the radius drawn through the point. Accompanying this shear stress there is an equal longitudinal shear stress on a radial plane and equal tensile and compressive stresses σ_t and σ_c at 45° (see Sec. 7.5). The deformation and stresses described are represented in Fig. 10.1.

In addition to these deformations and stresses, there is some longitudinal strain and stress. A solid circular cylinder wants to lengthen under twist, as shown experimentally by Poynting (Ref. 26). In any event, for elastic loading of metallic circular bars, neither longitudinal deformation nor stress is likely to be large enough to have engineering significance.

Formulas. Let $T =$ twisting moment, $l =$ length of the member, $r =$ outer radius of the section, $J =$ polar moment of inertia of the section, $\rho =$ radial distance from the center of the section to any point q, $\tau =$ the shear stress, $\theta =$ angle of twist (radians), $G =$ modulus of

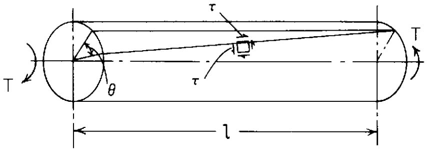


Figure 10.1

rigidity of the material, and $U =$ strain energy. Then

$$\theta = \frac{Tl}{JG} \tag{10.1-1}$$

$$\tau = \frac{T\rho}{J} \quad \text{(at point } q\text{)} \tag{10.1-2}$$

$$\tau_{\max} = \frac{Tr}{J} \quad \text{(at outer surface)} \tag{10.1-3}$$

$$U = \frac{1}{2}\frac{T^2 l}{JG} \tag{10.1-4}$$

By substituting for J in Eqs. (10.1-1) and (10.1-3) its value $2I$ from Table A.1, the formulas for cases 1 and 10 in Table 10.1 are readily obtained. If a solid or hollow circular shaft has a slight taper, the formulas above for shear stress are sufficiently accurate and the expressions for θ and U can be modified to apply to a differential length by replacing l by dl. If the change in section is abrupt, as at a shoulder with a small fillet, the maximum stress should be found by the use of a suitable factor of stress concentration K_t. Values of K_t are given in Table 17.1.

10.2 Bars of Noncircular Uniform Section under Pure Torsion

The formulas of this section are based on the same assumptions as those of Sec. 10.1 except that the cross section of the bar is not circular. It is important to note that the condition of loading implies that the end sections of the bar are free to warp, there being no constraining forces to hold them in their respective planes.

Behavior. The bar twists, each section rotating about its torsional center. Sections do not remain plane, but warp, and some radial lines through the torsional center do not remain straight. The distribution of shear stress on the section is not necessarily linear, and the direction of the shear stress is not necessarily normal to a radius.

Formulas. The torsional stiffness of the bar can be expressed by the general equations

$$T = \frac{\theta}{l} KG \quad \text{or} \quad \theta = \frac{Tl}{KG} \tag{10.2-1}$$

where K is a factor dependent on the form and dimensions of the cross section. For a *circular* section K is the polar moment of inertia J [Eq. (10.1-1)] for other sections K is less than J and may be only a very small fraction of J. The maximum stress is a function of the twisting moment and of the form and dimensions of the cross section. In Table 10.1, formulas are given for K and for max τ for a variety of sections. The formulas for cases 1 to 3, 5, 10, and 12 are based on rigorous mathematical analysis. The equations for case 4 are given in a simplified form involving an approximation, with a resulting error not greater than 4%. The K formulas for cases 13–21 and the stress formulas for cases 13–18 are based on mathematical analysis but are approximate (Ref. 2); their accuracy depends upon how nearly the actual section conforms to the assumptions indicated as to form. The K formulas for cases 22–26 and the stress formulas for cases 19–26 are based on the membrane analogy and are to be regarded as reasonably close approximations giving results that are rarely as much as 10% in error (Refs. 2–4 and 11).

It will be noted that formulas for K in cases 23–26 are based on the assumption of uniform flange thickness. For slightly tapering flanges, D should be taken as the diameter of the largest circle that can be inscribed in the actual section, and b as the average flange thickness. For sharply tapering flanges the method described by Griffith (Ref. 3) may be used. Charts relating especially to structural H- and I-sections are in Ref. 11.

Cases 7, 9, and 27–35 present the results of curve fitting to data from Isakower, Refs. 12 and 13. These data were obtained from running a computer code CLYDE (Ref. 14) based on a finite-difference solution using central differences with a constant-size square mesh. Reference 12 also suggests an extension of this work to include sections containing hollows. For some simple concentric hollows the results of solutions in Table 10.1 can be superposed to obtain closely approximate solutions if certain limitations are observed. See the examples at the end of this section.

The formulas of Table 10.1 make possible the calculation of the strength and stiffness of a bar of almost any form, but an understanding of the membrane analogy (Sec. 6.4) makes it possible to draw certain conclusions as to the *comparative* torsional properties of different sections by simply visualizing the bubbles that would be formed over holes of corresponding size and shape. From the volume relationship, it can be seen that of two sections having the same area, the one more nearly circular is the stiffer, and that although any extension whatever of the section increases its torsional stiffness, narrow outstanding flanges and similar protrusions have little effect. It is also apparent that any member having a narrow section, such as a thin plate, has practically the same torsional stiffness when flat as when bent into the form of an open tube or into channel or angle section.

From the slope relationship it can be seen that the greatest stresses (slopes) in a given section occur at the boundary adjacent to the thicker portions, and that the stresses are very low at the ends of outstanding flanges or protruding corners and very high at points where the boundary is sharply concave. Therefore a longitudinal slot or groove that is sharp at the bottom or narrow will cause high local stresses, and if it is deep will greatly reduce the torsional stiffness of the member. The direction of the shear stresses at any point is along the contour of the bubble surface at the corresponding point, and at points corresponding to local maximum and minimum elevations of the bubble having zero slopes in all directions the shear stress is zero. Therefore there may be several points of zero shear stress in a section. Thus for an I-section, there are high points of zero slope at the center of the largest inscribed circles (at the junction of web and flanges) and a low point of zero slope at the center of the web, and eight points of zero slope at the external corners. At these points in the section the shear stress is zero.

The preceding generalizations apply to solid sections, but it is possible to make somewhat similar generalizations concerning hollow or tubular sections from the formulas given for cases 10–16. These formulas show that the strength and stiffness of a hollow section depend largely upon the area inclosed by the median boundary. For this reason a circular tube is stiffer and stronger than one of any other form, and the more nearly the form of any hollow section approaches the circular, the greater will be its strength and stiffness. It is also apparent from the formulas for strength that even a local reduction in the thickness of the wall of a tube, such as would be caused by a longitudinal groove, may greatly increase the maximum shear stress, though if the groove is narrow the effect on stiffness will be small.

The torsional strengths and stiffnesses of thin-walled multicelled structures such as airplane wings and boat hulls can be calculated by the same procedures as for single-celled sections. The added relationships needed are developed from the fact that all cells twist at the same angular rate at a given section (Ref. 1).

EXAMPLES

1. It is required to compare the strength and stiffness of a circular steel tube, 4 in outside diameter and $\frac{5}{32}$ in thick, with the strength and stiffness of the same tube after it has been split by cutting full length along an element. No warping restraint is provided.

Solution. The strengths will be compared by comparing the twisting moments required to produce the same stress; the stiffnesses will be compared by comparing the values of K.

(a) For the tube (Table 10.1, case 10), $K = \frac{1}{2}\pi(r_o^4 - r_i^4) = \frac{1}{2}\pi[2^4 - (1\frac{27}{32})^4] = 6.98\,\text{in}^4$,

$$T = \tau \frac{\pi(r_o^4 - r_i^4)}{2r_o} = 3.49\tau \text{ lb-in}$$

(b) For the split tube (Table 10.1, case 17), $K = \frac{2}{3}\pi r t^3 = \frac{2}{3}\pi(1\frac{59}{64})(\frac{5}{32})^3 = 0.0154\,\text{in}^4$,

$$T = \tau \frac{4\pi^2 r^2 t^2}{6\pi r + 1.8t} = 0.097\tau \text{ lb-in}$$

The closed section is therefore more than 400 times as stiff as the open section and more than 30 times as strong.

2. It is required to determine the angle through which an airplane-wing spar of spruce, 8 ft long and having the section shown in Fig. 10.2, would be twisted by end torques of 500 lb-in, and to find the maximum resulting stress. For the material in question, $G = 100{,}000\,\text{lb/in}^2$ and $E = 1{,}500{,}000\,\text{lb/in}^2$.

Solution. All relevant dimensions are shown in Fig. 10.2, with notation corresponding to that used in the formulas. The first step is to compute K by the formulas given for case 26 (Table 10.1), and we have

$$K = 2K_1 + K_2 + 2\alpha D^4$$

$$K_1 = 2.75(1.045^3)\left\{\frac{1}{3} - \frac{0.21(1.045)}{2.75}\left[1 - \frac{1.045^4}{12(2.75^4)}\right]\right\} = 0.796\,\text{in}^4$$

$$K_2 = \tfrac{1}{3}(2.40)(0.507^3) = 0.104\,\text{in}^4$$

$$\alpha = \frac{0.507}{1.045}\left[0.150 + \frac{0.1(0.875)}{1.045}\right] = 0.1133$$

Thus

$$K = 2(0.796) + 0.104 + 2(0.1133)(1.502^4) = 2.85\,\text{in}^4$$

Therefore

$$\theta = \frac{Tl}{KG} = \frac{500(96)}{2.85(100{,}000)} = 0.168 \text{ rad} = 9.64^\circ$$

The maximum stress will probably be at P, the point where the largest inscribed circle touches the section boundary at a fillet. The formula is

$$\tau_{\max} = \frac{T}{K} C$$

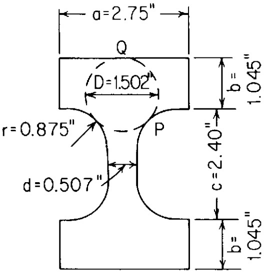


Figure 10.2

where

$$C = \frac{1.502}{1 + \frac{\pi^2(1.502^4)}{16(7.63^2)}} \left\{ 1 + \left[0.118 \ln\left(1 - \frac{1.502}{2(-0.875)}\right) - 0.238 \frac{1.502}{2(-0.875)} \right] \right.$$
$$\left. \times \tanh \frac{2(\pi/2)}{\pi} \right\} = 1.73\,\text{in}$$

Substituting the values of T, C, and K, it is found that

$$\tau_{\max} = \frac{500}{2.85}(1.73) = 303\,\text{lb/in}^2$$

It will be of interest to compare this stress with the stress at Q, the other point where the maximum inscribed circle touches the boundary. Here the formula that applies is

$$\tau = \frac{T}{K} C$$

where

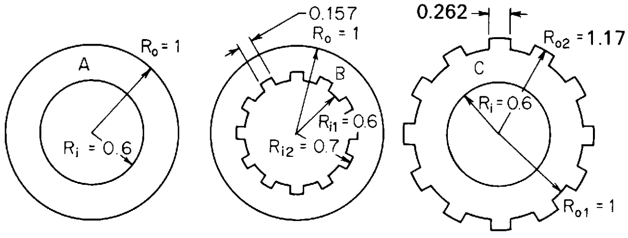
$$C = \frac{1.502}{1 + \pi^2 \frac{1.502^4}{16(7.63^2)}} \left[1 + 0.15\left(\frac{\pi^2(1.502^4)}{16(7.63^2)} - \frac{1.502}{\infty} \right) \right] = 1.437\,\text{in}$$

(Here $r =$ infinity because the boundary is straight.)

Substituting the values of $T, C,$ and K as before, it is found that $\tau = 252\,\text{lb/in}^2$.

3. For each of the three cross sections shown in Fig. 10.3, determine the numerical relationships between torque and maximum shear stress and between torque and rate of twist.

Solution. To illustrate the method of solution, superposition will be used for section A despite the availability of a solution from case 10 of Table 10.1. For a shaft, torque and angle of twist are related in the same way that a soap-film volume under the film is related to the pressure which inflates the film, provided the same cross section is used for each. See the discussion of the membrane analogy in Sec. 6.4. One can imagine then a soap film blown over a circular hole of radius R_o and then imagine the removal of that portion of the



(a) (b) (c)

Figure 10.3

volume extending above the level of the soap film for which the radius is R_i. Doing the equivalent operation with the shaft assumes that the rate of twist is θ/L and that it applies to both the outer round shaft of radius R_o and the material removed with radius R_i. The resulting torque T_R is then the difference of the two torques or

$$T_R = T_o - T_i = \frac{K_o G\theta}{L} - \frac{K_i G\theta}{L}$$

where from case 1

$$K_o = \frac{\pi R_o^4}{2} \quad \text{and} \quad K_i = \frac{\pi R_i^4}{2}$$

or

$$T_R = \frac{\pi/2(R_o^4 - R_i^4)G\theta}{L}$$

Case 10 for the hollow round section gives this same relationship. Slicing off the top of the soap film would not change the maximum slope of the bubble which is present along the outer radius. Thus the maximum shear stress on the hollow shaft with torque T_R is the same as the shear stress produced on a solid round section by the torque T_o. From case 1

$$\tau = \frac{2T_o}{\pi R_o^3} = \frac{2T_R}{\pi R_o^3}\frac{T_o}{T_R} = \frac{2T_R}{\pi R_o^3}\frac{R_o^4}{R_o^4 - R_i^4} = \frac{2T_R R_o}{\pi(R_o^4 - R_i^4)}$$

which again checks the expression for shear stress in case 10. Inserting the numerical values for R_o and R_i gives

$$T_R = \frac{1.3672G\theta}{L} \quad \text{and} \quad \tau = 0.7314T_R$$

It is important to note that the exact answer was obtained because there was a concentric circular contour line on the soap film blown over the large hole. Had the hole in the shaft been slightly off center, none of the equivalent contour lines on the soap film would have been absolutely circular and the answers just obtained would be slightly in error.

Now apply this same technique to the hollow shaft with section B having a 12-spline internal hole. To solve this case exactly, one would need to find a contour line on the equivalent soap film blown over the circular hole of radius R_o. Such a contour does not exist, but it can be created as discussed in Sec. 6.4. Imagine a massless fine wire bent into the shape of the internal spline in a single plane and allow this wire to float at a constant elevation in the soap film in equilibrium with the surface tension. The volume which is to be removed from the soap film is now well-defined and will be worked out using cases 1 and 33, but the unanswered question is what the addition of the wire did to the total volume of the original soap film. It is changed by a small amount. This information comes from Ref. 12, where Isakower shows numerical solutions to some problems with internal noncircular holes. The amount of the error depends upon the number and shape of the splines and upon how close they come to the outer boundary of the shaft. Ignoring these errors, the solution can be carried out as before.

The torque carried by the solid round shaft is the same as for section A. For the 12-point internal spline, one needs to use case 33 three times since case 33 carries only four splines and then remove the extra material added by using case 1 twice for the material internal to the splines. For case 33 let $r = 0.6$, $b = 0.1$, and $a = 0.157/2$, which gives $b/r = 0.167$ and $a/b = 0.785$. Using the equations in case 33 one finds $C = 0.8098$ and for each of the three four-splined sections, $K = 2(0.8098)(0.6)^4 = 0.2099$. For each of the two central circular sections removed, use case 1 with $r = 0.6$, getting $K = \pi(0.6^4)/2 = 0.2036$. Therefore, for the splined hole the value of $K_i = 3(0.2099) - 2(0.2036) = 0.2225$. For the solid shaft with the splined hole removed, $K_R = \pi(1)^4/2 - 0.2225 = 1.3483$ so that $T_R = 1.3483G\theta/L$.

Finding the maximum shear stress is a more difficult task for this cross section. If, as stated before, the total volume of the original soap film is changed a small amount when the spline-shaped wire is inserted, one might expect the meridional slope of the soap film at the outer edge and the corresponding stress at the outer surface of the shaft A to change slightly. However, if one ignores this effect, this shear stress can be found as (case 1)

$$\tau_o = \frac{2T_o}{\pi R_o^3} = \frac{2T_R}{\pi R_o^3}\frac{T_o}{T_R} = \frac{2T_R}{\pi R_o^3}\frac{K_o}{K_R} = \frac{2T_R}{\pi(1)^3}\frac{\pi/2}{1.3378} = 0.7475T_R$$

For this section, however, there is a possibility that the maximum shear stress and maximum slope of soap film will be bound at the outer edge of an internal spline. No value for this is known, but it would be close to the maximum shear stress on the inner edge of the spline for the material removed. This is given in case 33 as $\tau_i = TB/r^3$, where B can be found from the equations to be $B = 0.6264$, so that $\tau_i = T(0.6264)/0.6^3 = 2.8998T$. Since the torque here is the torque necessary to give a four-splined shaft a rate of twist θ/L, which is common to all the elements used, both positive and negative,

$$\tau_i = 2.8998\frac{KG\theta}{L} = 2.8998(0.2099)\frac{G\theta}{L} = 0.6087\frac{G\theta}{L}$$
$$= 0.6087\frac{T_R}{1.3483} = 0.454T_R$$

Any errors in this calculation would not be of consequence unless the stress concentrations in the corners of the splines raise the peak shear stresses above $\tau_o = 0.7475T_R$. Since this is possible, one would want to carry out a more complete analysis if considerations of fatigue or brittle fracture were necessary.

Using the same arguments already presented for the first two sections, one can find K_R for section C by using three of the 4-splined sections from case 33 and removing twice the solid round material with radius 1.0 and once the solid round material with radius 0.6. For case 33, $r = 1.0$, $b = 0.17$, $a = 0.262/2$,

$b/r = 0.17$, and $a/b = 0.771$. Using these data, one finds that $C = 0.8100$, $B = 0.6262$, and $K = 1.6200$. This gives then for the hollow section C,

$$K_R = 1.6200(3) - \frac{2\pi(1^4)}{2} - \frac{\pi(0.6^4)}{2} = 1.5148 \quad \text{and} \quad T_R = \frac{1.5148G\theta}{L}$$

Similarly,

$$\tau_{\max} = \frac{T(0.6262)}{1^3} = \frac{0.6262(1.6200)G\theta}{L} = \frac{1.0144G\theta}{L}$$
$$= 1.0144\frac{T_R}{1.5148} = 0.6697T_R$$

Again one would expect the maximum shear stress to be somewhat larger with twelve splines than with four and again the stress concentrations in the corners of the splines must be considered.

10.3 Effect of End Constraint

It was pointed out in Sec. 10.2 that when noncircular bars are twisted, the sections do not remain plane but warp, and that the formulas of Table 10.1 are based on the assumption that this warping is not prevented. If one or both ends of a bar are so fixed that warping is prevented, or if the torque is applied to a section other than at the ends of a bar, the stresses and the angle of twist produced by the given torque are affected. In compact sections the effect is slight, but in the case of open thin-walled sections the effect may be considerable.

Behavior. To visualize the additional support created by warping restraint, consider a very thin rectangular cross section and an I-beam having the same thickness and the same total cross-sectional area as the rectangle. With no warping restraint the two sections will have essentially the same stiffness factor K (see Table 10.1, cases 4 and 26). With warping restraint provided at one end of the bar, the rectangle will be stiffened very little but the built-in flanges of the I-beam act as cantilever beams. The shear forces developed in the flanges as a result of the bending of these cantilevers will assist the torsional shear stresses in carrying the applied torque and greatly increase the stiffness of the bar unless the length is very great.

Formulas. Table 10.2 gives formulas for the warping stiffness factor C_w, the torsional stiffness factor K, the location of the shear center, the magnitudes and locations within the cross section of the maximum shear stresses due to simple torsion, the maximum shear stresses due to warping, and the maximum bending stresses due to warping. All the cross sections listed are assumed to have thin walls and the same thickness throughout the section unless otherwise indicated.

Table 10.3 provides the expressions necessary to evaluate the angle of rotation θ and the first three derivatives of θ along the span for a variety of loadings and boundary restraints. The formulas in this table are based on deformations from bending stresses in the thin-walled open cross sections due to warping restraint and consequently, since the transverse shear deformations of the beam action are neglected, are not applicable to cases where the torsion member is short or where the torsional loading is applied close to a support which provides warping restraint.

In a study of the effect on seven cross sections, all of which were approximately 4 in deep and had walls approximately 0.1 in thick, Schwabenlender (Ref. 28) tested them with one end fixed and the other end free to twist but not warp with the torsional loading applied to the latter end. He found that the effect of the transverse shear stress noticeably reduced the torsional stiffness of cross sections such as those shown in Table 10.2, cases 1 and 6–8, when the length was less than six times the depth; for sections such as those in cases 2–5 (Table 10.2), the effect became appreciable at even greater lengths. To establish an absolute maximum torsional stiffness constant we note that for any cross section, when the length approaches zero, the effective torsional stiffness constant K' cannot exceed J, the polar moment of inertia, where $J = I_x + I_y$ for axes through the centroid of the cross section. (Example 1 illustrates this last condition.)

Reference 19 gives formulas and graphs for the angle of rotation and the first three derivatives for 12 cases of torsional loading of open cross sections. Payne (Ref. 15) gives the solution for a box girder that is fixed at one end and has a torque applied to the other. (This solution was also presented in detail in the fourth edition of this book.) Chu (Ref. 29) and Vlasov (Ref. 30) discuss solutions for cross sections with both open and closed parts. Kollbrunner and Basler (Ref. 31) discuss the warping of continuous beams and consider the multicellular box section, among other cross sections.

EXAMPLES

1. A steel torsion member has a cross section in the form of a twin channel with flanges inward as dimensioned in Fig. 10.4. Both ends of this channel are rigidly welded to massive steel blocks to provide full warping restraint. A torsional load is to be applied to one end block while the other is fixed. Determine the angle of twist at the loaded end for an applied torque of 1000 lb-in for lengths of 100, 50, 25, and 10 in. Assume $E = 30(10^6)\,\text{lb/in}^2$ and $\nu = 0.285$.

Solution. First determine cross-sectional constants, noting that $b = 4 - 0.1 = 3.9\,\text{in}$, $b_1 = 1.95 - 0.05 = 1.9\,\text{in}$, $h = 4 - 0.1 = 3.9\,\text{in}$, and $t = 0.1\,\text{in}$.

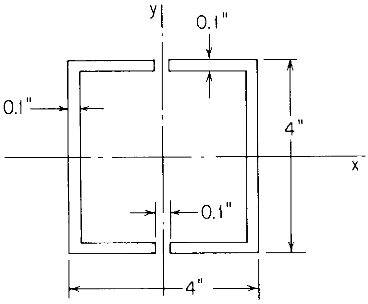


Figure 10.4

From Table 10.2, case 4,

$$K = \frac{t^3}{3}(2b + 4b_1) = \frac{0.1^3}{3}[2(3.9) + 4(1.9)] = 0.005133\,\text{in}^4$$

$$C_w = \frac{tb^2}{24}(8b_1^3 + 6h^2b_1 + h^2b + 12b_1^2h) = 28.93\,\text{in}^6$$

$$G = \frac{E}{2(1+\nu)} = \frac{30(10^6)}{2(1+0.285)} = 11.67(10^6)\,\text{lb/in}^2$$

$$\beta = \left(\frac{KG}{C_wE}\right)^{1/2} = \left[\frac{0.005133(11.67)(10^6)}{28.93(30)(10^6)}\right]^{1/2} = 0.00831\,\text{in}^{-1}$$

From Table 10.3, case 1d, when $a = 0$, the angular rotation at the loaded end is given as

$$\theta = \frac{T_o}{C_wE\beta^3}\left(\beta l - 2\tanh\frac{\beta l}{2}\right)$$

If we were to describe the total angle of twist at the loaded end in terms of an equivalent torsional stiffness constant K' in the expression

$$\theta = \frac{T_o l}{K'G}$$

then

$$K' = \frac{T_o l}{G\theta} \quad \text{or} \quad K' = K\frac{\beta l}{\beta l - 2(\beta l/2)}$$

The following table gives both K' and θ for the several lengths:

l	βl	$\frac{\beta l}{2}$	K'	θ
200	1.662	0.6810	0.0284	34.58°
100	0.831	0.3931	0.0954	5.15°
50	0.416	0.2048	0.3630	0.68°
25	0.208	0.1035	1.4333	0.09°
10	0.083	0.0415	8.926	0.006°

The stiffening effect of the fixed ends of the flanges is obvious even at a length of 200 in where $K' = 0.0284$ as compared with $K = 0.00513$. The warping restraint increases the stiffness more than five times. The large increases in K' at the shorter lengths of 25 and 10 in must be examined carefully. For this cross section $I_x = 3.88$ and $I_y = 3.96$, and so $J = 7.84\,\text{in}^4$. The calculated stiffness $K' = 8.926$ at $l = 10$ in is beyond the limiting value of 7.84, and so it is known to be in error because shear deformation was not included; therefore we would suspect the value of $K' = 1.433$ at $l = 25$ in as well. Indeed, Schwabenlender (Ref. 28) found that for a similar cross section the effect of shear deformation in the flanges reduced the stiffness by approximately 25% at a length of 25 in and by more than 60% at a length of 10 in.

2. A small cantilever crane rolls along a track welded from three pieces of 0.3-in-thick steel, as shown in Fig. 10.5. The 20-ft-long track is solidly welded to a rigid foundation at the right end and simply supported 4 ft from the left end, which is free. The simple support also provides resistance to rotation about the beam axis but provides no restraint against warping or rotation in horizontal and vertical planes containing the axis.

The crane weighs 300 lb and has a center of gravity which is 20 in out from the web of the track. A load of 200 lb is carried at the end of the crane 60 in from the web. It is desired to determine the maximum flexure stress and the maximum shear stress in the track and also the angle of inclination of the crane when it is located 8 ft from the welded end of the track.

Solution. The loading will be considered in two stages. First consider a vertical load of 500 lb acting 8 ft from the fixed end of a 16-ft beam fixed at one end and simply supported at the other. The following constants are needed:

$$\bar{y} = \frac{4(0.3)(10) + 9.7(0.3)(5)}{(4 + 9.7 + 8)(0.3)} = 4.08\,\text{in}$$

$$E = 30(10^6)\,\text{lb/in}^2$$

$$\begin{aligned} I_x &= \frac{4(0.3^3)}{12} + 4(0.3)(10 - 4.08)^2 + \frac{0.3(9.7^3)}{12} + 9.7(0.3)(5 - 4.08)^2 \\ &\quad + \frac{8(0.3^3)}{12} + 8(0.3)(4.08^2) \\ &= 107.3\,\text{in}^4 \end{aligned}$$

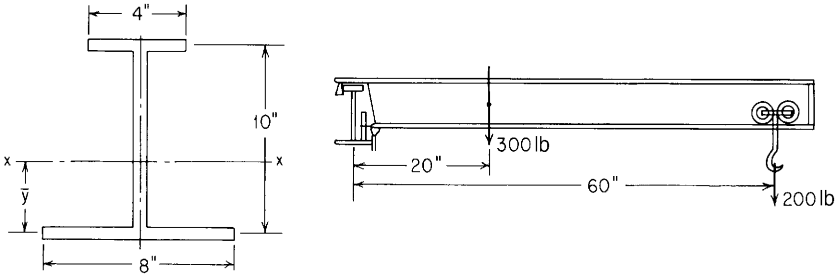
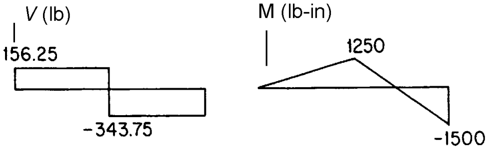


Figure 10.5

From Table 8.1, case 1c, $a = 8\,\text{ft}$, $l = 16\,\text{ft}$, $W = 500\,\text{lb}$, $M_A = 0$, and $R_A = [500/2(16^3)](16-8)^2[2(16)+8] = 156.2\,\text{lb}$. Now construct the shear and moment diagrams.



The second portion of the solution considers a beam fixed at the right end against both rotation and warping and free at the left end. It is loaded at a point 4 ft from the left end with an unknown torque T_c, which makes the angle of rotation zero at that point, and a known torque of $300(20)+200(60) = 18{,}000\,\text{lb-in}$ at a point 12 ft from the left end. Again evaluate the following constants; assume $G = 12(10^6)\,\text{lb/in}^2$:

$$K = \tfrac{1}{3}(4+8+10)(0.3^3) = 0.198\,\text{in}^4 \quad \text{(Table 10.2, case 7)}$$

$$C_w = \frac{(10^2)(0.3)(4^3)(8^3)}{(12)(4^3+8^3)} = 142.2\,\text{in}^6 \quad \text{(Table 10.2, case 7)}$$

$$\beta = \left(\frac{KG}{C_w E}\right)^{1/2} = \left[\frac{(0.198)(12)(10^6)}{(142.2)(30)(10^6)}\right]^{1/2} = 0.0236\,\text{in}^{-1}$$

Therefore $\beta l = 0.0236(20)(12) = 5.664$, $\beta(l-a) = 0.0236(20-12)(12) = 2.2656$ for $a = 12\,\text{ft}$, and $\beta(l-a) = 0.0236(20-4)(12) = 4.5312$ for $a = 4\,\text{ft}$.

From Table 10.3, case 1b, consider two torsional loads: an unknown torque T_c at $a = 4\,\text{ft}$ and a torque of 18,000 lb-in at $a = 12\,\text{ft}$. The following constants are needed:

$$C_1 = \cosh \beta l = \cosh 5.664 = 144.1515$$

$$C_2 = \sinh \beta l = \sinh 5.664 = 144.1480$$

For $a = 4\,\text{ft}$,

$$C_{a3} = \cosh \beta(l-a) - 1 = \cosh 4.5312 - 1 = 45.4404$$

$$C_{a4} = \sinh \beta(l-a) - \beta(l-a) = \sinh 4.5312 - 4.5312 = 41.8984$$

For $a = 12\,\text{ft}$,

$$C_{a3} = \cosh 2.2656 - 1 = 3.8703$$

$$C_{a4} = \sinh 2.2656 - 2.2656 = 2.5010$$

At the left end $T_A = 0$ and $\theta''_A = 0$:

$$\begin{aligned}\theta_A &= \frac{18{,}000}{(142.2)(30)(10^6)(0.0236^3)}\left[\frac{(144.1480)(3.8703)}{144.1515} - 2.5010\right]\\ &\quad + \frac{T_c}{(142.2)(30)(10^6)(0.0236^3)}\left[\frac{(144.1480)(45.4404)}{144.1515} - 41.8984\right]\\ &= 0.43953 + 6.3148(10^{-5})T_c\end{aligned}$$

Similarly, $\theta'_A = -0.0002034 - 1.327(10^{-7})T_c$.

To evaluate T_c the angle of rotation at $x = 4\,\text{ft}$ is set equal to zero:

$$\theta_c = 0 = \theta_A + \frac{\theta'_A}{\beta}F_{2(c)}$$

where

$$F_{2(c)} = \sinh[0.0236(48)] = \sinh 1.1328 = 1.3911$$

or

$$0 = 0.43953 + 6.3148(10^{-5})T_c - \frac{(0.0002034)(1.3911)}{0.0236} - \frac{(1.327)(10^{-7})(1.3911)}{0.0236}T_c$$

This gives $T_c = -7728\,\text{lb-in}$, $\theta_A = -0.04847\,\text{rad}$, and $\theta'_A = 0.0008221\,\text{rad/in}$.

To locate positions of maximum stress it is desirable to sketch curves of θ', θ'', and θ''' versus the position x:

$$\begin{aligned}\theta' &= \theta'_A F_1 + \frac{T_c}{C_w E\beta^2}F_{a3(c)} + \frac{T_o}{C_w E\beta^2}F_{a3}\\ &= 0.0008221\cosh\beta x - \frac{7728}{(142.2)(30)(10^6)(0.0236^2)}[\cosh\beta(x-48) - 1]\langle x-48\rangle^0\\ &\quad + \frac{18{,}000}{(142.2)(30)(10^6)(0.0236^2)}[\cosh\beta(x-144) - 1]\langle x-144\rangle^0\end{aligned}$$

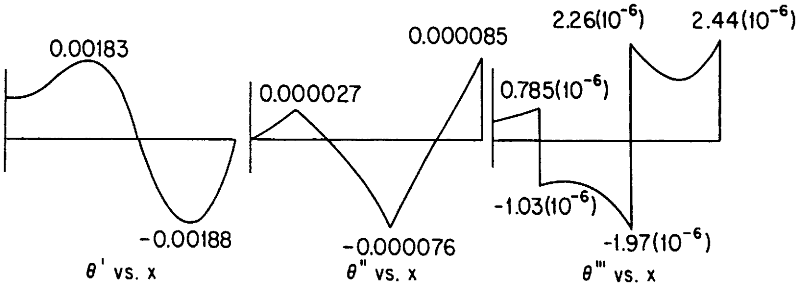
This gives

$$\begin{aligned}\theta' &= 0.0008221\cosh\beta x - 0.003253[\cosh\beta(x-48) - 1]\langle x-48\rangle^0\\ &\quad + 0.0007575[\cosh\beta(x-144) - 1]\langle x-144\rangle^0\end{aligned}$$

Similarly,

$$\begin{aligned}\theta'' &= 0.00001940\sinh\beta x - 0.00007676\sinh\beta\langle x-48\rangle\\ &\quad + 0.00001788\sinh\beta\langle x-144\rangle\\ \theta''' &= 10^{-6}[0.458\cosh\beta x - 1.812\cosh\beta(x-48)\langle x-48\rangle^0\\ &\quad + 422\cosh\beta(x-144)\langle x-144\rangle^0]\end{aligned}$$

Maximum bending stresses are produced by the beam bending moments of $+1250\,\text{lb-ft}$ at $x = 12\,\text{ft}$ and $-1500\,\text{lb-ft}$ at $x = 20\,\text{ft}$ and by maximum values of θ'' of -0.000076 at $x = 12\,\text{ft}$ and $+0.000085$ at $x = 20\,\text{ft}$. Since the largest



θ' vs. x   θ'' vs. x   θ''' vs. x

magnitudes of both M_x and θ'' occur at $x = 20$ ft, the maximum bending stress will be at the wall. Therefore, at the wall,

$$\sigma_A = \frac{1500(12)(10 - 4.08 + 0.15)}{107.26} - \frac{10(4)}{2}\frac{0.3(8^3)}{0.3(4^3) + 0.3(8^3)}(30)(10^6)(0.000085)$$

$$= 970 - 45{,}300 = -44{,}300\,\text{lb/in}^2$$

$$\sigma_B = 970 + 45{,}300 = 46{,}300\,\text{lb/in}^2$$

$$\sigma_C = \frac{-1500(12)(4.08 + 0.15)}{107.26} + \frac{10(8)}{2}\frac{0.3(4^3)}{0.3(4^3) + 0.3(8^3)}(30)(10^6)(0.000085)$$

$$= -700 + 11{,}300 = 10{,}600\,\text{lb/in}^2$$

$$\sigma_D = -700 - 11{,}300 = -12{,}000\,\text{lb/in}^2$$

Maximum shear stresses are produced by θ', θ''', and beam shear V. The shear stress due to θ''' is maximum at the top of the web, that due to θ' is maximum on the surface anywhere, and that due to V is maximum at the neutral axis but is not much smaller at the top of the web. The largest shear stress in a given cross section is therefore found at the top of the web and is the sum of the absolute values of the three components at one of four possible locations at the top of the web. This gives

$$|\tau_{\max}| = \left|\frac{1}{8}\frac{10(0.3)(8^3)(4^2)}{0.3(4^3 + 8^3)}30(10^6)\theta'''\right| + |0.3(12)(10^6)\theta'|$$

$$+ \left|\frac{(2 - 0.15)(0.3)(10 - 4.08)}{107.26(0.3)}V\right|$$

$$= |533.3(10^6)\theta'''| + |3.6(10^6)\theta'| + |0.1021V|$$

The following maximum values of θ''', θ', V, and $\tau_{\max}$ are found at the given values of the position x:

x	θ'''	θ'	V	$\lvert\tau_{\max}\rvert$, lb/in^2
48+	$-1.03(10^{-6})$	$1.41(10^{-3})$	156.3	5633
79.2	$-0.80(10^{-6})$	$1.83(10^{-3})$	156.3	7014
144−	$-1.97(10^{-6})$	$-0.28(10^{-3})$	156.3	2073
144+	$2.26(10^{-6})$	$-0.28(10^{-3})$	−343.7	2247
191.8	$1.37(10^{-6})$	$-1.88(10^{-3})$	−343.7	7522
240	$2.44(10^{-6})$	0	−343.7	1335

To obtain the rotation of the crane at the load, substitute $x = 144$ into

$$\begin{aligned}\theta &= \theta_A + \frac{\theta'_A}{\beta}F_2 + \frac{T_c}{C_w E\beta^3}F_{a4(c)}\\ &= -0.04847 + \frac{0.0008221}{0.0236}14.9414 - \frac{7728(4.7666 - 2.2656)}{142.2(30)(10^6)(0.0236^3)} = 0.1273\\ &= 7.295^\circ\end{aligned}$$

10.4 Effect of Longitudinal Stresses

It was pointed out in Sec. 10.1 that the elongation of the outer fibers consequent upon twist caused longitudinal stresses, but that in a bar of circular section these stresses were negligible. In a flexible bar, the section of which comprises one or more narrow rectangles, the stresses in the longitudinal fibers may become large; and since after twisting these fibers are inclined, the stresses in them have components, normal to the axis of twist, which contribute to the torsional resistance of the member.

The stress in the longitudinal fibers of a thin twisted strip and the effect of these stresses on torsional stiffness have been considered by Timoshenko (Ref. 5), Green (Ref. 6), Cook and Young (Ref. 1), and others. The following formulas apply to this case: Let $2a =$ width of strip; $2b =$ thickness of strip; τ, σ_t, and $\sigma_c =$ maximum shear, maximum tensile, and maximum compressive stress due to twisting, respectively; $T =$ applied twisting moment; and $\theta/l =$ angle of twist per unit length. Then

$$\sigma_t = \frac{E\tau^2}{12G^2}\left(\frac{a}{b}\right)^2 \tag{10.4-1}$$

$$\sigma_c = \tfrac{1}{2}\sigma_t \tag{10.4-2}$$

$$T = KG\frac{\theta}{l} + \frac{8}{45}E\left(\frac{\theta}{l}\right)^3 ba^5 \tag{10.4-3}$$

The first term on the right side of Eq. (10.4-3), $KG\theta/l$, represents the part of the total applied torque T that is resisted by torsional shear; the second term represents the part that is resisted by the tensile stresses in the (helical) longitudinal fibers. It can be seen that this second part is small for small angles of twist but increases rapidly as θ/l increases.

To find the stresses produced by a given torque T, first the value of θ/l is found by Eq. (10.4-3), taking K as given for Table 10.1, case 4. Then τ is found by the stress formula for case 4, taking $KG\theta/l$ for the twisting moment. Finally σ_t and σ_c can be found by Eqs. (10.4-1) and (10.4-2).

This stiffening and strengthening effect of induced longitudinal stress will manifest itself in any bar having a section composed of narrow rectangles, such as a I-, T-, or channel, provided that the parts are so thin as to permit a large unit twist without overstressing. At the same time the accompanying longitudinal compression [Eq. (10.4-2)] may cause failure through elastic instability (see Table 15.1).

If a thin strip of width a and maximum thickness b is *initially* twisted (as by cold working) to a helical angle β, then there is an initial stiffening effect in torsion that can be expressed by the ratio of effective K to nominal K (as given in Table 10.3):

$$\frac{\text{Effective}\,K}{\text{Nominal}\,K} = 1 + C(1+\nu)\beta^2\left(\frac{a}{b}\right)^2$$

where C is a numerical coefficient that depends on the shape of the cross section and is $\frac{2}{15}$ for a rectangle, $\frac{1}{8}$ for an ellipse, $\frac{1}{10}$ for a lenticular form, and $\frac{7}{60}$ for a double wedge (Ref. 22).

If a bar of any cross section is independently loaded in tension, then the corresponding longitudinal tensile stress σ_t similarly will provide a resisting torque that again depends on the angle of twist, and the total applied torque corresponding to any angle of twist θ is $T = (KG + \sigma_t J)\theta/l$, where J is the centroidal polar moment of inertia of the cross section. If the longitudinal loading causes a compressive stress σ_c, the equation becomes

$$T = (KG - \sigma_c J)\frac{\theta}{l}$$

Bending also influences the torsional stiffness of a rod unless the cross section has (1) two axes of symmetry, (2) point symmetry, or (3) one axis of symmetry that is normal to the plane of bending. (The influences of longitudinal loading and bending are discussed in Ref. 23.)

10.5 Ultimate Strength of Bars in Torsion

When twisted to failure, bars of ductile material usually break in shear, the surface of fracture being normal to the axis and practically flat. Bars of brittle material usually break in tension, the surface of fracture being helicoidal.

Circular sections. The formulas of Sec. 10.1 apply only when the maximum stress does not exceed the elastic limit. If Eq. (10.1-3) is used with T equal to the twisting moment at failure, a fictitious value of τ is obtained, which is called the *modulus of rupture in torsion* and which for convenience will be denoted here by τ'. For solid bars of steel, τ' slightly exceeds the ultimate tensile strength when the length is only about twice the diameter but drops to about 80% of the tensile strength when the length becomes 25 times the diameter. For solid bars of aluminum, τ' is about 90% of the tensile strength.

For tubes, the modulus of rupture decreases with the ratio of diameter D to wall thickness t. Younger (Ref. 7) gives the following approximate formula, applicable to tubes of steel and aluminum:

$$\tau' = \frac{1600\tau'_0}{(D/t - 2)^2 + 1600}$$

where τ' is the modulus of rupture in torsion of the tube and τ'_0 is the modulus of rupture in torsion of a solid circular bar of the same material. (Curves giving τ' as a function of D/t for various steels and light alloys may be found in Ref. 18.)

10.6 Torsion of Curved Bars; Helical Springs

The formulas of Secs. 10.1 and 10.2 can be applied to slightly curved bars without significant error, but for sharply curved bars, such as helical springs, account must be taken of the influence of curvature and slope. Among others, Wahl (Ref. 8) and Ancker and Goodier (Ref. 24) have discussed this problem, and the former presents charts which greatly facilitate the calculation of stress and deflection for springs of non-circular section. Of the following formulas cited, those for round wire were taken from Ref. 24, and those for square and rectangular wire from Ref. 8 (with some changes of notation).

Let $R =$ radius of coil measured from spring axis to center of section (Fig. 10.6), $d =$ diameter of circular section, $2b =$ thickness of square section, $P =$ load (either tensile or compressive), $n =$ number of active turns in spring, $\alpha =$ pitch angle of spring, $f =$ total stretch or shortening of spring, and $\tau =$ maximum shear stress produced. Then for a spring of *circular* wire,

$$f = \frac{64PR^3n}{Gd^4}\left[1 - \frac{3}{64}\left(\frac{d}{R}\right)^2 + \frac{3+\nu}{2(1+\nu)}(\tan\alpha)^2\right] \qquad (10.6\text{-}1)$$

$$\tau = \frac{16PR}{\pi d^3}\left[1 + \frac{5}{8}\frac{d}{R} + \frac{7}{32}\left(\frac{d}{R}\right)^2\right] \qquad (10.6\text{-}2)$$

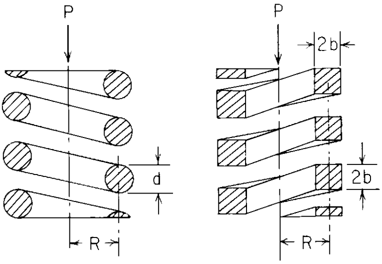


Figure 10.6

For a spring of *square* wire,

$$f = \frac{2.789PR^3n}{Gb^4} \quad \text{for } c > 3 \tag{10.6-3}$$

$$\tau = \frac{4.8PR}{8b^3}\left(1 + \frac{1.2}{c} + \frac{0.56}{c^2} + \frac{0.5}{c^3}\right) \tag{10.6-4}$$

where $c = R/b$.

For a spring of *rectangular* wire, section $2a \times 2b$ where $a > b$,

$$f = \frac{3\pi PR^3n}{8Gb^4}\frac{1}{a/b - 0.627[\tanh(\pi b/2a) + 0.004]} \tag{10.6-5}$$

for $c > 3$ if the long dimension $2a$ is parallel to the spring axis or for $c > 5$ if the long dimension $2a$ is perpendicular to the spring axis,

$$\tau = \frac{PR(3b + 1.8a)}{8b^2a^2}\left(1 + \frac{1.2}{c} + \frac{0.56}{c^2} + \frac{0.5}{c^3}\right) \tag{10.6-6}$$

It should be noted that in each of these cases the maximum stress is given by the ordinary formula for the section in question (from Table 10.1) multiplied by a corrective factor that takes account of curvature, and these corrective factors can be used for any curved bar of the corresponding cross section. Also, for compression springs with the end turns ground down for even bearing, n, should be taken as the actual number of turns (including the tapered end turns) less 2. For tension springs n should be taken as the actual number of turns or slightly more.

Unless laterally supported, compression springs that are relatively long will buckle when compressed beyond a certain critical deflection. This critical deflection depends on the ratio of L, the free length, to D, the mean diameter, and is indicated approximately by the following tabulation, based on Ref. 27. Consideration of coil closing before reaching the critical deflection is necessary.

L/D	1	2	3	4	5	6	7	8
Critical deflection/L	0.72	0.71	0.68	0.63	0.53	0.39	0.27	0.17

Precise formula. For very accurate calculation of the extension of a spring, as is necessary in designing precision spring scales, account must be taken of the change in slope and radius of the coils caused by stretching. Sayre (Ref. 9) gives a formula which takes into account not only the effect of this change in form but also the deformation due to direct transverse shear and flexure. This formula can be written as

$$f = P\left\{\left[\frac{R_0^2 L}{GK} - \frac{R_0^2 H_0^2}{GKL}\left(1 - \frac{GK}{EI}\right) + \frac{FL}{AG}\right] - \left[\frac{R_0^2}{3GKL}\left(3 - \frac{2GK}{EI}\right)(H^2 + HH_0 - 2H_0^2\right]\right\} \qquad (10.6\text{-}7)$$

where f = stretch of the spring; P = load; R_0 = initial radius of the coil; H = variable length of the effective portion of the stretched spring; H_0 = initial value of H; L = actual developed length of the wire of which the spring is made; A = cross-sectional area of this wire; K = the torsional-stiffness factor for the wire section, as given in Table 10.1 ($K = \frac{1}{2}\pi r^4$ for a circle; $K = 2.25a^4$ for a square; etc.); F = the section factor for shear deformation [Eq. (8.10-1); $F = \frac{10}{9}$ for a circle or ellipse, $F = \frac{6}{5}$ for a square or rectangle]; and I = moment of inertia of the wire section about a central axis parallel to the spring axis. The first term in brackets represents the initial rate of stretch, and the second term in brackets represents the change in this rate due to change in form consequent upon stretch. The final expression shows that f is not a linear function of P.

10.7 Tables

TABLE 10.1 Formulas for torsional deformation and stress

GENERAL FORMULAS: $\theta = TL/KG$ and $\tau = T/Q$, where $\theta =$ angle of twist (radians); $T =$ twisting moment (force-length); $L =$ length, $\tau =$ unit shear stress (force per unit area); $G =$ modulus of rigidity (force per unit area); K (length to the fourth) and Q (length cubed) are functions of the cross section

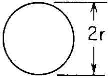
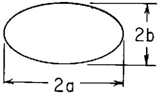
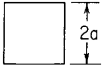
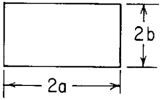
Form and dimensions of cross sections, other quantities involved, and case no.	Formula for K in $\theta = \dfrac{TL}{KG}$	Formula for shear stress
1. Solid circular section 	$K = \frac{1}{2}\pi r^4$	$\tau_{\max} = \dfrac{2T}{\pi r^3}$ at boundary
2. Solid elliptical section 	$K = \dfrac{\pi a^3 b^3}{a^2 + b^2}$	$\tau_{\max} = \dfrac{2T}{\pi a b^2}$ at ends of minor axis
3. Solid square section 	$K = 2.25a^4$	$\tau_{\max} = \dfrac{0.601T}{a^3}$ at midpoint of each side
4. Solid rectangular section 	$K = ab^3\left[\dfrac{16}{3} - 3.36\dfrac{b}{a}\left(1 - \dfrac{b^4}{12a^4}\right)\right]$ for $a \geqslant b$	$\tau_{\max} = \dfrac{3T}{8ab^2}\left[1 + 0.6095\dfrac{b}{a} + 0.8865\left(\dfrac{b}{a}\right)^2 - 1.8023\left(\dfrac{b}{a}\right)^3 + 0.9100\left(\dfrac{b}{a}\right)^4\right]$ at the midpoint of each longer side for $a \geqslant b$

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

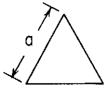
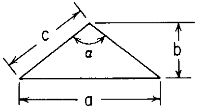
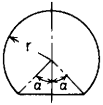
Form and dimensions of cross sections, other quantities involved, and case no.	Formula for K in $\theta = \frac{TL}{KG}$	Formula for shear stress
5. Solid triangular section (equilateral)  	$K = \frac{a^4\sqrt{3}}{80}$	$\tau_{\max} = \frac{20T}{a^3}$ at midpoint of each side
6. Isosceles triangle  (*Note:* See also Ref. 21 for graphs of stress magnitudes and locations and stiffness factors)	For $\frac{2}{3} < a/b < \sqrt{3}$ $(39^\circ < \alpha < 82^\circ)$ $K = \frac{a^3b^3}{15a^2 + 20b^2}$ approximate formula which is exact at $\alpha = 60^\circ$ where $K = 0.02165c^4$. For $\sqrt{3} < a/b < 2\sqrt{3}$ $(82^\circ < \alpha < 120^\circ)$ $K = 0.0915b^4\left(\frac{a}{b} - 0.8592\right)$ approximate formula which is exact at $\alpha = 90^\circ$ where $K = 0.0261c^4$ (errors $< 4\%$) (Ref. 20)	For $39^\circ < \alpha < 120^\circ$ $Q = \frac{K}{b[0.200 + 0.309a/b - 0.0418(a/b)^2]}$ approximate formula which is exact at $\alpha = 60^\circ$ and $\alpha = 90^\circ$ For $\alpha = 60^\circ$ $Q = 0.0768b^3 = 0.0500c^3$ For $\alpha = 90^\circ$ $Q = 0.1604b^3 = 0.0567c^3$ $\tau_{\max}$ at center of longest side
7. Circular segmental section  [*Note:* $h = r(1 - \cos\alpha)$]	$K = 2Cr^4$ where C varies with $\frac{h}{r}$ as follows. For $0 \leqslant \frac{h}{r} \leqslant 1.0$: $C = 0.7854 - 0.0333\frac{h}{r} - 2.6183\left(\frac{h}{r}\right)^2 + 4.1595\left(\frac{h}{r}\right)^3 - 3.0769\left(\frac{h}{r}\right)^4 + 0.9299\left(\frac{h}{r}\right)^5$	$\tau_{\max} = \frac{TB}{r^3}$ where B varies with $\frac{h}{r}$ as follows. For $0 \leqslant \frac{h}{r} \leqslant 1.0$: $B = 0.6366 + 1.7598\frac{h}{r} - 5.4897\left(\frac{h}{r}\right)^2 + 14.062\left(\frac{h}{r}\right)^3 - 14.510\left(\frac{h}{r}\right)^4 + 6.434\left(\frac{h}{r}\right)^5$ (Data from Refs. 12 and 13)

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

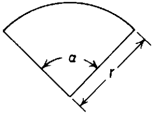
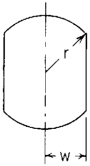
8. Circular sector  (*Note:* See also Ref. 21)	$K = Cr^4$ where C varies with $\frac{\alpha}{\pi}$ as follows. For $0.1 \leqslant \frac{\alpha}{\pi} \leqslant 2.0$: $C = 0.0034 - 0.0697\frac{\alpha}{\pi} + 0.5825\left(\frac{\alpha}{\pi}\right)^2$ $-0.2950\left(\frac{\alpha}{\pi}\right)^3 + 0.0874\left(\frac{\alpha}{\pi}\right)^4 - 0.0111\left(\frac{\alpha}{\pi}\right)^5$	$\tau_{\max} = \frac{T}{Br^3}$ on a radial boundary. B varies with $\frac{\alpha}{\pi}$ as follows. For $0.1 \leqslant \frac{\alpha}{\pi} \leqslant 1.0$: $B = 0.0117 - 0.2137\frac{\alpha}{\pi} + 2.2475\left(\frac{\alpha}{\pi}\right)^2$ $-4.6709\left(\frac{\alpha}{\pi}\right)^3 + 5.1764\left(\frac{\alpha}{\pi}\right)^4 - 2.2000\left(\frac{\alpha}{\pi}\right)^5$ (Data from Ref. 17)
9. Circular shaft with opposite sides flattened  (*Note:* $h = r - w$)	$K = 2Cr^4$ where C varies with $\frac{h}{r}$ as follows. For two flat sides where $0 \leqslant \frac{h}{r} \leqslant 0.8$: $C = 0.7854 - 0.4053\frac{h}{r} - 3.5810\left(\frac{h}{r}\right)^2$ $+ 5.2708\left(\frac{h}{r}\right)^3 - 2.0772\left(\frac{h}{r}\right)^4$ For four flat sides where $0 \leqslant \frac{h}{r} \leqslant 0.293$: $C = 0.7854 - 0.7000\frac{h}{r} - 7.7982\left(\frac{h}{r}\right)^2 + 14.578\left(\frac{h}{r}\right)^3$	$\tau_{\max} = \frac{TB}{r^3}$ where B varies with $\frac{h}{r}$ as follows. For two flat sides where $0 \leqslant \frac{h}{r} \leqslant 0.6$: $B = 0.6366 + 2.5303\frac{h}{r} - 11.157\left(\frac{h}{r}\right)^2 + 49.568\left(\frac{h}{r}\right)^3$ $- 85.886\left(\frac{h}{r}\right)^4 + 69.849\left(\frac{h}{r}\right)^5$ For four flat sides where $0 \leqslant \frac{h}{r} \leqslant 0.293$: $B = 0.6366 + 2.6298\frac{h}{r} - 5.6147\left(\frac{h}{r}\right)^2 + 30.853\left(\frac{h}{r}\right)^3$ (Data from Refs. 12 and 13)

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

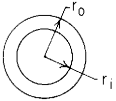
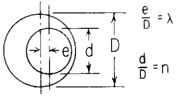
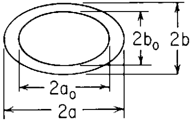
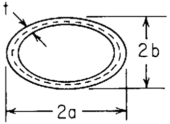
Form and dimensions of cross sections, other quantities involved, and case no.	Formula for K in $\theta = \frac{TL}{KG}$	Formula for shear stress
10. Hollow concentric circular section  	$K = \frac{1}{2}\pi(r_0^4 - r_i^4)$	$\tau_{\max} = \frac{2Tr_o}{\pi(r_o^4 - r_i^4)}$ at outer boundary
11. Eccentric hollow circular section  	$K = \frac{\pi(D^4 - d^4)}{32C}$ where $C = 1 + \frac{16n^2}{(1-n^2)(1-n^4)}\lambda^2 + \frac{384n^4}{(1-n^2)^2(1-n^4)^4}\lambda^4$	$\tau_{\max} = \frac{16TDF}{\pi(D^4 - d^4)}$ $F = 1 + \frac{4n^2}{1-n^2}\lambda + \frac{32n^2}{(1-n^2)(1-n^4)}\lambda^2 + \frac{48n^2(1+2n^2+3n^4+2n^6)}{(1-n^2)(1-n^4)(1-n^6)}\lambda^3$ $+ \frac{64n^2(2+12n^2+19n^4+28n^6+18n^8+14n^{10}+3n^{12})}{(1-n^2)(1-n^4)(1-n^6)(1-n^8)}\lambda^4$ (Ref. 10)
12. Hollow elliptical section, outer and inner boundaries similar ellipses  	$K = \frac{\pi a^3 b^3}{a^2 + b^2}(1 - q^4)$ where $q = \frac{a_o}{a} = \frac{b_o}{b}$ (*Note:* The wall thickness is not constant)	$\tau_{\max} = \frac{2T}{\pi ab^2(1 - q^4)}$ at ends of minor axis on outer surface
13. Hollow, thin-walled section of uniform thickness; $U =$ length of elliptical median boundary, shown dashed: $U = \pi(a+b-t)\left[1 + 0.258\frac{(a-b)^2}{(a+b-t)^2}\right]$ (approximately)  	$K = \frac{4\pi^2 t[(a - \frac{1}{2}t)^2(b - \frac{1}{2}t)^2]}{U}$	$\tau_{\text{average}} = \frac{T}{2\pi t(a - \frac{1}{2}t)(b - \frac{1}{2}t)}$ (stress is nearly uniform if t is small)

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

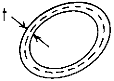
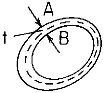
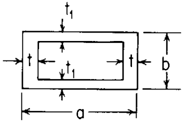
14. Any thin tube of uniform thickness; $U =$ length of median boundary; $A =$ mean of areas enclosed by outer and inner boundaries, or (approximate) area within median boundary  	$K = \dfrac{4A^2 t}{U}$	$\tau_{\text{average}} = \dfrac{T}{2tA}$ (stress is nearly uniform if t is small)
15. Any thin tube. U and A as for case 14; $t =$ thickness at any point  	$K = \dfrac{4A^2}{\int dU/t}$	τ_{average} on any thickness $AB = \dfrac{T}{2tA}$ ($\tau_{\max}$ where t is a minimum)
16. Hollow rectangle, thin-walled  (*Note:* For thick-walled hollow rectangles see Refs. 16 and 25. Reference 25 illustrates how to extend the work presented to cases with more than one enclosed region.)	$K = \dfrac{2tt_1(a-t)^2(b-t_1)^2}{at + bt_1 - t^2 - t_1^2}$	$\tau_{\text{average}} = \begin{cases} \dfrac{T}{2t(a-t)(b-t_1)} & \text{near midlength of short sides} \\ \dfrac{T}{2t_1(a-t)(b-t_1)} & \text{near midlength of long sides} \end{cases}$ (There will be higher stresses at inner corners unless fillets of fairly large radius are provided)

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

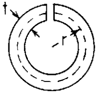
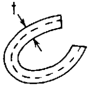
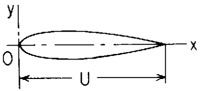
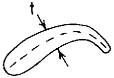
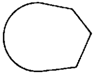
Form and dimensions of cross sections, other quantities involved, and case no.	Formula for K in $\theta = \frac{TL}{KG}$	Formula for shear stress
17. Thin circular open tube of uniform thickness; $r =$ mean radius 	$K = \frac{2}{3}\pi r t^3$	$\tau_{\max} = \frac{T(6\pi r + 1.8t)}{4\pi^2 r^2 t^2}$ along both edges remote from ends (this assumes t is small comopared with mean radius)
18. Any thin open tube of uniform thickness; $U =$ length of median line, shown dashed 	$K = \frac{1}{3}Ut^3$	$\tau_{\max} = \frac{T(3U + 1.8t)}{U^2 t^2}$ along both edges remote from ends (this assumes t small compared wtih least radius of curvature of median line; otherwise use the formulas given for cases 19–26)
19. Any elongated section with axis of symmetry OX; $U =$ length, $A =$ area of section, $I_x =$ moment of inertia about axis of symmetry 	$K = \frac{4I_x}{1 + 16I_x/AU^2}$	For all solid sections of irregular form (cases 19–26 inclusive) the maximum shear stress occurs at or very near one of the points where the largest inscribed circle touches the boundary,* and of these, at the one where the curvature of the boundary is algebraically least. (Convexity represents positive and concavity negative curvature of the boundary.) At a point where the curvature is positive (boundary of section straight or convex) this maximum stress is given approximately by $\tau_{\max} = G\frac{\theta}{L}C$ or $\tau_{\max} = \frac{T}{K}C$ where $C = \frac{D}{1 + \frac{\pi^2 D^4}{16A^2}}\left[1 + 0.15\left(\frac{\pi^2 D^4}{16A^2} - \frac{D}{2r}\right)\right]$ $D =$ diameter of largest inscribed circle $r =$ radius of curvature of boundary at the point (positive for this case) $A =$ area of the section *Unless at some point on the boundary there is a sharp reentant angle, causing high local stress.
20. Any elongated section or thin open tube; $dU =$ elementary length along median line, $t =$ thickness normal to median line, $A =$ area of section 	$K = \frac{F}{3 + 4F/AU^2}$ where $F = \int_0^U t^3\,dU$	
21. Any solid, fairly compact section without reentrant angles, $J =$ polar moment of inertia about centroid axis, $A =$ area of section 	$K = \frac{A^4}{40J}$	

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

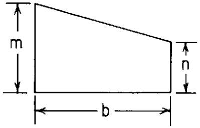
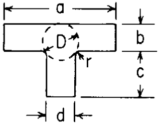
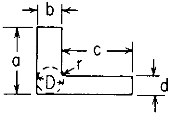
22. Trapezoid  	$K = \frac{1}{12}b(m+n)(m^2+n^2) - V_L m^4 - V_s n^4$ where $V_L = 0.10504 - 0.10s + 0.0848s^2 - 0.06746s^3 + 0.0515s^4$ $V_s = 0.10504 + 0.10s + 0.0848s^2 + 0.06746s^3 + 0.0515s^4$ $s = \frac{m-n}{b}$ (Ref. 11)	At a point where the curvature is negative (boundary of section concave or reentrant), this maximum stress is given approximately by $\tau_{\max} = G\frac{\theta}{L}C$ or $\tau_{\max} = \frac{T}{K}C$ where $C = \frac{D}{1 + \frac{\pi^2 D^4}{16A^2}}\left\{1 + \left[0.118\ln\left(1 - \frac{D}{2r}\right) - 0.238\frac{D}{2r}\right]\tanh\frac{2\phi}{\pi}\right\}$ and D, A, and r have the same meaning as before and $\phi =$ a positive angle through which a tangent to the boundary rotates in turning or traveling around the reentrant portion, measured in radians (here r is *negative*). The preceding formulas should also be used for cases 17 and 18 when t is relatively large compared with radius of median line.
23. T-section, flange thickness uniform. For definitions of r, D, t, and t_1, see case 26.  	$K = K_1 + K_2 + \alpha D^4$ where $K_1 = ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$ $K_2 = cd^3\left[\frac{1}{3} - 0.105\frac{d}{c}\left(1 - \frac{d^4}{192c^4}\right)\right]$ $\alpha = \frac{t}{t_1}\left(0.15 + 0.10\frac{r}{b}\right)$ $D = \frac{(b+r)^2 + rd + d^2/4}{(2r+b)}$ for $d < 2(b+r)$	
24. L-section; $b \geqslant d$. For definitions of r and D, see case 26.  	$K = K_1 + K_2 + \alpha D^4$ where $K_1 = ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$ $K_2 = cd^3\left[\frac{1}{3} - 0.105\frac{d}{c}\left(1 - \frac{d^4}{192c^4}\right)\right]$ $\alpha = \frac{d}{b}\left(0.07 + 0.076\frac{r}{b}\right)$ $D = 2[d + b + 3r - \sqrt{2(2r+b)(2r+d)}]$ for $b < 2(d+r)$	

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

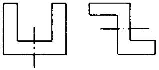
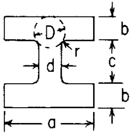
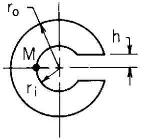
Form and dimensions of cross sections, other quantities involved, and case no.	Formula for K in $\theta = \dfrac{TL}{KG}$	Formula for shear stress
25. U- or Z-section 	$K =$ sum of K's of constituent L-sections computed as for case 24	
26. I-section, flange thickness uniform; $r =$ fillet radius, $D =$ diameter largest inscribed circle, $t = b$ if $b < d$; $t = d$ if $d < b$; $t_1 = b$ if $b > d$; $t_1 = d$ if $d > b$ 	$K = 2K_1 + K_2 + 2\alpha D^4$ where $K_1 = ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$ $K_2 = \frac{1}{3}cd^3$ $\alpha = \frac{t}{t_1}\left(0.15 + 0.1\frac{r}{b}\right)$ Use expression for D from case 23	
27. Split hollow shaft 	$K = 2Cr_o^4$ where C varies with $\frac{r_i}{r_o}$ as follows. For $0.2 \leqslant \frac{r_i}{r_o} \leqslant 0.6$: $C = K_1 + K_2\frac{r_i}{r_o} + K_3\left(\frac{r_i}{r_o}\right)^2 + K_4\left(\frac{r_i}{r_o}\right)^3$ where for $0.1 \leqslant h/r_i \leqslant 1.0$, $K_1 = 0.4427 + 0.0064\frac{h}{r_i} - 0.0201\left(\frac{h}{r_i}\right)^2$ $K_2 = -0.8071 - 0.4047\frac{h}{r_i} + 0.1051\left(\frac{h}{r_i}\right)^2$ $K_3 = -0.0469 + 1.2063\frac{h}{r_i} - 0.3538\left(\frac{h}{r_i}\right)^2$ $K_4 = 0.5023 - 0.9618\frac{h}{r_i} + 0.3639\left(\frac{h}{r_i}\right)^2$	At M, $\tau = \frac{TB}{r_o^3}$ where B varies with $\frac{r_i}{r_o}$ as follows. For $0.2 \leqslant \frac{r_i}{r_o} \leqslant 0.6$: $B = K_1 + K_2\frac{r_i}{r_o} + K_3\left(\frac{r_i}{r_o}\right)^2 + K_4\left(\frac{r_i}{r_o}\right)^3$ where fore $0.1 \leqslant h/r_i \leqslant 1.0$, $K_1 = \quad 2.0014 - 0.1400\frac{h}{r_i} - 0.3231\left(\frac{h}{r_i}\right)^3$ $K_2 = \quad 2.9047 + 3.0069\frac{h}{r_i} + 4.0500\left(\frac{h}{r_i}\right)^2$ $K_3 = -15.721 - 6.5077\frac{h}{r_i} - 12.496\left(\frac{h}{r_i}\right)^2$ $K_4 = \quad 29.553 + 4.1115\frac{h}{r_i} + 18.845\left(\frac{h}{r_i}\right)^2$ (Data from Refs. 12 and 13)

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

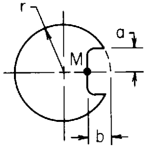
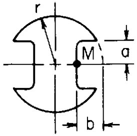
28. Shaft with one keyway  	$K = 2Cr^4$ where C varies with $\frac{b}{r}$ as follows. For $0 \leqslant \frac{b}{r} \leqslant 0.5$: $C = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.3 \leqslant a/b \leqslant 1.5$, $K_1 = 0.7854$ $K_2 = -0.0848 + 0.1234\frac{a}{b} - 0.0847\left(\frac{a}{b}\right)^2$ $K_3 = -0.4318 - 2.2000\frac{a}{b} + 0.7633\left(\frac{a}{b}\right)^2$ $K_4 = -0.0780 + 2.0618\frac{a}{b} - 0.5234\left(\frac{a}{b}\right)^2$	At M, $\tau = \frac{TB}{r^3}$ where B varies with $\frac{b}{r}$ as follows. For $0.2 \leqslant \frac{b}{r} \leqslant 0.5$: $B = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.5 \leqslant a/b \leqslant 1.5$, $K_1 = 1.1690 - 0.3168\frac{a}{b} + 0.0490\left(\frac{a}{b}\right)^2$ $K_2 = 0.43490 - 1.5096\frac{a}{b} + 0.8677\left(\frac{a}{b}\right)^2$ $K_3 = -1.1830 + 4.2764\frac{a}{b} - 1.7024\left(\frac{a}{b}\right)^2$ $K_4 = 0.8812 - 0.2627\frac{a}{b} - 0.1897\left(\frac{a}{b}\right)^2$ (Data from Refs. 12 and 13)
29. Shaft with two keyways  	$K = 2Cr^4$ where C varies with $\frac{b}{r}$ as follows. For $0 \leqslant \frac{b}{r} \leqslant 0.5$: $C = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.3 \leqslant a/b \leqslant 1.5$, $K_1 = 0.7854$ $K_2 = -0.0795 + 0.1286\frac{a}{b} - 0.1169\left(\frac{a}{b}\right)^2$ $K_3 = -1.4126 - 3.8589\frac{a}{b} + 1.3292\left(\frac{a}{b}\right)^2$ $K_4 = 0.7098 + 4.1936\frac{a}{b} - 1.1053\left(\frac{a}{b}\right)^2$	At M, $\tau = \frac{TB}{r^3}$ where B varies with $\frac{b}{r}$ as follows. For $0.2 \leqslant \frac{b}{r} \leqslant 0.5$: $B = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.5 \leqslant a/b \leqslant 1.5$, $K_1 = 1.2512 - 0.5406\frac{a}{b} + 0.0387\left(\frac{a}{b}\right)^2$ $K_2 = -0.9385 + 2.3450\frac{a}{b} + 0.3256\left(\frac{a}{b}\right)^2$ $K_3 = 7.2650 - 15.338\frac{a}{b} + 3.1138\left(\frac{a}{b}\right)^2$ $K_4 = -11.152 + 33.710\frac{a}{b} - 10.007\left(\frac{a}{b}\right)^2$ (Data from Refs. 12 and 13)

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

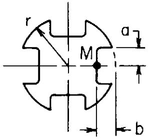
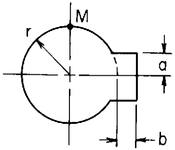
Form and dimensions of cross sections, other quantities involved, and case no.	Formula for K in $\theta = \frac{TL}{KG}$	Formula for shear stress
30. Shaft with four keyways  	$K = 2Cr^4$ where C varies with $\frac{b}{r}$ as follows. For $0 \leqslant \frac{b}{r} \leqslant 0.4$: $C = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.3 \leqslant a/b \leqslant 1.2$, $K_1 = 0.7854$ $K_2 = -0.1496 + 0.2773\frac{a}{b} - 0.2110\left(\frac{a}{b}\right)^2$ $K_3 = -2.9138 - 8.2354\frac{a}{b} + 2.5782\left(\frac{a}{b}\right)^2$ $K_4 = 2.2991 + 12.097\frac{a}{b} - 2.2838\left(\frac{a}{b}\right)^2$	At M, $\tau = \frac{TB}{r^3}$ where B varies with $\frac{b}{r}$ as follows. For $0.2 \leqslant \frac{b}{r} \leqslant 0.4$, $B = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.5 \leqslant a/b \leqslant 1.2$, $K_1 = 1.0434 + 1.0449\frac{a}{b} - 0.2977\left(\frac{a}{b}\right)^2$ $K_2 = 0.0958 - 9.8401\frac{a}{b} + 1.6847\left(\frac{a}{b}\right)^2$ $K_3 = 15.749 - 6.9650\frac{a}{b} + 14.222\left(\frac{a}{b}\right)^2$ $K_4 = -35.878 + 88.696\frac{a}{b} - 47.545\left(\frac{a}{b}\right)^2$ (Data from Refs. 12 and 13)
31. Shaft with one spline  	$K = 2Cr^4$ where C varies with $\frac{b}{r}$ as follows. For $0 \leqslant \frac{b}{r} \leqslant 0.5$: $C = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.2 \leqslant a/b \leqslant 1.4$, $K_1 = 0.7854$ $K_2 = 0.0264 - 0.1187\frac{a}{b} + 0.0868\left(\frac{a}{b}\right)^2$ $K_3 = -0.2017 + 0.9019\frac{a}{b} - 0.4947\left(\frac{a}{b}\right)^2$ $K_4 = 0.2911 - 1.4875\frac{a}{b} + 2.0651\left(\frac{a}{b}\right)^2$	At M, $\tau = \frac{TB}{r^3}$ where B varies with $\frac{b}{r}$ as follows. For $0 \leqslant \frac{b}{r} \leqslant 0.5$, $B = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.2 \leqslant a/b \leqslant 1.4$, $K_1 = 0.6366$ $K_2 = -0.0023 + 0.0168\frac{a}{b} + 0.0093\left(\frac{a}{b}\right)^2$ $K_3 = 0.0052 + 0.0225\frac{a}{b} - 0.3300\left(\frac{a}{b}\right)^2$ $K_4 = 0.0984 - 0.4936\frac{a}{b} + 0.2179\left(\frac{a}{b}\right)^2$ (Data from Refs. 12 and 13)

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

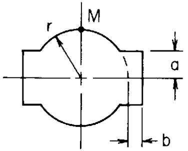
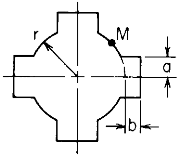
32. Shaft with two splines  	$K = 2Cr^4$ where C varies with $\frac{b}{r}$ as follows. For $0 \leqslant \frac{b}{r} \leqslant 0.5$: $C = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.2 \leqslant a/b \leqslant 1.4$, $K_1 = 0.7854$ $K_2 = 0.0204 - 0.1307\frac{a}{b} + 0.1157\left(\frac{a}{b}\right)^2$ $K_3 = -0.2075 + 1.1544\frac{a}{b} - 0.5937\left(\frac{a}{b}\right)^2$ $K_4 = 0.3608 - 2.2582\frac{a}{b} + 3.7336\left(\frac{a}{b}\right)^2$	At M, $\tau = \frac{TB}{r^3}$ where B varies with $\frac{b}{r}$ as follows. For $0 \leqslant \frac{b}{r} \leqslant 0.5$, $B = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.2 \leqslant a/b \leqslant 1.4$, $K_1 = 0.6366$ $K_2 = 0.0069 - 0.0229\frac{a}{b} + 0.0637\left(\frac{a}{b}\right)^2$ $K_3 = -0.0675 + 0.3996\frac{a}{b} - 1.0514\left(\frac{a}{b}\right)^2$ $K_4 = 0.3582 - 1.8324\frac{a}{b} + 1.5393\left(\frac{a}{b}\right)^2$ (Data from Refs. 12 and 13)
33. Shaft with four splines  	$K = 2Cr^4$ where C varies with $\frac{b}{r}$ as follows. For $0 \leqslant \frac{b}{r} \leqslant 0.5$: $C = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.2 \leqslant a/b \leqslant 1.0$, $K_1 = 0.7854$ $K_2 = 0.0595 - 0.3397\frac{a}{b} + 0.3239\left(\frac{a}{b}\right)^2$ $K_3 = -0.6008 + 3.1396\frac{a}{b} - 2.0693\left(\frac{a}{b}\right)^2$ $K_4 = 1.0869 - 6.2451\frac{a}{b} + 9.4190\left(\frac{a}{b}\right)^2$	At M, $\tau = \frac{TB}{r^3}$ where B varies with $\frac{b}{r}$ as follows. For $0 \leqslant \frac{b}{r} \leqslant 0.5$, $B = K_1 + K_2\frac{b}{r} + K_3\left(\frac{b}{r}\right)^2 + K_4\left(\frac{b}{r}\right)^3$ where for $0.2 \leqslant a/b \leqslant 1.0$, $K_1 = 0.6366$ $K_2 = 0.0114 - 0.0789\frac{a}{b} + 0.1767\left(\frac{a}{b}\right)^2$ $K_3 = -0.1207 + 1.0291\frac{a}{b} - 2.3589\left(\frac{a}{b}\right)^2$ $K_4 = 0.5132 - 3.4300\frac{a}{b} + 4.0226\left(\frac{a}{b}\right)^2$ (Data from Refs. 12 and 13)

TABLE 10.1 Formulas for torsional deformation and stress (*Continued*)

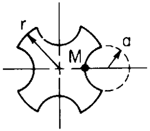
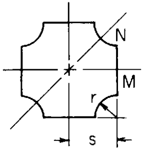
Form and dimensions of cross sections, other quantities involved, and case no.	Formula for K in $\theta = \dfrac{TL}{KG}$	Formula for shear stress
34. Pinned shaft with one, two, or four grooves  	$K = 2Cr^4$ where C varies with $\dfrac{a}{r}$ over the range $0 \leqslant \dfrac{a}{r} \leqslant 0.5$ as follows. For one groove: $C = 0.7854 - 0.0225\dfrac{a}{r} - 1.4154\left(\dfrac{a}{r}\right)^2 + 0.9167\left(\dfrac{a}{r}\right)^3$ For two grooves: $C = 0.7854 - 0.0147\dfrac{a}{r} - 3.0649\left(\dfrac{a}{r}\right)^2 + 2.5453\left(\dfrac{a}{r}\right)^3$ For four grooves: $C = 0.7854 - 0.0409\dfrac{a}{r} - 6.2371\left(\dfrac{a}{r}\right)^2 + 7.2538\left(\dfrac{a}{r}\right)^3$	At $M, \tau = \dfrac{TB}{r^3}$ where B varies with $\dfrac{a}{r}$ over the range $0.1 \leqslant \dfrac{a}{r} \leqslant 0.5$ as follows. For one groove: $B = 1.0259 + 1.1802\dfrac{a}{r} - 2.7897\left(\dfrac{a}{r}\right)^2 + 3.7092\left(\dfrac{a}{r}\right)^3$ For two grooves: $B = 1.0055 + 1.5427\dfrac{a}{r} - 2.9501\left(\dfrac{a}{r}\right)^2 + 7.0534\left(\dfrac{a}{r}\right)^3$ For four grooves: $B = 1.2135 - 2.9697\dfrac{a}{r} + 33.713\left(\dfrac{a}{r}\right)^2 - 99.506\left(\dfrac{a}{r}\right)^3 + 130.49\left(\dfrac{a}{r}\right)^4$ (Data from Refs. 12 and 13)
35. Cross shaft  	$K = 2Cs^4$ where C varies with $\dfrac{r}{s}$over the range $0 \leqslant \dfrac{r}{s} \leqslant 0.9$ as follows: $C = 1.1266 - 0.3210\dfrac{r}{s} + 3.1519\left(\dfrac{r}{s}\right)^2 - 14.347\left(\dfrac{r}{s}\right)^3 + 15.223\left(\dfrac{r}{s}\right)^4 - 4.7767\left(\dfrac{r}{s}\right)^5$	At $M, \tau = \dfrac{B_M T}{s^3}$ where B_M varies with $\dfrac{r}{s}$ over the range $0 \leqslant \dfrac{r}{s} \leqslant 0.5$ as follows: $B_M = 0.6010 + 0.1059\dfrac{r}{s} - 0.9180\left(\dfrac{r}{s}\right)^2 + 3.7335\left(\dfrac{r}{s}\right)^3 - 2.8686\left(\dfrac{r}{s}\right)^4$ At $N, \tau = \dfrac{B_N T}{s^3}$ where B_N varies with $\dfrac{r}{s}$ over the range $0.3 \leqslant \dfrac{r}{s} \leqslant 0.9$ as follows: $B_N = -0.3281 + 9.1405\dfrac{r}{s} - 42.520\left(\dfrac{r}{s}\right)^2 + 109.04\left(\dfrac{r}{s}\right)^3 - 133.95\left(\dfrac{r}{s}\right)^4 + 66.054\left(\dfrac{r}{s}\right)^5$ (*Note:* $B_N > B_M$ for $r/s > 0.32$) (Data from Refs. 12 and 13)

TABLE 10.2 Formulas for torsional properties and stresses in thin-walled open cross sections

NOTATION: Point 0 indicates the shear center. $e =$ distance from a reference to the shear center; $K =$ torsional stiffness constant (length to the fourth power); $C_w =$ warping constant (length to the sixth power); $\tau_1 =$ shear stress due to torsional rigidity of the cross section (force per unit area); $\tau_2 =$ shear stress due to warping rigidity of the cross section (force per unit area); $\sigma_x =$ bending stress due to warping rigidity of the cross section (force per unit area); $E =$ modulus of elasticity of the material (force per unit area); and $G =$ modulus of rigidity (shear modulus) of the material (force per unit area)

The appropriate values of θ', θ'', and θ''' are found in Table 10.3 for the loading and boundary restraints desired

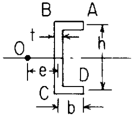
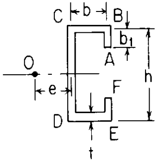
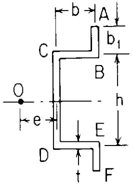
Cross section, reference no.	Constants	Selected maximum values
1. Channel 	$e = \frac{3b^2}{h+6b}$ $K = \frac{t^3}{3}(h+2b)$ $C_w = \frac{h^2b^3t}{12}\frac{2h+3b}{h+6b}$	$(\sigma_x)_{\max} = \frac{hb}{2}\frac{h+3b}{h+6b}E\theta''$ throughout the thickness at corners A and D $(\tau_2)_{\max} = \frac{hb^2}{4}\left(\frac{h+3b}{h+6b}\right)^2 E\theta'''$ throughout the thickness at a distance $b\frac{h+3b}{h+6b}$ from corners A and D $(\tau_1)_{\max} = tG\theta'$ at the surface everywhere
2. C-section 	$e = b\frac{3h^2b+6h^2b_1-8b_1^3}{h^3+6h^2b+6h^2b_1+8b_1^3-12hb_1^2}$ $K = \frac{t^3}{3}(h+2b+2b_1)$ $C_w = t\left[\frac{h^2b^2}{2}\left(b_1+\frac{b}{3}-e-\frac{2eb_1}{b}+\frac{2b_1^2}{h}\right)\right.$ $\left.+\frac{h^2e^2}{2}\left(b+b_1+\frac{h}{6}-\frac{2b_1^2}{h}\right)+\frac{2b_1^3}{3}(b+e)^2\right]$	$(\sigma_x)_{\max} = \left[\frac{h}{2}(b-e)+b_1(b+e)\right]E\theta''$ throughout the thickness at corners A and F $(\tau_2)_{\max} = \left[\frac{h}{4}(b-e)(2b_1+b-e)+\frac{b_1^2}{2}(b+e)\right]E\theta'''$ throughout the thickness on the top and bottom flanges at a distance e from corners C and D $(\tau_1)_{\max} = tG\theta'$ at the surface everywhere
3. Hat section 	$e = b\frac{3h^2b+6h^2b_1-8b_1^3}{h^3+6h^2b+6h^2b_1+8b_1^3+12hb_1^2}$ $K = \frac{t^3}{3}(h+2b+2b_1)$ $C_w = t\left[\frac{h^2b^2}{2}\left(b_1+\frac{b}{3}-e-\frac{2eb_1}{b}-\frac{2b_1^2}{h}\right)\right.$ $\left.+\frac{h^2e^2}{2}\left(b+b_1+\frac{h}{6}+\frac{2b_1^2}{h}\right)+\frac{2b_1^3}{3}(b+e)^2\right]$	$\sigma_x = \left[\frac{h}{2}(b-e)-b_1(b+e)\right]E\theta''$ throughout the thickness at corners A and F $\sigma_x = \frac{h}{2}(b-e)E\theta''$ throughout the thickness at corners B and E $\tau_2 = \left[\frac{h^2(b-e)^2}{8(b+e)}+\frac{b_1^2}{2}(b+e)-\frac{hb_1}{2}(b-e)\right]E\theta'''$ throughout the thickness at a distance $\frac{h(b-e)}{2(b+e)}$ from corner B toward corner A $\tau_2 = \left[\frac{b_1^2}{2}(b+e)-\frac{hb_1}{2}(b-e)-\frac{h}{4}(b-e)^2\right]E\theta'''$ throughout the thickness at a distance e from corner C toward corner B $\tau_1 = tG\theta'$ at the surface everywhere

TABLE 10.2 Formulas for torsional properties and stresses in thin-walled open cross sections (*Continued*)

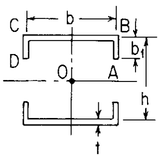
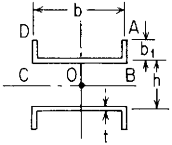
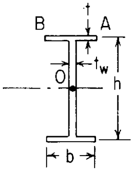
Cross section, reference no.	Constants	Selected maximum values
4. Twin channel with flanges inward 	$K = \frac{t^3}{3}(2b+4b_1)$ $C_w = \frac{tb^2}{24}(8b_1^3+6h^2b_1+h^2b+12b_1^2h)$	$(\sigma_x)_{\max} = \frac{b}{2}\left(b_1+\frac{h}{2}\right)E\theta''$ throughout the thickness at points A and D $(\tau_2)_{\max} = \frac{-b}{16}(4b_1^2+4b_1h+hb)E\theta'''$ throughout the thickness midway between corners B and C $(\tau_1)_{\max} = tG\theta'$ at the surface everywhere
5. Twin channel with flanges outward 	$K = \frac{t^3}{3}(2b+4b_1)$ $C_w = \frac{tb^2}{24}(8b_1^3+6h^2b_1+h^2b-12b_1^2h)$	$(\sigma_x)_{\max} = \frac{hb}{4}E\theta''$ throughout the thickness at points B and C if $h > b_1$ $(\sigma_x)_{\max} = \left(\frac{hb}{4}-\frac{bb_1}{2}\right)E\theta''$ throughout the thickness at points A and D if $h < b_1$ $(\tau_2)_{\max} = \frac{b}{4}\left(\frac{h}{2}-b_1\right)^2 E\theta'''$ throughout the thickness at a distance $\frac{h}{2}$ from corner B toward point A if $b_1 > \frac{h}{2}\left(1+\sqrt{\frac{1}{2}+\frac{b}{2h}}\right)$ $(\tau_2)_{\max} = \frac{b}{4}\left(b_1^2-\frac{hb}{4}-hb_1\right)E\theta'''$ throughout the thickness at a point midway between corners B and C if $b_1 < \frac{h}{2}\left(1+\sqrt{\frac{1}{2}+\frac{b}{2h}}\right)$ $(\tau_1)_{\max} = tG\theta'$ at the surface everywhere
6. Wide flanged beam with equal flanges 	$K = \frac{1}{3}(2t^3b+t_w^3h)$ $C_w = \frac{h^2tb^3}{24}$	$(\sigma_x)_{\max} = \frac{hb}{4}E\theta''$ throughout the thickness at points A and B $(\tau_2)_{\max} = -\frac{hb^2}{16}E\theta'''$ throughout the thickness at a point midway between A and B $(\tau_1)_{\max} = tG\theta'$ at the surface everywhere

TABLE 10.2 Formulas for torsional properties and stresses in thin-walled open cross sections (*Continued*)

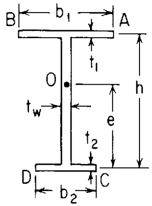
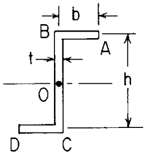
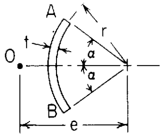
7. Wide flanged beam with unequal flanges  	$e = \frac{t_1 b_1^3 h}{t_1 b_1^3 + t_2 b_2^3}$ $K = \frac{1}{3}(t_1^3 b_1 + t_2^3 b_2 + t_w^3 h)$ $C_w = \frac{h^2 t_1 t_2 b_1^3 b_2^3}{12(t_1 b_1^3 + t_2 b_2^3)}$	$(\sigma_x)_{\max} = \frac{hb_1}{2}\frac{t_2 b_2^3}{t_1 b_1^3 + t_2 b_2^3}E\theta''$ throughout the thickness at points A and B if $t_2 b_2^2 > t_1 b_1^2$ $(\sigma_x)_{\max} = \frac{hb_2}{2}\frac{t_1 b_1^3}{t_1 b_1^3 + t_2 b_2^3}E\theta''$ throughout the thickness at points C and D if $t_2 b_2^2 < t_1 b_1^2$ $(\tau_2)_{\max} = \frac{-1}{8}\frac{h t_2 b_2^3 b_1^2}{t_1 b_1^3 + t_2 b_2^3}E\theta'''$ throughout the thickness at a point midway between A and B if $t_2 b_2 > t_1 b_1$ $(\tau_2)_{\max} = \frac{-1}{8}\frac{h t_1 b_1^3 b_2^2}{t_1 b_1^3 + t_2 b_2^3}E\theta'''$ throughout the thickness at a point midway between C and D if $t_2 b_2 < t_1 b_1$ $(\tau_1)_{\max} = t_{\max} G\theta'$ at the surface on the thickest portion
8. *Z*-section  	$K = \frac{t^3}{3}(2b + h)$ $C_w = \frac{th^2 b^3}{12}\left(\frac{b + 2h}{2b + h}\right)$	$(\sigma_x)_{\max} = \frac{hb}{2}\frac{b + h}{2b + h}E\theta''$ throughout the thickness at points A and D $(\tau_2)_{\max} = \frac{-hb^2}{4}\left(\frac{b + h}{2b + h}\right)^2 E\theta'''$ throughout the thickness at a distance $\frac{b(b + h)}{2b + h}$ from point A $(\tau_1)_{\max} = tG\theta'$ at the surface everywhere
9. Segment of a circular tube  (*Note:* If t/r is small, α can be larger than π to evaluate constants for the case when the walls overlap)	$e = 2r\frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha}$ $K = \frac{2}{3}t^3 r\alpha$ $C_w = \frac{2tr^5}{3}\left[\alpha^3 - 6\frac{(\sin\alpha - \alpha\cos\alpha)^2}{\alpha - \sin\alpha\cos\alpha}\right]$	$(\sigma_x)_{\max} = (r^2\alpha - re\sin\alpha)E\theta''$ throughout the thickness at points A and B $(\tau_2)_{\max} = r^2\left[e(1 - \cos\alpha) - \frac{r\alpha^2}{2}\right]E\theta'''$ throughout the thickness at midlength $(\tau_1)_{\max} = tG\theta'$ at the surface everywhere

TABLE 10.2 Formulas for torsional properties and stresses in thin-walled open cross sections (*Continued*)

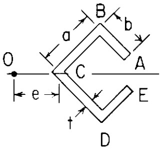
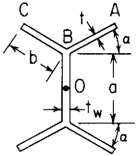
Cross section, reference no.	Constants	Selected maximum values
10. 	$e = 0.707ab^2 \dfrac{3a - 2b}{2a^3 - (a-b)^3}$ $K = \frac{2}{3}t^3(a+b)$ $C_w = \dfrac{ta^4b^3}{6} \dfrac{4a+3b}{2a^3 - (a-b)^3}$	$(\sigma_x)_{\max} = \dfrac{a^2b}{2} \dfrac{2a^2 + 3ab - b^2}{2a^3 - (a-b)^3} E\theta''$ throughout the thickness at points A and E $\tau_2 = \dfrac{a^2b^2}{4} \dfrac{a^2 - 2ab - b^2}{2a^3 - (a-b)^3} E\theta'''$ throughout the thickness at point C $(\tau_1)_{\max} = tG\theta'$ at the surface everywhere
11. 	$K = \frac{1}{3}(4t^3b + t_w^3a)$ $C_w = \dfrac{a^2b^3t}{3}\cos^2\alpha$ (*Note:* Expressions are equally valid for + and $-\alpha$)	$(\sigma_x)_{\max} = \dfrac{ab}{2}\cos\alpha E\theta''$ throughout the thickness at points A and C $(\tau_2)_{\max} = \dfrac{-ab^2}{4}\cos\alpha E\theta'''$ throughout the thickness at point B $(\tau_1)_{\max} = tG\theta'$ at the surface everywhere

TABLE 10.3 Formulas for the elastic deformations of uniform thin-walled open members under torsional loading

NOTATION: T_o = applied torsional load (force-length); t_o = applied distributed torsional load (force-length per unit length); T_A and T_B are the reaction end torques at the left and right ends, respectively. θ = angle of rotation at a distance x from the left end (radians). θ', θ'', and θ''' are the successive derivatives of θ with respect to the distance x. All rotations, applied torsional loads, and reaction end torques are positive as shown (*CCW* when viewed from the right end of the member). E is the modulus of elasticity of the material; C_w is the warping constant for the cross section; K is the torsional constant (see Table 10.2 for expressions for C_w and K); and G is the modulus of rigidity (shear modulus) of the material.

The following constants and functions are hereby defined in order to permit condensing the tabulated formulas which follow. See page 131 for a definition of $\langle x-a\rangle^n$. The function $\sinh \beta\langle x-a\rangle$ is also defined as zero if x is less than a. $\beta = (KG/C_w E)^{1/2}$

$$F_1 = \cosh \beta x \qquad F_{a1} = \langle x-a\rangle^0 \cosh \beta(x-a) \qquad C_1 = \cosh \beta l \qquad C_{a1} = \cosh \beta(l-a) \qquad A_1 = \cosh \beta a$$

$$F_2 = \sinh \beta x \qquad F_{a2} = \sinh \beta\langle x-a\rangle \qquad C_2 = \sinh \beta l \qquad C_{a2} = \sinh \beta(l-a) \qquad A_2 = \sinh \beta a$$

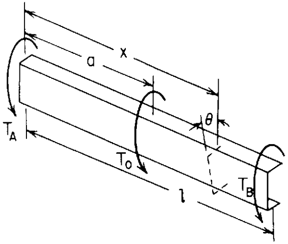
$$F_3 = \cosh \beta x - 1 \qquad F_{a3} = \langle x-a\rangle^0[\cosh \beta(x-a) - 1] \qquad C_3 = \cosh \beta l - 1 \qquad C_{a3} = \cosh \beta(l-a) - 1$$

$$F_4 = \sinh \beta x - \beta x \qquad F_{a4} = \sinh \beta\langle x-a\rangle - \beta\langle x-\alpha\rangle \qquad C_4 = \sinh \beta l - \beta l \qquad C_{a4} = \sinh \beta(l-a) - \beta(l-a)$$

$$F_{a5} = F_{a3} - \frac{\beta^2\langle x-a\rangle^2}{2} \qquad C_{a5} = C_{a3} - \frac{\beta^2(l-a)^2}{2}$$

$$F_{a6} = F_{a4} - \frac{\beta^3\langle x-a\rangle^3}{6} \qquad C_{a6} = C_{a4} - \frac{\beta^3(l-a)^3}{6}$$

1. Concentrated intermediate torque



$$\theta = \theta_A + \frac{\theta'_A}{\beta}F_2 + \frac{\theta''_A}{\beta^2}F_3 + \frac{T_A}{C_w E\beta^3}F_4 + \frac{T_o}{C_w E\beta^3}F_{a4}$$

$$\theta' = \theta'_A F_1 + \frac{\theta''_A}{\beta}F_2 + \frac{T_A}{C_w E\beta^2}F_3 + \frac{T_o}{C_w E\beta^2}F_{a3}$$

$$\theta'' = \theta''_A F_1 + \theta'_A \beta F_2 + \frac{T_A}{C_w E\beta}F_2 + \frac{T_o}{C_w E\beta}F_{a2}$$

$$\theta''' = \theta'_A \beta^2 F_1 + \theta''_A \beta F_2 + \frac{T_A}{C_w E}F_1 + \frac{T_o}{C_w E}F_{a1}$$

$$T = T_A + T_o\langle x-a\rangle^0$$

TABLE 10.3 Formulas for the elastic deformations of uniform thin-walled open members under torsional loading (*Continued*)

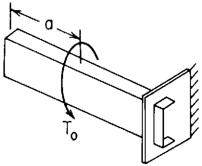
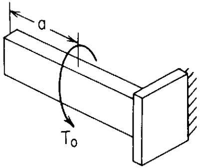
End restraints, reference no.	Boundary values	Selected special cases and maximum values
1a. Left end free to twist and warp, right end free to warp but not twist  	$T_A = 0 \quad \theta''_A = 0$ $\theta_A = \frac{T_o}{C_w E\beta^2}(l-a)$ $\theta'_A = \frac{-T_o}{C_w E\beta^2}\frac{C_{a2}}{C_2}$ $\theta_B = 0, \quad \theta''_B = 0, \quad T_B = -T_o$ $\theta'_B = \frac{-T_o}{C_w E\beta^2}\left(1-\frac{A_2}{C_2}\right)$ $\theta'''_B = \frac{T_o}{C_w E}\frac{A_2}{C_2}$	$\theta_{\max} = \theta_A$; max possible value $= \frac{T_o l}{C_w E\beta^2}$ when $a = 0$ $\theta'_{\max} = \theta'_B$; max possible value $= \frac{-T_o}{C_w E\beta^2}$ when $a = 0$ $\theta''_{\max} = \frac{-T_o}{C_w E\beta}\frac{C_{a2}}{C_2}A_2$ at $x = a$; max possible value $= \frac{-T_o}{2C_w E\beta}\tanh\frac{\beta l}{2}$ when $a = l/2$ $(-\theta''')_{\max} = \frac{-T_o}{C_w E}\frac{C_{a2}}{C_2}A_1$ just left of $x = a$ $(+\theta''')_{\max} = \frac{T_o}{C_w E}\left(1-\frac{C_{a2}}{C_2}A_1\right)$ just right of $x = a$; max possible value $= \frac{T_o}{C_w E}$ when a approaches l If $a = 0$ (torque applied at the left end), $\theta = \frac{T_o}{KG}(l-x), \quad \theta' = \frac{-T_o}{KG}, \quad \theta'' = 0, \quad \theta''' = 0$
1b. Left end free to twist and warp right end fixed (no twist or warp)  	$T_A = 0 \quad \theta''_A = 0$ $\theta_A = \frac{T_o}{C_w E\beta^3}\left(\frac{C_2 C_{a3}}{C_1} - C_{a4}\right)$ $\theta'_A = \frac{-T_o}{C_w E\beta^2}\frac{C_{a3}}{C_1}$ $\theta_B = 0, \quad \theta'_B = 0, \quad T_B = -T_o$ $\theta''_B = \frac{-T_o}{C_w E\beta}\frac{A_2 - C_2}{C_1}$ $\theta'''_B = \frac{T_o}{C_w E}$	If $a = 0$ (torque applied at the left end), $\theta = \frac{T_o}{C_w E\beta^3}\left[\beta(l-x) - \tanh\beta l + \frac{\sinh\beta x}{\cosh\beta l}\right]$ $\theta' = \frac{-T_o}{C_w E\beta^2}\left(1-\frac{\cosh\beta x}{\cosh\beta l}\right), \quad \theta'' = \frac{T_o}{C_w E\beta}\frac{\sinh\beta x}{\cosh\beta}$ $\theta''' = \frac{T_o}{C_w E}\frac{\cosh\beta x}{\cosh\beta l}$ $\theta_{\max} = \frac{T_o}{C_w E\beta^3}(\beta l - \tanh\beta l)$ at $x = 0$ $\theta'_{\max} = \frac{-T_o}{C_w E\beta^2}\left(1-\frac{1}{\cosh\beta l}\right)$ at $x = 0$ $\theta''_{\max} = \frac{T_o}{C_w E\beta}\tanh\beta l$ at $x = l$ $\theta'''_{\max} = \frac{T_o}{C_w E}$ at $x = l$

TABLE 10.3 Formulas for the elastic deformations of uniform thin-walled open members under torsional loading (*Continued*)

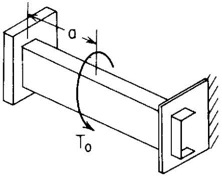
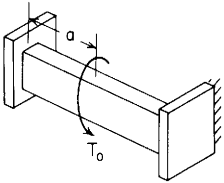
1c. Left end free to twist but not warp, right end free to warp but not twist 	$T_A = 0, \quad \theta'_A = 0$ $\theta_A = \frac{T_o}{C_w E\beta^3}\left(\frac{C_3 C_{a2}}{C_1} - C_{a4}\right)$ $\theta''_A = \frac{-T_o}{C_w E\beta}\frac{C_{a2}}{C_1}$ $\theta_B = 0, \quad \theta''_B = 0, \quad T_B = -T_o$ $\theta'_B = \frac{-T_o}{C_w E\beta^2}\left(\frac{C_2 C_{a2}}{C_1} - C_{a3}\right)$ $\theta'''_B = \frac{-T_o}{C_w E}\left(\frac{C_2 C_{a2}}{C_1} - C_{a1}\right)$	If $a = 0$ (torque applied at the left end), $\theta = \frac{T_o}{C_w E\beta^3}[\sinh\beta x - \tanh\beta l \cosh\beta x + \beta(l - x)]$ $\theta' = \frac{T_o}{C_w E\beta^2}(\cosh\beta x - \tanh\beta l \sinh\beta x - 1)$ $\theta'' = \frac{T_o}{C_w E\beta}(\sinh\beta x - \tanh\beta l \cosh\beta x), \qquad \theta''' = \frac{T_o}{C_w E}(\cosh\beta x - \tanh\beta l \sinh\beta x)$ $\theta_{\max} = \frac{T_o}{C_w E\beta^3}(\beta l - \tanh\beta l) \quad \text{at } x = 0$ $\theta'_{\max} = \frac{-T_o}{C_w E\beta^2}\left(\frac{-1}{\cosh\beta l} + 1\right) \quad \text{at } x = l$ $\theta''_{\max} = \frac{-T_o}{C_w E\beta}\tanh\beta l \quad \text{at } x = 0$ $\theta'''_{\max} = \frac{T_o}{C_w E} \quad \text{at } x = 0$
1d. Left end free to twist but not warp, right end fixed (no twist or warp) 	$T_A = 0, \quad \theta'_A = 0$ $\theta_A = \frac{T_o}{C_w E\beta^3}\left(\frac{C_3 C_{a3}}{C_2} - C_{a4}\right)$ $\theta''_A = \frac{-T_o}{C_w E\beta}\frac{C_{a3}}{C_2}$ $\theta_B = 0, \quad \theta'_B = 0, \quad T_B = -T_o$ $\theta''_B = \frac{-T_o}{C_w E\beta}\left(\frac{C_1 C_{a3}}{C_2} - C_{a2}\right)$ $\theta'''_B = \frac{T_o}{C_w E}$	If $a = 0$ (torque applied at the left end), $\theta = \frac{T_o}{C_w E\beta^3}\left[\sinh\beta x + \beta(l - x) - \tanh\frac{\beta l}{2}(1 + \cosh\beta x)\right]$ $\theta' = \frac{T_o}{C_w E\beta^2}\left(\cosh\beta x - 1 - \tanh\frac{\beta l}{2}\sinh\beta x\right)$ $\theta'' = \frac{T_o}{C_w E\beta}\left(\sinh\beta x - \tanh\frac{\beta l}{2}\cosh\beta x\right), \qquad \theta''' = \frac{T_o}{C_w E}\left(\cosh\beta x - \tanh\frac{\beta l}{2}\sinh\beta x\right)$ $\theta_{\max} = \frac{T_o}{C_w E\beta^3}\left(\beta l - 2\tanh\frac{\beta l}{2}\right) \quad \text{at } x = 0$ $\theta'_{\max} = \frac{T_o}{C_w E\beta^2}\left[\frac{1}{\cosh(\beta l/2)} - 1\right] \quad \text{at } x = \frac{l}{2}$ $\theta''_{\max} = \frac{\mp T_o}{C_w E\beta}\tanh\frac{\beta l}{2} \quad \text{at } x = 0 \text{ and } x = l, \text{ respectively}$ $\theta'''_{\max} = \frac{T_o}{C_w E} \quad \text{at } x = 0 \text{ and } x = l$

TABLE 10.3 Formulas for the elastic deformations of uniform thin-walled open members under torsional loading (*Continued*)

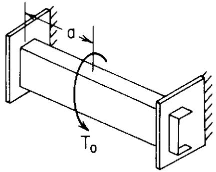
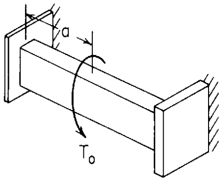
End restraints, reference no.	Boundary values	Selected special cases and maximum values
1e. Both ends free to warp but not twist  	$\theta_A = 0, \quad \theta''_A = 0$ $T_A = -T_o\left(1 - \frac{a}{l}\right)$ $\theta'_A = \frac{T_o}{C_w E\beta^2}\left(1 - \frac{a}{l} - \frac{C_{a2}}{C_2}\right)$ $\theta_B = 0, \quad \theta''_B = 0$ $\theta'_B = \frac{-T_o}{C_w E\beta^2}\left(\frac{a}{l} - \frac{A_2}{C_2}\right)$ $\theta'''_B = \frac{T_o}{C_w E}\frac{A_2}{C_2}$ $T_B = -T_o\frac{a}{l}$	If $a = l/2$ (torque applied at midlength), $\theta = \frac{T_o}{C_w E\beta^3}\left[\frac{\beta x}{2} - \frac{\sinh\beta x}{2\cosh(\beta l/2)} + \sinh\beta\langle x - \frac{1}{2}\rangle - \beta\langle x - \frac{l}{2}\rangle\right]$ $\theta' = \frac{T_o}{C_w E\beta^2}\left\{\frac{1}{2} - \frac{\cosh\beta x}{2\cosh(\beta l/2)} + \langle x - \frac{l}{2}\rangle^0\left[\cosh\beta\left(x - \frac{l}{2}\right) - 1\right]\right\}$ $\theta'' = \frac{-T_o}{C_w E\beta}\left[\frac{\sinh\beta x}{2\cosh(\beta l/2)} - \sinh\beta\langle x - \frac{l}{2}\rangle\right]$ $\theta''' = \frac{-T_o}{C_w E}\left[\frac{\cosh\beta x}{2\cosh(\beta l/2)} - \langle x - \frac{l}{2}\rangle^0\cosh\beta\left(x - \frac{l}{2}\right)\right]$ $\theta_{\max} = \frac{T_o}{2C_w E\beta^3}\left(\frac{\beta l}{2} - \tanh\frac{\beta l}{2}\right)$ at $x = \frac{l}{2}$ $\theta'_{\max} = \frac{\pm T_o}{2C_w E\beta^2}\left[1 - \frac{1}{\cosh(\beta l/2}\right]$ at $x = 0$ and $x = l$, respectively $\theta''_{\max} = \frac{-T_o}{2C_w E\beta}\tanh\frac{\beta l}{2}$ at $x = \frac{l}{2}$ $\theta'''_{\max} = \frac{\mp T_o}{2C_w E}$ just left and just right of $x = \frac{l}{2}$, respectively
1f. Left end free to warp but not twist, right end fixed (no twist or warp)  	$\theta_A = 0, \quad \theta''_A = 0$ $T_A = -T_o\frac{C_1 C_{a4} - C_2 C_{a3}}{C_1 C_4 - C_2 C_3}$ $\theta'_A = \frac{T_o}{C_w E\beta^2}\frac{C_3 C_{a4} - C_4 C_{a3}}{C_1 C_4 - C_2 C_3}$ $\theta_B = 0, \quad \theta'_B = 0$ $\theta''_B = \frac{T_o}{C_w E\beta}\frac{\beta l A_2 - \beta a C_2}{C_1 C_4 - C_2 C_3}$ $\theta'''_B = \frac{T_o}{C_w E}\frac{A_2 - \beta a C_1}{C_1 C_4 - C_2 C_3}$ $T_B = -T_o - T_A$	If $a = l/2$ (torque applied at midlength), $T_A = -T_o\frac{\sinh\beta l - (\beta l/2)\cosh\beta l - \sinh(\beta l/2)}{\sinh\beta l - \beta l\cosh\beta l}$ $\theta'_A = \frac{T_o}{C_w E\beta^2}\frac{2\sinh(\beta l/2) - \beta l\cosh(\beta l/2)}{\sinh\beta l - \beta l\cosh\beta l}\left(\cosh\frac{\beta l}{2} - 1\right)$

TABLE 10.3 Formulas for the elastic deformations of uniform thin-walled open members under torsional loading (*Continued*)

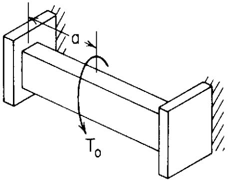
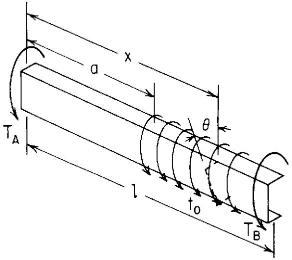
1g. Both ends fixed (no twist or warp)  	$\theta_A = 0, \quad \theta'_A = 0$ $\theta''_A = \frac{T_o}{C_wE\beta}\frac{C_3C_{a4} - C_4C_{a3}}{C_2C_4 - C_3^2}$ $T_A = -T_o\frac{C_2C_{a4} - C_3C_{a3}}{C_2C_4 - C_3^2}$ $\theta_B = 0, \quad \theta'_B = 0$ $\theta''_B = \theta''_A C_1 + \frac{T_A}{C_wE\beta}C_2 + \frac{T_o}{C_wE\beta}C_{a2}$ $\theta'''_B = \theta''_A \beta C_2 + \frac{T_A}{C_wE}C_1 + \frac{T_o}{C_wE}C_{a1}$ $T_B = -T_o - T_A$	If $a = l/2$ (torque applied at midlength), $T_A = T_B = \frac{-T_o}{2}$ $\theta_{\max} = \frac{T_o}{C_wE\beta^3}\left(\frac{\beta l}{4} - \tanh\frac{\beta l}{4}\right)$ at $x = \frac{l}{2}$ $\theta'_{\max} = \frac{T_o}{2C_wE\beta^2}\left[1 - \frac{1}{\cosh(\beta l/4)}\right]$ at $x = \frac{l}{4}$ $\theta''_{\max} = \substack{\pm \\ \mp}\frac{T_o}{2C_wE\beta}\tanh\frac{\beta l}{4}$ at $x = 0, x = \frac{l}{2}$, and $x = l$, respectively $(-\theta''')_{\max} = \frac{-T_o}{2C_wE}$ at $x = 0$ and just left of $x = \frac{l}{2}$ $(+\theta''')_{\max} = \frac{T_o}{2C_wE}$ at $x = l$ and just right of $x = \frac{l}{2}$
2. Uniformly distributed torque from a to l  	$\theta = \theta_A + \frac{\theta'_A}{\beta}F_2 + \frac{\theta''_A}{\beta^2}F_3 + \frac{T_A}{C_wE\beta^3}F_4 + \frac{t_o}{C_wE\beta^4}F_{a5}$ $\theta' = \theta'_A F_1 + \frac{\theta''_A}{\beta}F_2 + \frac{T_A}{C_wE\beta^2}F_3 + \frac{t_o}{C_wE\beta^3}F_{a4}$ $\theta'' = \theta''_A F_1 + \theta'_A\beta F_2 + \frac{T_A}{C_wE\beta}F_2 + \frac{t_o}{C_wE\beta^2}F_{a3}$ $\theta''' = \theta'_A\beta^2 F_1 + \theta''_A\beta F_2 + \frac{T_A}{C_wE}F_1 + \frac{t_o}{C_wE\beta}F_{a2}$ $T = T_A + t_o\langle x - a\rangle$	

TABLE 10.3 Formulas for the elastic deformations of uniform thin-walled open members under torsional loading (*Continued*)

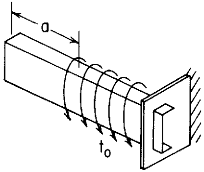
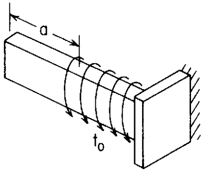
End restraints, reference no.	Boundary values	Selected special cases and maximum values
2a. Left end free to twist and warp, right end free to warp but not twist  	$T_A = 0, \quad \theta''_A = 0$ $\theta_A = \frac{t_o}{2C_wE\beta^2}(l-a)^2$ $\theta'_A = \frac{-t_o}{C_wE\beta^3}\frac{C_{a3}}{C_2}$ $\theta_B = 0, \quad \theta''_B = 0$ $\theta'_B = \frac{-t_o}{C_wE\beta^3}\left[\frac{A_1-C_1}{C_2}+\beta(l-a)\right]$ $\theta'''_B = \frac{-t_o}{C_wE\beta}\frac{A_1-C_1}{C_2}$ $T_B = -t_o(l-a)$	If $a = 0$ (uniformly distributed torque over entire span), $\theta = \frac{t_o}{C_wE\beta^4}\left[\frac{\beta^2(l^2-x^2)}{2}+\frac{\sinh\beta(l-x)+\sinh\beta x}{\sinh\beta l}-1\right]$ $\theta' = \frac{-t_o}{C_wE\beta^3}\left[\beta x+\frac{\cosh\beta(l-x)-\cosh\beta x}{\sinh\beta l}\right]$ $\theta'' = \frac{-t_o}{C_wE\beta^2}\left[1-\frac{\sinh\beta(l-x)+\sinh\beta x}{\sinh\beta l}\right]$ $\theta''' = \frac{-t_o}{C_wE\beta}\frac{\cosh\beta(l-x)-\cosh\beta x}{\sinh\beta l}$ $\theta_{\max} = \frac{t_o l^2}{2C_wE\beta^2}$ at $x = 0$ $\theta'_{\max} = \frac{t_o}{C_wE\beta^3}\left(\beta l-\tanh\frac{\beta l}{2}\right)$ at $x = l$ $\theta''_{\max} = \frac{-t_o}{C_wE\beta^2}\left[1-\frac{1}{\cosh(\beta l/2)}\right]$ at $x = \frac{l}{2}$ $\theta'''_{\max} = \frac{\mp t_o}{C_wE\beta}\tanh\frac{\beta l}{2}$ at $x = 0$ and $x = l$, respectively
2b. Left end free to twist and warp, right end fixed (no twist or warp)  	$T_A = 0, \quad \theta''_A = 0$ $\theta_A = \frac{t_o}{C_wE\beta^4}\left(\frac{C_2C_{a4}}{C_1}-C_{a5}\right)$ $\theta'_A = \frac{-t_o}{C_wE\beta^3}\frac{C_{a4}}{C_1}$ $\theta_B = 0, \quad \theta'_B = 0$ $\theta''_B = \frac{-t_o}{C_wE\beta^2}\left(\frac{C_2C_{a4}}{C_1}-C_{a3}\right)$ $\theta'''_B = \frac{t_o}{C_wE}(l-a)$ $T_B = -t_o(l-a)$	If $a = 0$ (uniformly distributed torque over entire span) $\theta = \frac{-t_o}{C_wE\beta^4}\left[\frac{1-\cosh\beta(l-x)+\beta l(\sinh\beta l-\sinh\beta x)}{\cosh\beta l}-\frac{\beta^2(l^2-x^2)}{2}\right]$ $\theta' = \frac{-t_o}{C_wE\beta^3}\left[\frac{\sinh\beta(l-x)-\beta l\cosh\beta x}{\cosh\beta l}+\beta x\right]$ $\theta'' = \frac{-t_o}{C_wE\beta^2}\left[1-\frac{\cosh\beta(l-x)+\beta l\sinh\beta x}{\cosh\beta l}\right]$ $\theta''' = \frac{-t_o}{C_wE\beta}\frac{\sinh\beta(l-x)-\beta l\cosh\beta x}{\cosh\beta l}$ $\theta_{\max} = \frac{t_o}{C_wE\beta^4}\left(1+\frac{\beta^2l^2}{2}-\frac{1+\beta l\sinh\beta l}{\cosh\beta l}\right)$ at $x = 0$ The location of max θ' depends upon βl $\theta''_{\max} = \frac{t_o}{C_wE\beta^2}\left(\frac{1+\beta l\sinh\beta l}{\cosh\beta l}-1\right)$ at $x = l$ $\theta'''_{\max} = \frac{t_o l}{C_wE}$ at $x = l$

TABLE 10.3 Formulas for the elastic deformations of uniform thin-walled open members under torsional loading (*Continued*)

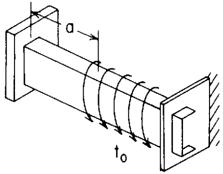
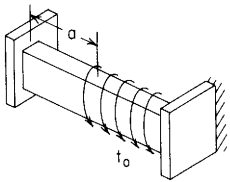
2c. Left end free to twist but not warp, right end free to warp but not twist  	$T_A = 0, \quad \theta'_A = 0$ $\theta_A = \frac{t_o}{C_w E\beta^4}\left(\frac{C_3 C_{a3}}{C_1} - C_{a5}\right)$ $\theta''_A = \frac{-t_o}{C_w E\beta^2}\frac{C_{a3}}{C_1}$ $\theta_B = 0, \quad \theta''_B = 0$ $\theta'_B = \frac{-t_o}{C_w E\beta^3}\left(\frac{C_2 C_{a3}}{C_1} - C_{a4}\right)$ $\theta'''_B = \frac{-t_o}{C_w E\beta}\left(\frac{C_2 C_{a3}}{C_1} - C_{a2}\right)$ $T_B = -t_o(l-a)$	If $a = 0$ (uniformly distributed torque over entire span), $\theta = \frac{t_o}{C_w E\beta^4}\left[\frac{\beta^2(l^2-x^2)}{2} + \frac{\cosh\beta x}{\cosh\beta l} - 1\right]$ $\theta' = \frac{-t_o}{C_w E\beta^3}\left(\beta x - \frac{\sinh\beta x}{\cosh\beta l}\right)$ $\theta'' = \frac{-t_o}{C_w E\beta^2}\left(1 - \frac{\cosh\beta x}{\cosh\beta l}\right), \quad \theta''' = \frac{t_o}{C_w E\beta}\frac{\sinh\beta x}{\cosh\beta l}$ $\theta_{\max} = \frac{t_o}{C_w E\beta^4}\left(\frac{\beta^2 l^2}{2} + \frac{1}{\cosh\beta l} - 1\right)$ at $x = 0$ $\theta'_{\max} = \frac{-t_o}{C_w E\beta^3}(\beta l - \tanh\beta l)$ at $x = l$ $\theta''_{\max} = \frac{-t_o}{C_w E\beta^2}\left(1 - \frac{1}{\cosh\beta l}\right)$ at $x = 0$ $\theta'''_{\max} = \frac{t_o}{C_w E\beta}\tanh\beta l$ at $x = l$
2d. Left end free to twist but not warp, right end fixed (no twist or warp)  	$T_A = 0, \quad \theta'_A = 0$ $\theta_A = \frac{t_o}{C_w E\beta^4}\left(\frac{C_3 C_{a4}}{C_2} - C_{a5}\right)$ $\theta''_A = \frac{-t_o}{C_w E\beta^2}\frac{C_{a4}}{C_2}$ $\theta_B = 0, \quad \theta'_B = 0$ $\theta''_B = \frac{-t_o}{C_w E\beta^2}\left(\frac{C_1 C_{a4}}{C_2} - C_{a3}\right)$ $\theta'''_B = \frac{t_o}{C_w E}(l-a)$ $T_B = -t_o(l-a)$	If $a = 0$ (uniformly distributed torque over entire span), $\theta = \frac{t_o}{C_w E\beta^4}\left[\frac{\beta^2}{2}(l^2-x^2) + \beta l\frac{\cosh\beta x - \cosh\beta l}{\sinh\beta l}\right]$ $\theta' = \frac{-t_o}{C_w E\beta^3}\left(\beta x - \frac{\beta l\sinh\beta x}{\sinh\beta l}\right)$ $\theta'' = \frac{-t_o}{C_w E\beta^2}\left(1 - \frac{\beta l\cosh\beta x}{\sinh\beta l}\right), \quad \theta''' = \frac{t_o l}{C_w E}\frac{\sinh\beta x}{\sinh\beta l}$ $\theta_{\max} = \frac{t_o l}{C_w E\beta^3}\left(\frac{\beta l}{2} - \tanh\frac{\beta l}{2}\right)$ at $x = 0$ $(+\theta'')_{\max} = \frac{t_o}{C_w E\beta^2}\left(\frac{\beta l}{\tanh\beta l} - 1\right)$ at $x = l$ $(-\theta'')_{\max} = \frac{-t_o}{C_w E\beta^2}\left(1 - \frac{\beta l}{\sinh\beta l}\right)$ at $x = 0$ $\theta'''_{\max} = \frac{t_o l}{C_w E}$ at $x = l$

TABLE 10.3 Formulas for the elastic deformations of uniform thin-walled open members under torsional loading (*Continued*)

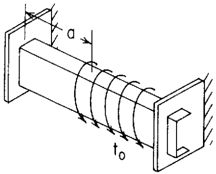
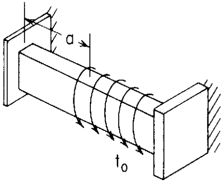
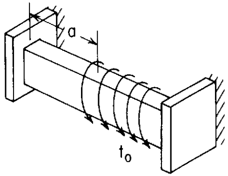
End restraints, reference no.	Boundary values	Selected special cases and maximum values
2e. Both ends free to warp but not twist  	$\theta_A = 0, \quad \theta''_A = 0$ $T_A = \frac{-t_o}{2l}(l-a)^2$ $\theta'_A = \frac{t_o}{C_w E \beta^3}\left[\frac{\beta}{2l}(l-a)^2 - \frac{C_{a3}}{C_2}\right]$ $\theta_B = 0, \quad \theta''_B = 0$ $\theta'_B = \frac{t_o}{C_w E \beta^3}\left[\frac{\beta}{2l}(l-a)^2 - \frac{C_1 C_{a3}}{C_2} + C_{a4}\right]$ $\theta'''_B = \frac{-t_o}{C_w E \beta}\frac{\cosh \beta a - \cosh \beta l}{\sinh \beta l}$ $T_B = \frac{-t_o}{2l}(l^2 - a^2)$	If $a = 0$ (uniformly distributed torque over entire span), $\theta = \frac{t_o}{C_w E \beta^4}\left[\frac{\beta^2 x(l-x)}{2} + \frac{\cosh \beta(x - l/2)}{\cosh(\beta l/2)} - 1\right]$ $\theta' = \frac{t_o}{C_w E \beta^3}\left[\frac{\sinh \beta(x - l/2)}{\cosh(\beta l/2)} - \beta(x - l/2)\right]$ $\theta'' = \frac{t_o}{C_w E \beta^2}\left[\frac{\cosh(x - l/2)}{\cosh(\beta l/2)} - 1\right]$ $\theta''' = \frac{t_o}{C_w E \beta}\frac{\sinh \beta(x - l/2)}{\cosh(\beta l/2)}$ $\theta_{\max} = \frac{t_o}{C_w E \beta^4}\left[\frac{\beta^2 l^2}{8} + \frac{1}{\cosh(\beta l/2)} - 1\right]$ at $x = \frac{l}{2}$ $\theta'_{\max} = \frac{\pm t_o}{C_w E \beta^3}\left(\frac{\beta l}{2} - \tanh \frac{\beta l}{2}\right)$ at $x = 0$ and $x = l$, respectively $\theta''_{\max} = \frac{t_o}{C_w E \beta^2}\left[\frac{1}{\cosh(\beta l/2)} - 1\right]$ at $x = \frac{l}{2}$ $\theta'''_{\max} = \frac{\mp t_o}{C_w E \beta}\tanh \frac{\beta l}{2}$ at $x = 0$ and $x = l$, respectively
2f. Left end free to warp but not twist, right end fixed (no twist or warp)  	$\theta_A = 0, \quad \theta''_A = 0$ $T_A = \frac{-t_o}{\beta}\frac{C_1 C_{a5} - C_2 C_{a4}}{C_1 C_4 - C_2 C_3}$ $\theta'_A = \frac{t_o}{C_w E \beta^3}\frac{C_3 C_{a5} - C_4 C_{a4}}{C_1 C_4 - C_2 C_3}$	If $a = 0$ (uniformly distributed torque over entire span), $\theta_{\max}$ occurs very close to $x = 0.425l$ $\theta'_{\max} = \frac{t_o}{C_w E \beta^3}\frac{2 - \beta^2 l^2/2 + 2\beta l \sinh \beta l - (2 + \beta^2 l^2/2)\cosh \beta l}{\sinh \beta l - \beta l \cosh \beta l}$ at $x = 0$ $\theta''_{\max} = \frac{-t_0}{C_w E \beta^2}\frac{(\beta^2 l^2/2)\sinh \beta l + \beta l(1 - \cosh \beta l)}{\sinh \beta l - \beta l \cosh \beta l}$ at $x = l$ $(-\theta''')_{\max} = \frac{t_o}{C_w E \beta}\frac{1 - \beta^2 l^2/2 - \cosh \beta l + \beta l \sinh \beta l}{\sinh \beta l - \beta l \cosh \beta l}$ at $x = 0$ $(+\theta''')_{\max} = \frac{t_o}{C_w E \beta}\frac{(1 - \beta^2 l^2/2)\cosh \beta l - 1}{\sinh \beta l - \beta l \cosh \beta l}$ at $x = l$

TABLE 10.3 Formulas for the elastic deformations of uniform thin-walled open members under torsional loading (*Continued*)

2g. Both ends fixed (no twist or warp)  	$\theta_A = 0, \quad \theta'_A = 0$ $\theta''_A = \dfrac{t_o}{C_w E \beta^2} \dfrac{C_3 C_{a5} - C_4 C_{a4}}{C_2 C_4 - C_3^2}$ $T_A = \dfrac{-t_o}{\beta} \dfrac{C_2 C_{a5} - C_3 C_{a4}}{C_2 C_4 - C_3^2}$ $\theta'''_A = \dfrac{T_A}{C_w E}$	If $a = 0$ (uniformly distributed torque over entire span), $T_A = \dfrac{-t_o l}{2}$ $\theta = \dfrac{t_o l}{2 C_w E \beta^3} \left[\dfrac{\beta x}{l}(l - x) + \dfrac{\cosh \beta(x - l/2) - \cosh(\beta l/2)}{\sinh(\beta l/2)} \right]$ $\theta' = \dfrac{t_o l}{2 C_w E \beta^2} \left[1 - \dfrac{2x}{l} + \dfrac{\sinh \beta(x - l/2)}{\sinh(\beta l/2)} \right]$ $\theta'' = \dfrac{t_o}{C_w E \beta^2} \left[\dfrac{\beta l \cosh \beta(x - l/2)}{2 \sinh(\beta l/2)} - 1 \right]$ $\theta''' = \dfrac{t_o}{C_w E \beta} \dfrac{\beta l \sinh \beta(x - l/2)}{2 \sinh(\beta l/2)}$ $\theta_{\max} = \dfrac{t_o l}{2 C_w E \beta^3} \left(\dfrac{\beta l}{4} - \tanh \dfrac{\beta l}{4} \right) \quad \text{at } x = \dfrac{l}{2}$ $(-\theta'')_{\max} = \dfrac{-t_o}{C_w E \beta^2} \left[1 - \dfrac{\beta l}{2 \sinh(\beta l/2)} \right] \quad \text{at } x = \dfrac{l}{2}$ $(+\theta'')_{\max} = \dfrac{t_o}{C_w E \beta^2} \left[\dfrac{\beta l}{2 \tanh(\beta l/2)} - 1 \right] \quad \text{at } x = 0 \text{ and } x = l$ $\theta'''_{\max} = \dfrac{\mp t_o l}{2 C_w E} \quad \text{at } x = 0 \text{ and } x = l, \text{ respectively}$

10.8 References

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Chapter 11

Flat Plates

11.1 Common Case

The formulas of this section are based on the following assumptions: (1) The plate is flat, of uniform thickness, and of homogeneous isotropic material; (2) the thickness is not more than about one-quarter of the least transverse dimension, and the maximum deflection is not more than about one-half the thickness; (3) all forces—loads and reactions—are normal to the plane of the plate; and (4) the plate is nowhere stressed beyond the elastic limit. For convenience in discussion, it will be assumed further that the plane of the plate is horizontal.

Behavior. The plate deflects. The middle surface (halfway between top and bottom surfaces) remains unstressed; at other points there are biaxial stresses in the plane of the plate. Straight lines in the plate that were originally vertical remain straight but become inclined; therefore the intensity of either principal stress at points on any such line is proportional to the distance from the middle surface, and the maximum stresses occur at the outer surfaces of the plate.

Formulas. Unless otherwise indicated the formulas given in Tables 11.2†–11.4 are based on very closely approximate mathematical analysis and may be accepted as sufficiently accurate so long as the assumptions stated hold true. Certain additional facts of importance in relation to these formulas are as follows.

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†*Note:* Table 11.1 contains numerical values for functions used in Table 11.2

Concentrated loading. It will be noted that all formulas for maximum stress due to a load applied over a small area give very high values when the radius of the loaded area approaches zero. Analysis by a more precise method (Ref. 12) shows that the actual maximum stress produced by a load concentrated on a very small area of radius r_o can be found by replacing the actual r_o by a so-called *equivalent radius* r'_o, which depends largely upon the thickness of the plate t and to a lesser degree on its least transverse dimension. Holl (Ref. 13) shows how r'_o varies with the width of a flat plate. Westergaard (Ref. 14) gives an approximate expression for this equivalent radius:

$$r'_o = \sqrt{1.6r_o^2 + t^2} - 0.675t \qquad (11.1\text{-}1)$$

This formula, which applies to a plate of any form, may be used for all values of r_o less than $0.5t$; for larger values the actual r_o may be used.

Use of the equivalent radius makes possible the calculation of the finite maximum stresses produced by a (nominal) point loading whereas the ordinary formula would indicate that these stresses were infinite.

Edge conditions. The formulas of Tables 11.2–11.4 are given for various combinations of edge support: free, guided (zero slope but free to move vertically), and simply supported or fixed. No exact edge condition is likely to be realized in ordinary construction, and a condition of true edge fixity is especially difficult to obtain. Even a small horizontal force at the line of contact may appreciably reduce the stress and deflection in a simply supported plate; however, a very slight yielding at nominally fixed edges will greatly relieve the stresses there while increasing the deflection and center stresses. For this reason it is usually advisable to design a fixed-edged plate that is to carry uniform load for somewhat higher center stresses than are indicated by theory.

11.2 Bending of Uniform-Thickness Plates with Circular Boundaries

In Table 11.2, cases 1–5 consider annular and solid circular plates of constant thickness under *axisymmetric* loading for several combinations of boundary conditions. In addition to the formulas, tabulated values of deformation and moment coefficients are given for many common loading cases. The remaining cases include concentrated loading and plates with some circular and straight boundaries. Only the deflections due to bending strains are included; in Sec. 11.3, the additional deflections due to shear strains are considered.

Formulas. For cases 1–15 (Table 11.2), expressions are given for deformations and reactions at the edges of the plates as well as general equations which allow the evaluation of *deflections*, *slopes*, *moments* and *shears* at any point in the plate. The several axisymmetric loadings include uniform, uniformly increasing, and parabolically increasing normal pressure over a portion of the plate. This permits the approximation of any reasonable axisymmetric distributed loading by fitting an approximate second-order curve to the variation in loading and solving the problem by superposition. (See the Examples at the end of this section.)

In addition to the usual loadings, Table 11.2 also includes loading cases that may be described best as *externally applied conditions which force a lack of flatness into the plate*. For example, in cases 6 and 14, expressions are given for a manufactured concentrated change in slope in a plate, which could also be used if a plastic hinge were to develop in a plate and the change in slope at the plastic hinge is known or assumed. Similarly, case 7 treats a plate with a small step manufactured into the otherwise flat surface and gives the reactions which develop when this plate is forced to conform to the specified boundary conditions. These cases are also useful when considering known boundary rotations or lateral displacements. (References 46, 47, 57, and 58 present tables and graphs for many of the loadings given in these cases.)

The use of the constants C_1 to C_9 and the functions F_1 to F_9, L_1 to L_{19}, and G_1 to G_{19} in Table 11.2 appears to be a formidable task at first. However, when we consider the large number of cases it is possible to present in a limited space, the reason for this method of presentation becomes clear. With careful inspection, we find that the constants and functions with *like subscripts* are the same except for the change in variable. We also note the use of the *singularity function* $\langle r - r_o \rangle^0$, which is given a value of 0 for $r < r_o$ and a value of 1 for $r > r_o$. In Table 11.1, values are listed for all the preceding functions for several values of the variables b/r, b/a, r_o/a, and r_o/r; also listed are five of the most used denominators for the several values of b/a. (Note that these values are for $\nu = 0.30$.)

EXAMPLES

1. A solid circular steel plate, 0.2 in thick and 20 in in diameter, is simply supported along the edge and loaded with a uniformly distributed load of 3 lb/in^2. It is required to determine the center deflection, the maximum stress, and the deflection equation. Given: $E = 30(10^6)$ lb/in^2 and $\nu = 0.285$.

Solution. This plate and loading are covered in Table 11.2, case 10a. The following constants are obtained:

$$D = \frac{30(10^6)(0.2^3)}{12(1 - 0.285^2)} = 21{,}800, \quad q = 3, \quad a = 10, \quad r_o = 0$$

Since $r_o = 0$,

$$y_c = \frac{-qa^4}{64D}\frac{5+\nu}{1+\nu} = \frac{-3(10^4)(5.285)}{64(21{,}800)(1.285)} = -0.0833\,\text{in}$$

and $$M_{\max} = M_c = \frac{qa^3}{16}(3+\nu) = \frac{3(10^2)(3.285)}{16} = 61.5\,\text{lb-in/in}$$

Therefore $$\sigma_{\max} = \frac{6M_c}{t^2} = \frac{6(61.5)}{0.2^2} = 9240\,\text{lb/in}^2$$

The general deflection equation for these several cases is

$$y = y_c + \frac{M_c r^2}{2D(1+\nu)} + LT_y$$

where for this case $LT_y = (-qr^4/D)G_{11}$. For $r_o = 0$, $G_{11} = \frac{1}{64}$ (note that $r > r_o$ everywhere in the plate, so that $\langle r - r_o\rangle^0 = 1$); therefore,

$$\begin{aligned} y &= -0.0833 + \frac{61.5r^2}{2(21{,}800)(1.285)} - \frac{3r^4}{21{,}800(64)} \\ &= -0.0883 + 0.001098r^2 - 0.00000215r^4 \end{aligned}$$

As a check, the deflection at the outer edge can be evaluated as

$$\begin{aligned} y_a &= -0.0883 + 0.001098(10^2) - 0.00000215(10^4) \\ &= -0.0883 + 0.1098 - 0.0215 = 0 \end{aligned}$$

2. An annular aluminum plate with an outer radius of 20 in and an inner radius of 5 in is to be loaded with an annular line load of 40 lb/in at a radius of 10 in. Both the inner and outer edges are simply supported, and it is required to determine the maximum deflection and maximum stress as a function of the plate thickness. Given: $E = 10(10^6)\,\text{lb/in}^2$ and $\nu = 0.30$.

Solution. The solution to this loading and support condition is found in Table 11.2, case 1c, where $b/a = 0.25$, $r_o/a = 0.50$, $a = 20$ in, and $w = 40$ lb/in. No numerical solutions are presented for this combination of b/a and r_o/a, and so either the equations for C_1, C_3, C_7, C_9, L_3, and L_9 must be evaluated or values for these coefficients must be found in Table 11.1. Since the values of C are found for the variable b/a, from Table 11.1, under the column headed 0.250, the following coefficients are determined.

$$C_1 = 0.881523, \quad C_3 = 0.033465, \quad C_7 = 1.70625$$
$$C_9 = 0.266288, \quad C_1C_9 - C_3C_7 = 0.177640$$

The values of L are found for the variable r_o/a, and so from Table 11.1, under the column headed 0.500, the following coefficients are determined:

$$L_3 = 0.014554 \quad \text{and} \quad L_9 = 0.290898$$

Whether the numbers in Table 11.1 can be interpolated and used successfully depends upon the individual problem. In some instances, where lesser degrees of accuracy are required, interpolation can be used; in other instances, requiring greater degrees of accuracy, it would be better to solve the problem for values of b and r_o that do fit Table 11.1 and then interpolate between the values of the final deflections or stresses.

Using the preceding coefficients, the reaction force and slope can be determined at the inside edge and the deflection equation developed (note that $y_b = 0$ and $M_{rb} = 0$):

$$\theta_b = \frac{-wa^2}{D}\frac{C_3L_9 - C_9L_3}{C_1C_9 - C_3C_7} = \frac{-40(20)^2}{D}\frac{0.033465(0.290898) - 0.266288(0.014554)}{0.177640}$$

$$= \frac{-527.8}{D}\text{rad}$$

$$Q_b = w\frac{C_1L_9 - C_7L_3}{C_1C_9 - C_3C_7} = 40\frac{0.881523(0.290898) - 1.70625(0.014554)}{0.177640}$$

$$= 52.15\,\text{lb/in}$$

Therefore $\qquad y = 0 - \dfrac{527.8r}{D}F_1 + 0 + \dfrac{52.15r^3}{D}F_3 - \dfrac{40r^3}{D}G_3$

Substituting the appropriate expressions for F_1, F_3, and G_3 would produce an equation for y as a function of r, but a reduction of this equation to simple form and an evaluation to determine the location and magnitude of maximum deflection would be extremely time-consuming. Table 11.1 can be used again to good advantage to evalute y at specific values of r, and an excellent approximation to the maximum deflection can be obtained.

b/r	r	F_1	$-527.8rF_1$	F_3	$52.15r^3F_3$	r_o/r	G_3	$-40r^3G_3$	$y(D)$
1.00	5.000	0.000	0.0	0.000	0.0		0.000	0.0	0.0
0.90	5.555	0.09858	−289.0	0.000158	1.4		0.000	0.0	−287.6
0.80	6.250	0.194785	−642.5	0.001191	15.2		0.000	0.0	−627.3
0.70	7.143	0.289787	−1092.0	0.003753	71.3		0.000	0.0	−1020.7
0.60	8.333	0.385889	−1697.1	0.008208	247.7		0.000	0.0	−1449.4
0.50	10.000	0.487773	−2574.2	0.014554	759.0	1.00	0.000	0.0	−1815.2
0.40	12.500	0.605736	−3996.0	0.022290	2270.4	0.80	0.001191	−93.0	−1818.6
0.33	15.000	0.704699	−5578.6	0.027649	4866.4	0.67	0.005019	−677.6	−1389.8
0.30	16.667	0.765608	−6734.2	0.030175	7285.4	0.60	0.008208	−1520.0	−968.8
0.25	20.000	0.881523	−9304.5	0.033465	13961.7	0.50	0.014554	−4657.3	−0.1

An examination of the last column on the right shows the deflection at the outer edge to be approximately zero and indicates that the maximum deflection is located at a radius near 11.25 in and has a value of approximately

$$\frac{-1900}{D} = \frac{-1900(12)(1 - 0.3^2)}{10(10^6)t^3} = \frac{-0.00207}{t^3}\text{ in}$$

The maximum bending moment will be either a tangential moment at the inside edge or a radial moment at the load line:

$$M_{tb} = \frac{\theta_b D(1 - \nu^2)}{b} = \frac{-527.8(1 - 0.3^2)}{5} = -96.2\,\text{lb-in/in}$$

$$M_{r(r_o)} = \theta_b \frac{D}{r} F_{7(r_o)} + Q_b r F_{9(r_o)}$$

where at $r = r_o$, $b/r = 0.5$. Therefore

$$F_{7(r_o)} = 0.6825$$

$$F_{9(r_o)} = 0.290898$$

$$\begin{aligned} M_{r(r_o)} &= \frac{-527.8}{10}(0.6825) + 52.15(10)(0.290898) \\ &= -36.05 + 151.5 = 115.45\,\text{lb-in/in} \end{aligned}$$

The maximum bending stress in the plate is

$$\sigma = \frac{6(115.45)}{t^2} = \frac{693}{t^2}\,\text{lb/in}^2$$

3. A flat phosphor bronze disk with thickness of 0.020 in and a diameter of 4 in is upset locally in a die to produce an abrupt change in slope in the radial direction of 0.05 rad at a radius of $\frac{3}{4}$ in. It is then clamped between two flat dies as shown in Fig. 11.1. It is required to determine the maximum bending stress due to the clamping. Given: $E = 16(10^6)\,\text{lb/in}^2$ and $\nu = 0.30$.

Solution. This example of forcing a known change in slope into a plate clamped at both inner and outer edges is covered in Table 11.2, case 6h, where $\theta_o = 0.05$, $b/a = 0.10$, and $r_o/a = 0.50$. These dimensions were chosen to fit the tabulated data for a case where $\nu = 0.30$. For this case $M_{rb} = -2.054(0.05)(11.72)/1.5 = -0.803\,\text{lb-in/in}$, $Q_b = -0.0915(0.05)(11.72)/1.5^2 = -0.0238\,\text{lb/in}$, $y_b = 0$, and $\theta_b = 0$. The expression for M_r then becomes

$$M_r = -0.803F_8 - 0.0238rF_9 + \frac{0.05(11.72)}{r} G_7$$

An examination of the numerical values of F_8 and F_9 shows that F_8 decreases slightly less than F_9 increases as r increases, but the larger coefficient of the first term indicates that M_{rb} is indeed the maximum moment. The maximum stress is therefore $\sigma = 0.803(6)/0.02^2 = 12{,}050\,\text{lb/in}^2$ in tension on the top surface at the inner edge. The maximum deflection is at $r_o = 0.75$ in and equals $-0.1071(0.05)(1.5) = -0.00803$ in.

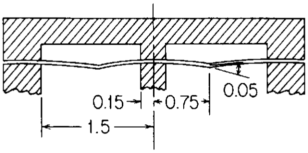


Figure 11.1

4. A circular steel plate 2 in thick and 20 ft in diameter is simply supported at the outer edge and supported on a center support which can be considered to provide uniform pressure over a diameter of 1.8 in. The plate is loaded in an axisymmetric manner with a distributed load which increases linearly with radius from a value of 0 at $r = 4$ ft to a value of 2000 lb/ft^2 at the outer edge. Determine the maximum bending stress. Given: $E = 30(10^6)$ lb/in^2 and $\nu = 0.30$.

Solution. Table 11.2, case 11a, deals with this loading and a simply supported outer edge. For this problem $q = \frac{2000}{144} = 13.9$ lb/in^2, $a = 120$ in, and $r_o = 48$ in, and so $r_o/a = 0.4$. From the tabulated data for these quantities, $K_{y_c} = -0.01646$, $K_{\theta_a} = 0.02788$, and $K_{M_c} = 0.04494$. Therefore

$$y_c = \frac{-0.01646(13.9)(120^4)}{D} = \frac{-0.475(10^8)}{D} \text{ in}$$

$$M_c = 0.04494(13.9)(120^2) = 9000 \text{ lb-in/in}$$

Case 16 (Table 11.2) considers the center load over a small circular area. It is desired to determine W such that the max $y = 0.475(10^8)/D$. Therefore

$$-\frac{W120^2}{16\pi D}\frac{3+0.3}{1+0.3} = \frac{0.475(10^8)}{D}$$

which gives $W = -65{,}000$ lb. The maximum moment is at the center of the plate where

$$M_r = \frac{W}{4\pi}\left[(1+\nu)\ln\frac{a}{b} + 1\right]$$

The equivalent radius r'_o is given by [Eq. (11.1-1)]

$$r'_o = \sqrt{1.6r_o^2 + t^2} - 0.675t = \sqrt{1.6(0.9^2) + 2^2} - 0.675(2) = 0.95 \text{ in}$$

Therefore $$M_{\max} = \frac{-65{,}000}{4\pi}\left(1.3\ln\frac{120}{0.95} + 1\right) = -37{,}500 \text{ lb-in/in}$$

The maximum stress is at the center of the plate where

$$\sigma = \frac{6M}{t^2} = \frac{6(-37{,}500 + 9000)}{2^2} = -43{,}200 \text{ lb/in}^2 \quad \text{(tension on the top surface)}$$

11.3 Circular-Plate Deflection due to Shear

The formulas for deflection given in Table 11.2 take into account bending stresses only; there is, in every case, some additional deflection due to shear. Usually this is so slight as to be negligible, but in circular pierced plates with large openings the deflection due to shear may constitute a considerable portion of the total deflection. Wahl (Ref. 19) suggests that this is the case when the thickness is greater than one-third the difference in inner and outer diameters for plates with simply supported edges, or greater than one-sixth this difference for plates with one or both edges fixed.

Table 11.3 gives formulas for the additional deflection due to shear in which the form factor F has been taken equal to 1.2, as in Sec. 8.10. All the cases listed have shear forces which are statically determinate. For the *indeterminate* cases, the shear deflection, along with the bending deflection, must be considered in the determination of the reactions if shear deflection is significant.

Essenburg and Gulati (Ref. 61) discuss the problem in which two plates when loaded touch over a portion of the surface. They indicate that the consideration of shear deformation is essential in developing the necessary expressions. Two examples are worked out.

EXAMPLE

An annular plate with an inner radius of 1.4 in, an outer radius of 2 in, and a thickness of 0.50 in is simply supported at the inner edge and loaded with an annular line load of 800 lb/in at a radius of 1.8 in. The deflection of the free outer edge is desired. Given: $E = 18(10^6)\,\text{lb/in}^2$ and $\nu = 0.30$.

Solution. To evaluate the deflection due to bending one can refer to Table 11.2, case 1k. Since $b/a = 0.7$, in Table 11.1, under the column headed 0.700, we obtain the following constants

$$C_1 = 0.2898, \quad C_3 = 0.003753, \quad C_7 = 0.3315, \quad C_9 = 0.2248$$

Similarly, $r_o/a = 0.9$, and again in Table 11.1, under the column headed 0.900, we obtain the additional constants $L_3 = 0.0001581$ and $L_9 = 0.09156$.

The plate constant $D = Et^3/12(1-\nu^2) = 18(10^6)(0.5)^3/12(1-0.3^2) = 206{,}000\,\text{lb-in}$, and the shear modulus $G = E/2(1+\nu) = 18(10^6)/2(1+0.3) = 6.92(10^6)\,\text{lb/in}^2$. The bending deflection of the outer edge is given by

$$\begin{aligned}
y_a &= \frac{-wa^3}{D}\left[\frac{C_1}{C_7}\left(\frac{r_oC_9}{b} - L_9\right) - \frac{r_oC_3}{b} + L_3\right] \\
&= \frac{-800(2)^3}{206{,}000}\left\{\frac{0.2898}{0.3315}\left[\frac{1.8(0.2248)}{1.4} - 0.09156\right] - \frac{1.8(0.003753)}{1.4} + 0.0001581\right\} \\
&= \frac{-800(2)^3}{206{,}000}(0.16774) = -0.00521\,\text{in}
\end{aligned}$$

For the deflection due to shear we refer to Table 11.3, case 1k, and obtain

$$y_{sa} = \frac{-wa}{tG}\left(1.2\frac{r_o}{a}\ln\frac{r_o}{b}\right) = \frac{800(2)}{0.5(6.92)(10^6)}\left[1.2(0.9)\ln\frac{1.8}{1.4}\right] = -0.000125\,\text{in}$$

Thus, the total deflection of the outer edge is $-0.00521 - 0.000125 = -0.00534$ in. Note that the thickness 0.50 is somewhat more than one-third the difference in inner and outer diameters 1.2, and the shear deflection is only 2.4% of the bending deflection.

11.4 Bimetallic Plates

A very wide beam of rectangular cross section can be treated as a beam if E is replaced by $E/(1-\nu^2)$ and I by $t^3/12$ (see Sec. 8.11). It can also be treated as a plate with two opposite edges free as shown in Figs. 8.16 and 11.2. For details see Ref. 88.

To use the beam equations in Tables 8.1, 8.5, 8.6, 8.8, and 8.9 for plates like that shown in Fig. 11.2 with two opposite edges free, the loadings must be uniformly distributed across the plate parallel to side b as shown. At every position in such a plate, except close to the free edges a, there will be bending moments $M_z = \nu M_x$. If the plate is isotropic and homogeneous, and in the absence of any in-plane loading, there will be no change in length of any line parallel to side b. The response of a *bimetallic plate* differs from that of the homogeneous plate in one important respect. If the values of Poisson's ratio differ for the two materials, there will be a change in length of those lines parallel to side b due to an in-plane strain ε_z developed from the loadings shown in Fig. 11.2. Using the notations from Figs. 11.2 and 11.3 and from the expression for K_{2p} in the next paragraph,

$$\varepsilon_z = \frac{6M_x(1-\nu_a^2)}{E_b t_b^2 K_{2p}} \frac{(t_b/t_a)(1+t_b/t_a)(\nu_a-\nu_b)}{(1+E_a t_a/E_b t_b)^2 - (\nu_a + \nu_b E_a t_a/E_b t_b)^2}$$

For the moment loading M_o in Fig. 11.2(a), the value of ε_z will be everywhere the same and the plate will merely expand or contract in the z direction. For the line loading shown in Fig. 11.2(c), however, the unit strains ε_z will differ from place to place depending upon the value of M_x, and consequently in-plane stresses will be developed. For more general analyses of this type of problem see Refs. 89 and 90.

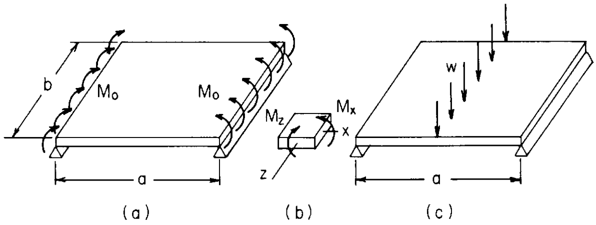


Figure 11.2

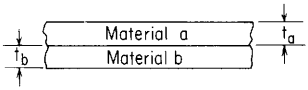


Figure 11.3

Bimetallic circular plates. Applying this same reasoning to a bimetallic circular plate leads to the following conclusions.

1. If the Poisson's ratios for the two materials are equal, any of the cases in Table 11.2 can be used if the following equivalent value of the plate stiffness constant D_e is substituted for D.
2. If the Poisson's ratios differ by a significant amount the equivalent values of D_e and ν_e may be used for any combination of loading and edge restraints which deform the plate into a spherical surface providing the edge restraints do not prevent motion parallel to the surface of the plate. This restriction assures that bending moments are constant in magnitude at all locations in the plate and in all directions. Thus one can use cases 8a, 8f, 8h, and 15 with either a uniform temperature rise or a temperature variation through the thickness which is the same everywhere in the plate. Obviously one needs also an equivalent temperature coefficient of expansion or an equivalent loading expression for each such temperature loading as well as the equivalent material constants D_e and ν_e.

$$\text{Equivalent}\, D_e = \frac{E_a t_a^3}{12(1-\nu_a^2)} K_{2p} \tag{11.3-1}$$

where

$$K_{2p} = 1 + \frac{E_b t_b^3 (1-\nu_a^2)}{E_a t_a^3 (1-\nu_b^2)} + \frac{3(1-\nu_a^2)(1+t_b/t_a)^2(1+E_a t_a/E_b t_b)}{(1+E_a t_a/E_b t_b)^2 - (\nu_a + \nu_b E_a t_a/E_b t_b)^2} \tag{11.3-2}$$

$$\text{Equivalent}\, \nu_e = \nu_a \frac{K_{3p}}{K_{2p}} \tag{11.3-3}$$

where

$$K_{3p} = 1 + \frac{\nu_b E_b t_b^3 (1-\nu_a^2)}{\nu_a E_a t_a^3 (1-\nu_b^2)} + \frac{3(1-\nu_a^2)(1+t_b/t_a)^2(1+\nu_b E_a t_a/\nu_a E_b t_b)}{(1+E_a t_a/E_b t_b)^2 - (\nu_a + \nu_b E_a t_a/E_b t_b)^2} \tag{11.3-4}$$

A bimetallic plate deforms laterally into a spherical surface when its uniform temperature differs from T_o, the temperature at which the plate is flat. Cases 8 and 15 (Table 11.2) can be used to solve for reaction moments and forces as well as the deformations of a bimetallic plate subjected to a uniform temperature T provided that any guided and/or fixed edges are not capable of developing in-plane resisting forces but instead allow the plate to expand or contract in its plane as necessary. To use these cases we need only to replace the term $\gamma(1+\nu)\Delta T/t$ by an equivalent expression

$$\left[\frac{\gamma(1+\nu)\Delta T}{t}\right]_e = \frac{6(\gamma_b-\gamma_a)(T-T_o)(t_a+t_b)(1+\nu_e)}{t_b^2 K_{1p}} \tag{11.3-5}$$

where

$$K_{1p} = 4 + 6\frac{t_a}{t_b} + 4\left(\frac{t_a}{t_b}\right)^2 + \frac{E_a t_a^3(1-\nu_b)}{E_b t_b^3(1-\nu_a)} + \frac{E_b t_b(1-\nu_a)}{E_a t_a(1-\nu_b)} \tag{11.3-6}$$

and replace D by the equivalent stiffness D_e given previously.

After the moments and deformations have been determined, the flexural stresses can be evaluated. The stresses due to the bending moments caused by restraints and any applied loads are given by the following expressions: In the top surface of material a, in the direction of any moment M

$$\sigma = \frac{-6M}{t_a^2 K_{2p}}\left[1 + \frac{(1-\nu_a^2)(1+t_b/t_a)(1+E_a t_a/E_b t_b)}{(1+E_a t_a/E_b t_b)^2 - (\nu_a + \nu_b E_a t_a/E_b t_b)^2}\right] \tag{11.3-7}$$

In the bottom surface of material b,

$$\sigma = \frac{6M}{t_a^2 K_{2p}}\left[\frac{E_b t_b(1-\nu_a^2)}{E_a t_a(1-\nu_b^2)} + \frac{t_a}{t_b}\frac{(1-\nu_a^2)(1+t_b/t_a)(1+E_a t_a/E_b t_b)}{(1+E_a t_a/E_b t_b)^2 - (\nu_a + \nu_b E_a t_a/E_b t_b)^2}\right] \tag{11.3-8}$$

Even when no restraints are imposed, the distortion of a bimetallic plate due to a temeprature change is accompanied by flexural stresses in the two materials. This differs from the plate made of a single material, which deforms free of stress when subjected to a linear temperature variation through the thickness when there are no restraints. Therefore, the following stresses must be added algebraically to the preceding stresses due to bending moments, if any: In the top surface of material a, in all directions

$$\sigma = \frac{-(\gamma_b-\gamma_a)(T-T_o)E_a}{(1-\nu_a)K_{1p}}\left[3\frac{t_a}{t_b} + 2\left(\frac{t_a}{t_b}\right)^2 - \frac{E_b t_b(1-\nu_a)}{E_a t_a(1-\nu_b)}\right] \tag{11.3-9}$$

In the bottom surface of material b,

$$\sigma = \frac{(\gamma_b-\gamma_a)(T-T_o)E_b}{(1-\nu_b)K_{1p}}\left[3\frac{t_a}{t_b} + 2 - \frac{E_a t_a^3(1-\nu_b)}{E_b t_b^3(1-\nu_a)}\right] \tag{11.3-10}$$

EXAMPLE

An annular bimetallic plate has a 3-in outer diameter and a 2.1-in inner diameter; the top portion is 0.020-in-thick stainless steel, and the bottom is 0.030-in-thick titanium (see Fig. 11.4). For the stainless steel $E = 28(10^6)\,\text{lb/in}^2$, $\nu = 0.3$, and $\gamma = 9.6(10^{-6})\,\text{in/in/}^\circ\text{F}$; for the titanium $E = 17(10^6)\,\text{lb/in}^2$, $\nu = 0.3$, and $\gamma = 5.7(10^{-6})\,\text{in/in/}^\circ\text{F}$. The outer edge is simply supported, and the inner edge is elastically supported by a spring which develops 500 lb of load for each inch of deflection. It is necessary to determine the center deflection and the maximum stress for a temperature rise of 50°F.

Solution. First evaluate the constants K_{1p}, K_{2p}, and K_{3p}, the equivalent stiffness D_e, and the equivalent Poisson's ratio ν_e. From Eq. (11.3-6),

$$K_{1p} = 4 + 6\frac{0.02}{0.03} + 4\left(\frac{2}{3}\right)^2 + \frac{28}{17}\left(\frac{2}{3}\right)^3\left(\frac{1-0.3}{1-0.3}\right) + \frac{17}{28}\left(\frac{3}{2}\right)\left(\frac{1-0.3}{1-0.3}\right)$$
$$= 11.177$$

Since $\nu_a = \nu_b$ for this example, $K_{3p} = K_{2p} = 11.986$ and the equivalent Poisson's ratio $\nu_e = 0.3$. From Eq. (11.3-1),

$$D_e = \frac{28(10^6)(0.02^3)}{12(1-0.3^2)}(11.986) = 246\,\text{lb-in}$$

Table 11.2, case 8a, treats an annular plate with the inner edge free and the outer edge simply supported. As in Eq. (11.3-5), the term $\gamma\Delta T/t$ must be replaced by

$$\frac{6(\gamma_b - \gamma_a)(T - T_o)(t_a + t_b)}{t_b^2 K_{1p}} = \frac{6(5.7 - 9.6)(10^{-6})(50)(0.02 + 0.03)}{(0.03^2)(11.177)} = -0.00582$$

Since $b/a = 1.05/1.5 = 0.7$ and $\nu_e = 0.3$, the tabulated data can be used and $K_{yb} = -0.255$ and $K_{\theta b} = 0.700$. Therefore, $y_b = -0.255(-0.00582)(1.5^2) = 0.00334$ in and $\theta_b = 0.7(-0.00582)(1.5) = -0.0061$ rad. There are no moments or edge loads in the plate, and so $M_{rb} = 0$, and $Q_b = 0$. Case 1a treats an annular plate with an annular line load. For $r_o = b$ and $b/a = 0.7$, $K_{yb} = -0.1927$ and $K_{\theta b} = 0.6780$. Therefore, $y_b = -0.1927w(1.5^3)/246 = -0.002645w$, $\theta_b = -0.678w(1.5^2)/246 = 0.0062w$ rad, $M_{rb} = 0$, and $Q_b = 0$.

Equating the deflection of the inner edge of the plate to the deflection of the elastic support gives $y_b = 0.00334 - 0.002645w = 2\pi(1.05)w/500 = 0.0132w$. Solving for w, we obtain $w = 0.211$ lb/in for a total center load of 1.39 lb. The deflection of the inner edge is $y_b = 0.0132(0.211) = 0.00279$ in. The maximum moment developed in the plate is the tangential moment at the inner edge:

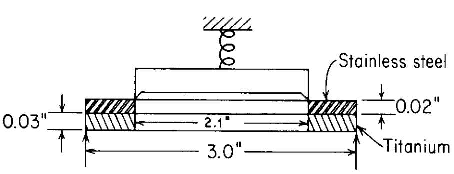


Figure 11.4

$M_{tb} = 0.8814(0.211)(1.5) = 0.279\,\text{lb-in}$. The stresses can now be computed. On the top surface of the stainless steel combining Eqs. (11.3-7) and (11.3-9) yields

$$\sigma = \frac{-6(0.279)}{0.02^2(11.986)}\left\{1 + \frac{(1-0.3^2)(1+3/2)[1+28(2)/17(3)]}{[1+28(2)/17(3)]^2 - [0.3+0.3(28)(2)/17(3)]^2}\right\}$$
$$-\frac{(5.7-9.6)(10^{-6})(50)(28)(10^6)}{(1-0.3)(11.177)}\left[3\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)^2-\frac{17}{28}\left(\frac{3}{2}\right)\right]$$
$$= -765 + 1381 = 616\,\text{lb/in}^2$$

Similarly, on the bottom surface of the titanium, Eqs. (11.3-8) and (11.3-10) give

$$\sigma = 595 - 1488 = -893\,\text{lb/in}^2$$

11.5 Nonuniform Loading of Circular Plates

The case of a circular plate under a nonuniformly distributed loading symmetrical about the center can be solved by treating the load as a series of elementary ring loadings and summing the stresses and deflections produced by such loadings. The number of ring loadings into which the actual load should be resolved depends upon the rate at which the distributed load varies along the radius and the accuracy desired. In general, a division of the load into rings each having a width equal to one-fifth the loaded length of the radius should be sufficient.

If the nonuniformly distributed loading can be reasonably approximated by a second-order curve, the loadings in Table 11.2, cases 2–4, can be superimposed in the necessary proportions. (This technique is illustrated in Sec. 11.6.) Heap (Ref. 48) gives tabular data for circular plates loaded with a lateral pressure varying inversely with the square of the radius.

Concentrated loads. In Refs. 60 and 75–79 similar numerical techniques are discussed for concentrated loads on either of two concentric annular plates in combination with edge beams in some cases. The numerical data presented are limited but are enough to enable the reader to approximate many other cases.

11.6 Circular Plates on Elastic Foundations

Discussions of the theory of bending of circular plates on elastic foundations can be found in Refs. 21 and 46, and in Ref. 41 of Chap. 8. The complexity of these solutions prohibits their inclusion in this handbook, but a simple *iterative* approach to this problem is possible.

The procedure consists in evaluating the deflection of the loaded plate without the elastic foundation and then superimposing a given fraction of the foundation reaction resulting from this deflection until finally the given fraction increases to 1 and the assumed and calculated foundation reactions are equal.

EXAMPLE

Given the same problem stated in Example 1 of Sec. 11.2, but in addition to the simply supported edge an elastic foundation with a modulus of 20 lb/in^2/in is present under the entire plate.

Solution. An examination of the deflection equation resulting from the uniform load shows that the term involving r^4 is significant only near the outer edge where the effect of foundation pressure would not be very large. We must also account for the fact that the foundation reactions will reduce the plate deflections or the procedure described may not converge. Therefore, for a first trial let us assume that the foundation pressure is given by

$$q_f = 20(-0.0883 + 0.001098r^2)(0.50) = -0.883 + 0.01098r^2$$

The total loading on the plate then consists of a uniform load of $3 - 0.883 = 2.117$ lb/in^2 and a parabolically increasing load of 1.098 lb/in^2 maximum value. From Table 11.2, case 10a,

$$y_c = \frac{-qa^4(5+\nu)}{64D(1+\nu)} = \frac{-2.117(10^4)(5.285)}{64(21{,}800)(1.285)} = -0.063\ \text{in}$$

$$M_c = \frac{qa^2}{16}(3+\nu) = \frac{2.117(10^2)(3.285)}{16} = 43.5\ \text{lb-in/in}$$

$$LT_y = \frac{-qr^4}{D}G_{11} = \frac{-2.117r^4}{21{,}800}\frac{1}{64} = -1.517(10^{-6})r^4$$

From Table 11.2, case 12a,

$$y_c = \frac{-qa^4(7+\nu)}{288D(1+\nu)} = \frac{-1.098(10^4)(7.285)}{288(21{,}800)(1.285)} = -0.00992\ \text{in}$$

$$M_c = \frac{qa^2(5+\nu)}{96} = \frac{1.098(10^2)(5.285)}{96} = 6.05\ \text{lb-in/in}$$

$$LT_y = \frac{-qr^6}{Da^2}G_{13} = \frac{-1.098r^6}{21{,}800(10^2)}\frac{25}{14{,}400} = -8.75(10^{-10})r^6$$

Using these values, the deflection equation can be written

$$y = -0.0623 - 0.00992 + \frac{(43.5+6.05)r^2}{2(21{,}800)1.285} - 1.517(10^{-6})r^4 - 8.75(10^{-10})r^6$$
$$= -0.0722 + 0.000885r^2 - 1.517(10^{-6})r^4 - 8.75(10^{-10})r^6$$

This deflection would create a foundation reaction

$$q_f = 20(-0.0722 + 0.000885r^2) = -1.445 + 0.0177r^2$$

if the higher-order terms were neglected. Again applying a 50% factor to the difference between the assumed and calculated foundation pressure gives an improved loading from the foundation

$$q_f = -1.164 + 0.01434r^2$$

Repeating the previous steps again, we obtain

$$\begin{aligned} y_c &= -0.0623\frac{3-1.164}{2.117} - 0.00992\frac{0.01434}{0.01098} = -0.0671\,\text{in} \\ M_c &= 43.5\frac{3-1.164}{2.117} + 6.05\frac{0.01434}{0.01098} = 45.61\,\text{lb-in/in} \\ y &= -0.0671 + 0.000813r^2 \\ q_f &= -1.342 + 0.01626r^2 \end{aligned}$$

Successive repetitions of the previous steps give improved values for q_f:

$$q_f = -1.306 + 0.1584r^2, \quad q_f = -1.296 + 0.1566r^2, \quad q_f = -1.290 + 0.1566r^2$$

Using values from the last iteration, the final answers are

$$y_c = -0.0645\,\text{in}, \quad M_c = 43.8\,\text{lb-in/in}, \quad \text{and}\,\sigma_{\max} = 6580\,\text{psi}$$

An exact analysis using expressions from Ref. 46 gives

$$y_c = -0.0637\,\text{in} \quad \text{and} \quad M_c = 43.3\,\text{lb-in/in}$$

11.7 Circular Plates of Variable Thickness

For any circular plate of variable thickness, loaded symmetrically with respect to the center, the stresses and deflections can be found as follows: The plate is divided into an arbitrary number of concentric rings, each of which is assumed to have a uniform thickness equal to its mean thickness. Each such ring is loaded by radial moments M_a and M_b at its outer and inner circumferences, respectively, by vertical shears at its inner and outer circumferences, and by whatever load is distributed over its surface. The shears are known, each being equal to the total load on the plate within the corresponding circumference. The problem is to determine the edge moments, and this is done by making use of the fact that the slope of each ring at its inner circumference is equal to the slope of the next inner ring at its outer circumference. This condition, together with the known slope (or moment) at the outer edge of the plate and the known slope (or moment) at the inside edge or center of the plate, enables as many equations to be written as there are unknown quantities M. Having found all the edge moments, stresses and deflections can be calculated for each ring by the appropriate formulas of Table 11.2 and the deflections added to find the deflection of the plate.

A more direct solution (Ref. 21) is available if the plate is of such form that the variation in thickness can be expressed fairly closely by the equation $t = t_o e^{-nx^2/6}$, where t is the thickness at any point a distance r from the center, t_o is the thickness at the center, e is the base for the napierian system of logarithms (2.178), x is the ratio r/a, and n is a number chosen so as to make the equation agree with the actual variation in thickness. The constant n is positive for a plate that decreases in thickness toward the edge and negative for a plate that increases in thickness toward the edge. For a plate of uniform thickness, $n = 0$; and for a plate twice as thick at the center as at the edge, $n = +4.16$. The maximum stress and deflection for a uniformly loaded circular plate are given by $\sigma_{\max} = \beta q a^2/t_o^2$ and $y_{\max} = \alpha q a^4/E t_o^3$, respectively, where β and α depend on n, where $\nu = 0.3$, and for values of n from 4 to -4 can be found by interpolation from the following table:

		n								
Edge conditions		+4	+3	+2	+1	0	−1	−2	−3	−4
Edges supported	β	1.63	1.55	1.45	1.39	1.24	1.16	1.04	0.945	0.855
Case 10a, $r_o = 0$	α	1.220	1.060	0.924	0.804	0.695	0.600	0.511	0.432	0.361
Edges fixed	β	2.14	1.63	1.31	0.985	0.75	0.55	0.43	0.32	0.26
Case 10b, $r_o = 0$	α	0.4375	0.3490	0.276	0.217	0.1707	0.1343	0.1048	0.0830	0.0653

For the loadings in the preceding table as well as for a simply supported plate with an edge moment, Ref. 46 gives graphs and tables which permit the evaluation of radial and tangential stresses throughout the plate. This same reference gives extensive tables of moment and shear coefficients for a variety of loadings and support conditions for plates in which the thickness varies as $t = t_a (r/a)^{-n/3}$, where t_a is the thickness at the outer edge: Values are tabulated for $n = 0, 1, 1.5$, and 2 and for $\nu = \frac{1}{6}$.

Stresses and deflections for plates with thicknesses varying linearly with radius are tabulated in Refs. 46 and 57. Annular plates with the outer edges fixed and the inner edges guided and with thicknesses increasing linearly with the radii from zero at the center are discussed in Ref. 36 and tabulated in previous editions of this handbook. A uniformly loaded circular plate with a fixed edge and a thickness varying linearly along a diameter is discussed by Strock and Yu (Ref. 65). Conway (Ref. 66) considers the necessary proportions for a rib along the diameter of a uniformly loaded, clamped circular plate to affect a minimum weight design for a given maximum stress.

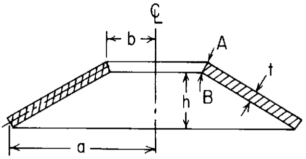


Figure 11.5

Perforated plates. Slot and O'Donnell (Ref. 62) present the relationship between the effective elastic constants for *thick perforated plates* under bending and *thin perforated plates* under in-plane loading. Numerical results are presented in the form of tables and graphs, and many references are listed.

11.8 Disk Springs

The *conical disk*, or *Belleville spring* (Fig. 11.5), is not a flat plate, of course, but it may appropriately be considered in this chapter because it bears a superficial resemblance to a flat ring and is sometimes erroneously analyzed by the formulas for case 1a. The stress and deflection produced in a spring of this type are not proportional to the applied load because the change in form consequent upon deflection markedly changes the load-deflection and load-stress relationships. This is indeed the peculiar advantage of this form of spring because it makes it possible to secure almost any desired variation of "spring rate" and also possible to obtain a considerable range of deflection under almost constant load. The largest stresses occur at the inner edge.

Formulas for deflection and stress at points A and B are (Ref. 27)

$$P = \frac{E\delta}{(1-\nu^2)Ma^2}\left[(h-\delta)\left(h-\frac{\delta}{2}\right)t + t^3\right]$$

$$\sigma_A = \frac{-E\delta}{(1-\nu^2)Ma^2}\left[C_1\left(h-\frac{\delta}{2}\right) + C_2 t\right]$$

$$\sigma_B = \frac{-E\delta}{(1-\nu^2)Ma^2}\left[C_1\left(h-\frac{\delta}{2}\right) - C_2 t\right]$$

where $P =$ total applied load; $E =$ modulus of elasticity; $\delta =$ deflection; $h =$ cone height of either inner or outer surface; $t =$ thickness; a and b are the outer and inner radii of the middle surface; and M, C_1, and C_2 are constants whose values are functions of a/b and are given in the following table:

a/b	M	C_1	C_2
1.0	0		
1.2	0.31	1.02	1.05
1.4	0.46	1.07	1.14
1.6	0.57	1.14	1.23
1.8	0.64	1.18	1.30
2.0	0.70	1.23	1.39
2.2	0.73	1.27	1.46
2.6	0.76	1.35	1.60
3.0	0.78	1.43	1.74
3.4	0.80	1.50	1.88
3.8	0.80	1.57	2.00
4.2	0.80	1.64	2.14
4.6	0.80	1.71	2.26
5.0	0.79	1.77	2.38

The formulas for stress may give either positive or negative results, depending upon δ; a negative result indicates compressive stress, and a positive result a tensile stress. It is to be noted that P also may become negative.

Wempner (Refs. 67 and 68) derives more exacting expressions for the conical spring. Unless the center hole is small or the cone angle is outside the range normally used for disk springs, however, the differences are slight. Reference 69 presents useful design curves based on Ref. 27.

Conical spring washers can be stacked to operate in either series or parallel. One must be careful to consider the effect of friction, however, when using them in the parallel configuration.

11.9 Narrow Ring under Distributed Torque about Its Axis

When the inner radius b is almost as great as the outer radius a, the loading for cases 1a, 1k, 2a, 2k, and so on, becomes almost equivalent to that shown in Fig. 11.6, which represents a ring subjected to a uniformly distributed torque of M (force-length/unit length) about that circumference passing through the centroids at the radius R. An

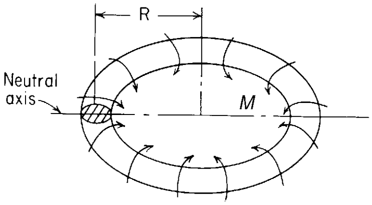


Figure 11.6

approximation to this type of loading also occurs in clamping, or "follower," rings used for joining pipe; here the bolt forces and the balancing gasket or flange pressure produce the distributed torque, which obviously tends to "roll" the ring, or turn it inside out, so to speak.

Under this loading the ring, whatever the shape of its cross section (as long as it is reasonably compact) is subjected to a bending moment at every section equal to MR, the neutral axis being the central axis of the cross section in the plane of the ring. The maximum resulting stress occurs at the extreme fiber and is given by Eq. (8.1-12); that is,

$$\sigma = \frac{MR}{I/c} \tag{11.9-1}$$

The ring does not bend, and there is no twisting, but every section rotates in its own plane about its centroid through an angle

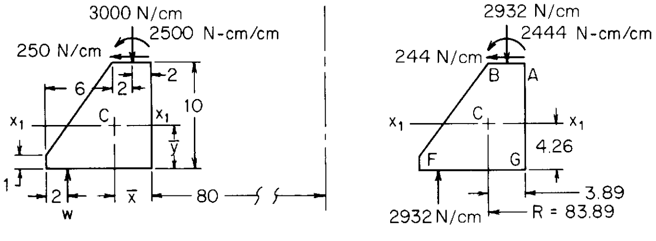
$$\theta = \frac{MR^2}{EI} = \frac{\sigma R}{Ec} \tag{11.9-2}$$

These formulas may be used to obtain approximate results for the cases of flat-plate loading listed previously when the difference between a and b is small, as well as for pipe flanges, etc. Paul (Ref. 70) discusses the collapse or inversion of rings due to plastic action.

EXAMPLE

The cross section shown in Fig. 11.7 is from a roll-forged ring of steel used to support the bottom of a large shell. The modulus of elasticity is 207 GPa, or $20.7(10^6)$ N/cm^2, and Poisson's ratio is 0.285. The loadings from the shell are shown in Fig. 11.7(a) and are unit loads at a radius of 82 cm where they are applied.

Solution. In the equations for stress and angular rotation the moment distributed around the ring must be evaluated as that moment acting upon a segment of the ring having a unit length along the circumference at the radius of the centroid of the cross section. In Fig. 11.7(b) these appropriate loadings are shown. Before they could be found, however, the centroid and the



(a) (b)

Figure 11.7 (All dimensions in centimeters) moment of inertia about the x axis through this centroid must have been evaluated. This was done as follows.

$$A = 10(10) - \frac{9(6)}{2} = 73\,\text{cm}^2$$

$$\bar{y} = \frac{100(5) - 27(7)}{73} = 4.26\,\text{cm}, \quad \bar{x} = \frac{100(5) - 27(8)}{73} = 3.89\,\text{cm}$$

$$I_{x1} = \frac{10^4}{12} + 100(5 - 4.26)^2 - \frac{6(9^3)}{36} - 27(7 - 4.26)^2 = 563.9\,\text{cm}^4$$

First calculate the value of w which will put into equilibrium at a radius of 88 cm the vertical load of 3000 N/cm at a radius of 82 cm. This is 2795 N/cm. Next convert all these loads to the values they will have when applied to a free-body diagram consisting of a segment that is 1 cm long at the centroidal radius of 83.89 cm. For the loads on the top of the free-body diagram the length upon which they act is $82/83.89 = 0.9775$ cm so that the desired couple is then $2500(0.9775) = 2444$ N-cm/cm. All the remaining forces were computed in a similar manner.

Using the loads shown in Fig. 11.7(*b*), the clockwise moment about the centroid is found to be $M = 2932(6) - 2444 - 244(10 - 4.26) = 13{,}747$ N-cm. This gives the section a clockwise rotation of $\theta = 13{,}747(83.89^2)/20.7(10^6)(563.9) = 0.00829$ rad. All material in the section lying above the x_1 axis will then move toward the central axis and be in compression. The stresses at positions A and B will then be $\sigma = -13{,}747(83.89)(5.74)/563.9 = -11{,}739\,\text{N/cm}^2$. Similarly, the stresses at positions F and G are $\sigma = 13{,}747(83.89)(4.26)/563.9 = 8712\,\text{N/cm}^2$.

In addition to the stresses caused by the rotation of the cross section, the radially outward shear force of 244 N/cm produces everywhere in the cross section a circumferential tensile stress of $\sigma = 244(83.89)/73 = 280\,\text{N/cm}^2$. Note that a tacit assumption has been made that no radially directed friction forces exist at the bottom of the ring.

11.10 Bending of Uniform-Thickness Plates with Straight Boundaries

Formulas. No general expression for deflection as a function of position in a plate is given since solutions for plates with straight boundaries are generally obtained numerically for specific ratios of plate dimensions, load location, and boundary conditions. In a few instances Poisson's ratio is included in the expressions given, but in most cases a specific value of Poisson's ratio has been used in obtaining the tabulated numerical results and the value used is indicated. Reference 47 includes results obtained using several values of Poisson's ratio and shows the range of values that can be expected as this ratio is changed. Errors in deflection should not exceed 7 or 8% and in maximum stress 15% for values of Poisson's ratio in the range from 0.15 to 0.30. Since much of the data are obtained using finite-difference approximations for the plate differential equations and a limited number of elements have been used, it is not always possible to identify maximum values if they occur at points between the chosen grid points.

Table 11.4 presents maximum values where possible and the significant values otherwise for deflections normal to the plate surface, bending stresses, and in many cases the boundary reaction forces R. For rectangular plates with simply supported edges the maximum stresses are shown to be near the center of the plate. There are, however, stresses of similar magnitude near the corners if the corners are held down as has been assumed for all cases presented. Reference 21 discusses the increase in stress at the center of the plate when the corners are permitted to rise. For a uniformly loaded square plate this increase in stress is approximately 35%.

It is impractical to include plates of all possible shapes and loadings, but many more cases can be found in the literature. Bareś (Ref. 47) presents tabulated values of bending moments and deflections for a series of plates in the form of *isoceles triangles* and *symmetric trapezoids* for linearly varying lateral pressures and for values of Poisson's ratio of 0.0 and 0.16. Tabulated values are given for *skew* plates with uniform lateral loading and concentrated lateral loads for the support conditions where two opposite edges are simply supported and two edges are free; the value of Poisson's ratio used was zero. In addition to many cases also included in Table 11.4, Marguerre and Woernle (Ref. 50) give results for line loading and uniform loading on a narrow strip across a rectangular plate. They also discuss the case of a rectangular plate supported within the span by elastic cross beams. Morley (Ref. 51) discusses solutions of problems involving *parallelogram*, or *skew*, plates and box structures. A few graphs and tables of results are given.

For plates with boundary shapes or restraints not discussed in the literature, we can only approximate an answer or resort to a direct numerical solution of the problem at hand. All numerical methods are approximate but can be carried to any degree of accuracy desired at the expense of time and computer costs. There are many numerical techniques used to solve plate problems, and the choice of a method for a given problem can be difficult. Leissa et al. (Ref. 56) have done a very complete and competent job of comparing and rating 9 approximate numerical methods on the basis of 11 different criteria. Szilard (Ref. 84) discusses both classical and numerical methods and tabulates many solutions.

Variable thickness. Petrina and Conway (Ref. 63) give numerical data for two sets of boundary conditions, three aspect ratios and two nearly linear tapers in plate thickness. The loading was uniform and they found that the center deflection and center moment differed little from the same uniform-thickness case using the average thickness; the location and magnitude of maximum stress, however, did vary.

11.11 Effect of Large Deflection; Diaphragm Stresses

When the deflection becomes larger than about one-half the thickness, as may occur in thin plates, the middle surface becomes appreciably strained and the stress in it cannot be ignored. This stress, called *diaphragm* stress, or *direct* stress, enables the plate to carry part of the load as a diaphragm in direct tension. This tension may be balanced by radial tension at the edges if the edges are *held* or by circumferential compression if the edges are not horizontally restrained. In thin plates this circumferential compression may cause buckling.

When this condition of large deflection exists, the plate is stiffer than indicated by the ordinary theory and the load-deflection and load-stress relations are nonlinear. Stresses for a given load are less and stresses for a given deflection are generally greater than the ordinary theory indicates.

Circular plates. Formulas for stress and deflection when middle surface stresses are taken into account are given below. These formulas should be used whenever the maximum deflection exceeds half the thickness if accurate results are desired. The following table gives the necessary constants for the several loadings and support conditions listed.

Let $t =$ thickness of plate; $a =$ outer radius of plate; $q =$ unit lateral pressure; $y =$ maximum deflection; $\sigma_b =$ bending stress; $\sigma_d =$ diaphragm stress; $\sigma = \sigma_b + \sigma_d =$ maximum stress due to flexure and diaphragm tension combined. Then the following formulas apply:

$$\frac{qa^4}{Et^4} = K_1 \frac{y}{t} + K_2 \left(\frac{y}{t}\right)^3 \tag{11.11-1}$$

$$\frac{\sigma a^2}{Et^2} = K_3 \frac{y}{t} + K_4 \left(\frac{y}{t}\right)^2 \tag{11.11-2}$$

First solve for y in Eq. (11.11-1) and then obtain the stresses from Eq. (11.11-2).

EXAMPLE

For the plate of Example 1 of Sec. 11.2, it is desired to determine the maximum deflection and maximum stress under a load of 10 lb/in^2.

Solution. If the linear theory held, the stresses and deflections would be directly proportional to the load, which would indicate a maximum stress of $9240(10)/3 = 30{,}800$ lb/in^2 and a maximum deflection of $0.0883(10)/3 =$

0.294 in. Since this deflection is much more than half the thickness, Eqs. (11.11-3) and (11.11-2) with the constants from case 1 in the table will be used to solve for the deflection and stress. From Eq. (11.11-1), we obtain

$$\frac{10(10^4)}{30(10^6)(0.2^4)} = \frac{1.016}{1-0.3}\frac{y}{t} + 0.376\left(\frac{y}{t}\right)^3$$

$$2.0833 = 1.4514\frac{y}{t} + 0.376\left(\frac{y}{t}\right)^3$$

Starting with a trial value for y somewhat less than 0.294 in, a solution is found when $y = 0.219$ in. From Eq. (11.11-2) the maximum stress is found to be 27,500 lb/in^2.

Warshawsky (Ref. 3) fitted Eqs. (11.11-1) and (11.11-2) to the data presented by Mah in Ref. 71, and cases 5–9 in the following table give these results. Chia in Ref. 91 has a chapter on nonlinear bending of isotropic nonrectangular plates in which he covers in great detail the derivations, plotted results, and formulas similar to Eqs. (11.11-1) and (11.11-2) for distributed loadings, concentrated center loads, applied edge moments, and combined loadings for circular plates with various boundary conditions. The uniformly loaded circular plate on an elastic foundation is discussed and results presented for several boundary conditions. He also treats annular plates, elliptical plates, and skew plates under uniform loading. Reference 54 presents the results of a study of the large deflections of clamped annular sector plates for sector angles from 30 to 90° in steps of 30° and for ratios of inner to outer radii from 0 to 0.6 in steps of 0.2.

Circular plates under distributed load producing large deflections

Case no., edge condition	Constants
1. Simply supported (neither fixed nor held). Uniform pressure q over entire plate.	$K_1 = \frac{1.016}{1-\nu}$ $K_2 = 0.376$ $K_3 = \frac{1.238}{1-\nu}$ $K_4 = 0.294$ (Ref. 5)
2. Fixed but not held (no edge tension). Uniform pressure q over entire plate.	$K_1 = \frac{5.33}{1-\nu^2}$ $K_2 = 0.857$ (At center) $K_3 = \frac{2}{1-\nu}$ $K_4 = 0.50$ (At edge) $K_3 = \frac{4}{1-\nu^2}$ $K_4 = 0.0$ (Ref. 5)
3. Fixed and held. Uniform pressure q over entire plate.	$K_1 = \frac{5.33}{1-\nu^2}$ $K_2 = \frac{2.6}{1-\nu^2}$ (At center) $K_3 = \frac{2}{1-\nu}$ $K_4 = 0.976$ (At edge) $K_3 = \frac{4}{1-\nu^2}$ $K_4 = 1.73$ (Refs. 15 and 16)

Circular plates under distributed load producing large deflections (*Continued*)

<table>
<tr><th>Case no., edge condition</th><th>Constants</th></tr>
<tr><td>4. Diaphragm without flexural stiffness, edge held. Uniform pressure q over entire plate.</td><td>$K_1 = 0.0 \quad K_2 = 3.44$
(At center) $K_3 = 0.0 \quad K_4 = 0.965$
(At edge) $K_3 = 0.0 \quad K_4 = 0.748$
(At r from the center) $y = y_{\max}\left(1 - 0.9\frac{r^2}{a^2} - 0.1\frac{r^5}{a^5}\right)$
(Refs. 18 and 29)</td></tr>
<tr><td>5. Fixed and held. Uniform pressure q over a central area of radius r_o. $\nu = 0.3$</td><td>
<table>
<tr><th rowspan="2">r_o/a</th><th rowspan="2">K_1</th><th rowspan="2">K_2</th><th colspan="2">At edge</th><th colspan="2">At center</th></tr>
<tr><th>K_3</th><th>K_4</th><th>K_3</th><th>K_4</th></tr>
<tr><td>1.00</td><td>5.86</td><td>3.32</td><td>4.40</td><td>1.73</td><td></td><td></td></tr>
<tr><td>0.75</td><td>6.26</td><td>3.45</td><td>3.80</td><td>1.32</td><td></td><td></td></tr>
<tr><td>0.50</td><td>9.17</td><td>5.50</td><td></td><td></td><td>3.38</td><td>0.76</td></tr>
<tr><td>0.25</td><td>27.1</td><td>13.9</td><td></td><td></td><td>4.62</td><td>1.18</td></tr>
</table>
(Ref. 3)</td></tr>
<tr><td>6. Simply supported and held radially. Uniform pressure q over a central area of radius r_o. $\nu = 0.3$</td><td>
<table>
<tr><th rowspan="2">r_o/a</th><th rowspan="2">K_1</th><th rowspan="2">K_2</th><th colspan="2">At center</th></tr>
<tr><th>K_3</th><th>K_4</th></tr>
<tr><td>0.75</td><td>1.71</td><td>3.21</td><td>1.84</td><td>0.81</td></tr>
<tr><td>0.50</td><td>2.95</td><td>5.07</td><td>2.06</td><td>0.95</td></tr>
<tr><td>0.25</td><td>9.95</td><td>13.8</td><td>2.60</td><td>1.31</td></tr>
</table>
(Ref. 3)</td></tr>
<tr><td>7. Fixed and held with a central support. Uniform pressure q over entire plate. $\nu = 0.3$</td><td>$y_{\max}$ at $r = 0.45a$
$K_1 = 36.4 \quad K_2 = 20.0$
(Ref. 3)</td></tr>
<tr><td>8. Annular plate fixed and held at both inner and outer edges. Uniform pressure q over entire annular plate. $\nu = 0.3$</td><td>For inner edge radius $= 0.2a$, max deflection y at $r = 0.576a$
$K_1 = 84.0 \quad K_2 = 63.5$
For stress at $r = 0.2a$,
$K_3 = 36.0 \quad K_4 = 25.8$
(Ref. 3)</td></tr>
<tr><td>9. Annular plate simply supported and held radially at both inner and outer edges. Uniform pressure q over entire annular plate. $\nu = 0.3$</td><td>For inner edge radius $= 0.2a$, max deflection y at $r = 0.576a$
$K_1 = 20.3 \quad K_2 = 51.8$
For stress at $r = 0.2a$,
$K_3 = 12.14 \quad K_4 = 2.41$
For inner edge radius $= 0.4a$, max deflection y at $r = 0.688a$
$K_1 = 57.0 \quad K_2 = 159$
For stress at $r = 0.664a$,
$K_3 = 14.52 \quad K_4 = 6.89$
(Ref. 3)</td></tr>
</table>

Elliptical plates. Nash and Cooley (Ref. 72) present graphically the results of a uniform pressure on a clamped elliptical plate for $a/b = 2$. Their method of solution is presented in detail, and the numerical solution is compared with experimental results and with previous solutions they have referenced. Ng (Ref. 73) has tabulated the values of center deflection for clamped elliptical plates on elastic foundations for ratios of a/b from 1 to 2 and for a wide range of foundation moduli. Large deflections are also graphed for two ratios a/b (1.5 and 2) for the same range of foundation moduli.

Rectangular plates. Analytical solutions for uniformly loaded rectangular plates with large deflections are given in Refs. 30–34, where the relations among load, deflection, and stress are expressed by numerical values of the dimensionless coefficients y/t, qb^4/Et^4, and $\sigma b^2/Et^2$. The values of these coefficients given in the following table are taken from these references and are for $\nu = 0.316$. In this table, a, b, q, E, y, and t have the same meaning as in Table 11.4, σ_d is the diaphragm stress, and σ is the total stress found by adding the diaphragm stress and the bending stress. See also Ref. 17.

In Ref. 35 experimentally determined deflections are given and compared with those predicted by theory. In Ref. 74 a numerical solution for uniformly loaded rectangular plates with simply supported edges is discussed, and the results for a square plate are compared with previous approximate solutions. Graphs are presented to show how stresses and deflections vary across a square plate.

Chia in Ref. 91 includes a chapter on moderately large deflections of isotropic rectangular plates. Not only are the derivations presented but the results of most cases are presented in the form of graphs usable for engineering calculations. Cases of initially deflected plates are included, and the comprehensive list of references is useful. Aalami and Williams in Ref. 92 present 42 tables of large-deflection reduction coefficients over a range of length ratios a/b and for a variety—three bending and four membrane—of symmetric and nonsymmetric boundary conditions. Loadings include overall uniform and linearly varying pressures as well as pressures over limited areas centered on the plates.

Parallelogram plates. Kennedy and Ng (Ref. 53) present several graphs or large elastic deflections and the accompanying stresses for uniformly loaded skew plates with clamped edges. Several apsect ratios and skew angles are represented.

11.12 Plastic Analysis of Plates

The onset of yielding in plates may occur before the development of appreciable diaphragm stress if the plate is relatively thick. For

Rectangular plates under uniform load producing large deflection

a/b	Edges and point of max σ	Coef.	qb^4/Et^4										
			0	12.5	25	50	75	100	125	150	175	200	250
1	Held, not fixed	y/t	0	0.430	0.650	0.930	1.13	1.26	1.37	1.47	1.56	1.63	1.77
	At center of plate	$\sigma_d b^2/Et^2$	0	0.70	1.60	3.00	4.00	5.00	6.10	7.00	7.95	8.60	10.20
		$\sigma b^2/Et^2$	0	3.80	5.80	8.70	10.90	12.80	14.30	15.60	17.00	18.20	20.50
1	Held and riveted	y/t	0	0.406	0.600	0.840	1.00	1.13	1.23	1.31	1.40	1.46	1.58
	At center of plate	$\sigma_d b^2/Et^2$	0	0.609	1.380	2.68	3.80	4.78	5.75	6.54	7.55	8.10	9.53
		$\sigma b^2/Et^2$	0	3.19	5.18	7.77	9.72	11.34	12.80	14.10	15.40	16.40	18.40
1	Held and fixed	y/t	0	0.165	0.32	0.59	0.80	0.95	1.08	1.19	1.28	1.38	1.54
	At center of long edges	$\sigma_d b^2/Et^2$	0	0.070	0.22	0.75	1.35	2.00	2.70	3.30	4.00	4.60	5.90
		$\sigma b^2/Et^2$	0	3.80	6.90	14.70	21.0	26.50	31.50	36.20	40.70	45.00	53.50
	At center of plate	$\sigma_d b^2/Et^2$	0	0.075	0.30	0.95	1.65	2.40	3.10	3.80	4.50	5.20	6.50
		$\sigma b^2/Et^2$	0	1.80	3.50	6.60	9.20	11.60	13.0	14.50	15.80	17.10	19.40
1.5	Held, not fixed	y/t	0	0.625	0.879	1.18	1.37	1.53	1.68	1.77	1.88	1.96	2.12
	At center of plate	$\sigma_d b^2/Et^2$	0	1.06	2.11	3.78	5.18	6.41	7.65	8.60	9.55	10.60	12.30
		$\sigma b^2/Et^2$	0	4.48	6.81	9.92	12.25	14.22	16.0	17.50	18.90	20.30	22.80
2 to ∞	Held, not fixed	y/t	0	0.696	0.946	1.24	1.44	1.60	1.72	1.84	1.94	2.03	2.20
	At center of plate	$\sigma_d b^2/Et^2$	0	1.29	2.40	4.15	5.61	6.91	8.10	9.21	10.10	10.90	12.20
		$\sigma b^2/Et^2$	0	4.87	7.16	10.30	12.60	14.60	16.40	18.00	19.40	20.90	23.60
1.5 to ∞	Held and fixed	y/t	0	0.28	0.51	0.825	1.07	1.24	1.40	1.50	1.63	1.72	1.86
	At center of long edges	$\sigma_d b^2/Et^2$	0	0.20	0.66	1.90	3.20	4.35	5.40	6.50	7.50	8.50	10.30
		$\sigma b^2/Et^2$	0	5.75	11.12	20.30	27.8	35.0	41.0	47.0	52.50	57.60	67.00

thinner plates, the nonlinear increase in stiffness due to diaphragm stresses is counteracted by the decrease in stiffness which occurs when the material starts to yield (Refs. 52 and 80). Save and Massonnet (Ref. 81) discuss the effect of the several yield criteria on the response of circular and rectangular plates under various loadings and give an extensive list of references. They also compare the results of theory with referenced experiments which have been performed. *Orthotropy* in plates can be caused by cold-forming the material or by the positioning of stiffeners. The effect of this orthotropic behavior on the yielding of circular plates is discussed by Save and Massonnet (Ref. 81) as well as by Markowitz and Hu (Ref. 82).

Crose and Ang (Ref. 83) describe an iterative solution scheme which first solves the elastic case and then increments the loading upward to allow a slow expansion of the yielded volume after it forms. The results of a test on a clamped plate are compared favorably with a theoretical solution.

11.13 Ultimate Strength

Plates of brittle material fracture when the actual maximum tensile stress reaches the ultimate tensile strength of the material. A flat-plate modulus of rupture, analogous to the modulus of rupture of a beam, may be determined by calculating the (fictitious) maximum stress corresponding to the breaking load, using for this purpose the appropriate formula for elastic stress. This flat-plate modulus of rupture is usually greater than the modulus of rupture determined by testing a beam of rectangular section.

Plates of ductile material fail by excessive plastic deflection, as do beams of similar material. For a number of cases the load required to produce collapse has been determined analytically, and the results for some of the simple loadings are summarized as follows.

1. *Circular plate; uniform load, edges simply supported*

$$W_u = \sigma_y(\tfrac{3}{2}\pi t^2) \quad \text{(Ref. 43)}$$

2. *Circular plate; uniform load, fixed edges*

$$W_u = \sigma_y(2.814\pi t^2) \quad \text{(Ref. 43)}$$

(For collapse loads on partially loaded orthotropic annular plates see Refs. 81 and 82.)

3. *Rectangular plate, length a, width b; uniform load, edges supported*

$$W_u = \beta\sigma_y t^2$$

where β depends on the ratio of b to a and has the following values (Ref. 44):

b/a	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2
β	5.48	5.50	5.58	5.64	5.89	6.15	6.70	7.68	9.69

4. *Plate of any shape and size, any type of edge support, concentrated load at any point*

$$W_u = \sigma_y(\tfrac{1}{2}\pi t^2) \quad \text{(Ref. 45)}$$

In each of the above cases W_u denotes the total load required to collapse the plate, t the thickness of the plate, and σ_y the yield point of the material. Accurate prediction of W_u is hardly to be expected; the theoretical error in some of the formulas may range up to 30%, and few experimental data seem to be available.

11.14 Tables

TABLE 11.1 Numerical values for functions used in Table 11.2

Numerical values for the plate coefficients F, C, L, and G for values of b/r, b/a, r_o/a, and r_o/r, respectively, from 0.05 to 1.0. Poisson's ratio is 0.30. The table headings are given for G_1 to G_{19} for the various values of r_o/r.† Also listed in the last five lines are values for the most used denominators for the ratios b/a

r_o/r	1.000	0.900	0.800	0.750	0.700	$\frac{2}{3}$	0.600	0.500
G_1	0.000	0.098580346	0.19478465	0.2423283	0.2897871	0.3215349	0.3858887	0.487773
G_2	0.000	0.004828991	0.01859406	0.0284644	0.0401146	0.0487855	0.0680514	0.100857
G_3	0.000	0.000158070	0.00119108	0.0022506	0.0037530	0.0050194	0.0082084	0.014554
G_4	1.000	0.973888889	0.95750000	0.9541667	0.9550000	0.9583333	0.9733333	1.025000
G_5	0.000	0.095000000	0.18000000	0.2187500	0.2550000	0.2777778	0.3200000	0.375000
G_6	0.000	0.004662232	0.01725742	0.0258495	0.0355862	0.0425624	0.0572477	0.079537
G_7	0.000	0.096055556	0.20475000	0.2654167	0.3315000	0.3791667	0.4853333	0.682500
G_8	1.000	0.933500000	0.87400000	0.8468750	0.8215000	0.8055556	0.7760000	0.737500
G_9	0.000	0.091560902	0.16643465	0.1976669	0.2247621	0.2405164	0.2664220	0.290898
G_{11}	0.000	0.000003996	0.00006104	0.0001453	0.0002935	0.0004391	0.0008752	0.001999
G_{12}	0.000	0.000000805	0.00001240	0.0000297	0.0000603	0.0000905	0.0001820	0.000422
G_{13}	0.000	0.000000270	0.00000418	0.0000100	0.0000205	0.0000308	0.0000623	0.000146
G_{14}	0.000	0.000158246	0.00119703	0.0022693	0.0038011	0.0051026	0.0084257	0.015272
G_{15}	0.000	0.000039985	0.00030618	0.0005844	0.0009861	0.0013307	0.0022227	0.004111
G_{16}	0.000	0.000016107	0.00012431	0.0002383	0.0004039	0.0005468	0.0009196	0.001721
G_{17}	0.000	0.004718219	0.01775614	0.0268759	0.0374539	0.0452137	0.0621534	0.090166
G_{18}	0.000	0.001596148	0.00610470	0.0093209	0.0131094	0.0159275	0.0221962	0.032948
G_{19}	0.000	0.000805106	0.00310827	0.0047694	0.0067426	0.0082212	0.0115422	0.017341
$C_1C_6 - C_3C_4$	0.000	0.000305662	0.00222102	0.0041166	0.0067283	0.0088751	0.0141017	0.023878
$C_1C_9 - C_3C_7$	0.000	0.009010922	0.03217504	0.0473029	0.0638890	0.0754312	0.0988254	0.131959
$C_2C_6 - C_3C_5$	0.000	0.000007497	0.00010649	0.0002435	0.0004705	0.0006822	0.0012691	0.002564
$C_2C_9 - C_3C_8$	0.000	0.000294588	0.00205369	0.0037205	0.0059332	0.0076903	0.0117606	0.018605
$C_4C_9 - C_6C_7$	0.000	0.088722311	0.15582772	0.1817463	0.2028510	0.2143566	0.2315332	0.243886

TABLE 11.1 Numerical values for functions used in Table 11.2 (*Continued*)

r_o/r	0.400	$\frac{1}{3}$	0.300	0.250	0.200	0.125	0.100	0.050
G_1	0.605736	0.704699	0.765608	0.881523	1.049227	1.547080	1.882168	3.588611
G_2	0.136697	0.161188	0.173321	0.191053	0.207811	0.229848	0.235987	0.245630
G_3	0.022290	0.027649	0.030175	0.033465	0.035691	0.035236	0.033390	0.025072
G_4	1.135000	1.266667	1.361667	1.562500	1.880000	2.881250	3.565000	7.032500
G_5	0.420000	0.444444	0.455000	0.468750	0.480000	0.492187	0.495000	0.498750
G_6	0.099258	0.109028	0.112346	0.114693	0.112944	0.099203	0.090379	0.062425
G_7	0.955500	1.213333	1.380167	1.706250	2.184000	3.583125	4.504500	9.077250
G_8	0.706000	0.688889	0.681500	0.671875	0.664000	0.655469	0.653500	0.650875
G_9	0.297036	0.289885	0.282550	0.266288	0.242827	0.190488	0.166993	0.106089
G_{11}	0.003833	0.005499	0.006463	0.008057	0.009792	0.012489	0.013350	0.014843
G_{12}	0.000827	0.001208	0.001435	0.001822	0.002266	0.003027	0.003302	0.003872
G_{13}	0.000289	0.000427	0.000510	0.000654	0.000822	0.001121	0.001233	0.001474
G_{14}	0.024248	0.031211	0.034904	0.040595	0.046306	0.054362	0.056737	0.060627
G_{15}	0.006691	0.008790	0.009945	0.011798	0.013777	0.016917	0.017991	0.020139
G_{16}	0.002840	0.003770	0.004290	0.005138	0.006065	0.007589	0.008130	0.009252
G_{17}	0.119723	0.139340	0.148888	0.162637	0.175397	0.191795	0.196271	0.203191
G_{18}	0.044939	0.053402	0.057723	0.064263	0.070816	0.080511	0.083666	0.089788
G_{19}	0.023971	0.028769	0.031261	0.035098	0.039031	0.045057	0.047086	0.051154
$C_1C_6 - C_3C_4$	0.034825	0.041810	0.044925	0.048816	0.051405	0.051951	0.051073	0.047702
$C_1C_9 - C_3C_7$	0.158627	0.170734	0.174676	0.177640	0.176832	0.168444	0.163902	0.153133
$C_2C_6 - C_3C_5$	0.004207	0.005285	0.005742	0.006226	0.006339	0.005459	0.004800	0.002829
$C_2C_9 - C_3C_8$	0.024867	0.027679	0.028408	0.028391	0.026763	0.020687	0.017588	0.009740
$C_4C_9 - C_6C_7$	0.242294	0.234900	0.229682	0.220381	0.209845	0.193385	0.188217	0.179431

† To obtain a value of either C_i, L_i, or F_i for a corresponding value of either b/a, r_o/a, or b/r, respectively, use the tabulated value of G_i for the corresponding value of r_o/r.

TABLE 11.2 Formulas for flat circular plates of constant thickness

NOTATION: W = total applied load (force); w = unit line load (force per unit of circumferential length); q = load per unit area; M_o = unit applied line moment loading (force-length per unit of circumferential length); θ_o = externally applied change in radial slope (radians); y_o = externally applied radial step in the vertical deflection (length); y = vertical deflection of plate (length); θ = radial slope of plate; M_r = unit radial bending moment; M_t = unit tangential bending moment; Q = unit shear force (force per unit of circumferential length); E = modulus of elasticity (force per unit area); ν = Poisson's ratio; γ = temperature coefficient of expansion (unit strain per degree); a = outer radius; b = inner radius for annular plate; t = plate thickness; r = radial location of quantity being evaluated; r_o = radial location of unit line loading or start of a distributed load. F_1 to F_9 and G_1 to G_{19} are the several functions of the radial location r. C_1 to C_9 are plate constants dependent upon the ratio a/b. L_1 to L_{19} are loading constants dependent upon the ratio a/r_o. When used as subscripts, r and t refer to radial and tangential directions, respectively. When used as subscripts, a, b, and o refer to an evaluation of the quantity subscripted at the outer edge, inner edge, and the position of the loading or start of distributed loading, respectively. When used as a subscript, c refers to an evaluation of the quantity subscripted at the center of the plate.

Positive signs are associated with the several quantities in the following manner: Deflections y and y_o are positive upward; slopes θ and θ_o are positive when the deflection y increases positively as r increases; moments M_r, M_t, and M_o are positive when creating compression on the top surface; and the shear force Q is positive when acting upward on the inner edge of a given annular section

Bending stresses can be found from the moments M_r and M_t by the expression $\sigma = 6M/t^2$. The plate constant $D = Et^3/12(1-\nu^2)$. The singularity function brackets $\langle\ \rangle$ indicate that the expression contained within the brackets must be equated to zero unless $r > r_o$, after which they are treated as any other brackets. Note that Q_b, Q_a, M_{rb}, and M_{ra} are reactions, not loads. They exist only when necessary edge restraints are provided.

General Plate Functions and Constants for Solid and Annular Circular Plates

$$F_1 = \frac{1+\nu a}{2}\frac{b}{r}\ln\frac{r}{b} + \frac{1-\nu}{4}\left(\frac{r}{b}-\frac{b}{r}\right)$$

$$F_2 = \frac{1}{4}\left[1-\left(\frac{b}{r}\right)^2\left(1+2\ln\frac{r}{b}\right)\right]$$

$$F_3 = \frac{b}{4r}\left\{\left[\left(\frac{b}{r}\right)^2+1\right]\ln\frac{r}{b}+\left(\frac{b}{r}\right)^2-1\right\}$$

$$F_4 = \frac{1}{2}\left[(1+\nu)\frac{b}{r}+(1-\nu)\frac{r}{b}\right]$$

$$F_5 = \frac{1}{2}\left[1-\left(\frac{b}{r}\right)^2\right]$$

$$F_6 = \frac{b}{4r}\left[\left(\frac{b}{r}\right)^2-1+2\ln\frac{r}{b}\right]$$

$$F_7 = \frac{1}{2}(1-\nu^2)\left(\frac{r}{b}-\frac{b}{r}\right)$$

$$F_8 = \frac{1}{2}\left[1+\nu+(1-\nu)\left(\frac{b}{r}\right)^2\right]$$

$$F_9 = \frac{b}{r}\left\{\frac{1+\nu}{2}\ln\frac{r}{b}+\frac{1-\nu}{4}\left[1-\left(\frac{b}{r}\right)^2\right]\right\}$$

$$C_1 = \frac{1+\nu}{2}\frac{b}{a}\ln\frac{a}{b} + \frac{1-\nu}{4}\left(\frac{a}{b}-\frac{b}{a}\right)$$

$$C_2 = \frac{1}{4}\left[1-\left(\frac{b}{a}\right)^2\left(1+2\ln\frac{a}{b}\right)\right]$$

$$C_3 = \frac{b}{4a}\left\{\left[\left(\frac{b}{a}\right)^2+1\right]\ln\frac{a}{b}+\left(\frac{b}{a}\right)^2-1\right\}$$

$$C_4 = \frac{1}{2}\left[(1+\nu)\frac{b}{a}+(1-\nu)\frac{a}{b}\right]$$

$$C_5 = \frac{1}{2}\left[1-\left(\frac{b}{a}\right)^2\right]$$

$$C_6 = \frac{b}{4a}\left[\left(\frac{b}{a}\right)^2-1+2\ln\frac{a}{b}\right]$$

$$C_7 = \frac{1}{2}(1-\nu^2)\left(\frac{a}{b}-\frac{b}{a}\right)$$

$$C_8 = \frac{1}{2}\left[1+\nu+(1-\nu)\left(\frac{b}{a}\right)^2\right]$$

$$C_9 = \frac{b}{a}\left\{\frac{1+\nu}{2}\ln\frac{a}{b}+\frac{1-\nu}{4}\left[1-\left(\frac{b}{a}\right)^2\right]\right\}$$

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

$$L_1 = \frac{1+\nu}{2}\frac{r_o}{a}\ln\frac{r_0}{a} + \frac{1-\nu}{4}\left(\frac{a}{r_o} - \frac{r_o}{a}\right)$$

$$L_2 = \frac{1}{4}\left[1 - \left(\frac{r_o}{a}\right)^2\left(1 + 2\ln\frac{a}{r_o}\right)\right]$$

$$L_3 = \frac{r_o}{4a}\left\{\left[\left(\frac{r_o}{a}\right)^2 + 1\right]\ln\frac{a}{r_o} + \left(\frac{r_o}{a}\right)^2 - 1\right\}$$

$$L_4 = \frac{1}{2}\left[(1+\nu)\frac{r_o}{a} + (1-\nu)\frac{a}{r_o}\right]$$

$$L_5 = \frac{1}{2}\left[1 - \left(\frac{r_o}{a}\right)^2\right]$$

$$L_6 = \frac{r_o}{4a}\left[\left(\frac{r_o}{a}\right)^2 - 1 + 2\ln\frac{a}{r_o}\right]$$

$$L_7 = \frac{1}{2}(1-\nu^2)\left(\frac{a}{r_o} - \frac{r_o}{a}\right)$$

$$L_8 = \frac{1}{2}\left[1 + \nu + (1-\nu)\left(\frac{r_o}{a}\right)^2\right]$$

$$L_9 = \frac{r_o}{a}\left\{\frac{1+\nu}{2}\ln\frac{a}{r_o} + \frac{1-\nu}{4}\left[1 - \left(\frac{r_o}{a}\right)^2\right]\right\}$$

$$L_{11} = \frac{1}{64}\left\{1 + 4\left(\frac{r_o}{a}\right)^2 - 5\left(\frac{r_o}{a}\right)^4 - 4\left(\frac{r_o}{a}\right)^2\left[2 + \left(\frac{r_o}{a}\right)^2\right]\ln\frac{a}{r_o}\right\}$$

$$L_{12} = \frac{a}{14{,}400(a-r_o)}\left\{64 - 225\frac{r_o}{a} - 100\left(\frac{r_o}{a}\right)^3 + 261\left(\frac{r_o}{a}\right)^5 + 60\left(\frac{r_o}{a}\right)^3\left[3\left(\frac{r_o}{a}\right)^2 + 10\right]\ln\frac{a}{r_o}\right\}$$

$$L_{13} = \frac{a^2}{14{,}400(a-r_o)^2}\left\{25 - 128\frac{r_o}{a} + 225\left(\frac{r_o}{a}\right)^2 - 25\left(\frac{r_o}{a}\right)^4 - 97\left(\frac{r_o}{a}\right)^6 - 60\left(\frac{r_o}{a}\right)^4\left[5 + \left(\frac{r_o}{a}\right)^2\right]\ln\frac{a}{r_o}\right\}$$

$$L_{14} = \frac{1}{16}\left[1 - \left(\frac{r_o}{a}\right)^4 - 4\left(\frac{r_o}{a}\right)^2\ln\frac{a}{r_o}\right]$$

$$L_{15} = \frac{a}{720(a-r_o)}\left[16 - 45\frac{r_o}{a} + 9\left(\frac{r_o}{a}\right)^5 + 20\left(\frac{r_o}{a}\right)^3\left(1 + 3\ln\frac{a}{r_o}\right)\right]$$

$$L_{16} = \frac{a^2}{1440(a-r_o)^2}\left[15 - 64\frac{r_o}{a} + 90\left(\frac{r_o}{a}\right)^2 - 6\left(\frac{r_o}{a}\right)^6 - 5\left(\frac{r_o}{a}\right)^4\left(7 + 12\ln\frac{a}{r_o}\right)\right]$$

$$L_{17} = \frac{1}{4}\left\{1 - \frac{1-\nu}{4}\left[1 - \left(\frac{r_o}{a}\right)^4\right] - \left(\frac{r_o}{a}\right)^2\left[1 + (1+\nu)\ln\frac{a}{r_o}\right]\right\}$$

$$G_1 = \left[\frac{1+\nu}{2}\frac{r_o}{r}\ln\frac{r}{r_0} + \frac{1-\nu}{4}\left(\frac{r}{r_o} - \frac{r_o}{r}\right)\right]\langle r - r_o\rangle^0$$

$$G_2 = \frac{1}{4}\left[1 - \left(\frac{r_o}{r}\right)^2\left(1 + 2\ln\frac{r}{r_o}\right)\right]\langle r - r_o\rangle^0$$

$$G_3 = \frac{r_o}{4r}\left\{\left[\left(\frac{r_o}{r}\right)^2 + 1\right]\ln\frac{r}{r_o} + \left(\frac{r_o}{r}\right)^2 - 1\right\}\langle r - r_o\rangle^0$$

$$G_4 = \frac{1}{2}\left[(1+\nu)\frac{r_o}{r} + (1-\nu)\frac{r}{r_o}\right]\langle r - r_o\rangle^0$$

$$G_5 = \frac{1}{2}\left[1 - \left(\frac{r_o}{r}\right)^2\right]\langle r - r_o\rangle^0$$

$$G_6 = \frac{r_o}{4r}\left[\left(\frac{r_o}{r}\right)^2 - 1 + 2\ln\frac{r}{r_o}\right]\langle r - r_o\rangle^0$$

$$G_7 = \frac{1}{2}(1-\nu^2)\left(\frac{r}{r_o} - \frac{r_o}{r}\right)\langle r - r_o\rangle^0$$

$$G_8 = \frac{1}{2}\left[1 + \nu + (1-\nu)\left(\frac{r_o}{r}\right)^2\right]\langle r - r_o\rangle^0$$

$$G_9 = \frac{r_o}{r}\left\{\frac{1+\nu}{2}\ln\frac{r}{r_o} + \frac{1-\nu}{4}\left[1 - \left(\frac{r_o}{r}\right)^2\right]\right\}\langle r - r_o\rangle^0$$

$$G_{11} = \frac{1}{64}\left\{1 + 4\left(\frac{r_o}{r}\right)^2 - 5\left(\frac{r_o}{r}\right)^4 - 4\left(\frac{r_o}{r}\right)^2\left[2 + \left(\frac{r_o}{r}\right)^2\right]\ln\frac{r}{r_o}\right\}\langle r - r_o\rangle^0$$

$$G_{12} = \frac{r\langle r - r_o\rangle^0}{14{,}400(r-r_o)}\left\{64 - 225\frac{r_o}{r} - 100\left(\frac{r_o}{r}\right)^3 + 261\left(\frac{r_o}{r}\right)^5 + 60\left(\frac{r_o}{r}\right)^3\left[3\left(\frac{r_o}{r}\right)^2 + 10\right]\ln\frac{r}{r_o}\right\}$$

$$G_{13} = \frac{r^2\langle r - r_o\rangle^0}{14{,}400(r-r_o)^2}\left\{25 - 128\frac{r_o}{r} + 225\left(\frac{r_o}{r}\right)^2 - 25\left(\frac{r_o}{r}\right)^4 - 97\left(\frac{r_o}{r}\right)^6 - 60\left(\frac{r_o}{r}\right)^4\left[5 + \left(\frac{r_o}{r}\right)^2\right]\ln\frac{r}{r_o}\right\}$$

$$G_{14} = \frac{1}{16}\left[1 - \left(\frac{r_o}{r}\right)^4 - 4\left(\frac{r_o}{r}\right)^2\ln\frac{r}{r_o}\right]\langle r - r_o\rangle^0$$

$$G_{15} = \frac{r\langle r - r_o\rangle^0}{720(r-r_o)}\left[16 - 45\frac{r_o}{r} + 9\left(\frac{r_o}{r}\right)^5 + 20\left(\frac{r_o}{r}\right)^3\left(1 + 3\ln\frac{r}{r_o}\right)\right]$$

$$G_{16} = \frac{r^2\langle r - r_o\rangle^0}{1440(r-r_o)^2}\left[15 - 64\frac{r_o}{a} + 90\left(\frac{r_o}{r}\right)^2 - 6\left(\frac{r_o}{r}\right)^6 - 5\left(\frac{r_o}{r}\right)^4\left(7 + 12\ln\frac{r}{r_o}\right)\right]$$

$$G_{17} = \frac{1}{4}\left\{1 - \frac{1-\nu}{4}\left[1 - \left(\frac{r_o}{r}\right)^4\right] - \left(\frac{r_o}{r}\right)^2\left[1 + (1+\nu)\ln\frac{r}{r_o}\right]\right\}\langle r - r_o\rangle^0$$

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

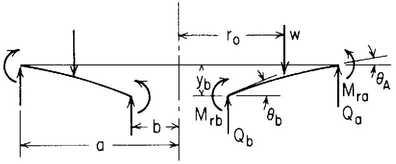
$$L_{18} = \frac{a}{720(a-r_o)}\left\{\left[20\left(\frac{r_o}{a}\right)^3+16\right](4+\nu) - 45\frac{r_o}{a}(3+\nu) - 9\left(\frac{r_o}{a}\right)^5(1-\nu) + 60\left(\frac{r_o}{a}\right)^3(1+\nu)\ln\frac{a}{r_o}\right\}$$

$$G_{18} = \frac{r\langle r-r_o\rangle^0}{720(r-r_o)}\left\{\left[20\left(\frac{r_o}{r}\right)^3+16\right](4+\nu) - 45\frac{r_o}{r}(3+\nu) - 9\left(\frac{r_o}{r}\right)^5(1-\nu) + 60\left(\frac{r_o}{r}\right)^3(1+\nu)\ln\frac{r}{r_o}\right\}$$

$$L_{19} = \frac{a^2}{1440(a-r_o)^2}\left[15(5+\nu) - 64\frac{r_o}{a}(4+\nu) + 90\left(\frac{r_o}{a}\right)^2(3+\nu) - 5\left(\frac{r_o}{a}\right)^4(19+7\nu) + 6\left(\frac{r_o}{a}\right)^6(1-\nu) - 60\left(\frac{r_o}{a}\right)^4(1+\nu)\ln\frac{a}{r_o}\right]$$

$$G_{19} = \frac{r^2\langle r-r_o\rangle^0}{1440(r-r_o)^2}\left[15(5+\nu) - 64\frac{r_o}{r}(4+\nu) + 90\left(\frac{r_o}{r}\right)^2(3+\nu) - 5\left(\frac{r_o}{r}\right)^4(19+7\nu) + 6\left(\frac{r_o}{r}\right)^6(1-\nu) - 60\left(\frac{r_o}{r}\right)^4(1+\nu)\ln\frac{r}{r_o}\right]$$

Case 1. Annular plate with a uniform annular line load w at a radius r_o



General expressions for deformations, moments, and shears:

$$y = y_b + \theta_b r F_1 + M_{rb}\frac{r^2}{D}F_2 + Q_b\frac{r^3}{D}F_3 - w\frac{r^3}{D}G_3$$

$$\theta = \theta_b F_4 + M_{rb}\frac{r}{D}F_5 + Q_b\frac{r^2}{D}F_6 - w\frac{r^2}{D}G_6$$

$$M_r = \theta_b\frac{D}{r}F_7 + M_{rb}F_8 + Q_b r F_9 - wrG_9$$

$$M_t = \frac{\theta D(1-\nu^2)}{r} + \nu M_r$$

$$Q = Q_b\frac{b}{r} - w\frac{r_o}{r}\langle r-r_o\rangle^0$$

For the numerical data given below, $\nu = 0.3$

$$y = K_y\frac{wa^3}{D}, \quad \theta = K_\theta\frac{wa^2}{D}, \quad M = K_M wa, \quad Q = K_Q w$$

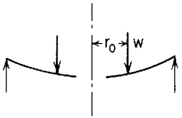
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>1. Outer edge simply supported, inner edge free

 </td>
<td>$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{-wa^3}{D}\left(\frac{C_1L_9}{C_7} - L_3\right)$
$\theta_b = \frac{wa^2}{DC_7}L_9$
$\theta_a = \frac{wa^2}{D}\left(\frac{C_4L_9}{C_7} - L_6\right)$
$Q_a = -w\frac{r_o}{a}$</td>
<td>$y_{\max} = y_b \quad M_{\max} = M_{tb}$
If $r_o = b$ (load at inner edge),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>−0.0364</td><td>−0.1266</td><td>−0.1934</td><td>−0.1927</td><td>−0.0938</td></tr>
<tr><td>K_{θ_b}</td><td>0.0371</td><td>0.2047</td><td>0.4262</td><td>0.6780</td><td>0.9532</td></tr>
<tr><td>K_{θ_a}</td><td>0.0418</td><td>0.1664</td><td>0.3573</td><td>0.6119</td><td>0.9237</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>0.3374</td><td>0.6210</td><td>0.7757</td><td>0.8814</td><td>0.9638</td></tr>
</table>
</td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

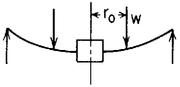
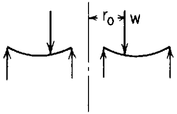
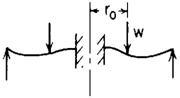
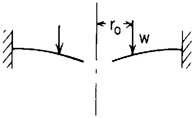
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>1b. Outer edge simply supported, inner edge guided

 </td>
<td>$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{-wa^3}{D}\left(\frac{C_2 L_9}{C_8} - L_3\right)$
$M_{rb} = \frac{wa}{C_8} L_9$
$\theta_a = \frac{wa^2}{D}\left(\frac{C_5 L_9}{C_8} - L_6\right)$
$Q_a = -w\frac{r_o}{a}$</td>
<td>$y_{\max} = y_b \quad M_{\max} = M_{rb}$
If $r_o = b$ (load at inner edge),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>−0.0269</td><td>−0.0417</td><td>−0.0252</td><td>−0.0072</td><td>−0.0003</td></tr>
<tr><td>K_{θ_a}</td><td>0.0361</td><td>0.0763</td><td>0.0684</td><td>0.0342</td><td>0.0047</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>0.2555</td><td>0.4146</td><td>0.3944</td><td>0.2736</td><td>0.0981</td></tr>
</table></td>
</tr>
<tr>
<td>1c. Outer edge simply supported, inner edge simply supported

 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad M_{ra} = 0$
$\theta_b = \frac{-wa^2}{D}\frac{C_3 L_9 - C_9 L_3}{C_1 C_9 - C_3 C_7}$
$Q_b = w\frac{C_1 L_9 - C_7 L_3}{C_1 C_9 - C_3 C_7}$
$\theta_a = \theta_b C_4 + Q_b \frac{a^2}{D} C_6 - \frac{wa^2}{D} L_6$
$Q_a = Q_b \frac{b}{a} - \frac{wr_o}{a}$</td>
<td><table>
<tr><th>b/a</th><th>0.1</th><th></th><th>0.5</th><th></th><th>0.7</th></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0102</td><td>−0.0113</td><td>−0.0023</td><td>−0.0017</td><td>−0.0005</td></tr>
<tr><td>K_{θ_a}</td><td>0.0278</td><td>0.0388</td><td>0.0120</td><td>0.0122</td><td>0.0055</td></tr>
<tr><td>K_{θ_b}</td><td>−0.0444</td><td>−0.0420</td><td>−0.0165</td><td>−0.0098</td><td>−0.0048</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.4043</td><td>−0.3819</td><td>−0.0301</td><td>−0.0178</td><td>−0.0063</td></tr>
<tr><td>$K_{M_{ro}}$</td><td>0.1629</td><td>0.1689</td><td>0.1161</td><td>0.0788</td><td>0.0662</td></tr>
<tr><td>K_{Q_b}</td><td>2.9405</td><td>2.4779</td><td>0.8114</td><td>0.3376</td><td>0.4145</td></tr>
</table></td>
</tr>
<tr>
<td>1d. Outer edge simply supported, inner edge fixed

 </td>
<td>$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$M_{rb} = -wa\frac{C_3 L_9 - C_9 L_3}{C_2 C_9 - C_3 C_8}$
$Q_b = w\frac{C_2 L_9 - C_8 L_3}{C_2 C_9 - C_3 C_8}$
$\theta_a = M_{rb}\frac{a}{D} C_5 + Q_b \frac{a^2}{D} C_6 - \frac{wa^2}{D} L_6$
$Q_a = Q_b \frac{b}{a} - \frac{wr_o}{a}$</td>
<td><table>
<tr><th>b/a</th><th>0.1</th><th></th><th>0.5</th><th></th><th>0.7</th></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0066</td><td>−0.0082</td><td>−0.0010</td><td>−0.0010</td><td>−0.0003</td></tr>
<tr><td>K_{θ_a}</td><td>0.0194</td><td>0.0308</td><td>0.0056</td><td>0.0084</td><td>0.0034</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>−0.4141</td><td>−0.3911</td><td>−0.1172</td><td>−0.0692</td><td>−0.0519</td></tr>
<tr><td>K_{Q_b}</td><td>3.3624</td><td>2.8764</td><td>1.0696</td><td>0.4901</td><td>0.5972</td></tr>
</table></td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

1e. Outer edge fixed, inner edge free



$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$

$$y_b = \frac{-wa^3}{D}\left(\frac{C_1 L_6}{C_4} - L_3\right)$$

$$\theta_b = \frac{wa^2}{DC_4} L_6$$

$$M_{ra} = -wa\left(L_9 - \frac{C_7 L_6}{C_4}\right)$$

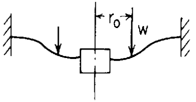
$$Q_a = \frac{-wr_o}{a}$$

If $r_o = b$ (load at inner edge),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_b}	−0.0143	−0.0330	−0.0233	−0.0071	−0.0003
K_{θ_b}	0.0254	0.0825	0.0776	0.0373	0.0048
$K_{M_{ra}}$	−0.0528	−0.1687	−0.2379	−0.2124	−0.0911
$K_{M_{tb}}$	0.2307	0.2503	0.1412	0.0484	0.0048

(*Note:* $|M_{ra}| > |M_{tb}|$ if $b/a > 0.385$)

1f. Outer edge fixed, inner edge guided



$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$

$$y_b = \frac{-wa^3}{D}\left(\frac{C_2 L_6}{C_5} - L_3\right)$$

$$M_{rb} = \frac{wa}{C_5} L_6$$

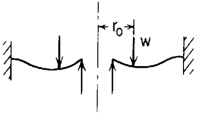
$$M_{ra} = -wa\left(L_9 - \frac{C_8 L_6}{C_5}\right)$$

$$Q_a = \frac{-wr_o}{a}$$

If $r_o = b$ (load at inner edge),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_b}	−0.0097	−0.0126	−0.0068	−0.0019	−0.0001
$K_{M_{rb}}$	0.1826	0.2469	0.2121	0.1396	0.0491
$K_{M_{ra}}$	−0.0477	−0.1143	−0.1345	−0.1101	−0.0458

1g. Outer edge fixed, inner edge simply supported



$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad \theta_a = 0$

$$\theta_b = \frac{-wa^2}{D} \frac{C_3 L_6 - C_6 L_3}{C_1 C_6 - C_3 C_4}$$

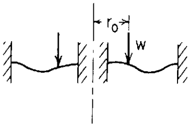
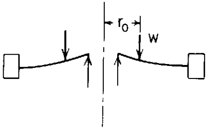
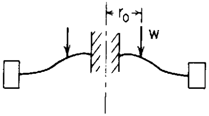
$$Q_b = w \frac{C_1 L_6 - C_4 L_3}{C_1 C_6 - C_3 C_4}$$

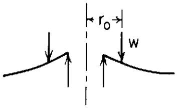
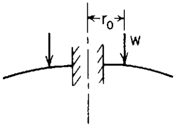
$$M_{ra} = \theta_b \frac{D}{a} C_7 + Q_b a C_9 - waL_9$$

$$Q_a = Q_b \frac{b}{a} - \frac{wr_o}{a}$$

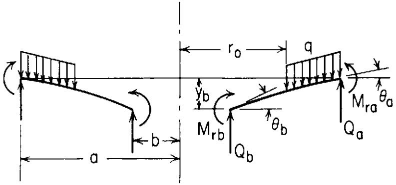
b/a	0.1		0.5		0.7
r_o/a	0.5	0.7	0.7	0.9	0.9
$K_{y_{\max}}$	−0.0053	−0.0041	−0.0012	−0.0004	−0.0002
K_{θ_b}	−0.0262	−0.0166	−0.0092	−0.0023	−0.0018
$K_{M_{tb}}$	−0.2388	−0.1513	−0.0167	−0.0042	−0.0023
$K_{M_{ro}}$	0.1179	0.0766	0.0820	0.0208	0.0286
$K_{M_{ra}}$	−0.0893	−0.1244	−0.0664	−0.0674	−0.0521
K_{Q_b}	1.9152	1.0495	0.5658	0.0885	0.1784

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>1h. Outer edge fixed, inner edge fixed

 </td>
<td>$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad \theta_a = 0$
$M_{rb} = -wa \dfrac{C_3 L_6 - C_6 L_3}{C_2 C_6 - C_3 C_5}$
$Q_b = w \dfrac{C_2 L_6 - C_5 L_3}{C_2 C_6 - C_3 C_5}$
$M_{ra} = M_{rb} C_8 + Q_b a C_9 - waL_9$
$Q_a = Q_b \dfrac{b}{a} - \dfrac{wr_o}{a}$</td>
<td>
<table>
<tr><th>b/a</th><th>0.1</th><th></th><th>0.5</th><th></th><th>0.7</th></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0038</td><td>−0.0033</td><td>−0.0006</td><td>−0.0003</td><td>−0.0001</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>−0.2792</td><td>−0.1769</td><td>−0.0856</td><td>−0.0216</td><td>−0.0252</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.0710</td><td>−0.1128</td><td>−0.0404</td><td>−0.0608</td><td>−0.0422</td></tr>
<tr><td>$K_{M_{ro}}$</td><td>0.1071</td><td>0.0795</td><td>0.0586</td><td>0.0240</td><td>0.0290</td></tr>
<tr><td>K_{Q_b}</td><td>2.4094</td><td>1.3625</td><td>0.8509</td><td>0.1603</td><td>0.3118</td></tr>
</table>
</td>
</tr>
<tr>
<td>1i. Outer edge guided, inner edge simply supported

 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad \theta_a = 0, \quad Q_a = 0$
$\theta_b = \dfrac{-wa^2}{DC_4} \left(\dfrac{r_o C_6}{b} - L_6 \right)$
$Q_b = \dfrac{wr_o}{b}$
$y_a = \dfrac{-wa^3}{D} \left[\dfrac{C_1}{C_4} \left(\dfrac{r_o C_6}{b} - L_6 \right) - \dfrac{r_o C_3}{b} + L_3 \right]$
$M_{ra} = wa \left[\dfrac{C_7}{C_4} \left(L_6 - \dfrac{r_o C_6}{b} \right) + \dfrac{r_o C_9}{b} - L_9 \right]$</td>
<td>If $r_o = a$ (load at outer edge),
$y_{\max} = y_a = \dfrac{-wa^4}{bD} \left(\dfrac{C_1 C_6}{C_4} - C_3 \right)$
$M_{\max} = M_{ra} = \dfrac{wa^2}{b} \left(C_9 - \dfrac{C_6 C_7}{C_4} \right) \quad \text{if } \dfrac{b}{a} > 0.385$
$M_{\max} = M_{tb} = \dfrac{-wa^3}{b^2} (1 - \nu^2) \dfrac{C_6}{C_4} \quad \text{if } \dfrac{b}{a} < 0.385$
(For numerical values see case 1e after computing the loading at the inner edge)</td>
</tr>
<tr>
<td>1j. Outer edge guided, inner edge fixed

 </td>
<td>$y_b = 0, \quad \theta_b = 0, \quad \theta_a = 0, \quad Q_a = 0$
$M_{rb} = \dfrac{-wa}{C_5} \left(\dfrac{r_o C_6}{b} - L_6 \right)$
$Q_b = \dfrac{wr_o}{b}$
$y_a = \dfrac{-wa^3}{D} \left[\dfrac{C_2}{C_5} \left(\dfrac{r_o C_6}{b} - L_6 \right) - \dfrac{r_o C_3}{b} + L_3 \right]$
$M_{ra} = wa \left[\dfrac{C_8}{C_5} \left(L_6 - \dfrac{r_o C_6}{b} \right) + \dfrac{r_o C_9}{b} - L_9 \right]$</td>
<td>If $r_o = a$ (load at outer edge),
$y_{\max} = y_a = \dfrac{-wa^4}{bD} \left(\dfrac{C_2 C_6}{C_5} - C_3 \right)$
$M_{\max} = M_{rb} = \dfrac{-wa^2 C_6}{bC_5}$
(For numerical values see case 1f after computing the loading at the inner edge)</td>
</tr>
</table>

1k. Outer edge free, inner edge simply supported 	$y_b = 0, \quad M_{rb} = 0, \quad M_{ra} = 0, \quad Q_a = 0$ $\theta_b = \frac{-wa^2}{DC_7}\left(\frac{r_o C_9}{b} - L_9\right)$ $Q_b = \frac{wr_o}{b}$ $y_a = \frac{-wa^3}{D}\left[\frac{C_1}{C_7}\left(\frac{r_o C_9}{b} - L_9\right) - \frac{r_o C_3}{b} + L_3\right]$ $\theta_a = \frac{-wa^2}{D}\left[\frac{C_4}{C_7}\left(\frac{r_o C_9}{b} - L_9\right) - \frac{r_o C_6}{b} + L_6\right]$	If $r_o = a$ (load at outer edge), $y_{\max} = y_a = \frac{-wa^4}{bD}\left(\frac{C_1 C_9}{C_7} - C_3\right)$ $M_{\max} = M_{tb} = \frac{-wa^3}{b^2}(1 - \nu^2)\frac{C_9}{C_7}$ (For numerical values see case 1a after computing the loading at the inner edge)
1l. Outer edge free, inner edge fixed 	$y_b = 0, \quad \theta_b = 0, \quad M_{ra} = 0, \quad Q_a = 0$ $M_{rb} = \frac{-wa}{C_8}\left(\frac{r_o C_9}{b} - L_9\right)$ $Q_b = \frac{wr_o}{b}$ $y_a = \frac{-wa^3}{D}\left[\frac{C_2}{C_8}\left(\frac{r_o C_9}{b} - L_9\right) - \frac{r_o C_3}{b} + L_3\right]$ $\theta_a = \frac{-wa^2}{D}\left[\frac{C_5}{C_8}\left(\frac{r_o C_9}{b} - L_9\right) - \frac{r_o C_6}{b} + L_6\right]$	If $r_o = a$ (load at outer edge), $y_{\max} = y_a = \frac{-wa^4}{bD}\left(\frac{C_2 C_9}{C_8} - C_3\right)$ $M_{\max} = M_{rb} = \frac{-wa^2}{b}\frac{C_9}{C_8}$ (For numerical values see case 1b after computing the loading at the inner edge)

Case 2. Annular plate with a uniformly distributed pressure q over the portion from r_o to a



General expressions for deformations, moments, and shears:

$$y = y_b + \theta_b r F_1 + M_{rb}\frac{r^2}{D}F_2 + Q_b\frac{r^3}{D}F_3 - q\frac{r^4}{D}G_{11}$$

$$\theta = \theta_b F_4 + M_{rb}\frac{r}{D}F_5 + Q_b\frac{r^2}{D}F_6 - q\frac{r^3}{D}G_{14}$$

$$M_r = \theta_b\frac{D}{r}F_7 + M_{rb}F_8 + Q_b r F_9 - qr^2 G_{17}$$

$$M_t = \frac{\theta D(1 - \nu^2)}{r} + \nu M_r$$

$$Q = Q_b\frac{b}{r} - \frac{q}{2r}(r^2 - r_o^2)\langle r - r_o\rangle^0$$

For the numerical data given below, $\nu = 0.3$

$$y = K_y\frac{qa^4}{D}, \quad \theta = K_\theta\frac{qa^3}{D}, \quad M = K_M qa^2, \quad Q = K_Q qa$$

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

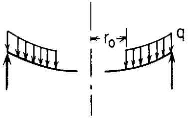
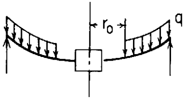
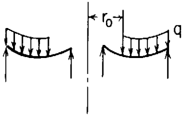
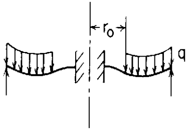
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>2a. Outer edge simply supported, inner edge free
 </td>
<td>$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{-qa^4}{D}\left(\frac{C_1 L_{17}}{C_7} - L_{11}\right)$
$\theta_b = \frac{qa^3}{DC_7} L_{17}$
$\theta_a = \frac{qa^3}{D}\left(\frac{C_4 L_{17}}{C_7} - L_{14}\right)$
$Q_a = \frac{-q}{2a}(a^2 - r_o^2)$</td>
<td>$y_{\max} = y_b, \quad M_{\max} = M_{tb}$
If $r_o = b$ (uniform load over entire plate),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>−0.0687</td><td>−0.0761</td><td>−0.0624</td><td>−0.0325</td><td>−0.0048</td></tr>
<tr><td>K_{θ_a}</td><td>0.0986</td><td>0.1120</td><td>0.1201</td><td>0.1041</td><td>0.0477</td></tr>
<tr><td>K_{θ_b}</td><td>0.0436</td><td>0.1079</td><td>0.1321</td><td>0.1130</td><td>0.0491</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>0.3965</td><td>0.3272</td><td>0.2404</td><td>0.1469</td><td>0.0497</td></tr>
</table></td>
</tr>
<tr>
<td>2b. Outer edge simply supported, inner edge guided
 </td>
<td>$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{-qa^4}{D}\left(\frac{C_2 L_{17}}{C_8} - L_{11}\right)$
$M_{rb} = \frac{qa^2}{C_8} L_{17}$
$\theta_a = \frac{qa^3}{D}\left(\frac{C_5 L_{17}}{C_8} - L_{14}\right)$
$Q_a = \frac{-q}{2a}(a^2 - r_o^2)$</td>
<td>$y_{\max} = y_b, \quad M_{\max} = M_{rb}$
If $r_o = b$ (uniform load over entire plate),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>−0.0575</td><td>−0.0314</td><td>−0.0103</td><td>−0.0015</td><td>−0.00002</td></tr>
<tr><td>K_{θ_a}</td><td>0.0919</td><td>0.0645</td><td>0.0306</td><td>0.0078</td><td>0.00032</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>0.3003</td><td>0.2185</td><td>0.1223</td><td>0.0456</td><td>0.00505</td></tr>
</table></td>
</tr>
<tr>
<td>2c. Outer edge simply supported, inner edge simply supported
 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad M_{ra} = 0$
$\theta_b = \frac{-qa^3}{D} \frac{C_3 L_{17} - C_9 L_{11}}{C_1 C_9 - C_3 C_7}$
$Q_b = qa \frac{C_1 L_{17} - C_7 L_{11}}{C_1 C_9 - C_3 C_7}$
$\theta_a = \theta_b C_4 + Q_b \frac{a^2}{D} C_6 - \frac{qa^3}{D} L_{14}$
$Q_a = Q_b \frac{b}{a} - \frac{q}{2a}(a^2 - r_o^2)$</td>
<td>If $r_o = b$ (uniform load over entire plate),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0060</td><td>−0.0029</td><td>−0.0008</td><td>−0.0001</td></tr>
<tr><td>K_{θ_b}</td><td>−0.0264</td><td>−0.0153</td><td>−0.0055</td><td>−0.0012</td></tr>
<tr><td>K_{θ_a}</td><td>0.0198</td><td>0.0119</td><td>0.0047</td><td>0.0011</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.2401</td><td>−0.0463</td><td>−0.0101</td><td>−0.0015</td></tr>
<tr><td>$K_{M_{r\max}}$</td><td>0.0708</td><td>0.0552</td><td>0.0300</td><td>0.0110</td></tr>
<tr><td>K_{Q_b}</td><td>1.8870</td><td>0.6015</td><td>0.3230</td><td>0.1684</td></tr>
</table></td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

2d. Outer edge simply supported, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad M_{ra} = 0$

$$M_{rb} = -qa^2 \frac{C_3 L_{17} - C_9 L_{11}}{C_2 C_9 - C_3 C_8}$$

$$Q_b = qa \frac{C_2 L_{17} - C_8 L_{11}}{C_2 C_9 - C_3 C_8}$$

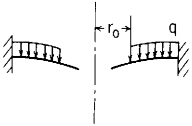
$$\theta_a = M_{rb} \frac{a}{D} C_5 + Q_b \frac{a^2}{D} C_6 - \frac{qa^3}{D} L_{14}$$

$$Q_a = Q_b \frac{b}{a} - \frac{q}{2a}(a^2 - r_o^2)$$

If $r_o = b$ (uniform load over entire plate),

b/a	0.1	0.3	0.5	0.7
$K_{y_{\max}}$	−0.0040	−0.0014	−0.0004	−0.00004
K_{θ_a}	0.0147	0.0070	0.0026	0.00056
$K_{M_{rb}}$	−0.2459	−0.0939	−0.0393	−0.01257
K_{Q_b}	2.1375	0.7533	0.4096	0.21259

2e. Outer edge fixed, inner edge free



$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$

$$y_b = \frac{-qa^4}{D}\left(\frac{C_1 L_{14}}{C_4} - L_{11}\right)$$

$$\theta_b = \frac{qa^3 L_{14}}{DC_4}$$

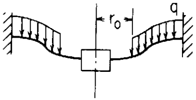
$$M_{ra} = -qa^2\left(L_{17} - \frac{C_7}{C_4} L_{14}\right)$$

$$Q_a = \frac{-q}{2a}(a^2 - r_o^2)$$

If $r_o = b$ (uniform load over entire plate),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_b}	−0.0166	−0.0132	−0.0053	−0.0009	−0.00001
K_{θ_b}	0.0159	0.0256	0.0149	0.0040	0.00016
$K_{M_{ra}}$	−0.1246	−0.1135	−0.0800	−0.0361	−0.00470
$K_{M_{tb}}$	0.1448	0.0778	0.0271	0.0052	0.00016

2f. Outer edge fixed, inner edge guided



$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$

$$y_b = \frac{-qa^4}{D}\left(\frac{C_2 L_{14}}{C_5} - L_{11}\right)$$

$$M_{rb} = \frac{qa^2 L_{14}}{C_5}$$

$$M_{ra} = -qa^2\left(L_{17} - \frac{C_8}{C_5} L_{14}\right)$$

$$Q_a = \frac{-q}{2a}(a^2 - r_o^2)$$

If $r_o = b$ (uniform load over entire plate),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_b}	−0.0137	−0.0068	−0.0021	−0.0003	
$K_{M_{rb}}$	0.1146	0.0767	0.0407	0.0149	0.00167
$K_{M_{ra}}$	−0.1214	−0.0966	−0.0601	−0.0252	−0.00316

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

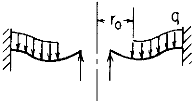
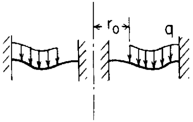
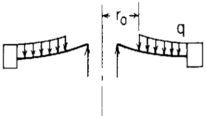
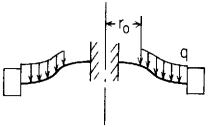
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>2g. Outer edge fixed, inner edge simply supported
 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad \theta_a = 0$
$\theta_B = \frac{-qa^3}{D} \frac{C_3 L_{14} - C_6 L_{11}}{C_1 C_6 - C_3 C_4}$
$Q_b = qa \frac{C_1 L_{14} - C_4 L_{11}}{C_1 C_6 - C_3 C_4}$
$M_{ra} = \theta_b \frac{D}{a} C_7 + Q_b a C_9 - qa^2 L_{17}$
$Q_a = Q_b \frac{b}{a} - \frac{q}{2a}(a^2 - r_o^2)$</td>
<td>If $r_o = b$ (uniform load over entire plate),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0025</td><td>−0.0012</td><td>−0.0003</td><td></td><td></td></tr>
<tr><td>K_{θ_b}</td><td>−0.0135</td><td>−0.0073</td><td>−0.0027</td><td>−0.0006</td><td></td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.1226</td><td>−0.0221</td><td>−0.0048</td><td>−0.0007</td><td></td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.0634</td><td>−0.0462</td><td>−0.0262</td><td>−0.0102</td><td>−0.0012</td></tr>
<tr><td>K_{Q_b}</td><td>1.1591</td><td>0.3989</td><td>0.2262</td><td>0.1221</td><td>0.0383</td></tr>
</table></td>
</tr>
<tr>
<td>2h. Outer edge fixed, inner edge fixed
 </td>
<td>$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad \theta_a = 0$
$M_{rb} = -qa^2 \frac{C_3 L_{14} - C_6 L_{11}}{C_2 C_6 - C_3 C_5}$
$Q_b = qa \frac{C_2 L_{14} - C_5 L_{11}}{C_2 C_6 - C_3 C_5}$
$M_{ra} = M_{rb} C_8 + Q_b a C_9 - qa^2 L_{17}$
$Q_a = Q_b \frac{b}{a} - \frac{q}{2a}(a^2 - r_o^2)$</td>
<td>If $r_o = b$ (uniform load over entire plate),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0018</td><td>−0.0006</td><td>−0.0002</td><td></td></tr>
<tr><td>$K_{M_{rb}}$</td><td>−0.1433</td><td>−0.0570</td><td>−0.0247</td><td>−0.0081</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.0540</td><td>−0.0347</td><td>−0.0187</td><td>−0.0070</td></tr>
<tr><td>K_{Q_b}</td><td>1.4127</td><td>0.5414</td><td>0.3084</td><td>0.1650</td></tr>
</table></td>
</tr>
<tr>
<td>2i. Outer edge guided, inner edge simply supported
 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad \theta_a = 0, \quad Q_a = 0$
$\theta_b = \frac{-qa^3}{DC_4}\left[\frac{C_6}{2ab}(a^2 - r_o^2) - L_{14}\right]$
$Q_b = \frac{q}{2b}(a^2 - r_o^2)$
$y_a = \theta_b a C_1 + Q_b \frac{a^3}{D} C_3 - \frac{qa^4}{D} L_{11}$
$M_{ra} = \theta_b \frac{D}{a} C_7 + Q_b a C_9 - qa^2 L_{17}$</td>
<td>If $r_o = b$ (uniform load over entire plate),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_a}</td><td>−0.0543</td><td>−0.0369</td><td>−0.0122</td><td>−0.0017</td><td>−0.00002</td></tr>
<tr><td>K_{θ_b}</td><td>−0.1096</td><td>−0.0995</td><td>−0.0433</td><td>−0.0096</td><td>−0.00034</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>0.1368</td><td>0.1423</td><td>0.0985</td><td>0.0412</td><td>0.00491</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.9971</td><td>−0.3018</td><td>−0.0788</td><td>−0.0125</td><td>−0.00035</td></tr>
</table></td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

2j. Outer edge guided, inner edge fixed



$$y_b = 0, \quad \theta_b = 0, \quad \theta_a = 0, \quad Q_a = 0$$

$$M_{rb} = \frac{-qa^2}{C_5}\left[\frac{C_6}{2ab}(a^2 - r_o^2) - L_{14}\right]$$

$$Q_b = \frac{q}{2b}(a^2 - r_o^2)$$

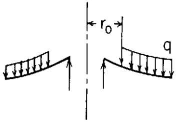
$$y_a = M_{rb}\frac{a^2}{D}C_2 + Q_b\frac{a^3}{D}C_3 - \frac{qa^4}{D}L_{11}$$

$$M_{ra} = M_{rb}C_8 + Q_b a C_9 - qa^2 L_{17}$$

If $r_o = b$ (uniform load over entire plate),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_a}	−0.0343	−0.0123	−0.0030	−0.0004	
$K_{M_{rb}}$	−0.7892	−0.2978	−0.1184	−0.0359	−0.00351
$K_{M_{ra}}$	0.1146	0.0767	0.0407	0.0149	0.00167

2k. Outer edge free, inner edge simply supported



$$y_b = 0, \quad M_{rb} = 0, \quad M_{ra} = 0, \quad Q_a = 0$$

$$\theta_b = \frac{-qa^3}{DC_7}\left[\frac{C_9}{2ab}(a^2 - r_o^2) - L_{17}\right]$$

$$Q_b = \frac{q}{2b}(a^2 - r_o^2)$$

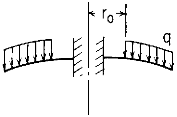
$$y_a = \theta_b a C_1 + Q_b\frac{a^3}{D}C_3 - \frac{qa^4}{D}L_{11}$$

$$\theta_a = \theta_b C_4 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{14}$$

If $r_o = b$ (uniform load over entire plate),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_a}	−0.1115	−0.1158	−0.0826	−0.0378	−0.0051
K_{θ_b}	−0.1400	−0.2026	−0.1876	−0.1340	−0.0515
K_{θ_a}	−0.1082	−0.1404	−0.1479	−0.1188	−0.0498
$K_{M_{tb}}$	−1.2734	−0.6146	−0.3414	−0.1742	−0.0521

2l. Outer edge free, inner edge fixed



$$y_b = 0, \quad \theta_b = 0, \quad M_{ra} = 0, \quad Q_a = 0$$

$$M_{rb} = \frac{-qa^2}{C_8}\left[\frac{C_9}{2ab}(a^2 - r_o^2) - L_{17}\right]$$

$$Q_b = \frac{q}{2b}(a^2 - r_o^2)$$

$$y_a = M_{rb}\frac{a^2}{D}C_2 + Q_b\frac{a^3}{D}C_3 - \frac{qa^4}{D}L_{11}$$

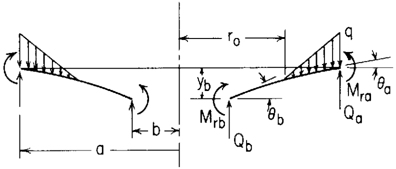
$$\theta_a = M_{rb}\frac{a}{D}C_5 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{14}$$

If $r_o = b$ (uniform load over entire plate),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_a}	−0.0757	−0.0318	−0.0086	−0.0011	
K_{θ_a}	−0.0868	−0.0512	−0.0207	−0.0046	−0.00017
$K_{M_{rb}}$	−0.9646	−0.4103	−0.1736	−0.0541	−0.00530

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

Case 3. Annular plate with a distributed pressure increasing linearly from zero at r_o to q at a



General expressions for deformations, moments, and shears:

$$y = y_b + \theta_b r F_1 + M_{rb}\frac{r^2}{D}F_2 + Q_b\frac{r^3}{D}F_3 - q\frac{r^4}{D}\frac{r-r_o}{a-r_o}G_{12}$$

$$\theta = \theta_b F_4 + M_{rb}\frac{r}{D}F_5 + Q_b\frac{r^2}{D}F_6 - q\frac{r^3}{D}\frac{r-r_o}{a-r_o}G_{15}$$

$$M_r = \theta_b\frac{D}{r}F_7 + M_{rb}F_8 + Q_b r F_9 - qr^2\frac{r-r_o}{a-r_o}G_{18}$$

$$M_t = \frac{\theta D(1-\nu^2)}{r} + \nu M_r$$

$$Q = Q_b\frac{b}{r} - \frac{q}{6r(a-r_o)}(2r^3 - 3r_o r^2 + r_o^3)\langle r-r_o\rangle^0$$

For the numerical data given below, $\nu = 0.3$

$$y = K_y\frac{qa^4}{D}, \quad \theta = K_\theta\frac{qa^3}{D}, \quad M = K_M qa^2, \quad Q = K_Q qa$$

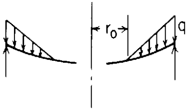
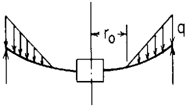
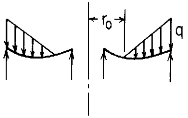
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>3a. Outer edge simply supported, inner edge free
 </td>
<td>$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{-qa^4}{D}\left(\frac{C_1 L_{18}}{C_7} - L_{12}\right)$
$\theta_b = \frac{qa^3}{DC_7}L_{18}$
$\theta_a = \frac{qa^3}{D}\left(\frac{C_4 L_{18}}{C_7} - L_{15}\right)$
$Q_a = \frac{-q}{6a}(2a^2 - r_o a - r_o^2)$</td>
<td>$y_{\max} = y_b \quad M_{\max} = M_{tb}$
If $r_o = b$ (linearly increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>−0.0317</td><td>−0.0306</td><td>−0.0231</td><td>−0.0114</td><td>−0.0016</td></tr>
<tr><td>K_{θ_a}</td><td>0.0482</td><td>0.0470</td><td>0.0454</td><td>0.0368</td><td>0.0161</td></tr>
<tr><td>K_{θ_b}</td><td>0.0186</td><td>0.0418</td><td>0.0483</td><td>0.0396</td><td>0.0166</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>0.1690</td><td>0.1269</td><td>0.0879</td><td>0.0514</td><td>0.0168</td></tr>
</table></td>
</tr>
<tr>
<td>3b. Outer edge simply supported, inner edge guided
 </td>
<td>$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{-qa^4}{D}\left(\frac{C_2 L_{18}}{C_8} - L_{12}\right)$
$M_{rb} = \frac{qa^2 L_{18}}{C_8}$
$\theta_a = \frac{qa^3}{D}\left(\frac{C_5 L_{18}}{C_8} - L_{15}\right)$
$Q_a = \frac{-q}{6a}(2a^2 - r_o a - r_o^2)$</td>
<td>$y_{\max} = y_b \quad M_{\max} = M_{rb}$
If $r_o = b$ (linearly increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>−0.0269</td><td>−0.0133</td><td>−0.0041</td><td>−0.0006</td><td>−0.00001</td></tr>
<tr><td>K_{θ_a}</td><td>0.0454</td><td>0.0286</td><td>0.0126</td><td>0.0031</td><td>0.00012</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>0.1280</td><td>0.0847</td><td>0.0447</td><td>0.0160</td><td>0.00171</td></tr>
</table></td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

3c. Outer edge simply supported, inner edge simply supported



$$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad M_{ra} = 0$$

$$\theta_b = \frac{-qa^3}{D}\frac{C_3L_{18} - C_9L_{12}}{C_1C_9 - C_3C_7}$$

$$Q_b = qa\frac{C_1L_{18} - C_7L_{12}}{C_1C_9 - C_3C_7}$$

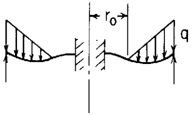
$$\theta_a = \theta_b C_4 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{15}$$

$$Q_a = Q_b\frac{b}{a} - \frac{q}{6a}(2a^2 - r_oa - r_o^2)$$

If $r_o = b$ (linearly increasing load from b to a),

b/a	0.1	0.3	0.5	0.7
$K_{y_{\max}}$	−0.0034	−0.0015	−0.0004	−0.0001
K_{θ_b}	−0.0137	−0.0077	−0.0027	−0.0006
K_{θ_a}	0.0119	0.0068	0.0026	0.0006
$K_{M_{tb}}$	−0.1245	−0.0232	−0.0049	−0.0007
$K_{M_{r\max}}$	0.0407	0.0296	0.0159	0.0057
K_{Q_b}	0.8700	0.2417	0.1196	0.0591

3d. Outer edge simply supported, inner edge fixed



$$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad M_{ra} = 0$$

$$M_{rb} = -qa^2\frac{C_3L_{18} - C_9L_{12}}{C_2C_9 - C_3C_8}$$

$$Q_b = qa\frac{C_2L_{18} - C_8L_{12}}{C_2C_9 - C_3C_8}$$

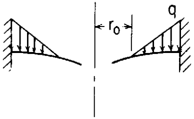
$$\theta_a = M_{rb}\frac{a}{D}C_5 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{15}$$

$$Q_a = Q_b\frac{b}{a} - \frac{q}{6a}(2a^2 - r_oa - r_o^2)$$

If $r_o = b$ (linearly increasing load from b to a),

b/a	0.1	0.3	0.5	0.7
$K_{y_{\max}}$	−0.0024	−0.0008	−0.0002	−0.00002
K_{θ_a}	0.0093	0.0044	0.0016	0.00034
$K_{M_{rb}}$	−0.1275	−0.0470	−0.0192	−0.00601
K_{Q_b}	0.9999	0.3178	0.1619	0.08029

3e. Outer edge fixed, inner edge free



$$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$$

$$y_b = \frac{-qa^4}{D}\left(\frac{C_1L_{15}}{C_4} - L_{12}\right)$$

$$\theta_b = \frac{qa^3L_{15}}{DC_4}$$

$$M_{ra} = -qa^2\left(L_{18} - \frac{C_7}{C_4}L_{15}\right)$$

$$Q_a = \frac{-q}{6a}(2a^2 - r_oa - r_o^2)$$

If $r_o = b$ (linearly increasing load from b to a),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_b}	−0.0062	−0.0042	−0.0015	−0.00024	
K_{θ_b}	0.0051	0.0073	0.0040	0.00103	0.00004
$K_{M_{ra}}$	−0.0609	−0.0476	−0.0302	−0.01277	−0.00159
$K_{M_{tb}}$	0.0459	0.0222	0.0073	0.00134	0.00004

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

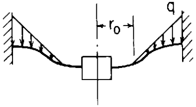
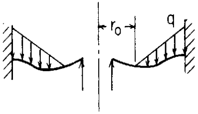
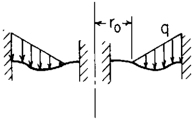
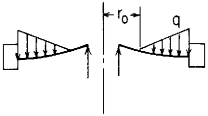
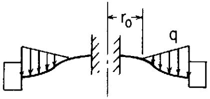
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>3f. Outer edge fixed, inner edge guided
 </td>
<td>$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$
$y_b = \frac{-qa^4}{D}\left(\frac{C_2 L_{15}}{C_5} - L_{12}\right)$
$M_{rb} = \frac{qa^2 L_{15}}{C_5}$
$M_{ra} = -qa^2\left(L_{18} - \frac{C_8}{C_5} L_{15}\right)$
$Q_a = \frac{-q}{6a}(2a^2 - r_o a - r_o^2)$</td>
<td>If $r_o = b$ (linearly increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>−0.0053</td><td>−0.0024</td><td>−0.0007</td><td>−0.0001</td><td></td></tr>
<tr><td>$K_{M_{rb}}$</td><td>0.0364</td><td>0.0219</td><td>0.0110</td><td>0.0039</td><td>0.00042</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.0599</td><td>−0.0428</td><td>−0.0249</td><td>−0.0099</td><td>−0.00120</td></tr>
</table></td>
</tr>
<tr>
<td>3g. Outer edge fixed, inner edge simply supported
 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad \theta_a = 0$
$\theta_b = \frac{-qa^3}{D}\frac{C_3 L_{15} - C_6 L_{12}}{C_1 C_6 - C_3 C_4}$
$Q_b = qa\frac{C_1 L_{15} - C_4 L_{12}}{C_1 C_6 - C_3 C_4}$
$M_{ra} = \theta_b \frac{D}{a} C_7 + Q_b a C_9 - qa^2 L_{18}$
$Q_a = Q_b \frac{b}{a} - \frac{q}{6a}(2a^2 - r_o a - r_o^2)$</td>
<td>If $r_o = b$ (linearly increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0013</td><td>−0.0005</td><td>−0.0002</td><td></td><td></td></tr>
<tr><td>K_{θ_b}</td><td>−0.0059</td><td>−0.0031</td><td>−0.0011</td><td>−0.0002</td><td></td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.0539</td><td>−0.0094</td><td>−0.0020</td><td>−0.0003</td><td></td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.0381</td><td>−0.0264</td><td>−0.0145</td><td>−0.0056</td><td>−0.0006</td></tr>
<tr><td>K_{Q_b}</td><td>0.4326</td><td>0.1260</td><td>0.0658</td><td>0.0339</td><td>0.0104</td></tr>
</table></td>
</tr>
<tr>
<td>3h. Outer edge fixed, inner edge fixed
 </td>
<td>$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad \theta_a = 0$
$M_{rb} = -qa^2 \frac{C_3 L_{15} - C_6 L_{12}}{C_2 C_6 - C_3 C_5}$
$Q_b = qa\frac{C_2 L_{15} - C_5 L_{12}}{C_2 C_6 - C_3 C_5}$
$M_{ra} = M_{rb} C_8 + Q_b a C_9 - qa^2 L_{18}$
$Q_a = Q_b \frac{b}{a} - \frac{q}{6a}(2a^2 - r_o a - r_o^2)$</td>
<td>If $r_o = b$ (linearly increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0009</td><td>−0.0003</td><td>−0.0001</td><td></td><td></td></tr>
<tr><td>$K_{M_{rb}}$</td><td>−0.0630</td><td>−0.0242</td><td>−0.0102</td><td>−0.0033</td><td>−0.00035</td></tr>
<tr><td>$K_{M_{rn}}$</td><td>−0.0340</td><td>−0.0215</td><td>−0.0114</td><td>−0.0043</td><td>−0.00048</td></tr>
<tr><td>K_{Q_b}</td><td>0.5440</td><td>0.1865</td><td>0.0999</td><td>0.0514</td><td>0.01575</td></tr>
</table></td>
</tr>
<tr>
<td>3i. Outer edge guided, inner edge simply supported
 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad \theta_a = 0, \quad Q_a = 0$
$\theta_b = \frac{-qa^3}{DC_4}\left[\frac{C_6}{6ab}(2a^2 - r_o a - r_o^2) - L_{15}\right]$
$Q_b = \frac{q}{6b}(2a^2 - r_o a - r_o^2)$
$y_a = \theta_b a C_1 + Q_b \frac{a^3}{D} C_3 - \frac{qa^4}{D} L_{12}$
$M_{ra} = \theta_b \frac{D}{a} C_7 + Q_b a C_9 - qa^2 L_{18}$</td>
<td>If $r_o = b$ (linearly increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_a}</td><td>−0.0389</td><td>−0.0254</td><td>−0.0082</td><td>−0.0011</td><td>−0.00001</td></tr>
<tr><td>K_{θ_b}</td><td>−0.0748</td><td>−0.0665</td><td>−0.0283</td><td>−0.0062</td><td>−0.00022</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>0.1054</td><td>0.1032</td><td>0.0689</td><td>0.0282</td><td>0.00330</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.6808</td><td>−0.2017</td><td>−0.0516</td><td>−0.0080</td><td>−0.00022</td></tr>
</table></td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

3j. Outer edge guided, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad \theta_a = 0, \quad Q_a = 0$

$$M_{rb} = \frac{-qa^2}{C_5}\left[\frac{C_6}{6ab}(2a^2 - r_o a - r_o^2) - L_{15}\right]$$

$$Q_b = \frac{q}{6b}(2a^2 - r_o a - r_o^2)$$

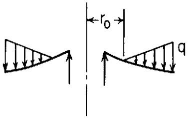
$$y_a = M_{rb}\frac{a^2}{D}C_2 + Q_b\frac{a^3}{D}C_3 - \frac{qa^4}{D}L_{12}$$

$$M_{ra} = M_{rb}C_8 + Q_b a C_9 - qa^2 L_{18}$$

If $r_o = b$ (linearly increasing load from b to a),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_a}	−0.0253	−0.0089	−0.0022	−0.0003	
$K_{M_{rb}}$	−0.5388	−0.1990	−0.0774	−0.0231	−0.00221
$K_{M_{ra}}$	0.0903	0.0594	0.0312	0.0113	0.00125

3k. Outer edge free, inner edge simply supported



$y_b = 0, \quad M_{rb} = 0, \quad M_{ra} = 0, \quad Q_a = 0$

$$\theta_b = \frac{-qa^3}{DC_7}\left[\frac{C_9}{6ab}(2a^2 - r_o a - r_o^2) - L_{18}\right]$$

$$Q_b = \frac{q}{6b}(2a^2 - r_o a - r_o^2)$$

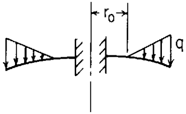
$$y_a = \theta_b a C_1 + Q_b\frac{a^3}{D}C_3 - \frac{qa^4}{D}L_{12}$$

$$\theta_a = \theta_b C_4 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{15}$$

If $r_o = b$ (linearly increasing load from b to a),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_a}	−0.0830	−0.0826	−0.0574	−0.0258	−0.0034
K_{θ_b}	−0.0982	−0.1413	−0.1293	−0.0912	−0.0346
K_{θ_a}	−0.0834	−0.1019	−0.1035	−0.0812	−0.0335
$K_{M_{tb}}$	−0.8937	−0.4286	−0.2354	−0.1186	−0.0350

3l. Outer edge free, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad M_{ra} = 0, \quad Q_a = 0$

$$M_{rb} = \frac{-qa^2}{C_8}\left[\frac{C_9}{6ab}(2a^2 - r_o a - r_o^2) - L_{18}\right]$$

$$Q_b = \frac{q}{6b}(2a^2 - r_o a - r_o^2)$$

$$y_a = M_{rb}\frac{a^2}{D}C_2 + Q_b\frac{a^3}{D}C_3 - \frac{qa^4}{D}L_{12}$$

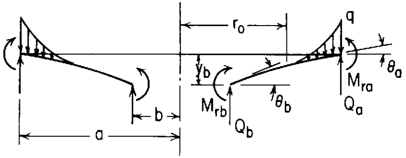
$$\theta_a = M_{rb}\frac{a}{D}C_5 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{15}$$

If $r_o = b$ (linearly increasing load from b to a),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_a}	−0.0579	−0.0240	−0.0064	−0.0008	
K_{θ_a}	−0.0684	−0.0397	−0.0159	−0.0035	−0.00013
$K_{M_{rb}}$	−0.6769	−0.2861	−0.1197	−0.0368	−0.00356

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

Case 4. Annular plate with a distributed pressure increasing parabolically from zero at r_o to q at a



General expressions for deformations, moments, and shears:

$$y = y_b + \theta_b r F_1 + M_{rb}\frac{r^2}{D}F_2 + Q_b\frac{r^3}{D}F_3 - a\frac{r^4}{D}\left(\frac{r-r_o}{a-r_o}\right)^2 G_{13}$$

$$\theta = \theta_b F_4 + M_{rb}\frac{r}{D}F_5 + Q_b\frac{r^2}{D}F_6 - q\frac{r^3}{D}\left(\frac{r-r_o}{a-r_o}\right)^2 G_{16}$$

$$M_r = \theta_b\frac{D}{r}F_7 + M_{rb}F_8 + Q_b r F_9 - qr^2\left(\frac{r-r_o}{a-r_o}\right)^2 G_{19}$$

$$M_t = \frac{\theta D(1-\nu^2)}{r} + \nu M_r$$

$$Q = Q_b\frac{b}{r} - \frac{q}{12r(a-r_o)^2}(3r^4 - 8r_o r^3 + 6r_0^2 r^2 - r_o^4)\langle r - r_o\rangle^0$$

For the numerical data given below, $\nu = 0.3$

$$y = K_y\frac{qa^4}{D}, \quad \theta = K_\theta\frac{qa^3}{D}, \quad M = K_M qa^2, \quad Q = K_Q qa$$

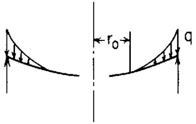
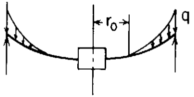
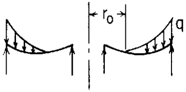
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>4a. Outer edge simply supported, inner edge free

 </td>
<td>$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{-qa^4}{D}\left(\frac{C_1 L_{19}}{C_7} - L_{13}\right)$
$\theta_b = \frac{qa^3}{DC_7}L_{19}$
$\theta_a = \frac{qa^3}{D}\left(\frac{C_4 L_{19}}{C_7} - L_{16}\right)$
$Q_a = \frac{-q}{12a}(3a^2 - 2ar_o - r_o^2)$</td>
<td>$y_{\max} = y_b \quad M_{\max} = M_{tb}$
If $r_o = b$ (parabolically increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>–0.0184</td><td>–0.0168</td><td>–0.0122</td><td>–0.0059</td><td>–0.0008</td></tr>
<tr><td>K_{θ_a}</td><td>0.0291</td><td>0.0266</td><td>0.0243</td><td>0.0190</td><td>0.0082</td></tr>
<tr><td>K_{θ_b}</td><td>0.0105</td><td>0.0227</td><td>0.0254</td><td>0.0203</td><td>0.0084</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>0.0951</td><td>0.0687</td><td>0.0462</td><td>0.0264</td><td>0.0085</td></tr>
</table></td>
</tr>
<tr>
<td>4b. Outer edge simply supported, inner edge guided

 </td>
<td>$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{-qa^4}{D}\left(\frac{C_2 L_{19}}{C_8} - L_{13}\right)$
$M_{rb} = \frac{qa^2 L_{19}}{C_8}$
$\theta_a = \frac{qa^3}{D}\left(\frac{C_5 L_{19}}{C_8} - L_{16}\right)$
$Q_a = \frac{-q}{12a}(3a^2 - 2ar_o - r_o^2)$</td>
<td>$y_{\max} = y_b \quad M_{\max} = M_{rb}$
If $r_o = b$ (parabolically increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>–0.0158</td><td>–0.0074</td><td>–0.0022</td><td>–0.0003</td><td></td></tr>
<tr><td>K_{θ_a}</td><td>0.0275</td><td>0.0166</td><td>0.0071</td><td>0.0017</td><td>0.00007</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>0.0721</td><td>0.0459</td><td>0.0235</td><td>0.0082</td><td>0.00086</td></tr>
</table></td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

4c. Outer edge simply supported, inner edge simply supported



$$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad M_{ra} = 0$$

$$\theta_b = \frac{-qa^3}{D}\frac{C_3L_{19} - C_9L_{13}}{C_1C_9 - C_3C_7}$$

$$Q_b = qa\frac{C_1L_{19} - C_7L_{13}}{C_1C_9 - C_3C_7}$$

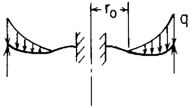
$$\theta_a = \theta_b C_4 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{16}$$

$$Q_a = Q_b\frac{b}{a} - \frac{q}{12a}(3a^2 - 2ar_o - r_o^2)$$

If $r_o = b$ (parabolically increasing load from b to a),

b/a	0.1	0.3	0.5	0.7
$K_{y_{\max}}$	−0.0022	−0.0009	−0.0003	
K_{θ_b}	−0.0083	−0.0046	−0.0016	−0.0003
K_{θ_a}	0.0080	0.0044	0.0017	0.0004
$K_{M_{tb}}$	−0.0759	−0.0139	−0.0029	−0.0004
$K_{M_{r\max}}$	0.0267	0.0185	0.0098	0.0035
K_{Q_b}	0.5068	0.1330	0.0633	0.0305

4d. Outer edge simply supported, inner edge fixed



$$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad M_{ra} = 0$$

$$M_{rb} = -qa^2\frac{C_3L_{19} - C_9L_{13}}{C_2C_9 - C_3C_8}$$

$$Q_b = qa\frac{C_2L_{19} - C_8L_{13}}{C_2C_9 - C_3C_8}$$

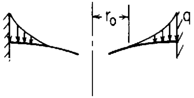
$$\theta_a = M_{rb}\frac{a}{D}C_5 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{16}$$

$$Q_a = Q_b\frac{b}{a} - \frac{q}{12a}(3a^2 - 2ar_o - r_o^2)$$

If $r_o = b$ (parabolically increasing load from b to a),

b/a	0.1	0.3	0.5	0.7
$K_{y_{\max}}$	−0.0016	−0.0005	−0.0001	−0.00002
K_{θ_a}	0.0064	0.0030	0.0011	0.00023
$K_{M_{rb}}$	−0.0777	−0.0281	−0.0113	−0.00349
K_{Q_b}	0.5860	0.1785	0.0882	0.04276

4e. Outer edge fixed, inner edge free



$$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$$

$$y_b = \frac{-qa^4}{D}\left(\frac{C_1L_{16}}{C_4} - L_{13}\right)$$

$$\theta_b = \frac{qa^3L_{16}}{DC_4}$$

$$M_{ra} = -qa^2\left(L_{19} - \frac{C_7}{C_4}L_{16}\right)$$

$$Q_a = \frac{-q}{12a}(3a^2 - 2ar_o - r_o^2)$$

If $r_o = b$ (parabolically increasing load from b to a),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_b}	−0.0031	−0.0019	−0.0007	−0.0001	
K_{θ_b}	0.0023	0.0032	0.0017	0.0004	0.00002
$K_{M_{ra}}$	−0.0368	−0.0269	−0.0162	−0.0066	−0.00081
$K_{M_{tb}}$	0.0208	0.0096	0.0031	0.0006	0.00002

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

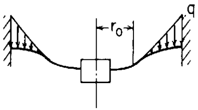
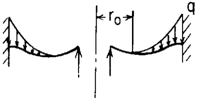
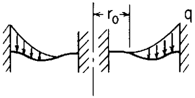
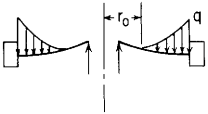
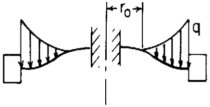
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>4f. Outer edge fixed, inner edge guided
 </td>
<td>$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$
$y_b = \frac{-qa^4}{D}\left(\frac{C_2 L_{16}}{C_5} - L_{13}\right)$
$M_{rb} = \frac{qa^2 L_{16}}{C_5}, \quad M_{ra} = -qa^2\left(L_{19} - \frac{C_8}{C_5} L_{16}\right)$
$Q_a = \frac{-q}{12a}(3a^2 - 2ar_o - r_o^2)$</td>
<td>If $r_o = b$ (parabolically increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>−0.0026</td><td>−0.0011</td><td>−0.0003</td><td></td><td></td></tr>
<tr><td>$K_{M_{rb}}$</td><td>0.0164</td><td>0.0094</td><td>0.0046</td><td>0.0016</td><td>0.00016</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.0364</td><td>−0.0248</td><td>−0.0140</td><td>−0.0054</td><td>−0.00066</td></tr>
</table></td>
</tr>
<tr>
<td>4g. Outer edge fixed, inner edge simply supported
 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad \theta_a = 0$
$\theta_b = \frac{-qa^3}{D}\frac{C_3 L_{16} - C_6 L_{13}}{C_1 C_6 - C_3 C_4}$
$Q_b = qa\frac{C_1 L_{16} - C_4 L_{13}}{C_1 C_6 - C_3 C_4}$
$M_{ra} = \theta_b \frac{D}{a} C_7 + Q_b a C_9 - qa^2 L_{19}$
$Q_a = Q_b \frac{b}{a} - \frac{q}{12a}(3a^2 - 2ar_o - r_o^2)$</td>
<td>If $r_o = b$ (parabolically increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0007</td><td>−0.0003</td><td>−0.0001</td><td></td></tr>
<tr><td>K_{θ_b}</td><td>−0.0031</td><td>−0.0016</td><td>−0.0006</td><td>−0.00012</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.0285</td><td>−0.0049</td><td>−0.0010</td><td>−0.00015</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.0255</td><td>−0.0172</td><td>−0.0093</td><td>−0.00352</td></tr>
<tr><td>K_{Q_b}</td><td>0.2136</td><td>0.0577</td><td>0.0289</td><td>0.01450</td></tr>
</table></td>
</tr>
<tr>
<td>4h. Outer edge fixed, inner edge fixed
 </td>
<td>$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad \theta_a = 0$
$M_{rb} = -qa^2\frac{C_3 L_{16} - C_6 L_{13}}{C_2 C_6 - C_3 C_5}$
$Q_b = qa\frac{C_2 L_{16} - C_5 L_{13}}{C_2 C_6 - C_3 C_5}$
$M_{ra} = M_{rb} C_8 + Q_b a C_9 - qa^2 L_{19}$
$Q_a = Q_b \frac{b}{a} - \frac{q}{12a}(3a^2 - 2ar_o - r_o^2)$</td>
<td>If $r_o = b$ (parabolically increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0005</td><td>−0.0002</td><td>−0.00005</td><td></td></tr>
<tr><td>$K_{M_{rb}}$</td><td>−0.0333</td><td>−0.0126</td><td>−0.00524</td><td>−0.00168</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.0234</td><td>−0.0147</td><td>−0.00773</td><td>−0.00287</td></tr>
<tr><td>K_{Q_b}</td><td>0.2726</td><td>0.0891</td><td>0.04633</td><td>0.02335</td></tr>
</table></td>
</tr>
<tr>
<td>4i. Outer edge guided, inner edge simply supported
 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad \theta_a = 0, \quad Q_a = 0$
$\theta_b = \frac{-qa^3}{DC_4}\left[\frac{C_6}{12ab}(3a^2 - 2ar_o - r_o^2) - L_{16}\right]$
$Q_b = \frac{q}{12b}(3a^2 - 2ar_o - r_o^2)$
$y_a = \theta_b a C_1 + Q_b \frac{a^3}{D} C_3 - \frac{qa^4}{D} L_{13}$
$M_{ra} = \theta_b \frac{D}{a} C_7 + Q_b a C_9 - qa^2 L_{19}$</td>
<td>If $r_o = b$ (parabolically increasing load from b to a),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_a}</td><td>−0.0302</td><td>−0.0193</td><td>−0.0061</td><td>−0.0008</td><td>−0.00001</td></tr>
<tr><td>K_{θ_b}</td><td>−0.0567</td><td>−0.0498</td><td>−0.0210</td><td>−0.0045</td><td>−0.00016</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>0.0859</td><td>0.0813</td><td>0.0532</td><td>0.0215</td><td>0.00249</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.5156</td><td>−0.1510</td><td>−0.0381</td><td>−0.0059</td><td>−0.00016</td></tr>
</table></td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

4j. Outer edge guided, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad \theta_a = 0, \quad Q_a = 0$

$$M_{rb} = \frac{-qa^2}{C_5}\left[\frac{C_6}{12ab}(3a^2 - 2ar_o - r_o^2) - L_{16}\right]$$

$$Q_b = \frac{q}{12b}(3a^2 - 2ar_o - r_o^2)$$

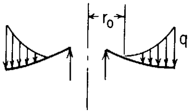
$$y_a = M_{rb}\frac{a^2}{D}C_2 + Q_b\frac{a^3}{D}C_3 - \frac{qa^4}{D}L_{13}$$

$$M_{ra} = M_{rb}C_8 + Q_b aC_9 - qa^2L_{19}$$

If $r_o = b$ (parabolically increasing load from b to a),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_a}	−0.0199	−0.0070	−0.0017	−0.0002	
$K_{M_{rb}}$	−0.4081	−0.1490	−0.0573	−0.0169	−0.00161
$K_{M_{rа}}$	0.0745	0.0485	0.0253	0.0091	0.00100

4k. Outer edge free, inner edge simply supported



$y_b = 0, \quad M_{rb} = 0, \quad M_{ra} = 0, \quad Q_a = 0$

$$\theta_a = \frac{-qa^3}{DC_7}\left[\frac{C_9}{12ab}(3a^2 - 2ar_o - r_o^2) - L_{19}\right]$$

$$Q_b = \frac{q}{12b}(3a^2 - 2ar_o - r_o^2)$$

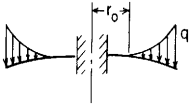
$$y_a = \theta_b aC_1 + Q_b\frac{a^3}{D}C_3 - \frac{qa^4}{D}L_{13}$$

$$\theta_a = \theta_b C_4 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{16}$$

If $r_o = b$ (parabolically increasing load from b to a),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_b}	−0.0662	−0.0644	−0.0441	−0.0196	−0.0026
K_{θ_b}	−0.0757	−0.1087	−0.0989	−0.0693	−0.0260
K_{θ_a}	−0.0680	−0.0802	−0.0799	−0.0618	−0.0252
$K_{M_{tb}}$	−0.6892	−0.3298	−0.1800	−0.0900	−0.0263

4l. Outer edge free, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad M_{ra} = 0, \quad Q_a = 0$

$$M_{rb} = \frac{-qa^2}{C_8}\left[\frac{C_9}{12ab}(3a^2 - 2ar_o - r_o^2) - L_{19}\right]$$

$$Q_b = \frac{q}{12b}(3a^2 - 2ar_o - r_o^2)$$

$$y_a = M_{rb}\frac{a^2}{D}C_2 + Q_b\frac{a^3}{D}C_3 - \frac{qa^4}{D}L_{13}$$

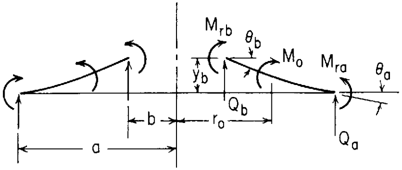
$$\theta_a = M_{rb}\frac{a}{D}C_5 + Q_b\frac{a^2}{D}C_6 - \frac{qa^3}{D}L_{16}$$

If $r_o = b$ (parabolically increasing load from b to a),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_a}	−0.0468	−0.0193	−0.0051	−0.0006	−0.00001
K_{θ_a}	−0.0564	−0.0324	−0.0128	−0.0028	−0.00010
$K_{M_{rb}}$	−0.5221	−0.2202	−0.0915	−0.0279	−0.00268

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

Case 5. Annular plate with a uniform line moment M_o at a radius r_o



Note: If the loading M_o is on the inside edge, $r > r_o$ everywhere, so $\langle r - r_o \rangle^0 = 1$ everywhere

General expressions for deformations, moments, and shears:

$$y = y_b + \theta_b r F_1 + M_{rb}\frac{r^2}{D}F_2 + Q_b\frac{r^3}{D}F_3 + M_o\frac{r^2}{D}G_2$$

$$\theta = \theta_b F_4 + M_{rb}\frac{r}{D}F_5 + Q_b\frac{r^2}{D}F_6 + M_o\frac{r}{D}G_5$$

$$M_r = \theta_b\frac{D}{r}F_7 + M_{rb}F_8 + Q_b r F_9 + M_o G_8$$

$$M_t = \frac{\theta D(1-v^2)}{r} + vM_r$$

$$Q = Q_b\frac{b}{r}$$

For the numerical data given below, $v = 0.3$

$$y = K_y\frac{M_o a^2}{D}, \quad \theta = K_\theta\frac{M_o a}{D}, \quad M = K_M M_o, \quad Q = K_Q\frac{M_o}{a}$$

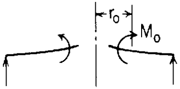
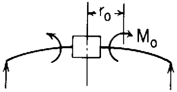
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>5a. Outer edge simply supported, inner edge free
 </td>
<td>$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{M_o a^2}{D}\left(\frac{C_1 L_8}{C_7} - L_2\right)$
$\theta_b = \frac{-M_o a}{DC_7}L_8$
$\theta_a = \frac{-M_o a}{D}\left(\frac{C_4 L_8}{C_7} - L_5\right)$
$Q_a = 0$</td>
<td>$y_{\max} = y_b, \quad M_{\max} = M_{tb}$
If $r_o = b$ (moment M_o at the inner edge),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>0.0371</td><td>0.2047</td><td>0.4262</td><td>0.6780</td><td>0.9532</td></tr>
<tr><td>K_{θ_a}</td><td>-0.0222</td><td>-0.2174</td><td>-0.7326</td><td>-2.1116</td><td>-9.3696</td></tr>
<tr><td>K_{θ_b}</td><td>-0.1451</td><td>-0.4938</td><td>-1.0806</td><td>-2.4781</td><td>-9.7183</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>-1.0202</td><td>-1.1978</td><td>-1.6667</td><td>-2.9216</td><td>-9.5263</td></tr>
</table>
If $r_o = a$ (moment M_o at the outer edge),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>0.4178</td><td>0.5547</td><td>0.7147</td><td>0.8742</td><td>1.0263</td></tr>
<tr><td>K_{θ_a}</td><td>-0.7914</td><td>-0.9866</td><td>-1.5018</td><td>-2.8808</td><td>-10.1388</td></tr>
<tr><td>K_{θ_b}</td><td>-0.2220</td><td>-0.7246</td><td>-1.4652</td><td>-3.0166</td><td>-10.4107</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>-2.0202</td><td>-2.1978</td><td>-2.6667</td><td>-3.9216</td><td>-10.5263</td></tr>
</table>
</td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

5b. Outer edge simply supported, inner edge guided



$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$

$$y_b = \frac{M_o a^2}{D}\left(\frac{C_2 L_8}{C_8} - L_2\right)$$

$$M_{rb} = \frac{-M_o L_8}{C_8}$$

$$\theta_a = \frac{-M_o a}{D}\left(\frac{C_5 L_8}{C_8} - L_5\right)$$

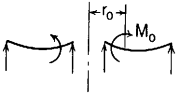
$$Q_a = 0$$

$y_{\max} = y_b \quad M_{\max} = M_{rb}$

If $r_o = a$ (moment M_o at the outer edge),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_b}	0.3611	0.2543	0.1368	0.0488	0.0052
K_{θ_a}	−0.7575	−0.6676	−0.5085	−0.3104	−0.1018
$K_{M_{rb}}$	−1.5302	−1.4674	−1.3559	−1.2173	−1.0712

5c. Outer edge simply supported, inner edge simply supported



$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad M_{ra} = 0$

$$\theta_b = \frac{M_o a}{D}\,\frac{C_3 L_8 - C_9 L_2}{C_1 C_9 - C_3 C_7}$$

$$Q_b = \frac{-M_o}{a}\,\frac{C_1 L_8 - C_7 L_2}{C_1 C_9 - C_3 C_7}$$

$$\theta_a = \theta_b C_4 + Q_b \frac{a^2}{D} C_6 + \frac{M_o a}{D} L_5$$

$$Q_a = Q_b \frac{b}{a}$$

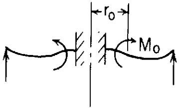
If $r_o = b$ (moment M_o at the inner edge),

b/a	0.1	0.3	0.5	0.7	0.9
$K_{y_{\max}}$	−0.0095	−0.0167	−0.0118	−0.0050	−0.0005
K_{θ_a}	0.0204	0.0518	0.0552	0.0411	0.0158
K_{θ_b}	−0.1073	−0.1626	−0.1410	−0.0929	−0.0327
$K_{M_{tb}}$	−0.6765	−0.1933	0.0434	0.1793	0.2669
K_{Q_b}	−1.0189	−1.6176	−2.2045	−3.5180	−10.1611

If $r_o = a$ (moment M_o at the outer edge),

b/a	0.1	0.3	0.5	0.7	0.9
$K_{y_{\max}}$	0.0587	0.0390	0.0190	0.0063	0.0004
K_{θ_a}	−0.3116	−0.2572	−0.1810	−0.1053	−0.0339
K_{θ_b}	0.2037	0.1728	0.1103	0.0587	0.0175
$K_{M_{tb}}$	1.8539	0.5240	0.2007	0.0764	0.0177
K_{Q_b}	−11.4835	−4.3830	−3.6964	−4.5358	−10.9401

5d. Outer edge simply supported, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad M_{ra} = 0$

$$M_{rb} = M_o \frac{C_3 L_8 - C_9 L_2}{C_2 C_9 - C_3 C_8}$$

$$Q_b = \frac{-M_o}{a}\,\frac{C_2 L_8 - C_8 L_2}{C_2 C_9 - C_3 C_8}$$

$$\theta_a = M_{rb} \frac{a}{D} C_5 + Q_b \frac{a^2}{D} C_6 + \frac{M_o a}{D} L_5$$

$$Q_a = Q_b \frac{b}{a}$$

If $r_o = a$ (moment M_o at the outer edge),

b/a	0.1	0.3	0.5	0.7	0.9
$K_{y_{\max}}$	0.0449	0.0245	0.0112	0.0038	0.0002
K_{θ_a}	−0.2729	−0.2021	−0.1378	−0.0793	−0.0255
$K_{M_{rb}}$	1.8985	1.0622	0.7823	0.6325	0.5366
K_{Q_b}	−13.4178	−6.1012	−5.4209	−6.7611	−16.3923

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

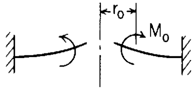
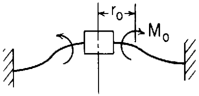
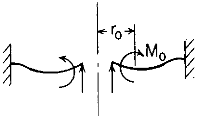
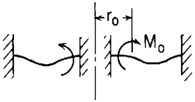
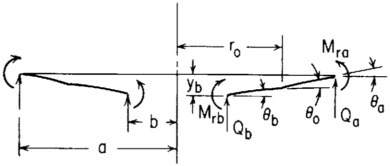
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>5e. Outer edge fixed, inner edge free
 </td>
<td>$M_{rb}=0,\quad Q_b=0,\quad y_a=0,\quad \theta_a=0$
$y_b=\frac{M_oa^2}{D}\left(\frac{C_1L_5}{C_4}-L_2\right)$
$\theta_b=\frac{-M_oa}{DC_4}L_5$
$M_{ra}=M_o\left(L_8-\frac{C_7}{C_4}L_5\right)$
$Q_a=0$</td>
<td>If $r_o=b$ (moment M_o at the inner edge),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>0.0254</td><td>0.0825</td><td>0.0776</td><td>0.0373</td><td>0.0048</td></tr>
<tr><td>K_{θ_b}</td><td>−0.1389</td><td>−0.3342</td><td>−0.3659</td><td>−0.2670</td><td>−0.0976</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.9635</td><td>−0.7136</td><td>−0.3659</td><td>−0.0471</td><td>0.2014</td></tr>
</table></td>
</tr>
<tr>
<td>5f. Outer edge fixed, inner edge guided
 </td>
<td>$\theta_b=0,\quad Q_b=0,\quad y_a=0,\quad \theta_a=0$
$y_b=\frac{M_oa^2}{D}\left(\frac{C_2L_5}{C_5}-L_2\right)$
$M_{rb}=\frac{-M_o}{C_5}L_5$
$M_{ra}=M_o\left(L_8-\frac{C_8}{C_5}L_5\right)$
$Q_a=0$</td>
<td><table>
<tr><th>b/a</th><th>0.1</th><th></th><th>0.5</th><th></th><th>0.7</th></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>K_{y_b}</td><td>0.0779</td><td>0.0815</td><td>0.0285</td><td>0.0207</td><td>0.0101</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>−0.7576</td><td>−0.5151</td><td>−0.6800</td><td>−0.2533</td><td>−0.3726</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>0.2424</td><td>0.4849</td><td>0.3200</td><td>0.7467</td><td>0.6274</td></tr>
</table></td>
</tr>
<tr>
<td>5g. Outer edge fixed, inner edge simply supported
 </td>
<td>$y_b=0,\quad M_{rb}=0,\quad y_a=0,\quad \theta_a=0$
$\theta_b=\frac{M_oa}{D}\frac{C_3L_5-C_6L_2}{C_1C_6-C_3C_4}$
$Q_b=\frac{-M_o}{a}\frac{C_1L_5-C_4L_2}{C_1C_6-C_3C_4}$
$M_{ra}=\theta_b\frac{D}{a}C_7+Q_baC_9+M_oL_8$
$Q_a=Q_b\frac{b}{a}$</td>
<td>If $r_o=b$ (moment M_o at the inner edge),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>$K_{y_{\max}}$</td><td>−0.0067</td><td>−0.0102</td><td>−0.0066</td><td>−0.0029</td><td>−0.0002</td></tr>
<tr><td>K_{θ_b}</td><td>−0.0940</td><td>−0.1278</td><td>−0.1074</td><td>−0.0699</td><td>−0.0245</td></tr>
<tr><td>K_{Q_b}</td><td>−1.7696</td><td>−2.5007</td><td>−3.3310</td><td>−5.2890</td><td>−15.2529</td></tr>
</table></td>
</tr>
<tr>
<td>5h. Outer edge fixed, inner edge fixed
 </td>
<td>$y_b=0,\quad \theta_b=0,\quad y_a=0,\quad \theta_a=0$
$M_{rb}=M_o\frac{C_3L_5-C_6L_2}{C_2C_6-C_3C_5}$
$Q_b=\frac{-M_o}{a}\frac{C_2L_5-C_5L_2}{C_2C_6-C_3C_5}$
$M_{ra}=M_{rb}C_8+Q_baC_9+M_oL_8$
$Q_a=Q_b\frac{b}{a}$</td>
<td><table>
<tr><th>b/a</th><th>0.1</th><th></th><th>0.5</th><th></th><th>0.7</th></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>0.7096</td><td>1.0185</td><td>0.2031</td><td>0.3895</td><td>0.3925</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.1407</td><td>0.0844</td><td>−0.2399</td><td>0.3391</td><td>0.0238</td></tr>
<tr><td>$K_{M_{ro}}$</td><td>−0.5045</td><td>−0.5371</td><td>−0.4655</td><td>−0.4671</td><td>−0.5540</td></tr>
<tr><td>$K_{M_{ro}}$</td><td>0.4955</td><td>0.4629</td><td>0.5345</td><td>0.5329</td><td>0.4460</td></tr>
<tr><td>K_{Q_b}</td><td>−8.0354</td><td>−8.3997</td><td>−4.1636</td><td>−3.0307</td><td>−5.4823</td></tr>
</table>
(*Note:* the two values of $K_{M_{ro}}$ are for positions just before and after the applied moment M_o.)</td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

Case 6. Annular plate with an externally applied change in slope θ_o on an annulus with a radius r_o



General expressions for deformations, moments, and shears:

$$y = y_b + \theta_b r F_1 + M_{rb}\frac{r^2}{D}F_2 + Q_b\frac{r^3}{D}F_3 + \theta_o r G_1$$

$$\theta = \theta_b F_4 + M_{rb}\frac{r}{D}F_5 + Q_b\frac{r^2}{D}F_6 + \theta_o G_4$$

$$M_r = \theta_b\frac{D}{r}F_7 + M_{rb}F_8 + Q_b r F_9 + \frac{\theta_o D}{r}G_7$$

$$M_t = \frac{\theta D(1-\nu^2)}{r} + \nu M_r$$

$$Q = Q_b\frac{b}{r}$$

For the numerical data given below, $\nu = 0.3$, and all values given for $K_{M_{to}}$ are found just outside r_o

$$y = K_y\theta_o a, \quad \theta = K_\theta\theta_o, \quad M = K_M\theta_o\frac{D}{a}, \quad Q = K_Q\theta_o\frac{D}{a^2}$$

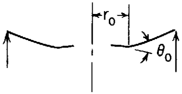
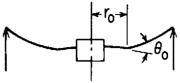
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>6a. Outer edge simply supported, inner edge free
 </td>
<td>$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \theta_o a\left(\frac{C_1 L_7}{C_7} - L_1\right)$
$\theta_b = \frac{-\theta_o}{C_7}L_7$
$\theta_a = -\theta_o\left(\frac{C_4 L_7}{C_7} - L_4\right)$
$Q_a = 0$</td>
<td>
<table>
<tr><td>b/a</td><td colspan="2">0.1</td><td colspan="2">0.5</td><td>0.7</td></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>K_{y_b}</td><td>−0.2026</td><td>−0.1513</td><td>−0.0529</td><td>−0.0299</td><td>−0.0146</td></tr>
<tr><td>K_{y_o}</td><td>−0.2821</td><td>−0.2224</td><td>−0.1468</td><td>−0.0844</td><td>−0.0709</td></tr>
<tr><td>K_{θ_b}</td><td>−0.1515</td><td>−0.0736</td><td>−0.4857</td><td>−0.1407</td><td>−0.2898</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−1.3788</td><td>−0.6697</td><td>−0.8840</td><td>−0.2562</td><td>−0.3767</td></tr>
<tr><td>$K_{M_{to}}$</td><td>1.1030</td><td>0.9583</td><td>0.6325</td><td>0.8435</td><td>0.7088</td></tr>
</table>
</td>
</tr>
<tr>
<td>6b. Outer edge simply supported, inner edge guided
 </td>
<td>$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \theta_o a\left(\frac{C_2 L_7}{C_8} - L_1\right)$
$M_{rb} = \frac{-\theta_o D L_7}{a C_8}$
$\theta_a = -\theta_o\left(\frac{C_5 L_7}{C_8} - L_4\right)$
$Q_a = 0$</td>
<td>
<table>
<tr><td>b/a</td><td colspan="2">0.1</td><td colspan="2">0.5</td><td>0.7</td></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>K_{y_b}</td><td>−0.2413</td><td>−0.1701</td><td>−0.2445</td><td>−0.0854</td><td>−0.0939</td></tr>
<tr><td>K_{θ_a}</td><td>0.5080</td><td>0.7039</td><td>0.7864</td><td>0.9251</td><td>0.9441</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>−1.0444</td><td>−0.5073</td><td>−0.4495</td><td>−0.1302</td><td>−0.1169</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−0.3133</td><td>−0.1522</td><td>−0.1349</td><td>−0.0391</td><td>−0.0351</td></tr>
</table>
</td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

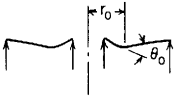
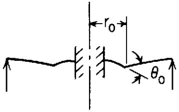
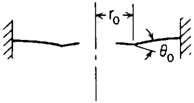
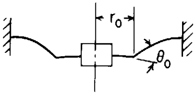
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>6c. Outer edge simply supported, inner edge simply supported
 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad M_{ra} = 0$
$\theta_b = \theta_o \dfrac{C_3 L_7 - C_9 L_1}{C_1 C_9 - C_3 C_7}$
$Q_b = \dfrac{-\theta_o D}{a^2} \dfrac{C_1 L_7 - C_7 L_1}{C_1 C_9 - C_3 C_7}$
$\theta_a = \theta_b C_4 + Q_b \dfrac{a^2}{D} C_6 + \theta_o L_4$
$Q_a = Q_b \dfrac{b}{a}$</td>
<td>
<table>
<tr><td>b/a</td><td colspan="2">0.1</td><td colspan="2">0.5</td><td>0.7</td></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>K_{y_o}</td><td>−0.1629</td><td>−0.1689</td><td>−0.1161</td><td>−0.0788</td><td>−0.0662</td></tr>
<tr><td>K_{θ_b}</td><td>−0.3579</td><td>−0.2277</td><td>−0.6023</td><td>−0.2067</td><td>−0.3412</td></tr>
<tr><td>K_{θ_a}</td><td>0.2522</td><td>0.5189</td><td>0.3594</td><td>0.7743</td><td>0.6508</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−3.2572</td><td>−2.0722</td><td>−1.0961</td><td>−0.3762</td><td>−0.4435</td></tr>
<tr><td>$K_{M_{to}}$</td><td>0.6152</td><td>0.6973</td><td>0.4905</td><td>0.7851</td><td>0.6602</td></tr>
<tr><td>K_{Q_b}</td><td>5.5679</td><td>4.1574</td><td>0.2734</td><td>0.1548</td><td>0.0758</td></tr>
</table>
</td>
</tr>
<tr>
<td>6d. Outer edge simply supported, inner edge fixed
 </td>
<td>$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$M_{rb} = \dfrac{\theta_o D}{a} \dfrac{C_3 L_7 - C_9 L_1}{C_2 C_9 - C_3 C_8}$
$Q_b = \dfrac{-\theta_o D}{a^2} \dfrac{C_2 L_7 - C_8 L_1}{C_2 C_9 - C_3 C_8}$
$\theta_a = M_{rb} \dfrac{a}{D} C_5 + Q_b \dfrac{a^2}{D} C_6 + \theta_o L_4$
$Q_a = Q_b \dfrac{b}{a}$</td>
<td>
<table>
<tr><td>b/a</td><td colspan="2">0.1</td><td colspan="2">0.5</td><td>0.7</td></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>K_{y_o}</td><td>−0.1333</td><td>−0.1561</td><td>−0.0658</td><td>−0.0709</td><td>−0.0524</td></tr>
<tr><td>K_{θ_a}</td><td>0.1843</td><td>0.4757</td><td>0.1239</td><td>0.6935</td><td>0.4997</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>−3.3356</td><td>−2.1221</td><td>−4.2716</td><td>−1.4662</td><td>−3.6737</td></tr>
<tr><td>K_{Q_b}</td><td>8.9664</td><td>6.3196</td><td>9.6900</td><td>3.3870</td><td>12.9999</td></tr>
</table>
</td>
</tr>
<tr>
<td>6e. Outer edge fixed, inner edge free
 </td>
<td>$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$
$y_b = \theta_o a \left(\dfrac{C_1 L_4}{C_4} - L_1 \right)$
$\theta_b = \dfrac{-\theta_o L_4}{C_4}$
$M_{ra} = \dfrac{\theta_o D}{a} \left(L_7 - \dfrac{C_7}{C_4} L_4 \right)$
$Q_a = 0$</td>
<td>
<table>
<tr><td>b/a</td><td colspan="2">0.1</td><td colspan="2">0.5</td><td>0.7</td></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>K_{y_b}</td><td>0.0534</td><td>0.2144</td><td>0.1647</td><td>0.3649</td><td>0.1969</td></tr>
<tr><td>K_{y_o}</td><td>−0.0975</td><td>−0.0445</td><td>−0.0155</td><td>−0.0029</td><td>−0.0013</td></tr>
<tr><td>K_{θ_b}</td><td>−0.2875</td><td>−0.2679</td><td>−0.9317</td><td>−0.9501</td><td>−1.0198</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>−2.6164</td><td>−2.4377</td><td>−1.6957</td><td>−1.7293</td><td>−1.3257</td></tr>
</table>
</td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

6f. Outer edge fixed, inner edge guided



$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$

$$y_b = \theta_o a\left(\frac{C_2 L_4}{C_5} - L_1\right)$$

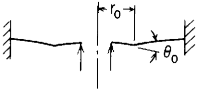
$$M_{rb} = \frac{-\theta_o D L_4}{a C_5}$$

$$M_{ra} = \frac{\theta_o D}{a}\left(L_7 - \frac{C_8}{C_5} L_4\right)$$

$Q_a = 0$

b/a	0.1		0.5		0.7
r_o/a	0.5	0.7	0.7	0.9	0.9
K_{y_b}	0.0009	0.1655	−0.0329	0.1634	0.0546
K_{y_o}	−0.1067	−0.0472	−0.0786	−0.0094	−0.0158
$K_{M_{rb}}$	−2.0707	−1.9293	−2.5467	−2.5970	−3.8192
$K_{M_{ra}}$	−0.6707	−0.9293	−1.5467	−1.8193	−3.0414

6g. Outer edge fixed, inner edge simply supported



$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad \theta_a = 0$

$$\theta_b = \theta_o \frac{C_3 L_4 - C_6 L_1}{C_1 C_6 - C_3 C_4}$$

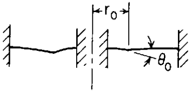
$$Q_b = \frac{-\theta_o D}{a^2} \frac{C_1 L_4 - C_4 L_1}{C_1 C_6 - C_3 C_4}$$

$$M_{ra} = \theta_b \frac{D}{a} C_7 + Q_b a C_9 + \frac{\theta_o D}{a} L_7$$

$$Q_a = Q_b \frac{b}{a}$$

b/a	0.1		0.5		0.7
r_o/a	0.5	0.7	0.7	0.9	0.9
K_{y_o}	−0.1179	−0.0766	−0.0820	−0.0208	−0.0286
K_{θ_b}	−0.1931	0.1116	−0.3832	0.2653	0.0218
$K_{M_{ra}}$	−0.8094	−1.6653	−1.9864	−4.2792	−6.1794
$K_{M_{tb}}$	−1.7567	1.0151	−0.6974	0.4828	0.0284
K_{Q_b}	−3.7263	−14.9665	−7.0690	−15.6627	−27.9529

6h. Outer edge fixed, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad \theta_a = 0$

$$M_{rb} = \frac{\theta_o D}{a} \frac{C_3 L_4 - C_6 L_1}{C_2 C_6 - C_3 C_5}$$

$$Q_b = \frac{-\theta_o D}{a^2} \frac{C_2 L_4 - C_5 L_1}{C_2 C_6 - C_3 C_5}$$

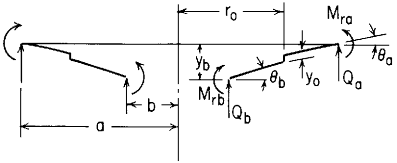
$$M_{ra} = M_{rb} C_8 + Q_b a C_9 + \theta_o \frac{D}{a} L_7$$

$$Q_a = Q_b \frac{b}{a}$$

b/a	0.1		0.5		0.7
r_o/a	0.5	0.7	0.7	0.9	0.9
K_{y_o}	−0.1071	−0.0795	−0.0586	−0.0240	−0.0290
$K_{M_{rb}}$	−2.0540	1.1868	−3.5685	2.4702	0.3122
$K_{M_{ra}}$	−0.6751	−1.7429	−0.8988	−5.0320	−6.3013
K_{Q_b}	−0.0915	−17.067	4.8176	−23.8910	−29.6041

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

Case 7. Annular plate with an externally applied vertical deformation y_o at a radius r_o



General expressions for deformations, moments, and shears:

$$y = y_b + \theta_b r F_1 + M_{rb}\frac{r^2}{D}F_2 + Q_b\frac{r^3}{D}F_3 + y_o\langle r - r_o\rangle^0$$

$$\theta = \theta_b F_4 + M_{rb}\frac{r}{D}F_5 + Q_b\frac{r^2}{D}F_6$$

$$M_r = \theta_b\frac{D}{r}F_7 + M_{rb}F_8 + Q_b r F_9$$

$$M_t = \frac{\theta D(1-\nu^2)}{r} + \nu M_r$$

$$Q = Q_b\frac{b}{r}$$

For the numerical data given below, $\nu = 0.3$

$$y = K_y y_o, \qquad \theta = K_\theta\frac{y_o}{a}, \qquad M = K_M y_o\frac{D}{a^2}, \qquad Q = K_Q y_o\frac{D}{a^3}$$

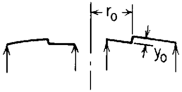
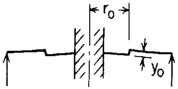
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>7c. Outer edge simply supported, inner edge simply supported
 </td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad M_{ra} = 0$
$\theta_b = \frac{-y_o C_9}{a(C_1C_9 - C_3C_7)}$
$Q_b = \frac{y_o D C_7}{a^3(C_1C_9 - C_3C_7)}$
$\theta_a = \frac{y_o}{a}\frac{C_7C_6 - C_9C_4}{C_1C_9 - C_3C_7}$
$Q_a = Q_b\frac{b}{a}$</td>
<td>
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{θ_b}</td><td>– 1.0189</td><td>– 1.6176</td><td>– 2.2045</td><td>– 3.5180</td><td>– 10.1611</td></tr>
<tr><td>K_{θ_a}</td><td>– 1.1484</td><td>– 1.3149</td><td>– 1.8482</td><td>– 3.1751</td><td>– 9.8461</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>– 9.2716</td><td>– 4.9066</td><td>– 4.0121</td><td>– 4.5734</td><td>– 10.2740</td></tr>
<tr><td>K_{Q_b}</td><td>27.4828</td><td>7.9013</td><td>5.1721</td><td>5.1887</td><td>10.6599</td></tr>
</table>
(<i>Note:</i> Constants given are valid for all values of $r_o > b$)
</td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

7d. Outer edge simply supported, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad M_{ra} = 0$

$$M_{rb} = \frac{-y_o D C_9}{a^2(C_2C_9 - C_3C_8)}$$

$$Q_b = \frac{y_o D C_8}{a^3(C_2C_9 - C_3C_8)}$$

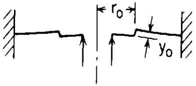
$$\theta_a = \frac{y_o}{a}\frac{C_6C_8 - C_5C_9}{C_2C_9 - C_3C_8}$$

$$Q_a = Q_b\frac{b}{a}$$

b/a	0.1	0.3	0.5	0.7	0.9
K_{θ_a}	−1.3418	−1.8304	−2.7104	−4.7327	−14.7530
$K_{M_{rb}}$	−9.4949	−9.9462	−15.6353	−37.8822	−310.808
K_{Q_b}	37.1567	23.9899	39.6394	138.459	3186.83

(*Note:* Constants given are valid for all values of $r_o > b$)

7g. Outer edge fixed, inner edge simply supported



$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad \theta_a = 0$

$$\theta_b = \frac{-y_o C_6}{a(C_1C_6 - C_3C_4)}$$

$$Q_b = \frac{y_o D C_4}{a^3(C_1C_6 - C_3C_4)}$$

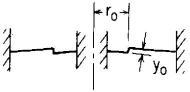
$$M_{ra} = \frac{y_o D}{a^2}\frac{C_4C_9 - C_6C_7}{C_1C_6 - C_3C_4}$$

$$Q_a = Q_b\frac{b}{a}$$

b/a	0.1	0.3	0.5	0.7	0.9
K_{θ_b}	−1.7696	−2.5008	−3.3310	−5.2890	−15.2528
$K_{M_{ra}}$	3.6853	5.1126	10.2140	30.1487	290.2615
$K_{M_{tb}}$	−16.1036	−7.5856	−6.0624	−6.8757	−15.4223
K_{Q_b}	69.8026	30.3098	42.9269	141.937	3186.165

(*Note:* Constants given are valid for all values of $r_o > b$)

7h. Outer edge fixed, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad \theta_a = 0$

$$M_{rb} = \frac{-y_o D C_6}{a^2(C_2C_6 - C_3C_5)}$$

$$Q_b = \frac{y_o D C_5}{a^3(C_2C_6 - C_3C_5)}$$

$$M_{ra} = \frac{y_o D}{a^2}\frac{C_5C_9 - C_6C_8}{C_2C_6 - C_3C_5}$$

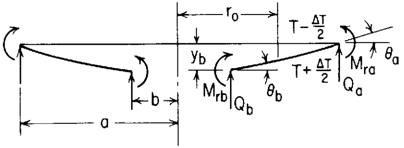
$$Q_a = Q_b\frac{b}{a}$$

b/a	0.1	0.3	0.5	0.7	0.9
$K_{M_{rb}}$	−18.8284	−19.5643	−31.0210	−75.6312	−621.8586
$K_{M_{ra}}$	4.9162	9.0548	19.6681	59.6789	579.6755
K_{Q_b}	103.1218	79.2350	146.258	541.958	12671.35

(*Note:* Constants given are valid for all values of $r_o > b$)

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

Case 8. Annular plate with, from r_o to a, a uniform temperature differential ΔT between the bottom and the top surface (the midplane temperature is assumed to be unchanged, and so no in-plane forces develop)



Note: If the temperature difference ΔT occurs over the entire plate, $r > r_o$ everywhere, so $\langle r - r_o \rangle^0 = 1$ everywhere, therefore, all numerical data for K_{Mtb} are given at a radius just greater than b.

General expressions for deformations, moments, and shears:

$$y = y_b + \theta_b r F_1 + M_{rb} \frac{r^2}{D} F_2 + Q_b \frac{r^3}{D} F_3 + \frac{\gamma(1+\nu)\Delta T}{t} r^2 G_2$$

$$\theta = \theta_b F_4 + M_{rb} \frac{r}{D} F_5 + Q_b \frac{r^2}{D} F_6 + \frac{\gamma(1+\nu)\Delta T}{t} r G_5$$

$$M_r = \theta_b \frac{D}{r} F_7 + M_{rb} F_8 + Q_b r F_9 + \frac{\gamma(1+\nu)\Delta T}{t} D(G_8 - \langle r - r_o \rangle^0)$$

$$M_t = \frac{\theta D(1-\nu^2)}{r} + \nu M_r - \frac{\gamma(1-\nu^2)\Delta T D}{t} \langle r - r_o \rangle^0$$

$$Q = Q_b \frac{b}{r}$$

For the numerical data given below, $\nu = 0.3$

$$y = K_y \frac{\gamma \Delta T a^2}{t}, \quad \theta = K_\theta \frac{\gamma \Delta T a}{t}, \quad M = K_M \frac{\gamma \Delta T D}{t}, \quad Q = K_Q \frac{\gamma \Delta T D}{a t}$$

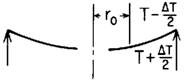
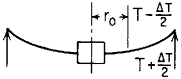
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>8a. Outer edge simply supported, inner edge free
 </td>
<td>$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$
$y_b = \frac{-\gamma(1+\nu)\Delta T a^2}{t} \left[L_2 + \frac{C_1}{C_7}(1 - L_8) \right]$
$\theta_b = \frac{\gamma(1+\nu)\Delta T a}{t C_7}(1 - L_8)$
$Q_a = 0$
$\theta_a = \frac{\gamma(1+\nu)\Delta T a}{t} \left[L_5 + \frac{C_4}{C_7}(1 - L_8) \right]$</td>
<td>If $r_o = b$ (ΔT over entire plate),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>−0.4950</td><td>−0.4550</td><td>−0.3750</td><td>−0.2550</td><td>−0.0950</td></tr>
<tr><td>K_{θ_a}</td><td>1.0000</td><td>1.0000</td><td>1.0000</td><td>1.0000</td><td>1.0000</td></tr>
<tr><td>K_{θ_b}</td><td>0.1000</td><td>0.3000</td><td>0.5000</td><td>0.7000</td><td>0.9000</td></tr>
</table>
(*Note:* There are no moments in the plate)</td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

8b. Outer edge simply supported, inner edge guided



$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad M_{ra} = 0$

$$y_b = \frac{-\gamma(1+\nu)\Delta T a^2}{t}\left[L_2 + \frac{C_2}{C_8}(1-L_8)\right]$$

$$M_{rb} = \frac{\gamma(1+\nu)\Delta T D}{t C_8}(1-L_8)$$

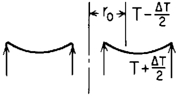
$$\theta_a = \frac{\gamma(1+\nu)\Delta T a}{t}\left[L_5 + \frac{C_5}{C_8}(1-L_8)\right]$$

$$Q_a = 0$$

If $r_o = b$ (ΔT over entire plate),

b/a	0.1	0.3	0.5	0.7	0.9
K_{y_b}	−0.4695	−0.3306	−0.1778	−0.0635	−0.0067
K_{θ_a}	0.9847	0.8679	0.6610	0.4035	0.1323
$K_{M_{rb}}$	0.6893	0.6076	0.4627	0.2825	0.0926
$K_{M_{tb}}$	−0.7032	−0.7277	−0.7712	−0.8253	−0.8822

8c. Outer edge simply supported, inner edge simply supported



$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad M_{ra} = 0$

$$\theta_b = \frac{-\gamma(1+\nu)\Delta T a}{t}\,\frac{C_9 L_2 + C_3(1-L_8)}{C_1 C_9 - C_3 C_7}$$

$$Q_b = \frac{\gamma(1+\nu)\Delta T D}{a t}\,\frac{C_7 L_2 + C_1(1-L_8)}{C_1 C_9 - C_3 C_7}$$

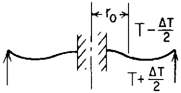
$$\theta_a = \theta_b C_4 + Q_b \frac{a^2}{D} C_6 + \frac{\gamma(1+\nu)\Delta T a}{t} L_5$$

$$Q_a = Q_b \frac{b}{a}$$

If $r_o = b$ (ΔT over entire plate),

b/a	0.1	0.3	0.5	0.7	0.9
$K_{y_{\max}}$	−0.0865	−0.0701	−0.0388	−0.0142	
K_{θ_b}	−0.4043	−0.4360	−0.3267	−0.1971	−0.0653
K_{θ_a}	0.4316	0.4017	0.3069	0.1904	0.0646
$K_{M_{tb}}$	−4.5894	−2.2325	−1.5045	−1.1662	−0.9760
K_{Q_b}	13.6040	3.5951	1.9395	1.3231	1.0127

8d. Outer edge simply supported, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad M_{ra} = 0$

$$M_{rb} = \frac{-\gamma(1+\nu)\Delta T D}{t}\,\frac{C_9 L_2 + C_3(1-L_8)}{C_2 C_9 - C_3 C_8}$$

$$Q_b = \frac{\gamma(1+\nu)\Delta T D}{a t}\,\frac{C_8 L_2 + C_2(1-L_8)}{C_2 C_9 - C_3 C_8}$$

$$\theta_a = M_{rb}\frac{a}{D} C_5 + Q_b \frac{a^2}{D} C_6 + \frac{\gamma(1+\nu)\Delta T a}{t} L_5$$

$$Q_a = Q_b \frac{b}{a}$$

If $r_o = b$ (ΔT over entire plate),

b/a	0.1	0.3	0.5	0.7	0.9
$K_{y_{\max}}$	−0.0583	−0.0318	−0.0147	−0.0049	
K_{θ_a}	0.3548	0.2628	0.1792	0.1031	0.0331
$K_{M_{rb}}$	−3.7681	−2.6809	−2.3170	−2.1223	−1.9975
K_{Q_b}	17.4431	7.9316	7.0471	8.7894	21.3100

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

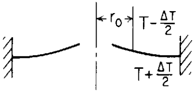
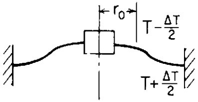
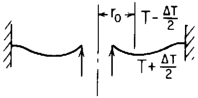
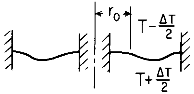
<table>
<tr><th>Case no., edge restraints</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>8e. Outer edge fixed, inner edge free

</td>
<td>$M_{rb} = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$
$y_b = \frac{-\gamma(1+\nu)\Delta T a^2}{t}\left(L_2 - \frac{C_1}{C_4}L_5\right)$
$\theta_b = \frac{-\gamma(1+\nu)\Delta T a}{tC_4}L_5$
$M_{ra} = \frac{-\gamma(1+\nu)\Delta T D}{t}\left(\frac{C_7}{C_4}L_5 + 1 - L_8\right)$
$Q_a = 0$</td>
<td>If $r_o = b$ (ΔT over entire plate),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>K_{y_b}</td><td>0.0330</td><td>0.1073</td><td>0.1009</td><td>0.0484</td><td>0.0062</td></tr>
<tr><td>K_{θ_b}</td><td>− 0.1805</td><td>− 0.4344</td><td>− 0.4756</td><td>− 0.3471</td><td>− 0.1268</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>− 1.2635</td><td>− 1.0136</td><td>− 0.6659</td><td>− 0.3471</td><td>− 0.0986</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>− 2.5526</td><td>− 2.2277</td><td>− 1.7756</td><td>− 1.3613</td><td>− 1.0382</td></tr>
</table></td>
</tr>
<tr>
<td>8f. Outer edge fixed, inner edge guided

</td>
<td>$\theta_b = 0, \quad Q_b = 0, \quad y_a = 0, \quad \theta_a = 0$
$y_b = \frac{-\gamma(1+\nu)\Delta T a^2}{t}\left(L_2 - \frac{C_2}{C_5}L_5\right)$
$M_{rb} = \frac{-\gamma(1+\nu)\Delta T D}{tC_5}L_5$
$M_{ra} = \frac{-\gamma(1+\nu)\Delta T D}{t}\left(\frac{C_8}{C_5}L_5 + 1 - L_8\right)$
$Q_a = 0$</td>
<td>If $r_o = b$ (ΔT over entire plate), all deflections are zero and $K_{M_r} = K_{M_t} = -1.30$ everywhere in the plate. If $r_o > b$, the following tabulated values apply.
<table>
<tr><th>b/a</th><th colspan="2">0.1</th><th colspan="2">0.5</th><th>0.7</th></tr>
<tr><td>r_o/a</td><td>0.5</td><td>0.7</td><td>0.7</td><td>0.9</td><td>0.9</td></tr>
<tr><td>K_{y_b}</td><td>0.1013</td><td>0.1059</td><td>0.0370</td><td>0.0269</td><td>0.0132</td></tr>
<tr><td>$K_{M_{rb}}$</td><td>− 0.9849</td><td>− 0.6697</td><td>− 0.8840</td><td>− 0.3293</td><td>− 0.4843</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>− 0.9849</td><td>− 0.6697</td><td>− 0.8840</td><td>− 0.3293</td><td>− 0.4843</td></tr>
<tr><td>$K_{M_{to}}$</td><td>− 1.5364</td><td>− 1.3405</td><td>− 1.3267</td><td>− 1.0885</td><td>− 1.1223</td></tr>
</table></td>
</tr>
<tr>
<td>8g. Outer edge fixed, inner edge simply supported

</td>
<td>$y_b = 0, \quad M_{rb} = 0, \quad y_a = 0, \quad \theta_a = 0$
$\theta_b = \frac{-\gamma(1+\nu)\Delta T a}{t}\frac{C_6L_2 - C_3L_5}{C_1C_6 - C_3C_4}$
$Q_b = \frac{\gamma(1+\nu)\Delta T D}{at}\frac{C_4L_2 - C_1L_5}{C_1C_6 - C_3C_4}$
$M_{ra} = \theta_B\frac{D}{a}C_7 + Q_b a C_9 - \frac{\gamma(1+\nu)\Delta T D}{t}(1 - L_8)$
$Q_a = Q_b\frac{b}{a}$</td>
<td>If $r_o = b$ (ΔT over entire plate),
<table>
<tr><th>b/a</th><th>0.1</th><th>0.3</th><th>0.5</th><th>0.7</th><th>0.9</th></tr>
<tr><td>$K_{y_{\max}}$</td><td>− 0.0088</td><td>− 0.0133</td><td>− 0.0091</td><td>− 0.0039</td><td></td></tr>
<tr><td>K_{θ_b}</td><td>− 0.1222</td><td>− 0.1662</td><td>− 0.1396</td><td>− 0.0909</td><td>− 0.0319</td></tr>
<tr><td>$K_{M_{tb}}$</td><td>− 2.0219</td><td>− 1.4141</td><td>− 1.1641</td><td>− 1.0282</td><td>− 0.9422</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>− 1.3850</td><td>− 1.5620</td><td>− 1.6962</td><td>− 1.8076</td><td>− 1.9050</td></tr>
<tr><td>K_{Q_b}</td><td>− 2.3005</td><td>− 3.2510</td><td>− 4.3303</td><td>− 6.8757</td><td>− 19.8288</td></tr>
</table></td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

8h. Outer edge fixed, inner edge fixed



$y_b = 0, \quad \theta_b = 0, \quad y_a = 0, \quad \theta_a = 0$

$$M_{rb} = \frac{-\gamma(1+\nu)\Delta T D}{t} \frac{C_6 L_2 - C_3 L_5}{C_2 C_6 - C_3 C_5}$$

$$Q_b = \frac{\gamma(1+\nu)\Delta T D}{at} \frac{C_5 L_2 - C_2 L_5}{C_2 C_6 - C_3 C_5}$$

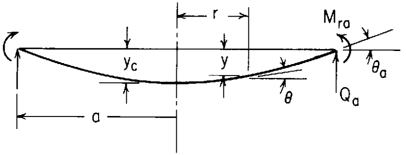
$$M_{ra} = M_{rb} C_8 + Q_b a C_9 - \frac{\gamma(1+\nu)\Delta T D}{t}(1 - L_8)$$

$$Q_a = Q_b \frac{b}{a}$$

If $r_o = b$ (ΔT over entire plate), all deflections are zero and $K_{M_r} = M_{M_t} = -1.30$ everywhere in the plate. If $r_o > b$, the following tabulated values apply.

b/a	0.1		0.5		0.7
r_o/a	0.5	0.7	0.7	0.9	0.9
$K_{M_{rb}}$	0.9224	1.3241	0.2640	0.5063	0.5103
$K_{M_{ra}}$	−1.4829	−1.1903	−1.6119	−0.8592	−1.2691
$K_{M_{ta}}$	−1.3549	−1.2671	−1.3936	−1.1677	−1.2907
K_{Q_b}	−10.4460	−10.9196	−5.4127	−3.9399	−7.1270

Cases 9 to 15. Solid circular plate under the several indicated loadings



General expressions for deformations, moments, and shears:

$$y = y_c + \frac{M_c r^2}{2D(1+\nu)} + LT_y$$ (*Note:* y_c is the center deflection)

$$\theta = \frac{M_c r}{D(1+\nu)} + LT_\theta$$ (*Note:* M_c is the moment at the center)

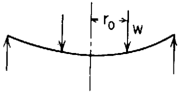
$$M_r = M_c + LT_M$$

$$M_t = \frac{\theta D(1-\nu^2)}{r} + \nu M_r$$ (*Note:* For $r < r_o$, $M_t = M_r = M_c$)

$$Q_r = LT_Q$$

For the numerical data given below, $\nu = 0.3$ (*Note:* ln = natural logarithm)

Case no., loading, load terms	Edge restraint	Boundary values	Special cases
9. Uniform annular line load	9a. Simply supported	$y_a = 0, \quad M_{ra} = 0$	$y = K_y \frac{wa^3}{D}, \quad \theta = K_\theta \frac{wa^2}{D}, \quad M = K_M wa$



$$LT_y = \frac{-wr^3}{D} G_3$$

$y_a = 0, \quad M_{ra} = 0$

$$y_c = \frac{-wa^3}{2D}\left(\frac{L_9}{1+\nu} - 2L_3\right)$$

$$M_c = waL_9$$

$$Q_a = -w\frac{r_o}{a}$$

$$\theta_a = \frac{wr_o(a^2 - r_o^2)}{2D(1+\nu)a}$$

$y = K_y \frac{wa^3}{D}, \quad \theta = K_\theta \frac{wa^2}{D}, \quad M = K_M wa$

r_o/a	0.2	0.4	0.6	0.8
K_{y_c}	−0.05770	−0.09195	−0.09426	−0.06282
K_{θ_a}	0.07385	0.12923	0.14769	0.11077
K_{M_c}	0.24283	0.29704	0.26642	0.16643

(*Note:* If r_o approaches 0, see case 16)

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

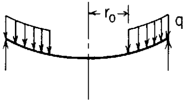
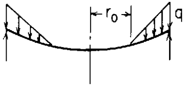
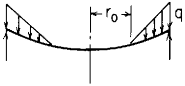
<table>
<tr><th>Case no., loading,
load terms</th><th>Edge restraint</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td>$LT_\theta = \frac{-wr^2}{D} G_6$
$LT_M = -wrG_9$
$LT_Q = \frac{-wr_o}{r} \langle r - r_o \rangle^0$</td>
<td>9b. Fixed</td>
<td>$y_c = \frac{-wa^3}{2D}(L_6 - 2L_3)$
$M_c = wa(1+\nu)L_6$
$M_{ra} = \frac{-wr_o}{2a^2}(a^2 - r_o^2)$
$y_a = 0, \quad \theta_a = 0$</td>
<td>
<table>
<tr><th>r_o/a</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th></tr>
<tr><td>K_{y_c}</td><td>−0.02078</td><td>−0.02734</td><td>−0.02042</td><td>−0.00744</td></tr>
<tr><td>K_{M_c}</td><td>0.14683</td><td>0.12904</td><td>0.07442</td><td>0.02243</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.09600</td><td>−0.16800</td><td>−0.19200</td><td>−0.14400</td></tr>
</table>
(*Note:* If r_o approaches 0, see case 17)
</td>
</tr>
<tr>
<td rowspan="2">10. Uniformly distributed pressure from r_o to a

$LT_y = \frac{-qr^4}{D} G_{11}$
$LT_\theta = \frac{-qr^3}{D} G_{14}$
$LT_M = -qr^2 G_{17}$
$LT_Q = \frac{-q}{24}(r^2 - r_o^2)\langle r - r_o \rangle^0$</td>
<td>10a. Simply supported</td>
<td>$y_a = 0, \quad M_{ra} = 0$
$y_c = \frac{-qa^4}{2D}\left(\frac{L_{17}}{1+\nu} - 2L_{11}\right)$
$M_c = qa^2 L_{17}$
$\theta_a = \frac{q}{8Da(1+\nu)}(a^2 - r_o^2)^2$
$Q_a = \frac{-q}{2a}(a^2 - r_o^2)$</td>
<td>
$y = K_y \frac{qa^4}{D}, \quad \theta = K_\theta \frac{qa^3}{D}, \quad M = K_M qa^2$
<table>
<tr><th>r_o/a</th><th>0.0</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th></tr>
<tr><td>K_{y_c}</td><td>−0.06370</td><td>−0.05767</td><td>−0.04221</td><td>−0.02303</td><td>−0.00677</td></tr>
<tr><td>K_{θ_a}</td><td>0.09615</td><td>0.08862</td><td>0.06785</td><td>0.03939</td><td>0.01246</td></tr>
<tr><td>K_{M_c}</td><td>0.20625</td><td>0.17540</td><td>0.11972</td><td>0.06215</td><td>0.01776</td></tr>
</table>
Note: If $r_o = 0, \quad G_{11} = \frac{1}{64}, \quad G_{14} = \frac{1}{16}, \quad G_{17} = \frac{(3+\nu)}{16}$
$y_c = \frac{-qa^4(5+\nu)}{64D(1+\nu)}, \quad M_c = \frac{qa^2(3+\nu)}{16}, \quad \theta_a = \frac{qa^3}{8D(1+\nu)}$
</td>
</tr>
<tr>
<td>10b. Fixed</td>
<td>$y_a = 0, \quad \theta_a = 0$
$y_c = \frac{-qa^4}{2D}(L_{14} - 2L_{11})$
$M_c = qa^2(1+\nu)L_{14}$
$M_{ra} = \frac{-q}{8a^2}(a^2 - r_o^2)^2$</td>
<td>
<table>
<tr><th>r_o/a</th><th>0.0</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th></tr>
<tr><td>K_{y_c}</td><td>−0.01563</td><td>−0.01336</td><td>−0.00829</td><td>−0.00344</td><td>−0.00054</td></tr>
<tr><td>K_{M_c}</td><td>0.08125</td><td>0.06020</td><td>0.03152</td><td>0.01095</td><td>0.00156</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.12500</td><td>−0.11520</td><td>−0.08820</td><td>−0.05120</td><td>−0.01620</td></tr>
</table>
Note: If $r_o = 0, \quad G_{11} = \frac{1}{64}, \quad G_{14} = \frac{1}{16}, \quad G_{17} = \frac{(3+\nu)}{16}$
$y_c = \frac{-qa^4}{64D}, \quad M_c = \frac{qa^2(1+\nu)}{16}, \quad M_{ra} = \frac{-qa^2}{8}$
</td>
</tr>
<tr>
<td>11. Linearly increasing pressure from r_o to a

$LT_y = \frac{-qr^4}{D}\frac{r - r_o}{a - r_o} G_{12}$
$LT_\theta = \frac{-qr^3}{D}\frac{r - r_o}{a - r_o} G_{15}$</td>
<td>11a. Simply supported</td>
<td>$M_{ra} = 0, \quad y_a = 0$
$y_c = \frac{-qa^4}{2D}\left(\frac{L_{18}}{1+\nu} - 2L_{12}\right)$
$M_c = qa^2 L_{18}$
$\theta_a = \frac{qa^3}{D}\left(\frac{L_{18}}{1+\nu} - L_{15}\right)$
$Q_a = \frac{-q}{6a}(2a^2 - r_o a - r_o^2)$</td>
<td>
$y = K_y \frac{qa^4}{D}, \quad \theta = K_\theta \frac{qa^3}{D}, \quad M = K_M qa^2$
<table>
<tr><th>r_o/a</th><th>0.0</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th></tr>
<tr><td>K_{y_c}</td><td>−0.03231</td><td>−0.02497</td><td>−0.01646</td><td>−0.00836</td><td>−0.00234</td></tr>
<tr><td>K_{θ_a}</td><td>0.05128</td><td>0.04070</td><td>0.02788</td><td>0.01485</td><td>0.00439</td></tr>
<tr><td>K_{M_c}</td><td>0.09555</td><td>0.07082</td><td>0.04494</td><td>0.02220</td><td>0.00610</td></tr>
</table>
Note: If $r_o = 0, \quad G_{12} = \frac{1}{225}, \quad G_{15} = \frac{1}{45}, \quad G_{18} = \frac{(4+\nu)}{45}$
$y_c = \frac{-qa^4(6+\nu)}{150D(1+\nu)}, \quad M_c = \frac{qa^2(4+\nu)}{45}, \quad \theta_a = \frac{qa^3}{15D(1+\nu)}$
</td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

<table>
<tr>
<td>$LT_M = -qr^2 \frac{r - r_o}{a - r_o} G_{18}$
$LT_Q = \frac{-q(2r^3 - 3r_o r^2 + r_o^3)}{6r(a - r_o)} \times \langle r - r_o \rangle^0$</td>
<td>11b. Fixed</td>
<td>$y_a = 0, \quad \theta_a = 0$
$y_c = \frac{-qa^4}{2D}(L_{15} - 2L_{12})$
$M_c = qa^2(1 + \nu)L_{15}$
$M_{ra} = -qa^2[L_{18} - (1 + \nu)L_{15}]$</td>
<td>
<table>
<tr><th>r_o/a</th><th>0.0</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th></tr>
<tr><td>K_{y_c}</td><td>−0.00667</td><td>−0.00462</td><td>−0.00252</td><td>−0.00093</td><td>−0.00014</td></tr>
<tr><td>K_{M_c}</td><td>0.02889</td><td>0.01791</td><td>0.00870</td><td>0.00289</td><td>0.00040</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.06667</td><td>−0.05291</td><td>−0.03624</td><td>−0.01931</td><td>−0.00571</td></tr>
</table>
Note: If $r_o = 0, \quad G_{12}\frac{1}{225}, \quad G_{15} = \frac{1}{45}, \quad G_{18} = \frac{(4 + \nu)}{45}$
$y_c = \frac{-qa^4}{150D}, \quad M_c = \frac{qa^2(1 + \nu)}{45}, \quad M_{ra} = \frac{-qa^2}{15}$
</td>
</tr>
<tr>
<td rowspan="2">12. Parabolically increasing pressure from r_o to a


$LT_y = \frac{-qr^4}{D}\left(\frac{r - r_o}{a - r_o}\right)^2 G_{13}$
$LT_\theta = \frac{-qr^3}{D}\left(\frac{r - r_o}{a - r_o}\right)^2 G_{16}$
$LT_M = -qr^2\left(\frac{r - r_o}{a - r_o}\right)^2 G_{19}$
$LT_Q = \frac{-q(3r^4 - 8r_o r^3 + 6r_o^2 r^2 - r_o^4)}{12r(a - r_o)^2} \times \langle r - r_o \rangle^0$</td>
<td>12a. Simply supported</td>
<td>$y_a = 0, \quad M_{ra} = 0$
$y_c = \frac{-qa^4}{2D}\left(\frac{L_{19}}{1 + \nu} - 2L_{13}\right)$
$M_c = qa^2 L_{19}$
$\theta_a = \frac{qa^3}{D}\left(\frac{L_{19}}{1 + \nu} - L_{16}\right)$
$Q_a = \frac{-q}{12a}(3a^2 - 2ar_o - r_o^2)$</td>
<td>
$y = K_y \frac{qa^4}{D}, \quad \theta = K_\theta \frac{qa^3}{D}, \quad M = K_M qa^2$
<table>
<tr><th>r_o/a</th><th>0.0</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th></tr>
<tr><td>K_{y_c}</td><td>−0.01949</td><td>−0.01419</td><td>−0.00893</td><td>−0.00438</td><td>−0.00119</td></tr>
<tr><td>K_{θ_a}</td><td>0.03205</td><td>0.02396</td><td>0.01560</td><td>0.00796</td><td>0.00227</td></tr>
<tr><td>K_{M_c}</td><td>0.05521</td><td>0.03903</td><td>0.02397</td><td>0.01154</td><td>0.00311</td></tr>
</table>
Note: If $r_o = 0, \quad G_{13} = \frac{1}{576}, \quad G_{16} = \frac{1}{96}, \quad G_{19} = \frac{(5 + \nu)}{96}$
$y_c = \frac{-qa^4(7 + \nu)}{288D(1 + \nu)}, \quad M_c = \frac{qa^2(5 + \nu)}{96}, \quad \theta_a = \frac{qa^3}{24D(1 + \nu)}$
</td>
</tr>
<tr>
<td>12b. Fixed</td>
<td>$y_a = 0, \quad \theta_a = 0$
$y_c = \frac{-qa^4}{2D}(L_{16} - 2L_{13})$
$M_c = qa^2(1 + \nu)L_{16}$
$M_{ra} = -qa^2[L_{19} - (1 + \nu)L_{16}]$</td>
<td>
<table>
<tr><th>r_o/a</th><th>0.0</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th></tr>
<tr><td>K_{y_c}</td><td>−0.00347</td><td>−0.00221</td><td>−0.00113</td><td>−0.00040</td><td>−0.000058</td></tr>
<tr><td>K_{M_c}</td><td>0.01354</td><td>0.00788</td><td>0.00369</td><td>0.00120</td><td>0.000162</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.04167</td><td>−0.03115</td><td>−0.02028</td><td>−0.01035</td><td>−0.002947</td></tr>
</table>
Note: If $r_o = 0, \quad G_{13} = \frac{1}{576}, \quad G_{16} = \frac{1}{96}, \quad G_{19} = \frac{(5 + \nu)}{96}$
$y_c = \frac{-qa^4}{288D}, \quad M_c = \frac{qa^2(1 + \nu)}{96}, \quad M_{ra} = \frac{-qa^2}{24}$
</td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

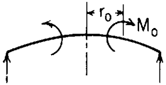
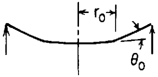
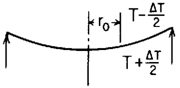
<table>
<tr><th>Case no., loading, load terms</th><th>Edge restraint</th><th>Boundary values</th><th>Special cases</th></tr>
<tr>
<td rowspan="2">13. Uniform line moment at r_o

$LT_y = \frac{M_o r^2}{D} G_2$
$LT_\theta = \frac{M_o r}{D} G_5$
$LT_M = M_o G_8$
$LT_Q = 0$</td>
<td>13a. Simply supported</td>
<td>$y_a = 0, \quad M_{ra} = 0, \quad Q_a = 0$
$y_c = \frac{M_o r_o^2}{2D}\left(\frac{1}{1+\nu} + \ln\frac{a}{r_o}\right)$
$M_c = -M_o L_8$
$\theta_a = \frac{-M_o r_o^2}{Da(1+\nu)}$</td>
<td>$y = K_y \frac{M_o a^2}{D}, \quad \theta = K_\theta \frac{M_o a}{D}, \quad M = K_M M_o$
<table>
<tr><th>r_o/a</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th><th>1.0</th></tr>
<tr><td>K_{y_c}</td><td>0.04757</td><td>0.13484</td><td>0.23041</td><td>0.31756</td><td>0.38462</td></tr>
<tr><td>K_{θ_a}</td><td>−0.03077</td><td>−0.12308</td><td>−0.27692</td><td>−0.49231</td><td>−0.76923</td></tr>
<tr><td>K_{M_c}</td><td>−0.66400</td><td>−0.70600</td><td>−0.77600</td><td>−0.87400</td><td>−1.00000</td></tr>
</table></td>
</tr>
<tr>
<td>13b. Fixed</td>
<td>$y_a = 0, \quad \theta_a = 0, \quad Q_a = 0$
$y_c = \frac{M_o r_o^2}{2D} \ln\frac{a}{r_o}$
$M_c = \frac{-M_o(1+\nu)}{2a^2}(a^2 - r_o^2)$
$M_{ra} = \frac{M_o r_o^2}{a^2}$</td>
<td><table>
<tr><th>r_o/a</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th></tr>
<tr><td>K_{y_c}</td><td>0.03219</td><td>0.07330</td><td>0.09195</td><td>0.07141</td></tr>
<tr><td>K_{M_c}</td><td>−0.62400</td><td>−0.54600</td><td>−0.41600</td><td>−0.23400</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>0.04000</td><td>0.16000</td><td>0.36000</td><td>0.64000</td></tr>
</table></td>
</tr>
<tr>
<td rowspan="2">14. Externally applied change in slope at a radius r_o

$LT_y = \theta_o r G_1$
$LT_\theta = \theta_o G_4$
$LT_M = \frac{\theta_o D}{r} G_7$
$LT_Q = 0$</td>
<td>14a. Simply supported</td>
<td>$y_a = 0, \quad M_{ra} = 0, \quad Q_a = 0$
$y_c = \frac{-\theta_o r_o(1+\nu)}{2}\ln\frac{a}{r_o}$
$M_c = \frac{-\theta_o D(1-\nu^2)}{2r_o a^2}(a^2 - r_o^2)$
$\theta_a = \frac{\theta_o r_o}{a}$</td>
<td>$y = K_y \theta_o a, \quad \theta = K_\theta \theta_o, \quad M = K_M \theta_o \frac{D}{a}$
<table>
<tr><th>r_o/a</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th></tr>
<tr><td>K_{y_c}</td><td>−0.20923</td><td>−0.23824</td><td>−0.19922</td><td>−0.11603</td></tr>
<tr><td>K_{θ_a}</td><td>0.20000</td><td>0.40000</td><td>0.60000</td><td>0.80000</td></tr>
<tr><td>K_{M_c}</td><td>−2.18400</td><td>−0.95550</td><td>−0.48533</td><td>−0.20475</td></tr>
<tr><td>$K_{M_{to}}$</td><td>2.33600</td><td>1.31950</td><td>1.03133</td><td>0.93275</td></tr>
</table></td>
</tr>
<tr>
<td>14b. Fixed</td>
<td>$y_a = 0, \quad \theta_a = 0, \quad Q_a = 0$
$y_c = \frac{\theta_o r_o}{2}\left[1 - (1+\nu)\ln\frac{a}{r_o}\right]$
$M_c = \frac{-\theta_o D(1+\nu)}{a} L_4$
$M_{ra} = \frac{-\theta_o D r_o}{a^2}(1+\nu)$</td>
<td><table>
<tr><th>r_o/a</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th><th>1.0</th></tr>
<tr><td>K_{y_c}</td><td>−0.10923</td><td>−0.03824</td><td>0.10078</td><td>0.28396</td><td>0.50000</td></tr>
<tr><td>K_{M_c}</td><td>−2.44400</td><td>−1.47550</td><td>−1.26533</td><td>−1.24475</td><td>−1.30000</td></tr>
<tr><td>$K_{M_{ra}}$</td><td>−0.26000</td><td>−0.52000</td><td>−0.78000</td><td>−1.04000</td><td>−1.30000</td></tr>
</table></td>
</tr>
</table>

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

15. Uniform temperature differential ΔT between the bottom and top surface from r_o to a



$$LT_y = \frac{\gamma(1+v)\Delta T}{t} r^2 G_2$$

$$LT_\theta = \frac{\gamma(1+v)\Delta T}{t} r G_5$$

$$LT_M = \frac{\gamma D(1+v)\Delta T}{t}(G_8 - \langle r - r_o \rangle^0)$$

$$LT_Q = 0$$

Note: Values for $K_{M_{to}}$ are given at a radius just greater than r_o

15a. Simply supported

$$y_a = 0, \quad M_{ra} = 0, \quad Q_a = 0$$

$$y_c = \frac{-\gamma \Delta T}{2t}\left[a^2 - r_o^2 - r_o^2(1+v)\ln\frac{a}{r_o}\right]$$

$$M_c = \frac{\gamma D(1+v)\Delta T}{t}(1 - L_8)$$

$$\theta_a = \frac{\gamma \Delta T}{ta}(a^2 - r_o^2)$$

$$y = K_y \frac{\gamma \Delta T a^2}{t}, \quad \theta = K_\theta \frac{\gamma \Delta T a}{t}, \quad M = K_M \frac{\gamma \Delta T D}{t}$$

r_o/a	0.0	0.2	0.4	0.6	0.8
K_{y_c}	−0.50000	−0.43815	−0.32470	−0.20047	−0.08717
K_{θ_a}	1.00000	0.96000	0.84000	0.64000	0.36000
K_{M_c}	0.00000	0.43680	0.38220	0.29120	0.16380
$K_{M_{to}}$		−0.47320	−0.52780	−0.61880	−0.74620

Note: When the entire plate is subjected to the temperature differential, there is no stress anywhere in the plate.

15b. Fixed

$$y_a = 0, \quad \theta_a = 0, \quad Q_a = 0$$

$$y_c = \frac{\gamma(1+v)\Delta T}{2t} r_o^2 \ln\frac{a}{r_o}$$

$$M_c = \frac{-\gamma D(1+v)^2 \Delta T}{2ta^2}(a^2 - r_o^2)$$

$$M_{ra} = \frac{-\gamma D(1+v)\Delta T}{ta^2}(a^2 - r_o^2)$$

r_o/a	0.0	0.2	0.4	0.6	0.8
K_{y_c}	0.00000	0.04185	0.09530	0.11953	0.09283
K_{M_c}	−1.30000	−0.81120	−0.70980	−0.54080	−0.30420
$K_{M_{ro}}$		−1.24800	−1.09200	−0.83200	−0.46800
$K_{M_{to}}$		−1.72120	−1.61980	−1.45080	−1.21420

Note: When the entire plate is subjected to the temperature differential, the moments are the same everywhere in the plate and there are no deflections.

Note: the term $\frac{-\gamma(1-v^2)\Delta TD}{t}\langle r - r_o\rangle^0$ must be added to M_t for this case 15. Also, if $r_o = 0$, then $G_2 = \frac{1}{4}$, $G_5 = \frac{1}{2}$, $G_8 = (1+v)/2$, and $\langle r - r_o \rangle^0 = 1$ for all values of r.

Cases 16 to 31. The following cases include loadings on circular plates or plates bounded by some circular boundaries (each case is complete in itself) (*Note:* ln = natural logarithm)

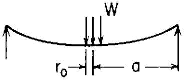
Case no., loading, restraints	Formulas	Special cases
16. Uniform load over a very small central circular area of radius r_o; edge simply supported  $W = q\pi r_o^2$	For $r > r_o$ $y = \frac{-W}{16\pi D}\left[\frac{3+v}{1+v}(a^2 - r^2) - 2r^2 \ln\frac{a}{r}\right]$ $\theta = \frac{Wr}{4\pi D}\left(\frac{1}{1+v} + \ln\frac{a}{r}\right)$ $M_r = \frac{W}{16\pi}\left[4(1+v)\ln\frac{a}{r} + (1-v)\left(\frac{a^2 - r^2}{a^2}\right)\frac{r_o'^2}{r^2}\right]$ where $r_o' = \sqrt{1.6r_o^2 + t^2} - 0.675t$ if $r_o < 0.5t$ or $r_o' = r_o$ if $r_o > 0.5t$ $M_t = \frac{W}{16\pi}\left[4(1+v)\ln\frac{a}{r} + (1-v)\left(4 - \frac{r_o'^2}{r^2}\right)\right]$	$y_{\max} = \frac{-Wa^2}{16\pi D}\frac{3+v}{1+v}$ at $r = 0$ $\theta_{\max} = \frac{Wa}{4\pi D(1+v)}$ at $r = a$ $(M_r)_{\max} = \frac{W}{4\pi}\left[(1+v)\ln\frac{a}{r_o'} + 1\right]$ at $r = 0$ $(M_t)_{\max} = (M_r)_{\max}$ at $r = 0$

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

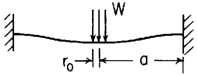
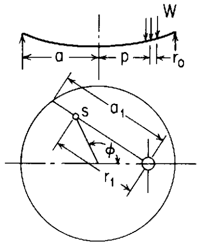
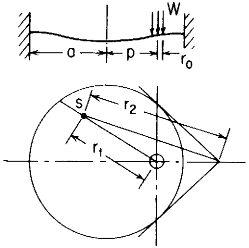
Case no., loading, restraints	Formulas	Special cases
17. Uniform load over a very small central circular area of radius r_o; edge fixed  $W = q\pi r_o^2$	For $r > r_o'$ $y = \frac{-W}{16\pi D}\left[a^2 - r^2\left(1+2\ln\frac{a}{r}\right)\right]$ $\theta = \frac{Wr}{4\pi D}\ln\frac{a}{r}$ $M_r = \frac{W}{4\pi}\left[(1+\nu)\ln\frac{a}{r} - 1 + \frac{(1-\nu)r_o'^2}{4r^2}\right]$ where $r_o' = \sqrt{1.6r_o^2+t^2} - 0.675t$ if $r_o < 0.5t$ or $r_o' = r_o$ if $r_o \geqslant 0.5t$ $M_t = \frac{W}{4\pi}\left[(1+\nu)\ln\frac{a}{r} - \nu + \frac{\nu(1-\nu)r_o'^2}{4r^2}\right]$	$y_{\max} = \frac{-Wa^2}{16\pi D}$ at $r = 0$ $\theta_{\max} = 0.0293\frac{Wa}{D}$ at $r = 0.368a$ $(+M_r)_{\max} = \frac{W}{4\pi}(1+\nu)\ln\frac{a}{r_o'}$ at $r = 0$ $(-M_r)_{\max} = \frac{-W}{4\pi}$ at $r = a$ $(+M_t)_{\max} = (+M_r)_{\max}$ at $r = 0$ $(-M_t)_{\max} = \frac{-\nu W}{4\pi}$ at $r = a$
18. Uniform load over a small eccentric circular area of radius r_o; edge simply supported  *Note:* $r_o' = \sqrt{1.6r_o^2+t^2} - 0.675t$ if $r_o < 0.5t$ or $r_o' = r_o$ if $r_o \geqslant 0.5t$	$(M_r)_{\max} = (M_t)_{\max} = \frac{W}{4\pi}\left[1+(1+\nu)\ln\left(\frac{a-p}{r_o'}\right) - \frac{(1-\nu)r_o'^2}{4(a-p)^2}\right]$ at the load At any point s, $M_r = (M_r)_{\max}\frac{(1+\nu)\ln(a_1/r_1)}{1+(1+\nu)\ln(a_1/r_o')}, \quad M_t = (M_t)_{\max}\frac{(1+\nu)\ln(a_1/r_1)+1-\nu}{1+(1+\nu)\ln(a_1/r_o')}$ $y = -[K_o(r^3 - b_o ar^2 + c_o a^3) + K_1(r^4 - b_1 ar^3 + c_1 a^3 r)\cos\phi + K_2(r^4 - b_2 ar^3 + c_2 a^2 r^2)\cos 2\phi]$ where $K_o = \frac{W}{\pi Da^4}\frac{2(1+\nu)}{9(5+\nu)}(p^3 - b_o ap^2 + c_o a^3)$ $K_1 = \frac{W}{\pi Da^6}\frac{2(3+\nu)}{3(9+\nu)}(p^4 - b_1 ap^3 + c_1 a^3 p)$ $K_2 = \frac{W}{\pi Da^6}\frac{(4+\nu)^2}{(9+\nu)(5+\nu)}(p^4 - b_2 ap^3 + c_2 a^2 p^2)$ where $b_o = \frac{3(2+\nu)}{2(1+\nu)}, \quad b_1 = \frac{3(4+\nu)}{2(3+\nu)}, \quad b_2 = \frac{2(5+\nu)}{4+\nu}, \quad c_o = \frac{4+\nu}{2(1+\nu)}, \quad c_1 = \frac{6+\nu}{2(3+\nu)}, \quad c_2 = \frac{6+\nu}{4+\nu}$ (See Ref. 1)	

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

19. Uniform load over a small eccentric circular area of radius r_o; edge fixed



At any point s,

$$y = \frac{-W}{16\pi D}\left[\frac{p^2 r_2^2}{a^2} - r_1^2\left(1 + 2\ln\frac{pr_2}{ar_1}\right)\right] \quad (\text{Note: As } p \to 0, pr_2 \to a^2)$$

At the load point,

$$y = \frac{-W}{16\pi D}\frac{(a^2 - p^2)^2}{a^2}$$

$$M_r = \frac{W(1+\nu)}{16\pi}\left[4\ln\left(\frac{a-p}{r'_o}\right) + \left(\frac{r'_o}{a-p}\right)^2\right] = M_{\max} \quad \text{if } r'_o < 0.6(a-p) \quad (\textit{Note}: r'_o \text{ defined in case 18})$$

At the near edge,

$$M_r = \frac{-W}{8\pi}\left[2 - \left(\frac{r'_o}{a-p}\right)^2\right] = M_{\max} \quad \text{if } r_o > 0.6(a-p)$$

[Formulas due to Michell (Ref. 2). See Ref. 60 for modified boundary conditions]

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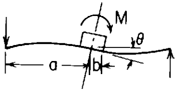
20. Central couple on an annular plate with a simply supported outer edge (trunnion loading)



(*Note:* For eccentric trunnions loaded with vertical loads, couples, and pressure on the plate, see Refs. 86 and 87)

20a. Trunnion simply supported to plate. For $\nu = 0.3$

$$\theta = \frac{\alpha M}{Et^3}, \quad \tau_{\max} = \tau_{rt} = \frac{\lambda M}{at^2} \text{ at } r = b \text{ at } 90^\circ \text{ to the plane of } M$$

$$\sigma_{\max} = \sigma_t = \frac{\gamma M}{at^2} \text{ at } r = b \text{ in the plane of } M$$

b/a	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80
λ	9.475	6.256	4.630	3.643	2.976	2.128	1.609	1.260	1.011	0.827
γ	12.317	8.133	6.019	4.735	3.869	2.766	2.092	1.638	1.314	1.075
α	2.624	2.256	1.985	1.766	1.577	1.257	0.984	0.743	0.528	0.333

(Ref. 85)

20b. Trunnion fixed to the plate

$$\theta = \frac{\alpha M}{Et^3}, \quad (\sigma_r)_{\max} = \frac{\beta M}{at^2} \text{ at } r = b \text{ in the plane of } M$$

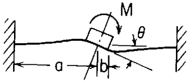
b/a	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80
β	9.478	6.252	4.621	3.625	2.947	2.062	1.489	1.067	0.731	0.449
α	1.403	1.058	0.820	0.641	0.500	0.301	0.169	0.084	0.035	0.010

(Ref. 21)

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

Case no., loading, load terms

21. Central couple on an annular plate with a fixed outer edge (trunnion loading)



(*Note:* For eccentric trunnions see note in case 20 above)

21a. Trunnion simply supported to plate. For $\nu = 0.3$

$$\theta = \frac{\alpha M}{Et^3}, \quad \tau_{\max} = \tau_{rt} = \frac{\lambda M}{at^2} \text{ at } r = b \text{ at } 90^\circ \text{ to the plane of } M$$

$$(\sigma_r)_{\max} = \frac{\beta M}{at^2} \text{ at } r = a \text{ in the plane of } M \quad \max \sigma_t = \frac{\gamma M}{at^2} \text{ at } r = b \text{ in the plane of } M$$

b/a	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80
λ	9.355	6.068	4.367	3.296	2.539	1.503	0.830	0.405	0.166	0.053
β	0.989	1.030	1.081	1.138	1.192	1.256	1.205	1.023	0.756	0.471
γ	12.161	7.889	5.678	4.285	3.301	1.954	1.079	0.526	0.216	0.069
α	2.341	1.949	1.645	1.383	1.147	0.733	0.405	0.184	0.064	0.015

(Ref. 85)

21b. Trunnion fixed to the plate. For $\nu = 0.3$

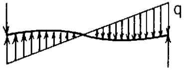
$$\theta = \frac{\alpha M}{Et^3} \quad \sigma_{\max} = \sigma_r = \frac{\beta M}{at^2} \text{ at } r = b \text{ in the plane of } M$$

$$\sigma_r = \frac{\beta b M}{a^2 t^2} \text{ at } r = a \text{ in the plane of } M$$

b/a	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80
β	9.36	6.08	4.41	3.37	2.66	1.73	1.146	0.749	0.467	0.262
α	1.149	0.813	0.595	0.439	0.320	0.167	0.081	0.035	0.013	0.003

(Ref. 22)

22. Linearly distributed load symmetrical about a diameter; edge simply supported



$$(M_r)_{\max} = \frac{qa^2(5+\nu)}{72\sqrt{3}} \quad \text{at } r = 0.577a$$

$$(M_t)_{\max} = \frac{qa^2(5+\nu)(1+3\nu)}{72(3+\nu)} \quad \text{at } r = 0.675a$$

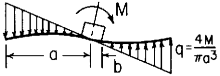
$$\text{Max edge reaction per linear inch} = \frac{qa}{4}$$

$$y_{\max} = 0.042\frac{qa^4}{Et^3} \quad \text{at } r = 0.503a \text{ (for } \nu = 0.3\text{)}$$

(Refs. 20 and 21)

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

23. Central couple balanced by linearly distributed pressure (footing)

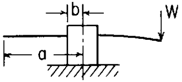


(At inner edge) $(\sigma_r)_{\max} = \beta \frac{M}{at^2}$ where β is given in the following table:

a/b	1.25	1.50	2.00	3.00	4.00	5.00
β	0.1625	0.4560	1.105	2.250	3.385	4.470

(Values for $\nu = 0.3$) (Ref. 21)

24. Concentrated load applied at the outer edge of an annular plate with a fixed inner edge



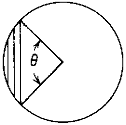
(At inner edge) $(\sigma_r)_{\max} = \beta \frac{W}{t^2}$ where β is given in the following table:

a/b	1.25	1.50	2.00	3.00	4.00	5.00
β	3.665	4.223	5.216	6.904	8.358	9.667

(Values for $\nu = 0.3$) (Ref. 93)

(See Ref. 64 for this loading on a plate with radially varying thickness. See graphs in Ref. 59 for the load distributed over an arc at the edge. See Ref. 60 for the load W placed away from the edge)

25. Solid circular plate with a uniformly distributed load q over the shaded segment



$\sigma_{\max} = (\sigma_r)_{\max} = \beta \frac{qa^2}{t^2}$

$y_{\max} = \alpha \frac{qa^4}{Et^3}$ on the symmetrical diameter at the value of r given in the table

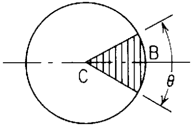
Edge	Coefficient	θ		
		90°	120°	180°
Supported	α	0.0244, $r = 0.39a$	0.0844, $r = 0.30a$	0.345, $r = 0.15a$
	β	0.306, $r = 0.60a$		
Fixed	α	0.00368, $r = 0.50a$	0.0173, $r = 0.4a$	0.0905, $r = 0.20a$
	β	0.285, $r = a$		

Values for $\nu = \frac{1}{3}$ (Ref. 39)

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

Case no., loading, load terms

26. Solid circular plate, uniform load q over the shaded sector



For simply supported edges:

$\sigma_{\max} = \sigma_r$ near the center along the loaded radius of symmetry (values not given)

σ_r at the center $= \dfrac{\theta}{360}\sigma_r$ at the center of a fully loaded plate

$y_{\max} = -\alpha_1 \dfrac{qa^4}{Et^3}$ at approximately $\frac{1}{4}$ the radius from center along the radius of symmetry (α_1 given in table)

For fixed edges:

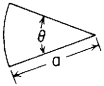
$\sigma_{\max} = \sigma_r$ at point $B = \beta \dfrac{qa^2}{t^2}$

$y_{\max} = -\alpha_2 \dfrac{qa^4}{Et^3}$ at approximately $\frac{1}{4}$ the radius from center along the radius of symmetry (β and α_2 given in table)

Edge condition	Coefficient	θ					
		30°	60°	90°	120°	150°	180°
Simply supported	α_1	0.061	0.121	0.179	0.235	0.289	0.343
Fixed	α_2	0.017	0.034	0.050	0.064	0.077	0.089
	β	0.240	0.371	0.457	0.518	0.564	0.602

[*Note:* For either edge condition $y_c = (\theta/360)y_c$ for a fully loaded plate] (Ref. 38)

27. Solid circular sector, uniformly distributed load q over the entire surface; edges simply supported



$(\sigma_r)_{\max} = \beta \dfrac{qa^2}{t^2}, \quad (\sigma_t)_{\max} = \beta_1 \dfrac{qa^2}{t^2}, \quad y_{\max} = \alpha \dfrac{qa^4}{Et^3}$

θ	45°	60°	90°	180°
β	0.102	0.147	0.240	0.522
β_1	0.114	0.155	0.216	0.312
α	0.0054	0.0105	0.0250	0.0870

(Values for $\nu = 0.3$) (Ref. 21)

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

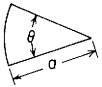
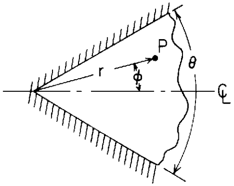
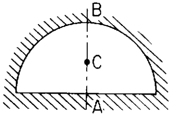
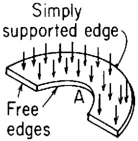
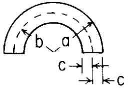
<table>
<tr>
<td>28. Solid circular sector, uniformly distributed load q over the entire surface; straight edges simply supported, curved edge fixed
 </td>
<td colspan="2">$\sigma_{\max} = \sigma_r$ at curved edge $= \beta \dfrac{qa^2}{t^2}$, $\quad y_{\max} = \alpha \dfrac{qa^4}{Et^3}$
<table><tr><th>θ</th><th>45°</th><th>60°</th><th>90°</th><th>180°</th></tr><tr><td>β</td><td>0.1500</td><td>0.2040</td><td>0.2928</td><td>0.4536</td></tr><tr><td>α</td><td>0.0035</td><td>0.0065</td><td>0.0144</td><td>0.0380</td></tr></table>(Values for $\nu = 0.3$)
(Ref. 21)</td>
</tr>
<tr>
<td>29. Solid circular sector of infinite radius, uniformly distributed load q over entire surface; straight edges fixed
 </td>
<td>At point P:
$$\sigma_r = \frac{9qr^2}{8t^2}\left[\frac{3+\nu}{3} - \frac{4\cos\theta\cos 2\phi - (1-\nu)\cos 4\phi}{2\cos^2\theta + 1}\right]$$
$$\sigma_t = \frac{9qr^2}{8t^2}\left[\frac{1+3\nu}{3} - \frac{4\nu\cos\theta\cos 2\phi + (1-\nu)\cos 4\phi}{2\cos^2\theta + 1}\right]$$
$$y = \frac{-3(1-\nu^2)qr^4}{16Et^3}\left(1 + \frac{\cos 4\phi - 4\cos\theta\cos 2\phi}{2\cos^2\theta + 1}\right)$$
(*Note:* θ should not exceed 60°)</td>
<td>At the edge, $\phi = \pm\theta/2$:
$$\sigma_t = \frac{3qr^2}{2t^2}\frac{\sin^2\theta}{1+2\cos^2\theta}$$
$\sigma_r = \nu\sigma_t$
Along the center line, $\phi = 0$
$$\sigma_r = \frac{3qr^2}{4t^2}\frac{3(1-\cos\theta)^2 - \nu\sin^2\theta}{1+2\cos^2\theta}$$
$$\sigma_t = \frac{3qr^2}{4t^2}\frac{3\nu(1-\cos\theta)^2 - \sin^2\theta}{1+2\cos^2\theta}$$
$$y = \frac{-3(1-\nu^2)qr^4}{8Et^3}\frac{(1-\cos\theta)^2}{1+2\cos^2\theta}$$
(Ref. 37)</td>
</tr>
<tr>
<td>30. Solid semicircular plate, uniformly distributed load q over the entire surface; all edges fixed
 </td>
<td colspan="2">$\sigma_{\max} = \sigma_r$ at $A = -0.42\dfrac{qa^2}{t^2}$ (values for $\nu = 0.2$)
σ_r at $B = -0.36\dfrac{qa^2}{t^2}$
σ_r at $C = 0.21\dfrac{qa^2}{t^2}$
(Ref. 40)</td>
</tr>
</table>

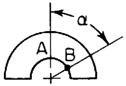
TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

Case no., loading, load terms

31. Semicircular annular plate, uniformly loaded over entire surface; outer edge supported, all other edges free







Formulas valid for $b \geqslant 0.7a$

(At A) $\sigma_t = \dfrac{6qcb}{t^2}\left(\dfrac{b}{c} - \dfrac{1}{3}\right)\left[c_1\left(1 - \gamma_1^2\dfrac{c}{b}\right) + c_2\left(1 - \gamma_2^2\dfrac{c}{b}\right) + \dfrac{c}{b}\right]K$ max stress occurs on inner edge over central 60°

(At B) $y = \dfrac{-24qc^2b^2}{Et^3}\left(\dfrac{b}{c} - \dfrac{1}{3}\right)\left[c_1 \cosh \gamma_1 \alpha + c_2 \cosh \gamma_2 \alpha + \dfrac{c}{b}\right]$ max deflection occurs when $\alpha = \dfrac{\pi}{2}$

where $c_1 = \dfrac{1}{\left(\dfrac{b}{c} - \gamma_1^2\right)(\lambda - 1)\cosh\dfrac{\gamma_1\pi}{2}}$, $\quad c_2 = \dfrac{1}{\left(\dfrac{b}{c} - \gamma_2^2\right)\left(\dfrac{1}{\lambda} - 1\right)\cosh\dfrac{\gamma_2\pi}{2}}$

$$\gamma_1 = \frac{\gamma}{\sqrt{2}}\sqrt{1 + \sqrt{1 - \frac{4b^2}{c^2\gamma^4}}}, \qquad \gamma_2 = \frac{\gamma}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \frac{4b^2}{c^2\gamma^4}}}$$

$$\gamma = \sqrt{\frac{2b}{c} + 4\left(1 - \frac{0.625t}{2c}\right)\frac{G}{E}\left(1 + \frac{b}{c}\right)^2}, \qquad \lambda = \frac{\gamma_1\left(\frac{b}{c} - \gamma_1^2 + \lambda_1\right)\left(\frac{b}{c} - \gamma_2^2\right)\tanh\frac{\gamma_1\pi}{2}}{\gamma_2\left(\frac{b}{c} - \gamma_2^2 + \lambda_1\right)\left(\frac{b}{c} - \gamma_1^2\right)\tanh\frac{\gamma_2\pi}{2}}, \qquad \lambda_1 = 4\left(1 - \frac{0.625t}{2c}\right)\frac{G}{E}\left(1 + \frac{b}{c}\right)^2$$

K = function of $(b - c)/(b + c)$ and has values as follows:

$(b - c)/(b + c)$	0.4	0.5	0.6	0.7	0.8	0.9	1.0
K	1.58	1.44	1.32	1.22	1.13	1.06	1.0

[Formulas due to Wahl (Ref. 10)]

TABLE 11.2 Formulas for flat circular plates of constant thickness (*Continued*)

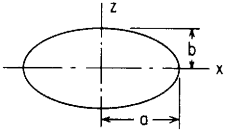
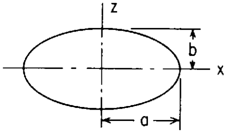
32. Elliptical plate, uniformly distributed load q over entire surface  $\alpha = \frac{b}{a}$	32a. Simply supported	At the center: $\sigma_{\max} = \sigma_z = -[2.816 + 1.581\nu - (1.691 + 1.206\nu)\alpha]\frac{qb^2}{t^2}$ $y_{\max} = -[2.649 + 0.15\nu - (1.711 + 0.75\nu)\alpha]\frac{qb^4(1-\nu^2)}{Et^3}$ [Approximate formulas for $0.2 < \alpha < 1.0$ (see numerical data in Refs. 21 and 56)]
	32b. Fixed	At the edge of span b: $\sigma_{\max} = \sigma_z = \frac{6qb^2}{t^2(3 + 2\alpha^2 + 3\alpha^4)}$ At the edge of span a: $\sigma_x = \frac{6qb^2\alpha^2}{t^2(3 + 2\alpha^2 + 3\alpha^4)}$ At the center: $\sigma_z = \frac{-3qb^2(1 + \nu\alpha^2)}{t^2(3 + 2a^2 + 3\alpha^4)}, \quad \sigma_x = \frac{-3qb^2(\alpha^2 + \nu)}{t^2(3 + 2\alpha^2 + 3\alpha^4)}, \quad y_{\max} = \frac{-3qb^4(1-\nu^2)}{2Et^3(3 + 2\alpha^2 + 3\alpha^4)}$ (Ref. 5)
33. Elliptical plate, uniform load over a small concentric circular area of radius r_o (note definition of r'_o in case 18)  $\alpha = \frac{b}{a}$	33a. Simply supported	At the center: $\sigma_{\max} = \sigma_z = \frac{-3W}{2\pi t^2}\left[(1+\nu)\ln\frac{b}{r'_o} + \nu(6.57 - 2.57\alpha)\right]$ $y_{\max} = \frac{-Wb^2}{Et^3}(0.76 - 0.18\alpha) \quad \text{for } \nu = 0.25$ [Approximate formulas by interpolation between cases of circular plate and infinitely long narrow strip (Ref. 4)]
	33b. Fixed	At the center: $\sigma_z = \frac{-3W(1+\nu)}{2\pi t^2}\left(\ln\frac{2b}{r'_o} - 0.317\alpha - 0.376\right)$ $y_{\max} = \frac{-Wb^2}{Et^3}(0.326 - 0.104\alpha) \quad \text{for } \nu = 0.25$ [Approximate formulas by interpolation between cases of circular plate and infinitely long narrow strip (ref. 6).]

TABLE 11.3 Shear deflections for flat circular plates of constant thickness

NOTATION: y_{sb}, y_{sa}, and y_{sr_o} are the deflections at b, a, and r_o, respectively, caused by transverse shear stresses. K_{sb}, K_{sa}, and K_{sr_o} are deflection coefficients defined by the relationships $y_s = K_s wa/tG$ for an annular line load and $y_s = K_s qa^2/tG$ for all distributed loadings (see Table 11.2 for all other notation and for the loading cases referenced)

Case no.	Shear deflection coefficients	Tabulated values for specific cases					
1a, 1b, 1e, 1f, 9	$K_{sr_o} = K_{sb} = -1.2\frac{r_o}{a}\ln\frac{a}{r_o}$ (*Note:* $r_o > 0$)	r_o/a	0.1	0.3	0.5	0.7	0.9
		K_{sb}	-0.2763	-0.4334	-0.4159	-0.2996	-0.1138
2a, 2b, 2e, 2f, 10	$K_{sr_o} = K_{sb} = -0.30\left[1-\left(\frac{r_o}{a}\right)^2\left(1+2\ln\frac{a}{r_o}\right)\right]$	K_{sb}	-0.2832	-0.2080	-0.1210	-0.0481	-0.0058
3a, 3b, 3e, 3f, 11	$K_{sr_o} = K_{sb} = \frac{-a}{30(a-r_o)}\left[4-9\frac{r_o}{a}+\left(\frac{r_o}{a}\right)^3\left(5+6\ln\frac{a}{r_o}\right)\right]$	K_{sb}	-0.1155	-0.0776	-0.0430	-0.0166	-0.0019
4a, 4b, 4e, 4f, 12	$K_{sr_o} = K_{sb} = \frac{-a^2}{120(a-r_o)^2}\left[9-32\frac{r_o}{a}+36\left(\frac{r_o}{a}\right)^2-\left(\frac{r_o}{a}\right)^4\left(13+12\ln\frac{a}{r_o}\right)\right]$	K_{sb}	-0.0633	-0.0411	-0.0223	-0.0084	-0.00098
1i, 1j, 1k, 1l	$K_{sr_o} = K_{sa} = -1.2\frac{r_o}{a}\ln\frac{r_o}{b}$ (*Note:* $b > 0$)	b/a \ r_o/a	0.2	0.4	0.6	0.8	1.0
		0.1	-0.1664	-0.6654	-1.2901	-1.9963	-2.7631
		0.3		-0.1381	-0.4991	-0.9416	-1.4448
		0.5			-0.1313	-0.4512	-0.8318
		0.7	Values of K_{sa}			-0.1282	-0.4280
		0.9					-0.1264
2i, 2j, 2k, 2l	$K_{sr_o} = -0.60\left[1-\left(\frac{r_o}{a}\right)^2\right]\ln\frac{r_o}{b}$ (*Note:* $b > 0$)	b/a \ r_o/a	0.1	0.3	0.5	0.7	0.9
		0.1	-0.0000	-0.5998	-0.7242	-0.5955	-0.2505
		0.3		-0.0000	-0.2299	-0.2593	-0.1252
		0.5			-0.0000	-0.1030	-0.0670
		0.7	Values of K_{sr_o}			-0.0000	-0.0287
		0.9					-0.0000
	$K_{sa} = -0.30\left[2\ln\frac{a}{b}-1+\left(\frac{r_o}{a}\right)^2\left(1-2\ln\frac{r_o}{b}\right)\right]$ (*Note:* $b > 0$)	0.1	-1.0846	-1.0493	-0.9151	-0.6565	-0.2567
		0.3		-0.4494	-0.4208	-0.3203	-0.1315
		0.5			-0.1909	-0.1640	-0.0732
		0.7	Values of K_{sa}			-0.0610	-0.0349
		0.9					-0.0062

TABLE 11.3 Shear deflections for flat circular plates of constant thickness (*Continued*)

		b/a \ r_o/a	0.1	0.3	0.5	0.7	0.9
3i, 3j, 3k, 3l	$K_{sr_o} = -0.20\left[2 - \frac{r_o}{a} - \left(\frac{r_o}{a}\right)^2\right]\ln\frac{r_o}{b}$ (*Note:* $b > 0$)	0.1	-0.0000	-0.3538	-0.4024	-0.3152	-0.1274
		0.3		-0.0000	-0.1277	-0.1373	-0.0637
		0.5			-0.0000	-0.0545	-0.0341
		0.7	Values of K_{sr_o}			-0.0000	-0.0146
		0.9					-0.0000
	$K_{sa} = \frac{-a}{30(a-r_o)}\left[6\left(2-3\frac{r_o}{a}\right)\ln\frac{a}{b} - 4 + 9\frac{r_o}{a} - \left(\frac{r_o}{a}\right)^3(5 - 6\ln\frac{r_o}{b}\Bigr)\right]$ (*Note:* $b > 0$)	0.1	-0.7549	-0.6638	-0.5327	-0.3565	-0.1316
		0.3		-0.3101	-0.2580	-0.1785	-0.0679
		0.5			-0.1303	-0.0957	-0.0383
		0.7	Values of K_{sa}			-0.0412	-0.0187
		0.9					-0.0042
4i, 4j, 4k, 4l	$K_{sr_o} = -0.10\left[3 - 2\frac{r_o}{a} - \left(\frac{r_o}{a}\right)^2\right]\ln\frac{r_o}{b}$ (*Note:* $b > 0$)	0.1	-0.0000	-0.2538	-0.2817	-0.2160	-0.0857
		0.3		-0.0000	-0.0894	-0.0941	-0.0428
		0.5			-0.0000	-0.0373	-0.0229
		0.7	Values of K_{sr_o}			-0.0000	-0.0098
		0.9					-0.0000
	$K_{sa} = \frac{-a^2}{120(a-r_o)^2}\left\{12\left[3 - 8\frac{r_o}{a} + 6\left(\frac{r_o}{a}\right)^2\right]\ln\frac{a}{b} - 9 + 32\frac{r_o}{a} - 36\left(\frac{r_o}{a}\right)^2 + \left(\frac{r_o}{a}\right)^4\left(13 - 12\ln\frac{r_o}{b}\right)\right\}$ (*Note:* $b > 0$)	0.1	-0.5791	-0.4908	-0.3807	-0.2472	-0.0888
		0.3		-0.2370	-0.1884	-0.1252	-0.0460
		0.5			-0.0990	-0.0685	-0.0261
		0.7	Values of K_{sa}			-0.0312	-0.0129
		0.9					-0.0031

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness

NOTATION: The notation for Table 11.2 applies with the following modifications: a and b refer to plate dimensions, and when used as subscripts for stress, they refer to the stresses in directions parallel to the sides a and b, respectively. σ is a bending stress which is positive when tensile on the bottom and compressive on the top if loadings are considered vertically downward. R is the reaction force per unit length normal to the plate surface exerted by the boundary support on the edge of the plate. r'_o is the equivalent radius of contact for a load concentrated on a very small area and is given by $r'_o = \sqrt{1.6r_o^2 + t^2} - 0.675t$ if $r_o < 0.5t$ and $r'_o = r_o$ if $r_o \geqslant 0.5t$

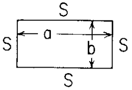
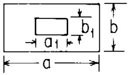
<table>
<tr><th>Case no., shape, and supports</th><th>Case no., loading</th><th>Formulas and tabulated specific values</th></tr>
<tr><td rowspan="2">1. Rectangular plate; all edges simply supported
 </td><td>1a. Uniform over entire plate</td><td>(At center) $\sigma_{\max} = \sigma_b = \dfrac{\beta q b^2}{t^2}$ and $y_{\max} = \dfrac{-\alpha q b^4}{Et^3}$
(At center of long sides) $R_{\max} = \gamma q b$
<table>
<tr><th>a/b</th><th>1.0</th><th>1.2</th><th>1.4</th><th>1.6</th><th>1.8</th><th>2.0</th><th>3.0</th><th>4.0</th><th>5.0</th><th>∞</th></tr>
<tr><td>β</td><td>0.2874</td><td>0.3762</td><td>0.4530</td><td>0.5172</td><td>0.5688</td><td>0.6102</td><td>0.7134</td><td>0.7410</td><td>0.7476</td><td>0.7500</td></tr>
<tr><td>α</td><td>0.0444</td><td>0.0616</td><td>0.0770</td><td>0.0906</td><td>0.1017</td><td>0.1110</td><td>0.1335</td><td>0.1400</td><td>0.1417</td><td>0.1421</td></tr>
<tr><td>γ</td><td>0.420</td><td>0.455</td><td>0.478</td><td>0.491</td><td>0.499</td><td>0.503</td><td>0.505</td><td>0.502</td><td>0.501</td><td>0.500</td></tr>
</table>
(Ref. 21 for $\nu = 0.3$)</td></tr>
<tr><td>1b. Uniform over small concentric circle of radius r_o (note definition of r'_o)</td><td>(At center) $\sigma_{\max} = \dfrac{3W}{2\pi t^2}\left[(1+\nu)\ln\dfrac{2b}{\pi r'_o} + \beta\right]$
$y_{\max} = \dfrac{-\alpha W b^2}{Et^3}$
<table>
<tr><th>a/b</th><th>1.0</th><th>1.2</th><th>1.4</th><th>1.6</th><th>1.8</th><th>2.0</th><th>∞</th></tr>
<tr><td>β</td><td>0.435</td><td>0.650</td><td>0.789</td><td>0.875</td><td>0.927</td><td>0.958</td><td>1.000</td></tr>
<tr><td>α</td><td>0.1267</td><td>0.1478</td><td>0.1621</td><td>0.1715</td><td>0.1770</td><td>0.1805</td><td>0.1851</td></tr>
</table>
(Ref. 21 for $\nu = 0.3$)</td></tr>
</table>

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

1c. Uniform over central rectangular area

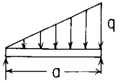


(At center) $\sigma_{\max} = \sigma_b = \dfrac{\beta W}{t^2}$ where $W = qa_1b_1$

a_1/b	$a = b$						$a = 1.4b$						$a = 2b$					
b_1/b	0	0.2	0.4	0.6	0.8	1.0	0	0.2	0.4	0.8	1.2	1.4	0	0.4	0.8	1.2	1.6	2.0
0		1.82	1.38	1.12	0.93	0.76		2.0	1.55	1.12	0.84	0.75		1.64	1.20	0.97	0.78	0.64
0.2	1.82	1.28	1.08	0.90	0.76	0.63	1.78	1.43	1.23	0.95	0.74	0.64	1.73	1.31	1.03	0.84	0.68	0.57
0.4	1.39	1.07	0.84	0.72	0.62	0.52	1.39	1.13	1.00	0.80	0.62	0.55	1.32	1.08	0.88	0.74	0.60	0.50
0.6	1.12	0.90	0.72	0.60	0.52	0.43	1.10	0.91	0.82	0.68	0.53	0.47	1.04	0.90	0.76	0.64	0.54	0.44
0.8	0.92	0.76	0.62	0.51	0.42	0.36	0.90	0.76	0.68	0.57	0.45	0.40	0.87	0.76	0.63	0.54	0.44	0.38
1.0	0.76	0.63	0.52	0.42	0.35	0.30	0.75	0.62	0.57	0.47	0.38	0.33	0.71	0.61	0.53	0.45	0.38	0.30

(Values from charts of Ref. 8; $\nu = 0.3$.)

1d. Uniformly increasing along length

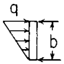


$\sigma_{\max} = \dfrac{\beta q b^2}{t^2}$ and $y_{\max} = \dfrac{-\alpha q b^4}{Et^3}$

a/b	1	1.5	2.0	2.5	3.0	3.5	4.0
β	0.16	0.26	0.34	0.38	0.43	0.47	0.49
α	0.022	0.043	0.060	0.070	0.078	0.086	0.091

(Values from charts of Ref. 8; $\nu = 0.3$.)

1e. Uniformly increasing along width



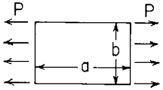
$\sigma_{\max} = \dfrac{\beta q b^2}{t^2}$ and $y_{\max} = \dfrac{-\alpha q b^4}{Et^3}$

a/b	1	1.5	2.0	2.5	3.0	3.5	4.0
β	0.16	0.26	0.32	0.35	0.37	0.38	0.38
α	0.022	0.042	0.056	0.063	0.067	0.069	0.070

(Values from charts of Ref. 8; $\nu = 0.3$.)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

Case no., shape, and supports	Case no., loading	Formulas and tabulated specific values
	1f. Uniform over entire plate plus uniform tension or compression P lb/linear in applied to short edges	$y_{\max} = \alpha \frac{qb^4}{Et^3}$ $(\sigma_a)_{\max} = \beta_x \frac{qb^2}{t^2}$ $(\sigma_b)_{\max} = \beta_y \frac{qb^2}{t^2}$. Here α, β_x, and β_y depend on ratios $\frac{a}{b}$ and $\frac{P}{P_E}$, where $P_E = \frac{\pi^2 Et^3}{3(1-\nu^2)b^2}$, and have the following values:

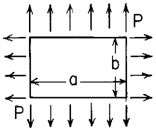


Coef.	P/P_E a/b	0	0.15	0.25	0.50	0.75	1	2	3	4	5
					P, Tension						
α	1	0.044	0.039		0.030		0.023	0.015	0.011	0.008	0.0075
	$1\frac{1}{2}$	0.084	0.075		0.060		0.045	0.0305	0.024	0.019	0.0170
	2	0.110	0.100		0.084		0.067	0.0475	0.0375	0.030	0.0260
	3	0.133	0.125		0.1135		0.100	0.081	0.066	0.057	0.0490
	4	0.140	0.136		0.1280		0.118	0.102	0.089	0.080	0.072
β_x	1	0.287					0.135	0.096	0.072	0.054	0.045
	$1\frac{1}{2}$	0.300					0.150	0.105	0.078	0.066	0.048
	2	0.278					0.162	0.117	0.093	0.075	0.069
	3	0.246					0.180	0.150	0.126	0.105	0.093
	4	0.222					0.192	0.168	0.156	0.138	0.124
β_y	1	0.287					0.132	0.084	0.054	0.036	0.030
	$1\frac{1}{2}$	0.487					0.240	0.156	0.114	0.090	0.072
	2	0.610					0.360	0.258	0.198	0.162	0.138
	3	0.713					0.510	0.414	0.348	0.294	0.258
	4	0.741					0.624	0.540	0.480	0.420	0.372
					P, Compression						
α	1	0.044		0.060	0.094	0.180					
	$1\frac{1}{2}$	0.084		0.109	0.155	0.237					
	2	0.110		0.139	0.161	0.181					
	3	0.131		0.145	0.150	0.150					
	4	0.140		0.142	0.142	0.138					
β_x	1	0.287		0.372	0.606	1.236					
	$1\frac{1}{2}$	0.300		0.372	0.522	0.846					
	2	0.278		0.330	0.390	0.450					
	3	0.246		0.228	0.228	0.210					
	4	0.222		0.225	0.225	0.225					
β_y	1	0.287		0.420	0.600	1.260					
	$1\frac{1}{2}$	0.487		0.624	0.786	1.380					
	2	0.610		0.720	0.900	1.020					
	3	0.713		0.750	0.792	0.750					
	4	0.741		0.750	0.750	0.750					

In the above formulas σ_a and σ_b are stresses due to bending only; add direct stress P/t to σ_a (Ref. 41)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

1g. Uniform over entire plate plus uniform tension P lb/linear in applied to all edges

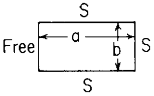


$y_{\max} = \alpha \dfrac{qb^4}{Et^3}$ $(\sigma_a)_{\max} = \beta_x \dfrac{qb^2}{t^2}$ $(\sigma_b)_{\max} = \beta_y \dfrac{qb^2}{t^2}$. Here α, β_x, and β_y depend on ratios $\dfrac{a}{b}$ and $\dfrac{P}{P_E}$, where $P_E = \dfrac{\pi^2 E t^3}{3(1-\nu^2)b^2}$, and have the following values:

Coef.	P/P_E \ a/b	0	0.15	0.5	1	2	3	4	5
α	1	0.044	0.035	0.022	0.015	0.008	0.006	0.004	0.003
	$1\frac{1}{2}$	0.084	0.060	0.035	0.022	0.012	0.008	0.006	0.005
	2	0.110	0.075	0.042	0.025	0.014	0.010	0.007	0.006
	3	0.133	0.085	0.045	0.026	0.016	0.011	0.008	0.007
	4	0.140	0.088	0.046	0.026	0.016	0.011	0.008	0.007
β_x	1	0.287	0.216	0.132	0.084	0.048	0.033	0.026	0.021
	$1\frac{1}{2}$	0.300	0.204	0.117	0.075	0.045	0.031	0.024	0.020
	2	0.278	0.189	0.111	0.072	0.044	0.031	0.024	0.020
	3	0.246	0.183	0.108	0.070	0.043	0.031	0.025	0.020
	4	0.222	0.183	0.108	0.074	0.047	0.032	0.027	0.024
β_y	1	0.287	0.222	0.138	0.090	0.051	0.036	0.030	0.024
	$1\frac{1}{2}$	0.487	0.342	0.186	0.108	0.066	0.042	0.036	0.030
	2	0.610	0.302	0.216	0.132	0.072	0.051	0.042	0.036
	3	0.713	0.444	0.234	0.141	0.078	0.054	0.042	0.036
	4	0.741	0.456	0.240	0.144	0.078	0.054	0.042	0.036

In the above formulas σ_a and σ_b are stresses due to bending only; add direct stress P/t to σ_a and σ_b. (Ref. 42)

2. Rectangular plate; three edges simply supported, one edge (b) free



2a. Uniform over entire plate

$\sigma_{\max} = \dfrac{\beta q b^2}{t^2}$ and $y_{\max} = \dfrac{-\alpha q b^4}{Et^3}$

a/b	0.50	0.667	1.0	1.5	2.0	4.0
β	0.36	0.45	0.67	0.77	0.79	0.80
α	0.080	0.106	0.140	0.160	0.165	0.167

(Ref. 8 for $\nu = 0.3$)

2d. Uniformly increasing along the a side

$\sigma_{\max} = \dfrac{\beta q b^2}{t^2}$ and $y_{\max} = \dfrac{-\alpha q b^4}{Et^3}$

a/b	0.50	0.667	1.0	1.5	2.0	2.5	3.0	3.5	4.0
β	0.11	0.16	0.20	0.28	0.32	0.35	0.36	0.37	0.37
α	0.026	0.033	0.040	0.050	0.058	0.064	0.067	0.069	0.070

(Ref. 8 for $\nu = 0.3$)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

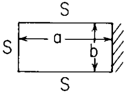
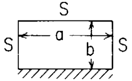
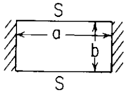
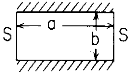
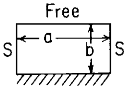
<table>
<tr><th>Case no., shape, and supports</th><th>Case no., loading</th><th>Formulas and tabulated specific values</th></tr>
<tr><td>3. Rectangular plate; three edges simply supported, one short edge (b) fixed
 </td><td>3a. Uniform over entire plate</td><td>$\sigma_{\max} = \frac{\beta q b^2}{t^2}$ and $y_{\max} = \frac{-\alpha q b^4}{E t^3}$
<table>
<tr><th>a/b</th><th>1</th><th>1.5</th><th>2.0</th><th>2.5</th><th>3.0</th><th>3.5</th><th>4.0</th></tr>
<tr><td>β</td><td>0.50</td><td>0.67</td><td>0.73</td><td>0.74</td><td>0.75</td><td>0.75</td><td>0.75</td></tr>
<tr><td>α</td><td>0.030</td><td>0.071</td><td>0.101</td><td>0.122</td><td>0.132</td><td>0.137</td><td>0.139</td></tr>
</table>
(Values from charts of Ref. 8; $\nu = 0.3$)</td></tr>
<tr><td>4. Rectangular plate; three edges simply supported, one long edge (a) fixed
 </td><td>4a. Uniform over entire plate</td><td>$\sigma_{\max} = \frac{\beta q b^2}{t^2}$ and $y_{\max} = \frac{-\alpha q b^4}{E t^3}$
<table>
<tr><th>a/b</th><th>1</th><th>1.5</th><th>2.0</th><th>2.5</th><th>3.0</th><th>3.5</th><th>4.0</th></tr>
<tr><td>β</td><td>0.50</td><td>0.66</td><td>0.73</td><td>0.74</td><td>0.74</td><td>0.75</td><td>0.75</td></tr>
<tr><td>α</td><td>0.030</td><td>0.046</td><td>0.054</td><td>0.056</td><td>0.057</td><td>0.058</td><td>0.058</td></tr>
</table>
(Values from charts of Ref. 8; $\nu = 0.3$)</td></tr>
<tr><td>5. Rectangular plate; two long edges simply supported, two short edges fixed
 </td><td>5a. Uniform over entire plate</td><td>(At center of short edges) $\sigma_{\max} = \frac{-\beta q b^2}{t^2}$
(At center) $y_{\max} = \frac{-\alpha q b^4}{E t^3}$
<table>
<tr><th>a/b</th><th>1</th><th>1.2</th><th>1.4</th><th>1.6</th><th>1.8</th><th>2</th><th>∞</th></tr>
<tr><td>β</td><td>0.4182</td><td>0.5208</td><td>0.5988</td><td>0.6540</td><td>0.6912</td><td>0.7146</td><td>0.750</td></tr>
<tr><td>α</td><td>0.0210</td><td>0.0349</td><td>0.0502</td><td>0.0658</td><td>0.0800</td><td>0.0922</td><td></td></tr>
</table>
(Ref. 21)</td></tr>
<tr><td>6. Rectangular plate; two long edges fixed, two short edges simply supported
 </td><td>6a. Uniform over entire plate</td><td>(At center of long edges) $\sigma_{\max} = \frac{-\beta q b^2}{t^2}$
(At center) $y_{\max} = \frac{-\alpha q b^4}{E t^3}$
<table>
<tr><th>a/b</th><th>1</th><th>1.2</th><th>1.4</th><th>1.6</th><th>1.8</th><th>2</th><th>∞</th></tr>
<tr><td>β</td><td>0.4182</td><td>0.4626</td><td>0.4860</td><td>0.4968</td><td>0.4971</td><td>0.4973</td><td>0.500</td></tr>
<tr><td>α</td><td>0.0210</td><td>0.0243</td><td>0.0262</td><td>0.0273</td><td>0.0280</td><td>0.0283</td><td>0.0285</td></tr>
</table>
(Ref. 21)</td></tr>
</table>

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

7. Rectangular plate; one edge fixed, opposite edge free, remaining edges simply supported



7a. Uniform over entire plate

(At center of fixed edge) $\sigma = \dfrac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

(At center of free edge) $\sigma = \dfrac{\beta_2 q b^2}{t^2}$

(At end of free edge) $R = \gamma_2 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.044	0.176	0.380	0.665	1.282	1.804	2.450
β_2	0.048	0.190	0.386	0.565	0.730	0.688	0.434
γ_1	0.183	0.368	0.541	0.701	0.919	1.018	1.055
γ_2	0.131	0.295	0.526	0.832	1.491	1.979	2.401

(Ref. 49 for $\nu = 0.2$)

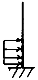
7aa. Uniform over $\frac{2}{3}$ of plate from fixed edge

(At center of fixed edge) $\sigma = \dfrac{-\beta q b^2}{t^2}$ and $R = \gamma q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β	0.044	0.161	0.298	0.454	0.730	0.932	1.158
γ	0.183	0.348	0.466	0.551	0.645	0.681	0.689

(Ref. 49 for $\nu = 0.2$)

7aaa. Uniform over $\frac{1}{3}$ of plate from fixed edge



(At center of fixed edge) $\sigma = \dfrac{-\beta q b^2}{t^2}$ and $R = \gamma q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β	0.040	0.106	0.150	0.190	0.244	0.277	0.310
γ	0.172	0.266	0.302	0.320	0.334	0.338	0.338

(Ref. 49 for $\nu = 0.2$)

7d. Uniformly decreasing from fixed edge to free edge

(At center of fixed edge) $\sigma = \dfrac{-\beta q b^2}{t^2}$ and $R = \gamma q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β	0.037	0.120	0.212	0.321	0.523	0.677	0.866
γ	0.159	0.275	0.354	0.413	0.482	0.509	0.517

(Ref. 49 for $\nu = 0.2$)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

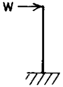
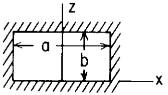
<table>
<tr><th>Case no., shape, and supports</th><th>Case no., loading</th><th>Formulas and tabulated specific values</th></tr>
<tr><td></td><td>7dd. Uniformly decreasing from fixed edge to zero at $\frac{2}{3}b$
</td><td>(At center of fixed edge) $\sigma = \frac{-\beta q b^2}{t^2}$ and $R = \gamma q b$
<table>
<tr><th>a/b</th><th>0.25</th><th>0.50</th><th>0.75</th><th>1.0</th><th>1.5</th><th>2.0</th><th>3.0</th></tr>
<tr><td>β</td><td>0.033</td><td>0.094</td><td>0.146</td><td>0.200</td><td>0.272</td><td>0.339</td><td>0.400</td></tr>
<tr><td>γ</td><td>0.148</td><td>0.233</td><td>0.277</td><td>0.304</td><td>0.330</td><td>0.339</td><td>0.340</td></tr>
</table>
(Ref. 49 for $\nu = 0.2$)</td></tr>
<tr><td></td><td>7ddd. Uniformly decreasing from fixed edge to zero at $\frac{1}{3}b$
</td><td>(At center of fixed edge) $\sigma = \frac{-\beta q b^2}{t^2}$ and $R = \gamma q b$
<table>
<tr><th>a/b</th><th>0.25</th><th>0.50</th><th>0.75</th><th>1.0</th><th>1.5</th><th>2.0</th><th>3.0</th></tr>
<tr><td>β</td><td>0.023</td><td>0.048</td><td>0.061</td><td>0.073</td><td>0.088</td><td>0.097</td><td>0.105</td></tr>
<tr><td>γ</td><td>0.115</td><td>0.149</td><td>0.159</td><td>0.164</td><td>0.167</td><td>0.168</td><td>0.168</td></tr>
</table>
(Ref. 49 for $\nu = 0.2$)</td></tr>
<tr><td></td><td>7f. Distributed line load w lb/in along free edge

 </td><td>(At center of fixed edge) $\sigma_b = \frac{-\beta_1 w b}{t^2}$ and $R = \gamma_1 w$
(At center of free edge) $\sigma_a = \frac{\beta_2 w b}{t^2}$
(At ends of free edge) $R = \gamma_2 w$
<table>
<tr><th>a/b</th><th>0.25</th><th>0.50</th><th>0.75</th><th>1.0</th><th>1.5</th><th>2.0</th><th>3.0</th></tr>
<tr><td>β_1</td><td>0.000</td><td>0.024</td><td>0.188</td><td>0.570</td><td>1.726</td><td>2.899</td><td>4.508</td></tr>
<tr><td>β_2</td><td>0.321</td><td>0.780</td><td>1.204</td><td>1.554</td><td>1.868</td><td>1.747</td><td>1.120</td></tr>
<tr><td>γ_1</td><td>0.000</td><td>0.028</td><td>0.160</td><td>0.371</td><td>0.774</td><td>1.004</td><td>1.119</td></tr>
<tr><td>γ_2</td><td>1.236</td><td>2.381</td><td>3.458</td><td>4.510</td><td>6.416</td><td>7.772</td><td>9.031</td></tr>
</table>
(Ref. 49 for $\nu = 0.2$)</td></tr>
<tr><td>8. Rectangular plate, all edges fixed

 </td><td>8a. Uniform over entire plate</td><td>(At center of long edge) $\sigma_{\max} = \frac{-\beta_1 q b^2}{t^2}$
(At center) $\sigma = \frac{\beta_2 q b^2}{t^2}$ and $y_{\max} = \frac{\alpha q b^4}{E t^3}$
<table>
<tr><th>a/b</th><th>1.0</th><th>1.2</th><th>1.4</th><th>1.6</th><th>1.8</th><th>2.0</th><th>∞</th></tr>
<tr><td>β_1</td><td>0.3078</td><td>0.3834</td><td>0.4356</td><td>0.4680</td><td>0.4872</td><td>0.4974</td><td>0.5000</td></tr>
<tr><td>β_2</td><td>0.1386</td><td>0.1794</td><td>0.2094</td><td>0.2286</td><td>0.2406</td><td>0.2472</td><td>0.2500</td></tr>
<tr><td>α</td><td>0.0138</td><td>0.0188</td><td>0.0226</td><td>0.0251</td><td>0.0267</td><td>0.0277</td><td>0.0284</td></tr>
</table>
(Refs. 7 and 25 and Ref. 21 for $\nu = 0.3$)</td></tr>
</table>

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

8b. Uniform over small concentric circle of radius r_o (note definition of r'_o)

(At center) $\sigma_b = \dfrac{3W}{2\pi t^2}\left[(1+\nu)\ln\dfrac{2b}{\pi r'_o} + \beta_1\right]$ and $y_{\max} = \dfrac{\alpha W b^2}{Et^3}$

(At center of long edge) $\sigma_b = \dfrac{-\beta_2 W}{t^2}$

a/b	1.0	1.2	1.4	1.6	1.8	2.0	∞
β_1	−0.238	−0.078	0.011	0.053	0.068	0.067	0.067
β_2	0.7542	0.8940	0.9624	0.9906	1.0000	1.004	1.008
α	0.0611	0.0706	0.0754	0.0777	0.0786	0.0788	0.0791

(Ref. 26 and Ref. 21 for $\nu = 0.3$)

8d. Uniformly decreasing parallel to side b

(At $x = 0, z = 0$ $\quad (\sigma_b)_{\max} = \dfrac{-\beta_1 q b^2}{t^2}$

(At $x = 0, z = 0.4b$) $\quad \sigma_b = \dfrac{\beta_2 q b^2}{t^2}$ and $\sigma_a = \dfrac{\beta_3 q b^2}{t^2}$

(At $x = 0, z = b$) $\quad \sigma_b = \dfrac{-\beta_4 q b^2}{t^2}$

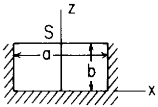
$\left(\text{At } x = \pm\dfrac{a}{2}, z = 0.45b\right)$ $\quad (\sigma_a)_{\max} = \dfrac{-\beta_5 q b^2}{t^2}$

$y_{\max} = \dfrac{-\alpha q b^4}{Et^3}$

a/b	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
β_1	0.1132	0.1778	0.2365	0.2777	0.3004	0.3092	0.3100	0.3068
β_2	0.0410	0.0633	0.0869	0.1038	0.1128	0.1255	0.1157	0.1148
β_3	0.0637	0.0688	0.0762	0.0715	0.0610	0.0509	0.0415	0.0356
β_4	0.0206	0.0497	0.0898	0.1249	0.1482	0.1615	0.1680	0.1709
β_5	0.1304	0.1436	0.1686	0.1800	0.1845	0.1874	0.1902	0.1908
α	0.0016	0.0047	0.0074	0.0097	0.0113	0.0126	0.0133	0.0136

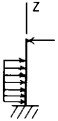
(Ref. 28 for $\nu = 0.3$)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

Case no., shape, and supports	Case no., loading	Formulas and tabulated specific values
9. Rectangular plate, three edges fixed, one edge (a) simply supported  	9a. Uniform over entire plate	(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$ (At $x = 0, z = 0.6b$) $\sigma_b = \frac{\beta_2 q b^2}{t^2}$ and $\sigma_a = \frac{\beta_3 q b^2}{t^2}$ (At $x = 0, z = b$) $R = \gamma_2 q b$ $\left(\text{At } x = \pm\frac{a}{2}, z = 0.6b\right)$ $\sigma_a = \frac{-\beta_4 q b^2}{t^2}$ and $R = \gamma_3 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.020	0.081	0.173	0.307	0.539	0.657	0.718
β_2	0.004	0.018	0.062	0.134	0.284	0.370	0.422
β_3	0.016	0.061	0.118	0.158	0.164	0.135	0.097
β_4	0.031	0.121	0.242	0.343	0.417	0.398	0.318
γ_1	0.115	0.230	0.343	0.453	0.584	0.622	0.625
γ_2	0.123	0.181	0.253	0.319	0.387	0.397	0.386
γ_3	0.125	0.256	0.382	0.471	0.547	0.549	0.530

(Ref. 49 for $\nu = 0.2$)

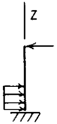
Case no., shape, and supports	Case no., loading	Formulas and tabulated specific values
	9aa. Uniform over $\frac{2}{3}$ of plate from fixed edge  	(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$ (At $x = 0, z = 0.6b$) $\sigma_b = \frac{\beta_2 q b^2}{t^2}$ and $\sigma_a = \frac{\beta_3 q b^2}{t^2}$ (At $x = 0, z = b$) $R = \gamma_2 q b$ $\left(\text{At } x = \pm\frac{a}{2}, z = 0.4b\right)$ $\sigma_a = \frac{-\beta_4 q b^2}{t^2}$ and $R = \gamma_3 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.020	0.080	0.164	0.274	0.445	0.525	0.566
β_2	0.003	0.016	0.044	0.093	0.193	0.252	0.286
β_3	0.012	0.043	0.081	0.108	0.112	0.091	0.066
β_4	0.031	0.111	0.197	0.255	0.284	0.263	0.204
γ_1	0.115	0.230	0.334	0.423	0.517	0.542	0.543
γ_2	0.002	0.015	0.048	0.088	0.132	0.139	0.131
γ_3	0.125	0.250	0.345	0.396	0.422	0.417	0.405

(Ref. 49 for $\nu = 0.2$)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

9aaa. Uniform over $\frac{1}{3}$ of plate from fixed edge



(At $x = 0, z = 0$) $\quad (\sigma_b)_{\max} = \dfrac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

(At $x = 0, z = 0.2b$) $\quad \sigma_b = \dfrac{\beta_2 q b^2}{t^2}$ and $\sigma_a = \dfrac{\beta_3 q b^2}{t^2}$

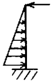
(At $x = 0, z = b$) $\quad R = \gamma_2 q b$

$\left(\text{At } x = \pm\dfrac{a}{2}, z = 0.2b\right) \quad \sigma_a = \dfrac{-\beta_4 q b^2}{t^2}$ and $R = \gamma_3 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.020	0.068	0.108	0.148	0.194	0.213	0.222
β_2	0.005	0.026	0.044	0.050	0.047	0.041	0.037
β_3	0.013	0.028	0.031	0.026	0.016	0.011	0.008
β_4	0.026	0.063	0.079	0.079	0.068	0.056	0.037
γ_1	0.114	0.210	0.261	0.290	0.312	0.316	0.316
γ_2	0.000	0.000	0.004	0.011	0.020	0.021	0.020
γ_3	0.111	0.170	0.190	0.185	0.176	0.175	0.190

(Ref. 49 for $v = 0.2$)

9d. Uniformly decreasing from fixed edge to simply supported edge



(At $x = 0, z = 0$ $\quad (\sigma_b)_{\max} = \dfrac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

$\left(\text{At } x = \pm\dfrac{a}{2}, z = 0.4b)\right) \quad \sigma_a = \dfrac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.018	0.064	0.120	0.192	0.303	0.356	0.382
β_2	0.019	0.068	0.124	0.161	0.181	0.168	0.132
γ_1	0.106	0.195	0.265	0.323	0.383	0.399	0.400
γ_2	0.075	0.152	0.212	0.245	0.262	0.258	0.250

(Ref. 49 for $v = 0.2$)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

Case no., shape, and supports	Case no., loading	Formulas and tabulated specific values
	9dd. Uniformly decreasing from fixed edge to zero at $\frac{2}{3}b$ 	(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$ $\left(\text{At } x = \pm\frac{a}{2}, z = 0.4b \text{ if } a \geqslant b \text{ or } z = 0.2b \text{ if } a < b\right)$ $\sigma_a = \frac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.017	0.056	0.095	0.140	0.201	0.228	0.241
β_2	0.019	0.050	0.068	0.098	0.106	0.097	0.074
γ_1	0.101	0.177	0.227	0.262	0.294	0.301	0.301
γ_2	0.082	0.129	0.146	0.157	0.165	0.162	0.158

(Ref. 49 for $\nu = 0.2$)

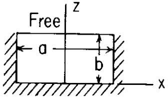
Case no., shape, and supports	Case no., loading	Formulas and tabulated specific values
	9ddd. Uniformly decreasing from fixed edge to zero at $\frac{1}{3}b$ 	(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$ $\left(\text{At } x = \pm\frac{a}{2}, z = 0.2b\right)$ $\sigma_a = \frac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.014	0.035	0.047	0.061	0.075	0.080	0.082
β_2	0.010	0.024	0.031	0.030	0.025	0.020	0.013
γ_1	0.088	0.130	0.146	0.155	0.161	0.162	0.162
γ_2	0.046	0.069	0.079	0.077	0.074	0.074	0.082

(Ref. 49 for $\nu = 0.2$

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

10. Rectangular plate; three edges fixed, one edge (a) free



10a. Uniform over entire plate

(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

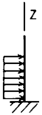
(At $x = 0, z = b$) $\sigma_a = \frac{\beta_2 q b^2}{t^2}$

$\left(\text{At } x = \pm\frac{a}{2}, z = b\right)$ $\sigma_a = \frac{-\beta_3 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.020	0.081	0.173	0.321	0.727	1.226	2.105
β_2	0.016	0.066	0.148	0.259	0.484	0.605	0.519
β_3	0.031	0.126	0.286	0.511	1.073	1.568	1.982
γ_1	0.114	0.230	0.341	0.457	0.673	0.845	1.012
γ_2	0.125	0.248	0.371	0.510	0.859	1.212	1.627

(Ref. 49 for $v = 0.2$)

10aa. Uniform over $\frac{2}{3}$ of plate from fixed edge



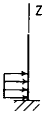
(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

$\left(\text{At } x = \pm\frac{a}{2}, z = 0.6b \text{ for } a > b \text{ or } z = 0.4b \text{ for } a \leqslant b\right)$ $\sigma_a = \frac{-\beta_2 q b^2}{t^2}$ and $r = \gamma_2 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.020	0.080	0.164	0.277	0.501	0.710	1.031
β_2	0.031	0.110	0.198	0.260	0.370	0.433	0.455
γ_1	0.115	0.230	0.334	0.424	0.544	0.615	0.674
γ_2	0.125	0.250	0.344	0.394	0.399	0.409	0.393

(Ref. 49 for $v = 0.2$)

10aaa. Uniform over $\frac{1}{3}$ of plate from fixed edge



(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

$\left(\text{At } x = \pm\frac{a}{2}, z = 0.2b\right)$ $\sigma_a = \frac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.020	0.068	0.110	0.148	0.202	0.240	0.290
β_2	0.026	0.063	0.084	0.079	0.068	0.057	0.040
γ_1	0.115	0.210	0.257	0.291	0.316	0.327	0.335
γ_2	0.111	0.170	0.194	0.185	0.174	0.170	0.180

(Ref. 49 for $v = 0.2$)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

<table>
<tr><th>Case no., shape and supports</th><th>Case no., loading</th><th>Formulas and tabulated specific values</th></tr>
<tr><td></td><td>10d. Uniformly decreasing from fixed edge to zero at free edge
 </td><td>(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$
$\left(\text{At } x = \pm\frac{a}{2}, z = b \text{ if } a > b \text{ or } z = 0.4b \text{ if } a < b\right)$ $\sigma_a = \frac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$
<table>
<tr><th>a/b</th><th>0.25</th><th>0.50</th><th>0.75</th><th>1.0</th><th>1.5</th><th>2.0</th><th>3.0</th></tr>
<tr><td>β_1</td><td>0.018</td><td>0.064</td><td>0.120</td><td>0.195</td><td>0.351</td><td>0.507</td><td>0.758</td></tr>
<tr><td>β_2</td><td>0.019</td><td>0.068</td><td>0.125</td><td>0.166</td><td>0.244</td><td>0.387</td><td>0.514</td></tr>
<tr><td>γ_1</td><td>0.106</td><td>0.195</td><td>0.265</td><td>0.324</td><td>0.406</td><td>0.458</td><td>0.505</td></tr>
<tr><td>γ_2</td><td>0.075</td><td>0.151</td><td>0.211</td><td>0.242</td><td>0.106</td><td>0.199</td><td>0.313</td></tr>
</table>
(Ref. 49 for $\nu = 0.2$)</td></tr>
<tr><td></td><td>10dd. Uniformly decreasing from fixed edge to zero at $\frac{2}{3}b$
 </td><td>(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$
$\left(\text{At } x = \pm\frac{a}{2}, z = 0.4b \text{ if } a \geqslant b \text{ or } z = 0.2b \text{ if } a < b\right)$ $\sigma_b = \frac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$
<table>
<tr><th>a/b</th><th>0.25</th><th>0.50</th><th>0.75</th><th>1.0</th><th>1.5</th><th>2.0</th><th>3.0</th></tr>
<tr><td>β_1</td><td>0.017</td><td>0.056</td><td>0.095</td><td>0.141</td><td>0.215</td><td>0.277</td><td>0.365</td></tr>
<tr><td>β_2</td><td>0.019</td><td>0.050</td><td>0.068</td><td>0.099</td><td>0.114</td><td>0.113</td><td>0.101</td></tr>
<tr><td>γ_1</td><td>0.102</td><td>0.177</td><td>0.227</td><td>0.263</td><td>0.301</td><td>0.320</td><td>0.336</td></tr>
<tr><td>γ_2</td><td>0.082</td><td>0.129</td><td>0.146</td><td>0.157</td><td>0.163</td><td>0.157</td><td>0.146</td></tr>
</table>
(Ref. 49 for $\nu = 0.2$)</td></tr>
</table>

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

10ddd. Uniformly decreasing from fixed edge to zero at $\frac{1}{3}b$

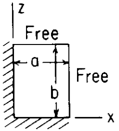
(At $x = 0, z = 0$) $(\sigma_b)_{\max} = \dfrac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

$\left(\text{At } x = \pm\dfrac{a}{2}, z = 0.2b\right)$ $\sigma_a = \dfrac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.25	0.50	0.75	1.0	1.5	2.0	3.0
β_1	0.014	0.035	0.047	0.061	0.076	0.086	0.100
β_2	0.010	0.024	0.031	0.030	0.025	0.020	0.014
γ_1	0.088	0.130	0.146	0.156	0.162	0.165	0.167
γ_2	0.046	0.069	0.079	0.077	0.073	0.073	0.079

(Ref. 49 for $\nu = 0.2$)

11. Rectangular plate; two adjacent edges fixed, two remaining edges free



11a. Uniform over entire plate

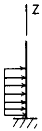
(At $x = a, z = 0$) $(\sigma_b)_{\max} = \dfrac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

$\left(\text{At } x = 0, z = b \text{ if } a > \dfrac{b}{2} \text{ or } a = 0.8b \text{ if } a \leqslant \dfrac{b}{2}\right)$ $\sigma_a = \dfrac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.125	0.25	0.375	0.50	0.75	1.0
β_1	0.050	0.182	0.353	0.631	1.246	1.769
β_2	0.047	0.188	0.398	0.632	1.186	1.769
γ_1	0.312	0.572	0.671	0.874	1.129	1.183
γ_2	0.127	0.264	0.413	0.557	0.829	1.183

(Ref. 49 for $\nu = 0.2$)

11aa. Uniform over plate from $z = 0$ to $z = \frac{2}{3}b$



(At $x = a, z = 0$) $(\sigma_b)_{\max} = \dfrac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

$\left(\text{At } x = 0, z = 0.6b \text{ if } a > \dfrac{b}{2} \text{ or } z = 0.4b \text{ if } a \leqslant \dfrac{b}{2}\right)$ $\sigma_a = \dfrac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

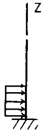
a/b	0.125	0.25	0.375	0.50	0.75	1.0
β_1	0.050	0.173	0.297	0.465	0.758	0.963
β_2	0.044	0.143	0.230	0.286	0.396	0.435
γ_1	0.311	0.543	0.563	0.654	0.741	0.748
γ_2	0.126	0.249	0.335	0.377	0.384	0.393

(Ref. 49 for $\nu = 0.2$)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

Case no., shape, and supports	Case no., loading	Formulas and tabulated specific values

11aaa. Uniform over plate from $z = 0$ to $z = \frac{1}{3}b$



(At $x = a, z = 0$) $(\sigma_b)_{\max} = \dfrac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

$\left(\text{At } x = 0, z = 0.4b \text{ if } a > \dfrac{b}{2} \text{or } z = 0.2b \text{ if } a \leqslant \dfrac{b}{2}\right)$ $\sigma_a = \dfrac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.125	0.25	0.375	0.50	0.75	1.0
β_1	0.034	0.099	0.143	0.186	0.241	0.274
β_2	0.034	0.068	0.081	0.079	0.085	0.081
γ_1	0.222	0.311	0.335	0.343	0.349	0.347
γ_2	0.109	0.162	0.180	0.117	0.109	0.105

(Ref. 49 for $\nu = 0.2$)

11d. Uniformly decreasing from $z = 0$ to $z = b$

(At $x = a, z = 0$) $(\sigma_b)_{\max} = \dfrac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

$\left(\text{At } x = 0, z = b \text{ if } a = b, \text{ or } z = 0.6b \text{ if } \dfrac{b}{2} \leqslant a < b, \text{ or } z = 0.4b \text{ if } a < \dfrac{b}{2}\right)$ $\sigma_a = \dfrac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.125	0.25	0.375	0.50	0.75	1.0
β_1	0.043	0.133	0.212	0.328	0.537	0.695
β_2	0.028	0.090	0.148	0.200	0.276	0.397
γ_1	0.271	0.423	0.419	0.483	0.551	0.559
γ_2	0.076	0.151	0.205	0.195	0.230	0.192

(Ref. 49 for $\nu = 0.2$)

11dd. Uniformly decreasing from $z = 0$ to $z = \frac{2}{3}b$

(At $x = a, z = 0$) $(\sigma_b)_{\max} = \dfrac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$

(At $x = 0, z = 0.4b$ if $a \geqslant 0.375b$, or $z = 0.2b$ if $a < 0.375b$) $\sigma_a = \dfrac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$

a/b	0.125	0.25	0.375	0.50	0.75	1.0
β_1	0.040	0.109	0.154	0.215	0.304	0.362
β_2	0.026	0.059	0.089	0.107	0.116	0.113
γ_1	0.250	0.354	0.316	0.338	0.357	0.357
γ_2	0.084	0.129	0.135	0.151	0.156	0.152

(Ref. 49 for $\nu = 0.2$)

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

<table>
<tr>
<td></td>
<td>11ddd. Uniformly decreasing from $z = 0$ to $z = \frac{1}{3}b$
 </td>
<td>(At $x = a, z = 0$) $(\sigma_b)_{\max} = \frac{-\beta_1 q b^2}{t^2}$ and $R = \gamma_1 q b$
(At $x = 0, z = 0.2b$) $\sigma_a = \frac{-\beta_2 q b^2}{t^2}$ and $R = \gamma_2 q b$
<table>
<tr><th>a/b</th><th>0.125</th><th>0.25</th><th>0.375</th><th>0.50</th><th>0.75</th><th>1.0</th></tr>
<tr><td>β_1</td><td>0.025</td><td>0.052</td><td>0.071</td><td>0.084</td><td>0.100</td><td>0.109</td></tr>
<tr><td>β_2</td><td>0.014</td><td>0.028</td><td>0.031</td><td>0.029</td><td>0.025</td><td>0.020</td></tr>
<tr><td>γ_1</td><td>0.193</td><td>0.217</td><td>0.170</td><td>0.171</td><td>0.171</td><td>0.171</td></tr>
<tr><td>γ_2</td><td>0.048</td><td>0.072</td><td>0.076</td><td>0.075</td><td>0.072</td><td>0.072</td></tr>
</table>
(Ref. 49 for $\nu = 0.2$)</td>
</tr>
<tr>
<td>12. Continuous plate; supported at equal intervals a on circular supports of radius r_o</td>
<td>12a. Uniform over entire surface</td>
<td>(At edge of support)
$\sigma_a = \frac{0.15q}{t^2}\left(a - \frac{4}{3}r_o\right)^2\left(\frac{1}{n} + 4\right)$ when $0.15 \leqslant n < 0.30$ (Ref. 9)
or $\sigma_a = \frac{3qa^2}{2\pi t^2}\left[(1+\nu)\ln\frac{a}{r_o} - 21(1-\nu)\frac{r_o^2}{a^2} - 0.55 - 1.50\nu\right]$ when $n < 0.15$ (Ref. 11)
where $n = \frac{2r_o}{a}$</td>
</tr>
<tr>
<td>13. Continuous plate; supported continuously on an elastic foundation of modulus k (lb/in^2/in)</td>
<td>13b. Uniform over a small circle of radius r_o, remote from edges</td>
<td>(Under the load)
$\sigma_{\max} = \frac{3W(1+\nu)}{2\pi t^2}\left(\ln\frac{L_e}{r_o} + 0.6159\right)$ where $L_e = \sqrt[4]{\frac{Et^3}{12(1-\nu^2)k}}$
Max foundation pressure $q_o = \frac{W}{8L_e^2}$
$y_{\max} = \frac{-W}{8kL_e^2}$
(Ref. 14)</td>
</tr>
</table>

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

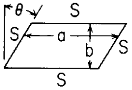
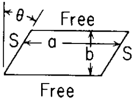
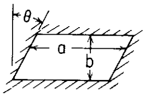
<table>
<tr><th>Case no., shape, and supports</th><th>Case no., loading</th><th>Formulas and tabulated specific values</th></tr>
<tr><td></td><td>13bb. Uniform over a small circle of radius r_o, adjacent to edge but remote from corner</td><td>(Under the load)
$\sigma_{\max} = \frac{0.863W(1+\nu)}{t^2}\left(\ln\frac{L_e}{r_o}+0.207\right)$
$y_{\max} = 0.408(1+0.4\nu)\frac{W}{kL_e^2}$
(Ref. 14)</td></tr>
<tr><td></td><td>13bbb. Uniform over a small circle of radius r_o, adjacent to a corner</td><td>(At the corner) $y_{\max} = \left(1.1-1.245\frac{r_o}{L_e}\right)\frac{W}{kL_e^2}$
(At a distance = $2.38\sqrt{r_oL_e}$ from the corner along diagonal)
$\sigma_{\max} = \frac{3W}{t^2}\left[1-1.083\left(\frac{r_o}{L_e}\right)^{0.6}\right]$
(Ref. 14)</td></tr>
<tr><td>14. Parallelogram plate (skew slab); all edges simply supported
 </td><td>14a. Uniform over entire plate</td><td>(At center of plate) $\sigma_{\max} = \frac{\beta qb^2}{t^2}$ and $y_{\max} = \frac{\alpha qb^4}{Et^3}$
For $a/b = 2.0$
<table>
<tr><th>θ</th><th>0°</th><th>30°</th><th>45°</th><th>60°</th><th>75°</th></tr>
<tr><td>β</td><td>0.585</td><td>0.570</td><td>0.539</td><td>0.463</td><td>0.201</td></tr>
<tr><td>α</td><td>0.119</td><td>0.118</td><td>0.108</td><td>0.092</td><td>0.011</td></tr>
</table>
(Ref. 24 for $\nu = 0.2$)</td></tr>
<tr><td>15. Parallelogram plate (skew slab); shorter edges simply supported, longer edges free
 </td><td>15a. Uniform over entire plate</td><td>(Along free edge) $\sigma_{\max} = \frac{\beta_1 qb^2}{t^2}$ and $y_{\max} \frac{\alpha_1 qb^4}{Et^3}$
(At center of plate) $\sigma_{\max} = \frac{\beta_2 qb^2}{t^2}$ and $y_{\max} = \frac{\alpha_2 qb^4}{Et^3}$
For $a/b = 2.0$
<table>
<tr><th>θ</th><th>0°</th><th>30°</th><th>45°</th><th>60°</th></tr>
<tr><td>β_1</td><td>3.05</td><td>2.20</td><td>1.78</td><td>0.91</td></tr>
<tr><td>β_2</td><td>2.97</td><td>2.19</td><td>1.75</td><td>1.00</td></tr>
<tr><td>α_1</td><td>2.58</td><td>1.50</td><td>1.00</td><td>0.46</td></tr>
<tr><td>α_2</td><td>2.47</td><td>1.36</td><td>0.82</td><td>0.21</td></tr>
</table>
(Ref. 24 for $\nu = 0.2$)</td></tr>
</table>

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

16. Parallelogram plate (skew slab); all edges fixed



16a. Uniform over entire plate

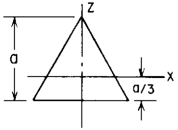
Along longer edge toward obtuse angle) $\sigma_{\max} = \dfrac{\beta_1 q b^2}{t^2}$

(At center of plate) $\sigma = \dfrac{\beta_2 q b^2}{t^2}$ and $y_{\max} = \dfrac{\alpha q b^4}{E t^3}$

θ	a/b	1.00	1.25	1.50	1.75	2.00	2.25	2.50	3.00
0°	β_1	0.308	0.400	0.454	0.481	0.497			
	β_2	0.138	0.187	0.220	0.239	0.247			
	α	0.0135	0.0195	0.0235	0.0258	0.0273			
15°	β_1	0.320	0.412	0.483	0.531	0.553			
	β_2	0.135	0.200	0.235	0.253	0.261			
	α	0.0127	0.0189	0.0232	0.0257	0.0273			
30°	β_1		0.400	0.495	0.547	0.568	0.580		
	β_2		0.198	0.221	0.235	0.245	0.252		
	α		0.0168	0.0218	0.0249	0.0268	0.0281		
45°	β_1			0.394	0.470	0.531	0.575	0.601	
	β_2			0.218	0.244	0.260	0.265	0.260	
	α			0.0165	0.0208	0.0242	0.0265	0.0284	
60°	β_1					0.310	0.450	0.538	0.613
	β_2					0.188	0.204	0.214	0.224
	α					0.0136	0.0171	0.0198	0.0245

(Ref. 53 for $\nu = \frac{1}{3}$)

17. Equilateral triangle; all edges simply supported



17a. Uniform over entire plate

(At $x = 0, z = -0.062a$) $(\sigma_z)_{\max} = \dfrac{0.1488 q a^2}{t^2}$

(At $x = 0, z = 0.129a$) $(\sigma_x)_{\max} = \dfrac{0.1554 q a^2}{t^2}$

(At $x = 0, z = 0$) $y_{\max} = \dfrac{-q a^4 (1 - \nu^2)}{81 E t^3}$

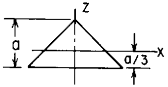
(Refs. 21 and 23 for $\nu = 0.3$)

17b. Uniform over small circle of radius r_o at $x = 0, z = 0$

(At $x = 0, z = 0$) $\sigma_{\max} = \dfrac{3W}{2\pi t^2}\left[\dfrac{1-\nu}{2} + (1+\nu)\ln\dfrac{0.377a}{r'_o}\right]$

$$y_{\max} = 0.069 W (1 - \nu^2) a^2 / E t^3$$

TABLE 11.4 Formulas for flat plates with straight boundaries and constant thickness (*Continued*)

<table>
<tr><th>Case no.,
shape, and supports</th><th>Case no., loading</th><th>Formulas and tabulated specific values</th></tr>
<tr>
<td>18. Right-angle isosceles triangle; all edges simply supported
 </td>
<td>18a. Uniform over entire plate</td>
<td>$\sigma_{\max} = \sigma_z = \dfrac{0.262qa^2}{t^2}$
$(\sigma_x)_{\max} = \dfrac{0.225qa^2}{t^2}$
$y_{\max} = \dfrac{0.038qa^4}{Et^3}$
(Ref. 21 for $\nu = 0.3$)</td>
</tr>
<tr>
<td>19. Regular polygonal plate; all edges simply supported

Number of sides $= n$</td>
<td>19a. Uniform over entire plate</td>
<td>(At center) $\sigma = \dfrac{\beta qa^2}{t^2}$ and $y_{\max} = \dfrac{-\alpha qa^4}{Et^3}$
(At center of straight edge) Max slope $= \dfrac{\xi qa^3}{Et^3}$
<table>
<tr><th>n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>15</th><th>∞</th></tr>
<tr><td>β</td><td>1.302</td><td>1.152</td><td>1.086</td><td>1.056</td><td>1.044</td><td>1.038</td><td>1.038</td><td>1.044</td><td>1.074</td><td>1.236</td></tr>
<tr><td>α</td><td>0.910</td><td>0.710</td><td>0.635</td><td>0.599</td><td>0.581</td><td>0.573</td><td>0.572</td><td>0.572</td><td>0.586</td><td>0.695</td></tr>
<tr><td>ξ</td><td>1.535</td><td>1.176</td><td>1.028</td><td>0.951</td><td>0.910</td><td>0.888</td><td>0.877</td><td>0.871</td><td>0.883</td><td>1.050</td></tr>
</table>
(Ref. 55 for $\nu = 0.3$)</td>
</tr>
<tr>
<td>20. Regular polygonal plate; all edges fixed

Number of sides $= n$</td>
<td>20a. Uniform over entire plate</td>
<td>(At center) $\sigma = \dfrac{\beta_1 qa^2}{t^2}$ and $y_{\max} = \dfrac{-\alpha qa^4}{Et^3}$
(At center of straight edge) $\sigma_{\max} = \dfrac{-\beta_2 qa^2}{t^2}$
<table>
<tr><th>n</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>∞</th></tr>
<tr><td>β</td><td>0.589</td><td>0.550</td><td>0.530</td><td>0.518</td><td>0.511</td><td>0.506</td><td>0.503</td><td>0.500</td><td>0.4875</td></tr>
<tr><td>β_2</td><td>1.423</td><td>1.232</td><td>1.132</td><td>1.068</td><td>1.023</td><td>0.990</td><td>0.964</td><td>0.944</td><td>0.750</td></tr>
<tr><td>α</td><td>0.264</td><td>0.221</td><td>0.203</td><td>0.194</td><td>0.188</td><td>0.184</td><td>0.182</td><td>0.180</td><td>0.171</td></tr>
</table>
(Ref. 55 for $\nu = 0.3$)</td>
</tr>
</table>

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Chapter 12

Columns and Other Compression Members

12.1 Columns; Common Case

The formulas and discussion of this section are based on the following assumptions: (1) The column is nominally straight and is subjected only to nominally concentric and axial end loads, and such crookedness and eccentricity as may occur are accidental and not greater than is consistent with standard methods of fabrication and ordinary conditions of service; (2) the column is homogeneous and of uniform cross section; (3) if the column is made up of several longitudinal elements, these elements are so connected as to act integrally; (4) there are no parts so thin as to fail by local buckling before the column as a whole has developed its full strength.

End conditions. The strength of a column is in part dependent on the *end conditions*, that is, the degree of end fixity or constraint. A column with ends that are supported and fixed, so that there can be neither lateral displacement nor change in slope at either end, is called *fixed-ended*. A column with ends that are supported against lateral displacement but not constrained against change in slope is called *round-ended*. A column with one end fixed and the other end neither laterally supported nor otherwise constrained is called *free-ended*. A column with both end surfaces that are flat and normal to the axis and that bear evenly against rigid loading surfaces is called *flat-ended*. A column with ends that bear against transverse pins is called *pin-ended*.

Truly fixed-ended and truly round-ended columns practically never occur in practice; the actual conditions are almost always intermediate. The greatest degree of fixity is found in columns with ends that are riveted or welded to relatively rigid parts that are also fixed.

Theoretically a flat-ended column is equivalent to a fixed-ended column until the load reaches a certain critical value at which the column “kicks out” and bears only on one edge of each end surface instead of on the whole surface. Actually, flat-ended columns have a degree of end constraint considerably less than that required to produce fixity. The nearest approach to round-ended conditions is found in pin-ended columns subject to vibration or other imposed motion. The degree of end fixity may be expressed by the *coefficient of constraint* [explained following Eq. (12.1-1)] or by the *free* or *effective* length, which is the length measured between points of counterflexure or the length of a round-ended column of equal strength.

Behavior. If sufficiently slender, a column will fail by elastic instability (see Chap. 15). In this case the maximum unit stress sustained is less than the proportional limit of the material; it depends on the modulus of elasticity, the slenderness ratio (ratio of the length of the column to the least radius of gyration of the section), and the end conditions and is independent of the strength of the material. Columns which fail in this way are called *long columns*.

Columns that are too short to fail by elastic instability are called *short columns*; such a column will fail when the maximum fiber stress due to direct compression and to the bending that results from accidental crookedness and eccentricity reaches a certain value. For structural steel this value is about equal to the tensile yield point; for light alloys it is about equal to the compressive yield strength; and for wood it lies between the flexural elastic limit and the modulus of rupture.

For a given material and given end conditions, there is a certain slenderness ratio which marks the dividing point between long and short columns called the *critical slenderness ratio*.

Formulas for long columns. The unit stress at which a long column fails by elastic instability is given by the Euler formula

$$\frac{P}{A}=\frac{C\pi^2 E}{(L/r)^2} \tag{12.1-1}$$

where $P =$ total load, $A =$ area of section, $E =$ modulus of elasticity, $L/r =$ slenderness ratio, and C is the coefficient of constraint, which depends on end conditions. For round ends, $C = 1$; for fixed ends, $C = 4$; and for the end conditions that occur in practice, C can rarely be assumed greater than 2. It is generally not considered good practice to employ long columns in building and bridge construction, but they are used in aircraft. (Formulas for the loads producing elastic instability of uniform and tapered bars under a wide variety of conditions of loading and support are given in Table 15.1.)

Formulas for short columns. It is not possible to calculate with accuracy the maximum stress produced in a short column by a nominally concentric load because of the large influence of the indeterminate crookedness and eccentricity. The maximum unit stress that a column will sustain, however, can be expressed by any of a number of formulas, each of which contains one or more terms that is empirically adjusted to secure conformity with test results. Of such formulas, those given below are the best known and provide the basis for most of the design formulas used in American practice. In these equations P denotes the load at failure, A the cross-sectional area, L the length, and r the least radius of gyration of the section; the meaning of other symbols used is explained in the discussion of each formula.

Secant formula

$$\frac{P}{A} = \frac{\sigma}{1 + \dfrac{ec}{r^2} \sec\left(\dfrac{KL}{2r}\sqrt{\dfrac{P}{AE}}\right)} \tag{12.1-2}$$

This formula is adapted from the formula for stress due to eccentric loading [Eq. (12.4-1)]. Here σ denotes the maximum fiber stress at failure (usually taken as the yield point for steel and as the yield strength for light alloys); e denotes the *equivalent eccentricity* (that eccentricity which in a perfectly straight column would cause the same amount of bending as the actual eccentricity and crookedness); c denotes the distance from the central axis about which bending occurs to the extreme fiber on the concave or compression side of the bent column; and K is a numerical coefficient, dependent on end conditions, such that KL is the effective length of the column, or distance between points of inflection. The term ec/r^2 is called the *eccentric ratio*, and a value is assumed for this ratio which makes the formula agree with the results of tests on columns of the type under consideration. For example, tests on structural steel columns of conventional design indicate that the average value of the eccentric ratio is 0.25. In using the secant formula, P/A must be solved for by trial or by the use of prepared charts.

Rankine formula

$$\frac{P}{A} = \frac{\sigma}{1 + \phi(L/r)^2} \tag{12.1-3}$$

This is a semirational formula. The value of σ is sometimes taken as the ultimate strength of the material and the value of ϕ as $\sigma/C\pi^2E$, thus making the formula agree with the results of tests on short prisms when L/r is very small and with Euler's equation when L/r is very large. More often σ and ϕ are adjusted empirically to make the equation agree with the results of tests through the L/r range of most importance.

Simple polynomial formula

$$\frac{P}{A} = \sigma - k\left(\frac{L}{r}\right)^n \qquad (12.1\text{-}4)$$

This is an empirical formula. For most steels the exponent n is chosen as 2 and σ is chosen as the yield point to give the well-known parabolic formula. The constant k is generally chosen to make the parabola intersect tangent to the Euler curve for a long column. See Refs. 1 to 4.

For cast irons and for many of the aluminum alloys (Refs. 1 and 5) the exponent n is taken as unity to give a *straight-line* formula. If σ were here taken as the maximum fiber stress at failure and the straight line made tangent to the Euler curve, the formula would give values well below experimental values. For this reason the straight-line formula is generally used for *intermediate* lengths with σ and k modified to make the straight line pass through the experimental data of interest. For columns shorter than intermediate in length, a constant value of P/A is given; for those longer than intermediate, the Euler curve is specified.

For timber the exponent n is usually chosen as 4 (Refs. 1 and 6) and then treated in the same manner as was the parabolic formula. The value used for σ is generally the ultimate compressive strength of the timber.

Exponential formula

$$\frac{P}{A} = C_1^{\lambda^2}\sigma \qquad \text{where } \lambda = \frac{KL}{r\pi}\left(\frac{\sigma}{E}\right)^{1/2} \qquad (12.1\text{-}5)$$

The American Institute of Steel Construction in Ref. 7 suggests the use of a formula of the form shown in Eq. (12.1-5), where K, L, r, and σ are as defined for the secant formula. The constant λ combines the column dimensions L and r, the degree of end fixity indicated by K, and the material properties σ and E. The secant, Rankine, parabolic, exponential, and Euler formulas can all be expressed in terms of λ and a simple tabulation used to compare them.

Parabolic formula $\quad \dfrac{P/A}{\sigma} = 1 - \dfrac{\lambda^2}{4} \quad$ for $\lambda < 1.414$

Exponential formula $\quad \dfrac{P/A}{\sigma} = 0.6922^{\lambda^2} \quad$ for $\lambda < 1.649$

Rankine formula $$\frac{P/A}{\sigma}=\frac{1}{1+\lambda^2} \qquad \text{for all } \lambda$$

Secant formula $$\frac{P/A}{\sigma}=\frac{1}{1+0.25\sec\left(\dfrac{\pi\lambda}{2}\sqrt{\dfrac{P/A}{\sigma}}\right)} \qquad \text{for all } \lambda$$

The Euler formula becomes $(P/A)/\sigma = 1/\lambda^2$, and both the parabolic and exponential formulas given above have been derived to be tangent to the Euler formulas where they intersect. In the table on the next page are shown the values of $(P/A)/\sigma$ for each of the given equations or sets of equations.

In the case of the secant formula at very short lengths, the lower values can be attributed to the chosen eccentricity ratio of 0.25.

By applying a proper factor of safety a *safe* load can be calculated. Many codes make use of safety factors which vary with the effective L/r ratios, so the safe-load formulas may differ somewhat in form from the ultimate-load formulas just discussed. See Ref. 1 for extensive listings of applicable codes and design specifications.

Calculation of stress. The best way to compute the probable value of the maximum fiber stress in a short column, caused by the imposition of a nominally concentric load that is less than the ultimate load, is to use the secant formula [Eq. (12.1-2)] with an assumed value of e or ec/r^2. However, by transposing terms, any one of Eqs. (12.1-3)–(12.1-5) can be written so as to give the maximum stress σ in terms of the load P. Such procedure is logical only when σ is the fiber stress at failure and P the ultimate load; but if the maximum stress due to some load that is less than the ultimate load is thus computed, the result, although probably considerably in error, is almost sure to be greater than the true stress and hence the method errs on the side of safety.

12.2 Local Buckling

If a column is composed wholly or partially of thin material, local buckling may occur at a unit load less than that required to cause failure of the column as a whole. When such local buckling occurs at a unit stress less than the proportional limit, it represents elastic instability; the critical stress at which this occurs can be determined by mathematical analysis. Formulas for the critical stress at which bars and thin plates exhibit elastic instability, under various conditions of loading and support, are given in Tables 15.1 and 15.2. All such formulas are based upon assumptions as to homogeneity of material, regularity of form, and boundary conditions that are never

Values of $(P/A)/\sigma$ for four different equations or sets of equations

	λ															
Equations	0.000	0.200	0.400	0.600	0.800	1.000	1.200	1.400	1.600	1.800	2.000	2.200	2.400	2.600	2.800	3.000
Parabolic-Euler	1.000	0.990	0.960	0.910	0.840	0.750	0.640	0.510	0.391	0.309	0.250	0.207	0.174	0.148	0.128	0.111
Exponential-Euler	1.000	0.985	0.943	0.876	0.790	0.692	0.589	0.486	0.390	0.309	0.250	0.207	0.174	0.148	0.128	0.111
Rankine	1.000	0.962	0.862	0.735	0.610	0.500	0.410	0.338	0.281	0.236	0.200	0.171	0.148	0.129	0.113	0.100
Secant	0.800	0.794	0.773	0.734	0.673	0.589	0.494	0.405	0.331	0.273	0.227	0.191	0.163	0.140	0.122	0.107

realized in practice; the critical stress to be expected under any actual set of circumstances is nearly always less than that indicated by the corresponding theoretical formula and can be determined with certainty only by test. This is also true of the ultimate load that will be carried by such parts as they buckle since elastic buckling is not necessarily attended by failure and thin flanges and webs, by virtue of the support afforded by attached parts, may carry a load considerably in excess of that at which buckling occurs (see Sec. 12.6).

In the following paragraphs, the more important facts and relations that have been established concerning local buckling are stated, insofar as they apply to columns of more or less conventional design. In the formulas given, b represents the unsupported width of the part under consideration, t its thickness, σ_y the yield point or yield strength, and E and ν have their usual meanings.

Outstanding flanges. For a long flange having one edge fixed and the other edge free, the theoretical formula for buckling stress is

$$\sigma' = \frac{1.09E}{1-\nu^2}\left(\frac{t}{b}\right)^2 \tag{12.2-1}$$

and for a flange having one edge simply supported and the other edge free, the corresponding formula is

$$\sigma' = \frac{0.416E}{1-\nu^2}\left(\frac{t}{b}\right)^2 \tag{12.2-2}$$

(See Table 15.2.)

For the outstanding flange of a column, the edge condition is intermediate, the degree of constraint depending upon the torsional rigidity of the main member and on the way in which the flange is attached. The conclusions of the ASCE Column Research Committee (Ref. 8) on this point may be summed up as follows: For columns of structural steel having a proportional limit of 30,000 lb/in^2, an outstanding flange riveted between two angles, each having a thickness equal to that of the flange, will not fail by elastic buckling if b/t is less than 15, b being measured from the free edge of the flange to the first row of rivets; for wider flanges, the formula for buckling stress is

$$\sigma' = 0.4E\left(\frac{t}{b}\right)^2 \tag{12.2-3}$$

If the thickness of each supporting angle is twice that of the flange, elastic buckling will not occur if b/t is less than 20, b in this case being measured from the free edge of the flange to the toe of the angle; for wider flanges, the formula for buckling stress is

$$\sigma' = 0.6E\left(\frac{t}{b}\right)^2 \tag{12.2-4}$$

The ultimate strength of an outstanding flange is practically equal to the area times the yield point up to a b/t ratio of 15; for wider flanges the ultimate load is not appreciably greater, and so there is no substantial gain in load-carrying capacity when the width of a flange is increased to more than 15 times the thickness. In Ref. 2 are given recommended limiting values of width/thickness ratios in terms of σ_y for webs, flanges, and other parts subject to buckling.

In the case of aluminum, the *allowable* unit stress on an outstanding flange may be found by the formulas

$$(\text{Allowable, lb/in}^2)\quad \sigma = \begin{cases} 15{,}000 - 123k\dfrac{b}{t} & \text{when } k\dfrac{b}{t} < 81 \quad (12.2\text{-}5) \\[2ex] \dfrac{33{,}000{,}000}{\left(k\dfrac{b}{t}\right)^2} & \text{when } k\dfrac{b}{t} > 81 \quad (12.2\text{-}6) \end{cases}$$

Here k is to be taken as 4 when the outstanding flange is one leg of an angle T or other section having relatively little torsional rigidity, and may be taken as 3 when the flange is part of or firmly attached to a heavy web or other part that offers relatively great edge constraint.

A formula (Ref. 13) for the ultimate strength of short compression members consisting of single angles, which takes into account both local and general buckling, is

$$\frac{P}{A} = \sigma \tanh\left[K\left(\frac{t}{b}\right)^2\right] \tag{12.2-7}$$

where $K = 149.1 + 0.1(L/r - 47)^2$ and σ, which depends on L/r, has the following values:

L/r	0	20	40	60	80
σ	40,000	38,000	34,000	27,000	18,000

This formula is for an alloy (24ST) having a yield strength of 43,000 lb/in^2 and a modulus of elasticity of 10,500,000 lb/in^2, and is for round-ended columns ($c = 1$). A more general formula for thin sections other than angles is

$$\frac{P}{A} = \sigma \tanh(Kt) \tag{12.2-8}$$

Here $\sigma = \sigma_y(1+B)/(1+B+B^2)$, where $B = \sigma_y(L/r)^2/(c\pi^2 E)$, and $K = K_o(\sigma_y/\sigma)^{1/2}$, where K_o is a *shape factor*, the value of which is found from Eq. (12.2-8), P/A being experimentally determined by testing columns of the section in question that have a slenderness ratio of about 20. For a closed box or "hat" section, $K_o = 15.6$; for a section with flat flanges with a width that is not more than 25 times the thickness, $K_o = 10.8$; for a section of oval form or having wholly or partially curved flanges, K_o ranges from 12 to 32 (Ref. 13). (An extensive discussion of design procedures and buckling formulas for aluminum columns and other structural elements is found in Ref. 5.)

For spruce and other wood of similar properties, Trayer and March (Chap. 14, Ref. 3) give as the formula for buckling stress

$$\sigma' = 0.07E\left(\frac{t}{b}\right)^2 \tag{12.2-9}$$

when the edge constraint is as great as normally can be expected in all-wood construction and

$$\sigma' = 0.044E\left(\frac{t}{b}\right)^2 \tag{12.2-10}$$

when conditions are such as to make the edge constraint negligible.

Thin webs. For a long thin web that is fixed along each edge, the theoretical formula for buckling stress is

$$\sigma' = \frac{5.73E}{1-\nu^2}\left(\frac{t}{b}\right)^2 \tag{12.2-11}$$

and for a web that is simply supported along each edge, the corresponding formula is

$$\sigma' = \frac{3.29E}{1-\nu^2}\left(\frac{t}{b}\right)^2 \tag{12.2-12}$$

(See Table 15.2.)

For structural steel columns, the conclusion of the ASCE Column Research Committee (Ref. 8) is that elastic buckling will not occur at b/t ratios less than 30. Tests made by the Bureau of Standards (Ref. 14) on steel members consisting of wide webs riveted between edge angles indicate that this conclusion is conservative and that b/t may be safely as great as 35 if b is taken as the width between rivet lines.

For aluminum columns, the same formulas for allowable stress on a thin web are suggested as are given previously for the outstanding flange [(Eqs. (12.2-5) and (12.2-6)] but with $k = 1.2$. (For discussion of the ultimate strength developed by a thin web, see Sec. 12.6.)

Thin cylindrical tubes. For a thin cylindrical tube, the theoretical formula for the critical stress at which buckling occurs is

$$\sigma' = \frac{E}{\sqrt{3}\sqrt{1-\nu^2}}\,\frac{t}{R} \tag{12.2-13}$$

when R denotes the mean radius of the tube (see Table 15.2). Tests indicate that the critical stress actually developed is usually only 40–60% of this theoretical value.

Much recent work has been concerned with measuring initial imperfections in manufactured cylindrical tubes and correlating these imperfections with measured critical loads. For more detailed discussions and recommendations refer to Refs. 1–5 in this chapter and to Refs. 101–109 in Chap. 15.

Attached plates. When the flanges or web of a column are formed by riveting a number of plates placed flat against one another, there is a possibility that the outer plate or plates will buckle between points of attachment if the unsupported length is too great compared with the thickness. If the full yield strength σ_y of an outer plate is to be developed, the ratio of unsupported length a to thickness t should not exceed the value indicated by the formula

$$\frac{a}{t} = 0.52\sqrt{\frac{E}{\sigma_y}} \tag{12.2-14}$$

(Ref. 17). Some specifications (Ref. 10) guard against the possibility of such buckling by limiting the maximum distance between rivets (in the direction of the stress) to 16 times the thickness of the thinnest outside plate and to 20 times the thickness of the thinnest inside plate; the ratio 16 is in agreement with Eq. (12.2-14).

Local buckling of latticed columns. To guard against the possibility that the longitudinal elements of a latticed column will buckle individually between points of support, some specifications (Ref. 10) limit the slenderness ratio of such parts between points of attachment of lacing bars to 40 or to two-thirds the slenderness ratio of the column as a whole, whichever is less.

Lacing bars. In a column composed of channels or other structural shapes connected by lacing bars, the function of the lacing bars is to resist the transverse shear due to initial obliquity and that consequent upon such bending as may occur under load. The amount of this shear is conjectural since the obliquity is accidental and indeterminate.

Salmon (Ref. 17) shows that with the imperfections usually to be expected, the transverse shear will be at least 1% of the axial load. Moore and Talbot (Ref. 18) found that for certain experimental columns the shear amounted to from 1% to 3% of the axial load. Some specifications require that in buildings the lacing be designed to resist a shear equal to 2% of the axial load (Ref. 2), and that in bridges it be designed to resist a shear V given by

$$V = \frac{P}{100}\left[\frac{100}{(L/r) + 10} + \frac{L/r}{100}\right]$$

where P is the allowable axial load and r is the radius of gyration of the column section with respect to the central axis perpendicular to the plane of the lacing (Ref. 10).

The strength of individual lacing bars as columns has been investigated experimentally. For a bar of rectangular section with a single rivet at each end, the ultimate strength (in psi) is given by

$$\frac{P}{A} = 25{,}000 - 50\frac{L}{r} \qquad \text{(Ref. 8)}$$

or

$$\frac{P}{A} = 21{,}400 - 45\frac{L}{r} \qquad \text{(Ref. 18)}$$

For bars of angle or channel section, these formulas are conservative. For flat bars used as double lacing, the crossed bars being riveted together, tests show that the effective L is about half the actual distance between end rivets. Some specifications (Refs. 2 and 10) require lacing bars of any section to be designed by the regular column formula, L being taken as the distance between end rivets for single lacing and as 70% of that distance for double lacing. There are additional limitations as to slope of lacing, minimum section, and method of riveting.

12.3 Strength of Latticed Columns

Although it is customary to assume that a latticed column acts integrally and develops the full strength of the nominal section, tests show that when bending occurs in the plane of the lacing, the column is less stiff than would be the case if this assumption were valid. For a column so designed that buckling occurs in a plane normal to that of the lacing, this fact is unimportant; but in long open columns laced on all sides, such as are often used for derrick booms and other light construction, it may be necessary to take it into account.

For any assumed transverse loading, it is easy to calculate that part of the deflection of a latticed member which is due to strains in the lacing bars and thus to derive a value for what may be called the *reduced modulus of elasticity KE.* Such calculations agree reasonably well with the results of tests (see Ref. 8), but K, of course, varies with the nature of the assumed transverse loading or with the form of the assumed elastic curve, which amounts to the same thing. For uniformly distributed loading and end support, and for the type of lacing shown in Fig. 12.1(a), K is given by

$$K = \frac{1}{1 + \dfrac{4.8I}{AL^2 \cos^2\theta \sin\theta}} \tag{12.3-1}$$

where $L =$ length of the column, $I =$ moment of inertia of the column cross section about the principal axis which is normal to the plane of battens, and $A =$ cross-sectional area of a single lacing bar. For double lacing, 2.4 should be used in place of 4.8. If KE is used in place of E, the effect of reduced stiffness on the strength of a long column will be approximately allowed for. The method is theoretically inexact mainly because the form of elastic curve assumed is not identical with that taken by the column, but the error due to this is small.

Timoshenko (Ref. 19) gives formulas based upon the assumption that the elastic curve of the column is a sinusoid, from which the following expressions for K may be derived: For the arrangement shown in Fig. 12.1(a),

$$K = \frac{1}{1 + \dfrac{4.93I}{AL^2 \cos^2\theta \sin\theta}} \tag{12.3-2}$$

For the arrangement shown in Fig. 12.1(b),

$$K = \frac{1}{1 + \dfrac{4.93I}{A_1L^2 \cos^2\theta \sin\theta} + \dfrac{4.93I}{A_2L^2 \tan\theta}} \tag{12.3-3}$$

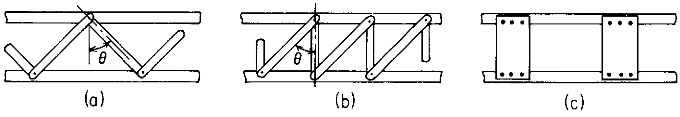


Figure 12.1

where A_1 = cross-sectional area of each diagonal bar and A_2 = cross-sectional area of each transverse bar. For the channel and batten-plate arrangement shown in Fig. 12.1(*c*),

$$K = \frac{1}{1 + \dfrac{\pi^2 I}{L^2}\left(\dfrac{ab}{12I_2} + \dfrac{a^2}{24I_1}\right)} \tag{12.3-4}$$

where a = center-to-center distance between battens; b = length of a batten between rivets; I_1 = moment of inertia of a single-channel section (or any similar section being used for each column leg) about its own centroidal axis normal to the plane of the battens, and I_2 = moment of inertia of a pair of batten plates (that is, $I_2 = 2tc^3/12$, where t is the batten-plate thickness and c is the batten-plate width measured parallel to the length of the column).

In all the preceding expressions for K, it is assumed that all parts have the same modulus of elasticity, and only the additional deflection due to longitudinal strain in the lacing bars and to secondary flexure of channels and batten plates is taken into account. For fairly long columns laced over practically the entire length, the values of K given by Eqs. (12.3-1)–(12.3-3) are probably sufficiently accurate. More elaborate formulas for shear deflection, in which direct shear stress in the channels, bending of the end portions of channels between stay plates, and rivet deformation, as well as longitudinal strains in lacing bars, are taken into account, are given in Ref. 8; these should be used when calculating the deflection of a short latticed column under direct transverse loading.

The use of K as a correction factor for obtaining a reduced value of E is convenient in designing long latticed columns; for short columns the correction is best made by replacing L in whatever column formula is selected by $\sqrt{(1/K)}L$.

Several failure modes and the critical loads associated with these modes are discussed in Refs. 11 and 12.

12.4 Eccentric Loading; Initial Curvature

When a round-ended column is loaded eccentrically with respect to one of the principal axes of the section (here called axis 1), the formula for the maximum stress produced is

$$\sigma = \frac{P}{A}\left\{1 + \frac{ec}{r^2}\sec\left[\frac{P}{4EA}\left(\frac{L}{r}\right)^2\right]^{1/2}\right\} \tag{12.4-1}$$

where $e =$ eccentricity, $c =$ distance from axis 1 to the extreme fiber on the side nearest the load, and $r =$ radius of gyration of the section with respect to axis 1. (This equation may be derived from the formula for case 3e, Table 8.8, by putting $M_1 = Pe$.)

If a column with fixed ends is loaded eccentrically, as is assumed here, the effect of the eccentricity is merely to increase the constraining moments at the ends; the moment at midlength and the buckling load are not affected. If the ends are *partially* constrained, as by a frictional moment M, this constraint may be taken into account by considering the actual eccentricity e reduced to $e - M/P$. If a free-ended column is loaded eccentrically, as is assumed here, the formula for the maximum stress is

$$\sigma = \frac{P}{A}\left\{1 + \frac{ec}{r^2}\sec\left[\frac{P}{EA}\left(\frac{L}{r}\right)^2\right]^{1/2}\right\} \tag{12.4-2}$$

where the notation is the same as for Eq. (12.4-1).

When a round-ended column is loaded eccentrically with respect to *both* principal axes of the section (here called axes 1 and 2), the formula for the maximum stress is

$$\sigma = \frac{P}{A}\left\{1 + \frac{e_1c_1}{r_1^2}\sec\left[\frac{P}{4EA}\left(\frac{L}{r_1}\right)^2\right]^{1/2} + \frac{e_2c_2}{r_2^2}\sec\left[\frac{P}{4EA}\left(\frac{L}{r_2}\right)^2\right]^{1/2}\right\} \tag{12.4-3}$$

where the subscripts 1 and 2 have reference to axes 1 and 2 and the notation is otherwise the same as for Eq. (12.4-1). [The use of Eq. (12.4-1) is illustrated in the example below, which also shows the use of Eq. (12.3-1) in obtaining a reduced modulus of elasticity to use with a latticed column.]

If a round-ended column is initially curved in a plane that is perpendicular to principal axis 1 of the section, the formula for the maximum stress produced by concentric end loading is

$$\sigma = \frac{P}{A}\left(1 + \frac{dc}{r^2}\,\frac{8EA}{P(L/r)^2}\left\{\sec\left[\frac{P}{4EA}\left(\frac{L}{r}\right)^2\right]^{1/2} - 1\right\}\right) \tag{12.4-4}$$

where $d =$ maximum initial deflection, $c =$ distance from axis 1 to the extreme fiber on the concave side of the column, and $r =$ radius of gyration of the section with respect to axis 1. If the column is initially curved in a plane that is not the plane of either of the principal axes 1 and 2 of the section, the formula for the maximum stress is

$$\sigma = \frac{P}{A}\left(1 + \frac{d_1 c_1}{r_1^2}\,\frac{8EA}{P(L/r_1)^2}\left\{\sec\left[\frac{P}{4EA}\left(\frac{L}{r_1}\right)^2\right]^{1/2} - 1\right\} + \frac{d_2 c_2}{r_2^2}\,\frac{8EA}{P(L/r_2)^2}\left\{\sec\left[\frac{P}{4EA}\left(\frac{L}{r_2}\right)^2\right]^{1/2} - 1\right\}\right) \tag{12.4-5}$$

where $d_1 =$ the component of the initial deflection perpendicular to the plane of axis 1, $d_2 =$ the component of the initial deflection perpendicular to the plane of axis 2, and c_1, c_2, r_1, and r_2 each has reference to the axis indicated by the subscript.

Eccentrically loaded columns and columns with initial curvature can also be designed by the interaction formulas given in Sec. 12.5.

EXAMPLE

Figure 12.2 represents the cross section of a structural steel column composed of two 10-in, 35-lb channels placed 12 in back to back and latticed together. The length of the column is 349.3 in, and it has single lacing, the bars being of rectangular section, $2\frac{1}{2}$ by $\frac{1}{4}$ in, and inclined at 45°. This column is loaded eccentrically, the load being applied on axis 2 but 2.40 in from axis 1. With respect to bending in the plane of the eccentricity, the column is round-ended. It is required to calculate the maximum fiber stress in the column when a load of 299,000 lb, or 14,850 lb/in^2, is thus applied.

Solution. For axis 1, $r = 5.38$ in, $c = 6.03$ in (measured), and $e = 2.40$ in. Since the bending due to eccentricity is in the plane of the lacing, a reduced E is used. K is calculated by Eq. (12.3-1), where $I = 583\text{in}^4$, $A = 2\frac{1}{2} \times \frac{1}{4} = 0.625\,\text{in}^2$, $L = 349.3$ in, and $\theta = 45°$. Therefore

$$K = \frac{1}{1 + \dfrac{(4.8)(583)}{(0.625)(349.3^2)(0.707^2)(0.707)}} = 0.94$$

and using the secant formula [Eq. (23)], we have

$$\sigma = 14{,}850\left\{1 + \frac{(2.40)(6.03)}{5.38^2}\sec\left[\frac{14{,}850}{(4)(0.94)(30{,}000{,}000)}\left(\frac{349.3}{5.38}\right)^2\right]^{1/2}\right\}$$
$$= 25{,}300\ \text{lb/in}^2$$

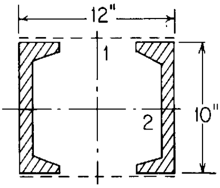


Figure 12.2

[This column was actually tested under the loading just described, and the maximum stress (as determined by strain-gage measurements) was found to be 25,250 lb/in^2. Such close agreement between measured and calculated stress must be regarded as fortuitous, however.]

12.5 Columns under Combined Compression and Bending

A column bent by lateral forces or by couples presents essentially the same problem as a beam under axial compression, and the stresses produced can be found by the formulas of Table 8.8 provided the end conditions are determinable. Because these and other uncertainties generally preclude precise solution, it is common practice to rely upon some interaction formula, such as one of those given below. The column may be considered safe for the given loading when the relevant equations are satisfied.

The following notation is common to all the equations; other terms are defined as introduced:

F_a = allowable value of P/A for the member considered as a concentrically loaded column

F_b = allowable value of compressive fiber stress for the member considered as a beam under bending only

$f_a = P/A$ = average compressive stress due to the axial load P

f_b = computed maximum bending stress due to the transverse loads, applied couples, or a combination of these

L = Unbraced length in plane of bending

L/r = slenderness ratio for buckling in that plane

For structural steel

$$\frac{f_a}{F_a} + \frac{C_m f_b}{(1 - f_a/F_e)F_b} \leqslant 1 \qquad \text{when } \frac{f_a}{F_a} > 0.15$$

$$\frac{f_a}{F_a} + \frac{f_b}{F_b} \leqslant 1 \qquad \text{when } \frac{f_a}{F_a} < 0.15$$

for sections between braced points, and

$$\frac{f_a}{0.6F_y} + \frac{f_b}{F_b} \leqslant 1$$

for sections at braced points only. Here $F_e = 149{,}000{,}000/(L/r)^2$ and F_y = yield point of steel; $C_m = 0.85$ except that for restrained compression members in frames braced against joint translation and without transverse loading between joints; $C_m = 0.6 + 0.4(M_1/M_2)$, where M_1 is the smaller and M_2 the larger of the moments at the ends of the critical unbraced length of the member. M_1/M_2 is positive when the unbraced length is bent in single curvature and negative when it is bent in reverse curvature. For such members with transverse loading between joints, C_m may be determined by rational analysis, or the appropriate formula from Table 8.8 may be used. (Formulas are adapted from Ref. 2, with F_e given in English units.)

For structural aluminum

$$\frac{f_a}{F_a} + \frac{f_b}{F_b(1 - f_a/F_e)} \leqslant 1$$

Here $F_e = 51{,}000{,}000/(L/r)^2$ for building structures and $F_e = 45{,}000{,}000/(L/r)^2$ for bridge structures. (Formulas are taken from Ref. 9 with some changes of notation and with F_e given in English units.)

For wood (solid rectangular). When $L/d \leqslant \sqrt{0.3E/F_a}$,

$$\frac{f_b}{F_b} + \frac{f_a}{F_a} \leqslant 1$$

When $L/d > \sqrt{0.3E/F_a}$:

1. Concentric end loads plus lateral loads,

$$\frac{f_b}{F_b - f_a} + \frac{f_a}{F_a} \leqslant 1$$

2. Eccentric end load,

$$\frac{1.25 f_b}{F_b - f_a} + \frac{f_a}{F_a} \leqslant 1$$

3. Eccentric end load plus lateral loads,

$$\frac{f_{bl} + 1.25 f_{be}}{F_b - f_a} + \frac{f_a}{F_a} \leqslant 1$$

Here $d =$ dimension of the section in the plane of bending, $f_{bl} =$ computed bending stress due to lateral loads, and $f_{be} =$ computed bending stress due to the eccentric moment. (Formulas are taken from Ref. 16 with some changes of notation.)

12.6 Thin Plates with Stiffeners

Compression members and compression flanges of flexural members are sometimes made of a very thin sheet reinforced with attached stiffeners; this construction is especially common in airplanes, where both wings and fuselage are often of the "stressed-skin" type.

When a load is applied to such a combination, the portions of the plate not very close to the stiffeners buckle elastically at a very low unit stress, but those portions immediately adjacent to the stiffeners develop the same stress as do the latter, and portions a short distance from the stiffeners develop an intermediate stress. In calculating the part of any applied load that will be carried by the plate or in calculating the strength of the combination, it is convenient to make use of the concept of "effective," or "apparent," width, i.e., the width of that portion of the sheet which, if it developed the same stress as the stiffener, would carry the same load as is actually carried by the entire sheet.

For a flat, rectangular plate that is supported but not fixed along each of two opposite edges and subjected to a uniform shortening parallel to those edges, the theoretical expression (Ref. 20) for the effective width is

$$w = \frac{\pi t}{2\sqrt{3(1-\nu^2)}}\sqrt{\frac{E}{\sigma}} \qquad (12.6\text{-}1)$$

where $w =$ the effective width along each supported edge, $t =$ the thickness of the plate, and $\sigma =$ the unit stress at the supported edge. Since the maximum value of σ is σ_y (the yield point or yield strength), the maximum load that can be carried by the effective strip or by the whole plate (which amounts to the same thing) is

$$P = \frac{\pi t^2}{\sqrt{3(1-\nu^2)}}\sqrt{E\sigma_y} \qquad (12.6\text{-}2)$$

This formula can be written

$$P = Ct^2\sqrt{E\sigma_y} \qquad (12.6\text{-}3)$$

where C is an empirical constant to be determined experimentally for any given material and manner of support. Tests (Ref. 21) made on single plates of various metals, supported at the edges, gave values for C ranging from 1.18 to 1.67; its theoretical value from Eq. (12.6-2) (taking $\nu = 0.25$) is 1.87.

Sechler (Ref. 22) represents C as a function of $\lambda = (t/b)\sqrt{E/\sigma_y}$, where b is the panel width, and gives a curve showing experimentally determined values of C plotted against λ. The following table of corresponding values is taken from Sechler's corrected graph:

λ	0.02	0.05	0.1	0.15	0.2	0.3	0.4	0.5	0.6	0.8
C	2.0	1.76	1.62	1.50	1.40	1.28	1.24	1.20	1.15	1.10

The effective width at failure can be calculated by the relation

$$w = \tfrac{1}{2} Ct\sqrt{\frac{E}{\sigma_y}} = \tfrac{1}{2} Cb\lambda$$

In the case of a cylindrical panel loaded parallel to the axis, the effective width at failure can be taken as approximately equal to that for a flat sheet, but the increase in the buckling stress in the central portion of the panel due to curvature must be taken into account. Sechler shows that the contribution of this central portion to the strength of the panel may be allowed for by using for C in the formula $P = Ct^2\sqrt{E\sigma_y}$, a value given by

$$C = C_f - 0.3C_f\lambda\eta + 0.3\eta$$

where $\lambda = (t/b)\sqrt{E/\sigma_y}$, $\eta = (b/r)\sqrt{E/\sigma_y}$, and C_f, is the value of C for a flat sheet, as given by the above table.

The above formulas and experimental data refer to single sheets supported along each edge. In calculating the load carried by a flat sheet with longitudinal stiffeners at any given stiffener stress σ_s, the effective width corresponding to that stress is found by

$$w = b(0.25 + 0.91\lambda^2)$$

where $\lambda = (t/b)\sqrt{E/\sigma_s}$ and $b =$ distance between the stiffeners (Ref. 23). The total load carried by n stiffeners and the supported plate is then

$$P = n(A_s + 2wt)\sigma_s$$

where A_s is the section area of one stiffener. When σ_s is the maximum unit load the stiffener can carry as a column, P becomes the ultimate load for the reinforced sheet.

In calculating the ultimate load on a curved sheet with stiffeners, the strength of each unit or panel may be found by adding to the buckling strength of the central portion of the panel the strength of a column made up of the stiffener and the effective width of the attached sheet, this effective width being found by

$$w = \tfrac{1}{2} C_f t\sqrt{\frac{E}{\sigma_c}}$$

where C_f is the flat-sheet coefficient corresponding to $\lambda = (t/b)\sqrt{E/\sigma_c}$ and σ_c is the unit load that the stiffener-and-sheet column will carry before failure, determined by an appropriate column formula. [For the type of thin section often used for stiffeners in airplane construction, σ_c may be found by Eqs. (12.2-7) or (12.2-8).] Since the unit load σ_c and the effective width w are interdependent (because of the effect of w on the column radius of gyration), it is necessary to assume a value of σ_c, calculate the corresponding w, and then ascertain if the value of σ_c is consistent (according to the column formula used) with this w. (This procedure may have to be repeated several times before agreement is reached.) Then, σ_c and w being known, the strength of the stiffener-and-sheet combination is calculated as

$$P = n[\sigma_c(A_s + 2wt) + (b - 2w)t\sigma']$$

where n is the number of stiffeners, A_s is the section area of one stiffener, b is the distance between stiffeners (rivet line to rivet line) and σ' is the critical buckling stress for the central portion of the sheet, taken as $\sigma' = 0.3Et/r$ (r being the radius of curvature of the sheet).

Methods of calculating the strength of stiffened panels and thin columns subject to local and torsional buckling are being continually modified in the light of current study and experimentation. A more extensive discussion than is appropriate here can be found in books on airplane stress analysis, as well as in Refs. 4 and 5.

12.7 Short Prisms under Eccentric Loading

When a compressive or tensile load is applied eccentrically to a short prism (i.e., one so short that the effect of deflection is negligible), the resulting stresses are readily found by superposition. The eccentric load P is replaced by an equal axial load P' and by couples Pe_1 and Pe_2, where e_1 and e_2 denote the eccentricities of P with respect to the principal axes 1 and 2, respectively. The stress at any point, or the maximum stress, is then found by superposing the direct stress P'/A due to the axial load and the bending stresses due to the couples Pe_1 and Pe_2, these being found by the ordinary flexure formula (Sec. 8.1).

If, however, the prism is composed of a material that can withstand compression only (masonry) or tension only (very thin shell), this method cannot be employed when the load acts *outside the kern* because the reversal of stress implied by the flexure formula cannot occur. By assuming a linear stress distribution and making use of the facts that the volume of the stress solid must equal the applied load P and that the center of gravity of the stress solid must lie on the line of action of P, formulas can be derived for the position of the neutral axis (line of zero stress) and for maximum fiber stress in a prism of any given cross section. A number of such formulas are given in Table 12.1, together with the dimensions of the kern for each of the sections considered. For any section that is symmetrical about the axis of eccentricity, the maximum stress $K(P/A)$ is just twice the average stress P/A when the load is applied at the edge of the kern and increases as the eccentricity increases, becoming (theoretically) infinite when the load is applied at the extreme fiber. A prism made of material incapable of sustaining both tension and compression will fail completely when the resultant of the loads falls outside the boundary of any cross section, and will crack (under tension) or buckle (under compression) part way across any section through which the resultant of the loads passes at a point lying outside the kern.

For any section not shown in Table 12.1, a chart may be constructed showing the relation between e and x; this is done by assuming successive positions of the neutral axis (parallel to one principal axis) and solving for the corresponding eccentricity by the relation $b = I/M$, where $b =$ distance from the neutral axis to the point of application of the load (assumed to be on the other principal axis), $I =$ moment of inertia, and $M =$ the statical moment about the neutral axis of that part of the section carrying stress. The position of the neutral axis for any given eccentricity being known, the maximum stress can be found by the relation $\sigma_{\max} = Px/M$. These equations simply express the facts stated above—that the center of gravity of the stress solid lies on the line of action of P, and that the volume of the stress solid is equal to P. The procedure outlined is simple in principle but rather laborious when applied to any except the simpler type of section since both M and I may have to be determined by graphical integration.

The method of solution just outlined and all the formulas of Table 12.1 are based on the assumption that the load is applied on one of the principal axes of the section. If the load is applied outside the kern and on neither principal axis, solution is more difficult because neither the position nor the direction of the neutral axis corresponding to a given position of the load is known. The following graphical method, which involves successive trials, may be used for a section of any form.

Let a prism of any section (Fig. 12.3) be loaded at any point P. Guess the position of the neutral axis NN. Draw from NN to the most remote fiber q the perpendicular aq. That part of the section on the load side of NN is under compression, and the intensity of stress varies linearly from 0 at NN to σ at q. Divide the stressed area into narrow strips of uniform width dy running parallel to NN. The total stress on any strip acts at the center of that strip and is proportional to the area of the strip $w\,dy$ and to its distance y from NN. Draw the locus of the centers of the strips bcq and mark off a length of strip extending $\frac{1}{2}wy/x$ to each side of this locus. This portion of the strip, if it sustained a unit stress σ, would carry the same total load as does the whole strip when sustaining the actual unit stress $\sigma y/x$ and may be called the *effective*

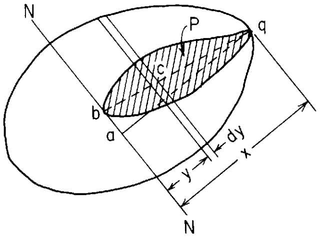


Figure 12.3

portion of the strip. The effective portions of all strips combine to form the *effective area*, shown as the shaded portion of Fig. 12.3. Now if the assumed position of NN is correct, the centroid of this effective area will coincide with the point P and the maximum stress σ will then be equal to the load P divided by the effective area.

To ascertain whether or not the centroid of the effective area does coincide with P, trace its outline on stiff cardboard; then cut out the piece so outlined and balance it on a pin thrust through at P. If the piece balances in any position, P is, of course, the centroid. Obviously the chance of guessing the position of NN correctly at the first attempt is remote, and a number of trials are likely to be necessary. Each trial, however, enables the position of NN to be estimated more closely, and the method is less tedious than might be supposed.

For a solid rectangular section, Esling (Ref. 15) explains a special method of analysis and gives tabulated constants which greatly facilitate solution for this particular case. The coefficient K, by which the average stress P/A is multiplied to give the maximum stress σ, is given as a function of the eccentric ratios e_1/d and e_2/b, where the terms have the meaning shown by Fig. 12.4. The values of K, taken from Esling's paper, are as shown in the accompanying table.

e_1/d e_2/b	0	0.05	0.10	0.15	0.175	0.200	0.225	0.250	0.275	0.300	0.325	0.350	0.375	0.400
0	1.0	1.30	1.60	1.90	2.05	2.22	2.43	2.67	2.96	3.33	3.81	4.44	5.33	6.67
0.05	1.30	1.60	1.90	2.21	2.38	2.58	2.81	3.09	3.43	3.87	4.41	5.16	6.17	7.73
0.10	1.60	1.90	2.20	2.56	2.76	2.99	3.27	3.60	3.99	4.48	5.14	5.99	7.16	9.00
0.15	1.90	2.21	2.56	2.96	3.22	3.51	3.84	4.22	4.66	5.28	6.03	7.04	8.45	10.60
0.175	2.05	2.38	2.76	3.22	3.50	3.81	4.16	4.55	5.08	5.73	6.55	7.66	9.17	11.50
0.200	2.22	2.58	2.99	3.51	3.81	4.13	4.50	4.97	5.54	6.24	7.12	8.33	9.98	
0.225	2.43	2.81	3.27	3.84	4.16	4.50	4.93	5.18	6.05	6.83	7.82	9.13	10.90	
0.250	2.67	3.09	3.60	4.22	4.55	4.97	5.48	6.00	6.67	7.50	8.57	10.0	12.00	
0.275	2.96	3.43	3.99	4.66	5.08	5.54	6.05	6.67	7.41	8.37	9.55	11.1		
0.300	3.33	3.87	4.48	5.28	5.73	6.24	6.83	7.50	8.37	9.37	10.80			
0.325	3.81	4.41	5.14	6.03	6.55	7.12	7.82	8.57	9.55	10.80				
0.350	4.44	5.16	5.99	7.04	7.66	8.33	9.13	10.00	11.10					
0.375	5.33	6.17	7.16	8.45	9.17	9.98	10.90	12.00						
0.400	6.67	7.73	9.00	10.60	11.50									

By double linear interpolation, the value of K for any eccentricity within the limits of the table may readily be found.

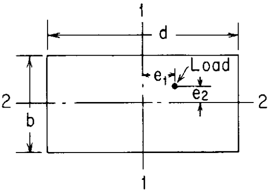


Figure 12.4

EXAMPLE

A bridge pier of masonry, 80 ft high, is rectangular in section, measuring at the base 20 by 10 ft, the longer dimension being parallel to the track. This pier is subjected to a vertical load P (including its own weight) of 1500 tons, a horizontal braking load (parallel to the track) of 60 tons, and a horizontal wind load P_z (transverse to the track) of 50 tons. It is required to determine the maximum compressive stress at the base of the pier, first assuming that the masonry can sustain tension and, second, that it cannot.

Solution. For convenience in numerical work, the ton will be retained as the unit of force and the foot as the unit of distance.

(*a*) *Masonry can sustain tension:* Take $d = 20$ ft, and $b = 10$ ft, and take axes 1 and 2 as shown in Fig. 12.5. Then, with respect to axis 1, the bending moment $M_1 = 60 \times 80 = 4800$ ton-ft, and the section modulus $(I/c)_1 = \frac{1}{6}(10)(20^2) = 667\,\text{ft}^3$. With respect to axis 2, the bending moment $M_2 = 50 \times 80 = 4000$ ton-ft, and the section modulus $(I/c)_2 = \frac{1}{6}(20)(10^2) = 333\,\text{ft}^3$. The section area is $10 \times 20 = 200\,\text{ft}^2$. The maximum stress obviously occurs at the corner where both bending moments cause compression and is

$$\sigma = \frac{1500}{200} + \frac{4800}{667} + \frac{4000}{333} = 7.5 + 7.2 + 12 = 26.7\ \text{tons/ft}^2$$

(*b*) *Masonry cannot sustain tension:* The resultant of the loads pierces the base section of the pier at point P, at a distance $e_1 = 60 \times 80/1500 = 3.2$ ft from axis 1 at $e_2 = 50 \times 80/1500 = 2.67$ ft from axis 2. This resultant is resolved at point P into rectangular components, the only one of which causing compression is the vertical component, equal to 1500, with eccentricities e_1 and e_2. The eccentric ratios are $e_1/d = 0.16$ and $e_2/b = 0.267$. Referring to the tabulated values of K, linear interpolation between $e_1/d = 0.15$ and 0.175 at $e_2/b = 0.250$ gives $K = 4.35$. Similar interpolation at $e_2/b = 0.275$ gives $K = 4.83$. Linear interpolation between these values gives $K = 4.68$ as the true value at $e_2/b = 0.267$. The maximum stress is therefore

$$\sigma = \frac{KP}{A} = 4.68 \times \frac{1500}{200} = 35.1\ \text{tons/ft}^2$$

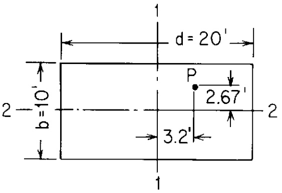


Figure 12.5

12.8 Tables

TABLE 12.1 Formulas for short prisms loaded eccentrically; stress reversal impossible

NOTATION: m and n are dimensions of the kern which is shown shaded in each case; x is the distance from the most stressed fiber to the neutral axis; and A is the net area of the section. Formulas for x and for maximum stress assume the prism to be subjected to longitudinal load P acting outside the kern on one principal axis of the section and at a distance e from the other principal axis

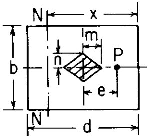
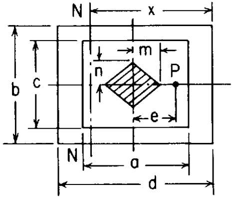
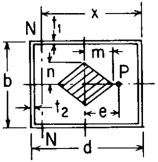
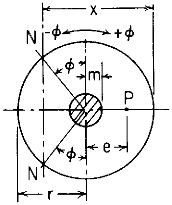
Form of section, form of kern, and case no.	Formulas for m, n, x, and $\sigma_{\max}$
1. Solid rectangular section  	$m = \frac{1}{6}d \qquad n = \frac{1}{6}b$ $x = 3(\frac{1}{2}d - e)$ $\sigma_{\max} = \frac{P}{A}\,\frac{4d}{3d - 6e}$
2. Hollow rectangular section  	$m = \frac{1}{6}\,\frac{bd^3 - ca^3}{d(db - ac)}, \qquad n = \frac{1}{6}\,\frac{db^3 - ac^3}{b(db - ac)}$ $x \text{ satisfies } \frac{e}{d} = \begin{cases} \frac{1}{2} - \frac{\frac{1}{6}bx^3 - \frac{1}{2}a^2c(\frac{1}{2}d - \frac{1}{6}a) - \frac{1}{2}acd(x - \frac{1}{2}a - \frac{1}{2}d)}{d[\frac{1}{2}bx^2 - ac(x - \frac{1}{2}d)]} & \text{if } \frac{a+d}{2} < x < d \\ \frac{1}{2} - \frac{\frac{1}{6}bx^3 - \frac{1}{2}c(x - \frac{1}{2}d - \frac{1}{2}a)^2(\frac{1}{3}x + \frac{1}{6}d - \frac{1}{6}a)}{d[\frac{1}{2}bx^2 - \frac{1}{2}c(x - \frac{1}{2}d - \frac{1}{2}a)^2]} & \text{if } \frac{d-a}{2} < x < \frac{a+d}{2} \end{cases}$ $\sigma_{\max} = \begin{cases} \frac{P}{\frac{1}{2}bx - ac(x - \frac{1}{2}d)/x} & \text{if } \frac{a+d}{2} < x < d \\ \frac{P}{\frac{1}{2}bx - c(x - \frac{1}{2}d + \frac{1}{2}a)^2/2x} & \text{if } \frac{d-a}{2} < x < \frac{a+d}{2} \end{cases}$
3. Thin-walled rectangular shell  	$m = \frac{1}{6}d\left(\frac{dt_1 + 3bt_2}{dt_1 + bt_2}\right), \qquad n = \frac{1}{6}b\left(\frac{bt_2 + 3dt_1}{dt_1 + bt_2}\right)$ $x = \frac{1}{2}\left(\frac{3}{2}d - 3e\right) + \sqrt{b\left(\frac{3}{2}d\frac{t_2}{t_1} - 3e\frac{t_2}{t_1}\right) + \frac{1}{4}\left(\frac{3}{2}d - 3e\right)^2}$ $\sigma_{\max} = \frac{P}{xt_1 + bt_2}$

TABLE 12.1 Formulas for short prisms loaded eccentrically; stress reversal impossible (*Continued*)

4. Solid circular section



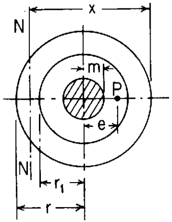
$$m = \frac{r}{4}$$

$$x = r(1 - \sin\phi) \qquad \text{where } \phi \text{ satisfies} \qquad \frac{e}{r} = \frac{\frac{1}{8}\pi - \frac{1}{4}\phi - \frac{5}{12}\sin\phi\cos\phi + \frac{1}{6}\sin^3\phi\cos\phi}{\cos\phi - \frac{1}{3}\cos^3\phi - \frac{1}{2}\pi\sin\phi + \phi\sin\phi}$$

$$\sigma_{\max} = \frac{P}{A}\,\frac{\pi(1 - \sin\phi)}{A\cos\phi - \frac{1}{3}\cos^3\phi - \frac{1}{2}\pi\sin\phi + \phi\sin\phi} \qquad \text{or} \qquad \sigma_{\max} = \frac{P}{A}K \qquad \text{where } K \text{ is given by following table:}$$

e/r	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90
x/r	2.00	1.82	1.66	1.51	1.37	1.23	1.10	0.97	0.84	0.72	0.60	0.47	0.35	0.24
K	2.00	2.21	2.46	2.75	3.11	3.56	4.14	4.90	5.94	7.43	9.69	13.4	20.5	37.5

5. Hollow circular section



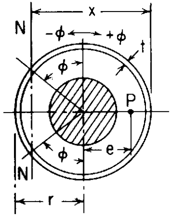
$$m = \frac{\frac{1}{4}(r^2 + r_1^2)}{r}$$

$$\sigma_{\max} = \frac{P}{A}K \qquad \text{where } K \text{ is given by following table:}$$

		e/r															
r_1/r		0.29	0.30	0.34	0.35	0.40	0.41	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90
0.4	x/r	2.00	1.97		1.81	1.67		1.53	1.38	1.22	1.05	0.88	0.73	0.60	0.48	0.35	0.24
	K	2.00	2.03		2.22	2.43		2.68	2.99	3.42	4.03	4.90	6.19	8.14	11.3	17.3	31.5
0.6	x/r			2.00	1.97	1.84		1.71	1.56	1.39	1.21	1.02	0.82	0.64	0.48	0.35	0.24
	K			2.00	2.03	2.18		2.36	2.58	2.86	3.24	3.79	4.64	6.04	8.54	13.2	24.0
0.8	x/r						2.00	1.91	1.78	1.62	1.45	1.26	1.05	0.84	0.63	0.41	0.24
	K						2.00	2.10	2.24	2.42	2.65	2.94	3.34	3.95	4.98	7.16	13.4
1.0	x/r								2.00	1.87	1.71	1.54	1.35	1.15	0.94	0.72	0.48
	K								2.00	2.12	2.26	2.43	2.63	2.90	3.27	3.79	4.68

TABLE 12.1 Formulas for short prisms loaded eccentrically; stress reversal impossible (*Continued*)

6. Thin-walled circular shell

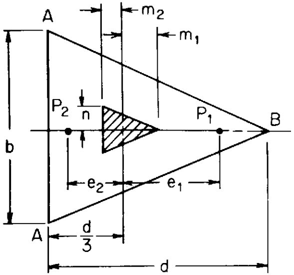


$m = \frac{1}{2}r$

$x = r(1 - \sin\phi)$ where ϕ satisfies $\dfrac{e}{r} = \dfrac{\frac{1}{2}\pi - \phi - \sin\phi\cos\phi}{2\cos\phi - \pi\sin\phi + 2\phi\sin\phi}$

$\sigma_{\max} = \dfrac{P}{tr}\dfrac{1 - \sin\phi}{2\cos\phi - \pi\sin\phi + 2\phi\sin\phi}$ or $\sigma_{\max} = \dfrac{P}{A}K$ where K is given by the above table (case 5) for $\dfrac{r_1}{r} = 1$

7. Solid isosceles triangle



$m_1 = \dfrac{d}{6}$, $m_2 = \dfrac{d}{12}$, $n = \dfrac{b}{8}$, $A = \dfrac{bd}{2}$

For the load right of centroid and kern, that is, $e_1 > m_1$:

Distance from vertex to neutral axis $= \dfrac{4d}{3} - 2e_1$

$\sigma_{\max} = \sigma_B = \dfrac{P_1}{A}\dfrac{6.75}{(2 - 3e_1/d)^2}$

For the load left of centroid and kern, that is, $e_2 > m_2$:

Distance from base to neutral axis $= x_2$ where x_2 satisfies

$$\frac{x_2}{2d}\frac{2 - x_2/d}{3 - x_2/d} = \frac{1}{3} - \frac{e_2}{d}$$

$\sigma_{\max} = \sigma_A = \dfrac{P_2}{A}\dfrac{3d}{x_2(3 - x_2/d)}$ or $\sigma_{\max} = \dfrac{P}{A}K$

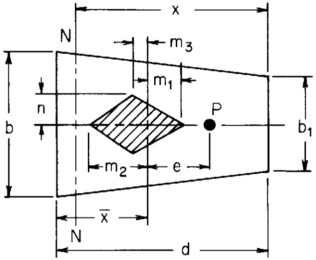
where K is given by following table:

e_2/d	0.083	0.100	0.125	0.150	0.175	0.200	0.225	0.250	0.275	0.300	0.325
x_2/d	1.000	0.885	0.750	0.635	0.532	0.437	0.348	0.263	0.181	0.102	0.025
K	1.500	1.602	1.778	1.997	2.283	2.677	3.252	4.173	5.886	10.17	40.18

(*continued*)

TABLE 12.1 Formulas for short prisms loaded eccentrically; stress reversal impossible (*Continued*)

8. Solid trapezoidal section



$$q = \frac{b_1}{b}, \qquad \bar{x} = \frac{d}{3}\,\frac{1+2q}{1+q}, \qquad n = \frac{b}{8}\,\frac{1+q+q^2+q^3}{1+q+q^2}$$

$$m_1 = \frac{d}{6}\,\frac{1+4q+q^2}{1+3q+2q^2}, \qquad m_2 = \frac{d}{6}\,\frac{1+4q+q^2}{2+3q+q^2}, \qquad m_3 = \frac{d}{12}\,\frac{1+3q-3q^2-q^3}{1+2q+2q^2+q^3}$$

$$A = \frac{d(b+b_1)}{2}$$

$$\sigma_{\max} = \frac{P}{A}\,\frac{3(1+q)}{3qx/d+(1-q)(x/d)^2} \qquad \text{where } x \text{ satisfies} \qquad \frac{e}{d} = \frac{2+q}{3(1+q)} - \frac{x}{2d}\,\frac{2q+(1-q)x/d}{3q+(1-q)x/d}$$

or $\sigma_{\max} = \dfrac{P}{A}K$ where K is given by following table:

q	$\bar{x}/d$	m_1/d	m_2/d	m_3/d	n/b	x/d	1.000	0.900	0.800	0.700	0.600	0.500	0.400	0.300	0.200
0.40	0.429	0.183	0.137	0.063	0.130	e/d	0.183	0.225	0.267	0.308	0.348	0.388	0.427	0.465	0.502
						K	2.333	2.682	3.125	3.704	4.487	5.600	7.292	10.14	15.91
0.80	0.481	0.172	0.160	0.018	0.151	e/d	0.172	0.208	0.244	0.279	0.314	0.349	0.383	0.417	0.451
						K	2.077	2.326	2.637	3.037	3.571	4.320	5.444	7.317	11.07
1.25	0.519	0.160	0.172	−0.018	0.189	e/d	0.160	0.191	0.222	0.254	0.286	0.318	0.350	0.383	0.415
						K	1.929	2.128	2.377	2.697	3.135	3.724	4.623	6.122	9.122
2.50	0.571	0.137	0.183	−0.063	0.325	e/d	0.137	0.161	0.187	0.214	0.242	0.271	0.301	0.332	0.363
						K	1.750	1.897	2.083	2.326	2.652	3.111	3.804	4.965	7.292

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Chapter

13

Shells of Revolution; Pressure Vessels; Pipes

13.1 Circumstances and General State of Stress

The discussion and formulas in this section apply to any vessel that is a figure of revolution. For convenience of reference, a line that represents the intersection of the wall and a plane containing the axis of the vessel is called a *meridian*, and a line representing the intersection of the wall and a plane normal to the axis of the vessel is called a *circumference*. Obviously the meridian through any point is perpendicular to the circumference through that point.

When a vessel of the kind under consideration is subjected to a distributed loading, such as internal or external pressure, the predominant stresses are membrane stresses, i.e., stresses constant through the thickness of the wall. There is a meridional membrane stress σ_1 acting parallel to the meridian, a circumferential, or hoop, membrane stress σ_2 acting parallel to the circumference, and a generally small radial stress σ_3 varying through the thickness of the wall. In addition, there may be bending and/or shear stresses caused by loadings or physical characteristics of the shell and its supporting structure. These include (1) concentrated loads, (2) line loads along a meridian or circumference, (3) sudden changes in wall thickness or an abrupt change in the slope of the meridian, (4) regions in the vessel where a meridian becomes normal to or approaches being normal to the axis of the vessel, and (5) wall thicknesses greater than those considered thin-walled, resulting in variations of σ_1 and σ_2 through the wall.

In consequence of these stresses, there will be meridional, circumferential, and radial strains leading to axial and radial deflections and changes in meridional slope. If there is axial symmetry of both the loading and the vessel, there will be no tendency for any circumference to depart from the circular form unless buckling occurs.

13.2 Thin Shells of Revolution under Distributed Loadings Producing Membrane Stresses Only

If the walls of the vessel are relatively thin (less than about one-tenth the smaller principal radius of curvature) and have no abrupt changes in thickness, slope, or curvature and if the loading is uniformly distributed or smoothly varying and axisymmetric, the stresses σ_1 and σ_2 are practically uniform throughout the thickness of the wall and are the only important ones present; the radial stress σ_3 and such bending stresses as occur are negligibly small. Table 13.1 gives formulas for the stresses and deformations under loadings such as those just described for cylindrical, conical, spherical, and toroidal vessels as well as for general smooth figures of revolution as listed under case 4.

If two thin-walled shells are joined to produce a vessel, and if it is desired to have no bending stresses at the joint under uniformly distributed or smoothly varying loads, then it is necessary to choose shells for which the radial deformations and the rotations of the meridians are the same for each shell at the point of connection. For example, a *cylindrical* shell under uniform internal pressure will have a radial deformation of $qR^2(1 - \nu/2)/Et$ while a *hemispherical* head of equal thickness under the same pressure will have a radial deformation of $qR^2(1 - \nu)/2Et$; the meridian rotation ψ is zero in both cases. This mismatch in radial deformation will produce bending and shear stresses in the near vicinity of the joint. An examination of case 4a (Table 13.1) shows that if R_1 is infinite at $\theta = 90^\circ$ for a smooth figure of revolution, the radial deformation and the rotation of the meridian will match those of the cylinder.

Flügge (Ref. 5) points out that the family of *cassinian* curves has the property just described. He also discusses in some detail the *ogival* shells, which have a constant radius of curvature R_1 for the meridian but for which R_2 is a variable. If R_2 is everywhere less than R_1, the ogival shell has a pointed top, as shown in Fig. 13.1(*a*). If R_2 is infinite, as it is at point A in Fig. 13.1(*b*), the center of the shell must be supported to avoid large bending stresses although some bending stresses are still present in the vicinity of point A. For more details of these deviations from membrane action see Refs. 66 and 74–76.

For very thin shells where bending stresses are negligible, a nonlinear membrane theory can provide more realistic values near the crown, point A. Rossettos and Sanders have carried out such a solution (Ref. 52). Chou and Johnson (Ref. 57) have examined large

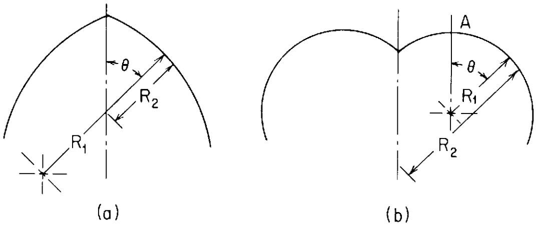


Figure 13.1

deflections of elastic toroidal membranes of a type used in some sensitive pressure-measuring devices.

Galletly in Ref. 67 shows that simple membrane theory is not adequate for the stress analysis of most torispherical pressure vessels. Ranjan and Steele in Ref. 68 have worked with asymptotic expansions and give a simple design formula for the maximum stress in the toroidal segment which is in good agreement with experimental and numerical studies. They present a simple condition that gives the optimum knuckle radius for prescribed spherical cap and cylinder geometries and also give expressions leading to a lower limit for critical internal pressure at which wrinkles are formed due to circumferential compression in the toroid.

Baker, Kovalevsky, and Rish (Ref. 6) give formulas for toroidal segments, ogival shells, elliptical shells, and Cassini shells under various loadings; all these cases can be evaluated from case 4 of Table 13.1 once R_1 and R_2 are calculated. In addition to the axisymmetric shells considered in this chapter, Refs. 5, 6, 45, 59, 66, 74, 81, and 82 discuss in some detail the membrane stresses in nonaxisymmetric shells, such as barrel vaults, elliptic cylinders, and hyperbolic paraboloids.

EXAMPLES

1. A segment of a toroidal shell shown in Fig. 13.2 is to be used as a transition between a cylinder and a head closure in a thin-walled pressure vessel. To properly match the deformations, it is desired to know the change in radius and the rotation of the meridian at both ends of the toroidal segment under an internal pressure loading of 200 lb/in^2. Given: $E = 30(10^6)$ lb/in^2, $\nu = 0.3$, and the wall thickness $t = 0.1$ in.

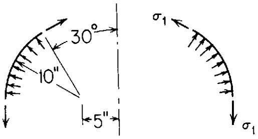


Figure 13.2

Solution. Since this particular case is not included in Table 13.1, the general case 4a can be used. At the upper end $\theta = 30^\circ$, $R_1 = 10\,\text{in}$, and $R_2 = 10 + 5/\sin 30^\circ = 20\,\text{in}$; therefore,

$$\Delta R_{30^\circ} = \frac{200(20^2)(0.5)}{2(30)(10^6)(0.1)}\left(2 - \frac{20}{10} - 0.3\right) = -0.002\ \text{in}$$

Since R_1 is a constant, $dR_1/d\theta = 0$ throughout the toroidal segment; therefore,

$$\psi_{30^\circ} = \frac{200(20^2)}{2(30)(10^6)(0.1)(10)(0.577)}\left[3\frac{10}{20} - 5 + \frac{20}{10}(2+0)\right] = 0.00116\ \text{rad}$$

At the lower end, $\theta = 90^\circ$, $R_1 = 10\,\text{in}$, and $R_2 = 15\,\text{in}$; therefore,

$$\Delta R_{90^\circ} = \frac{200(15^2)(1)}{2(30)(10^6)(0.1)}\left(2 - \frac{15}{10} - 0.3\right) = 0.0015\ \text{in}$$

Since $\tan 90^\circ = \text{infinity}$ and $dR_1/d\theta = 0$, $\psi_{90^\circ} = 0$. In this problem $R_2/R_1 \leqslant 2$, so the value of σ_2 is never compressive, but this is not always true. One must check for the possibility of circumferential buckling.

2. The truncated thin-walled cone shown in Fig. 13.3 is supported at its base by the membrane stress σ_1. The material in the cone weighs $0.10\,\text{lb/in}^3$, $t = 0.25\,\text{in}$, $E = 10(10^6)\,\text{lb/in}^2$, and Poisson's ratio is 0.3. Find the stress σ_1 at the base, the change in radius at the base, and the change in height of the cone if the cone is subjected to an acceleration parallel to its axis of $399g$.

Solution. Since the formulas for a cone loaded by its own weight are given only for a complete cone, superposition will have to be used. From Table 13.1 cases 2c and 2d will be applicable. First take a complete cone loaded by its own weight with its density multiplied by 400 to account for the acceleration. Since the vertex is up instead of down, a negative value can be used for δ. $R = 20\,\text{in}$, $\delta = -40.0$, and $\alpha = 15^\circ$; therefore,

$$\sigma_1 = \frac{-40(20)}{2\cos 15^\circ \cos 15^\circ} = -1600\ \text{lb/in}^2$$

$$\Delta R = \frac{-40(20^2)}{10(10^6)\cos 15^\circ}\left(\sin 15^\circ - \frac{0.3}{2\sin 15^\circ}\right) = 0.000531\ \text{in}$$

$$\Delta y = \frac{-40(20^2)}{10(10^6)\cos^2 15^\circ}\left(\frac{1}{4\sin^2 15^\circ} - \sin^2 15^\circ\right) = -0.00628\ \text{in}$$

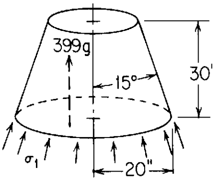


Figure 13.3

Next we find the radius of the top as 11.96 in and calculate the change in length and effective weight of the portion of the complete cone to be removed. $R = 11.96$ in, $\delta = -40.0$, and $\alpha = 15^\circ$; therefore,

$$\Delta y = -0.00628\left(\frac{11.96}{20}\right)^2 = -0.00225 \text{ in}$$

The volume of the removed cone is

$$\frac{11.96}{\sin 15^\circ}\,\frac{11.96(2\pi)}{2}(0.25) = 434 \text{ in}^3$$

and the effective weight of the removed cone is 434(0.1)(400) = 17,360 lb.

Removing the load of 17,360 lb can be accounted for by using case 2d, where $P = 17{,}360$, $R = 20$ in, $r = 11.96$ in, $h = 30$ in, and $\alpha = 15^\circ$:

$$\begin{aligned}
\sigma_1 &= \frac{17{,}360}{2\pi(20)(0.25)\cos 15^\circ} = 572 \text{ lb/in}^2 \\
\Delta R &= \frac{-0.3(17{,}360)}{2\pi(10)(10^6)(0.25)\cos 15^\circ} = -0.000343 \text{ in} \\
\Delta h &= \frac{17{,}360\ln(20/11.96)}{2\pi(10)(10^6)(0.25)\sin 15^\circ \cos^2 15^\circ} = 0.002353 \text{ in}
\end{aligned}$$

Therefore, for the truncated cone,

$$\begin{aligned}
\sigma_1 &= -1600 + 572 = -1028 \text{ lb/in}^2 \\
\Delta R &= 0.000531 - 0.000343 = 0.000188 \text{ in} \\
\Delta h &= -0.00628 + 0.00225 + 0.002353 = -0.00168 \text{ in}
\end{aligned}$$

13.3 Thin Shells of Revolution under Concentrated or Discontinuous Loadings Producing Bending and Membrane Stresses

Cylindrical shells. Table 13.2 gives formulas for forces, moments, and displacements for several *axisymmetric* loadings on both *long* and *short* thin-walled *cylindrical* shells having free ends. These expressions are based on differential equations similar in form to those used to develop the formulas for beams on elastic foundations in Chap. 8. To avoid excessive redundancy in the presentation, only the free-end cases are given in this chapter, but all of the loadings and boundary conditions listed in Tables 8.5 and 8.6 as well as the tabulated data in Tables 8.3 and 8.4 are directly applicable to cylindrical shells by substituting the shell parameters λ and D for the beam parameters β and EI, respectively. (This will be demonstrated in the examples which follow.) Since many loadings on cylindrical shells occur at the ends, note carefully on page 148 the modified numerators to be used in the equations in Table 8.5 for the condition when $a = 0$. A special application of this would be the situation where one end of a cylindrical shell is forced to increase a known amount in radius while maintaining zero slope at that same end. This reduces to an application of an externally created concentrated lateral displacement Δ_0 at $a = 0$ (Table 8.5, case 6) with the left end fixed. (See Example 4.)

Pao (Ref. 60) has tabulated influence coefficients for short cylindrical shells under edge loads with wall thicknesses varying according to $t = Cx^n$ for values of $n = \frac{1}{4}(\frac{1}{4})(2)$ and for values of t_1/t_2 of 2, 3, and 4. Various degrees of taper are considered by representing data for $k = 0.2(0.2)(1.0)$ where $k^4 = 3(1 - \nu^2)x_1^4/R^2t_1^2$. Stanek (Ref. 49) has tabulated similar coefficients for constant-thickness cylindrical shells.

A word of caution is in order at this point. The original differential equations used to develop the formulas presented in Table 13.2 were based on the assumption that radial deformations were small. If the magnitude of the radial deflection approaches the wall thickness, the accuracy of the equations declines. In addition, if axial loads are involved on a relatively short shell, the moments of these axial loads might have an appreciable effect if large deflections are encountered. The effects of these moments are not included in the expressions given.

EXAMPLES

1. A steel tube with a 4.180-in outside diameter and a 0.05-in wall thickness is free at both ends and is 6 in long. At a distance of 2 in from the left end a steel ring with a circular cross section is shrunk onto the outside of the tube such as to compress the tube radially inward a distance of 0.001 in. The maximum tensile stress in the tube is desired. Given: $E = 30(10^6)$ lb/in^2 and $\nu = 0.30$.

Solution. We calculate the following constants:

$$R = 2.090 - 0.025 = 2.065$$

$$\lambda = \left[\frac{3(1 - 0.3^2)}{2.065^2(0.05^2)}\right]^{1/4} = 4.00$$

$$D = \frac{30(10^6)(0.05^3)}{12(1 - 0.3^2)} = 344$$

Since $6/\lambda = 6/4.0 = 1.5$ in and the closest end of the tube is 2 in from the load, this can be considered a very long tube. From Table 13.2, case 15 indicates that both the maximum deflection and the maximum moment are under the load, so that

$$-0.001 = \frac{-p}{8(344)(4.00^3)} \qquad \text{or} \qquad p = 176 \text{ lb/in}$$

$$M_{\max} = \frac{176}{4(4)} = 11.0 \text{ in-lb/in}$$

At the cross section under the load and on the inside surface, the following stresses are present:

$$\begin{aligned}
\sigma_1 &= 0\\
\sigma_1' &= \frac{6M}{t^2} = \frac{6(11.0)}{0.05^2} = 26{,}400\ \text{lb/in}^2\\
\sigma_2 &= \frac{yE}{R} + \nu\sigma_1 = \frac{-0.001(30)(10^6)}{2.065} = -14{,}500\ \text{lb/in}^2\\
\sigma_2' &= 0.30(26{,}400) = 7920\ \text{lb/in}^2
\end{aligned}$$

The principal stresses on the inside surface are 26,400 and −6580 lb/in^2.

2. Given the same tube and loading as in Example 1, except the tube is only 1.2 in long and the ring is shrunk in place 0.4 in from the left end, the maximum tensile stress is desired.

Solution. Since both ends are closer than $6/\lambda = 1.5$ in from the load, the free ends influence the behavior of the tube under the load. From Table 13.2, case 2 applies in this example, and since the deflection under the load is the given value from which to work, we must evaluate y at $x = a = 0.4$ in. Note that $\lambda l = 4.0(1.2) = 4.8$, $\lambda x = \lambda a = 4.0(0.4) = 1.6$, and $\lambda(l - a) = 4.0(1.2 - 0.4) = 3.2$. Also,

$$\begin{aligned}
y &= y_A F_1 + \frac{\psi_A}{2\lambda} + LT_y\\
y_A &= \frac{-p}{2D\lambda^3}\frac{C_3C_{a2} - C_4C_{a1}}{C_{11}}\\
\psi_A &= \frac{p}{2D\lambda^2}\frac{C_3C_{a2} - C_4C_{a1}}{C_{11}}
\end{aligned}$$

where

$C_3 = -60.51809$ (from Table 8.3, under F_3 for $\beta x = 4.8$)
$C_{a2} = -12.94222$ (from Table 8.3, under F_2 for $\beta x = 3.2$)
$C_4 = -65.84195$
$C_{11} = 3689.703$
$C_{a1} = -12.26569$
$C_2 = -55.21063$

Also F_1 (at $x = a$) $= -0.07526$ and F_2 (at $x = a$) $= 2.50700$; therefore,

$$\begin{aligned}
y_A &= \frac{-p}{2(344)(4.0^3)}\ \frac{-60.52(-12.94) - (-65.84)(-12.27)}{3689.7} = 0.154(10^{-6})p\\
\psi_A &= \frac{-p}{2(344)(4.0^2)}\ \frac{-55.21(-12.94) - 2(-60.52)(-12.27)}{3689.7} = -19.0(10^{-6})p
\end{aligned}$$

and $LT_y = 0$ since x is not greater than a. Substituting into the expression for y at $x = a$ gives

$$-0.001 = 0.154(10^{-6})p(-0.07526) - \frac{19.0(10^{-6})p(2.507)}{2(4.0)} = -5.96(10^{-6})p$$

or $p = 168$ lb/in, $y_A = 0.0000259$ in, and $\psi_A = -0.00319$ rad.

Although the position of the maximum moment depends upon the position of the load, the maximum moment in this case would be expected to be under the load since the load is some distance from the free end:

$$M = -y_A 2D\lambda^2 F_3 - \psi_A D\lambda F_4 + LT_M$$

and at $x = a$, $F_3 = 2.37456$, $F_4 = 2.64573$, and $LT_M = 0$ since x is not greater than a. Therefore,

$$\begin{aligned} M_{\max} &= -(0.0000259)(2)(344)(4.0^2)(2.375) - (-0.00319)(344)(4.0)(2.646) \\ &= 10.92 \text{ lb-in/in} \end{aligned}$$

At the cross section under the load and on the inside surface the following stresses are present:

$$\sigma_1 = 0, \qquad \sigma_2 = \frac{-0.001(30)(10^6)}{2.065} = -14{,}500 \text{ lb/in}^2$$

$$\sigma_1' = \frac{6(10.92)}{0.05^2} = 26{,}200 \text{ lb/in}^2, \qquad \sigma_2' = 0.30(26{,}200) = 7{,}860 \text{ lb/in}^2$$

The small change in the maximum stress produced in this shorter tube points out how localized the effect of a load on a shell can be. Had the radial load been the same, however, instead of the radial deflection, a greater difference might have been noted and the stress σ_2 would have increased in magnitude instead of decreasing.

3. A cylindrical aluminum shell is 10 in long and 15 in in diameter and must be designed to carry an internal pressure of 300 lb/in^2 without exceeding a maximum tensile stress of 12,000 lb/in^2. The ends are capped with massive flanges, which are sufficiently clamped to the shell to effectively resist any radial or rotational deformation at the ends. Given: $E = 10(10^6)$ lb/in^2 and $\nu = 0.3$.

First solution. Case 1c from Table 13.1 and cases 1 and 3 or cases 8 and 10 from Table 13.2 can be superimposed to find the radial end load and the end moment which will make the slopes and deflections at both ends zero. Figure 13.4 shows the loadings applied to the shell. First we evaluate the necessary constants:

$$R = 7.5 \text{ in}, \qquad l = 10 \text{ in}, \qquad D = \frac{10(10^6)t^3}{12(1-0.3^2)} = 915{,}800t^3$$

$$\lambda = \left[\frac{3(1-0.3^2)}{7.5^2 t^2}\right]^{1/4} = \frac{0.4694}{t^{0.5}}, \qquad \lambda l = \frac{4.694}{t^{0.5}}$$

Since the thickness is unknown at this step in the calculation, we can only estimate whether the shell must be considered long or short, i.e., whether the loads at one end will have any influence on the deformations at the other. To make an estimate of this effect we can calculate the wall thickness necessary for just the internal pressure. From case 1c of Table 13.1, the value of the hoop stress $\sigma_2 = qR/t$ can be equated to 12,000 lb/in^2 and the expression solved for the thickness:

$$t = \frac{300(7.5)}{12{,}000} = 0.1875 \text{ in}$$

Using this value for t gives $\lambda l = 10.84$, which would be a very long shell.

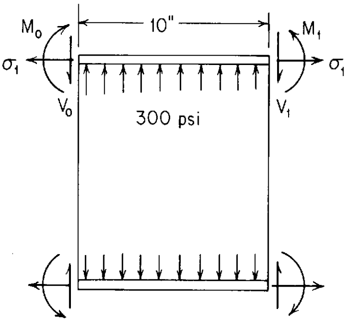


Figure 13.4

For a trial solution the assumption will be made that the radial load and bending moment at the right end do not influence the deformations at the left end. Owing to the rigidity of the end caps, the radial deformation and the angular rotation of the left end will be set equal to zero. From Table 13.1, case 1c,

$$\sigma_1 = \frac{qR}{2t} = \frac{300(7.5)}{2t} = \frac{1125}{t}, \qquad \sigma_2 = \frac{qR}{t} = \frac{2250}{t}, \qquad \psi = 0$$

$$\Delta R = \frac{qR^2}{Et}\left(1 - \frac{\nu}{2}\right) = \frac{300(7.5^2)}{10(10^6)t}\left(1 - \frac{0.3}{2}\right) = \frac{0.001434}{t}$$

From Table 13.2, case 8,

$$y_A = \frac{-V_o}{2D\lambda^3} = \frac{-V_o}{2(915{,}800t^3)(0.4694/t^{1/2})^3} = \frac{-5.279(10^{-6})V_o}{t^{3/2}}$$

$$\psi_A = \frac{V_o}{2D\lambda^2} = \frac{2.478(10^{-6})V_o}{t^{5/2}}$$

From Table 13.2, case 10,

$$y_A = \frac{M_o}{2D\lambda^2} = \frac{2.478(10^{-6})M_o}{t^2}$$

$$\psi_A = \frac{-M_o}{D\lambda} = \frac{-2.326(10^{-6})M_o}{t^{5/2}}$$

Summing the radial deformations to zero gives

$$\frac{0.001434}{t} - \frac{5.279(10^{-6})V_o}{t^{3/2}} + \frac{2.478(10^{-6})M_o}{t^2} = 0$$

Similarly, summing the end rotations to zero gives

$$\frac{2.478(10^{-6})V_o}{t^2} - \frac{2.326(10^{-6})M_o}{t^{5/2}} = 0$$

Solving these two equations gives

$$V_o = 543t^{1/2} \qquad \text{and} \qquad M_o = 579t$$

A careful examination of the problem reveals that the maximum bending stress will occur at the end, and so the following stresses must be combined: From Table 13.1, case 1c,

$$\sigma_1 = \frac{1125}{t}, \qquad \sigma_2 = \frac{2250}{t}$$

From Table 13.2, case 8,

$$\sigma_1 = 0, \qquad \sigma_1' = 0, \qquad \sigma_2' = 0$$

$$\sigma_2 = \frac{-2V_o\lambda R}{t} = \frac{-2(543t^{1/2})(0.4694/t^{1/2})(7.5)}{t} = \frac{-3826}{t}$$

From Table 13.2, case 10,

$$\sigma_1 = 0$$

$$\sigma_2 = \frac{2M_o\lambda^2 R}{t} = \frac{2(579t)(0.4694/t^{1/2})^2(7.5)}{t} = \frac{1913}{t}$$

and on the inside surface

$$\sigma_1' = \frac{6M_o}{t^2} = \frac{3473}{t}$$

$$\sigma_2' = \nu\sigma_1' = \frac{1042}{t}$$

Therefore, at the end of the cylinder the maximum longitudinal tensile stress is $1125/t + 3473/t = 4598/t$; similarly the maximum circumferential tensile stress is $2250/t - 3826/t + 1913/t + 1042/t = 1379/t$.

Since the allowable tensile stress was 12,000 lb/in^2, we can evaluate $4598/t = 12{,}000$ to obtain $t = 0.383$ in. This allows λl to be calculated as 7.59, which verifies the assumption that the shell can be considered a long shell for this loading and support.

Second solution. This loaded shell represents a case where both ends are fixed and a uniform radial pressure is applied over the entire length. Since the shell is considered long, we can find the expressions for R_A and M_A in Table 8.6, case 2, under the condition of the left end fixed and where the distance $a = 0$ and b can be considered infinite:

$$R_A = \frac{-2w}{\beta}(B_1 - A_1) \qquad \text{and} \qquad M_A = \frac{w}{\beta^2}(B_4 - A_4)$$

If $-V_o$ is substituted for R_A, M_o for M_A, λ for β, and D for EI, the solution should apply to the problem at hand. Care must be exercised when substituting for the distributed load w. A purely radial pressure would produce a radial deformation $\Delta R = qR^2/Et$, while the effect of the axial pressure on the ends reduces this to $\Delta R = qR^2(1 - \nu/2)/Et$. Therefore, for w we must substitute

$-300(1 - \nu/2) = -255\,\mathrm{lb/in^2}$. Also note that for $a = 0$, $A_1 = A_4 = 0.5$, and for $b = \infty$, $B_1 = B_4 = 0$. Therefore,

$$V_o = \frac{-2(255)}{\lambda}(0 - 0.5) = \frac{255}{\lambda} = 543t^{1/2}$$

$$M_o = \frac{-255}{\lambda^2}(0 - 0.5) = \frac{127.5}{\lambda^2} = 579t$$

which verifies the results of the first solution.

If we examine case 2 of Table 8.5 under the condition of both ends fixed, we find the expression

$$M_o = M_A = \frac{w}{2\lambda^2}\,\frac{2C_3C_5 - C_4^2}{C_{11}}$$

Substituting for the several constants and reducing the expression to a simple form, we obtain

$$M_o = \frac{-w}{2\lambda^2}\,\frac{\sinh\lambda l - \sin\lambda l}{\sinh\lambda l + \sin\lambda l}$$

The hyperbolic sine of 7.59 is 989, and so for all practical purposes

$$M_o = \frac{-w}{2\lambda^2} = 579t$$

which, of course, is the justification for the formulas in Table 8.6.

4. A 2-in length of steel tube described in Example 1 is heated, and rigid plugs are inserted $\frac{1}{2}$ in into each end. The rigid plugs have a diameter equal to the inside diameter of the tube plus 0.004 in at room temperature. Find the longitudinal and circumferential stresses at the outside of the tube adjacent to the end of the plug and the diameter at midlength after the tube is shrunk into the plugs.

Solution. The most straightforward solution would consist of assuming that the portion of the tube outside the rigid plug is, in effect, displaced radially a distance of 0.002 in and owing to symmetry the midlength has zero slope. A steel cylindrical shell $\frac{1}{2}$ in in length, fixed on the left end with a radial displacement of 0.002 in at $a = 0$ and with the right end guided, i.e., slope equal to zero, is the case to be solved.

From Example 1, $R = 2.065$ in, $\lambda = 4.00$, and $D = 344$; $\lambda l = 4.0(0.5) = 2.0$. From Table 8.5, case 6, for the left end fixed and the right end guided, we find the following expressions when $a = 0$:

$$R_A = \Delta_o 2EI\beta^3\,\frac{C_4^2 + C_2^2}{C_{12}} \qquad \text{and} \qquad M_A = \Delta_o 2EI\beta^2\,\frac{C_1C_4 - C_3C_2}{C_{12}}$$

Replace EI with D and β with λ; $\Delta_o = 0.002$, and from page 148 note that $C_4^2 + C_2^2 = 2C_{14}$ and $C_1C_4 - C_3C_2 = -C_{13}$. From Table 8.3, for $\lambda l = 2.00$, we find that $2C_{14} = 27.962$, $-C_{13} = -14.023$, and $C_{12} = 13.267$. Therefore,

$$R_A = 0.002(2)(344)(4.0^3)\frac{27.962}{13.267} = 185.6 \text{ lb/in}$$

$$M_A = 0.002(2)(344)(4.0^2)\frac{-14.023}{13.267} = -23.27 \text{ in-lb/in}$$

To find the deflection at the midlength of the shell, which is the right end of the half-shell being used here, we solve for y at $x = 0.5$ in and $\lambda x = 4.0(0.5) = 2.0$. Note that $y_A = 0$ because the deflection of 0.002 in was forced into the shell just beyond the end in the solution being considered here. Therefore,

$$y = \frac{M_A}{2D\lambda^2}F_3 + \frac{R_A}{2D\lambda^3}F_4 + \Delta_o F_{a1}$$

where from Table 8.3 at $\lambda x = 2.0$

$$F_3 = 3.298, \qquad F_4 = 4.930, \qquad F_{a1} = F_1 = -1.566 \text{ since } a = 0$$

$$y_{x=0.5} = \frac{-23.27}{2(344)(4.0^2)}3.298 + \frac{185.6}{4(344)(4.0^3)}4.93 + 0.002(-1.566) = 0.00029 \text{ in}$$

For a partial check on the solution we can calculate the slope at midlength. From Table 8.5, case 6,

$$\theta = \frac{M_A}{2EI\beta}F_2 + \frac{R_A}{2EI\beta^2}F_3 - \Delta_o \beta F_{a4}$$

where $F_2 = 1.912$ and $F_{a4} = F_4$ since $a = 0$. Therefore,

$$\theta = \frac{-23.27}{2(344)(4.0)}1.912 + \frac{185.6}{2(344)(4.0^2)}3.298 - 0.002(4.0)(4.930) = 0.00000$$

Now from Table 13.2,

$$\sigma_1' = \frac{-6M}{t^2} = \frac{-6(-23.27)}{0.05^2} = 55{,}850 \text{ lb/in}^2$$

Since $\sigma_1 = 0$,

$$\sigma_2 = \frac{0.002(30)(10^6)}{2.065} = 29{,}060 \text{ lb/in}^2$$

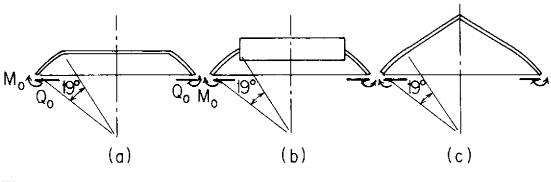
$$\sigma_2' = 0.3(55{,}800) = 16{,}750 \text{ lb/in}^2$$

On the outside surface at the cross section adjacent to the plug the longitudinal stress is 55,850 lb/in^2 and the circumferential stress is $29{,}060 + 16{,}750 = 45{,}810$ lb/in^2. Since a rigid plug is only hypothetical, the actual stresses present would be smaller when a solid but elastic plug is used. External clamping around the shell over the plugs would also be necessary to fulfill the assumed fixed-end condition. The stresses calculated are, therefore, maximum possible values and would be conservative.

Spherical shells. The format used to present the formulas for the finite-length cylindrical shells could be adapted for finite portions of open spherical and conical shells with both edge loads and loads applied within the shells if we were to accept the approximate solutions based on equivalent cylinders. Baker, Kovalevsky, and Rish (Ref. 6) present formulas based on this approximation for open spherical and conical shells under edge loads and edge displacements. For partial spherical shells under axisymmetric loading, Hetényi, in an earlier work (Ref. 14), discusses the errors introduced by this same approximation and compares it with a better approximate solution derived therein. Table 13.3, case 1, gives formulas based on Hetényi's work, and although it is estimated that the calculational effort is twice that of the simpler approximation, the errors in maximum stresses are decreased substantially, especially when the opening angle ϕ is much different from 90°.

Stresses and deformations due to edge loads decrease exponentially away from the loaded edges of axisymmetric shells, and consequently boundary conditions or other restraints are not important if they are far enough from the loaded edge. For example, the exponential term decreases to approximately 1% when the product of the spherical shell parameter β (see Table 13.3, case 1) and the angle ω (in radians) is greater than 4.5; similarly it reduces to approximately 5% at $\beta\omega = 3$. This means that a spherical shell with a radius/thickness ratio of 50, for which $\beta \approx 9$, can have an opening angle ϕ as small as $\frac{1}{3}$rad, or 19°, and still be solved with formulas for cases 1 with very little error. Figure 13.5 shows three shells, for which R/t is approximately 50, which would respond similarly to the edge loads M_o and Q_o. In fact, the conical portion of the shell in Fig. 13.5(c) could be extended much closer than 19° to the loaded edge since the conical portion near the junction of the cone and sphere would respond in a similar way to the sphere. (Hetényi discusses this in Ref. 14.)

Similar bounds on *nonspherical* but axisymmetric shells can be approximated by using closely matching equivalent spherical shells (Ref. 6). (We should note that the angle ϕ in Table 13.3, case 1, is not limited to a maximum of 90°, as will be illustrated in the examples at the end of this section.)



(a) (b) (c)

Figure 13.5

For shallow spherical shells where ϕ is small, Gerdeen and Niedenfuhr (Ref. 46) have developed influence coefficients for uniform pressure and for edge loads and moments. Shells with central cutouts are also included as are loads and moments on the edge of the cutouts. Many graphs as well as tabulated data are presented, which permits the solution of a wide variety of problems by superposition.

Cheng and Angsirikul (Ref. 80) present the results of an elasticity solution for edge-loaded spherical domes with thick walls and with thin walls.

Conical shells. Exact solutions to the differential equations for both long and short thin-walled truncated conical shells are described in Refs. 30, 31, 64, and 65. Verifications of these expressions by tests are described in Ref. 32, and applications to reinforced cones are described in Ref. 33. In Table 13.3, case 4 for long cones, where the loads at one end do not influence the displacements at the other, is based on the solution described by Taylor (Ref. 65) in which the Kelvin functions and their derivatives are replaced by asymptotic formulas involving negative powers of the conical shell parameter k (presented here in a modified form):

$$k = \frac{2}{\sin\alpha}\left[\frac{12(1-\nu^2)R^2}{t^2\sec^2\ \alpha}\right]^{1/4}$$

These asymptotic formulas will give three-place accuracy for the Kelvin functions for all values of $k > 5$. To appreciate this fully, one must understand that a truncated thin-walled cone with an R/t ratio of 10 at the small end, a semiapex angle of 80°, and a Poisson's ratio of 0.3 will have a value of $k = 4.86$. For problems where k is much larger than 5, fewer terms can be used in the series, but a few trial calculations will soon indicate the number of terms it is necessary to carry. If only displacements and stresses at the loaded edge are needed, the simpler forms of the expressions can be used. (See the example at the end of Sec. 13.4.)

Baltrukonis (Ref. 64) obtains approximations for the influence coefficients which give the edge displacements for *short* truncated conical shells under axisymmetric edge loads and moments; this is done by using one-term asymptotic expressions for the Kelvin functions. Applying the multiterm asymptotic expressions suggested by Taylor to a short truncated conical shell leads to formulas that are too complicated to present in a reasonable form. Instead, in Table 13.3, case 5 tabulates numerical coefficients based upon this more accurate formulation but evaluated by a computer for the case where Poisson's ratio is 0.3. Because of limited space, only five values of k and six values of the length parameter $\mu_D = |k_A - k_B|/\sqrt{2}$ are presented. If μ_D is greater than 4, the end loads do not interact appreciably and the formulas from case 4 may be used.

Tsui (Ref. 58) derives expressions for deformations of conical shells for which the thickness tapers linearly with distance along the meridian; influence coefficients are tabulated for a limited range of shell parameters. Blythe and Kyser (Ref. 50) give formulas for thin-walled conical shells loaded in torsion.

Toroidal shells. Simple closed-form solutions for toroidal shells are generally valid for a rather limited range of parameters, so that usually it is necessary to resort to numerical solutions. Osipova and Tumarkin (Ref. 18) present extensive tables of functions for the asymptotic method of solution of the differential equations for toroidal shells; this reference also contains an extensive bibliography of work on toroidal shells. Tsui and Massard (Ref. 43) tabulate the results of numerical solutions in the form of influence coefficients and influence functions for internal pressure and edge loadings on finite portions of segments of toroidal shells. Segments having *positive* and *negative* gaussian curvatures are considered; when both positive and negative curvatures are present in the same shell, the solutions can be obtained by matching slopes and deflections at the junction. References 29, 51, and 61 describe similar solutions.

Stanley and Campbell (Ref. 77) present the principal test results on 17 full-size, production-quality torispherical ends and compare them to theory. Kishida and Ozawa (Ref. 78) compare results arrived at from elasticity, photoelasticity, and shell theory. References 67 and 68 discuss torispherical shells and present design formulas. See the discussion in Sec. 13.2 on this topic.

Jordon (Ref. 53) works with the shell-equilibrium equations of a *deformed* shell to examine the effect of pressure on the stiffness of an axisymmetrically loaded toroidal shell.

Kraus (Ref. 44), in addition to an excellent presentation of the theory of thin elastic shells, devotes one chapter to numerical analysis under static loadings and another to numerical analysis under dynamic loadings. Comparisons are made among results obtained by finite-element methods, finite-difference methods, and analytic solutions. Numerical techniques, element sizes, and techniques of shell subdivision are discussed in detail. It would be impossible to list here all the references describing the finite-element computer programs available for solving shell problems, but Perrone (Ref. 62) has presented an excellent summary and Bushnell (Ref. 63) describes work on shells in great detail.

EXAMPLES

1. Two partial spheres of aluminum are to be welded together as shown in Fig. 13.6 to form a pressure vessel to withstand an internal pressure of 200 lb/in^2. The mean radius of each sphere is 2 ft, and the wall thickness is 0.5 in. Calculate the stresses at the seam. Given: $E = 10(10^6)$ lb/in^2 and $\nu = 0.33$.

Solution. The edge loading will be considered in three parts, as shown in Fig. 13.6(*b*). The tangential edge force T will be applied to balance the internal pressure and, together with the pressure, will cause only membrane stresses and the accompanying change in circumferential radius ΔR; this loading will produce no rotation of the meridian. Owing to the symmetry of the two shells, there is no resultant radial load on the edge, and so Q_o is added to eliminate that component of T. M_o is needed to ensure no edge rotation.

First apply the formulas from Table 13.1, case 3a:

$$\sigma_1 = \sigma_2 = \frac{qR_2}{2t} = \frac{200(24)}{2(0.5)} = 4800 \text{ lb/in}^2$$

$$\Delta R = \frac{qR_2^2(1-\nu)\sin\theta}{2Et} = \frac{200(24^2)(1-0.33)\sin 120^\circ}{2(10)(10^6)(0.5)} = 0.00668 \text{ in}$$

$$T = \sigma_1 t = 4800(0.5) = 2400 \text{ lb/in}$$

$$\psi = 0$$

Next apply case 1a from Table 13.3:

$$Q_o = T\sin 30^\circ = 2400(0.5) = 1200 \text{ lb/in}$$

$$\phi = 120^\circ$$

$$\beta = \left[3(1-\nu^2)\left(\frac{R_2}{t}\right)^2\right]^{1/4} = \left[3(1-0.33^2)\left(\frac{24}{0.5}\right)^2\right]^{1/4} = 8.859$$

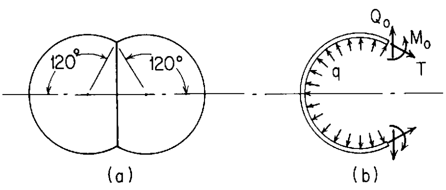


Figure 13.6

At the edge where $\omega = 0$,

$$K_1 = 1 - \frac{1-2\nu}{2\beta}\cot\phi = 1 - \frac{1-2(0.33)}{2(8.859)}\cot 120^\circ = 1.011$$

$$K_2 = 1 - \frac{1+2\nu}{2\beta}\cot\phi = 1.054$$

$$\Delta R = \frac{Q_o R_2 \beta \sin^2\ \phi}{EtK_1}(1 + K_1K_2) = \frac{1200(24)(8.859)\sin^2\ 120^\circ}{10(10^6)(0.5)(1.011)}[1 + 1.011(1.054)]$$

$$= 0.0782 \text{ in}$$

$$\psi = \frac{Q_o 2\beta^2 \sin\phi}{EtK_1} = \frac{1200(2)(8.859^2)\sin 120^\circ}{10(10^6)(0.5)(1.011)} = 0.0323 \text{ rad}$$

$$\sigma_1 = \frac{Q_o \cos\phi}{t} = \frac{1200\cos 120^\circ}{0.5} = -1200 \text{ lb/in}^2$$

$$\sigma_1' = 0$$

$$\sigma_2 = \frac{Q_o\beta\sin\phi}{2t}\left(\frac{2}{K_1} + K_1 + K_2\right) = \frac{1200(8.859)\sin 120^\circ}{2(0.5)}\left(\frac{2}{1.011} + 1.011 + 1.054\right)$$

$$= 37{,}200 \text{ lb/in}^2$$

$$\sigma_2' = \frac{-Q_o\beta^2\cos\phi}{K_1R_2} = \frac{-1200(8.859^2)\cos 120^\circ}{1.011(24)} = 1940 \text{ lb/in}^2$$

Now apply case 1b from Table 13.3:

$$\Delta R = \frac{M_o 2\beta^2 \sin\phi}{EtK_1} = 0.00002689 M_o$$

$$\psi = \frac{M_o 4\beta^3}{EtR_2K_1} = \frac{M_o 4(8.859)^3}{10(10^6)(0.5)(24)(1.011)} = 0.00002292 M_o$$

Since the combined edge rotation ψ must be zero,

$$0 = 0 + 0.0323 + 0.00002292 M_o \qquad \text{or} \qquad M_o = -1409 \text{ lb-in/in}$$

and

$$\Delta R = 0.00668 + 0.0782 + 0.00002689(-1409) = 0.04699 \text{ in}$$

$$\sigma_1 = 0$$

$$\sigma_1' = \frac{-6(-1409)}{0.05^2} = 33{,}800 \text{ lb/in}^2$$

$$\sigma_2 = \frac{M_o 2\beta^2}{R_2K_1t} = \frac{-1409(2)(8.859^2)}{24(1.011)(0.5)} = -18{,}200 \text{ lb/in}^2$$

$$M_2 = \frac{M_o}{2\nu K_1}[(1+\nu^2)(K_1+K_2) - 2K_2]$$

$$= \frac{-1409}{2(0.33)(1.011)}[(1 + 0.33^2)(1.011 + 1.054) - 2(1.054)] = -384 \text{ lb-in/in}$$

$$\sigma_2' = \frac{-6(-384)}{0.5^2} = 9220 \text{ lb/in}^2$$

The superimposed stresses at the joint are, therefore,

$$\begin{aligned}
\sigma_1 &= 4800 - 1200 + 0 = 3600 \text{ lb/in}^2 \\
\sigma_1' &= 0 + 0 + 33{,}800 = 33{,}800 \text{ lb/in}^2 \\
\sigma_2 &= 4800 + 37{,}200 - 18{,}200 = 23{,}800 \text{ lb/in}^2 \\
\sigma_2' &= 0 + 1940 + 9220 = 11{,}160 \text{ lb/in}^2
\end{aligned}$$

The maximum stress is a tensile meridional stress of 37,400 lb/in^2 on the outside surface at the joint. A further consideration would be given to any stress concentrations due to the shape of the weld cross section.

2. To reduce the high stresses in Example 1, it is proposed to add to the joint a reinforcing ring of aluminum having a cross-sectional area A. Calculate the optimum area to use.

Solution. If the ring could be designed to expand in circumference by the same amount that the sphere does under membrane loading only, then all bending stresses could be eliminated. Therefore, let a ring be loaded radially with a load of $2Q_o$ and have the radius increase by 0.00668 in. Since $\Delta R/R = 2Q_oR/AE$, then

$$A = \frac{2Q_oR^2}{E\Delta R} = \frac{2(1200)(24^2)\sin^2 60^\circ}{10(10^6)(0.00668)} = 15.5 \text{ in}^2$$

With this large an area required, the simple expression just given for $\Delta R/R$ based on a thin ring is not adequate; furthermore, there is not enough room to place such a ring external to the shell. An internal reinforcement seems more reasonable. If a 6-in-diameter hole is required for passage of the fluid, the internal reinforcing disk can have an outer radius of 20.78 in, an inner radius of 3 in, and a thickness t_1 to be determined. The loading on the disk is shown in Fig. 13.7. The change in the outer radius is desired.

From Table 13.5, case 1a, the effect of the 200 lb/in^2 internal pressure can be evaluated:

$$\Delta a = \frac{q}{E}\frac{2ab^2}{a^2 - b^2} = \frac{200}{10(10^6)}\frac{2(20.78)(3^2)}{20.78^2 - 3^2} = 0.0000177 \text{ in}$$

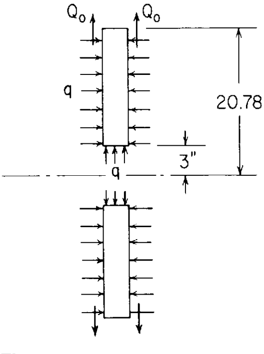


Figure 13.7

From Table 13.5, case 1c, the effect of the loads Q_o can be determined if the loading is modeled as an outward pressure of $-2Q_o/t_1$. Therefore,

$$\Delta a = \frac{-qa}{E}\left(\frac{a^2+b^2}{a^2-b^2}-\nu\right) = \frac{2(1200)(20.78)}{t_1 10(10^6)}\left(\frac{20.78^2+3^2}{20.78^2-3^2}-0.33\right) = \frac{0.00355}{t_1}$$

The longitudinal pressure of 200 lb/in^2 will cause a small lateral expansion in the outer radius of

$$\Delta_a = \frac{200(0.33)(20.78)}{10(10^6)} = 0.000137 \text{ in}$$

Summing the changes in the outer radius to the desired value gives

$$0.00668 = 0.0000177 + 0.000137 + \frac{0.00355}{t_1} \qquad \text{or} \qquad t_1 = 0.545 \text{ in}$$

(Undoubtedly further optimization could be carried out on the volume of material required and the ease of welding the joint by varying the thickness of the disk and the size of the internal hole.)

3. A truncated cone of aluminum with a uniform wall thickness of 0.050 in and a semiapex angle of 55° has a radius of 2 in at the small end and 2.5 in at the large end. It is desired to know the radial loading at the small end which will increase the radius by half the wall thickness. Given: $E = 10(10^6)$ lb/in^2 and $\nu = 0.33$.

Solution. Evaluate the distances from the apex along a meridian to the two ends of the shell and then obtain the shell parameters:

$$\begin{aligned}
R_A &= 2.5 \text{ in} \\
R_B &= 2.0 \text{ in} \\
k_A &= \frac{2}{\sin 55^\circ}\left[\frac{12(1-0.33^2)(2.5^2)}{0.050^2 \sec^2 55^\circ}\right]^{1/4} = 23.64 \\
k_B &= 21.15 \\
\mu_D &= \frac{23.64-21.15}{2} = 1.76 \\
\beta &= [12(1-0.33^2)]^{1/2} = 3.27
\end{aligned}$$

From Table 13.3, case 6c, tabulated constants for shell forces, moments, and deformations can be found when a radial load is applied to the small end. For the present problem the value of $K_{\Delta R}$ at the small end ($\Omega = 1.0$) is needed when $\mu_D = 1.76$ and $k_A = 23.64$. Interpolation from the following data gives $K_{\Delta R} = 1.27$:

k_A	10.0				20.0				40.0			
μ_n	0.8	1.2	1.6	3.2	0.8	1.2	1.6	3.2	0.8	1.2	1.6	3.2
$K_{\Delta R}$	2.085	1.610	1.343	1.113	2.400	1.696	1.351	1.051	2.491	1.709	1.342	1.025

At $\Omega = 1.0$

Therefore,

$$\Delta R_B = \frac{-Q_B(2.0)\sin 55^\circ}{10(10^6)(0.050)}\,\frac{21.15}{\sqrt{2}}(1.27) = -0.00006225Q_B$$

Since $\Delta R_B = 0.050/2$ (half the thickness), $Q_B = -402\,\text{lb/in}$ (outward).

13.4 Thin Multielement Shells of Revolution

The discontinuity stresses at the junctions of shells or shell elements due to changes in thickness or shape are not serious under static loading of ductile materials; however, they are serious under conditions of cyclic or fatigue loading. In Ref. 9, discontinuity stresses are discussed with a numerical example; also, allowable levels of the membrane stresses due to internal pressure are established, as well as allowable levels of membrane and bending stresses due to discontinuities under both static and cyclic loadings.

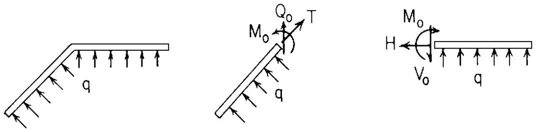
Langer (Ref. 10) discusses four modes of failure of a pressure vessel—bursting due to general yielding, ductile tearing at a discontinuity, brittle fracture, and creep rupture—and the way in which these modes are affected by the choice of material and wall thickness; he also compares pressure-vessel codes of several countries. Zaremba (Ref. 47) and Johns and Orange (Ref. 48) describe in detail the techniques for accurate deformation matching at the intersections of axisymmetric shells. See also Refs. 74 and 75.

The following example illustrates the use of the formulas in Tables 13.1–13.3 to determine discontinuity stresses.

EXAMPLE

The vessel shown in quarter longitudinal section in Fig. 13.8(*a*) consists of a cylindrical shell ($R = 24\,\text{in}$ and $t = 0.633\,\text{in}$) with conical ends ($\alpha = 45^\circ$ and $t = 0.755\,\text{in}$). The parts are welded together, and the material is steel, for which $E = 30(10^6)\,\text{lb/in}^2$ and $\nu = 0.25$. It is required to determine the maximum stresses at the junction of the cylinder and cone due to an internal pressure of $300\,\text{lb/in}^2$. (This vessel corresponds to one for which the results of a supposedly precise analysis and experimentally determined stress values are available. See Ref. 17.)

Solution. For the cone, case 2a in Table 13.1 and cases 4a and 4b in Table 13.3 can be used: $R = 24\,\text{in}$, $\alpha = 45^\circ$, and $t = 0.755\,\text{in}$. The following conditions exist at the end of the cone: From Table 13.1, case 2a, for the load T and



(a) (b) (c)

Figure 13.8

pressure q,

$$\sigma_1 = \frac{300(24)}{2(0.755)\cos 45^\circ} = 6740 \text{ lb/in}^2, \qquad T = 6740(0.755) = 5091 \text{ lb/in}$$

$$\sigma_2 = 13{,}480 \text{ lb/in}^2, \qquad \sigma_1' = 0, \qquad \sigma_2' = 0$$

$$\Delta R = \frac{300(24^2)}{30(10^6)(0.755)\cos 45^\circ}\left(1 - \frac{0.25}{2}\right) = 0.00944 \text{ in}$$

$$\psi = \frac{3(300)(24)(1)}{2(30)(10^6)(0.755)\cos 45^\circ} = 0.000674 \text{ rad}$$

From Table 13.3, case 4a, for the radial edge load Q_o,

$$R_A = 24 \text{ in}$$

$$k_A = \frac{2}{\sin 45^\circ}\left[\frac{12(1-0.25^2)(24^2)}{0.755^2 \sec^2\ 45^\circ}\right]^{1/4} = 24.56$$

$$\beta = [12(1-0.25^2)]^{1/2} = 3.354$$

Only values at $R = R_A$ are needed for this solution. Therefore, the series solutions for the constants can be used to give

$$F_{9A} = C_1 = 0.9005, \qquad F_{1A} = 0, \qquad F_{3A} = 0, \qquad F_{2A} = 0.8977$$

$$F_{4A} = 0.8720, \qquad F_{5A} = F_{8A} = 0.8746, \qquad F_{10A} = F_{7A} = F_{6A} = 0.8947$$

$$\Delta R_A = \frac{Q_o 24(0.7071)(24.56)}{30(10^6)(0.755)(\sqrt{2})(0.9005)}\left[0.8720 - \frac{4(0.25^2)}{24.56^2}0.8977\right] = 12.59(10^{-6})Q_o$$

$$\psi_A = \frac{Q_o 24(3.354)}{30(10^6)(0.755^2)(0.9005)}(0.8947) = 4.677(10^{-6})Q_o$$

$$N_{1A} = 0.7071 Q_o, \qquad M_{1A} = 0$$

$$N_{2A} = \frac{Q_o(0.7071)(24.56)}{2(0.9005)}\left[0.8720 + \frac{2(0.25)}{24.56}0.8746\right] = 12.063 Q_o$$

$$M_{2A} = \frac{Q_o(0.7071)(1-0.25^2)(0.755)}{3.354(0.9005)}0.8947 = 0.1483 Q_o$$

From Table 13.3, case 4b, for the edge moment M_A,

$\Delta R_A = 4.677(10^{-6})M_o$ (same coefficient shown for ψ_A for the loading Q_o as would be expected from Maxwell′s theorem)

$$\psi_A = \frac{M_o 2\sqrt{2}(3.354^2)(24)}{30(10^6)(0.755^3)(24.56)(0.7071)}\,\frac{0.8977}{0.9005} = 3.395(10^{-6})M_o$$

$$N_{1A} = 0, \qquad N_{2A} = M_o\frac{3.354(0.8947)}{0.755(0.9005)} = 4.402 M_o$$

$$M_{1A} = M_o, \qquad M_{2A} = M_o\left[0.25 + \frac{2(2)(1-0.25^2)(0.8977)}{24.56(0.9005)}\right] = 0.3576 M_o$$

For the cylinder, case 1c in Table 13.1 and cases 8 and 10 in Table 13.2 can be used (it is assumed that the other end of the cylinder is far enough away so as to not affect the deformations and stresses at the cone-cylinder junction):

$R = 24$ in; $t = 0.633$ in; $\lambda = [3(1 - 0.25^2)/24^2/0.633^2]^{1/4} = 0.3323$; and $D = 30(10^6)(0.633^3)/12(1 - 0.25^2) = 6.76(10^5)$. The following conditions exist at the end of the cylinder: From Table 13.1, case 1c, for the axial load H and the pressure q,

$$\sigma_1 = \frac{300(24)}{2(0.633)} = 5690 \text{ lb/in}^2, \qquad H = 5690(0.633) = 3600 \text{ lb/in}$$

$$\sigma_2 = 11{,}380 \text{ lb/in}^2, \qquad \sigma_1' = 0, \qquad \sigma_2' = 0$$

$$\Delta R = \frac{300(24^2)}{30(10^6)(0.633)}\left(1 - \frac{0.25}{2}\right) = 0.00796 \text{ in}$$

$$\psi = 0$$

From Table 13.2, case 8, for the radial end load V_o,

$$\psi_A = \frac{V_o}{2(6.76)(10^5)(0.3323^2)} = 6.698(10^{-6})V_o$$

$$\Delta R_A = y_A = \frac{-V_o}{2(6.76)(10^5)(0.3323^2)} = -20.16(10^{-6})V_o$$

$$\sigma_1 = 0, \qquad \sigma_2 = \frac{yE}{R} = \frac{-20.16(10^{-6})V_o(30)(10^6)}{24} = -25.20V_o$$

$$\sigma_1' = 0, \qquad \sigma_2' = 0$$

From Table 13.2, case 10, for the end moment M_o,

$$\psi_A = \frac{-M_o}{6.76(10^5)(0.3323)} = -4.452(10^{-6})M_o$$

$$\Delta R_A = y_A = \frac{M_o}{2(6.76)(10^5)(0.3323^2)} = 6.698(10^{-6})M_o$$

$$\sigma_1 = 0, \qquad \sigma_2 = \frac{2M_o\lambda^2R}{t} = \frac{2M_o(0.3323^2)(24)}{0.633} = 8.373M_o$$

$$\sigma_1' = \frac{-6M_o}{t^2} = \frac{-6M_o}{0.633^2} = -14.97M_o, \qquad \sigma_2' = \nu\sigma_1' = -3.74M_o$$

Summing the radial deflections for the end of the cone and equating to the sum for the cylinder gives

$$0.00944 + 12.59(10^{-6})Q_o + 4.677(10^{-6})M_o = 0.00796 - 20.16(10^{-6})V_o + 6.698(10^{-6})M_o$$

Doing the same with the meridian rotations gives

$$0.000674 + 4.677(10^{-6})Q_o + 3.395(10^{-6})M_o = 0 + 6.698(10^{-6})V_o - 4.452(10^{-6})M_o$$

Finally, equating the radial forces gives

$$Q_o + 5091\cos 45^\circ = V_o$$

Solving the three equations simultaneously yields

$$Q = -2110 \text{ lb/in}, \qquad V_o = 1490 \text{ lb/in}, \qquad M_o = 2443 \text{ lb-in/in}$$

In the cylinder,

$$\sigma_1 = 5690 + 0 + 0 = 5690 \text{ lb/in}^2$$
$$\sigma_2 = 11{,}380 - 25.20(1490) + 8.373(2443) = -5712 \text{ lb/in}^2$$
$$\sigma_1' = 0 + 0 - 14.97(2443) = -36{,}570 \text{ lb/in}^2$$
$$\sigma_2' = 0 + 0 - 3.74(2443) = -9140 \text{ lb/in}^2$$

Combined hoop stress on outside $= -5712 - 9140 = -14{,}852 \text{ lb/in}^2$

Combined hoop stress on inside $= -5712 + 9140 = 3428 \text{ lb/in}^2$

Combined meridional stress on outside $= 5690 - 36{,}570 = -30{,}880 \text{ lb/in}^2$

Combined meridional stress on inside $= 5690 + 36{,}570 = 42{,}260 \text{ lb/in}^2$

Similarly, in the cone,

$$\sigma_1 = 6740 + \frac{0.7071(-2110)}{0.755} + 0 = 4764 \text{ lb/in}^2$$
$$\sigma_2 = 13{,}480 + \frac{12.063(-2110)}{0.755} + \frac{4.402(2443)}{0.755} = -5989 \text{ lb/in}^2$$
$$\sigma_1' = 0 + 0 - \frac{2443(6)}{0.755^2} = -25{,}715 \text{ lb/in}^2$$
$$\sigma_2' = 0 - \frac{0.1483(-2110)(6)}{0.755^2} - \frac{0.3576(2443)(6)}{0.755^2} = -5902 \text{ lb/in}^2$$

Combined hoop stress on outside $= -5989 - 5902 = -11{,}891 \text{ lb/in}^2$

Combined hoop stress on inside $= -5989 + 5902 = -87 \text{ lb/in}^2$

Combined meridional stress on outside $= 4764 - 25{,}715 = -20{,}951 \text{ lb/in}^2$

Combined meridional stress on inside $= 4764 + 25{,}715 = 30{,}480 \text{ lb/in}^2$

These stress values are in substantial agreement with the computed and experimental values cited in Refs. 17 and 26. Note that the radial deflections are much less than the wall thicknesses. See the discussion in the third paragraph of Sec. 13.3.

In the problem just solved by the method of deformation matching only two shells met at their common circumference. The method, however, can be extended to cases where more than two shells meet in this manner. The primary source of difficulty encountered when setting up the equations to carry out such a solution is the rigor needed when labeling the several edge loads and the establishment of

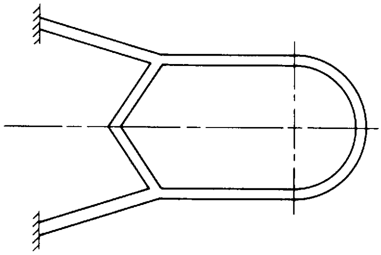


Figure 13.9

the proper signs for the radial and rotational deformations. An additional problem arises when the several shells intersect not at a single circumference but at two or more closely spaced circumferences.

Figure 13.9 illustrates two conical shells and a spherical shell joined together by a length of cylindrical shell. The length of this central cylinder is a critical dimension in determining how the cylinder is treated. If the length is small enough for a given radius and wall thickness, it may be sufficient to treat it as a narrow ring whose cross section deflects radially and rotates with respect to the original meridian but whose cross section does not change shape. For an example as to how these narrow rings are treated see Sec. 11.9. For a longer cylinder the cross section does change shape and it is treated as a short cylinder, using expressions from Table 13.2. Here there are two circumferences where slopes and deflections are to be matched but the loads on each end of the cylinder influence the deformations at the other end. Finally, if the cylinder is long enough, $\lambda l > 6$, for example, the ends are far enough apart so that two separate problems may be solved.

Table 13.3 presents formulas and tabulated data for several combinations of thin shells of revolution and thin circular plates joined two at a time at a common circumference. All shells are assumed long enough so that the end interactions can be neglected. Loadings include axial load, a loading due to a rotation at constant angular velocity about the central axis, and internal or external pressure where the pressure is either constant or varying linearly along the axis of the shell. For the pressure loading the equations represent the case where the junction of the shells carries no axial loading such as when a cylindrical shell carries a frictionless piston which is supported axially by structures other than the cylinder walls. *The decision to present the pressure loadings in this form was based primarily on the ease of presentation. When used for closed pressure vessels, the deformations and stresses for the axial load must be superposed on those for the pressure loading.*

The reasons for presenting the tabulated data in this table are several. (1) In many instances one merely needs to know whether the stresses and deformations at such discontinuities are important to the safety and longevity of a structure. Using interpolation one can explore quickly the tables of tabulated data to make such a determination. (2) The tabulated data also allow those who choose to solve the formulas to verify their results.

The basic information in Table 13.4 can be developed as needed from formulas in the several preceding tables, but the work has been extended a few steps further by modifying the expressions in order to make them useful for shells with somewhat thicker walls.

In the sixth edition of this book, correction terms were presented to account for the fact that internal pressure loading acts on the inner surface, not at the mid-thickness. For external pressure, the proper substitutions are indicated by notes for the several cases. This has already been accounted for in the general pressure loadings on the several shell types, but there is an additional factor to account for at the junction of the shells. In Fig. 13.10(*a*), the internal pressure is shown acting all the way to the hypothetical end of the left-hand shell. The general equations in Table 13.1 assumes this to be the case, and the use of these equations in Table 13.4 makes this same assumption. The correction terms in the sixth edition of this book added or subtracted, depending upon the signs of α_1, α_2, and q, the pressure loading over the length x shown in Fig. 13.10(*b*). These corrections included the effects of the radial components, the axial components, and the moments about point A of this local change in loading. The complexity of these corrections may seem out of proportion to the benefits derived, and, depending upon their needs, users will have to decide whether or not to include them in their calculations. To assist users in making this decision, the following example will compare results with and without the correction terms and show the relative

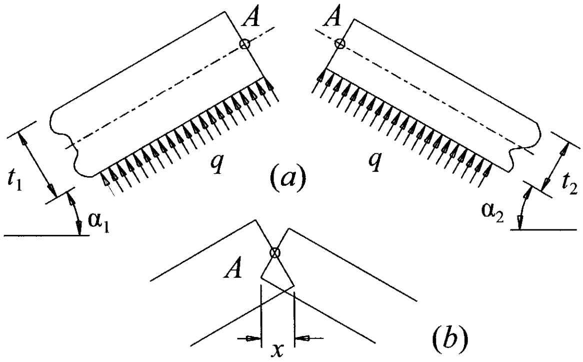


Figure 13.10

effect of using only the radial component of the change in the local pressure loading at the junction of a cone and cylinder.

EXAMPLE

For this example, the pressurized shell is that of the previous example shown in Fig. 13.8. The calculations for that example were carried out using equations from Tables 13.1 and 13.3. The stresses in the cylinder at the junction are given at the end of the solution, and the radial deflection and the rotation at the junction can be calculated from the expressions given just before the stress calculations. The following results table lists these stresses and deflections in column [1]. As stated above, the equations used in Table 13.4 to solve for the shell junction stresses were those given in Tables 13.1–13.3, but modified somewhat to make them more accurate for shells with thicker walls. Using cases 2a and 2b from Table 13.4 gives the results shown in columns [2]–[7] in the results table. The axial load used for column [2] was $P = \pi(24 - 0.633/2)^2(300) = 528{,}643$ lb. All of the stresses in the results table are those found in the cylinder at the junction. Column [3] gives data for the internal pressure loading with no correction factors and column [4] is the sums for the axial load and internal pressure, columns [2] plus [3]. Column [5] is for the internal pressure corrected for the change in loading at the joint. Column [7], is the difference in the numbers of columns [4] and [6], and gives the changes due to the correction factors in Table 13.4.

Column [8] shows the changes due to the radial component of the correction in the joint loading which are calculated as follows. Figure 13.11(*a*) shows the joint being considered, with the dimensions. The value of $x = 0.2669$ in, and when this is multiplied by the internal pressure of 300 lb/in^2, one obtains $Q_1 + Q_2 = 80.07$ lb/in, the radially inward load needed to compensate for the radial component of the internal pressure *not* acting on the joint.

Using already evaluated expressions from the previous example, the following equations can be written.

For the cone:

$$\Delta R_A = -12.59(10^{-6})Q_1 + 4.677(10^{-6})M_1$$
$$\psi_A = -4.677(10^{-6})Q_1 + 3.395(10^{-6})M_1$$

For the cylinder:

$$\Delta R_A = -20.16(10^{-6})Q_2 + 6.698(10^{-6})M_1$$
$$\psi_A = 6.698(10^{-6})Q_2 - 4.452(10^{-6})M_1$$

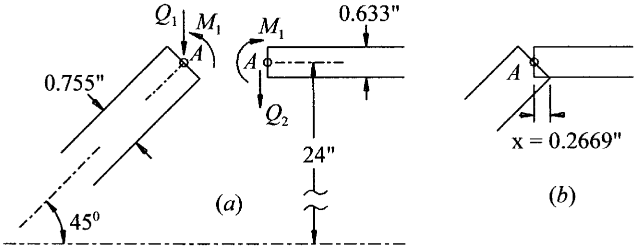


Figure 13.11

Equating the equations for ΔR_A and ψ_A with $Q_1 + Q_2 = 80.07\,\text{lb/in}$ yields $Q_1 = 48.83\,\text{lb/in}$, $Q_2 = 31.24\,\text{lb/in}$, $M_1 = 55.77\,\text{lb-in/in}$, $\Delta R_A = -0.000354\,\text{in}$, and $\psi_A = -0.000039\,\text{rad}$.

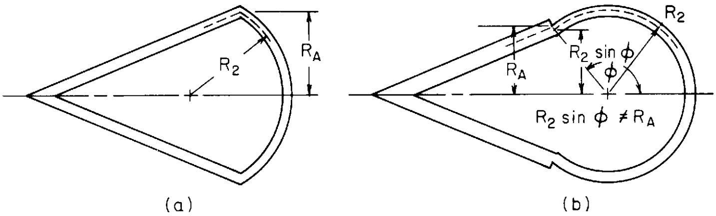
As would be expected, the radial component is a major contributor for the joint being discussed and would be for most pressure vessel joints.

RESULTS table (stresses in lb/in^2, deflection in inches, rotation in radians)

		From Table 13.4						
	From previous example [1]	Case 2b Axial Load [2]	Case 2a Internal Pressure without corrections [3]	Sum [2] + [3] [4]	Case 2a Internal Pressure with corrections [5]	Sum [2] + [5] [6]	Change due to the use of the correction terms [7]	Change due to the approx. corrections given above [8]
σ_1	5,690	5,538	0	5,538	0	5,538	0	0
σ_2	−5,712	−17,038	11,647	−5,391	11,252	−5,786	−395	−321
σ_1'	−36,570	−3,530	1,927	−35,604	1,030	−36,500	−896	−835
σ_2'	−9,140	−9,382	482	−8,900	258	−9,124	−224	−209
ΔR_A	−0.005699	−0.01474	0.00932	−0.00542	0.00900	−0.00574	−0.00032	−0.000354
ψ_A	−0.000900	0.001048	−0.000174	0.000874	−0.000146	0.000902	−0.000028	−0.000039

Most shell intersections have a common circumference, identified by the radius R_A, and defined as the intersection of the midsurfaces of the shells. If the two shells have meridional slopes which differ substantially at this intersection, the shape of the joint is easily described. See Fig. 13.12(*a*). If, however, these slopes are very nearly the same and the shell thicknesses differ appreciably, the intersection of the two midsurfaces could be far away from an actual joint, and the midthickness radius must be defined for each shell. See Fig. 13.12(*b*).

For this reason there are two sets of correction terms based on these two joint contours. All correction terms are treated as external loads on the right-hand member. The appropriate portion of this loading is transferred back to the left-hand member by small changes in the radial load V_1 and the moment M_1 which are found by equating the deformations in the two shells at the junction. In each case the formulas for the stresses at the junction are given only for the left-



(a) (b)

Figure 13.12

hand member. Stresses are computed on the assumption that each member ends abruptly at the joint with the end cross section normal to the meridian. No stress concentrations are considered, and no reduction in stress due to any added weld material or joint reinforcement has been made. The examples show how such corrections can be made for the stresses.

While the discussion above has concentrated primarily on the stresses at or very near the junction of the members, there are cases where stresses at some distance from the junction can be a source of concern. Although a toroidal shell is not included in Table 13.4, the presence of large circumferential compressive stress in the toroidal region of a torispherical head on a pressure vessel is known to create buckling instabilities when such a vessel is subjected to internal pressure. Section 15.4 describes this problem and others of a similar nature such as a truncated spherical shell under axial tension.

EXAMPLES

1. The shell consisting of a cone and a partial sphere shown in Fig. 13.13 is subjected to an internal pressure of 500 N/cm^2. The junction deformations and the circumferential and meridional stress components at the inside surface of the junction are required. Use $E = 7(10^6)$ N/cm^2 and $\nu = 0.3$ for the material in both portions of the shell. All the linear dimensions will be given and used in centimeters.

Solution. The meridional slopes of the cone and sphere are the same at the junction, and the sphere is not truncated nor are any penetrations present at any other location, so $\theta_2 = \phi_2 = 105°$. Using case 6 from Table 13.4, the cone and shell parameters and the limiting values for which the given equations are acceptable are now evaluated.

For the cone using Table 13.3, case 4:

$$\alpha_1 = 15°, \qquad R_A = R_1 = 50 \sin 105° = 48.296$$

$$k_A = \frac{2}{\sin 15°}\left[\frac{12(1-0.3^2)(48.296^2)}{1.2^2 \sec^2\ 15°}\right]^{0.25} = 87.58$$

Where $\mu = 4$, the value of $k_B = 87.58 - 4\sqrt{2} = 81.93$ and $R_B = 42.26$. Since both k_A and k_B are greater than 5 and both R_A and R_B are greater than 5

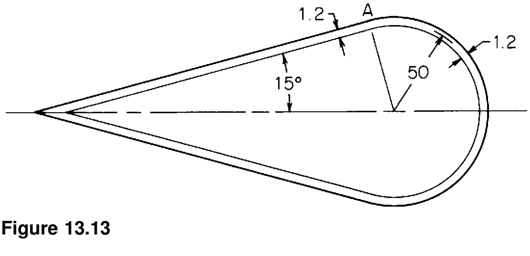


Figure 13.13

$(1.2\cos 15^\circ)$, the cone parameters are within the acceptable range for use with the equations. $b_1 = 48.296 - 1.2\cos 15^\circ/2 = 47.717$, $\alpha_1 = 48.876$, and $\beta_1 = [12(1-0.3^2)]^{0.5} = 3.305$.

For the sphere using Table 13.3, case 1:

$$\beta_2 = \left[3(1-0.3^2)\left(\frac{50}{1.2}\right)^2\right]^{0.25} = 8.297$$

$$b_2 = 50 - \frac{1.2}{2} = 49.40, \qquad a_2 = 50.60$$

and, at the edge, where $\omega = 0$,

$$K_{12} = 1 - [1-2(0.3)]\frac{(\cot 105^\circ)/2}{8.297} = 1.0065 \qquad \text{and} \qquad K_{22} = 1.0258$$

Since $3/\beta_2 = 0.3616$ and $\pi - 3/\beta_2 = 2.78$, the value of $\phi_2 = 1.833\,\text{rad}$ lies within this range, so the spherical shell parameters are also acceptable.

Next the several junction constants are determined from the shell parameters just found and from any others required. Again from Table 13.3, case 4:

$$F_{2A} = 1 - \frac{2.652}{87.58} + \frac{3.516}{87.58^2} - \frac{2.610}{87.58^3} + \frac{0.038}{87.58^4} = 0.9702$$

Similarly,

$$F_{4A} = 0.9624, \qquad F_{7A} = 0.9699, \qquad \text{and} \qquad C_1 = 0.9720$$

Using these values, $C_{AA1} = 638.71$, $C_{AA2} = 651.39$, $C_{AA} = 1290.1$, $C_{AB1} = -132.72$, $C_{AB2} = 132.14$, $C_{AB} = -0.5736$, $C_{BB1} = 54.736$, $C_{BB2} = 54.485$, and $C_{BB} = 109.22$.

Turning now to the specific loadings needed, one uses case 6a for internal pressure with *no axial load on the junction* and case 6b with an axial load $P = 500\pi(47.72)^2 = 3.577(10^6)\,\text{N}$.

For case 6a: Although the tables of numerical data include $\alpha_1 = 15^\circ$ and $\phi_2 = 105^\circ$ as a given pair of parameters, the value of $R_1/t_1 = 40.25$ is not one of the values for which data are given. The load terms are

$$LT_{A1} = \frac{47.72(48.3)}{1.2^2\cos 15^\circ} = 1656.8, \qquad LT_{A2} = -1637.0, \qquad LT_{B1} = -22.061,$$
$$LT_{B2} = 0$$

In this example the junction meridians are tangent and the inside surface is smooth, so there are no correction terms to consider. Had the radii and the thicknesses been such that the welded junction had an internal step, either abrupt or tapered, the internal pressure acting upon this step would be accounted for by the appropriate correction terms (see the next example).

Now combining the shell and load terms, $LT_A = 1656.8 - 1637.0 + 0 = 19.8$, $LT_B = -22.06$, $K_{V1} = 0.0153$, $K_{M1} = -0.2019$, $V_1 = 9.193$, $M_1 = -145.4$, $N_1 =$

-2.379, $\Delta R_A = 0.1389$, $\psi_A = 641(10^{-6})$, $\sigma_1 = -1.98$, $\sigma_2 = 20{,}128$, $\sigma_1' = 605.7$, and $\sigma_2' = 196.2$.

For case 6b:

$$LT_{A1} = \frac{-0.3(48.30^2)}{2(1.2^2)\cos 15^\circ} = -251.5, \qquad LT_{A2} = 1090.0$$

$$LT_{B1} = 5.582, \qquad LT_{B2} = 0, \qquad LT_{AC} = 0, \qquad LT_{BC} = 0$$

$$LT_A = 838.5, \qquad LT_B = 5.582$$

Again combine the shell and load terms to get $K_{V1} = 0.6500$, $K_{M1} = 0.0545$, $V_1 = 380.7$, $M_1 = 38.33$, $N_1 = 12{,}105$, $\Delta R_A = -0.0552$, $\psi_A = -0.0062$, $\sigma_1 = 10{,}087$, $\sigma_2 = -4972$, $\sigma_1' = -159.7$, and $\sigma_2' = -188$.

The final step is to sum the deformations and stresses. $\Delta R_A = 0.0837$, $\psi_A = -0.0056$, $\sigma_1 = 10{,}085$, $\sigma_2 = 15{,}156$, $\sigma_1' = 446.0$, $\sigma_2' = 8.2$. A check of these values against the tabulated constants shows that reasonable values could have been obtained by interpolation. At the junction the shell moves outward a distance of 0.0837 cm, and the upper meridian as shown in Fig. 13.13 rotates 0.0056 rad clockwise. On the inside surface of the junction the circumferential stress is 15,164 N/cm^2 and the meridional stress is 10,531 N/cm^2.

As should have been expected by the smooth transition from a conical to a spherical shell of the same thickness, the bending stresses are very small. In the next example the smooth inside surface will be retained but the cone and sphere will be different in thickness, and external pressure will be applied to demonstrate the use of the terms which correct for the pressure loading on the step in the wall thickness at the junction.

2. The only changes from Example 1 will be to make the pressure external at 1000 N/cm^2 and to increase the cone thickness to 4 cm and the sphere thickness to 2 cm. The smooth inside surface will be retained. If the correction terms are not used in this example, the external pressure will be presumed to act on the outer surface of the cone up to the junction and on the external spherical surface of the sphere. There will be no consideration given for the external pressure acting upon the 2-cm-wide external shoulder at the junction. The correction terms treat the additional axial and radial loadings and the added moment due to this pressure loading on the shoulder. If a weld fillet were used at the junction, the added pressure loading would be the same, so the correction terms are still applicable but *no* consideration is made for the added stiffness due to the extra material in the weld fillet. If the meridians for the cone and for the sphere intersect at an angle more than about 5°, a different correction term is used. This second correction term assumes that no definite step occurs on either the inner or the outer surface. See the discussion in Sec. 13.4 related to Fig. 13.12.

Solution. For the cone using Table 13.3, case 4: $\alpha_1 = 15^\circ$, $R_A = R_1 = 51\sin 105^\circ = 49.262$, and $k_A = 48.449$ when the radius and thickness are changed to the values shown in Fig. 13.14. Where $\mu = 4$, the value of $k_B = 42.793$ and $R_B = 38.4$, which is greater than 5 ($4\cos 15^\circ$). Thus, again k_A and k_B are greater than 5 and the cone parameters are within the acceptable range for use with the equations. In a similar manner the parameters for the sphere are found to be within the range for which the equations are acceptable. Repeating the calculations as was done for the first example, with and without correction terms, one finds the following stresses:

	Case 6a ($q = -1000\,\text{N/cm}^2$)		
	Without correction terms	With correction terms	Case 6b ($P = -8{,}233{,}600\,\text{N}$)
ΔR_A	−0.1244	−0.1262	0.0566
γ_A	−0.00443	−0.00425	0.00494
σ_1	−110.15	−133.64	−6739.1
σ_2	−17,710	−17,972	6021.3
σ_1'	1317.2	342.3	−1966.2
σ_2'	283.7	−99.0	−454.3

The effect of the correction terms is apparent but does not cause large changes in the larger stresses or the deformations. Summing the values for cases 6a with correction terms and for 6b gives the desired results as follows. The radial deflection at the junction is 0.0696 cm inward, the upper meridian rotates 0.00069 rad clockwise, on the inside of the junction the circumferential stress is $-12{,}504\,\text{N/cm}^2$, and the meridional stress is $-8497\,\text{N/cm}^2$.

3. The vessel shown in Fig. 13.15 is conical with a flat-plate base supported by an annular line load at a radius of 35 in. The junction deformations and the meridional and circumferential stresses on the outside surface at the junction of the cone and plate are to be found. The only loading to be considered is the hydrostatic loading due to the water contained to a depth of 50 in. Use $E = 10(10^6)\,\text{lb/in}^2$ and $\nu = 0.3$ for the material in the shell and in the plate. All linear dimensions will be given and used in inches.

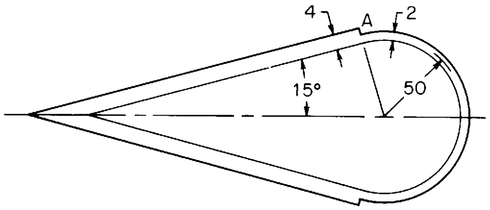


Figure 13.14

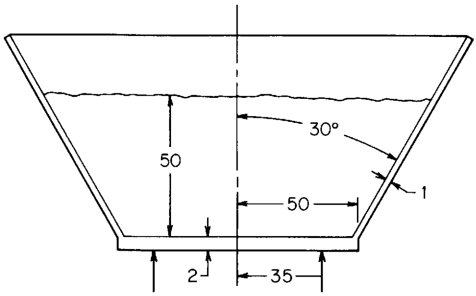


Figure 13.15

Solution. The proportions chosen in this example are ones matching the tabulated data in cases 7b and 7c from Table 13.4 in order to demonstrate the use of this tabulated information.

From case 7c with the loading an internal hydrostatic pressure with *no axial load on the junction* and for $\alpha = -30^\circ$, $R_1 = 50$, $t_1 = 1$, $R_1/t_1 = 50$, $t_2 = 2$, $t_2/t_1 = 2$, $x_1 = 50$, $x_1/R_1 = 1$, $R_2/R_1 = \frac{35}{50} = 0.7$, $\nu_1 = \nu_2 = 0.3$; and for $a_2 = a_1$ due to the plate extending only to the outer surface of the cone at the junction, we find from the table the following coefficients: $K_{V1} = 3.3944$, $K_{M1} = 2.9876$, $K_{\Delta RA} = 0.1745$, $K_{\psi A} = -7.5439$, and $K_{\sigma 2} = 0.1847$. Since water has a weight of $62.4\,\text{lb/ft}^3$, the internal pressure q_1 at the junction is $62.4(50)/1728 = 1.806\,\text{lb/in}^2$. Using these coefficients and the dimensions and the material properties we find that

$$\begin{aligned}
V_1 &= 1.806(1)(3.3944) = 6.129\\
M_1 &= 1.806(1^2)(2.9876) = 5.394\\
N_1 &= -6.129 \sin -30^\circ = 3.064\\
\Delta R_A &= \frac{1.806(50^2)}{10(10^6)(1)} 0.1745 = 78.77(10^{-6})\\
\psi_A &= \frac{1.806(50)}{10(10^6)(1)}(-7.5439) = -68.10(10^{-6})\\
\sigma_1 &= \frac{3.064}{1} = 3.064\\
\sigma_2 &= 78.77(10^{-6})\frac{10(10^6)}{50} + 0.3(3.064) = 16.673\\
\sigma_1' &= \frac{-6(5.394)}{1^2} = -32.364\\
\sigma_2' &= -2.895 \quad (\textit{Note}\text{: The extensive calculations are not shown})
\end{aligned}$$

In the above calculations no correction terms were used. When the correction terms are included and the many calculations carried out, the deformations and stresses are found to be

$$\begin{aligned}
&\Delta R_A = 80.13(10^{-6}), \qquad \psi_A = -68.81(10^{-6})\\
&\sigma_1 = 3.028, \qquad \sigma_2 = 16.934, \qquad \sigma_1' = -30.595, \qquad \sigma_2' = -2.519
\end{aligned}$$

There is not a great change due to the correction terms.

For case 7b the axial load to be used must now be calculated. The radius of the fluid at the plate is $50 - 0.5/\cos 30^\circ = 49.423$. The radius of the fluid at the top surface is $49.423 + 50 \tan 30^\circ = 78.290$. The vertical distance below the top of the plate down to the tip of the conical inner surface is $49.423/\tan 30^\circ = 85.603$. The volume of fluid $= \pi(78.290)^2(85.603 + 50)/3 - \pi(49.423)^2(85.603)/3 = 651{,}423\,\text{in}^3$. The total weight of the fluid $= 651{,}423(62.4)/1728 = 23{,}524\,\text{lb}$. The axial load acting on the plate in case $7\text{c} = q_1 \pi b_1^2 = 1.806\pi(49.56)^2 = 13{,}940\,\text{lb}$. Using case 7b with the axial compressive load $P = 13{,}940 - 23{,}524 = -9584$ gives the following results:

$$\begin{aligned}
&\Delta R_A = 496.25(10^{-6}), \qquad \psi_A = -606.03(10^{-6})\\
&\sigma_1 = -57.973, \qquad \sigma_2 = 81.858, \qquad \sigma_1' = 1258.4, \qquad \sigma_2' = 272.04
\end{aligned}$$

Summing the results from case 7c with correction terms and from case 7b produces $\Delta R_A = 576.38(10^{-6})$, $\psi_A = -674.84(10^{-6})$, $\sigma_1 = -54.945$, $\sigma_2 = 98.792$, $\sigma_1' = 1227.8$, and $\sigma_2' = 269.5$.

The junction moves radially outward a distance of 0.00058 in, and the junction meridian on the right in Fig. 13.15 rotates 0.000675 rad clockwise. On the outside of the cone at the junction, the circumferential stress is 368.3 lb/in^2 and the meridional stress is 1173 lb/in^2.

13.5 Thin Shells of Revolution under External Pressure

All formulas given in Tables 13.1 and 13.3 for thin vessels under distributed pressure are for internal pressure, but they will apply equally to cases of external pressure if q is given a negative sign. The formulas in Table 13.2 for distributed pressure are for external pressure in order to correspond to similar loadings for beams on elastic foundations in Chap. 8. It should be noted with care that the application of external pressure may cause an instability failure due to stresses lower than the elastic limit, and in such a case the formulas in this chapter do not apply. This condition is discussed in Chap. 15, and formulas for the critical pressures or stresses producing instability are given in Table 15.2.

A vessel of moderate thickness may collapse under external pressure at stresses just below the yield point, its behavior being comparable to that of a short column. The problem of ascertaining the pressure that produces failure of this kind is of special interest in connection with cylindrical vessels and pipe. For external loading such as that in Table 13.1, case 1c, the external collapsing pressure can be given by

$$q' = \frac{t}{R}\,\frac{\sigma_y}{1 + (4\sigma_y/E)(R/t)^2} \qquad \text{(see Refs. 1, 7, and 8)}$$

In Refs. 8 and 9, charts are given for designing vessels under external pressure.

A special instability problem should be considered when designing long cylindrical vessels or even *relatively short corrugated tubes* under internal pressure. Haringx (Refs. 54 and 55) and Flügge (Ref. 5) have shown that vessels of this type will buckle laterally if the ends are restrained against longitudinal displacement and if the product of the internal pressure and the cross-sectional area reaches the Euler load for the column as a whole. For cylindrical shells this is seldom a critical factor, but for corrugated tubes or bellows this is recognized as a so-called *squirming instability.* To determine the Euler load for a bellows, an equivalent thin-walled circular cross section can be established which will have a radius equal to the mean radius of the bellows and a product Et, for which the equivalent cylinder will have the same axial deflection under end load as would the bellows. The overall bending moment of inertia I of the very thin equivalent cylinder can then be used in the expression $P_u = K\pi^2 EI/l^2$ for the Euler load. In a similar way Seide (Ref. 56) discusses the effect of pressure on the lateral bending of a bellows.

EXAMPLE

A corrugated-steel tube has a mean radius of 5 in, a wall thickness of 0.015 in, and 60 semicircular corrugations along its 40-in length. The ends are rigidly fixed, and the internal pressure necessary to produce a squirming instability is to be calculated. Given: $E = 30(10^6)$ lb/in^2 and $\nu = 0.3$.

Solution. Refer to Table 13.3, case 6b: $a = 5$ in, length $= 40$ in, $b = \frac{40}{120} = 0.333$ in, and $t = 0.015$ in

$$\mu = \frac{b^2}{at}\sqrt{12(1-\nu^2)} = \frac{0.333^2}{5(0.015)}\sqrt{12(1-0.3^2)} = 4.90$$

$$\text{Axial stretch} = \frac{-0.577Pbn\sqrt{1-\nu^2}}{Et^2} = \frac{0.577P(0.333)(60)\sqrt{0.91}}{30(10^6)(0.015^2)} = -0.00163P$$

If a cylinder with a radius of 5 in and product $E_1 t_1$ were loaded in compression with a load P, the stretch would be

$$\text{Stretch} = \frac{-Pl}{A_1 E_1} = \frac{-P(40)}{2\pi 5 t_1 E_1} = -0.00163P$$

or

$$t_1 E_1 = \frac{40}{2(5)(0.00163)} = 780.7 \text{ lb/in}$$

The bending moment of inertia of such a cylinder is $I_1 = \pi R^3 t_1$ (see Table A.1, case 13). The Euler load for fixed ends is

$$P_{cr} = \frac{4\pi^2 E_1 I_1}{l^2} = \frac{4\pi^2 E_1 \pi R^3 t_1}{l^2} = \frac{4\pi^3 5^3 (780.7)}{40^2} = 7565 \text{ lb}$$

The internal pressure is therefore

$$q' = \frac{P_{cr}}{\pi R^2} = \frac{7565}{\pi 5^2} = 96.3 \text{ lb/in}^2$$

From Table 13.3, case 6c, the maximum stresses caused by this pressure are

$$(\sigma_2)_{\max} = 0.955(96.3)(0.91)^{1/6}\left[\frac{5(0.333)}{0.015^2}\right]^{2/3} = 34{,}400 \text{ lb/in}^2$$

$$(\sigma')_{\max} = 0.955(96.3)(0.91)^{-1/3}\left[\frac{5(0.333)}{0.015^2}\right]^{2/3} = 36{,}060 \text{ lb/in}^2$$

If the yield strength is greater than 36,000 lb/in^2, the corrugated tube should buckle laterally, that is, squirm, at an internal pressure of 96.3 lb/in^2.

13.6 Thick Shells of Revolution

If the wall thickness of a vessel is more than about one-tenth the radius, the meridional and hoop stresses cannot be considered uniform throughout the thickness of the wall and the radial stress cannot be considered negligible. These stresses in thick vessels, called *wall stresses*, must be found by formulas that are quite different from those used in finding membrane stresses in thin vessels.

It can be seen from the formulas for cases 1a and 1b of Table 13.5 that the stress σ_2 at the inner surface of a thick cylinder approaches q as the ratio of outer to inner radius approaches infinity. It is apparent, therefore, that if the stress is to be limited to some specified value σ, the pressure must never exceed $q = \sigma$, no matter how thick the wall is made. To overcome this limitation, the material at and near the inner surface must be put into a state of initial compression. This can be done by shrinking on one or more jackets (as explained in Sec. 3.12 and in the example which follows) or by subjecting the vessel to a high internal pressure that stresses the inner part into the plastic range and, when removed, leaves residual compression there and residual tension in the outer part. This procedure is called *autofrettage*, or *self-hooping*. If many successive jackets are superimposed on the original tube by shrinking or wrapping, the resulting structure is called a *multilayer vessel*. Such a construction has certain advantages, but it should be noted that the formulas for hoop stresses are based on the assumption that an isotropic material is used. In a multilayered vessel the effective radial modulus of elasticity is less than the tangential modulus, and in consequence the hoop stress at and near the outer wall is less than the formula would indicate; therefore, the outer layers of material contribute less to the strength of the vessel than might be supposed.

Cases 1e and 1f if in Table 13.5 represent radial body-force loading, which can be superimposed to give results for centrifugal loading, etc. (see Sec. 16.2). Case 1f is directly applicable to thick-walled disks with embedded electrical conductors used to generate magnetic fields. In many such cases the magnetic field varies linearly through the wall to zero at the outside. If there is a field at the outer turn, cases 1e and 1f can be superimposed in the necessary proportions.

The tabulated formulas for elastic wall stresses are accurate for both thin and thick vessels, but formulas for predicted yield pressures do not always agree closely with experimental results (Refs. 21, 34–37, and 39). The expression for q_y given in Table 13.5 is based on the minimum strain-energy theory of elastic failure. The expression for bursting pressure

$$q_u = 2\sigma_u \frac{a-b}{a+b} \tag{13.6-1}$$

commonly known as the *mean diameter formula*, is essentially empirical but agrees reasonably well with experiment for both thin and thick cylindrical vessels and is convenient to use. For very thick vessels the formula

$$q_u = \sigma_u \ln \frac{a}{b} \tag{13.6-2}$$

is preferable. Greater accuracy can be obtained by using with this formula a multiplying factor that takes into account the strain-hardening properties of the material (Refs. 10, 20, and 37). With the same objective, Faupel (Ref. 39) proposes (with different notation) the formula

$$q_u = \frac{2\sigma_y}{\sqrt{3}} \left(2 - \frac{\sigma_y}{\sigma_u} \right) \ln \frac{a}{b} \tag{13.6-3}$$

A rather extensive discussion of bursting pressure is given in Ref. 38, which presents a tabulated comparison between bursting pressures as calculated by a number of different formulas and as determined by actual experiment.

EXAMPLE

At the powder chamber, the inner radius of a 3-in gun tube is 1.605 in and the outer radius is 2.425 in. It is desired to shrink a jacket on this tube to produce a radial pressure between the tube and jacket of 7600 lb/in^2. The outer radius of this jacket is 3.850 in. It is required to determine the difference between the inner radius of the jacket and the outer radius of the tube in order to produce the desired pressure, calculate the stresses in each part when assembled, and calculate the stresses in each part when the gun is fired, generating a powder pressure of 32,000 lb/in^2.

Solution. Using the formulas for Table 13.5, case 1c, it is found that for an external pressure of 7600 lb/in^2, the stress σ_2 at the outer surface of the tube is $-19{,}430$ lb/in^2, the stress σ_2 at the inner surface is $-27{,}050$ lb/in^2, and the change in outer radius $\Delta a = -0.001385$ in; for an internal pressure of 7600 lb/in^2, the stress σ_2 at the inner surface of the jacket is $+17{,}630$ lb/in^2, the stress σ_2 at the outer surface is $+10{,}050$ lb/in^2, and the change in inner radius $\Delta b = +0.001615$ in. (In making these calculations the inner radius of the jacket is assumed to be 2.425 in.) The initial difference between the inner radius of the jacket and the outer radius of the tube must be equal to the sum of the radial deformations they suffer, or $0.001385 + 0.001615 = 0.0030$ in; therefore the initial radius of the jacket should be $2.425 - 0.0030 = 2.422$ in.

The stresses produced by the powder pressure are calculated at the inner surface of the tube, at the common surface of tube and jacket ($r = 2.425$ in), and at the outer surface of the jacket. These stresses are then superimposed on those found previously. The calculations are as follows: For the tube at the inner surface,

$$\sigma_2 = +32{,}000\frac{3.85^2 + 1.605^2}{3.85^2 - 1.605^2} = 45{,}450 \text{ lb/in}^2$$

$$\sigma_3 = -32{,}000 \text{ lb/in}^2$$

For tube and jacket at the interface,

$$\sigma_2 = +32{,}000\frac{1.605^2}{2.425^2}\,\frac{3.85^2 + 2.425^2}{3.85^2 - 1.605^2} = +23{,}500 \text{ lb/in}^2$$

$$\sigma_3 = -32{,}000\frac{1.605^2}{2.425^2}\,\frac{3.85^2 - 2.425^2}{3.85^2 - 1.605^2} = -10{,}200 \text{ lb/in}^2$$

For the jacket at the outer surface,

$$\sigma_2 = +32{,}000\frac{1.605^2}{3.85^2}\,\frac{3.85^2 + 3.85^2}{3.85^2 - 1.605^2} = +13{,}500 \text{ lb/in}^2$$

These are the stresses due to the powder pressure. Superimposing the stresses due to the shrinkage, we have as the resultant stresses:

At inner surface of tube,

$$\sigma_2 = -27{,}050 + 45{,}450 = +18{,}400 \text{ lb/in}^2$$

$$\sigma_3 = 0 - 32{,}000 = -32{,}000 \text{ lb/in}^2$$

At outer surface of tube,

$$\sigma_2 = -19{,}430 + 23{,}500 = +4070 \text{ lb/in}^2$$

$$\sigma_3 = -7600 - 10{,}200 = -17{,}800 \text{ lb/in}^2$$

At inner surface of jacket,

$$\sigma_2 = +17{,}630 + 23{,}500 = +41{,}130 \text{ lb/in}^2$$

$$\sigma_3 = -7600 - 10{,}200 = -17{,}800 \text{ lb/in}^2$$

At outer surface of jacket,

$$\sigma_2 + 10{,}050 + 13{,}500 = +23{,}550 \text{ lb/in}^2$$

13.7 Pipe on Supports at Intervals

For a pipe or cylindrical tank supported at intervals on saddles or pedestals and filled or partly filled with liquid, the stress analysis is difficult and the results are rendered uncertain by doubtful boundary conditions. Certain conclusions arrived at from a study of tests (Refs. 11 and 12) may be helpful in guiding design: See also Ref. 75.

1. For a circular pipe or tank supported at intervals and held circular at the supports by rings or bulkheads, the ordinary theory of flexure is applicable if the pipe is completely filled.
2. If the pipe is only partially filled, the cross section at points between supports becomes out of round and the distribution of longitudinal fiber stress is neither linear nor symmetrical across the section. The highest stresses occur for the half-full condition; then the maximum longitudinal compressive stress and the maximum circumferential bending stresses occur at the ends of the horizontal diameter, the maximum longitudinal tensile stress occurs at the bottom, and the longitudinal stress at the top is practically zero. According to theory (Ref. 4), the greatest of these stresses is the longitudinal compression, which is equal to the maximum longitudinal stress for the full condition divided by

$$K = \left(\frac{L}{R} \sqrt{\frac{t}{R}} \right)^{1/2}$$

   where $R =$ pipe radius, $t =$ thickness, and $L =$ span. The maximum circumferential stress is about one-third of this. Tests (Ref. 11) on a pipe having $K = 1.36$ showed a longitudinal stress that is somewhat less and a circumferential stress that is considerably greater than indicated by this theory.
3. For an unstiffened pipe resting in saddle supports, there are high local stresses, both longitudinal and circumferential, adjacent to the tips of the saddles. These stresses are less for a large saddle angle β (total angle subtended by arc of contact between pipe and saddle) than for a small angle, and for the ordinary range of dimensions they are practically independent of the thickness of the saddle, i.e., its dimension parallel to the pipe axis. For a pipe that fits the saddle well, the maximum value of these localized stresses will probably not exceed that indicated by the formula

$$\sigma_{\max} = k \frac{P}{t^2} \ln \frac{R}{t}$$

   where $P =$ total saddle reaction, $R =$ pipe radius, $t =$ pipe thickness, and $k =$ coefficient given by

$$k = 0.02 - 0.00012(\beta - 90)$$

   where β is in degrees. This stress is almost wholly due to circumferential bending and occurs at points about 15° above the saddle tips.

4. The maximum value of P the pipe can sustain is about 2.25 times the value that will produce a maximum stress equal to the yield point of the pipe material, according to the formula given above.
5. The comments in conclusion 3 above are based on the results of tests performed on very thin-walled pipe. Evces and O'Brien in Ref. 73 describe similar tests on thicker-walled ductile-iron pipe for which R/t does not normally exceed 50. They found that optimum saddle angles lie in the range $90^\circ > \beta > 120^\circ$ and that for $R/t \geqslant 28$ the formulas for $\sigma_{\max}$ can be used if the value of k is given by

$$k = 0.03 - 0.00017(\beta - 90)$$

   The maximum stress will be located within ±15° of the tip if the pipe fits the saddle well.
6. For a pipe supported in flexible slings instead of on rigid saddles, the maximum local stresses occur at the points of tangency of sling and pipe section; in general, they are less than the corresponding stresses in the saddle-supported pipe but are of the same order of magnitude.

A different but closely related support system for horizontal cylindrical tanks consists of a pair of longitudinal line loads running the full length of the vessel. If the tank wall is thin, accounting for the deformations, which are normally ignored in standard stress formulas, it shows that the stresses are significantly lower. Cook in Ref. 79 uses a nonlinear analysis to account for deformations and reports results for various positions of the supports, radius/thickness ratios, and depths of fill in the tank.

13.8 Tables

TABLE 13.1 Formulas for membrane stresses and deformations in thin-walled pressure vessels

NOTATION: P = axial load (force); p = unit load (force per unit length); q and w = unit pressures (force per unit area); δ = density (force per unit volume); σ_1 = meridional stress; σ_2 = circumferential, or hoop, stress; R_1 = radius of curvature of a meridian, a principal radius of curvature of the shell surface; R_2 = length of the normal between the point on the shell and the axis of rotation, the second principal radius of curvature; R = radius of curvature of a circumference; ΔR = radial displacement of a circumference; Δy = change in the height dimension y; y = length of cylindrical or conical shell and is also used as a vertical position coordinate, positive upward, from an indicated origin in some cases; ψ = rotation of a meridian from its unloaded position, positive when that meridional rotation represents an increase in ΔR when y or θ increases; E = modulus of elasticity; and ν = Poisson's ratio

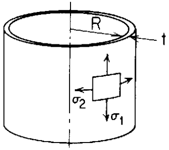
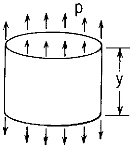
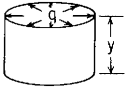
Case no., form of vessel	Manner of loading	Formulas
1. Cylindrical  $\frac{R}{t} > 10$	1a. Uniform axial load, p force/unit length 	$\sigma_1 = \frac{p}{t}$ $\sigma_2 = 0$ $\Delta R = \frac{-p\nu R}{Et}$ $\Delta y = \frac{py}{Et}$ $\psi = 0$
	1b. Uniform radial pressure, q force/unit area 	$\sigma_1 = 0$ $\sigma_2 = \frac{qR}{t}$ $\Delta R = \frac{qR^2}{Et}$ $\Delta y = \frac{-qR\nu y}{Et}$ $\psi = 0$

TABLE 13.1 Formulas for membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

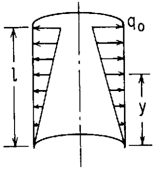
1c. Uniform internal or external pressure, q force/unit area (ends capped)	At points away from the ends $\sigma_1 = \frac{qR}{2t}$ $\sigma_2 = \frac{qR}{t}$ $\Delta R = \frac{qR^2}{Et}\left(1 - \frac{\nu}{2}\right)$ $\Delta y = \frac{qRy}{Et}(0.5 - \nu)$ $\psi = 0$
1d. Linearly varying radial pressure, q force/unit area 	$q = \frac{q_0 y}{l}$ (where y must be measured from a free end. If pressure starts away from the end, see case 6 in Table 13.2) $\sigma_1 = 0$ $\sigma_2 = \frac{qR}{t} = \frac{q_0 Ry}{lt}$ $\Delta R = \frac{qR^2}{Et} = \frac{q_0 R^2 y}{Etl}$ $\Delta y = \frac{-q_0 R\nu y^2}{2Etl}$ $\psi = \frac{q_0 R^2}{Etl}$
1e. Own weight, δ force/unit volume; top edge support, bottom edge free	$\sigma_1 = \delta y$ $\sigma_2 = 0$ $\Delta R = \frac{-\delta \nu R y}{E}$ $\Delta y = \frac{\delta y^2}{2E}$ $\psi = \frac{-\delta \nu R}{E}$

TABLE 13.1 Formulas for membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

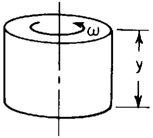
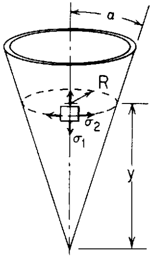
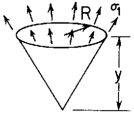
Case no., form of vessel	Manner of loading	Formulas
	1f. Uniform rotation, ω rad/s, about central axis  δ_m = mass density	$\sigma_1 = 0$ $\sigma_2 = \delta_m R^2 \omega^2$ $\Delta R = \frac{\delta_m R^3 \omega^2}{E}$ $\Delta y = \frac{-\nu \delta_m R^2 \omega^2 y}{E}$ $\psi = 0$
2. Cone  $\frac{R_2}{t} > 10$	2a. Uniform internal or external pressure, q force/unit area; tangential edge support  	$\sigma_1 = \frac{qR}{2t \cos \alpha}$ $\sigma_2 = \frac{qR}{t \cos \alpha}$ $\Delta R = \frac{qR^2}{Et \cos \alpha}\left(1 - \frac{\nu}{2}\right)$ $\Delta y = \frac{qR^2}{4Et \sin \alpha}(1 - 2\nu - 3 \tan^2 \alpha)$ $\psi = \frac{3qR \tan \alpha}{2Et \cos \alpha}$

TABLE 13.1 Formulas for membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

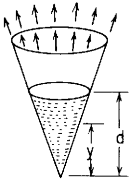
2b. Filled to depth d with liquid of density δ force/unit volume; tangential edge support  	At any level y below the liquid surface $y \leqslant d$ $\sigma_1 = \frac{\delta y \tan\alpha}{2t\cos\alpha}\left(d - \frac{2y}{3}\right), \qquad (\sigma_1)_{\max} = \frac{3\delta^2 \tan\alpha}{16t\cos\alpha} \qquad \text{when } y = \frac{3d}{4}$ $\sigma_2 = \frac{y(d-y)\delta\tan\alpha}{t\cos\alpha}, \qquad (\sigma_2)_{\max} = \frac{\delta d^2 \tan\alpha}{4t\cos\alpha} \qquad \text{when } y = \frac{d}{2}$ $\Delta R = \frac{\delta y^2 \tan^2\alpha}{Et\cos\alpha}\left[d\left(1-\frac{\nu}{2}\right) - y\left(1-\frac{\nu}{3}\right)\right]$ $\Delta y = \frac{\delta y^2 \sin\alpha}{Et\cos^4\alpha}\left\{\frac{d}{4}(1-2\nu) - \frac{y}{9}(1-3\nu) - \sin^2\alpha\left[\frac{d}{2}(2-\nu) - \frac{y}{3}(3-\nu)\right]\right\}$ $\psi = \frac{\delta y \sin^2\alpha}{6Et\cos^3\alpha}(9d - 16y)$ At any level y above the liquid level $\sigma_1 = \frac{\delta d^3 \tan\alpha}{6ty\cos\alpha}, \qquad \sigma_2 = 0, \qquad \Delta R = \frac{-\nu\delta d^3 \tan^2\alpha}{6Et\cos\alpha}$ $\Delta y = \frac{\delta d^3 \sin\alpha}{6Et\cos^4\alpha}\left[\frac{5}{6} - \nu(1-\sin^2\alpha) + \ln\frac{y}{d}\right]$ $\psi = \frac{-\delta d^3 \sin^2\alpha}{6Et\cos^3\alpha}\,\frac{1}{y}$ (*Note:* There is a discontinuity in the rate of increase in fluid pressure at the top of the liquid. This leads to some bending in this region and is indicated by a discrepancy in the two expressions for the meridional slope at $y = d$.)
2c. Own weight, δ force/unit volume tangential top edge support	$\sigma_1 = \frac{\delta R}{2\sin\alpha\cos\alpha}$ $\sigma_2 = \delta R \tan\alpha$ $\Delta R = \frac{\delta R^2}{E\cos\alpha}\left(\sin\alpha - \frac{\nu}{2\sin\alpha}\right)$ $\Delta y = \frac{\delta R^2}{E\cos^2\alpha}\left(\frac{1}{4\sin^2\alpha} - \sin^2\alpha\right)$ $\psi = \frac{2\delta R}{E\cos^2\alpha}\left[\sin^2\alpha\left(1+\frac{\nu}{2}\right) - \frac{1}{4}(1+2\nu)\right]$

TABLE 13.1 Formulas for membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

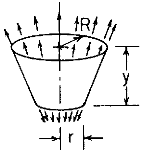
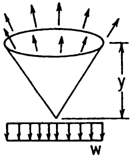
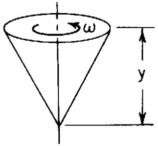
Case no., form of vessel	Manner of loading	Formulas
	2d. Tangential loading only; resultant load $= P$  	$\sigma_1 = \dfrac{P}{2\pi Rt\cos\alpha}$ $\sigma_2 = 0$ $\Delta R = \dfrac{-\nu P}{2\pi Et\cos\alpha}$ $\Delta y = \dfrac{P}{2\pi Et\sin\alpha\cos^2\alpha}\ln\dfrac{R}{r}$ $\psi = \dfrac{-P\sin\alpha}{2\pi ERt\cos^2\alpha}$ r must be finite to avoid infinite stress and $r/t > 10$ to be considered thin-walled
	2e. Uniform loading, force/unit area; on the horizontal projected area; tangential top edge support  	$\sigma_1 = \dfrac{wR}{2t\cos\alpha}$ $\sigma_2 = \dfrac{wR\sin^2\alpha}{t\cos\alpha}$ $\Delta R = \dfrac{wR^2}{Et\cos\alpha}\left(\sin^2\alpha - \dfrac{\nu}{2}\right)$ $\Delta y = \dfrac{wR^2}{2Et\cos^2\alpha}\left[\dfrac{1}{2\sin\alpha} + \nu(1 - \sin\alpha) - 2\sin^2\alpha\right]$ $\psi = \dfrac{wR\sin\alpha}{2Et\cos^2\alpha}(4\sin^2\alpha - 1 - 2\nu\cos^2\alpha)$
	2f. Uniform rotation, ω rad/s, about central axis  $\delta_m =$ mass density	$\sigma_1 = 0$ $\sigma_2 = \delta_m R^2\omega^2$ $\Delta R = \dfrac{\delta_m R^3\omega^2}{E}$ $\Delta y = \dfrac{-\delta_m R^3\omega^2}{E\cos\alpha}\left(\sin\alpha + \dfrac{\nu}{3\sin\alpha}\right)$ $\psi = \dfrac{\delta_m R^2\omega^2\tan\alpha}{E}(3 + \nu)$

TABLE 13.1 Formulas for membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

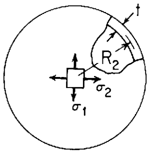
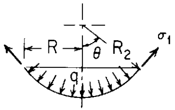
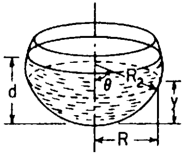
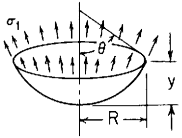
3. Spherical  $\frac{R_2}{t} > 10$	3a. Uniform internal or external pressure, q force/unit area; tangential edge support 	$\sigma_1 = \sigma_2 = \frac{qR_2}{2t}$ $\Delta R = \frac{qR_2^2(1-\nu)\sin\theta}{2Et}$ $\Delta R_2 = \frac{qR_2^2(1-\nu)}{2Et}$ $\Delta y = \frac{qR_2^2(1-\nu)(1-\cos\theta)}{2Et}$ $\psi = 0$
	3b. Filled to depth d with liquid of density δ force/unit volume; tangential edge support 	At any level y below the liquid surface, $y < d$ $\sigma_1 = \frac{\delta R_2^2}{6t}\left(3\frac{d}{R_2} - 1 + \frac{2\cos^2\theta}{1+\cos\theta}\right)$ $\sigma_2 = \frac{\delta R_2^2}{6t}\left[3\frac{d}{R_2} - 5 + \frac{(3+2\cos\theta)2\cos\theta}{1+\cos\theta}\right]$ (*Note:* See the note in case 2b regarding the discrepancy in the meridional slope at $y = d$) $\Delta R = \frac{\delta R_2^3 \sin\theta}{6Et}\left[3(1-\nu)\frac{d}{R_2} - 5 + \nu + 2\cos\theta\frac{3+(2-\nu)\cos\theta}{1+\cos\theta}\right]$ $\Delta y = \frac{\delta R_2^3}{6Et}\left\{3(1-\nu)\left[\frac{d}{R_2}(1-\cos\theta)+\cos\theta\right] - (2-\nu)\cos^2\theta + (1+\nu)\left(1 + 2\ln\frac{2}{1+\cos\theta}\right)\right\}$ $\psi = \frac{-\delta R_2^2}{Et}\sin\theta$ Weight of liquid $= P = \delta\pi d^2\left(R_2 - \frac{d}{3}\right)$ At any level y above the liquid level use case 3d with the load equal to the entire weight of the liquid
	3c. Own weight, δ force/unit volume; tangential top edge support 	$\sigma_1 = \frac{\delta R_2}{1+\cos\theta}, \quad \sigma_2 = -\delta R_2\left(\frac{1}{1+\cos\theta} - \cos\theta\right)$ Max tensile $\sigma_2 = \frac{\delta R_2}{2}$ at $\theta = 0$ $\sigma_2 = 0$ at $\theta = 51.83^\circ$ $\Delta R = \frac{-\delta R_2^2 \sin\theta}{E}\left(\frac{1+\nu}{1+\cos\theta} - \cos\theta\right)$ $\Delta y = \frac{\delta R_2^2}{E}\left[\sin^2\theta + (1+\nu)\ln\frac{2}{1+\cos\theta}\right]$ $\psi = \frac{-\delta R_2}{E}(2+\nu)\sin\theta$

TABLE 13.1 Formulas for membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

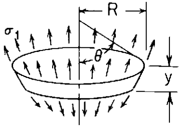
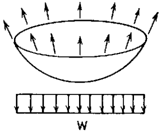
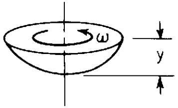
Case no., form of vessel	Manner of loading	Formulas
	3d. Tangential loading only; resultant load $= P$  	$\sigma_1 = \frac{P}{2\pi R_2 t \sin^2 \theta}, \qquad \sigma_2 = -\sigma_1$ $\Delta R = \frac{-P(1+\nu)}{2\pi E t \sin \theta}$ $\Delta y = \frac{P(1+\nu)}{2\pi E t}\left[\ln\left(\tan\frac{\theta}{2}\right) - \ln\left(\tan\frac{\theta_o}{2}\right)\right]$ $\psi = 0$ (*Note:* θ_o is the angle to the lower edge and cannot go to zero without local bending occurring in the shell)
	3e. Uniform loading, w force/unit area; on the horizontal projected area; tangential top edge support  	For $\theta \leqslant 90^\circ$ $\sigma_1 = \frac{wR_2}{2t}$ $\sigma_2 = \frac{wR_2}{2t}\cos 2\theta$ $\Delta R = \frac{wR_2^2 \sin\theta}{2Et}(\cos 2\theta - \nu)$ $\Delta y = \frac{wR_2^2}{2Et}[2(1-\cos^3\theta) + (1+\nu)(1-\cos\theta)]$ $\psi = \frac{-wR_2}{Et}(3+\nu)\sin\theta\cos\theta$
	3f. Uniform rotation, ω rad/s, about central axis  δ_m = mass density	$\sigma_1 = 0$ $\sigma_2 = \delta_m R^2 \omega^2$ $\Delta R = \frac{\delta_m R^3 \omega^2}{E}$ $\Delta y = \frac{-\delta_m R_2^3 \omega^2}{E}(1 + \nu - \nu\cos\theta - \cos^3\theta)$ $\psi = \frac{\delta_m R_2^2 \omega^2 \sin\theta\cos\theta}{E}(3+\nu)$

TABLE 13.1 Formulas for membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

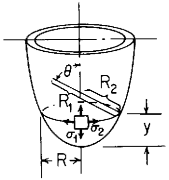
4. Any smooth figure of revolution if R_2 is less than infinity  $\frac{R_2}{t} > 10$	4a. Uniform internal or external pressure, q force/unit area; tangential edge support	$\sigma_1 = \frac{qR_2}{2t}$ $\sigma_2 = \frac{qR_2}{2t}\left(2 - \frac{R_2}{R_1}\right)$ $\Delta R = \frac{qR_2^2 \sin\theta}{2Et}\left(2 - \frac{R_2}{R_1} - \nu\right)$ $\psi = \frac{qR_2^2}{2EtR_1 \tan\theta}\left[3\frac{R_1}{R_2} - 5 + \frac{R_2}{R_1}\left(2 + \frac{1}{R_1}\frac{dR_1}{d\theta}\tan\theta\right)\right]$
	4b. Filled to depth d with liquid of density δ force/unit volume; tangential edge support. W = weight of liquid contained to a depth y	At any level y below the liquid surface, $y < d$, $\sigma_1 = \frac{W}{2\pi R_2 t \sin^2\theta} + \frac{\delta R_2(d-y)}{2t}$ $\sigma_2 = \frac{-W}{2\pi R_1 t \sin^2\theta} + \frac{\delta R_2(d-y)}{2t}\left(2 - \frac{R_2}{R_1}\right)$ $\Delta R = \frac{R_2 \sin\theta}{E}(\sigma_2 - \nu\sigma_1)$ At any level y above the liquid level use case 4d with the load equal to the entire weight of the liquid
	4c. Own weight, δ force/unit volume; tangential top edge support. W = weight of vessel below the level y	$\sigma_1 = \frac{W}{2\pi R_2 t \sin^2\theta}$ $\sigma_2 = \frac{W}{2\pi R_1 t \sin^2\theta} + \delta R_2 \cos\theta$ $\Delta R = \frac{R_2 \sin\theta}{E}(\sigma_2 - \nu\sigma_1)$
	4d. Tangential loading only, resultant load = P	$\sigma_1 = \frac{P}{2\pi R_2 t \sin^2\theta}$ $\sigma_2 = \frac{-P}{2\pi R_1 t \sin^2\theta}$ $\Delta R = \frac{-P}{2\pi Et \sin\theta}\left(\frac{R_2}{R_1} + \nu\right)$ $\psi = \frac{-P}{2\pi EtR_1 \sin^2\theta}\left[\frac{1}{\tan\theta}\left(1 + \frac{R_1}{R_2} - 2\frac{R_2}{R_1}\right) - \frac{R_2}{R_1^2}\frac{dR_1}{d\theta}\right]$

TABLE 13.1 Formulas for membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

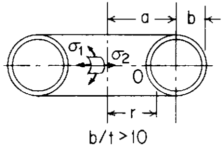
Case no., form of vessel	Manner of loading	Formulas
	4e. Uniform loading, w force/unit area, on the horizontal projected area; tangential top edge support	For $\theta \leqslant 90^\circ$ $\sigma_1 = \frac{wR_2}{2t}$ $\sigma_2 = \frac{wR_2}{2t}\left(2\cos^2\theta - \frac{R_2}{R_1}\right)$ $\Delta R = \frac{wR_2^2 \sin\theta}{2Et}\left(2\cos^2\theta - \frac{R_2}{R_1} - \nu\right)$ $\psi = \frac{w}{2EtR_1 \tan\theta}\left[R_1R_2(4\cos^2\theta - 1 - 2\nu\sin^2\theta) - R_2^2(7 - 2\cos\theta) + \frac{R_2^3}{R_1}\left(2 + \frac{\tan\theta}{R_1}\frac{dR_1}{d\theta}\right)\right]$
	4f. Uniform rotation, ω rad/s, about central axis δ_m = mass density	$\sigma_1 = 0$ $\sigma_2 = \delta_m R^2 \omega^2$ $\Delta R = \frac{\delta_m R^3 \omega^2}{E}$ $\psi = \frac{\delta_m R^2 \omega^2}{E \tan\theta}(3 + \nu)$
5. Toroidal shell 	5a. Uniform internal or external pressure, q force/unit area	$\sigma_1 = \frac{qb}{2t}\frac{r + a}{r}$ $(\sigma_1)_{\max} = \frac{qb}{2t}\frac{2a - b}{a - b}$ at point O $\sigma_2 = \frac{qb}{2t}$ (throughout) $\Delta r = \frac{qb}{2Et}[r - \nu(r + a)]$ [*Note:* There are some bending stresses at the top and bottom where R_2 (see case 4) is infinite (see Ref. 42)]

TABLE 13.2 Shear, moment, slope, and deflection formulas for long and short thin-walled cylindrical shells under axisymmetric loading

NOTATION: V_o, H, and p = unit loads (force per unit length); q = unit pressure (force per unit area); M_o = unit applied couple (force-length per unit length); all loads are positive as shown. At a distance x from the left end, the following quantities are defined: V = meridional radial shear, positive when acting outward on the right hand portion; M = meridional bending moment, positive compressive on the outside; ψ = meridional slope (radians), positive when the deflection increases with x; y = radial deflection, positive outward. σ_1 and σ_2 = meridional and circumferential membrane stresses; positive when tensile; σ'_1 and σ'_2 = meridional and circumferential bending stresses, positive when tensile on the outside; τ = meridional radial shear stress; E = modulus of elasticity; ν = Poisson's ratio; R = mean radius; t = wall thickness.

The following constants and functions are hereby defined in order to permit condensing the tabulated formulas which follow:

$$\lambda = \left[\frac{3(1-\nu^2)}{R^2t^2}\right]^{1/4} \qquad D = \frac{Et^3}{12(1-\nu^2)}$$

(*Note:* See page 131 for a definition of $\langle x-a\rangle^n$; also all hyperbolic and trigonometric functions of the argument $\langle x-a\rangle$ are also defined as zero if $x < a$) (Note when also the limitations on maximum deflections discussed in paragraph 3 of Sec 13.3)

$F_1 = \cosh\lambda x \cos\lambda x$

$F_2 = \cosh\lambda x \sin\lambda x + \sinh\lambda x \cos\lambda x$

$F_3 = \sinh\lambda x \sin\lambda x$

$F_4 = \cosh\lambda x \sin\lambda x - \sinh\lambda x \cos\lambda x$

$C_1 = \cosh\lambda l \cos\lambda l$

$C_2 = \cosh\lambda l \sin\lambda l + \sinh\lambda l \cos\lambda l$

$C_3 = \sinh\lambda l \sin\lambda l$

$C_4 = \cosh\lambda l \sin\lambda l - \sinh\lambda l \cos\lambda l$

$C_{11} = \sinh^2 \lambda l - \sin^2 \lambda l$

$C_{12} = \cosh\lambda l \sinh\lambda l + \cos\lambda l \sin\lambda l$

$C_{13} = \cosh\lambda l \sinh\lambda l - \cos\lambda l \sin\lambda l$

$C_{14} = \sinh^2 \lambda l + \sin^2 \lambda l$

$F_{a1} = \langle x-a\rangle^0 \cosh\lambda\langle x-a\rangle \cos\lambda\langle x-a\rangle$

$F_{a2} = \cosh\lambda\langle x-a\rangle \sin\lambda\langle x-a\rangle + \sinh\lambda\langle x-a\rangle \cos\lambda\langle x-a\rangle$

$F_{a3} = \sinh\lambda\langle x-a\rangle \sin\lambda\langle x-a\rangle$

$F_{a4} = \cosh\lambda\langle x-a\rangle \sin\lambda\langle x-a\rangle - \sinh\lambda\langle x-a\rangle \cos\lambda\langle x-a\rangle$

$F_{a5} = \langle x-a\rangle^0 - F_{a1}$

$F_{a6} = 2\lambda(x-a)\langle x-a\rangle^0 - F_{a2}$

$C_{a1} = \cosh\lambda(l-a) \cos\lambda(l-a)$

$C_{a2} = \cosh\lambda(l-a) \sin\lambda(l-a) + \sinh\lambda(l-a) \cos\lambda(l-a)$

$C_{a3} = \sinh\lambda(l-a) \sin\lambda(l-a)$

$C_{a4} = \cosh\lambda(l-a) \sin\lambda(l-a) - \sinh\lambda(l-a) \cos\lambda(l-a)$

$C_{a5} = 1 - C_{a1}$

$C_{a6} = 2\lambda(l-a) - C_{a2}$

$A_1 = \frac{1}{2}e^{-\lambda a} \cos\lambda a$

$A_2 = \frac{1}{2}e^{-\lambda a}(\sin\lambda a - \cos\lambda a)$

$A_3 = -\frac{1}{2}e^{-\lambda a} \sin\lambda a$

$A_4 = \frac{1}{2}e^{-\lambda a}(\sin\lambda a + \cos\lambda a)$

$B_1 = \frac{1}{2}e^{-\lambda b} \cos\lambda b$

$B_2 = \frac{1}{2}e^{-\lambda b}(\sin\lambda b - \cos\lambda b)$

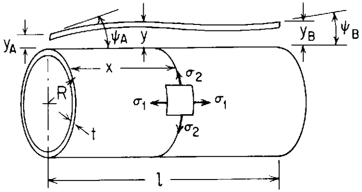
$B_3 = -\frac{1}{2}e^{-\lambda b} \sin\lambda b$

$B_4 = \frac{1}{2}e^{-\lambda b}(\sin\lambda b + \cos\lambda b)$

Numerical values of F_1, F_2, F_3, and F_4 for λx ranging from 0 to 6 are tabulated in Table 8.3; numerical values of C_{11}, C_{12}, C_{13}, and C_{14} are tabulated in Table 8.4.

TABLE 13.2 Shear, moment, slope, and deflection formulas for long and short thin-walled cylindrical shells under axisymmetric loading (*Continued*)

Short shells with free ends



$R/t > 10$

Meridional radial shear $= V = -y_A 2D\lambda^3 F_2 - \psi_A 2D\lambda^2 F_3 + LT_V$

Meridional bending moment $= M = -y_A 2D\lambda^2 F_3 - \psi_A D\lambda F_4 + LT_M$ (*Note:* The load terms LT_V, LT_M, etc., are given for each of the following cases)

Meridional slope $= \psi = \psi_A F_1 - y_A \lambda F_4 + LT_\psi$

Radial deflection $= y = y_A F_1 + \frac{\psi_A}{2\lambda} F_2 + LT_y$

Circumferential membrane stress $= \sigma_2 = \frac{yE}{R} + \nu\sigma_1$

Meridional bending stress $= \sigma_1' = \frac{-6M}{t^2}$

Circumferential bending stress $= \sigma_2' = \nu\sigma_1'$

Meridional radial shear stress $= \tau = \frac{V}{t}$ (average value)

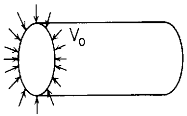
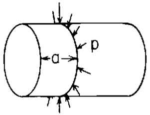
Loading and case no.	End deformations	Load terms or load and deformation equations	Selected values
1. Radial end load, V_o lb/in  If $\lambda l > 6$, see case 8	$\psi_A = \frac{V_o}{2D\lambda^2}\frac{C_{14}}{C_{11}}$ $y_A = \frac{-V_o}{2D\lambda^3}\frac{C_{13}}{C_{11}}$ $\psi_B = \frac{V_o}{2D\lambda^2}\frac{2C_3}{C_{11}}$ $y_B = \frac{V_o}{2D\lambda^3}\frac{C_4}{C_{11}}$	$LT_V = -V_o F_1$ $LT_M = \frac{-V_o}{2\lambda}F_2$ $LT_\psi = \frac{-V_o}{2D\lambda^2}F_3$ $LT_y = \frac{-V_o}{4D\lambda^3}F_4$	$\sigma_1 = 0 \quad (\sigma_2)_{\max} = \frac{y_A E}{R}$ $\psi_{\max} = \psi_A$ $y_{\max} = y_A$
2. Intermediate radial load, p lb/in  If $\lambda l > 6$, consider case 9	$\psi_A = \frac{p}{2D\lambda^2}\frac{C_2C_{a2} - 2C_3C_{a1}}{C_{11}}$ $y_A = \frac{-p}{2D\lambda^3}\frac{C_3C_{a2} - C_4C_{a1}}{C_{11}}$ $\psi_B = \psi_A C_1 - y_A\lambda C_4 - \frac{p}{2D\lambda^2}C_{a3}$ $y_B = y_A C_1 + \frac{\psi_A C_2}{2\lambda} - \frac{p}{4D\lambda^3}C_{a4}$	$LT_V = -pF_{a1}$ $LT_M = \frac{-p}{2\lambda}F_{a2}$ $LT_\psi = \frac{-p}{2D\lambda^2}F_{a3}$ $LT_y = \frac{-p}{4D\lambda^3}F_{a4}$	$\sigma_1 = 0$

TABLE 13.2 Shear, moment, slope, and deflection formulas for long and short thin-walled cylindrical shells under axisymmetric loading (*Continued*)

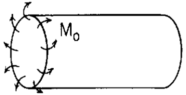
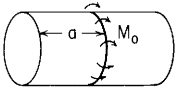
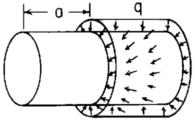
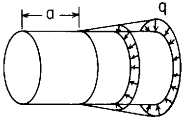
3. End moment, M_o lb-in/in  If $\lambda l > 6$, see case 10	$\psi_A = \dfrac{-M_o}{D\lambda}\dfrac{C_{12}}{C_{11}}$ $y_A = \dfrac{M_o}{2D\lambda^2}\dfrac{C_{14}}{C_{11}}$ $\psi_B = \dfrac{-M_o}{D\lambda}\dfrac{C_2}{C_{11}}$ $y_B = \dfrac{-M_o}{D\lambda^2}\dfrac{C_3}{C_{11}}$	$LT_V = -M_o\lambda F_4$ $LT_M = M_o F_1$ $LT_\psi = \dfrac{M_o}{2D\lambda}F_2$ $LT_y = \dfrac{M_o}{2D\lambda^2}F_3$	$\sigma_1 = 0$ $(\sigma_2)_{\max} = \dfrac{y_A E}{R}$ $M_{\max} = M_o \quad (\text{at } x = 0)$ $\psi_{\max} = \psi_A$ $y_{\max} = y_A$
4. Intermediate applied moment, M_o lb-in/in  If $\lambda l > 6$, consider case 11	$\psi_A = \dfrac{-M_o}{D\lambda}\dfrac{C_2C_{a1} + C_3C_{a4}}{C_{11}}$ $y_A = \dfrac{M_o}{2D\lambda^2}\dfrac{2C_3C_{a1} + C_4C_{a4}}{C_{11}}$ $\psi_B = \psi_A C_1 - y_A\lambda C_4 + \dfrac{M_o}{2D\lambda}C_{a2}$ $y_B = y_A C_1 + \dfrac{\psi_A}{2\lambda}C_2 + \dfrac{M_o}{2D\lambda^2}C_{a3}$	$LT_V = -M_o\lambda F_{a4}$ $LT_M = M_o F_{a1}$ $LT_\psi = \dfrac{M_o}{2D\lambda}F_{a2}$ $LT_y = \dfrac{M_o}{2D\lambda^2}F_{a3}$	
5. Uniform radial pressure from a to l  If $\lambda l > 6$, consider case 12	$\psi_A = \dfrac{q}{2D\lambda^3}\dfrac{C_2C_{a3} - C_3C_{a2}}{C_{11}}$ $y_A = \dfrac{-q}{4D\lambda^4}\dfrac{2C_3C_{a3} - C_4C_{a2}}{C_{11}}$ $\psi_B = \psi_A C_1 - y_A\lambda C_4 - \dfrac{q}{4D\lambda^3}C_{a4}$ $y_B = y_A C_1 + \dfrac{\psi_A}{2\lambda}C_2 - \dfrac{q}{4D\lambda^4}C_{a5}$	$LT_V = \dfrac{-q}{2\lambda}F_{a2}$ $LT_M = \dfrac{-q}{2\lambda^2}F_{a3}$ $LT_\psi = \dfrac{-q}{4D\lambda^3}F_{a4}$ $LT_y = \dfrac{-q}{4D\lambda^4}F_{a5}$	
6. Uniformly increasing pressure from a to l 	$\psi_A = \dfrac{-q}{4D\lambda^4(l-a)}\dfrac{2C_3C_{a3} - C_2C_{a4}}{C_{11}}$ $y_A = \dfrac{-q}{4D\lambda^5(l-a)}\dfrac{C_3C_{a4} - C_4C_{a3}}{C_{11}}$ $\psi_B = \psi_A C_1 - y_A\lambda C_4 - \dfrac{qC_{a5}}{4D\lambda^4(l-a)}$ $y_B = y_A C_1 + \dfrac{\psi_A}{2\lambda}C_2 - \dfrac{qC_{a6}}{8D\lambda^5(l-a)}$	$LT_V = \dfrac{-q}{2\lambda^2(l-a)}F_{a3}$ $LT_M = \dfrac{-q}{4\lambda^3(l-a)}F_{a4}$ $LT_\psi = \dfrac{-q}{4D\lambda^4(l-a)}F_{a5}$ $LT_y = \dfrac{-q}{8D\lambda^5(l-a)}F_{a6}$	$\sigma_1 = 0$ $(\sigma_2)_{\max} = \dfrac{y_B E}{R}$ $y_{\max} = y_B$ $\psi_{\max} = \psi_B$

TABLE 13.2 Shear, moment, slope, and deflection formulas for long and short thin-walled cylindrical shells under axisymmetric loading (*Continued*)

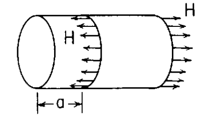
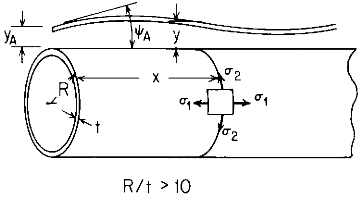
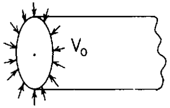
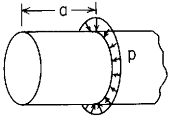
Loading and case no.	End deformations	Load terms or load and deformation equations	Selected values
7. Axial load along the portion from a to l only  	$\psi_A = \frac{\nu H}{2D\lambda^3 R}\,\frac{C_2 C_{a3} - C_3 C_{a2}}{C_{11}}$ $y_A = \frac{-\nu HR}{Et}\,\frac{2C_3 C_{a3} - C_3 C_{a2}}{C_{11}}$ $\psi_B = \psi_A C_1 - y_A \lambda C_4 - \frac{\nu HR\lambda}{Et} C_{a4}$ $y_B = y_A C_1 + \frac{\psi_A C_2}{2\lambda} - \frac{\nu HR}{Et} C_{a5}$	$LT_V = \frac{-\nu H}{2\lambda R} F_{a2}$ $LT_M = \frac{-\nu H}{2\lambda^2 R} F_{a3}$ $LT_\psi = \frac{-\nu HR\lambda}{Et} F_{a4}$ $LT_y = \frac{-\nu HR}{Et} F_{a5}$	$\sigma_1 = \frac{H}{t}\langle x - a\rangle^0$
Long shells with the left end free (right end more than $6/\lambda$ units of length from the closest load)  	Meridional radial shear $= V = -y_A 2D\lambda^3 F_2 - \psi_A 2D\lambda^2 F_3 + LT_V$ Meridional bending moment $= M = -y_A 2D\lambda^2 F_3 - \psi_A D\lambda F_4 + LT_M$ Meridional slope $= \psi = \psi_A F_1 - y_A \lambda F_4 + LT_\psi$ Radial deflection $= y = y_A F_1 + \frac{\psi_A}{2\lambda} F_2 + LT_y$ Circumferential membrane stress $= \sigma_2 = \frac{yE}{R} + \nu\sigma_1$ Meridional bending stress $= \sigma_1' = \frac{-6M}{t^2}$ Circumferential bending stress $= \sigma_2' = \nu\sigma_1'$ Meridional radial shear stress $= \tau = \frac{V}{t}$ (average value)		(*Note:* The load terms LT_V, LT_M, etc., are given where needed in the following cases)
8. Radial end load, V_o lb/in  	$\psi_A = \frac{V_o}{2D\lambda^2}$ $y_A = \frac{-V_o}{2D\lambda^3}$	$V = -V_o e^{-\lambda x}(\cos\lambda x - \sin\lambda x)$ $M = \frac{-V_o}{\lambda} e^{-\lambda x}\sin\lambda x$ $\psi = \frac{V_o}{2D\lambda^2} e^{-\lambda x}(\cos\lambda x + \sin\lambda x)$ $y = \frac{-V_o}{2D\lambda^3} e^{-\lambda x}\cos\lambda x$	$V_{\max} = -V_o$ at $x = 0$ $M_{\max} = -0.3224\frac{V_o}{\lambda}$ at $x = \frac{\pi}{4\lambda}$ $\psi_{\max} = \psi_A$ $y_{\max} = y_A$ $\sigma_1 = 0$ $(\sigma_2)_{\max} = \frac{-2V_o\lambda R}{t}$ at $x = 0$
9. Intermediate radial load, p lb/in  If $\lambda a > 3$, consider case 15	$\psi_A = \frac{-p}{D\lambda^2} A_2$ $y_A = \frac{-p}{D\lambda^3} A_1$	$LT_V = -pF_{a1}$ $LT_M = \frac{-p}{2\lambda} F_{a2}$ $LT_\psi = \frac{-p}{2D\lambda^2} F_{a3}$ $LT_y = \frac{-p}{4D\lambda^3} F_{a4}$	$\sigma_1 = 0$

TABLE 13.2 Shear, moment, slope, and deflection formulas for long and short thin-walled cylindrical shells under axisymmetric loading (*Continued*)

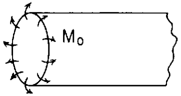
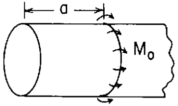
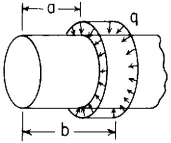
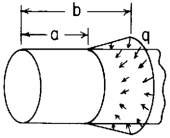
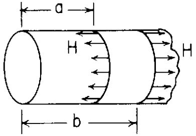
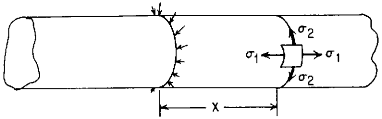
10. End moment, M_o lb-in/in 	$\psi_A = \dfrac{-M_o}{D\lambda}$ $y_A = \dfrac{M_o}{2D\lambda^2}$	$V = -2M_o\lambda e^{-\lambda x}\sin\lambda x$ $M = M_o e^{-\lambda x}(\cos\lambda x + \sin\lambda x)$ $\psi = \dfrac{-M_o}{D\lambda}e^{-\lambda x}\cos\lambda x$ $y = \dfrac{-M_o}{2D\lambda^2}e^{-\lambda x}(\sin\lambda x - \cos\lambda x)$	$V_{\max} = -0.6448M_o\lambda \quad \text{at } x = \dfrac{\pi}{4\lambda}$ $M_{\max} = M_o \quad \text{at } x = 0$ $\psi_{\max} = \psi_A, \quad y_{\max} = y_A$ $\sigma_1 = 0$ $(\sigma_2)_{\max} = \dfrac{2M_o\lambda^2 R}{t} \quad \text{at } x = 0$
11. Intermediate applied moment, M_o lb-in/in  If $\lambda a > 3$, consider case 16	$\psi_A = \dfrac{-2M_o}{D\lambda}A_1$ $y_A = \dfrac{M_o}{D\lambda^2}A_4$	$LT_V = -M_o\lambda F_{a4}$ $LT_M = M_o F_{a1}$ $LT_\psi = \dfrac{M_o}{2D\lambda}F_{a2}$ $LT_y = \dfrac{M_o}{2D\lambda^2}F_{a3}$	$\sigma_1 = 0$
12. Uniform radial pressure from a to b 	$\psi_A = \dfrac{-q}{D\lambda^3}(B_3 - A_3)$ $y_A = \dfrac{-q}{2D\lambda^4}(B_2 - A_2)$	$LT_V = \dfrac{-q}{2\lambda}(F_{a2} - F_{b2})$ $LT_M = \dfrac{-q}{2\lambda^2}(F_{a3} - F_{b3})$ $LT_\psi = \dfrac{-q}{4D\lambda^3}(F_{a4} - F_{b4})$ $LT_y = \dfrac{-q}{4D\lambda^4}(F_{a5} - F_{b5})$ For values of F_{b1} to F_{b6} substitute b for a in the expressions for F_{a1} to F_{a6}	$\sigma_1 = 0$
13. Uniformly increasing pressure from a to b 	$\psi_A = \dfrac{q}{D}\left[\dfrac{B_4 - A_4}{2\lambda^4(b-a)} - \dfrac{B_3}{\lambda^3}\right]$ $y_A = \dfrac{q}{2D}\left[\dfrac{B_3 - A_3}{\lambda^5(b-a)} - \dfrac{B_2}{\lambda^4}\right]$	$LT_V = \dfrac{-q}{2}\left[\dfrac{F_{a3} - F_{b3}}{\lambda^2(b-a)} - \dfrac{F_{b2}}{\lambda}\right]$ $LT_M = \dfrac{-q}{2}\left[\dfrac{F_{a4} - F_{b4}}{2\lambda^3(b-a)} - \dfrac{F_{b3}}{\lambda^2}\right]$ $LT_\psi = \dfrac{-q}{4D}\left[\dfrac{F_{a5} - F_{b5}}{\lambda^4(b-a)} - \dfrac{F_{b4}}{\lambda^3}\right]$ $LT_y = \dfrac{-q}{4D}\left[\dfrac{F_{a6} - F_{b6}}{2\lambda^5(b-a)} - \dfrac{F_{b5}}{\lambda^4}\right]$ See note in case 12	$\sigma_1 = 0$

TABLE 13.2 Shear, moment, slope, and deflection formulas for long and short thin-walled cylindrical shells under axisymmetric loading (*Continued*)

Loading and case no.	End deformations	Load terms or load and deformation equations	Selected values
14. Axial load along the portion from a to b 	$\psi_A = \frac{-\nu H}{RD\lambda^3}(B_3 - A_3)$ $y_A = \frac{-\nu H}{2RD\lambda^4}(B_2 - A_2)$	$LT_V = \frac{-\nu H}{2R\lambda}(F_{a2} - F_{b2})$ $LT_M = \frac{-\nu H}{2R\lambda^2}(F_{a3} - F_{b3})$ $LT_\psi = \frac{-\nu H}{4RD\lambda^3}(F_{a4} - F_{b4})$ $LT_y = \frac{-\nu H}{4RD\lambda^4}(F_{a5} - F_{b5})$	$\sigma_1 = \frac{H}{t}\langle x - a\rangle^0 - \frac{H}{t}\langle x - b\rangle^0$

Very long shells (both ends more than $6/\lambda$ units of length from the nearest loading)



$R/t > 10$

Circumferential membrane stress $= \sigma_2 = \frac{yE}{R} + \nu\sigma_1$

Meridional bending stress $= \sigma_1' = \frac{-6M}{t^2}$

Circumferential bending stress $= \sigma_2' = \nu\sigma_1'$

Meridional radial shear stress $= \tau = \frac{V}{t}$

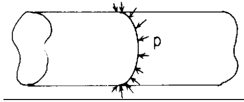
Loading and case no.	Load and deformation equations	Selected values
15. Concentrated radial load, p (lb/linear in of circumference) 	$V = \frac{-p}{2}e^{-\lambda x}\cos\lambda x$ $M = \frac{p}{4\lambda}e^{-\lambda x}(\cos\lambda x - \sin\lambda x)$ $\psi = \frac{p}{4D\lambda^2}e^{-\lambda x}\sin\lambda x$ $y = \frac{-p}{8D\lambda^3}e^{-\lambda r}(\cos\lambda x + \sin\lambda x)$	$V_{\max} = \frac{-p}{2}$ at $x = 0$, $\sigma_1 = 0$ $M_{\max} = \frac{p}{4\lambda}$ at $x = 0$ $\psi_{\max} = 0.0806\frac{p}{D\lambda^2}$ at $x = \frac{\pi}{4\lambda}$ $y_{\max} = \frac{-p}{8D\lambda^3}$ at $x = 0$

TABLE 13.2 Shear, moment, slope, and deflection formulas for long and short thin-walled cylindrical shells under axisymmetric loading (*Continued*)

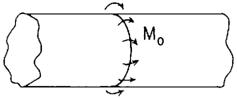
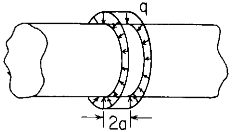
16. Applied moment  	$V = \frac{-M_o\lambda}{2} e^{-\lambda x}(\cos\lambda x + \sin\lambda x)$ $M = \frac{M_o}{2} e^{-\lambda x}\cos\lambda x$ $\psi = \frac{-M_o}{4D\lambda} e^{-\lambda x}(\cos\lambda x - \sin\lambda x)$ $y = \frac{-M_o}{4D\lambda^2} e^{-\lambda x}\sin\lambda x$	$V_{\max} = \frac{-M_o\lambda}{2}$ at $x = 0$, $\sigma_1 = 0$ $M_{\max} = \frac{M_o}{2}$ at $x = 0$ $\psi_{\max} = \frac{-M_o}{4D\lambda}$ at $x = 0$ $y_{\max} = -0.0806\frac{M_o}{D\lambda^2}$ at $x = \frac{\pi}{4\lambda}$
17. Uniform pressure over a band of width $2a$  	Superimpose cases 10 and 12 to make ψ_A (at $x = 0$) $= 0$ [Note: x is measured from the midlength of the loaded band]	$M_{\max} = \frac{q}{2\lambda^2} e^{-\lambda a}\sin\lambda a$ at $x = 0$ $y_{\max} = \frac{-q}{4D\lambda^4}(1 - e^{-\lambda a}\cos\lambda a)$ at $x = 0$ $\sigma_1 = 0$

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels

NOTATION: Q_o and p unit loads (force per unit length); $q =$ unit pressure (force per unit area); $M_o =$ unit applied couple (force-length per unit length); $\psi_o =$ applied edge rotation (radians); $\Delta_o =$ applied edge displacement, all loads are positive as shown. $V =$ meridional transverse shear, positive as shown; M_1 and $M_2 =$ meridional and circumferential bending moments, respectively, positive when compressive on the outside; $\psi =$ change in meridional slope (radians), positive when the change is in the same direction as a positive M_1; $\Delta R =$ change in circumferential radius, positive outward; σ_1 and $\sigma_2 =$ meridional and circumferential membrane stresses, positive when tensile; σ_1' and $\sigma_2' =$ meridional and circumferential bending stresses, positive when tensile on the outside. $E =$ modulus of elasticity; $\nu =$ Poisson's ratio; and $D = Et^3/12(1-\nu^2)$

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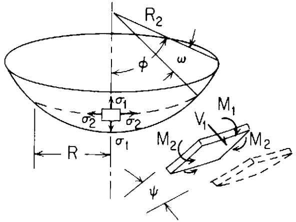
1. Partial spherical shells



$$\beta = \left[3(1-\nu^2)\left(\frac{R_2}{t}\right)^2\right]^{1/4}$$

$$K_1 = 1 - \frac{1-2\nu}{2\beta}\cot(\phi-\omega)$$

$$K_2 = 1 - \frac{1+2\nu}{2\beta}\cot(\phi-\omega)$$

$$K_3 = \frac{e^{-\beta\omega}}{\sqrt{\sin(\phi-\omega)}}$$

$$\frac{R_2}{t} > 10$$

Meridional radial shear $= V_1 = -CK_3\sin(\beta\omega+\zeta)$ (*Note:* Expressions for C and ζ are given below for the several loads)

Meridional bending moment $= M_1 = \dfrac{CR_2K_3}{2\beta}[K_1\cos(\beta\omega+\zeta)+\sin(\beta\omega+\zeta)]$

Circumferential bending moment $= M_2 = \dfrac{CR_2K_3}{2\beta}\left\{\nu\sin(\beta\omega+\zeta)+\left[2-\left(1-\dfrac{\nu}{2}\right)(K_1+K_2)\right]\cos(\beta\omega+\zeta)\right\}$

Change in meridional slope $= \psi = \dfrac{C2\beta^2K_3}{Et}\cos(\beta\omega+\zeta)$

Change in radius of circumference $= \Delta R = \dfrac{CR_2\beta K_3\sin(\phi-\omega)}{Et}[\cos(\beta\omega+\zeta)-K_2\sin(\beta\omega+\zeta)]$

Meridional membrane stress $= \sigma_1 = \dfrac{-CK_3\cot(\phi-\omega)}{t}\sin(\beta\omega+\zeta)$

Meridional bending stress $= \sigma_1' = \dfrac{-6M_1}{t^2}$

Circumferential membrane stress $= \sigma_2 = \dfrac{C\beta K_3}{2t}[2\cos(\beta\omega+\zeta)-(K_1+K_2)\sin(\beta\omega+\zeta)]$

Circumferential bending stress $= \sigma_2' = \dfrac{-6M_2}{t^2}$

Meridional radial shear stress $= \tau = \dfrac{V_1}{t}$ (average value)

[*Note:* For reasonable accuracy $3/\beta < \phi < \pi - 3/\beta$. Deformations and stresses due to edge loads and displacements are essentially zero when $\omega > 3/\beta$ (see discussion)]

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

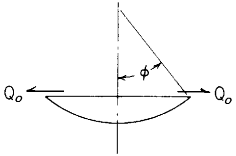
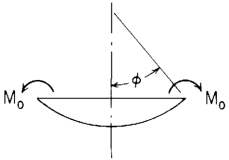
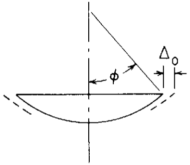
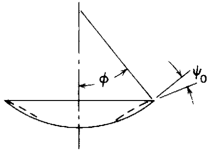
Case no., loading	Formulas
1a. Uniform radial force Q_o at the edge  	$C = \frac{Q_o(\sin\phi)^{3/2}\sqrt{1+K_1^2}}{K_1}$ and $\zeta = \tan^{-1}(-K_1)$ where $-\frac{\pi}{2} < \zeta < \frac{\pi}{2}$ Max value of M_1 occurs at $\omega = \pi/4\beta$ At the edge where $\omega = 0$, $V_1 = Q_o\sin\phi, \quad M_1 = 0, \quad \sigma_1' = 0, \quad \sigma_1 = \frac{Q_o\cos\phi}{t}$ $\sigma_2 = \frac{Q_o\beta\sin\phi}{2t}\left(\frac{2}{K_1} + K_1 + K_2\right) = (\sigma_2)_{\max}$ $M_2 = \frac{Q_o t^2 B^2\cos\phi}{6K_1R_2}, \quad \sigma_2' = \frac{-Q_oB^2\cos\phi}{K_1R_2}$ $\psi = \frac{Q_o 2\beta^2\sin\phi}{EtK_1}, \quad \Delta R = \frac{Q_oR_2\beta\sin^2\phi}{EtK_1}(1+K_1K_2)$ (Refs. 14 and 42)
1b. Uniform edge moment M_o 	$C = \frac{M_o 2\beta\sqrt{\sin\phi}}{R_2K_1}, \quad \zeta = 0$ At the edge where $\omega = 0$, $V_1 = 0, \quad \sigma_1 = 0, \quad M_1 = M_o, \quad \sigma_1' = \frac{-6M_o}{t^2}$ $\sigma_2 = \frac{M_o 2\beta^2}{R_2K_1t}, \quad M_2 = \frac{M_o}{2\nu K_1}[(1+\nu^2)(K_1+K_2) - 2K_2], \quad \sigma_2' = \frac{-6M_2}{t^2}$ $\psi = \frac{M_o 4\beta^3}{EtR_2K_1}, \quad \Delta R = \frac{M_o 2\beta^2\sin\phi}{EtK_1}$ (Refs. 14 and 42)
1c. Radial displacement Δ_o; no edge rotation 	$C = \frac{-\Delta_o Et}{R_2\beta K_2\sqrt{\sin\phi}}, \quad \zeta = \frac{\pi}{2} = 90^\circ$ At the edge where $\omega = 0$, $V_1 = \frac{\Delta_o Et}{R_2\beta K_2\sin\phi}, \quad \sigma_1 = \frac{\Delta_o E\cos\phi}{R_2\beta K_2\sin^2\phi}$ Resultant radial edge force $= \frac{\Delta_o Et}{R_2\beta K_2\sin^2\phi}$ $M_1 = \frac{-\Delta_o Et}{2\beta^2K_2\sin\phi}, \quad \sigma_1' = \frac{3\Delta_o E}{t\beta^2K_2\sin\phi}$ $\sigma_2 = \frac{\Delta_o E(K_1+K_2)}{2R_2K_2\sin\phi}, \quad M_2 = \frac{-\Delta_o Et\nu}{2\beta^2K_2\sin\phi}, \quad \sigma_2' = \frac{3\Delta_o E\nu}{t\beta^2K_2\sin\phi}$ $\psi = 0, \quad \Delta R = \Delta_o$

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

<table>
<tr><th>Case no., loading</th><th>Formulas</th></tr>
<tr><td>1d. Edge rotation, ψ_o rad; no edge displacement
 </td><td>

$$C = \frac{\psi_o Et\sqrt{(1+K_2^2)\sin\phi}}{2\beta^2 K_2}, \qquad \zeta = \tan^{-1}\frac{1}{K_2} \qquad \text{where } 0 < \zeta < \pi$$

At the edge where $\omega = 0$,

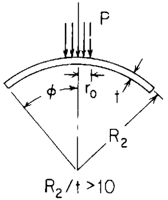
$$V_1 = \frac{-\psi_o Et}{2\beta^2 K_2}, \qquad \sigma_1 = \frac{-\psi_o E\cot\phi}{2\beta^2 K_2}$$

$$\text{Resultant radial edge force} = \frac{-\psi_o Et}{2\beta^2 K_2 \sin\phi}$$

$$M_1 = \frac{\psi_o EtR_2}{4\beta^3}\left(K_1 + \frac{1}{K_2}\right), \qquad \sigma_1' = \frac{-6M_1}{t^2}$$

$$\sigma_2 = \frac{-\psi_o E}{4\beta K_2}(K_1 - K_2), \qquad M_2 = \frac{\psi_o EtR_2}{8\nu\beta^3}\left[(1+\nu^2)(K_1+K_2) - 2K_2 + \frac{2\nu^2}{K_2}\right], \qquad \sigma_2' = \frac{-6M_2}{t^2}$$

$$\psi = \psi_o, \qquad \Delta R = 0$$

</td></tr>
<tr><td>2. Partial spherical shell, load P concentrated on small circular area of radius r_o at pole; any edge support


Note: $r_o' = \sqrt{1.6t_0^2 + t^2} - 0.675t$ if $r_o < 0.5t$; $r_o' = r_o$ if $r_o \geqslant 0.5t$ (see Sec. 11.1).
For $\phi > \sin^{-1}(1.65\sqrt{t/R_2})$</td><td>

Note: The deflection for this case is measured locally relative to the undeformed shell. It does not include any deformations due to the edge supports or membrane stresses remote from the loading. The formulas for deflection and stress are applicable also to off-axis loads if no edge is closer than $\phi = \sin^{-1}(1.65\sqrt{t/R_2})$. If an edge were as close as half this angle, the results would be modified very little.

$$\text{Deflection under the center of the load} = -A\frac{PR_2\sqrt{1-\nu^2}}{Et^2}$$

$$\text{Max membrane stress under the center of the load } \sigma_1 = \sigma_2 = -B\frac{P\sqrt{1-\nu^2}}{t^2}$$

Note: See also Ref. 72

$$\text{Max bending stress under the center of the load } \sigma_1' = \sigma_2' = -C\frac{P(1+\nu)}{t^2}$$

Here A, B, and C are numerical coefficients that depend upon

$\mu = r_o'\left[\frac{12(1-\nu^2)}{R_2^2 t^2}\right]^{1/4}$ and have the values tabulated below

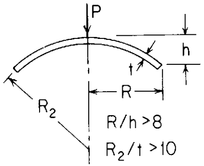
<table>
<tr><th>μ</th><th>0</th><th>0.1</th><th>0.2</th><th>0.4</th><th>0.6</th><th>0.8</th><th>1.0</th><th>1.2</th><th>1.4</th></tr>
<tr><td>A</td><td>0.433</td><td>0.431</td><td>0.425</td><td>0.408</td><td>0.386</td><td>0.362</td><td>0.337</td><td>0.311</td><td>0.286</td></tr>
<tr><td>B</td><td>0.217</td><td>0.215</td><td>0.212</td><td>0.204</td><td>0.193</td><td>0.181</td><td>0.168</td><td>0.155</td><td>0.143</td></tr>
<tr><td>C</td><td>∞</td><td>1.394</td><td>1.064</td><td>0.739</td><td>0.554</td><td>0.429</td><td>0.337</td><td>0.266</td><td>0.211</td></tr>
</table>

(Ref. 15)

</td></tr>
</table>

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

3. Shallow spherical shell, point load P at the pole



$h < \frac{R}{8}$

$t < \frac{R_2}{10}$

3a. Edge vertically supported and guided

Max deflection $y = -A_1 \frac{PR^2}{16\pi D}$

Edge moment $M_o = -B_1 \frac{P}{4\pi}$

3b. Edge fixed and held

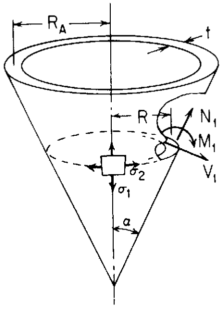
Max deflection $y = -A_2 \frac{PR^2}{16\pi D}$

Edge moment $M_o = -B_2 \frac{P}{4\pi}$

Here A and B are numerical coefficients that depend upon $\alpha = 2\left[\frac{3(1-\nu^2)h^2}{t^2}\right]^{1/4}$ and have the values tabulated below

α	0	1	2	3	4	5	6	7	8	9	10
A_1	1.000	0.996	0.935	0.754	0.406	0.321	0.210	0.148	0.111	0.085	0.069
B_1	1.000	0.995	0.932	0.746	0.498	0.324	0.234	0.192	0.168	0.153	0.140
A_2	1.000	0.985	0.817	0.515	0.320	0.220	0.161	0.122	0.095	0.075	0.061
B_2	1.000	0.975	0.690	0.191	−0.080	−0.140	−0.117	−0.080	−0.059	−0.034	−0.026

4. Long conical shells with edge loads. Expressions are accurate if $R/(t\cos\alpha) > 10$ and $|k| > 5$ everywhere in the region from the loaded end to the position where $\mu = 4$



$$k = \frac{2}{\sin\alpha}\left[\frac{12(1-\nu^2)R^2}{t^2\sec^2\alpha}\right]^{1/4}$$

$$\mu = \left|\frac{k_A - k}{\sqrt{2}}\right|$$

$$\beta = [12(1-\nu^2)]^{1/2}$$

$$l = 1 - \frac{1.326}{k} - \frac{0.218}{k^3} - \frac{0.317}{k^4}$$

$$m = \frac{1.326}{k} - \frac{0.820}{k^2} - \frac{0.218}{k^3}$$

$$s = 1 - \frac{1.679}{k} + \frac{1.233}{k^3} + \frac{0.759}{k^4}$$

$$f = \frac{1.679}{k} - \frac{3.633}{k^2} + \frac{1.233}{k^3}$$

$$C_1 = l_A(s_A - f_A) + m_A(s_A + f_A) + \frac{2\sqrt{2}\nu}{k_A}(l_A^2 + m_A^2)$$

(*Note:* The subscript A denotes that the quantity subscripted is evaluated at $R = R_A$)

$$\sigma_1 = \frac{N_1}{t}, \qquad \sigma_2 = \frac{N_2}{t}, \qquad \sigma_1' = \frac{-6M_1}{t^2}, \qquad \sigma_2' = \frac{-6M_2}{t^2}$$

$$V_1 = \frac{N_1}{\tan\alpha}$$

(*Note:* If the cone increases in radius below the section A, the angle α is negative, making k negative as well. As indicated for a position α, the positive values of N_1 and M_1 are as shown and V_1 is still positive when acting outward on the lower portion.)

$$F_1 = ml_A - lm_A$$
$$F_2 = ll_A + mm_A$$
$$F_3 = fs_A - sf_A$$
$$F_4 = ss_A + ff_A$$
$$F_5 = l(s_A - f_A) + m(s_A + f_A)$$
$$F_6 = l(s_A + f_A) - m(s_A - f_A)$$
$$F_7 = s(l_A - m_A) + f(l_A + m_A)$$
$$F_8 = s(l_A + m_A) - f(l_A - m_A)$$
$$F_9 = F_5 + \frac{2\sqrt{2}\nu}{k_A}F_2$$
$$F_{10} = F_6 - \frac{2\sqrt{2}\nu}{k_A}F_1$$
$$F_{11} = F_4 + \frac{\sqrt{2}\nu}{k_A}F_8$$
$$F_{12} = F_3 + \frac{\sqrt{2}\nu}{k_A}F_7$$

(*Note:* At sections where $\mu > 4$, the deformations and stresses have decreased to negligible values)

(Ref. 65)

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

For use at the loaded end where $R = R_A$,

$$F_{1A} = F_{3A} = 0$$

$$F_{4A} = 1 - \frac{3.359}{k_A} + \frac{5.641}{k_A^2} - \frac{9.737}{k_A^3} + \frac{14.716}{k_A^4}$$

$$F_{10A} = F_{7A} = F_{6A} = 1 - \frac{2.652}{k_A} + \frac{1.641}{k_A^2} - \frac{0.290}{k_A^3} - \frac{2.211}{k_A^4}$$

$$F_{2A} = 1 - \frac{2.652}{k_A} + \frac{3.516}{k_A^2} - \frac{2.610}{k_A^3} + \frac{0.038}{k_A^4}$$

$$F_{8A} = F_{5A} = 1 - \frac{3.359}{k_A} + \frac{7.266}{k_A^2} - \frac{10.068}{k_A^3} + \frac{5.787}{k_A^4}$$

$$F_{9A} = C_1 = F_{5A} + \frac{2\sqrt{2}\nu}{k_A} F_{2A}$$

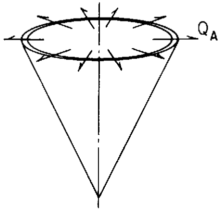
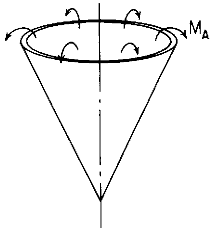
Case no., loading	Formulas
4a. Uniform radial force Q_A  	$N_1 = Q_A \sin\alpha \left(\frac{k_A}{k}\right)^{5/2} \frac{e^{-\mu}}{C_1}(F_9 \cos\mu - F_{10} \sin\mu)$ $N_2 = Q_A \sin\alpha \left(\frac{k_A}{k}\right)^{3/2} \frac{k_A}{\sqrt{2}} \frac{e^{-\mu}}{C_1}(F_{11} \cos\mu + F_{12} \sin\mu)$ $M_1 = Q_A \sin\alpha \left(\frac{k_A}{k}\right)^{3/2} \frac{k_A t}{\sqrt{2}\beta} \frac{e^{-\mu}}{C_1}\left[-\left(F_{12} - \frac{\sqrt{2}\nu}{k} F_{10}\right)\cos\mu + \left(F_{11} + \frac{\sqrt{2}\nu}{k} F_9\right)\sin\mu\right]$ $M_2 = Q_A \nu \sin\alpha \left(\frac{k_A}{k}\right)^{3/2} \frac{k_A t}{\sqrt{2}\beta} \frac{e^{-\mu}}{C_1}\left[-\left(F_{12} - \frac{\sqrt{2}}{\nu k} F_{10}\right)\cos\mu + \left(F_{11} + \frac{\sqrt{2}}{\nu k} F_9\right)\sin\mu\right]$ $\Delta R = \frac{Q_A R}{Et} \sin\alpha \left(\frac{k_A}{k}\right)^{3/2} \frac{k_A}{\sqrt{2}} \frac{e^{-\mu}}{C_1}\left[\left(F_{11} - \frac{\sqrt{2}\nu}{k} F_9\right)\cos\mu + \left(F_{12} + \frac{\sqrt{2}\nu}{k} F_{10}\right)\sin\mu\right]$ $\psi = \frac{Q_A R_A \beta}{Et^2}\left(\frac{k_A}{k}\right)^{1/2} \frac{e^{-\mu}}{C_1}(F_{10} \cos\mu + F_9 \sin\mu)$ At the loaded end where $R = R_A$, $N_{1A} = Q_A \sin\alpha, \qquad N_{2A} = Q_A \sin\alpha \frac{k_A}{\sqrt{2}C_1}\left(F_{4A} + \frac{\sqrt{2}\nu}{k_A} F_{8A}\right)$ $M_{1A} = 0, \quad M_{2A} = Q_A \sin\alpha(1 - \nu^2)\frac{t}{\beta C_1} F_{10A}$ $\Delta R_A = \frac{Q_A R_A \sin\alpha}{Et} \frac{k_A}{\sqrt{2}C_1}\left(F_{4A} - \frac{4\nu^2}{k_A^2} F_{2A}\right), \qquad \psi_A = \frac{Q_A R_A \beta}{Et^2 C_1} F_{10A}$

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

4b. Uniform edge moment M_A



$$N_1 = M_A\left(\frac{k_A}{k}\right)^{5/2}\frac{2\sqrt{2}\beta}{tk_A}\,\frac{e^{-\mu}}{C_1}(F_1\cos\mu - F_2\sin\mu)$$

$$N_2 = M_A\left(\frac{k_A}{k}\right)^{3/2}\frac{\beta}{t}\,\frac{e^{-\mu}}{C_1}(F_7\cos\mu - F_8\sin\mu)$$

$$M_1 = M_A\left(\frac{k_A}{k}\right)^{3/2}\frac{e^{-\mu}}{C_1}\left[\left(F_8 + \frac{2\sqrt{2}v}{k}F_2\right)\cos\mu + \left(F_7 + \frac{2\sqrt{2}v}{k}F_1\right)\sin\mu\right]$$

$$M_2 = M_A\left(\frac{k_A}{k}\right)^{3/2} v\frac{e^{-\mu}}{C_1}\left[\left(F_8 + \frac{2\sqrt{2}}{vk}F_2\right)\cos\mu + \left(F_7 + \frac{2\sqrt{2}}{vk}F_1\right)\sin\mu\right]$$

$$\Delta R = M_A\left(\frac{k_A}{k}\right)^{3/2}\frac{\beta R}{Et^2}\,\frac{e^{-\mu}}{C_1}\left[\left(F_7 - \frac{2\sqrt{2}v}{k}F_1\right)\cos\mu - \left(F_8 - \frac{2\sqrt{2}v}{k}F_2\right)\sin\mu\right]$$

$$\psi = M_A\left(\frac{k_A}{k}\right)^{1/2}\frac{2\sqrt{2}\beta^2 R_A}{Et^3 k_A\sin\alpha}\,\frac{e^{-\mu}}{C_1}(F_2\cos\mu + F_1\sin\mu)$$

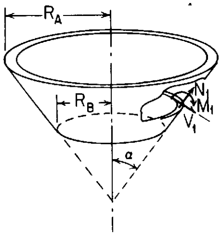
At the loaded end where $R = R_A$,

$$N_{1A} = 0, \qquad N_{2A} = M_A\frac{\beta}{tC_1}F_{7A}$$

$$M_{1A} = M_A, \qquad M_{2A} = M_A\left[v + \frac{2\sqrt{2}(1-v^2)}{k_A C_1}F_{2A}\right]$$

$$\Delta R_A = M_A\frac{\beta R_A}{Et^2 C_1}F_{7A}, \qquad \psi_A = M_A\frac{2\sqrt{2}\beta^2 R_A}{Et^3 k_A C_1\sin\alpha}F_{2A}$$

5. Short conical shells. Expressions are accurate if $R/(t\cos\alpha) > 10$ and $|k| > 5$ everywhere in the cone

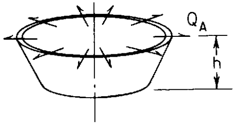


$$k = \frac{2}{\sin\alpha}\left[\frac{12(1-v^2)R^2}{t^2\sec^2\alpha}\right]^{1/4}, \qquad \sigma_1 = \frac{N_1}{t}, \qquad \sigma_2 = \frac{N_2}{t}$$

$$\mu_D = \left|\frac{k_A - k_B}{\sqrt{2}}\right|, \qquad \sigma_1' = \frac{-6M_1}{t^2}, \qquad \sigma_2' = \frac{-6M_2}{t^2}$$

$$\beta = [12(1-v^2)]^{1/2}, \qquad V_1 = \frac{N_1}{\tan\alpha}$$

5a. Uniform radial force Q_A at the large end



$$N_1 = Q_A\sin\alpha K_{N1}, \qquad N_2 = Q_A\sin\alpha\frac{k_A}{\sqrt{2}}K_{N2}$$

$$M_1 = Q_A\sin\alpha\frac{k_A t}{\sqrt{2}\beta}K_{M1}, \qquad M_2 = Q_A v\sin\alpha\frac{k_A t}{\sqrt{2}\beta}K_{M2}, \qquad \Delta h = \frac{Q_A R_A}{Et}K_{\Delta h1} - \frac{Q_A R_A\sin^2\alpha}{Et\cos\alpha}\,\frac{k_A}{\sqrt{2}}K_{\Delta h2}$$

$$\Delta R = \frac{Q_A R_A\sin\alpha}{Et}\,\frac{k_A}{\sqrt{2}}K_{\Delta R}, \qquad \psi = \frac{Q_A R_A\beta}{Et^2}K_\psi$$

For $R_B/(t\cos\alpha) > 10$ and $k_B > 5$ and for $v = 0.3$, the following tables give the values of K at several locations along the shell $[\Omega = (R_A - R)/(R_A - R_B)]$

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 9.434$	$K_{\Delta h1} = -0.266$	$K_{\Delta h2} = 2.979$		$\mu_D = 0.6$	$k_B = 9.151$	$K_{\Delta h1} = -0.266$		$K_{\Delta h2} = 2.899$	$\mu_D = 0.8$	$k_B = 8.869$	$K_{\Delta h1} = -0.269$		$K_{\Delta h2} = 2.587$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
10	K_{N1}	1.000	0.548	0.216	0.025	0.000	1.000	0.450	0.068	−0.102	0.000	1.000	0.392	−0.029	−0.192	0.000
	K_{N2}	2.748	2.056	1.319	0.533	−0.307	2.304	1.643	0.919	0.123	−0.761	1.952	1.371	0.720	−0.016	−0.862
	K_{M1}	0.000	0.047	0.054	0.034	0.000	0.000	0.076	0.084	0.047	0.000	0.000	0.106	0.114	0.060	0.000
	K_{M2}	3.279	3.445	3.576	3.690	3.801	2.151	2.340	2.467	2.566	2.675	1.468	1.671	1.780	1.848	1.941
	$K_{\Delta R}$	2.706	1.977	1.238	0.489	−0.274	2.262	1.558	0.842	0.111	−0.637	1.909	1.282	0.644	−0.006	−0.678
	K_{ψ}	7.644	7.703	7.759	7.819	7.887	5.014	5.064	5.106	5.156	5.223	3.421	3.454	3.470	3.500	3.559

		$\mu_D = 1.2$	$k_B = 8.303$	$K_{\Delta h1} = -0.278$	$K_{\Delta h2} = 1.986$		$\mu_D = 1.6$	$k_B = 7.737$	$K_{\Delta h1} = -0.287$		$K_{\Delta h2} = 1.571$	$\mu_D = 3.2$	$k_B = 5.475$	$K_{\Delta h1} = -0.323$		$K_{\Delta h2} = 0.953$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
10	K_{N1}	1.000	0.339	−0.133	−0.308	0.000	1.000	0.315	−0.185	−0.377	0.000	1.000	0.156	−0.301	−0.361	0.000
	K_{N2}	1.470	1.023	0.514	−0.080	−0.810	1.197	0.815	0.394	−0.082	−0.696	0.940	0.466	0.088	−0.097	−0.182
	K_{M1}	0.000	0.165	0.176	0.088	0.000	0.000	0.220	0.235	0.117	0.000	0.000	0.347	0.318	0.131	0.000
	K_{M2}	0.810	1.040	1.105	1.098	1.147	0.555	0.813	0.832	0.746	0.743	0.406	0.729	0.538	0.191	0.010
	$K_{\Delta R}$	1.428	0.931	0.439	−0.051	−0.559	1.155	0.721	0.321	−0.046	−0.416	0.898	0.379	0.066	−0.039	−0.055
	K_{ψ}	1.887	1.880	1.829	1.806	1.843	1.294	1.242	1.112	1.025	1.038	0.948	0.734	0.333	0.066	0.007

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 19.434$	$K_{\Delta h1} = -0.293$	$K_{\Delta h2} = 5.488$		$\mu_D = 0.6$	$k_B = 19.151$	$K_{\Delta h1} = -0.294$		$K_{\Delta h2} = 4.247$	$\mu_D = 0.8$	$k_B = 18.869$	$K_{\Delta h1} = -0.296$		$K_{\Delta h2} = 3.368$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
20	K_{N1}	1.000	0.346	−0.053	−0.175	0.000	1.000	0.281	−0.149	−0.256	0.000	1.000	0.255	−0.193	−0.296	0.000
	K_{N2}	4.007	2.674	1.298	−0.123	−1.591	2.975	1.954	0.885	−0.234	−1.410	2.334	1.528	0.677	−0.244	−1.186
	K_{M1}	0.000	0.052	0.052	0.025	0.000	0.000	0.082	0.078	0.035	0.000	0.000	0.111	0.106	0.045	0.000
	K_{M2}	2.974	3.079	3.133	3.163	3.198	1.546	1.668	1.704	1.703	1.717	0.933	1.072	1.093	1.064	1.059
	$K_{\Delta R}$	3.985	2.629	1.263	−0.114	−1.052	2.954	1.908	0.852	−0.214	−1.293	2.313	1.481	0.644	−0.200	−1.056
	K_ψ	13.866	13.918	13.967	14.020	14.079	7.210	7.241	7.263	7.293	7.338	4.349	4.359	4.351	4.360	4.393

		$\mu_D = 1.2$	$k_B = 18.303$	$K_{\Delta h1} = -0.298$	$K_{\Delta h2} = 2.334$		$\mu_D = 1.6$	$k_B = 17.737$	$K_{\Delta h1} = -0.300$		$K_{\Delta h2} = 1.777$	$\mu_D = 3.2$	$k_B = 15.475$	$K_{\Delta h1} = -0.305$		$K_{\Delta h2} = 1.023$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
20	K_{N1}	1.000	0.236	−0.230	−0.337	0.000	1.000	0.222	−0.247	−0.352	0.000	1.000	0.049	−0.274	−0.215	0.000
	K_{N2}	1.629	1.059	0.460	−0.175	−0.867	1.283	0.808	0.334	−0.139	−0.655	0.977	0.390	0.027	−0.094	−0.112
	K_{M1}	0.000	0.168	0.160	0.066	0.000	0.000	0.222	0.211	0.087	0.000	0.000	0.342	0.253	0.079	0.000
	K_{M2}	0.458	0.639	0.634	0.553	0.514	0.299	0.519	0.489	0.359	0.289	0.211	0.507	0.322	0.089	−0.002
	$K_{\Delta R}$	1.608	1.011	0.427	−0.147	−0.726	1.262	0.761	0.303	−0.111	−0.515	0.956	0.350	0.026	−0.062	−0.067
	K_ψ	2.137	2.105	2.031	1.992	2.005	1.392	1.313	1.160	1.065	1.061	0.938	0.692	0.259	0.032	−0.007

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 39.434$	$K_{\Delta h1} = -0.299$	$K_{\Delta h2} = 6.866$		$\mu_D = 0.6$	$k_B = 39.151$	$K_{\Delta h1} = -0.299$	$K_{\Delta h2} = 4.776$		$\mu_D = 0.8$	$k_B = 38.869$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 3.634$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
40	K_{N1}	1.000	0.238	−0.192	−0.275	0.000	1.000	0.216	−0.225	−0.304	0.000	1.000	0.209	−0.238	−0.317	0.000
	K_{N2}	4.692	2.998	1.277	−0.471	−2.248	3.234	1.060	0.860	−0.366	−1.621	2.459	1.565	0.649	−0.289	−1.255
	K_{M1}	0.000	0.055	0.051	0.021	0.000	0.000	0.084	0.077	0.030	0.000	0.000	0.112	0.103	0.040	0.000
	K_{M2}	1.851	1.922	1.934	1.921	1.917	0.863	0.957	0.959	0.924	0.906	0.498	0.617	0.611	0.555	0.524
	$K_{\Delta R}$	4.681	2.974	1.261	−0.458	−2.185	3.223	2.036	0.844	−0.351	−1.553	2.448	1.541	0.634	−0.273	−1.185
	K_{ψ}	17.256	17.286	17.311	17.341	17.377	8.048	8.058	8.059	8.069	8.092	4.645	4.637	4.612	4.603	4.617

		$\mu_D = 1.2$	$k_B = 38.303$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 2.446$		$\mu_D = 1.6$	$k_B = 37.737$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 1.847$		$\mu_D = 3.2$	$k_B = 35.475$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 1.053$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
40	K_{N1}	1.000	0.202	−0.249	−0.327	0.000	1.000	0.189	−0.253	−0.324	0.000	1.000	0.012	−0.245	−0.165	0.000
	K_{N2}	1.677	1.055	0.430	−0.200	−0.850	1.309	0.792	0.304	−0.153	−0.616	0.991	0.352	0.007	−0.087	−0.092
	K_{M1}	0.000	0.168	0.154	0.060	0.000	0.000	0.221	0.200	0.077	0.000	0.000	0.333	0.223	0.063	0.000
	K_{M2}	0.237	0.405	0.385	0.291	0.236	0.152	0.367	0.331	0.201	0.126	0.107	0.408	0.250	0.066	−0.001
	$K_{\Delta R}$	1.666	1.031	0.415	−0.184	−0.780	1.299	0.768	0.290	−0.137	−0.548	0.981	0.333	0.009	−0.072	−0.072
	K_{ψ}	2.206	2.159	2.071	2.021	2.022	1.419	1.324	1.160	1.061	1.048	0.994	0.665	0.229	0.023	−0.009

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 79.434$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 7.324$		$\mu_D = 0.6$	$k_B = 79.151$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 4.935$		$\mu_D = 0.8$	$k_B = 78.869$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 3.713$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
80	K_{N1}	1.000	0.202	−0.235	−0.305	0.000	1.000	0.196	−0.245	−0.313	0.000	1.000	0.194	−0.248	−0.317	0.000
	K_{N2}	4.917	3.098	1.264	−0.584	−2.446	3.309	2.083	0.846	−0.403	−1.667	2.494	1.568	0.635	−0.306	−1.259
	K_{M1}	0.000	0.056	0.050	0.019	0.000	0.000	0.084	0.076	0.029	0.000	0.000	0.112	0.101	0.038	0.000
	K_{M2}	0.985	1.045	1.043	1.017	1.002	0.444	0.531	0.524	0.479	0.454	0.253	0.367	0.355	0.293	0.257
	$K_{\Delta R}$	4.912	3.086	1.256	−0.576	−2.412	3.303	2.071	0.838	−0.395	−1.631	2.489	1.556	0.627	−0.298	−1.224
	K_{ψ}	18.365	18.378	18.386	18.399	18.416	8.287	8.285	8.272	8.270	8.280	4.725	4.707	4.672	4.654	4.658

		$\mu_D = 1.2$	$k_B = 78.303$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 2.483$		$\mu_D = 1.6$	$k_B = 77.737$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 1.873$		$\mu_D = 3.2$	$k_B = 75.475$	$K_{\Delta h1} = -0.298$	$K_{\Delta h2} = 1.067$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
80	K_{N1}	1.000	0.189	−0.251	−0.318	0.000	1.000	0.176	−0.252	−0.308	0.000	1.000	−0.002	−0.230	−0.145	0.000
	K_{N2}	1.691	1.047	0.415	−0.206	−0.832	1.318	0.781	0.290	−0.157	−0.593	0.997	0.334	0.000	−0.083	−0.084
	K_{M1}	0.000	0.168	0.150	0.057	0.000	0.000	0.220	0.194	0.073	0.000	0.000	0.327	0.208	0.056	0.000
	K_{M2}	0.119	0.285	0.264	0.169	0.113	0.076	0.292	0.258	0.132	0.059	0.053	0.363	0.221	0.057	−0.001
	$K_{\Delta R}$	1.686	1.035	0.408	−0.198	−0.797	1.313	0.769	0.283	−0.149	−0.560	0.992	0.324	0.001	−0.076	−0.075
	K_{ψ}	2.224	2.170	2.075	2.019	2.014	1.426	1.324	1.155	1.053	1.037	0.998	0.651	0.216	0.020	−0.009

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 159.434$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 7.452$		$\mu_D = 0.6$	$k_B = 159.151$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 4.980$		$\mu_D = 0.8$	$k_B = 158.869$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 3.738$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
160	K_{N1}	1.000	0.192	−0.247	−0.312	0.000	1.000	0.190	−0.249	−0.314	0.000	1.000	0.189	−0.250	−0.315	0.000
	K_{N2}	4.979	3.121	1.257	−0.164	−2.493	3.329	2.086	0.839	−0.413	−1.671	2.505	1.566	0.628	−0.311	−1.254
	K_{M1}	0.000	0.056	0.050	0.019	0.000	0.000	0.084	0.075	0.028	0.000	0.000	0.112	0.100	0.038	0.000
	K_{M2}	0.500	0.558	0.552	0.522	0.504	0.224	0.309	0.300	0.253	0.225	0.127	0.240	0.227	0.164	0.127
	$K_{\Delta R}$	4.977	3.115	1.253	−0.610	−2.475	3.327	2.080	0.835	−0.409	−1.654	2.502	1.560	0.624	−0.307	−1.236
	K_ψ	18.666	18.668	18.667	18.670	18.678	8.350	8.341	8.322	8.313	8.316	4.746	4.724	4.683	4.660	4.660

		$\mu_D = 1.2$	$k_B = 158.303$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 2.497$		$\mu_D = 1.6$	$k_B = 157.737$	$K_{\Delta h1} = -0.300$	$K_{\Delta h2} = 1.884$		$\mu_D = 3.2$	$k_B = 155.475$	$K_{\Delta h1} = -0.298$	$K_{\Delta h2} = 1.073$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
160	K_{N1}	1.000	0.185	−0.251	−0.312	0.000	1.000	0.171	−0.250	−0.300	0.000	1.000	−0.008	−0.223	−0.136	0.000
	K_{N2}	1.696	1.042	0.408	−0.208	−0.821	1.322	0.775	0.283	−0.158	−0.582	1.000	0.325	−0.003	−0.081	−0.080
	K_{M1}	0.000	0.168	0.149	0.056	0.000	0.000	0.219	0.191	0.071	0.000	0.000	0.324	0.201	0.053	0.000
	K_{M2}	0.060	0.226	0.205	0.111	0.055	0.038	0.255	0.223	0.100	0.028	0.027	0.341	0.207	0.054	−0.000
	$K_{\Delta R}$	1.693	1.036	0.404	−0.204	−0.804	1.319	0.769	0.280	−0.154	−0.565	0.998	0.320	−0.003	−0.077	−0.076
	K_ψ	2.230	2.171	2.074	2.015	2.006	1.429	1.323	1.151	1.048	1.030	0.999	0.644	0.210	0.018	−0.010

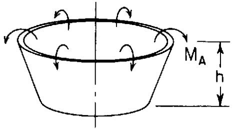
Case no., loading	Formulas
5b. Uniform edge moment M_A at the large end  	$N_1 = M_A \frac{2\sqrt{2}\beta}{tk_A} K_{N1}, \qquad N_2 = M_A \frac{\beta}{t} K_{N2}$ $M_1 = M_A K_{M1}, \qquad M_2 = M_A \nu K_{M2}$ $\Delta R = M_A \frac{\beta R_A}{Et^2} K_{\Delta R}, \qquad \psi = M_A \frac{2\sqrt{2}\beta^2 R_A}{Et^3 k_A \sin\alpha} K_\psi, \qquad \Delta h = \frac{M_A R_A \beta}{Et^2 \sin\alpha} K_{\Delta h1} - \frac{M_A \beta R_A \sin\alpha}{Et^2 \cos\alpha} K_{\Delta h2}$ For $R_B/(t\cos\alpha) > 10$ and $k_B > 5$ and for $\nu = 0.3$, the following tables give the values of K at several locations along the shell $[\Omega = (R_A - R)/(R_A - R_B)]$

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 9.434$	$K_{\Delta h1} = -0.017$	$K_{\Delta h2} = 15.507$		$\mu_D = 0.6$	$k_B = 9.151$	$K_{\Delta h1} = -0.026$		$K_{\Delta h2} = 9.766$	$\mu_D = 0.8$	$k_B = 8.869$	$K_{\Delta h1} = -0.031$		$K_{\Delta h2} = 6.554$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
10	K_{N1}	0.000	−0.584	−0.819	−0.647	0.000	0.000	−0.579	−0.830	−0.672	0.000	0.000	−0.527	−0.772	−0.641	0.000
	K_{N2}	7.644	4.020	0.170	−3.934	−8.328	5.014	2.687	0.163	−2.604	−5.674	3.421	1.851	0.138	−1.780	−3.983
	K_{M1}	1.000	0.816	0.538	0.238	0.000	1.000	0.844	0.554	0.229	0.000	1.000	0.865	0.568	0.226	0.000
	K_{M2}	17.470	17.815	18.122	18.475	18.961	8.300	8.428	8.481	8.581	8.867	4.850	4.836	4.718	4.644	4.792
	$K_{\Delta R}$	7.644	3.957	0.227	−3.559	−7.413	5.014	2.625	0.215	−2.236	−4.752	3.421	1.795	0.181	−1.449	−3.133
	K_{ψ}	19.197	19.268	19.368	19.503	19.671	8.509	8.480	8.489	8.548	8.656	4.488	4.381	4.320	4.325	4.393

		$\mu_D = 1.2$	$k_B = 8.303$	$K_{\Delta h1} = -0.038$	$K_{\Delta h2} = 3.401$		$\mu_D = 1.6$	$k_B = 7.737$	$K_{\Delta h1} = -0.042$		$K_{\Delta h2} = 2.086$	$\mu_D = 3.2$	$k_B = 5.475$	$K_{\Delta h1} = -0.050$		$K_{\Delta h2} = 0.953$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
10	K_{N1}	0.000	−0.429	−0.642	−0.554	0.000	0.000	−0.370	−0.547	−0.476	0.000	0.000	−0.375	−0.389	−0.193	0.000
	K_{N2}	1.887	0.993	0.070	−0.951	−2.196	1.294	0.602	−0.004	−0.594	−1.323	0.948	0.127	−0.236	−0.240	−0.018
	K_{M1}	1.000	0.890	0.590	0.227	0.000	1.000	0.899	0.596	0.225	0.000	1.000	0.816	0.395	0.071	0.000
	K_{M2}	2.623	2.444	2.119	1.834	1.814	2.052	1.772	1.319	0.904	0.784	1.833	1.258	0.479	−0.058	−0.262
	$K_{\Delta R}$	1.887	0.949	0.106	−0.694	−1.514	1.294	0.570	0.034	−0.387	−0.792	0.948	0.131	−0.132	−0.106	−0.005
	K_{ψ}	1.892	1.670	1.506	1.436	1.458	1.226	0.915	0.674	0.553	0.547	0.971	0.425	0.064	−0.071	−0.091

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 19.434$	$K_{\Delta h1} = -0.008$	$K_{\Delta h2} = 27.521$		$\mu_D = 0.6$	$k_B = 19.151$	$K_{\Delta h1} = -0.009$	$K_{\Delta h2} = 14.217$		$\mu_D = 0.8$	$k_B = 18.869$	$K_{\Delta h1} = -0.010$	$K_{\Delta h2} = 8.477$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
20	K_{N1}	0.000	−1.050	−1.435	−1.104	0.000	0.000	−0.821	−1.135	−0.883	0.000	0.000	−0.660	−0.919	−0.722	0.000
	K_{N2}	13.866	7.110	0.149	−7.036	−14.461	7.210	3.726	0.107	−3.672	−7.642	4.349	2.246	0.072	−2.210	−4.638
	K_{M1}	1.000	0.832	0.517	0.191	0.000	1.000	0.848	0.524	0.183	0.000	1.000	0.858	0.530	0.181	0.000
	K_{M2}	15.905	15.971	15.910	15.860	15.967	6.219	6.150	5.930	5.721	5.697	3.421	3.286	2.988	2.703	2.617
	$K_{\Delta R}$	13.866	7.055	0.204	−6.696	−13.655	7.210	3.683	0.149	−3.408	−7.007	4.349	2.212	0.105	−1.999	−4.128
	K_{ψ}	34.745	34.800	34.881	34.998	35.145	12.165	12.103	12.079	12.107	12.178	5.643	5.504	5.415	5.395	5.431

		$\mu_D = 1.2$	$k_B = 18.303$	$K_{\Delta h1} = -0.010$	$K_{\Delta h2} = 3.962$		$\mu_D = 1.6$	$k_B = 17.737$	$K_{\Delta h1} = -0.011$	$K_{\Delta h2} = 2.326$		$\mu_D = 3.2$	$k_B = 15.475$	$K_{\Delta h1} = -0.011$	$K_{\Delta h2} = 0.978$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
20	K_{N1}	0.000	−0.476	−0.662	−0.524	0.000	0.000	−0.389	−0.519	−0.401	0.000	0.000	−0.357	−0.279	−0.091	0.000
	K_{N2}	2.137	1.051	0.008	−1.045	−2.179	1.392	0.587	−0.055	−0.611	−1.187	0.983	0.048	−0.240	−0.177	0.010
	K_{M1}	1.000	0.867	0.537	0.180	0.000	1.000	0.865	0.529	0.175	0.000	1.000	0.730	0.287	0.040	0.000
	K_{M2}	1.899	1.692	1.314	0.954	0.814	1.552	1.292	0.861	0.466	0.305	1.425	0.906	0.297	−0.011	−0.067
	$K_{\Delta R}$	2.137	1.027	0.033	−0.898	−1.825	1.392	0.571	−0.029	−0.499	−0.934	0.983	0.057	−0.182	−0.121	0.006
	K_{ψ}	2.096	1.846	1.664	1.585	1.588	1.286	0.943	0.690	0.571	0.559	0.992	0.369	0.019	−0.083	−0.094

		$\mu_D = 0.4$	$k_B = 39.434$	$K_{\Delta h1} = -0.002$	$K_{\Delta h2} = 34.371$		$\mu_D = 0.6$	$k_B = 39.151$	$K_{\Delta h1} = -0.002$		$K_{\Delta h2} = 15.957$	$\mu_D = 0.8$	$k_B = 38.869$	$K_{\Delta h1} = -0.002$		$K_{\Delta h2} = 9.123$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
40	K_{N1}	0.000	−1.300	−1.755	−1.333	0.000	0.000	−0.910	−1.234	−0.942	0.000	0.000	−0.698	−0.948	−0.726	0.000
	K_{N2}	17.256	8.734	0.089	−8.689	−17.610	8.048	4.080	0.052	−4.054	−8.255	4.645	2.339	0.024	−2.327	−4.743
	K_{M1}	1.000	0.842	0.508	0.168	0.000	1.000	0.849	0.511	0.166	0.000	1.000	0.852	0.514	0.165	0.000
	K_{M2}	10.272	10.179	9.917	9.657	9.577	3.910	3.768	3.449	3.134	3.009	2.290	2.125	1.780	1.441	1.298
	$K_{\Delta R}$	17.256	8.700	0.124	−8.478	−17.115	8.048	4.056	0.076	−3.907	−7.909	4.645	2.321	0.043	−2.214	−4.478
	K_{ψ}	43.228	43.226	43.250	43.310	43.397	13.566	13.468	13.409	13.403	13.438	6.015	5.852	5.741	5.700	5.714

		$\mu_D = 1.2$	$k_B = 38.303$	$K_{\Delta h1} = -0.003$	$K_{\Delta h2} = 4.136$		$\mu_D = 1.6$	$k_B = 37.737$	$K_{\Delta h1} = -0.003$		$K_{\Delta h2} = 2.404$	$\mu_D = 3.2$	$k_B = 34.475$	$K_{\Delta h1} = -0.003$		$K_{\Delta h2} = 0.986$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
40	K_{N1}	0.000	−0.484	−0.649	−0.494	0.000	0.000	−3.388	−0.494	−0.362	0.000	0.000	−0.341	−0.233	−0.065	0.000
	K_{N2}	2.206	1.043	−0.026	−1.055	−2.105	1.419	0.565	−0.079	−0.603	−1.107	0.994	0.014	−0.231	−0.150	0.011
	K_{M1}	1.000	0.853	0.513	0.164	0.000	1.000	0.845	0.498	0.156	0.000	1.000	0.684	0.243	0.030	0.000
	K_{M2}	1.462	1.267	0.892	0.532	0.375	1.280	1.053	0.654	0.289	0.133	1.214	0.762	0.244	0.009	−0.025
	$K_{\Delta R}$	2.206	1.032	−0.012	−0.980	−1.930	1.419	0.557	−0.065	−0.547	−0.985	0.994	0.020	−0.201	−0.125	0.009
	K_{ψ}	2.154	1.888	1.696	1.610	1.603	1.303	0.944	0.687	0.569	0.552	0.999	0.342	0.005	−0.084	−0.091

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

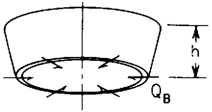
		$\mu_D = 0.4$	$k_B = 79.434$	$K_{\Delta h1} = -0.001$	$K_{\Delta h2} = 36.643$		$\mu_D = 0.6$	$k_B = 79.151$	$K_{\Delta h1} = -0.001$		$K_{\Delta h2} = 16.473$	$\mu_D = 0.8$	$k_B = 78.869$	$K_{\Delta h1} = -0.001$		$K_{\Delta h2} = 9.313$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
80	K_{N1}	0.000	−1.380	−1.851	−1.397	0.000	0.000	−0.934	−1.254	−0.948	0.000	0.000	−0.707	−0.948	−0.716	0.000
	K_{N2}	18.365	9.234	0.044	−9.211	−18.539	8.287	4.159	0.019	−4.150	−8.363	4.725	2.348	−0.001	−2.348	−4.720
	K_{M1}	1.000	0.845	0.504	0.160	−0.000	1.000	0.847	0.505	0.160	0.000	1.000	0.848	0.506	0.160	0.000
	K_{M2}	5.934	5.791	5.466	5.142	5.004	2.498	2.340	1.998	1.657	1.507	1.656	1.489	1.138	0.791	0.636
	$K_{\Delta R}$	18.365	9.216	0.063	−9.099	−18.277	8.287	4.147	0.032	−4.075	−8.186	4.725	2.339	0.009	−2.291	−4.588
	K_{ψ}	46.009	45.963	45.944	45.961	46.004	13.967	13.848	13.768	13.742	13.755	6.117	5.941	5.818	5.765	5.767

		$\mu_D = 1.2$	$k_B = 78.303$	$K_{\Delta h1} = -0.001$	$K_{\Delta h2} = 4.192$		$\mu_D = 1.6$	$k_B = 77.737$	$K_{\Delta h1} = -0.001$		$K_{\Delta h2} = 2.432$	$\mu_D = 3.2$	$k_B = 75.475$	$K_{\Delta h1} = -0.001$		$K_{\Delta h2} = 0.988$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
80	K_{N1}	0.000	−0.485	−0.637	−0.476	0.000	0.000	−0.386	−0.479	−0.342	0.000	0.000	−0.331	−0.212	−0.055	0.000
	K_{N2}	2.224	1.031	−0.043	−1.052	−2.054	1.426	0.551	−0.091	−0.596	−1.065	0.998	−0.001	−0.224	−0.138	0.010
	K_{M1}	1.000	0.845	0.501	0.157	0.000	1.000	0.834	0.483	0.148	0.000	1.000	0.660	0.224	0.027	0.000
	K_{M2}	1.233	1.051	0.687	0.335	0.179	1.140	0.937	0.558	0.211	0.062	1.107	0.697	0.223	0.017	−0.011
	$K_{\Delta R}$	2.224	1.025	−0.035	−1.014	−1.968	1.426	0.547	−0.083	−0.568	−1.005	0.998	0.002	−0.210	−0.126	0.009
	K_{ψ}	2.170	1.896	1.699	1.609	1.598	1.309	0.942	0.683	0.565	0.547	1.002	0.329	−0.001	−0.083	−0.089

		$\mu_D = 0.4$	$k_B = 159.434$	$K_{\Delta h1} = -0.000$	$K_{\Delta h2} = 37.272$		$\mu_D = 0.6$	$k_B = 159.151$	$K_{\Delta h1} = -0.000$	$K_{\Delta h2} = 16.619$		$\mu_D = 0.8$	$k_B = 158.869$	$K_{\Delta h1} = -0.000$	$K_{\Delta h2} = 9.371$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
160	K_{N1}	0.000	−1.401	−1.873	−1.409	−0.000	0.000	−0.939	−1.225	−0.944	0.000	0.000	−0.708	−0.944	−0.709	0.000
	K_{N2}	18.666	9.353	0.019	−9.342	−18.739	8.350	4.170	0.002	−4.169	−8.357	4.746	2.343	−0.014	−2.350	−4.691
	K_{M1}	1.000	0.845	0.502	0.158	0.000	1.000	0.845	0.502	0.158	0.000	1.000	0.845	0.502	0.157	0.000
	K_{M2}	3.508	3.353	3.012	2.672	2.520	1.755	1.595	1.249	0.904	0.749	1.329	1.166	0.817	0.470	0.314
	$K_{\Delta R}$	18.666	9.344	0.029	−9.285	−18.607	8.350	4.164	0.009	−4.131	−8.269	4.746	2.338	−0.009	−2.321	−4.625
	K_ψ	46.763	46.693	46.650	46.643	46.662	14.074	13.943	13.852	13.815	13.818	6.145	5.962	5.833	5.775	5.770

		$\mu_D = 1.2$	$k_B = 158.303$	$K_{\Delta h1} = -0.000$	$K_{\Delta h2} = 4.213$		$\mu_D = 1.6$	$k_B = 157.737$	$K_{\Delta h1} = -0.000$	$K_{\Delta h2} = 2.443$		$\mu_D = 3.2$	$k_B = 155.475$	$K_{\Delta h1} = -0.000$	$K_{\Delta h2} = 0.990$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
160	K_{N1}	0.000	−0.484	−0.630	−0.466	0.000	1.000	−0.385	−0.471	−0.332	0.000	0.000	−0.325	−0.202	−0.051	0.000
	K_{N2}	2.230	1.022	−0.051	−1.048	−2.026	1.429	0.543	−0.096	−0.592	−1.044	0.999	−0.008	−0.221	−0.133	0.010
	K_{M1}	1.000	0.841	0.495	0.154	0.000	1.000	0.829	0.475	0.144	0.000	1.000	0.649	0.214	0.025	0.000
	K_{M2}	1.117	0.943	0.587	0.242	0.087	1.070	0.879	0.512	0.175	0.030	1.054	0.666	0.214	0.020	−0.005
	$K_{\Delta R}$	2.230	1.020	−0.047	−1.029	−1.983	1.429	0.542	−0.092	−0.577	−1.014	0.999	−0.007	−0.214	−0.127	0.009
	K_ψ	2.175	1.897	1.698	1.606	1.593	1.311	0.940	0.680	0.562	0.543	1.003	0.323	−0.004	−0.083	−0.088

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

Case no., loading	Formulas
5c. Uniform radial force Q_B, at the small end 	$N_1 = Q_B \sin\alpha K_{N1}$, $N_2 = -Q_B \sin\alpha \dfrac{k_B}{\sqrt{2}} K_{N2}$ $M_1 = -Q_B \sin\alpha \dfrac{k_B t}{\sqrt{2}\beta} K_{M1}$, $M_2 = -Q_B \nu \sin\alpha \dfrac{k_B t}{\sqrt{2}\beta} K_{M2}$, $\Delta h = \dfrac{Q_B R_B}{Et} K_{\Delta h1} - \dfrac{Q_B R_B \sin^2\alpha}{Et\cos\alpha} \dfrac{k_B}{\sqrt{2}} K_{\Delta h2}$ $\Delta R = \dfrac{-Q_B R_B \sin\alpha}{Et} \dfrac{k_B}{\sqrt{2}} K_{\Delta R}$, $\psi = \dfrac{Q_B R_B \beta}{Et^2} K_\psi$ For $R_B/(t\cos\alpha) > 10$ and $k_B > 5$ and for $\nu = 0.3$, the following tables give the values of K at several locations along the shell $[\Omega = (R_A - R)/(R_A - R_B)]$

		$\mu_D = 0.4$	$k_B = 9.434$	$K_{\Delta h1} = 0.335$	$K_{\Delta h2} = 3.162$		$\mu_D = 0.6$	$k_B = 9.151$	$K_{\Delta h1} = 0.036$	$K_{\Delta h2} = 3.129$		$\mu_D = 0.8$	$k_B = 8.869$	$K_{\Delta h1} = 0.336$	$K_{\Delta h2} = 2.850$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
10	K_{N1}	0.000	0.016	0.172	0.491	1.000	0.000	−0.075	0.033	0.372	1.000	0.000	−0.122	−0.048	0.292	1.000
	K_{N2}	−0.258	0.436	1.178	1.973	2.827	−0.583	0.047	0.744	1.519	2.387	−0.601	−0.077	0.523	1.219	2.037
	K_{M1}	0.000	0.026	0.046	0.043	0.000	0.000	0.032	0.066	0.069	0.000	0.000	0.037	0.085	0.095	0.000
	K_{M2}	−3.000	−3.086	−3.188	−3.321	−3.502	−1.866	−1.939	−2.026	−2.159	−2.376	−1.192	−1.246	−1.310	−1.434	−1.678
	$K_{\Delta R}$	−0.290	0.478	1.259	2.057	2.872	−0.696	0.050	0.818	1.611	2.433	−0.764	−0.100	0.591	1.317	2.085
	K_ψ	7.413	7.478	7.550	7.627	7.702	4.752	4.817	4.895	4.985	5.070	3.133	3.192	3.274	3.376	3.469

		$\mu_D = 1.2$	$k_B = 8.303$	$K_{\Delta h1} = 0.336$	$K_{\Delta h2} = 2.283$		$\mu_D = 1.6$	$k_B = 7.737$	$K_{\Delta h1} = 0.340$	$K_{\Delta h2} = 1.882$		$\mu_D = 3.2$	$k_B = 5.475$	$K_{\Delta h1} = 0.431$	$K_{\Delta h2} = 1.213$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
10	K_{N1}	0.000	−0.148	−0.120	0.197	1.000	0.000	−0.135	−0.138	0.134	1.000	0.000	−0.026	−0.065	−0.039	1.000
	K_{N2}	−0.464	−0.125	0.295	0.837	1.559	−0.322	−0.113	0.172	0.604	1.288	−0.030	−0.034	−0.020	0.137	1.036
	K_{M1}	0.000	0.043	0.155	0.144	0.000	0.000	0.045	0.134	0.185	0.000	0.000	0.014	0.073	0.190	0.000
	K_{M2}	−0.539	−0.560	−0.583	−0.691	−0.992	−0.263	−0.263	−0.258	−0.358	−0.729	−0.001	0.010	0.034	−0.019	−0.721
	$K_{\Delta R}$	−0.673	−0.177	0.354	0.943	1.610	−0.538	−0.181	0.220	0.713	1.343	−0.100	−0.098	−0.053	0.213	1.113
	K_ψ	1.514	1.562	1.657	1.798	1.920	0.792	0.833	0.945	1.142	1.314	0.005	0.016	0.108	0.422	0.920

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 19.434$	$K_{\Delta h1} = 0.308$	$K_{\Delta h2} = 5.616$		$\mu_D = 0.6$	$k_B = 19.151$	$K_{\Delta h1} = 0.306$		$K_{\Delta h2} = 4.390$	$\mu_D = 0.8$	$k_B = 18.869$	$K_{\Delta h1} = 0.306$		$K_{\Delta h2} = 3.518$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
20	K_{N1}	0.000	−0.157	−0.057	0.322	1.000	0.000	−0.216	−0.145	0.247	1.000	0.000	−0.235	−0.181	0.211	1.000
	K_{N2}	−1.460	−0.151	1.201	2.600	4.048	−1.238	−0.255	0.778	1.867	3.018	−0.996	−0.242	0.565	1.435	2.377
	K_{M1}	0.000	0.022	0.048	0.051	0.000	0.000	0.029	0.071	0.079	0.000	0.000	0.036	0.093	0.107	0.000
	K_{M2}	−2.846	−2.877	−2.906	−2.962	−3.072	−1.439	−1.450	−1.454	−1.495	−1.624	−0.835	−0.831	−0.813	−0.844	−0.996
	$K_{\Delta R}$	−1.546	−0.162	1.235	2.645	4.070	−1.350	−0.278	0.810	1.915	3.040	−1.119	−0.271	0.595	1.484	2.400
	K_ψ	13.655	13.714	13.781	13.851	13.918	7.007	7.055	7.116	7.188	7.251	4.128	4.169	4.232	4.314	4.381

		$\mu_D = 1.2$	$k_B = 18.303$	$K_{\Delta h1} = 0.305$	$K_{\Delta h2} = 2.490$		$\mu_D = 1.6$	$k_B = 17.737$	$K_{\Delta h1} = 0.306$		$K_{\Delta h2} = 1.932$	$\mu_D = 3.2$	$k_B = 15.475$	$K_{\Delta h1} = 0.313$		$K_{\Delta h2} = 1.138$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
20	K_{N1}	0.000	−0.236	−0.205	0.171	1.000	0.000	−0.217	−0.208	0.138	1.000	0.000	−0.069	−0.144	−0.046	1.000
	K_{N2}	−0.664	−0.189	0.344	0.959	1.673	−0.457	−0.149	0.221	0.701	1.327	−0.052	−0.059	−0.022	0.234	1.023
	K_{M1}	0.000	0.048	0.133	0.160	0.000	0.000	0.057	0.165	0.207	0.000	0.000	0.030	0.137	0.277	0.000
	K_{M2}	−0.358	−0.332	−0.281	−0.302	−0.506	−0.178	−0.138	−0.065	−0.080	−0.341	0.001	0.028	0.105	0.145	−0.276
	$K_{\Delta R}$	−0.793	−0.223	0.372	1.010	1.696	−0.581	−0.185	0.246	0.752	1.351	−0.087	−0.092	−0.035	0.272	1.051
	K_ψ	1.824	1.859	1.940	2.066	2.160	0.934	0.967	1.078	1.268	1.410	−0.006	0.012	0.155	0.557	0.994

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 39.434$	$K_{\Delta h1} = 0.301$	$K_{\Delta h2} = 6.940$		$\mu_D = 0.6$	$k_B = 39.151$	$K_{\Delta h1} = 0.301$		$K_{\Delta h2} = 4.853$	$\mu_D = 0.8$	$k_B = 38.869$	$K_{\Delta h1} = 0.301$		$K_{\Delta h2} = 3.711$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
40	K_{N1}	0.000	−0.260	−0.189	0.227	1.000	0.000	−0.279	−0.219	0.200	1.000	0.000	−0.282	−0.229	0.188	1.000
	K_{N2}	−2.154	−0.480	1.222	2.952	4.713	−1.520	−0.374	0.803	2.012	3.255	−1.152	−0.296	0.591	1.516	2.480
	K_{M1}	0.000	0.020	0.049	0.055	0.000	0.000	0.028	0.073	0.083	0.000	0.000	0.036	0.097	0.111	0.000
	K_{M2}	−1.810	−1.807	−1.795	−1.808	−1.881	−0.830	−0.814	−0.782	−0.787	−0.884	−0.467	−0.440	−0.391	−0.391	−0.515
	$K_{\Delta R}$	−2.216	−0.493	1.237	2.976	4.724	−1.587	−0.389	0.818	2.036	3.266	−1.220	−0.312	0.606	1.540	2.491
	K_{ψ}	17.115	17.153	17.196	17.244	17.288	7.909	7.937	7.978	8.029	8.071	4.478	4.502	4.549	4.614	4.662

		$\mu_D = 1.2$	$k_B = 38.303$	$K_{\Delta h1} = 0.301$	$K_{\Delta h2} = 2.523$		$\mu_D = 1.6$	$k_B = 37.737$	$K_{\Delta h1} = 0.301$		$K_{\Delta h2} = 1.923$	$\mu_D = 3.2$	$k_B = 35.475$	$K_{\Delta h1} = 0.300$		$K_{\Delta h2} = 1.107$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
40	K_{N1}	0.000	−0.275	−0.234	0.170	1.000	0.000	−0.256	−0.232	0.149	1.000	0.000	−0.096	−0.182	−0.033	1.000
	K_{N2}	−0.747	−0.206	0.372	1.004	1.698	−0.517	−0.157	0.249	0.739	1.331	−0.064	−0.070	−0.016	0.278	1.013
	K_{M1}	0.000	0.051	0.141	0.164	0.000	0.000	0.063	0.177	0.214	0.000	0.000	0.040	0.167	0.303	0.000
	K_{M2}	−0.198	−0.154	−0.076	−0.070	−0.248	−0.100	−0.043	0.057	0.070	−0.162	0.001	0.039	0.148	0.235	−0.121
	$K_{\Delta R}$	−0.814	−0.223	0.386	1.029	1.709	−0.581	−0.175	0.261	0.764	1.342	−0.081	−0.086	−0.020	0.296	1.025
	K_{ψ}	1.930	1.954	2.026	2.140	2.218	0.985	1.012	1.120	1.303	1.428	−0.009	0.014	0.180	0.602	1.001

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 79.434$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 7.362$		$\mu_D = 0.6$	$k_B = 79.151$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 4.974$		$\mu_D = 0.8$	$k_B = 78.869$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 3.752$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
80	K_{N1}	0.000	−0.296	−0.233	0.197	1.000	0.000	−0.300	−0.241	0.188	1.000	0.000	−0.299	−0.243	0.184	1.000
	K_{N2}	−2.395	−0.588	1.235	3.073	4.928	−1.614	−0.407	0.817	2.059	3.319	−1.207	−0.309	0.606	1.543	2.505
	K_{M1}	0.000	0.019	0.050	0.056	0.000	0.000	0.028	0.074	0.084	0.000	0.000	0.037	0.098	0.112	0.000
	K_{M2}	−0.973	−0.959	−0.933	−0.932	−0.993	−0.434	−0.410	−0.367	−0.362	−0.450	−0.243	−0.208	−0.151	−0.142	−0.257
	$K_{\Delta R}$	−2.429	−0.595	1.243	3.085	4.933	−1.649	−0.415	0.824	2.071	3.325	−1.241	−0.318	0.613	1.556	2.510
	K_{ψ}	18.277	18.298	18.324	18.355	18.382	8.186	8.202	8.231	8.270	8.299	4.588	4.603	4.640	4.695	4.734

		$\mu_D = 1.2$	$k_B = 78.303$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 2.522$		$\mu_D = 1.6$	$k_B = 77.737$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 1.911$		$\mu_D = 3.2$	$k_B = 75.475$	$K_{\Delta h1} = 0.298$	$K_{\Delta h2} = 1.093$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
80	K_{N1}	0.000	−0.291	−0.244	0.174	1.000	0.000	−0.274	−0.241	0.156	1.000	0.000	−0.111	−0.199	−0.024	1.000
	K_{N2}	−0.780	−0.209	0.386	1.022	1.702	−0.544	−0.159	0.262	0.755	1.329	−0.070	−0.075	−0.011	0.298	1.008
	K_{M1}	0.000	0.053	0.144	0.166	0.000	0.000	0.066	0.183	0.216	0.000	0.000	0.045	0.181	0.313	0.000
	K_{M2}	−0.103	−0.052	0.034	0.049	−0.122	−0.052	0.012	0.122	0.145	−0.079	0.000	0.044	0.171	0.278	−0.057
	$K_{\Delta R}$	−0.815	−0.218	0.394	1.034	1.707	−0.577	−0.167	0.269	0.767	1.334	−0.079	−0.082	−0.013	0.307	1.014
	K_{ψ}	1.968	1.986	2.053	2.160	2.230	1.005	1.029	1.135	1.314	1.431	−0.009	0.015	0.192	0.621	1.001

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 159.434$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 7.471$		$\mu_D = 0.6$	$k_B = 159.151$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 5.000$		$\mu_D = 0.8$	$k_B = 158.869$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 3.758$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
160	K_{N1}	0.000	−0.307	−0.245	0.189	1.000	0.000	−0.307	−0.247	0.186	1.000	0.000	−0.306	−0.248	0.184	1.000
	K_{N2}	−2.466	−0.616	1.242	3.109	4.984	−1.645	−0.415	0.824	2.074	3.335	−1.227	−0.312	0.613	1.554	2.510
	K_{M1}	0.000	0.019	0.050	0.056	0.000	0.000	0.028	0.075	0.084	0.000	0.000	0.037	0.099	0.112	0.000
	K_{M2}	−0.497	−0.480	−0.450	−0.445	−0.502	−0.221	−0.193	−0.148	−0.140	−0.225	−0.123	−0.087	−0.026	−0.015	−0.128
	$K_{\Delta R}$	−2.484	−0.620	1.246	3.115	4.987	−1.662	−0.419	0.828	2.080	3.337	−1.245	−0.371	0.617	1.560	2.513
	K_ψ	18.607	18.618	18.635	18.656	18.673	8.269	8.279	8.301	8.333	8.355	4.625	4.635	4.667	4.718	4.751

		$\mu_D = 1.2$	$k_B = 158.303$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 2.516$		$\mu_D = 1.6$	$k_B = 157.737$	$K_{\Delta h1} = 0.300$	$K_{\Delta h2} = 1.903$		$\mu_D = 3.2$	$k_B = 155.475$	$K_{\Delta h1} = 0.298$	$K_{\Delta h2} = 1.086$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
160	K_{N1}	0.000	−0.299	−0.247	0.177	1.000	0.000	−0.283	−0.245	0.161	1.000	0.000	−0.119	−0.207	−0.019	1.000
	K_{N2}	−0.796	−0.210	0.394	1.030	1.701	−0.557	−0.159	0.269	0.762	1.327	−0.074	−0.077	−0.009	0.307	1.006
	K_{M1}	0.000	0.054	0.146	0.167	0.000	0.000	0.068	0.186	0.217	0.000	0.000	0.048	0.188	0.317	0.000
	K_{M2}	−0.053	0.001	0.090	0.108	−0.061	−0.027	0.040	0.155	0.182	−0.039	0.000	0.047	0.183	0.300	−0.028
	$K_{\Delta R}$	−0.813	−0.214	0.397	1.036	1.704	−0.573	−0.163	0.272	0.768	1.330	−0.078	−0.081	−0.010	0.311	1.008
	K_ψ	1.983	1.998	2.062	2.166	2.233	1.014	1.036	1.142	1.318	1.431	−0.009	0.016	0.198	0.629	1.001

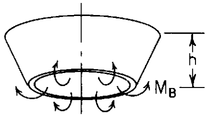
Case no., loading	Formulas
5d. Uniform edge moment M_B at the small end  	$N_1 = -M_B \dfrac{2\sqrt{2}\beta}{tk_B} K_{N1}, \qquad N_2 = M_B \dfrac{\beta}{t} K_{N2}$ $M_1 = M_B K_{M1}, \qquad M_2 = M_B \nu K_{M2}, \qquad \Delta h = \dfrac{-M_B R_B \beta}{Et^2 \sin\alpha} K_{\Delta h1} - \dfrac{M_B \beta R_B \sin\alpha}{Et^2 \cos\alpha} K_{\Delta h2}$ $\Delta R = M_B \dfrac{\beta R_B}{Et^2} K_{\Delta R}, \qquad \psi = -M_B \dfrac{2\sqrt{2}\beta^2 R_B}{Et^3 k_B \sin\alpha} K_\psi$ For $R_B/(t\cos\alpha) > 10$ and $k_B > 5$ and for $\nu = 0.3$, the following tables give the values of K at several locations along the shell $[\Omega = (R_A - R)/(R_A - R_B)]$

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 9.434$	$K_{\Delta h1} = -0.018$	$K_{\Delta h2} = -15.589$		$\mu_D = 0.6$	$k_B = 9.151$	$K_{\Delta h1} = -0.027$		$K_{\Delta h2} = -10.293$	$\mu_D = 0.8$	$k_B = 8.869$	$K_{\Delta h1} = -0.034$		$K_{\Delta h2} = -7.029$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
10	K_{N1}	0.000	−0.507	−0.712	−0.563	0.000	0.000	−0.465	−0.670	−0.546	0.000	0.000	−0.389	−0.578	−0.487	0.000
	K_{N2}	−7.020	−3.707	−0.170	3.622	7.702	−4.374	−2.378	−0.172	2.290	5.070	−2.800	−1.571	−0.164	1.486	3.469
	K_{M1}	0.000	0.187	0.461	0.758	1.000	0.000	0.160	0.443	0.761	1.000	0.000	0.141	0.425	0.757	1.000
	K_{M2}	−15.021	−15.396	−15.742	−16.134	−16.657	−6.220	−6.412	−6.550	−6.740	−7.110	−2.965	−3.052	−3.069	−3.143	−3.433
	$K_{\Delta R}$	−7.887	−4.100	−0.248	3.681	7.702	−5.223	−2.773	−0.257	2.348	5.070	−3.559	−1.936	−0.249	1.537	3.469
	K_{ψ}	18.558	18.722	18.917	19.149	19.416	7.922	8.031	8.183	8.390	8.651	3.896	3.973	4.101	4.305	4.582

		$\mu_D = 1.2$	$k_B = 8.303$	$K_{\Delta h1} = -0.043$	$K_{\Delta h2} = -3.764$		$\mu_D = 1.6$	$k_B = 7.737$	$K_{\Delta h1} = -0.049$		$K_{\Delta h2} = -2.352$	$\mu_D = 3.2$	$k_B = 5.475$	$K_{\Delta h1} = -0.068$		$K_{\Delta h2} = -0.927$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
10	K_{N1}	0.000	−0.255	−0.407	−0.376	0.000	0.000	−0.161	−0.285	−0.301	0.000	0.000	−0.009	−0.051	−0.155	0.000
	K_{N2}	−1.271	−0.774	−0.153	0.690	1.920	−0.621	−0.429	−0.154	0.335	1.314	−0.002	−0.045	−0.107	−0.905	0.920
	K_{M1}	0.000	0.113	0.381	0.729	1.000	0.000	0.088	0.327	0.681	1.000	0.000	0.005	0.066	0.330	1.000
	K_{M2}	−0.862	−0.855	−0.771	−0.750	−1.004	−0.281	−0.245	−0.136	−0.092	−0.390	0.024	0.035	0.084	0.158	−0.517
	$K_{\Delta R}$	−1.843	−1.071	−0.239	0.725	1.920	−1.038	−0.672	−0.247	0.353	1.314	−0.007	−0.128	−0.248	−0.188	0.920
	K_{ψ}	1.210	1.254	1.366	1.592	1.939	0.423	0.451	0.558	0.814	1.254	−0.050	−0.052	−0.024	0.174	0.968

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 19.434$	$K_{\Delta h1} = -0.008$	$K_{\Delta h2} = 27.998$		$\mu_D = 0.6$	$k_B = 19.151$	$K_{\Delta h1} = -0.009$		$K_{\Delta h2} = -14.589$	$\mu_D = 0.8$	$k_B = 18.869$	$K_{\Delta h1} = -0.010$		$K_{\Delta h2} = -8.774$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
20	K_{N1}	0.000	−0.979	−1.340	−1.031	0.000	0.000	−0.737	−1.022	−0.799	0.000	0.000	−0.567	−0.799	−0.636	0.000
	K_{N2}	−13.294	−6.833	−0.156	6.755	13.918	−6.729	−3.513	−0.130	3.448	7.251	−3.910	−2.081	−0.117	2.021	4.381
	K_{M1}	0.000	0.171	0.483	0.806	1.000	0.000	0.155	0.474	0.811	1.000	0.000	0.146	0.465	0.809	1.000
	K_{M2}	−14.236	−14.330	−14.303	−14.289	−14.426	−4.790	−4.771	−4.612	−4.465	−4.496	−2.074	−2.009	−1.798	−1.601	−1.594
	$K_{\Delta R}$	−14.079	−7.180	−0.221	6.809	13.918	−7.338	−3.787	−0.183	3.489	7.251	−4.393	−2.301	−0.162	2.054	4.381
	K_{ψ}	34.151	34.302	34.481	34.697	34.944	11.662	11.744	11.867	12.045	12.269	5.124	5.178	5.284	5.463	5.705

		$\mu_D = 1.2$	$k_B = 18.303$	$K_{\Delta h1} = -0.011$	$K_{\Delta h2} = -4.165$		$\mu_D = 1.6$	$k_B = 17.737$	$K_{\Delta h1} = -0.011$		$K_{\Delta h2} = -2.471$	$\mu_D = 3.2$	$k_B = 15.475$	$K_{\Delta h1} = -0.013$		$K_{\Delta h2} = -0.988$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
20	K_{N1}	0.000	−0.362	−0.533	−0.448	0.000	0.000	−0.242	−0.385	−0.355	0.000	0.000	−0.024	−0.199	−0.259	0.000
	K_{N2}	−1.679	−0.948	−0.117	0.887	2.160	−0.835	−0.527	−0.136	0.452	1.410	0.004	−0.085	−0.175	−0.067	0.994
	K_{M1}	0.000	0.131	0.440	0.793	1.000	0.000	0.113	0.403	0.766	1.000	0.000	0.013	0.135	0.516	1.000
	K_{M2}	−0.571	−0.477	−0.241	−0.016	0.003	−0.189	−0.096	0.132	0.359	0.369	0.024	0.039	0.144	0.394	0.443
	$K_{\Delta R}$	−2.005	−1.105	−0.155	0.908	2.160	−1.061	−0.648	−0.175	0.464	1.410	0.007	−0.130	−0.242	−0.095	0.994
	K_{ψ}	1.454	1.486	1.593	1.810	2.127	0.496	0.520	0.635	0.898	1.305	−0.073	−0.073	−0.021	0.257	1.005

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 39.434$	$K_{\Delta h1} = -0.002$	$K_{\Delta h2} = -34.665$		$\mu_D = 0.6$	$k_B = 39.151$	$K_{\Delta h1} = -0.002$	$K_{\Delta h2} = -16.163$		$\mu_D = 0.8$	$k_B = 38.869$	$K_{\Delta h1} = -0.003$	$K_{\Delta h2} = -9.280$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
40	K_{N1}	0.000	−1.256	−1.696	−1.289	0.000	0.000	−0.861	−1.172	−0.898	0.000	0.000	−0.644	−0.885	−0.685	0.000
	K_{N2}	−16.889	−8.564	−0.100	8.514	17.288	−7.752	−3.966	−0.080	3.926	8.071	−4.360	−2.259	−0.075	2.221	4.662
	K_{M1}	0.000	0.159	0.492	0.830	1.000	0.000	0.153	0.487	0.831	1.000	0.000	0.149	0.482	0.828	1.000
	K_{M2}	−9.047	−8.972	−8.731	−8.493	−8.432	−2.761	−2.648	−2.363	−2.082	−1.985	−1.157	−1.031	−0.734	−0.440	−0.335
	$K_{\Delta R}$	−17.377	−8.778	−0.139	8.548	17.288	−8.092	−4.116	−0.107	3.950	8.071	−4.617	−2.373	−0.097	2.238	4.662
	K_{ψ}	42.783	42.880	43.003	43.163	43.351	13.153	13.203	13.292	13.437	13.623	5.552	5.586	5.672	5.832	6.048

		$\mu_D = 1.2$	$k_B = 38.303$	$K_{\Delta h1} = -0.003$	$K_{\Delta h2} = -4.240$		$\mu_D = 1.6$	$k_B = 37.737$	$K_{\Delta h1} = -0.003$	$K_{\Delta h2} = -2.476$		$\mu_D = 3.2$	$k_B = 35.475$	$K_{\Delta h1} = -0.003$	$K_{\Delta h2} = -0.992$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
40	K_{N1}	0.000	−0.412	−0.584	−0.471	0.000	0.000	−0.283	−0.427	−0.372	0.000	0.000	−0.034	−0.156	−0.294	0.000
	K_{N2}	−1.854	−1.007	−0.090	0.961	2.218	−0.933	−0.562	−0.120	0.498	1.428	0.007	−0.107	−0.199	−0.042	1.001
	K_{M1}	0.000	0.140	0.466	0.818	1.000	0.000	0.127	0.437	0.797	1.000	0.000	0.018	0.171	0.583	1.000
	K_{M2}	−0.315	−0.187	0.110	0.408	0.514	−0.105	0.014	0.295	0.594	0.701	0.015	0.033	0.174	0.518	0.757
	$K_{\Delta R}$	−2.022	−1.085	−0.107	0.972	2.218	−1.048	−0.621	−0.138	0.505	1.428	0.009	−0.129	−0.231	−0.052	1.001
	K_{ψ}	1.535	1.560	1.660	1.870	2.170	0.521	0.544	0.660	0.923	1.313	−0.081	−0.079	−0.015	0.289	1.006

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 79.434$	$K_{\Delta h1} = -0.001$	$K_{\Delta h2} = 36.799$		$\mu_D = 0.6$	$k_B = 79.151$	$K_{\Delta h1} = -0.001$	$K_{\Delta h2} = -16.579$		$\mu_D = 0.8$	$k_B = 78.869$	$K_{\Delta h1} = -0.001$	$K_{\Delta h2} = -9.392$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
80	K_{N1}	0.000	−1.356	−1.820	−1.374	0.000	0.000	−0.907	−1.222	−0.927	0.000	0.000	−0.675	−0.916	−0.700	0.000
	K_{N2}	−18.157	−9.146	−0.056	9.116	18.382	−8.105	−4.105	−0.048	4.081	8.299	−4.527	−2.314	−0.051	2.288	4.734
	K_{M1}	0.000	0.156	0.496	0.839	1.000	0.000	0.154	0.493	0.838	1.000	0.000	0.152	0.490	0.836	1.000
	K_{M2}	−4.864	−4.731	−4.417	−4.104	−3.976	−1.444	−1.301	−0.977	−0.653	−0.517	−0.601	−0.456	−0.130	0.195	0.333
	$K_{\Delta R}$	−18.417	−9.258	−0.076	9.134	18.382	−8.280	−4.181	−0.062	4.093	8.299	−4.658	−2.372	−0.062	2.297	4.734
	K_{ψ}	45.679	45.733	45.813	45.930	46.074	13.610	13.639	13.709	13.832	13.997	5.685	5.708	5.783	5.931	6.133

		$\mu_D = 1.2$	$k_B = 78.303$	$K_{\Delta h1} = -0.001$	$K_{\Delta h2} = -4.244$		$\mu_D = 1.6$	$k_B = 77.737$	$K_{\Delta h1} = -0.001$	$K_{\Delta h2} = -2.468$		$\mu_D = 3.2$	$k_B = 75.475$	$K_{\Delta h1} = -0.001$	$K_{\Delta h2} = -0.992$	
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
80	K_{N1}	0.000	−0.435	−0.605	−0.478	−0.000	0.000	−0.303	−0.446	−0.378	0.000	0.000	−0.041	−0.175	−0.308	0.000
	K_{N2}	−1.929	−1.028	−0.075	0.990	2.230	−0.979	−0.575	−0.111	0.518	1.431	0.008	−0.117	−0.209	−0.028	1.001
	K_{M1}	0.000	0.146	0.478	0.828	1.000	0.000	0.134	0.452	0.811	1.000	0.000	0.020	0.188	0.611	1.000
	K_{M2}	−0.164	−0.023	0.297	0.623	0.761	−0.055	0.075	0.381	0.709	0.855	0.008	0.029	0.190	0.578	0.886
	$K_{\Delta R}$	−2.014	−1.066	−0.083	0.995	2.230	−1.037	−0.604	−0.199	0.521	1.431	0.009	−0.128	−0.224	−0.033	1.001
	K_{ψ}	1.564	1.585	1.682	1.887	2.178	0.531	0.552	0.670	0.931	1.314	−0.084	−0.081	−0.011	0.303	1.005

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

		$\mu_D = 0.4$	$k_B = 159.434$	$K_{\Delta h1} = -0.000$	$K_{\Delta h2} = -37.352$		$\mu_D = 0.6$	$k_B = 159.151$	$K_{\Delta h1} = -0.000$		$K_{\Delta h2} = -16.672$	$\mu_D = 0.8$	$k_B = 158.869$	$K_{\Delta h1} = -0.000$		$K_{\Delta h2} = -9.411$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
160	K_{N1}	0.000	−1.388	−1.857	−1.398	0.000	0.000	−0.923	−1.239	−0.936	0.000	0.000	−0.688	−0.923	−0.705	0.000
	K_{N2}	−18.547	−9.310	−0.033	9.292	18.673	−8.228	−4.146	−0.032	4.130	8.355	−4.594	−2.333	−0.039	2.313	4.751
	K_{M1}	0.000	0.156	0.498	0.842	1.000	0.000	0.155	0.496	0.841	1.000	0.000	0.153	0.494	0.839	1.000
	K_{M2}	−2.484	−2.335	−2.000	−1.666	−1.518	−0.733	−0.581	−0.245	0.092	0.241	−0.305	−0.154	0.182	0.519	0.668
	$K_{\Delta R}$	−18.679	−9.367	−0.043	9.301	18.673	−8.316	−4.185	−0.039	4.136	8.355	−4.660	−2.361	−0.044	2.317	4.751
	K_{ψ}	46.497	46.527	46.584	46.677	46.796	13.745	13.764	13.823	13.935	14.089	5.729	5.746	5.816	5.957	6.153

		$\mu_D = 1.2$	$k_B = 158.303$	$K_{\Delta h1} = -0.000$	$K_{\Delta h2} = -4.239$		$\mu_D = 1.6$	$k_B = 157.737$	$K_{\Delta h1} = -0.000$		$K_{\Delta h2} = -2.461$	$\mu_D = 3.2$	$k_B = 155.475$	$K_{\Delta h1} = -0.000$		$K_{\Delta h2} = -0.991$
k_A	Ω	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000	0.000	0.250	0.500	0.750	1.000
160	K_{N1}	0.000	−0.446	−0.614	−0.480	0.000	0.000	−0.313	−0.455	−0.381	0.000	0.000	−0.044	−0.184	−0.314	0.000
	K_{N2}	−1.964	−1.036	−0.067	1.002	2.233	−1.001	−0.581	−0.106	0.527	1.431	0.009	−0.122	−0.213	−0.022	1.001
	K_{M1}	0.000	0.148	0.484	0.833	1.000	0.000	0.137	0.460	0.817	1.000	0.000	0.022	0.197	0.624	1.000
	K_{M2}	−0.084	0.063	0.393	0.731	0.882	−0.028	0.107	0.424	0.767	0.929	0.004	0.026	0.198	0.608	0.945
	$K_{\Delta R}$	−2.006	−1.055	−0.071	1.005	2.233	−1.030	−0.596	−0.110	0.529	1.431	0.010	−0.128	−0.221	−0.024	1.001
	K_{ψ}	1.576	1.594	1.689	1.892	2.179	0.536	0.556	0.674	0.935	1.313	−0.085	−0.081	−0.008	0.310	1.005

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

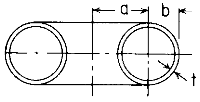
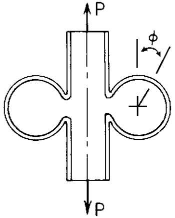
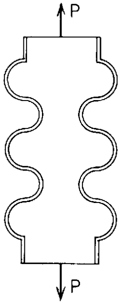
<table>
<tr><th>Case no., loading</th><th></th><th>Formulas</th></tr>
<tr>
<td>6. Toroidal shells

$\mu = \dfrac{b^2}{at}\sqrt{12(1-\nu^2)}$
$\dfrac{t}{b} < \dfrac{1}{10}$</td>
<td>6a. Split toroidal shell under axial load P (omega joint)

(Refs. 16 and 40)</td>
<td>For $4 < \mu < 40$,
Stretch $= \dfrac{3.47Pb\sqrt{1-\nu^2}}{Et^2}$
$(\sigma_2)_{\max} = \dfrac{2.15P}{2\pi at}\left[\dfrac{ab(1-\nu^2)}{t^2}\right]^{1/3}$ for $\phi = 0^\circ,\ 180^\circ$
$(\sigma_1')_{\max} = \dfrac{2.99P}{2\pi at}\left[\dfrac{ab}{t^2\sqrt{1-\nu^2}}\right]^{1/3}$ Range* $\mu = 4,\ \phi = 50^\circ$: $\mu = 40,\ \phi = 20^\circ$
If $\mu < 4$, the following values for stretch should be used, where $\Delta = \dfrac{Pb^3}{2aD}$:
<table>
<tr><th>μ</th><th><1</th><th>1</th><th>2</th><th>3</th></tr>
<tr><td>Stretch</td><td>1.00Δ</td><td>0.95Δ</td><td>0.80Δ</td><td>0.66Δ</td></tr>
</table>
* Within range, ϕ is approximately linear versus $\log \mu$</td>
</tr>
<tr>
<td></td>
<td>6b. Corrugated tube under axial load P
 </td>
<td>For $4 < \mu < 40$,
Stretch $= \dfrac{0.577Pbn\sqrt{1-\nu^2}}{Et^2}$ where n is the number of semicircular corrugations (five shown in figure)
$(\sigma_2)_{\max} = \dfrac{0.925P}{2\pi at}\left[\dfrac{ab(1-\nu^2)}{t^2}\right]^{1/3}$
$(\sigma_1')_{\max} = \dfrac{1.63P}{2\pi at}\left[\dfrac{ab}{t^2\sqrt{1-\nu^2}}\right]^{1/3}$ (Ref. 16)
If $\mu < 4$, let $\Delta = \dfrac{Pb^3n}{4aD}$ and use the tabulated values from case 6a
For U-shaped corrugations where a flat annular plate separates the inner and outer semicircles, see Ref. 41</td>
</tr>
</table>

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

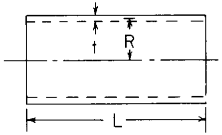
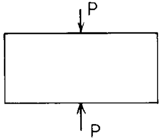
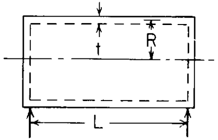
	6c. Corrugated tube under internal pressure, q. If internal pressure on the ends must be carried by the walls, calculate the end load and use case 6b in addition (see Ref. 55 and Sec. 13.5 for a discussion of a possible instability due to internal pressure in a long bellows)	For $4 < \mu < 40$, Stretch per semicircular corrugation $= \pm 2.45(1-\nu^2)^{1/3}\left(\frac{a}{t}\right)^{4/3}\left(\frac{b}{t}\right)^{1/3}\frac{bq}{E}$ Total stretch $= 0$ if there are an equal number of inner and outer corrugations $(\sigma_2)_{\max} = 0.955q(1-\nu^2)^{1/6}\left(\frac{ab}{t^2}\right)^{2/3}$ $(\sigma'_1)_{\max} = 0.955q(1-\nu^2)^{-1/3}\left(\frac{ab}{t^2}\right)^{2/3}$ If $\mu < 1$, the stretch per semicircular corrugation $= \pm 3.28(1-\nu^2)\frac{b^4 q}{Et^3}$ For U-shaped corrugations, see Ref. 41 (Ref. 16)
7. Cylindrical shells with open ends 	7a. Diametrically opposite and equal concentrated loads, P at mid-length 	For $1 < L/R < 18$ and $R/t > 10$, Deflection under the load $= 6.5\frac{P}{Et}\left(\frac{R}{t}\right)^{3/2}\left(\frac{L}{R}\right)^{-3/4}$ For $L/R > 18$, the maximum stresses and deflections are approximately the same as for case 8a For loads at the extreme ends, the maximum stresses are approximately four times as great as for loading at midlength See Refs. 24 and 25

TABLE 13.3 Formulas for bending and membrane stress and deformations in thin-walled pressure vessels (*Continued*)

Case no., loading		Formulas
8. Cylindrical shells with closed ends and end support 	8a. Radial load P uniformly distributed over small area A, approximately square or round, located near midspan	Maximum stresses are circumferential stresses at center of loaded area and can be found from the following table. Values given are for $L/R = 8$ but may be used for L/R ratios between 3 and 40. [Coefficients adapted from Bjilaard (Refs. 22, 23, 28)]

	A/R^2													
R/t	0.0004	0.0016	0.0036	0.0064	0.010	0.0144	0.0196	0.0256	0.0324	0.040	0.0576	0.090	0.160	0.25
	Values of $\sigma_2'(t^2/P)$													
300	1.475	1.11	0.906	0.780	0.678	0.600	0.522	0.450	0.390	0.348	0.264	0.186	0.120	0.078
100		1.44	1.20	1.044	0.918	0.840	0.750	0.666	0.600	0.540	0.444	0.342	0.240	0.180
50			1.44	1.254	1.11	1.005	0.900	0.840	0.756	0.720	0.600	0.480	0.360	0.264
15										0.990	0.888	0.780	0.600	0.468
	Values of $\sigma_2(Rt/P)$													
300	58	53.5	49	44.5	40	35.5	32	28	24	21	16	11	6	4
100		33.5	30.5	27.6	25	25.5	20	17.5	15	13	10	7	4.2	3.6
50					9.6	9	8.5	8.0	7.7	7.5	6.5	5.6	4.1	3.1
15										3.25	3.0	2.4	2.0	1.56

For A very small (nominal point loading) at point of load

$$\sigma_2 = \frac{0.4P}{t^2} \qquad \sigma_2' = \frac{2.4P}{t^2} \qquad y = \frac{P}{Et}\left[0.48\left(\frac{L}{R}\right)^{1/2}\left(\frac{R}{t}\right)^{1.22}\right]$$

(Approximate empirical formulas which are based on tests of Refs. 2 and 19)

For a more extensive presentation of Bjilaard's work in graphic form over an extended range of parameters, see Refs. 27 and 60 to 71

TABLE 13.3 Formulas for bending and membrane stresses and deformations in thin-walled pressure vessels (*Continued*)

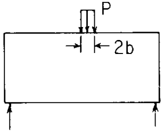
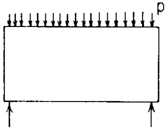
8b. Center load, P lb, concentrated on a very short length $2b$  	At the top center, $(\sigma_2)_{\max} = -0.130BPR^{3/4}b^{-3/2}t^{-5/4}$ $(\sigma_2')_{\max} = -1.56B^{-1}PR^{1/4}b^{-1/2}t^{-7/4}$ $(\sigma_1)_{\max} = -0.153B^3PR^{1/4}b^{-1/2}t^{-7/4}$ Deflection $= 0.0820B^5PR^{3/4}L^{1/2}t^{-9/4}E^{-1}$ where $B = [12(1-\nu^2)]^{1/8}$ (Ref. 13)
8c. Uniform load, P lb/in, over entire length of top element  	At the top center, $(\sigma_2)_{\max} = -0.492BpR^{3/4}L^{-1/2}t^{-5/4}$ $(\sigma_2')_{\max} = -1.217B^{-1}pR^{1/4}L^{1/2}t^{-7/4}$ $(\sigma_1)_{\max} = -0.1188B^3pR^{3/4}L^{3/2}t^{-9/4}E^{-1}$ Deflection $= 0.0305B^5pR^{3/4}L^{3/2}t^{-9/4}E^{-1}$ where B is given in case 8b Quarter-span deflection $= 0.774$ midspan deflection (Ref. 13)

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates

NOTATION: R_A = radius of common circumference; ΔR_A is the radial deflection of the common circumference, positive outward; ψ_A is the rotation of the meridian at the common circumference, positive as indicated. The notation used in Tables 11.2 and 13.1–13.3 is retained where possible with added subscripts 1 and 2 used for left and right members, respectively, when needed for clarification. There are some exceptions in using the notation from the other tables when differences occur from one table to another

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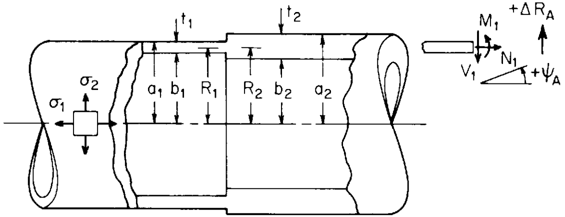
1\. Cylindrical shell connected to another cylindrical shell. Expressions are accurate if $R/t > 5$. E_1 and E_2 are the moduli of elasticity and ν_1 and ν_2 the Poisson's ratios for the left and right cylinders, respectively. See Table 13.2 for formulas for D_1 and λ_1. $R_A = R_1$, $b_1 = R_1 - t_1/2$, and $a_1 = R_1 + t_1/2$. Similar expressions hold for b_2, a_2, D_2, and λ_2.



$$K_{V1} = \frac{LT_A C_{BB} - LT_B C_{AB}}{C_{AA} C_{BB} - C_{AB}^2}, \qquad K_{M1} = \frac{LT_B C_{AA} - LT_A C_{AB}}{C_{AA} C_{BB} - C_{AB}^2},$$

$$\left.\begin{aligned} LT_A &= LT_{A1} + LT_{A2} + LT_{AC} \\ LT_B &= LT_{B1} + TL_{B2} + LT_{BC} \end{aligned}\right\}$$ See cases 1a to 1d for these load terms

$$C_{AA} = C_{AA1} + C_{AA2}, \qquad C_{AA1} = \frac{E_1}{2D_1\lambda_1^3}, \qquad C_{AA2} = \frac{R_1 E_1}{2R_2 D_2 \lambda_2^3}$$

$$C_{AB} = C_{AB1} + C_{AB2}, \qquad C_{AB1} = \frac{-E_1 t_1}{2D_1\lambda_1^2}, \qquad C_{AB2} = \frac{R_1 E_1 t_1}{2R_2 D_2 \lambda_2^2}$$

$$C_{BB} = C_{BB1} + C_{BB2}, \qquad C_{BB1} = \frac{E_1 t_1^2}{D_1\lambda_1}, \qquad C_{BB2} = \frac{R_1 E_1 t_1^2}{R_2 D_2 \lambda_2}$$

The stresses in the left cylinder at the junction are given by

$$\sigma_1 = \frac{N_1}{t_1}$$

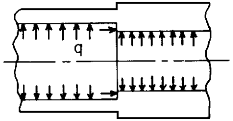
$$\sigma_2 = \frac{\Delta R_A E_1}{R_A} + \nu_1 \sigma_1$$

$$\sigma_1' = \frac{-6M_1}{t_1^2}$$

$$\sigma_2' = \nu_1 \sigma_1'$$

Note: The use of joint load correction terms LT_{AC} and LT_{BC} depends upon the accuracy desired and the relative values of the thickness and the radii. Read Sec. 13.3 carefully. For thin-walled shells, $R/t > 10$, they can be neglected.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
1a. Internal* pressure q  *Note:* There is no axial load on the left cylinder. A small axial load on the right cylinder balances any axial pressure on the joint. For an enclosed pressure vessel superpose an axial load $P = q\pi b_1^2$ using case 1b.	$LT_{A1} = \frac{b_1 R_1}{t_1^2}$ $LT_{A2} = \frac{-b_2 R_2 E_1}{E_2 t_1 t_2}$ $LT_{AC} = \frac{E_1(b_1^2 - b_2^2)}{8t_1}\left(\frac{a_2 - b_1}{R_2 D_2 \lambda_2^2} - \frac{4\nu_2}{E_2 t_2}\right)$ $LT_{B1} = 0, \quad LT_{B2} = 0$ $LT_{BC} = E_1(b_1^2 - b_2^2)\frac{a_2 - b_1}{4R_2 D_2 \lambda_2}$ At the junction of the two cylinders, $V_1 = qt_1 K_{V1}, \quad M_1 = qt_1^2 K_{M1}, \quad N_1 = 0$ $\Delta R_A = \frac{qt_1}{E_1}(LT_{A1} - K_{V1}C_{AA1} - K_{M1}C_{AB1})$ $\psi_A = \frac{q}{E_1}(K_{V1}C_{AB1} + K_{M1}C_{BB1})$	For internal pressure, $b_1 = b_2$ (smooth internal wall), $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, and for $R/t > 5$. $\Delta R_A = \frac{qR_1^2}{E_1 t_1}K_{\Delta RA}, \quad \psi_A = \frac{qR_1}{E_1 t_1}K_{\psi A}, \quad \sigma_2 = \frac{qR_1}{t_1}K_{\sigma 2}$

		R_1/t_1					
	$\frac{t_2}{t_1}$	10	15	20	30	50	100
K_{V1}	1.1	0.0542	0.0688	0.0808	0.1007	0.1318	0.1884
	1.2	0.1066	0.1353	0.1589	0.1981	0.2593	0.3705
	1.5	0.2574	0.3269	0.3843	0.4791	0.6273	0.8966
	2.0	0.4945	0.6286	0.7392	0.9220	1.2076	1.7264
	3.0		1.1351	1.3356	1.6667	2.1840	3.1231
K_{M1}	1.1	0.0065	0.0101	0.0137	0.0208	0.0352	0.0711
	1.2	0.0246	0.0382	0.0518	0.0790	0.1334	0.2695
	1.5	0.1295	0.2012	0.2730	0.4166	0.7038	1.4221
	2.0	0.3891	0.6050	0.8211	1.2535	2.1186	4.2815
	3.0		1.4312	1.9436	2.9691	5.0207	10.1505
$K_{\Delta RA}$	1.1	0.9080	0.9232	0.9308	0.9383	0.9444	0.9489
	1.2	0.8715	0.8853	0.8922	0.8991	0.9046	0.9087
	1.5	0.7835	0.7940	0.7992	0.8043	0.8084	0.8115
	2.0	0.6765	0.6827	0.6857	0.6887	0.6910	0.6927
	3.0		0.5285	0.5283	0.5281	0.5278	0.5275
$K_{\psi A}$	1.1	−0.1618	−0.2053	−0.2412	−0.3006	−0.3934	−0.5620
	1.2	−0.2862	−0.3633	−0.4269	−0.5321	−0.6965	−0.9952
	1.5	−0.5028	−0.6390	−0.7515	−0.9372	−1.2275	−1.7547
	2.0	−0.5889	−0.7501	−0.8831	−1.1026	−1.4454	−2.0676
	3.0		−0.6118	−0.7216	−0.9026	−1.1850	−1.6971
$K_{\sigma 2}$	1.1	0.9080	0.9232	0.9308	0.9383	0.9444	0.9489
	1.2	0.8715	0.8853	0.8922	0.8991	0.9046	0.9087
	1.5	0.7835	0.7940	0.7992	0.8043	0.8084	0.8115
	2.0	0.6765	0.6827	0.6857	0.6887	0.6910	0.6927
	3.0		0.5285	0.5283	0.5281	0.5278	0.5275

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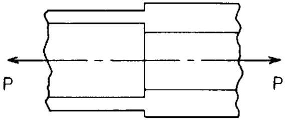
* For external pressure, substitute $-q$ for q, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in the load terms.

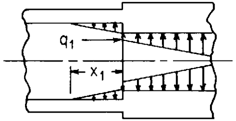
TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
1b. Axial load P 	$LT_{A1} = \frac{-\nu_1 R_1^2}{2t_1^2}, \qquad LT_{AC} = 0$ $LT_{A2} = \frac{E_1 R_1^2}{2t_1}\left(\frac{\nu_2}{E_2 t_2} - \frac{R_2 - R_1}{2R_2 D_2 \lambda_2^2}\right)$ $LT_{B1} = 0, \qquad LT_{BC} = 0$ $LT_{B2} = \frac{-(R_2 - R_1)R_1^2 E_1}{2R_2 D_2 \lambda_2}$	For axial tension, $b_1 = b_2$ (smooth internal wall), $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, and for $R/t > 5$. $\Delta R_A = \frac{P\nu_1}{2\pi E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{P\nu_1}{2\pi E_1 t_1^2} K_{\psi A}, \qquad \sigma_2 = \frac{P}{2\pi R_1 t_1} K_{\sigma 2}$
	At the junction of the two cylinders, $V_1 = \frac{Pt_1 K_{V1}}{\pi R_1^2}, \quad M_1 = \frac{Pt_1^2 K_{M1}}{\pi R_1^2}, \quad N_1 = \frac{P}{2\pi R_1}$ $\Delta R_A = \frac{Pt_1}{E_1 \pi R_1^2}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$ $\psi_A = \frac{P}{E_1 \pi R_1^2}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$	

	$\frac{t_2}{t_1}$	R_1/t_1					
		10	15	20	30	50	100
K_{V1}	1.1	−0.0583	−0.0715	−0.0825	−0.1011	−0.1306	−0.1847
	1.2	−0.1129	−0.1383	−0.1598	−0.1958	−0.2529	−0.3577
	1.5	−0.2517	−0.3089	−0.3571	−0.4378	−0.5657	−0.8005
	2.0	−0.4084	−0.5022	−0.5811	−0.7132	−0.9222	−1.3058
	3.0		−0.6732	−0.7800	−0.9586	−1.2411	−1.7589
K_{M1}	1.1	−0.1170	−0.1756	−0.2341	−0.3513	−0.5856	−1.1714
	1.2	−0.2194	−0.3294	−0.4394	−0.6594	−1.0995	−2.1995
	1.5	−0.4564	−0.6863	−0.9163	−1.3761	−2.2957	−4.5949
	2.0	−0.6892	−1.0389	−1.3887	−2.0883	−3.4876	−6.9860
	3.0		−1.2874	−1.7242	−2.5981	−4.3462	−8.7167
$K_{\Delta RA}$	1.1	−0.9416	−0.9416	−0.9416	−0.9416	−0.9416	−0.9416
	1.2	−0.8717	−0.8716	−0.8716	−0.8716	−0.8715	−0.8715
	1.5	−0.6414	−0.6410	−0.6408	−0.6406	−0.6405	−0.6404
	2.0	−0.3047	−0.3033	−0.3027	−0.3020	−0.3015	−0.3011
	3.0		0.0883	0.0900	0.0918	0.0931	0.0942
$K_{\psi A}$	1.1	−0.0810	−0.0662	−0.0573	−0.0468	−0.0363	−0.0257
	1.2	−0.1443	−0.1180	−0.1022	−0.0835	−0.0647	−0.0458
	1.5	−0.2630	−0.2154	−0.1869	−0.1528	−0.1185	−0.0839
	2.0	−0.3345	−0.2752	−0.2392	−0.1961	−0.1524	−0.1080
	3.0		−0.2663	−0.2326	−0.1915	−0.1494	−0.1062
$K_{\sigma 2}$	1.1	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175
	1.2	0.0385	0.0385	0.0385	0.0385	0.0385	0.0385
	1.5	0.1076	0.1077	0.1077	0.1078	0.1079	0.1079
	2.0	0.2086	0.2090	0.2092	0.2094	0.2096	0.2097
	3.0		0.3265	0.3270	0.3275	0.3279	0.3283

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

1c. Hydrostatic internal* pressure q_1 at the junction for $x_1 > 3/\lambda_1$†



Note: There is no axial load on the left cylinder. A small axial load on the right cylinder balances any axial pressure on the joint.

$$LT_{A1} = \frac{b_1 R_1}{t_1^2}$$

$$LT_{A2} = \frac{-b_2 R_2 E_1}{E_2 t_1 t_2}$$

For LT_{AC} use the expression from case 1a

$$LT_{B1} = \frac{-b_1 R_1}{x_1 t_1}, \qquad LT_{B2} = \frac{b_2 R_2 E_1}{x_1 E_2 t_2}$$

For LT_{BC} use the expression from case 1a

At the junction of the two cylinders,

$$V_1 = q_1 t_1 K_{V1}, \qquad M_1 = q_1 t_1^2 K_{M1}, \qquad N_1 = 0$$

$$\Delta R_A = \frac{q_1 t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For internal pressure, $b_1 = b_2$ (smooth internal wall), $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, and for $R/t > 5$.

$$\Delta R_A = \frac{q_1 R_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{q_1 R_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{q_1 R_1}{t_1} K_{\sigma 2}$$

		R_1/t_1					
		10			20		
		x_1/R_1			x_1/R_1		
	$\frac{t_2}{t_1}$	1	2	4	1	2	4
K_{V1}	1.1	0.0536	0.0539	0.0541	0.0802	0.0805	0.0807
	1.2	0.1041	0.1054	0.1060	0.1564	0.1577	0.1583
	1.5	0.2445	0.2509	0.2542	0.3707	0.3775	0.3809
	2.0	0.4556	0.4751	0.4848	0.6982	0.7187	0.7290
	3.0				1.2385	1.2870	1.3113
K_{M1}	1.1	−0.0107	−0.0021	0.0022	−0.0119	0.0009	0.0073
	1.2	−0.0103	0.0071	0.0158	−0.0002	0.0258	0.0388
	1.5	0.0406	0.0850	0.1073	0.1407	0.2068	0.2399
	2.0	0.2152	0.3022	0.3456	0.5625	0.6918	0.7565
	3.0				1.4882	1.7159	1.8298
$K_{\Delta RA}$	1.1	0.9029	0.9055	0.9068	0.9269	0.9289	0.9298
	1.2	0.8619	0.8667	0.8691	0.8851	0.8886	0.8904
	1.5	0.7647	0.7741	0.7788	0.7852	0.7922	0.7957
	2.0	0.6507	0.6636	0.6701	0.6666	0.6761	0.6809
	3.0				0.5090	0.5187	0.5235
$K_{\psi A}$	1.1	0.7442	0.2912	0.0647	0.6875	0.2231	−0.0091
	1.2	0.5781	0.1460	−0.0701	0.4580	0.0155	−0.2057
	1.5	0.2511	−0.1259	−0.3143	0.0173	−0.3671	−0.5593
	2.0	0.0226	−0.2831	−0.4360	−0.2637	−0.5734	−0.7282
	3.0				−0.2905	−0.5060	−0.6138
$K_{\sigma 2}$	1.1	0.9029	0.9055	0.9068	0.9269	0.9289	0.9298
	1.2	0.8619	0.8667	0.8691	0.8851	0.8886	0.8904
	1.5	0.7647	0.7741	0.7788	0.7852	0.7922	0.7957
	2.0	0.6507	0.6636	0.6701	0.6666	0.6761	0.6809
	3.0				0.5090	0.5187	0.5235

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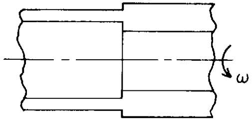
* For external pressure, substitute $-q$ for q, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in the load terms.

† If pressure increases right to left, substitute $-x_1$ for x_1 and verify that $|x_1| > 3/\lambda_2$.

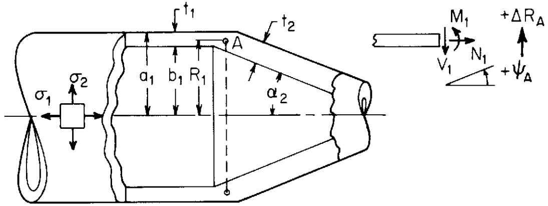
TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
1d. Rotation around the axis of symmetry at ω rad/s  *Note:* δ = mass/unit volume	$LT_{A1} = \dfrac{R_1^2}{t_1^2}$ $LT_{A2} = \dfrac{-\delta_2 R_2^3 E_1}{\delta_1 R_1 E_2 t_1^2}$ $LT_{AC} = 0$ $LT_{B1} = 0, \qquad LT_{B2} = 0$ $LT_{BC} = 0$ At the junction of the two cylinders, $V_1 = \delta_1 \omega^2 R_1 t_1^2 K_{V1}, \qquad M_1 = \delta_1 \omega^2 R_1 t_1^3 K_{M1}$ $N_1 = 0$ $\Delta R_A = \dfrac{\delta_1 \omega^2 R_1 t_1^2}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$ $\psi_A = \dfrac{\delta_1 \omega^2 R_1 t_1}{E_1}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$	For $b_1 = b_2$ (smooth internal wall), $\delta_1 = \delta_2$, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, and for $R/t > 5$. $\Delta R_A = \dfrac{\delta_1 \omega^2 R_1^3}{E_1} K_{\Delta RA}, \qquad \psi_A = \dfrac{\delta_1 \omega^2 R_1^2}{E_1} K_{\psi A}, \qquad \sigma_2 = \delta_1 \omega^2 R_1^2 K_{\sigma 2}$

		R_1/t_1					
	$\dfrac{t_2}{t_1}$	10	15	20	30	50	100
K_{V1}	1.1	−0.0100	−0.0081	−0.0070	−0.0057	−0.0044	−0.0031
	1.2	−0.0215	−0.0175	−0.0151	−0.0123	−0.0095	−0.0067
	1.5	−0.0658	−0.0534	−0.0461	−0.0375	−0.0289	−0.0204
	2.0	−0.1727	−0.1391	−0.1196	−0.0970	−0.0747	−0.0526
	3.0		−0.3893	−0.3322	−0.2673	−0.2046	−0.1434
K_{M1}	1.1	−0.0012	−0.0012	−0.0012	−0.0012	−0.0012	−0.0012
	1.2	−0.0049	−0.0049	−0.0049	−0.0049	−0.0049	−0.0049
	1.5	−0.0331	−0.0328	−0.0327	−0.0326	−0.0325	−0.0324
	2.0	−0.1359	−0.1339	−0.1328	−0.1318	−0.1310	−0.1304
	3.0		−0.4908	−0.4834	−0.4761	−0.4703	−0.4660
$K_{\Delta RA}$	1.1	1.0077	1.0051	1.0038	1.0026	1.0015	1.0008
	1.2	1.0158	1.0105	1.0079	1.0052	1.0031	1.0016
	1.5	1.0426	1.0282	1.0211	1.0140	1.0084	1.0042
	2.0	1.0955	1.0628	1.0468	1.0310	1.0185	1.0092
	3.0		1.1503	1.1111	1.0730	1.0433	1.0215
$K_{\psi A}$	1.1	0.0297	0.0242	0.0210	0.0171	0.0133	0.0094
	1.2	0.0577	0.0470	0.0406	0.0331	0.0256	0.0181
	1.5	0.1285	0.1043	0.0900	0.0733	0.0566	0.0400
	2.0	0.2057	0.1659	0.1429	0.1159	0.0894	0.0630
	3.0		0.2098	0.1795	0.1447	0.1110	0.0779
$K_{\sigma 2}$	1.1	1.0077	1.0051	1.0038	1.0026	1.0015	1.0008
	1.2	1.0158	1.0105	1.0079	1.0052	1.0031	1.0016
	1.5	1.0426	1.0282	1.0211	1.0140	1.0084	1.0042
	2.0	1.0955	1.0628	1.0468	1.0310	1.0185	1.0092
	3.0		1.1503	1.1111	1.0730	1.0433	1.0215

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

2. Cylindrical shell connected to a conical shell.* To ensure accuracy, evaluate k_A and the value of k in the cone at the position where $\mu = 4$. The absolute values of k at both positions should be greater than 5. $R/(t_2 \cos \alpha_2)$ should also be greater than 5 at both of these positions. E_1 and E_2 are the moduli of elasticity and ν_1 and ν_2 the Poisson's ratios for the cylinder and cone, respectively. See Table 13.2 for formulas for D_1 and λ_1. $b_1 = R_1 - t_1/2$ and $a_1 = R_1 + t_1/2$. $b_2 = R_2 - (t_2 \cos \alpha_2)/2$ and $a_2 = R_2 + (t_2 \cos \alpha_2)/2$, where R_2 is the mid-thickness radius of the cone at the junction. $R_A = R_1$. See Table 13.3, case 4, for formulas for k_A, β, μ, C_1, and the F functions.



$$K_{V1} = \frac{LT_A C_{BB} - LT_B C_{AB}}{C_{AA} C_{BB} - C_{AB}^2}, \qquad K_{M1} = \frac{LT_B C_{AA} - LT_A C_{AB}}{C_{AA} C_{BB} - C_{AB}^2} \qquad \left. \begin{aligned} LT_A &= LT_{A1} + LT_{A2} + LT_{AC} \\ LT_B &= LT_{B1} + LT_{B2} + LT_{BC} \end{aligned} \right\} \text{See cases 2a–2d for these load terms}$$

$$C_{AA} = C_{AA1} + C_{AA2}, \qquad C_{AA1} = \frac{E_1}{2D_1 \lambda_1^3}, \qquad C_{AA2} = \frac{R_1 E_1 k_A \sin \alpha_2}{E_2 t_2 \sqrt{2} C_1} \left(F_{4A} - \frac{4\nu_2^2}{k_A^2} F_{2A} \right)$$

$$C_{AB} = C_{AB1} + C_{AB2}, \qquad C_{AB1} = \frac{-E_1 t_1}{2D_1 \lambda_1^2}, \qquad C_{AB2} = \frac{R_1 E_1 t_1 \beta F_{7A}}{E_2 t_2^2 C_1}$$

$$C_{BB} = C_{BB1} + C_{BB2}, \qquad C_{BB1} = \frac{E_1 t_1^2}{D_1 \lambda_1}, \qquad C_{BB2} = \frac{R_1 E_1 t_1^2 2\sqrt{2} \beta^2 F_{2A}}{E_2 t_2^3 k_A C_1 \sin \alpha_2}$$

The stresses in the left cylinder at the junction are given by

$$\sigma_1 = \frac{N_1}{t_1}$$

$$\sigma_2 = \frac{\Delta R_A E_1}{R_A} + \nu_1 \sigma_1$$

$$\sigma_1' = \frac{-6M_1}{t_1^2}$$

$$\sigma_2' = \nu_1 \sigma_1'$$

Note: The use of joint load correction terms LT_{AC} and LT_{BC} depends upon the accuracy desired and the relative values of the thicknesses and the radii and the cone angle α. Read Sec. 13.4 carefully. For thin-walled shells, $R/t > 10$ at the junction, they can be neglected.

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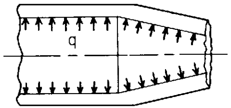
* If the conical shell increases in radius away from the junction, substitute $-\alpha$ for α in all the formulas above and in those used from case 4, Table 13.3.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
2a. Internal* pressure q  *Note:* There is no axial load on the cylinder. An axial load on the right end of the cone balances any axial component of the pressure on the cone and the joint. For an enclosed pressure vessel superpose an axial load $P = q\pi b_1^2$ using case 2b.	$LT_{A1} = \dfrac{b_1 R_1}{t_1^2}$ $LT_{A2} = \dfrac{-b_2 E_1 R_2}{E_2 t_1 t_2 \cos\alpha_2}$ $LT_{AC} = \dfrac{t_2 \sin\alpha_2}{2t_1} C_{AA2} + \dfrac{t_2^2 - t_1^2}{8t_1^2} C_{AB2} + \dfrac{E_1 \nu_2 R_1 (t_1 - t_2 \cos\alpha_2)}{2E_2 t_1 t_2 \cos\alpha_2} \dagger$ $LT_{B1} = 0$ $LT_{B2} = \dfrac{-2E_1 b_2 \tan\alpha_2}{E_2 t_2 \cos\alpha_2}$ $LT_{BC} = \dfrac{t_2 \sin\alpha_2}{2t_1} C_{AB2} + \dfrac{t_2^2 - t_1^2}{8t_1^2} C_{BB2} + \dfrac{E_1 \sin\alpha_2}{E_2 t_2 \cos^2\alpha_2}(t_1 - t_2 \cos\alpha_2) \dagger$ At the junction of the cylinder and cone $V_1 = qt_1 K_{V1}, \quad M_1 = qt_1^2 K_{M1}, \quad N_1 = 0$ $\Delta R_A = \dfrac{qt_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$ $\psi_A = \dfrac{q}{E_1}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$	For internal pressure, $R_1 = R_2$, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, and for $R/t > 5$. (*Note:* No correction terms are used) $\Delta R_A = \dfrac{qR_1^2}{E_1 t_1} K_{\Delta RA}, \quad \psi_1 = \dfrac{qR_1}{E_1 t_1} K_{\psi A}, \quad \sigma_2 = \dfrac{qR_1}{t_1} K_{\sigma 2}$

		R_1/t_1					
		20			40		
		α_2			α_2		
	$\dfrac{t_2}{t_1}$	−30	30	45	−30	30	45
K_{V1}	0.8	−0.3602	−0.3270	−0.5983	−0.5027	−0.4690	−0.8578
	1.0	−0.1377	−0.1419	−0.4023	−0.1954	−0.1994	−0.5633
	1.2	0.0561	0.0212	−0.2377	0.0725	0.0374	−0.3165
	1.5	0.3205	0.2531	−0.0083	0.4399	0.3715	0.0225
	2.0	0.7208	0.6267	0.3660	1.0004	0.9040	0.5628
K_{M1}	0.8	0.4218	−0.1915	−0.4167	0.6693	−0.2053	−0.4749
	1.0	0.3306	−0.3332	−0.6782	0.4730	−0.4756	−0.9638
	1.2	0.3562	−0.3330	−0.7724	0.5175	−0.4696	−1.1333
	1.5	0.5482	−0.1484	−0.6841	0.8999	−0.1011	−0.9487
	2.0	1.0724	0.4091	−0.1954	1.9581	0.9988	0.0084
$K_{\Delta RA}$	0.8	1.2517	1.1314	1.2501	1.2471	1.1612	1.2969
	1.0	1.1088	1.0015	1.0942	1.1060	1.0293	1.1368
	1.2	1.0016	0.9078	0.9840	1.0008	0.9335	1.0225
	1.5	0.8813	0.8050	0.8667	0.8830	0.8281	0.9000
	2.0	0.7378	0.6823	0.7323	0.7426	0.7026	0.7594
$K_{\psi A}$	0.8	1.9915	0.7170	1.1857	2.5602	1.2739	2.1967
	1.0	1.0831	−0.1641	0.0411	1.2812	0.0201	0.5668
	1.2	0.4913	−0.7028	−0.6817	0.4554	−0.7544	−0.4765
	1.5	−0.0178	−1.1184	−1.2720	−0.2449	−1.3635	−1.3488
	2.0	−0.3449	−1.2939	−1.5808	−0.6758	−1.6457	−1.8483
$K_{\sigma 2}$	0.8	1.2517	1.1314	1.2501	1.2471	1.1612	1.2969
	1.0	1.1088	1.0015	1.0942	1.1060	1.0293	1.1368
	1.2	1.0016	0.9078	0.9840	1.0008	0.9335	1.0225
	1.5	0.8813	0.8050	0.8667	0.8830	0.8281	0.9000
	2.0	0.7378	0.6823	0.7323	0.7426	0.7026	0.7594

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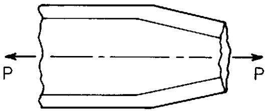
* For external pressure, substitute $-q$ for q, a_1 for b_1, and a_2 for b_2 in the load terms.

† If α_2 approaches 0, use the correction term from case 1a.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

2b. Axial load P



$$LT_{A1} = \frac{-v_1 R_1^2}{2t_1^2}$$

$$LT_{A2} = \frac{v_2 R_1^2 E_1}{2E_2 t_1 t_2 \cos\alpha_2} + \frac{R_1 C_{AA2}}{2t_1} \tan\alpha_2$$

$$LT_{AC} = 0$$

$$LT_{B1} = 0$$

$$LT_{B2} = \frac{E_1 R_1^2 \tan\alpha_2}{2E_2 R_2 t_2 \cos\alpha_2} + \frac{R_1 C_{AB2}}{2t_1} \tan\alpha_2$$

$$LT_{BC} = 0$$

At the junction of the cylinder and cone,

$$V_1 = \frac{Pt_1 K_{V1}}{\pi R_1^2}, \quad M_1 = \frac{Pt_1^2 K_{M1}}{\pi R_1^2}, \quad N_1 = \frac{P}{2\pi R_1}$$

$$\Delta R_A = \frac{Pt_1}{E_1 \pi R_1^2}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

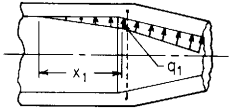
$$\psi_A = \frac{P}{E_1 \pi R_1^2}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For axial tension, $R_1 = R_2$, $E_1 = E_2$, $v_1 = v_2 = 0.3$, and for $R/t > 5$.

$$\Delta R_A = \frac{Pv_1}{2\pi E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{Pv_1}{2\pi E_1 t_1^2} K_{\psi A}, \qquad \sigma_2 = \frac{P}{2\pi R_1 t_1} K_{\sigma 2}$$

		R_1/t_1					
		20			40		
		α_2			α_2		
	$\frac{t_2}{t_1}$	−30	30	45	−30	30	45
K_{V1}	0.8	−2.9323	2.9605	4.8442	−5.8775	5.9175	9.7022
	1.0	−2.7990	2.7667	4.5259	−5.5886	5.5429	9.0767
	1.2	−2.6963	2.6095	4.2750	−5.3637	5.2411	8.5862
	1.5	−2.5522	2.3879	3.9263	−5.0493	4.8171	7.9088
	2.0	−2.3074	2.0205	3.3492	−4.5212	4.1161	6.7938
K_{M1}	0.8	−4.5283	4.5208	7.4287	−12.7003	12.6851	20.8134
	1.0	−4.9208	4.9206	8.0936	−13.8055	13.8053	22.6838
	1.2	−5.0832	5.0531	8.3299	−14.2489	14.1890	23.3674
	1.5	−5.0660	4.9455	8.1850	−14.1627	13.9229	23.0120
	2.0	−4.7100	4.3860	7.3095	−13.0797	12.4341	20.6655
$K_{\Delta RA}$	0.8	5.2495	−7.3660	−11.3820	7.9325	−10.0493	−15.8287
	1.0	4.3064	−6.1827	−9.4297	6.5410	−8.4173	−13.1035
	1.2	3.7337	−5.4346	−8.2078	5.6872	−7.3879	−11.3978
	1.5	3.2008	−4.7036	−7.0309	4.8827	−6.3855	−9.7578
	2.0	2.6544	−3.9120	−5.7838	4.0481	−5.3060	−8.0287
$K_{\psi A}$	0.8	0.3626	−0.3985	−0.6321	0.3938	−0.4193	−0.6840
	1.0	−0.0327	0.0683	0.1395	−0.0127	0.0378	0.0792
	1.2	−0.2488	0.3252	0.5655	−0.2359	0.2900	0.5024
	1.5	−0.3965	0.5013	0.8580	−0.3897	0.4639	0.7959
	2.0	−0.4408	0.5517	0.9392	−0.4381	0.5167	0.8847
$K_{\sigma 2}$	0.8	1.8748	−1.9098	−3.1146	2.6797	−2.7148	−4.4486
	1.0	1.5919	−1.5548	−2.5289	2.2623	−2.2252	−3.6310
	1.2	1.4201	−1.3304	−2.1623	2.0062	−1.9164	−3.1193
	1.5	1.2602	−1.1111	−1.8093	1.7648	−1.6156	−2.6273
	2.0	1.0963	−0.8736	−1.4351	1.5144	−1.2918	−2.1086

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
2c. Hydrostatic internal* pressure q_1 at the junction for $x_1 > 3/\lambda_1$†. If $x_1 < 3/\lambda_1$ the discontinuity in pressure gradient introduces small deformations at the junction.  *Note:* There is no axial load on the cylinder. An axial load on the right end of the cone balances any axial component of pressure in the cone and the joint.	$LT_{A1} = \frac{b_1 R_1}{t_1^2}$ $LT_{A2} = \frac{-b_2 R_2 E_1}{E_2 t_1 t_2 \cos \alpha_2}$ For LT_{AC} use the expression from case 2a $LT_{B1} = \frac{-b_1 R_1}{x_1 t_1}$ $LT_{B2} = \frac{E_1 b_2}{E_2 t_2 \cos \alpha_2}\left(\frac{R_2}{x_1} - 2 \tan \alpha_2\right)$ For LT_{BC} use the expression from case 2a At the junction of the cylinder and cone $V_1 = q_1 t_1 K_{V1}, \quad M_1 = q_1 t_1^2 K_{M1}, \quad N_1 = 0$ $\Delta R_A = \frac{q_1 t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$ $\psi_A = \frac{q_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$	For internal pressure, $R_1 = R_2$, $x_1 = R_1$, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, and for $R/t > 5$. (*Note:* No correction terms are used) $\Delta R_A = \frac{q_1 R_1^2}{E_1 t_1} K_{\Delta RA}, \quad \psi_A = \frac{q_1 R_1}{E_1 t_1} K_{\psi A}, \quad \sigma_2 = \frac{q_1 R_1}{t_1} K_{\sigma 2}$

		R_1/t_1					
		20			40		
		α_2			α_2		
	$\frac{t_2}{t_1}$	− 30	30	45	− 30	30	45
K_{V1}	0.8	−0.3660	−0.3329	−0.6082	−0.5085	−0.4748	−0.8678
	1.0	−0.1377	−0.1418	−0.4020	−0.1954	−0.1994	−0.5631
	1.2	0.0555	0.0206	−0.2346	0.0719	0.0368	−0.3134
	1.5	0.3107	0.2429	−0.0109	0.4300	0.3614	0.0199
	2.0	0.6843	0.5890	0.3392	0.9639	0.8666	0.5363
K_{M1}	0.8	0.5047	−0.1081	−0.2814	0.7866	−0.0876	−0.2840
	1.0	0.3698	−0.2938	−0.5779	0.5285	−0.4200	−0.8224
	1.2	0.3442	−0.3451	−0.7175	0.5005	−0.4866	−1.0559
	1.5	0.4542	−0.2430	−0.7070	0.7670	−0.2347	−0.9811
	2.0	0.8455	0.1802	−0.3526	1.6371	0.6759	−0.2128
$K_{\Delta RA}$	0.8	1.2688	1.1485	1.2781	1.2592	1.1733	1.3168
	1.0	1.1153	1.0080	1.1106	1.1106	1.0338	1.1485
	1.2	1.0000	0.9062	0.9913	0.9996	0.9323	1.0277
	1.5	0.8715	0.7952	0.8645	0.8761	0.8212	0.8983
	2.0	0.7213	0.6662	0.7218	0.7310	0.6911	0.7519
$K_{\psi A}$	0.8	3.1432	1.8696	2.4504	3.7245	2.4390	3.4736
	1.0	2.1326	0.8856	1.2057	2.3432	1.0822	1.7437
	1.2	1.4455	0.2515	0.3872	1.4221	0.2123	0.6049
	1.5	0.8113	−0.2892	−0.3320	0.5967	−0.5218	−0.3962
	2.0	0.3197	−0.6291	−0.8157	0.0014	−0.9683	−1.0706
$K_{\sigma 2}$	0.8	1.2688	1.1485	1.2781	1.2592	1.1733	1.3168
	1.0	1.1153	1.0080	1.1106	1.1106	1.0338	1.1485
	1.2	1.0000	0.9062	0.9913	0.9996	0.9323	1.0277
	1.5	0.8715	0.7952	0.8645	0.8761	0.8212	0.8983
	2.0	0.7213	0.6662	0.7218	0.7310	0.6911	0.7519

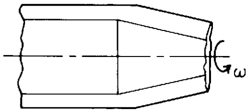
---

* For external pressure, substitute $-q_1$ for q_1, a_1 for b_1, and a_2 for b_2 in the load terms.

† If pressure increases right to left, substitute $-x_1$ for x_1 and verify that $\lvert x_1 \rvert$ is large enough to extend into the cone as far as the position where $\lvert \mu \rvert = 4$.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

2d. Rotation around the axis of symmetry at ω rad/s



Note: δ = mass/unit volume

$$LT_{A1} = \frac{R_1^2}{t_1^2}$$

$$LT_{A2} = \frac{-\delta_2 R_2^3 E_1}{\delta_1 t_1^2 E_2 R_1}$$

$$LT_{AC} = 0$$

$$LT_{B1} = 0, \qquad LT_{B2} = \frac{-\delta_2 R_2^2 E_1 (3 + \nu_2) \tan \alpha_2}{\delta_1 E_2 t_1 R_1}$$

$$LT_{BC} = 0$$

At the junction of the cylinder and cone,

$$V_1 = \delta_1 \omega^2 R_1 t_1^2 K_{V1}, \qquad M_1 = \delta_1 \omega^2 R_1 t_1^3 K_{M1}$$

$$N_1 = 0$$

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1 t_1^2}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{\delta_1 \omega^2 R_1 t_1}{E_1}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

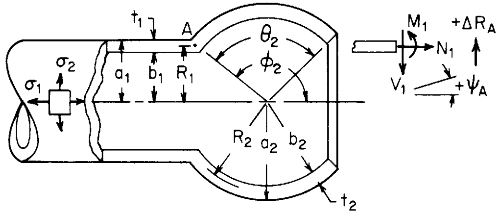
For $R_1 = R_2$, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $\delta_1 = \delta_2$, and for $R/t > 5$.

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1^3}{E_1} K_{\Delta RA}, \qquad \psi_A = \frac{\delta_1 \omega^2 R_1^2}{E_1} K_{\psi A}, \qquad \sigma_2 = \delta_1 \omega^2 R_1^2 K_{\sigma 2}$$

		R_1/t_1					
		20			40		
		α_2			α_2		
	$\frac{t_2}{t_1}$	−30	30	45	−30	30	45
K_{V1}	0.8	−0.0248	0.0250	0.0425	−0.0249	0.0251	0.0428
	1.0	−0.0000	−0.0006	−0.0022	−0.0001	−0.0004	−0.0013
	1.2	0.0308	−0.0325	−0.0581	0.0308	−0.0320	−0.0565
	1.5	0.0816	−0.0849	−0.1507	0.0817	−0.0840	−0.1480
	2.0	0.1645	−0.1701	−0.3023	0.1649	−0.1687	−0.2979
K_{M1}	0.8	0.3565	−0.3583	−0.5815	0.5043	−0.5061	−0.8207
	1.0	0.4828	−0.4851	−0.7988	0.6829	−0.6852	−1.1269
	1.2	0.6056	−0.6089	−1.0149	0.8566	−0.8599	−1.4309
	1.5	0.7777	−0.7831	−1.3237	1.1000	−1.1053	−1.8646
	2.0	1.0229	−1.0320	−1.7708	1.4469	−1.4559	−2.4920
$K_{\Delta RA}$	0.8	1.0731	0.9264	0.8795	1.0518	0.9480	0.9148
	1.0	1.0798	0.9202	0.8693	1.0565	0.9435	0.9074
	1.2	1.0824	0.9181	0.8657	1.0582	0.9420	0.9047
	1.5	1.0816	0.9194	0.8679	1.0576	0.9428	0.9061
	2.0	1.0744	0.9273	0.8812	1.0525	0.9483	0.9152
$K_{\psi A}$	0.8	0.7590	−0.7633	−1.2450	0.7597	−0.7628	−1.2438
	1.0	0.9173	−0.9194	−1.5101	0.9176	−0.9192	−1.5094
	1.2	1.0488	−1.0494	−1.7360	1.0488	−1.0493	−1.7354
	1.5	1.2078	−1.2071	−2.0164	1.2074	−0.2070	−2.0156
	2.0	1.3995	−1.3984	−2.3649	1.3988	−1.3981	−2.3631
$K_{\sigma 2}$	0.8	1.0731	0.9264	0.8795	1.0518	0.9480	0.9148
	1.0	1.0798	0.9202	0.8693	1.0565	0.9435	0.9074
	1.2	1.0824	0.9181	0.8657	1.0582	0.9420	0.9047
	1.5	1.0816	0.9194	0.8679	1.0576	0.9428	0.9061
	2.0	1.0744	0.9273	0.8812	1.0525	0.9483	0.9152

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

3. Cylindrical shell connected to a spherical shell. To ensure accuracy $R/t > 5$ and the junction angle for the spherical shell must lie within the range $3/\beta_1 < \phi_2 < (\pi - 3/\beta_2)$. The spherical shell must also extend with no interruptions such as a second junction or a cutout, such that $\theta_2 > 3/\beta_2$. See the discussion on page 565. E_1 and E_2 are the moduli of elasticity and ν_1 and ν_2 the Poisson's ratios for the cylinder and sphere, respectively. See Table 13.2 for formulas for D_1 and λ_1. $b_1 = R_1 - t_1/2$ and $a_1 = R_1 + t_1/2$. See Table 13.3, case 1, for formulas for K_{12},* K_{22},* and β_2 for the spherical shell. $b_2 = R_2 - t_2/2$ and $a_2 = R_2 + t_2/2$. $R_A = R_1$ and normally $R_2 \sin\phi_2 = R_1$ but if $\phi_2 = 90^\circ$ or is close to 90° the midthickness radii at the junction may not be equal. Under this condition different correction terms will be used if necessary.



$$K_{V1} = \frac{LT_A C_{BB} - LT_B C_{AB}}{C_{AA}C_{BB} - C_{AB}^2}, \qquad K_{M1} = \frac{LT_B C_{AA} - LT_A C_{AB}}{C_{AA}C_{BB} - C_{AB}^2}, \qquad \left.\begin{aligned} LT_A &= LT_{A1} + LT_{A2} + LT_{AC} \\ LT_B &= LT_{B1} + LT_{B2} + LT_{BC} \end{aligned}\right\} \text{See cases 3a–3d for these load terms}$$

$$C_{AA} = C_{AA1} + C_{AA2}, \qquad C_{AA1} = \frac{E_1}{2D_1\lambda_1^3}, \qquad C_{AA2} = \frac{R_A E_1 \beta_2 \sin\phi_2}{E_2 t_2}\left(\frac{1}{K_{12}} + K_{22}\right)$$

$$C_{AB} = C_{AB1} + C_{AB2}, \qquad C_{AB1} = \frac{-E_1 t_1}{2D_1\lambda_1^2}, \qquad C_{AB2} = \frac{2E_1 R_A t_1 \beta_2^2}{E_2 R_2 t_2 K_{12}}$$

$$C_{BB} = C_{BB1} + C_{BB2}, \qquad C_{BB1} = \frac{E_1 t_1^2}{D_1\lambda_1}, \qquad C_{BB2} = \frac{4E_1 R_A t_1^2 \beta_2^3}{R_2^2 E_2 t_2 K_{12} \sin\phi_2}$$

The stresses in the cylinder at the junction are given by

$$\sigma_1 = \frac{N_1}{t_1}$$

$$\sigma_2 = \frac{\Delta R_A E_1}{R_A} + \nu_1\sigma_1$$

$$\sigma_1' = \frac{-6M_1}{t_1^2}$$

$$\sigma_2' = \nu_1\sigma_1'$$

Note: The use of joint load correction terms LT_{AC} and LT_{BC} depends upon the accuracy desired and the relative values of the thicknesses and the radii. Read Sec. 13.4 carefully. For thin-walled shells, $R/t > 10$, they can be neglected.

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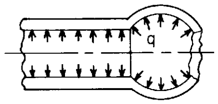
*The second subscript is added to refer to the right-hand shell. Evaluate K at the junction where $\omega = 0$.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values

3a. Internal* pressure q



Note: There is no axial load on the cylinder. An axial load on the right end of the sphere balances any axial component of the pressure on the sphere and the joint. For an enclosed pressure vessel superpose an axial load $P = q\pi b_1^2$ using case 3b.

$$LT_{A1} = \frac{b_1 R_1}{t_1^2}$$

$$LT_{A2} = \frac{-b_2^2 E_1 \sin\phi_2}{E_2 t_1 t_2}$$

$$LT_{AC} = \frac{t_2\cos\phi_2}{2t_1} C_{AA2} + \frac{t_2^2 - t_1^2}{8t_1^2} C_{AB2} + \frac{E_1 R_1 (1+\nu_2)(t_1 - t_2\sin\phi_2)}{2E_2 t_1 t_2 \sin\phi_2}\dagger$$

$$LT_{B1} = 0, \qquad LT_{B2} = 0$$

$$LT_{BC} = \frac{t_2\cos\phi_2}{2t_1} C_{AB2} + \frac{t_2^2 - t_1^2}{8t_1^2} C_{BB2} + \frac{E_1 \cos\phi_2 (t_1 - t_2\sin\phi_2)}{E_2 t_2 \sin^2\phi_2}\dagger$$

At the junction of the cylinder and sphere,

$$V_1 = q t_1 K_{V1}, \qquad M_1 = q t_1^2 K_{M1}, \qquad N_1 = 0$$

$$\Delta R_A = \frac{q t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q}{E_1}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $R_2 \sin\phi_2 = R_1$, and for $R/t > 5$. (*Note:* No correction terms are used)

$$\Delta R_A = \frac{q R_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{q R_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{q R_1}{t_1} K_{\sigma 2}$$

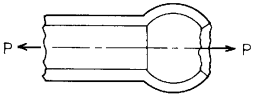
		R_1/t_1					
		10			20		
		ϕ_2			ϕ_2		
	$\frac{t_2}{t_1}$	60	90	120	60	90	120
K_{V1}	0.5	−0.5344	−0.3712	−0.5015	−0.7630	−0.5382	−0.7294
	0.8	−0.2216	−0.1062	−0.2115	−0.3299	−0.1676	−0.3192
	1.0	−0.0694	0.0292	−0.0668	−0.1190	0.0212	−0.1158
	1.2	0.0633	0.1517	0.0614	0.0650	0.1917	0.0636
	1.5	0.2472	0.3263	0.2410	0.3200	0.4344	0.3142
K_{M1}	0.5	0.2811	0.2058	0.2591	0.5660	0.4219	0.5344
	0.8	0.0560	0.0267	0.0483	0.1162	0.0597	0.1047
	1.0	0.0011	0.0000	−0.0010	0.0019	0.0000	−0.0018
	1.2	0.0130	0.0349	0.0148	0.0194	0.0624	0.0212
	1.5	0.1172	0.1636	0.1225	0.2169	0.3081	0.2238
$K_{\Delta RA}$	0.5	1.4774	1.3197	1.4433	1.5071	1.3541	1.4826
	0.8	1.1487	1.0452	1.1379	1.1838	1.0812	1.1758
	1.0	1.0068	0.9263	1.0040	1.0437	0.9628	1.0413
	1.2	0.9028	0.8382	0.9050	0.9408	0.8751	0.9420
	1.5	0.7878	0.7388	0.7946	0.8269	0.7762	0.8313
$K_{\psi A}$	0.5	2.5212	1.7793	2.3533	3.5965	2.5800	3.4255
	0.8	0.8827	0.4228	0.8287	1.3109	0.6674	1.2536
	1.0	0.2323	−0.0965	0.2180	0.3967	−0.0701	0.3792
	1.2	−0.1743	−0.4076	−0.1631	−0.1780	−0.5151	−0.1698
	1.5	−0.5021	−0.6387	−0.4671	−0.6453	−0.8504	−0.6132
$K_{\sigma 2}$	0.5	1.4774	1.3197	1.4433	1.5071	1.3541	1.4826
	0.8	1.1487	1.0452	1.1379	1.1838	1.0812	1.1758
	1.0	1.0068	0.9263	1.0040	1.0437	0.9628	1.0413
	1.2	0.9028	0.8382	0.9050	0.9408	0.8751	0.9420
	1.5	0.7878	0.7388	0.7946	0.8269	0.7762	0.8313

* For external pressure, substitute $-q$ for q, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in the load terms.

† If $\phi_2 = 90^\circ$ or is close to 90° the following correction terms should be used:

$$LT_{AC} = \frac{b_1^2 - b_2^2}{4t_1^2}\left[\frac{a_2 - b_1}{R_1} C_{AB2} - \frac{2E_1 t_1 (1+\nu_2)}{E_2 t_2}\right], \qquad LT_{BC} = \frac{b_1^2 - b_2^2}{4t_1^2}\,\frac{a_2 - b_1}{R_1} C_{BB2}$$

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
3b. Axial load P 	$LT_{A1} = \frac{-\nu_1 R_1^2}{2t_1^2}$ $LT_{A2} = \frac{R_1^2 E_1(1+\nu_2)}{2E_2 t_1 t_2 \sin\phi_2} + \frac{R_1 C_{AA2}}{2t_1 \tan\phi_2}$ $LT_{AC} = 0$ * $LT_{B1} = 0$ $LT_{B2} = \frac{R_1 C_{AB2}}{2t_1 \tan\phi_2}$ $LT_{BC} = 0$ *	For axial tension, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $R_2 \sin\phi_2 = R_1$, and for $R/t > 5$. $\Delta R_A = \frac{P\nu_1}{2\pi E_1 t_1} K_{\Delta RA}$, $\psi_A = \frac{P\nu_1}{2\pi E_1 t_1^2} K_{\psi A}$, $\sigma_2 = \frac{P}{2\pi R_1 t_1} K_{\sigma 2}$

At the junction of the cylinder and sphere,

$$V_1 = \frac{Pt_1 K_{V1}}{\pi R_1^2}, \quad M_1 = \frac{Pt_1^2 K_{M1}}{\pi R_1^2}, \quad N_1 = \frac{P}{2\pi R_1}$$

$$\Delta R_A = \frac{Pt_1}{E_1 \pi R_1^2}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{P}{E_1 \pi R_1^2}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

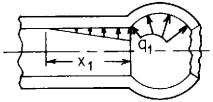
		R_1/t_1					
		10			20		
		ϕ_2			ϕ_2		
	$\frac{t_2}{t_1}$	60	90	120	60	90	120
K_{V1}	0.5	2.2519	0.4487	−1.2194	4.1845	0.6346	−2.7251
	0.8	1.8774	0.3483	−1.0921	3.5154	0.4925	−2.4052
	1.0	1.7427	0.3075	−1.0535	3.2769	0.4349	−2.3023
	1.2	1.6409	0.2781	−1.0186	3.0945	0.3933	−2.2145
	1.5	1.5083	0.2435	−0.9620	2.8531	0.3444	−2.0804
K_{M1}	0.5	0.8275	−0.2487	−1.3712	2.5815	−0.4975	−3.6682
	0.8	1.4700	−0.0877	−1.6631	4.2478	−0.1754	−4.6337
	1.0	1.7061	−0.0000	−1.7064	4.8334	−0.0000	−4.8338
	1.2	1.8303	0.0640	−1.6870	5.1238	0.1280	−4.8367
	1.5	1.8775	0.1221	−1.5987	5.1991	0.2442	−4.6411
$K_{\Delta RA}$	0.5	−11.3815	−3.9800	2.5881	−14.1929	−3.9800	5.4031
	0.8	−7.9365	−3.0808	1.2550	−9.7933	−3.0808	3.1133
	1.0	−6.6866	−2.6667	0.9502	−8.2341	−2.6667	2.4986
	1.2	−5.8608	−2.3665	0.8041	−7.2151	−2.3665	2.1590
	1.5	−5.0386	−2.0509	0.6916	−6.2071	−2.0509	1.8605
$K_{\psi A}$	0.5	−3.4789	−1.4341	0.2306	−2.9746	−1.0140	0.6790
	0.8	−1.5032	−0.9243	−0.5727	−1.1826	−0.6536	−0.2848
	1.0	−0.7837	−0.6775	−0.7353	−0.5490	−0.4790	−0.5248
	1.2	−0.3368	−0.4982	−0.7774	−0.1642	−0.3532	−0.6233
	1.5	0.0397	−0.3178	−0.7440	0.1494	−0.2247	−0.6472
$K_{\sigma 2}$	0.5	−3.1144	−0.8940	1.0764	−3.9579	−0.8940	1.9209
	0.8	−2.0809	−0.6242	0.6765	−2.6380	−0.6242	1.2340
	1.0	−1.7060	−0.5000	0.5851	−2.1702	−0.5000	1.0496
	1.2	−1.4582	−0.4099	0.5412	−1.8645	−0.4099	0.9477
	1.5	−1.2116	−0.3153	0.5075	−1.5621	−0.3153	0.8581

* If $\phi_2 = 90°$ or is close to 90° the following correction terms should be used:

$$LT_{AC} = \frac{-R_1(R_2 - R_1)C_{AB2}}{2t_1^2}, \qquad LT_{BC} = \frac{-R_1(R_2 - R_1)C_{BB2}}{2t_1^2}$$

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

3c. Hydrostatic internal* pressure q_1 at the junction for $x_1 > 3/\lambda_1$.† If $x_1 < 3/\lambda_1$ the discontinuity in pressure gradient introduces small deformations at the junction.



Note: There is no axial load on the cylinder. An axial load on the right end of the sphere balances the axial component of pressure in the sphere and on the joint.

$$LT_{A1} = \frac{b_1 R_1}{t_1^2}$$

$$LT_{A2} = \frac{-b_2^2 E_1 \sin \phi_2}{E_2 t_1 t_2}$$

For LT_{AC} use the expressions from case 3a

$$LT_{B1} = \frac{-b_1 R_1}{x_1 t_1}$$

$$LT_{B2} = \frac{E_1 b_2 R_2 \sin \phi_2}{E_2 t_2 x_1}$$

For LT_{BC} use the expressions from case 3a

At the junction of the cylinder and sphere,

$$V_1 = q_1 t_1 K_{V1}, \qquad M_1 = q_1 t_1^2 K_{M1}, \qquad N_1 = 0$$

$$\Delta R_A = \frac{q_1 t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For internal pressure, $x_1 = R_1$, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $R_2 \sin \phi_2 = R_1$, and for $R/t > 5$. (*Note:* No correction terms are used)

$$\Delta R_A = \frac{q_1 R_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{q_1 R_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{q_1 R_1}{t_1} K_{\sigma 2}$$

		R_1/t_1					
		10			20		
		ϕ_2			ϕ_2		
	$\frac{t_2}{t_1}$	60	90	120	60	90	120
K_{V1}	0.5	−0.5636	−0.3928	−0.5284	−0.7918	−0.5598	−0.7566
	0.8	−0.2279	−0.1095	−0.2169	−0.3360	−0.1710	−0.3247
	1.0	−0.0696	−0.0292	−0.0667	−0.1191	0.0212	−0.1157
	1.2	0.0628	0.1490	0.0607	0.0644	0.1890	0.0629
	1.5	0.2375	0.3119	0.2308	0.3101	0.4201	0.3041
K_{M1}	0.5	0.3642	0.2737	0.3430	0.6837	0.5180	0.6528
	0.8	0.1140	0.0615	0.1078	0.1986	0.1088	0.1885
	1.0	0.0285	0.0000	0.0271	0.0408	0.0000	0.0378
	1.2	0.0047	−0.0037	0.0062	0.0075	0.0078	0.0091
	1.5	0.0515	0.0653	0.0552	0.1236	0.1691	0.1289
$K_{\Delta RA}$	0.5	1.5286	1.3598	1.4929	1.5431	1.3824	1.5178
	0.8	1.1730	1.0593	1.1620	1.2010	1.0913	1.1928
	1.0	1.0160	0.9263	1.0131	1.0502	0.9628	1.0477
	1.2	0.9005	0.8276	0.9027	0.9392	0.8676	0.9403
	1.5	0.7740	0.7180	0.7806	0.8171	0.7615	0.8215
$K_{\psi A}$	0.5	3.7909	2.9832	3.6176	4.8904	3.8089	4.7155
	0.8	2.0095	1.4769	1.9564	2.4627	1.7466	2.4061
	1.0	1.2564	0.8535	1.2431	1.4459	0.9049	1.4291
	1.2	0.7551	0.4477	0.7659	0.7764	0.3652	0.7842
	1.5	0.3036	0.0949	0.3355	0.1849	−0.0919	0.2149
$K_{\sigma 2}$	0.5	1.5286	1.3598	1.4929	1.5431	1.3824	1.5178
	0.8	1.1730	1.0593	1.1620	1.2010	1.0913	1.1928
	1.0	1.0160	0.9263	1.0131	1.0502	0.9628	1.0477
	1.2	0.9005	0.8276	0.9027	0.9392	0.8676	0.9403
	1.5	0.7740	0.7180	0.7806	0.8171	0.7615	0.8215

---

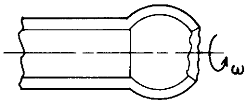
* For external pressure, substitute $-q$ for q_1, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in the load terms.

† If pressure increases right to left, substitute $-x_1$ for x_1 and verify that $|x_1|$ is large enough to extend into the sphere as far as the position where $\theta_2 = 3/\beta_2$.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values

3d. Rotation around the axis of symmetry at ω rad/s



Note: δ = mass/unit volume

Load terms

$$LT_{A1} = \frac{R_1^2}{t_1^2}$$

$$LT_{A2} = \frac{-\delta_2 R_2^3 E_1 \sin^3\ \phi_2}{\delta_1 R_1 E_2 t_1^2}$$

$$LT_{AC} = 0$$

$$LT_{B1} = 0$$

$$LT_{B2} = \frac{-\delta_2 R_2^2 E_1 (3+\nu_2)\sin\phi_2 \cos\phi_2}{\delta_1 R_1 E_2 t_1}$$

$$LT_{BC} = 0$$

At the junction of the cylinder and sphere,

$$V_1 = \delta_1 \omega^2 R_1 t_1^2 K_{V1}, \qquad M_1 = \delta_1 \omega^2 R_1 t_1^3 K_{M1}$$

$$N_1 = 0$$

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1 t_1^2}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{\delta_1 \omega^2 R_1 t_1}{E_1}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

Selected values

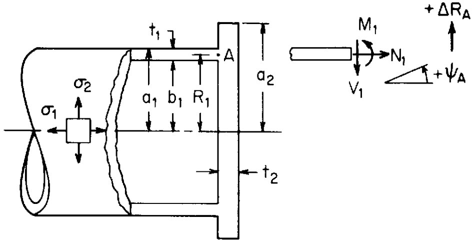
For $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $\delta_1 = \delta_2$, $R_2 \sin\phi_2 = R_1$, and for $R/t > 5$.

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1^3}{E_1} K_{\Delta RA}, \qquad \psi_A = \frac{\delta_1 \omega^2 R_1^2}{E_1} K_{\psi A}, \qquad \sigma_2 = \delta_1 \omega^2 R_1^2 K_{\sigma 2}$$

		R_1/t_1					
		10			20		
		ϕ_2			ϕ_2		
	$\frac{t_2}{t_1}$	60	90	120	60	90	120
K_{V1}	0.5	0.0425	0.0000	−0.0392	0.0420	0.0000	−0.0396
	0.8	0.0270	0.0000	−0.0233	0.0264	0.0000	−0.0238
	1.0	0.0020	0.0000	0.0018	0.0014	0.0000	0.0013
	1.2	−0.0292	0.0000	0.0330	−0.0298	0.0000	0.0325
	1.5	−0.0806	0.0000	0.0843	−0.0812	0.0000	0.0838
K_{M1}	0.5	−0.1209	0.0000	0.1220	−0.1712	0.0000	0.1723
	0.8	−0.2495	0.0000	0.2556	−0.3541	0.0000	0.3602
	1.0	−0.3371	0.0000	0.3465	−0.4788	0.0000	0.4881
	1.2	−0.4228	0.0000	0.4347	−0.6005	0.0000	0.6123
	1.5	−0.5437	0.0000	0.5576	−0.7719	0.0000	0.7858
$K_{\Delta RA}$	0.5	0.9255	1.0000	1.0722	0.9476	1.0000	1.0513
	0.8	0.8956	1.0000	1.1034	0.9263	1.0000	1.0732
	1.0	0.8870	1.0000	1.1130	0.9201	1.0000	1.0799
	1.2	0.8840	1.0000	1.1168	0.9179	1.0000	1.0825
	1.5	0.8859	1.0000	1.1157	0.9191	1.0000	1.0817
$K_{\psi A}$	0.5	−0.4653	0.0000	0.4573	−0.4639	0.0000	0.4583
	0.8	−0.7594	0.0000	0.7636	−0.7600	0.0000	0.7629
	1.0	−0.9122	0.0000	0.9248	−0.9140	0.0000	0.9229
	1.2	−1.0394	0.0000	1.0585	−1.0422	0.0000	1.0557
	1.5	−1.1943	0.0000	1.2195	−1.1980	0.0000	1.2158
$K_{\sigma 2}$	0.5	0.9255	1.0000	1.0722	0.9476	1.0000	1.0513
	0.8	0.8956	1.0000	1.1034	0.9263	1.0000	1.0732
	1.0	0.8870	1.0000	1.1130	0.9201	1.0000	1.0799
	1.2	0.8840	1.0000	1.1168	0.9179	1.0000	1.0825
	1.5	0.8859	1.0000	1.1157	0.9191	1.0000	1.0817

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

4. Cylindrical shell connected to a circular plate. Expressions are accurate if $R_1/t_1 > 5$ and $R_1/t_2 > 4$. E_1 and E_2 are the moduli of elasticity and ν_1 and ν_2 the Poisson's ratios for the cylinder and plate, respectively. See Table 13.2 for formulas for D_1 and λ_1. $b_1 = R_1 - t_1/2$ and $a_1 = R_1 + t_1/2$. See Table 11.2 for the formula for D_2.



$$K_{P1} = 1 + \frac{R_1^2(1-\nu_2)}{a_2^2(1+\nu_2)}, \qquad R_A = R_1$$

$$K_{V1} = \frac{LT_A C_{BB} - LT_B C_{AB}}{C_{AA}C_{BB} - C_{AB}^2}, \qquad K_{M1} = \frac{LT_B C_{AA} - LT_A C_{AB}}{C_{AA}C_{BB} - C_{AB}^2}, \qquad \left.\begin{aligned} LT_A &= LT_{A1} + LT_{A2} + LT_{AC} \\ LT_B &= LT_{B1} + LT_{B2} + LT_{BC} \end{aligned}\right\} \text{ See cases 4a–4d for these load terms}$$

$$C_{AA} = C_{AA1} + C_{AA2}, \qquad C_{AA1} = \frac{E_1}{2D_1\lambda_1^3}, \qquad C_{AA2} = \frac{E_1 t_2^2 R_1 K_{P1}}{6D_2}$$

$$C_{AB} = C_{AB1} + C_{AB2}, \qquad C_{AB1} = \frac{-E_1 t_1}{2D_1\lambda_1^2}, \qquad C_{AB2} = \frac{E_1 t_1 t_2 R_1 K_{P1}}{4D_2}$$

$$C_{BB} = C_{BB1} + C_{BB2}, \qquad C_{BB1} = \frac{E_1 t_1^2}{D_1\lambda_1}, \qquad C_{BB2} = \frac{E_1 t_1^2 R_1 K_{P1}}{2D_2}$$

The stresses in the left cylinder at the junction are given by

$$\sigma_1 = \frac{N_1}{t_1}$$

$$\sigma_2 = \frac{\Delta R_A E_1}{R_A} + \nu_1\sigma_1$$

$$\sigma_1' = \frac{-6M_1}{t_1^2}$$

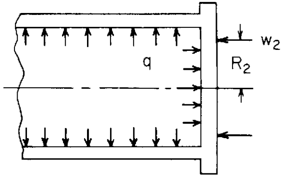
$$\sigma_2' = \nu_1\sigma_1'$$

Note: The use of joint load correction terms LT_{AC} and LT_{BC} depends upon the accuracy desired and the relative values of the thicknesses and the radii. Read Sec. 13.4 carefully. For thin-walled shells, $R/t > 10$, they can be neglected.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values

4a. Internal pressure* q



Note: There is no axial load on the cylinder. The axial load on the plate is reacted by the annular line load $w_2 = qb_1^2/(2R_2)$ at a radius R_2. For an enclosed pressure vessel superpose an axial load $P = q\pi b_1^2$ using case 4b.

$$LT_{A1} = \frac{b_1 R_1}{t_1^2}, \qquad LT_{AC} = 0$$

$$LT_{A2} = \frac{E_1 t_2 b_1^2}{32 D_2 t_1 R_1} K_{P2}$$

where

$$K_{P2} = \begin{cases} (2R_2^2 - b_1^2)K_{P1} & \text{for } R_2 \leqslant R_1 \\ (2R_2^2 - b_1^2)K_{P1} - 2(R_2^2 - R_1^2) & \\ \qquad +4R_1^2 \ln \dfrac{R_2}{R_1} & \text{for } R_2 \geqslant R_1 \end{cases}$$

$$LT_{B1} = 0, \qquad LT_{BC} = 0$$

$$LT_{B2} = \frac{E_1 b_1^2}{16 D_2 R_1} K_{P2}$$

At the junction of the cylinder and plate,

$$V_1 = qt_1 K_{V1}, \qquad M_1 = qt_1^2 K_{M1}, \qquad N_1 = 0$$

$$\Delta R_A = \frac{qt_1}{E_1}(LT_{A1} - K_{V1}C_{AA1} - K_{M1}C_{AB1})$$

$$\psi_A = \frac{q}{E_1}(K_{V1}C_{AB1} + K_{M1}C_{BB1})$$

For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $a_2 = a_1$, $R_2 = 0.7R_1$, and for $R_1/t_1 > 5$ and $R_1/t_2 > 4$.

$$\Delta R_A = \frac{qR_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{qR_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{qR_1}{t_1} K_{\sigma 2}$$

		R_1/t_1					
	$\frac{t_2}{t_1}$	15	20	30	40	80	100
K_{V1}	1.5	1.5578	1.8088	2.1689	2.4003	2.5659	2.3411
	2.0	1.7087	1.9850	2.3934	2.6724	3.0336	2.9105
	2.5	1.8762	2.1839	2.6518	2.9896	3.5949	3.6011
	3.0	2.0287	2.3667	2.8931	3.2891	4.1422	4.2830
	4.0		2.6452	3.2629	3.7513	5.0100	5.3783
K_{M1}	1.5	1.1277	1.3975	1.6600	1.5275	−3.2234	−8.2255
	2.0	1.5904	2.0300	2.6646	2.9496	0.3139	−3.3803
	2.5	2.0582	2.6818	3.7244	4.4723	4.2328	2.0517
	3.0	2.4593	3.2474	4.6606	5.8356	7.8744	7.1711
	4.0		4.0612	6.0209	7.8344	13.3902	15.0391
$K_{\Delta RA}$	1.5	0.1811	0.1661	0.1482	0.1380	0.1231	0.1213
	2.0	0.1828	0.1693	0.1535	0.1449	0.1348	0.1350
	2.5	0.1747	0.1627	0.1489	0.1417	0.1353	0.1370
	3.0	0.1618	0.1510	0.1388	0.1327	0.1284	0.1309
	4.0		0.1254	0.1151	0.1099	0.1069	0.1093
$K_{\psi A}$	1.5	−2.6740	−3.3226	−4.5926	−5.8801	−11.5407	−14.7243
	2.0	−2.1579	−2.7034	−3.7762	−4.8690	−9.7264	−12.4897
	2.5	−1.6854	−2.1223	−2.9864	−3.8720	−7.8591	−10.1571
	3.0	−1.3096	−1.6521	−2.3314	−3.0303	−6.2088	−8.0614
	4.0		−1.0265	−1.4437	−1.8728	−3.8376	−4.9966
$K_{\sigma 2}$	1.5	0.1811	0.1661	0.1482	0.1380	0.1231	0.1213
	2.0	0.1828	0.1693	0.1535	0.1449	0.1348	0.1350
	2.5	0.1747	0.1627	0.1489	0.1417	0.1353	0.1370
	3.0	0.1618	0.1510	0.1388	0.1327	0.1284	0.1309
	4.0		0.1254	0.1151	0.1099	0.1069	0.1093

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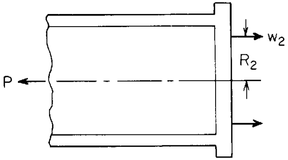
* For external pressure, substitute $-q$ for q and a_1 for b_1 in the load terms.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

4b. Axial load P



$$w_2 = \frac{P}{2\pi R_2}$$

$$LT_{A1} = \frac{-\nu_1 R_1^2}{2t_1^2}, \qquad LT_{AC} = 0$$

$$LT_{A2} = \frac{E_1 t_2 R_1^3}{16 D_2 t_1} K_{P2}$$

where

$$K_{P2} = \begin{cases} \left(1 - \dfrac{R_2^2}{R_1^2}\right) K_{P1} & \text{for } R_2 \leqslant R_1 \\ -\dfrac{1-\nu_2}{1+\nu_2}\,\dfrac{R_2^2 - R_1^2}{a_2^2} - 2\ln\dfrac{R_2}{R_1} & \\ & \text{for } R_2 \geqslant R_1 \end{cases}$$

$$LT_{B1} = 0, \qquad LT_{BC} = 0$$

$$LT_{B2} = \frac{E_1 R_1^3}{8 D_2} K_{P2}$$

At the junction of the cylinder and plate,

$$V_1 = \frac{P t_1 K_{V1}}{\pi R_1^2}, \quad M_1 = \frac{P t_1^2 K_{M1}}{\pi R_1^2}, \quad N_1 = \frac{P}{2\pi R_1}$$

$$\Delta R_A = \frac{P t_1}{E_1 \pi R_1^2}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{P}{E_1 \pi R_1^2}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

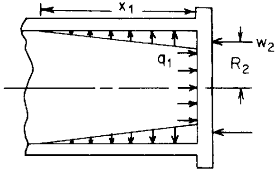
For axial tension, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $a_2 = a_1$, and for $R_1/t_1 > 5$ and $R_1/t_2 > 4$.

$$\Delta R_A = \frac{P\nu_1}{2\pi E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{P\nu_1}{2\pi E_1 t_1^2} K_{\psi A}, \qquad \sigma_2 = \frac{P}{2\pi R_1 t_1} K_{\sigma 2}$$

		R_2/R_1					
		0.8			0.9		
		R_1/t_1			R_1/t_1		
	$\frac{t_2}{t_1}$	15	20	30	15	20	30
K_{V1}	1.5	4.4625	7.2335	14.1255	2.2612	3.7073	7.3171
	2.0	3.4224	5.6661	11.3633	1.6969	2.8629	5.8397
	2.5	2.5276	4.2700	8.7738	1.2087	2.1078	4.4512
	3.0	1.8389	3.1711	6.6622	0.8311	1.5116	3.3167
	4.0		1.7432	3.8418		0.7339	1.7981
K_{M1}	1.5	12.0771	22.7118	54.8202	6.3440	11.9494	28.8827
	2.0	8.8772	17.0664	42.4231	4.6078	8.9085	22.2520
	2.5	6.3757	12.4854	31.7955	3.2429	6.4308	16.5534
	3.0	4.5632	9.0843	23.5956	2.2494	4.5854	12.1481
	4.0		4.9077	13.2133		2.3106	6.5578
$K_{\Delta RA}$	1.5	−3.0099	−3.7039	−4.9434	−1.6891	−2.0452	−2.6862
	2.0	−3.1072	−3.9154	−5.4038	−1.7418	−2.1590	−2.9323
	2.5	−2.8215	−3.6112	−5.1054	−1.5858	−1.9944	−2.7723
	3.0	−2.4354	−3.1463	−4.5192	−1.3741	−1.7421	−2.4573
	4.0		−2.2748	−3.3184		−1.2673	−1.8107
$K_{\psi A}$	1.5	5.2198	6.4135	8.5221	2.8636	3.4828	4.5818
	2.0	3.6278	4.5653	6.2776	1.9999	2.4873	3.3814
	2.5	2.5032	3.2024	4.5161	1.3863	1.7502	2.4369
	3.0	1.7479	2.2592	3.2405	0.9723	1.2385	1.7516
	4.0		1.1874	1.7331		0.6547	0.9399
$K_{\sigma 2}$	1.5	−0.6030	−0.8112	−1.1830	−0.2067	−0.3135	−0.5059
	2.0	−0.6322	−0.8746	−1.3211	−0.2225	−0.3477	−0.5797
	2.5	−0.5464	−0.7834	−1.2316	−0.1757	−0.2983	−0.5317
	3.0	−0.4306	−0.6439	−1.0558	−0.1122	−0.2226	−0.4372
	4.0		−0.3824	−0.6955		−0.0802	−0.2432

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
4c. Hydrostatic internal* pressure q_1 at the junction for $x_1 > 3/\lambda_1$.†	$LT_{A1} = \frac{b_1 R_1}{t_1^2}, \quad LT_{AC} = 0$	For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $x_1 = R_1$, $a_2 = a_1$, $R_2 = 0.7R_1$, and for $R_1/t_1 > 5$ and $R_1/t_2 > 4$.



Note: There is no axial load on the cylinder. The axial load on the plate is reacted by the annular line load $w_2 = q_1(b_1^2/2R_2)$ at a radius R_2.

$$LT_{A1} = \frac{b_1 R_1}{t_1^2}, \qquad LT_{AC} = 0$$

$$LT_{A2} = \frac{E_1 t_2 b_1^2}{32 D_2 t_1 R_1} K_{P2}$$

where

$$K_{P2} = \begin{cases} (2R_2^2 - b_1^2)K_{P1} & \text{for } R_2 \leqslant R_1 \\ (2R_2^2 - b_1^2)K_{P1} - 2(R_2^2 - R_1^2) & \\ \quad +4R_1^2 \ln \frac{R_2}{R_1} & \text{for } R_2 \geqslant R_1 \end{cases}$$

$$LT_{B1} = \frac{-b_1 R_1}{x_1 t_1}, \qquad LT_{BC} = 0$$

$$LT_{B2} = \frac{E_1 b_1^2}{16 D_2 R_1} K_{P2}$$

At the junction of the cylinder and plate,

$$V_1 = q_1 t_1 K_{V1}, \quad M_1 = q_1 t_1^2 K_{M1}, \quad N_1 = 0$$

$$\Delta R_A = \frac{q_1 t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

$$\Delta R_A = \frac{q_1 R_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{q_1 R_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{q_1 R_1}{t_1} K_{\sigma 2}$$

		R_1/t_1					
	$\frac{t_2}{t_1}$	15	20	30	40	80	100
K_{V1}	1.5	1.5304	1.7832	2.1457	2.3788	2.5486	2.3251
	2.0	1.6383	1.9170	2.3295	2.6119	2.9820	2.8620
	2.5	1.7652	2.0747	2.5466	2.8880	3.5041	3.5144
	3.0	1.8841	2.2228	2.7519	3.1509	4.0143	4.1593
	4.0		2.4528	3.0704	3.5599	4.8249	5.1962
K_{M1}	1.5	0.9101	1.1645	1.4064	1.2602	−3.5206	−8.5313
	2.0	1.2404	1.6448	2.2289	2.4780	−0.2425	−3.9630
	2.5	1.5946	2.1613	3.1188	3.8031	3.4026	1.1684
	3.0	1.9073	2.6197	3.9156	4.9999	6.7971	6.0099
	4.0		3.2916	5.0872	6.7688	11.9491	13.4589
$K_{\Delta RA}$	1.5	0.1513	0.1424	0.1311	0.1247	0.1158	0.1153
	2.0	0.1525	0.1448	0.1355	0.1305	0.1266	0.1283
	2.5	0.1463	0.1395	0.1316	0.1278	0.1271	0.1301
	3.0	0.1362	0.1300	0.1230	0.1198	0.1207	0.1243
	4.0		0.1089	0.1025	0.0997	0.1005	0.1039
$K_{\psi A}$	1.5	−2.0945	−2.7054	−3.9258	−5.1808	−10.7721	−13.9361
	2.0	−1.7263	−2.2355	−3.2576	−4.3151	−9.0908	−11.8293
	2.5	−1.3687	−1.7750	−2.5946	−3.4475	−7.3540	−9.6258
	3.0	−1.0757	−1.3940	−2.0371	−2.7086	−5.8157	−7.6438
	4.0		−0.8776	−1.2726	−1.6843	−3.6009	−4.7423
$K_{\sigma 2}$	1.5	0.1513	0.1424	0.1311	0.1247	0.1158	0.1153
	2.0	0.1525	0.1448	0.1355	0.1305	0.1266	0.1283
	2.5	0.1463	0.1395	0.1316	0.1278	0.1271	0.1301
	3.0	0.1362	0.1300	0.1230	0.1198	0.1207	0.1243
	4.0		0.1089	0.1025	0.0997	0.1005	0.1039

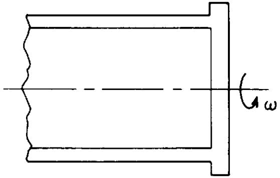
---

* For external pressure, substitute $-q$ for q_1 and a_1 for b_1 in the load terms.

† If pressure increases right to left, substitute $-x_1$ for x_1.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

4d Rotation around the axis of symmetry at ω rad/s



Note: $\delta =$ mass/unit volume

$$LT_{A1} = \frac{R_1^2}{t_1^2}, \qquad LT_{AC} = 0$$

$$LT_{A2} = \frac{-E_1 \delta_2 t_2^3}{96 D_2 \delta_1 t_1^2}\left[\frac{a_2^2(3+\nu_2)}{1+\nu_2} - R_1^2\right]$$

$$LT_{B1} = 0$$

$$LT_{B2} = 0$$

$$LT_{BC} = 0$$

At the junction of the cylinder and plate,

$$V_1 = \delta_1 \omega^2 R_1 t_1^2 K_{V1}, \quad M_1 = \delta_1 \omega^2 R_1 t_1^3 K_{M1}$$

$$N_1 = 0$$

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1 t_1^2}{E_1} \times (LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{\delta_1 \omega^2 R_1 t_1}{E_1}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

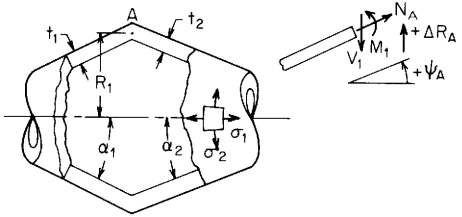
For $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $\delta_1 = \delta_2$, $a_2 = a_1$, and for $R_1/t_1 > 5$ and $R_1/t_2 > 4$.

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1^3}{E_1} K_{\Delta RA}, \qquad \psi_A = \frac{\delta_1 \omega^2 R_1^2}{E_1} K_{\psi A}, \qquad \sigma_2 = \delta_1 \omega^2 R_1^2 K_{\sigma 2}$$

		R_1/t_1					
	$\frac{t_2}{t_1}$	15	20	30	40	80	100
K_{V1}	1.5	1.0683	1.2627	1.5887	1.8632	2.7105	3.0514
	2.0	1.2437	1.4585	1.8143	2.1105	3.0101	3.3674
	2.5	1.4257	1.6676	2.0656	2.3946	3.3815	3.7690
	3.0	1.5852	1.8538	2.2957	2.6604	3.7505	4.1763
	4.0		2.1297	2.6424	3.0676	4.3459	4.8472
K_{M1}	1.5	0.3416	0.4269	0.5778	0.7109	1.1434	1.3236
	2.0	0.8797	1.1304	1.5883	2.0037	3.4081	4.0109
	2.5	1.3881	1.8157	2.6188	3.3673	6.0009	7.1688
	3.0	1.8077	2.3920	3.5118	4.5776	8.4561	10.2267
	4.0		3.1982	4.7871	6.3383	12.2402	15.0458
$K_{\Delta RA}$	1.5	0.3661	0.3447	0.3180	0.3014	0.2682	0.2593
	2.0	0.3683	0.3483	0.3234	0.3076	0.2756	0.2668
	2.5	0.3595	0.3414	0.3189	0.3048	0.2759	0.2680
	3.0	0.3460	0.3295	0.3093	0.2968	0.2713	0.2643
	4.0		0.3042	0.2871	0.2767	0.2565	0.2511
$K_{\psi A}$	1.5	−2.7811	−3.3618	−4.3536	−5.2023	−7.8711	−8.9591
	2.0	−2.1801	−2.6725	−3.5321	−4.2829	−6.7099	−7.7203
	2.5	−1.6664	−2.0614	−2.7641	−3.3900	−5.4746	−6.3645
	3.0	−1.2732	−1.5822	−2.1392	−2.6428	−4.3620	−5.1128
	4.0		−0.9623	−1.3069	−1.6231	−2.7354	−3.2359
$K_{\sigma 2}$	1.5	0.3661	0.3447	0.3180	0.3014	0.2682	0.2593
	2.0	0.3683	0.3483	0.3234	0.3076	0.2756	0.2668
	2.5	0.3595	0.3414	0.3189	0.3048	0.2759	0.2680
	3.0	0.3460	0.3295	0.3093	0.2968	0.2713	0.2643
	4.0		0.3042	0.2871	0.2767	0.2565	0.2511

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

5. Conical shell connected to another conical shell.* To ensure accuracy, for each cone evaluate k_A and the value of k in that cone at the position where $\mu = 4$. The absolute values of k at all four positions should be greater than 5. $R/(t \cos \alpha)$ should also be greater than 5 at all these positions. E_1 and E_2 are the moduli of elasticity and ν_1 and ν_2 the Poisson's ratios for the left and right cones, respectively. $b_1 = R_1 - (t_1 \cos \alpha_1)/2$, $a_1 = R_1 + (t_1 \cos \alpha_1)/2$. $R_A = R_1$. Similar expressions are used for the right-hand cone. See Table 13.3, case 4, for formulas for k_A, β, μ, C_1, and the F functions for each of the two cones. Normally $R_2 = R_1$, but if $\alpha_1 + \alpha_2$ is close to zero the midthickness radii may not be equal at the junction. Under this condition a different set of correction terms will be used if they are necessary. Note that rather than use an additional level of subscripting in the following equations, use has been made of subscripted parentheses or brackets to denote which cone the coefficients refer to.



$$K_{V1} = \frac{LT_A C_{BB} - LT_B C_{AB}}{C_{AA} C_{BB} - C_{AB}^2}, \qquad K_{M1} = \frac{LT_B C_{AA} - LT_A C_{AB}}{C_{AA} C_{BB} - C_{AB}^2}, \qquad \left.\begin{aligned} LT_A &= LT_{A1} + LT_{A2} + LT_{AC} \\ LT_B &= LT_{B1} + LT_{B2} + LT_{BC} \end{aligned}\right\} \text{ See cases 5a – 5d for these load terms}$$

$$C_{AA} = C_{AA1} + C_{AA2}, \qquad C_{AA1} = \frac{R_1 \sin \alpha}{t_1 \sqrt{2}} \left[\frac{k_A}{C_1} \left(F_{4A} - \frac{4\nu_1^2}{k_A^2} F_{2A} \right) \right]_1, \qquad C_{AA2} = \frac{R_1 E_1 \sin \alpha_2}{E_2 t_2 \sqrt{2}} \left[\frac{k_A}{C_1} \left(F_{4A} - \frac{4\nu_2^2}{k_A^2} F_{2A} \right) \right]_2$$

$$C_{AB} = C_{AB1} + C_{AB2}, \qquad C_{AB1} = \frac{-R_1}{t_1} \left(\frac{\beta F_{7A}}{C_1} \right)_1, \qquad C_{AB2} = \frac{R_1 E_1 t_1}{E_2 t_2^2} \left(\frac{\beta F_{7A}}{C_1} \right)_2$$

$$C_{BB} = C_{BB1} + C_{BB2}, \qquad C_{BB1} = \frac{R_1 2\sqrt{2}}{t_1 \sin \alpha_1} \left(\frac{\beta^2 F_{2A}}{k_A C_1} \right)_1, \qquad C_{BB2} = \frac{R_1 E_1 t_1^2 2\sqrt{2}}{E_2 t_2^3 \sin \alpha_2} \left(\frac{\beta^2 F_{2A}}{k_A C_1} \right)_2$$

The stresses in the left-hand cone at the junction are given by

$$\sigma_1 = \frac{N_1}{t_1}, \qquad \sigma_2 = \frac{\Delta R_A E_1}{R_A} + \nu_1 \sigma_1$$

$$\sigma_1' = \frac{-6M_1}{t_1^2}$$

$$\sigma_2' = \frac{-6V_1 \sin \alpha_1}{t_1 \beta_1} (1 - \nu_1^2) \left(\frac{F_{7A}}{C_1} \right)_1 + \sigma_1' \left[\nu_1 + \frac{2\sqrt{2}}{k_A C_1} (1 - \nu_1^2) F_{2A} \right]_1$$

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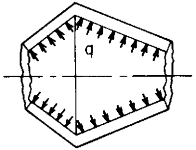
* *Note:* If either conical shell increases in radius away from the junction, substitute $-\alpha$ for α for that cone in all the appropriate formulas above and in those used from case 4, Table 13.3.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

5a. Internal* pressure q



Note: There is no axial load on the junction. An axial load on the left end of the left cone balances any axial component of the pressure on the left cone, and an axial load on the right end of the right cone balances any axial component of the pressure on the cone and the joint. For an enclosed pressure vessel superpose an axial load $P = q\pi b_1^2$ using case 5b.

$$LT_{A1} = \frac{b_1 R_1}{t_1^2 \cos\alpha_1}$$

$$LT_{A2} = \frac{-b_2 R_2 E_1}{E_2 t_1 t_2 \cos\alpha_2}$$

$$LT_{AC} = \frac{t_2 \sin\alpha_2 + t_1 \sin\alpha_1}{2t_1} C_{AA2} + \frac{t_2^2 - t_1^2}{8t_1^2} C_{AB2} + \frac{E_1 R_1 \nu_2 (t_1 \cos\alpha_1 - t_2 \cos\alpha_2)}{2E_2 t_1 t_2 \cos\alpha_2} \dagger$$

$$LT_{B1} = \frac{-2b_1 \tan\alpha_1}{t_1 \cos\alpha_1}$$

$$LT_{B2} = \frac{-2E_1 b_2 \tan\alpha_2}{E_2 t_2 \cos\alpha_2}$$

$$LT_{BC} = \frac{t_2 \sin\alpha_2 + t_1 \sin\alpha_1}{2t_1} C_{AB2} + \frac{t_2^2 - t_1^2}{8t_1^2} C_{BB2} + \frac{E_1 \tan\alpha_2 (t_1 \cos\alpha_1 - t_2 \cos\alpha_2)}{E_2 t_2 \cos\alpha_2} \dagger$$

At the junction of the two cones,

$$V_1 = qt_1 K_{V1}, \quad M_1 = qt_1^2 K_{M1}, \quad N_1 = -V_1 \sin\alpha_1$$

$$\Delta R_A = \frac{qt_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $R_1 = R_2$, and for $R/t \cos\alpha > 5$. (*Note:* No correction terms are used)

$$\Delta R_A = \frac{qR_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{qR_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{qR_1}{t_1} K_{\sigma 2}$$

		α_1					
		−30			15		
		R_1/t_1			R_1/t_1		
	α_2	10	20	50	10	20	50
K_{V1}	−30.0	0.0000	0.0000	0.0000	−0.0764	−0.1081	−0.1711
	−15.0	0.0746	0.1065	0.1696	0.0000	0.0000	0.0000
	15.0	0.0764	0.1081	0.1711	0.0000	0.0000	0.0000
	30.0	0.0000	0.0000	0.0000	−0.0800	−0.1114	−0.1742
	45.0	−0.1847	−0.2583	−0.4056	−0.2708	−0.3746	−0.5832
K_{M1}	−30.0	0.4408	0.6374	1.0216	0.1333	0.1924	0.3079
	−15.0	0.3200	0.4634	0.7433	0.0000	0.0000	0.0000
	15.0	0.1333	0.1924	0.3079	−0.1940	−0.2809	−0.4502
	30.0	0.0000	0.0000	0.0000	−0.3238	−0.4672	−0.7472
	45.0	−0.2441	−0.3482	−0.5529	−0.5603	−0.8025	−1.2770
$K_{\Delta RA}$	−30.0	1.2502	1.2351	1.2123	1.0889	1.1022	1.1062
	−15.0	1.1519	1.1479	1.1356	0.9853	1.0103	1.0253
	15.0	1.0889	1.1022	1.1062	0.9215	0.9640	0.9956
	30.0	1.1047	1.1297	1.1447	0.9414	0.9954	1.0377
	45.0	1.1689	1.2139	1.2477	1.0131	1.0868	1.1476
$K_{\psi A}$	−30.0	0.0000	0.0000	0.0000	1.1418	1.2683	1.4897
	−15.0	−0.5956	−0.7081	−0.9212	0.5280	0.5414	0.5494
	15.0	−1.1418	−1.2683	−1.4897	0.0000	0.0000	0.0000
	30.0	−1.2756	−1.3045	−1.3218	−0.0904	0.0078	0.2123
	45.0	−1.3724	−1.1627	−0.6949	−0.1067	0.2286	0.9146
$K_{\sigma 2}$	−30.0	1.2502	1.2351	1.2123	1.0894	1.1026	1.1065
	−15.0	1.1530	1.1487	1.1361	0.9853	1.0103	1.0253
	15.0	1.0900	1.1030	1.1067	0.9215	0.9640	0.9956
	30.0	1.1047	1.1297	1.1447	0.9421	0.9958	1.0380
	45.0	1.1661	1.2120	1.2465	1.0152	1.0882	1.1485

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* For external pressure, substitute $-q$ for q, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in the load terms.

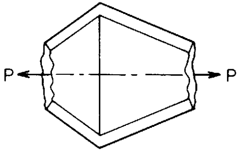
† If $\alpha_1 + \alpha_2$ is zero or close to zero the following correction terms should be used:

$$LT_{AC} = \frac{b_1^2 - b_2^2}{4t_1^2 \cos^2 \alpha_2}\left(\frac{a_2 - b_1}{R_1} C_{AB2} - \frac{2\nu_2 E_1 t_1 \cos\alpha_2}{E_2 t_2}\right), \qquad LT_{BC} = \frac{b_1^2 - b_2^2}{4t_1^2 \cos^2 \alpha_2}\left(\frac{a_2 - b_1}{R_1} C_{BB2} - \frac{2E_1 t_1^2 \sin\alpha_2}{E_2 R_2 t_2}\right)$$

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values

5b. Axial load P



$$LT_{A1}=\frac{-\nu_1R_1^2}{2t_1^2\cos\alpha_1}$$

$$LT_{A2}=\frac{\nu_2R_1^2E_1}{2E_2t_1t_2\cos\alpha_2}+\frac{R_1C_{AA2}}{2t_1}(\tan\alpha_1+\tan\alpha_2)$$

$$LT_{AC}=0\,*$$

$$LT_{B1}=\frac{R_1\tan\alpha_1}{2t_1\cos\alpha_1}$$

$$LT_{B2}=\frac{E_1R_1^2\tan\alpha_2}{2E_2R_2t_2\cos\alpha_2}+\frac{R_1C_{AB2}}{2t_1}(\tan\alpha_1+\tan\alpha_2)$$

$$LT_{BC}=0\,*$$

At the junction of the two cones,

$$V_1=\frac{Pt_1K_{V1}}{\pi R_1^2},\qquad M_1=\frac{Pt_1^2K_{M1}}{\pi R_1^2}$$

$$N_1=\frac{P}{2\pi R_1\cos\alpha_1}-V_1\sin\alpha_1$$

$$\Delta R_A=\frac{Pt_1}{E_1\pi R_1^2}(LT_{A1}-K_{V1}C_{AA1}-K_{M1}C_{AB1})$$

$$\psi_A=\frac{P}{E_1\pi R_1^2}(-LT_{B1}+K_{V1}C_{AB1}+K_{M1}C_{BB1})$$

For axial tension, $E_1=E_2$, $\nu_1=\nu_2=0.3$, $t_1=t_2$, $R_1=R_2$, and for $R/t\cos\alpha>5$.

$$\Delta R_A=\frac{P\nu_1}{2\pi E_1t_1}K_{\Delta RA},\qquad \psi_A=\frac{P\nu_1}{2\pi E_1t_1^2}K_{\psi A},\qquad \sigma_2=\frac{P}{2\pi R_1t_1}K_{\sigma2}$$

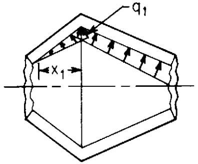
		α_1					
		−30			15		
		R_1/t_1			R_1/t_1		
	α_2	10	20	50	10	20	50
K_{V1}	−30.0	−2.8868	−5.7735	−14.4338	−0.7618	−1.5181	−3.7830
	−15.0	−2.1629	−4.3306	−10.8367	0.0000	0.0000	0.0000
	15.0	−0.7852	−1.5760	−3.9521	1.3397	2.6795	6.6987
	30.0	0.0000	0.0000	0.0000	2.0473	4.0996	10.2597
	45.0	0.9808	1.9747	4.9659	2.8992	5.8125	14.5630
K_{M1}	−30.0	−3.4070	−9.4911	−37.1735	−0.9420	−2.6184	−10.2418
	−15.0	−2.5628	−7.1386	−27.9573	0.0000	0.0000	0.0000
	15.0	−0.9420	−2.6184	−10.2418	1.6690	4.6507	18.2184
	30.0	0.0000	0.0000	0.0000	2.5628	7.1386	27.9573
	45.0	1.2012	3.3225	12.9563	3.6553	10.1672	39.7823
$K_{\Delta RA}$	−30.0	6.4309	9.5114	15.5525	0.8751	1.7262	3.3919
	−15.0	4.5065	6.8241	11.3678	−1.0353	−1.0353	−1.0353
	15.0	0.8751	1.7262	3.3919	−4.3681	−5.8778	−8.8384
	30.0	−1.1547	−1.1547	−1.1547	−6.1013	−8.4188	−12.9625
	45.0	−3.6360	−4.7187	−6.8292	−8.1516	−11.4538	−17.9219
$K_{\psi A}$	−30.0	0.0000	0.0000	0.0000	−0.1324	−0.0619	−0.0213
	−15.0	0.0362	0.0144	0.0028	−0.0925	−0.0462	−0.0185
	15.0	0.1324	0.0619	0.0213	0.0000	0.0000	0.0000
	30.0	0.2222	0.1111	0.0444	0.0713	0.0394	0.0187
	45.0	0.3819	0.2012	0.0885	0.1870	0.1045	0.0503
$K_{\sigma2}$	−30.0	2.1891	3.1132	4.9256	0.5849	0.8402	1.3399
	−15.0	1.6335	2.3287	3.6917	0.0000	0.0000	0.0000
	15.0	0.5854	0.8406	1.3403	−1.0207	−1.4735	−2.3617
	30.0	0.0000	0.0000	0.0000	−1.5516	−2.2469	−3.6100
	45.0	−0.7150	−1.0396	−1.6725	−2.1799	−3.1707	−5.1112

* If $\alpha_1+\alpha_2$ is zero or close to zero the following correction terms should be used:

$$LT_{AC}=\frac{-R_1(R_2-R_1)C_{AB2}}{2t_1^2\cos^2\ \alpha_2},\qquad LT_{BC}=\frac{-R_1(R_2-R_1)C_{BB2}}{2t_1^2\cos^2\ \alpha_2}$$

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

5c. Hydrostatic internal* pressure q_1 at the junction if $|\mu| > 4$† at the position of zero pressure. If $|\mu| < 4$ at this position the discontinuity in pressure gradient introduces deformations at the junction.



Note: There is no axial load on the junction. An axial load on the left end of the left cone balances any axial component of the pressure on the left cone, and an axial load on the right end of the right cone balances the axial component of pressure on the right cone and on the joint.

$$LT_{A1} = \frac{b_1 R_1}{t_1^2 \cos\alpha_1}$$

$$LT_{A2} = \frac{-b_2 R_2 E_1}{E_2 t_1 t_2 \cos\alpha_2}$$

For LT_{AC} use the expressions from case 5a

$$LT_{B1} = \frac{-b_1}{t_1 \cos\alpha_1}\left(\frac{R_1}{x_1} + 2\tan\alpha_1\right)$$

$$LT_{B2} = \frac{E_1 b_2}{E_2 t_2 \cos\alpha_2}\left(\frac{R_2}{x_1} - 2\tan\alpha_2\right)$$

For LT_{BC} use the expressions from case 5a

At the junction of the two cones,

$$V_1 = q_1 t_1 K_{V1}, \quad M_1 = q_1 t_1^2 K_{M1}$$

$$N_1 = -V_1 \sin\alpha_1$$

$$\Delta R_A = \frac{q_1 t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For internal pressure, $x_1 = R_1$, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $R_1 = R_2$, and for $R/t \cos\alpha > 5$.
(*Note:* No correction terms are used)

$$\Delta R_A = \frac{q_1 R_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{q_1 R_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{q_1 R_1}{t_1} K_{\sigma 2}$$

		α_1					
		−30			15		
		R_1/t_1			R_1/t_1		
	α_2	10	20	50	10	20	50
K_{V1}	−30.0	0.0000	0.0000	0.0000	−0.0764	−0.1081	−0.1712
	−15.0	0.0746	0.1065	0.1696	0.0000	0.0000	0.0000
	15.0	0.0763	0.1081	0.1711	0.0000	0.0000	0.0000
	30.0	0.0000	0.0000	0.0000	−0.0799	−0.1114	−0.1742
	45.0	−0.1844	−0.2582	−0.4055	−0.2704	−0.3744	−0.5831
K_{M1}	−30.0	0.4408	0.6374	1.0216	0.1546	0.2224	0.3554
	−15.0	0.2988	0.4334	0.6958	0.0000	0.0000	0.0000
	15.0	0.1120	0.1623	0.2604	−0.1940	−0.2809	−0.4502
	30.0	0.0000	0.0000	0.0000	−0.3024	−0.4370	−0.6995
	45.0	−0.2010	−0.2875	−0.4572	−0.4955	−0.7113	−1.1335
$K_{\Delta RA}$	−30.0	1.2502	1.2351	1.2123	1.0959	1.1071	1.1093
	−15.0	1.1449	1.1429	1.1324	0.9853	1.0103	1.0253
	15.0	1.0818	1.0972	1.1031	0.9215	0.9640	0.9956
	30.0	1.1047	1.1297	1.1447	0.9484	1.0003	1.0408
	45.0	1.1829	1.2239	1.2540	1.0341	1.1016	1.1570
$K_{\psi A}$	−30.0	1.1047	1.1297	1.1447	2.1850	2.3365	2.5730
	−15.0	0.4478	0.3602	0.1621	1.5133	1.5517	1.5747
	15.0	−0.0986	−0.2000	−0.4064	0.9853	1.0103	1.0253
	30.0	−0.1709	−0.1748	−0.1771	0.9530	1.0762	1.2957
	45.0	−0.1440	0.0905	0.5732	1.0535	1.4138	2.1147
$K_{\sigma 2}$	−30.0	1.2502	1.2351	1.2123	1.0965	1.1075	1.1096
	−15.0	1.1460	1.1437	1.1329	0.9853	1.0103	1.0253
	15.0	1.0830	1.0980	1.1036	0.9215	0.9640	0.9956
	30.0	1.1047	1.1297	1.1447	0.9491	1.0008	1.0411
	45.0	1.1801	1.2219	1.2528	1.0362	1.1031	1.1579

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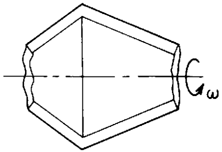
* For external pressure, substitute $-q_1$ for q_1, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in the load terms.

† If pressure increases right to left, substitute $-x_1$ for x_1 and verify that $|x_1|$ is large enough to extend into the right cone as far as the position where $|\mu| = 4$.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
5d. Rotation around the axis of symmetry at ω rad/s  *Note:* δ = mass/unit volume	$LT_{A1}=\dfrac{R_1^2}{t_1^2}$ $LT_{A2}=\dfrac{-\delta_2 R_2^3 E_1}{\delta_1 R_1 E_2 t_1^2}$ $LT_{AC}=0$ $LT_{B1}=\dfrac{-R_1(3+\nu_1)\tan\alpha_1}{t_1}$ $LT_{B2}=\dfrac{-\delta_2 R_2^2 E_1(3+\nu_2)\tan\alpha_2}{\delta_1 E_2 R_1 t_1}$ $LT_{BC}=0$	For $E_1=E_2$, $\nu_1=\nu_2=0.3$, $t_1=t_2$, $R_1=R_2$, $\delta_1=\delta_2$, and for $R/t\cos\alpha>5$. $\Delta R_A=\dfrac{\delta_1\omega^2R_1^3}{E_1}K_{\Delta RA}$, $\psi_A=\dfrac{\delta_1\omega^2R_1^2}{E_1}K_{\psi A}$, $\sigma_2=\delta_1\omega^2R_1^2K_{\sigma 2}$
	At the junction of the two cones, $V_1=\delta_1\omega^2R_1t_1^2K_{V1}$, $N_1=V_1\sin\alpha_1$ $M_1=\delta_1\omega^2R_1t_1^3K_{M1}$ $\Delta R_A=\dfrac{\delta_1\omega^2R_1t_1^2}{E_1}(LT_{A1}-K_{V1}C_{AA1}-K_{M1}C_{AB1})$ $\psi_A=\dfrac{\delta_1\omega^2R_1t_1}{E_1}(-LT_{B1}+K_{V1}C_{AB1}+K_{M1}C_{BB1})$	

		α_1					
		−30			15		
		R_1/t_1			R_1/t_1		
	α_2	10	20	50	10	20	50
K_{V1}	−30.0	0.0000	0.0000	0.0000	−0.0001	−0.0002	−0.0001
	−15.0	−0.0003	−0.0001	0.0000	0.0000	0.0000	0.0000
	15.0	0.0001	0.0002	0.0001	0.0000	0.0000	0.0000
	30.0	0.0000	0.0000	0.0000	−0.0011	−0.0006	−0.0003
	45.0	−0.0016	−0.0010	−0.0005	−0.0043	−0.0023	−0.0011
K_{M1}	−30.0	0.6583	0.9309	1.4726	0.1818	0.2569	0.4061
	−15.0	0.4951	0.7004	1.1079	0.0000	0.0000	0.0000
	15.0	0.1818	0.2569	0.4061	−0.3249	−0.4588	−0.7245
	30.0	0.0000	0.0000	0.0000	−0.5001	−0.7053	−1.1129
	45.0	−0.2315	−0.3260	−0.5141	−0.7149	−1.0061	−1.5851
$K_{\Delta RA}$	−30.0	1.2173	1.1539	1.0975	1.0599	1.0424	1.0268
	−15.0	1.1636	1.1158	1.0733	1.0000	1.0000	1.0000
	15.0	1.0599	1.0424	1.0268	0.8932	0.9245	0.9522
	30.0	1.0000	1.0000	1.0000	0.8364	0.8842	0.9267
	45.0	0.9248	0.9466	0.9662	0.7683	0.8356	0.8958
$K_{\psi A}$	−30.0	0.0000	0.0000	0.0000	1.3796	1.3799	1.3803
	−15.0	−0.4714	−0.4717	−0.4719	0.8842	0.8842	0.8842
	15.0	−1.3796	−1.3799	−1.3803	0.0000	0.0000	0.0000
	30.0	−1.9053	−1.9053	−1.9053	−0.4735	−0.4732	−0.4730
	45.0	−2.5700	−2.5693	−2.5686	−1.0477	−1.0473	−1.0466
$K_{\sigma 2}$	−30.0	1.2173	1.1539	1.0975	1.0599	1.0424	1.0268
	−15.0	1.1636	1.1158	1.0733	1.0000	1.0000	1.0000
	15.0	1.0599	1.0424	1.0268	0.8932	0.9245	0.9522
	30.0	1.0000	1.0000	1.0000	0.8364	0.8842	0.9267
	45.0	0.9248	0.9466	0.9662	0.7683	0.8356	0.8958

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

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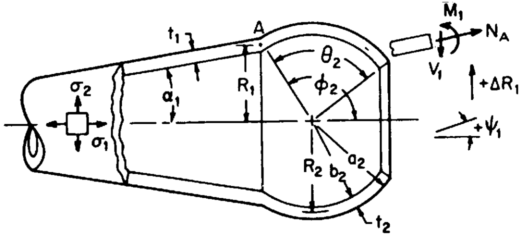
6. Conical shell connected to a spherical shell.* To ensure accuracy, evaluate k_A and the value of k in the cone at the position where $\mu = 4$. The absolute values of k at both positions should be greater than 5. $R/(t_1 \cos\alpha_1)$ should also be greater than 5 at both these positions. The junction angle for the spherical shell must lie within the range $3/\beta_2 < \phi_2 < \pi - 3/\beta_2$. The spherical shell must also extend with no interruptions such as a second junction or a cutout, such that $\theta_2 > 3/\beta_2$. See the discussion on page 565. E_1 and E_2 are the moduli of elasticity and ν_1 and ν_2 the Poisson's ratios for the cone and the sphere, respectively. $b_1 = R_1 - (t_1 \cos\alpha_1)/2$, $a_1 = R_1 + (t_1 \cos\alpha_1)/2$, and $R_A = R_1$. See Table 13.3, case 4, for formulas for k_A, β_1, μ, C_1, and the F functions for the cone. See Table 13.3, case 1, for formulas for K_{12},†, K_{22},† and β_2 for the spherical shell. $b_2 = R_2 - t_2/2$ and $a_2 = R_2 + t_2/2$. Normally $R_2 \sin\phi_2 = R_1$, but if $\phi_2 - \alpha_1 = 90°$ or is close to 90° the midthickness radii at the junction may not be equal. Under this condition different correction terms will be used if necessary.



$$K_{V1} = \frac{LT_A C_{BB} - LT_B C_{AB}}{C_{AA} C_{BB} - C_{AB}^2}, \qquad K_{M1} = \frac{LT_B C_{AA} - LT_A C_{AB}}{C_{AA} C_{BB} - C_{AB}^2}, \qquad \left.\begin{aligned} LT_A &= LT_{A1} + LT_{A2} + LT_{AC} \\ LT_B &= LT_{B1} + LT_{B2} + LT_{BC} \end{aligned}\right\} \text{ See cases 6a – 6d for these load terms}$$

$$C_{AA} = C_{AA1} + C_{AA2}, \qquad C_{AA1} = \frac{R_1 \sin\alpha_1}{t_1 \sqrt{2}} \left[\frac{k_A}{C_1} \left(F_{4A} - \frac{4\nu_1^2}{k_A^2} F_{2A} \right) \right], \qquad C_{AA2} = \frac{R_A E_1 \beta_2 \sin\phi_2}{E_2 t_2} \left(\frac{1}{K_{12}} + K_{22} \right)$$

$$C_{AB} = C_{AB1} + C_{AB2}, \qquad C_{AB1} = \frac{-R_1}{t_1} \frac{\beta_1 F_{7A}}{C_1}, \qquad C_{AB2} = \frac{2E_1 R_A t_1 \beta_2^2}{E_2 R_2 t_2 K_{12}}$$

$$C_{BB} = C_{BB1} + C_{BB2}, \qquad C_{BB1} = \frac{R_1 2\sqrt{2}}{t_1 \sin\alpha_1} \frac{\beta_1^2 F_{2A}}{k_A C_1}, \qquad C_{BB2} = \frac{4E_1 R_A t_1^2 \beta_2^3}{R_2^2 E_2 t_2 K_{12} \sin\phi_2}$$

The stresses in the left-hand cone at the junction are given by

$$\sigma_1 = \frac{N_1}{t_1}, \qquad \sigma_2 = \frac{\Delta R_A E_1}{R_A} + \nu_1 \sigma_1$$

$$\sigma_1' = \frac{-6M_1}{t_1^2}, \qquad \sigma_2' = \frac{-6V_1 \sin\alpha_1}{t_1 \beta_1} (1 - \nu_1^2) \frac{F_{7A}}{C_1} + \sigma_1' \left[\nu_1 + \frac{2\sqrt{2}}{k_A C_1} (1 - \nu_1^2) F_{2A} \right]$$

Note: The use of joint load correction terms LT_{AC} and LT_{BC} depends upon the accuracy desired and the relative values of the thicknesses and the radii. Read Sec. 13.4 carefully. For thin-walled shells, $R/t > 10$, they can be neglected.

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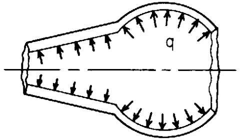
* If the conical shell increases in radius away from the junction, substitute $-\alpha$ for α for the cone in all of the appropriate formulas above and in those used from case 4, Table 13.3.

† The second subscript is added to refer to the right-hand shell.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values

6a. Internal* pressure q



Note: There is no axial load on the junction. An axial load on the left end of the cone balances any axial component of the pressure on the cone, and an axial load on the right end of the sphere balances any axial component of the pressure on the sphere and on the joint. For an enclosed pressure vessel superpose an axial load $P = q\pi b_1^2$ using case 6b.

$$LT_{A1} = \frac{b_1 R_1}{t_1^2 \cos\alpha_1}$$

$$LT_{A2} = \frac{-b_2^2 E_1 \sin\phi_2}{E_2 t_1 t_2}$$

$$LT_{AC} = \frac{t_2\cos\phi_2 + t_1\sin\alpha_1}{2t_1} C_{AA2} + \frac{t_2^2 - t_1^2}{8t_1^2} C_{AB2} + \frac{E_1 R_1 (1+\nu_2)(t_1\cos\alpha_1 - t_2\sin\phi_2)}{2E_2 t_1 t_2 \sin\phi_2} \dagger$$

$$LT_{B1} = \frac{-2b_1 \tan\alpha_1}{t_1\cos\alpha_1}$$

$$LT_{B2} = 0$$

$$LT_{BC} = \frac{t_2\cos\phi_2 + t_1\sin\alpha_1}{2t_1} C_{AB2} + \frac{t_2^2 - t_1^2}{8t_1^2} C_{BB2} + \frac{E_1 \cos\phi_2 (t_1\cos\alpha_1 - t_2\sin\phi_2)}{E_2 t_2 \sin^2\ \phi_2} \dagger$$

At the junction of the cone and sphere,

$$V_1 = q t_1 K_{V1}, \quad M_1 = q t_1^2 K_{M1}, \quad N_1 = -V_1 \sin\alpha_1$$

$$\Delta R_A = \frac{q t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $R_2 \sin\phi_2 = R_1$, and for $R/t \cos\alpha_1 > 5$ and $R_2/t_2 > 5$. (*Note:* No correction terms are used)

$$\Delta R_A = \frac{q R_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{q R_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{q R_1}{t_1} K_{\sigma 2}$$

		α_1					
		−30			15		
		R_1/t_1			R_1/t_1		
	ϕ_2	10	20	50	10	20	50
K_{V1}	45.0	−0.1504	−0.2338	−0.3900	−0.2313	−0.3470	−0.5661
	60.0	0.0303	0.0220	0.0141	−0.0468	−0.0877	−0.1592
	90.0	0.1266	0.1594	0.2327	0.0520	0.0528	0.0630
	105.0	0.1050	0.1287	0.1839	0.0296	0.0215	0.0138
	120.0	0.0316	0.0232	0.0150	−0.0460	−0.0861	−0.1570
K_{M1}	45.0	0.2084	0.3034	0.4889	−0.0811	−0.1182	−0.1908
	60.0	0.2172	0.3158	0.5081	−0.0920	−0.1335	−0.2146
	90.0	0.2285	0.3306	0.5301	−0.0978	−0.1416	−0.2270
	105.0	0.2288	0.3303	0.5285	−0.0973	−0.1408	−0.2255
	120.0	0.2240	0.3224	0.5146	−0.0960	−0.1388	−0.2221
$K_{\Delta RA}$	45.0	1.2913	1.3082	1.3113	1.1396	1.1841	1.2132
	60.0	1.1526	1.1698	1.1735	0.9917	1.0371	1.0675
	90.0	1.0809	1.0969	1.0997	0.9124	0.9576	0.9880
	105.0	1.0979	1.1137	1.1164	0.9301	0.9751	1.0055
	120.0	1.1538	1.1703	1.1736	0.9898	1.0354	1.0662
$K_{\psi A}$	45.0	−0.1762	0.0897	0.6014	1.0674	1.4565	2.1836
	60.0	−0.7470	−0.7309	−0.7111	0.4315	0.5731	0.8128
	90.0	−1.0321	−1.1548	−1.4058	0.0909	0.0946	0.0646
	105.0	−0.9601	−1.0540	−1.2466	0.1658	0.1993	0.2288
	120.0	−0.7317	−0.7212	−0.7058	0.4181	0.5574	0.7961
$K_{\sigma 2}$	45.0	1.2891	1.3064	1.3101	1.1414	1.1854	1.2140
	60.0	1.1531	1.1700	1.1735	0.9920	1.0375	1.0678
	90.0	1.0828	1.0981	1.1004	0.9120	0.9574	0.9879
	105.0	1.0995	1.1146	1.1169	0.9299	0.9750	1.0055
	120.0	1.1543	1.1705	1.1736	0.9901	1.0357	1.0665

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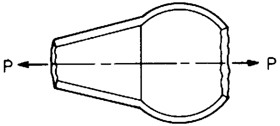
* For external pressure, substitute $-q$ for q, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in the load terms.

† If $\phi_2 - \alpha_1 = 90^\circ$ or is close to 90° the following correction terms should be used:

$$LT_{AC} = \frac{b_1^2 - b_2^2\sin^2\phi_2}{4t_1^2\sin^2\phi_2}\left[\frac{a_2\sin\phi_2 - b_1}{R_1} C_{AB2} - \frac{2E_1 t_1(1+\nu_2)\sin\phi_2}{E_2 t_2}\right], \qquad LT_{BC} = \frac{b_1^2 - b_2^2\sin^2\phi_2}{4t_1^2\sin^2\phi_2}\ \frac{a_2\sin\phi_2 - b_1}{R_1} C_{BB2}$$

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

6b. Axial load P



$$LT_{A1} = \frac{-\nu_1 R_1^2}{2t_1^2 \cos\alpha_1}$$

$$LT_{A2} = \frac{R_1^2 E_1 (1+\nu_2)}{2E_2 t_1 t_2 \sin\phi_2} + \frac{R_1 C_{AA2}}{2t_1}\left(\tan\alpha_1 + \frac{1}{\tan\phi_2}\right)$$

$$LT_{AC} = 0$$

$$LT_{B1} = \frac{R_1 \tan\alpha_1}{2t_1 \cos\alpha_1}$$

$$LT_{BC} = 0$$

$$LT_{B2} = \frac{R_1 C_{AB2}}{2t_1}\left(\tan\alpha_1 + \frac{1}{\tan\phi_2}\right)$$

At the junction of the cone and sphere,

$$V_1 = \frac{P t_1 K_{V1}}{\pi R_1^2}, \qquad M_1 = \frac{P t_1^2 K_{M1}}{\pi R_1^2}$$

$$N_1 = \frac{P}{2\pi R_1 \cos\alpha_1} - V_1 \sin\alpha_1$$

$$\Delta R_A = \frac{P t_1}{E_1 \pi R_1^2}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

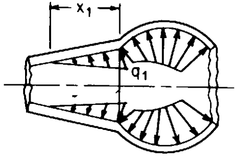
$$\psi_A = \frac{P}{E_1 \pi R_1^2}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For axial tension, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $R_2 \sin\phi_2 = R_1$, and for $R/t \cos\alpha_1 > 5$ and $R_2/t_2 > 5$.

$$\Delta R_A = \frac{P\nu_1}{2\pi E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{P\nu_1}{2\pi E_1 t_1^2} K_{\psi A}, \qquad \sigma_2 = \frac{P}{2\pi R_1 t_1} K_{\sigma 2}$$

		α_1					
		−30			15		
		R_1/t_1			R_1/t_1		
	ϕ_2	10	20	50	10	20	50
K_{V1}	45.0	1.4637	2.6550	6.0381	3.3529	6.4470	15.5559
	60.0	0.3826	0.5404	0.8537	2.4117	4.6104	11.0611
	90.0	−1.1708	−2.5296	−6.7444	0.9917	1.7969	4.0811
	105.0	−1.8475	−3.8810	−10.1209	0.3241	0.4582	0.7244
	120.0	−2.5422	−5.2810	−13.6476	−0.4001	−1.0054	−2.9706
K_{M1}	45.0	1.0818	3.1542	12.6906	3.5439	10.0102	39.5349
	60.0	−0.0630	−0.0887	−0.1397	2.5015	7.0525	27.8219
	90.0	−1.7682	−4.9205	−19.2610	0.8411	2.3448	9.1872
	105.0	−2.5375	−7.1021	−27.8984	0.0271	0.0384	0.0608
	120.0	−3.3516	−9.4119	−37.0470	−0.8798	−2.5305	−10.1027
$K_{\Delta RA}$	45.0	−6.4214	−7.3936	−9.4058	−10.7636	−13.9818	−20.3748
	60.0	−3.2923	−3.2299	−3.1745	−8.1367	−10.4098	−14.9142
	90.0	1.0715	2.6784	5.8176	−4.3666	−5.1327	−6.6300
	105.0	2.9144	5.2194	9.7519	−2.6671	−2.6945	−2.7189
	120.0	4.7524	7.7965	13.8050	−0.8758	−0.0808	1.5350
$K_{\psi A}$	45.0	−0.9108	−0.6635	−0.4304	−1.0097	−0.6930	−0.4266
	60.0	−0.7412	−0.5450	−0.3563	−0.8384	−0.5768	−0.3558
	90.0	−0.6135	−0.4575	−0.3026	−0.7396	−0.5126	−0.3183
	105.0	−0.6090	−0.4562	−0.3028	−0.7536	−0.5247	−0.3272
	120.0	−0.6513	−0.4888	−0.3250	−0.8121	−0.5683	−0.3560
$K_{\sigma 2}$	45.0	−1.5361	−1.8318	−2.4391	−2.9706	−3.9340	−5.8502
	60.0	−0.6298	−0.6144	−0.6008	−2.1679	−2.8482	−4.1980
	90.0	0.6327	1.1120	2.0512	−1.0148	−1.2432	−1.6911
	105.0	1.1653	1.8540	3.2113	−0.4946	−0.5013	−0.5073
	120.0	1.6959	2.6062	4.4060	0.0540	0.2942	0.7803

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
6c. Hydrostatic internal* pressure q_1 at the junction when $\lvert\mu\rvert > 4$† at the position of zero pressure. If $\lvert\mu\rvert < 4$ at this position the discontinuity in pressure gradient introduces small deformations at the junction.  	$LT_{A1} = \dfrac{b_1 R_1}{t_1^2 \cos\alpha_1}$ $LT_{A2} = \dfrac{-b_2^2 E_1 \sin\phi_2}{E_2 t_1 t_2}$ For LT_{AC} use the expressions from case 6a $LT_{B1} = \dfrac{-b_1}{t_1 \cos\alpha_1}\left(\dfrac{R_1}{x_1} + 2\tan\alpha_1\right)$ $LT_{B2} = \dfrac{E_1 b_2 R_2 \sin\phi_2}{E_2 t_2 x_1}$ For LT_{BC} use the expressions from case 6a	For internal pressure, $x_1 = R_1$, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $R_2 \sin\phi_2 = R_1$, and for $R/t \cos\alpha_1 > 5$ and $R_2/t_2 > 5$. (*Note:* No correction terms are used) $\Delta R_A = \dfrac{q_1 R_1^2}{E_1 t_1} K_{\Delta RA}, \quad \psi_A = \dfrac{q_1 R_1}{E_1 t_1} K_{\psi A}, \quad \sigma_2 = \dfrac{q_1 R_1}{t_1} K_{\sigma 2}$
Note: There is no axial load on the junction. An axial load on the left end of the cone balances any axial component of the pressure on the cone, and an axial load on the right end of the sphere balances any axial component of the pressure on the sphere and on the joint.	At the junction of the cone and sphere, $V_1 = q_1 t_1 K_{V1}, \quad M_1 = q_1 t_1^2 K_{M1}$ $N_1 = -V_1 \sin\alpha_1$ $\Delta R_A = \dfrac{q_1 t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$ $\psi_A = \dfrac{q_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$	

		α_1					
		−30			15		
		R_1/t_1			R_1/t_1		
	ϕ_2	10	20	50	10	20	50
K_{V1}	45.0	−0.1508	−0.2341	−0.3902	−0.2320	−0.3475	−0.5664
	60.0	0.0303	0.0220	0.0141	−0.0469	−0.0878	−0.1593
	90.0	0.1266	0.1594	0.2327	0.0520	0.0528	0.0630
	105.0	0.1050	0.1287	0.1839	0.0296	0.0215	0.0138
	120.0	0.0316	0.0232	0.0150	−0.0459	−0.0860	−0.1569
K_{M1}	45.0	0.2501	0.3628	0.5833	−0.0185	−0.0292	−0.0492
	60.0	0.2172	0.3158	0.5081	−0.0710	−0.1037	−0.1673
	90.0	0.2008	0.2914	0.4681	−0.1043	−0.1508	−0.2416
	105.0	0.2074	0.3002	0.4809	−0.0973	−0.1408	−0.2255
	120.0	0.2240	0.3224	0.5146	−0.0744	−0.1084	−0.1743
$K_{\Delta RA}$	45.0	1.3054	1.3182	1.3176	1.1607	1.1990	1.2226
	60.0	1.1526	1.1698	1.1735	0.9987	1.0421	1.0707
	90.0	1.0718	1.0904	1.0956	0.9103	0.9560	0.9870
	105.0	1.0909	1.1087	1.1133	0.9301	0.9751	1.0055
	120.0	1.1538	1.1703	1.1736	0.9968	1.0403	1.0694
$K_{\psi A}$	45.0	1.0506	1.3418	1.8688	2.2254	2.6400	3.3827
	60.0	0.3577	0.3988	0.4336	1.4744	1.6411	1.8960
	90.0	−0.0076	−0.1053	−0.3413	1.0584	1.0871	1.0721
	105.0	0.0830	0.0141	−0.1634	1.1511	1.2096	1.2541
	120.0	0.3730	0.4085	0.4389	1.4618	1.6261	1.8797
$K_{\sigma 2}$	45.0	1.3031	1.3164	1.3164	1.1625	1.2004	1.2235
	60.0	1.1531	1.1700	1.1735	0.9990	1.0425	1.0709
	90.0	1.0737	1.0916	1.0963	0.9099	0.9558	0.9869
	105.0	1.0925	1.1097	1.1138	0.9299	0.9750	1.0055
	120.0	1.1543	1.1705	1.1736	0.9971	1.0407	1.0696

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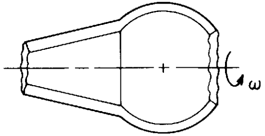
* For external pressure, substitute $-q_1$ for q_1, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in load terms.

† If pressure increases right to left, substitute $-x_1$ for x_1 and verify that $\lvert x_1\rvert$ is large enough to extend into the sphere as far as the position where $\theta_2 = 3/\beta_2$.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

6d. Rotation around the axis of symmetry at ω rad/s



Note: $\delta =$ mass/unit volume

$$LT_{A1} = \frac{R_1^2}{t_1^2}$$

$$LT_{A2} = \frac{-\delta_2 R_2^3 E_1 \sin^3\ \phi_2}{\delta_1 R_1 E_2 t_1^2}$$

$$LT_{AC} = 0$$

$$LT_{B1} = \frac{-R_1(3+\nu_1)\tan\alpha_1}{t_1}$$

$$LT_{B2} = \frac{-\delta_2 R_2^2 E_1 (3+\nu_2)\sin\phi_2\cos\phi_2}{\delta_1 R_1 t_1 E_2}$$

$$LT_{BC} = 0$$

At the junction of the cone and sphere,

$$V_1 = \delta_1 \omega^2 R_1 t_1^2 K_{V1}, \qquad M_1 = \delta_1 \omega^2 R_1 t_1^3 K_{M1}$$

$$N_1 = -V_1 \sin\alpha_1$$

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1 t_1^2}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{\delta_1 \omega^2 R_1 t_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $\delta_1 = \delta_2$, $R_2 \sin\phi_2 = R_1$, and for $R/t\cos\alpha_1 > 5$ and $R_2/t_2 > 5$.

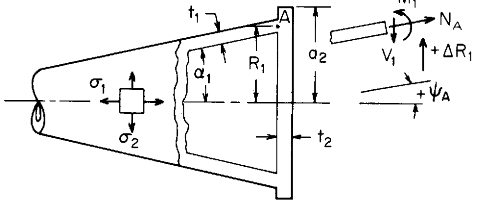
$$\Delta R_A = \frac{\delta_1 \omega^2 R_1^3}{E_1} K_{\Delta RA}, \qquad \psi_A = \frac{\delta_1 \omega^2 R_1^2}{E_1} K_{\psi A}, \qquad \sigma_2 = \delta_1 \omega^2 R_1^2 K_{\sigma 2}$$

		α_1					
		−30			15		
		R_1/t_1			R_1/t_1		
	ϕ_2	10	20	50	10	20	50
K_{V1}	45.0	0.0023	0.0015	0.0009	0.0078	0.0053	0.0033
	60.0	0.0000	0.0000	0.0000	0.0035	0.0024	0.0015
	90.0	−0.0001	0.0000	0.0001	0.0002	0.0001	0.0001
	105.0	0.0012	0.0010	0.0007	0.0000	0.0000	0.0000
	120.0	0.0034	0.0027	0.0019	0.0008	0.0006	0.0004
K_{M1}	45.0	−0.2240	−0.3189	−0.5073	−0.6913	−0.9836	−1.5636
	60.0	0.0000	0.0000	0.0000	−0.4906	−0.6961	−1.1039
	90.0	0.3414	0.4828	0.7636	−0.1635	−0.2310	−0.3651
	105.0	0.4988	0.7041	1.1117	0.0000	0.0000	0.0000
	120.0	0.6678	0.9408	1.4829	0.1845	0.2597	0.4090
$K_{\Delta RA}$	45.0	0.9243	0.9464	0.9661	0.7667	0.8350	0.8957
	60.0	1.0000	1.0000	1.0000	0.8360	0.8840	0.9267
	90.0	1.1127	1.0798	1.0505	0.9461	0.9619	0.9759
	105.0	1.1637	1.1159	1.0733	1.0000	1.0000	1.0000
	120.0	1.2177	1.1541	1.0975	1.0600	1.0424	1.0268
$K_{\psi A}$	45.0	−2.5613	−2.5631	−2.5646	−1.0229	−1.0292	−1.0349
	60.0	−1.9053	−1.9053	−1.9053	−0.4625	−0.4653	−0.4679
	90.0	−0.9169	−0.9173	−0.9177	0.4386	0.4385	0.4384
	105.0	−0.4656	−0.4676	−0.4694	0.8842	0.8842	0.8842
	120.0	0.0161	0.0113	0.0071	1.3838	1.3829	1.3821
$K_{\sigma 2}$	45.0	0.9243	0.9464	0.9661	0.7666	0.8350	0.8957
	60.0	1.0000	1.0000	1.0000	0.8360	0.8840	0.9267
	90.0	1.1127	1.0798	1.0505	0.9461	0.9619	0.9759
	105.0	1.1637	1.1159	1.0733	1.0000	1.0000	1.0000
	120.0	1.2177	1.1541	1.0975	1.0600	1.0424	1.0268

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no. Load terms Selected values

7. Conical shell connected to a circular plate.* To ensure accuracy, evaluate k_A and the value of k in the cone at the position where $\mu = 4$. The absolute values of k at both positions should be greater than 5. $R/(t_1 \cos \alpha_1)$ should also be greater than 5 at both these positions. E_1 and E_2 are the moduli of elasticity and ν_1 and ν_2 the Poisson's ratios for the cone and the plate, respectively. $b_1 = R_1 - (t_1 \cos \alpha_1)/2$, $a_1 = R_1 + (t_1 \cos \alpha_1)/2$, and $R_A = R_1$. See Table 13.3, case 4, for formulas k_A, β_1, μ, C_1, and the F functions for the cone. See Table 11.2 for the formulas for D_2.



$$K_{P1} = 1 + \frac{R_1^2(1-\nu_2)}{a_2^2(1+\nu_2)}$$

$$K_{V1} = \frac{LT_A C_{BB} - LT_B C_{AB}}{C_{AA} C_{BB} - C_{AB}^2}, \qquad K_{M1} = \frac{LT_B C_{AA} - LT_A C_{AB}}{C_{AA} C_{BB} - C_{AB}^2}, \qquad \left.\begin{aligned} LT_A &= LT_{A1} + LT_{A2} + LT_{AC} \\ LT_B &= LT_{B1} + LT_{B2} + LT_{BC} \end{aligned}\right\} \text{ See cases 7a–7d for these load terms}$$

$$C_{AA} = C_{AA1} + C_{AA2}, \qquad C_{AA1} = \frac{R_1 \sin \alpha_1}{t_1 \sqrt{2}} \left[\frac{k_A}{C_1} \left(F_{4A} - \frac{4\nu_1^2}{k_A^2} F_{2A} \right) \right], \qquad C_{AA2} = \frac{E_1 t_2^2 R_1 K_{P1}}{6D_2}$$

$$C_{AB} = C_{AB1} + C_{AB2}, \qquad C_{AB1} = \frac{-R_1}{t_1} \left(\frac{\beta F_{7A}}{C_1} \right), \qquad C_{AB2} = \frac{E_1 t_1 t_2 R_1 K_{P1}}{4D_2}$$

$$C_{BB} = C_{BB1} + C_{BB2}, \qquad C_{BB1} = \frac{R_1 2\sqrt{2}}{t_1 \sin \alpha_1} \left(\frac{\beta^2 F_{2A}}{k_A C_1} \right)_1, \qquad C_{BB2} = \frac{E_1 t_1^2 R_1 K_{P1}}{2D_2}$$

The stresses in the left-hand cone at the junction are given by

$$\sigma_1 = \frac{N_1}{t_1}, \qquad \sigma_2 = \frac{\Delta R_A E_1}{R_A} + \nu_1 \sigma_1$$

$$\sigma_1' = \frac{-6M_1}{t_1^2}, \qquad \sigma_2' = \frac{-6V_1 \sin \alpha_1}{t_1 \beta_1} (1-\nu_1^2) \left(\frac{F_{7A}}{C_1} \right)_1 + \sigma_1' \left[\nu_1 + \frac{2\sqrt{2}}{k_A C_1} (1-\nu_1^2) F_{2A} \right]_1$$

Note: The use of joint load correction terms LT_{AC} and LT_{BC} depends upon the accuracy desired and the relative values of the thicknesses and the radii. Read Sec. 13.4 carefully. For thin-walled shells, $R/t > 10$, they can be neglected.

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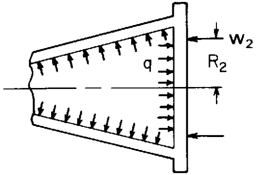
* *Note:* If the conical shell increases in radius away from the junction, substitute $-\alpha$ for α for the cone in all of the appropriate formulas above and in those used from case 4, Table 13.3.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
7a. Internal* pressure q 	$LT_{A1} = \dfrac{b_1 R_1}{t_1^2 \cos\alpha_1}, \quad LT_{A2} = \dfrac{E_1 t_2 b_1^2}{32 D_2 t_1 R_1} K_{P2}$ where $K_{P2} = \begin{cases} (2R_2^2 - b_1^2)K_{P1} & \text{for } R_2 \leqslant R_1 \\ (2R_2^2 - b_1^2)K_{P1} - 2(R_2^2 - R_1^2) + 4R_1^2 \ln\dfrac{R_2}{R_1} & \text{for } R_2 \geqslant R_1 \end{cases}$ $LT_{AC} = \dfrac{E_1 b_1 t_2 \sin\alpha_1}{12 D_2}\left(t_2 - \dfrac{3t_1 \sin\alpha_1}{8}\right)K_{P1}$ $LT_{B1} = \dfrac{-2b_1 \tan\alpha_1}{t_1 \cos\alpha_1}$ $LT_{B2} = \dfrac{E_1 b_1^2}{16 D_2 R_1} K_{P2}$ $LT_{BC} = \dfrac{E_1 b_1 t_1 \sin\alpha_1}{8 D_2}\left(t_2 - \dfrac{t_1 \sin\alpha_1}{2}\right)K_{P1}$	For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $a_2 = a_1$, $R_2 = 0.7R_1$, and for $R/t \cos\alpha_1 > 5$ and $R_1/t_2 > 4$. (*Note:* No correction terms are used) $\Delta R_A = \dfrac{qR_1^2}{E_1 t_1} K_{\Delta RA}, \quad \psi_A = \dfrac{qR_1}{E_1 t_1} K_{\psi A}, \quad \sigma_2 = \dfrac{qR_1}{t_1} K_{\sigma 2}$
Note: There is no axial load on the junction. An axial load on the left end of the cone balances any axial component of the pressure on the cone, and an axial load on the plate is reacted by the annular line load $w_2 = qb_1^2/(2R_2)$ at a radius R_2. For an enclosed pressure vessel superpose an axial load $P = q\pi b_1^2$ using case 7b.	At the junction of the cone and plate, $V_1 = qt_1 K_{V1}, \qquad M_1 = qt_1^2 K_{M1}$ $N_1 = -V_1 \sin\alpha_1$ $\Delta R_A = \dfrac{qt_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$ $\psi_A = \dfrac{q}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$	

		α_1					
		−30			30		
		R_1/t_1			R_1/t_1		
	$\dfrac{t_2}{t_1}$	15	30	50	15	30	50
K_{V1}	1.5	1.8080	2.5258	2.9931	1.8323	2.5642	3.0351
	2.0	2.0633	2.8778	3.4621	1.9988	2.8319	3.4256
	2.5	2.3269	3.2629	3.9945	2.1871	3.1396	3.8816
	3.0	2.5567	3.6105	4.4891	2.3608	3.4267	4.3139
	4.0		4.1282	5.2397		3.8687	4.9843
K_{M1}	1.5	1.3216	1.7736	0.9500	0.7803	1.1393	0.2492
	2.0	2.0621	3.2566	3.5429	1.2435	2.2323	2.3554
	2.5	2.7563	4.7394	6.2463	1.7200	3.3817	4.6146
	3.0	3.3251	6.0046	8.6287	2.1337	4.3961	6.6473
	4.0		7.7864	12.0641		5.8744	9.6442
$K_{\Delta RA}$	1.5	0.2625	0.2066	0.1777	0.1994	0.1699	0.1533
	2.0	0.2632	0.2130	0.1880	0.2011	0.1754	0.1622
	2.5	0.2458	0.2047	0.1837	0.1927	0.1701	0.1594
	3.0	0.2276	0.1891	0.1712	0.1793	0.1587	0.1496
	4.0		0.1546	0.1403		0.1321	0.1241
$K_{\psi A}$	1.5	−4.1463	−6.7069	−9.9970	−2.8696	−5.2149	−8.3411
	2.0	−3.2400	−5.3937	−8.1941	−2.3306	−4.2811	−6.9166
	2.5	−2.4705	−4.1900	−6.4582	−1.8320	−3.3853	−5.5109
	3.0	−1.8858	−3.2264	−5.0124	−1.4327	−2.6460	−4.3184
	4.0		−1.9618	−3.0515		−1.6459	−2.6691
$K_{\sigma 2}$	1.5	0.2806	0.2192	0.1867	0.1811	0.1570	0.1442
	2.0	0.2838	0.2274	0.1984	0.1811	0.1613	0.1519
	2.5	0.2717	0.2211	0.1957	0.1708	0.1544	0.1477
	3.0	0.2532	0.2072	0.1847	0.1557	0.1416	0.1366
	4.0		0.1753	0.1560		0.1127	0.1092

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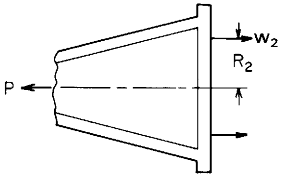
* For external pressure, substitute $-q$ for q and a_1 for b_1 in the load terms.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values

7b. Axial load P



$$w_2 = \frac{P}{2\pi R_2}$$

$$LT_{A1} = \frac{-\nu_1 R_1^2}{2t_1^2 \cos\alpha_1}, \qquad LT_{AC} = 0$$

$$LT_{A2} = \frac{E_1 t_2 R_1^3}{16 D_2 t_1} K_{P2} + \frac{R_1 C_{AA2}}{2t_1} \tan\alpha_1$$

where

$$K_{P2} = \begin{cases} \left(1 - \dfrac{R_2^2}{R_1^2}\right) K_{P1} & \text{for } R_2 \leqslant R_1 \\ -\dfrac{1-\nu_2}{1+\nu_2}\,\dfrac{R_2^2 - R_1^2}{a_2^2} - 2\ln\dfrac{R_2}{R_1} & \\ & \text{for } R_2 \geqslant R_1 \end{cases}$$

$$LT_{B1} = \frac{R_1 \tan\alpha_1}{2t_1 \cos\alpha_1}, \qquad LT_{BC} = 0$$

$$LT_{B2} = \frac{E_1 R_1^3}{8 D_2} K_{P2} + \frac{R_1 C_{AB2}}{2t_1} \tan\alpha_1$$

At the junction of the cone and plate,

$$V_1 = \frac{P t_1 K_{V1}}{\pi R_1^2}, \qquad M_1 = \frac{P t_1^2 K_{M1}}{\pi R_1^2}$$

$$N_1 = \frac{P}{2\pi R_1} \cos\alpha_1 - V_1 \sin\alpha_1$$

$$\Delta R_A = \frac{P t_1}{E_1 \pi R_1^2}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{P}{E_1 \pi R_1^2}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

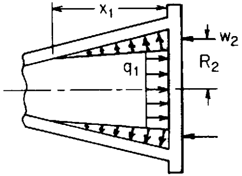
For axial tension, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $a_2 = a_1$, $R_2 = 0.8R_1$, and for $R/t \cos\alpha_1 > 5$ and $R_1/t_2 > 4$.

$$\Delta R_A = \frac{P\nu_1}{2\pi E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{P\nu_1}{2\pi E_1 t_1^2} K_{\psi A}, \qquad \sigma_2 = \frac{P}{2\pi R_1 t_1} K_{\sigma 2}$$

		α_1					
		−30			30		
		R_1/t_1			R_1/t_1		
	$\frac{t_2}{t_1}$	15	30	50	15	30	50
K_{V1}	1.5	3.5876	13.1267	31.6961	5.7802	16.6732	36.7478
	2.0	2.4577	10.0731	25.5337	4.6739	13.7167	30.7925
	2.5	1.5630	7.3736	19.6429	3.6712	10.8769	24.7596
	3.0	0.9203	5.2737	14.7796	2.8640	8.5120	19.5405
	4.0		2.6122	8.2677		5.2609	12.1583
K_{M1}	1.5	9.6696	48.9553	152.7700	13.8101	58.1862	169.2630
	2.0	6.3808	36.0554	118.6220	10.7213	46.0816	137.0810
	2.5	4.0214	25.6494	88.6891	8.1816	35.4643	107.1740
	3.0	2.4296	18.0038	65.2524	6.2589	27.1071	82.6250
	4.0		8.8324	35.4261		16.2264	49.6067
$K_{\Delta RA}$	1.5	−2.1662	−4.2130	−6.4572	−4.0595	−6.0789	−8.2689
	2.0	−2.2052	−4.6128	−7.3879	−4.1508	−6.5095	−9.1996
	2.5	−1.8764	−4.2328	−7.0820	−3.8563	−6.1865	−8.9613
	3.0	−1.4888	−3.6069	−6.2667	−3.4421	−5.5641	−8.1742
	4.0		−2.4320	−4.4893		−4.2647	−6.3131
$K_{\psi A}$	1.5	5.0355	8.6001	12.4137	5.8059	9.2022	12.8653
	2.0	3.2406	6.0555	9.2419	4.2033	6.8989	9.9584
	2.5	2.0765	4.1764	6.6773	3.0207	5.0584	7.4756
	3.0	1.3485	2.8815	4.7801	2.1953	3.7043	5.5547
	4.0		1.4330	2.5079		2.0669	3.1302
$K_{\sigma 2}$	1.5	−0.2317	−0.7862	−1.4006	−0.9870	−1.6440	−2.3547
	2.0	−0.2660	−0.9367	−1.7168	−0.9923	−1.7436	−2.5982
	2.5	−0.1853	−0.8497	−1.6603	−0.8839	−1.6183	−2.4905
	3.0	−0.0818	−0.6829	−1.4449	−0.7435	−1.4079	−2.2231
	4.0		−0.3571	−0.9508		−0.9856	−1.6205

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

7c. Hydrostatic internal* pressure q_1 at the junction when $|\mu| > 4$† at position of zero pressure. If $|\mu| < 4$ at this position, the discontinuity in pressure gradient introduces small deformations at the junction.



Note: There is no axial load on the junction. An axial load on the left end of the cone balances any axial component of the pressure on the cone, and the axial load on the plate is reacted by the annular line load $w_2 = q_1 b_1^2/(2R_2)$ at a radius R_2.

$$LT_{A1} = \frac{b_1 R_1}{t_1^2 \cos\alpha_1}$$

$$LT_{A2} = \frac{E_1 t_2 b_1^2}{32 D_2 t_1 R_1}$$

For K_{P2} use the expressions from case 7a

For LT_{AC} use the expression from case 7a

$$LT_{B1} = \frac{-b_1}{t_1 \cos\alpha_1}\left(\frac{R_1}{x_1} + 2\tan\alpha_1\right)$$

$$LT_{B2} = \frac{E_1 b_1^2}{16 D_2 R_1} K_{P2}$$

For LT_{BC} use the expression from case 7a

At the junction of the cone and plate,

$$V_1 = q_1 t_1 K_{V1}, \qquad M_1 = q_1 t_1^2 K_{M1}$$

$$N_1 = -V_1 \sin\alpha_1$$

$$\Delta R_A = \frac{q_1 t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $x_1 = R_1$, $a_2 = a_1$, $R_2 = 0.7R_1$, and for $R/t\cos\alpha > 5$ and $R_1/t_2 > 4$. (*Note:* No correction terms are used.)

$$\Delta R_A = \frac{q_1 R_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{q_1 R_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{q_1 R_1}{t_1} K_{\sigma 2},$$

		α_1					
		−30			30		
		R_1/t_1			R_1/t_1		
	$\frac{t_2}{t_1}$	15	30	50	15	30	50
K_{V1}	1.5	1.7764	2.4985	2.9693	1.7997	2.5362	3.0107
	2.0	1.9832	2.8038	3.3944	1.9143	2.7550	3.3559
	2.5	2.2020	3.1423	3.8805	2.0538	3.0132	3.7634
	3.0	2.3954	3.4503	4.3338	2.1871	3.2577	4.1520
	4.0		3.9122	5.0243		3.6388	4.7582
K_{M1}	1.5	1.0786	1.4887	0.6370	0.5358	0.8534	−0.0645
	2.0	1.6780	2.7748	2.9876	0.8544	1.7465	1.7969
	2.5	2.2541	4.0782	5.4554	1.2072	2.7112	3.8157
	3.0	2.7328	5.1990	7.6390	1.5247	3.5749	5.6432
	4.0		6.7888	10.7995		4.8492	8.3528
$K_{\Delta RA}$	1.5	0.2290	0.1873	0.1652	0.1656	0.1505	0.1407
	2.0	0.2295	0.1929	0.1745	0.1667	0.1550	0.1486
	2.5	0.2173	0.1856	0.1706	0.1604	0.1505	0.1461
	3.0	0.1997	0.1718	0.1591	0.1501	0.1408	0.1372
	4.0		0.1410	0.1307		0.1179	0.1142
$K_{\psi A}$	1.5	−3.4953	−5.9545	−9.1782	−2.2131	−4.4588	−7.5197
	2.0	−2.7619	−4.8159	−7.5439	−1.8429	−3.6961	−6.2610
	2.5	−2.1234	−3.7579	−5.9594	−1.4738	−2.9444	−5.0049
	3.0	−1.6313	−2.9042	−4.6343	−1.1675	−2.3150	−3.9328
	4.0		−1.7760	−2.8303		−1.4529	−2.4417
$K_{\sigma 2}$	1.5	0.2467	0.1998	0.1741	0.1476	0.1379	0.1317
	2.0	0.2493	0.2069	0.1847	0.1475	0.1413	0.1385
	2.3	0.2393	0.2013	0.1822	0.1399	0.1355	0.1348
	3.0	0.2236	0.1890	0.1721	0.1282	0.1245	0.1248
	4.0		0.1606	0.1458		0.0997	0.0999

---

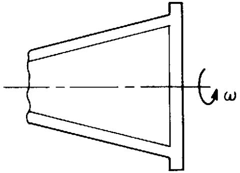
* For external pressure, substitute $-q_1$ for q_1 and a_1 for b_1 in the load terms.

† If pressure increases right to left, substitute $-x_1$ for x_1.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values

7d. Rotation around the axis of symmetry at ω rad/s



Note: δ = mass/unit volume

$$LT_{A1} = \frac{R_1^2}{t_1^2}, \qquad LT_{AC} = 0$$

$$LT_{A2} = \frac{-E_1 \delta_2 t_2^3}{96 D_2 \delta_1 t_1^2} \left[\frac{a_2^2 (3 + \nu_2)}{1 + \nu_2} - R_1^2 \right]$$

$$LT_{B1} = \frac{-R_1 (3 + \nu_1) \tan \alpha_1}{t_1}, \qquad LT_{B2} = 0$$

$$LT_{BC} = 0$$

At the junction of the cone and plate,

$$V_1 = \delta_1 \omega^2 R_1 t_1^2 K_{V1}, \qquad M_1 = \delta_1 \omega^2 R_1 t_1^3 K_{M1}$$

$$N_1 = -V_1 \sin \alpha_1$$

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1 t_1^2}{E_1} (LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{\delta_1 \omega^2 R_1 t_1}{E_1} (-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

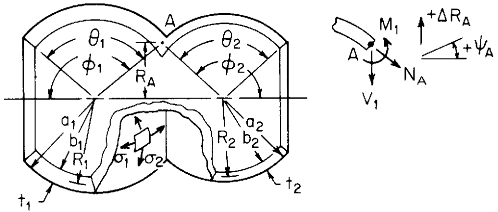
For $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $\delta_1 = \delta_2$, $a_2 = a_1$, and for $R/t \cos \alpha_1 > 5$ and $R_1/t_2 > 4$.

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1^3}{E_1} K_{\Delta RA}, \qquad \psi_A = \frac{\delta_1 \omega^2 R_1^2}{E_1} K_{\psi A}, \qquad \sigma_2 = \delta_1 \omega^2 R_1^2 K_{\sigma 2}$$

		α_1					
		−30			30		
		R_1/t_1			R_1/t_1		
	$\frac{t_2}{t_1}$	15	30	50	15	30	50
K_{V1}	1.5	1.1636	1.7101	2.2561	1.1171	1.6863	2.2484
	2.0	1.4271	2.0268	2.6125	1.2301	1.8620	2.4739
	2.5	1.6871	2.3651	3.0170	1.3556	2.0639	2.7440
	3.0	1.9078	2.6661	3.3926	1.4704	2.2525	3.0036
	4.0		3.1083	3.9624		2.5427	3.4113
K_{M1}	1.5	0.7537	1.0646	1.3760	−0.0626	0.1247	0.3507
	2.0	1.5184	2.3995	3.3469	0.2520	0.8420	1.5667
	2.5	2.2034	3.7021	5.4006	0.5697	1.5965	2.9044
	3.0	2.7495	4.7976	7.2098	0.8428	2.2625	4.1251
	4.0		6.3200	9.8177		3.2332	5.9476
$K_{\Delta RA}$	1.5	0.4259	0.3543	0.3148	0.3206	0.2950	0.2765
	2.0	0.4268	0.3602	0.3226	0.3218	0.2986	0.2816
	2.5	0.4122	0.3529	0.3193	0.3162	0.2951	0.2799
	3.0	0.3922	0.3394	0.3099	0.3073	0.2876	0.2740
	4.0		0.3100	0.2864		0.2701	0.2585
$K_{\psi A}$	1.5	−3.9694	−5.7840	−7.5904	−1.8922	−3.4239	−5.0395
	2.0	−3.0329	−4.6016	−6.2200	−1.5261	−2.8111	−4.2173
	2.5	−2.2734	−3.5442	−4.9013	−1.1937	−2.2231	−3.3850
	3.0	−1.7119	−2.7097	−3.8033	−0.9300	−1.7377	−2.6689
	4.0		−1.6292	−2.3148		−1.0810	−1.6660
$K_{\sigma 2}$	1.5	0.4375	0.3629	0.3216	0.3094	0.2865	0.2697
	2.0	0.4410	0.3703	0.3304	0.3095	0.2893	0.2742
	2.5	0.4291	0.3647	0.3284	0.3026	0.2848	0.2717
	3.0	0.4113	0.3527	0.3200	0.2926	0.2764	0.2650
	4.0		0.3255	0.2983		0.2574	0.2483

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

8. Spherical shell connected to another spherical shell. To ensure accuracy, $R/t > 5$ and the junction angles for each of the spherical shells must lie within the range $3/\beta < \phi < \pi - 3/\beta$. Each spherical shell must also extend with no interruptions such as a second junction or a cutout, such that $\theta > 3/\beta$. See the discussion on page 565. E_1 and E_2 are the moduli of elasticity and ν_1 and ν_2 the Poisson's ratios for the left and right spheres, respectively. $b_1 = R_1 - t_1/2$, $a_2 = R_1 + t_1/2$, and $R_A = R_1 \sin\phi_1$. See Table 13.3, case 1, for formulas K_{11},* K_{21},* and β_1 for the left-hand spherical shell. Similar expressions hold for b_2, a_2, β_2 and K_{12}* and K_{22}* for the right- hand sphere. Normally $R_2 \sin\phi_2 = R_A$, but if $\phi_1 + \phi_2 = 180^\circ$ or close to 180° the midthickness radii may not be equal at the junction. Under this condition a different set of correction terms will be used if necessary.



$$K_{V1} = \frac{LT_A C_{BB} - LT_B C_{AB}}{C_{AA}C_{BB} - C_{AB}^2}, \qquad K_{M1} = \frac{LT_B C_{AA} - LT_A C_{AB}}{C_{AA}C_{BB} - C_{AB}^2}, \qquad \left.\begin{aligned} LT_A &= LT_{A1} + LT_{A2} + LT_{AC} \\ LT_B &= LT_{B1} + LT_{B2} + LT_{BC} \end{aligned}\right\} \text{See cases 8a–8d for these load terms}$$

$$C_{AA} = C_{AA1} + C_{AA2}, \qquad C_{AA1} = \frac{R_1\beta_1 \sin^2\phi_1}{t_1}\left(\frac{1}{K_{11}} + K_{21}\right), \qquad C_{AA2} = \frac{R_A E_1 \beta_2 \sin\phi_2}{E_2 t_2}\left(\frac{1}{K_{12}} + K_{22}\right)$$

$$C_{AB} = C_{AB1} + C_{AB2}, \qquad C_{AB1} = \frac{-2\beta_1^2 \sin\phi_1}{K_{11}}, \qquad C_{AB2} = \frac{2E_1 R_A t_1 \beta_2^2}{E_2 R_2 t_2 K_{12}}$$

$$C_{BB} = C_{BB1} + C_{BB2}, \qquad C_{BB1} = \frac{4t_1\beta_1^3}{R_1 K_{11}}, \qquad C_{BB2} = \frac{4E_1 R_A t_1^2 \beta_2^3}{R_2^2 E_2 t_2 K_{12} \sin\phi_2}$$

The stresses in the left sphere at the junction are given by

$$\sigma_1 = \frac{N_1}{t_1}$$

$$\sigma_2 = \frac{\Delta R_A E_1}{R_A} + \nu_1\sigma_1$$

$$\sigma_1' = \frac{-6M_1}{t_1^2}$$

$$\sigma_2' = \frac{V_1\beta_1^2 \cos\phi_1}{K_{11}R_1} - \frac{6M_1}{t_1^2 K_{11}}\left(\nu_1 + \frac{1 - \nu_1/2}{\beta_1 \tan\phi_1}\right)$$

Note: The use of joint load correction terms LT_{AC} and LT_{BC} depends upon the accuracy desired and the relative values of the thicknesses and the radii. Read Sec. 13.4 carefully. For thin-walled shells, $R/t > 10$, they can be neglected.

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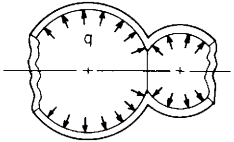
*The second subscript refers to the left-hand (1) or right-hand (2) shell.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
8a. Internal* pressure q  *Note:* There is no axial load on the junction. An axial load on the left end of the left sphere balances any axial component of the pressure on the left sphere, and an axial load on the right end of the right sphere balances any axial component of the pressure on the right sphere and on the joint. For an enclosed pressure vessel superpose an axial load $P = q\pi b_1^2 \sin^2\phi_1$ using case 8b.	$LT_{A1} = \frac{b_1^2 \sin\phi_1}{t_1^2}$ $LT_{A2} = \frac{-b_2^2 E_1 \sin\phi_2}{E_2 t_1 t_2}$ $LT_{AC} = \frac{t_2\cos\phi_2 + t_1\cos\phi_1}{2t_1} C_{AA2} + \frac{t_2^2 - t_1^2}{8t_1^2} C_{AB2} + \frac{E_1 R_1 \nu_2 (t_1\sin\phi_1 - t_2\sin\phi_2)}{2E_2 t_1 t_2 \sin\phi_2}\dagger$ $LT_{B1} = 0, \qquad LT_{B2} = 0$ $LT_{BC} = \frac{t_2\cos\phi_2 + t_1\cos\phi_1}{2t_1} C_{AB2} + \frac{t_2^2 - t_1^2}{8t_2^2} C_{BB2} + \frac{E_1\cos\phi_2 (t_1\sin\phi_1 - t_2\sin\phi_2)}{E_2 t_2 \sin^2\phi_2}\dagger$ At the junction of the two spheres, $V_1 = qt_1 K_{V1}, \qquad M_1 = qt_1^2 K_{M1}$ $N_1 = -V_1\cos\phi_1$ $\Delta R_A = \frac{qt_1}{E_1}(LT_{A1} - K_{V1}C_{AA1} - K_{M1}C_{AB1})$ $\psi_A = \frac{q}{E_1}(-LT_{B1} + K_{V1}C_{AB1} + K_{M1}C_{BB1})$	For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $R_1\sin\phi_1 = R_2\sin\phi_2$, and for $R/t > 5$. (*Note:* No correction terms are used) $\Delta R_A = \frac{qR_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{qR_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{qR_1}{t_1} K_{\sigma 2}$

		ϕ_1					
		75			120		
		R_1/t_1			R_1/t_1		
	ϕ_2	10	20	50	10	20	50
K_{V1}	45.0	−0.2610	−0.3653	−0.5718	−0.1689	−0.2386	−0.3765
	60.0	−0.0776	−0.1089	−0.1710	0.0000	0.0000	0.0000
	75.0	0.0000	0.0000	0.0000	0.0705	0.1003	0.1590
	90.0	0.0217	0.0306	0.0483	0.0896	0.1278	0.2032
	135.0	−0.2462	−0.3505	−0.5570	−0.1587	−0.2282	−0.3660
K_{M1}	45.0	0.0042	0.0059	0.0092	0.0063	0.0088	0.0139
	60.0	0.0006	0.0009	0.0014	0.0000	0.0000	0.0000
	75.0	0.0000	0.0000	0.0000	−0.0016	−0.0023	−0.0037
	90.0	0.0002	0.0002	0.0004	−0.0014	−0.0020	−0.0032
	135.0	−0.0074	−0.0105	−0.0168	−0.0012	−0.0018	−0.0029
$K_{\Delta RA}$	45.0	1.0672	1.1124	1.1395	0.8907	0.9305	0.9542
	60.0	0.9296	0.9760	1.0043	0.7816	0.8233	0.8488
	75.0	0.8717	0.9182	0.9467	0.7363	0.7784	0.8044
	90.0	0.8557	0.9021	0.9305	0.7242	0.7662	0.7921
	135.0	1.0524	1.1019	1.1329	0.8822	0.9244	0.9503
$K_{\psi A}$	45.0	0.8559	1.1883	1.8470	0.4864	0.6858	1.0805
	60.0	0.2526	0.3527	0.5506	0.0000	0.0000	0.0000
	75.0	0.0000	0.0000	0.0000	−0.2005	−0.2856	−0.4538
	90.0	−0.0696	−0.0982	−0.1546	−0.2530	−0.3621	−0.5780
	135.0	0.7762	1.1091	1.7683	0.4383	0.6369	1.0309
$K_{\sigma 2}$	45.0	1.1069	1.1530	1.1806	1.0260	1.0726	1.1007
	60.0	0.9630	1.0108	1.0400	0.9025	0.9506	0.9801
	75.0	0.9025	0.9506	0.9801	0.8512	0.8996	0.9293
	90.0	0.8857	0.9338	0.9632	0.8376	0.8857	0.9153
	135.0	1.0915	1.1421	1.1737	1.0162	1.0657	1.0963

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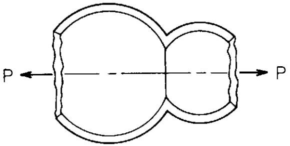
* For external pressure, substitute $-q$ for q, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in the load terms.

† If $\phi_1 + \phi_2 = 180^\circ$ or is close to 180° the following correction terms should be used:

$$LT_{AC} = \frac{b_1^2 - b_2^2}{4t_1^2}\left[\frac{a_2 - b_1}{R_1} C_{AB2} - \frac{2E_1 t_1}{E_2 t_2}(1 + \nu_2)\sin\phi_2\right], \qquad LT_{BC} = \frac{b_1^2 - b_2^2}{4t_1^2 R_1}(a_2 - b_1) C_{BB2}$$

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

8b. Axial load P



$$LT_{A1} = \frac{-R_1^2(1+\nu_1)}{2t_1^2 \sin\phi_1}$$

$$LT_{A2} = \frac{R_1^2 E_1(1+\nu_2)}{2E_2 t_1 t_2 \sin\phi_2} + \frac{R_1 C_{AA2}}{2t_1 \sin\phi_1}\left(\frac{1}{\tan\phi_1} + \frac{1}{\tan\phi_2}\right)$$

$$LT_{AC} = 0\,^*$$

$$LT_{B1} = 0$$

$$LT_{B2} = \frac{R_1 C_{AB2}}{2t_1 \sin\phi_1}\left(\frac{1}{\tan\phi_1} + \frac{1}{\tan\phi_2}\right)$$

$$LT_{BC} = 0\,^*$$

At the junction of the two spheres,

$$V_1 = \frac{P t_1 K_{V1}}{\pi R_1^2}$$

$$M_1 = \frac{P t_1^2 K_{M1}}{\pi R_1^2}$$

$$N_1 = \frac{P}{2\pi R_1 \sin^2\ \phi_1} - V_1 \cos\phi_1$$

$$\Delta R_A = \frac{P t_1}{E_1 \pi R_1^2}(LT_{A1} - K_{V1}C_{AA1} - K_{M1}C_{AB1})$$

$$\psi_A = \frac{P}{E_1 \pi R_1^2}(-LT_{B1} + K_{V1}C_{AB1} + K_{M1}C_{BB1})$$

For axial tension, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $R_1 \sin\phi_1 = R_2 \sin\phi_2$, and for $R/t > 5$.

$$\Delta R_A = \frac{P\nu_1}{2\pi E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{P\nu_1}{2\pi E_1 t_1^2} K_{\psi A}, \qquad \sigma_2 = \frac{P}{2\pi R_1 t_1} K_{\sigma 2}$$

		ϕ_1					
		75			120		
		R_1/t_1			R_1/t_1		
	ϕ_2	10	20	50	10	20	50
K_{V1}	45.0	3.1103	6.1700	15.3153	1.2155	2.3973	5.9187
	60.0	2.1514	4.2888	10.6917	0.0000	0.0000	0.0000
	75.0	1.3870	2.7740	6.9350	−0.9414	−1.8690	−4.6414
	90.0	0.6936	1.3907	3.4844	−1.7581	−3.4981	−8.7042
	135.0	−1.6776	−3.3955	−8.5790	−4.2542	−8.5470	−21.4561
K_{M1}	45.0	3.5944	10.1684	40.1986	1.2318	3.4870	13.7907
	60.0	2.5283	7.1514	28.2692	0.0000	0.0000	0.0000
	75.0	1.6480	4.6612	18.4250	−0.9769	−2.7638	−10.9267
	90.0	0.8311	2.3507	9.2922	−1.8388	−5.2015	−20.5626
	135.0	−2.0739	−5.8688	−23.2056	−4.6045	−13.0241	−51.4841
$K_{\Delta RA}$	45.0	−12.1536	−15.3196	−21.6051	−7.8717	−8.8421	−10.7727
	60.0	−9.6988	−11.9270	−16.3487	−5.0037	−5.0037	−5.0037
	75.0	−7.8081	−9.2607	−12.1428	−2.8247	−2.0535	−0.5219
	90.0	−6.1324	−6.8649	−8.3183	−0.9612	0.4913	3.3743
	135.0	−0.6421	1.1826	4.8086	4.4902	8.1282	15.3477
$K_{\psi A}$	45.0	−0.1836	−0.1273	−0.0791	−0.1098	−0.0073	−0.0487
	60.0	−0.0505	−0.0352	−0.0220	0.0000	0.0000	0.0000
	75.0	0.0000	0.0000	0.0000	0.0453	0.0322	0.0204
	90.0	0.0126	0.0089	0.0056	0.0594	0.0424	0.0271
	135.0	−0.1463	−0.1044	−0.0665	−0.1265	−0.0917	−0.0594
$K_{\sigma 2}$	45.0	−3.5015	−4.4844	−6.4362	−2.2904	−2.6270	−3.2962
	60.0	−2.7242	−3.4161	−4.7893	−1.3333	−1.3333	−1.3333
	75.0	−2.1251	−2.5762	−3.4713	−0.6067	−0.3394	0.1913
	90.0	−1.5939	−1.8214	−2.2728	0.0143	0.5177	1.5167
	135.0	0.1482	0.7152	1.8417	1.8278	3.0875	5.5879

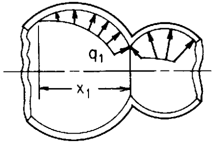
* If $\phi_1 + \phi_2 = 180^\circ$ or is close to 180° the following correction terms should be used:

$$LT_{AC} = \frac{-R_1(R_2 - R_1)C_{AB2}}{2t_1^2 \sin^2\phi_2}, \qquad LT_{BC} = \frac{-R_1(R_2 - R_1)C_{BB2}}{2t_1^2 \sin^2\phi_2}$$

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values

8c. Hydrostatic internal* pressure q_1 at the junction where the angle to the position of zero pressure, $\theta_1 > 3/\beta_1$.† If $\theta_1 < 3/\beta_1$ the discontinuity in pressure gradient introduces small deformations at the junction.



Note: There is no axial load on the junction. An axial load on the left end of the left sphere balances any axial component of the pressure on the left sphere, and an axial load on the right end of the right sphere balances any axial component of the pressure on the right sphere and on the joint.

$$LT_{A1} = \frac{b_1^2 \sin\phi_1}{t_1^2}$$

$$LT_{A2} = \frac{-b_2^2 E_1 \sin\phi_2}{E_2 t_1 t_2}$$

For LT_{AC} use the expressions from case 8a

$$LT_{B1} = \frac{-b_1 R_1 \sin\phi_1}{x_1 t_1}$$

$$LT_{B2} = \frac{E_1 b_2 R_2 \sin\phi_2}{E_2 t_2 x_1}$$

For LT_{BC} use the expressions from case 8a

At the junction of the two spheres,

$$V_1 = q_1 t_1 K_{V1}$$

$$M_1 = q_1 t_1^2 K_{M1}$$

$$N_1 = -V_1 \cos\phi_1$$

$$\Delta R_A = \frac{q_1 t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For internal pressure, $x_1 = R_1$, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $R_1 \sin\phi_1 = R_2 \sin\phi_2$, and for $R/t > 5$. (*Note:* No correction terms are used)

$$\Delta R_A = \frac{q_1 R_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{q_1 R_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{q_1 R_1}{t_1} K_{\sigma 2}$$

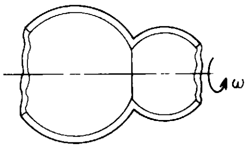
		ϕ_1					
		75			120		
		R_1/t_1			R_1/t_1		
	ϕ_2	10	20	50	10	20	50
K_{V1}	45.0	−0.2614	−0.3656	−0.5720	−0.1695	−0.2390	−0.3768
	60.0	−0.0776	−0.1090	−0.1710	0.0000	0.0000	0.0000
	75.0	0.0000	0.0000	0.0000	0.0707	0.1004	0.1591
	90.0	0.0216	0.0306	0.0483	0.0898	0.1279	0.2033
	135.0	−0.2454	−0.3500	−0.5567	−0.1586	−0.2281	−0.3660
K_{M1}	45.0	0.0632	0.0900	0.1431	0.0403	0.0571	0.0905
	60.0	0.0204	0.0290	0.0461	0.0000	0.0000	0.0000
	75.0	0.0000	0.0000	0.0000	−0.0189	−0.0267	−0.0421
	90.0	−0.0060	−0.0085	−0.0134	−0.0241	−0.0339	−0.0535
	135.0	0.0543	0.0763	0.1199	0.0344	0.0481	0.0752
$K_{\Delta RA}$	45.0	1.0866	1.1261	1.1482	0.9006	0.9375	0.9586
	60.0	0.9361	0.9805	1.0072	0.7816	0.8233	0.8488
	75.0	0.8717	0.9182	0.9467	0.7314	0.7749	0.8022
	90.0	0.8537	0.9007	0.9296	0.7178	0.7617	0.7893
	135.0	1.0718	1.1156	1.1416	0.8920	0.9314	0.9547
$K_{\psi A}$	45.0	1.9353	2.2922	2.9657	1.3998	1.6213	2.0294
	60.0	1.2243	1.3485	1.5610	0.8227	0.8444	0.8574
	75.0	0.9176	0.9418	0.9563	0.5767	0.5131	0.3578
	90.0	0.8314	0.8270	0.7850	0.5101	0.4224	0.2194
	135.0	1.8593	2.2156	2.8887	1.3538	1.5740	1.9809
$K_{\sigma 2}$	45.0	1.1270	1.1672	1.1896	1.0374	1.0807	1.1058
	60.0	0.9697	1.0156	1.0430	0.9025	0.9506	0.9801
	75.0	0.9025	0.9506	0.9801	0.8456	0.8956	0.9268
	90.0	0.8837	0.9323	0.9623	0.8302	0.8805	0.9120
	135.0	1.1115	1.1563	1.1827	1.0276	1.0737	1.1014

* For external pressure, substitute $-q_1$ for q_1, a_1 for b_1, b_2 for a_2, and a_2 for b_2 in the load terms.

† If pressure increases right to left, substitute $-x_1$ for x_1 and verify that $|x_1|$ is large enough to extend into the right hand sphere as far as the position where $\theta_2 = 3/\beta_2$.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

8d. Rotation around the axis of symmetry at ω rad/s



Note: δ = mass/unit volume

$$LT_{A1} = \frac{R_1^2 \sin^3 \phi_1}{t_1^2}$$

$$LT_{A2} = \frac{-\delta_2 R_2^3 E_1 \sin^3 \phi_2}{\delta_1 R_1 E_2 t_1^2}$$

$$LT_{AC} = 0$$

$$LT_{B1} = \frac{-R_1 \sin\phi_1 \cos\phi_1 (3+\nu_1)}{t_1}$$

$$LT_{B2} = \frac{-\delta_2 R_2^2 E_1 (3+\nu_2) \sin\phi_2 \cos\phi_2}{\delta_1 R_1 t_1 E_2}$$

$$LT_{BC} = 0$$

At the junction of the two spheres,

$$V_1 = \delta_1 \omega^2 R_1 t_1^2 K_{V1}$$

$$M_1 = \delta_1 \omega^2 R_1 t_1^3 K_{M1}$$

$$N_1 = -V_1 \cos\phi_1$$

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1 t_1^2}{E_1} (LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{\delta_1 \omega^2 R_1 t_1}{E_1} (-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

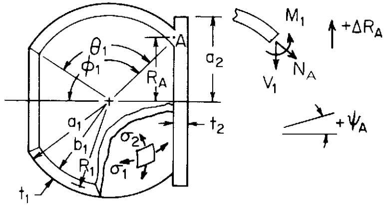
For $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $t_1 = t_2$, $R_1 \sin\phi_1 = R_2 \sin\phi_2$, $\delta_1 = \delta_2$, and for $R/t > 5$.

$$\Delta R_A = \frac{\delta_1 \omega^2 R_1^3}{E_1} K_{\Delta RA}, \qquad \psi_A = \frac{\delta_1 \omega^2 R_1^2}{E_1} K_{\psi A}, \qquad \sigma_2 = \delta_1 \omega^2 R_1^2 K_{\sigma 2}$$

		ϕ_1					
		75			120		
		R_1/t_1			R_1/t_1		
	ϕ_2	10	20	50	10	20	50
K_{V1}	45.0	0.0047	0.0033	0.0020	0.0034	0.0024	0.0015
	60.0	0.0015	0.0010	0.0006	0.0000	0.0000	0.0000
	75.0	0.0000	0.0000	0.0000	−0.0014	−0.0010	−0.0006
	90.0	−0.0004	−0.0003	−0.0002	−0.0017	−0.0012	−0.0008
	135.0	0.0047	0.0034	0.0021	0.0025	0.0018	0.0012
K_{M1}	45.0	−0.6507	−0.9283	−1.4790	−0.1829	−0.2595	−0.4114
	60.0	−0.4616	−0.6568	−1.0441	0.0000	0.0000	0.0000
	75.0	−0.3030	−0.4302	−0.6826	0.1475	0.2082	0.3285
	90.0	−0.1538	−0.2180	−0.3453	0.2795	0.3937	0.6201
	135.0	0.3931	0.5534	0.8714	0.7152	1.0009	1.5678
$K_{\Delta RA}$	45.0	0.6872	0.7500	0.8056	0.5965	0.6121	0.6259
	60.0	0.7508	0.7949	0.8340	0.6495	0.6495	0.6495
	75.0	0.8032	0.8319	0.8574	0.6915	0.6792	0.6683
	90.0	0.8518	0.8663	0.8791	0.7284	0.7053	0.6848
	135.0	1.0249	0.9886	0.9565	0.8470	0.7891	0.7378
$K_{\psi A}$	45.0	−0.9617	−0.9655	−0.9689	−1.9161	−1.9188	−1.9213
	60.0	−0.4365	−0.4377	−0.4388	−1.4289	−1.4289	−1.4289
	75.0	0.0000	0.0000	0.0000	−1.0397	−1.0385	−1.0374
	90.0	0.4076	0.4080	0.4083	−0.6940	−0.6924	−0.6911
	135.0	1.8800	1.8754	1.8713	0.4324	0.4299	0.4278
$K_{\sigma 2}$	45.0	0.7115	0.7764	0.8340	0.6888	0.7068	0.7227
	60.0	0.7773	0.8229	0.8634	0.7500	0.7500	0.7500
	75.0	0.8315	0.8613	0.8876	0.7984	0.7842	0.7716
	90.0	0.8818	0.8968	0.9101	0.8411	0.8144	0.7907
	135.0	1.0610	1.0235	0.9902	0.9780	0.9112	0.8519

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

9\. Spherical shell connected to a circular plate. Expressions are accurate if $R_1/t_1 > 5$ and $R_1/t_2 > 4$. The junction angle for each the spherical shells must lie within the range $3/\beta < \phi_1 < \pi - 3/\beta$. The spherical shell must also extend with no interruptions such as a second junction or a cutout, such that $\theta_1 > 3/\beta$. See the discussion on page 565. E_1 and E_2 are the moduli of elasticity and ν_1 and ν_2 the Poisson's ratios for the sphere and plate, respectively. $b_1 = R_1 - t_1/2$, $a_1 = R_1 + t_1/2$, and $\sin\phi_1 = R_A/R_1$. See Table 13.3, case 1, for formulas for K_1, K_2, and β for the spherical shell. See Table 11.2 for the formula for D_2.



$$K_{P1} = 1 + \frac{R_A^2(1-\nu_2)}{a_2^2(1+\nu_2)}$$

$$K_{V1} = \frac{LT_A C_{BB} - LT_B C_{AB}}{C_{AA}C_{BB} - C_{AB}^2}, \qquad K_{M1} = \frac{LT_B C_{AA} - LT_A C_{AB}}{C_{AA}C_{BB} - C_{AB}^2}, \qquad \left.\begin{aligned} LT_A &= LT_{A1} + LT_{A2} + LT_{AC} \\ LT_B &= LT_{B1} + LT_{B2} + LT_{BC} \end{aligned}\right\} \text{See cases 9a–9d for these load terms}$$

$$C_{AA} = C_{AA1} + C_{AA2}, \qquad C_{AA1} = \frac{R_1 \beta \sin^2\phi_1}{t_1}\left(\frac{1}{K_1} + K_2\right), \qquad C_{AA2} = \frac{E_1 t_2^2 R_A K_{P1}}{6D_2}$$

$$C_{AB} = C_{AB1} + C_{AB2}, \qquad C_{AB1} = \frac{-2\beta^2 \sin\phi_1}{K_1}, \qquad C_{AB2} = \frac{E_1 t_1 t_2 R_A K_{P1}}{4D_2}$$

$$C_{BB} = C_{BB1} + C_{BB2}, \qquad C_{BB1} = \frac{4t_1\beta^3}{R_1 K_1}, \qquad C_{BB2} = \frac{E_1 t_1^2 R_A K_{P1}}{2D_2}$$

The stresses in the left sphere at the junction are given by

$$\sigma_1 = \frac{N_1}{t_1}$$

$$\sigma_2 = \frac{\Delta R_A E_1}{R_A} + \nu_1\sigma_1$$

$$\sigma_1' = \frac{-6M_1}{t_1^2}$$

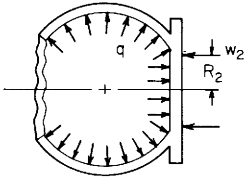
$$\sigma_2' = \frac{V_1\beta^2\cos\phi_1}{K_1 R_1} - \frac{6M_1}{t_1^2 K_1}\left(\nu_1 + \frac{1-\nu_1/2}{\beta\tan\phi_1}\right)$$

Note: The use of joint load correction terms LT_{AC} and LT_{BC} depends upon the accuracy desired and the relative values of the thicknesses and the radii. Read Sec. 13.4 carefully. For thin-walled shells, $R/t > 10$, they can be neglected.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values

9a. Internal* pressure q



Note: There is no axial load on the junction. An axial load on the left end of the left sphere balances any axial component of the pressure on sphere, and the axial load on the plate is reacted by the annular line load $w_2 = qb_1^2\sin^2\phi_1/(2R_2)$ at a radius R_2. For an enclosed pressure vessel superpose an axial load $P = q\pi b_1^2\sin^2\phi_1$ using case 9b.

$$LT_{A1} = \frac{b_1^2\sin\phi_1}{t_1^2}. \qquad LT_{A2} = \frac{E_1 t_2 b_1^2\sin\phi_1}{32D_2 t_1 R_1}K_{p_2}$$

where $K_{P2} =$

$$\begin{cases} (2R_2^2 - b_1^2\sin^2\phi_1)K_{P1} & \text{for } R_2 \leqslant R_A \\ (2R_2^2 - b_1^2\sin^2\phi_1)K_{P1} - 2(R_2^2 - R_A^2) + 4R_A^2\ln\dfrac{R_2}{R_A} & \text{for } R_2 \geqslant R_A \end{cases}$$

$$LT_{AC} = \frac{E_1 b_1 t_2\sin\phi_1\cos\phi_1}{12D_2}\left(t_2 - \frac{3t_1\cos\phi_1}{8}\right)K_{P1}$$

$$LT_{B1} = 0$$

$$LT_{B2} = \frac{E_1 b_1^2\sin\phi_1}{16D_2R_1}K_{P2}$$

$$LT_{BC} = \frac{E_1 b_1 t_1\sin\phi_1\cos\phi_1}{8D_2}\left(t_2 - \frac{t_1\cos\phi_1}{2}\right)K_{P1}$$

At the junction of the sphere and plate,

$$V_1 = qt_1K_{V1}, \qquad M_1 = qt_1^2K_{M1}$$

$$N_1 = -V_1\cos\phi_1$$

$$\Delta R_A = \frac{qt_1}{E_1}(LT_{A1} - K_{V1}C_{AA1} - K_{M1}C_{AB1})$$

$$\psi_A = \frac{q}{E_1}(-LT_{B1} + K_{V1}C_{AB1} + K_{M1}C_{BB1})$$

For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $a_2 = a_1\sin\phi_1$, $R_2 = 0.8a_2$, and for $R/t > 5$ and $R_A/t_2 > 4$. (*Note:* No correction terms are used.)

$$\Delta R_A = \frac{qR_1^2}{E_1t_1}K_{\Delta RA}, \qquad \psi_A = \frac{qR_1}{E_1t_1}K_{\psi A}, \qquad \sigma_2 = \frac{qR_1}{t_1}K_{\sigma 2}$$

		ϕ_1					
		60			120		
		R_1/t_1			R_1/t_1		
	$\frac{t_2}{t_1}$	15	30	50	15	30	50
K_{V1}	1.5	3.7234	8.1706	15.4808	3.4307	7.6706	14.6997
	2.0	3.4877	7.3413	13.6666	3.1733	6.8420	12.9155
	2.5	3.3436	6.6565	11.9903	3.0014	6.1506	11.2632
	3.0	3.2675	6.1516	10.6380	2.8981	5.6384	9.9338
	4.0		5.5526	8.8672		5.0264	8.2023
K_{M1}	1.5	5.2795	19.5856	53.6523	5.2058	19.4818	53.5198
	2.0	4.6874	16.5098	44.7285	4.4942	16.1774	44.2151
	2.5	4.3589	14.1907	37.1658	4.0654	13.6690	36.3338
	3.0	4.1955	12.5748	31.3788	3.8241	11.9177	30.3314
	4.0		10.7573	24.1731		9.9411	22.9142
$K_{\Delta RA}$	1.5	0.0508	−0.0690	−0.1820	0.0121	−0.1011	−0.2113
	2.0	0.0484	−0.0825	−0.2140	0.0121	−0.1124	−0.2414
	2.5	0.0535	−0.0722	−0.2040	0.0207	−0.0984	−0.2277
	3.0	0.0582	−0.0556	−0.1778	0.0290	−0.0780	−0.1975
	4.0		−0.0261	−0.1218		−0.0427	−0.1355
$K_{\psi A}$	1.5	0.9467	7.1118	20.4155	1.5646	8.1319	21.9534
	2.0	0.3077	4.6745	14.8148	0.7591	5.4222	15.9567
	2.5	−0.0078	3.0100	10.4710	0.3206	3.5412	11.2759
	3.0	−0.1516	1.9308	7.3488	0.0922	2.3109	7.9116
	4.0		0.8078	3.7130		1.0178	4.0052
$K_{\sigma 2}$	1.5	0.0214	−0.1205	−0.2566	0.0483	−0.0784	−0.1999
	2.0	0.0210	−0.1320	−0.2881	0.0457	−0.0956	−0.2400
	2.5	0.0283	−0.1167	−0.2716	0.0539	−0.0829	−0.2991
	3.0	0.0345	−0.0949	−0.2372	0.0624	−0.0619	−0.1982
	4.0		−0.0579	−0.1673		−0.0242	−0.1319

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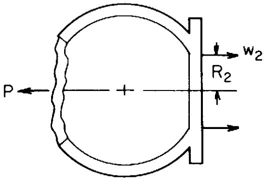
* For external pressure, substitute $-q$ for q and a_1 for b_1 in the load terms.

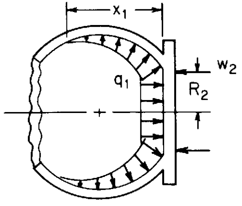
TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
9b. Axial load P  $w_2 = \frac{P}{2\pi R_2}$	$LT_{A1} = \frac{-R_1^2(1+\nu_1)}{2t_1^2 \sin\phi_1}, \qquad LT_{AC} = 0$ $LT_{A2} = \frac{E_1 t_2 R_A R_1^2}{16 D_2 t_1} K_{P2} + \frac{R_1 C_{AA2}}{2t_1 \sin\phi_1} \frac{1}{\tan\phi_1}$ where $K_{P2} = \begin{cases} \left(1 - \frac{R_2^2}{R_A^2}\right) K_{P1} & \text{for } R_2 \leqslant R_A \\ -\frac{1-\nu_2}{1+\nu_2} \frac{R_2^2 - R_A^2}{a_2^2} - 2\ln\frac{R_2}{R_A} & \\ & \text{for } R_2 \geqslant R_A \end{cases}$ $LT_{B1} = 0, \qquad LT_{BC} = 0$ $LT_{B2} = \frac{E_1 R_A R_1^2}{8D_2} K_{P2} + \frac{R_1 C_{AB2}}{2t_1 \sin\phi_1} \frac{1}{\tan\phi_1}$ At the junction of the sphere and plate, $V_1 = \frac{P t_1 K_{V1}}{\pi R_1^2}, \qquad M_1 = \frac{P t_1^2 K_{M1}}{\pi R_1^2}$ $N_1 = \frac{P}{2\pi R_1 \sin^2\ \phi_1} - V_1 \cos\phi_1$ $\Delta R_A = \frac{P t_1}{E_1 \pi R_1^2}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$ $\psi_A = \frac{P}{E_1 \pi r_1^2}(K_{V1} C_{AB1} + K_{M1} C_{BB1})$	For axial tension, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $a_2 = a_1 \sin\phi_1$, $R_2 = 0.8a_2$, and for $R/t > 5$ and $R_A/t_2 > 4$. $\Delta R_A = \frac{P\nu_1}{2\pi E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{P\nu_1}{2\pi E_1 t_1^2} K_{\psi A}, \qquad \sigma_2 = \frac{P}{2\pi R_1 t_1} K_{\sigma 2}$

		ϕ_1					
		60			120		
		R_1/t_1			R_1/t_1		
	$\frac{t_2}{t_1}$	15	30	50	15	30	50
K_{V1}	1.5	4.6437	15.5843	36.3705	1.9491	10.9805	29.5359
	2.0	3.3547	12.2448	29.6890	0.7536	7.7075	22.8663
	2.5	2.1645	9.0429	22.9862	−0.2002	4.8369	16.5737
	3.0	1.1889	6.3731	17.2246	−0.8920	2.6161	11.4344
	4.0		2.6779	9.0973		−0.1930	4.6167
K_{M1}	1.5	11.8774	53.5919	160.3030	7.5277	43.6643	142.3870
	2.0	8.6537	41.2277	127.4680	4.2295	30.6183	107.6120
	2.5	5.9474	30.3930	97.2405	1.8532	20.2072	77.5990
	3.0	3.8585	21.8516	72.5898	0.2383	12.6155	54.3952
	4.0		10.6527	39.5330		3.5467	25.1934
$K_{\Delta RA}$	1.5	−4.4181	−6.0318	−7.9101	−2.3416	−4.0897	−6.0730
	2.0	−4.4852	−6.3806	−8.6849	−2.3351	−4.3814	−6.8196
	2.5	−4.1953	−6.0555	−8.4140	−2.0124	−3.9922	−6.4701
	3.0	−3.7890	−5.4660	−7.6662	−1.6431	−3.4017	−5.6919
	4.0		−4.2468	−5.9485		−2.3204	−4.0641
$K_{\psi A}$	1.5	5.8073	8.7035	11.9527	4.7495	7.9366	11.3940
	2.0	4.2685	6.5301	9.2075	3.0929	5.5605	8.4063
	2.5	3.1172	4.8036	6.8936	2.0145	3.8261	6.0299
	3.0	2.3029	3.5366	5.1207	1.3359	2.6412	4.2959
	4.0		2.0015	2.8983		1.3235	2.2456
$K_{\sigma 2}$	1.5	−1.2233	−1.8453	−2.5584	−0.3722	−0.9069	−1.5265
	2.0	−1.2208	−1.9328	−2.7867	−0.3938	−1.0407	−1.8252
	2.5	−1.0966	−1.7881	−2.6526	−0.3011	−0.9346	−1.7419
	3.0	−0.9363	−1.5572	−2.3590	−0.1870	−0.7522	−1.5031
	4.0		−1.0979	−1.7152		−0.4057	−0.9801

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

9c. Hydrostatic internal* pressure q_1 at the junction where the angle to the position of zero pressure, $\theta > 3/\beta$.† If $\theta < 3/\beta$ the discontinuity in pressure gradient introduces small deformations at the junction.



Note: There is no axial load on the junction. An axial load on the left end of the sphere balances any axial component of the pressure on the sphere, and the axial load on the plate is reacted by the annular line load $w_2 = q_1 b_1^2 \sin^2 \phi_1/(2R_2)$ at a radius R_2.

$$LT_{A1} = \frac{b_1^2 \sin\phi_1}{t_1^2}$$

$$LT_{A2} = \frac{E_1 t_2 b_1^2 \sin\phi_1}{32 D_2 t_1 R_1} K_{P2}$$ For K_{P2} use the expressions from case 9a

For LT_{AC} use the expression from case 9a

$$LT_{B1} = \frac{-b_1 R_1 \sin\phi_1}{x_1 t_1}$$

$$LT_{B2} = \frac{E_1 b_1^2 \sin\phi_1}{16 D_2 R_1} K_{P2}$$

For LT_{BC} use the expression from case 9a

At the junction of the sphere and plate,

$$V_1 = q_1 t_1 K_{V1}, \qquad M_1 = q_1 t_1^2 K_{M1}$$

$$N_1 = -V_1 \cos\phi_1$$

$$\Delta R_A = \frac{q_1 t_1}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$$

$$\psi_A = \frac{q_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$$

For internal pressure, $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $x_1 = R_1$, $a_2 = a_1 \sin\phi_1$, $R_2 = 0.8a_2$, and for $R/t > 5$ and $R_A/t_2 > 4$. (*Note:* No correction terms are used.)

$$\Delta R_A = \frac{q_1 R_1^2}{E_1 t_1} K_{\Delta RA}, \qquad \psi_A = \frac{q_1 R_1}{E_1 t_1} K_{\psi A}, \qquad \sigma_2 = \frac{q_1 R_1}{t_1} K_{\sigma 2}$$

		ϕ_1					
		60			120		
		R_1/t_1			R_1/t_1		
	$\frac{t_2}{t_1}$	15	30	50	15	30	50
K_{V1}	1.5	3.6924	8.1443	15.4580	3.4040	7.6470	14.6788
	2.0	3.4106	7.2712	13.6031	3.1050	6.7779	12.8563
	2.5	3.2236	6.5428	11.8838	2.8953	6.0468	11.1641
	3.0	3.1120	6.0007	10.4934	2.7617	5.5011	9.7996
	4.0		5.3486	8.6670		4.8425	8.0177
K_{M1}	1.5	5.0776	19.3466	53.3882	5.0013	19.2411	53.2544
	2.0	4.3702	16.1090	44.2642	4.1753	15.7756	43.7503
	2.5	3.9440	13.6425	36.5081	3.6524	13.1232	35.6786
	3.0	3.7048	11.9073	30.5580	3.3402	11.2576	29.5179
	4.0		9.9290	23.1258		9.1306	21.8848
$K_{\Delta RA}$	1.5	0.0262	−0.0831	−0.1912	−0.0122	−0.1152	−0.2205
	2.0	0.0235	−0.0973	−0.2239	−0.0122	−0.1270	−0.2512
	2.5	0.0302	−0.0864	−0.2137	−0.0016	−0.1122	−0.2371
	3.0	0.0371	−0.0685	−0.1867	0.0090	−0.0904	−0.2062
	4.0		−0.0364	−0.1290		−0.0525	−0.1424
$K_{\psi A}$	1.5	1.4212	7.6631	21.0176	2.0380	8.6826	22.5550
	2.0	0.6584	5.0979	15.2914	1.1033	5.8410	16.4299
	2.5	0.2494	3.3278	10.8366	0.5689	3.8521	11.6362
	3.0	0.0391	2.1689	7.6264	0.2736	2.5416	8.1831
	4.0		0.9467	3.8764		1.1502	4.1631
$K_{\sigma 2}$	1.5	−0.0067	−0.1367	−0.2671	0.0199	−0.0948	−0.2106
	2.0	−0.0070	−0.1487	−0.2994	0.0170	−0.1128	−0.2515
	2.5	0.0026	−0.1325	−0.2824	0.0271	−0.0993	−0.2403
	3.0	0.0117	−0.1091	−0.2470	0.0380	−0.0769	−0.2087
	4.0		−0.0688	−0.1749		−0.0364	−0.1404

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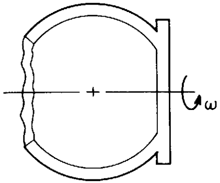
* For external pressure, substitute $-q_1$ for q_1 and a_1 for b_1 in the load terms.

† If pressure increases right to left, substitute $-x_1$ for x_1.

TABLE 13.4 Formulas for discontinuity stresses and deformations at the junctions of shells and plates (*Continued*)

Loading and case no.	Load terms	Selected values
9d. Rotation around the axis of symmetry at ω rad/s  *Note:* $\delta =$ mass/unit volume	$LT_{A1} = \dfrac{R_1^2 \sin^3 \phi_1}{t_1^2}, \qquad LT_{AC} = 0$ $LT_{A2} = \dfrac{-E_1 \delta_2 t_2^3}{96 D_2 \delta_1 t_1^2}\left[\dfrac{a_2^2(3+\nu_2)}{1+\nu_2} - R_A^2\right]$ $LT_{B1} = \dfrac{-R_1 \sin\phi_1 \cos\phi_1 (3+\nu_1)}{t_1}, \qquad LT_{B2} = 0$ $LT_{BC} = 0$ At the junction of the sphere and plate, $V_1 = \delta_1 \omega^2 R_1 t_1^2 K_{V1}, \qquad M_1 = \delta_1 \omega^2 R_1 t_1^3 K_{M1}$ $N_1 = -V_1 \cos\phi_1$ $\Delta R_A = \dfrac{\delta_1 \omega^2 R_1 t_1^2}{E_1}(LT_{A1} - K_{V1} C_{AA1} - K_{M1} C_{AB1})$ $\psi_A = \dfrac{\delta_1 \omega^2 R_1 t_1}{E_1}(-LT_{B1} + K_{V1} C_{AB1} + K_{M1} C_{BB1})$	For $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $\delta_1 = \delta_2$, $a_2 = a_1 \sin\phi_1$, and for $R/t > 5$ and $R_A/t_2 > 4$. $\Delta R_A = \dfrac{\delta_1 \omega^2 R_1^3}{E_1} K_{\Delta RA}, \qquad \psi_A = \dfrac{\delta_1 \omega^2 R_1^2}{E_1} K_{\psi A}, \qquad \sigma_2 = \delta_1 \omega^2 R_1^2 K_{\sigma 2}$

		ϕ_1					
		60			120		
		R_1/t_1			R_1/t_1		
	$\frac{t_2}{t_1}$	15	30	50	15	30	50
K_{V1}	1.5	0.8493	1.2962	1.7363	0.8885	1.3155	1.7419
	2.0	0.9373	1.4358	1.9173	1.0957	1.5667	2.0264
	2.5	1.0345	1.5946	2.1310	1.2971	1.8307	2.3441
	3.0	1.1233	1.7419	2.3346	1.4665	2.0631	2.6355
	4.0		1.9683	2.6525		2.4016	3.0728
K_{M1}	1.5	−0.0650	0.0730	0.2432	0.5903	0.8287	1.0685
	2.0	0.1543	0.5888	1.1308	1.1604	1.8275	2.5488
	2.5	0.3751	1.1257	2.0941	1.6617	2.7842	4.0628
	3.0	0.5652	1.5967	2.9649	2.0568	3.5782	5.3777
	4.0		2.2823	4.2570		4.6700	7.2494
$K_{\Delta RA}$	1.5	0.2286	0.2115	0.1992	0.3018	0.2528	0.2258
	2.0	0.2292	0.2135	0.2022	0.3013	0.2559	0.2304
	2.5	0.2256	0.2111	0.2009	0.2910	0.2504	0.2277
	3.0	0.2200	0.2062	0.1969	0.2774	0.2411	0.2211
	4.0		0.1949	0.1868		0.2215	0.2053
$K_{\psi A}$	1.5	−1.2055	−2.2272	−3.3080	−2.6485	−3.8683	−5.0838
	2.0	−0.9706	−1.8202	−2.7526	−2.0059	−3.0513	−4.1317
	2.5	−0.7594	−1.4356	−2.1999	−1.4946	−2.3348	−3.2333
	3.0	−0.5928	−1.1213	−1.7304	−1.1213	−1.7774	−2.4967
	4.0		−0.6988	−1.0794		−1.0645	−1.5120
$K_{\sigma 2}$	1.5	0.2554	0.2377	0.2248	0.3573	0.2984	0.2660
	2.0	0.2552	0.2394	0.2277	0.3589	0.3033	0.2722
	2.5	0.2501	0.2358	0.2255	0.3490	0.2983	0.2700
	3.0	0.2428	0.2294	0.2203	0.3350	0.2888	0.2632
	4.0		0.2153	0.2077		0.2678	0.2463

TABLE 13.5 Formulas for thick-walled vessels under internal and external loading

NOTATION: q = unit pressure (force per unit area); δ and δ_b = radial body forces (force per unit volume); a = outer radius; b = inner radius; σ_1, σ_2, and σ_3 are normal stresses in the longitudinal, circumferential, and radial directions, respectively (positive when tensile); E = modulus of elasticity; ν = Poisson's ratio. Δa, Δb, and Δl are the changes in the radii a and b and in the length l, respectively. ε_1 = unit normal strain in the longitudinal direction

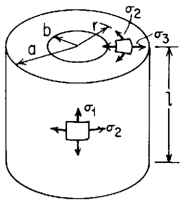
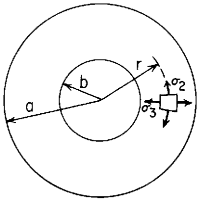
Case no., form of vessel	Case no., manner of loading	Formulas
1. Cylindrical disk or shell  	1a. Uniform internal radial pressure q, longitudinal pressure zero or externally balanced; for a disk or a shell	$\sigma_1 = 0$ $\sigma_2 = \dfrac{qb^2(a^2+r^2)}{r^2(a^2-b^2)}, \qquad (\sigma_2)_{\max} = q\dfrac{a^2+b^2}{a^2-b^2}, \qquad \text{at } r = b$ $\sigma_3 = \dfrac{-qb^2(a^2-r^2)}{r^2(a^2-b^2)}, \qquad (\sigma_3)_{\max} = -q, \qquad \text{at } r = b$ $\tau_{\max} = \dfrac{\sigma_2-\sigma_3}{2} = q\dfrac{a^2}{a^2-b^2}, \qquad \text{at } r = b$ $\Delta a = \dfrac{q}{E}\dfrac{2ab^2}{a^2-b^2}, \qquad \Delta b = \dfrac{qb}{E}\left(\dfrac{a^2+b^2}{a^2-b^2}+\nu\right), \qquad \Delta l = \dfrac{-q\nu l}{E}\dfrac{2b^2}{a^2-b^2}$
	1b. Uniform internal pressure q, in all directions; ends capped; for a disk or a shell	$\sigma_1 = \dfrac{qb^2}{a^2-b^2} \qquad [\sigma_2, \sigma_3, (\sigma_2)_{\max}, (\sigma_3)_{\max}, \text{ and } \tau_{\max} \text{ are the same as for case 1a}]$ $\Delta a = \dfrac{qa}{E}\dfrac{b^2(2-\nu)}{a^2-b^2}$ $\Delta b = \dfrac{qb}{E}\dfrac{a^2(1+\nu)+b^2(1-2\nu)}{a^2-b^2}$ $\Delta l = \dfrac{ql}{E}\dfrac{b^2(1-2\nu)}{a^2-b^2}$
	1c. Uniform external radial pressure q, longitudinal pressure zero or externally balanced; for a disk or a shell	$\sigma_1 = 0$ $\sigma_2 = \dfrac{-qa^2(b^2+r^2)}{r^2(a^2-b^2)}, \qquad (\sigma_2)_{\max} = \dfrac{-q2a^2}{a^2-b^2}, \qquad \text{at } r = b$ $\sigma_3 = \dfrac{-qa^2(r^2-b^2)}{r^2(a^2-b^2)}, \qquad (\sigma_3)_{\max} = -q, \qquad \text{at } r = a$ $\tau_{\max} = \dfrac{(\sigma_2)_{\max}}{2} = \dfrac{qa^2}{a^2-b^2} \qquad \text{at } r = b$ $\Delta a = \dfrac{-qa}{E}\left(\dfrac{a^2+b^2}{a^2-b^2}-\nu\right), \qquad \Delta b = \dfrac{-q}{E}\dfrac{2a^2}{a^2-b^2}, \qquad \Delta l = \dfrac{q\nu l}{E}\dfrac{2a^2}{a^2-b^2}$

TABLE 13.5 Formulas for thick-walled vessels under internal and external loading (*Continued*)

Case no., form of vessel	Case no., manner of loading	Formulas
	1d. Uniform external pressure q in all directions; ends capped; for a disk or a shell	$\sigma_1 = \frac{-qa^2}{a^2 - b^2}$ $[\sigma_2, \sigma_3, (\sigma_2)_{\max}, (\sigma_3)_{\max},$ and $\tau_{\max}$ are the same as for case 1c] $\Delta a = \frac{-qa}{E}\frac{a^2(1-2\nu) + b^2(1+\nu)}{a^2 - b^2}, \quad \Delta b = \frac{-qb}{E}\frac{a^2(2-\nu)}{a^2-b^2}$ $\Delta l = \frac{-ql}{E}\frac{a^2(1-2\nu)}{a^2-b^2}$
	1e. Uniformly distributed radial body force δ acting outward throughout the wall; for a disk only	$\sigma_1 = 0$ $\sigma_2 = \frac{\delta(2+\nu)}{3(a+b)}\left[a^2 + ab + b^2 - (a+b)\left(\frac{1+2\nu}{2+\nu}\right)r + \frac{a^2b^2}{r^2}\right]$ $(\sigma_2)_{\max} = \frac{\delta a^2}{3}\left[\frac{2(2+\nu)}{a+b} + \frac{b}{a^2}(1-\nu)\right]$ at $r = b$ $\sigma_3 = \frac{\delta(2+\nu)}{3(a+b)}\left[a^2 + ab + b^2 - (a+b)r - \frac{a^2b^2}{r^2}\right]$ (*Note:* $\sigma_3 = 0$ at both $r = b$ and $r = a$.) $\tau_{\max} = \frac{(\sigma_2)_{\max}}{2}$ at $r = b$ $\Delta a = \frac{\delta a^2}{3E}\left[1 - \nu + \frac{2(2+\nu)b^2}{a(a+b)}\right], \quad \Delta b = \frac{\delta ab}{3E}\left[\frac{b}{a}(1-\nu) + \frac{2a(2+\nu)}{a+b}\right]$ $\varepsilon_1 = \frac{-\delta a\nu}{E}\left[\frac{2(a^2+ab+b^2)}{3a(a+b)}(2+\nu) - \frac{r}{a}(1+\nu)\right]$
	1f. Linearly varying radial body force from δ_b outward at $r = b$ to zero at $r = a$; for a disk only	$\sigma_1 = 0$ $\sigma_2 = \delta_b\left[\frac{(7+5\nu)a^4 - 8(2+\nu)ab^3 + 3(3+\nu)b^4}{24(a-b)(a^2-b^2)} - \frac{(1+2\nu)a}{3(a-b)}r + \frac{1+3\nu}{8(a-b)}r^2 + \frac{b^2a^2}{24r^2}\frac{(7+5\nu)a^2 - 8(2+\nu)ab + 3(3+\nu)b^2}{(a-b)(a^2-b^2)}\right]$ $\sigma_3 = \delta_b\left[\frac{(7+5\nu)a^4 - 8(2+\nu)ab^3 + 3(3+\nu)b^4}{24(a-b)(a^2-b^2)} - \frac{(2+\nu)a}{3(a-b)}r + \frac{(3+\nu)}{8(a-b)}r^2 - \frac{b^2a^2}{24r^2}\frac{(7+5\nu)a - 3(3+\nu)b}{a^2-b^2}\right]$ (*Note:* $\sigma_3 = 0$ at both $r = b$ and $r = a$) $(\sigma_2)_{\max} = \frac{\delta_b}{12}\frac{2a^4 + (1+\nu)a^2(5a^2 - 12ab + 6b^2) - (1-\nu)b^3(4a-3b)}{(a-b)(a^2-b^2)}$ at $r = b$

TABLE 13.5 Formulas for thick-walled vessels under internal and external loading (*Continued*)

		$\tau_{\max} = \frac{(\sigma_2)_{\max}}{2}$ at $r = b$ $\Delta a = \frac{\delta_b a}{12E} \frac{(1-\nu)a^4 - 8(2+\nu)ab^3 + 3(3+\nu)b^4 + 6(1+\nu)a^2b^2}{(a-b)(a^2-b^2)}$ $\Delta b = (\sigma_2)_{\max} \frac{b}{E}$ $\varepsilon_1 = \frac{-\delta_b \nu}{E}\left[\frac{(7+5\nu)a^4 - 8(2+\nu)ab^3 + 3(3+\nu)b^4}{12(a-b)(a^2-b^2)} - \frac{1+\nu}{a-b}\left(a - \frac{r}{2}\right)r\right]$
2. Spherical  	2a. Uniform internal pressure q	$\sigma_1 = \sigma_2 = \frac{qb^3}{2r^3} \frac{a^3 + 2r^3}{a^3 - b^3}$, $(\sigma_1)_{\max} = (\sigma_2)_{\max} = \frac{q}{2} \frac{a^3 + 2b^3}{a^3 - b^3}$ at $r = b$ $\sigma_3 = \frac{-qb^3}{r^3} \frac{a^3 - r^3}{a^3 - b^3}$, $(\sigma_3)_{\max} = -q$ at $r = b$ $\tau_{\max} = \frac{q3a^3}{4(a^3 - b^3)}$ at $r = b$ The inner surface yields at $q = \frac{2\sigma_y}{3}\left(1 - \frac{b^3}{a^3}\right)$ (Ref. 20) $\Delta a = \frac{qa}{E} \frac{3(1-\nu)b^3}{2(a^3 - b^3)}$, $\Delta b = \frac{qb}{E}\left[\frac{(1-\nu)(a^3 + 2b^3)}{2(a^3 - b^3)} + \nu\right]$ (Ref. 3)
	2b. Uniform external pressure q	$\sigma_1 = \sigma_2 = \frac{-qa^3}{2r^3} \frac{b^3 + 2r^3}{a^3 - b^3}$, $(\sigma_1)_{\max} = (\sigma_2)_{\max} = \frac{-q3a^3}{2(a^3 - b^3)}$ at $r = b$ $\sigma_3 = \frac{-qa^3}{r^3} \frac{r^3 - b^3}{a^3 - b^3}$, $(\sigma_3)_{\max} = -q$ at $r = a$ $\Delta a = \frac{-qa}{E}\left[\frac{(1-\nu)(b^3 + 2a^3)}{2(a^3 - b^3)} - \nu\right]$, $\Delta b = \frac{-qb}{E} \frac{3(1-\nu)a^3}{2(a^3 - b^3)}$ (Ref. 3)

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Chapter

14

Bodies under Direct Bearing and Shear Stress

14.1 Stress due to Pressure between Elastic Bodies

The stresses caused by the pressure between elastic bodies are of importance in conection with the design or investigation of ball and roller bearings, gears, trunnions, expansion rollers, track stresses, etc. Hertz (Ref. 1) developed the mathematical theory for the surface stresses and deformations produced by pressure between curved bodies, and the results of his analysis are supported by experiment. Formulas based on this theory give the maximum compressive stresses, which occur at the center of the surfaces of contact, but not the maximum shear stresses, which occur in the interiors of the compressed parts, nor the maximum tensile stress, which occurs at the boundary of the contact area and is normal thereto.

Both surface and subsurface stresses were studied by Belajef (Refs. 28 and 29), and some of his results are cited in Ref. 6. A tabulated summary of surface and subsurface stresses, greatly facilitating calculation, is given in Ref. 33. For a cylinder on a plane and for crossed cylinders Thomas and Hoersch (Ref. 2) investigated mathematically surface compression and internal shear and checked the calculated value of the latter experimentally. The stresses due to the pressure of a sphere on a plate (Ref. 3) and of a cylinder on a plate (Ref. 4) have also been investigated by photoelasticity. The deformation and contact area for a ball in a race were measured by Whittemore and Petrenko (Ref. 8) and compared with the theoretical values. Additionally, investigations have considered the influence of tangential loading combined with normal loading (Refs. 35, 47–49, and 58).

In Table 14.1, formulas are given for the elastic stress and deformation produced by pressure between bodies of various forms, and for the dimensions of the circular, elliptical, or rectangular area of contact formed by the compressed surfaces. Except where otherwise indicated, these equations are based on Hertz's theory, which assumes the length of the cylinder and dimensions of the plate to be infinite. For a very short cylinder and for a plate having a width less than five or six times that of the contact area or a thickness less than five or six times the depth to the point of maximum shear stress, the actual stresses may vary considerably from the values indicated by the theory (see Refs. 4, 45, and 50). Tu (Ref. 50) discusses the stresses and deformations for a plate pressed between two identical elastic spheres with no friction; graphs are also presented. Pu and Hussain (Ref. 51) consider the unbonded contact between a flat plate and an elastic half-space when a normal load is applied to the plate. Graphs of the contact radii are presented for a concentrated load and two distributed loadings on circular areas.

Hertz (Ref. 1) based his work on the assumption that the contact area was small compared with the radius of the ball or cylinder; Goodman and Keer (Ref. 52) compare the work of Hertz with a solution which permits the contact area to be larger, such as the case when the negative radius of one surface is only very slightly larger (1.01 to 1) than the positive radius of the other. Cooper (Ref. 53) presents some reformulated hertzian coefficients in a more easily interpolated form and also points out some numerical errors in the coefficients originally published by Hertz. Dundurs and Stippes (Ref. 54) discuss the effect of Poisson's ratio on contact stress problems.

The use of the formulas of Table 14.1 is illustrated in the example at the end of this section. The general formula for case 4 can be used, as in the example, for any contact-stress problems involving any geometrically regular bodies except parallel cylinders, but for bearing calculations use should be made of charts such as those given in Refs. 33 and 34, which not only greatly facilitate calculations but provide for influences not taken into account in the formulas.

Because of the very small area involved in what initially approximates a point or line contact, contact stresses for even light loads are very high; but as the formulas show, the stresses do not increase in proportion to the loading. Furthermore, because of the facts that the stress is highly localized and triaxial, the actual stress intensity can be very high without producing apparent damage. To make use of the Hertz formulas for purposes of design or safe-load determination, it is necessary to know the relationship between theoretical stresses and likelihood of failure, whether from excessive deformation or fracture. In discussing this relationship, it is convenient to refer to the computed stress as the Hertz stress, whether the elastic range has been exceeded or not. Some of the available information showing the Hertz stress corresponding to loadings found to be safe and to loadings that produced excessive deformations or fracture may be summarized as follows.

Static or near-static conditions

Cylinder. The American Railway Engineering Association gives as the allowable loading for a steel cylinder on a flat steel plate the formulas

$$p = \begin{cases} \dfrac{\sigma_y - 13{,}000}{20{,}000} 600d & \text{for } d < 25 \text{ in} \\ \dfrac{\sigma_y - 13{,}000}{20{,}000} 3000\sqrt{d} & \text{for } 25 < d < 125 \text{ in} \end{cases}$$

Here (and in subsequent equations) p is the load per linear inch in pounds, d is the diameter of the cylinder in inches, and σ_y is the tensile yield point of the steel in the roller or plate, whichever is lower. If σ_y is taken as 32,000 lb/in^2, the Hertz stress corresponding to this loading is constant at 76,200 lb/in^2 for any diameter up to 25 in and decreases as $d^{-1/4}$ to 50,900 at $d = 125$ in. See Ref. 10.

Wilson (Refs. 7, 11, and 32) carried out several series of static and slow-rolling tests on large rollers. From static tests on rollers of medium-grade cast steel having diameters of 120 to 720 in, he concluded that the load per linear inch required to produce appreciable permanent set could be represented by the empirical formula $p = 500 + 110d$, provided the bearing plates were 3 in thick or more. He found that p increased with the axial length of the roller up to a length of 6 in, after which it remained practically constant (Ref. 32). Slow-rolling tests (Ref. 11) undertaken to determine the load required to produce a permanent elongation or spread of 0.001 in/in in the bearing plate led to the empirical formula

$$p = (18{,}000 + 120d)\frac{\sigma_y - 13{,}000}{23{,}000}$$

for rollers with $d > 120$ in. Wilson's tests indicated that the average pressure on the area of contact required to produce set was greater for small rollers than for large rollers, and that there was little difference in bearing capacity under static and slow-rolling conditions, though the latter showed more tendency to produce surface deterioration.

Jensen (Ref. 4), making use of Wilson's test results and taking into account the three-dimensional aspect of the problem, proposed for the load-producing set the formula

$$p = \left(1 + \frac{1.78}{1 + d^2/800L^2}\right)\frac{\sigma_y^2 d\pi}{E}$$

where L is the length of the cylinder in inches and E is the modulus of elasticity in pounds per square inch. For values of the ratio d/L from 0.1 to 10, the corresponding Hertz stress ranges from $1.66\sigma_y$ to $1.72\sigma_y$.

Whittemore (Ref. 8) found that the elastic limit load for a flexible roller of hardened steel (tensile strength about 265,000 lb/in^2) tested between slightly hardened races corresponded to a Hertz stress of about 436,000 lb/in^2. The roller failed before the races.

Sphere. Tests reported in Whittemore and Petrenko (Ref. 8) gave, for balls 1, $1\frac{1}{4}$, and $1\frac{1}{2}$ in in diameter, tested between straight races, Hertz stresses of 239,000, 232,000, and 212,000 lb/in^2, respectively, at loads producing a permanent strain of 0.0001. The balls were of steel having sclerescope hardness of 60 to 68, and the races were of approximately the same hardness. The critical strain usually occurred first in the races.

From the results of crushing tests of a sphere between two similar spheres, SKF derived the empirical formula $P = 1960(8d)^{1.75}$, where P is the crushing load in pounds and d is the diameter of the sphere in inches. The test spheres were made of steel believed to be of hardness 64 to 66 Rockwell C, and the formula corresponds to a Hertz stress of about $4{,}000{,}000 \times d^{-1/12}$.

Knife-edge. Knife-edge pivots are widely used in scales and balances, and if accuracy is to be maintained, the bearing loads must not cause excessive deformation. It is impossible for a truly sharp edge to bear against a flat plane without suffering plastic deformation, and so pivots are not designed on the supposition that the contact stresses will be elastic; instead, the maximum load per inch consistent with the requisite degree of accuracy in weighing is determined by experience or by testing. In Wilson et al. (Ref. 9), the National Bureau of Standards is quoted as recommending that for heavy service the load per linear inch should not exceed 5000 lb/in for high-carbon steels or 6000 for special alloy steels; for light service the values can be increased to 6000 and 7000, respectively. In the tests described in Ref. 9, the maximum load that could be sustained without damage—the so-called *critical load*—was defined as the load per linear inch that produced an increase in the edge width of 0.0005 in or a sudden increase in the load rate of vertical deformation. The two methods gave about the same results when the bearing was harder than the pivot, as it should be for good operation. The conclusions drawn from the reported tests may be summarized as follows.

The bearing value of a knife-edge or pivot varies approximately with the wedge angle for angles of 30°–120°, the bearing value of a flat pivot varies approximately with the width of the edge for widths of 0.004–0.04 in, and the bearing value of pivots increases with the hardness for variations in hardness of 45–60 on the Rockwell C scale. Successive applications of a load less than the critical load will cause no plastic flow; the edge of a pivot originally sharp will increase in width with the load, but no further plastic deformation is produced by successive applications of the same or smaller loads. The application of a load greater than the critical load will widen the edge at the first application, but additional applications of the same load will not cause additional flow; the average unit pressure on 90° pivots having a hardness represented by Rockwell C numbers of 50–60 is about 400,000–500,000 $\mathrm{lb/in^2}$ at the critical load. This critical unit pressure appears to be independent of the width of the edge but increases with the pivot angle and the hardness of the material (Ref. 9).

These tests and the quoted recommendations relate to applications involving heavy loads (thousands of pounds) and reasonable accuracy. For light loads and extreme accuracy, as in analytical balances, the pressures are limited to much smaller values. Thus, in Ref. 39, on the assumptions that an originally sharp edge indents the bearing and that the common surface becomes cylindrical, it is stated that the radius of the loaded edge must not exceed 0.25 μm (approximately 0.00001 in) if satisfactory accuracy is to be attained, and that the corresponding loading would be about 35,000 $\mathrm{lb/in^2}$ of contact area.

Dynamic conditions. If the motion involved is a true rolling motion without any slip, then under conditions of slow motion (expansion rollers, bascules, etc.) the stress conditions are comparable with those produced by static loading. This is indicated by a comparison of the conclusions reached in Ref. 7, where the conditions are truly static, with those reached in Ref. 11, where there is a slow-rolling action. If there is even a slight amount of slip, however, the conditions are very much more severe and failure is likely to occur through mechanical wear. The only guide to proper design against wear is real or simulated service testing (Refs. 24, 41, and 46).

When the motion involved is at high speed and produces cyclic loading, as in ball and roller bearings, fatigue is an important consideration. A great many tests have been made to determine the fatigue properties of bearings, especially ball bearings, and such tests have been carried out to as many as 1 billion cycles and with Hertz stresses up to 750,000 $\mathrm{lb/in^2}$ (Ref. 37). The number of cycles to damage (either spalling or excessive deformation) has been found to be inversely proportional to the cube of the load for point contact (balls) and to the fourth power for line contact; this would be inversely proportional to the ninth and eighth powers, respectively, of the Hertz stress. Styri (Ref. 40) found the cycles to failure to vary as the ninth power of the Hertz stress and was unable to establish a true endurance limit. Some of these tests show that ball bearings can run for a great number of cycles at very high stresses; for example, $\frac{1}{2}$-in balls of SAE 52,100 steel (RC 63–64) withstood 17,500,000 cycles at a stress of 174,000 lb/in^2 before 10% failures occurred, and withstood 700,000,000 cycles at that stress before 90% failures occurred.

One difficulty in correlating different tests on bearings is the difference in criteria for judging damage; some experimenters have defined failure as a certain permanent deformation, others as visible surface damage through spalling. Palmgren (Ref. 36) states that a permanent deformation at any one contact point of rolling element and bearing ring combined equal to 0.001 times the diameter of the rolling element has no significant influence on the functioning of the bearing. In the tests of Ref. 37, spalling of the surface was taken as the sign of failure; this spalling generally originated on plates of maximum shear stress below the surface.

Large-diameter bearings, usually incorporating integral gearing, are heat-treated to produce a hardened case to resist wear and fatigue and a tough machinable core. Sague in Ref. 56 describes how high subsurface shear stresses have produced yielding in the core with subsequent failure of the case due to lack of support.

It is apparent from the foregoing discussion that the practical design of parts that sustain direct bearing must be based largely on experience since this alone affords a guide as to whether, at any given load and number of stress cycles, there is enough deformation or surface damage to interfere with proper functioning. The rated capacities of bearings and gears are furnished by the manufacturers, with proper allowance indicated for the conditions of service and recommendations as to proper lubrication (Ref. 38). Valid and helpful conclusions, however, can often be drawn from a comparison of service records with calculated stresses.

EXAMPLE

A ball 1.50 in in diameter, in a race which has a diameter of 10 in and a groove radius of 0.80 in, is subjected to a load of 2000 lb. It is required to find the dimensions of the contact area, the combined deformation of ball and race at the contact, and the maximum compressive stress.

Solution. The formulas and table of case 4 (Table 14.1) are used. The race is taken as body 1 and the ball as body 2; hence $R_1 = -0.80$ in, $R_1' = -5$ in, and $R_2 = R_2' = 0.75$ in. Taking $E_1 = E_2 = 30{,}000{,}000\ \text{lb/in}^2$ and $\nu_1 = \nu_2 = 0.3$, we have

$$C_E = \frac{2(1-0.09)}{30(10^6)} = 6.067(10^{-8})\ \text{in}^2/\text{lb}$$

$$K_D = \frac{1.5}{-1.25 + 1.33 - 0.20 + 1.33} = 1.233\ \text{in}$$

$$\cos\theta = \frac{1.233}{1.5}\sqrt{(-1.25+0.20)^2 + 0 + 0} = 0.863$$

From the table, by interpolation

$$\alpha = 2.710, \qquad \beta = 0.495, \qquad \lambda = 0.546$$

Then

$$c = 2.710\sqrt[3]{2000(1.233)(6.067)(10^{-8})} = 0.144 \text{ in}$$

$$d = 0.495\sqrt[3]{2000(1.233)(6.067)(10^{-8})} = 0.0263 \text{ in}$$

$$(\sigma_c)_{\max} = \frac{1.5(20000)}{0.144(0.0263)\pi} = 252{,}000 \text{ lb/in}^2$$

$$y = 0.546\sqrt[3]{\frac{2000^2(6.067^2)(10^{-16})}{1.233}} = 0.00125 \text{ in}$$

Therefore the contact area is an ellipse with a major axis of 0.288 in and a minor axis of 0.0526 in.

14.2 Rivets and Riveted Joints

Although the actual state of stress in a riveted joint is complex, it is customary—and experience shows it is permissible—to ignore such considerations as the stress concentration at the edges of rivet holes, unequal division of load among rivets, and nonuniform distribution of shear stress across the section of the rivet and of the bearing stress between rivet and plate. Simplifying assumptions are made, which may be summarized as follows: (1) The applied load is assumed to be transmitted entirely by the rivets, friction between the connected plates being ignored; (2) when the centroid of the rivet areas is on the line of action of the load, all the rivets of the joint are assumed to carry equal parts of the load if they are of the same size, or to be loaded proportionally to their respective section areas if they are of different sizes; (3) the shear stress is assumed to be uniformly distributed across the rivet section; (4) the bearing stress between plate and rivet is assumed to be uniformly distributed over an area equal to the rivet diameter times the plate thickness; (5) the stress in a tension member is assumed to be uniformly distributed over the net area; and (6) the stress in a compression member is assumed to be uniformly distributed over the gross area.

The design of riveted joints on the basis of these assumptions is the accepted practice, although none of them is strictly correct and methods of stress calculation that are supposedly more accurate have been proposed (Ref. 12).

Details of design and limitations. The possibility of secondary failure due to secondary causes, such as the shearing or tearing out of a plate between the rivet and the edge of a plate or between adjacent rivets, the bending or insufficient upsetting of long rivets, or tensile failure along a zigzag line when rivets are staggered, is guarded against in standard specifications (Ref. 13) by detailed rules for edge clearance, maximum grip of rivets, maximum pitch, and computing the net width of riveted parts. Provision is made for the use of high-strength bolts in place of rivets under certain circumstances (Ref. 42). Joints may be made by welding instead of riveting, but the use of welding in conjunction with riveting is not approved on new work; the division of the load as between the welds and the rivets would be indeterminate.

Tests on riveted joints. In general, tests on riveted joints show that although under working loads the stress conditions may be considerably at variance with the usual assumptions, the ultimate strength may be closely predicted by calculations based thereon. Some of the other conclusions drawn from such tests may be summarized as follows.

In either lap or double-strap butt joints in very wide plates, the unit tensile strength developed by the net section is greater than that developed by the plate itself when tested in full width and is practically equal to that developed by narrow tension specimens cut from the plate. The rivets in lap joints are as strong relative to undriven rivets tested in shear as are the rivets in butt joints. Lap joints bend sufficiently at stresses below the usual design stresses to cause opening of caulked joints (Ref. 14).

Although it is frequently specified that rivets shall not be used in tension, tests show that hot-driven buttonhead rivets develop a strength in direct tension greater than the strength of the rod from which they are made, and that they may be relied upon to develop this strength in every instance. Although the initial tension in such rivets due to cooling usually amounts to 70% or more of the yield strength, this initial tension does not reduce the ability of the rivets to resist an applied tensile load (see also Sec. 3.12). Unless a joint is subjected to reversals of primary load, the use of rivets in tension appears to be justified; but when the primary load producing shear in the rivets is reversed, the reduction in friction due to simultaneous rivet tension may permit slip to occur, with possible deleterious effects (Ref. 15).

With respect to the form of the rivet head, the rounded or buttonhead type is standard; but countersunk rivets are often used, and tests show that these develop the same ultimate strength, although they permit much more slip and deformation at working loads than do the buttonhead rivets (Ref. 16).

In designing riveted joints in very thin metals, especially the light alloys, it may be necessary to take into account factors that are not usually considered in ordinary structural-steel work, such as the radial stresses caused at the hole edges by closing pressure and the buckling of the plates under rivet pressure (Ref. 17).

Eccentric loading. When the rivets of a joint are so arranged that the centroid G of the areas of the rivet group lies not on the line of action of the load but at a distance e therefrom, the load P can be replaced by an equal and parallel load P' acting through G and a couple Pe. The load on any one of the n rivets is then found by *vectorially* adding the load P/n due to P' and the load Q due to the couple Pe. This load Q acts normal to the line from G to the rivet and is given by the equation $Q = PeA_1r_1/J$, where A_1 is the area of the rivet in question, r_1 is its distance from G, and $J = \sum Ar^2$ for all the rivets of the group. When all rivets are of the same size, as is usually the case, the formula becomes $Q = Per_1/\sum r^2$. Charts and tables are available which greatly facilitate the labor of the calculation involved, and which make possible direct design of the joint without recourse to trial and error (Ref. 18). (The direct procedure, as outlined previously, is illustrated in the following example.)

The stiffness or resistance to angular displacement of a riveted joint determines the degree of fixity that should be assumed in the analysis of beams with riveted ends or of rectangular frames. Tests (Ref. 19) have shown that although joints made with wide gusset plates are practically rigid, joints made by simply riveting through clip angles are not even approximately so. A method of calculating the elastic constraint afforded by riveted joints of different types, based on an extensive series of tests, has been proposed by Rathbun (Ref. 20). Brombolich (Ref. 55) describes the use of a finite-element-analysis procedure to determine the effect of yielding, interference fits, and load sequencing on the stresses near fastener holes.

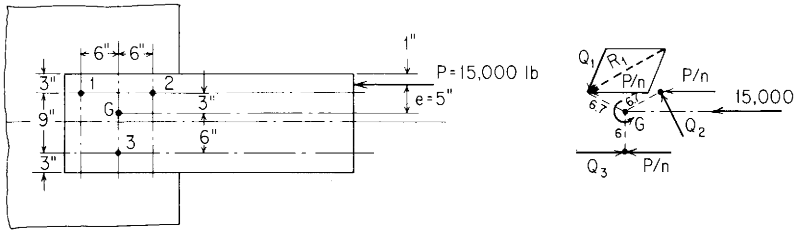
EXAMPLE

Figure 14.1 represents a lap joint in which three 1-in rivets are used to connect a 15-in channel to a plate. The channel is loaded eccentrically as shown. It is required to determine the maximum shear stress in the rivets. (This is not, of course, a properly designed joint intended to develop the full strength of the channel. It represents a simple arrangement of rivets assumed for the purpose of illustrating the calculation of rivet stress due to a moment.)

Solution. The centroid of the rivet areas is found to be at G. The applied load is replaced by an equal load through G and a couple equal to

$$15{,}000 \times 5 = 75{,}000 \text{ lb-in}$$

as shown in Fig. 14.1(b). The distances r_1, r_2, and r_3 of rivets 1, 2, and 3, respectively, from G are as shown; the value of $\sum r^2$ is 126. The loads on the



(a) (b)

Figure 14.1

rivets due to the couple of 75,000 lb-in are therefore

$$Q_1 = Q_2 = \frac{(75{,}000)(6.7)}{126} = 3990 \text{ lb}$$

$$Q_3 = \frac{(75{,}000)(6)}{126} = 3570 \text{ lb}$$

These loads act on the rivets in the directions shown. In addition, each rivet is subjected to a load in the direction of P' of $P/n = 5000$ lb. The resultant load on each rivet is then found by graphically (or algebraically) solving for the resultant of Q and P/n as shown. The resultant loads are $R_1 = R_2 = 7670$ lb; $R_3 = 1430$ lb. The maximum shear stress occurs in rivets 1 and 2 and is $\tau = 7670/0.785 = 9{,}770 \text{ lb/in}^2$.

14.3 Miscellaneous Cases

In most instances, the stress in bodies subjected to direct shear or pressure is calculated on the basis of simplifying assumptions such as are made in analyzing a riveted joint. Design is based on rules justified by experience rather than exact theory, and a full discussion does not properly come within the scope of this book. However, a brief consideration of a number of cases is given here; a more complete treatment of these cases may be found in books on machine and structural design and in the references cited.

Pins and bolts. These are designed on the basis of shear and bearing stress calculated in the same way as for rivets. In the case of pins bearing on wood, the allowable bearing stress must be reduced to provide for nonuniformity of pressure when the length of bolt is more than five or six times its diameter. When the pressure is inclined to the grain, the safe load is found by the formula

$$N = \frac{PQ}{P \sin^2 \theta + Q \cos^2 \theta}$$

where N is the safe load for the case in question, P is the safe load applied parallel to the grain, Q is the safe load applied transverse to the grain, and θ is the angle N makes with the direction of the grain (Ref. 21).

Hollow pins and rollers are thick-walled but can be analyzed as circular rings by the appropriate use of the formulas of Table 9.2. The loading is essentially as shown in Fig. 14.2, and the greatest circumferential stresses, which occur at points 1–4, may be found by the formula

$$\sigma = K\frac{2p}{\pi b}$$

where p = load/unit length of the pin and the numerical coefficient K depends on the ratio a/b and has the following values [a plus sign for K indicates tensile stress and a minus sign compressive stress (Ref. 30)]:

	a/b							
Point	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
1	−5.0	−5.05	−5.30	−5.80	−7.00	−9.00	−12.9	−21.4
2	+3.0	+3.30	+3.80	+4.90	+7.00	+10.1	+16.0	+31.0
3	0	+0.06	+0.20	+1.0	+1.60	+3.0	+5.8	+13.1
4	+0.5	+0.40	0	−0.50	−1.60	−3.8	−8.4	−19.0

For changes in the mean vertical and horizontal diameters see case 1 in Table 9.2. Durelli and Lin in Ref. 59 have made extensive use of Nelson's equations for diametrically loaded hollow circular cylinders from Ref. 60 and present, in graphical form, stress factors and radial displacements at all angular positions along both inner and outer boundaries. Results are plotted for the radius ratio a/b from near zero to 0.92.

Gear teeth. Gear teeth may be investigated by considering the tooth as a cantilever beam, the critical stress being the tensile bending stress at the base. This stress can be calculated by the modified Heywood formula for a very short cantilever beam (Sec. 8.10) or by a combination of the modified Lewis formula and stress concentration factor given for case 21 in Table 17.1 (see also Refs. 22–24). The allowable stress is reduced according to speed of operation by one of

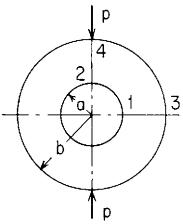


Figure 14.2

several empirical formulas (Ref. 24). Under certain conditions, the bearing stress between teeth may become important (especially as this stress affects wear), and this stress may be calculated by the formula for case 2b, Table 14.1. The total deformation of the tooth, the result of direct compression at the point of contact and of beam deflection and shear, may be calculated by the formula of case 2b and the methods of Sec. 8.1 (Ref. 23).

Keys. Keys are designed for a total shearing force $F = T/r$ (Fig. 14.3), where T represents the torque transmitted. The shear stress is assumed to be uniformly distributed over the horizontal section AB, and the bearing stress is assumed to be uniformly distributed over half the face. These assumptions lead to the following formulas: $\tau = F/Lb$; $\sigma_b = 2F/tL$ on the sides; and $\sigma_b = 2Ft/b^2L$ on top and bottom. Here L is the length of the key; in conventional design $4b < L < 16b$. As usually made, $b \geqslant t$; hence the bearing stress on the sides is greater than that on the top and bottom.

Photoelastic analysis of the stresses in square keys shows that the shear stress is not uniform across the breadth b but is greatest at A and B, where it may reach a value of from two or four times the average value (Ref. 25). Undoubtedly the shear stress also varies in intensity along the length of the key. The bearing stresses on the surfaces of the key are also nonuniform, that on the sides being greatest near the common surface of shaft and hub, and that on the top and bottom being greatest near the corners C and D. When conservative working stresses are used, however, and the proportions of the key are such as have been found satisfactory in practice, the approximate methods of stress calculation that have been indicated result in satisfactory design.

Fillet welds. These are successfully designed on the basis of uniform distribution of shear stress on the longitudinal section of least area, although analysis and tests show that there is considerable variation in the intensity of shear stress along the length of the fillet (Refs. 26 and 27). (Detailed recommendations for the design of welded structural joints are given in Ref. 13.)

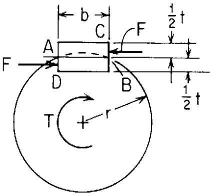


Figure 14.3

Screwthreads. The strength of screwthreads is of great importance in the design of joints, where the load is transferred to a bolt or stud by a nut. A major consideration is the load distribution. The load is not transferred uniformly along the engaged thread length; both mathematical analysis and tests show that the maximum load per linear inch of thread, which occurs near the loaded face of the nut, is several times as great as the average over the engaged length. This ratio, called the *thread-load concentration factor* and denoted by H, is often 2, 3, or even 4 (Ref. 43). The maximum load per linear inch on a screwthread is therefore the total load divided by the helical length of the engaged screwthread times H. The maximum stress due to this loading can be computed by the Heywood–Kelley–Pedersen formula for a short cantilever, as given in Sec. 8.10. It is important to note that in some cases the values of k_f given in the literature are for loading through a nut, and so include H, while in other cases (as in rotating-beam tests) this influence is absent. Because of the combined effects of reduced area, nonuniform load distribution, and stress concentration, the efficiency of a bolted joint under reversed repeated loading is likely to be quite small. In Ref. 28 of Chap. 3, values from 18% (for a 60,000 lb/in^2 steel with rolled threads) to 6.6% (for a 200,000 lb/in^2 steel with machine-cut threads) are cited.

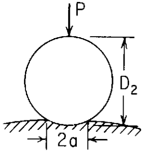
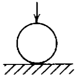
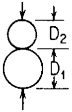
The design of bolted connections has received much study, and an extensive discussion and bibliography are given in Heywood (Chap. 3, Ref. 28) and in some of the papers of Ref. 32 of Chap. 3.

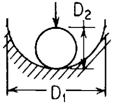
14.4 TABLES

TABLE 14.1 Formulas for stress and strain due to pressure on or between elastic bodies

NOTATION: P = total load; p = load per unit length; a = radius of circular contact area for case 1; b = width of rectangular contact area for case 2; c = major semiaxis and d = minor semiaxis of elliptical contact area for cases 3 and 4; y = relative motion of approach along the axis of loading of two points, one in each of the two contact bodies, remote from the contact zone; ν = Poisson's ratio; E = modulus of elasticity. Subscripts 1 and 2 refer to bodies 1 and 2, respectively. To simplify expressions let

$$C_E = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$

<table>
<tr><th>Conditions and case no.</th><th colspan="2">Formulas</th></tr>
<tr>
<td rowspan="3">1. Sphere

 </td>
<td rowspan="3">$a = 0.721\sqrt[3]{PK_DC_E}$
$(\sigma_c)_{\max} = 1.5\frac{P}{\pi a^2}$
$= 0.918\sqrt[3]{\frac{P}{K_D^2C_E^2}}$
$y = 1.040\sqrt[3]{\frac{P^2C_E^2}{K_D}}$
If $E_1 = E_2 = E$ and $\nu_1 = \nu_2 = 0.3$, then
$a = 0.881\sqrt[3]{\frac{PK_D}{E}}$
$(\sigma_c)_{\max} = 0.616\sqrt[3]{\frac{PE^2}{K_D^2}}$
$y = 1.55\sqrt[3]{\frac{P^2}{E^2K_D}}$
(Note: 50% of y occurs within a distance of 1.2 times the contact radius a and 90% within a distance $7a$ from the contact zone)
$(\sigma_t)_{\max} \approx 0.133(\sigma_c)_{\max}$ radially at the edge of contact area
$\tau_{\max} \approx \frac{1}{3}(\sigma_c)_{\max}$ at a point on the load line a distance $a/2$ below the contact surface
(Approximate stresses from Refs. 3 and 6)
For graphs of subsurface stress variations see Refs. 6 and 57</td>
<td>1a. Sphere on a flat plate

$K_D = D_2$</td>
</tr>
<tr><td>1b. Sphere on a sphere


$K_D = \frac{D_1D_2}{D_1 + D_2}$</td></tr>
<tr><td>1c. Sphere in a spherical socket


$K_D = \frac{D_1D_2}{D_1 - D_2}$</td></tr>
</table>

TABLE 14.1 Formulas for stress and strain due to pressure on or between elastic bodies (*Continued*)

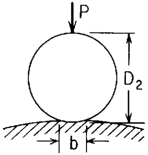
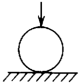
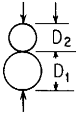
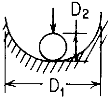
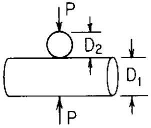
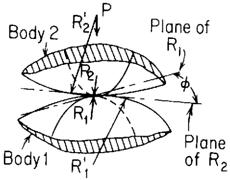
<table>
<tr>
<td rowspan="3">2. Cylinder of length L large as compared with D; $p =$ load per unit length $= P/L$
 </td>
<td rowspan="3">$b = 1.60\sqrt{pK_D C_E}$
$(\sigma_c)_{\max} = 0.798\sqrt{\frac{p}{K_D C_E}}$
If $E_1 = E_2 = E$ and $\nu_1 = \nu_2 = 0.3$, then
$b = 2.15\sqrt{\frac{pK_D}{E}}$
$(\sigma_c)_{\max} = 0.591\sqrt{\frac{pE}{K_D}}$
For a cylinder between two flat plates
$\Delta D_2 = \frac{4p(1-\nu^2)}{\pi E}\left(\frac{1}{3}+\ln\frac{2D}{b}\right)$ Refs. 5 and 44
For a cylinder on a cylinder the distance between centers is reduced by
$\frac{2p(1-\nu^2)}{\pi E}\left(\frac{2}{3}+\ln\frac{2D_1}{b}+\ln\frac{2D_2}{b}\right)$ Ref. 31
For graphs of subsurface stress variations see Refs. 6 and 56</td>
<td>2a. Cylinder on a flat plate

$K_D = D_2$
$\tau_{\max} \approx \frac{1}{3}(\sigma_c)_{\max}$ at a depth of $0.4b$ below the surface of the plane</td>
</tr>
<tr>
<td>2b. Cylinder on a cylinder

$K_D = \frac{D_1 D_2}{D_1 + D_2}$</td>
</tr>
<tr>
<td>2c. Cylinder in a cylindrical socket

$K_D = \frac{D_1 D_2}{D_1 - D_2}$</td>
</tr>
<tr>
<td>3. Cylinder on a cylinder; axes at right angles
 </td>
<td colspan="2">$c = \alpha\sqrt[3]{PK_D C_E}$ $K_D = \frac{D_1 D_2}{D_1 + D_2}$ and α, β, and λ depend upon $\frac{D_1}{D_2}$ as shown
$d = \beta\sqrt[3]{PK_D C_E}$
$(\sigma_c)_{\max} = \frac{1.5P}{\pi c d}$
$y = \lambda\sqrt[3]{\frac{P^2 C_E^2}{K_D}}$
$\tau_{\max} = \frac{1}{3}(\sigma_c)_{\max}$
<table>
<tr><th>D_1/D_2</th><th>1</th><th>1.5</th><th>2</th><th>3</th><th>4</th><th>6</th><th>10</th></tr>
<tr><td>α</td><td>0.908</td><td>1.045</td><td>1.158</td><td>1.350</td><td>1.505</td><td>1.767</td><td>2.175</td></tr>
<tr><td>β</td><td>0.908</td><td>0.799</td><td>0.732</td><td>0.651</td><td>0.602</td><td>0.544</td><td>0.481</td></tr>
<tr><td>λ</td><td>0.825</td><td>0.818</td><td>0.804</td><td>0.774</td><td>0.747</td><td>0.702</td><td>0.641</td></tr>
</table>
</td>
</tr>
</table>

TABLE 14.1 Formulas for stress and strain due to pressure on or between elastic bodies (*Continued*)

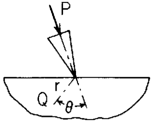
<table>
<tr><th>Conditions and case no.</th><th>Formulas</th></tr>
<tr><td>4. General case of two bodies in contact; $P=$ total load
 </td><td>At point of contact minimum and maximum radii of curvature are R_1 and R'_1 for body 1, and R_2 and R'_2 for body 2. Then $1/R_1$ and $1/R'_1$ are principal curvatures of body 1, and $1/R_2$ and $1/R'_2$ of body 2; and in each body the principal curvatures are mutually perpendicular. The radii are positive if the center of curvature lies within the given body, i.e., the surface is convex, and negative otherwise. The plane containing curvature $1/R_1$ in body 1 makes with the plane containing curvature $1/R_2$ in body 2 the angle ϕ. Then:

$$c=\alpha\sqrt[3]{PK_DC_E} \qquad d=\beta\sqrt[3]{PK_DC_E} \qquad (\sigma_c)_{\max}=\frac{1.5P}{\pi cd} \qquad \text{and} \qquad y=\lambda\sqrt[3]{\frac{P^2C_E^2}{K_D}} \qquad \text{where } K_D=\frac{1.5}{1/R_1+1/R_2+1/R'_1+1/R'_2}$$

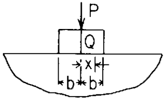
and α, β, and λ are given by the following table in which

$$\cos\theta=\frac{K_D}{1.5}\sqrt{\left(\frac{1}{R_1}-\frac{1}{R'_1}\right)^2+\left(\frac{1}{R_2}-\frac{1}{R'_2}\right)^2+2\left(\frac{1}{R_1}-\frac{1}{R'_1}\right)\left(\frac{1}{R_2}-\frac{1}{R'_2}\right)\cos 2\phi}$$

<table>
<tr><th>$\cos\theta$</th><th>0.00</th><th>0.10</th><th>0.20</th><th>0.30</th><th>0.40</th><th>0.50</th><th>0.60</th><th>0.70</th><th>0.75</th><th>0.80</th><th>0.85</th><th>0.90</th><th>0.92</th><th>0.94</th><th>0.96</th><th>0.98</th><th>0.99</th></tr>
<tr><td>α</td><td>1.000</td><td>1.070</td><td>1.150</td><td>1.242</td><td>1.351</td><td>1.486</td><td>1.661</td><td>1.905</td><td>2.072</td><td>2.292</td><td>2.600</td><td>3.093</td><td>3.396</td><td>3.824</td><td>4.508</td><td>5.937</td><td>7.774</td></tr>
<tr><td>β</td><td>1.000</td><td>0.936</td><td>0.878</td><td>0.822</td><td>0.769</td><td>0.717</td><td>0.664</td><td>0.608</td><td>0.578</td><td>0.544</td><td>0.507</td><td>0.461</td><td>0.438</td><td>0.412</td><td>0.378</td><td>0.328</td><td>0.287</td></tr>
<tr><td>λ</td><td>0.750</td><td>0.748</td><td>0.743</td><td>0.734</td><td>0.721</td><td>0.703</td><td>0.678</td><td>0.644</td><td>0.622</td><td>0.594</td><td>0.559</td><td>0.510</td><td>0.484</td><td>0.452</td><td>0.410</td><td>0.345</td><td>0.288</td></tr>
</table>

(Ref. 8)</td></tr>
<tr><td>5. Rigid knife-edge across edge of semi-infinite plate; load $p=P/t$ where t is plate thickness
 </td><td>At any point Q.

$$\sigma_c=\frac{2p\cos\theta}{\pi r} \qquad \text{in the direction of the radius } r$$

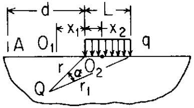
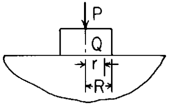
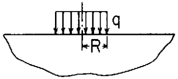
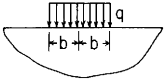
(Ref. 6)</td></tr>
<tr><td>6. Rigid block of width $2b$ across edge of semi-infinite plate; load $p=P/t$ where t is plate thickness
 </td><td>At any point Q on surface of contact,

$$\sigma_c=\frac{p}{\pi\sqrt{b^2-x^2}}$$

(For loading on block of finite width and influence of distance of load from corner see Ref. 45)

(Ref. 6)</td></tr>
</table>

TABLE 14.1 Formulas for stress and strain due to pressure on or between elastic bodies (*Continued*)

7. Uniform pressure q over length L across edge of semi-infinite plate  	At any point O_1 outside loaded area, $y = \frac{2q}{\pi E}\left[(L+x_1)\ln\frac{d}{L+x_1} - x_1\ln\frac{d}{x_1}\right] + qL\frac{1-\nu}{\pi E}$ At any point O_2 inside loaded area, $y = \frac{2q}{\pi E}\left[(L-x_2)\ln\frac{d}{L-x_2} + x_2\ln\frac{d}{x_2}\right] + qL\frac{1-\nu}{\pi E}$ where $y =$ deflection relative to a remote point A a distance d from edge of loaded area At any point Q, $\sigma_c = 0.318q(\alpha + \sin\alpha)$ $\tau = 0.318q\sin\alpha$ (Ref. 6)
8. Rigid cylindrical die of radius R on surface of semi-infinite body; total load P  	$y = \frac{P(1-\nu^2)}{2RE}$ At any point Q on surface of contact $\sigma_c = \frac{P}{2\pi R\sqrt{R^2 - r^2}}$ $(\sigma_c)_{\max} = \infty$ at edge (theoretically) $(\sigma_c)_{\min} = \frac{P}{2\pi R^2}$ at center (Ref. 6)
9. Uniform pressure q over circular area of radius R on surface of semi-infinite body  	$y_{\max} = \frac{2qR(1-\nu^2)}{E}$ at center y at edge $= \frac{4qR(1-\nu^2)}{\pi E}$ $\tau_{\max} = 0.33q$ at point $0.638R$ below center of loaded area
10. Uniform pressure q over square area of sides $2b$ on the surface of semi-infinite body  	$y_{\max} = \frac{2.24qb(1-\nu^2)}{E}$ at center $y = \frac{1.12qb(1-\nu^2)}{E}$ at corners $y_{\text{average}} = \frac{1.90qb(1-\nu^2)}{E}$ (Ref. 6)

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Chapter

15

Elastic Stability

15.1 General Considerations

Failure through elastic instability has been discussed briefly in Sec. 3.13, where it was pointed out that it may occur when the bending or twisting effect of an applied load is proportional to the deformation it produces. In this chapter, formulas for the critical load or critical unit stress at which such failure occurs are given for a wide variety of members and conditions of loading.

Such formulas can be derived mathematically by integrating the differential equation of the elastic curve or by equating the strain energy of bending to the work done by the applied load in the corresponding displacement of its point of application, the form of the elastic curve being assumed when unknown. Of all possible forms of the curve, that which makes the critical load a minimum is the correct one; but almost any reasonable assumption (consistent with the boundary conditions) can be made without gross error resulting, and for this reason the strain-energy method is especially adapted to the approximate solution of difficult cases. A very thorough discussion of the general problem, with detailed solutions of many specified cases, is given in Timoshenko and Gere (Ref. 1), from which many of the formulas in this chapter are taken. Formulas for many cases are also given in Refs. 35 and 36; in addition Ref. 35 contains many graphs of numerically evaluated coefficients.

At one time, most of the problems involving elastic stability were of academic interest only since engineers were reluctant to use compression members so slender as to fail by buckling at elastic stresses and danger of corrosion interdicted the use of very thin material in exposed structures. The requirements for minimum-weight construction in the fields of aerospace and transportation, however, have given great impetus to the theoretical and experimental investigation of elastic stability and to the use of parts for which it is a governing design consideration.

There are certain definite advantages in lightweight construction, in which stability determines strength. One is that since elastic buckling may occur without damage, part of a structure—such as the skin of an airplane wing or web of a deep beam—may be used safely at loads that cause local buckling, and under these circumstances the resistance afforded by the buckled part is definitely known. Furthermore, members such as Euler columns may be loaded experimentally to their maximum capacity without damage or permanent deformation and subsequently incorporated in a structure.

15.2 Buckling of Bars

In Table 15.1, formulas are given for the critical loads on columns, beams, and shafts. In general, the theoretical values are in good agreement with test results as long as the assumed conditions are reasonably well-satisfied. It is to be noted that even slight changes in the amount of end constraint have a marked effect on the critical loads, and therefore it is important that such constraint be closely estimated. Slight irregularities in form and small accidental eccentricities are less likely to be important in the case of columns than in the case of thin plates. For latticed columns or columns with tie plates, a reduced value of E may be used, calculated as shown in Sec. 12.3. Formulas for the elastic buckling of bars may be applied to conditions under which proportional limit is exceeded if a reduced value of E corresponding to the actual stress is used (Ref. 1), but the procedure requires a stress-strain diagram for the material and, in general, is not practical.

In Table 15.1, cases 1–3, the tabulated buckling coefficients are worked out for various combinations of concentrated and distributed axial loads. Tensile end loads are included so that the effect of axial end restraint under axial loading within the column length can be considered (see the example at the end of this section). Carter and Gere (Ref. 46) present graphs of buckling coefficients for columns with single tapers for various end conditions, cross sections, and degrees of taper. Culver and Preg (Ref. 47) investigate and tabulate buckling coefficients for singly tapered beam-columns in which the effect of torsion, including warping restraint, is considered for the case where the loading is by end moments in the stiffer principal plane.

Kitipornchai and Trahair describe (Ref. 55) the lateral stability of singly tapered cantilever and doubly tapered simple I-beams, including the effect of warping restraint; experimental results are favorably compared with numerical solutions. Morrison (Ref. 57) considers the effect of lateral restraint of the tensile flange of a beam under lateral buckling; example calculations are presented. Massey and McGuire (Ref. 54) present graphs of buckling coefficients for both stepped and tapered cantilever beams; good agreement with experiments is reported. Tables of lateral stability constants for laminated timber beams are presented in Fowler (Ref. 53) along with two design examples.

Clark and Hill (Ref. 52) derive a general expression for the lateral stability of unsymmetrical I-beams with boundary conditions based on both bending and warping supports; tables of coefficients as well as nomographs are presented. Anderson and Trahair (Ref. 56) present tabulated lateral buckling coefficients for uniformly loaded and end-loaded cantilevers and center- and uniformly loaded simply supported beams having unsymmetric I-beam cross sections; favorable comparisons are made with extensive tests on cantilever beams.

The Southwell plot is a graph in which the lateral deflection of a column or any other linearly elastic member undergoing a manner of loading which will produce buckling is plotted versus the lateral deflection divided by the load; the slope of this line gives the critical load. For columns and some frameworks, significant deflections do occur within the range where small-deflection theory is applicable. If the initial imperfections are such that experimental readings of lateral deflection must be taken beyond the small-deflection region, then the Southwell procedure is not adequate. Roorda (Ref. 93) discusses the extension of this procedure into the nonlinear range.

Bimetallic beams. Burgreen and Manitt (Ref. 48) and Burgreen and Regal (Ref. 49) discuss the analysis of bimetallic beams and point out some of the difficulties in predicting the *snap-through instability* of these beams under changes in temperature. The thermal expansion of the support structure is an important design factor.

Rings and arches. Austin (Ref. 50) tabulates in-plane buckling coefficients for circular, parabolic, and catenary arches for pinned and fixed ends as well as for the three-hinged case; he considers cases where the cross section varies with the position in the span as well as the usual case of a uniform cross section. Uniform loads, unsymmetric distributed loads, and concentrated center loads are considered, and the stiffening effect of tying the arch to the girder with columns is also evaluated. (The discussion referenced with the paper gives an extensive bibliography of work on arch stability.)

A thin ring shrunk by cooling and inserted into a circular cavity usually will yield before buckling unless the radius/thickness ratio is very large and the elastic-limit stress is high. Chicurel (Ref. 51) derives approximate solutions to this problem when the effect of friction is considered. He suggests a conservative expression for the *no-friction* condition: $P_o/AE = 2.67(k/r)^{1.2}$, where P_o is the prebuckling hoop compressive force, A is the hoop cross-sectional area, E is the modulus of elasticity, k is the radius of gyration of the cross section, and r is the radius of the ring.

EXAMPLE

A 4-in steel pipe is to be used as a column to carry 8000 lb of transformers centered axially on a platform 20 ft above the foundation. The factor of safety *FS* is to be determined for the following conditions, based on elastic buckling of the column.

(a) The platform is supported only by the pipe fixed at the foundation.

(b) A $3\frac{1}{2}$-in steel pipe is to be slipped into the top of the 4-in pipe a distance of 4 in, welded in place, and extended 10 ft to the ceiling above, where it will extend through a close-fitting hole in a steel plate.

(c) This condition is the same as in (b) except that the $3\frac{1}{2}$-in pipe will be welded solidly into a heavy steel girder passing 10 ft above the platform.

Solution. A 4-in steel pipe has a cross-sectional area of 3.174 in^2 and a bending moment of inertia of 7.233 in^4. For a $3\frac{1}{2}$-in pipe these are 2.68 in^2 and 4.788 in^4, respectively.

(a) This case is a column fixed at the bottom and free at the top with an end load only. In Table 15.1, case la, for $I_2/I_1 = 1.00$ and $P_2/P_1 = 0$, K_1 is given as 0.25. Therefore,

$$P_1' = 0.25\frac{\pi^2 30(10^6)(7.233)}{240^2} = 9295 \text{ lb}$$

$$FS = \frac{9295}{8000} = 1.162$$

(b) This case is a column fixed at the bottom and pinned at the top with a load at a distance of two-thirds the 30-ft length from the bottom: $I_1 = 4.788$ in^4, $I_2 = 7.233$ in^4, and $I_2/I_1 = 1.511$. In Table 15.1, case 2d, for $E_2I_2/E_1I_1 = 1.5$, $P_1/P_2 = 0$, and $a/l = \frac{2}{3}$, K_2 is given as 6.58. Therefore,

$$P_2' = 6.58\frac{\pi^2 30(10^6)(4.788)}{360^2} = 72{,}000 \text{ lb}$$

$$FS = \frac{72{,}000}{8000} = 9$$

(c) This case is a column fixed at both ends and subjected to an upward load on top and a downward load at the platform. The upward load depends to some extent on the stiffness of the girder to which the top is welded, and so we can only bracket the actual critical load. If we assume the girder is infinitely rigid and permits no vertical deflection of the top, the elongation of the upper 10 ft would equal the reduction in length of the lower 20 ft.

Equating these deformations gives

$$\frac{P_1(10)(12)}{2.68(30)(10^6)} = \frac{(P_2 - P_1)(20)(12)}{3.174(30)(10^6)} \qquad \text{or} \qquad P_1 = 0.628P_2$$

From Table 15.1, case 2e, for $E_2I_2/E_1I_1 = 1.5$ and $a/l = \frac{2}{3}$, we find the following values of K_2 for the several values of P_1/P_2:

P_1/P_2	0	0.125	0.250	0.375	0.500
K_2	8.34	9.92	12.09	15.17	19.86

By extrapolation, for $P_1/P_2 = 0.628$, $K_2 = 26.5$.

If we assume the girder provides no vertical load but does prevent rotation of the top, then $K_2 = 8.34$. Therefore, the value of P_2 ranges from 91,200 to 289,900 lb, and the factor of safety lies between 11.4 and 36.2. A reasonable estimate of the rotational and vertical stiffness of the girder will allow a good estimate to be made of the actual factor of safety from the values calculated.

15.3 Buckling of Flat and Curved Plates

In Table 15.2, formulas are given for the critical loads and critical stresses on plates and thin-walled members. Because of the greater likelihood of serious geometrical irregularities and their greater relative effect, the critical stresses actually developed by such members usually fall short of the theoretical values by a wider margin than in the case of bars. The discrepancy is generally greater for pure compression (thin tubes under longitudinal compression or external pressure) than for tension and compression combined (thin tubes under torsion or flat plates under edge shear), and increases with the thinness of the material. The critical stress or load indicated by any one of the theoretical formulas should therefore be regarded as an upper limit, approached more or less closely according to the closeness with which the actual shape of the member approximates the geometrical form assumed. In Table 15.2, the approximate discrepancy to be expected between theory and experiment is indicated wherever the data available have made this possible.

Most of the theoretical analyses of the stability of plates and shells require a numerical evaluation of the resulting equations. Considering the variety of shapes and combinations of shapes as well as the multiplicity of boundary conditions and loading combinations, it is not possible in the limited space available to present anything like a comprehensive coverage of plate and shell buckling. As an alternative, Table 15.2 contains many of the simpler loadings and shapes. The following paragraphs and the References contain some, but by no means all, of the more easily acquired sources giving results in tabular or graphic form that can be applied directly to specific problems. See also Refs. 101–104, and 109–111.

Rectangular plates. Stability coefficients for *orthotropic* rectangular plates with several combinations of boundary conditions and several ratios of the bending stiffnesses parallel to the sides of the plate are tabulated in Shuleshko (Ref. 60); these solutions were obtained by reducing the problem of plate buckling to that of an isotropic bar that is in a state of vibration and under tension. Srinivas and Rao (Ref. 63) evaluate the effect of shear deformation on the stability of simply supported rectangular plates under edge loads parallel to one side; the effect becomes noticeable for $h/b > 0.05$ and is greatest when the loading is parallel to the short side.

Skew plates. Ashton (Ref. 61) and Durvasula (Ref. 64) consider the buckling of skew (parallelogram) plates under combinations of edge compression, edge tension, and edge shear. Since the loadings evaluated are generally parallel to orthogonal axes and not to both sets of the plate edges, we would not expect to find the particular case desired represented in the tables of coefficients; the general trend of results is informative.

Circular plates. Vijayakumar and Joga Rao (Ref. 58) describe a technique for solving for the radial buckling loads on a *polar orthotropic annular plate*. They give graphs of stability coefficients for a wide range of rigidity ratios and for the several combinations of free, simply supported, and fixed inner and outer edges for the radius ratio (outer to inner) 2 : 1. Two loadings are presented: outer edge only under uniform compression and inner and outer edges under equal uniform compression.

Amon and Widera (Ref. 59) present graphs showing the effect of an edge beam on the stability of a circular plate of uniform thickness.

Sandwich plates. There is a great amount of literature on the subject of sandwich construction. References 38 and 100 and the publications listed in Ref. 39 provide initial sources of information.

15.4 Buckling of Shells

Baker, Kovalevsky, and Rish (Ref. 97) discuss the stability of unstiffened orthotropic composite, stiffened, and sandwich shells. They represent data based on theory and experiment which permit the designer to choose a loading or pressure with a 90% probability of no stability failure; the work is extensively referenced. For similar collected data see Refs. 41 and 42.

Stein (Ref. 95) discusses some comparisons of theory with experimentation in shell buckling. Rabinovich (Ref. 96) describes in some detail the work in structural mechanics, including shell stability, in the U.S.S.R. from 1917 to 1957.

In recent years, there have been increasing development and application of the finite-element method for the numerical solution of shell problems. Navaratna, Pian, and Witmer (Ref. 94) describe a finite-element method of solving axisymmetric shell problems where the element considered is either a conical frustum or a frustum with a curved meridian; examples are presented of cylinders with uniform or tapered walls under axial load, a truncated hemisphere under axial tension, and a conical shell under torsion. Bushnell (Ref. 99) presents a very general finite-element program for shell analysis and Perrone (Ref. 98) gives a compendium of such programs. See also Refs. 101 to 108.

Cylindrical and conical shells. In general, experiments to determine the axial loads required to buckle cylindrical shells yield results that are between one-half and three-fourths of the classical buckling loads predicted by theory. The primary causes of these discrepancies are the deviations from a true cylindrical form in most manufactured vessels and the inability to accurately define the boundary conditions. Hoff (Refs. 67 and 68) shows that removing the in-plane shear stress at the boundary of a simply supported cylindrical shell under axial compression can reduce the theoretical buckling load by a factor of 2 from that predicted by the more usual boundary conditions associated with a simply supported edge. Baruch, Harari, and Singer (Ref. 84) find similar low-buckling loads for simply supported conical shells under axial load but for a different modification of the boundary support. Tani and Yamaki (Ref. 83) carry out further work on this problem, including the effect of clamped edges.

The random nature of manufacturing deviations leads to the use of the statistical approach, as mentioned previously (Ref. 97) and as Hausrath and Dittoe have done for conical shells (Ref. 77). Weingarten, Morgan, and Seide (Ref. 80) have developed empirical expressions for lower bounds of stability coefficients for cylindrical and conical shells under axial compression with references for the many data they present.

McComb, Zender, and Mikulas (Ref. 44) discuss the effects of internal pressure on the bending stability of very thin-walled cylindrical shells. Internal pressure has a stabilizing effect on axially and/or torsionally loaded cylindrical and conical shells. This subject is discussed in several references: Seide (Ref. 75), Weingarten (Ref. 76), and Weingarten, Morgan, and Seide (Ref. 82) for conical and cylindrical shells; Ref. 97 contains much information on this subject as well.

Axisymmetric snap-buckling of conical shells is discussed by Newman and Reiss (Ref. 73), which leads to the concept of the Belleville spring for the case of shallow shells. (See also Sec. 11.8.)

External pressure as a cause of buckling is examined by Singer (Ref. 72) for cones and by Newman and Reiss (Ref. 73) and Yao and Jenkins (Ref. 69) for elliptic cylinders. External pressure caused by pretensioned filament winding on cylinders is analyzed by Mikulas and Stein (Ref. 66); they point out that material compressibility in the thickness direction is important in this problem.

The combination of external pressure and axial loads on cylindrical and conical shells is very thoroughly examined and referenced by Radkowski (Ref. 79) and Weingarten and Seide (Ref. 81). The combined loading on orthotropic and stiffened conical shells is discussed by Singer (Ref. 74).

Attempts to manufacture nearly perfect shells in order to test the theoretical results have led to the construction of thin-walled shells by electroforming; Sendelbeck and Singer (Ref. 85) and Arbocz and Babcock (Ref. 91) describe the results of such tests.

A very thorough survey of buckling theory and experimentation for conical shells of constant thickness is presented by Seide (Ref. 78).

Spherical shells. Experimental work is described by Loo and Evan-Iwanowski on the effect of a concentrated load at the apex of a spherical cap (Ref. 90) and the effect of multiple concentrated loads (Ref. 89). Carlson, Sendelbeck, and Hoff (Ref. 70) report on the experimental study of buckling of electroformed complete spherical shells; they report experimental critical pressures of up to 86% of those predicted by theory and the correlation of flaws with lower test pressures.

Burns (Ref. 92) describes tests of static and dynamic buckling of thin spherical caps due to external pressure; both elastic and plastic buckling are considered and evaluated in these tests. Wu and Cheng (Ref. 71) discuss in detail the buckling due to circumferential hoop compression which is developed when a truncated spherical shell is subjected to an axisymmetric tensile load.

Toroidal shells. Stein and McElman (Ref. 86) derive nonlinear equations of equilibrium and buckling equations for segments of toroidal shells; segments that are symmetric with the equator are considered for both inner and outer diameters, as well as segments centered at the crown. Sobel and Flügge (Ref. 87) tabulate and graph the minimum buckling external pressures on full toroidal shells. Almroth, Sobel, and Hunter (Ref. 88) compare favorably the theory in Ref. 87 with experiments they performed.

Corrugated tubes or bellows. An instability can develop when a corrugated tube or bellows is subjected to an internal pressure with the ends partially or totally restrained against axial displacement. (This instability can also occur in very long cylindrical vessels under similar restraints.) For a discussion and an example of this effect, see Sec. 13.5.

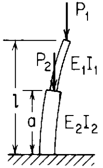
15.5 Tables

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams

NOTATION: P' = critical load (force); p' = critical unit load (force per unit length); T' = critical torque (force-length); M' = critical bending moment (force-length); E = modulus of elasticity (force per unit area); and I = moment of inertia of cross section about central axis perpendicular to plane of buckling

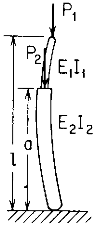
Reference number, form of bar, and manner of loading and support

1a. Stepped straight bar under end load P_1 and intermediate load P_2; upper end free, lower end fixed $P_1' = K_1 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	0.250	0.250	0.250	0.250	0.250	0.279	0.312	0.342	0.364	0.373	0.296	0.354	0.419	0.471	0.496
0.5	0.249	0.243	0.228	0.208	0.187	0.279	0.306	0.317	0.306	0.279	0.296	0.350	0.393	0.399	0.372
1.0	0.248	0.237	0.210	0.177	0.148	0.278	0.299	0.295	0.261	0.223	0.296	0.345	0.370	0.345	0.296
2.0	0.246	0.222	0.178	0.136	0.105	0.277	0.286	0.256	0.203	0.158	0.295	0.335	0.326	0.267	0.210
4.0	0.242	0.195	0.134	0.092	0.066	0.274	0.261	0.197	0.138	0.099	0.294	0.314	0.257	0.184	0.132
8.0	0.234	0.153	0.088	0.056	0.038	0.269	0.216	0.132	0.084	0.057	0.290	0.266	0.174	0.112	0.076

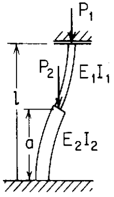
1b. Stepped straight bar under end load P_1 and intermediate load P_2; both ends pinned $P_1' = K_1 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	1.000	1.000	1.000	1.000	1.000	1.010	1.065	1.180	1.357	1.479	1.014	1.098	1.297	1.633	1.940
0.5	0.863	0.806	0.797	0.789	0.740	0.876	0.872	0.967	1.091	1.098	0.884	0.908	1.069	1.339	1.452
1.0	0.753	0.672	0.663	0.646	0.584	0.769	0.736	0.814	0.908	0.870	0.776	0.769	0.908	1.126	1.153
2.0	0.594	0.501	0.493	0.473	0.410	0.612	0.557	0.615	0.676	0.613	0.621	0.587	0.694	0.850	0.814
4.0	0.412	0.331	0.325	0.307	0.256	0.429	0.373	0.412	0.442	0.383	0.438	0.397	0.470	0.566	0.511
8.0	0.254	0.197	0.193	0.180	0.147	0.267	0.225	0.248	0.261	0.220	0.272	0.240	0.284	0.336	0.292

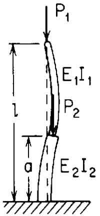
TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

1c. Stepped straight bar under end load P_1 and intermediate load P_2; upper end guided, lower end fixed $P_1' = K_1 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	1.000	1.000	1.000	1.000	1.000	1.113	1.208	1.237	1.241	1.309	1.184	1.367	1.452	1.461	1.565
0.5	0.986	0.904	0.792	0.711	0.672	1.105	1.117	1.000	0.897	0.885	1.177	1.288	1.192	1.063	1.063
1.0	0.972	0.817	0.650	0.549	0.507	1.094	1.026	0.830	0.697	0.669	1.171	1.206	1.000	0.832	0.805
2.0	0.937	0.671	0.472	0.377	0.339	1.073	0.872	0.612	0.482	0.449	1.156	1.047	0.745	0.578	0.542
4.0	0.865	0.480	0.304	0.231	0.204	1.024	0.642	0.397	0.297	0.270	1.126	0.794	0.486	0.358	0.327
8.0	0.714	0.299	0.176	0.130	0.114	0.910	0.406	0.232	0.169	0.151	1.042	0.511	0.284	0.203	0.182

1d. Stepped straight bar under end load P_1 and intermediate load P_2; upper end pinned, lower end fixed $P_1' = K_1 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below

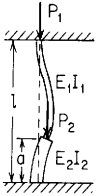


E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	2.046	2.046	2.046	2.046	2.046	2.241	2.289	2.338	2.602	2.976	2.369	2.503	2.550	2.983	3.838
0.5	1.994	1.814	1.711	1.700	1.590	2.208	2.071	1.991	2.217	2.344	2.344	2.286	2.196	2.570	3.066
1.0	1.938	1.613	1.464	1.450	1.290	2.167	1.869	1.727	1.915	1.918	2.313	2.088	1.915	2.250	2.525
2.0	1.820	1.300	1.130	1.111	0.933	2.076	1.535	1.355	1.506	1.390	2.250	1.742	1.518	1.796	1.844
4.0	1.570	0.918	0.773	0.753	0.594	1.874	1.107	0.941	1.042	0.891	2.097	1.277	1.065	1.270	1.184
8.0	1.147	0.569	0.469	0.454	0.343	1.459	0.697	0.582	0.643	0.514	1.727	0.812	0.664	0.796	0.686

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

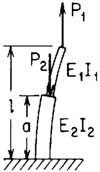
Reference number, form of bar, and manner of loading and support

1e. Stepped straight bar under end load P_1 and intermediate load P_2; both ends fixed $P_1' = K_1 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	4.000	4.000	4.000	4.000	4.000	4.389	4.456	4.757	5.359	5.462	4.657	4.836	5.230	6.477	6.838
0.5	3.795	3.298	3.193	3.052	2.749	4.235	3.756	3.873	4.194	3.795	4.545	4.133	4.301	5.208	4.787
1.0	3.572	2.779	2.647	2.443	2.094	4.065	3.211	3.254	3.411	2.900	4.418	3.568	3.648	4.297	3.671
2.0	3.119	2.091	1.971	1.734	1.414	3.679	2.459	2.459	2.452	1.968	4.109	2.766	2.782	3.136	2.496
4.0	2.365	1.388	1.297	1.088	0.857	2.921	1.659	1.649	1.555	1.195	3.411	1.882	1.885	2.008	1.523
8.0	1.528	0.826	0.769	0.623	0.479	1.943	1.000	0.992	0.893	0.671	2.334	1.138	1.141	1.158	0.854

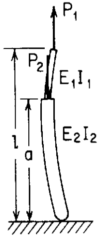
2a. Stepped straight bar under end load P_1 and intermediate load P_2; upper end free, lower end fixed $P_2' = K_2 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	9.00	2.25	1.00	0.56	0.36	13.50	3.38	1.50	0.84	0.54	18.00	4.50	2.00	1.13	0.72
0.125	15.55	3.75	1.48	0.74	0.44	21.87	5.36	2.19	1.11	0.65	27.98	6.92	2.89	1.48	0.87
0.250	21.33	5.30	2.19	1.03	0.55	29.51	7.36	3.13	1.53	0.82	37.30	9.31	4.02	2.02	1.10
0.375	29.02	7.25	3.13	1.52	0.74	39.89	9.97	4.37	2.21	1.10	50.10	12.52	5.52	2.86	1.46
0.500	40.50	10.12	4.46	2.31	1.08	55.66	13.92	6.16	3.28	1.60	69.73	17.43	7.73	4.18	2.12

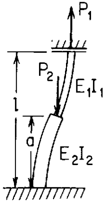
TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

2b. Stepped straight bar under tensile end load P_1 and intermediate load P_2; both ends pinned $P_2' = K_2 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	2.60	1.94	1.89	1.73	1.36	2.77	2.24	2.47	2.54	2.04	2.86	2.41	2.89	3.30	2.72
0.125	3.51	2.49	2.43	2.14	1.62	3.81	2.93	3.26	3.18	2.43	3.98	3.21	3.89	4.19	3.24
0.250	5.03	3.41	3.32	2.77	1.99	5.63	4.15	4.64	4.15	2.99	5.99	4.65	5.75	5.52	3.98
0.375	7.71	5.16	4.96	3.76	2.55	8.98	6.61	7.26	5.63	3.82	9.80	7.67	9.45	7.50	5.09
0.500	12.87	9.13	8.00	5.36	3.48	15.72	12.55	12.00	7.96	5.18	17.71	15.45	16.00	10.54	6.87

2c. Stepped straight bar under tensile end load P_1 and intermediate load P_2; upper end guided, lower end fixed $P_2' = K_2 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below

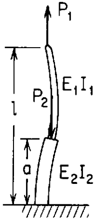


E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	10.40	3.08	1.67	1.19	1.03	14.92	4.23	2.21	1.55	1.37	19.43	5.37	2.73	1.88	1.65
0.125	15.57	4.03	2.03	1.40	1.18	21.87	5.57	2.71	1.82	1.57	27.98	7.07	3.36	2.21	1.90
0.250	21.33	5.37	2.54	1.67	1.38	29.52	7.40	3.42	2.20	1.84	37.32	9.34	4.26	2.68	2.24
0.375	29.02	7.26	3.31	2.08	1.67	39.90	9.97	4.50	2.76	2.24	50.13	12.53	5.61	3.39	2.73
0.500	40.51	10.12	4.53	2.72	2.10	55.69	13.91	6.21	3.66	2.84	69.76	17.43	7.76	4.52	3.47

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

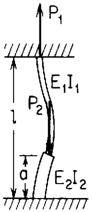
Reference number, form of bar, and manner of loading and support

2d. Stepped straight bar under tensile end load P_1 and intermediate load P_2; upper end pinned, lower end fixed $P'_2 = K_2 \frac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l \ P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	13.96	5.87	4.80	4.53	3.24	18.66	7.33	6.04	6.58	4.86	23.26	8.64	6.98	8.40	6.48
0.125	20.21	7.93	6.50	5.84	3.91	27.12	10.12	8.43	8.71	5.86	33.71	12.06	9.92	11.51	7.81
0.250	28.58	11.35	9.64	7.68	4.87	38.32	14.82	13.13	11.50	7.27	47.36	17.85	15.96	15.30	9.65
0.375	41.15	17.64	15.82	10.15	6.26	55.43	23.67	23.44	14.96	9.30	68.40	28.88	30.65	19.66	12.28
0.500	62.90	31.73	23.78	13.58	8.42	85.64	44.28	34.97	19.80	12.40	106.27	55.33	45.81	25.83	16.27

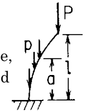
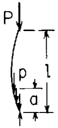
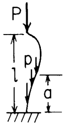
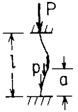
2e. Stepped straight bar under tensile end load P_1 and intermediate load P_2; both ends fixed $P'_2 = K_2 \frac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l \ P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	16.19	8.11	7.54	5.79	4.34	21.06	9.93	9.89	8.34	6.09	25.75	11.44	11.55	10.87	7.78
0.125	21.83	10.37	9.62	6.86	5.00	28.74	12.93	13.03	9.92	7.05	35.28	15.06	15.55	12.96	9.01
0.250	30.02	14.09	12.86	8.34	5.91	39.81	17.99	18.36	12.09	8.35	48.88	21.25	22.98	15.79	10.69
0.375	42.72	20.99	17.62	10.47	7.19	57.14	27.66	26.02	15.17	10.20	70.23	33.29	34.36	19.79	13.11
0.500	64.94	36.57	24.02	13.70	9.16	86.23	50.39	35.09	19.86	13.07	102.53	61.71	45.87	25.86	16.85

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

3a. Uniform straight bar under end load P and a uniformly distributed load p over a lower portion of the length; several end conditions. $(pa)' = K\frac{\pi^2 EI}{l^2}$ where K is tabulated below (a negative value for P/pa means the end load is tensile)

End conditions	Upper end free, lower end fixed 				Both ends pinned 				Upper end pinned, lower end fixed 				Both ends fixed 			
a/l P/pa	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
−0.25		11.31	5.18	2.38	9.03	5.32	4.25	3.30		27.9	17.4	11.3		31.3	19.4	13.4
0.00	12.74	3.185	1.413	0.795	3.52	2.53	2.22	1.88	22.2	9.46	7.13	5.32	25.3	13.0	9.78	7.56
0.25	0.974	0.825	0.614	0.449	1.97	1.59	1.46	1.30	6.83	4.70	3.98	3.30	11.2	7.50	6.25	5.20
0.50	0.494	0.454	0.383	0.311	1.34	1.15	1.08	0.98	3.76	3.03	2.71	2.37	6.75	5.18	4.54	3.94
1.00	0.249	0.238	0.218	0.192	0.81	0.73	0.70	0.66	1.97	1.75	1.64	1.51	3.69	3.17	2.91	2.65

3b. Uniform straight bar under end load P and a uniformly distributed load p over an upper portion of the length; several end conditions. $(pa)' = K\frac{\pi^2 EI}{l^2}$ where K is tabulated below (a negative value for P/pa means the end load is tensile)

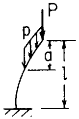
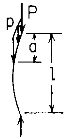
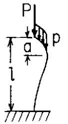
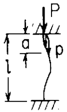
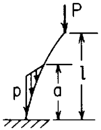
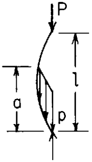
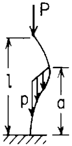
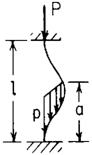
End conditions	Upper end free, lower end fixed 			Both ends pinned 			Upper end pinned, lower end fixed 			Both ends fixed 		
a/l P/pa	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
−0.25	0.481	0.745	1.282	1.808	2.272	2.581	4.338	5.937	7.385	5.829	7.502	9.213
0.00	0.327	0.440	0.600	1.261	1.479	1.611	2.904	3.586	4.160	4.284	5.174	5.970
0.25	0.247	0.308	0.380	0.963	1.088	1.159	2.164	2.529	2.815	3.384	3.931	4.383
0.50	0.198	0.236	0.276	0.778	0.859	0.903	1.720	1.943	2.111	2.796	3.164	3.453
1.00	0.142	0.161	0.179	0.561	0.603	0.624	1.215	1.323	1.400	2.073	2.273	2.419

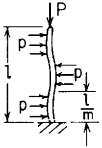
TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

Reference number, form of bar, and manner of loading and support

3c. Stepped straight bar under end load P and a distributed load of maximum value p at the bottom linearly decreasing to zero at a distance a from the bottom. $(pa)' = K\frac{\pi^2 EI}{l^2}$ where K is tabulated below (a negative value for P/pa means the end load is tensile)

End conditions	Upper end free, lower end fixed 				Both ends pinned 				Upper end pinned, lower end fixed 				Both ends fixed 			
a/l P/pa	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
−0.250						58.9	41.1	30.4								
−0.125			26.7	15.5	31.9	15.7	12.0	9.41			62.1	43.8		113.0	70.2	48.7
0.000	52.4	13.1	5.80	3.26	9.66	6.31	5.32	4.72		30.3	20.6	16.1		38.9	27.8	21.9
0.125	1.98	1.85	1.58	1.29	4.65	3.66	3.29	3.03	15.2	11.7	9.73	8.50	27.3	18.9	15.6	13.4
0.250	0.995	0.961	0.887	0.787	2.98	2.54	2.35	2.22	7.90	6.92	6.18	5.66	14.9	12.1	10.6	9.53
0.500	0.499	0.490	0.471	0.441	1.72	1.56	1.49	1.43	4.02	3.77	3.54	3.36	7.73	6.95	6.43	6.00
1.000	0.250	0.248	0.243	0.235	0.93	0.88	0.86	0.84	2.03	1.96	1.90	1.85	3.93	3.73	3.57	3.44

4. Uniform straight bar under end load P; both ends hinged and bar elastically supported by lateral pressure p proportional to deflection ($p = ky$, where k = lateral force per unit length per unit of deflection)



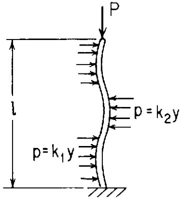
$$P' = \frac{\pi^2 EI}{l^2}\left(m^2 + \frac{kl^4}{m^2\pi^4 EI}\right)$$

where m represents the number of half-waves into which the bar buckles and is equal to the lowest integer greater than

$$\frac{1}{2}\left(\sqrt{1 + \frac{4l^2}{\pi^2}\sqrt{\frac{k}{EI}}} - 1\right)$$

(Ref. 1)

5. Uniform straight bar under end load P; both ends hinged and bar elastically supported by lateral pressure p proportional to deflection but where the constant of proportionality depends upon the direction of the deflection ($p = k_1 y$ for deflection toward the softer foundation; $p = k_2 y$ for deflection toward the harder foundation); these are also called unattached foundations

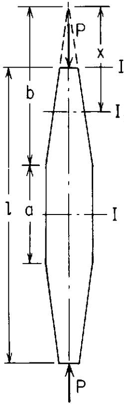


$$P' = \frac{\pi^2 EI}{l^2}\left(m^2 + \frac{k_2 l^4}{m^2 \pi^4 EI}\phi^\alpha\right) \quad \text{where } \phi = \frac{k_1}{k_2} \text{ and } \alpha \text{ depends upon } m \text{ as given below}$$

m	α
1	1
2	$1 + \phi(0.23 - 0.017l^2\sqrt{k_2/EI})$
3	$0.75 - 0.56\phi$

This is an empirical expression which closely fits numerical solutions found in Ref. 45 and is valid only over the range $0 \leqslant l^2\sqrt{k_2/EI} \leqslant 120$. Solutions for P' are carried out for values of $m = 1$, 2, and 3, and the lowest one governs

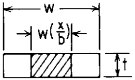
6. Straight bar, middle portion uniform, end portions tapered and alike; end load; I = moment of inertia of cross section of middle portion; I_0 = moment of inertia of end cross sections; I_x = moment of inertia of section x



(For singly tapered columns see Ref. 46.)

6a. $I_x = I\frac{x}{b}$

for example, rectangular section tapering uniformly in width



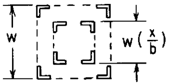
$P' = \frac{KEI}{l^2}$ where K depends on $\frac{I_0}{I}$ and $\frac{a}{l}$ and may be found from the following table:

I_0/I → a/l ↓	K for ends hinged							K for ends fixed			
	0	0.01	0.10	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0	5.78	5.87	6.48	7.01	7.86	8.61	9.27	20.36	26.16	31.04	35.40
0.2	7.04	7.11	7.58	7.99	8.59	9.12	9.53	22.36	27.80	32.20	36.00
0.4	8.35	8.40	8.63	8.90	9.19	9.55	9.68	23.42	28.96	32.92	36.36
0.6	9.36	9.40	9.46	9.73	9.70	9.76	9.82	25.44	30.20	33.80	36.84
0.8	9.80	9.80	9.82	9.82	9.83	9.85	9.86	29.00	33.08	35.80	37.84

(Ref. 5)

6b. $I_x = I\left(\frac{x}{b}\right)^2$

for example, section of four slender members latticed together



$P' = \frac{KEI}{l^2}$ where K may be found from the following table:

I_0/I → a/l ↓	K for ends hinged							K for ends fixed			
	0	0.01	0.10	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0	1.00	3.45	5.40	6.37	7.61	8.51	9.24	18.94	25.54	30.79	35.35
0.2	1.56	4.73	6.67	7.49	8.42	9.04	9.50	21.25	27.35	32.02	35.97
0.4	2.78	6.58	8.08	8.61	9.15	9.48	9.70	22.91	28.52	32.77	36.34
0.6	6.25	8.62	9.25	9.44	9.63	9.74	9.82	24.29	29.69	33.63	36.80
0.8	9.57	9.71	9.79	9.81	9.84	9.85	9.86	27.67	32.59	35.64	37.81

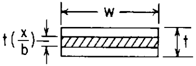
(Ref. 5)

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

Reference number, form of bar, and manner of loading and support

6c. $I_x = I\left(\frac{x}{b}\right)^3$

for example, rectangular section tapering uniformly in thickness



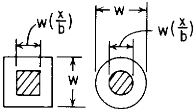
$P' = \frac{KEI}{l^2}$ where K may be found from the following table:

I_0/I	K for ends hinged						K for ends fixed			
a/l	0.01	0.10	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0	2.55	5.01	6.14	7.52	8.50	9.23	18.48	25.32	30.72	35.32
0.2	3.65	6.32	7.31	8.38	9.02	9.50	20.88	27.20	31.96	35.96
0.4	5.42	7.84	8.49	9.10	9.46	9.69	22.64	28.40	32.72	36.32
0.6	7.99	9.14	9.39	9.62	9.74	9.81	23.96	29.52	33.56	36.80
0.8	9.63	9.77	9.81	9.84	9.85	9.86	27.24	32.44	35.60	37.80

(Ref. 5)

6d. $I_x = I\left(\frac{x}{b}\right)^4$

for example, end portions pyramidal or conical



$P' = \frac{KEI}{l^2}$ where K may be found from the following table:

I_0/I	K for ends hinged						K for ends fixed			
a/l	0.01	0.10	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0	2.15	4.81	6.02	7.48	8.47	9.23	18.23	25.23	30.68	35.33
0.2	3.13	6.11	7.20	8.33	9.01	9.49	20.71	27.13	31.94	35.96
0.4	4.84	7.68	8.42	9.10	9.45	9.69	22.49	28.33	32.69	36.32
0.6	7.53	9.08	9.38	9.62	9.74	9.81	23.80	29.46	33.54	36.78
0.8	9.56	9.77	9.80	9.84	9.85	9.86	27.03	32.35	35.56	37.80

(Ref. 5)

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

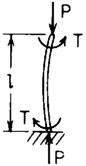
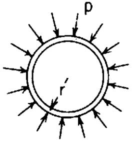
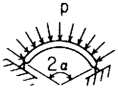
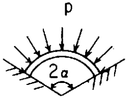
<table>
<tr>
<td>7. Uniform straight bar under end loads P and end twisting couples T; cross section of bar has same I for all central axes; both ends hinged

 </td>
<td>Critical combination of P and T is given by
$$\frac{T^2}{4(EI)^2}+\frac{P}{EI}=\frac{\pi^2}{l^2}$$ (Ref. 1)
If $P=0$, the formula gives critical twisting moment T' which, acting alone, would cause buckling
If for a given value of T the formula gives a negative value for P, $T>T'$ and P represents tensile load required to prevent buckling
For thin circular tube of diameter D and thickness t under torsion only, critical shear stress
$$\tau=\frac{\pi ED}{l(1-\nu)}\left(1-\frac{t}{D}+\frac{1}{3}\frac{t^2}{D^2}\right)$$ for helical buckling only (not for shell-type buckling in the thin wall) (Ref. 2)</td>
</tr>
<tr>
<td>8. Uniform circular ring under uniform radial pressure p lb/in; mean radius of ring r

 </td>
<td>$$p'=\frac{3EI}{r^3}$$ (Ref. 1)</td>
</tr>
<tr>
<td>9. Uniform circular arch under uniform radial pressure p lb/in; mean radius r; ends hinged

 </td>
<td>$$p'=\frac{EI}{r^3}\left(\frac{\pi^2}{\alpha^2}-1\right)$$ (Ref. 1)
(For symmetrical arch of any form under central concentrated loading, see Ref. 40; for parabolic and catenary arches, see Ref. 50)</td>
</tr>
<tr>
<td>10. Uniform circular arch under uniform radial pressure p lb/in; mean radius r; ends fixed

 </td>
<td>$$p'=\frac{EI}{r^3}(k^2-1)$$ (Ref. 1)
where k depends on α and is found by trial from the equation: $k\tan\alpha\cot k\alpha=1$ or from the following table:
<table>
<tr><th>α</th><th>15°</th><th>30°</th><th>45°</th><th>60°</th><th>75°</th><th>90°</th><th>120°</th><th>180°</th></tr>
<tr><td>k</td><td>17.2</td><td>8.62</td><td>5.80</td><td>4.37</td><td>3.50</td><td>3.00</td><td>2.36</td><td>2.00</td></tr>
</table>
(For parabolic and catenary arches, see Ref. 50)</td>
</tr>
</table>

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

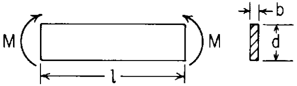
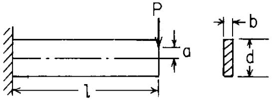
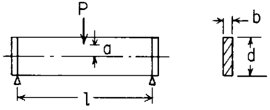
	Reference number, form of bar, and manner of loading and support
11. Straight uniform beam of narrow rectangular section under pure bending  	For ends held vertical but not fixed in horizontal plane: $M' = \dfrac{\pi b^3 d\sqrt{EG\left(1-0.63\dfrac{b}{d}\right)}}{6l}$ For ends held vertical and fixed in horizontal plane $M' = \dfrac{2\pi b^3 d\sqrt{EG\left(1-0.63\dfrac{b}{d}\right)}}{6l}$ (Refs. 1, 3, 4)
12. Straight uniform cantilever beam of narrow rectangular section under end load applied at a point a distance a above (a positive) or below (a negative) centroid of section  	$P' = \dfrac{0.669 b^3 d\sqrt{\left(1-0.63\dfrac{b}{d}\right)EG}}{l^2}\left[1-\dfrac{a}{2l}\sqrt{\dfrac{E}{G\left(1-0.63\dfrac{b}{d}\right)}}\right]$ For a load W uniformly distributed along the beam the critical load, $W' = 3P'$ (approximately) (For tapered and stepped beams, see Ref. 54) (Refs. 1, 3, 4)
13. Straight uniform beam of narrow rectangular section under center load applied at a point a distance a above (a positive) or below (a negative) centroid of section: ends of beam simply supported and constrained against twisting  	$P' = \dfrac{2.82 b^3 d\sqrt{\left(1-0.63\dfrac{b}{d}\right)EG}}{l^2}\left[1-\dfrac{1.74a}{l}\sqrt{\dfrac{E}{G\left(1-0.63\dfrac{b}{d}\right)}}\right]$ For a uniformly distributed load, the critical load $W' = 1.67P'$ (approximately) If P is applied at an intermediate point, a distance C from one end, its critical value is practically the same as for central loading if $0.4l < C < 0.5l$: if $C < 0.4l$, the critical load is given approximately by multiplying the P' for central loading by $0.36 + 0.28\dfrac{l}{C}$ If the ends of the beam are fixed and the load P is applied at the centroid of the middle cross section, $P' = \dfrac{4.43 b^3 d}{l^2}\sqrt{\left(1-0.63\dfrac{b}{d}\right)EG}$ (Refs. 1, 3, 4)

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

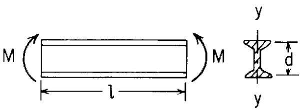
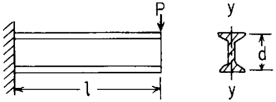
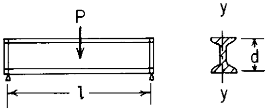
14. Straight uniform I-beam under pure bending; d = depth center to center of flange; ends constrained against twisting  	$M' = \frac{\pi\sqrt{EI_yKG}}{l}\sqrt{1+\pi^2\frac{I_fEd^2}{2KGl^2}}$ where I_y is the moment of inertia of the cross section about its vertical axis of symmetry, I_f is the moment of inertia of one flange about this axis, and KG is the torsional rigidity of the section (see Table 10.1, case 26) (For tapered I-beams, see Ref. 47.) (Refs. 1, 3)
15. Straight uniform cantilever beam of I-section under end load applied at centroid of cross section; d = depth center to center of flanges  	$P' = \frac{m\sqrt{EI_yKG}}{l^2}$ where m is approximately equal to $4.01 + 11.7\sqrt{\frac{I_fEd^2}{2KGl^2}}$ and I_y, I_f, and KG have the same significance as in case 14 (For unsymmetric I-beams, see Refs. 52 and 56; for tapered I-beams, see Ref. 55.) (Refs. 1, 3)
16. Straight uniform I-beam loaded at centroid of middle section; ends simply supported and constrained against twisting  	$P' = \frac{m\sqrt{EI_yKG}}{l^2}$ where m is approximately equal to $16.93 + 45\left(\frac{I_fEd^2}{2KGl^2}\right)^{0.8}$ and I_y, I_f, and KG have same significance as in case 14. (For unsymmetric I-beams, see Refs. 52 and 56; for tapered I-beams, see Ref. 55.) (Refs. 1, 3)

TABLE 15.2 Formulas for elastic stability of plates and shells

NOTATION: E = modulus of elasticity; ν = Poisson's ratio; and t = thickness for all plates and shells. All angles are in radians. Compression is positive; tension is negative. For the plates, the smaller width should be greater than 10 times the thickness unless otherwise specified.

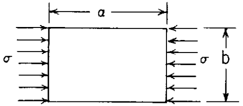
<table>
<tr><th>Form of plate or shell and manner of loading</th><th>Manner of support</th><th>Formulas for critical unit compressive stress σ', unit shear stress τ', load P', bending moment M', or unit external pressure q' at which elastic buckling occurs</th></tr>
<tr><td rowspan="5">1. Rectangular plate under equal uniform compression on two opposite edges b
 </td>
<td>1a. All edges simply supported</td>
<td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
Here K depends on ratio $\frac{a}{b}$ and may be found from the following table:
<table>
<tr><td>$\frac{a}{b}$</td><td>0.2</td><td>0.3</td><td>0.4</td><td>0.6</td><td>0.8</td><td>1.0</td><td>1.2</td><td>1.4</td><td>1.6</td><td>1.8</td><td>2.0</td><td>2.2</td><td>2.4</td><td>2.7</td><td>3.0</td><td>∞</td></tr>
<tr><td>K</td><td>22.2</td><td>10.9</td><td>6.92</td><td>4.23</td><td>3.45</td><td>3.29</td><td>3.40</td><td>3.68</td><td>3.45</td><td>3.32</td><td>3.29</td><td>3.32</td><td>3.40</td><td>3.32</td><td>3.29</td><td>3.29</td></tr>
</table>
(For unequal end compressions, see Ref. 33) (Refs. 1, 6)</td></tr>
<tr><td>1b. All edges clamped</td>
<td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
<table>
<tr><td>$\frac{a}{b}$</td><td>1</td><td>2</td><td>3</td><td>∞</td></tr>
<tr><td>K</td><td>7.7</td><td>6.7</td><td>6.4</td><td>5.73</td></tr>
</table>
(Refs. 1, 6, 7)</td></tr>
<tr><td>1c. Edges b simply supported, edges a clamped</td>
<td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
<table>
<tr><td>$\frac{a}{b}$</td><td>0.4</td><td>0.5</td><td>0.6</td><td>0.7</td><td>0.8</td><td>1.0</td><td>1.2</td><td>1.4</td><td>1.6</td><td>1.8</td><td>2.1</td><td>∞</td></tr>
<tr><td>K</td><td>7.76</td><td>6.32</td><td>5.80</td><td>5.76</td><td>6.00</td><td>6.32</td><td>5.80</td><td>5.76</td><td>6.00</td><td>5.80</td><td>5.76</td><td>5.73</td></tr>
</table>
(Refs. 1, 6)</td></tr>
<tr><td>1d. Edges b simply supported, one edge a simply supported, other edge a free</td>
<td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
<table>
<tr><td>$\frac{a}{b}$</td><td>0.5</td><td>1.0</td><td>1.2</td><td>1.4</td><td>1.6</td><td>1.8</td><td>2.0</td><td>2.5</td><td>3.0</td><td>4.0</td><td>5.0</td></tr>
<tr><td>K</td><td>3.62</td><td>1.18</td><td>0.934</td><td>0.784</td><td>0.687</td><td>0.622</td><td>0.574</td><td>0.502</td><td>0.464</td><td>0.425</td><td>0.416</td></tr>
</table>
(Ref. 1)</td></tr>
<tr><td>1e. Edges b simply supported, one edge a clamped, other edge a free</td>
<td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
<table>
<tr><td>$\frac{a}{b}$</td><td>1</td><td>1.1</td><td>1.2</td><td>1.3</td><td>1.4</td><td>1.5</td><td>1.6</td><td>1.7</td><td>1.8</td><td>1.9</td><td>2.0</td><td>2.2</td><td>2.4</td></tr>
<tr><td>K</td><td>1.40</td><td>1.28</td><td>1.21</td><td>1.16</td><td>1.12</td><td>1.10</td><td>1.09</td><td>1.09</td><td>1.10</td><td>1.12</td><td>1.14</td><td>1.19</td><td>1.21</td></tr>
</table>
(Ref. 1)</td></tr>
</table>

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

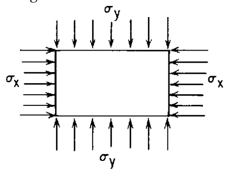
1f. Edges b clamped, edges a simply supported

$$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$$

$\frac{a}{b}$	0.6	0.8	1.0	1.2	1.4	1.6	1.7	1.8	2.0	2.5	3.0
K	11.0	7.18	5.54	4.80	4.48	4.39	4.39	4.26	3.99	3.72	3.63

(Ref. 1)

2. Rectangular plate under uniform compression (or tension) σ_x on edges b and uniform compression (or tension) σ_y on edges a



2a. All edges simply supported

$$\sigma'_x\frac{m^2}{a^2} + \sigma'_y\frac{n^2}{b^2} = 0.823\frac{E}{1-\nu^2}t^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2$$

Here m and n signify the number of half-waves in the buckled plate in the x and y directions, respectively. To find σ'_y for a given σ_x, take $m = 1$, $n = 1$ if $C\left(1 - 4\frac{a^4}{b^4}\right) < \sigma_x < C\left(5 + 2\frac{a^2}{b^2}\right)$, where $C = \frac{0.823Et^2}{(1-\nu^2)a^2}$.

If σ_x is too large to satisfy this inequality, take $n = 1$ and m to satisfy:

$C\left(2m^2 - 2m + 1 + 2\frac{a^2}{b^2}\right) < \sigma_x < C\left(2m^2 + 2m + 1 + 2\frac{a^2}{b^2}\right)$. If σ_x is too small to satisy the first inequality, take $m = 1$ and n to satisfy:

$$C\left[1 - n^2(n-1)^2\frac{a^4}{b^4}\right] > \sigma_x > C\left[1 - n^2(n+1)^2\frac{a^4}{b^4}\right]$$

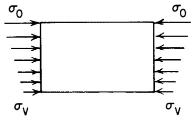
(Refs. 1, 6)

2b. All edges clamped

$$\sigma'_x + \frac{a^2}{b^2}\sigma'_y = 1.1\frac{Et^2a^2}{1-\nu^2}\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right)$$

(This equation is approximate and is most accurate when the plate is nearly square and σ_x and σ_y nearly equal (Ref. 1)

3. Rectangular plate under linearly varying stress on edges b (bending or bending combined with tension or compression)



3a. All edges simply supported

$$\sigma'_o = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$$

Here K depends on $\frac{a}{b}$ and on $\alpha = \frac{\sigma_o}{\sigma_o - \sigma_v}$ and may be found from the following table:

	$\frac{a}{b} = 0.4$	0.5	0.6	0.667	0.75	0.8	0.9	1.0	1.5
$\alpha = 0.5$	$K = 23.9$	21.1	19.8	19.7	19.8	20.1	21.1	21.1	19.8
0.75	15.4		10.6		9.5	9.2		9.1	9.5
1.00	12.4		8.0		6.9	6.7		6.4	6.9
1.25	10.95		6.8		5.8	5.7		5.4	5.8
1.50	8.9		5.3		5.0	4.9		4.8	5.0
∞ (pure compression)	6.92		4.23			3.45		3.29	3.57

(Refs. 1, 6)

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

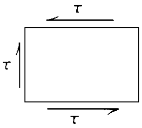
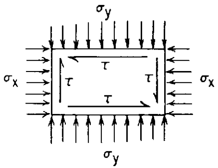
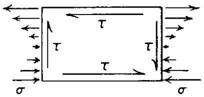
<table>
<tr><th>Form of plate or shell and manner of loading</th><th>Manner of support</th><th>Formulas for critical unit compressive stress σ', unit shear stress τ', load P', bending moment M', or unit external pressure q' at which elastic buckling occurs</th></tr>
<tr><td>4. Rectangular plate under uniform shear on all edges

 </td><td>4a. All edges simply supported</td><td>$\tau' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
<table>
<tr><td>$\frac{a}{b}$</td><td>1.0</td><td>1.2</td><td>1.4</td><td>1.5</td><td>1.6</td><td>1.8</td><td>2.0</td><td>2.5</td><td>3.0</td><td>∞</td></tr>
<tr><td>K</td><td>7.75</td><td>6.58</td><td>6.00</td><td>5.84</td><td>5.76</td><td>5.59</td><td>5.43</td><td>5.18</td><td>5.02</td><td>4.40</td></tr>
</table>
(Refs. 1, 6, 8, 22)</td></tr>
<tr><td></td><td>4b. All edges clamped</td><td>$\tau' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
<table>
<tr><td>$\frac{a}{b}$</td><td>1</td><td>2</td><td>∞</td></tr>
<tr><td>K</td><td>12.7</td><td>9.5</td><td>7.38</td></tr>
</table>
Test results indicate a value for K of about 4.1 for very large values of $\frac{a}{b}$ (Ref. 9)
(For continuous panels, see Ref. 30)</td></tr>
<tr><td>5. Rectangular plate under uniform shear on all edges; compression (or tension) σ_x on edges b; compression (or tension) σ_y on edges a; a/b very large

 </td><td>5a. All edges simply supported</td><td>$\tau' = \sqrt{C^2\left(2\sqrt{1-\frac{\sigma_y}{C}}+2-\frac{\sigma_x}{C}\right)\left(2\sqrt{1-\frac{\sigma_y}{C}}+6-\frac{\sigma_x}{C}\right)}$
where $C = \frac{0.823}{1-\nu^2}\left(\frac{t}{b}\right)^2 E$ (Refs. 1, 6, 23, and 31)</td></tr>
<tr><td></td><td>5b. All edges clamped</td><td>$\tau' = \sqrt{C^2\left(2.31\sqrt{4-\frac{\sigma_y}{C}}+\frac{4}{3}-\frac{\sigma_x}{C}\right)\left(2.31\sqrt{4-\frac{\sigma_y}{C}}+8-\frac{\sigma_x}{C}\right)}$
where $C = \frac{0.823}{1-\nu^2}\left(\frac{t}{b}\right)^2 E$
(σ_x and σ_y are negative when tensile) (Ref. 6)</td></tr>
<tr><td>6. Rectangular plate under uniform shear on all edges and bending stresses on edges b

 </td><td>6a. All edges simply supported</td><td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
Here K depends on $\frac{\tau}{\tau'}$ (ratio of actual shear stress to shear stress that, acting alone, would be critical) and on $\frac{a}{b}$. K varies less than 10% for values $\frac{a}{b}$ from 0.5 to 1, and for $\frac{a}{b} = 1$ is approximately as follows:
<table>
<tr><td>$\frac{\tau}{\tau'}$</td><td>0</td><td>0.2</td><td>0.3</td><td>0.4</td><td>0.5</td><td>0.6</td><td>0.7</td><td>0.8</td><td>0.9</td><td>1.0</td></tr>
<tr><td>K</td><td>21.1</td><td>20.4</td><td>19.6</td><td>18.5</td><td>17.7</td><td>16.0</td><td>14.0</td><td>11.9</td><td>8.20</td><td>0</td></tr>
</table>
(Refs. 1, 10)</td></tr>
</table>

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

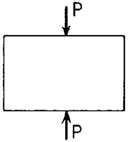
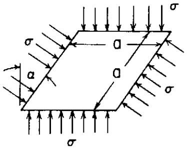
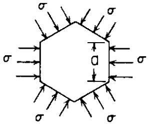
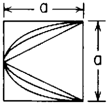
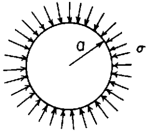
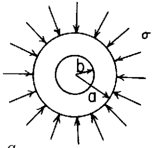
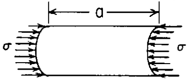
<table>
<tr>
<td rowspan="2">7. Rectangular plate under concentrated center loads on two opposite edges
 </td>
<td>7a. All edges simply supported</td>
<td>$P' = \frac{\pi}{3}\frac{Et^3}{(1-\nu^2)b} \quad \left(\text{for } \frac{a}{b} > 2\right)$ (Ref. 1)</td>
</tr>
<tr>
<td>7b. Edges b simply supported, edges a clamped</td>
<td>$P' = \frac{2\pi}{3}\frac{Et^3}{(1-\nu^2)b} \quad \left(\text{for } \frac{a}{b} > 2\right)$ (Ref. 1)</td>
</tr>
<tr>
<td>8. Rhombic plate under uniform compression on all edges
 </td>
<td>8a. All edges simply supported</td>
<td>$\sigma' = K\frac{Et^2}{a^2(1-\nu^2)}$
<table>
<tr><td>α</td><td>0°</td><td>9°</td><td>18°</td><td>27°</td><td>36°</td><td>45°</td></tr>
<tr><td>K</td><td>1.645</td><td>1.678</td><td>1.783</td><td>1.983</td><td>2.338</td><td>2.898</td></tr>
</table>
Ref. 65)</td>
</tr>
<tr>
<td>9. Polygon plate under uniform compression on all edges

N = number of sides</td>
<td>9a. All edges simply supported</td>
<td>$\sigma' = K\frac{Et^2}{a^2(1-\nu^2)}$
<table>
<tr><td>N</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td></tr>
<tr><td>K</td><td>4.393</td><td>1.645</td><td>0.916</td><td>0.597</td><td>0.422</td><td>0.312</td></tr>
</table>
(Ref. 65)</td>
</tr>
<tr>
<td rowspan="2">10. Parabolic and semielliptic plates under uniform compression on all edges
 </td>
<td>10a. All edges simply supported</td>
<td rowspan="2">$\sigma' = K\frac{Et^2}{a^2(1-\nu^2)}$
where K is tabulated below for the several shapes and boundary conditions for $\nu = \frac{1}{3}$:
<table>
<tr><td></td><td>Square</td><td>Semiellipse</td><td>Parabola</td><td>Triangle</td></tr>
<tr><td>Simply supported</td><td>1.65</td><td>1.86</td><td>2.50</td><td>3.82</td></tr>
<tr><td>Fixed</td><td>4.36</td><td>5.57</td><td>7.22</td><td>10.60</td></tr>
</table>
(Ref. 62)</td>
</tr>
<tr>
<td>10b. All edges fixed</td>
</tr>
</table>

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

<table>
<tr><th>Form of plate or shell and manner of loading</th><th>Manner of support</th><th>Formulas for critical unit compressive stress σ', unit shear stress τ', load P', bending moment M', or unit external pressure q' at which elastic buckling occurs</th></tr>
<tr><td rowspan="2">11. Isotropic circular plate under uniform radial edge compression

 </td><td>11a. Edges simply supported</td><td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{a}\right)^2$
<table>
<tr><th>ν</th><th>0</th><th>0.1</th><th>0.2</th><th>0.3</th><th>0.4</th></tr>
<tr><td>K</td><td>0.282</td><td>0.306</td><td>0.328</td><td>0.350</td><td>0.370</td></tr>
</table>
(Ref. 1)</td></tr>
<tr><td>11b. Edges clamped</td><td>$\sigma' = 1.22\frac{E}{1-\nu^2}\left(\frac{t}{a}\right)^2$ (Ref. 1)
For elliptical plate with major semiaxis a, minor semiaxis b, $\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$, where K has values as follows:
<table>
<tr><th>$\frac{a}{b}$</th><th>1.0</th><th>1.1</th><th>1.2</th><th>1.3</th><th>2.0</th><th>5.0</th></tr>
<tr><td>K</td><td>1.22</td><td>1.13</td><td>1.06</td><td>1.01</td><td>0.92</td><td>0.94</td></tr>
</table>
(Ref. 21)</td></tr>
<tr><td rowspan="2">12. Circular plate with concentric hole under uniform radial compression on outer edge


$\frac{a}{t} > 10$</td><td>12a. Outer edge simply supported, inner edge free</td><td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{a}\right)^2$
Here K depends on $\frac{b}{a}$ and is given approximately by following table:
<table>
<tr><th>$\frac{b}{a}$</th><th>0</th><th>0.1</th><th>0.2</th><th>0.3</th><th>0.4</th><th>0.5</th><th>0.6</th><th>0.7</th><th>0.8</th><th>0.9</th></tr>
<tr><td>K</td><td>0.35</td><td>0.33</td><td>0.30</td><td>0.27</td><td>0.23</td><td>0.21</td><td>0.19</td><td>0.18</td><td>0.17</td><td>0.16</td></tr>
</table>
(Ref. 1)</td></tr>
<tr><td>12b. Outer edge clamped, inner edge free</td><td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{a}\right)^2$
Here K depends on $\frac{b}{a}$ and is given approximately by following table:
<table>
<tr><th>$\frac{b}{a}$</th><th>0</th><th>0.1</th><th>0.2</th><th>0.3</th><th>0.4</th><th>0.5</th></tr>
<tr><td>K</td><td>1.22</td><td>1.17</td><td>1.11</td><td>1.21</td><td>1.48</td><td>2.07</td></tr>
</table>
(Ref. 1)</td></tr>
<tr><td>13. Curved panel under uniform compression on curved edges b (b = width of panel measured on arc; r = radius of curvature)


$\frac{b}{t} > 10$</td><td>13a. All edges simply supported</td><td>$\sigma' = \frac{1}{6}\frac{E}{1-\nu^2}\left[\sqrt{12(1-\nu^2)\left(\frac{t}{r}\right)^2+\left(\frac{\pi t}{b}\right)^4}+\left(\frac{\pi t}{b}\right)^2\right]$
(*Note:* With $a > b$, the solution does not depend upon a)
or $\sigma' = 0.6E\frac{t}{r}$ if $\frac{b}{r}$ (central angle of curve) is less than $\frac{1}{2}$ and b and a are nearly equal (Refs. 1 and 6)
(For compression combined with shear, see Refs. 28 and 34.)</td></tr>
</table>

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

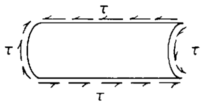
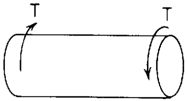
14. Curved panel under uniform shear on all edges  $b/t > 10$. See case 13 for b and r.	14a. All edges simply supported	$\tau' = 0.1E\frac{t}{r} + 5E\left(\frac{t}{b}\right)^2$ (Refs. 6, 27, 29)
	14b. All edges clamped	$\tau' = 0.1E\frac{t}{r} + 7.5E\left(\frac{t}{b}\right)^2$ (Ref. 6) Tests show $\tau' = 0.075E\frac{t}{r}$ for panels curved to form quadrant of a circle (Ref. 11) (See also Refs. 27, 29)
15. Thin-walled circular tube under uniform longitudinal compression (radius of tube $= r$)  $\frac{r}{t} > 10$	15a. Ends not constrained	$\sigma' = \frac{1}{\sqrt{3}}\frac{E}{\sqrt{1-\nu^2}}\frac{t}{r}$ (Refs. 6, 12, 13, 24) Most accurate for very long tubes, but applicable if length is several times as great as $1.72\sqrt{rt}$, which is the length of a half-wave of buckling. Tests indicate an actual buckling strength of 40–60% of this theoretical value, or $\sigma' = 0.3Et/r$ approximately
16. Thin-walled circular tube under a transverse bending moment M (radius of tube $= r$)  $\frac{r}{t} > 10$	16a. No constraint	$M' = K\frac{E}{1-\nu^2}rt^2$ Here the theoretical value of K for pure bending and long tubes is 0.99. The average value of K determined by tests is 1.14, and the minimum value is 0.72. Except for very short tubes, length effect is negligible and a small transverse shear produces no appreciable reduction in M'. A very short cylinder under transverse (beam) shear may fail by buckling at neutral axis when shear stress there reaches a value of about $1.25\tau'$ for case 17a (Refs. 6, 14, 15)
17. Thin-walled circular tube under a twisting moment T that produces a uniform circumferential shear stress: $\tau = \frac{T}{2\pi r^2 t}$ (length of tube $= l$; radius of tube $= r$)  $\frac{r}{t} > 10$	17a. Ends hinged, i.e., wall free to change angle with cross section, but circular section maintained	$\tau' = \frac{E}{1-\nu^2}\left(\frac{t}{l}\right)^2(1.27 + \sqrt{9.64 + 0.466H^{1.5}})$ where $H = \sqrt{1-\nu^2}\frac{l^2}{tr}$ Tests indicate that the actual buckling stress is 60–75% of this theoretical value, with the majority of the data points nearer 75% (Refs. 6, 16, 18, 25)
	17b. Ends clamped, i.e., wall held perpendicular to cross section and circular section maintained	$\tau' \frac{E}{1-\nu^2}\left(\frac{t}{l}\right)^2(-2.39 + \sqrt{96.9 + 0.605H^{1.5}})$ where H is given in part 17a. The statement in part a regarding actual buckling stress applies here as well (Refs. 6, 16, 18, 25)

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

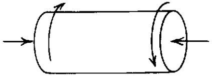
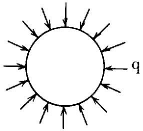
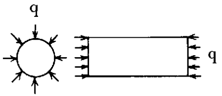
Form of plate or shell and manner of loading	Manner of support	Formulas for critical unit compressive stress σ', unit shear stress τ', load P', bending moment M', or unit external pressure q' at which elastic buckling occurs
18. Thin-walled circular tube under uniform longitudinal compression σ and uniform circumferential shear τ due to torsion (case 15 combined with case 17)  $\frac{r}{t} > 10$	18a. Edges hinged as in case 17a. 18b. Edges clamped as in case 17b.	The equation $1-\frac{\sigma'}{\sigma'_o}=\left(\frac{\tau'}{\tau'_o}\right)^n$ holds, where σ' and τ' are the critical compressive and shear stresses for the combined loading, σ'_o is the critical compressive stress for the cylinder under compression alone (case 15), and τ'_o is the critical shear stress for the cylinder under torsion alone (case 17a or 17b according to end conditions). Tests indicate that n is approximately 3. If σ is tensile, then σ' should be considered negative. (Ref. 6) (See also Ref. 26. For square tube, see Ref. 32)
19. Thin tube under uniform lateral external pressure (radius of tube = r)  $\frac{r}{t} > 10$	19a. Very long tube with free ends; length l	$q'=\frac{1}{4}\frac{E}{1-\nu^2}\frac{t^3}{r^3}$ Applicable when $l > 4.90r\sqrt{\frac{r}{t}}$ (Ref. 19)
	19b. Short tube, of length l, ends held circular, but not otherwise constrained, or long tube held circular at intervals l	$q'=0.807\frac{Et^2}{lr}\sqrt[4]{\left(\frac{1}{1-\nu^2}\right)^3\frac{t^2}{r^2}}$ approximate formula (Ref. 19)
20. Thin tube with closed ends under uniform external pressure, lateral and longitudinal (length of tube = l; radius of tube = r)  $\frac{r}{t} > 10$	20a. Ends held circular	$q'=\frac{E\frac{t}{r}}{1+\frac{1}{2}\left(\frac{\pi r}{nl}\right)^2}\left\{\frac{1}{n^2\left[1+\left(\frac{nl}{\pi r}\right)^2\right]^2}+\frac{n^2t^2}{12r^2(1-\nu^2)}\left[1+\left(\frac{\pi r}{nl}\right)^2\right]^2\right\}$ (Refs. 19, 20) where n = number of lobes formed by the tube in buckling. To determine q' for tubes of a given t/r, plot a group of curves, one curve for each integral value of n of 2 or more, with l/r as ordinates and q' as abscissa; that curve of the group which gives the least value of q' is then used to find the q' corresponding to a given l/r. If $60 < \left(\frac{l}{r}\right)^2\left(\frac{r}{t}\right) < 2.5\left(\frac{r}{t}\right)^2$, the critical pressure can be approximated by $q'=\frac{0.92E}{\left(\frac{l}{r}\right)\left(\frac{r}{t}\right)^{2.5}}$ (Ref. 81) For other approximations see ref. 109 Values of experimentally determined critical pressures range 20% above and below the theoretical values given by the expressions above. A recommended probable minimum critical pressure is 0.80q'.

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

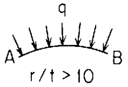
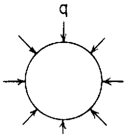
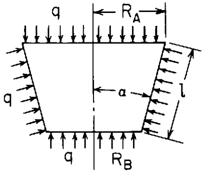
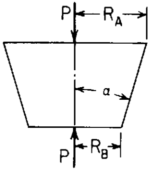
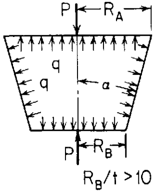
21. Curved panel under uniform radial pressure (radius of curvature r, central angle 2α, when $2\alpha = \text{arc } AB/r$) 	21a. Curved edges free, straight edges at A and B simply supported (i.e., hinged)	$q' = \dfrac{Et^3\left(\dfrac{\pi^2}{\alpha^2} - 1\right)}{12r^3(1-\nu^2)}$ (Ref. 1)
	21b. Curved edges free, straight edges at A and B clamped	Here k is found from the equation $k \tan\alpha \cot k\alpha = 1$ and has the following values: $q' = \dfrac{Et^3(k^2-1)}{12r^3(1-\nu^2)}$ $\begin{array}{c\|ccccccc} \alpha & 15^\circ & 30^\circ & 60^\circ & 90^\circ & 120^\circ & 150^\circ & 180^\circ \\ \hline k & 17.2 & 8.62 & 4.37 & 3.0 & 2.36 & 2.07 & 2.0 \end{array}$ (Ref. 1)
22. Thin sphere under uniform external pressure (radius of sphere $= r$)  r/t > 10	22a. No constraint	$q' = \dfrac{2Et^2}{r^2\sqrt{3(1-\nu^2)}}$ (for ideal case) (Refs. 1, 37) $q' = \dfrac{0.365Et^2}{r^2}$ (probable actual minimum q') For spherical cap, half-central angle ϕ between 20 and 60°, R/t between 400 and 2000, $q' = [1 - 0.00875(\phi^\circ - 20^\circ)]\left(1 - 0.000175\dfrac{R}{t}\right)(0.3E)\left(\dfrac{t}{R}\right)^2$ (Empirical formula, Ref. 43)
23. Thin truncated conical shell with closed ends under external pressure (both lateral and longitudinal pressure)  $R_B/t > 10$	23a. Ends held circular	q' can be found from the formula of case 20a if the slant length of the cone is substituted for the length of the cylinder and if the average radius of curvature of the wall of the cone normal to the meridian $(R_A + R_B)/(2\cos\alpha)$ is substituted for the radius of the cylinder. The same recommendation of a probable minimum critical pressure of $0.8q'$ is made from the examination of experimental data for cones. (Refs. 78, 81)
24. Thin truncated conical shell under axial load  $R_B/t > 10$	24a. Ends held circular	$P' = \dfrac{2\pi Et^2\cos^2\alpha}{\sqrt{3(1-\nu^2)}}$ (theoretical) Tests indicate an actual buckling strength of from 40 to 60% of the above theoretical value, or $P' = 0.3(2\pi Et^2\cos^2\alpha)$ approximately. (Ref. 78) In Ref. 77 it is stated that $P' = 0.277(2\pi Et^2\cos^2\alpha)$ will give 95% confidence in at least 90% of the cones carrying more than this critical load. This is based on 170 tests.

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

<table>
<tr><th>Form of plate or shell and manner of loading</th><th>Manner of support</th><th>Formulas for critical unit compressive stress σ', unit shear stress τ', load P', bending moment M', or unit external pressure q' at which elastic buckling occurs</th></tr>
<tr><td>25. Thin truncated conical shell under combined axial load and internal pressure


</td><td>25a. Ends held circular</td><td>

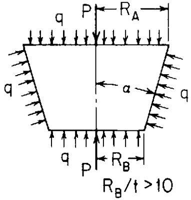
$$P' - q\pi R_B^2 = K_A 2\pi E t^2 \cos^2 \alpha$$

The probable minimum values of K_A are tabulated for several values of $K_P = \frac{q}{E}\left(\frac{R_B}{t\cos\alpha}\right)^2$.

$$k_B = 2\left[\frac{12(1-\nu^2)R_B^2}{t^2\tan^2\alpha\sin^2\alpha}\right]^{1/4}$$

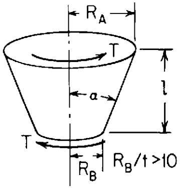
<table>
<tr><th>K_p</th><th>0.00</th><th>0.25</th><th>0.50</th><th>1.00</th><th>1.50</th><th>2.00</th><th>3.00</th></tr>
<tr><td>for $k_B \leqslant 150$</td><td>0.30</td><td>0.52</td><td>0.60</td><td>0.68</td><td>0.73</td><td>0.76</td><td>0.80</td></tr>
<tr><td>for $k_B > 150$</td><td>0.20</td><td>0.36</td><td>0.48</td><td>0.60</td><td>0.64</td><td>0.66</td><td>0.69</td></tr>
</table>

(Ref. 78))</td></tr>
<tr><td>26. Thin truncated conical shell under combined axial load and external pressure


</td><td>26a. Ends held circular</td><td>The following conservative interaction formula may be used for design. It is applicable equally to theoretical values or to minimum probable values of critical load and pressure.

$$\frac{P'}{P'_{\text{case 24}}} + \frac{q'}{q'_{\text{case 23}}} = 1$$

This expression can be used for cylinders if the angle α is set equal to zero or use is made of cases 15 and 20. For small values of $P'/P'_{\text{case 24}}$ the external pressure required to collapse the shell is greater than that required to initiate buckling. See Ref. 78.</td></tr>
<tr><td>27. Thin truncated conical shell under torsion


</td><td>27a. Ends held circular</td><td>Let $T = \tau' 2\pi r_e^2 t$ and for τ' use the formulas for thin-walled circular tubes, case 17, substituting for the radius r of the tube the equivalent radius r_e, where $r_e = R_B \cos\alpha \left\{1 + \left[\frac{1}{2}\left(1+\frac{R_A}{R_B}\right)\right]^{1/2} - \left[\frac{1}{2}\left(1+\frac{R_A}{R_B}\right)\right]^{-1/2}\right\}$. l and t remain the axial length and wall thickness, respectively.

(Ref.17)</td></tr>
</table>

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Chapter

16

Dynamic and Temperature Stresses

16.1 Dynamic Loadings; General Conditions

Dynamic loading was defined in Chap. 3 as any loading during which the parts of the body cannot be considered to be in static equilibrium. It was further pointed out that two kinds of dynamic loading can be distinguished: (1) that in which the body has imposed upon it a particular kind of motion involving known accelerations, and (2) impact, of which sudden loading may be considered a special case. In the following sections, specific cases of each kind of dynamic loading will be considered.

16.2 Body in a Known State of Motion

The acceleration a of each particle of mass dm being known, the effective force on each particle is $dm \times a$, directed like a. If to each particle a force equal and opposite to the effective force were applied, equilibrium would result. If then such reversed effective forces are assumed to be applied to all the constituent particles of the body, the body may be regarded as being in equilibrium under these forces and the actual forces (loads and reactions) that act upon it, and the resulting stresses can be found exactly as for a body at rest. The reversed effective forces are *imaginary* forces exerted *on* the particles but are equal to and directed like the actual reactions the particles exert on whatever gives them their acceleration, i.e., in general, on the rest of the body. Since these reactions are due to the inertia of the particles, they are called *inertia forces*, and the body may be thought of as loaded by these inertia forces. Similarly, any attached mass will exert on a body inertia forces equal and opposite to the forces which the body has to exert on the attached mass to accelerate it.

The results of applying this method of analysis to a number of more or less typical problems are given below. In all cases, in finding the accelerations of the particles, it has been assumed that the effect of deformation could be ignored; i.e., the acceleration of each particle has been found as though the body were rigid. For convenience, stresses, bending moments, and shears due to inertia forces only are called *inertia* stresses, moments, and shears; they are calculated as though the body were outside the field of gravitation. Stresses, moments, and shears due to balanced forces (including gravity) may be superimposed thereon. The gravitational acceleration constant g depends upon the units used for the imposed acceleration.

1. A slender uniform rod of weight W, length L, section area A, and modulus of elasticity E is given a motion of translation with an acceleration of a parallel to its axis by a pull (push) applied at one end. The maximum tensile (compressive) stress occurs at the loaded end and is $\sigma = Wa/gA$. The elongation (shortening) due to the inertia stresses is

$$e = \frac{1}{2}\frac{W}{g}\frac{aL}{AE}$$

2. The rod described in problem 1 is given a motion of translation with an acceleration of a normal to its axis by forces applied at each end. The maximum inertia bending moment occurs at the middle of the bar and is $M = \frac{1}{8}WaL/g$. The maximum inertia vertical (transverse) shear occurs at the ends and is $V = \frac{1}{2}Wa/g$.
3. The rod described in problem 1 is made to rotate about an axis through one end normal to its length at a uniform angular velocity of ω rad/s. The maximum tensile inertia stress occurs at the pinned end and is

$$\sigma = \frac{1}{2}\frac{W}{g}\frac{L\omega^2}{A}$$

The elongation due to inertia stress is

$$e = \frac{1}{3}\frac{W}{g}\frac{L^2\omega^2}{AE}$$

4. The rod described in problem 1 is pinned at the lower end and allowed to swing down under the action of gravity from an initially vertical position. When the rod reaches a position where it makes with the vertical the angle θ, it is subjected to a positive bending moment (owing to its weight and the inertia forces) which has its

maximum value at a section a distance $\frac{1}{3}L$ from the pinned end. This maximum value is $M = \frac{1}{27}WL\sin\theta$. The maximum positive inertia shear occurs at the pinned end and is $V = \frac{1}{4}W\sin\theta$. The maximum negative inertia shear occurs at a section a distance $\frac{2}{3}L$ from the pinned end and is $V = -\frac{1}{12}W\sin\theta$. The axial force at any section x in from the pinned end is given by

$$H = \frac{3W}{2}\left(1 - \frac{x^2}{L^2}\right) - \frac{W\cos\theta}{2}\left(5 - 2\frac{x}{L} - 3\frac{x^2}{L^2}\right)$$

and becomes tensile near the free end when θ exceeds 41.4°. (This case represents approximately the conditions existing when a chimney or other slender structure topples over, and the bending moment M explains the tendency of such a structure to break near the one-third point while falling.)

5. The rod described in problem 1 is pinned at the lower end and, while in the vertical position, has imposed upon its lower end a horizontal acceleration of a. The maximum inertia bending moment occurs at a section a distance $\frac{1}{3}L$ from the lower end and is $M = \frac{1}{27}Wla/g$. The maximum inertia shear is in the direction of the acceleration, is at the lower end, and is $V = \frac{1}{4}Wa/g$. The maximum inertia shear in the opposite direction occurs at a section a distance $\frac{2}{3}L$ from the lower end and is $V = \frac{1}{12}Wa/g$. (This case represents approximately the conditions existing when a chimney or other slender structure without anchorage is subjected to an earthquake shock.)
6. A uniform circular ring of mean radius R and weight per unit volume δ, having a thickness in the plane of curvature that is very small compared with R, rotates about its own axis with a uniform angular velocity of ω rad/s. The ring is subjected to a uniform tangential inertial stress

$$\sigma = \frac{\delta R^2 \omega^2}{g}$$

7. A solid homogeneous circular disk of uniform thickness (or a solid cylinder) of radius R, Poisson's ratio ν, and weight per unit volume δ rotates about its own axis with a uniform angular velocity of ω rad/s. At any point a distance r from the center there is a radial tensile inertial stress

$$\sigma_r = \frac{1}{8}\frac{\delta\omega^2}{g}[(3+\nu)(R^2 - r^2)] \qquad (16.2\text{-}1)$$

and a tangential tensile inertia stress

$$\sigma_t = \frac{1}{8}\frac{\delta\omega^2}{g}[(3+\nu)R^2 - (1+3\nu)r^2] \qquad (16.2\text{-}2)$$

The maximum radial stress and maximum tangential stress are equal, occur at the center, and are

$$(\sigma_r)_{\max} = (\sigma_t)_{\max} = \frac{1}{8}\frac{\delta\omega^2}{g}(3+\nu)R^2 \qquad (16.2\text{-}3)$$

8. A homogeneous annular disk of uniform thickness outer radius R, and weight per unit volume δ, with a central hole of radius R_0, rotates about its own axis with a uniform angular velocity of ω rad/s. At any point a distance r from the center there is a radial tensile inertia stress

$$\sigma_r = \frac{3+\nu}{8}\frac{\delta\omega^2}{g}\left(R^2 + R_0^2 - \frac{R^2R_0^2}{r^2} - r^2\right) \qquad (16.2\text{-}4)$$

and a tangential tensile inertia stress

$$\sigma_t = \frac{1}{8}\frac{\delta\omega^2}{g}\left[(3+\nu)\left(R^2 + R_0^2 + \frac{R^2R_0^2}{r^2}\right) - (1+3\nu)r^2\right] \qquad (16.2\text{-}5)$$

The maximum radial stress occurs at $r = \sqrt{RR_0}$ and is

$$(\sigma_r)_{\max} = \frac{3+\nu}{8}\frac{\delta\omega^2}{g}(R - R_0)^2 \qquad (16.2\text{-}6)$$

and the maximum tangential stress occurs at the perimeter of the hole and is

$$(\sigma_t)_{\max} = \frac{1}{4}\frac{\delta\omega^2}{g}[(3+\nu)R^2 + (1-\nu)R_0^2] \qquad (16.2\text{-}7)$$

The change in the outer radius is

$$\Delta R = \frac{1}{4}\frac{\delta\omega^2}{g}\frac{R}{E}[(1-\nu)R^2 + (3+\nu)R_0^2] \qquad (16.2\text{-}8)$$

and the change in the inner radius is

$$\Delta R_0 = \frac{1}{4}\frac{\delta\omega^2}{g}\frac{R_0}{E}[(3+\nu)R^2 + (1-\nu)R_0^2] \qquad (16.2\text{-}9)$$

If there are radial pressures or pulls distributed uniformly along either the inner or outer perimeter of the disk, such as a radial

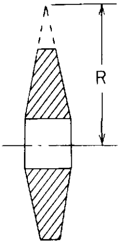

Figure 16.1

pressure from the shaft or a centrifugal pull from parts attached to the rim, the stresses due thereto can be found by the formula for thick cylinders (Table 13.5) and superimposed upon the inertia stresses given by the preceding formulas.

9. A homogeneous circular disk of conical section (Fig. 16.1) of density δ lb/in^3 rotates about its own axis with a uniform angular velocity of N rpm. At any point a distance r from the center, the tensile inertia stresses σ_r and σ_t are given, in lb/in^2, by

$$\sigma_r = TK_r + Ap_1 + Bp_2 \tag{16.2-10}$$

$$\sigma_t = TK_t + Aq_1 + Bq_2 \tag{16.2-11}$$

Tabulated values of coefficients

r/R	K_r	K_t	p_1	q_1	p_2	q_2
0.00	0.1655	0.1655	1.435	1.435	∞	∞
0.05	0.1709	0.1695	1.475	1.497	−273.400	288.600
0.10	0.1753	0.1725	1.559	1.518	−66.620	77.280
0.15	0.1782	0.1749	1.627	1.565	−28.680	36.550
0.20	0.1794	0.1763	1.707	1.617	−15.540	21.910
0.25	0.1784	0.1773	1.796	1.674	−9.553	14.880
0.30	0.1761	0.1767	1.898	1.738	−6.371	10.890
0.35	0.1734	0.1757	2.015	1.809	−4.387	8.531
0.40	0.1694	0.1739	2.151	1.890	−3.158	6.915
0.45	0.1635	0.1712	2.311	1.983	−2.328	5.788
0.50	0.1560	0.1675	2.501	2.090	−1.743	4.944
0.55	0.1465	0.1633	2.733	2.217	−1.309	4.301
0.60	0.1355	0.1579	3.021	2.369	−0.9988	3.816
0.65	0.1229	0.1525	3.390	2.556	−0.7523	3.419
0.70	0.1094	0.1445	3.860	2.794	−0.5670	3.102
0.75	0.0956	0.1370	4.559	3.111	−0.4161	2.835
0.80	0.0805	0.1286	5.563	3.557	−0.2971	2.614
0.85	0.0634	0.1193	7.263	4.276	−0.1995	2.421
0.90	0.0442	0.1100	10.620	5.554	−0.1203	2.263
0.95	0.0231	0.0976	20.645	8.890	−0.0555	2.140
1.00	0.0000	0.0840	∞	∞	−0.0000	2.051

where $T = 0.0000282R^2N^2\delta$ (or for steel, $T = 0.000008R^2N^2$); K_r, K_t, p_1, p_2, q_1, and q_2 are given by the preceding table; and A and B are constants which may be found by setting σ_r equal to its known or assumed values at the outer perimeter and solving the resulting equations simultaneously for A and B, as in the example on pages 750 and 751. [See papers by Hodkinson and Rushing (Refs. 1 and 2) from which Eqs. (16.2-8) and (16.2-9) and the tabulated coefficients are taken.]

10. A homogeneous circular disk of hyperbolic section (Fig. 16.2) of density δ lb/in^3 rotates about its own axis with uniform angular velocity ω rad/s. The equation $t = cr^a$ defines the section, where if t_1 = thickness at radius r_1 and t_2 at radius r_2,

$$a = \frac{\ln(t_1/t_2)}{\ln(r_1/r_2)}$$

and

$$\ln c = \ln t_1 - a \ln r_1 = \ln t_2 - a \ln r_2$$

(For taper toward the rim, a is negative; and for uniform t, $a = 0$.) At any point a distance r in from the center the tensile inertia stresses σ_r and σ_t, in lb/in^2, are

$$\sigma_r = \frac{E}{1-\nu^2}[(3+\nu)Fr^2 + (m_1+\nu)Ar^{m_1-1} + (m_2+\nu)Br^{m_2-1}] \tag{16.2-12}$$

$$\sigma_t = \frac{E}{1-\nu^2}[(1+3\nu)Fr^2 + (1+m_1\nu)Ar^{m_1-1} + (1+m_2\nu)Br^{m_2-1}] \tag{16.2-13}$$

$$\text{where } F = \frac{-(1-\nu^2)\delta\omega^2/386.4}{E[8+(3+\nu)a]}$$

$$m_1 = -\frac{a}{2} - \sqrt{\frac{a^2}{4} - a\nu + 1}$$

$$m_2 = -\frac{a}{2} + \sqrt{\frac{a^2}{4} - a\nu + 1}$$

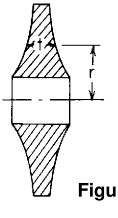

Figure 16.2

A and B are constants, found by setting σ_r equal to its known or assumed values at the inner and outer perimeters and solving the two resulting equations simultaneously for A and B. [Equations (16.2-12) and (16.2-13) are taken from Stodola (Ref. 3) with some changes in notation.]

11. A homogeneous circular disk with section bounded by curves and straight lines (Fig. 16.3) rotates about its own axis with a uniform angular velocity N rpm. The disk is imagined divided into annular rings of such width that each ring can be regarded as having a section with hyperbolic outline, as in problem 10. For each ring, a is calculated by the formulas of problem 10, using the inner and outer radii and the corresponding thicknesses. Then, if r_1 and r_2 represent, respectively, the inner and outer radii of any ring, the tangential stresses σ_{t_1} and σ_{t_2} at the inner and outer boundaries of the ring are related to the corresponding radial stresses σ_{r_1} and σ_{r_2}, in lb/in^2, as follows:

$$\sigma_{t_1} = Ar_2^2 - B\sigma_{r_1} + C\sigma_{r_2} \qquad (16.2\text{-}14)$$

$$\sigma_{t_2} = Dr_2^2 - E\sigma_{r_1} + F\sigma_{r_2} \qquad (16.2\text{-}15)$$

where
$$B = \frac{m_2 K^{m_1-1} - m_1 K^{m_2-1}}{K^{m_2-1} - K^{m_1-1}}$$

$$K = \frac{r_1}{r_2}$$

$$E = -\frac{m_2 - m_1}{K^{m_2-1} - K^{m_1-1}}$$

$$C = \frac{E}{K^{a+2}}$$

$$F = B + a$$

$$A = -\frac{7.956(N/1000)^2}{8 + 3.3a}[1.9K^2 + 3.3(K^2B - C)]$$

$$D = -\frac{7.956(N/1000)^2}{8 + 3.3a}[1.9 + 3.3(K^2E - F)]$$

$$m_1 = -\frac{a}{2} - \sqrt{\frac{a^2}{4} - 0.3a + 1}$$

$$m_2 = -\frac{a}{2} + \sqrt{\frac{a^2}{4} - 0.3a + 1}$$

The preceding formulas, which are given by Loewenstein (Ref. 4), are directly applicable to steel, for which the values $\nu = 0.3$ and $\delta = 0.28$ lb/in^3 have been assumed.

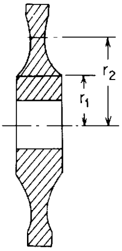

Figure 16.3

Two values of σ_r are known or can be assumed, viz. the values at the inner and outer perimeters of the disk. Then, by setting the tangential stress at the outer boundary of each ring equal to the tangential stress at the inner boundary of the adjacent larger ring, one equation in σ_r will be obtained for each common ring boundary. In this case, the modulus of elasticity is the same for adjacent rings and the radial stress σ_r at the boundary is common to both rings, and so the tangential stresses can be equated instead of the tangential strains [Eq. (16.2-15) for the smaller ring equals Eq. (16.2-14) for the larger ring]. Therefore there are as many equations as there are unknown boundary radial stresses, and hence the radial stress at each boundary can be found. The tangential stresses can then be found by Eqs. (16.2-14) and (16.2-15), and then the stresses at any point in a ring can be found by using, in Eq. (16.2-14), the known values of σ_{t_1} and σ_{r_1} and substituting for σ_{r_2} the unknown radial stress σ_r, and for r_2 the corresponding radius r.

A fact of importance with reference to turbine disks or other rotating bodies is that geometrically similar disks of different sizes will be equally stressed at corresponding points when running at the same *peripheral* velocity. Furthermore, for any given peripheral velocity, the axial and radial dimensions of a rotating body may be changed independently of each other and in any ratio without affecting the stresses at similarly situated points.

EXAMPLE

The conical steel disk shown in section in Fig. 16.4 rotates at 2500 rpm. To its rim it has attached buckets whose aggregate mass amounts to $w = 0.75$ lb/linear in of rim; this mass may be considered to be centered 30 in from the axis. it is desired to determine the stresses at a point 7 in from the axis.

Solution. From the dimensions of the section, R is found to be 28 in. The values of r/R for the inner and outer perimeters and for the circumference $r = 7$ are calculated, and the corresponding coefficients K_r, K_t, etc., are determined from the table on page 747 by graphic interpolation. The results are tabulated here for convenience:

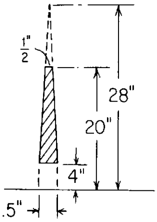

Figure 16.4

	r/R	K_r	K_t	p_1	q_1	p_2	q_2
Inner rim	0.143	0.1780	0.1747	1.616	1.558	−32.5	40.5
Outer rim	0.714	0.1055	0.1425	4.056	2.883	−0.534	3.027
$r = 7$ in	0.25	0.1784	0.1773	1.796	1.674	−9.553	14.88

The attached mass exerts on the rim outward inertia forces which will be assumed to be uniformly distributed; the amount of force per linear inch is

$$p = \frac{w}{g}\omega^2 r = \frac{0.75}{386.4}(261.5^2)(30) = 3980 \text{ lb/linear in}$$

Therefore at the outer rim $\sigma_r = 7960 \text{ lb/in}^2$.

It is usual to design the shrink fit so that in operation the hub pressure is a few hundred pounds per square inch; it will be assumed that the radial stress at the inner rim $\sigma_r = -700 \text{ lb/in}^2$. The value of $T = 0.000008(28^2)(2500^2) = 39{,}200$. Having two values of σ_r, Eq. (16.2-10) can now be written

$$-700 = (39{,}200)(0.1780) + A(1.616) + B(-32.5) \qquad \text{(inner rim)}$$
$$7960 = (39{,}200)(0.1055) + A(4.056) + B(-0.534) \qquad \text{(outer rim)}$$

The solution gives

$$A = 973, \qquad B = 285$$

The stresses at $r = 7$ are now found by Eqs. (16.2-10) and (16.2-11) to be

$$\sigma_r = (39{,}200)(0.1784) + (973)(1.796) + (285)(-9.553) = 6020 \text{ lb/in}^2$$
$$\sigma_t = (39{,}200)(0.1773) + (973)(1.674) + (285)(14.88) = 12{,}825 \text{ lb/in}^2$$

Bursting speed. The formulas given above for stresses in rotating disks presuppose *elastic* conditions; when the elastic limit is exceeded, plastic yielding tends to equalize the stress intensity along a diametral plane. Because of this, the average stress σ_a on such a plane is perhaps a better criterion of margin of safety against bursting than is the maximum stress computed for elastic conditions. For a solid disk of uniform thickness (case 7),

$$\sigma_a = \frac{\delta\omega^2 R^2}{3g}$$

For a pierced disk (case 8),

$$\sigma_a = \frac{\delta\omega^2 (R^3 - R_0^3)}{3g(R - R_0)}$$

Tests (Refs. 12 and 13) have shown that for some materials, rupture occurs in both solid and pierced disks when σ_a, computed for the original dimensions, becomes equal to the ultimate tensile strength of the material as determined by a conventional test. On the other hand, some materials fail at values of σ_a as low as 61.5% of the ultimate strength, and the lowest values have been observed in tests of solid disks. The ratio of σ_a at failure to the ultimate strength does not appear to be related in any consistent way to the ductility of the material; it seems probable that it depends on the form of the stress-strain diagram. In none of the tests reported did the weakening effect of a central hole prove to be nearly as great as the formulas for elastic stress would seem to indicate.

16.3 Impact and Sudden Loading

When a force is suddenly applied to an elastic body (as by a blow), a wave of stress is propagated, which travels through the body with a velocity

$$V = \sqrt{\frac{gE}{\delta}} \tag{16.3-1}$$

where E is the modulus of elasticity of the material and δ is the weight of the material per unit volume.

Bar with free ends. When one end of an unsupported uniform elastic bar is subjected to longitudinal impact from a rigid body moving with velocity v, a wave of compressive stress of intensity

$$\sigma = \frac{v}{V} E = v\sqrt{\frac{\delta E}{g}} \tag{16.3-2}$$

is propagated. The intensity of stress is seen to be independent of the mass of the moving body, but the length of the stressed zone, or volume of material simultaneously subjected to this stress, does depend on the mass of the moving body. If this mass is infinite (or very large compared with that of the bar), the wave of compression is reflected back from the free end of the bar as a wave of tension and returns to the struck end after a period $t_1 = 2L/V$ s, where L is the length of the bar and the period t_1 is the duration of contact between bar and body.

If the impinging body is very large compared with the bar (so that its mass may be considered infinite), the bar, after breaking contact, moves with a velocity $2v$ in the direction of the impact and is free of stress. If the mass of the impinging body is μ times the mass of the bar, the average velocity of the bar after contact is broken is

$$\mu v(1 - e^{-2/\mu})$$

and it is left vibrating with a stress of intensity

$$\sigma = \frac{v}{V} E e^{-\beta t_1}$$

where $\beta = A\sqrt{\delta E g}/W$, A being the section area of the bar and W the weight of the moving body.

Bar with one end fixed. If one end of a bar is fixed, the wave of compressive stress resulting from impact on the free end is reflected back unchanged from the fixed end and combines with advancing waves to produce a maximum stress very nearly equal to

$$\sigma_{\max} = \frac{v}{V} E \left(1 + \sqrt{\mu + \frac{2}{3}}\right) \tag{16.3-3}$$

where, as before, μ denotes the ratio of the mass of the moving body to the mass of the bar. The total time of contact is approximately

$$t_1 = \frac{L}{V}\left[\pi\sqrt{\mu + \frac{1}{2}} - \frac{1}{2}\right] \mathrm{s}$$

[The above formulas are taken from the paper by Donnell (Ref. 5); see also Ref. 17.]

Sudden loading. If a dead load is suddenly transferred to the free end of a bar, the other end being fixed, the resulting state of stress is characterized by waves, as in the case of impact. The space-average value of the pull exerted by the bar on the load is not half the maximum tension, as is usually assumed, but is somewhat greater than that, and therefore the maximum stress that results from sudden loading is somewhat less than twice that which results from static loading. Love (Ref. 6) shows that if μ (the ratio of the mass of the load to that of the bar) is 1, sudden loading causes 1.63 times as much stress as static loading; for $\mu = 2$, the ratio is 1.68; for $\mu = 4$, it is 1.84; and it approaches 2 as a limit as μ increases. It can be seen that the ordinary assumption that sudden loading causes twice as much stress and deflection as static loading is always a safe one to make.

Moving load on beam. If a constant *force* moves at uniform speed across a beam with simply supported ends, the maximum deflection produced exceeds the static deflection that the same force would produce. If v represents the velocity of the force, l the span, and ω the lowest natural vibration frequency of the (unloaded) beam, then theoretically the maximum value of the ratio of dynamic to static deflection is 1.74; it occurs for $v = \omega l/1.64\pi$ and at the instant when the force has progressed at a distance $0.757l$ along the span (Refs. 15 and 16).

If a constant *mass* W moves across a simple beam of relatively negligible mass, then the maximum ratio of dynamic to static deflection is $[1 + (v^2/g)(Wl/3EI)]$ (see Ref. 30). (Note that consistent units must be used in the preceding equations.)

Vibration. A very important type of dynamic loading occurs when an elastic body vibrates under the influence of a periodic impulse. This occurs whenever a rotating or reciprocating mass is unbalanced and also under certain conditions of fluid flow. The most serious situation arises when the impulse synchronizes (or nearly synchronizes) with the natural period of vibration, and it is of the utmost importance to guard against this condition of resonance (or near resonance). There is always some resistance to vibration, whether natural or introduced; this is called *damping* and tends to prevent vibrations of excessive amplitude. In the absence of effective damping, the amplitude y for near-resonance vibration will much exceed the deflection y_s that would be produced by the same force under static conditions. The ratio y/y_s, called the *relative amplification factor*, in the absence of damping, is $1/[1 - (f/f_n)^2]$, where f is the frequency of the forcing impulse and f_n is the natural frequency of the elastic system. Obviously, it is necessary to know at least approximately the natural period of vibration of a member in order to guard against resonance.

Thomson and Dahleh (Ref. 19) describes in detail analytical and numerical techniques for determining resonant frequencies for systems with single and multiple degrees of freedom; they also describe methods and gives numerous examples for torsional and lateral vibrations of rods and beams. Huang (Ref. 22) has tabulated the first five resonant frequencies as well as deflections, slopes, bending moments, and shearing forces for each frequency at intervals of $0.02l$ for uniform beams; these are available for six combinations of boundary conditions. He has also included the first five resonant frequencies for all combinations of 7 different amounts of correction for rotary inertia and 10 different amounts of correction for lateral shear deflection; many mode shapes for these corrections are also included. In Table 16.1, the resonant frequencies and nodal locations are listed for several boundary conditions with no corrections for rotary inertia or shear deflection. (Corrections for rotary inertia and shear deflection have a relatively small effect on the fundamental frequency but a proportionally greater effect on the higher modes.)

Leissa (Ref. 20) has compiled, compared, and in some cases extended most of the known work on the vibration of plates; where possible, mode shapes are given in addition to the many resonant frequencies. Table 16.1 lists only a very few simple cases. Similarly, Leissa (Ref. 21) has done an excellent job of reporting the known work on the vibration of shells. Since, in general, this work must involve three additional variables—the thickness/radius ratio, length/radius ratio, and Poisson's ratio—no results are included here. Blevins, in Ref. 24, gives excellent coverage, by both formulas and tables, of resonant frequencies and mode shapes for cables, straight and curved beams, rings, plates, and shells.

A simple but close approximation for the fundamental frequency of a uniform thin plate of arbitrary shape having any combination of fixed, partially fixed, or simply supported boundaries is given by Jones in Ref. 23. The equation

$$f = \frac{1.2769}{2\pi}\sqrt{\frac{g}{\delta_{\max}}}$$

is based on his work where $\delta_{\max}$ is the maximum static deflection produced by the weight of the plate and any uniformly distributed mass attached to the plate and vibrating with it. It is based on the expression for the fundamental frequency of a clamped elliptical plate but, as Jones points out with several examples of triangular, rectangular, and circular plates having various combinations of boundary conditions, it should hold equally well for all uniform plates having no free boundaries. In the 16 examples he presents, the maximum error in frequency is about 3%.

16.4 Impact and Sudden Loading; Approximate Formulas

If it is assumed that the stresses due to impact are distributed throughout any elastic body exactly as in the case of static loading, then it can be shown that the vertical deformation d_i and stress σ_i produced in any such body (bar, beam, truss, etc.) by the vertical impact of a body falling from a height of h are greater than the deformation d and stress σ produced by the weight of the same body applied as a static load in the ratio

$$\frac{d_i}{d} = \frac{\sigma_i}{\sigma} = 1 + \sqrt{1 + 2\frac{h}{d}} \tag{16.4-1}$$

If $h = 0$, we have the case of sudden loading, and $d_i/d = \sigma_i/\sigma = 2$, as is usually assumed.

If the impact is horizontal instead of vertical, the impact deformation and stress are given by

$$\frac{d_i}{d} = \frac{\sigma_i}{\sigma} = \sqrt{\frac{v^2}{gd}} \tag{16.4-2}$$

where, as before, d is the deformation the weight of the moving body would produce if applied as a static load in the direction of the velocity and v is the velocity of impact.

Energy losses. The above approximate formulas are derived on the assumptions that impact strains the elastic body in the same way (though not in the same degree) as static loading and that all the kinetic energy of the moving body is expended in producing this strain. Actually, on impact, some kinetic energy is dissipated; and this loss, which can be found by equating the momentum of the entire system before and after impact, is most conveniently taken into account by multiplying the available energy (measured by h or by v^2) by a factor K, the value of which is as follows for a number of simple cases involving members of uniform section:

1. A moving body of mass M strikes axially one end of a bar of mass M_1, the other end of which is fixed. Then

$$K = \frac{1 + \frac{1}{3}\frac{M_1}{M}}{\left(1 + \frac{1}{2}\frac{M_1}{M}\right)^2}$$

   If there is a body of mass M_2 attached to the struck end of the bar, then

$$K = \frac{1 + \frac{1}{3}\frac{M_1}{M} + \frac{M_2}{M}}{\left(1 + \frac{1}{2}\frac{M_1}{M} + \frac{M_2}{M}\right)^2}$$

2. A moving body of mass M strikes transversely the center of a simple beam of mass M_1. Then

$$K = \frac{1 + \frac{17}{25}\frac{M_1}{M}}{\left(1 + \frac{5}{8}\frac{M_1}{M}\right)^2}$$

If there is a body of mass M_2 attached to the beam at its center, then

$$K = \frac{1 + \frac{17}{35}\frac{M_1}{M} + \frac{M_2}{M}}{\left(1 + \frac{5}{8}\frac{M_1}{M} + \frac{M_2}{M}\right)^2}$$

3. A moving body of mass M strikes transversely the end of a cantilever beam of mass M_1. Then

$$K = \frac{1 + \frac{33}{140}\frac{M_1}{M}}{\left(1 + \frac{3}{8}\frac{M_1}{M}\right)^2}$$

If there is a body of mass M_2 attached to the beam at the struck end, then

$$K = \frac{1 + \frac{33}{140}\frac{M_1}{M} + \frac{M_2}{M}}{\left(1 + \frac{3}{8}\frac{M_1}{M} + \frac{M_2}{M}\right)^2}$$

4. A moving body of mass M strikes transversely the center of a beam with fixed ends and of mass M_1. Then

$$K = \frac{1 + \frac{13}{35}\frac{M_1}{M}}{\left(1 + \frac{1}{2}\frac{M_1}{M}\right)^2}$$

If there is a body of mass M_2 attached to the beam at the center, then

$$K = \frac{1 + \frac{13}{35}\frac{M_1}{M} + \frac{M_2}{M}}{\left(1 + \frac{1}{2}\frac{M_1}{M} + \frac{M_2}{M}\right)^2}$$

16.5 Remarks on Stress due to Impact

It is improbable that in any actual case of impact the stresses can be calculated accurately by any of the methods or formulas given above. Equation (16.3-3), for instance, is supposedly very nearly precise if the conditions assumed are realized, but those conditions—perfect elasticity of the bar, rigidity of the moving body, and simultaneous contact of the moving body with all points on the end of the rod—are obviously unattainable. On the one hand, the damping of the initial stress wave by elastic hysteresis in the bar and the diminution of the intensity of that stress wave by the cushioning effect of the actually nonrigid moving body would serve to make the actual maximum stress less than the theoretical value; on the other hand, uneven contact between the moving body and the bar would tend to make the stress conditions nonuniform across the section and would probably increase the maximum stress.

The formulas given in Sec. 16.4 are based upon an admittedly false assumption, viz. that the distribution of stress and strain under impact loading is the same as under static loading. It is known, for instance, that the elastic curve of a beam under impact is different from that under static loading. Such a difference exists in any case, but it is less marked for low than for high velocities of impact, and Eqs. (16.4-1) and (16.4-2) probably give reasonably accurate values for the deformation and stress (especially the deformation) resulting from the impact of a relatively heavy body moving at low velocity. The lenitive effect of the inertia of the body struck and of attached bodies, as expressed by K, is greatest when the masses of these parts are large compared with that of the moving body. When this is the case, impact can be serious only if the velocity is relatively high, and under such circumstances the formulas probably give only a rough indication of the actual stresses and deformations to be expected. (See Ref. 18.)

16.6 Temperature Stresses

Whenever the expansion or contraction that would normally result from the heating or cooling of a body is prevented, stresses are developed that are called *thermal*, or *temperature*, *stresses*. It is convenient to distinguish two different sets of circumstances under which thermal stresses occur: (1) The form of the body and the temperature conditions are such that there would be no stresses except for the *constraint of external forces*; in any such case, the stresses may be found by determining the shape and dimensions the body would assume if unconstrained and then calculating the stresses produced by forcing it back to its original shape and dimensions (see Sec. 7.2, Example 2). (2) The form of the body and the temperature conditions are such that stresses are produced in the *absence of external constraint* solely because of the incompatibility of the natural expansions or contractions of the different parts of the body.

A number of representative examples of each type of thermal stress will now be considered.* In all instances the modulus of elasticity E and the coefficient of thermal expansion γ are assumed to be constant for the temperature range involved and the increment or difference in temperature ΔT is assumed to be positive; when ΔT is negative, the stress produced is of the opposite kind. Also, it is assumed that the compressive stresses produced do not produce buckling and that yielding does not occur; if either buckling or yielding is indicated by the stress levels found, then the solution must be modified by appropriate methods discussed in previous chapters.

Stresses due to external constraint

1. A uniform straight bar is subjected to a temperature change ΔT throughout while held at the ends; the resulting unit stress is $\Delta T \gamma E$ (compression). (For other conditions of end restraint see Table 8.2 cases 1q–12q.)
2. A uniform flat plate is subjected to a temperature change ΔT throughout while held at the edges; the resulting unit stress is $\Delta T \gamma E/(1 - \nu)$ (compression).
3. A solid body of any form is subjected to a temperature change ΔT throughout while held to the same form and volume; the resulting stress is $\Delta T \gamma E/(1 - 2\nu)$ (compression).
4. A uniform bar of rectangular section has one face at a uniform temperature T and the opposite face at a uniform temperature $T + \Delta T$, the temperature gradient between these faces being linear. The bar would normally curve in the arc of the circle of radius $d/\Delta T \gamma$, where d is the distance between the hot and cold faces. If the ends are fixed, the bar will be held straight by end couples $EI \Delta T \gamma/d$, and the maximum resulting bending stress will be $\frac{1}{2} \Delta T \gamma E$ (compression on the hot face; tension on the cold face). (For many other conditions of end restraint and partial heating, see Table 8.1, cases 6a–6f; Table 8.2, cases 1r–12r; Table 8.5, case 7; Table 8.6, case 7; Table 8.8, cases 6a–6f; and Table 8.9 cases 6a–6f.)
5. A flat plate of uniform thickness t and of any shape has one face at a uniform temperature T and the other face at a uniform tempera-

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*Most of the formulas given here are taken from the papers by Goodier (Refs. 7 and 14), Maulbetsch (Ref. 8). and Kent (Ref. 9).

ture $T + \Delta T$, the temperature gradient between the faces being linear. The plate would normally assume a spherical curvature with radius $t/(\Delta T\gamma)$. If the edges are fixed, the plate will be held flat by uniform edge moments and the maximum resulting bending stress will be $\frac{1}{2}\Delta T\gamma E/(1-\nu)$ (compression on the hot face; tension on the cold face). (For many other conditions of edge restraint and axisymmetric partial heating, see Table 11.2, cases 8a–8h; a more general treatment of the solid circular plate is given in Table 11.2, case 15.)

6. If the plate described in case 5 is circular, no stress is produced by supporting the edges in a direction normal to the plane of the plate.
7. If the plate described in case 5 has the shape of an equilateral triangle of altitude a (sides $2a/\sqrt{3}$) and the edges are rigidly supported so as to be held in a plane, the supporting reactions will consist of a uniform load $\frac{1}{8}\Delta T\gamma Et^2/a$ per unit length along each edge against the hot face and a concentrated load $\sqrt{3}\Delta T\gamma Et^2/12$ at each corner against the cold face. The maximum resulting bending stress is $\frac{3}{4}\Delta T\gamma E$ at the corners (compression on the hot face; tension on the cold face). There are also high shear stresses near the corners (Ref. 8).
8. If the plate described in case 5 is square, no simple formula is available for the reactions necessary to hold the edges in their original plane. The maximum bending stress occurs near the edges, and its value approaches $\frac{1}{2}\Delta T\gamma E$. There are also high shear stresses near the corners (Ref. 8).

Stresses due to internal constraint

9. Part or all of the surface of a solid body is suddenly subjected to a temperature change ΔT; a compressive stress $\Delta T\gamma E/(1-\nu)$ is developed in the surface layer of the heated part (Ref. 7).
10. A thin circular disk at uniform temperature has the temperature changed ΔT throughout a comparatively small central circular portion of radius a. Within the heated part there are radial and tangential compressive stresses $\sigma_r = \sigma_t = \frac{1}{2}\Delta T\gamma E$. At points outside the heated part a distance r from the center of the disk but still close to the central portion, the stresses are $\sigma_r = \frac{1}{2}\Delta T\gamma Ea^2/r^2$ (compression) and $\sigma_t = \frac{1}{2}\Delta T\gamma Ea^2/r^2$ (tension); at the edge of the heated portion, there is a maximum shear stress $\frac{1}{2}\Delta T\gamma E$ (Ref. 7).
11. If the disk of case 10 is heated uniformly throughout a small central portion of elliptical instead of circular outline, the maxi-

mum stress is the tangential stress at the ends of the ellipse and is $\sigma_t = \Delta T\gamma E/[1 + (b/a)]$, where a is the major and b the minor semi-axis of the ellipse (Ref. 7).

12. If the disk of case 10 is heated symmetrically about its center and uniformly throughout its thickness so that the temperature is a function of the distance r from the center only, the radial and tangential stresses at any point a distance r_1 from the center are

$$\sigma_{r_1} = \gamma E\left(\frac{1}{R^2}\int_0^R Tr\,dr - \frac{1}{r_1^2}\int_0^{r_1} Tr\,dr\right)$$

$$\sigma_{t_1} = \gamma E\left(-T + \frac{1}{R^2}\int_0^R Tr\,dr + \frac{1}{r_1^2}\int_0^{r_1} Tr\,dr\right)$$

where R is the radius of the disk and T is the temperature at any point a distance r from the center minus the temperature of the coldest part of the disk. [In the preceding expressions, the negative sign denotes compressive stress (Ref. 7).]

13. A rectangular plate or strip $ABCD$ (Fig. 16.5) is heated along a transverse line FG uniformly throughout the thickness and across the width so that the temperature varies only along the length with x. At FG the temperature is T_1; the minimum temperature in the plate is T_0. At any point along the edges of the strip where the temperature is T, a tensile stress $\sigma_x = E\gamma(T - T_0)$ is developed; this stress has its maximum value at F and G, where it becomes $E\gamma(T_1 - T_0)$. Halfway between F and G, a compressive stress σ_y of equal intensity is developed along line FG (Ref. 7).

14. The plate of case 13 is heated as described except that the lower face of the plate is cooler than the upper face, the maximum temperature there being T_2 and the temperature gradient through the thickness being linear. The maximum tensile stress at F and G is (see Ref. 7)

$$\sigma_x = \frac{1}{2}E\gamma\left[T_1 + T_2 - 2T_0 + \frac{1-\nu}{3+\nu}(T_1 - T_2)\right]$$

15. A long hollow cylinder with thin walls has the outer surface at the uniform temperature T and the inner surface at the uniform

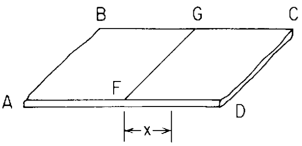

Figure 16.5

temperature $T + \Delta T$. The temperature gradient through the thickness is linear. At points remote from the ends, the maximum circumferential stress is $\frac{1}{2}\Delta T\gamma E/(1-\nu)$ (compression at the inner surface; tension at the outer surface) and the longitudinal stress is $\frac{1}{2}\Delta T\gamma E/(1-\nu)$ (compression at the inside; tension at the outside). (These formulas apply to a thin tube of any cross section.) At the ends, if these are free, the maximum tensile stress in a tube of circular section is about 25% greater than the value given by the formula (Ref. 7).

16. A hollow cylinder with thick walls of inner radius b and outer radius c has the outer surface at the uniform temperature T and the inner surface at the uniform temperature $T + \Delta T$. After steady-state heat flow is established the temperature decreases logarithmically with r and then the maximum stresses, which are circumferential and which occur at the inner and outer surfaces, are

(Outer surface)

$$\sigma_t = \frac{\Delta T\gamma E}{2(1-\nu)\ln(c/b)}\left(1 - \frac{2b^2}{c^2-b^2}\ln\frac{c}{b}\right) \qquad \text{tension}$$

(Inner surface)

$$\sigma_t = \frac{\Delta T\gamma E}{2(1-\nu)\ln(c/b)}\left(1 - \frac{2c^2}{c^2-b^2}\ln\frac{c}{b}\right) \qquad \text{compression}$$

At the inner and outer surfaces, the longitudinal stresses are equal to the tangential stresses (Ref. 7).

17. If the thick tube of case 16 has the temperature of the outer surface raised at the uniform rate of m°/s then, after a steady rate of temperature rise has been reached throughout, the maximum tangential stresses are

(Outer surface)

$$\sigma_t = \frac{E\gamma m}{8A(1-\nu)}\left(3b^2 - c^2 - \frac{4b^4}{c^2-b^2}\ln\frac{c}{b}\right) \qquad \text{compression}$$

(Inner surface)

$$\sigma_t = \frac{E\gamma m}{8A(1-\nu)}\left(b^2 + c^2 - \frac{4b^2c^2}{c^2-b^2}\ln\frac{c}{b}\right) \qquad \text{tension}$$

where A is the coefficient of thermal diffusivity equal to the coefficient of thermal conductivity divided by the product of density of the material and its specific heat. (For steel, A may be taken as 0.027 in^2/s at moderate temperatures.) [At the inner and outer surfaces, the longitudinal stresses are equal to the tangential stresses (Ref. 9).] The stated conditions in this case 17 as well as those in cases 19 to 21 are difficult to create in a short time except for small parts heated or cooled in liquids.

18. A solid rod of circular section is heated or cooled symmetrically with respect to its axis, the condition being uniform along the length, so that the temperature is a function of r (the distance from the axis) only. The stresses are equal to those given by the formulas for case 12 divided by $1 - \nu$ (Ref. 7).

19. If the solid rod of case 18 has the temperature of its convex surface raised at the uniform rate of $m°$/s, then, after a steady rate of temperature rise has been reached throughout, the radial, tangential, and longitudinal stresses at any point a distance r from the center are

$$\sigma_r = \frac{E\gamma m}{1-\nu}\frac{c^2 - r^2}{16A}$$

$$\sigma_t = \frac{E\gamma m}{1-\nu}\frac{c_2 - 3r^2}{16A}$$

$$\sigma_x = \frac{E\gamma m}{1-\nu}\frac{c^2 - 2r^2}{8A}$$

Here A has the same meaning as in case 17 and c is the radius of the shaft, [A negative result indicates compression, a positive result tension (Ref. 9).]

20. A solid sphere of radius c is considered instead of a solid cylinder but with all other conditions kept the same as in case 19. The radial and tangential stresses produced at any point a distance r from the center are

$$\sigma_r = \frac{E\gamma m}{15A(1-\nu)}(c^2 - r^2)$$

$$\sigma_t = \frac{E\gamma m}{15A(1-\nu)}(c^2 - 2r^2)$$

[A negative result indicates compression, a positive result tension (Ref. 9).]

21. If the sphere is hollow, with outer radius c and inner radius b, and with all other conditions kept as stated in case 17, the stresses at

any point are

$$\sigma_r = \frac{E\gamma m}{15A(1-\nu)}\left(-r^2 - \frac{5b^3}{r} + \phi - \psi\right)$$

$$\sigma_t = \frac{E\gamma m}{15A(1-\nu)}\left(-2r^2 - \frac{5b^3}{2r} + \phi + \frac{\psi}{2}\right)$$

where

$$\phi = \frac{c^5 + 5c^2b^3 - 6b^5}{c^3 - b^3}$$

$$\psi = \frac{c^5b^3 - 6c^3b^5 + 5c^2b^6}{r^3(c^3 - b^3)}$$

[A negative result indicates compression, a positive result tension (Ref. 9).]

Other problems involving thermal stress, the solutions of which cannot be expressed by simple formulas, are considered in the references cited above and in Refs. 3, 10 and 25 to 29; charts for the solution of thermal stresses in tubes are given in Ref. 11. Derivations for many of the thermal loadings shown above along with thermal loadings on many other examples of bars, rings, plates, and cylindrical and spherical shells are given in Ref. 28.

16.7 Tables

TABLE 16.1 Natural frequencies of vibration for continuous members

NOTATION: f = natural frequency (cycles per second); K_n = constant where n refers to the mode of vibration; g = gravitational acceleration (units consistent with length dimension); E = modulus of elasticity; I = area moment of inertia; $D = Et^3/12(1 - \nu^2)$

<table>
<tr><th>Case no. and description</th><th></th><th>Natural frequencies</th></tr>
<tr><td rowspan="3">1. Uniform beam; both ends simply supported</td><td>1a. Center load W, beam weight negligible</td><td>$f_1 = \dfrac{6.93}{2\pi}\sqrt{\dfrac{EIg}{Wl^3}}$</td></tr>
<tr><td>1b. Uniform load w per unit length including beam weight</td><td>$f_n = \dfrac{K_n}{2\pi}\sqrt{\dfrac{EIg}{wl^4}}$
<table>
<tr><th>Mode</th><th>K_n</th><th colspan="6">Nodal position/l</th></tr>
<tr><td>1</td><td>9.87</td><td>0.0</td><td>1.00</td><td></td><td></td><td></td><td></td></tr>
<tr><td>2</td><td>39.5</td><td>0.0</td><td>0.50</td><td>1.00</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td>88.8</td><td>0.0</td><td>0.33</td><td>0.67</td><td>1.00</td><td></td><td></td></tr>
<tr><td>4</td><td>158</td><td>0.0</td><td>0.25</td><td>0.50</td><td>0.75</td><td>1.00</td><td></td></tr>
<tr><td>5</td><td>247</td><td>0.0</td><td>0.20</td><td>0.40</td><td>0.60</td><td>0.80</td><td>1.00</td></tr>
</table>
Ref. 22</td></tr>
<tr><td>1c. Uniform load w per unit length plus a center load W</td><td>$f_1 = \dfrac{6.93}{2\pi}\sqrt{\dfrac{EIg}{Wl^3 + 0.486wl^4}}$ approximately</td></tr>
<tr><td rowspan="3">2. Uniform beam; both ends fixed</td><td>2a. Center load W, beam weight negligible</td><td>$f_1 = \dfrac{13.86}{2\pi}\sqrt{\dfrac{EIg}{Wl^3}}$</td></tr>
<tr><td>2b. Uniform load w per unit length including beam weight</td><td>$f_n = \dfrac{K_n}{2\pi}\sqrt{\dfrac{EIg}{wl^4}}$
<table>
<tr><th>Mode</th><th>K_n</th><th colspan="6">Nodal position/l</th></tr>
<tr><td>1</td><td>22.4</td><td>0.0</td><td>1.00</td><td></td><td></td><td></td><td></td></tr>
<tr><td>2</td><td>61.7</td><td>0.0</td><td>0.50</td><td>1.00</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td>121</td><td>0.0</td><td>0.36</td><td>0.64</td><td>1.00</td><td></td><td></td></tr>
<tr><td>4</td><td>200</td><td>0.0</td><td>0.28</td><td>0.50</td><td>0.72</td><td>1.00</td><td></td></tr>
<tr><td>5</td><td>299</td><td>0.0</td><td>0.23</td><td>0.41</td><td>0.59</td><td>0.77</td><td>1.00</td></tr>
</table>
Ref. 22</td></tr>
<tr><td>2c. Uniform load w per unit length plus a center load W</td><td>$f_1 = \dfrac{13.86}{2\pi}\sqrt{\dfrac{EIg}{Wl^3 + 0.383wl^4}}$ approximately</td></tr>
<tr><td rowspan="2">3. Uniform beam; left end fixed, right end free (cantilever)</td><td>3a. Right end load W, beam weight negligible</td><td>$f_1 = \dfrac{1.732}{2\pi}\sqrt{\dfrac{EIg}{Wl^3}}$</td></tr>
<tr><td>3b. Uniform load w per unit length including beam weight</td><td>$f_n = \dfrac{K_n}{2\pi}\sqrt{\dfrac{EIg}{wl^4}}$
<table>
<tr><th>Mode</th><th>K_n</th><th colspan="5">Nodal position/l</th></tr>
<tr><td>1</td><td>3.52</td><td>0.0</td><td></td><td></td><td></td><td></td></tr>
<tr><td>2</td><td>22.0</td><td>0.0</td><td>0.783</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td>61.7</td><td>0.0</td><td>0.504</td><td>0.868</td><td></td><td></td></tr>
<tr><td>4</td><td>121</td><td>0.0</td><td>0.358</td><td>0.644</td><td>0.905</td><td></td></tr>
<tr><td>5</td><td>200</td><td>0.0</td><td>0.279</td><td>0.500</td><td>0.723</td><td>0.926</td></tr>
</table>
Ref. 22</td></tr>
</table>

TABLE 16.1 Natural frequencies of vibration for continuous members (*Continued*)

<table>
<tr><th>Case no. and description</th><th></th><th>Natural frequencies</th></tr>
<tr><td></td><td>3c. Uniform load w per unit length plus an end load W</td><td>$f_1 = \frac{1.732}{2\pi}\sqrt{\frac{EIg}{Wl^3 + 0.236wl^4}}$ approximately</td></tr>
<tr><td>4. Uniform beam; both ends free</td><td>4a. Uniform load w per unit length including beam weight</td><td>$f_n = \frac{K_n}{2\pi}\sqrt{\frac{EIg}{wl^4}}$
<table>
<tr><th>Mode</th><th>K_n</th><th colspan="6">Nodal position/l</th></tr>
<tr><td>1</td><td>22.4</td><td>0.224</td><td>0.776</td><td></td><td></td><td></td><td></td></tr>
<tr><td>2</td><td>61.7</td><td>0.132</td><td>0.500</td><td>0.868</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td>121</td><td>0.095</td><td>0.356</td><td>0.644</td><td>0.905</td><td></td><td></td></tr>
<tr><td>4</td><td>200</td><td>0.074</td><td>0.277</td><td>0.500</td><td>0.723</td><td>0.926</td><td></td></tr>
<tr><td>5</td><td>299</td><td>0.060</td><td>0.226</td><td>0.409</td><td>0.591</td><td>0.774</td><td>0.940</td></tr>
</table>
Ref. 22</td></tr>
<tr><td>5. Uniform beam; left end fixed, right end hinged</td><td>5a. Uniform load w per unit length including beam weight</td><td>$f_n = \frac{K_n}{2\pi}\sqrt{\frac{EIg}{wl^4}}$
<table>
<tr><th>Mode</th><th>K_n</th><th colspan="6">Nodal position/l</th></tr>
<tr><td>1</td><td>15.4</td><td>0.0</td><td>1.000</td><td></td><td></td><td></td><td></td></tr>
<tr><td>2</td><td>50.0</td><td>0.0</td><td>0.557</td><td>1.000</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td>104</td><td>0.0</td><td>0.386</td><td>0.692</td><td>1.000</td><td></td><td></td></tr>
<tr><td>4</td><td>178</td><td>0.0</td><td>0.295</td><td>0.529</td><td>0.765</td><td>1.000</td><td></td></tr>
<tr><td>5</td><td>272</td><td>0.0</td><td>0.239</td><td>0.428</td><td>0.619</td><td>0.810</td><td>1.000</td></tr>
</table>
Ref. 22</td></tr>
<tr><td>6. Uniform beam; left end hinged, right end free</td><td>6a. Uniform load w per unit length including beam weight</td><td>$f_n = \frac{K_n}{2\pi}\sqrt{\frac{EIg}{wl^4}}$
<table>
<tr><th>Mode</th><th>K_n</th><th colspan="6">Nodal position/l</th></tr>
<tr><td>1</td><td>15.4</td><td>0.0</td><td>0.736</td><td></td><td></td><td></td><td></td></tr>
<tr><td>2</td><td>50.0</td><td>0.0</td><td>0.446</td><td>0.853</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td>104</td><td>0.0</td><td>0.308</td><td>0.617</td><td>0.898</td><td></td><td></td></tr>
<tr><td>4</td><td>178</td><td>0.0</td><td>0.235</td><td>0.471</td><td>0.707</td><td>0.922</td><td></td></tr>
<tr><td>5</td><td>272</td><td>0.0</td><td>0.190</td><td>0.381</td><td>0.571</td><td>0.763</td><td>0.937</td></tr>
</table>
Ref. 22</td></tr>
<tr><td>7. Uniform bar or spring vibrating along its longitudinal axis; upper end fixed, lower end free</td><td>7a. Weight W at lower end, bar weight negligible</td><td>$f_1 = \frac{1}{2\pi}\sqrt{\frac{kg}{W}}$ for a spring where k is the spring constant
$f_1 = \frac{1}{2\pi}\sqrt{\frac{AEg}{Wl}}$ for a bar where A is the area, l the length, and E the modulus</td></tr>
<tr><td></td><td>7b. Uniform load w per unit length including bar weight</td><td>$f_n = \frac{K_n}{2\pi}\sqrt{\frac{AEg}{wl^2}}$ where $K_1 = 1.57$ $K_2 = 4.71$ $K_3 = 7.85$</td></tr>
<tr><td></td><td>7c. Uniform load w per unit length plus a load W at the lower end</td><td>$f_1 = \frac{1}{2\pi}\sqrt{\frac{kg}{W + wl/3}}$ approximately for a spring where k is the spring constant
$f_1 = \frac{1}{2\pi}\sqrt{\frac{AEg}{Wl + wl^2/3}}$ approximately for a bar where A is the area</td></tr>
</table>

TABLE 16.1 Natural frequencies of vibration for continuous members (*Continued*)

8. Uniform shaft or bar in torsional vibration; one end fixed, the other end free	8a. Concentrated end mass of J mass moment of inertia, shaft weight negligible	$f_1 = \frac{1}{2\pi}\sqrt{\frac{GK}{Jl}}$ G is the shear modulus of elasticity and K is the torsional stiffness constant (see Chap. 10)
	8b. Uniform distribution of mass moment of inertia along shaft; J_s = total distributed mass moment of inertia	$f_n = \frac{K_n}{2\pi}\sqrt{\frac{GK}{J_s l}}$ where $K_1 = 1.57$ $K_2 = 4.71$ $K_3 = 7.85$
	8c. Uniformly distributed inertia plus a concentrated end mass	$f_1 = \frac{1}{2\pi}\sqrt{\frac{GK}{(J + J_s/3)l}}$ approximately
9. String vibrating laterally under a tension T with both ends fixed	9a. Uniform load w per unit length including own weight	$f = \frac{K_n}{2\pi}\sqrt{\frac{Tg}{wl^2}}$ where $K_1 = \pi$ $K_2 = 2\pi$ $K_3 = 3\pi$
10. Circular flat plate of uniform thickness t and radius r; edge fixed	10a. Uniform load w per unit area including own weight	$f = \frac{K_n}{2\pi}\sqrt{\frac{Dg}{wr^4}}$ where $K_1 = 10.2$ fundamental $K_2 = 21.3$ one nodal diameter $K_3 = 34.9$ two nodal diameters $K_4 = 39.8$ one nodal circle Ref. 20
11. Circular flat plate of uniform thickness t and radius r; edge simply supported	11a. Uniform load w per unit area including own weight; $\nu = 0.3$	$f = \frac{K_n}{2\pi}\sqrt{\frac{Dg}{wr^4}}$ where $K_1 = 4.99$ fundamental $K_2 = 13.9$ one nodal diameter $K_3 = 25.7$ two nodal diameters $K_4 = 29.8$ one nodal circle Ref. 20
12. Circular flat plate of uniform thickness t and radius r; edge free	12a. Uniform load w per unit area including own weight; $\nu = 0.33$	$f = \frac{K_n}{2\pi}\sqrt{\frac{Dg}{wr^4}}$ where $K_1 = 5.25$ two nodal diameters $K_2 = 9.08$ one nodal circle $K_3 = 12.2$ three nodal diameters $K_4 = 20.5$ one nodal diameter and one nodal circle Ref. 20

TABLE 16.1 Natural frequencies of vibration for continuous members (*Continued*)

<table>
<tr><th>Case no. and description</th><th></th><th>Natural frequencies</th></tr>
<tr>
<td>13. Circular flat plate of uniform thickness t and radius r; edge simply supported with an additional edge constraining moment $M = \beta\psi$ per unit circumference where ψ is the edge rotation</td>
<td>13a. Uniform load w per unit area including own weight; $\nu = 0.3$</td>
<td>$f = \frac{K_n}{2\pi}\sqrt{\frac{Dg}{wr^4}}$ where K_n is tabulated for various degrees of edge stiffness in the form of $\beta r/D$.
<table>
<tr><th rowspan="2">$\beta r/D$</th><th colspan="4">K_n</th></tr>
<tr><th>Fundamental</th><th>1 nodal diameter</th><th>2 nodal diameters</th><th>1 nodal circle</th></tr>
<tr><td>∞</td><td>10.2</td><td>21.2</td><td>34.8</td><td>39.7</td></tr>
<tr><td>1</td><td>10.2</td><td>21.2</td><td>34.8</td><td>39.7</td></tr>
<tr><td>0.1</td><td>10.0</td><td>20.9</td><td>34.2</td><td>39.1</td></tr>
<tr><td>0.01</td><td>8.76</td><td>18.6</td><td>30.8</td><td>35.2</td></tr>
<tr><td>0.001</td><td>6.05</td><td>15.0</td><td>26.7</td><td>30.8</td></tr>
<tr><td>0</td><td>4.93</td><td>13.9</td><td>25.6</td><td>29.7</td></tr>
</table>
Ref. 20</td>
</tr>
<tr>
<td>14. Elliptical flat plate of major radius, a, minor radius b, and thickness t; edge fixed</td>
<td>14a. Uniform load w per unit area including own weight</td>
<td>$f = \frac{K_1}{2\pi}\sqrt{\frac{Dg}{wa^4}}$ where K_1 is tabulated for various ratios of a/b
<table>
<tr><th>a/b</th><th>1.0</th><th>1.1</th><th>1.2</th><th>1.5</th><th>2.0</th><th>3.0</th></tr>
<tr><td>K_1</td><td>10.2</td><td>11.3</td><td>12.6</td><td>17.0</td><td>27.8</td><td>57.0</td></tr>
</table>
Ref. 20</td>
</tr>
<tr>
<td>15. Rectangular flat plate with short edge a, long edge, b, and thickness, t; all edges fixed</td>
<td>15a. Uniform load w per unit area including own weight</td>
<td>$f = \frac{K_1}{2\pi}\sqrt{\frac{Dg}{wa^2}}$ where K_1 is tabulated for various ratios of a/b
<table>
<tr><th>a/b</th><th>1</th><th>0.9</th><th>0.8</th><th>0.6</th><th>0.4</th><th>0.2</th><th>0</th></tr>
<tr><td>K_1</td><td>36.0</td><td>32.7</td><td>29.9</td><td>25.9</td><td>23.6</td><td>22.6</td><td>22.4</td></tr>
</table>
Ref. 20</td>
</tr>
<tr>
<td>16. Rectangular flat plate with short edge a, long edge b, and thickness t; all edges simply supported</td>
<td>16a. Uniform load w per unit area including own weight</td>
<td>$f = \frac{K_n}{2\pi}\sqrt{\frac{Dg}{wa^4}}$ where $K_n = \pi^2\left[m_a^2 + \left(\frac{a}{b}\right)^2 m_b^2\right]$
<table>
<tr><th></th><th>a/b</th><th>1.0</th><th>0.8</th><th>0.6</th><th>0.4</th><th>0.2</th><th>0.0</th></tr>
<tr><td>$(m_a = 1, m_b = 1)$</td><td>K_1</td><td>19.7</td><td>16.2</td><td>13.4</td><td>11.5</td><td>10.3</td><td>9.87</td></tr>
<tr><td>$(m_a = 1, m_b = 2)$</td><td>K_2</td><td>49.3</td><td>35.1</td><td>24.1</td><td>16.2</td><td>11.5</td><td></td></tr>
<tr><td>$(m_a = 2, m_b = 1)$</td><td>K_3</td><td>49.3</td><td>45.8</td><td></td><td></td><td></td><td></td></tr>
<tr><td>$(m_a = 1, m_b = 3)$</td><td>K_3</td><td></td><td></td><td>41.9</td><td>24.1</td><td>13.4</td><td></td></tr>
</table>
Ref. 20</td>
</tr>
<tr>
<td>17. Rectangular flat plate with two edges a fixed, one edge b fixed, and one edge b simply supported</td>
<td>17a. Uniform load w per unit area including own weight</td>
<td>$f = \frac{K_1}{2\pi}\sqrt{\frac{Dg}{wa^4}}$ where K_1 is tabulated for various ratios of $\frac{a}{b}$
<table>
<tr><th>a/b</th><th>3.0</th><th>2.0</th><th>1.6</th><th>1.2</th><th>1.0</th><th>0.8</th><th>0.6</th><th>0.4</th><th>.2</th><th>0</th></tr>
<tr><td>K_1</td><td>213</td><td>99</td><td>67</td><td>42.4</td><td>33.1</td><td>25.9</td><td>20.8</td><td>17.8</td><td>16.2</td><td>15.8</td></tr>
</table>
Ref. 22</td>
</tr>
</table>

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Chapter

17

Stress Concentration

When a large stress gradient occurs in a small, localized area of a structure, the high stress is referred to as a *stress concentration*. Near changes in geometry of a loaded structure, the *flow* of stress is interfered with, causing high stress gradients where the maximum stress and strain may greatly exceed the average or nominal values based on simple calculations. Contact stresses, as discussed in Chapter 14, also exhibit high stress gradients near the point of contact, which subside quickly as one moves away from the contact area. Thus, the two most common occurrences of stress concentrations are due to (1) discontinuities in continuum and (2) contact forces. Discontinuities in continuum include changes in geometry and material properties. This chapter is devoted to geometric changes.

Rapid geometry changes disrupt the smooth flow of stresses through the structure between load application areas. Plates in tension or bending with holes, notches, steps, etc. are simple examples involving direct normal stresses. Shafts in tension, bending, and torsion, with holes, notches, steps, keyways, etc., are simple examples involving direct and bending normal stresses and torsional shear stresses. More complicated geometries must be analyzed either by experimental or numerical techniques such as the finite element method. Other, less obvious, geometry changes include rough surface finishes and external and internal cracks.

Changes in material properties are discussed in Chap. 7, and demonstrated in an example where a change in modulus of elasticity drastically changed the stress distribution. Changes in material properties can occur both at macroscopic and microscopic levels which include alloy formulation, grain size and orientation, foreign materials, etc.

17.1 Static Stress and Strain Concentration Factors

Consider the plate shown in Fig. 17.1, loaded in tension by a force per unit area, σ. Although not drawn to scale, consider that the outer dimensions of the plate are infinite compared with the diameter of the hole, $2a$. It can be shown, from linear elasticity, that the tangential stress throughout the plate is given by (see Ref. 60)

$$\sigma_\theta = \frac{\sigma}{2}\left[1 + \frac{a^2}{r^2} - \left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta\right] \tag{17.1-1}$$

The maximum stress is $\sigma_\theta = 3\sigma$ at $r = a$ and $\theta = \pm 90^\circ$. Figure 17.2 shows how the tangential stress varies along the x and y axes of the plate. For the top (and bottom) of the hole, we see the stress gradient is extremely large compared with the nominal stress, and hence the term *stress concentratiom* applies. Along the surface of the hole, the tangential stress is $-\sigma$ at $\theta = 0^\circ$ and 180°, and increases, as θ increases, to 3σ at $\theta = 90^\circ$ and 270°.

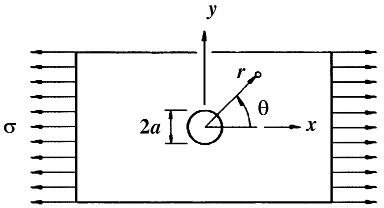


Figure 17.1 Circular hole in a plate loaded in tension.

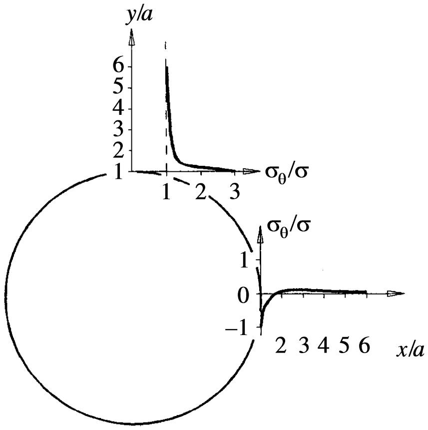


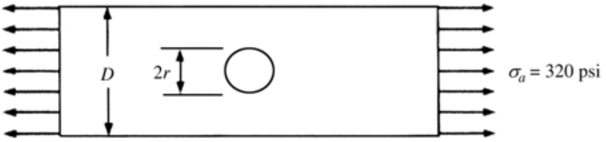
Figure 17.2 Tangential stress distribution for $\theta = 0^\circ$ and 90°.

The *static stress concentration factor* in the elastic range, K_t, is defined as the ratio of the maximum stress, σ_{max}, to the nominal stress, σ_{nom}. That is,

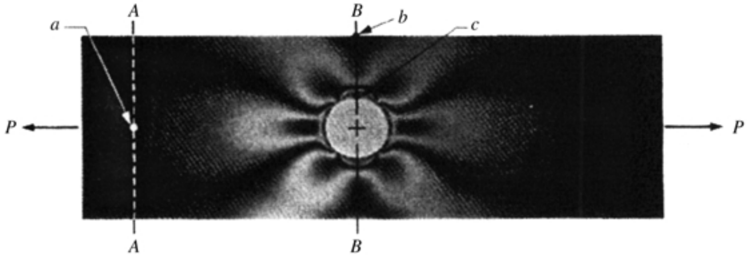
$$K_t = \frac{\sigma_{\text{max}}}{\sigma_{\text{nom}}} \tag{17.1-2}$$

For the infinite plate containing a hole and loaded in tension, $\sigma_{\text{nom}} = \sigma$, $\sigma_{\text{max}} = 3\sigma$, and thus $K_t = 3$.*

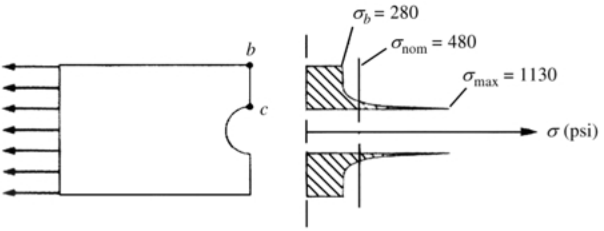
The analysis of the plate in tension with a hole just given is for a very wide plate (infinite in the limit). As the width of the plate decreases, the maximum stress becomes less than three times the nominal stress at the zone containing the hole. Figure 17.3(a) shows a plate of thickness $t = 0.125$ in, width $D = 1.50$ in, with a hole of diameter $2r = 0.50$ in, and an applied uniform stress of $\sigma_0 = 320$ psi.



(a) Loaded plate



(b) Photoelastic model



(c) Stress distribution

Figure 17.3 Stress distribution for a plate in tension containing a centrally located hole.

---

*See Case 7a of Table 17.1. As $2a/D \to 0$, $K_t \to 3.00$.

A photoelastic* model is shown in Fig. 17.3(*b*). From a photoelastic analysis, the stresses at points *a*, *b*, and *c* are found to be

$$\begin{array}{lll} \text{zone } A\text{–}A: & \sigma_a = 320 \text{ psi} & \\ \text{zone } B\text{–}B: & \sigma_b = 280 \text{ psi}, & \sigma_c = 1130 \text{ psi} \end{array}$$

The nominal stress in zone *B–B* is

$$\sigma_{\text{nom}} = \frac{D}{D - 2r}\sigma_0 = \frac{1.50}{1.50 - 0.5}320 = 480 \text{ psi}$$

If the stress was uniform from *b* to *c*, the stress would be 480 psi. However, the photoelastic analysis shows the stress to be nonuniform, ranging from 280 psi at *b* to a maximum stress at *c* of 1130 psi. Thus, for this example, the stress concentration factor is found to be

$$K_t = \frac{\sigma_{\max}}{\sigma_{\text{nom}}} = \frac{1130}{480} = 2.35$$

The static stress concentration factor for a plate containing a centrally located hole in which the plate is loaded in tension depends on the ratio $2r/D$ as given for case 7a of Table 17.1. For our example here, $2r/D = 0.5/1.5 = \frac{1}{3}$. The equation for K_t from Table 17.1 gives

$$K_t = 3.00 - 3.13(\tfrac{1}{3}) + 3.66(\tfrac{1}{3})^2 - 1.53(\tfrac{1}{3})^3 = 2.31$$

which is within 2% of the results from the photoelastic model.

Table 17.1 provides the means to evaluate the static stress concentration factors in the elastic range for many cases that apply to fundamental forms of geometry and loading conditions.

Neuber's Formula for Nonlinear Material Behavior. If the load on a structure exceeds the value for which the maximum stress at a stress concentration equals the elastic limit of the material, the stress distribution changes from that within the elastic range. Neuber (Ref. 61) presented a formula which includes stress and strain. Defining an effective stress concentration factor, $K_\sigma = \sigma_{\max}/\sigma_{\text{nom}}$, and an effective strain concentration factor, $K_\varepsilon = \varepsilon_{\max}/\varepsilon_{\text{nom}}$, Neuber established that K_t is the geometric means of the stress and strain factors. That is, $K_t = (K_\sigma K_\varepsilon)^{1/2}$, or

$$K_\sigma = \frac{K_t^2}{K_\varepsilon} \qquad (17.1\text{-}3)$$

---

*Photoelasticity is discussed at some length in Ref. 60.

In terms of the stresses and strains, Eq. (17.1-3) can be written as

$$\sigma_{\max}\varepsilon_{\max} = K_t^2 \sigma_{\text{nom}}\varepsilon_{\text{nom}} \tag{17.1-4}$$

K_t and σ_{nom} are obtained exactly the same as when the max stress is within the elastic range. The determination of ε_{nom} is found from the material's elastic stress-strain curve using the nominal stress.

EXAMPLE

A circular shaft with a square shoulder and fillet is undergoing bending (case 17b of Table 17.1). A bending moment of 500 N-m is being transmitted at the fillet section. For the shaft, $D = 50$ mm, $h = 9$ mm, and $r = 3$ mm. The stress–strain data for the shaft material is tabulated below and plotted in Fig. 17.4. Determine the maximum stress in the shaft.

$\varepsilon, 10^{-5}$	0	25	50	75	100	125	150	175	200	225	250	275	300	325	350	375	400
σ, (MPa)	0	50	100	150	200	235	252	263	267	272	276	279	282	285	287	289	290

Solution. From the given dimensions, $h/r = 9/3 = 3$. From case 17b of Table 17.1,

$$C_1 = 1.225 + 0.831\sqrt{3} - 0.010(3) = 2.634$$
$$C_2 = -3.790 + 0.958\sqrt{3} - 0.257(3) = -2.902$$
$$C_3 = 7.374 - 4.834\sqrt{3} + 0.862(3) = 1.587$$
$$C_4 = -3.809 + 3.046 - 0.595(3) = -0.3182$$

With $2h/D = 18/50 = 0.36$,

$$K_t = 2.634 - 2.902(0.36) + 1.587(0.36)^2 - 0.3182(0.36)^3 = 1.780$$

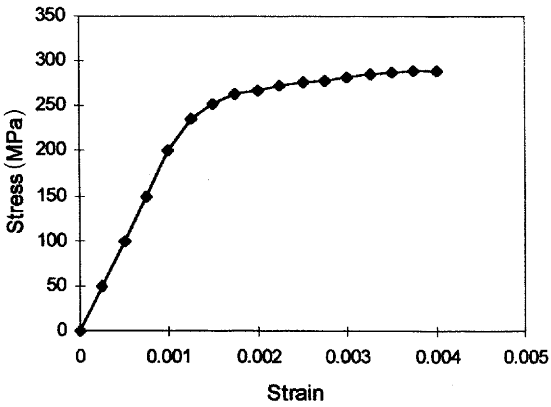


Figure 17.4

The nominal stress at the minor radius of the step shaft is

$$\sigma_{\text{nom}} = \frac{32M}{\pi(D - 2h)^3} = \frac{32(500)}{\pi[50 - 2(9)]^3(10^{-3})^3} = 155.4(10^6)\ \text{N/m}^2 = 155.4\ \text{MPa}$$

If σ_{max} is in the elastic range, then

$$\sigma_{\text{max}} = K_t \sigma_{\text{nom}} = 1.780(155.4) = 276.6\ \text{MPa}$$

However, as one can see from the stress–strain plot that this exceeds the elastic limit of 200 MPa. Thus, σ_{max} must be determined from Neuber's equation.

The modulus of elasticity in the elastic range of the material is $E = 20$ GPa. Thus, the nominal strain is found to be $\varepsilon_{\text{nom}} = \sigma_{\text{nom}}/E = 155.4\ (10^6)/20(10^9) = 77.7\ (10^{-5})$. Thus,

$$K_t^2 \sigma_{\text{nom}} \varepsilon_{\text{nom}} = (1.780)^2(155.4)(77.7)(10^{-5}) = 0.3826\ \text{MPa}$$

From the tabulated data, the product $\sigma\ \varepsilon$ can be tabulated as a function of σ. This results in the following:

σ (MPa)	0	50	100	150	200	235	252	263	267
$\sigma\ \varepsilon$ (MPa)	0	0.0125	0.05	0.1125	0.2	0.29375	0.378	0.46025	0.534

σ (MPa)	272	276	279	282	285	287	289	290
$\sigma\varepsilon$ (MPa)	0.612	0.69	0.76725	0.846	0.92625	1.0045	1.08375	1.16

Since, based on Eq. (17.4), we are looking for the value of $\sigma_{\text{max}}\ \varepsilon_{\text{max}} = 0.3826$, we will interpolate $\sigma\ \varepsilon$ between 0.378 and 0.46025. Thus,

$$\frac{\sigma_{\text{max}} - 252}{0.3826 - 0.378} = \frac{263 - 252}{0.46025 - 0.378}$$

This yields $\sigma_{\text{max}} = 252.6$ MPa.

For dynamic problems where loading is cycling, the fatigue stress concentration factor is more appropriate to use. See Sec. 3.20 for a discussion of this.

17.2 Stress Concentration Reduction Methods

Intuitive methods such as the *flow analogy* are sometimes helpful to the analyst faced with the task of reducing stress concentrations. When dealing with a situation where it is necessary to reduce the cross section abruptly, the resulting stress concentration can often be minimized by a further reduction of material. This is contrary to the common advice "if it is not strong enough, make it bigger." This can be explained by examining the flow analogy.

The governing field equations for ideal irrotational fluid flow are quite similar to those for stress. Thus, there exists an analogy between fluid flow lines, velocity, and pressure gradients on the one hand, and stress trajectories, magnitudes, and principal stresses on the other. The flow analogy for the plate in Fig. 17.3 is shown in Fig. 17.5(*a*), where stress-free surface boundaries are replaced by solid-channel boundaries for the fluid (wherever stress cannot exist, fluid flow cannot exist). The uniformly applied loads are replaced by a uniform

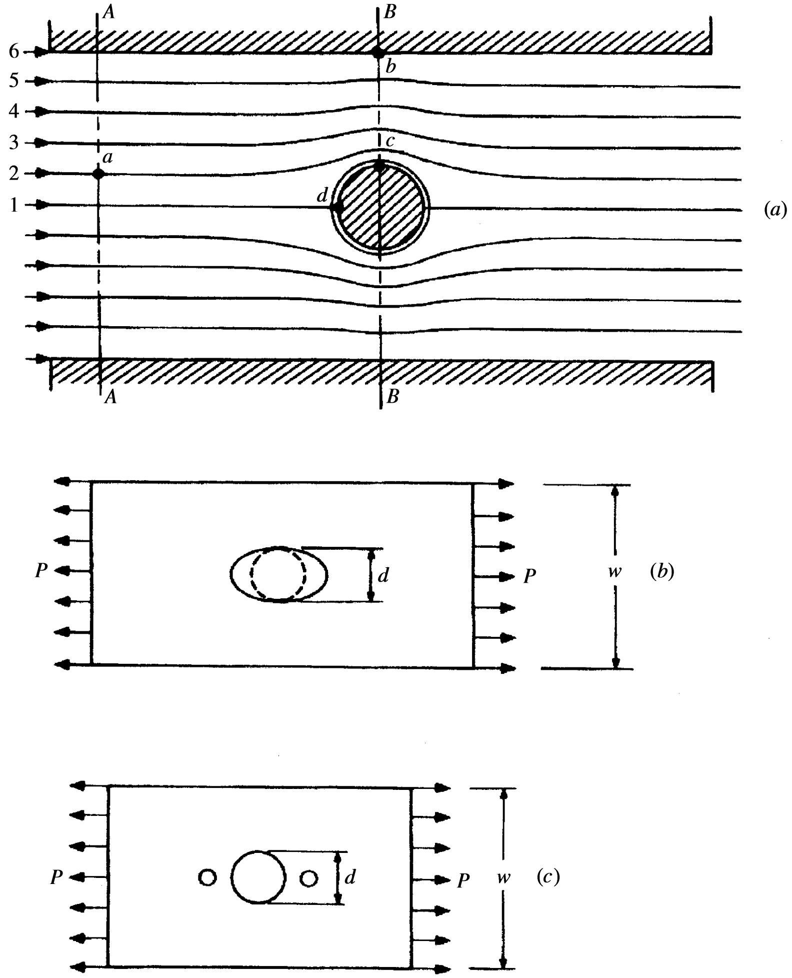


Figure 17.5 Stress–flow analogy.

fluid flow field. Along the entrance section A–A of Fig. 17.5(*a*), the flow is uniform, and, owing to symmetry, the flow is uniform at the exit of the channel. However, as the fluid particles approach section B–B, the streamlines need to adjust to move around the circular obstacle. In order to accomplish this, particles close to streamline 1 must make the greatest adjustment and must accelerate until they reach section B–B, where they reach maximum velocity, and then decelerate to their original uniform velocity some distance from B–B. Thus, the velocity at point c is the maximum. The compaction of the streamlines at c will lead to the development of a pressure gradient, which will actually cause the velocity of point b to be less than that of the incoming velocity of streamline 6 at A–A. Note also that when a particle on streamline 1 reaches point d, the particle theoretically takes on a velocity perpendicular to the net flow. This analogy agrees with that of the plate loaded in tension with a centrally located hole. The stress is a maximum at the edge of the hole corresponding to point c in Fig. 17.5(*a*). The stress in the plate corresponding to point b is lower than the applied stress, and for point d the stress in the plate is compressive perpendicular to the axial direction.

This analogy can be used to suggest improvements to reduce stress concentrations. For example, for the plate with the hole, the hole can be elongated to an ellipse as shown in Fig. 17.5(*b*), which will improve the flow transition into section B–B (note that this is a reduction of material). An ellipse, however, is not a practical solution, but it can effectively be approximated by drilling two smaller relief holes in line

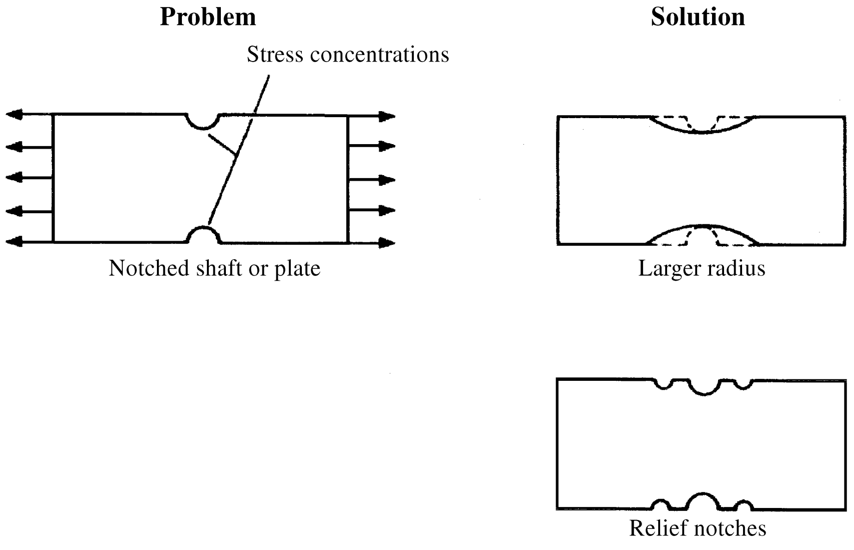


Figure 17.6

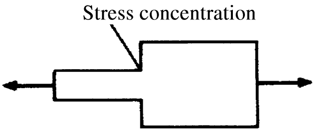
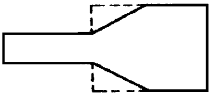
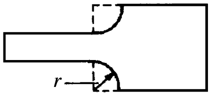


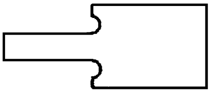
Plate or shaft with area reduction



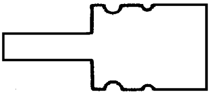
Taper



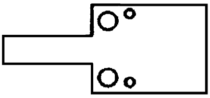
Radius



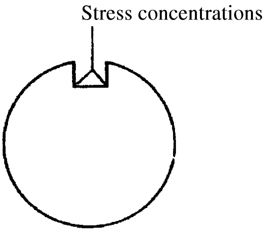
Undercut



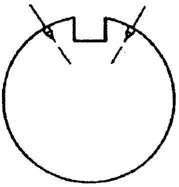
Relief grooves



Relief holes (plate)



Keyway in shaft under torsion



Drill holes on each side of keyway

Figure 17.6 (*continued*) Stress concentration reductions.

and in close proximity to the original hole as shown in Fig. 17.5(*c*). The material between the holes, provided the holes are close, will be a stagnation area where the flow (stresses) will be low. Consequently, the configuration acts much like that of an elliptical hole.

At first, this might not seem to make sense, since this is a reduction of more material—and if one hole weakens the part, obviously more holes will make things worse. One must keep in mind that the first hole increased the stress in two ways: (1) by reducing the cross-sectional area and (2) by changing the shape of the stress distribution. The two additional holes in Fig. 17.5(*c*) do not change the area reduction unless they are larger than the original hole. However, as stated, the additional holes will improve the flow transition, and consequently reduce the stress concentration. Another way of improving the plate with the hole is to elongate the hole in the axial direction to a slot.

Some other examples of situations where stress concentrations occur and possible methods of improvements are given in Fig. 17.6. Note that in each case, improvement is made by reducing material.

This is not a hard and fast rule, however; most reductions in high stress concentrations are made by removing material from adjacent low-stressed areas. This “draws” the high stresses away from the stress concentration area towards the low-stressed area, which decreases the stress in the high-stressed areas.

17.3 Table

TABLE 17.1 Stress concentration factors for elastic stress (K_t)

The elastic stress concentration factor K_t is the ratio of the maximum stress in the stress raiser to the nominal stress computed by the ordinary mechanics-of-materials formulas, using the dimensions of the net cross section unless defined otherwise in specific cases.

For those data presented in the form of equations, the equations have been developed to fit as closely as possible the many data points given in the literature referenced in each case. Over the majority of the ranges specified for the variables, the curves fit the data points with much less than a 5% error.

It is not possible to tabulate all the available values of stress concentration factors found in the literature, but the following list of topics and sources of data will be helpful. All the following references have extensive bibliographies.

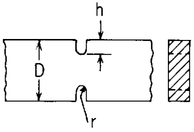
Fatigue stress concentration factors (K_f); see the 4th edition of this book, Ref. 23.

Stress concentration factors for rupture (K_r); see the 4th edition of this book, Ref. 23.

Stress concentration factors pertaining to odd-shaped holes in plates, multiple holes arranged in various patterns, and reinforced holes under multiple loads; see Refs. 1 and 24.

Stress concentration around holes in pressure vessels; see Ref. 1.

For a discussion of the effect of stress concentration on the response of machine elements and structures, see Ref. 25.

<table>
<tr><th>Type of form irregularity or stress raiser</th><th>Stress condition and manner of loading</th><th>Stress concentration factor K_t for various dimensions</th></tr>
<tr>
<td>1. Two U-notches in a member of rectangular section
 </td>
<td>1a. Elastic stress, axial tension</td>
<td>

$$K_t = C_1 + C_2\left(\frac{2h}{D}\right) + C_3\left(\frac{2h}{D}\right)^2 + C_4\left(\frac{2h}{D}\right)^3$$

where

<table>
<tr><th></th><th>$0.1 \leqslant h/r \leqslant 2.0$</th><th>$2.0 \leqslant h/r \leqslant 50.0$</th></tr>
<tr><td>C_1</td><td>$0.850 + 2.628\sqrt{h/r} - 0.413h/r$</td><td>$0.833 + 2.069\sqrt{h/r} - 0.009h/r$</td></tr>
<tr><td>C_2</td><td>$-1.119 - 4.826\sqrt{h/r} + 2.575h/r$</td><td>$2.732 - 4.157\sqrt{h/r} + 0.176h/r$</td></tr>
<tr><td>C_3</td><td>$3.563 - 0.514\sqrt{h/r} - 2.402h/r$</td><td>$-8.859 + 5.327\sqrt{h/r} - 0.320h/r$</td></tr>
<tr><td>C_4</td><td>$-2.294 + 2.713\sqrt{h/r} + 0.240h/r$</td><td>$6.294 - 3.239\sqrt{h/r} + 0.154h/r$</td></tr>
</table>

(Refs. 1–10)

For the semicircular notch ($h/r = 1$)

$$K_t = 3.065 - 3.370\left(\frac{2h}{D}\right) + 0.647\left(\frac{2h}{D}\right)^2 + 0.658\left(\frac{2h}{D}\right)^3$$

</td>
</tr>
</table>

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

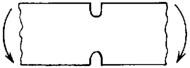
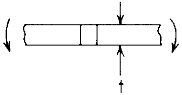
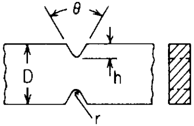
Type of form irregularity or stress raiser	Stress condition and manner of loading	Stress concentration factor K_t for various dimensions
	1b. Elastic stress, in-plane bending 	$K_t = C_1 + C_2\left(\frac{2h}{D}\right) + C_3\left(\frac{2h}{D}\right)^2 + C_4\left(\frac{2h}{D}\right)^3$ where $\begin{array}{c\|c\|c} & 0.25 \leqslant h/r \leqslant 2.0 & 2.0 \leqslant h/r \leqslant 50.0 \\ \hline C_1 & 0.723 + 2.845\sqrt{h/r} - 0.504h/r & 0.833 + 2.069\sqrt{h/r} - 0.009h/r \\ C_2 & -1.836 - 5.746\sqrt{h/r} + 1.314h/r & 0.024 - 5.383\sqrt{h/r} + 0.126h/r \\ C_3 & 7.254 - 1.885\sqrt{h/r} + 1.646h/r & -0.856 + 6.460\sqrt{h/r} - 0.199h/r \\ C_4 & -5.140 + 4.785\sqrt{h/r} - 2.456h/r & 0.999 - 3.146\sqrt{h/r} + 0.082h/r \end{array}$ (Refs. 1, 3, 8, 11, and 12) For the semicircular notch ($h/r = 1$) $K_t = 3.065 - 6.269\left(\frac{2h}{D}\right) + 7.015\left(\frac{2h}{D}\right)^2 - 2.812\left(\frac{2h}{D}\right)^3$
	1c. Elastic stress, out-of-plane bending  	$K_t = C_1 + C_2\left(\frac{2h}{D}\right) + C_3\left(\frac{2h}{D}\right)^2 + C_4\left(\frac{2h}{D}\right)^3$ where for $0.25 \leqslant h/r \leqslant 4.0$ and h/t is large $C_1 = 1.031 + 0.831\sqrt{h/r} + 0.014h/r$ $C_2 = -1.227 - 1.646\sqrt{h/r} + 0.117h/r$ $C_3 = 3.337 - 0.750\sqrt{h/r} + 0.469h/r$ $C_4 = -2.141 + 1.566\sqrt{h/r} - 0.600h/r$ (Refs. 1, 9, 13, and 14) For the semicircular notch ($h/r = 1$) $K_t = 1.876 - 2.756\left(\frac{2h}{D}\right) + 3.056\left(\frac{2h}{D}\right)^2 - 1.175\left(\frac{2h}{D}\right)^3$
2. Two V-notches in a member of rectangular section  	2a. Elastic stress, axial tension	The stress concentration factor for the V-notch, K_θ, is the smaller of the values $K_{t\theta} = K_{tu}$ or $K_{t\theta} = 1.11K_{tu} - \left[0.0275 + 0.000145\theta + 0.0164\left(\frac{\theta}{120}\right)^8\right]K_{tu}^2 \quad \text{for } \frac{2h}{D} = 0.40 \text{ and } \theta \leqslant 120^\circ$ or $K_{t\theta} = 1.11K_{tu} - \left[0.0275 + 0.00042\theta + 0.0075\left(\frac{\theta}{120}\right)^8\right]K_{tu}^2 \quad \text{for } \frac{2h}{D} = 0.667 \text{ and } \theta \leqslant 120^\circ$ where K_{tu} is the stress concentration factor for a U-notch, case 1a, when the dimensions h, r, and D are the same as for the V-notch and θ is the notch angle in degrees. (Refs. 1 and 15)

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

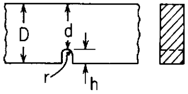
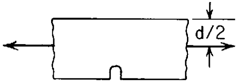
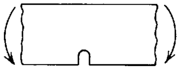
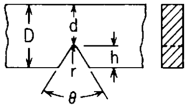
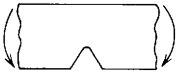
3. One U-notch in a member of rectangular section  	3a. Elastic stress, axial tension  	$K_t = C_1 + C_2\left(\frac{h}{D}\right) + C_3\left(\frac{h}{D}\right)^2 + C_4\left(\frac{h}{D}\right)^3$ where for $0.5 \leqslant h/r \leqslant 4.0$ $C_1 = 0.721 + 2.394\sqrt{h/r} - 0.127h/r$ $C_2 = 1.978 - 11.489\sqrt{h/r} + 2.211h/r$ $C_3 = -4.413 + 18.751\sqrt{h/r} - 4.596h/r$ $C_4 = 2.714 - 9.655\sqrt{h/r} + 2.512h/r$ (Refs. 16 and 17) For the semicircular notch ($h/r = 1$) $K_t = 2.988 - 7.300\left(\frac{h}{D}\right) + 9.742\left(\frac{h}{D}\right)^2 - 4.429\left(\frac{h}{D}\right)^3$
	3b. Elastic stress, in-plane bending 	$K_t = C_1 + C_2\left(\frac{h}{D}\right) + C_3\left(\frac{h}{D}\right)^2 + C_4\left(\frac{h}{D}\right)^3$ where for $0.5 \leqslant h/r \leqslant 4.0$ $C_1 = 0.721 + 2.394\sqrt{h/r} - 0.127h/r$ $C_2 = -0.426 - 8.827\sqrt{h/r} + 1.518h/r$ $C_3 = 2.161 + 10.968\sqrt{h/r} - 2.455h/r$ $C_4 = -1.456 - 4.535\sqrt{h/r} + 1.064h/r$ (Refs. 17 and 18) For the semicircular notch ($h/r = 1$) $K_t = 2.988 - 7.735\left(\frac{h}{D}\right) + 10.674\left(\frac{h}{D}\right)^2 - 4.927\left(\frac{h}{D}\right)^3$
4. One V-notch in a member of rectangular section  	4b. Elastic stress, in-plane bending 	The stress concentration factor for the V-notch, $K_{t\theta}$, is the smaller of the values $K_{t\theta} = K_{tu}$ or $K_{t\theta} = 1.11K_{tu} - \left[0.0275 + 0.1125\left(\frac{\theta}{150}\right)^4\right]K_{tu}^2 \quad \text{for } \theta \leqslant 150^\circ$ where K_{tu} is the stress concentration factor for a U-notch, case 3b, when the dimensions h, r and D are the same as for the V-notch and θ is the notch angle in degrees. (Ref. 18)

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

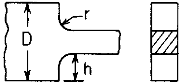
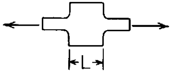
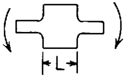
<table>
<tr><th>Type of form irregularity or stress raiser</th><th>Stress condition and manner of loading</th><th>Stress concentration factor K_t for various dimensions</th></tr>
<tr>
<td>5. Square shoulder with fillet in a member of rectangular section
 </td>
<td>5a. Elastic stress, axial tension
 </td>
<td>
$$K_t = C_1 + C_2\left(\frac{2h}{D}\right) + C_3\left(\frac{2h}{D}\right)^2 + C_4\left(\frac{2h}{D}\right)^3$$
where $\frac{L}{D} > \frac{3}{[r/(D-2h)]^{1/4}}$ and where
<table>
<tr><th></th><th>$0.1 \leqslant h/r \leqslant 2.0$</th><th>$2.0 \leqslant h/r \leqslant 20.0$</th></tr>
<tr><td>C_1</td><td>$1.007 + 1.000\sqrt{h/r} - 0.031h/r$</td><td>$1.042 + 0.982\sqrt{h/r} - 0.036h/r$</td></tr>
<tr><td>C_2</td><td>$-0.114 - 0.585\sqrt{h/r} + 0.314h/r$</td><td>$-0.074 - 0.156\sqrt{h/r} - 0.010h/r$</td></tr>
<tr><td>C_3</td><td>$0.241 - 0.992\sqrt{h/r} - 0.271h/r$</td><td>$-3.418 + 1.220\sqrt{h/r} - 0.005h/r$</td></tr>
<tr><td>C_4</td><td>$-0.134 + 0.577\sqrt{h/r} - 0.012h/r$</td><td>$3.450 - 2.046\sqrt{h/r} + 0.051h/r$</td></tr>
</table>
(Refs. 1, 8, 11, and 19)
For cases where $\frac{L}{D} < \frac{3}{[r/(D-2h)]^{1/4}}$ see Refs. 1, 21, and 22.
</td>
</tr>
<tr>
<td></td>
<td>5b. Elastic stress, in-plane bending
 </td>
<td>
$$K_t = C_1 + C_2\left(\frac{2h}{D}\right) + C_3\left(\frac{2h}{D}\right)^2 + C_4\left(\frac{2h}{D}\right)^3$$
where $\frac{L}{D} > \frac{0.8}{[r/(D-2h)]^{1/4}}$ and where
<table>
<tr><th></th><th>$0.1 \leqslant h/r \leqslant 2.0$</th><th>$2.0 \leqslant h/r \leqslant 20.0$</th></tr>
<tr><td>C_1</td><td>$1.007 + 1.000\sqrt{h/r} - 0.031h/r$</td><td>$1.042 + 0.982\sqrt{h/r} - 0.036h/r$</td></tr>
<tr><td>C_2</td><td>$-0.270 - 2.404\sqrt{h/r} + 0.749h/r$</td><td>$-3.599 + 1.619\sqrt{h/r} - 0.431h/r$</td></tr>
<tr><td>C_3</td><td>$0.677 + 1.133\sqrt{h/r} - 0.904h/r$</td><td>$6.084 - 5.607\sqrt{h/r} + 1.158h/r$</td></tr>
<tr><td>C_4</td><td>$-0.414 + 0.271\sqrt{h/r} + 0.186h/r$</td><td>$-2.527 + 3.006\sqrt{h/r} - 0.691h/r$</td></tr>
</table>
(Refs. 1, 11, and 20)
For cases where $\frac{L}{D} < \frac{0.8}{[r/(D-2h)]^{1/4}}$ see Refs. 1 and 20.
</td>
</tr>
</table>

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

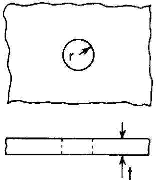
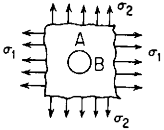
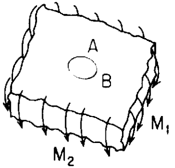
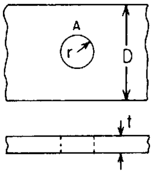
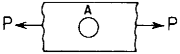
6. Circular hole in an infinite plate  	6a. Elastic stress, in-plane normal stress  	(a1) Uniaxial stress, $\sigma_2 = 0$ $\sigma_A = 3\sigma_1 \qquad \sigma_B = -\sigma_1$ (a2) Biaxial stress, $\sigma_2 = \sigma_1$ $\sigma_A = \sigma_B = 2\sigma_1$ (a3) Biaxial stress, $\sigma_2 = -\sigma_1$ (pure shear) $\sigma_A = -\sigma_B = 4\sigma_1$
	6b. Elastic stress, out-of-plane bending  Note: M_1 and M_2 are unit moments. See Table 24.	(b1) Simple bending, $M_2 = 0$ $\sigma_A = K_t \frac{6M_1}{t^2}$ where $K_t = 1.79 + \frac{0.25}{0.39 + (2r/t)} + \frac{0.81}{1 + (2r/t)^2} - \frac{0.26}{1 + (2r/t)^3}$ (b2) Cylindrical bending, $M_2 = \nu M_1$ $\sigma_A = K_t \frac{6M_1}{t^2}$ where $K_t = 1.85 + \frac{0.509}{0.70 + (2r/t)} - \frac{0.214}{1 + (2r/t)^2} + \frac{0.335}{1 + (2r/t)^3}$ for $\nu = 0.3$ (b3) Isotropic bending, $M_2 = M_1$ $\sigma_A = K_t \frac{6M_1}{t^2}$ where $K_t = 2$ (independent of r/t) (Refs. 1 and 26–30)
7. Central circular hole in a member of rectangular cross section  	7a. Elastic stress, axial tension  	$\sigma_{\max} = \sigma_A = K_t \sigma_{\text{nom}}$ where $\sigma_{\text{nom}} = \frac{P}{t(D - 2r)}$ $K_t = 3.00 - 3.13\left(\frac{2r}{D}\right) + 3.66\left(\frac{2r}{D}\right)^2 - 1.53\left(\frac{2r}{D}\right)^3$ (Refs. 5 and 25)

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

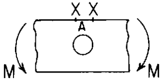
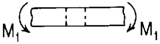
Type of form irregularity or stress raiser	Stress condition and manner of loading	Stress concentration factor K_t for various dimensions
	7b. Elastic stress, in-plane bending  	The maximum stress at the edge of the hole is $\sigma_A = K_t \sigma_{\text{nom}}$ where $\sigma_{\text{nom}} = \dfrac{12Mr}{t[D^3 - (2r)^3]}$ (at the edge of the hole) $K_t = 2$ (independent of r/D) The maximum stress at the edge of the plate is not directly above the hole but is found a short distance away in either side, points X. $\sigma_X = \sigma_{\text{nom}}$ where $\sigma_{\text{nom}} = \dfrac{6MD}{t[D^3 - (2r)^3]}$ (at the edge of the plate) (Refs. 1, 25, and 31)
	7c. Elastic stress, out-of-plane bending  *Note:* see case 6b for interpretation of M_2.	(c1) Simple bending, $M_2 = 0$ $\sigma_{\max} = \sigma_A = K_t \dfrac{6M_1}{t^2(D - 2r)}$ where $K_t = \left[1.79 + \dfrac{0.25}{0.39 + (2r/t)} + \dfrac{0.81}{1 + (2r/t)^2} - \dfrac{0.26}{1 + (2r/t)^3}\right]\left[1 - 1.04\left(\dfrac{2r}{D}\right) + 1.22\left(\dfrac{2r}{D}\right)^2\right]$ for $\dfrac{2r}{D} < 0.3$ (c2) Cylindrical bending (plate action), $M_2 = \nu M_1$ $\sigma_{\max} = \sigma_A = K_t \dfrac{6M_1}{t^2(D - 2r)}$ where $K_t = \left[1.85 + \dfrac{0.509}{0.70 + (2r/t)} - \dfrac{0.214}{1 + (2r/t)^2} + \dfrac{0.335}{1 + (2r/t)^3}\right]$ $\times \left[1 - 1.04\left(\dfrac{2r}{D}\right) + 1.22\left(\dfrac{2r}{D}\right)^2\right]$ for $\dfrac{2r}{D} < 0.3$ and $\nu = 0.3$ (Refs. 1 and 27 to 29)

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

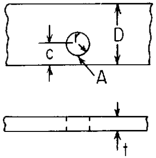
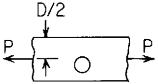
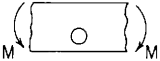
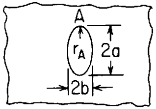
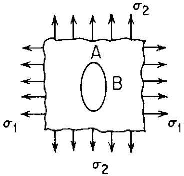
8. Off-center circular hole in a member of rectangular cross section  	8a. Elastic stress, axial tension  	$\sigma_{\max} = \sigma_A = K_t \sigma_{\text{nom}}$ where $\sigma_{\text{nom}} = \dfrac{P}{Dt}\dfrac{\sqrt{1-(r/c)^2}}{1-(r/c)}\dfrac{1-(c/D)}{1-(c/D)[2-\sqrt{1-(r/c)^2}]}$ $K_t = 3.00 - 3.13\left(\dfrac{r}{c}\right) + 3.66\left(\dfrac{r}{c}\right)^2 - 1.53\left(\dfrac{r}{c}\right)^3$ (Refs. 1 and 32)
	8b. Elastic stress, in-plane bending  	$\sigma_{\max} = \sigma_A = K_t \sigma_{\text{nom}}$ where $\sigma_{\text{nom}} = \dfrac{12M}{tD^3}\left(\dfrac{D}{2} - c + r\right)$ and $K_t = 3.0$ if $r/c < 0.05$ or $K_t = C_1 + C_2\left(\dfrac{2c}{D}\right) + C_3\left(\dfrac{2c}{D}\right)^2 + C_4\left(\dfrac{2c}{D}\right)^3$ where for $0.05 \leqslant r/c \leqslant 0.5$ $C_1 = 3.022 - 0.422r/c + 3.556(r/c)^2$ $C_2 = -0.569 + 2.664r/c - 4.397(r/c)^2$ $C_3 = 3.138 - 18.367r/c + 28.093(r/c)^2$ $C_4 = -3.591 + 16.125r/c - 27.252(r/c)^2$ (Ref. 31)
9. Elliptical hole in an infinite plate  $r_A = \dfrac{b^2}{a}$	9a. Elastic stress, in-plane normal stress  	(a1) Uniaxial stress, $\sigma_2 = 0$ $\sigma_A = \left(1 + \dfrac{2a}{b}\right)\sigma_1$ or $\sigma_A = \left(1 + 2\sqrt{\dfrac{a}{r_A}}\right)\sigma_1$ $\sigma_B = -\sigma_1$ (a2) Biaxial stress, $\sigma_2 = \sigma_1$ $\sigma_A = 2\dfrac{a}{b}\sigma_1$ $\sigma_B = 2\dfrac{b}{a}\sigma_1$ (a3) Biaxial stress, $\sigma_2 = -\sigma_1$ $\sigma_A = 2\left(1 + \dfrac{a}{b}\right)\sigma_1$ $\sigma_B = -2\left(1 + \dfrac{b}{a}\right)\sigma_1$ This stress condition would also be created by pure shear inclined at 45° to the axes of the ellipse.

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

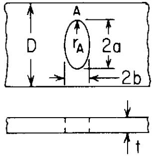
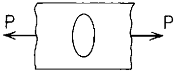
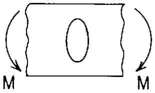
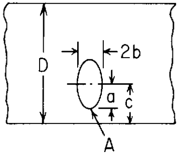
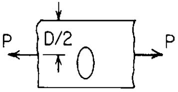
Type of form irregularity or stress raiser	Stress condition and manner of loading	Stress concentration factor K_t for various dimensions
10. Central elliptical hole in a member of rectangular cross section  $r_A = \dfrac{b^2}{a}$	10a. Elastic stress, axial tension  	$\sigma_{\max} = \sigma_A = K_t \sigma_{\text{nom}}$ where $\sigma_{\text{nom}} = \dfrac{P}{t(D-2a)}$ $K_t = C_1 + C_2\left(\dfrac{2a}{D}\right) + C_3\left(\dfrac{2a}{D}\right)^2 + C_4\left(\dfrac{2a}{D}\right)^3$ where for $0.5 \leqslant a/b \leqslant 10.0$ $C_1 = \quad 1.000 + 0.000\sqrt{a/b} + 2.000a/b$ $C_2 = -0.351 - 0.021\sqrt{a/b} - 2.483a/b$ $C_3 = \quad 3.621 - 5.183\sqrt{a/b} + 4.494a/b$ $C_4 = -2.270 + 5.204\sqrt{a/b} - 4.011a/b$ (Refs. 33–37)
	10b. Elastic stress, in-plane bending  	The maximum stress at the edge of the hole is $\sigma_A = K_t \sigma_{\text{nom}}$ where $\sigma_{\text{nom}} = \dfrac{12Ma}{t[D^3 - (2a)^3]}$ (at the edge of the hole) $K_t = C_1 + C_2\left(\dfrac{2a}{D}\right) + C_3\left(\dfrac{2a}{D}\right)^2$ where for $1.0 \leqslant a/b \leqslant 2.0$ and $0.4 \leqslant 2a/D \leqslant 1.0$ $C_1 = \quad 3.465 - 3.739\sqrt{a/b} + 2.274a/b$ $C_2 = -3.841 + 5.582\sqrt{a/b} - 1.741a/b$ $C_3 = \quad 2.376 - 1.843\sqrt{a/b} - 0.534a/b$ (Refs. 1, 36, and 37)
11. Off-center elliptical hole in a member of rectangular cross section  	11a. Elastic stress, axial tension  	$\sigma_{\max} = \sigma_A = K_t \sigma_{\text{nom}}$ The expression for σ_{nom} from case 8a can be used by substituting a/c for r/c. Use the expression for K_t from case 10a by substituting a/c for $2a/D$.

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

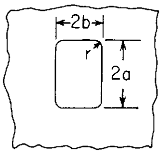
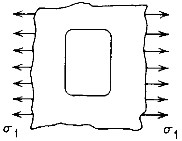
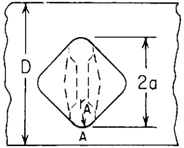
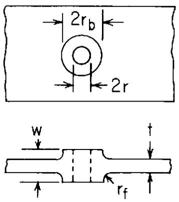
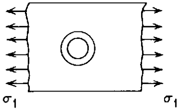
12. Rectangular hole with round corners in an infinite plate  	12a. Elastic stress, axial tension  	$\sigma_{\max} = K_t \sigma_1$ and $K_t = C_1 + C_2\left(\frac{b}{a}\right) + C_3\left(\frac{b}{a}\right)^2 + C_4\left(\frac{b}{a}\right)^3$ where for $\leqslant 0.2 \leqslant r/b \leqslant 1.0$ and $0.3 \leqslant b/a \leqslant 1.0$ $C_1 = 14.815 - 15.774\sqrt{r/b} + 8.149r/b$ $C_2 = -11.201 - 9.750\sqrt{r/b} + 9.600r/b$ $C_3 = 0.202 + 38.622\sqrt{r/b} - 27.374r/b$ $C_4 = 3.232 - 23.002\sqrt{r/b} + 15.482r/b$ (Refs. 38–40)
13. Lateral slot with circular ends in a member of rectangular section  The equivalent ellipse has a width $2b_{\text{eq}}$ where $b_{\text{eq}} = \sqrt{r_A a}$	13a. Elastic stress, axial tension	A very close approximation to the maximum stress σ_A can be obtained by using the maximum stress for the given loading with the actual slot replaced by an ellipse having the same overall dimension normal to the loading direction, $2a$, and the same end radius r_A. See cases 9a, 10a, and 11a.
	13b. Elastic stress, in-plane bending	As above, but see case 10b.
14. Reinforced circular hole in a wide plate  	14a. Elastic stress, axial tension  	$\sigma_{\max} = K_t \sigma_1$ where for $r_f \geqslant 0.6t$ and $w \geqslant 3t$ $K_t = 1.0 + \frac{1.66}{1+A} - \frac{2.182}{(1+A)^2} + \frac{2.521}{(1+A)^3}$ A is the ratio of the transverse area of the added reinforcement to the transverse area of the hole: $A = \frac{(r_b - r)(w - t) + 0.429r_f^2}{rt}$ (Ref. 41)

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

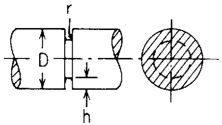
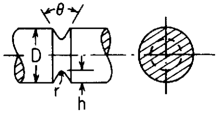
Type of form irregularity or stress raiser	Stress condition and manner of loading	Stress concentration factor K_t for various dimensions
15. U-notch in a circular shaft  	15a. Elastic stress, axial tension	$\sigma_{\max} = K_t \dfrac{4P}{\pi(D-2h)^2}$ where $K_t = C_1 + C_2\left(\dfrac{2h}{D}\right) + C_3\left(\dfrac{2h}{D}\right)^2 + C_4\left(\dfrac{2h}{D}\right)^3$ where $\begin{array}{l\|c\|c} & 0.25 \leqslant h/r \leqslant 2.0 & 2.0 \leqslant h/r \leqslant 50.0 \\ \hline C_1 & 0.455 + 3.354\sqrt{h/r} - 0.769h/r & 0.935 + 1.922\sqrt{h/r} + 0.004h/r \\ C_2 & 3.129 - 15.955\sqrt{h/r} + 7.404h/r & 0.537 - 3.708\sqrt{h/r} + 0.040h/r \\ C_3 & -6.909 + 29.286\sqrt{h/r} - 16.104h/r & -2.538 + 3.438\sqrt{h/r} - 0.012h/r \\ C_4 & 4.325 - 16.685\sqrt{h/r} + 9.469h/r & 2.066 - 1.652\sqrt{h/r} - 0.031h/r \end{array}$ (Refs. 1, 9, and 42) For the semicircular notch ($h/r = 1$) $K_t = 3.04 - 5.42\left(\dfrac{2h}{D}\right) + 6.27\left(\dfrac{2h}{D}\right)^2 - 2.89\left(\dfrac{2h}{D}\right)^3$
	15b. Elastic stress, bending	$\sigma_{\max} = K_t \dfrac{32M}{\pi(D-2h)^3}$ where $K_t = C_1 + C_2\left(\dfrac{2h}{D}\right) + C_3\left(\dfrac{2h}{D}\right)^2 + C_4\left(\dfrac{2h}{D}\right)^3$ where $\begin{array}{l\|c\|c} & 0.25 \leqslant h/r \leqslant 2.0 & 2.0 \leqslant h/r \leqslant 50.0 \\ \hline C_1 & 0.455 + 3.354\sqrt{h/r} - 0.769h/r & 0.935 + 1.922\sqrt{h/r} + 0.004h/r \\ C_2 & 0.891 - 12.721\sqrt{h/r} + 4.593h/r & -0.552 - 5.327\sqrt{h/r} + 0.086h/r \\ C_3 & 0.286 + 15.481\sqrt{h/r} - 6.392h/r & 0.754 + 6.281\sqrt{h/r} - 0.121h/r \\ C_4 & -0.632 - 6.115\sqrt{h/r} + 2.568h/r & -0.138 - 2.876\sqrt{h/r} + 0.031h/r \end{array}$ (Refs. 1 and 9) For the semicircular notch ($h/r = 1$) $K_t = 3.04 - 7.236\left(\dfrac{2h}{D}\right) + 9.375\left(\dfrac{2h}{D}\right)^2 - 4.179\left(\dfrac{2h}{D}\right)^3$
	15c. Elastic stress, torsion	$\sigma_{\max} = K_t \dfrac{16T}{\pi(D-2h)^3}$ where $K_t = C_1 + C_2\left(\dfrac{2h}{D}\right) + C_3\left(\dfrac{2h}{D}\right)^2 + C_4\left(\dfrac{2h}{D}\right)^3$ where $\begin{array}{l\|c\|c} & 0.25 \leqslant h/r \leqslant 2.0 & 2.0 \leqslant h/r \leqslant 50.0 \\ \hline C_1 & 1.245 + 0.264\sqrt{h/r} + 0.491h/r & 1.651 + 0.614\sqrt{h/r} + 0.040h/r \\ C_2 & -3.030 + 3.269\sqrt{h/r} - 3.633h/r & -4.794 - 0.314\sqrt{h/r} - 0.217h/r \\ C_3 & 7.199 - 11.286\sqrt{h/r} + 8.318h/r & 8.457 - 0.962\sqrt{h/r} + 0.389h/r \\ C_4 & -4.414 + 7.753\sqrt{h/r} - 5.176h/r & -4.314 + 0.662\sqrt{h/r} - 0.212h/r \end{array}$ (Refs. 1, 9, and 43–46) For the semicircular notch ($h/r = 1$) $K_t = 2.000 - 3.394\left(\dfrac{2h}{D}\right) + 4.231\left(\dfrac{2h}{D}\right)^2 - 1.837\left(\dfrac{2h}{D}\right)^3$

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

16. V-notch in a circular shaft



16c. Elastic stress, torsion

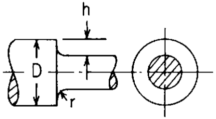
The stress concentration factor for the V-notch, $K_{t\theta}$, is the smaller of the values

$$K_{t\theta} = K_{tu} \qquad \text{or} \qquad K_{t\theta} = 1.065K_{tu} - \left[0.022 + 0.137\left(\frac{\theta}{135}\right)^2\right](K_{tu} - 1)K_{tu}$$

for $\frac{r}{D-2h} \leqslant 0.01$ and $\theta \leqslant 135^\circ$ where K_{tu} is the stress concentration factor for a U-notch, case 15c, when the dimensions h, r, and D are the same as for the V-notch and θ is the notch angle in degrees.

(Refs. 1 and 44)

17. Square shoulder with fillet in circular shaft



17a. Elastic stress, axial tension

$$\sigma_{\max} = K_t \frac{4P}{\pi(D-2h)^2} \quad \text{where } K_t = C_1 + C_2\left(\frac{2h}{D}\right) + C_3\left(\frac{2h}{D}\right)^2 + C_4\left(\frac{2h}{D}\right)^3$$

where

	$0.25 \leqslant h/r \leqslant 2.0$	$2.0 \leqslant h/r \leqslant 20.0$
C_1	$0.927 + 1.149\sqrt{h/r} - 0.086h/r$	$1.225 + 0.831\sqrt{h/r} - 0.010h/r$
C_2	$0.011 - 3.029\sqrt{h/r} + 0.948h/r$	$-1.831 - 0.318\sqrt{h/r} - 0.049h/r$
C_3	$-0.304 + 3.979\sqrt{h/r} - 1.737h/r$	$2.236 - 0.522\sqrt{h/r} + 0.176h/r$
C_4	$0.366 - 2.098\sqrt{h/r} + 0.875h/r$	$-0.630 + 0.009\sqrt{h/r} - 0.117h/r$

(Refs. 1, 19, and 47)

17b. Elastic stress, bending

$$\sigma_{\max} = K_t \frac{32M}{\pi(D-2h)^3} \quad \text{where } K_t = C_1 + C_2\left(\frac{2h}{D}\right) + C_3\left(\frac{2h}{D}\right)^2 + C_4\left(\frac{2h}{D}\right)^3$$

where

	$0.25 \leqslant h/r \leqslant 2.0$	$2.0 \leqslant h/r \leqslant 20.0$
C_1	$0.927 + 1.149\sqrt{h/r} - 0.086h/r$	$1.225 + 0.831\sqrt{h/r} - 0.010h/r$
C_2	$0.015 - 3.281\sqrt{h/r} + 0.837h/r$	$-3.790 + 0.958\sqrt{h/r} - 0.257h/r$
C_3	$0.847 + 1.716\sqrt{h/r} - 0.506h/r$	$7.374 - 4.834\sqrt{h/r} + 0.862h/r$
C_4	$-0.790 + 0.417\sqrt{h/r} - 0.246h/r$	$-3.809 + 3.046\sqrt{h/r} - 0.595h/r$

(Refs. 1, 20, and 48)

17c. Elastic stress, torsion

$$\sigma_{\max} = K_t \frac{16T}{\pi(D-2h)^3} \quad \text{where } K_t = C_1 + C_2\left(\frac{2h}{D}\right) + C_3\left(\frac{2h}{D}\right)^2 + C_4\left(\frac{2h}{D}\right)^3$$

where for $0.25 \leqslant h/r \leqslant 4.0$

$C_1 = 0.953 + 0.680\sqrt{h/r} - 0.053h/r$

$C_2 = -0.493 - 1.820\sqrt{h/r} + 0.517h/r$

$C_3 = 1.621 + 0.908\sqrt{h/r} - 0.529h/r$

$C_4 = -1.081 + 0.232\sqrt{h/r} + 0.065h/r$

(Refs. 1, 19, 46, 49, and 50)

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

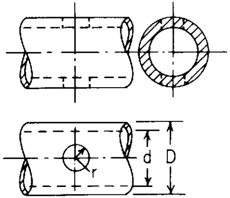
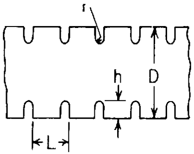
Type of form irregularity or stress raiser	Stress condition and manner of loading	Stress concentration factor K_t for various dimensions
18. Radial hole in a hollow or solid circular shaft  for a solid shaft $d = 0$	18a. Elastic stress, axial tension	$\sigma_{\max} = K_t \frac{4P}{\pi(D^2 - d^2)}$ where $K_t = C_1 + C_2\left(\frac{2r}{D}\right) + C_3\left(\frac{2r}{D}\right)^2 + C_4\left(\frac{2r}{D}\right)^3$ and where for $d/D \leqslant 0.9$ and $2r/D \leqslant 0.45$ $C_1 = 3.000$ $C_2 = 2.773 + 1.529d/D - 4.379(d/D)^2$ $C_3 = -0.421 - 12.782d/D + 22.781(d/D)^2$ $C_4 = 16.841 + 16.678d/D - 40.007(d/D)^2$ (Refs. 1, 43, 51, and 52)
	18b. Elastic stress, bending when hole is farthest from bending axis	$\sigma_{\max} = K_t \frac{32MD}{\pi(D^4 - d^4)}$ where $K_t = C_1 + C_2\left(\frac{2r}{D}\right) + C_3\left(\frac{2r}{D}\right)^2 + C_4\left(\frac{2r}{D}\right)^3$ and where for $d/D \leqslant 0.9$ and $2r/D \leqslant 0.3$ $C_1 = 3.000$ $C_2 = -6.690 - 1.620d/D + 4.432(d/D)^2$ $C_3 = 44.739 + 10.724d/D - 19.927(d/D)^2$ $C_4 = -53.307 - 25.998d/D + 43.258(d/D)^2$ (Refs. 1 and 51 to 54)
	18c. Elastic stress, torsional loading	$\sigma_{\max} = K_t \frac{16TD}{\pi(D^4 - d^4)}$ where $K_t = C_1 + C_2\left(\frac{2r}{D}\right) + C_3\left(\frac{2r}{D}\right)^2 + C_4\left(\frac{2r}{D}\right)^3$ and where for $d/D \leqslant 0.9$ and $2r/D \leqslant 0.4$ $C_1 = 4.000$ $C_2 = -6.793 + 1.133d/D - 0.126(d/D)^2$ $C_3 = 38.382 - 7.242d/D + 6.495(d/D)^2$ $C_4 = -44.576 - 7.428d/D + 58.656(d/D)^2$ (Refs. 1 and 51 to 53)
19. Multiple U-notches in a member of rectangular section  	19a. Elastic stress, axial tension, semicircular notches only, i.e., $h = r$	The stress concentration factor for the multiple semicircular U-notches, K_{tm}, is the *smaller* of the values $K_{tm} = K_{tu}$ or $K_{tm} = \left\{1.1 - \left[0.88 - 1.68\left(\frac{2r}{D}\right)\right]\frac{2r}{L} + \left[1.3\left(0.5 - \frac{2r}{D}\right)^2\right]\left(\frac{2r}{L}\right)^3\right\}K_{tu}$ for $\frac{2r}{L} < 1$ where K_{tu} is the stress concentration factor for a single pair of semicircular U-notches, case 1a. (Refs. 1, 55, and 56)

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

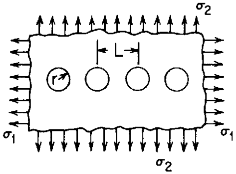
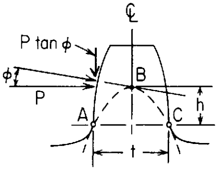
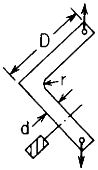
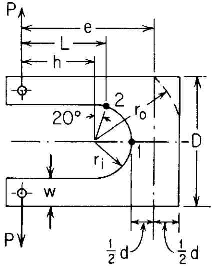
20. Infinite row of circular holes in an infinite plate  	20a. Elastic stress, axial tension parallel to the row of holes	$\sigma_{\max} = K_t\sigma_1, \qquad \sigma_2 = 0$ where $K_t = 3.0 - 1.061\left(\frac{2r}{L}\right) - 2.136\left(\frac{2r}{L}\right)^2 + 1.877\left(\frac{2r}{L}\right)^3$ (Refs. 1 and 57)
	20b. Elastic stress, axial tension normal to the row of holes	$\sigma_{\max} = K_t\sigma_2, \qquad \sigma_1 = 0$ where $K_t = 3.0 - 3.057\left(\frac{2r}{L}\right) + 0.214\left(\frac{2r}{L}\right)^2 + 0.843\left(\frac{2r}{L}\right)^3$ (Refs. 1 and 57)
21. Gear tooth  *A* and *C* are points of tangency of the inscribed parabola *ABC* with tooth profile, b = tooth width normal to plane of figure, r = minimum radius of tooth fillet.	Elastic stress, bending plus some compression	For 14.5° pressure angle: $K_t = 0.22 + \left(\frac{t}{r}\right)^{0.2}\left(\frac{t}{h}\right)^{0.4}$ For 20° pressure angle: $K_t = 0.18 + \left(\frac{t}{r}\right)^{0.15}\left(\frac{t}{h}\right)^{0.45}$ $K_t = [(\sigma_t)_{\max}$ by photoelastic analysis$]/\left[\text{calculated } (\sigma_t)_{\max} = \frac{6Ph}{bt^2} - \frac{P\tan\phi}{bt}\right]$ (Alternatively, the maximum stress can be found by the formula for a short cantilever, pages 168 and 169.) (Ref. 58)
22. Square or filleted corner in tension  	Elastic stress	$\frac{D}{d} = 5.5$ $\begin{array}{c\|ccccccccc} \frac{r}{d} & 0.125 & 0.15 & 0.20 & 0.25 & 0.30 & 0.40 & 0.50 & 0.70 & 1.00 \\ \hline K_t & 2.50 & 2.30 & 2.03 & 1.88 & 1.70 & 1.53 & 1.40 & 1.26 & 1.20 \end{array}$ (Ref. 17)

TABLE 17.1 Stress concentration factors for elastic stress (K_t) (*Continued*)

<table>
<tr><th>Type of form irregularity or stress raiser</th><th>Stress condition and manner of loading</th><th colspan="7">Stress concentration factor K_t for various dimensions</th></tr>
<tr><td rowspan="33">23. U-shaped member
 </td><td rowspan="33">Elastic stress, as shown</td><td colspan="7">K_{t1} is the ratio of actual to nominal bending stress at point 1, and K_{t2} is this ratio at point 2. Nominal bending stress $= Pey/I$ at point 1 and PLy/I at point 2, where $I/y =$ section modulus at the section in question</td></tr>
<tr><td></td><td colspan="6">Dimension ratios and values of K_t</td></tr>
<tr><td>Outer corners</td><td colspan="2">$\frac{e}{r_i} = \frac{e}{w} = \frac{e}{d}$</td><td colspan="2">$K_{t1}$</td><td colspan="2">$K_{t2}$</td></tr>
<tr><td>Square</td><td colspan="2">4.5</td><td colspan="2">1.24</td><td colspan="2">1.24</td></tr>
<tr><td></td><td colspan="2">3.5</td><td colspan="2">1.20</td><td colspan="2">1.24</td></tr>
<tr><td></td><td colspan="2">2.5</td><td colspan="2">1.30</td><td colspan="2">1.20</td></tr>
<tr><td></td><td colspan="2">1.5</td><td colspan="2">1.24</td><td colspan="2">1.61</td></tr>
<tr><td></td><td colspan="2">$\frac{e}{2r_i} = \frac{e}{2w} = \frac{e}{d}$</td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><td>Square</td><td colspan="2">2.5</td><td colspan="2">1.50</td><td colspan="2">1.29</td></tr>
<tr><td></td><td colspan="2">2.0</td><td colspan="2">1.52</td><td colspan="2">1.33</td></tr>
<tr><td></td><td colspan="2">1.5</td><td colspan="2">1.53</td><td colspan="2">1.22</td></tr>
<tr><td></td><td colspan="2">1.0</td><td colspan="2">1.46</td><td colspan="2">1.75</td></tr>
<tr><td></td><td colspan="2">$\frac{d}{r_i} = \frac{d}{w}$</td><td colspan="2">$h = \frac{3}{4}D$</td><td colspan="2">$h = \frac{1}{4}D$</td></tr>
<tr><td></td><td colspan="2"></td><td>K_{t1}</td><td>K_{t2}</td><td>K_{t1}</td><td>K_{t2}</td></tr>
<tr><td>Square</td><td colspan="2">2.0</td><td>1.50</td><td>1.29</td><td>1.53</td><td>1.22</td></tr>
<tr><td></td><td colspan="2">1.5</td><td>1.34</td><td>1.10</td><td>1.37</td><td>1.40</td></tr>
<tr><td></td><td colspan="2">1.25</td><td>1.29</td><td>1.23</td><td>1.33</td><td>1.41</td></tr>
<tr><td></td><td colspan="2">1.0</td><td>1.24</td><td>1.24</td><td>1.30</td><td>1.20</td></tr>
<tr><td></td><td colspan="2">0.75</td><td>1.21</td><td>1.10</td><td>1.24</td><td>1.22</td></tr>
<tr><td></td><td colspan="2">$\frac{r_0}{r_i} = \frac{r_0}{d} = \frac{r_0}{w}$</td><td></td><td></td><td></td><td></td></tr>
<tr><td>Rounded to radius r_0</td><td colspan="2">2.75</td><td>1.24</td><td>1.24</td><td>1.30</td><td>1.20</td></tr>
<tr><td></td><td colspan="2">2.37</td><td>1.18</td><td>1.21</td><td>1.18</td><td>1.22</td></tr>
<tr><td></td><td colspan="2">2.12</td><td>1.16</td><td>1.22</td><td>1.21</td><td>1.31</td></tr>
<tr><td></td><td colspan="2">2.0</td><td>1.27</td><td>1.42</td><td>1.31</td><td>1.56</td></tr>
<tr><td></td><td colspan="2">$\frac{d}{r_i} = \frac{w}{r_i}$</td><td></td><td></td><td></td><td></td></tr>
<tr><td>Square</td><td colspan="2">7.0</td><td>2.29</td><td>1.93</td><td>2.38</td><td>2.38</td></tr>
<tr><td></td><td colspan="2">3.0</td><td>1.72</td><td>1.59</td><td>1.76</td><td>1.62</td></tr>
<tr><td></td><td colspan="2">1.67</td><td>1.49</td><td>1.37</td><td>1.41</td><td>1.52</td></tr>
<tr><td></td><td colspan="2">1.0</td><td>1.24</td><td>1.24</td><td>1.30</td><td>1.20</td></tr>
<tr><td></td><td>$\frac{d}{r_i}$</td><td>$\frac{d}{w}$</td><td></td><td></td><td></td><td></td></tr>
<tr><td>Square</td><td>5</td><td>1.67</td><td>2.33</td><td>1.73</td><td>2.32</td><td>2.00</td></tr>
<tr><td></td><td>3</td><td>1.80</td><td>1.82</td><td>1.30</td><td>1.75</td><td>1.56</td></tr>
<tr><td></td><td>2</td><td>2.0</td><td>1.50</td><td>1.29</td><td>1.53</td><td>1.22</td></tr>
</table>

(Ref. 59)

17.4 References

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Appendix A

Properties of a Plane Area

Because of their importance in connection with the analysis of bending and torsion, certain relations for the *second-area moments*, commonly referred to as *moments of inertia*, are indicated in the following paragraphs. The equations given are in reference to Fig. A.1, and the notation is as follows:

A	area of the section
X, Y	rectangular axes in the plane of the section at arbitrary point O
x, y	rectangular axes in the plane of the section parallel to X, Y, respectively with origin at the centroid, C, of the section

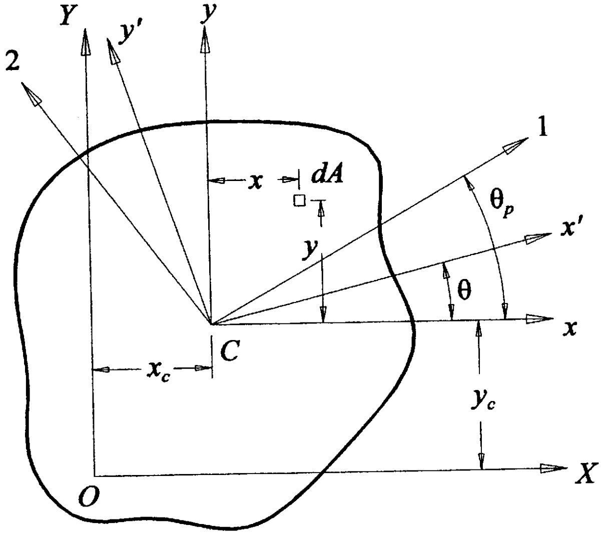


Figure A.1 Plane area.

z	polar axis through C
x', y'	rectangular axes in the plane of the section, with origin at C, inclined at a counterclockwise angle θ from x, y
1, 2	principal axes at C inclined at a counterclockwise angle θ_p from x, y
r	the distance from C to the dA element, $r = \sqrt{x^2 + y^2}$

By definition,

Moments of inertia: $I_x = \int_A y^2\, dA, \qquad I_y = \int_A x^2\, dA$

Polar moment of inertia:

$$I_z = J = \int_A r^2\, dA = I_x + I_y = I_{x'} + I_{y'} = I_1 + I_2$$

Product of inertia: $I_{xy} = \int_A xy\, dA$

Radii of gyration: $k_x = \sqrt{I_x/A}, \qquad k_y = \sqrt{I_y/A}$

Parallel axis theorem:

$$I_X = I_x + Ay_c^2, \qquad I_Y = I_y + Ax_c^2, \qquad I_{XY} = I_{xy} + Ax_c y_c$$

Transformation equations:

$$I_{x'} = I_x \cos^2\theta + I_y \sin^2\theta - I_{xy}\sin 2\theta$$
$$I_{y'} = I_x \sin^2\theta + I_y \cos^2\theta + I_{xy}\sin 2\theta$$
$$I_{x'y'} = \tfrac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy}\cos 2\theta$$

Principal moments of inertia and directions:

$$I_{1,2} = \tfrac{1}{2}\left[(I_x + I_y) \pm \sqrt{(I_y - I_x)^2 + 4I_{xy}^2}\right], \qquad I_{12} = 0,$$
$$\theta_p = \tfrac{1}{2}\tan^{-1}\left(\frac{2I_{xy}}{I_y - I_x}\right)$$

Upon the determination of the two principal moments of inertia, I_1 and I_2, two angles, 90° apart, can be solved for from the equation for θ_p. It may be obvious which angle corresponds to which principal moment of inertia. If not, one of the angles must be substituted into the equations $I_{x'}$ and $I_{y'}$ which will again yield the principal moments of inertia but also their orientation.

Note, if either one of the xy axes is an axis of symmetry, $I_{xy} = 0$, with I_x and I_y being the principal moments of inertia of the section.

If $I_1 = I_2$ for a set of principal axes through a point, it follows that the moments of inertia for all $x'y'$ axes through that point, in the same plane, are equal and $I_{x'y'} = 0$ regardless of θ. Thus the moment of inertia of a square, an equilateral triangle, or any section having two or more axes of identical symmetry is the same for any central axis.

The moment of inertia and radius of gyration of a section with respect to a centroidal axis are less than for any other axis parallel thereto.

The moment of inertia of a composite section (one regarded as made up of rectangles, triangles, circular segments, etc.) about an axis is equal to the sum of the moments of inertia of each component part about that axis. Voids are taken into account by subtracting the moment of inertia of the void area.

Expressions for the area, distances of centroids from edges, moments of inertia, and radii of gyration are given in Table A.1 for a number of representative sections. The moments of products of inertia for composite areas can be found by addition; the centroids of composite areas can be found by using the relation that the statical moment about any line of the entire area is equal to the sum of the statical moments of its component parts.

Although properties of structural sections—wide-flange beams, channels, angles, etc.—are given in structural handbooks, formulas are included in Table A.1 for similar sections. These are applicable to sections having a wider range of web and flange thicknesses than normally found in the rolled or extruded sections included in the handbooks.

Plastic or ultimate strength design is discussed in Secs. 8.15 and 8.16, and the use of this technique requires the value of the fully plastic bending moment—the product of the yield strength of a ductile material and the plastic section modulus Z. The last column in Table A.1 gives for many of the sections the value or an expression for Z and the location of the neutral axis under fully plastic pure bending. This neutral axis does not, in general, pass through the centroid, but instead divides the section into equal areas in tension and compression.

TABLE A.1 Properties of sections

NOTATION: $A =$ area (length)^2; $y =$ distance to extreme fiber (length); $I =$ moment of inertia (length^4); $r =$ radius of gyration (length); $Z =$ plastic section modulus (length^3); SF = shape factor. See Sec. 8.15 for applications of Z and SF

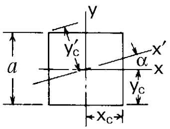
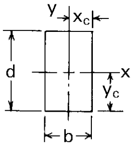
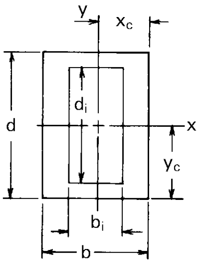
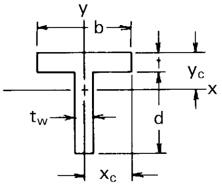
Form of section	Area and distances from centroid to extremities	Moments and products of inertia and radii of gyration about central axes	Plastic section moduli, shape factors, and locations of plastic neutral axes
1. Square 	$A = a^2$ $y_c = x_c = \frac{a}{2}$ $y'_c = 0.707a\cos\left(\frac{\pi}{4} - \alpha\right)$	$I_x = I_y = I'_x = \frac{1}{12}a^4$ $r_x = r_y = r'_x = 0.2887a$	$Z_x = Z_y = 0.25a^3$ $\text{SF}_x = \text{SF}_y = 1.5$
2. Rectangle 	$A = bd$ $y_c = \frac{d}{2}$ $x_c = \frac{b}{2}$	$I_x = \frac{1}{12}bd^3$ $I_y = \frac{1}{12}db^3$ $I_x > I_y \quad \text{if } d > b$ $r_x = 0.2887d$ $r_y = 0.2887b$	$Z_x = 0.25bd^2$ $Z_y = 0.25db^2$ $\text{SF}_x = \text{SF}_y = 1.5$
3. Hollow rectangle 	$A = bd - b_id_i$ $y_c = \frac{d}{2}$ $x_c = \frac{b}{2}$	$I_x = \frac{bd^3 - b_id_i^3}{12}$ $I_y = \frac{db^3 - d_ib_i^3}{12}$ $r_x = \left(\frac{I_x}{A}\right)^{1/2}$ $r_y = \left(\frac{I_y}{A}\right)^{1/2}$	$Z_x = \frac{bd^2 - b_id_i^2}{4}$ $\text{SF}_x = \frac{Z_xd}{2I_x}$ $Z_x = \frac{db^2 - d_ib_i^2}{4}$ $\text{SF}_y = \frac{Z_yb}{2I_y}$

TABLE A.1 Properties of sections (*Continued*)

4. Tee section

 	
	$A = tb + t_w d$
	$y_c = \frac{bt^2 + t_w d(2t+d)}{2(tb + t_w d)}$
	$x_c = \frac{b}{2}$
		$I_x = \frac{b}{3}(d+t)^3 - \frac{d^3}{3}(b - t_w) - A(d+t-y_c)^2$
		$I_y = \frac{tb^3}{12} + \frac{dt_w^3}{12}$
		$r_x = \left(\frac{I_x}{A}\right)^{1/2}$
		$r_y = \left(\frac{I_y}{A}\right)^{1/2}$
			If $t_w d \geqslant bt$, then
			$Z_x = \frac{d^2 t_w}{4} - \frac{b^2 t^2}{4t_w} + \frac{bt(d+t)}{2}$
			Neutral axis x is located a distance $(bt/t_w + d)/2$ from the bottom.
			If $t_w d \leqslant bt$, then
			$Z_x = \frac{t^2 b}{4} + \frac{t_w d(t + d - t_w d/2b)}{2}$
			Neutral axis x is located a distance $(t_w d/b + t)/2$ from the top.
			$\text{SF}_x = \frac{Z_x(d+t-y_c)}{I_1}$
			$Z_y = \frac{b^2 t + t_w^2 d}{4}$
			$\text{SF}_y = \frac{Z_y b}{2I_y}$

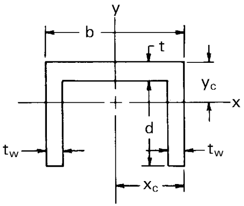
5. Channel section

 	
	$A = tb + 2t_w d$
	$y_c = \frac{bt^2 + 2t_w d(2t+d)}{2(tb + 2t_w d)}$
	$x_c = \frac{b}{2}$
		$I_x = \frac{b}{3}(d+t)^3 - \frac{d^3}{3}(b - 2t_w) - A(d+t-y_c)^2$
		$I_y = \frac{(d+t)b^3}{12} - \frac{d(b-2t_w)^3}{12}$
		$r_x = \left(\frac{I_x}{A}\right)^{1/2}$
		$r_y = \left(\frac{I_y}{A}\right)^{1/2}$
			If $2t_w d \geqslant bt$, then
			$Z_x = \frac{d^2 t_w}{2} - \frac{b^2 t^2}{8t_w} + \frac{bt(d+t)}{2}$
			Neutral axis x is located a distance $(bt/2t_w + d)/2$ from the bottom.
			If $2t_w d \leqslant bt$, then
			$Z_x = \frac{t^2 b}{4} + t_w d\left(t + d - \frac{t_w d}{b}\right)$
			Neutral axis x is located a distance $t_w d/b + t/2$ from the top.
			$\text{SF}_x = \frac{Z_x(d+t-y_c)}{I_x}$
			$Z_y = \frac{b^2 t}{4} + t_w d(b - t_w)$
			$\text{SF}_y = \frac{Z_y b}{2I_y}$

TABLE A.1 Properties of sections (*Continued*)

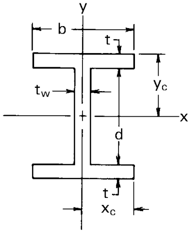
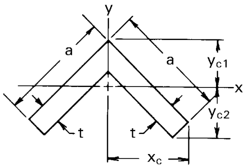
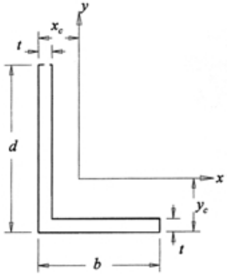
Form of section	Area and distances from centroid to extremities	Moments and products of inertia and radii of gyration about central axes	Plastic section moduli, shape factors, and locations of plastic neutral axes
6. Wide-flange beam with equal flanges  	$A = 2bt + t_w d$ $y_c = \frac{d}{2} + t$ $x_c = \frac{b}{2}$	$I_x = \frac{b(d+2t)^3}{12} - \frac{(b-t_w)d^3}{12}$ $I_y = \frac{b^3 t}{6} + \frac{t_w^3 d}{12}$ $r_x = \left(\frac{I_x}{A}\right)^{1/2}$ $r_y = \left(\frac{I_y}{A}\right)^{1/2}$	$Z_x = \frac{t_w d^2}{4} + bt(d+t)$ $\mathrm{SF}_x = \frac{Z_x y_c}{I_x}$ $Z_y = \frac{b^2 t}{2} + \frac{t_w^2 d}{4}$ $\mathrm{SF}_y = \frac{Z_y x_c}{I_y}$
7. Equal-legged angle  	$A = t(2a - t)$ $y_{c1} = \frac{0.7071(a^2 + at - t^2)}{2a - t}$ $y_{c2} = \frac{0.7071a^2}{2a - t}$ $x_c = 0.7071a$	$I_x = \frac{a^4 - b^4}{12} - \frac{0.5ta^2b^2}{a+b}$ $I_y = \frac{a^4 - b^4}{12}$ where $b = a - t$ $r_x = \left(\frac{I_x}{A}\right)^{1/2}$ $r_y = \left(\frac{I_y}{A}\right)^{1/2}$	Let y_p be the vertical distance from the top corner to the plastic neutral axis. If $t/a \geqslant 0.40$, then $y_p = a\left[\frac{t}{a} - \frac{(t/a)^2}{2}\right]^{1/2}$ $Z_x = A(y_{c1} - 0.6667y_p)$ If $t/a \leqslant 0.4$, then $y_p = 0.3536(a + 1.5t)$ $Z_x = Ay_{c1} - 2.8284y_p^2 t + 1.8856t^3$
8. Unequal-legged angle  	$A = t(b + d - t)$ $x_c = \frac{b^2 + dt - t^2}{2(b + d - t)}$ $y_c = \frac{d^2 + bt - t^2}{2(b + d - t)}$	$I_x = \frac{1}{3}[bd^3 - (b-t)(d-t)^3] - A(d - y_c)^2$ $I_y = \frac{1}{3}[db^3 - (d-t)(b-t)^3] - A(b - x_c)^2$ $I_{xy} = \frac{1}{4}[b^2d^2 - (b-t)^2(d-t)^2] - A(b - x_c)(d - y_c)$ $r_x = \left(\frac{I_x}{A}\right)^{1/2}$ $r_y = \left(\frac{I_y}{A}\right)^{1/2}$	

TABLE A.1 Properties of sections (*Continued*)

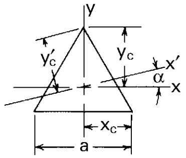
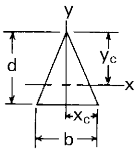
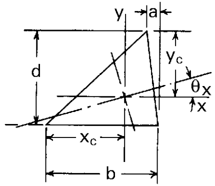
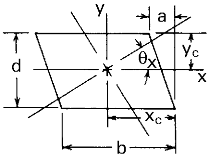
9. Equilateral triangle  	$A = 0.4330a^2$ $y_c = 0.5774a$ $x_c = 0.5000a$ $y'_c = 0.5774a\cos\alpha$	$I_x = I_y = I_{x'} = 0.01804a^4$ $r_x = r_y = r_{x'} = 0.2041a$	$Z_x = 0.0732a^3$, $Z_y = 0.0722a^3$ $\mathrm{SF}_x = 2.343$, $\mathrm{SF}_y = 2.000$ Neutral axis x is $0.2537a$ from the base.
10. Isosceles triangle  	$A = \dfrac{bd}{2}$ $y_c = \frac{2}{3}d$ $x_c = \dfrac{b}{2}$	$I_x = \frac{1}{36}bd^3$ $I_y = \frac{1}{48}db^3$ $I_x > I_y \quad \text{if } d > 0.866b$ $r_x = 0.2357d$ $r_y = 0.2041b$	$Z_x = 0.097bd^2$, $Z_y = 0.0833db^2$ $\mathrm{SF}_x = 2.343$, $\mathrm{SF}_y = 2.000$ Neutral axis x is $0.2929d$ from the base.
11. Triangle  	$A = \dfrac{bd}{2}$ $y_c = \frac{2}{3}d$ $x_c = \frac{2}{3}b - \frac{1}{3}a$	$I_x = \frac{1}{36}bd^3$ $I_y = \frac{1}{36}bd(b^2 - ab + a^2)$ $I_{xy} = \frac{1}{72}bd^2(b - 2a)$ $\theta_x = \dfrac{1}{2}\tan^{-1}\dfrac{d(b-2a)}{b^2 - ab + a^2 - d^2}$ $r_x = 0.2357d$ $r_y = 0.2357\sqrt{b^2 - ab + a^2}$	
12. Parallelogram  	$A = bd$ $y_c = \dfrac{d}{2}$ $x_c = \frac{1}{2}(b + a)$	$I_x = \frac{1}{12}bd^3$ $I_y = \frac{1}{12}bd(b^2 + a^2)$ $I_{xy} = -\frac{1}{12}abd^2$ $\theta_x = \dfrac{1}{2}\tan^{-1}\dfrac{-2ad}{b^2 + a^2 - d^2}$ $r_x = 0.2887d$ $r_y = 0.2887\sqrt{b^2 + a^2}$	

TABLE A.1 Properties of sections (*Continued*)

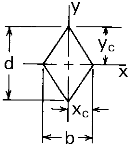
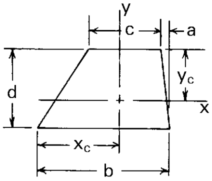
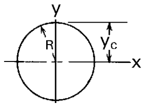
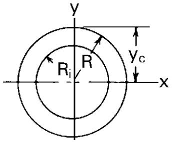
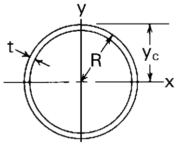
Form of section	Area and distances from centroid to extremities	Moments and products of inertia and radii of gyration about central axes	Plastic section moduli, shape factors, and locations of plastic neutral axes
13. Diamond  	$A = \frac{bd}{2}$ $y_c = \frac{d}{2}$ $x_c = \frac{b}{2}$	$I_x = \frac{1}{48}bd^3$ $I_y = \frac{1}{48}db^3$ $r_x = 0.2041d$ $r_y = 0.2041b$	$Z_x = 0.0833bd^2, \quad Z_y = 0.0833db^2$ $\mathrm{SF}_x = \mathrm{SF}_y = 2.000$
14. Trapezoid  	$A = \frac{d}{2}(b + c)$ $y_c = \frac{d}{3}\frac{2b + c}{b + c}$ $x_c = \frac{2b^2 + 2bc - ab - 2ac - c^2}{3(b + c)}$	$I_x = \frac{d^3}{36}\frac{b^2 + 4bc + c^2}{b + c}$ $I_y = \frac{d}{36(b + c)}[b^4 + c^4 + 2bc(b^2 + c^2)$ $- a(b^3 + 3b^2c - 3bc^2 - c^3)$ $+ a^2(b^2 + 4bc + c^2)]$ $I_{xy} = \frac{d^2}{72(b + c)}[c(3b^2 - 3bc - c^2)$ $+ b^3 - a(2b^2 + 8bc + 2c^2)]$	
15. Solid circle  	$A = \pi R^2$ $y_c = R$	$I_x = I_y = \frac{\pi}{4}R^4$ $r_x = r_y = \frac{R}{2}$	$Z_x = Z_y = 1.333R^3$ $\mathrm{SF}_x = 1.698$

TABLE A.1 Properties of sections (*Continued*)

16. Hollow circle  	$A = \pi(R^2 - R_i^2)$ $y_c = R$	$I_x = I_y = \frac{\pi}{4}(R^4 - R_i^4)$ $r_x = r_y = \frac{1}{2}\sqrt{R^2 + R_i^2}$	$Z_x = Z_y = 1.333(R^3 - R_i^3)$ $\text{SF}_x = 1.698\frac{R^4 - R_i^3 R}{R^4 - R_i^4}$
17. Very thin annulus  	$A = 2\pi Rt$ $y_c = R$	$I_x = I_y = \pi R^3 t$ $r_x = r_y = 0.707R$	$Z_x = Z_y = 4R^2 t$ $\text{SF}_x = \text{SF}_y = \frac{4}{\pi}$

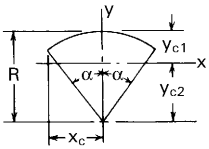
18. Sector of solid circle

 	
	$A = \alpha R^2$
	$y_{c1} = R\left(1 - \frac{2\sin\alpha}{3\alpha}\right)$
	$y_{c2} = \frac{2R\sin\alpha}{3\alpha}$
	$x_c = R\sin\alpha$
		$I_x = \frac{R^4}{4}\left(\alpha + \sin\alpha\cos\alpha - \frac{16\sin^2\alpha}{9\alpha}\right)$
		$I_y = \frac{R^4}{4}(\alpha - \sin\alpha\cos\alpha)$
		(*Note*: If α is small, $\alpha - \sin\alpha\cos\alpha = \frac{2}{3}\alpha^3 - \frac{2}{15}\alpha^5$)
		$r_x = \frac{R}{2}\sqrt{1 + \frac{\sin\alpha\cos\alpha}{\alpha} - \frac{16\sin^2\alpha}{9\alpha^2}}$
		$r_y = \frac{R}{2}\sqrt{1 - \frac{\sin\alpha\cos\alpha}{\alpha}}$
			If $\alpha \leqslant 54.3°$, then
			$Z_x = 0.6667R^3\left[\sin\alpha - \left(\frac{\alpha^3}{2\tan\alpha}\right)^{1/2}\right]$
			Neutral axis x is located a distance $R(0.5\alpha/\tan\alpha)^{1/2}$ from the vertex.
			If $\alpha \geqslant 54.3°$, then
			$Z_x = 0.6667R^3(2\sin^3\alpha_1 - \sin\alpha)$ where the expression $2\alpha_1 - \sin 2\alpha_1 = \alpha$ is solved for the value of α_1.
			Neutral axis x is located a distance $R\cos\alpha_1$ from the vertex.
			If $\alpha \leqslant 73.09°$, then $\text{SF}_x = \frac{Z_x y_{c2}}{I_x}$
			If $73.09° \leqslant \alpha \leqslant 90°$, then $\text{SF}_x = \frac{Z_x y_{c1}}{I_x}$
			$Z_y = 0.6667R^3(1 - \cos\alpha)$
			If $\alpha \leqslant 90°$, then
			$\text{SF}_y = 2.6667\sin\alpha\frac{1 - \cos\alpha}{\alpha - \sin\alpha\cos\alpha}$
			If $\alpha \geqslant 90°$, then
			$\text{SF}_y = 2.6667\frac{1 - \cos\alpha}{\alpha - \sin\alpha\cos\alpha}$

TABLE A.1 Properties of sections (*Continued*)

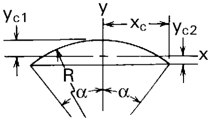
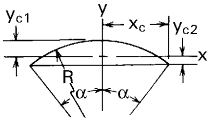
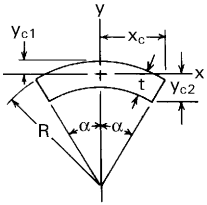
Form of section	Area and distances from centroid to extremities	Moments and products of inertia and radii of gyration about central axes	Plastic section moduli, shape factors, and locations of plastic neutral axes
19. Segment of solid circle (*Note:* If $\alpha \leqslant \pi/4$, use expressions from case 20)  	$A = R^2(\alpha - \sin\alpha\cos\alpha)$ $y_{c1} = R\left[1 - \frac{2\sin^3\alpha}{3(\alpha - \sin\alpha\cos\alpha)}\right]$ $y_{c2} = R\left[\frac{2\sin^3\alpha}{3(\alpha - \sin\alpha\cos\alpha)} - \cos\alpha\right]$ $x_c = R\sin\alpha$	$I_x = \frac{R^4}{4}\left[\alpha - \sin\alpha\cos\alpha + 2\sin^3\alpha\cos\alpha - \frac{16\sin^6\alpha}{9(\alpha - \sin\alpha\cos\alpha)}\right]$ $I_y = \frac{R^4}{12}(3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha\cos\alpha)$ $r_x = \frac{R}{2}\sqrt{1 + \frac{2\sin^3\alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha} - \frac{16\sin^6\alpha}{9(\alpha - \sin\alpha\cos\alpha)^2}}$ $r_y = \frac{R}{2}\sqrt{1 - \frac{2\sin^3\alpha\cos\alpha}{3(\alpha - \sin\alpha\cos\alpha)}}$	
20. Segment of solid circle (*Note:* Do not use if $\alpha > \pi/4$)  	$A = \frac{2}{3}R^2\alpha^3(1 - 0.2\alpha^2 + 0.019\alpha^4)$ $y_{c1} = 0.3R\alpha^2(1 - 0.0976\alpha^2 + 0.0028\alpha^4)$ $y_{c2} = 0.2R\alpha^2(1 - 0.0619\alpha^2 + 0.0027\alpha^4)$ $x_c = R\alpha(1 - 0.1667\alpha^2 + 0.0083\alpha^4)$	$I_x = 0.01143R^4\alpha^7(1 - 0.3491\alpha^2 + 0.0450\alpha^4)$ $I_y = 0.1333R^4\alpha^5(1 - 0.4762\alpha^2 + 0.1111\alpha^4)$ $r_x = 0.1309R\alpha^2(1 - 0.0745\alpha^2)$ $r_y = 0.4472R\alpha(1 - 0.1381\alpha^2 + 0.0184\alpha^4)$	
21. Sector of hollow circle  (*Note:* If t/R is small, α can exceed π to form an overlapped annulus)	$A = \alpha t(2R - t)$ $y_{c1} = R\left[1 - \frac{2\sin\alpha}{3\alpha}\left(1 - \frac{t}{R} + \frac{1}{2 - t/R}\right)\right]$ $y_{c2} = R\left[\frac{2\sin\alpha}{3\alpha(2 - t/R)} + \left(1 - \frac{t}{R}\right)\frac{2\sin\alpha - 3\alpha\cos\alpha}{3\alpha}\right]$ $x_c = R\sin\alpha$	$I_x = R^3t\left[\left(1 - \frac{3t}{2R} + \frac{t^2}{R^2} - \frac{t^3}{4R^3}\right) \times \left(\alpha + \sin\alpha\cos\alpha - \frac{2\sin^2\alpha}{\alpha}\right) + \frac{t^2\sin^2\alpha}{3R^2\alpha(2 - t/R)}\left(1 - \frac{t}{R} + \frac{t^2}{6R^2}\right)\right]$ $I_y = R^3t\left(1 - \frac{3t}{2R} + \frac{t^2}{R^2} - \frac{t^3}{4R^3}\right)(\alpha - \sin\alpha\cos\alpha)$ $r_x = \sqrt{\frac{I_x}{A}}, \qquad r_y = \sqrt{\frac{I_y}{A}}$	

TABLE A.1 Properties of sections (*Continued*)

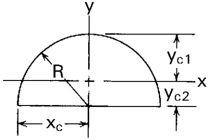
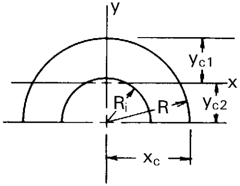
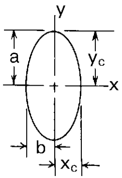
	Note: If α is small: $\frac{\sin\alpha}{\alpha}=1-\frac{\alpha^2}{6}+\frac{\alpha^4}{120}$, $\alpha-\sin\alpha\cos\alpha=\frac{2}{3}\alpha^3\left(1-\frac{\alpha^2}{5}+\frac{2\alpha^4}{105}\right)$, $\frac{\sin^2\alpha}{\alpha}=\alpha\left(1-\frac{\alpha^2}{3}+\frac{2\alpha^4}{45}\right)$ $\cos=1-\frac{\alpha^2}{2}+\frac{\alpha^4}{24}$, $\alpha+\sin\alpha\cos\alpha-\frac{2\sin^2\alpha}{\alpha}=\frac{2\alpha^5}{45}\left(1-\frac{\alpha^2}{7}+\frac{\alpha^4}{105}\right)$		
22. Solid semicircle 	$A=\frac{\pi}{2}R^2$ $y_{c1}=0.5756R$ $y_{c2}=0.4244R$ $x_c=R$	$I_x=0.1098R^4$ $I_y=\frac{\pi}{8}R^4$ $r_x=0.2643R$ $r_y=\frac{R}{2}$	$Z_x=0.3540R^3$, $Z_y=0.6667R^3$ $\text{SF}_x=1.856$, $\text{SF}_y=1.698$ Plastic neutral axis x is located a distance $0.4040R$ from the base.
23. Hollow semicircle  Note: $b=\frac{R+R_i}{2}$ $t=R-R_i$	$A=\frac{\pi}{2}(R^2-R_i^2)$ $y_{c2}=\frac{4}{3\pi}\frac{R^3-R_i^2}{R^2-R_i^2}$ or $y_{c2}=\frac{2b}{\pi}\left[1+\frac{(t/b)^2}{12}\right]$ $y_{c1}=R-y_{c2}$ $x_c=R$	$I_x=\frac{\pi}{8}(R^4-R_i^4)-\frac{8}{9\pi}\frac{(R^3-R_i^3)^2}{R^2-R_i^2}$ or $I_x=0.2976tb^3+0.1805bt^3-\frac{0.00884t^5}{b}$ $I_y=\frac{\pi}{8}(R^4-R_i^4)$ or $I_y=1.5708b^3t+0.3927bt^3$	Let y_p be the vertical distance from the bottom to the plastic neutral axis. $y_p=(0.7071-0.2716C-0.4299C^2+0.3983C^3)R$ $Z_x=(0.8284-0.9140C+0.7245C^2$ $-0.2850C^3)R^2t$ where $C=t/R$ $Z_y=0.6667(R^3-R_i^3)$
24. Solid ellipse 	$A=\pi ab$ $y_c=a$ $x_c=b$	$I_x=\frac{\pi}{4}ba^3$ $I_y=\frac{\pi}{4}ab^3$ $r_x=\frac{a}{2}$ $r_y=\frac{b}{2}$	$Z_x=1.333a^2b$, $Z_y=1.333b^2a$ $\text{SF}_x=\text{SF}_y=1.698$

TABLE A.1 Properties of sections (*Continued*)

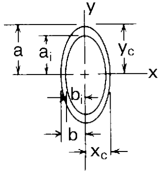
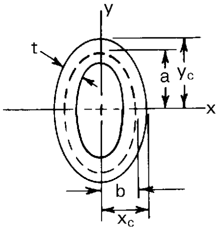
Form of section	Area and distances from centroid to extremities	Moments and products of inertia and radii of gyration about central axes	Plastic section moduli, shape factors, and locations of plastic neutral axes
25. Hollow ellipse  	$A = \pi(ab - a_i b_i)$ $y_c = a$ $x_c = b$	$I_x = \frac{\pi}{4}(ba^3 - b_i a_i^3)$ $I_y = \frac{\pi}{4}(ab^3 - a_i b_i^3)$ $r_x = \frac{1}{2}\sqrt{\frac{ba^3 - b_i a_i^3}{ab - a_i b_i}}$ $r_y = \frac{1}{2}\sqrt{\frac{ab^3 - a_i b_i^3}{ab - a_i b_i}}$	$Z_x = 1.333(a^2 b - a_i^2 b_i)$ $Z_y = 1.333(b^2 a - b_i^2 a_i)$ $\text{SF}_x = 1.698\frac{a^3 b - a_i^2 b_i a}{a^3 b - a_i^3 b_i}$ $\text{SF}_y = 1.698\frac{b^3 a - b_i^2 a_i b}{b^3 a - b_i^3 a_i}$
Note: For this case the inner and outer perimeters are both ellipses and the wall thickness is not constant. For a cross section with a constant wall thickness see case 26.			
26. Hollow ellipse with constant wall thickness t. The midthickness perimeter is an ellipse (shown dashed). $0.2 < a/b < 5$  See the note on maximum wall thickness in case 27.	$A = \pi t(a+b)\left[1 + K_1\left(\frac{a-b}{a+b}\right)^2\right]$ where $K_1 = 0.2464 + 0.002222\left(\frac{a}{b} + \frac{b}{a}\right)$ $y_c = a + \frac{t}{2}$ $x_c = b + \frac{t}{2}$	$I_x = \frac{\pi}{4}ta^2(a+3b)\left[1 + K_2\left(\frac{a-b}{a+b}\right)^2\right]$ $+\frac{\pi}{16}t^3(3a+b)\left[1 + K_3\left(\frac{a-b}{a+b}\right)^2\right]$ where $K_2 = 0.1349 + 0.1279\frac{a}{b} - 0.01284\left(\frac{a}{b}\right)^2$ $K_3 = 0.1349 + 0.1279\frac{b}{a} - 0.01284\left(\frac{b}{a}\right)^2$ For I_y interchange a and b in the expressions for I_x, K_2, and K_3	$Z_x = 1.3333ta(a+2b)\left[1 + K_4\left(\frac{a-b}{a+b}\right)^2\right] + \frac{t^3}{3}$ where $K_4 = 0.1835 + 0.895\frac{a}{b} - 0.00978\left(\frac{a}{b}\right)^2$ For Z_y interchange a and b in the expression for Z_x and K_4.

TABLE A.1 Properties of sections (*Continued*)

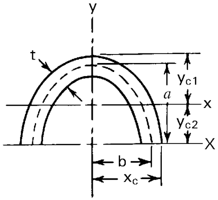
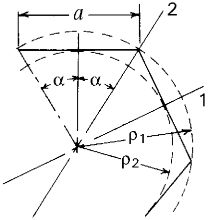
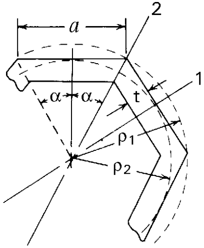
27. Hollow semiellipse with constant wall thickness t. The midthickness perimeter is an ellipse (shown dashed). $0.2 < a/b < 5$  *Note:* There is a limit on the maximum wall thickness allowed in this case. Cusps will form in the perimeter at the ends of the major axis if this maximum is exceeded. If $\frac{a}{b} \leqslant 1$, then $t_{\max} = \frac{2a^2}{b}$ If $\frac{a}{b} \geqslant 1$, then $t_{\max} = \frac{2b^2}{a}$	$A = \frac{\pi}{2}t(a+b)\left[1 + K_1\left(\frac{a-b}{a+b}\right)^2\right]$ where $K_1 = 0.2464 + 0.002222\left(\frac{a}{b} + \frac{b}{a}\right)$ $y_{c2} = \frac{2a}{\pi}K_2 + \frac{t^2}{6\pi a}K_3$ where $K_2 = 1 - 0.3314C + 0.0136C^2 + 0.1097C^3$ $K_3 = 1 + 0.9929C - 0.2287C^2 - 0.2193C^3$ Using $C = \frac{a-b}{a+b}$ $y_{c1} = a + \frac{t}{2} - y_{c2}$ $x_c = b + \frac{t}{2}$	$I_X = \frac{\pi}{8}ta^2(a+3b)\left[1 + K_4\left(\frac{a-b}{a+b}\right)^2\right]$ $+ \frac{\pi}{32}t^3(3a+b)\left[1 + K_5\left(\frac{a-b}{a+b}\right)^2\right]$ where $K_4 = 0.1349 + 0.1279\frac{a}{b} - 0.01284\left(\frac{a}{b}\right)^2$ $K_5 = 0.1349 + 0.1279\frac{b}{a} - 0.01284\left(\frac{b}{a}\right)^2$ $I_x = I_X - Ay_{c2}^2$ For I_y use one-half the value for I_y in case 26.	Let y_p be the vertical distance from the bottom to the plastic neutral axis. $y_p = \left[C_1 + \frac{C_2}{a/b} + \frac{C_3}{(a/b)^2} + \frac{C_4}{(a/b)^3}\right]a$ where if $0.25 < a/b \leqslant 1$, then $C_1 = 0.5067 - 0.5588D + 1.3820D^2$ $C_2 = 0.3731 + 0.1938D - 1.4078D^2$ $C_3 = -0.1400 + 0.0179D + 0.4885D^2$ $C_4 = 0.0170 - 0.0079D - 0.0565D^2$ or if $1 \leqslant a/b < 4$, then $C_1 = 0.4829 + 0.0725D - 0.1815D^2$ $C_2 = 0.1957 - 0.6608D + 1.4222D^2$ $C_3 = 0.0203 + 1.8999D - 3.4356D^2$ $C_4 = 0.0578 - 1.6666D + 2.6012D^2$ where $D = t/t_{\max}$ and where $0.2 < D \leqslant 1$ $Z_x = \left[C_5 + \frac{C_6}{a/b} + \frac{C_7}{(a/b)^2} + \frac{C_8}{(a/b)^3}\right]4a^2t$ where if $0.25 < a/b \leqslant 1$, then $C_5 = -0.0292 + 0.3749D^{1/2} + 0.0578D$ $C_6 = 0.3674 - 0.8531D^{1/2} + 0.3882D$ $C_7 = -0.1218 + 0.3563D^{1/2} - 0.1803D$ $C_8 = 0.0154 - 0.0448D^{1/2} + 0.0233D$ or if $1 \leqslant a/b < 4$, then $C_5 = 0.2241 - 0.3922D^{1/2} + 0.2960D$ $C_6 = -0.6637 + 2.7357D^{1/2} - 2.0482D$ $C_7 = 1.5211 - 5.3864D^{1/2} + 3.9286D$ $C_8 = -0.8498 + 2.8763D^{1/2} - 1.8874D$ For Z_y use one-half the value for Z_y in case 26.

TABLE A.1 Properties of sections *Continued*)

Form of section	Area and distances from centroid to extremities	Moments and products of inertia and radii of gyration about central axes	Plastic section moduli, shape factors, and locations of plastic neutral axes
28. Regular polygon with n sides  	$A=\dfrac{a^2 n}{4\tan\alpha}$ $\rho_1=\dfrac{a}{2\sin\alpha}$ $\rho_2=\dfrac{\alpha}{2\tan\alpha}$ If n is odd $y_1=y_2=\rho_1\cos\left[\alpha\left(\dfrac{n+1}{2}\right)-\dfrac{\pi}{2}\right]$ If $n/2$ is odd $y_1=\rho_1,\qquad y_2=\rho_2$ If $n/2$ is even $y_1=\rho_2,\qquad y_2=\rho_1$	$I_1=I_2=\frac{1}{24}A(6\rho_1^2-a^2)$ $r_1=r_2=\sqrt{\frac{1}{24}(6\rho_1^2-a^2)}$	For $n=3$, see case 9. For $n=4$, see cases 1 and 13. For $n=5$, $Z_1=Z_2=0.8825\rho_1^3$. For an axis perpendicular to axis 1, $Z=0.8838\rho_1^3$. The location of this axis is $0.7007a$ from that side which is perpendicular to axis 1. For $n\geqslant 6$, use the following expression for a neutral axis of any inclination: $Z=\rho_1^3\left[1.333-13.908\left(\dfrac{1}{n}\right)^2+12.528\left(\dfrac{1}{n}\right)^3\right]$
29. Hollow regular polygon with n sides  	$A=nat\left(1-\dfrac{t\tan\alpha}{a}\right)$ $\rho_1=\dfrac{a}{2\sin\alpha}$ $\rho_2=\dfrac{\alpha}{2\tan\alpha}$ If n is odd $y_1=y_2=\rho_1\cos\left(\alpha\dfrac{n+1}{2}-\dfrac{\pi}{2}\right)$ If $n/2$ is odd $y_1=\rho_1,\qquad y_2=\rho_2$ If $n/2$ is even $y_1=\rho_2,\qquad y_2=\rho_1$	$I_1=I_2=\dfrac{na^3t}{8}\left(\dfrac{1}{3}+\dfrac{1}{\tan^2\alpha}\right)$ $\times\left[1-3\dfrac{t\tan\alpha}{a}+4\left(\dfrac{t\tan\alpha}{a}\right)^2-2\left(\dfrac{t\tan\alpha}{a}\right)^3\right]$ $r_1=r_2=\dfrac{a}{\sqrt{8}}$ $\times\sqrt{\left(\dfrac{1}{3}\right)+\dfrac{1}{\tan^2\alpha}\left[1-2\dfrac{t\tan\alpha}{a}+2\left(\dfrac{t\tan\alpha}{a}\right)^2\right]}$	

Appendix B

Glossary: Definitions

The definitions given here apply to the terminology used throughout this book. Some of the terms may be defined differently by other authors; when this is the case, alternative terminology is noted. When two or more terms with identical or similar meaning are in general acceptance, they are given in the order of preference of the current writers.

Allowable stress (working stress): If a member is so designed that the maximum stress as calculated for the expected conditions of service is less than some limiting value, the member will have a proper margin of security against damage or failure. This limiting value is the allowable stress subject to the material and condition of service in question. The allowable stress is made less than the damaging stress because of uncertainty as to the conditions of service, nonuniformity of material, and inaccuracy of the stress analysis (see Ref. 1). The margin between the allowable stress and the damaging stress may be reduced in proportion to the certainty with which the conditions of the service are known, the intrinsic reliability of the material, the accuracy with which the stress produced by the loading can be calculated, and the degree to which failure is unattended by danger or loss. (Compare with *Damaging stress; Factor of safety; Factor of utilization; Margin of safety.* See Refs. 1–3.)

Apparent elastic limit (useful limit point): The stress at which the rate of change of strain with respect to stress is 50% greater than at zero stress. It is more definitely determinable from the stress–strain diagram than is the proportional limit, and is useful for comparing materials of the same general class. (Compare with *Elastic limit; Proportional limit; Yield point, Yield strength.*)

Apparent stress: The stress corresponding to a given unit strain on the assumption of uniaxial elastic stress. It is calculated by multiplying the unit strain by the modulus of elasticity, and may differ from the true stress because the effect of the transverse stresses is not taken into account.

Bending moment: Reference is to a simple straight beam, assumed for convenience to be horizontal and loaded and supported by forces, all of which lie in a vertical plane. The bending moment at any section of the beam is the moment of all forces that act on the beam to the left (or right) of that section, taken about the horizontal axis in the plane of the section. When considering the moment at the section due to the forces to the left of the section, the bending moment is positive when counterclockwise and negative when clockwise. The reverse is true when considering the moment due to forces to the right of the section. Thus, a positive bending moment bends the beam such that the beam deforms concave upward, and a negative bending moment bends it concave downward. The *bending moment equation* is an expression for the bending moment at any section in terms of x, the distance along the longitudinal axis of the beam to the section measured from an origin, usually taken to be the left end of the beam.

Bending moments as applied to straight beams in two-plane symmetric or unsymmetric bending, curved beams, or plates are a bit more involved, and are discussed in the appropriate sections of this book.

Bending stress (flexural stress): The tensile and compressive stress transmitted in a beam or plate that arises from the bending moment. (See *Flexure equation.*)

Boundary conditions: As used in structural analysis, the term usually refers to the condition of stress, displacement, or slope at the ends or edges of a member, where these conditions are apparent from the circumstances of the problem. For example, given a beam with fixed ends, the zero displacement and slope at each end are boundary conditions. For a plate with a freely supported edge, the zero-stress state is a boundary condition.

Brittle fracture: The tensile failure of a material with negligible plastic deformation. The material can inherently be a brittle material in its normal state such as glass, masonry, ceramic, cast iron, or high strength high-carbon steel (see Sec. 3.7); or can be a material normally considered ductile which contains imperfections exceeding specific limits, or in a low-temperature environment, or undergoing high strain rates, or any combination thereof.

Bulk modulus of elasticity: The ratio of a tensile or compressive stress, triaxial and equal in all directions (e.g., hydrostatic pressure) to the relative change it produces in volume.

Central axis (centroidal axis): A central axis of a line, area, or volume is one that passes through the *centroid*; in the case of an area, it is understood to lie in the plane of the area unless stated otherwise. When taken normal to the plane of the area, it is called the *central polar axis*.

Centroid of an area: That point in the plane of an area where the moment of the area is zero about any axis. The centroid coincides with the center of gravity in the plane of an infinitely thin homogeneous uniform plate.

Corrosion fatigue: Fatigue aggravated by corrosion, as in parts repeatedly stressed while exposed to a corrosive environment.

Creep: Continuous increase in deformation under constant or decreasing stress. The term is ordinarily used with reference to the behavior of metals under tension at elevated temperatures. The similar yielding of a material under compressive stress is called *plastic flow*, or *flow*. Creep at atmospheric temperature due to sustained elastic stress is sometimes called *drift*, or *elastic drift*. (See also *Relaxation*.)

Damaging stress: The least unit stress of a given kind and for a given material and condition of service that will render a member unfit for service before the end of its useful life. It may do this by excessive deformation, by excessive yielding or creep, or through fatigue cracking, excessive strain hardening, or rupture.

Damping capacity: The amount of energy dissipated into heat per unit of total strain energy present at maximum strain for a complete cycle. (See Ref. 4.)

Deformation: Change in the shape or dimensions of a body produced by stress. *Elongation* is often used for tensile deformation, *compression* or *shortening* for compressive deformation, and *distortion* for shear deformation. *Elastic deformation* is deformation that invariably disappears upon removal of stress, whereas *permanent deformation* is that which remains after the removal of stress. (Compare with *Set*.)

Eccentricity: A load or component of a load normal to a given cross section of a member is eccentric with respect to that section if it does not act through the centroid. The perpendicular distance from the line of action of the load to the central polar axis is the eccentricity with respect to that axis.

Elastic: Capable of sustaining stress without permanent deformation; the term is also used to denote conformity to the law of stress–

strain proportionality (Hooke's law). An elastic stress or strain is a stress or strain within the elastic limit.

Elastic axis: The elastic axis of a beam is the line, lengthwise of the beam, along which transverse loads must be applied to avoid torsion of the beam at any section. Strictly speaking, no such line exists except for a few conditions of loading. Usually the elastic axis is assumed to be the line through the elastic center of every section. The term is most often used with reference to an airplane wing of either the shell or multiple spar type. (Compare with *Torsional center*; *Flexural center*; *Elastic center*. See Ref. 5.)

Elastic center: The elastic center of a given section of a beam is that point in the plane of the section lying midway between the shear center and center of twist of that section. The three points may be identical—which is the normal assumption. (Compare with *Shear center*; *Torsional center*; *Elastic axis*. See Refs. 5 and 6.)

Elastic curve: The curve assumed by the longitudinal axis of an initially straight beam or column in bending where the stress is within the elastic limit.

Elastic instability (buckling): Unstable local or global elastic deformations caused by compressive stresses in members with large length to lateral dimensions. (See *Slenderness ratio*.)

Elastic, perfectly plastic material: A model that represents the stress–strain curve of a material as linear from zero stress and strain to the elastic limit. Beyond the elastic limit, the stress remains constant with strain.

Elastic limit: The least stress that will cause permanent set. (Compare with *Proportional limit*; *Apparent elastic limit, Yield point*; *Yield strength*. See Sec. 3.2 and Ref. 7.)

Elastic ratio: The ratio of the elastic limit to the ultimate strength.

Ellipsoid of strain: An ellipsoid that represents the state of strain at any given point in a body. It has the shape assumed under stress by a sphere centered at the point in question (Ref 8).

Ellipsoid of stress: An ellipsoid that represents the state of stress at any given point in a body; its semi-axes are vectors representing the principal stresses at the point, and any radius vector represents the resultant stress on a particular plane through the point. For a condition of plane stress, where one of the principal stresses is zero, the ellipsoid becomes the *ellipse of stress* (see Ref. 9).

Endurance limit (fatigue strength): The maximum stress amplitude of a purely reversing stress that can be applied to a material an indefinitely large number of cycles without producing fracture (see Sec. 3.8).

Endurance ratio: Ratio of the endurance limit to the ultimate static tensile strength.

Endurance strength: The maximum stress amplitude of a purely reversing stress that can be applied to a material for a specific number of cycles without producing fracture. (Compare with *Endurance limit*.)

Energy of rupture (modulus of toughness): The work done per unit volume in producing fracture. It is not practicable to establish a specific energy of rupture value for a given material, because the result obtained depends upon the form and proportions of the test specimen and the manner of loading. As determined by similar tests on similar specimens, the energy of rupture affords a criterion for comparing the toughness of different materials.

Equivalent bending moment: A bending moment that, acting alone, would produce in a circular shaft a normal (tensile or compressive) stress of the same magnitude as the maximum normal stress produced by a given bending moment and a given twisting moment acting simultaneously.

Equivalent twisting moment: A twisting moment that, acting alone, would produce in a circular shaft a shear stress of the same magnitude as the maximum shear stress produced by a given twisting moment and a given bending moment acting simultaneously.

Factor of safety: The intent of the factor of safety is to provide a safeguard to failure. The term usually refers to the ratio of the load that would cause failure of a member or structure to the load that is imposed upon it in service. The term may also be used to represent the ratio of the failure to service value of speed, deflection, temperature variation, or other stress-producing quantities. (Compare with *Allowable stress*; *Margin of safety*.)

Fatigue: The fracture of a material under many repetitions of a stress at a level considerably less than the ultimate strength of the material.

Fatigue strength: See *Endurance limit*.

Fixed (clamped): A support condition at the end of a beam or column or at the edge of a plate or shell that prevents *rotation and transverse displacement* of the edge of the neutral surface but permits *longitudinal displacement*. (Compare *Guided*; *Held*; *Simply-supported*.)

Flexural center: See *Shear center*.

Flexure equation: The equation for tensile and compressive stresses in beams undergoing bending, given by $\sigma = Mc/I$.

Flexural rigidity (beam, plate): A measure of the resistance of the bending deformation of a beam or plate. For a beam, the flexural rigidity is given by EI; whereas for a plate of thickness t, it is given by $Et^3/[12(1-\nu)]$.

Form factor: The term is applied to several situations pertaining to beams:

(1) Given a beam section of a given shape, the form factor is the ratio of the modulus of rupture of a beam having that particular section to the modulus of rupture of a beam otherwise similar but having a section adopted as a standard. This standard section is usually taken as rectangular or square; for wood it is a 2 in by 2 in square with edges horizontal and vertical (see Secs. 3.11 and 8.15).

(2) For the shear deflection of a beam due to transverse loading, the form factor is a correction factor that is the ratio of the actual shear deflection to the shear deflection calculated on the assumption of a uniform shear stress across the section (see Sec. 8.10).

(3) For a given maximum fiber stress within the elastic limit, the form factor is the ratio of the actual resisting moment of a wide-flanged beam to the resisting moment the beam would develop if the fiber stress were uniformly distributed across the entire width of the flanges. So used, the term expresses the strength-reducing effort of shear lag.

Fretting fatigue (chafing fatigue): Fatigue aggravated by surface rubbing, as in shafts with press-fitted collars.

Guided: A support condition at the end of a beam or column or at the edge of a plate or shell that prevents *rotation* of the edge of the neutral surface in the plane of bending but permits *longitudinal and transverse displacement*. (Compare with *Fixed*; *Held*; *Simply-supported*.)

Held: A support condition at the end of a beam or column or at the edge of a plate or shell that prevents *longitudinal and transverse displacement* of the edge of the neutral surface but permits *rotation* in the plane of bending. (Compare with *Fixed*; *Guided*; *Simply-supported*.)

Hertzian Stress (contact stress): Stress caused by the pressure between elastic bodies in contact.

Hysteresis: The dissipation of energy as heat during a stress cycle of a member.

Influence line: Usually pertaining to a particular section of a beam, an influence line is a curve drawn so that its ordinate at any point represents the value of the reaction, vertical shear, bending moment, or deflection produced at the particular section by a unit load applied at the point where the ordinate is measured. An influence line may be used to show the effect of load position on any quantity dependent thereon, such as the stress in a given truss member, the deflection of a truss, or the twisting moment in a shaft.

Isoclinic: A line (in a stressed body) at all points on which the corresponding principal stresses have the same direction.

Isotropic: Having the same properties in all directions. In discussions pertaining to strength of materials, isotropic usually means having the same strength and elastic properties (modulus of elasticity, modulus of rigidity, and Poisson's ratio) in all directions.

Kern (kernal): Reference is to some particular section of a member. The kern is that area in the plane of a section through which the line of action of a force must pass if that force is to produce, at all points in the given section, the same kind of normal stress, i.e., tension throughout or compression throughout.

Limit load: The fictitious theoretical load that the cross section of a member made of an elastic, perfectly plastic material reaches when the entire section goes into the plastic range.

Lüder's lines: See *Slip lines*.

Margin of Safety: As used in aeronautical design, margin of safety is the percentage by which the ultimate strength of a member exceeds the *design load*. The *design load* is the applied load, or maximum probable load, multiplied by a specified factor of safety. [The use of the terms margin of safety and design load in this sense is practically restricted to aeronautical engineering (see Ref. 11).]

Member: Any single part or element of a machine or structure, such as a beam, column, shaft, etc.

Modulus of elasticity, *E* (Young's modulus): The rate of change of normal stress, σ, to normal strain, ε, for the condition of uniaxial stress within the proportional limit of a given material. For most, but not all materials, the modulus of elasticity is the same for tension and compression. For nonisotropic materials such as wood, it is necessary to distinguish between the moduli of elasticity in different directions.

Modulus of resilience: The strain energy per unit volume absorbed up to the elastic limit under conditions of uniform uniaxial stress.

Modulus of rigidity, G (modulus of elasticity in shear): The rate of change of shear stress, τ, with respect to shear strain, γ, within the proportional limit of a given material. For nonisotropic materials such as wood, it is necessary to distinguish between the moduli of rigidity in different directions.

Modulus of rupture in bending (computed ultimate bending strength): The fictitious normal stress in the extreme fiber of a beam computed by the flexure equation $\sigma = M_R c/I$, where M_R is the bending moment that causes rupture.

Modulus of rupture in torsion (computed ultimate torsional strength): The fictitious shear stress at the outer radius of a circular shaft computed by the torsion equation $\tau = T_R r/J$, where T_R is the torsional moment that causes rupture.

Moment of an area (first moment of an area): With respect to an axis within the plane of an area, the sum of the products obtained by multiplying each element of the area dA by its distance, y, from the axis: it is therefore the quantity $\int y\, dA$.

Moment of inertia of an area (second moment of an area): With respect to an axis x within the xy plane of an area, the sum of the products obtained by multiplying each element of the area dA by the square of the distance y from the x axis: it is thus the quantity $I_x = \int y^2\, dA$ (see Appendix A).

Neutral axis: The line of zero fiber stress in any given section of a member subject to bending; it is the line formed by the intersection of the neutral surface and the section.

Neutral surface: The longitudinal surface of zero fiber stress in a member subject to bending; it contains the neutral axis of every section.

Notch-sensitivity factor: Used to compare the *stress concentration factor* K_t and *fatigue-strength reduction factor* K_f. The notch-sensitivity factor q is commonly defined as the ratio $(K_f - 1)/(K_t - 1)$, and varies from 0, for some soft ductile materials, to 1, for some hard brittle materials.

Plane strain: A condition where the normal and shear strains in a particular direction are zero; e.g., $\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$.

Plane stress: A condition where the normal and shear stresses in a particular direction are zero; e.g., $\sigma_z = \tau_{zx} = \tau_{zy} = 0$.

Plastic moment; plastic hinge; plastic section modulus: The maximum hypothetical bending moment for which the stresses in *all* fibers of a section of a ductile member in bending reach the lower yield point σ_y is called the *plastic moment*, M_p. Under this condition the section cannot accommodate any additional load, and a *plastic hinge* is said to form. The *section modulus* Z_p is defined as M_p/σ_y.

Plasticity: The property of sustaining appreciable permanent deformation without rupture. The term is also used to denote the property of yielding or flowing under steady load (Ref. 13).

Poisson's ratio, ν: The ratio of lateral to longitudinal strain under the condition of uniform and uniaxial longitudinal stress within the proportional limit.

Polar moment of inertia: With respect to an axis normal to the plane of an area, the sum of the products obtained by multiplying each element of the area dA by the square of the distance r from the axis; it is thus the quantity $\int r^2\, dA$ (see Appendix A).

Principal axes of inertia: The two mutually perpendicular axes in the plane of an area, centered at the centroid of the area, with moments of inertia that are maximum and minimum (see Appendix A).

Principal axes of stress: The three mutually perpendicular axes at a specific point within a solid where the state of stress on each surface normal to the axes contains a tensile or compressive stress and zero shear stress.

Principal moment of inertia: The moment of inertia of an area about a principal axis of inertia (see Appendix A).

Principal stresses: The tensile or compressive stresses acting along the principal axes of stress.

Product of inertia of an area: With respect to a pair of xy rectangular axes in the plane of an area, the sum of the products obtained by multiplying each element of area dA by the coordinates with respect to these axes; that is, $\int xy\, dA$ (see Appendix A). The product of inertia relative to the principal axes of inertia is zero.

Proof stress: Pertaining to acceptance tests of metals, a specified tensile stress that must be sustained without deformation in excess of a specified amount.

Proportional limit: The greatest stress that a material can sustain without deviating from the law of stress–strain proportionality (*Hookes' law.*) (Compare *Elastic limit*; *Apparent elastic limit*; *Yield point*; *Yield strength*. See Sec. 3.2 and Ref. 8.)

Radius of gyration, k: The *radius of gyration* of an area with respect to a given axis is the square root of the quantity obtained by dividing the moment of inertia of the area I with respect to that axis by the area A; that is, $k = \sqrt{I/A}$ (see Appendix A)

Reduction of area: The difference between the cross-sectional areas of a tensile specimen at the section of rupture before loading and after rupture.

Relaxation: The reduction in stress when the deformation is maintained constant. (Compare *Creep*.)

Rupture factor: Used in reference to brittle materials, i.e., materials in which failure occurs through tensile rupture rather than excessive deformation. For a member of given form, size and material, loaded and supported in a given manner, the *rupture factor* is the ratio of the fictitious maximum tensile stress at failure, as calculated by the appropriate formula for elastic stress, to the ultimate tensile strength of the material, as determined by a conventional tension test (Sec. 3.11.)

Saint-Venant's principle: If a load distribution is replaced by a statically equivalent force system, the distribution of stress throughout the body is possibly altered *only* near the regions of load application.

Section modulus (section factor), S: Pertaining to the cross section of a beam, the *section modulus* with respect to either principal axis of inertia is the moment of inertia with respect to that axis, I, divided by the distance from that axis to the most remote point of the section, c; that is, $S = I/c$. (Compare *Plastic section modulus*.)

Set (permanent deformation): Strain remaining after the removal of the applied loading.

Shakedown load (stabilizing load): The maximum load that can be applied to a beam or rigid frame and upon removal leave residual moments such that subsequent applications of the same or a smaller load will cause only elastic stresses.

Shape factor: The ratio of the plastic section modulus to the elastic section modulus.

Shear center (flexural center): With reference to a beam, the shear center of any section is that point in the plane of the section through which a transverse load, applied at the section, must act to produce bending deflection only and no twist of the section. (Compare with *Torsional center*; *Elastic center*; *Elastic axis*. See Refs. 5 and 10.)

Shear lag: Because of shear strain, the longitudinal tensile or compressive bending stresses in wide beam flanges decrease with the distance from the web(s), and this stress reduction is called *shear lag*.

Simply-supported: A support condition at the end of a beam or column or at the edge of a plate or shell that prevents *transverse displacement* of the edge of the neutral surface but permits *rotation and longitudinal displacement*. (Compare *Fixed*, *Guided*; *Held*.)

Singularity functions: A class of mathematical functions that can be used to describe discontinuous behavior using one equation. *Singularity functions* are commonly employed to represent shear forces, bending moments, slopes, and deformations as functions of position for discontinuous loading of beams, plates, and shells. The functions are written using bracket notation as $F_n = \langle x - a \rangle^n$, where $F_n = 0$ for $x \leqslant a$, and $F_n = (x - a)^n$ for $x > a$. (See Ref. 12.)

Slenderness ratio: The ratio of length of a uniform column to the minimum radius of gyration of the cross-section.

Slip lines (Lüder's lines): Lines that appear on the polished surface of a crystal or crystalline body that has been stressed beyond the elastic limit. They represent the intersection of the surface by planes on which shear stress has produced plastic slip (see Sec. 3.5 and Ref. 13).

Strain: Any forced change in the dimensions and/or shape of an elastic element. A stretch is a *tensile strain*; a shortening is a *compressive strain*; and an angular distortion is a *shear strain*.

Strain concentration factor: Localized peak strains develop in the presence of stress raisers. The strain concentration factor is the ratio of the *localized maximum strain* at a given location to the *nominal average strain* at that location. The nominal average strain is computed from the average stress and a knowledge of the stress–strain behavior of the material. In a situation where all stresses and strains are elastic, the stress and strain concentration factors are equal. (Compare with *Stress concentration factor*.)

Strain energy: Mechanical energy stored in a stressed material. Stress within the elastic limit is implied where the strain energy is equal to the work done by the external forces in producing the stress and is recoverable.

Strain rosette: At any point on the surface of a stressed body, strains measured along each of three intersecting gage lines make the calculation of the principal stresses possible. The gage lines and the corresponding strains are called *strain rosettes*.

Strength: Typically refers to a particular limiting value of stress for which a material ceases to behave according to some prescribed function. [Compare *Endurance limit (Fatigue strength)*; *Endurance strength*; *Ultimate strength*; *Yield strength*.]

Stress: Internal force per unit area exerted on a specified surface. When the force is tangential to the surface, the stress is called a *shear stress*; when the force is normal to the surface, the stress is called a *normal stress*; when the normal stress is directed toward the surface, it is called a *compressive stress*; and when the normal stress is directed away from the surface, it is called a *tensile stress*.

Stress concentration factor, K_t: Irregularities of form such as holes, screw threads, notches, and sharp shoulders, when present in a beam, shaft, or other member subject to loading, may produce high localized stresses. This phenomenon is called a *stress concentration*, and the form irregularities that cause it are called *stress raisers*. For the particular type of stress raiser in question, the ratio of the true maximum stress to the nominal stress calculated by the ordinary formulas of mechanics (P/A, Mc/I, Tc/J, etc.) is the stress concentration factor. The nominal stress calculation is based on the net section properties at the location of the stress raiser ignoring the redistribution of stress caused by the form irregularity. (See Sec. 3.10.)

Stress intensity factor: A term employed in fracture mechanics to describe the elastic stress field surrounding a crack tip.

Stress trajectory (isostatic): A line (in a stressed body) tangent to the direction of one of the principal stresses at every point through which it passes.

Superposition, principle of: With certain exceptions, the effect of a given combined loading on a structure may be resolved by determining the effects of each load separately and adding the results algebraically. The principle may be applied provided: (1) each effect is linearly related to the load that produces it, (2) a load does not create a condition which affects the result of another load, and (3) the deformations resulting from any specific load are not large enough to appreciably alter the geometric relations of the parts of the structural system. (See Sec. 4.2.)

Torsional center (center of twist): If a twisting couple is applied at a given section of a straight member, that section rotates about some point in its plane. This point, which does not move when the member twists, is the torsional center of that section. (See Refs. 5 and 6.)

Torsional moment (torque, twisting moment): At any section of a member, the moment of all forces that act on the member to the left (or right) of that section, taken about a polar axis through the *flexural center* of that section. For sections that are symmetrical about each principal axis, the flexural center coincides with the centroid (see Refs. 6 and 10).

Transformations of stress or strain: Conversions of stress or strain at a point from one three-dimensional coordinate system to another. (See Secs. 2.3 and 2.4)

Transverse shear force (vertical shear): Reference is to a simple straight beam, assumed for convenience to be horizontal and loaded and supported by forces, all of which lie in a vertical plane. The transverse shear force at any section of the beam is the vertical component of all forces that act on the beam to the left (or right) of that section. The *shear force equation* is an expression for the transverse shear at any section in terms of x, the distance to that section measured from a chosen origin, usually taken from the left of the beam.

Tresca stress: Based on the failure mode of a ductile material being due to shear stress, the *Tresca stress* is a single shear stress value, which is equivalent to an actual combined state of stress.

True strain: The summation (integral) of each infinitesimal elongation ΔL of successive values of a specific gage length L divided by that length. It is equal to $\int_{L_0}^{L}(dL/L) = \log_e (L/L_0) = \log_e (1 + \varepsilon)$, where L_0 is the original gage length and ε is the normal strain as ordinarily defined (Ref. 14).

True stress: For an axially loaded bar, the force divided by the actual cross-sectional area undergoing loading. It differs from the *engineering stress* defined in terms of the original area.

Ultimate elongation: The percentage of permanent deformation remaining after tensile rupture (measured over an arbitrary length including the section of rupture).

Ultimate strength: The ultimate strength of a material in uniaxial tension or compression, or pure shear, respectively, is the maximum tensile, compressive, or shear stress that the material can sustain calculated on the basis of the greatest load achieved prior to fracture and the original unstrained dimensions.

von Mises stress: Based on the failure mode of a ductile material being due to distortional energy caused by a stress state, the *von Mises* *stress* is a single normal stress value, which is equivalent to an actual combined state of stress.

Yield point: The stress at which the strain increases without an increase in stress. For some purposes, it is important to distinguish between *upper* and *lower* yield points. When they occur, the upper yield point is reached first and is a maxima that is followed by the lower yield point, a minima. Only a few materials exhibit a true yield point. For other materials the term is sometimes used synonymously with yield strength. (Compare *Yield strength*; *Elastic limit*; *Apparent elastic limit*; *Proportional limit*. See Ref. 7.)

Yield strength: The stress at which a material exhibits a specified permanent deformation or set. This stress is usually determined by the offset method, where the strain departs from the linear portion of the actual stress–strain diagram by an offset unit strain of 0.002. (See Ref. 7.)

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Appendix C

Composite Materials

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Composite members, as discussed in Secs. 8.3 and 8.8, are composed of more than one material where each material is of a continuous and homogeneous cross-section. The equivalent stiffnesses of these members are determined by a simple technique using an equivalent width. Composite materials, on the other hand, are made up of more than one material continuously dispersed at the macroscopic level at various angular orientations, and obtaining the combined material properties is much more complex. To this end, computer software is available. Once the material properties have been obtained from the software, the equations and tables provided in this book can be utilized. The following discussion is only intended to provide some introductory insight into the analysis of structures composed of

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*Analysis of composites is simplified if you have access to software specifically developed for this purpose. A free version of software for a limited number of composite materials is available from UTS at the web site, www.uts.com/composites/.

composite materials. Certainly, much more exposure is necessary before one can become proficient in this important topic.

C.1 Composite Materials

Composites are formed from two or more dissimilar materials, each of which contributes to the final properties. Unlike metallic alloys, the materials in a composite remain distinct from each other at the macroscopic level.

Most engineering composites consist of two materials: a reinforcement and a matrix. The reinforcement provides stiffness and strength; the matrix holds the material together and serves to transfer load among the discontinuous reinforcements. The most common reinforcements, illustrated in Figure C.1, are continuous fibers, either straight or woven, short chopped fibers, and particulates. The most common matrices are various plastic resins.

Metals and other traditional engineering materials are uniform, or isotropic, in nature. This means that material properties, such as strength, stiffness, and thermal conductivity, are independent of both position within the material and the choice of coordinate system. The discontinuous nature of composite reinforcements, though, means that material properties can vary with both position and direction. For example, an epoxy resin reinforced with continuous graphite fibers will have very high strength and stiffness in the direction of the fibers, but very low properties normal or transverse to the fibers.

This directionality increases the complexity of structural analyses. Isotropic materials are fully defined by two engineering constants: Young's modulus E and Poisson's ratio ν. A single ply of a composite material, however, requires four constants, defined with respect to the ply coordinate system shown in Figure C.2. The constants are two Young's moduli (the longitudinal modulus in the direction of the fibers, E_1, and the transverse modulus normal to the fibers, E_2, one Poisson's ratio ν_{12}, called the major Poisson's ratio, and one shear modulus G_{12}).

Particulate composite

Randomly oriented short fiber composite

Unidirectional continuous fiber composite

Woven fabric composite

Figure C.1 Composites categorized by reinforcement type

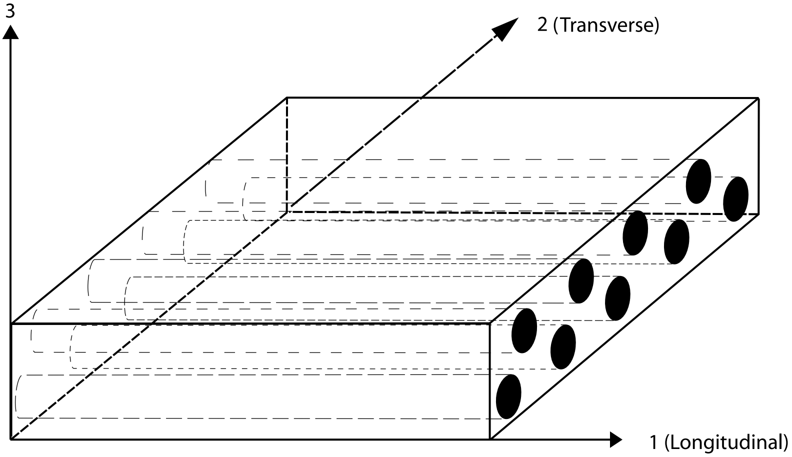

Figure C.2 Ply coordinate system

A fifth constant, the minor Poisson's ratio v_{21}, is determined from the other properties through the reciprocity relation

$$\frac{v_{12}}{E_1} = \frac{v_{21}}{E_2} \tag{C.1}$$

Table C.1 shows typical properties for a variety of composite systems. Each of the composites listed is a *unidirectional composite*, consisting of parallel fibers running in a single direction. The longitudinal modulus is largely a function of the fiber modulus, whereas the transverse and shear moduli are largely functions of the resin modulus. Thus, higher-modulus fibers will raise the longitudinal modulus of the composite, but will have a negligible effect on the other properties. The table also shows typical strengths for each material. As with the elastic properties, the strength is significantly greater in the longitudinal than in the transverse direction. Also note that compressive strengths are significantly lower than tensile strengths. This difference must be accounted for when analyzing composite structures for failure.

C.2 Laminated Composite Materials

The properties in Table C.1 are for a single ply or lamina of a composite. Because the transverse properties are so low, practical composite structures consist of laminates built up from a stack of laminae. To improve the transverse properties of the laminate, the plies are stacked so the fibers are rotated at various angles θ, defined relative to a convenient laminate coordinate system, as shown in Figure C.3. In the case of a beam, for example, the x-axis of the

TABLE C.1 Composite Material Systems

	T300/976	IM7/3501- 6	IM6/APC2	E/Epoxy	S2/381	K49/Epoxy	B/5505
Type	Low-mod. Gr/Epoxy	Int.-mod. Gr/Epoxy	Int.-mod. Gr/PEEK	E-Glass/ Epoxy	S-Glass/ Epoxy	Kevlar/ Epoxy	Boron/ Epoxy
Property							
E_1 (Msi)	19.6	20.2	21.6	5.7	6.93	12.6	29.2
E_2 (Msi)	1.34	1	1.28	1.24	1.84	0.8	3.15
G_{12} (Msi)	0.91	0.8	0.78	0.54	0.681	0.31	0.78
v_{12}	0.318	0.33	0.342	0.28	0.27*	0.34	0.17
F_{1t} (ksi)	211	350	350	157	255	185	200
F_{1c} (ksi)	188	234	167	90	172	49	232
F_{2t} (ksi)	5.66	8.1	9.41	5.7	8.7	4.2	8.1
F_{2c} (ksi)	30	35.7	25.8*	18.6	28.1*	22.9	18
F_{12} (ksi)	11.1	13.8	23.9	12.9	19.7	7.1	0.78
Ref.	6	7	6	1	6	1	1

Key: F_{1t} = longitudinal tensile strength; F_{1c} = longitudinal compressive strength; F_{2t} = transverse tensile strength; F_{2c} = transverse compressive strength; F_{12} = shear strength
* Estimated

laminate coordinate system might be chosen to coincide with the axis of the beam.

Rotating the plies increases the properties in the laminate *y*-direction, but this comes at the expense of a decrease in properties in the laminate *x*-direction. The greater the rotation angle, the greater the decrease in *x*-direction properties. Figure C.4, for example, shows how the modulus in the laminate *x*-direction decreases as the ply is rotated off-axis. The key to lightweight laminate design is to provide just enough off-axis stiffness or strength to handle the secondary loads, while orienting as many fibers as possible in the direction of the primary load.

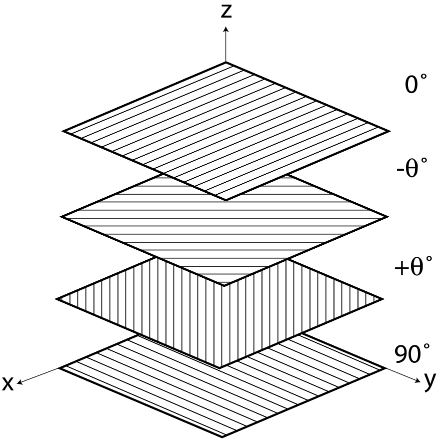

Figure C.3 Laminate coordinate system

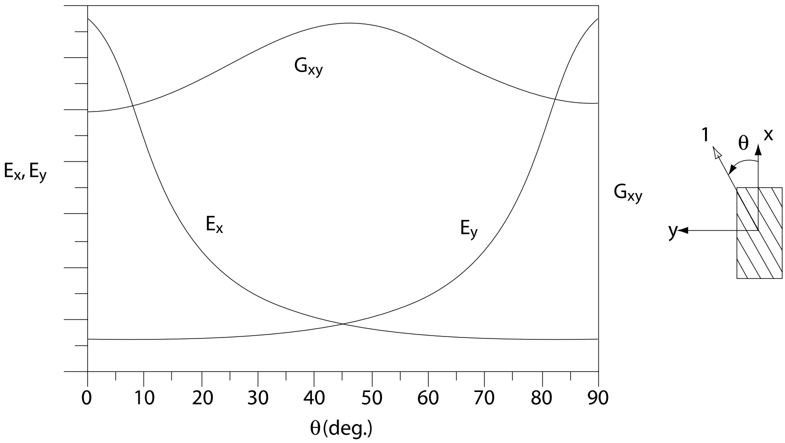

Figure C.4 Laminate modulus as a function of ply angle

Composite laminates are analyzed by determining the properties of the individual laminae and then calculating the effective properties of the laminate. Ply properties must be expressed in terms of the laminate coordinate system; this transformation is accomplished using a method similar to stress and strain transformations (e.g., Mohr's circle).

This analytical process is called *Classical Laminated Plate Theory* (CLPT). CLPT requires all lamina and laminate stiffness equations to be expressed in matrix form. Because of the large amount of matrix mathematics involved, CLPT solutions are usually handled by a computer software program. Users input the laminate stacking sequence, ply properties, and loads; the program outputs the stiffness matrix, engineering properties, and stresses and strains.

A list of shareware and commercial programs is available on the Internet (Ref. 5). Some of the more popular programs used for laminate analysis include:

- CompositePro
- ESAComp
- HyperSizer
- Laminator
- V-Lab

Although the programs handle all of the mathematics, it is important to at least understand the basic stress–strain relation for a laminate. The following discussion presents an overview of laminate theory;

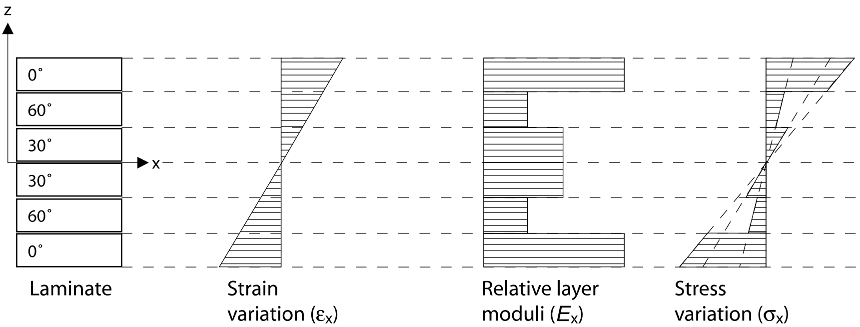

Figure C.5 Variation of strain, modulus, and stress in a laminate subjected to bending

more complete coverage can be found in any textbook on composite analysis, such as Daniel and Ishai (Ref. 1).

A general laminate can be subjected to both membrane (in-plane) and bending loads. CLPT assumes that the resulting strains and curvatures are uniform throughout the laminate. Because the plies are oriented in various directions, though, stresses are continuous only within the individual plies. Figure C.5 shows the variation in strain, modulus, and stress for a laminate subjected to a bending load. The strain variation is linear, as with any isotropic material. Because the ply moduli are discontinuous, however, ply stresses are also discontinuous. Stresses vary linearly within each ply, but are discontinuous at the ply boundaries.

Because the stresses are discontinuous, it is easier to define applied loads in terms of averaged stresses, or stress resultants. An element of a laminate can have up to six applied stress resultants, as shown in Figure C.6: three in-plane resultants N_i, and three bending resultants M_i. For an isotropic material, the resultants are simply the applied

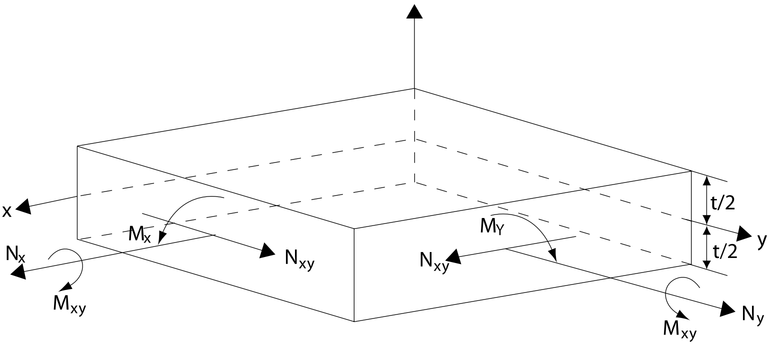

Figure C.6 Force and moment resultants stress or stress couple multiplied by the thickness. For laminated composites, the resultants are found by integrating the ply stresses through the thickness of the laminate. As with the stiffness properties, the CLPT computer programs handle the necessary integrations.

Although software makes it unnecessary to work directly with the stiffness matrices, it is important to understand what the different values mean. The general form of the laminate stress–strain relation is given by

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ \hline M_x \\ M_y \\ M_{xy} \end{bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{C.2}$$

which can be written more succinctly as

$$\begin{aligned} [N] &= [A][\varepsilon^0] + [B][\kappa] \\ [M] &= [B][\varepsilon^0] + [D][\kappa] \end{aligned} \tag{C.3}$$

The matrices $[N]$ and $[M]$ are the applied membrane and bending loads, expressed as stress resultants. $[\varepsilon^0]$ are the in-plane strains; the supercript 0 indicates the strains are referenced to the laminate midplane. $[\kappa]$ are the laminate curvatures. $[A]$, $[B]$, and $[D]$ are the matrix forms of the laminate stiffnesses. The matrix $[A]$ relates in-plane loads to in-plane strains, and the matrix $[D]$ relates bending loads to curvatures.

$[B]$ is known as the membrane-bending coupling matrix. It shows that, under the right conditions, a purely in-plane load can cause the laminate to warp, or a pure bending moment can cause the laminate to stretch. This can be seen by looking at one of the six stress–strain equations:

$$N_x = A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + A_{16}\gamma_{xy}^0 + B_{11}\kappa_x + B_{12}\kappa_y + B_{16}\kappa_{xy} \tag{C.4}$$

If B_{11} is non-zero, then pulling on the laminate in the x-direction also causes it to warp about the y-axis. This is the same effect that causes bimetallic beams (Sec. 8.2) to warp, and in fact the laminate stress–strain relation will reduce to the bimetallic beam solution if all of the plies are made from isotropic materials. Such coupling between in-plane and bending behavior is often undesirable. Fortunately, it is a simple matter to design a laminate where all $[B]$ terms are equal to zero.

Laminates are described by their stacking sequence. This lists the orientations of each ply in the stack, beginning with the top ply. If all plies are of the same material and thickness, then no further notation is needed. Numerical subscripts refer to the number of times a ply orientation is repeated. Superscripts may be used to denote different materials and thicknesses. For example, a $[0_2/90]$ laminate consists of three plies: two 0° plies, followed by one 90° ply. Often, the subscript "T" is used to denote "total." Thus, the previous laminate might be written as $[0_2/90]_T$.

A laminate is called symmetric if, for each ply on one side of the laminate midplane, there is a corresponding ply on the other side of the midplane at the same distance and of the same material, thickness and orientation. The subscript "S" is used to denote symmetry. The laminate in Figure C.7 can be written as [0/90/30/30/90/0], or more succinctly as $[0/90/30]_S$.

The matrix $[B]$ is identically zero for all symmetric laminates. This greatly reduces the stress–strain relations by uncoupling the membrane and bending terms:

$$\begin{aligned}[N] &= [A][\varepsilon^0] \\ [M] &= [D][\kappa]\end{aligned} \tag{C.5}$$

In a general symmetric laminate, the A_{12}, A_{26}, D_{16}, and D_{26} terms are nonzero. This means that there is coupling between extensional stresses and shear strains, and between bending and twisting. Once again, the N_x equation is

$$N_x = A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + A_{16}\gamma_{xy}^0 \tag{C.6}$$

where the A_{16} term shows that an extensional load is related to shear strain.

The A_{16} shear coupling terms can be made to vanish by requiring that the laminate consist only of plies oriented at 0° and 90°, or that all

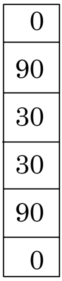

Figure C.7 General symmetric laminate with a stacking sequence for a $[0/90/30]_S$ laminate angle plies be balanced. Angle plies are balanced when, for each ply at $+\theta^\circ$, there is a corresponding ply at $-\theta^\circ$. If the laminate is both balanced and symmetric, then all coupling terms are zero, and the stress–strain relation reduces to

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix}$$
$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{C.7}$$

Figure C.8 shows the stacking sequence for a $[0/\pm 30/90/\pm 45]_T$ balanced laminate; Figure C.9 shows the stacking sequence for a $[0/\pm 30]_S$ balanced-symmetric laminate.

For balanced-symmetric laminates, it is also possible to calculate effective engineering properties. They are

$$\begin{aligned} E_x &= \frac{1}{h}\left[A_{11} - \frac{A_{12}^2}{A_{22}}\right] \\ E_y &= \frac{1}{h}\left[A_{22} - \frac{A_{12}^2}{A_{11}}\right] \\ v_{xy} &= \frac{A_{12}}{A_{22}} \\ G_{xy} &= \frac{A_{66}}{h} \end{aligned} \tag{C.8}$$

where h is the total laminate thickness.

There are several other types of special laminates, but the one of most interest is the *quasi-isotropic laminate*. Quasi-isotropic laminates are balanced-symmetric, and the ply angles are such that the laminate stiffness properties are independent of direction. In other

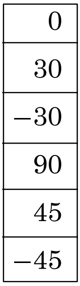

Figure C.8 $[0/\pm 30/90/\pm 45]_T$ balanced laminate

0
30
−30
−30
30
0

Figure C.9 $[0/\pm 30]_S$ balanced-symmetric laminate

words, the laminate behaves like an isotropic material. (Strength and bending stiffness however, still vary with direction – hence the term *quasi*-isotropic.)

Quasi-isotropic laminates are of the form (angles are in radians)

$$\left[0\Big/\frac{\pi}{n}\Big/\frac{2\pi}{n}\Big/\cdots\Big/\frac{(n-1)\pi}{n}\right]_S$$

or

$$\left[\frac{\pi}{n}\Big/\frac{2\pi}{n}\Big/\cdots\Big/\pi\right]_S$$

where n is any integer greater than 2. The two simplest and most common quasi-isotropic laminates are $[0/\pm 60]_S$ and $[0/\pm 45/90]_S$, illustrated in Figure C.10(a) and (b), respectively.

0
60
−60
−60
60
0

(a)

0
45
−45
90
90
−45
45
0

(b)

Figure C.10 Stacking sequences for the two simplest quasi- isotropic laminates

C.3 Laminated Composite Structures

Classical Laminate Plate Theory describes the behavior of laminated plate elements under in-plane and bending loads. The solution applies only to plate elements that are in load equilibrium – it does not account for the geometry of the plate or the boundary conditions.

In general, analytical solutions for composite structures are much more difficult to derive than for isotropic structures. Solutions for beams (Refs. 2 and 3), plates (Refs. 2 and 4), and shells (Ref. 2) can be found if the loads, geometry and boundary conditions remain simple. Beam solutions are the simplest and are illustrated in the two examples below.

Bending of Composite Beams

If a composite plate meets the geometric definition of a beam, namely that the width and height are much smaller than the length, then the assumptions of standard beam theory can be used. An equivalent modulus E^* is derived for composite beams under pure bending. The equivalent modulus is then used in conjunction with Table 8.1 of Chap. 8 to provide an approximate solution for the bending of laminated rectangular beams. The solution ignores transverse shear deformation, which could be significant, so results should be used for initial sizing only.

In addition to the standard beam theory assumptions, the following requirements also apply:

1. The beam is of rectangular cross-section with width b and height h.
2. Plies lie in the x–z or 1–2 plane.
3. The shear coupling terms $(\ )_{16}$ and $(\ )_{26}$ of the matrices $[A]$, $[B]$, and $[D]$ are zero.

The equivalent modulus is found by applying the beam theory assumptions to the constitutive equations, Eq. (C.2), which gives a simple expression for the moment resultant M_x in terms of the curvature κ_x (Refs. 2 and 3). This expression can then be substituted into the standard beam bending expression, $(EI)\kappa_x = M$ to find

$$E^* = \left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{12}{h^3} \tag{C.9}$$

If the lay-up is symmetric about the mid-plane, then $B_{11} = 0$, and the equivalent modulus reduces to

$$E^* = D_{11}\frac{12}{h^3} \tag{C.10}$$

Stresses are calculated on a ply-by-ply basis. In general, only the stresses along the beam axis will be of interest. The stress in ply k is given by

$$[\sigma_x]_k = z[\overline{Q}_{11}]_k \kappa_x \tag{C.11}$$

where

$$\kappa_x = M\left[\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)b\right]^{-1} \tag{C.12}$$

and $\overline{Q}$ is the transformed stiffness matrix of ply k. The components of $\overline{Q}$ are a function of the ply material properties and the orientation angle. $\overline{Q}$ is found while calculating the matrices [A], [B], and [D], and is usually output by whatever software package is being used.

EXAMPLE

A composite beam is made from T300/976 graphite epoxy with a $[0_3/\pm 45]_{4T}$ layup. It is 10 in. long, 0.5 in wide, cantilevered at the right end, and subjected to a concentrated load of 5 lb at the left end. It is desired to find the maximum deflection and the stresses at the mid-plane of the top five plies.

Solution

From Table C.1, the ply properties are $E_1 = 19.6$ Msi, $E_2 = 1.34$ Msi, $G_{12} = 0.91$ Msi, and $\nu_{12} = 0.318$. Normalized ply thickness is 0.005 in, for a total laminate thickness of 0.100 in.

Using a standard software package, the constitutive properties are found to be $A_{11} = 1.440 \times 10^6$ lb/in, $B_{11} = -4.002 \times 10^3$ lb and $D_{11} = 1.193 \times 10^3$ lb-in. These values give an equivalent modulus of $E^*I = 590.94$ lb-in^2. Table 8.1 of Chap. 8, Case 1a, then gives -0.705 in as the tip deflection and -50 in-lb as the moment.

The transformed stiffness properties, again from a standard software package, are $[\overline{Q}_{11}]^0 = 1.974 \times 10^7$ psi and $[\overline{Q}_{11}]^{\pm 45} = 6.396 \times 10^6$ psi. The curvature is $\kappa_x = -0.085\,\text{in}^{-1}$, giving the following for the ply stresses:

Ply	z (in)	σ_{11} (ksi)
1	– 0.0475	79.34
2	– 0.0425	70.98
3	– 0.0375	62.63
4	– 0.0325	17.59
5	– 0.0275	14.88

The stresses σ_{11} are in the laminate coordinate system. Because plies 4 and 5 are oriented at $+45^\circ$ and -45°, the σ_{11} stresses in the ply coordinate system have a shear component. Because the shear strength of composites is much lower than the tensile strength, this laminate would have to be checked for failure by comparing the resulting value of σ_{11} in plies 4 and 5 with the ultimate shear strength of T300/976.

Axial Tension or Compression of Composite Beams

The solution for beams under pure axial load is similar to beams under bending. Instead of an equivalent bending stiffness E^*I, an equivalent axial stiffness $(EA)^*$ is found. Again, the standard beam assumptions apply, with the following additional restrictions:

1. Plies lie in the x–z or 1–2 plane.
2. The shear coupling terms $(\ )_{16}$ and $(\ )_{26} = 0$.
3. The laminate is balanced-symmetric ($[B] = [0]$).

The requirement for a rectangular cross-section has been relaxed, but the laminate is now restricted to balanced symmetric lay-ups. Without that assumption, the membrane-bending coupling matrix $[B]$ is nonzero, and pure axial loads cause bending in the beam.

The process for calculating the equivalent beam stiffness is similar to the process for calculating beam bending stiffness (Refs. 3 and 4). The constitutive equations (C.2) are again simplified using the beam theory assumptions, and the standard beam equation $(EA)\varepsilon_x^0 = P$ is solved using the laminate properties:

$$(EA)^* = b^* A_{11} \tag{C.13}$$

where b^* is the equivalent width of the cross-section. For a rectangular beam, $b^* = b$, the width of the cross-section. For a thin-walled circular cross section, $b^* = \pi D$, the circumference of the cross section. If the beam is built up from a series of uniform cross sections (such as a box beam or an I-beam), then

$$(EA)^* = \sum b^* A_{11} \tag{C.14}$$

where the product $b^* A_{11}$ is summed over each cross-section.

Ply stresses are calculated from

$$[\sigma_x]_k = [\overline{Q}_{11}]_k \varepsilon_x^0 \tag{C.15}$$

where

$$\varepsilon_x^0 = \frac{P}{(EA)^*} \tag{C.16}$$

EXAMPLE

A composite I-beam is made from T300/976 graphite epoxy with a $[0/\pm 30]_{nS}$ lay-up (where $n = 5$ for each flange and $n = 3$ for the web). The flanges are 1 in wide and 0.15 in thick; the web is 2 in tall and 0.09 in thick. The length of the beam is 20 in. An axial tensile load of 1000 lb is applied to the beam. It is desired to find the axial deflection and the stresses in the plies.

Ply properties are the same as in the previous example. The constitutive properties are calculated using a standard software package, and are found to be

$$[A_{11}]_{\text{flange}} = 2.190 \times 10^6 \text{ lb/in}$$
$$[A_{11}]_{\text{web}} = 1.314 \times 10^6 \text{ lb/in}$$

The equivalent stiffness is

$$\begin{aligned}(EA)^* &= \sum bA_{11} = 2[bA_{11}]_{\text{flange}} + [bA_{11}]_{\text{web}} = 2[(1)(2.190 \times 10^6)] \\ &\quad + [(2)(1.314) \times 10^6)] \\ &= 7.008 \times 10^6 \text{ lb}\end{aligned}$$

Equation (7.1-3) of Chap. 7 then gives the total beam elongation as

$$\delta = Pl/(EA)^* = (1000)(20)/(7.008 \times 10^6) = 2.854 \times 10^{-3} \text{ in}$$

The transformed stiffness properties, once again from the software package, are $[\overline{Q}_{11}]^0 = 1.974 \times 10^7$ psi and $[\overline{Q}_{11}]^{\pm 30} = 1.203 \times 10^7$ psi. The strain in the beam is simply $\varepsilon_x^0 = \delta/l = 1.427 \times 10^{-4}$, giving

$$[\sigma_x]^0 = 2817 \text{ psi}$$
$$[\sigma_x]^{30} = 1717 \text{ psi}$$

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