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# ENGINEERING MECHANICS 

Vector and Classical Approach

## SECOND EDITION

## S S BHAVIKATTI



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# Vector and Classical Approach 

 SECOND EDITION
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## Preface to the Second Edition

In this edition, vector approach has been used and more number of typical problems are solved, apart from correcting print mistakes of the first edition.
S.S. Bhavikatti

## Preface to the First Edition

Engineering mechanics is a basic course taught to students of all branches in their very first year of undergraduate curriculum. In this course, students are taught to model actual field problems into engineering problems and find the solutions using laws of mechanics. The skill developed through this course helps in breaking a large problem into a set of small parts and finding the solution for each part maintaining the continuity. This enhances the analytical capability of the engineering students.

There are two approaches for the solution of engineering mechanics problems i.e., vector approach and classical approach. In this text, emphasis is on vector approach which is ideally suited for the analysis of three dimensional problems. However, classical approach gives physical feel of the structure and ideally suited for two dimensional problems. Hence, this approach is also used wherever necessary.

Key to successful solution to engineering mechanics problem is drawing correct and neat free body diagrams. Hence throughout the text, the author has given emphasis on this. In this text, SI units are used with standard notations as recommended by various National Codes. All problems are solved systematically without skipping steps, so that the reader picks up correct method of presenting solution. A large number of problems are solved and given for exercise, to help students to understand the subject thoroughly.

The matter presented meets the requirement of almost all universities. The author hopes that both teachers and students will find this book useful. The author thanks M/s. New Age International (P) Limited, Publishers for the neat and prompt work in publishing this book. Care has been taken to avoid the mistakes and misprints. However if any lapse is observed, the author will be thankful to readers if that is brought to his notice.

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## Chapter 1

## Basics

The state of rest and state of motion of the bodies under the action of different forces has engaged the attention of philosophers, mathematicians and scientists for many centuries. The branch of physical science that deals with the state of rest or the state of motion is termed as Mechanics. Starting from the analysis of rigid bodies under gravitational force and simple applied forces the mechanics has grown to the analysis of robotics, aircrafts, spacecrafts under dynamic forces, atmospheric forces, temperature forces etc.

Archemedes (287-212 BC), Galileo (1564-1642), Sir Issac Newton (1624-1727) and Einstein (1878-1955) have contributed a lot to the development of mechanics. Contributions by Varignon, Euler, D'Alembert are also substantial. The mechanics developed by these researchers may be grouped as:
(i) Classical mechanics/Newtonian mechanics
(ii) Relativistic mechanics
(iii) Quantum mechanics/Wave mechanics

Sir Issac Newton, the principal architect of mechanics, consolidated the philosophy and experimental findings developed around the state of rest and state of motion of the bodies and put forth them in the form of three laws of motion and law of gravitation. The mechanics based on these laws is called Classical mechanics or Newtonian mechanics.

Albert Einstein proved that Newtonian mechanics fails to explain the behaviour of high speed (speed of light) bodies. He put forth the theory of Relativistic Mechanics.

Schrodinger (1887-1961) and Broglie (1892-1965) showed that Newtonian mechanics fails to explain the behaviour of particles when atomic distances are concerned. They put forth the theory of Quantum Mechanics.

Engineers are keen to use the laws of mechanics to actual field problems. Application of laws of mechanics to field problem is termed as Engineering Mechanics. For all the problems between atomic distances to high speed distances, classical/Newtonian mechanics has stood the test of time and hence that is the mechanics used by engineers. Therefore, in this text classical mechanics is used for the analysis of engineering problems.

### 1.1 BASIC TERMINOLOGIES IN MECHANICS

The following are the terms basic to the study of mechanics, which should be understood clearly:

## Mass

The quantity of the matter possessed by a body is called mass. The mass of a body will not change unless the body is damaged and part of it is physically separated. When a body is taken out in a spacecraft, the mass will not change but its weight may change due to change in gravitational force. Even the body may become weightless when gravitational force vanishes but the mass remains the same.

## Time

Time is the measure of succession of events. The successive event selected is the rotation of Earth about its own axis and this is called a day. To have convenient units for various activities, a day is divided into 24 hours, an hour into 60 minutes and a minute into 60 seconds. Clocks are the instruments developed to measure time. To overcome difficulties due to irregularities in the Earth's rotation, the unit of time is taken as second which is defined as the duration of 9192631770 periods of radiation of the cesium-133 atom.

## Space

The geometric region in which study of body is involved is called 'space'. Any point in the space may be referred with respect to a predetermined point by a set of linear and angular measurements. The reference point is called the origin and set of measurements as 'coordinates'. If the coordinates involve only in mutually perpendicular directions they are known as Cartesian coordinates. If the coordinates involve angle and distances, it is termed as polar coordinate system.

## Length

It is a concept to measure linear distances. The diameter of a cylinder may be 300 mm , the height of a building may be 15 m . Actually metre is the unit of length. However depending upon the sizes involved micro, milli or kilo metre units are used for measurement. A metre is defined as length of the standard bar of platinumiradium kept at the International Bureau of Weights and Measures. To overcome difficulties of accessibility and reproduction, now metre is defined as 1690763.73 wave length of krypton-86 atom.

## Displacement

Displacement is defined as the distance moved by a body/particle in the specified direction. Referring to Fig 1.1, if a body moves from position $A$ to position $B$ in the $x-y$ plane shown, its displacement in $x$-direction is $A B^{\prime}$ and its displacement in $y$-direction is $-B^{\prime} B$.


Fig. 1.1

## Velocity

The rate of change of displacement with respect to time is defined as velocity.

## Acceleration

Acceleration is the rate of change of velocity with respect to time. Thus

$$
\begin{equation*}
a=\frac{d v}{d t}, \text { where } v \text { is velocity } \tag{1.1}
\end{equation*}
$$

## Momentum

The product of mass and velocity is called momentum. Thus

$$
\text { Momentum }=\text { Mass } \times \text { Velocity }
$$

## Continuum

A body consists of several matters. It is a well known fact that each particle can be subdivided into molecules, atoms and electrons. It is not possible to solve any engineering problem by treating a body as a conglomeration of such discrete particles. The body is assumed to consist of a continuous distribution of matter. In other words, the body is treated as continuит.

## Rigid Body

A body is said to be rigid, if the relative positions of any two particles do not change under the action of the forces. In Fig. 1.2 (a) points $A$ and $B$ are the original positions in a body. After application of a system of forces $F_{1}, F_{2}, F_{3}$, the body takes the position as shown in Fig 1.2 (b). $A^{\prime}$ and $B^{\prime}$ are the new positions of $A$ and $B$. If the body is treated as rigid, the relative position of $A^{\prime} B^{\prime}$ and $A B$ are the same. i.e.,

$$
A^{\prime} B^{\prime}=A B
$$


(a)

(b)

Fig.1.2
Many engineering problems can be solved satisfactorily by assuming bodies rigid.

## Particle

A particle may be defined as an object which has only mass and no size. Such a body cannot exist theoretically. However in dealing with problems involving distances considerably larger compared to the size of the body, the body may be treated as particle, without sacrificing accuracy. Examples of such situations are

- A bomber aeroplane is a particle for a gunner operating from the ground.
- A ship in mid sea is a particle in the study of its relative motion from a control tower.
- In the study of movement of the Earth in celestial sphere, Earth is treated as a particle.


### 1.2 LAWS OF MECHANICS

The following are the fundamental laws of mechanics:
Newton's first law
Newton's second law
Newton's third law
Newton's law of gravitation
Law of transmissibility of forces and
Parallelogram law of forces

## Newton's First Law

It states that every body continues to be in its state of rest or of uniform motion in a straight line unless it is compelled by an external agency acting on it. This leads to the definition of force as the external agency which changes or tends to change the condition of rest or uniform linear motion of the body.

## Newton's Second Law

It states that the rate of change of momentum of a body is directly proportional to
the impressed force and it takes place in the direction of the force acting on it. Thus according to this law:

Force $\propto$ rate of change of momentum. But momentum $=$ mass $\times$ velocity As mass do not change,

$$
\text { Force } \propto \text { mass } \times \text { rate of change of velocity }
$$

i.e., $\quad$ Force $\propto$ mass $\times$ acceleration

$$
F \propto m \times a
$$

## Newton's Third Law

It states that for every action there is an equal and opposite reaction. Consider the two bodies in contact with each other. Let one body apply a force $F$ on another. According to this law the second body develops a reactive force $R$ which is equal in magnitude to force $F$ and acts in the line same as $F$ but in the opposite direction. Figure 1.3 shows the action of the ball and the reaction from the floor. In Fig. 1.4 the action of the ladder on the wall and the floor and the reactions from the wall and floor are shown.

(a)

(b)

Fig. 1.3


$R_{2}=F_{2}$
Fig. 1.4

## Newton's Law of Gravitation

Every body attracts the other body. The force of attraction between any two bodies is directly proportional to their masses and inversely proportional to the
square of the distance between them. According to this law the force of attraction between the bodies of mass $m_{1}$ and $m_{2}$ at distance ' $d$ ' as shown in Fig. 1.5 is

$$
\begin{equation*}
F=G \frac{m_{1} m_{2}}{d^{2}} \tag{1.4}
\end{equation*}
$$

where $G$ is the constant of proportionality and is known as constant of gravitation.


Fig. 1.5

## Law of Transmissibility of Force

According to this law, the state of rest or motion of the rigid body is unaltered if a force acting on the body is replaced by another force of the same magnitude and direction but acting anywhere on the body along the line of action of the replaced force.

Let $F$ be the force acting on a rigid body at point $A$ as shown in Fig. 1.6. According to the law of transmissibility of force, this force has the same effect on the state of body as the force $F$ applied at point $B$.

In using law of transmissibility of


Fig. 1.6 forces it should be carefully noted that it is applicable only if the body can be treated as rigid. In this text, the engineering mechanics is restricted to study of state of rigid bodies and hence this law is frequently used. Same thing cannot be done in the subject 'solid mechanics' where the bodies are treated as deformable and internal forces in the body are studied.

## Parallelogram Law of Forces

The parallelogram law of forces enables us to determine the single force called resultant which can replace the two forces acting at a point with the same effect as that of the two forces. This law was formulated based on experimental results. Though Stevinces employed it in 1586, the credit of presenting it as a law goes to Varignon and Newton (1687). This law states that if two forces acting simultaneously on a body at a point are represented in magnitude and direction
by the two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection of the two sides representing the forces.

In Fig. 1.7 the force $F_{1}=4$ units and force $F_{2}=3$ units are acting on a body at point $A$. Then to get resultant of these forces, parallelogram $A B D C$ is constructed such that $A B$ is equal to 4 units to linear scale and $A C$ equal to 3 units. Then according to this law, the diagonal $A D$ represents the resultant in the direction and magnitude. Thus the resultant of the forces $F_{1}$ and $F_{2}$ on the body is equal to units corresponding to $A B$ in the direction $\alpha$ to $F_{1}$.

(a)

(b)

(c)

Fig. 1.7

### 1.3 UNITS

Length $(L)$, Mass $(M)$ and Time $(S)$ are the fundamental units in mechanics. The units of all other quantities may be expressed in terms of these basic units. The three commonly used systems in engineering are:

- Metre-Kilogramme-Second (MKS) system
- Centimetre-Gramme-Second (CGS) system, and
- Foot-Pound-Second (FPS) system.

The units of length, mass and time used in the system are used to name the systems. Using these basic units, the units for other quantities can be found. For example, in MKS the units for the various quantities are as shown below:

| Quantity | Unit | Notation |
| :--- | :--- | :--- |
| Area | Square metre | $\mathrm{m}^{2}$ |
| Volume | Cubic metre | $\mathrm{m}^{3}$ |
| Velocity | metre per second | $\mathrm{m} / \mathrm{s}$ |
| Acceleration | metre per second square | $\mathrm{m} / \mathrm{s}^{2}$ |

## Unit of Forces

Presently the whole world is in the process of switching over to SI system of units. SI stands for System Internationale $d$ ' units or International System of Units. Like in MKS system, in SI system also the fundamental units are metre for length, kilogramme for mass and second for time. The difference between MKS and SI system arises mainly in selecting the unit of force. From Eqn. (1.3), we have

$$
\begin{align*}
\text { Force } & \propto \text { Mass } \times \text { Acceleration } \\
& =k \times \text { Mass } \times \text { Acceleration } \tag{1}
\end{align*}
$$

In SI system unit of force is defined as that force which causes 1 kg mass to move with an acceleration of $1 \mathrm{~m} / \mathrm{s}^{2}$ and is termed as 1 Newton. Hence the constant of proportionality $k$ becomes unity. Unit of force can be derived from Eqn. 1.5 as

$$
\begin{aligned}
\text { Unit of Force } & =\mathrm{kg} \times \mathrm{m} / \mathrm{s}^{2} \\
& =\mathrm{kg}-\mathrm{m} / \mathrm{s}^{2}
\end{aligned}
$$

In MKS, the unit of force is defined as that force which makes a mass of 1 kg to move with gravitational acceleration ' g ' $\mathrm{m} / \mathrm{s}^{2}$. This unit of force is called kilogramme weight or kg-wt. Gravitational acceleration is $9.81 \mathrm{~m} / \mathrm{s}^{2}$ near the Earth surface. In all the problems encountered in engineering mechanics the variation in gravitational acceleration is negligible and may be taken as $9.81 \mathrm{~m} / \mathrm{s}^{2}$. Hence the constant of proportionality in Eqn. 1.5 is 9.81 , which means
$1 \mathrm{~kg}-\mathrm{wt}=9.81$ Newton
It may be noted that in public usage, kg -wt force is called as kg only.

## Unit of Constant of Gravitation

From Eqn. 1.4,

$$
\begin{array}{rlrl} 
& F & =G \frac{m_{1} m_{2}}{d^{2}} \\
\text { or } & G & =\frac{F d^{2}}{m_{1} m_{2}} \\
\therefore \quad & \text { Unit of } G & =\frac{\mathrm{N} \times \mathrm{m}^{2}}{\mathrm{~kg} \times \mathrm{kg}} \\
& =\mathrm{Nm}^{2} / \mathrm{kg}^{2}
\end{array}
$$

It has been proved by experimental results that the value of $G=6.673 \times 10^{-11} \mathrm{Nm}^{2} / \mathrm{kg}^{2}$. Thus if two bodies, one of mass 10 kg and the other of 5 kg are at a distance of 1 m , they exert a force

$$
\begin{aligned}
F & =\frac{6.673 \times 10^{-11} \times 10 \times 5}{1^{2}} \\
& =33.365 \times 10^{-10} \mathrm{~N}
\end{aligned}
$$

on each other.
Now let us find the force acting between 1 kg -mass near Earth surface and the Earth. Earth has a radius of $6371 \times 10^{3} \mathrm{~m}$ and has a mass $5.96506 \times 10^{24} \mathrm{~kg}$. Hence the force between the two bodies is

$$
\begin{aligned}
& =\frac{6.673 \times 10^{-11} \times 1 \times 5.96504 \times 10^{24}}{\left(6371 \times 10^{3}\right)^{2}} \\
& =9.80665 \mathrm{~N}^{(6)}
\end{aligned}
$$

In common usage we call the force exerted by Earth on a body as weight of the body. Thus weight of 1 kg mass on Earth surface is 9.80665 N , which is approximated as 9.81 N for all practical problems. Compared to this force the force exerted by two bodies near Earth surface is negligible as may be seen from the example of 10 kg and 5 kg mass bodies.

Denoting the weight of the body by $W$, from expression 1.4, we get

$$
W=\frac{G m M_{e}}{r^{2}}
$$

where $\quad m$ is the mass of the body $M_{e}$ is the mass of the Earth, and $r$ is the radius of the Earth
Denoting $\frac{G M_{e}}{r^{2}}$ by $g$, we get

$$
\begin{align*}
W & =m g \\
& =9.81 \mathrm{~m} \tag{1.7}
\end{align*}
$$

Unit of $g$ can be obtained as follows:

$$
g=\frac{G M_{e}}{r^{2}}
$$

Unit of

$$
g=\frac{\mathrm{Nm}^{2}}{(\mathrm{~kg})^{2}} \times \frac{\mathrm{kg}}{\mathrm{~m}^{2}}=\frac{\mathrm{N}}{\mathrm{~kg}}
$$

as unit of Newton force is $\mathrm{kg}-\mathrm{m} / \mathrm{s}^{2}$, we get

$$
\text { Unit of } g=\frac{\mathrm{kgm} / \mathrm{s}^{2}}{\mathrm{~kg}}=\mathrm{m} / \mathrm{s}^{2}
$$

Hence $g$ may be called as acceleration due to gravity. Any body falling freely near Earth surface experiences this acceleration. The value of $g$ is $9.81 \mathrm{~m} / \mathrm{s}^{2}$ near the Earth surface as can be seen from Eqn. 1.7.

The prefixes used in SI system when quantities are too big or too small are shown in Table 1.1.

Table 1.1 Prefixes and Symbols of Multiplying Factors in SI

| Multiplying Factor | Prefix | Symbol |
| :---: | :---: | :---: |
| $10^{12}$ | tera | $\mathbf{T}$ |
| $10^{9}$ | giga | $\mathbf{G}$ |
| $10^{6}$ | mega | $\mathbf{M}$ |
| $10^{3}$ | kilo | $\mathbf{K}$ |
| $10^{0}$ | - | - |
| $10^{-3}$ | milli | $\mathbf{m}$ |
| $10^{-6}$ | micro | $\boldsymbol{\mu}$ |
| $10^{-9}$ | nano | $\mathbf{n}$ |
| $10^{-12}$ | pico | $\mathbf{p}$ |
| $10^{-15}$ | femto | $\mathbf{f}$ |
| $10^{-18}$ | atto | $\mathbf{a}$ |

### 1.4 DIMENSIONS

The qualitative description of physical variable is known as dimension, while the quantitative description is known as unit. We come across several relations among the physical quantities. Some of the terms may be having dimensions and some others may be dimensionless. However in any equation dimensions of the terms on both sides must be the same. This is called dimensional homogenity. The branch of mathematics dealing with dimensions of quantities is called dimensional analysis.

There are two systems of dimensional analysis viz. absolute system and gravitational system. In absolute system the basic quantities selected are Mass, Length and Time. Hence it is known as MLT-system. In gravitational system the basic quantities are Force, Length and Time. Hence it is termed as FLT-system.

The dimension of acceleration is $\frac{L}{T^{2}}=L T^{-2}$ since its unit is $\mathrm{m} / \mathrm{s}^{2}$. From
Newton's law we have the physical relation

$$
\text { Force }=\text { Mass } \times \text { Acceleration }
$$

Hence the dimensional relation is,

$$
\begin{align*}
F & =\frac{M L}{T^{2}}  \tag{1.8a}\\
M & =\frac{F T^{2}}{L} \tag{1.8b}
\end{align*}
$$

Equation 1.8 helps in converting dimensions from one system to another. The dimensions of some of the physical quantities are listed in Table 1.2.

Table 1.2 Dimensions of Quantities

| Sr. No. | Quantity | MLT-system | FLT-system |
| :---: | :---: | :---: | :---: |
| 1 | Velocity | $L T^{-1}$ | $L T^{-1}$ |
| 2 | Acceleration | $L T^{-2}$ | $L T^{-2}$ |
| 3 | Momentum | $M L T^{-1}$ | $F T$ |
| 4 | Area | $L^{2}$ | $L^{2}$ |
| 5 | Volume | $L^{3}$ | $L^{3}$ |
| 6 | Force | $M L T^{-2}$ | $F$ |
| 7 | Gravitational Constant | $M^{-1} L^{3} T^{-2}$ | $F^{-1} L^{4} T^{-4}$ |

## Checking Dimensional Homogenity

As stated earlier all the terms in an equation to the left and right side should have the same dimension. In other words if,

$$
X=Y+Z
$$

the terms $X, Y$ and $Z$ should have same dimension. If,

$$
X=b Y
$$

and if $X$ and $Y$ do not have same dimension, ' $b$ ' is not a dimensionless constant. The value of this constant will be different in different system of units.

Example 1.1 Verify whether the following equation has dimensional homoginity:

$$
v^{2}-u^{2}=2 a s
$$

where $v$ is final velocity, $u$ is initial velocity, $a$ is acceleration and $s$ is the distance moved.
Solution. Dimension of velocity $=L T^{-1}$
Dimension of acceleration $=L T^{-2}$
and $\quad$ Dimension of distance $=L$
$\therefore \quad$ Dimension of $v^{2}=\left(L T^{-1}\right)^{2}$
Dimension of $u^{2}=\left(L T^{-1}\right)^{2}$
Dimension of right hand side $=L T^{-2} L=\left(L T^{-1}\right)^{2}$
Hence it is a dimensionally homogeneous equation.
Example 1.2 In the following equation verify, whether 9.81 is a dimensionless constant. If it is not so, what should be its dimension?
where

$$
\begin{aligned}
s & =u t+\frac{1}{2} 9.81 t^{2} \\
s & =\text { distance } \\
u & =\text { initial velocity } \\
t & =\text { time }
\end{aligned}
$$

Solution. Dimensions of various terms are

$$
\begin{aligned}
s & =L \\
u & =L T^{-1} \\
t & =T
\end{aligned}
$$

Substituting these in the given equation, we get

$$
\begin{aligned}
& L=L T^{-1} T+\frac{1}{2} \times 9.81 T^{2} \\
& L=L+\frac{1}{2} \times 9.81 T^{2}
\end{aligned}
$$

Hence 9.81 cannot be dimensionless constant. Its dimensions is given by

$$
L=\frac{1}{2} 9.81 T^{2}
$$

Therefore, 9.81 should have dimension $L T^{-2}$, same as that of acceleration. We know this is gravitational acceleration term in SI unit, i.e., it is in $\mathrm{m} / \mathrm{s}^{2}$ term. Hence the given equation cannot be straight way used in FPS or CGS system.

### 1.5 CHARACTERISTICS OF A FORCE

From Newton's first law, we defined the force as the agency which tries to change state of stress or state of uniform motion of the body. From Newton's second law of motion we arrived at practical definition of unit force as the force required to produce unit acceleration in a body of unit mass. Thus 1 Newton is the force required to produce an acceleration of $1 \mathrm{~m} / \mathrm{s}^{2}$ in a body of 1 kg mass. It may be noted that a force is completely specified only when the following four characteristics are specified:

- Magnitude
- Point of application
- Line of action and
- Direction.

In Fig 1.8, $A B$ is a ladder kept against a wall. At point $C$, a person weighing 600 N is standing. The force applied by the person on the ladder has the following characters:

- magnitude is 600 N
- the point of application is at $C$ which is 2 m from $A$ along the ladder
- the line of action is vertical and
- the direction is downward.


Fig. 1.8

Note that the magnitude of the force is written near the arrow. The line of the arrow shows the line of application and the arrow head represents the point of application and the direction of the force.

### 1.6 SYSTEM OF FORCES

When several forces act simultaneously on a body, they constitute a system of forces. If all the forces in a system do not lie in a single plane they constitute the system of forces in space. If all the forces in a system lie in a single plane, it is called a coplanar force system. If the line of action of all the forces in a system pass through a single point, it is called a concurrent force system. In a system of parallel forces all the forces are parallel to each other. If the line of action of all the forces lie along a single line then it is called a collinear force system. Various system of forces, their characteristics and examples are given in Table 1.3.

Table 1.3 System of Forces

| Force System | Characteristic | Examples |
| :---: | :---: | :---: |
| Collinear forces | Line of action of all the forces act along the same line | Forces on a rope in a tug of war |
| Coplanar parallel forces | All forces are parallel to each other and lie in a single plane. | System of forces acting on a beam subjected to vertical loads (including reactions) |
| Coplanar like parallel forces | All forces are parallel to each other, lie in a single plane and are acting in the same direction. | Weight of a stationary train on a rail when the track is straight |
| Coplanar concurrent forces | Line of action of all forces pass through a single point and forces lie in the same plane. | Forces on a rod resting against a wall |
| Coplanar nonconcurrent forces | All forces do not meet at a point, but lie in a single plane. | Forces on a ladder resting aganist a wall when a person stands on a rung which is not at its centre of gravity |
| Non-coplanar parallel forces | All the forces are parallel to each other, but not in same plane. | The weight of benches in a class room |
| Non-coplanar concurrent forces | All forces do not lie in the same plane, but their lines of action pass through a single point. | A tripod carrying a camera |
| Non-coplanar nonconcurrent forces | All forces do not lie in the same plane and their lines of action do not pass through a single point. | Forces acting on a moving bus |

### 1.7 VECTORS

Various quantities used in engineering mechanics may be grouped into scalars and vectors. A quantity is said to be scalar if it is completely defined by its magnitude alone. Examples of scalars are length, area, time and mass.

A quantity is said to be vector if it is completely defined only when its magnitude as well as direction are specified. Hence force is a vector. The other examples of vector are velocity, acceleration, momentum etc.

Vectors are represented by bold face letters in print and with bars above or below the letters when hand written. Thus $\bar{A}, \bar{B}$ or $\mathbf{A}, \mathbf{B}$ are vectorial representations. The absolute magnitude of vector may be represented by ordinary letters like $A, B$ or as $|A|,|B|$.

### 1.8 VECTORIAL REPRESENTATION OF FORCES AND MOMENTS

Vectorial representation of force by $\mathbf{F}$ and moment by $\mathbf{M}$ will not clearly indicate the direction of the vector. To make the direction clear, the vector is represented by its components in the cartesian coordinates. Referring to Fig. 1.9, the force $F$ in a plane is written as

$$
\begin{equation*}
\mathbf{F}=\mathbf{F}_{\mathrm{x}}+\mathbf{F}_{\mathrm{y}} \tag{1.9a}
\end{equation*}
$$

More convenient way for handling vectors is to represent it in terms of unit vectors in the cartesian coordinate directions. If $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the coordinate directions $x$ and $y$, then

$$
\mathbf{F}_{\mathbf{x}}=F_{x} \mathbf{i}
$$

and

$$
\mathbf{F}_{\mathbf{y}}=F_{y} \mathbf{j}
$$



Fig. 1.9
where $F_{x}$ and $F_{y}$ represent only magnitudes.

$$
\begin{equation*}
\mathbf{F}=F_{x} \mathbf{i}+F_{y} \mathbf{j} \tag{1.9b}
\end{equation*}
$$

Obviously from Fig. 1.9,

$$
\begin{align*}
F_{x} & =F \cos \theta_{x} \text { and } F_{y}=F \cos \theta_{y} \\
\therefore \quad \mathbf{F} & =F \cos \theta_{x} \mathbf{i}+F \cos \theta_{y} \mathbf{j} \\
& =F(l \mathbf{i}+m \mathbf{j}) \tag{1.9c}
\end{align*}
$$

where $l=\cos \theta_{x}$ and $m=\cos \theta_{y}$ are the direction cosines of the force. In two dimensional problems,
since

$$
\begin{aligned}
& \theta_{y}=90-\theta_{x} \\
& m=\cos \theta_{y}=\sin \theta_{x}
\end{aligned}
$$

$\therefore$ Magnitude of the force

$$
\begin{align*}
F & =\sqrt{F_{x}^{2}+F_{y}^{2}}=\sqrt{(F l)^{2}+(F m)^{2}} \\
& =F \sqrt{l^{2}+m^{2}} \tag{1.9~d}
\end{align*}
$$

Similarly a vector in three dimensions (Ref. Fig. 1.10) may be written as

$$
\begin{align*}
\mathbf{F} & =\mathbf{F}_{\mathbf{x}}+\mathbf{F}_{\mathbf{y}}+\mathbf{F}_{\mathbf{z}} \\
& =F_{x} \mathbf{i}+F_{y} \mathbf{j}+F_{z} \mathbf{k}  \tag{1.10a}\\
& =F \cos \theta_{x} \mathbf{i}+F \cos \theta_{y} \mathbf{j}+F \cos \theta_{z} \mathbf{k} \\
& =F(l \mathbf{i}+m \mathbf{j}+n \mathbf{k}) \tag{1.10b}
\end{align*}
$$



Fig. 1.10
where $l=\cos \theta_{x}, m=\cos \theta_{y}$ and $n=\cos \theta_{z}$ are called direction cosines and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are, unit vectors in the directions $x, y, z$. The magnitude of

$$
\begin{align*}
F & =\sqrt{F_{x}^{2}+F_{y}^{2}+F_{z}^{2}}=\sqrt{(F l)^{2}+(F m)^{2}+(F n)^{2}} \\
& =F \sqrt{l^{2}+m^{2}+n^{2}} \tag{1.10c}
\end{align*}
$$

If $n$ is the unit vector in the direction of the force, then

$$
\begin{equation*}
\mathbf{F}=F \mathbf{n} \tag{1.11}
\end{equation*}
$$

From Eqns. 1.9 and 1.10, we get

$$
\begin{align*}
F(l \mathbf{i}+m \mathbf{j}+n \mathbf{k}) & =F \mathbf{n} \\
\mathbf{n} & =l \mathbf{i}+m \mathbf{j}+n \mathbf{k} \tag{1.12}
\end{align*}
$$

or
Thus a vector can be represented as

$$
\begin{align*}
\mathbf{F} & =F \mathbf{n} \\
& =F(l \mathbf{i}+m \mathbf{j}+n \mathbf{k}) \tag{1.13}
\end{align*}
$$



Fig. 1.11
where $F$ is the magnitude of the vector, $n$ unit vector in the direction of vector and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the coordinate directions $x, y, z . F$ may be any vector like force, moment, velocity etc. Usually in the literature $F$ is used to represent the
force and $M$ for moments. Hence, if we refer $M$ to moment, its vectorial representation is

$$
\begin{aligned}
\mathbf{M} & =M \mathbf{n} \\
& =M(l \mathbf{i}+m \mathbf{j}+n \mathbf{k})
\end{aligned}
$$

with usual notations.
Example 1.3 Determine the vectorial form of forces for $F_{1}, F_{2}$, and $F_{3}$ shown in Fig. 1.12.


Fig. 1.12
Solution. For force $\mathbf{F}_{1}=$ Magnitude 800 N .

$$
\begin{array}{rlrl}
\theta_{x} & =40^{\circ} & \therefore \theta_{y} & =50^{\circ} \\
\therefore \quad F_{1 \mathrm{x}} & =800 \cos 40^{\circ} & F_{1 y} & =800 \cos 50^{\circ} \\
& & =612.84 \mathrm{~N} & \\
& & =514.23 \mathrm{~N}
\end{array}
$$

$\therefore \quad \mathbf{F}_{1}=612.84 \mathbf{i}+514.23 \mathbf{j}$
For force $\mathbf{F}_{2}$ : Magnitude $F_{2}=100 \mathrm{~N} \quad \theta_{x}=210^{\circ} \quad \theta_{y}=120^{\circ}$

$$
\begin{aligned}
& F_{2 x}=100 \cos 210^{\circ}=-86.60 \mathrm{~N} \\
& F_{2 y}=100 \cos 120^{\circ}=-50.0 \mathrm{~N} \\
\therefore \quad & \mathbf{F}_{2}=-86.6 \mathbf{i}-50.0 \mathbf{j}
\end{aligned}
$$

For force $\mathbf{F}_{3}$ : Magnitude $F_{3}=120 \mathrm{~N}, \theta_{x}=300^{\circ}, \theta_{y}=210^{\circ}$

$$
\begin{aligned}
\therefore \quad F_{2 x} & =120 \cos 300^{\circ} & F_{2 y} & =120 \cos 210^{\circ} \\
& =60 \mathrm{~N} & & =-103.92 \mathrm{~N}
\end{aligned}
$$

$$
\therefore \quad \mathbf{F}_{3}=60 \mathbf{i}-103.92 \mathbf{j}
$$

## Note:

The component of forces may be noted using the acute angles with $x$ and $y$ axis, but carefully observing for the sign of the component, e.g. for $F_{3}$,

$$
\begin{array}{ll} 
& F_{x}=120 \cos 60^{\circ}=60 \mathrm{~N} \\
& F_{y}=-120 \cos 30^{\circ}=-103.92 \mathrm{~N} \\
\therefore \quad & \mathbf{F}_{3}=60 \mathbf{i}-103.92 \mathbf{j}
\end{array}
$$

To get the direction of force move from tail to head of the force vector in the desired coordinate directions.

Example 1.4 A force of magnitude 800 N makes $20^{\circ}$ with the $y$-axis and its projection on $x-z$ plane makes $30^{\circ}$ with the $x$-axis. Express the force in the vector form. Find the direction cosines and the direction of the vector w.r.t. $x, y$ and $z$ directions.

Solution. Figure 1.13 shows the given force in $x, y, z$ coordinate system. [Note the coordinate system selected is always the right hand system i.e., the system in which the stretched thumb indicates $x$-direction, index finger indicates $y$-direction and the middle finger which is stretched at right angles to $x-y$ plane indicates $z$-direction].

$$
\begin{aligned}
F & =800 \mathrm{~N} \\
F_{y} & =800 \cos 20^{\circ}=751.8 \mathrm{~N} \\
O B & =800 \cos (90-20)^{\circ}=273.6 \mathrm{~N}
\end{aligned}
$$

Resolving $O B$ in $x-z$ plane, we get

$$
\begin{aligned}
F_{x} & =O B \cos 30^{\circ}=273.6 \cos 30^{\circ} \\
& =237 \mathrm{~N} \\
F_{z} & =O B \cos (90-30)^{\circ}=273.6 \sin 30^{\circ}=136.8 \mathrm{~N} \\
\therefore \quad \quad \mathbf{F} & =237 \mathbf{i}+751.8 \mathbf{j}+136.8 \mathbf{k}
\end{aligned}
$$



Fig. 1.13

Let $l, m, n$ be direction cosines. Then,

$$
\text { i.e., } \quad \cos \theta_{x}=0.296
$$

$$
\begin{aligned}
F_{x} & =l F \\
237 & =l 800 \text { or } l=0.296 \\
\text { os } \theta_{x} & =0.296 \\
\theta_{x} & =72.767^{\circ} \\
F_{y} & =m F
\end{aligned}
$$

or

$$
751.8=m 800 \text { or } m=0.9396
$$

i.e., $\quad \cos \theta_{y}=0.9396$ or $\theta_{y}=20^{\circ} \quad$ (given in this problem)
and $\quad F_{z}=n F$
$136.8=n 800$ or $n=0.171$
i.e., $\quad \cos \theta_{z}=n=0.171$
or $\quad \theta_{z}=80.154^{\circ}$
Thus the direction cosines are $l=0.296, m=0.9396$ and $n=0.1732$ and the directions are $\theta_{x}=72.767^{\circ}, \theta_{y}=20^{\circ}$ and $\theta_{z}=80.154^{\circ}$

Ans.

### 1.9 POSITION VECTOR

The position vector of a point in the space is defined as the vector represented by the line segment connecting the origin and the point. Cartesian coordinate system
used to locate a point in the space is always taken according to right hand rule. Figure 1.14 shows different orientations of cartesian system according to right hand rule.

(a)


(c)

Fig. 1.14 Different Orientations of Right Hand System of Coordinates
Let $A$ be a point in space and the coordinate system selected be $x, y, z$ as shown in Fig. 1.15.


Fig. 1.15
Then the position vector of point $A$ is $O A$ and is represented as $r_{O A}$. If $x, y$, $z$ are the coordinates of $A$ then the vector $\mathbf{r}_{\mathbf{O A}}$ is given by

$$
\begin{equation*}
\mathbf{r}_{\mathbf{O A}}=x \mathbf{i}+y \mathbf{j}+z \mathbf{k} \tag{1.14}
\end{equation*}
$$

Magnitude of this vector is given by

$$
\begin{equation*}
r_{O A}=\sqrt{x^{2}+y^{2}+z^{2}} \tag{1.15}
\end{equation*}
$$

In case of two dimensional problem the position vector will be (Ref. Fig. 1.16)
and

$$
\mathbf{r}_{\mathrm{OA}}=x \mathbf{i}+y \mathbf{j}
$$

$$
r_{O A}=\sqrt{x^{2}+y^{2}}
$$



Fig. 1.16

### 1.10 ADDITION OF VECTORS

The parallelogram law of forces holds good for any vector. According to this law, if $\mathbf{P}$ and $\mathbf{Q}$ are two vectors to be added, construct a parallelogram making these two vectors as adjacent sides, then the diagonal $R$ of the parallelogram passing through the intersection of these two points represent the addition of vectors $P$ and $Q$, the direction being from the intersection point towards the diagonally opposite point. Thus in Fig. 1.17.

$$
\begin{equation*}
\mathbf{R}=\mathbf{P}+\mathbf{Q} \tag{1.16}
\end{equation*}
$$



Fig. 1.17
From the parallelogram law of vectors, triangle law of vectors may be derived. Referring to Fig. 1.17, instead of drawing parallelogram $O A C B$ to get $R$, same result can be obtained by constructing the triangle $O A C$. In other words to get the vector addition of $P$ and $Q$ construct the triangle in tail to head fashion one after the other. Then the closing line of the triangle $O A C$ i.e., $O C$ represents the addition of $P$ and $Q$ as shown in Fig. 1.18. This is called triangle law of vectors.


Fig. 1.18

The triangle law of vectors may be used repeatedly applied to get addition of number of vectors. Let $P, Q, R$ and $S$ be the vectors to be added [Ref. Fig. 1.19(a)]


Fig. 1.19
From Fig 1.19(b) we find,

$$
\begin{align*}
& \mathbf{T}=\mathbf{P}+\mathbf{Q} \\
& \mathbf{U}=\mathbf{T}+\mathbf{R}=\mathbf{P}+\mathbf{Q}+\mathbf{R} \\
& \mathbf{V}=\mathbf{U}+\mathbf{S}=\mathbf{P}+\mathbf{Q}+\mathbf{R}+\mathbf{S} \tag{1.17}
\end{align*}
$$

Instead of drawing triangles to add two vectors at a time, from Fig 1.19 it may be observed that the same result can be obtained if the vectors are added in tail to head fashion one after the other. Then the closing line of the polygon shows the addition in the direction from the first point to the last point. This is called polygon law of vectors and may be stated as if vectors are drawn one after the other in tail to head fashion, the line joining the tail of the first vector and the head of the last vector represents the sum of all the vectors.

From Fig. 1.17, it may be observed that,

$$
\begin{equation*}
\mathbf{R}=\mathbf{P}+\mathbf{Q}=\mathbf{Q}+\mathbf{P} \tag{1.18}
\end{equation*}
$$

Thus the commutative law of addition holds good. Figure 1.20 shows addition of three vectors $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ using polygon law of forces. In this it may be seen that,

$$
\begin{equation*}
(\mathbf{P}+\mathbf{Q})+\mathbf{R}=\mathbf{P}+(\mathbf{Q}+\mathbf{R})=\mathbf{T}_{3} \tag{1.19}
\end{equation*}
$$

Since

$$
\mathbf{P}+\mathbf{Q}=\mathbf{T}_{1} \text { and } \mathbf{Q}+\mathbf{R}=\mathbf{T}_{\mathbf{2}}
$$



Fig. 1.20

Thus the associative law holds good.
All the above concepts have been developed using graphical method, since parallelogram law of vector addition is the base for vector addition. It is not convenient to use graphical method in the analysis since it needs drawing sheet and drawing accessories. Analytical method is more convenient for any design office. This method consists in expressing the quantities in the vectorial form as presented in equation 1.13. Then all the component with $i$ direction can be added algebraically since they are collinear. Similarly components in $j$ and $k$ directions may be added. Thus if
and

$$
\mathbf{P}=5 \mathbf{i}+6 \mathbf{j}-3 \mathbf{k}
$$

$\mathbf{Q}=7 \mathbf{i}-4 \mathbf{j}+5 \mathbf{k}$,
then

$$
\mathbf{R}=\mathbf{P}+\mathbf{Q}=(5+7) \mathbf{i}+(6-4) \mathbf{j}+(-3+5) \mathbf{k}
$$

The commutative law and associative law hold good.

### 1.11 SUBTRACTION OF VECTORS

If $Q$ is a vector, then ' $-Q$ ' is a vector of the same magnitude and line of action, but opposite in the sense. Hence to subtract $Q$ from $P$, we can make use of law of addition to add ' $P$ ' and ' $-Q$ '. Figure 1.21 shows use of parallelogram and triangle laws for this purpose.


Fig. 1.21
Analytically, if $\mathbf{Q}=a_{2} \mathbf{i}+b_{2} \mathbf{j}+c_{2} \mathbf{k}$, then ' $-\mathbf{Q}^{\prime}$ ' is $-a_{2} \mathbf{i}-b_{2} \mathbf{j}-c_{2} \mathbf{k}$
If

$$
\begin{align*}
\mathbf{P} & =a_{1} \mathbf{i}+b_{1} \mathbf{j}+c_{1} \mathbf{k}, \text { then } \\
\mathbf{P}-\mathbf{Q} & =\mathbf{P}+(-\mathbf{Q})=\left(a_{1}-a_{2}\right) \mathbf{i}+\left(b_{1}-b_{2}\right) \mathbf{j}+\left(c_{1}-c_{2}\right) \mathbf{k} \tag{1.20}
\end{align*}
$$

### 1.12 PRODUCT OF A VECTOR WITH A SCALAR QUANTITY

If we add vector $\mathbf{P}$ to itself, we get a vector of magnitude $2 \mathbf{P}$. If we add $\mathbf{P}$ to itself $m$ times, we get vector of magnitude $m$ times $\mathbf{P}$. i.e., it will be $m \mathbf{P}$. On the same line, we can see that

$$
\begin{align*}
(m+n) \mathbf{P} & =m \mathbf{P}+n \mathbf{P}  \tag{1.21a}\\
m(\mathbf{P}+\mathbf{Q}) & =m \mathbf{P}+m \mathbf{Q}  \tag{1.21b}\\
m(n \mathbf{P}) & =m n \mathbf{P} \tag{1.21c}
\end{align*}
$$

Example 1.5 $\mathbf{P}_{1}, \mathbf{P}_{2}$ and $\mathbf{P}_{3}$ are the vectors as given below:

$$
\begin{aligned}
& \mathbf{P}_{1}=8 \mathbf{i}+10 \mathbf{j}-6 \mathbf{k} \\
& \mathbf{P}_{2}=9 \mathbf{i}+3 \mathbf{j}+4 \mathbf{k} \\
& \mathbf{P}_{3}=\mathbf{i}-5 \mathbf{j}+4 \mathbf{k}
\end{aligned}
$$

Determine (i) $\mathbf{P}=\mathbf{P}_{1}+\mathbf{P}_{2}+\mathbf{P}_{3}$ and $P$
(ii) $\mathbf{P}=\mathbf{P}_{1}-\mathbf{P}_{2}+\mathbf{P}_{3}$ and $P$

## Solution.

(i)

$$
\begin{aligned}
\mathbf{P} & =\mathbf{P}_{1}+\mathbf{P}_{\mathbf{2}}+\mathbf{P}_{\mathbf{3}} \\
& =8 \mathbf{i}+10 \mathbf{j}-6 \mathbf{k}+9 \mathbf{i}+3 \mathbf{j}+4 \mathbf{k}+\mathbf{i}-5 \mathbf{j}+4 \mathbf{k} \\
& =(8+9+1) \mathbf{i}+(10+3-5) \mathbf{j}+(-6+4+4) \mathbf{k} \\
& =18 \mathbf{i}+8 \mathbf{j}+2 \mathbf{k}
\end{aligned}
$$

$$
\begin{aligned}
& =18 \mathbf{i}+8 \mathbf{j}+2 \mathbf{k} \\
\therefore & P
\end{aligned}
$$

Ans.
(ii)

$$
\begin{aligned}
\mathbf{P} & =\mathbf{P}_{\mathbf{1}}-\mathbf{P}_{\mathbf{2}}+\mathbf{P}_{\mathbf{3}} \\
& =8 \mathbf{i}+10 \mathbf{j}-6 \mathbf{k}-(9 \mathbf{i}+3 \mathbf{j}+4 \mathbf{k})+\mathbf{i}-5 \mathbf{j}+4 \mathbf{k} \\
& =(8-9+1) \mathbf{i}+(10-3-5) \mathbf{j}+(-6-4+4) \mathbf{k} \\
& =0 \mathbf{i}+2 \mathbf{j}-6 \mathbf{k}
\end{aligned}
$$

$$
\begin{aligned}
& =2 \mathbf{j}-6 \mathbf{k} \\
\therefore \quad P & =\sqrt{2^{2}+6^{2}}=6.325 \text { units }
\end{aligned}
$$

Ans.

Example 1.6 The vectors $\mathbf{P}_{1}, \mathbf{P}_{2}$ and $\mathbf{P}_{3}$ are as given below:

$$
\begin{aligned}
& \mathbf{P}_{1}=2 \mathbf{i}+3 \mathbf{j}-\mathbf{k} \\
& \mathbf{P}_{2}=\mathbf{i}-3 \mathbf{j}+2 \mathbf{k} \\
& \mathbf{P}_{3}=-\mathbf{i}+2 \mathbf{j}+3 \mathbf{k}
\end{aligned}
$$

Find $\mathbf{P}=2 \mathbf{P}_{1}+3 \mathbf{P}_{2}-2 \mathbf{P}_{3}$ and determine the magnitude of $P$.

## Solution.

$$
\begin{aligned}
\mathbf{P} & =2 \mathbf{P}_{1}+3 \mathbf{P}_{2}-2 \mathbf{P}_{3} \\
& =2(2 \mathbf{i}+3 \mathbf{j}-\mathbf{k})+3(\mathbf{i}-3 \mathbf{j}+2 \mathbf{k})-2(-\mathbf{i}+2 \mathbf{j}+3 \mathbf{k}) \\
& =(4+3+2) \mathbf{i}+(6-9-4) \mathbf{j}+(-2+6-6) \mathbf{k} \\
& =9 \mathbf{i}-7 \mathbf{j}-2 \mathbf{k}
\end{aligned}
$$

Ans.
Magnitude of $P$ is

$$
P=\sqrt{9^{2}+7^{2}+2^{2}}=11.576 \text { units }
$$

Ans.

### 1.13 DISPLACEMENT VECTOR

A vector in space with its tail at one point (say point 1) and the head at another point (say point 2) is termed as displacement vector. It represents the direction from point 1 to 2 and magnitude of the rectilinear distance between those two points. In Fig. 1.22,


Fig. 1.22
The vector $\mathbf{r}_{12}$ is displacement vector. Since $\mathbf{r}_{01}$ and $\mathbf{r}_{02}$ are referred as position vectors of point 1 and 2 ,

$$
\begin{align*}
\mathbf{r}_{12} & =\mathbf{r}_{02}-\mathbf{r}_{01} \\
& =x_{2} \mathbf{i}+y_{2} \mathbf{j}+z_{2} \mathbf{k}-\left(x_{1} \mathbf{i}+y_{1} \mathbf{j}+z_{1} \mathbf{k}\right) \\
& =\left(x_{2}-x_{1}\right) \mathbf{i}+\left(y_{2}-y_{1}\right) \mathbf{j}+\left(z_{2}-z_{1}\right) \mathbf{k}  \tag{1.22a}\\
\mathbf{r}_{12} & =\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}+\left(z_{2}-z_{1}\right)^{2}} \tag{1.22b}
\end{align*}
$$

and hence
Example 1.7 Determine the vector form of displacement vector $A B$ connecting points $A(15,-10)$ and $B(-8,20)$. Find its magnitude, direction cosines and directions also.

## Solution.

$$
\left.\begin{array}{rlrl}
A(15,-10) & , B(-8,20) & & \\
\mathbf{r}_{\mathbf{O A}} & =15 \mathbf{i}-10 \mathbf{j} \quad \text { and } \quad \mathbf{r}_{\mathbf{O B}}=-8 \mathbf{i}+20 \mathbf{j} \\
\mathbf{r}_{\mathbf{A B}} & =\mathbf{r}_{\mathbf{O B}}-\mathbf{r}_{\mathbf{O A}} \\
& =-8 \mathbf{i}+20 \mathbf{j}-(15 \mathbf{i}-10 \mathbf{j}) \\
& =-23 \mathbf{i}+30 \mathbf{j} & & \\
r_{A B} & =\sqrt{(-23)^{2}+30^{2}}=37.80 \text { units } & & \text { Ans. } \\
r_{A B} l & =-23 & & \text { Ans. } \\
r_{A B} m & =30 & & \therefore l
\end{array}\right)-\frac{23}{37.80}=-0.608 \quad \text { Ans. }
$$

Example 1.8 Determine the displacement vector $A B$ when the coordinates of $A$ and $B$ in mm units are given by $A(60,80,-20)$ and $B(100,-40,80)$. Find its magnitude, direction cosines and directions.

## Solution.

$$
\left.\begin{array}{rlrl} 
& \mathrm{A}(60,80,-20) & \mathrm{B}(100,-40,80) \\
\mathbf{r}_{\mathbf{O A}}=60 \mathbf{i}+80 \mathbf{j}-20 \mathbf{k} \quad \mathbf{r}_{\mathbf{O B}}=100 \mathbf{i}-40 \mathbf{j}+80 \mathbf{k}
\end{array}\right)
$$

### 1.14 DOT PRODUCT OF VECTORS

The dot product of two vectors $\mathbf{P}$ and $\mathbf{Q}$ is defined as

$$
\begin{equation*}
\mathbf{P} \cdot \mathbf{Q}=P Q \cos \theta \tag{1.23}
\end{equation*}
$$

where $\theta$ is the acute angle between the two vectors. The left hand side of Eqn. 1.23 is read as $P$ dot $Q$. Since the right hand side of dot product is a scalar, this product is also known as scalar product of vectors.

The graphical representation of dot product is shown in Fig. 1.23. The right hand side of Eqn. 1.23 may be looked as the magnitude of P multiplied by the projection of magnitude of $Q$ on $P$ [Fig. 1.23 (b)] or as magnitude of $Q$ multiplied by the projection of magnitude of $P$ on $Q$ [as seen from Fig. 1.23 (c)].


Fig. 1.23

Since $\theta$ is the angle between the two vectors involved in the dot product, we get

$$
\begin{aligned}
\mathbf{i} \cdot \mathbf{i} & =(1)(1) \cos 0=1 \\
\mathbf{j} \cdot \mathbf{j} & =(1)(1) \cos 0=1 \\
\mathbf{k} \cdot \mathbf{k} & =(1)(1) \cos 0=1 \\
\mathbf{i} \cdot \mathbf{j} & =(1)(1) \cos 90=0 \\
\text { Similarly } \quad & \mathbf{j} \cdot \mathbf{k}
\end{aligned}=\mathbf{k} \cdot \mathbf{i}=\mathbf{i} \cdot \mathbf{k}=\mathbf{k} \cdot \mathbf{j}=\mathbf{j} \cdot \mathbf{i}=0 \mathrm{l}=\mathrm{l}
$$

Thus in dot product

$$
\begin{equation*}
\mathbf{i} \cdot \mathbf{i}=\mathbf{j} \cdot \mathbf{j}=\mathbf{k} \cdot \mathbf{k}=\mathbf{1} \tag{1.24a}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{i} \cdot \mathbf{j}=\mathbf{j} \cdot \mathbf{k}=\mathbf{k} \cdot \mathbf{i}=\mathbf{i} \cdot \mathbf{k}=\mathbf{k} \cdot \mathbf{j}=\mathbf{j} \cdot \mathbf{i}=0 \tag{1.24b}
\end{equation*}
$$

The following useful results from dot product may be noted:

$$
\begin{array}{rlrl}
\text { (i) If } & \mathbf{P} \cdot \mathbf{Q} & =0 \\
& & P Q \cos \theta & =0 \\
\therefore & \cos \theta & =0 \tag{1.25}
\end{array}
$$

i.e., the vector $P$ and $Q$ are at right angle to each other.
(ii) Using the Eqn. 1.24,

$$
\begin{align*}
\mathbf{P} \cdot \mathbf{Q} & =\left(P_{x} \mathbf{i}+P_{y} \mathbf{j}+P_{z} \mathbf{k}\right)\left(Q_{x} \mathbf{i}+Q_{y} \mathbf{j}+Q_{z} \cdot \mathbf{k}\right) \\
& =P_{x} Q_{x}+P_{y} Q_{y}+P_{z} Q_{z}  \tag{1.26a}\\
\mathbf{P} . \mathbf{P} & =P_{x}^{2}+P_{y}^{2}+P_{z}^{2} \tag{1.26b}
\end{align*}
$$

and also
(iii) If $\mathbf{P}$ and $\mathbf{Q}$ are known, the angle between the two vectors is given by

$$
\begin{equation*}
\cos \theta=\frac{\mathbf{P} \cdot \mathbf{Q}}{P Q}=\frac{P_{x} Q_{x}+P_{y} Q_{y}+P_{z} Q_{z}}{P Q} \tag{1.27}
\end{equation*}
$$

If $l_{1}, m_{1}, n_{1}$ are the direction cosines of $\mathbf{P}$ and $l_{2}, m_{2}, n_{2}$ are the direction cosines of $\mathbf{Q}$ then

$$
\begin{align*}
\cos \theta & =\frac{P l_{1} Q l_{2}+P m_{1} Q m_{2}+P n_{1} Q n_{2}}{P Q} \\
& =l_{1} l_{2}+m_{1} m_{2}+n_{1} n_{2} \tag{1.28}
\end{align*}
$$

(iv) The projection of a vector may be obtained in any desired direction, since the projection of a vector $P$ in direction $\theta$ to it is equal to $P \cos \theta, \cos \theta$ being found by using Eqn. 1.28.
Example 1.9 Determine the dot product and the angle between the vectors
and

$$
\mathbf{F}_{1}=4 \mathbf{i}+\mathbf{j}-\mathbf{k}
$$

Solution. Noting that $\mathbf{i} \cdot \mathbf{i}=\mathbf{j} \cdot \mathbf{j}=\mathbf{k} \cdot \mathbf{k}=\mathbf{1} \& \mathbf{i} \cdot \mathbf{j}=\mathbf{j} \cdot \mathbf{k}$, etc. $=0$

$$
\begin{aligned}
\mathbf{F}_{1} \cdot \mathbf{F}_{2} & =(4 \mathbf{i}+\mathbf{j}-\mathbf{k}) \cdot(3 \mathbf{i}+2 \mathbf{j}+\mathbf{k}) \\
& =4 \times 3+1 \times 2-1 \times 1 \\
& =13
\end{aligned}
$$

Ans.

$$
\begin{aligned}
& F_{1}=\sqrt{4^{2}+1^{2}+(-1)^{2}}=4.243 \\
& F_{2}=\sqrt{3^{2}+2^{2}+1^{2}}=3.742
\end{aligned}
$$

From dot matrix rule, we know

$$
\begin{aligned}
& & \mathbf{F}_{1} \cdot \mathbf{F}_{2} & =F_{1} F_{2} \cos \theta \\
& \therefore & 13 & =(4.243)(3.742) \cos \theta . \\
& \therefore & \cos \theta & =0.8188 \text { or } \theta=35.037^{\circ}
\end{aligned}
$$

Ans.
Example 1.10 If $\mathbf{F}_{1}=5 \mathbf{i}+4 \mathbf{j}+3 \mathbf{k}$ and $\mathbf{F}_{2}=4 \mathbf{i}-2 \mathbf{j}-4 \mathbf{k}$. Show that $\mathbf{F}_{1}$ and $\mathbf{F}_{2}$ are at right angles to each other.

## Solution.

$$
\begin{aligned}
& F_{1}=\sqrt{5^{2}+4^{2}+3^{2}}=7.071 \\
& F_{2}=\sqrt{4^{2}+(-2)^{2}+(-4)^{2}}=6 \\
& \mathbf{F}_{1} \cdot \mathbf{F}_{2}=(5 \mathbf{i}+4 \mathbf{j}+3 \mathbf{k}) \cdot(4 \mathbf{i}-2 \mathbf{j}-4 \mathbf{k}) \\
&=5 \times 4-4 \times 2+3(-4)=0 . \\
& \text { i.e., } \quad F_{1} F_{2} \cos \theta=0, \text { i.e. }, \cos \theta=0 \\
& \therefore \quad \mathbf{F}_{1} \text { and } \mathbf{F}_{2} \text { are at right angles to each other }
\end{aligned}
$$

Ans.
Example 1.11 Determine the projection of force $\mathbf{F}_{1}=3 \mathbf{i}+2 \mathbf{j}-\mathbf{k}$ on the line of action of the force $\mathbf{F}_{2}=\mathbf{i}+4 \mathbf{j}+\mathbf{k}$.

## Solution.

$$
\begin{array}{rlrl}
\mathbf{F}_{1} & =3 \mathbf{i}+2 \mathbf{j}-\mathbf{k} \quad \therefore F_{1}=\sqrt{3^{2}+2^{2}+(-1)^{2}}=3.742 \\
\mathbf{F}_{2} & =\mathbf{i}+4 \mathbf{j}+\mathbf{k} \quad \therefore F_{2}=\sqrt{1^{2}+4^{2}+1^{2}}=4.2426 \\
\therefore \quad & \mathbf{F}_{1} \cdot \mathbf{F}_{2} & =(3 \mathbf{i}+2 \mathbf{j}-\mathbf{k}) \cdot(\mathbf{i}+4 \mathbf{j}+\mathbf{k}) \\
& =3 \times 1+2 \times 4-1 \times 1=10 \\
\text { i.e., } \quad & F_{1} F_{2} \cos \theta & =10
\end{array}
$$

$\therefore$ Projection of $\mathbf{F}_{1}$ on the line of action of $\mathbf{F}_{2}$

$$
F_{1} \cos \theta=\frac{F_{1} F_{2} \cos \theta}{F_{2}}=\frac{10}{4.2426}=2.357
$$

Ans.

### 1.15 CROSS PRODUCT OF VECTORS

Cross product of vectors is the another type of product of vectors used in vector operation. This is also called vector product of vectors. The cross product of two vectors $P$ and $Q$ is defined by

$$
\begin{equation*}
\mathbf{P} \times \mathbf{Q}=P Q \sin \theta \mathbf{n} \tag{1.29}
\end{equation*}
$$

The left hand side of Eqn. 1.29 is read as $P$ cross $Q$. In the right hand side ' $\theta$ ' is the acute angle between $P$ and $Q$, i.e., $\sin$ ' $\theta$ ' is always a positive quantity and $\mathbf{n}$ is the unit vector in the direction perpendicular to the plane of $P$ and $Q$. Its sense is given by the direction of advance of a right hand screw rotated from first vector to second vector. The vectors $\mathbf{P}, \mathbf{Q}$ and their cross product vector $\mathbf{R}=\mathbf{P} \times \mathbf{Q}$ are shown in Fig. 1.24. The following points of cross product may be carefully noted.
(i) The magnitude is $P Q \sin \theta$


Fig. 1.24
(ii) It is a vector, with direction at right angles to the plane of $P$ and $Q$ in the sense advance of right hand screw from first to second vector is positive.
Figure 1.25 shows the cross product $\mathbf{Q} \times \mathbf{P}$. Obviously its direction is opposite to that of $\mathbf{P} \times \mathbf{Q}$, but the magnitude is same as that of $\mathbf{P} \times \mathbf{Q}$.

Figure 1.26 shows the vectors $\mathbf{P}$ and $\mathbf{Q}$ drawn to the scale and the parallelogram with $\mathbf{P}$ and $\mathbf{Q}$ as sides, completed.


Fig. 1.25


Fig. 1.26

Now, the magnitude of the product $\mathbf{P} \times \mathbf{Q}$ is

$$
\begin{align*}
& =P Q \sin \theta \\
& =P(Q \sin \theta) \\
& =\text { Area of the parallelogram } \tag{1.30}
\end{align*}
$$

Thus the magnitude of $\mathbf{P}$ cross $\mathbf{Q}$ is equal to the area of parallelogram built with $\mathbf{P}$ and $\mathbf{Q}$ as sides.

From the cross product rule, we can note the following points:
(i) Since $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are unit vectors in three mutually perpendicular directions taken according to the right handed coordinate system, from the definition of cross product, we note

$$
\begin{array}{llll}
\mathbf{i} \times \mathbf{j}=\mathbf{k} & \mathbf{j} \times \mathbf{k}=\mathbf{i} & \text { and } & \mathbf{k} \times \mathbf{i}=\mathbf{j} \\
\mathbf{j} \times \mathbf{i}=-\mathbf{k} & \mathbf{k} \times \mathbf{j}=-\mathbf{i} & \text { and } & \mathbf{i} \times \mathbf{k}=-\mathbf{j} \tag{1.31~b}
\end{array}
$$

(ii) The cross product of a collinear vector must vanish since in such case, $\theta=0$.
i.e.,

$$
\begin{equation*}
\mathbf{i} \times \mathbf{i}=\mathbf{j} \times \mathbf{j}=\mathbf{k} \times \mathbf{k}=0 \tag{1.32}
\end{equation*}
$$

(iii)

$$
\begin{align*}
\text { Now } & \mathbf{P} \times \mathbf{Q}=\left(P_{x} \mathbf{i}+P_{y} \mathbf{j}+P_{z} \mathbf{k}\right) \times\left(Q_{x} \mathbf{i}+Q_{y} \mathbf{j}+Q_{z} \mathbf{k}\right) \\
& =P_{x} Q_{y} \mathbf{k}+P_{x} Q_{z}(-\mathbf{j})+P_{y} Q_{x}(-\mathbf{k})+P_{y} Q_{z} \mathbf{i}+P_{z} Q_{x} \mathbf{j}+P_{z} Q_{y}(-\mathbf{i}) \\
& =\left(P_{y} Q_{z}-P_{z} Q_{y}\right) \mathbf{i}+\left(P_{z} Q_{x}-P_{x} Q_{z}\right) \mathbf{j}+\left(P_{x} Q_{y}-P_{y} Q_{x}\right) \mathbf{k} \\
& =\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
P_{x} & P_{y} & P_{z} \\
Q_{x} & Q_{y} & Q_{z}
\end{array}\right| \tag{1.33}
\end{align*}
$$

(iv)The cross product may be used to find the angle between the two vectors. In this case

$$
\begin{equation*}
\sin \theta=\frac{|\mathbf{P} \times \mathbf{Q}|}{P Q} \tag{1.34}
\end{equation*}
$$

where $|\mathbf{P} \times \mathbf{Q}|$ is the absolute magnitude of $\mathbf{P} \times \mathbf{Q}$.
(v)The commutative law does not hold good
i.e.,

$$
\mathbf{P} \times \mathbf{Q} \neq \mathbf{Q} \times \mathbf{P}
$$

(vi) The distributive law holds good, i.e.,

$$
\begin{align*}
& \mathbf{P} \times(\mathbf{Q} \times \mathbf{R}) & =\mathbf{P} \times \mathbf{Q} \times \mathbf{R}  \tag{1.36a}\\
\text { and } & n(\mathbf{P} \times \mathbf{Q}) & =n \mathbf{P} \times \mathbf{Q}=\mathbf{P} \times n \mathbf{Q} \tag{1.36b}
\end{align*}
$$

Example 1.12 Using cross product determine the angle between the vectors $\mathbf{F}_{1}$ and $\mathbf{F}_{2}$ given in Example 1.9. Also determine the magnitude and direction cosines of the resultant vector.
Solution. In this problem

$$
\begin{aligned}
\mathbf{F}_{1} & =4 \mathbf{i}+\mathbf{j}-\mathbf{k} \text { and } \mathbf{F}_{2}=3 \mathbf{i}+2 \mathbf{j}+\mathbf{k} \\
\therefore \quad \mathbf{F}_{1} \times \mathbf{F}_{2} & =\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
4 & 1 & -1 \\
3 & 2 & 1
\end{array}\right| \\
& =\mathbf{i}(1+2)-\mathbf{j}(4+3)+\mathbf{k}(8-3) \\
& =3 \mathbf{i}-7 \mathbf{j}+5 \mathbf{k} \\
\therefore \quad\left|\mathbf{F}_{1} \times \mathbf{F}_{2}\right| & =\sqrt{3^{2}+(-7)^{2}+5^{2}}=9.110 \\
F_{1} & =\sqrt{4^{2}+1^{2}+(-1)^{2}}=4.243 \\
F_{2} & =\sqrt{3^{2}+2^{2}+1^{2}}=3.742
\end{aligned}
$$

$$
\begin{aligned}
& \therefore \quad \sin \theta=\frac{\left|\mathbf{F}_{1} \times \mathbf{F}_{2}\right|}{F_{1} F_{2}}=\frac{9.110}{(4.243)(3.742)}=0.5738 \\
& \therefore \quad \theta=35.014^{\circ} \\
& \text { Now } \quad \mathbf{R}=\mathbf{F}_{1} \times \mathbf{F}_{2}=3 \mathbf{i}-7 \mathbf{j}+5 \mathbf{k} \\
& \therefore \quad R=\sqrt{3^{2}+7^{2}+5^{2}}=9.110 \\
& \text { Now } \quad R_{x}=R l=3 \\
& l=\frac{3}{R}=\frac{3}{9.11}=-0.329 \\
& m=\frac{-7}{9.11}=-0.768 \\
& n=\frac{5}{9.11}=0.549
\end{aligned}
$$

$\therefore$ The unit vector of $R$ is

$$
\mathbf{n}=0.329 \mathbf{i}-0.768 \mathbf{j}+0.549 \mathbf{k}
$$

Example 1.13 Determine the cross product of $\mathbf{P}$ and $\mathbf{Q}$
where

$$
\mathbf{P}=2 \mathbf{i}+4 \mathbf{j}-3 \mathbf{k}
$$

and

$$
\mathbf{Q}=3 \mathbf{i}+5 \mathbf{j}+\mathbf{k}
$$

and hence find angle $\theta$ between those two vectors.

## Solution.

$$
\begin{aligned}
& \mathbf{R}=\mathbf{P} \times \mathbf{Q}=\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 4 & -3 \\
3 & 5 & 1
\end{array}\right| \\
& =\mathbf{i}(4+15)-\mathbf{j}(2+9)+\mathbf{k}(10-12) \\
& =19 \mathbf{i}-11 \mathbf{j}-2 \mathbf{k} \\
& \therefore \quad R=\sqrt{19^{2}+(-11)^{2}+(-2)^{2}}=22.045 \text { units } \\
& P=\sqrt{2^{2}+4^{2}+(-3)^{2}}=5.385 \text { units } \\
& \mathrm{Q}=\sqrt{3^{2}+5^{2}+1^{2}}=5.916 \text { units } \\
& \therefore \quad \sin \theta=\frac{R}{P Q}=\frac{|\mathbf{P} \times \mathbf{Q}|}{P Q}=\frac{22.045}{5.385 \times 5.916} \\
& =0.692 \\
& \therefore \quad \theta=43.788^{\circ}
\end{aligned}
$$

Ans.

Ans.

## IMPORTANT DEFINITIONS AND CONCEPTS

1. Displacement is defined as the distance moved by a body or particle in the specified direction.
2. The rate of change of displacement with time is called velocity.
3. Acceleration is the rate of change of velocity with respect to time.
4. The product of mass and velocity is called momentum.
5. A body is said to be treated as continuum, if it is assumed to consist of continuous distribution of matter.
6. A body is said to be rigid, if the relative position of any two particles in it do not change under the action of the forces.
7. Newton's first law states that everybody continues in its state of rest or of uniform motion in a straight line unless it is compelled by an external agency acting on it.
8. Newton's second law states that the rate of change of momentum of a body is directly proportional to the impressed force and it takes place in the direction of the force acting on it.
9. Newton's third law states that for every action there is an equal and opposite reaction.
10. Newton's law of gravitation states that everybody attracts the other body. The force of attraction between any two bodies is directly proportional to their mass and inversely proportional to the square of the distance between them.
11. According to the law of transmissibility of force, the state of rest or motion of a rigid body is unaltered, if a force acting on a body is replaced by another force of the same magnitude and direction but acting anywhere on the body along the line of action of the replaced force.
12. The parallelogram law of forces states that if two forces acting simultaneously on a body at a point are represented by the two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection of the two sides representing the forces.
13. The qualitative description of physical variable is known as dimension while the quantitative description is known as unit.
14. A quantity is said to be scalar, if it is completely defined by its magnitude alone.
15. A quantity is said to be vector if it is completely defined only when it's magnitude as well as direction are specified.
16. The position vector of a point in the space is defined as the vector represented by the line segment connecting the origin and the point. A vector in space with its tail at one point and the head at another point is termed as displacement vector.
17. The dot product of two vectors $\mathbf{P}$ and $\mathbf{Q}$ is defined as $\mathbf{P} \cdot \mathbf{Q}=P Q \cos \theta$ where $\theta$ is the acute angle between the two vectors.
18. The cross product of vectors $\mathbf{P}$ and $\mathbf{Q}$ is defined as $\mathbf{P} \times \mathbf{Q}=P Q \sin \theta \mathbf{n}$, where $\theta$ is the acute angle between $\mathbf{P}$ and $\mathbf{Q}$ and $\mathbf{n}$ is the unit vector in the direction perpendicular to the plane of $\mathbf{P}$ and $\mathbf{Q}$, its sense being the direction of advance of a right hand screw rotated from first to second ( $\mathbf{P}$ to $\mathbf{Q}$ ) vector.

## IMPORTANT FORMULAE

$$
\begin{aligned}
a & =\frac{d v}{d t} \\
\text { Momentum } & =\text { Mass } \times \text { Velocity } \\
F & =m a
\end{aligned}
$$

The force of attraction between two bodies

$$
\begin{aligned}
F & =G \frac{m_{1} m_{2}}{d^{2}} \\
g & =\frac{G M_{e}}{r^{2}}=9.80665 \simeq 9.81 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Dimensional relation of $F=\frac{M L}{T^{2}}$ or $M=\frac{F T^{2}}{L}$
Vectorial representation is given by

$$
\begin{aligned}
& \mathbf{F}=\mathbf{F}_{\mathrm{x}}+\mathbf{F}_{\mathrm{y}}+\mathbf{F}_{\mathrm{z}} \\
& =F_{x} \mathbf{i}+F_{y} \mathbf{j}+F_{z} \mathbf{k} \\
& \therefore \quad F=\sqrt{F_{x}^{2}+F_{y}^{2}+F_{z}^{2}} \\
& F_{x}=F \cos \theta_{x}, \quad F_{y}=F \cos \theta_{y}, \quad F_{z}=F \cos \theta_{z} \\
& F_{x}=F l \quad F_{y}=F m \quad F_{z}=F n \\
& \mathbf{F}=F(l \mathbf{i}+m \mathbf{j}+n \mathbf{k}) \\
& \mathbf{n}=l \mathbf{i}+m \mathbf{j}+n \mathbf{k} \\
& \text { Position vector } \quad \mathbf{r}_{\mathbf{O A}}=x \mathbf{i}+y \mathbf{j}+z \mathbf{k} \\
& r_{O A}=\sqrt{x^{2}+y^{2}+z^{2}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{P}+\mathbf{Q} & =\mathbf{Q}+\mathbf{P} \\
(\mathbf{P}+\mathbf{Q})+\mathbf{R} & =\mathbf{P}+(\mathbf{Q}+\mathbf{R}) \\
(m+n) \mathbf{P} & =m \mathbf{P}+n \mathbf{P} \\
m(\mathbf{P}+\mathbf{Q}) & =m \mathbf{P}+m \mathbf{Q} \\
m(n \mathbf{P}) & =m n \mathbf{P}
\end{aligned}
$$

Displacement vector $\mathbf{r}_{12}=\mathbf{r}_{02}-\mathbf{r}_{01}$

$$
\begin{aligned}
\mathbf{P} \cdot \mathbf{Q} & =P Q \cos \theta \\
\mathbf{i} \cdot \mathbf{i} & =\mathbf{j} \cdot \mathbf{j}=\mathbf{k} \cdot \mathbf{k}=1 \\
\mathbf{i} \cdot \mathbf{j} & =\mathbf{j} \cdot \mathbf{k}=\mathbf{k} \cdot \mathbf{i}=\mathbf{k} \cdot \mathbf{j}=\mathbf{j} \cdot \mathbf{i}=\mathbf{i} \cdot \mathbf{k}=0 \\
\mathbf{P} \times \mathbf{Q} & =P Q \sin \theta \mathbf{n}
\end{aligned}
$$

$$
=\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
P_{x} & P_{y} & P_{z} \\
Q_{x} & Q_{y} & Q_{z}
\end{array}\right|
$$

$$
\mathbf{P} \times \mathbf{Q} \neq \mathbf{Q} \times \mathbf{P}
$$

## PROBLEMS FOR EXERCISE

1.1 When a projectile is projected at angle $\alpha$ to the horizontal ground with an initial velocity $u \mathrm{~m} / \mathrm{s}$.
(i) its trajectory is given by $y=(4 \sin x) t-\frac{1}{2} g t^{2}$
(ii) time of flight $t=\frac{2 u \sin \alpha}{g}$, and
(iii) range $R=\frac{u^{2} \sin ^{2} \alpha}{g}$

Check the dimensional homogenity of the equations.
$\left[\right.$ Ans. (i) $L \equiv L T^{-1}, T \equiv L T^{-2} T^{2} ;$ (ii) $\left.T \equiv \frac{L T^{-1}}{L T^{-2}},(i i i) L \equiv \frac{\left(L T^{-1}\right)^{2}}{L T^{-2}}\right]$
In all the three, there is dimensional homogenity.
1.2 The discharge equation for a triangular notch in a canal is

$$
Q=\left(\frac{8}{15}\right) C d \sqrt{2 g} \tan \theta(\mathrm{H})^{5 / 2}
$$

where $Q$-discharge
$C d$-a dimensionless constant
$g$-acceleration due to gravity
$H$-Height of water level.
Check the dimensional homoginity of the equation.
$\left[\right.$ Ans. $L^{3} T^{-1} \equiv \sqrt{L T^{-2}}(L)^{5 / 2}$. Homogenious $]$
1.3 A force of magnitude 20 kN makes $30^{\circ}$ with the $y$-axis and its projection on $x$-z plane makes $45^{\circ}$ with $x$-axis. Write the force in vector form. Determine its direction cosines and directions with respect to $x, y, z$ axes.
[Ans. $\mathbf{F}=7.07 \mathbf{i}+17.321 \mathbf{j}+7.07 \mathbf{k} ; l=0.3535 m=0.866 n=0.3535$

$$
\left.Q_{x}=69.30^{\circ}, Q_{y}=30^{\circ}, Q_{z}=69.30^{\circ}\right]
$$

1.4 $\mathbf{F}_{1}, \mathbf{F}_{2}$ and $\mathbf{F}_{3}$ are the 3 forces as given below:

$$
\begin{aligned}
& \mathbf{F}_{1}=4 \mathbf{i}+3 \mathbf{j}+\mathbf{k} \\
& \mathbf{F}_{2}=2 \mathbf{i}-2 \mathbf{j}+3 \mathbf{k} \\
& \mathbf{F}_{3}=5 \mathbf{i}-\mathbf{j}+2 \mathbf{k}
\end{aligned}
$$

Determine
(i) $\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}$
(ii) $\mathbf{F}_{1}-\mathbf{F}_{2}-\mathbf{F}_{3}$
(iii) $2 \mathbf{F}_{1}+3 \mathbf{F}_{2}-2 \mathbf{F}_{3}$
[Ans. (i) $11 \mathbf{i}+6 \mathbf{k}$, (ii) $-3 \mathbf{i}+6 \mathbf{j}-4 \mathbf{k}$, (iii) $4 \mathbf{i}+2 \mathbf{j}+7 \mathbf{k}]$
1.5 Determine the displacement vector $A B$, given that $A(2,4,5), B(3,2,1)$ in metre units. Determine the vector form of $A B$, its direction cosines and directions w.r.t. to the $x, y, z$, axis.

$$
\begin{array}{r}
{\left[\text { Ans. } \mathbf{r}_{\mathrm{AB}}=\mathbf{i}-2 \mathbf{j}-4 \mathbf{k}, l=0.218, m=-0.436, n=-0.872,\right.} \\
\left.\theta_{x}=77.4^{\circ}, \theta_{y}=115.88^{\circ}, \theta_{z}=190.794^{\circ}\right]
\end{array}
$$

1.6 The two vectors $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$ are as given below:

$$
\begin{aligned}
& \mathbf{P}_{1}=\mathbf{i}+2 \mathbf{j}+3 \mathbf{k} \\
& \mathbf{P}_{2}=3 \mathbf{i}+2 \mathbf{j}+\mathbf{k}
\end{aligned}
$$

Determine
(i) $\mathbf{P}_{1} . \mathbf{P}_{2}$ and hence the acute angle between $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$.
(ii) $\mathbf{P}_{1} \times \mathbf{P}_{2}$ and hence the acute angle between $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$.
[Ans. (i) 10, $Q=44.415^{\circ}$; (ii) 9.798, $Q=44.415^{\circ}$ ]
1.7 Determine the projection of $\mathbf{P}_{1}$ on the line of action of $\mathbf{P}_{2}$, given

$$
\mathbf{P}_{1}=\mathbf{i}+2 \mathbf{j}+3 \mathbf{k}
$$

and

$$
\begin{equation*}
\mathbf{P}_{2}=3 \mathbf{i}+2 \mathbf{j}+\mathbf{k} \tag{Ans.2.676}
\end{equation*}
$$

### 1.8 If

$$
\begin{aligned}
& \mathbf{P}_{1}=4 \mathbf{i}+2 \mathbf{j}+\mathbf{k} \text { and } \\
& \mathbf{P}_{2}=2 \mathbf{i}-3 \mathbf{j}-2 \mathbf{k}
\end{aligned}
$$

prove that they are at right angles to each other.
[Hint: From dot product show that $\cos \theta=0$, hence $\theta=90^{\circ}$ ]
1.9 $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ are the three vectors as given below:

$$
\begin{aligned}
& \mathbf{P}=4 \mathbf{i}+2 \mathbf{j}+\mathbf{k} \\
& \mathbf{Q}=3 \mathbf{i}+2 \mathbf{j}-2 \mathbf{k} \\
& \mathbf{R}=\mathbf{i}-3 \mathbf{j}-3 \mathbf{k}
\end{aligned}
$$

and
determine
(i) $(\mathbf{P} \times \mathbf{Q}) \cdot \mathbf{R}$
(ii) $(\mathbf{P} \times \mathbf{Q}) \times \mathbf{R}$
[Ans. (i) - 25; (ii) 39i $+20 \mathbf{j}+7 \mathbf{k}$ ]

## chapter 2

## Statics of Particles

A particle is a body which has mass but no size. Hence the system of forces acting on a particle are concurrent forces. Actually there is no body which has no size. However there are many engineering problems in which the shape and size of the body do not significantly affect the analysis. In all these problems the system of forces can be considered as concurrent. The resolution, composition and equilibrium of concurrent force system in a plane is explained first and then the concept is extended to the system of forces in space. Many engineering problems are considered to make the analysis procedure clear.

### 2.1 RESOLUTION OF CONCURRENT COPLANAR FORCES

Resolution of the forces is the process of finding a number of component forces which will have the same effect on the body as the given single force. Thus the resolution of forces is exactly opposite to finding the resultant of two or more forces, for which we have seen parallelogram law of forces, triangular law of forces and polygon law of forces can be used.

In Fig 2.1 (a) the force $F$ is resolved into two components making angles $\alpha$ and $\beta$ with the force $F$. Use of parallelogram law and triangle law of forces in the reversed order is shown in the figure.


(a)

(b)

(c)

Fig. 2.1

In Fig. 2.1 (b) the force $F$ is resolved into four components $F_{1}, F_{2}, F_{3}$ and $F_{4}$ in which polygon law of forces is used in the reverse order.

In Fig. 2.1 (c) the force $F$ is resolved into its rectangular components $F_{x}$ and $F_{y}$, where $x$ and $y$ are cartesian coordinates.

It may be noted that the component forces act at the same point ' $O$ ' as the given force $F$. Resolution of a force into its rectangular components is more useful for the analysis. If $F_{x}$ and $F_{y}$ are its components in $x$ and $y$ coordinate directions and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in $x$ and $y$ coordinate direction, then

$$
\begin{align*}
\mathbf{F} & =F_{x}+F_{y} \\
& =F_{x} \mathbf{i}+F_{y} \mathbf{j}
\end{align*}
$$

Example 2.1 The guy wire of a electric pole shown in Fig. 2.2 makes $30^{\circ}$ to the pole and is applying a force of 12 kN . Find the horizontal and vertical component of the force. Express it in the vector form taking horizontal direction as $x$-axis and vertical as $y$-axis.


Fig. 2.2
Solution. To get the component of a force, assume the arrow marked to represent the force to some scale. Move from tail to head of the arrow in the coordinate directions. Thus the vertical component

$$
\begin{aligned}
F_{v} & =12 \cos 30^{\circ} \\
F_{H} & =12 \sin 30^{\circ}
\end{aligned}=6 \mathrm{kN} \text { towards left } \mathrm{kN}(\text { downward }) ~ \$
$$

and
Since the coordinates selected are as shown in Fig. 2.2 (c), taking $\mathbf{i}$ as unit vector in $x$-direction and $\mathbf{j}$ as unit vector in $y$-direction, we find

$$
\begin{array}{ll} 
& \mathbf{F}_{x}=-6 \mathbf{i} \text { and } \mathbf{F}_{y}=-10.392 \mathbf{j} \\
\therefore & \mathbf{F}=-6 \mathbf{i}-10.392 \mathbf{j}
\end{array}
$$

Ans.
Example 2.2 A block weighing $W=12 \mathrm{kN}$ is resting on an inclined plane as shown in Fig. 2.3 (a). Determine its components normal and parallel to the inclined plane. Using $x$ and $y$ axes as shown in figure, write the vector form of the force.


Fig. 2.3
Solution. Self weight of a body is due to gravitational attraction ' $g$ ' and is equal to ' $m g$ ', where $m$ is the mass of the body. It is always directed to the centre of the earth; in other words it is in the vertically downward direction. In Fig 2.3(b) the self weight $W$ is shown by line $A C$. By moving from $A$ (tail of the force line) to the head in the directions normal and parallel to the inclined plane direction, triangle $A B C$ is completed. Now $A B$ represents normal component and $B C$ represents component parallel to $A B$. By extending $B C, A C$ and drawing horizontal line $B^{\prime} A^{\prime}$, triangle $A^{\prime} B^{\prime} C$ is obtained.
Now, $\quad \triangle A B C$ III $\triangle A^{\prime} B^{\prime} C$, since $\angle A C B=\angle A^{\prime} C B^{\prime}$
and
$\angle A B C=\angle C A^{\prime} B^{\prime}=90^{\circ}$
$\therefore \quad \angle B A C=\angle A^{\prime} B^{\prime} C=\theta=40^{\circ}$
Thus normal to the plane makes angle $\theta$ with the vertical if the plane makes angle $\theta$ with the horizontal.
$\therefore$ component parallel to plane $=B C=W \sin \theta$,
where $W=m g$ is weight of the body.
and component normal to plane $=A B=W \cos \theta$.
If $x$ and $y$ directions are selected as given in Fig. 2.3(a).

$$
\begin{aligned}
& F_{x} \\
= & -W \sin \theta=-12 \sin 40^{\circ}=-7.713 \mathrm{kN} \\
& F_{y}
\end{aligned}=-W \cos \theta=-12 \cos 40^{\circ}=-9.193 \mathrm{kN}
$$

Ans.
Ans.

From the above two examples, the following points for finding component of forces and vector form may be noted:
(i) Imagine that arrow drawn to represent the force gives the magnitude of force to some scale.
(ii) Travel from the tail of the force line to arrow head in the direction of coordinates.
(iii) Then the direction of travel gives the direction of component forces.
(iv) From the triangle of forces $(\triangle A B C)$, the magnitude of the components can be found.
(v) Looking at the positive sense of the coordinate directions, the force may be expressed in the vector form.

### 2.2 COMPOSITION OF CONCURRENT COPLANAR FORCES

It is possible to find a single force which will have the same effect as that of a number of forces acting on a particle. Such single force is called Resultant force and the process of finding the resultant force is called Composition of Forces. Using parallelogram law, triangle law and polygon law the resultant of forces can be found. These are the graphical methods, which need drawing sheet and drawing accessories.

In the analytical method the components of each one of the forces in the coordinate directions are found and they are expressed in the form of vectors. Referring to Fig. 2.4, if $F_{1}, F_{2}, F_{3}$ and $F_{4}$ are the forces in the system, then we write

$$
\left.\begin{array}{l}
\mathbf{F}_{1}=F_{1 x} \mathbf{i}+F_{1 \mathbf{y}} \mathbf{j} \\
\mathbf{F}_{2}=F_{2 x} \mathbf{i}+F_{2 \mathbf{y}}^{\mathbf{j}}  \tag{2.2}\\
\mathbf{F}_{3}=F_{3 x} \mathbf{i}+F_{3 y} \mathbf{j} \\
\mathbf{F}_{4}=F_{4 x} \mathbf{i}+F_{4, \mathbf{j}}
\end{array}\right\}
$$



Fig. 2.4

## Note:

In the above expression $F_{2 x}, F_{3 x}, F_{3 y}$ and $F_{4 y}$ have negative values
Then the resultant $R$ is the vector sum of $F_{1}, F_{2}, F_{3}$ and $F_{4}$. i.e.,

$$
\begin{aligned}
\mathbf{R} & =\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4} \\
& =F_{1 x} \mathbf{i}+F_{1 y} \mathbf{j}+F_{2 x} \mathbf{i}+F_{2 y} \mathbf{j}+F_{3 x} \mathbf{i}+F_{3 y} \mathbf{j}+F_{4 x} \mathbf{i}+F_{4 y} \mathbf{j} \\
& =\left(F_{1 x}+F_{2 x}+F_{3 x}+F_{4 x}\right) \mathbf{i}+\left(F_{1 y}+F_{2 y}+F_{3 y}+F_{4 y}\right) \mathbf{j} \\
& =R_{x} \mathbf{i}+R_{y} \mathbf{j}
\end{aligned}
$$

where

$$
R_{x}=\Sigma F_{x} \text { and } R_{y}=\Sigma F_{y} .
$$

$$
\begin{array}{ll}
\therefore & \mathbf{R}=\Sigma F_{x} \mathbf{i}+\Sigma F_{y} \mathbf{j} \\
\therefore & R=\sqrt{\left(\Sigma F_{x}\right)^{2}+\left(\Sigma F_{y}\right)^{2}} \tag{2.4}
\end{array}
$$

If $\alpha$ is the angle between $x$-axis and the force, then

$$
\begin{equation*}
\tan \alpha=\frac{\Sigma F_{y}}{\Sigma F_{x}} \tag{2.5}
\end{equation*}
$$

Example 2.3 Determine the resultant of the three forces acting on a hook as shown in Fig. 2.5 (a).

(a)

(b)

Fig. 2.5

## Solution.

$$
\begin{aligned}
& \text { Force } \\
& F_{1}=100 \mathrm{~N} \\
& x \text {-component } \\
& 100 \cos 60^{\circ}=50 \mathrm{~N} \\
& y \text {-component } \\
& F_{2}=80 \mathrm{~N} \quad 80 \cos 30^{\circ}=69.28 \mathrm{~N} \quad 80 \sin 30^{\circ}=40.00 \mathrm{~N} \\
& F_{3}=60 \mathrm{~N} \quad 60 \cos 45^{\circ}=42.43 \mathrm{~N} \quad-60 \sin 45^{\circ}=-42.43 \mathrm{~N} \\
& \therefore \quad \mathbf{F}_{1}=50 \mathbf{i}+86.60 \mathbf{j} \\
& \mathbf{F}_{2}=69.28 \mathbf{i}+40.00 \mathbf{j} \\
& \mathbf{F}_{3}=42.43 \mathbf{i}-42.43 \mathbf{j} \\
& \therefore \quad \mathbf{R}=(50+69.28+42.43) \mathbf{i}+(86.60+40.00-42.43) \mathbf{j} \\
& =161.71 \mathbf{i}+84.17 \mathbf{j} \\
& \therefore \quad R=\sqrt{(161.71)^{2}+(84.17)^{2}}=182.3 \mathrm{~N} \\
& \tan \alpha=\frac{\Sigma F_{y}}{\Sigma F_{x}}=\frac{R_{y}}{R_{x}}=\frac{84.17}{161.71}=0.5205 \\
& \alpha=27.5^{\circ}
\end{aligned}
$$

Example 2.4 A system of four forces acting on a body is as shown in Fig. 2.6. Determine the resultant.

Solution. If $\theta_{1}$ is the inclination of 200 N force to $x$-axis, then

$$
\tan \theta_{1}=\frac{1}{2} \text { or } \theta_{1}=26.565^{\circ}
$$

Similarly if $\theta_{2}$ is the inclination of 120 N force to $x$-axis, as shown in Fig. 2.6,

$$
\tan \theta_{2}=\frac{4}{3} \quad \theta_{2}=53.13^{\circ}
$$


$\mathbf{F}_{1}=200\left(\cos \theta_{1} \mathbf{i}+\sin \theta_{1} \mathbf{j}\right) \quad=178.88 \mathbf{i}+89.44 \mathbf{j}$
$\mathbf{F}_{2}=-120 \cos \theta_{2} \mathbf{i}+120 \sin \theta_{2} \quad \mathbf{j}=-72 \mathbf{i}+96 \mathbf{j}$
$\mathbf{F}_{3}=-50 \cos 60^{\circ} \mathbf{i}-50 \sin 60^{\circ} \mathbf{j}=-25 \mathbf{i}-43.30 \mathbf{j}$
$\mathbf{F}_{4}=100 \sin 40^{\circ} \mathbf{i}-100 \cos 40^{\circ} \mathbf{j}=64.28 \mathbf{i}-76.60 \mathbf{j}$
$\therefore \quad \mathbf{R}=\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}$
$=(178.88-72-25+64.28) \mathbf{i}+(89.44+96-43.30-76.60) \mathbf{j}$

$$
=146.16 \mathbf{i}+65.54 \mathbf{j}
$$

i.e., $\quad R_{x}=146.16 \mathrm{~N}$ and $R_{y}=65.54 \mathrm{~N}$.
$\therefore \quad R=\sqrt{(146.16)^{2}+(65.54)^{2}}=160.18 \mathrm{~N}$
$\alpha=\tan ^{-1} \frac{R_{y}}{R_{x}}=\tan ^{-1} \frac{65.54}{146.16}=24.15^{\circ}$
Ans.

Ans.

Example 2.5 A system of forces acting on a body resting on an inclined plane is as shown in Fig. 2.7. Determine the resultant force if $\theta=60^{\circ}, W=1000 \mathrm{~N}$, $N=500 \mathrm{~N}, F=100 \mathrm{~N}$ and $T=1200 \mathrm{~N}$.
Solution. In this problem, it is convenient to select the coordinate system as parallel to and perpendicular to the plane. The vector form of various forces is as given below:

$$
\begin{aligned}
& \mathbf{T}=1200 \mathbf{i} \\
& \mathbf{F}=-100 \mathbf{i} \\
& \mathbf{N}=500 \mathbf{j} \\
& \mathbf{W}=-1000 \sin 60^{\circ} \mathbf{i}-1000 \cos 60^{\circ} \mathbf{j} \\
& =-866.03 \mathbf{i}-500 \mathbf{j} \\
& \therefore \quad \mathbf{R}=\mathbf{T}+\mathbf{F}+\mathbf{N}+\mathbf{W} \\
& =1200 \mathbf{i}-100 \mathbf{i}+500 \mathbf{j}-866.03 \mathbf{i}-500 \mathbf{j} \\
& \text { Fig. } 2.7
\end{aligned}
$$

$$
\begin{aligned}
& =(1200-100-866.03) \mathbf{i}+(500-500) \mathbf{j} \\
& =233.97 \mathbf{i} \\
R & =233.97 \mathrm{~N}
\end{aligned}
$$

Ans.
Ans.

$$
\begin{aligned}
\tan \alpha & =\frac{0}{233.97}=0 \\
\alpha & =0^{\circ}
\end{aligned}
$$

Ans.
Ans.
Example 2.6 Two forces acting on a body are 500 N and 1000 N as shown in Fig. 2.8(a). Determine the third force $F_{3}$ such that the resultant of all the three forces is 1000 N directed at $45^{\circ}$ to $x$-axis as shown is the figure.

(a)

(b)

Fig. 2.8
Solution. Let the third force $\mathbf{F}_{3}$ make an angle $\theta_{3}$ with $x$-axis
Now

$$
\begin{aligned}
\mathbf{F}_{1} & =500 \cos 30^{\circ} \mathbf{i}+500 \sin 30^{\circ} \mathbf{j} \\
& =433.01 \mathbf{i}+250 \mathbf{j} \\
\mathbf{F}_{2} & =1000 \sin 30^{\circ} \mathbf{i}+1000 \cos 30^{\circ} \mathbf{j} \\
& =500 \mathbf{i}+866.02 \mathbf{j} \\
\mathbf{F}_{3} & =F_{3 x} \mathbf{i}+F_{3 y} \mathbf{j} \\
& =F_{3} \cos \theta_{3} \mathbf{i}+F_{3} \sin \theta_{3} \mathbf{j} \\
\mathbf{R} & =1000 \cos 45^{\circ} \mathbf{i}+1000 \sin 45^{\circ} \mathbf{j} \\
& =707.11 \mathbf{i}+707.11 \mathbf{j} \\
\mathbf{R} & =\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}, \text { we get }
\end{aligned}
$$

Since $707.11 \mathbf{i}+707.11 \mathbf{j}=\left(433.01+500+F_{3} \cos \theta_{3}\right) \mathbf{i}+$ $\left(250+866.02+F_{3} \sin \theta_{3}\right) \mathbf{j}$
i.e., $\quad 707.11 \mathbf{i}+707.11 \mathbf{j}=\left(933.01+F_{3} \cos \theta\right) \mathbf{i}+\left(1116.02+F_{3} \sin \theta_{3}\right) \mathbf{j}$
$\therefore \quad 707.11=933.01+F_{3} \cos \theta$
and
$707.11=1116.02+F_{3} \sin \theta_{3}$

$$
\begin{array}{lcc}
\text { i.e., } & F_{3} \cos \theta_{3}=-225.90 \quad F_{3} \sin \theta_{3}=-408.91 & \text { Ans. } \\
\therefore & \mathbf{F}_{3}=-225.90 \mathbf{i}-408.91 \mathbf{j} \\
\therefore & F_{3}=\sqrt{(-255.90)^{2}+(408.91)^{2}}=467.16 \mathrm{~N} & \text { Ans. } \\
\therefore & \tan \theta_{3}=\frac{408.91}{225.9}=1.810 & \text { Ans. } \\
& \theta_{3}=61.08^{\circ} \text { as shown in Fig. } 2.8(\mathrm{~b}) & \text { Ans. }
\end{array}
$$

Ans.

## Ans.

Ans.
Ans.

Example 2.7 Three forces acting at a point are shown in Fig. 2.9. The direction of 300 N forces may vary, but the angle between them is always $40^{\circ}$. Determine the value of $\theta$ for which the resultant of the three forces is directed to $x$-axis.


Fig. 2.9

## Solution.

$$
\begin{aligned}
\mathbf{F}_{1} & =300 \cos \theta \mathbf{i}+300 \sin \theta \mathbf{j} \\
\mathbf{F}_{2} & =300 \cos \left(40^{\circ}+\theta\right) \mathbf{i}+300 \sin \left(40^{\circ}+\theta\right) \mathbf{j} \\
\mathbf{F}_{3} & =500 \cos 30^{\circ} \mathbf{i}-500 \sin 30^{\circ} \mathbf{j} \\
& =433.01 \mathbf{i}-250 \mathbf{j} \\
\therefore \quad & \mathbf{R}= \\
& =\left[\left(300 \operatorname{Fos} \theta+\mathbf{F}_{2}+\mathbf{F}_{3}\right.\right. \\
& \\
& \left.\left.\left.+\left(300 \sin \theta+300 \cos \left(40^{\circ}+\theta\right)+433.01\right) \mathbf{i}\left(40^{\circ}+\theta\right)-250\right) \mathbf{j}\right)\right]
\end{aligned}
$$

According to condition specified in the problem, component of resultant in $y$-direction is zero.
Hence $\quad 300 \sin \theta+300 \sin \left(40^{\circ}+\theta\right)-250=0$

$$
\begin{array}{ll} 
& \sin \theta+\sin \left(40^{\circ}+\theta\right)=\frac{250}{300}=0.8333 \\
\text { i.e., } & 2 \sin \left(\frac{40^{\circ}+\theta+\theta}{2}\right) \cos \left(\frac{40^{\circ}+\theta-\theta}{2}\right)=0.8333 \\
\text { i.e., } & 2 \sin \left(20^{\circ}+\theta\right) \cos 20^{\circ}=0.8333 \\
& \sin \left(20^{\circ}+\theta\right)=0.4434 \\
\therefore & 20^{\circ}+\theta=26.32^{\circ} \\
\text { or } & \theta=6.32^{\circ}
\end{array}
$$

Ans.

Example 2.8 The resultant of two forces, one of which is double the other is 260 N . If the direction of the larger force is reversed and the other remains unaltered, the resultant reduces to 180 N . Determine the magnitude of the forces and the angle between the forces.
Solution. Let $\mathbf{F}_{1}$ and $\mathbf{F}_{2}$ be the forces and the angle between them be $\theta$. Taking the direction of the force $\mathbf{F}_{1}$ as $x$-axis and normal to it as $y$-axis [Ref Fig. 2.10(a)].

$$
\begin{align*}
& \mathbf{F}_{1}=F_{1} \mathbf{i} \\
& \mathbf{F}_{2}=F_{2} \cos \theta \mathbf{i}+F_{2} \sin \theta \mathbf{j} \\
& =2 F_{1} \cos \theta \mathbf{i}+2 F_{1} \sin \theta \mathbf{j} \\
& \therefore \quad \mathbf{R}=\left(F_{1}+2 \mathrm{~F}_{1} \cos \theta\right) \mathbf{i}+2 F_{1} \sin \theta \mathbf{j} \\
& \therefore \quad R=\sqrt{\left(F_{1}+2 F_{1} \cos \theta\right)^{2}+\left(2 F_{1} \sin \theta\right)^{2}}=260 \\
& \text { i.e., } \quad F_{1}^{2}+4 F_{1} F_{1} \cos \theta+4 F_{1}^{2} \cos ^{2} \theta+4 F_{1}^{2} \sin ^{2} \theta=260^{2} \\
& \text { or } \quad F_{1}^{2}+4 F_{1}^{2} \cos \theta+4 F_{1}^{2}=260^{2} \\
& \text { since } \quad \cos ^{2} \theta+\sin ^{2} \theta=1 \\
& \text { i.e., } \\
& 5 F_{1}^{2}+4 F_{1}^{2} \cos \theta=260^{2} \tag{i}
\end{align*}
$$

when the direction of larger force is reversed, the system of forces is as shown in Fig. 2.10 (b).


Fig. 2.10

$$
\begin{align*}
& \text { Now } \\
& \\
& \\
& \\
& \\
& \\
& \therefore \\
&  \tag{ii}\\
& \therefore \\
& \\
& \therefore
\end{aligned} \begin{aligned}
\mathbf{F}_{2} & =-F_{1} \mathbf{i} \\
& =-2 F_{1} \cos \theta \mathbf{i}-F_{2} \sin \theta \mathbf{i}-2 F_{1} \sin \theta \mathbf{j} \\
& =\left(F_{1}-2 F_{1} \cos \theta\right) \mathbf{i}-2 F_{1} \sin \theta \mathbf{j} \\
\text { i.e., } & R
\end{align*}
$$

Adding eqns (i) and (ii), we get

$$
10 F_{1}^{2}=260^{2}+180^{2}
$$

$$
\therefore \quad F_{1}=100 \mathrm{~N} \quad \text { Ans. }
$$

$$
\text { Hence } \quad F_{2}=200 \mathrm{~N} \quad \text { Ans. }
$$

Substituting this value in Eqn. (i), we get

$$
\begin{aligned}
& 5(100)^{2}+4 \times 100^{2} \cos \theta & =260^{2} \\
\therefore & \cos \theta & =\frac{260^{2}-5 \times 100^{2}}{4 \times 100^{2}}=0.44 \\
\therefore & \theta & =63.896^{\circ}
\end{aligned}
$$

Ans.

### 2.3 EQUILIBRIANT OF SYSTEM OF FORCES

According to Newton's second law of motion a body moves with uniform acceleration, if it is acted upon by a force. Hence when a system of forces act upon a body, the resultant force makes the body to move with uniform acceleration in its direction. If a force equal in magnitude but opposite in sense to the resultant force is applied to the system of forces the body comes to rest. Such a force which is equal in magnitude but opposite to the resultant force is called Equilibriant. Denoting equilibriant by $\mathbf{E}$ and resultant by $\mathbf{R}$, we have,

$$
\begin{equation*}
\mathbf{E}+\mathbf{R}=0 \tag{2.6}
\end{equation*}
$$

Hence to find the equilibriant $\mathbf{E}$ of a force system, the resultant $\mathbf{R}$ of the force system is found as explained in Art 2.2 and then equilibriant is shown as -R. Thus in Example 2.4, we have

$$
\begin{aligned}
\mathbf{E} & =-\mathbf{R}=-(146.16 \mathbf{i}+65.54 \mathbf{j}) \\
\text { i.e., } \quad & E=-R=160.18 \mathrm{~N} \quad \text { and } \quad \theta=180+24.15=204.15^{\circ} .
\end{aligned}
$$

### 2.4 EQUILIBRIUM OF PARTICLES SUBJECT TO COPLANAR FORCES

A body is said to be in equilibrium when, it is at rest or continues to be in steady linear motion. According to Newton's second law of motion, it means the resultant of the force system acting on the body is zero. In graphical concept, it means the force polygon closes. Analytically, it means

$$
\begin{align*}
\mathbf{R} & =0 \\
\text { i.e., } & \Sigma F_{x} \mathbf{i}+\Sigma F_{y} \mathbf{j}
\end{align*}=0
$$

Equations 2.7 (a \& b) are called equations of equilibrium for concurrent coplanar system of forces.

While applying equilibrium conditions to a body it is necessary that all forces acting on the body are considered and marked in proper direction on the figure of the body, freed from all its contact surfaces. Hence in this article, first various forces to be considered is explained and then method of representing them on the free body is discussed. Before taking up the general cases, specific cases of two force system and three force system is presented.

## Types of Forces on a Body

The various forces acting on a body may be grouped as:

- Applied forces
- Non-applied forces


## Applied Forces

Applied forces are the forces applied externally to a body. If a person stands on a ladder, his weight is an applied force to the ladder. If a car is pulled with a rope, the force in the rope is applied force to the car.

## Non-applied Forces

There are two types of non-applied forces e.g. (i) Self weight (ii) Reactions from other bodies in contact.
Self Weight: Everybody is subjected to gravitational attraction and hence has got self weight

$$
W=m g
$$

where $m$ is the mass of the body and $g$ is acceleration due to gravity. The value of ' $g$ ' is $9.81 \mathrm{~m} / \mathrm{s}^{2}$ near the Earth surface.

Since self weight is due to gravitational attraction, it is always directed towards the centre of the Earth i.e., in vertically downward direction. In this analysis, self weight is treated as acting through the centre of gravity of the body. If self weight is small compared to the applied forces, in some engineering problems, it may be neglected.
Reactions: These are self adjusting forces exerted by the other bodies which are in contact with the body under consideration. According to Newton's third law of motion, the reactions are equal and opposite to the actions. For example, if a block weighing 2 kN is kept on the table, action of the block is 2 kN vertically downward on the table while the reaction from the table is 2 kN vertically upward. The reactions adjust themselves to bring the body to equilibrium.

If the surface of contact is smooth, the direction of the reaction is normal to the surface of contact. If the surface of contact is not smooth, there will be frictional reaction also. Hence the resultant reaction will not be normal to the surface of contact.

## Free Body Diagram (FBD)

In the analysis, it is necessary to isolate the body under consideration from the other bodies in contact and draw all the forces acting on the body. For this, first the body is drawn and then applied forces, self weight and the reactions are drawn. Such a diagram of the body in which the body under consideration is freed from all the contact surfaces and all the forces acting on it (including self weight and reactions) are drawn, is termed as Free Body Diagram (FBD). Free body diagrams are shown for a typical cases in Table 2.1.

Table 2.1 Free Body Diagrams (FBD) for a Few Typical Cases
Reacting bodies FBD required for

## Equilibrium of Two Force System

When a body is in equilibrium under the action of two forces $\mathbf{F}_{1}$ and $\mathbf{F}_{2}$ only, then the equilibrium condition is,

$$
\mathbf{R}=0
$$

i.e.,

$$
\begin{aligned}
\mathbf{F}_{1}+\mathbf{F}_{2} & =0 \\
\mathbf{F}_{1} & =-\mathbf{F}_{2}
\end{aligned}
$$

In other words the two forces must have same magnitude, direction but opposite sense as shown in Fig. 2.11.


Fig. 2.11

## Equilibrium of Three Force System/Lami's Theorem

If a body is in equilibrium under the action of only three forces, each force is proportional to the sine of the angle between the other two forces. This is known as Lami's theorem. Thus for the system of forces shown in Fig. 2.12(a).

$$
\frac{F_{1}}{\sin \alpha}=\frac{F_{2}}{\sin \beta}=\frac{F_{3}}{\sin \gamma}
$$


(a)

(b)

Fig. 2.12
The above theorem can be proved as given below:
Draw the three forces $F_{1}, F_{2}$ and $F_{3}$ from tail to head fashion as explained for drawing polygon of forces. If ' $a$ ' is the starting point of polygon of forces the last point of the polygon also must be $a$, since the resultant is zero.

Thus it results into triangle of forces $a b c$ as shown in Fig. 2.12(b). Now the external angles at $a, b$ and c are equal to $\beta, \gamma$ and $\alpha$, since

$$
\begin{aligned}
& a b \text { is } \| \text { to } F_{1} \\
& b c \text { is } \| \text { to } F_{2}
\end{aligned}
$$

and

$$
c a \text { is } \| \text { to } F_{3}
$$

In the triangle of forces $a b c$,
and

$$
\begin{aligned}
a b & =F_{1} \\
b c & =F_{2} \\
c a & =F_{3}
\end{aligned}
$$

Applying sine rule to triangle $a b c$, we get,

$$
\frac{a b}{\sin (180-\alpha)}=\frac{b c}{\sin (180-\beta)}=\frac{c a}{\sin (180-\gamma)}
$$

i.e.,

$$
\frac{F_{1}}{\sin \alpha}=\frac{F_{2}}{\sin \beta}=\frac{F_{3}}{\sin \gamma}
$$

Example 2.9 A sphere weighing 120 N is tied to a smooth wall by a string as shown in Fig. 2.13(a). Find the tension $T$ in the string and reaction $R$ of the wall.

(a)

(b)

(c)

Fig. 2.13
Solution. Free body diagram of the sphere is as shown in Fig. 2.13(b). Let $T$ be the tension in the string and $R$ be the reaction from the wall. Selecting the coordinate directions $x, y$ as shown in the figure, the vector form of various forces in the system are,

$$
\begin{aligned}
& \mathbf{F}_{1}=-T \sin 20^{\circ} \mathbf{i}+T \cos 20^{\circ} \mathbf{j} \\
& \mathbf{F}_{2}=\operatorname{Ri} \\
& \mathbf{F}_{3}=-120 \mathbf{j}
\end{aligned}
$$

Since the body is in equilibrium, the resultant $=0$

$$
\begin{align*}
& \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}=0 \\
& -T \sin 20^{\circ} \mathbf{i}+T \cos 20^{\circ} \mathbf{j}+R \mathbf{i}-120 \mathbf{j}=0 \\
& \left(-T \sin 20^{\circ}+R\right) \mathbf{i}+\left(T \cos 20^{\circ}-120\right) \mathbf{j}=0 \\
\text { i.e., } \quad & -T \sin 20^{\circ}+R=0 \ldots(i) \text { and } T \cos 20^{\circ}-120=0
\end{align*}
$$

From Eqn. (ii), $\quad T=\frac{120}{\cos 20^{\circ}}=127.70 \mathrm{~N}$
Ans.
From Eqn. (i),

$$
R=T \sin 20^{\circ}=127.7 \sin 20^{\circ}=43.68 \mathrm{~N}
$$

Ans.

Alternately, the above problem may be solved using Lami's theorem. Using law of transmissibility of forces, $R$ is shifted as shown in Fig. 2.13(c), so that all the forces emerge out from the centre of gravity of the sphere. Then applying Lami's theorem, we get

$$
\frac{T}{\sin 90^{\circ}}=\frac{R}{\sin \left(180-20^{\circ}\right)}=\frac{120}{\sin \left(90+20^{\circ}\right)}
$$

$\therefore \quad \mathrm{T}=127.70 \mathrm{~N}$
Ans.
and $\quad R=43.68 \mathrm{~N}$
Ans.
Note:

1. A flexible cable like string can pull a body, but cannot push. Hence the force in the string is always directed away from the body.
2. The reaction from the wall on the ball is a push but not a pull. Hence this reaction is directed towards the body at contact surface. The vertical wall being smooth, the reaction from it is horizontal.
3. However, if the sense of the reactions are taken opposite, the values of the reactions obtained will be negative. While interpreting the results one can take care of the negative signs.

Example 2.10 Determine the horizontal force $P$ to be applied to a block weighing 1500 N to hold it in the position shown in Fig. 2.14(a). The inclined plane is smooth and makes $30^{\circ}$ with the horizontal.


Solution. The free body diagram of the block is as shown in Fig. 2.14(b). It may be noted that since the plane makes $30^{\circ}$ to the horizontal, the reaction which is normal to the plane, makes $30^{\circ}$ to the vertical. Taking $x, y$, coordinates as shown in the figure, the vector form of the forces in the system are,

$$
\begin{aligned}
& \mathbf{F}_{1}=P \mathbf{i} \\
& \mathbf{F}_{2}=-R \sin 30 \mathbf{i}+R \cos 30^{\circ} \mathbf{j} \\
& \mathbf{F}_{3}=-1500 \mathbf{j}
\end{aligned}
$$

Since the body is in equilibrium the resultant is zero.
i.e.,

$$
\begin{aligned}
& \mathbf{F}_{1}+\mathbf{F}_{\mathbf{2}}+\mathbf{F}_{\mathbf{3}}=0 \\
& P \mathbf{i}-R \sin 30^{\circ} \mathbf{i}+R \cos 30^{\circ} \mathbf{j}-1500 \mathbf{j}=0 \\
& \quad\left(P-R \sin 30^{\circ}\right) \mathbf{i}+\left(R \cos 30^{\circ}-1500\right) \mathbf{j}=0
\end{aligned}
$$

i.e., $\quad P-R \sin 30^{\circ}=0$
and $\quad R \cos 30^{\circ}-1500=0$
From Eqn. (ii), $\quad R=\frac{1500}{\cos 30^{\circ}}=1732.05 \mathrm{~N}$
Ans.
Hence from Eqn. (i), $\quad P=R \sin 30^{\circ}=1732.05 \sin 30=866.03 \mathrm{~N}$
Ans.
Example 2.11 A roller weighing 10 kN rests on a smooth horizontal floor. It is connected to the floor using bar $A C$ as shown in Fig. 2.15(a). Determine the force in the bar $A C$ and reaction from the floor, if the roller is subjected to a horizontal force of 5 kN and an inclined force of 7 kN as shown in the figure.

(a)

(b)

Fig. 2.15
Solution. A bar can develop tensile force or compressive force. Let the force developed be a compressive force ' $S$ ' (push on the cylinder). Free body diagram of the cylinder is as shown in Fig. 2.15(b).

Since there are more than three forces in the system. Lami's equation cannot be applied

$$
\mathbf{R}=\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}=0
$$

i.e., $\quad 5 \mathbf{i}-7 \cos 45^{\circ} \mathbf{i}-7 \sin 45^{\circ} \mathbf{j}-10 \mathbf{j}+S \cos 30^{\circ} \mathbf{i}+S \sin 30^{\circ} \mathbf{j}+R \mathbf{j}=0$

$$
\begin{equation*}
\left(5-7 \cos 45^{\circ}+S \cos 30^{\circ}\right) \mathbf{i}+\left(-7 \sin 45^{\circ}-10+S \sin 30^{\circ}+R\right) \mathbf{j}=0 \tag{i}
\end{equation*}
$$

i.e., $\quad 5-7 \cos 45^{\circ}+S \cos 30^{\circ}=0$
and $\quad-7 \sin 45^{\circ}-10+S \sin 30^{\circ}+R=0$
From Eqn. (i)
$S=\frac{7 \cos 45^{\circ}-5}{\cos 30^{\circ}}=-0.058 \mathrm{kN}$
Ans
From Eqn. (ii), $\quad-7 \sin 45^{\circ}-10+\left(-0.058 \sin 30^{\circ}\right)+R=0$
or

$$
R=14.979 \mathrm{kN}
$$

Ans.
Since the magnitude of S is negative the force exerted by the bar is not a push but a pull i.e., the bar is under tension.

Example 2.12 A roller of radius $r=300 \mathrm{~mm}$, weighing 2000 N is to be pulled over a curb of height 150 mm [Fig. 2.16(a)] by horizontal force $F$ applied to the end of a string wound tightly around the circumference of the roller. Find the magnitude of $F$ required to start the roller move over the curb. What is the least $F$ through the centre of the roller to just turn it over the curb?


Fig. 2.16
Solution. When the roller is about to turn over the curb, the contact with the floor is lost and hence there is no reaction from the floor. When body is in equilibrium under action of only three forces, they must be concurrent forces (this will be proved in the next chapter in which concept of moment is developed). Hence the reaction $R$ from the curb must pass through the intersection of the other two forces i.e., $F$ and $W$. The FBD of the roller is as shown in Fig. 2.16(b).

Referring to Fig. 2.16(b),

$$
\begin{aligned}
\cos \alpha & =\frac{O C}{O A}=\frac{300-150}{300}=\frac{1}{2} \\
\alpha & =60^{\circ}
\end{aligned}
$$

In $\triangle A O B, \quad \angle O A B=\angle O B A$ since $O A=O B=r$
but $\quad \angle O A B+\angle O B A=\alpha=60^{\circ}$
$\therefore \quad 2 \angle O B A=60^{\circ}$
or $\quad \angle O B A=30^{\circ}$ i.e., reaction makes $30^{\circ}$ with vertical.
From the equilibrium condition, resultant $=0$

$$
R \sin 30^{\circ} \mathbf{i}+R \cos 30^{\circ} \mathbf{j}-F \mathbf{i}-2000 \mathbf{j}=0
$$

$$
\begin{equation*}
\left(R \sin 30^{\circ}-F\right) \mathbf{i}+\left(R \cos 30^{\circ}-2000\right) \mathbf{j}=0 \tag{i}
\end{equation*}
$$

i.e., $\quad R \sin 30^{\circ}-F=0$
and $\quad R \cos 30^{\circ}-2000=0$
From Eqn. (ii) $\quad R=\frac{2000}{\cos 30^{\circ}}=2309.4 \mathrm{~N}$
Hence from Eqn. (i), $\quad F=R \sin 30^{\circ}=2309.4 \sin 30^{\circ}$

$$
=1154.70 \mathrm{~N}
$$

Ans.

## Least Force Through the Centre of Roller

Now the reaction from the curb must pass through the centre of the wheel since the other two forces pass through that point. Its inclination to vertical is $60^{\circ}$, [Ref. Fig. 2.17(b)].

Let the direction of the least force $F$ makes an angle $\theta$ with the horizontal axis as shown in Fig. 2.17(a).

(a)


(b)

Fig. 2.17
The free body diagram of the roller when it is on the verge of rolling over the curb is as shown in Fig. 2.17(b). Since the body is in equilibrium,

Resultant Force $=0$
i.e.,
$R \sin 60^{\circ} \mathbf{i}+R \cos 60^{\circ} \mathbf{j}-F \cos \theta \mathbf{i}+F \sin \theta \mathbf{j}-2000 \mathbf{j}=0$
i.e., $\quad\left(R \sin 60^{\circ}-F \cos \theta\right) \mathbf{i}+\left(R \cos 60^{\circ}+F \sin \theta-2000\right) \mathbf{j}=0$
$\therefore \quad R \sin 60^{\circ}-F \cos \theta=0$
and

$$
\begin{equation*}
R \cos 60^{\circ}+F \sin \theta-2000=0 \tag{i}
\end{equation*}
$$

From Eqn. (i)

$$
\begin{equation*}
R=\frac{F \cos \theta}{\sin 60^{\circ}} \tag{ii}
\end{equation*}
$$

Hence from Eqn. (ii),

$$
\begin{aligned}
& \frac{F \cos \theta}{\sin 60^{\circ}} \cos 60^{\circ}+F \sin \theta & =2000 \\
\therefore \quad & F\left(\cos \theta \cot 60^{\circ}+\sin \theta\right) & =2000
\end{aligned}
$$

or

$$
F=\frac{2000}{\cos \theta \cot 60^{\circ}+\sin \theta}
$$

$F$ is least when $\quad \frac{d F}{d \theta}=0$
i.e., $\quad 2000 \frac{-\sin \theta \cdot \cot 60^{\circ}+\cos \theta}{\left(\cos \theta \cot 60^{\circ}+\sin \theta\right)^{2}}=0$
i.e., $\quad \cos \theta=\sin \theta \cot 60^{\circ}$
or $\quad \cot \theta=\cot 60^{\circ}$
i.e., $\quad \theta=60^{\circ}$
i.e., $\quad F$ is at right angle to reaction $R$.
$\therefore \quad F_{\text {least }}=\frac{2000}{\cos 60^{\circ} \cot 60^{\circ}+\sin 60^{\circ}}=1732.05 \mathrm{~N} \quad$ Ans.

## Equilibrium of Connected Bodies

When two or more bodies are in contact with each other, the system of forces appears to be non-concurrent force system. However when each body is considered separately, in many situations it turns out to be a set of concurrent force system. In such problems first the body subjected to only two unknown forces may be analysed (in case of planar problems) and then the others. This type of examples are solved below:

Example 2.13 A system of connected flexible cables shown in Fig. 2.18(a) is supporting two vertical forces 300 N and 400 N at points $B$ and $D$. Determine the forces in various segments of the cable.

(a)

(b)

Fig. 2.18
Solution. The free body diagrams of points $B$ and $D$ are as shown in Fig. 2.18(b). Let the tensile forces in the various segments of the flexible cables be $T_{1}, T_{2}, T_{3}$ and $T_{4}$ as shown in the figure.

From the equilibrium condition of point $D$,

$$
\begin{align*}
& \quad T_{1} \cos 45^{\circ} \mathbf{i}+T_{1} \sin 45^{\circ} \mathbf{j}-T_{2} \sin 60^{\circ} \mathbf{i}+T_{2} \cos 60^{\circ} \mathbf{j}-400 \mathbf{j}=0 \\
& \left(T_{1} \cos 45^{\circ}-T_{2} \sin 60^{\circ}\right) \mathbf{i}+\left(T_{1} \sin 45^{\circ}+T_{2} \cos 60^{\circ}-400\right) \mathbf{j}=0 \\
& \text { i.e., } \quad T_{1} \cos 45^{\circ}-T_{2} \sin 60^{\circ}=0  \tag{i}\\
& \text { and }  \tag{ii}\\
& \\
& \\
& \text { From Eqn. (i), } T_{1} \sin 45^{\circ}+T_{2} \cos 60^{\circ}-400=0 \\
& \\
& \text { Substituting it in Eqn. (ii), we get } \\
& \\
& 1.2247 T_{1}=\frac{\sin 60^{\circ}}{\cos 45^{\circ}} T_{2}=1.2247 T_{2} \\
& \text { or } \quad T_{2}+T_{2} \cos 60^{\circ}-400=0 \\
& T_{2}\left[1.2247 \sin 45^{\circ}+\cos 60^{\circ}\right]=400
\end{align*}
$$

$$
\begin{array}{lll}
\therefore & T_{2}=292.82 \mathrm{~N} & \text { Ans. } \\
\text { Hence } & T_{1}=1.2247 \times 292.82=358.62 \mathrm{~N} & \text { Ans. }
\end{array}
$$

Considering the equilibrium of point $B$, we get
$T_{2} \sin 60^{\circ} \mathbf{i}-T_{2} \cos 60^{\circ} \mathbf{j}+T_{3} \sin 30^{\circ} \mathbf{i}+T_{3} \cos 30^{\circ} \mathbf{j}-T_{4} \mathbf{i}-300 \mathbf{j}=0$
i.e., $\quad\left(T_{2} \sin 60^{\circ}+T_{3} \sin 30^{\circ}-T_{4}\right)=0$
and $\quad-T_{2} \cos 60^{\circ}+T_{3} \cos 30^{\circ}-300=0$
Substituting the value of $T_{2}$, from Eqn. (iv) we get

$$
\begin{array}{rlrl} 
& T_{3} \cos 30^{\circ} & =292.82 \cos 60^{\circ}+300 \\
\therefore \quad T_{3} & =515.47 \mathrm{~N}
\end{array}
$$

and then from Eqn. (iii), we get

$$
\begin{array}{cc} 
& 292.82 \sin 60^{\circ}+515.47 \sin 30^{\circ}-T_{4}=0 \\
\therefore & T_{4}=511.32 \mathrm{~N}
\end{array}
$$

Ans.
Example 2.14 A rope $A B, 4.5 \mathrm{~m}$ long is connected at two points $A$ and $B$ at the same level 4 m apart. A load of 1500 N is suspended from a point $C$ on the rope 1.5 m from $A$ as shown in Fig. 2.19(a). What load connected at point $D$ on the rope, 1 m from $B$ will be required to keep the position $C D$ horizontal?


Fig. 2.19
Solution. Drop perpendiculars $C E$ and $D F$ on $A B$.
Let

$$
\begin{equation*}
C E=y \text { and } A E=x \tag{i}
\end{equation*}
$$

From $\triangle A E C, \quad x^{2}+y^{2}=1.5^{2}=2.25$
It is given that

$$
\begin{array}{rlrl} 
& & A C+C D+B D & =4.5 \mathrm{~m} \\
\therefore & C D & =4.5-1.5-1.0=2.0 \mathrm{~m} \\
\therefore & B F & =A B-A E-E F \\
& & & 4.0-x-2.0=2-x \tag{ii}
\end{array}
$$

From triangle $B F D$, we have

$$
\begin{align*}
B F^{2}+D F^{2} & =1^{2} \\
(2-x)^{2}+y^{2} & =1 \tag{iii}
\end{align*}
$$

Subtracting Eqn. (iii) from Eqn. (i), we get

$$
\begin{array}{rlrl} 
& & x^{2}-(2-x)^{2} & =1.25 \\
\text { i.e., } & x^{2}-4+4 x-x^{2} & =1.25 \\
\therefore & x & =1.3125 \mathrm{~m} \\
\therefore & & =\cos ^{-1} \frac{1.3125}{1}=28.955^{\circ} \\
& & =\cos ^{-1} \frac{2-1.3125}{1}=46.567^{\circ}
\end{array}
$$

Considering the equilibrium of point $C$, and selecting cartesian coordinates $x, y$ as shown in Fig. 2.19(b).

$$
\begin{array}{lc} 
& -T_{1} \cos 28.955^{\circ} \mathbf{i}+T_{1} \sin 28.955^{\circ} \mathbf{j}+T_{2} \mathbf{i}-1500 \mathbf{j}=0 \\
\text { i.e., } & \left(-T_{1} \cos 28.955^{\circ}+T_{2}\right) \mathbf{i}+\left(T_{1} \sin 28.955^{\circ}-1500\right) \mathbf{j}=0 \\
\text { i.e., } & -T_{1} \cos 28.955^{\circ}+T_{2}=0 \\
\text { and } & T_{1} \sin 28.955^{\circ}-1500=0 \tag{v}
\end{array}
$$

From Eqn. (v),

$$
T_{1}=3098.39 \mathrm{~N}
$$

and hence from Eqn. (v), we get

$$
T_{2}=T_{1} \cos 28.955^{\circ}=2711.09 \mathrm{~N}
$$

Considering the equilibrium of point $D$, we get
$-T_{2} \mathbf{i}+T_{3} \cos 46.567^{\circ} \mathbf{i}+T_{3} \sin 46.567^{\circ} \mathbf{j}-W \mathbf{j}=0$
i.e., $\left(-T_{2}+T_{3} \cos 46.567^{\circ}\right) \mathbf{i}+\left(T_{3} \sin 46.567^{\circ}-W\right) \mathbf{j}=0$
i.e., $\quad-2711.09+T_{3} \cos 46.567^{\circ}=0$
and $\quad T_{3} \sin 46.567^{\circ}-W=0$
From Eqn. (vi), we get $T_{3}=\frac{2711.09}{\cos 46.567^{\circ}}=3943.37 \mathrm{~N}$
From Eqn. (vii), we get

$$
\begin{aligned}
W & =T_{3} \sin 46.567^{\circ} \\
& =3943.37 \sin 46.567^{\circ} \\
& =2863.59 \mathrm{~N}
\end{aligned}
$$

Ans.
Example 2.15 A wire rope is fixed at two points $A$ and $D$ as shown in Fig. 2.20(a). Two weights 20 kN and 30 kN are suspended from $B$ and $C$ respectively. The inclination of chords $A B$ and $B C$ are at $30^{\circ}$ and $50^{\circ}$ respectively to the vertical. Find the forces in segments $A B, B C$ and $C D$. Determine the inclination of the segment $C D$ to vertical.
Solution. Fig. 2.20(b) shows free body diagrams of the points $B$ and $C$. The coordinates $x, y$ are taken as indicated in the figure. From the equilibrium condition of point $B$ :


Fig. 2.20
$-T_{1} \sin 30^{\circ} \mathbf{i}+T_{1} \cos 30^{\circ} \mathbf{j}+T_{2} \sin 50^{\circ} \mathbf{i}-T_{2} \cos 50^{\circ} \mathbf{j}-20 \mathbf{j}=0$
i.e., $\quad\left(-T_{1} \sin 30^{\circ}+T_{2} \sin 50^{\circ}\right) \mathbf{i}+\left(T_{1} \cos 30^{\circ}-T_{2} \cos 50^{\circ}-20\right) \mathbf{j}=0$
i.e., $\quad-T_{1} \sin 30^{\circ}+T_{2} \sin 50^{\circ}=0$
and $\quad T_{1} \cos 30^{\circ}-T_{2} \cos 50^{\circ}-20=0$
From Eqn. (i),

$$
T_{2}=T_{1} \frac{\sin 30^{\circ}}{\sin 50^{\circ}}=0.6527 T_{1}
$$

Substituting it in Eqn. (ii),
$T_{1} \cos 30^{\circ}-0.6527 T_{1} \cos 50^{\circ}=20$

$$
\therefore \quad \mathrm{T}_{1}=\frac{20}{\cos 30^{\circ}-0.6527 \cos 50^{\circ}}=44.80 \mathrm{kN}
$$

$$
\therefore \quad T_{2}=0.6527 T_{1}=29.24 \mathrm{kN}
$$

Ans.
Ans.

From the equilibrium condition of point $C$, we get
i.e.,

$$
-T_{2} \sin 50^{\circ} \mathbf{i}+T_{2} \cos 50^{\circ} \mathbf{j}+T_{3} \sin \theta \mathbf{i}+T_{3} \cos \theta \mathbf{j}-30 \mathbf{j}=0
$$

$$
\left(-T_{2} \sin 50^{\circ}+T_{3} \sin \theta\right) \mathbf{i}+\left(T_{2} \cos 50^{\circ}+T_{3} \cos \theta-30\right) \mathbf{j}=0
$$

i.e.,

$$
\begin{equation*}
-T_{2} \sin 50^{\circ}+T_{3} \sin \theta=0 \tag{iii}
\end{equation*}
$$

$$
\begin{equation*}
T_{2} \cos 50^{\circ}+T_{3} \cos \theta=30 \tag{iv}
\end{equation*}
$$

From Eqn. (iii),

$$
\begin{align*}
T_{3} \sin \theta & =T_{2} \sin 50^{\circ} \\
& =29.24 \sin 50^{\circ} \\
& =22.4 \tag{v}
\end{align*}
$$

From Eqn. (iv), $\quad T_{3} \cos \theta=30-T_{2} \cos 50^{\circ}$
$=30-29.24 \cos 50^{\circ}$

$$
\begin{equation*}
=11.21 \tag{vi}
\end{equation*}
$$

From Eqns. (v) and (vi), we get

$$
\begin{array}{rlrl}
\tan \theta & =\frac{22.4}{11.21} \\
\therefore \quad \theta & \theta & =63.41^{\circ}
\end{array}
$$

## Ans.

Hence from Eqn. (v), $\quad T_{3}=\frac{22.4}{\sin 63.41^{\circ}}=25.05 \mathrm{kN} \quad$ Ans.
Example 2.16 A wire is fixed at two points $A$ and $D$ as shown in Fig. 2.21(a). Two weights 20 kN and 25 kN are suspended at $B$ and C respectively. When equilibrium is reached it is found that inclination of $A B$ is $30^{\circ}$ and that of $C D$ is $60^{\circ}$ to the vertical. Determine the tension in the segment $A B, B C$ and $C D$ of the rope and also the inclination of $B C$ to the vertical.


Fig. 2.21
Solution. Let $T_{1}, T_{2}$ and $T_{3}$ be the forces in the segments $A B, B C$ and $C D$. Let $\theta$ be the inclination of $B C$ to vertical. Figure 2.21(b) shows $F B D$ for point $B$ and $C$.

From the equilibrium condition at $B$.

$$
\begin{array}{ll} 
& -T_{1} \sin 30^{\circ} \mathbf{i}+T_{1} \cos 30^{\circ} \mathbf{j}+T_{2} \sin \theta \mathbf{i}-T_{2} \cos \theta \mathbf{j}-20 \mathbf{j}=0 \\
\text { i.e., } & \left(-T_{1} \sin 30^{\circ}+T_{2} \sin \theta\right) \mathbf{i}+\left(T_{1} \cos 30^{\circ}-T_{2} \cos \theta-20\right) \mathbf{j}=0 \\
\text { i.e., } & -T_{1} \sin 30^{\circ}+T_{2} \sin \theta=0 \\
\text { and } & T_{1} \cos 30^{\circ}-T_{2} \cos \theta-20=0
\end{array}
$$

From the equilibrium conditions at $C$, we get
i.e., $\quad\left(-T_{2} \sin \theta+T_{3} \sin 60^{\circ}\right) \mathbf{i}+\left(T_{2} \cos \theta+T_{3} \cos 60^{\circ}-25\right) \mathbf{j}=0$
or $\quad-T_{2} \sin \theta+T_{3} \sin 60^{\circ}=0$

$$
\begin{equation*}
T_{2} \cos \theta+T_{3} \cos 60^{\circ}-25=0 \tag{iii}
\end{equation*}
$$

From Eqns. (i) and (iii), we get

$$
\begin{align*}
-T_{1} \sin 30^{\circ}+T_{3} \sin 60^{\circ} & =0 \\
T_{1} & =\frac{\sin 60^{\circ}}{\sin 30^{\circ}} T_{3}=1.732 T_{3} \tag{v}
\end{align*}
$$

i.e.,

Adding Eqns. (ii) and (iv), we get

$$
T_{1} \cos 30^{\circ}+T_{3} \cos 60^{\circ}-45=0
$$

Substituting the value of $T_{1}$ from Eqn. (v) in this, we get

$$
\begin{aligned}
1.732 T_{3} \cos 30^{\circ}+T_{3} \cos 60^{\circ} & =45 \\
T_{3}\left[1.732 \cos 30^{\circ}+\cos 60^{\circ}\right] & =45
\end{aligned}
$$

or

$$
T_{3}=22.5 \mathrm{kN}
$$

Ans.
From Eqn. (v), $\quad T_{1}=1.732 T_{3}=38.97 \mathrm{kN}$

## Ans.

and from Eqn. (i), $T_{2} \sin \theta=T_{1} \sin 30^{\circ}=19.485$
and from Eqn. (iv), $T_{2} \cos \theta=25-T_{3} \cos 60^{\circ}$

$$
\begin{align*}
& =25-22.5 \cos 60^{\circ} \\
& =13.75 \tag{vii}
\end{align*}
$$

From Eqns. (vi) \& (vii), we get

$$
\tan \theta=\frac{19.485}{13.75} \quad \therefore \theta=54.78^{\circ}
$$

Ans.
Hence from Eqn. (vi), $\quad T_{2}=23.85 \mathrm{kN}$
Ans.
Example 2.17 A 600 N cylinder is supported by the frame $B C D$ as shown in Fig. 2.22(a). The frame is hinged at $D$. Determine the reactions at $A, B, C$, and $D$.


Fig. 2.22
Solution. Free body diagrams of sphere and the frame are shown in Fig. 2.22(b). The coordinates selected are also indicated.

From the equilibrium condition of the sphere,

$$
\begin{array}{lc} 
& -R_{A} \mathbf{i}+R_{B} \mathbf{j}+R_{C} \mathbf{i}-600 \mathbf{j}=0 \\
\text { i.e., } & \left(-R_{A}+R_{C}\right) \mathbf{i}+\left(R_{B}-600\right) \mathbf{j}=0 \\
\text { i.e., } & -R_{A}+R_{C}=0 \text { and } R_{B}-600=0 \\
\therefore & R_{A}=R_{C}  \tag{i}\\
\text { and } & R_{B}=600 \mathrm{~N}
\end{array}
$$

Consider the $F B D$ of the frame. Since the frame is in equilibrium under the action of three forces only, they must be concurrent. In other words the reaction at $D$ is in the direction $O D$. Hence its inclination to the horizontal is given by,

$$
\tan \alpha=\frac{450}{150}=3 \quad \text { or } \quad \alpha=71.565^{\circ}
$$

$\therefore$ The equation of equilibrium is,

$$
R_{D} \cos 71.565^{\circ} \mathbf{i}+R_{D} \sin 71.565^{\circ} \mathbf{j}-R_{B} \mathbf{j}-R_{C} \mathbf{i}=0
$$

$$
\left(R_{D} \cos 71.565^{\circ}-R_{C}\right) \mathbf{i}+\left(R_{D} \sin 71.565^{\circ}-R_{B}\right) \mathbf{j}=0
$$

$$
\begin{equation*}
\text { i.e., } R_{D} \cos 71.565^{\circ}-R_{C}=0 \tag{iii}
\end{equation*}
$$

and $R_{D} \sin 71.569^{\circ}-R_{B}=0$
From Eqn. (iv), $\quad R_{D}=\frac{R_{B}}{\sin 71.565^{\circ}}=\frac{600}{\sin 71.565^{\circ}}=632.46 \mathrm{~N}$
Ans.
$\therefore$ From Eqn. (iii), $\quad R_{C}=R_{D} \cos 71.565^{\circ}=632.46 \cos 71.565^{\circ}$

$$
=200 \mathrm{~N}
$$

Ans.
Example 2.18 Two identical rollers, each weighing 100 N are supported by an inclined plane and a vertical wall as shown in Fig. 2.23(a). Assuming all contact surfaces are smooth, find the reactions developed at the contact surfaces $A, B, \mathrm{C}$ and $D$.


Fig. 2.23
Solution. Free body diagrams of the two roller and cartesian coordinate system used for the analysis are shown in Fig. 2.23(b). It may be noted that $R_{A}$ is normal to the plane and hence at $30^{\circ}$ to vertical; $R_{B}$ is parallel to the plane. From the equilibrium condition of cylinder 1 .

$$
\begin{array}{r}
R_{B} \cos 30^{\circ} \mathbf{i}+R_{B} \sin 30^{\circ} \mathbf{j}-R_{A} \sin 30^{\circ} \mathbf{i}+R_{A} \cos 30^{\circ} \mathbf{j}-100 \mathbf{j}=0 \\
\left(R_{B} \cos 30^{\circ}-R_{A} \sin 30^{\circ}\right) \mathbf{i}+\left(R_{B} \sin 30^{\circ}+R_{A} \cos 30^{\circ}-100\right) \mathbf{j}=0
\end{array}
$$

i.e., $\quad R_{B} \cos 30^{\circ}-R_{A} \sin 30^{\circ}=0$
and $\quad R_{B} \sin 30^{\circ}+R_{A} \cos 30^{\circ}-100=0$
From Eqn. (i) $\quad R_{A}=R_{B} \cot 30^{\circ}$
Substituting it in Eqn. (ii), we get

$$
\begin{aligned}
& R_{B} \sin 30^{\circ}+R_{B} \cot 30^{\circ} \cos 30^{\circ}=100 \\
& R_{B} \frac{\sin ^{2} 30^{\circ}+\cos ^{2} 30^{\circ}}{\sin 30^{\circ}}=100 \\
\therefore \quad & R_{B}=50 \mathrm{~N} \\
\therefore & R_{A}=R_{B} \cot 30^{\circ}=86.67 \mathrm{~N}
\end{aligned}
$$

Ans.
Ans.
From the equilibrium condition of cylinder 2 ,
$-R_{B} \cos 30^{\circ} \mathbf{i}-R_{B} \sin 30^{\circ} \mathbf{j}-100 \mathbf{j}+R_{D} \mathbf{i}-R_{C} \sin 30^{\circ} \mathbf{i}+R_{C} \cos 30^{\circ} \mathbf{j}=0$
i.e., $\left(-R_{B} \cos 30^{\circ}+R_{D}-R_{C} \sin 30^{\circ}\right) \mathbf{i}+\left(-R_{B} \sin 30^{\circ}-100+R_{C} \cos 30\right)^{\circ} \mathbf{j}=0$
i.e., $\quad-R_{B} \cos 30^{\circ}+R_{D}-R_{C} \sin 30^{\circ}=0$
and $\quad-R_{B} \sin 30^{\circ}+R_{C} \cos 30^{\circ}-100=0$
Substituting the value of $R_{B}$ in Eqn. (iv), we get

$$
-50 \sin 30^{\circ}+R_{C} \cos 30^{\circ}=100
$$

$$
\therefore \quad R_{C}=\frac{100+50 \sin 30^{\circ}}{\cos 30^{\circ}}=144.34 \mathrm{~N} \quad \text { Ans. }
$$

Substituting it in Eqn. (iii), we get

$$
\begin{aligned}
-50 \cos 30^{\circ}+R_{D}-144.34 \sin 30^{\circ} & =0 \\
R_{D} & =144.34 \sin 30^{\circ}+50 \cos 30^{\circ} \\
& =115.47 \mathrm{~N}
\end{aligned}
$$

Ans.
Example 2.19 Two smooth spheres each of radius 100 mm , and weight 100 N , rest in a horizontal channel having vertical walls, the distance between the walls being 360 mm . Find the reactions at the points of contacts $A, B, C$ and $D$ as shown in Fig. 2.24(a).


Fig. 2.24

Solution. Let $O_{1}$ and $O_{2}$ be the centres of the first and second spheres respectively. Drop perpendicular $O_{1} P$ to the horizontal line through $O_{2}$. Free body diagrams of the spheres and the coordinate directions are shown in Fig. 2.24(b). Since the surfaces of contact are smooth, reaction at $B$ is at right angle to tangent at $B$ i.e., it is in the radical direction. Let it make angle $\alpha$ with horizontal. Then,

$$
\begin{aligned}
\cos \alpha & =\frac{O_{2} P}{O_{1} O_{2}}=\frac{360-O_{1} A-O_{2} D}{O_{1} B+O_{2} B}=\frac{360-100-100}{100+100} \\
& =0.8 \\
\alpha & =36.87^{\circ}
\end{aligned}
$$

Equation of equilibrium for first sphere is,

$$
-R_{A} \mathbf{i}-100 \mathbf{j}+R_{B} \cos 36.87^{\circ} \mathbf{i}+R_{B} \sin 36.87^{\circ} \mathbf{j}=0
$$

i.e., $\quad\left(-R_{A}+R_{B} \cos 36.87^{\circ}\right) \mathbf{i}+\left(R_{B} \sin 36.87^{\circ}-100\right) \mathbf{j}=0$
i.e., $\quad-R_{A}+R_{B} \cos 36.87^{\circ}=0$
and $\quad R_{B} \sin 36.87^{\circ}-100=0$
From Eqn. (ii), $\quad R_{B}=\frac{100}{\sin 36.87^{\circ}}=166.67 \mathrm{~N} \quad$ Ans.
From Eqn. (i), $\quad R_{A}=R_{B} \cos 36.87^{\circ}=166.67 \cos 36.87^{\circ}$

$$
=133.33 \mathrm{~N}
$$

Ans.
Consider the equilibrium of sphere 2.

$$
\begin{array}{lr} 
& -R_{B} \cos \alpha \mathbf{i}-R_{B} \sin \alpha \mathbf{j}+R_{D} \mathbf{i}-100 \mathbf{j}+R_{C} \mathbf{j}=0 \\
\text { i.e., } & \left(-R_{B} \cos 36.87^{\circ}+R_{D}\right) \mathbf{i}+\left(-R_{B} \sin \alpha-100+R_{C}\right) \mathbf{j}=0 \\
\text { i.e., } & R_{D}=R_{B} \cos 36.87^{\circ} \\
\text { and } \quad-R_{B} \sin 36.87^{\circ}+R_{C}=100  \tag{iv}\\
\text { From Eqn. (iii), } \quad R_{D}=166.67 \cos 36.87^{\circ}=133.33 \mathrm{kN} \\
\text { From Eqn. (iv), } \quad R_{C}=100+166.67 \sin 36.87^{\circ} \\
& =200 \mathrm{~N}
\end{array}
$$

Ans.

Ans.

### 2.5 RESOLUTION OF FORCES IN SPACE

To represent a force in space, we need three mutually perpendicular directions, say $x, y$ and $z$. In literature coordinate directions $x, y, z$ selected are as per right hand rule. As shown in Fig. 1.15 different orientation of coordinates are possible and in this book orientation shown in Fig. 1.15(a) is used. The equation derived in one orientation holds good for any other orientations provided right hand rule is followed.

In Fig. 2.25, $O A$ represents a force $F$. The projection of $O A$ on the plane of $x z$ is $O B . O C$ and $O D$ are the projection of $O B$ on $z$ and $x$ coordinates. Then,


Fig. 2.25

$$
\begin{align*}
F & =O A=\sqrt{O B^{2}+A B^{2}} \\
& =\sqrt{O C^{2}+O D^{2}+A B^{2}} \\
& =\sqrt{F_{z}^{2}+F_{x}^{2}+F_{y}^{2}} \\
& =\sqrt{F_{x}^{2}+F_{y}^{2}+F_{z}^{2}} \tag{2.8a}
\end{align*}
$$

If we consider vector summation,

$$
\begin{align*}
\mathbf{F} & =\mathbf{O A}=\mathbf{O B}+\mathbf{B A} \\
& =\mathbf{O D}+\mathbf{O C}+\mathbf{B A} \\
& =\mathbf{F}_{x}+\mathbf{F}_{z}+\mathbf{F}_{y} \\
& =F_{x} \mathbf{i}+F_{y} \mathbf{j}+F_{z} \mathbf{k} \tag{2.9}
\end{align*}
$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are unit vectors in the coordinate directions $x, y$, and $z$ respectively. $F_{x}, F_{y}$ and $F_{z}$ are the components of $F$ in $x, y$ and $z$ directions. If $\theta_{x}, \theta_{y}$ and $\theta_{z}$ are the angles made by vector with $x, y$ and $z$ directions respectively, then

$$
\begin{aligned}
\mathbf{F} & =F_{x} \cos \theta_{x} \mathbf{i}+F \cos \theta_{y} \mathbf{j}+F \cos \theta_{z} \mathbf{k} \\
& =\mathrm{F}(l \mathbf{i}+m \mathbf{j}+n \mathbf{k})
\end{aligned}
$$

If $\mathbf{n}$ is the direction of unit vector of $F$, then

$$
\begin{array}{ll} 
& \mathbf{F}=F \mathbf{n}=F(l \mathbf{i}+m \mathbf{j}+n \mathbf{k}) \\
\therefore & \mathbf{n}=l \mathbf{i}+m \mathbf{j}+n \mathbf{k} \tag{2.10}
\end{array}
$$

Example 2.20 Determine the components of 50 kN force shown in Fig. 2.26 and express the force in the vector form.
Solution. Let OA be of 50 kN force. Resolving the force in the plane containing $y$-axis and the force, we get

$$
\begin{aligned}
F_{y} & =50 \cos 20^{\circ}=46.985 \mathrm{kN} . \\
F_{O B} & =50 \cos (90-20)^{\circ}=17.101 \mathrm{kN}
\end{aligned}
$$



Fig. 2.26
Resolving $O B$ into its $x$ and $z$ components, we get

$$
\begin{aligned}
& F_{x}=O B \cos 30^{\circ}=17.101 \cos 30^{\circ}=14.810 \mathrm{kN} \\
& F_{z}=O B \cos (90-30)^{\circ}=17.101 \cos 60^{\circ}=8.550 \mathrm{kN} \\
& \therefore \quad \mathbf{F}=F_{x} \mathbf{i}+F_{y} \mathbf{j}+F_{z} \mathbf{k} \\
& =14.810 \mathbf{i}+46.985 \mathbf{j}+8.550 \mathbf{k}
\end{aligned}
$$

Example 2.21 Express force $\mathbf{F}$ shown in Fig. 2.27 in the vector form.


Fig. 2.27
Solution. $\quad O A=\sqrt{10^{2}+8^{2}+4^{2}}=13.416$
$\therefore l=\frac{10}{13.416}=0.745 \quad m=\frac{8}{13.416}=0.596 \quad n=\frac{4}{13.416}=0.298$
$\therefore F_{x}=F l=500 \times 0.745 ; \quad F_{y}=F m=500 \times 0.596 ; \quad F_{z}=F n=500 \times 0.298$

$$
=372.5 \mathrm{~N} \quad=298.14 \mathrm{~N} \quad=149.07 \mathrm{~N}
$$

$\therefore \mathbf{F}=372.5 \mathbf{i}+298.14 \mathbf{j}+149.07 \mathbf{k}$
Example 2.22 A rope is tied to a pole at a height of 5 m and is pulled with a force of 500 N by a person standing at a distance of 10 m in $\mathrm{S} 60^{\circ} \mathrm{E}$ (South $60^{\circ}$ East) direction. His hands are 1 m above the base of the pole. Determine the component of the pull exerted in vertical, horizontal East and horizontal South directions. Express the force in vector form.

Solution. Let $A$ be the point on the pole, where the rope is tied and $B$ be the point on the rope where force is applied (Ref. Fig. 2.28). Taking the coordinate system as shown in the figure, the coordinates of the points $A$ and $B$ are

$$
A(0,5,0)
$$

and $B\left(10 \cos 30^{\circ}, 1.0,10 \cos 60^{\circ}\right)$ i.e., $B(8.66,1.0,5.0)$


Fig. 2.28
$\therefore$ The position vector of $A B$ is,

$$
\begin{aligned}
\mathbf{r}_{\mathrm{AB}} & =\mathbf{r}_{\mathbf{O B}}-\mathbf{r}_{\mathbf{O A}}=(8.66-0) \mathbf{i}+(1.0-5) \mathbf{j}+(5-0) \mathbf{k} \\
& =8.66 \mathbf{i}-4 \mathbf{j}+5 \mathbf{k} \\
\therefore \quad r_{A B} & =\sqrt{8.66^{2}+(-4)^{2}+5^{2}}=10.77 \mathrm{~m}
\end{aligned}
$$

$\therefore$ The direction cosines are,

$$
l=\frac{8.66}{10.77}=0.804 ; m=\frac{-4}{10.77}=-0.371 ; n=\frac{5}{10.77}=0.464
$$

Since 500 N force is in the direction $A B$, its direction cosines are the same as that of $A B$. Hence

$$
\begin{aligned}
& F_{x}=500 \times(0.804)=402 \mathrm{~N} \\
F_{y} & =500 \times(-0.377)=-185.5 \mathrm{~N} \\
& F_{z}=500 \times(0.464)=232.12 \mathrm{~N} \\
\therefore \quad & \mathbf{F}
\end{aligned}=402 \mathbf{i}-185.5 \mathbf{j}+232.12 \mathbf{k}
$$

Ans.
Ans.
Ans.
Ans.

### 2.6 RESULTANT OF CONCURRENT FORCE SYSTEM IN SPACE

The system of forces in space acting on a particle constitute concurrent force system in space. The best way of finding their resultant force is to express each
force in vector form and then use vector addition. Thus, if $\mathbf{F}_{1}, \mathbf{F}_{2}$ and $\mathbf{F}_{3}$ are the three forces acting on a particle in space, their resultant is given by

$$
\begin{aligned}
\mathbf{R} & =\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3} \\
& =F_{1 x} \mathbf{i}+F_{1 y} \mathbf{j}+F_{1 z} \mathbf{k}+F_{2 x} \mathbf{i}+F_{2 y} \mathbf{j}+F_{2 z} \mathbf{k}+F_{3 x} \mathbf{i}+F_{3 y} \mathbf{j}+F_{3 z} \mathbf{k} \\
& =\left(F_{1 x}+F_{2 x}+F_{3 x}\right) \mathbf{i}+\left(F_{1 y}+F_{2 y}+F_{3 y}\right) \mathbf{j}+\left(F_{1 z}+F_{2 z}+F_{3 z}\right) \mathbf{k} \\
& =\Sigma F_{x} \mathbf{i}+\Sigma F_{y} \mathbf{j}+\Sigma F_{z} \mathbf{k}
\end{aligned}
$$

Example 2.23 The directions of the three forces acting at point $O$ are shown in Fig. 2.29 by drawing a parallelepiped of size 6,4 and 2. Determine the resultant of the three forces.


Fig. 2.29
Solution. The coordinates of points $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ are

$$
\begin{array}{ll} 
& A(6,4,0), B(6,4,2) \text { and } C(6,0,2) \\
\therefore \quad r_{O A} & =\sqrt{6^{2}+4^{2}+0^{2}}=7.211 \text { units } \\
r_{O B} & =\sqrt{6^{2}+4^{2}+2^{2}}=7.483 \text { units } \\
\text { and } \quad r_{O C} & =\sqrt{6^{2}+0^{2}+2^{2}}=6.325 \text { units } \\
\mathbf{r}_{\mathbf{O A}} & =6 \mathbf{i}+4 \mathbf{j}+0 \mathbf{k}=r_{O A}\left(l_{1} \mathbf{i}+m_{1} \mathbf{j}+n_{1} \mathbf{k}\right) \\
& =7.211\left(l_{1} \mathbf{i}+m_{1} \mathbf{j}+n_{1} \mathbf{k}\right) \\
\mathbf{r}_{\mathbf{O B}} & =6 \mathbf{i}+4 \mathbf{j}+2 \mathbf{k}=7.483\left(l_{2} \mathbf{i}+m_{2} \mathbf{j}+n_{2} \mathbf{k}\right) \\
\mathbf{r}_{\mathbf{O C}} & =6 \mathbf{i}+0 \mathbf{j}+2 \mathbf{k}=6.325\left(l_{3} \mathbf{i}+m_{3} \mathbf{j}+n_{3} \mathbf{k}\right)
\end{array}
$$

where $l_{i}, m_{i}$ and $n_{i}$ are the direction cosines of $i^{\text {th }}$ member

$$
\therefore \quad l_{1}=\frac{6}{7.211}=0.832, \quad m_{1}=\frac{4}{7.211}=0.558, \quad n_{1}=0
$$

$$
\begin{array}{rlr}
l_{2} & =\frac{6}{7.483}=0.802, \quad m_{2}=\frac{4}{7.483}=0.535, \quad n_{2}=\frac{2}{7.483}=0.267 \\
l_{3} & =\frac{6}{6.325}=0.947 \quad m_{3}=0 & n_{3}=\frac{2}{6.325}=0.316 \\
\mathbf{F}_{\mathbf{O A}} & =\mathbf{F}_{1}=F_{1} \mathbf{n}_{1}=200\left(l_{1} \mathbf{i}+m_{i} \mathbf{j}+n_{i} \mathbf{k}\right) \\
& =200(0.832 \mathbf{i}+0.558 \mathbf{j})=166.4 \mathbf{i}+111.6 \mathbf{j} \\
\mathbf{F}_{\mathbf{O B}} & =F_{2} \mathbf{n}_{2}=150(0.802 \mathbf{i}+0.535 \mathbf{j}+0.267 \mathbf{k}) \\
& =120.3 \mathbf{i}+80.25 \mathbf{j}+40.05 \mathbf{k} \\
\mathbf{F}_{\mathbf{O C}} & =F_{3} \mathbf{n}_{3}=100(0.947 \mathbf{i}+0.316 \mathbf{k}) \\
& =94.7 \mathbf{i}+31.6 \mathbf{k}
\end{array}
$$

$\therefore$ The resultant force $R$ is given by

$$
\begin{aligned}
\mathbf{R}= & \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3} \\
= & (166.4+120.3+94.7) \mathbf{i}+(111.6+80.25) \mathbf{j} \\
& +(0+40.05+31.6) \mathbf{k} \\
= & 326.6 \mathbf{i}+191.85 \mathbf{j}+71.65 \mathbf{k} \\
\therefore \quad R= & \sqrt{326.6^{2}+191.85^{2}+71.65^{2}}=385.5 \mathrm{~N} \\
l= & \frac{326.6}{385.5}=0.847, m=\frac{196.85}{385.5}=0.511, n=\frac{71.65}{385.5}=0.186 \quad \text { Ans. } \\
\theta_{x}= & 32.113, \theta_{y}=59.29^{\circ}, \theta_{z}=79.289^{\circ} \quad \text { Ans. }
\end{aligned}
$$

Example 2.24 Three forces are acting at point $A(4,6,-4)$. If $B(3,6,2)$ is a point on the line of action of the force $F_{1}=40 \mathrm{kN}, C(-3,4,-5)$ is a point on the line of action of the force $F_{2}=50 \mathrm{kN}$ and $D(-5,-6,8)$ is a point on the third force $F_{3}=60 \mathrm{kN}$, determine the resultant of the three forces.
Solution. We need direction cosines of the displacement vector $A B, A C$ and $A D$, which will be the same as of the forces $F_{1}, F_{2}$ and $F_{3}$.

$$
\begin{aligned}
\mathbf{r}_{\mathrm{AB}} & =\mathbf{r}_{\mathrm{OB}}-\mathbf{r}_{\mathbf{O A}}=(3-4) \mathbf{i}+(6-6) \mathbf{j}+2-(-4) \mathbf{k} \\
& =-\mathbf{i}+6 \mathbf{k} \\
\text { Similarly } \quad \mathbf{r}_{\mathrm{AC}} & =(-3-4) \mathbf{i}+(4-6) \mathbf{j}+(-5+4) \mathbf{k} \\
& =-7 \mathbf{i}-2 \mathbf{j}-\mathbf{k} \\
\mathbf{r}_{\mathrm{AD}} & =(-5-4) \mathbf{i}+(-6-6) \mathbf{j}+(8+4) \mathbf{k} \\
& =-9 \mathbf{i}-12 \mathbf{j}+12 \mathbf{k} \\
r_{A B} & =\sqrt{(-1)^{2}+6^{2}}=6.083 \\
r_{A C} & =\sqrt{(-7)^{2}+(-2)^{2}+(-1)^{2}}=7.348 \\
r_{A D} & =\sqrt{(-9)^{2}+(-12)^{2}+(12)^{2}}=19.209
\end{aligned}
$$

and
$\therefore$ The resultant force $R$ is given by

$$
\begin{aligned}
& \mathbf{R}= \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3} \\
&=(-6.56-47.65-28.14) \mathbf{i}+(0-13.6-37.5) \mathbf{j} \\
&+(39.454-6.8+37.5) \mathbf{k} \\
&=-82.35 \mathbf{i}-51.1 \mathbf{j}+70.154 \mathbf{k} \\
& R= \sqrt{82.35^{2}+51.1^{2}+70.154^{2}}=119.64 \mathrm{kN} \\
& l= \frac{-82.35}{119.64}=-0.688, m=\frac{-51.1}{119.64}=-0.427, n=\frac{70.154}{119.64}=0.586 \\
& \theta_{x}= 133.47^{\circ} \quad \theta_{y}=115.277^{\circ} \quad \theta_{z}=54.1^{\circ} \quad \text { Ans. } \\
& \text { Ans. } \\
& \text { Ans. }
\end{aligned}
$$

Example 2.25 After making a cut at ground level, a tree is to be pulled down with the help of two ropes tied at 8 m high above the ground level. Rope $O A$ is in $\mathrm{S} 60^{\circ} \mathrm{E}$ direction at a horizontal distance of 10 m and here a group of labourers are applying a force of 6 kN at a height of 1 m (Ref. Fig. 2.30). Rope $O^{\prime} B$ is in $\mathrm{N} 45^{\circ} \mathrm{E}$ at a distance of 8 m and another group of labourers is applying a force, 1 m above the ground level. What should be the magnitude of the force in rope $O B$ so that the tree falls in Eastern direction?
Solution. Coordinate system is selected with $O$ as origin, East direction as $x$-axis, South direction as $z$-axis and vertical upward direction as $y$-direction as shown in the figure. Hence the coordinates of $O^{\prime}, A$ and $B$ are

$$
O^{\prime}(0,8,0), A\left(10 \cos 30^{\circ}, 1.0,10 \cos 60^{\circ}\right), B\left(8 \cos 45^{\circ}, 1.0,-8 \cos 45^{\circ}\right)
$$

$$
\text { i.e., } \quad O^{\prime}(0,8,0), A(8.660,1.0,5.0) B(5.657,1.0,-5.657)
$$

$\therefore$

$$
\begin{aligned}
\mathbf{r}_{\mathbf{O}^{\prime} \mathbf{A}} & =\mathbf{r}_{\mathbf{O A}}-\mathbf{r}_{\mathbf{O O}} \\
& =8.660 \mathbf{i}+1.0 \mathbf{j}+5 \mathbf{k}-(0 \mathbf{i}+8.0 \mathbf{j}+0 \mathbf{k}) \\
& =8.66 \mathbf{i}-7.0 \mathbf{j}+5 \mathbf{k} \\
r_{O^{\prime} A}^{\prime} & =\sqrt{8.66^{2}+7.0^{2}+5^{2}}=12.207 \mathrm{~m} .
\end{aligned}
$$

$$
\begin{aligned}
& \therefore \quad l_{1}=\frac{-1}{6.083}=-0.164 ; m_{1}=0 \quad n_{1}=\frac{6}{6.083}=0.986 \\
& l_{2}=\frac{-7}{7.348}=-0.953 ; m_{2}=\frac{-2}{7.348}=-0.272, n_{2}=\frac{-1}{7.348}=-0.136 \\
& l_{3}=\frac{-9}{19.209}=-0.469 ; m_{3}=\frac{-12}{19.203}=-0.625, n_{3}=\frac{12}{19.209}=0.625 \\
& \therefore \quad \mathbf{F}_{1}=F_{1}\left(l_{1} \mathbf{i}+m_{1} \mathbf{j}+n_{1} \mathbf{k}\right) \\
& =40(-0.164 \mathbf{i}+0.986 \mathbf{k})=-6.56 \mathbf{i}+39.454 \mathbf{k} \\
& \text { Similarly, } \mathbf{F}_{2}=50(-0.953 \mathbf{i}-0.272 \mathbf{j}-0.136 \mathbf{k}) \\
& =-47.65 \mathbf{i}-13.6 \mathbf{j}-6.8 \mathbf{k} \\
& \mathbf{F}_{3}=60(-0.469 \mathbf{i}-0.625 \mathbf{j}+0.625 \mathbf{k}) \\
& =-28.14 \mathbf{i}-37.5 \mathbf{j}+37.5 \mathbf{k}
\end{aligned}
$$



Fig 2.30
$111^{1 \mathrm{y}}, \quad r_{O^{\prime} B}=\mathbf{r}_{\mathbf{O B}}-\mathbf{r}_{\mathbf{O O}}{ }^{\prime}=5.657 \mathbf{i}+1.0 \mathbf{j}-5.657 \mathbf{k}-(0 \mathbf{i}+8.0 \mathbf{j}+0 \mathbf{k})$

$$
=5.657 \mathbf{i}-7.0 \mathbf{j}-5.657 \mathbf{k}
$$

$\therefore \quad r_{O^{\prime} B}=\sqrt{5.657^{2}+7^{2}+5.657^{2}}=10.630 \mathrm{~m}$
$\therefore$ Direction cosines of $O^{\prime} A$ are

$$
l_{1}=\frac{8.66}{12.207}=0.709, m_{1}=\frac{-7.0}{12.207}=-0.573, n_{1}=\frac{5}{12.207}=0.410
$$

Direction cosines of $O^{\prime} B$ are

$$
l_{2}=\frac{5.657}{10.630}=0.532, m_{2}=\frac{-7.0}{10.630}=-0.659 n_{1}=\frac{5.657}{10.630}=0.532
$$

$\therefore$ Force in rope $O^{\prime} A$ is,

$$
\begin{aligned}
\mathbf{F}_{1} & =F_{1}\left(l_{1} \mathbf{i}+m_{1} \mathbf{j}+n_{1} \mathbf{k}\right) \\
& =6(0.709 \mathbf{i}-0.573 \mathbf{j}+0.410 \mathbf{k}) \\
& =4.254 \mathbf{i}-3.438 \mathbf{j}+2.460 \mathbf{k}
\end{aligned}
$$

Force in rope $O^{\prime} B$ is

$$
\mathbf{F}_{2}=F_{2}(0.532 \mathbf{i}-0.659 \mathbf{j}-0.532 \mathbf{k})
$$

The resultant force is,

$$
\begin{aligned}
\mathbf{R}= & \mathbf{F}_{1}+\mathbf{F}_{2} \\
= & \left(4.254+0.532 F_{2}\right) \mathbf{i}+\left(-3.438-0.659 F_{2}\right) \mathbf{j} \\
& +\left(2.460-0.532 F_{2}\right) \mathbf{k}
\end{aligned}
$$

If the tree has to fall in $x$-direction, the component of the resultant in $z$-direction should be zero. i.e.,

$$
\begin{aligned}
2.460-0.532 F_{2} & =0 \\
F_{2} & =4.624 \mathrm{kN}
\end{aligned}
$$

Ans.

### 2.7 EQUILIBRIUM OF PARTICLES SUBJECT TO SYSTEM OF FORCES IN SPACE

Under this title, we consider the equilibrium of bodies subject to concurrent forces. As explained in Art 2.4, when a body is in equilibrium, the resultant of the force system acting on it must be zero. Mathematically,
i.e.,

$$
\begin{aligned}
& \mathbf{R}=0 \\
& \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\ldots=0
\end{aligned}
$$

$$
F_{1 x} \mathbf{i}+F_{1 y} \mathbf{j}+F_{1 z} \mathbf{k}+F_{2 x} \mathbf{i}+F_{2 y} \mathbf{j}+F_{2 z} \mathbf{k}+\ldots=0
$$

$$
\begin{equation*}
\Sigma F_{x} \mathbf{i}+\Sigma F_{y} \mathbf{j}+\Sigma F_{z} \mathbf{k}=0 \tag{a}
\end{equation*}
$$

i.e.,

$$
\begin{equation*}
\Sigma F_{x}=0, \Sigma F_{y}=0 \text { and } \Sigma F_{z}=0 \tag{b}
\end{equation*}
$$

It is to be noted that while writing the equations of equilibrium, applied forces, self weight and all reactions are to be considered. However, in some problems, the self weight may be negligible compared to the other forces considered.

Example 2.26 A tripod is resting with its legs on a horizontal plane at points $A$, $B$ and $C$ as shown in Fig. 2.31. Its apex point $D$ is 4 m above the floor level and carries a vertically downward load of 12 kN . Determine the forces developed in the legs.


Fig. 2.31
Solution. Let the forces developed in the members $A D, B D$ and $C D$ be $F_{1}, F_{2}$ and $F_{3}$ respectively. The point $D$ is in equilibrium under the action of $F_{1}, F_{2}, F_{3}$ and 12 kN load.

Taking $O$, the projection of apex point $D$ as origin and the coordinate system as shown in Fig. 2.31, we have

$$
\begin{array}{ll} 
& A(-2,0,0), B(3,0,3), C(3,0,-3) \text { and } D(0,4,0) \\
\therefore & \mathbf{r}_{\mathrm{AD}}=[0-(-2)] \mathbf{i}+(4-0) \mathbf{j}+(0-0) \mathbf{k}=2 \mathbf{i}+4 \mathbf{j} \\
& \mathbf{r}_{\mathrm{BD}}=-3 \mathbf{i}+4 \mathbf{j}-3 \mathbf{k} \\
& \mathbf{r}_{\mathrm{CD}}=-3 \mathbf{i}+4 \mathbf{j}+3 \mathbf{k} \\
\therefore & \\
& r_{A D}=\sqrt{2^{2}+4^{2}}=4.472 \mathrm{~m} \\
& \\
& r_{B D}=\sqrt{(-3)^{2}+(4)^{2}+(-3)^{2}}=5.831 \mathrm{~m} \\
& \\
& r_{C D}=\sqrt{(-3)^{2}+(4)^{2}+(3)^{2}}=5.831 \mathrm{~m}
\end{array}
$$

$\therefore$ Unit vector of $A D$ is,

$$
\begin{aligned}
\mathbf{n}_{1} & =\frac{2}{4.472} \mathbf{i}+\frac{4}{4.472} \mathbf{j}+0 \mathbf{k} \\
& =0.447 \mathbf{i}+0.894 \mathbf{j}
\end{aligned}
$$

Unit vector of $B D$ is,

$$
\begin{aligned}
\mathbf{n}_{2} & =\frac{-3}{5.831} \mathbf{i}+\frac{4}{5.831} \mathbf{j}-\frac{3}{5.831} \mathbf{k} \\
& =-0.514 \mathbf{i}+0.686 \mathbf{j}-0.514 \mathbf{k}
\end{aligned}
$$

that of $C D$ is,

$$
\begin{aligned}
\mathbf{n}_{3} & =\frac{-3}{5.831} \mathbf{i}+\frac{4}{5.831} \mathbf{j}+\frac{3}{5.831} \mathbf{k} \\
& =-0.514 \mathbf{i}+0.686 \mathbf{j}+0.514 \mathbf{k}
\end{aligned}
$$

$\therefore$ Vector form of the forces in the legs are,

$$
\begin{aligned}
& \mathbf{F}_{1}=0.447 F_{1} \mathbf{i}+0.894 F_{1} \mathbf{j} \\
& \mathbf{F}_{2}=-0.514 F_{2} \mathbf{i}+0.686 F_{2} \mathbf{j}-0.514 F_{2} \mathbf{k} \\
& \mathbf{F}_{3}=-0.514 F_{3} \mathbf{i}+0.686 F_{3} \mathbf{j}+0.514 F_{3} \mathbf{k} \\
& \mathbf{F}_{4}=-12 \mathbf{j}
\end{aligned}
$$

The load is
Hence the equilibrium condition gives

$$
\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}=0
$$

$\left(0.447 F_{1}-0.514 F_{2}-0.514 F_{3}\right) \mathbf{i}+\left(0.894 F_{1}+0.686 F_{2}\right.$

$$
\begin{equation*}
\left.+0.686 F_{3}-12\right) \mathbf{j}+\left(-0.514 F_{2}+0.514 F_{3}\right) \mathbf{k}=0 \tag{i}
\end{equation*}
$$

i.e., $\quad 0.447 F_{1}-0.514 F_{2}-0.514 F_{3}=0$

$$
\begin{equation*}
0.894 F_{1}+0.686 F_{2}+0.686 F_{3}=12 \tag{ii}
\end{equation*}
$$

$$
\begin{equation*}
-0.514 F_{2}+0.514 F_{3}=0 \tag{iii}
\end{equation*}
$$

From Eqn. (iii),

$$
\begin{equation*}
F_{2}=F_{3} \tag{iv}
\end{equation*}
$$

Substituting it in Eqn. (i), we get

$$
\begin{align*}
0.447 F_{1}-2 \times 0.514 F_{2} & =0  \tag{v}\\
F_{1} & =2.30 F_{2}
\end{align*}
$$

or
Substituting for $F_{1}$ and $F_{3}$ in terms of $F_{2}$ in Eqn. (ii), we get

$$
\begin{array}{ll}
\text { i.e., } & F_{2}=3.50 \mathrm{kN} \\
\therefore & F_{1}=2.30 \times 3.50=8.05 \mathrm{kN} \\
\text { and } & F_{3}=3.50 \mathrm{kN}
\end{array}
$$

Ans.
Ans.

Example 2.27 A system of 3 cables $A B, A C$ and $A D$ shown in Fig. 2.32 is subjected to a force of 300 kN along the $x$-direction by using turn buckle $A E$. Find the forces developed in the cables.


Fig. 2.32
Solution. The coordinates of points $A, B, C$ and $D$ are $A(5,0,0), B(0,2,-4)$, $C(0,0,2)$ and $D(0,-3,0)$

$$
\begin{aligned}
& \mathbf{r}_{\mathrm{AB}} \\
& =-5 \mathbf{i}+2 \mathbf{j}-4 \mathbf{k}, \quad \mathbf{r}_{\mathrm{AC}}=-5 \mathbf{i}+0 \mathbf{j}+2 \mathbf{k} \\
\mathbf{r}_{\mathrm{AD}} & =-\mathbf{5} \mathbf{i}-\mathbf{3} \mathbf{j}+0 \mathbf{k} \\
\therefore \quad & r_{A B}
\end{aligned}=\sqrt{(-5)^{2}+(2)^{2}+(-4)^{2}}=6.708 \mathrm{~m}
$$

$\therefore$ unit vector in the direction of $A B$

$$
\begin{array}{ll} 
& \mathbf{n}_{1}=-0.745 \mathbf{i}+0.298 \mathbf{j}-0.596 \mathbf{k} \\
\therefore & \mathbf{F}_{1}=F_{1} \mathbf{n}_{1}=-0.745 F_{1} \mathbf{i}+0.298 F_{1} \mathbf{j}-0.596 F_{1} \mathbf{k} \\
\text { Similarly, } & r_{A C}=\sqrt{(-5)^{2}+(2)^{2}}=5.385 \mathrm{~m} \\
& \mathbf{n}_{2}=-0.929 \mathbf{i}+0 \mathbf{j}+0.371 \mathbf{k}
\end{array}
$$

$$
\left.\left.\begin{array}{rlrl} 
& \therefore & \mathbf{F}_{2} & =-0.929 F_{2} \mathbf{i}+0.371 F_{2} \mathbf{k} \\
\text { and } & & r_{A D} & =\sqrt{(-5)^{2}+(-3)^{2}+0}=5.831 \\
& & \therefore & \mathbf{n}_{3}
\end{array}\right)=-0.857 \mathbf{i}-0.514 \mathbf{j}\right)
$$

Let the force applied by turn buckle be $F_{4}$. Then

$$
F_{4}=300 \mathbf{i}
$$

From the equilibrium condition, we have,

$$
.
$$

From Eqn. (iii)

$$
F_{2}=\frac{0.596}{0.371} F_{1}=1.606 F_{1}
$$

From Eqn. (ii) $\quad F_{3}=\frac{0.298}{0.514} F_{1}=0.580 F_{1}$
$\therefore$ From Eqn. $(i),(0.745+0.929 \times 1.606+0.857 \times 0.580) F_{1}=300$
or

$$
\begin{aligned}
& F_{1}=109.73 \mathrm{kN} \\
& F_{2}=1.606 \times 109.73=176.223 \mathrm{kN}
\end{aligned}
$$

Ans.

Hence

$$
\text { and } \quad F_{3}=0.580 \times 109.73=63.64 \mathrm{kN}
$$

Ans.
Ans.

Example 2.28 A wire of length $1.5 r$ is tied to point $D$ on the surface of a sphere of radius $r$. Then the wire is tied to point $C$ which is the intersection of two smooth walls, the walls being at right angles to each other as shown in Fig. 2.33. If weight of sphere is $W$, determine the tension in the wire and the reactions from the wall. Assume all contact surface are smooth.

(a)

(b)

Fig. 2.33

Solution. Selecting the centre of sphere as origin and the coordinates system as shown in Fig. 2.33, the coordinates of the points of contact with walls are,

$$
A(-r, 0,0), B(0,0,-r) \text { and } C(-r, y,-r)
$$

Now

$$
O C=C D+r=1.5 r+r=2.5 r
$$

$\therefore$

$$
O C^{2}=(r)^{2}+y^{2}+(-r)^{2}
$$

$$
(2.5 r)^{2}=r^{2}+y^{2}+r^{2}
$$

$$
\therefore \quad y^{2}=4.25 r^{2}
$$

$$
y=2.062 r
$$

$\therefore \quad$ The coordinates of $C$ are $(-r, 2.06 r,-r)$.
Figure 2.33(b) shows $F B D$ of the sphere, which is in equilibrium under the action of reaction $R_{A}=F_{A O}$ and $R_{B}=F_{B O}$ from walls, tension $T=F_{O C}$ in wire and self weight $W$.

$$
\begin{aligned}
\mathbf{r}_{\mathrm{AO}} & =r \mathbf{i}+0 \mathbf{j}+0 \mathbf{k} \\
\mathbf{r}_{\mathbf{A O}} & =r \mathbf{i} \\
\mathbf{n}_{\mathbf{A O}} & =\mathbf{i}+0 \mathbf{j}+0 \mathbf{k} \\
\mathbf{F}_{\mathbf{A O}} & =F_{A O}(\mathbf{i}+0 \mathbf{j}+0 \mathbf{k}) \\
\mathbf{r}_{\mathbf{B O}} & =0 \mathbf{i}+0 \mathbf{j}+r \mathbf{k} \\
\mathbf{r}_{\mathbf{B O}} & =r \mathbf{k} \\
\mathbf{n}_{\mathbf{B O}} & =0 \mathbf{i}+0 \mathbf{j}+\mathbf{k} \\
\mathbf{F}_{\mathbf{B O}} & =F_{B O}(0 \mathbf{i}+0 \mathbf{j}+\mathbf{k}) \\
\mathbf{r}_{\mathbf{O C}} & =-r \mathbf{i}+2.062 r \mathbf{j}-r \mathbf{k} \\
\therefore \quad r_{O C} & =\sqrt{(-r)^{2}+(2.062 r)^{2}+(-r)^{2}}=2.5 r \\
& \mathbf{r}_{\mathbf{O C}} \\
& =-0.4 \mathbf{i}+0.825 \mathbf{j}-0.4 \mathbf{k} \\
\mathbf{F}_{\mathbf{O C}} & =F_{O C}(-0.4 \mathbf{i}+0.825 \mathbf{j}-0.4 \mathbf{k}) \\
\mathbf{W} & =-W \mathbf{j}
\end{aligned}
$$

$\therefore$ The equilibrium equation is,

$$
\begin{array}{cc}
\mathbf{F}_{\mathbf{A O}}+\mathbf{F}_{\mathbf{B O}}+\mathbf{F}_{\mathbf{O C}}+\mathbf{W}=0 \\
\left(F_{A O}+0-0.4 F_{O C}\right) \mathbf{i}+\left(0+0+0.825 F_{O C}-W\right) \mathbf{j} \\
& +\left(0+F_{B O}-0.4 F_{O C}\right) \mathbf{k}=0 \\
\text { i.e., } & F_{A O}-0.4 F_{O C}=0 \\
\text { i.e., } & 0.825 F_{O C}-W=0 \\
F_{B O}-0.4 F_{O C}=0
\end{array}
$$

From Eqn. (ii) $\quad F_{O C}=\frac{W}{0.825}=1.212 \mathrm{~W}$
Ans.
From Eqn. (i), $\quad F_{A O}=0.4 F_{O C}=0.485 \mathrm{~W}$
Ans.
From Eqn. (iii), $\quad F_{B O}=0.4 F_{O C}=0.485 \mathrm{~W}$
Ans.

Example 2.29 The system of cables supporting 100 kN load at $C$ is shown in Fig. 2.34(a). The coordinates of point $C$ with respect to ' $O$ ' as origin are $(0.2,-0.3,0)$. The points $C, D, E$ and $F$ are in $x-y$ plane. Determine the forces in various segments of cables.

(b)

Fig. 2.34
Solution. Consider the equilibrium condition at point $C$. Let the tensile forces developed in cables $C A, C B$ and $C D$ be $\mathbf{F}_{1}, \mathbf{F}_{2}$, and $\mathbf{F}_{3}$ respectively. The FBD of point C is shown in Fig. 2.34(b),

Now, $A(0,0,0.1), B(0,0,-0.1)$ and $C(0.2,-0.3,0)$

$$
\begin{aligned}
\mathbf{r}_{\mathbf{C A}} & =-0.2 \mathbf{i}+0.3 \mathbf{j}+0.1 \mathbf{k} \\
r_{C A} & =\sqrt{0.2^{2}+0.3^{2}+0.1^{2}}=0.374 \mathrm{~m} \\
\mathbf{n}_{1} & =-0.535 \mathbf{i}+0.802 \mathbf{j}+0.267 \mathbf{k} \\
\mathbf{F}_{1} & =F_{1}(-0.535 \mathbf{i}+0.802 \mathbf{j}+0.267 \mathbf{k}) \\
\mathbf{r}_{\mathbf{C B}} & =-0.2 \mathbf{i}+0.3 \mathbf{j}-0.1 \mathbf{k} \\
r_{C B} & =\sqrt{0.2^{2}+0.3^{2}+0.1^{2}}=0.374 \mathrm{~m} \\
\therefore \quad \mathbf{n}_{2} & =-0.535 \mathbf{i}+0.802 \mathbf{j}-0.267 \mathbf{k} \\
\mathbf{F}_{2} & =F_{2}(-0.537 \mathbf{i}+0.802 \mathbf{j}-0.267 \mathbf{k}) \\
\mathbf{F}_{3} & =F_{3} \mathbf{i} \text { since it is in } x \text {-direction } \\
\mathbf{F}_{4} & =-100 \mathbf{j}
\end{aligned}
$$

$\therefore$ Equilibrium equation is

$$
\begin{aligned}
& \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}=0 \\
&\left(-0.535 F_{1}-0.535 F_{2}+F_{3}\right) \mathbf{i}+\left(0.802 F_{1}+0.802 F_{2}-100\right) \mathbf{j} \\
&+\left(0.267 F_{1}-0.267 F_{2}\right) \mathbf{k}=0
\end{aligned}
$$

i.e.,

$$
\begin{align*}
-0.535 F_{1}-0.535 F_{2}+\mathrm{F}_{3} & =0  \tag{i}\\
0.802 F_{1}+0.802 F_{2} & =100  \tag{ii}\\
0.267 F_{1}-0.267 F_{2} & =0  \tag{iii}\\
F_{1} & =F_{2}
\end{align*}
$$

From Eqn. (iii)
Substituting it in Eqn. (ii), we get

$$
\begin{aligned}
(0.802+0.802) & F_{1}
\end{aligned}=100 ~ 子 ~\left(F_{1}=62.344 \mathrm{kN},\right.
$$

Ans.
Ans.

Ans.
Consider the equilibrium of point $D$. Let the forces in cables $D C, D E$ and $D F$ be $F_{3}, F_{4}$ and $F_{5}$ respectively. The FBD of point $D$ is shown in Fig. 2.34(b). Taking $D$ as origin,

$$
D(0,0,0), E(0.3,0.3,0) \text { and } F(0.3,-0.15,0)
$$

$$
\begin{aligned}
& r_{D E}=0.3 \mathbf{i}+0.3 \mathbf{j} \\
& r_{D E}=\sqrt{0.3^{2}+0.3^{2}}=0.424 \\
& \mathbf{n}_{4} \\
&=0.707 \mathbf{i}+0.707 \mathbf{j} \\
& \mathbf{F}_{4}=F_{4}(0.707 \mathbf{i}+0.707 \mathbf{j}) \\
& \therefore \quad r_{D F}=0.3 \mathbf{i}-0.15 \mathbf{j} \\
& \therefore \quad r_{D F}=\sqrt{0.3^{2}+0.15^{2}}=0.335 \mathrm{~m} \\
& \therefore \quad \mathbf{n}_{4}=0.896 \mathbf{i}-0.447 \mathbf{j} \\
& \mathbf{F}_{5}
\end{aligned}=F_{5}(0.896 \mathbf{i}-0.447 \mathbf{j})
$$

Equation of equilibrium is

$$
\begin{gather*}
\mathbf{F}_{3}+\mathbf{F}_{4}+\mathbf{F}_{5}=0 \\
\left(-F_{3}+0.707 F_{4}+0.896 F_{5}\right) \mathbf{i}+\left(0.707 F_{4}-0.447 \mathbf{F}_{5}\right) \mathbf{j}=0 \\
-F_{3}+0.707 F_{4}+0.896 F_{5}=0  \tag{iv}\\
0707 F_{4}-0.447 F_{5}=0 \tag{v}
\end{gather*}
$$

i.e.,

Subtracting Eqn. (v) from Eqn. (iv), we get

$$
\begin{aligned}
-F_{3}+(0.896+0.447) F_{5} & =0 \\
F_{5} & =\frac{F_{3}}{1.343}=\frac{66.708}{1.343}=49.671 \mathrm{kN}
\end{aligned}
$$

From Eqn. (v)

$$
F_{4}=\frac{0.447}{0.707} F_{5}=31.404 \mathrm{kN}
$$

Ans.

## IMPORTANT DEFINITIONS AND CONCEPTS

1. A particle is a body which has mass but no size and shape. However in engineering mechanics the bodies, the shape and size of which do not significantly affect the analysis, are treated as particles.
2. The system of forces acting on a particle constitute concurrent force system i.e., the system of forces whose line of action pass through single point.
3. Resolution of the forces is the process of finding a number of component forces which will have the same effect on the body as the given single force.
4. Self weight of a body is due to gravitational attraction and is always directed vertically downward.
5. The process of finding a single force (resultant force) which will have the same effect as the system of forces acting on the body is called composition of forces.
6. Equilibriant is a force which brings the body to state of equilibrium. It is equal in magnitude but opposite to the resultant force.
7. Reactions are the self adjusting forces exerted by the other bodies in contact with the body under consideration. If the surface of contact is smooth they are at right angles to the surface of contact.
8. The diagram of a body in which the body under consideration is freed from all contact surfaces and all the forces acting on it (including applied forces, self weight and reactions from the other bodies in contact) are drawn is termed as free body diagram.
9. If a body is in equilibrium under the action of only two forces, those two forces are collinear.
10. Lami's theorem states if a body is in equilibrium under the action of three forces, each force is proportional to the sine of the angle between the other two forces.

## IMPORTANT FORMULAE

Resultant of system of forces is given by

$$
\begin{array}{rlrl}
\mathbf{R} & =\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}+\ldots \\
& =\Sigma F_{x} \mathbf{i}+\Sigma F_{y} \mathbf{j}+\Sigma F_{z} \mathbf{k} & & \text { For system of concurrent forces in space. } \\
R & =\Sigma F_{x} \mathbf{i}+\Sigma F_{y} \mathbf{j} & & \text { For system of coplanar forces. } \\
R & =\sqrt{\left(\Sigma F_{x}\right)^{2}+\Sigma F_{y}^{2}+\Sigma F_{z}^{2}} & & \text { For system of concurrent forces in space. }
\end{array}
$$

and
and for system of forces in a plane

$$
R=\sqrt{\left(\Sigma F_{x}\right)^{2}+\left(\Sigma F_{y}\right)^{2}}
$$

$\tan \alpha=\frac{\Sigma F_{y}}{F_{x}}$ where $\alpha$ is angle with $x$-axis

If $A$ and $B$ are the two points on the line of action of force $F$ from $A$ to $B$, then

$$
\begin{aligned}
\mathbf{r}_{\mathbf{A B}} & =\mathbf{r}_{\mathbf{O B}}-\mathbf{r}_{\mathbf{O A}}=\left(x_{B}-x_{A}\right) \mathbf{i}+\left(y_{B}-y_{A}\right) \mathbf{j}+\left(Z_{B}-Z_{A}\right) \mathbf{k} \\
r_{A B} & =\sqrt{\left(x_{B}-x_{A}\right)^{2}+\left(y_{B}-y_{A}\right)^{2}+\left(z_{B}-z_{A}\right)^{2}} \\
\mathbf{n} & =\frac{x_{B}-x_{A}}{r_{A B}} \mathbf{i}+\frac{y_{B}-y_{A}}{r_{A B}} \mathbf{j}+\frac{z_{B}-z_{A}}{r_{A B}} \mathbf{k} \\
& =l \mathbf{i}+m \mathbf{j}+n \mathbf{k} \\
F & =F \mathbf{n}=F(l \mathbf{i}+m \mathbf{j}+n \mathbf{k})
\end{aligned}
$$

Equation of equilibrium is
i.e.,

$$
\begin{aligned}
& \mathbf{R}=0 \\
& \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\ldots=0 \\
& \Sigma F_{x} \mathbf{i}+\Sigma F_{y} \mathbf{j}+\Sigma F_{y} \mathbf{k}=0 \\
& \Sigma F_{x}=0 \Sigma F_{y}=0 \Sigma F_{z}=0
\end{aligned}
$$

In planar system $\quad \Sigma F_{x}=0, \Sigma F_{y}=0$.
Lami's equation for system of coplanar forces with 3 forces is,

$$
\begin{array}{ll}
\qquad \frac{F_{1}}{\sin \alpha} & =\frac{F_{2}}{\sin \beta}=\frac{F_{3}}{\sin \gamma} \\
\text { where } \quad & \alpha-\text { angle between } F_{2} \text { and } F_{3} \\
\text { and } \quad & \beta \text {-angle between } F_{3} \text { and } F_{1} \\
& \gamma-\text { angle between } F_{1} \text { and } F_{2}
\end{array}
$$

## PROBLEMS FOR EXERCISE

2.1 Determine the resultant of system of five coplanar forces acting on a particle as shown in Fig. 2.35.
[Ans. $\mathbf{R}=65.479 \mathbf{i}-178.963 \mathbf{j} ; \boldsymbol{R}=190.57 \mathrm{~N}, \alpha=-69.90^{\circ}$ ]
2.2 A body is subjected to the system of three forces as shown in Fig. 2.36. If possible determine the direction of the force $F$ so that the resultant is in $x$-direction, when (a) $F=10 \mathrm{kN}$ (b) $F=6 \mathrm{kN}$.
[Ans. (a) $\theta=36.87^{\circ}$; (b) Not possible]


Fig. 2.35


Fig. 2.36
2.3 A disabled car is pulled along a road by two ropes as shown in Fig. 2.37. Find the force $F_{1}$ and the resultant $R$ along the direction of the road.

$$
\text { [Ans. } \left.F_{1}=7.929 \mathrm{kN}, R=13.297 \mathrm{kN}\right]
$$

2.4 Determine the force $F$ shown in Fig. 2.38, such that the resultant of the three forces is 25 kN vertical downward on the pole.
[Ans. $\left.\alpha=19.467^{\circ}, F=20.687 \mathrm{kN}\right]$


Fig. 2.37


Fig. 2.38
2.5 A spherical ball weighing 100 N is held in the position by applying a force of 200 N as shown in Fig. 2.39. Determine the tension in the string and angle $\alpha$ made by the string with the vertical.
[Ans. $T=400 \mathrm{~N}, \theta=63.435^{\circ}$ ]
2.6 An electric light fixture weighing 150 N is supported by two wires as shown in Fig. 2.40. Determine the tensile forces developed in the wires.
[Ans. $F_{O A}=77.646 \mathrm{~N}, F_{O B}=109.808 \mathrm{~N}$ ]


Fig. 2.39


Fig. 2.40
2.7 The frictionless pulley $A$ shown in Fig. 2.41 is supported by two bars $A B$ and $A C$ which are hinged at $B$ and $C$ to a vertical wall. The flexible cable $D A$ hinged at $D$, goes over the pulley and supports a load of 25 kN at $A$. The angles between the various members are shown in the figure. Determine the forces in the bars $A B$ and $A C$. Neglect the size of pulley and treat it as frictionless.

$$
\left[\text { Ans. } F_{A B}=0, F_{A C}=43.301 \mathrm{kN}\right]
$$



Fig. 2.41
2.8 A three bar link mechanism is in equilibrium under the action of the forces shown in Fig. 2.42. Determine the magnitude of force $F$ and the forces developed in each bar.

$$
\text { [Ans. } F=304.72 \mathrm{~N} ; F_{B A}=163.30 \mathrm{~N} \text {, }
$$

$$
\left.F_{B C}=223.07 \mathrm{~N} \text { and } F_{C D}=273.20 \mathrm{~N}\right]
$$

2.9 Two cylinders of diameter 100 mm and 50 mm , weighing 200 N and 50 N respectively are placed in a trough as shown in Fig. 2.43. Assuming all contact surfaces smooth, find the reactions developed at contact surfaces 1, 2,3 , and 4 .
[Ans. $R_{1}=37.5 \mathrm{~N}, R_{2}=62.5 \mathrm{~N}, R_{3}=287.5 \mathrm{~N}$ and $\left.R_{4}=353.5 \mathrm{~N}\right]$


Fig. 2.42


Fig. 2.43
2.10 $A B$ is a guy wire as shown in Fig. 2.44 and is carrying a force of 40 kN . Determine its vector form, direction cosines and the angles with the coordinate directions $x, y$ and $z$.
[Ans. $-22.28 \mathbf{i}+14.84 \mathbf{j}+29.74 \mathbf{k}, l=-0.557$, $\left.m=0.371, n=0.742, \theta_{x}=123.85^{\circ}, \theta_{y}=68.22^{\circ}, \theta_{z}=42.01^{\circ}\right]$
2.11 Three concurrent forces acting on a parallelepiped at $b^{\prime}$ are as shown in Fig. 2.45. Determine the resultant force.
[Ans. $R=7.07 \mathbf{i}+17.67 \mathbf{j}$

$$
\left.+6.67 \mathbf{k}, R=20.197 \mathrm{kN}, \theta_{x}=69.51^{\circ}, \theta_{y}=28.57^{\circ}, \theta_{z}=70.45^{\circ}\right]
$$



Fig. 2.44


Fig. 2.45
2.12 The tripod shown in Fig. 2.46 is subjected to a vertical force of 40 kN and a horizontal force of 10 kN in $x$-direction at its apex point $D$. Determine the forces developed in its legs. Given $D$ is 3 m above the horizontal plane and $A, B, C$ are on the circumference of a horizontal circle of radius 3 m .
[Ans. $F_{A D}=9.429 \mathrm{kN}, F_{B D}=23.574 \mathrm{kN}, F_{C D}=23.574 \mathrm{kN}$ ]
[Note: $A(3,0,0), B(-1.5,0,-2.598), C(-1.5,0,2.598), D(0.3,0)]$


Fig. 2.46

## chapter 3

## Statics of Rigid Bodies

In many engineering problems the size of a body may not be so small as to neglect it and treat the body as particle. The system of forces acting on such bodies will not be concurrent. Hence the body is to be analysed for the system of non-concurrent forces. In engineering mechanics such bodies are treated as rigid and analysed. The force acting on a rigid body has not only the tendency to move the body but it has a tendency to rotate the body also. The measure of rotational effect of the force is called moment. In this chapter the concept of moment is developed and the method of finding it about a point and about an axis are explained. Finding the resultant of a non-concurrent force system is dealt in detail taking the system in a plane as well as in space.

After explaining the various forces acting on rigid bodies and the types of reactive forces developed for various support conditions, the method of analysing equilibrium conditions of rigid bodies is illustrated.

### 3.1 MOMENT OF A FORCE ABOUT A POINT

The moment of a force acting on a rigid body about a point in the body is defined as the product of the magnitude of the force and the perpendicular distance of the line of action of the force from the point.

In Fig. 3.1(a), $F$ is the force and $O$ is the point about which the moment is required. Taking the plane containing $F$ and $O$ as $x-z$ plane and $y$ as the axis at right angles to $x-z$ plane at $O$, the magnitude of the moment about $O$ is

(a)


$$
\begin{equation*}
M=F d \tag{3.1}
\end{equation*}
$$

where $d$ is the perpendicular distance of the line of action of $F$ from $O$.
Moment $M$ is directed in the anticlockwise direction (when seen from the arrow head) as shown in Fig. 3.1(a).

The point ' $O$ ' about which the moment is found is called moment centre. The perpendicular distance ' $d$ ' between the moment centre and the line of action of the force $F$ is called moment arm.

From equation 3.1 it is obvious that, if force is taken in Newtons and the distance ' $d$ ' in mm , the unit of moment is $\mathrm{N}-\mathrm{mm}$. The commonly used units in SI system are $\mathrm{N}-\mathrm{mm}, \mathrm{kN}-\mathrm{mm}, \mathrm{N}-\mathrm{m}, \mathrm{kN}-\mathrm{m}$.

Moment is a vector since it is associated with magnitude as well as direction (clockwise/anticlockwise). It can be proved that moment of a force $F$ about point $O$ is

$$
\begin{equation*}
\mathbf{M}=\mathbf{r}_{\mathbf{O A}} \times \mathbf{F} \tag{3.2}
\end{equation*}
$$

where $A$ is a point on the line of action of $F$.
Referring to Fig 3.1(b),
$r_{O A}$ - is position vector of $A$, a point on the line of action of $F$,
$\theta$ - is the acute angle between the position vector and the line of action of $F$. From the cross product rule, we have

$$
\begin{aligned}
\mathbf{r}_{\mathbf{O A}} \times \mathbf{F} & =r_{O A} F \sin \theta \\
& =F\left(r_{O A} \sin \theta\right) \\
& =F d \\
& =M
\end{aligned}
$$

From the cross product rule we also know that, the product $r_{O A} \times F$ has direction at right angles to the plane of $r_{O A}$ and $F$. Hence, this product is directed as shown in Fig. 3.1(b). We look at this direction as the axis about which the moment is acting. If thumb of right hand is stretched and the other four fingers
are closed, the thumb shows the axis about which moment is acting and the other four fingers indicate the direction of moment. Many times this moment is marked with double arrow heads as shown in Fig. 3.1(c).

It is necessary to remember that $\mathbf{M}=\mathbf{r}_{\mathbf{O A}} \times \mathbf{F}$ and not $\mathbf{F} \times \mathbf{r}_{\mathbf{O A}}$, since the cross product $\mathbf{r}_{\mathbf{O A}} \times \mathbf{F}$ is exactly opposite to $\mathbf{F} \times \mathbf{r}_{\mathbf{O A}}$ in direction. According to sign convention anticlockwise moment is the +ve moment and hence clockwise moment will be the negative moment.

It may be proved easily that any point on the line of action of force may be selected to get the moment about a point. Let point 1 and 2 be two points on the line of action of force $F$ as shown in Fig. 3.2. From the definition of moment about point $O$,


Fig. 3.2

$$
\mathbf{M}=\mathbf{r}_{01} \times \mathbf{F}
$$

Its magnitude is

$$
M=r_{01} F \sin \theta_{1}=F d
$$

and the direction is as shown in the figure.
If we consider point 2 , the vector $\mathbf{r}_{02} \times F$ has

$$
\begin{aligned}
\text { magnitude } & =r_{02} F \sin \theta_{2} \\
& =F d, \text { as before }
\end{aligned}
$$

and the direction is also same as before. Thus we can take the position vector of any point on the line of action of the force to find its moment about a moment centre. In otherwords, it proves law of transmissibility of force, which is one of the important laws of statics.

### 3.2 COMPONENTS OF MOMENTS

Imagine the building of a parallelepiped on any point $B$ on the line of action of force $F$ as shown in Fig. 3.3. Let the coordinates of point $B$ be $(x, y, z)$ and $F_{x}, F_{y}$ and $F_{z}$ be the component of force $F$. Noting that the force parallel to an axis can
not give any rotational effect (moment) about that axis, we get


Fig. 3.3

$$
\begin{align*}
& M_{x}=y F_{z}-z F_{y}  \tag{3.3a}\\
& M_{y}=z F_{x}-x F_{z}  \tag{3.3b}\\
& M_{z}=x F_{y}-y F_{x} \tag{3.3c}
\end{align*}
$$

From our earlier discussion, moment of force $F$ about point $O$ is,

$$
\begin{align*}
\mathbf{M} & =\mathbf{r}_{\mathbf{O B}} \times \mathbf{F} \\
& =\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
x & y & z \\
F_{x} & F_{y} & F_{z}
\end{array}\right| \\
& =\left(y F_{z}-z F_{y}\right) \mathbf{i}-\mathbf{j}\left(x F_{z}-z F_{x}\right)+\mathbf{k}\left(x F_{y}-y F_{x}\right) \\
& =M_{x} \mathbf{i}+M_{y} \mathbf{j}+M_{z} \mathbf{k} \tag{3.4}
\end{align*}
$$

Thus the moment of a force about a point is the vector sum of the moment of the force about the orthogonal axes through that point. $M$, the magnitude of this vector is equal to $\sqrt{M_{x}^{2}+M_{y}^{2}+M_{z}^{2}}$ and its unit vector is

$$
\frac{M_{x}}{M} \mathbf{i}+\frac{M_{y}}{M} \mathbf{j}+\frac{M_{z}}{M} \mathbf{k}
$$

### 3.3 MOMENT OF A FORCE ABOUT A GIVEN AXIS

If $F$ is the force and $O$ is the moment centre we know it produces $M_{0}$ about the moment centre at right angles to the plane containing the force and the moment centre (Ref. Fig. 3.4). In the form of components about $x, y, z$ plane this moment is given by

$$
\mathbf{M}=M_{x} \mathbf{i}+M_{y} \mathbf{j}+M_{z} \mathbf{k}
$$



Fig. 3.4
Now our interest is to find $\mathbf{M}_{O P}$ where $O P$ is an axis with its unit vectors

$$
\mathbf{n}=\cos \theta_{x} \mathbf{i}+\cos \theta_{y} \mathbf{j}+\cos \theta_{z} \mathbf{k}
$$

where $\theta_{x}, \theta_{y}$ and $\theta_{z}$ are the angles made by $O P$ with $x, y, z$-axis. Then $\mathbf{M}_{O P}$ is given by

$$
\begin{equation*}
\mathbf{M}_{\mathbf{O P}}=\mathbf{M} \cdot \mathbf{n} \tag{3.5}
\end{equation*}
$$

For examples, component of $M_{O P}$ about $x$-axis is given by

$$
\begin{aligned}
\mathbf{M}_{\mathbf{O P}} & =\left(M_{x} \mathbf{i}+M_{y} \mathbf{j}+M_{z} \mathbf{k}\right) \cdot \mathbf{i} \\
& =M_{x}
\end{aligned}
$$

### 3.4 VARIGNON'S THEOREM OF MOMENTS

The French mathematician Varignon (1654-1722) gave the following theorem which is known as Varignon's theorem of moments.
"The algebraic sum of the moments of a system of forces about a moment centre is equal to the moment of their resultant force about the same moment centre".

Actually, Varignon stated this theorem in the context of coplanar forces and proved it.

However it can be generalised to two as well as three dimensional problems.
Let $\mathbf{F}_{1}$ and $\mathbf{F}_{2}$ be the two component forces acting at $A$ and $\mathbf{R}$ be the resultant force.
Then,

$$
\begin{align*}
\mathbf{M}_{O} & =\mathbf{r}_{\mathbf{O A}} \times \mathbf{R}=\mathbf{r}_{\mathbf{O A}} \times\left(\mathbf{F}_{1}+\mathbf{F}_{2}\right) \\
& =\mathbf{r}_{\mathbf{O A}} \times \mathbf{F}_{1}+\mathbf{r}_{\mathbf{O A}} \times \mathbf{F}_{2} \\
& =\mathbf{M}_{1}+\mathbf{M}_{2}
\end{align*}
$$

where $\mathrm{M}_{1}$ and $M_{2}$ are the moments of component forces about $O$.

Example 3.1 Determine the moment of 400 N force acting at $B$ in $x-y$ plane about point $A$, as shown in Fig. 3.5.


Fig. 3.5
Solution. Taking $A$ as origin, the position vector of $B$ is

$$
\begin{aligned}
\mathbf{r}_{\mathrm{AB}} & =300 \mathbf{i}+400 \mathbf{j} \\
\mathbf{F} & =400 \cos 30^{\circ} \mathbf{i}+400 \sin 30^{\circ} \mathbf{j} \\
& =346.41 \mathbf{i}+200 \mathbf{j} \\
\therefore \quad \boldsymbol{M}_{A} & =\mathbf{r}_{\mathrm{AB}} \times \mathbf{F} \\
& =(300 \mathbf{i}+400 \mathbf{j}) \times(346.41 \mathbf{i}+200 \mathbf{j}) \\
& =60000 \mathbf{k}-138564 \mathbf{k} \\
& =-78564 \mathbf{k} \\
& =78564 \mathbf{N}-\mathrm{mm}, \text { clockwise moment. }
\end{aligned}
$$

Example 3.2 What will be the $y$ intercept of 5000 N force shown in Fig. 3.6, if its moment about $A$ is 8000 N-m clockwise. Determine its $x$ intercept also.
Solution. Let 5000 N force intercept $y$-axis at $B$ and its intercept be $y$. Then,

$$
\begin{aligned}
\mathbf{r}_{\mathbf{O B}} & =0 \mathbf{i}+\mathbf{y} \mathbf{j} \\
\theta & =\tan ^{-1} 3 / 4 \\
\therefore \quad \theta & =36.8694^{\circ} \\
\therefore \quad F_{x} & =5000 \cos \theta=4000 \mathrm{~N} \\
F_{y} & =5000 \sin \theta=3000 \mathrm{~N} \\
\therefore \quad \mathbf{F} & =4000 \mathbf{i}+3000 \mathbf{j} \\
\mathbf{M} & =\mathbf{r}_{\mathbf{O B}} \times \mathbf{F} \\
& =(0 \mathbf{i}+y \mathbf{j}) \times(4000 \mathbf{i}+3000 \mathbf{j}) \\
& =4000 y(-\mathbf{k})
\end{aligned}
$$

Now moment is given as 8000 N -m clockwise.
i.e., -8000 k. Hence
or

$$
\begin{aligned}
-8000 \mathbf{k} & =-4000 y \mathbf{k} \\
y & =2 \mathrm{~m}
\end{aligned}
$$

Ans.
If $x$ is the intercept on $x$-axis, then the position vector of the point of intersection of the force $F$ with $x$ axis is

$$
\mathbf{r}_{\mathbf{O C}}=x \mathbf{i}+0 \mathbf{j}
$$

$\therefore$ Using this position vector,

$$
\begin{aligned}
\mathbf{M}_{\mathbf{o}} & =(x \mathbf{i}+0 \mathbf{j}) \times(4000 \mathbf{i}+3000 \mathbf{j}) \\
& =3000 x \mathbf{k}
\end{aligned}
$$

Since this moment is $-8000 \mathrm{~N}-\mathrm{m}$, we get
or

$$
\begin{aligned}
-8000 \mathbf{k} & =3000 x \mathbf{k} \\
x & =-\frac{8}{3}=-2.667 \mathrm{~m}
\end{aligned}
$$

Ans.
Example $3.3 A(50,30,40)$ and $B(30,20,80)$ are the two points on the line of action of 400 N force which is acting in the direction $A$ to $B$. Find the moment of this force about the origin ' $O$ ' and about the point $C$ (20, 20, 20). Millimeter units are used for the coordinate system in space.
Solution. $\quad \mathbf{r}_{\mathbf{O A}}=50 \mathbf{i}+30 \mathbf{j}+40 \mathbf{k}$ and $\mathbf{r}_{\mathbf{O B}}=30 \mathbf{i}+20 \mathbf{j}+80 \mathbf{k}$

$$
\mathbf{r}_{\mathrm{AB}}=(30-50) \mathbf{i}+(20-30) \mathbf{j}+(80-40) \mathbf{k}=-20 \mathbf{i}-10 \mathbf{j}+40 \mathbf{k}
$$

$$
\therefore \quad r_{A B}=\sqrt{20^{2}+10^{2}+40^{2}}=45.825 \mathrm{~mm}
$$

$$
\therefore \quad \mathbf{n}=\frac{-20}{45.825} \mathbf{i}+\frac{(-10)}{45.825} \mathbf{j}+\frac{40}{45.825} \mathbf{k}
$$

$$
=-0.4364 \mathbf{i}-0.2182 \mathbf{j}+0.8729 \mathbf{k}
$$

$$
\therefore \quad \mathbf{F}=F \mathbf{n}=400(-0.4364 \mathbf{i}-0.2182 \mathbf{j}+0.8729 \mathbf{k})
$$

$$
=-174.56 \mathbf{i}-87.28 \mathbf{j}+349.12 \mathbf{k}
$$

$\therefore$ Moment of this force about ' $O$ ' is

$$
\begin{aligned}
\mathbf{M}_{\mathbf{0}} & =\mathbf{r}_{\mathbf{O A}} \times \mathbf{F} \\
& =(50 \mathbf{i}+30 \mathbf{j}+40 \mathbf{k}) \times(-174.56 \mathbf{i}-87.28 \mathbf{j}+349.12 \mathbf{k})
\end{aligned}
$$

$$
=\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
50 & 30 & 40 \\
-174.56 & -87.28 & 349.12
\end{array}\right|
$$

$$
=\mathbf{i}(30 \times 349.12+87.28 \times 40)-\mathbf{j}(50 \times 349.12+174.56 \times 40)
$$

$$
+\mathbf{k}(-50 \times 87.28+174.56 \times 30)
$$

$$
=13964.8 \mathbf{i}-24438.4 \mathbf{j}+872.8 \mathbf{k}
$$

i.e., $\quad M_{x}=13964.8 \mathrm{~N}-\mathrm{mm}, M_{y}=-24438.4 \mathrm{~N}-\mathrm{mm}$

Ans.
and $\quad M_{z}=872.8 \mathrm{~N}-\mathrm{mm}$

## Moment About C:

$$
\begin{aligned}
& \text { Now } \mathbf{r}_{\mathbf{C A}}=(50-20) \mathbf{i}+(30-20) \mathbf{j}+(40-20) \mathbf{k} \\
&=30 \mathbf{i}+10 \mathbf{j}+20 \mathbf{k} \\
& \mathbf{M}_{\mathbf{C}}= \mathbf{r}_{\mathbf{C A}} \times \mathbf{F} \\
& \mathbf{M}_{\mathbf{C}}=\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
30 & 10 & 20 \\
-174.56 & -87.28 & 349.12
\end{array}\right| \\
&= \mathbf{i}(10 \times 349.12+20 \times 87.28)-\mathbf{j}(30 \times 349.12+20 \times 174.56) \\
&+\mathbf{k}(-30 \times 87.28+10 \times 174.56) \\
&= 5236.8 \mathbf{i}-1394.8 \mathbf{j}-872.8 \mathbf{k} \\
& \text { i.e., } \quad M_{x}= \\
& \text { and } \quad\{236.8 \mathrm{~N}-\mathrm{mm}, \text { anticlockwise } \\
& M_{y}= 13965.9 \mathrm{~N}-\mathrm{mm}, \text { clockwise } \\
& \mathrm{M}_{z}= 872.8 \mathrm{~N}-\mathrm{mm}, \text { clockwise }
\end{aligned}
$$

Example 3.4 The coordinates of point $A$ and $B$ in metre units are $A(1,2,3)$ and $B(4,3,2)$. A force of magnitude 10 kN passes through these two points and is directed from $A$ towards $B$. Find the moment of this force
(i) about $C(1,1,1)$
(ii) about the axis passing through the points $D(2.5,1.5,4)$ and $E(4,3,5)$.
Solution. $\mathbf{r}_{\mathrm{AB}}=(4-1) \mathbf{i}+(3-2) \mathbf{j}+(2-3) \mathbf{k}$

$$
=3 \mathbf{i}+\mathbf{j}-\mathbf{k}
$$

$$
\therefore \quad r_{A B}=\sqrt{3^{2}+1^{2}+1^{2}}=3.3166
$$

$$
\therefore \quad \mathbf{n}_{\mathrm{AB}}=\frac{3}{3.3166} \mathbf{i}+\frac{1}{3.3166} \mathbf{j}-\frac{1}{3.3166} \mathbf{k}
$$

$$
=0.9045 \mathbf{i}+0.3015 \mathbf{j}-0.3015 \mathbf{k}
$$

$$
\therefore \quad F=10 \mathbf{n}_{\mathbf{A B}}=9.045 \mathbf{i}+3.015 \mathbf{j}-3.015 \mathbf{k}
$$

(i) Moment about $C$ :

Displacement vector of $C A$ is

$$
\begin{aligned}
\mathbf{r}_{\mathbf{C A}} & =(1-1) \mathbf{i}+(2-1) \mathbf{j}+(3-1) \mathbf{k} \\
& =0 \mathbf{i}+\mathbf{j}+2 \mathbf{k} \\
\therefore \quad & \mathbf{M}_{\mathbf{C}}
\end{aligned}=\mathbf{r}_{\mathbf{C A}} \times \mathbf{F} .
$$

$$
\begin{aligned}
& =\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 1 & 2 \\
9.045 & 3.015 & -3.015
\end{array}\right| \\
& =\mathbf{i}(-3.015-6.030)-\mathbf{j}(-18.090)+\mathbf{k}(-9.045) \\
& =-9.045 \mathbf{i}+18.090 \mathbf{j}-9.045 \mathbf{k} \\
M_{C} & =\sqrt{(9.045)^{2}+(18.090)^{2}+(9.045)^{2}}=22.16 \mathrm{kN}-\mathrm{m}
\end{aligned}
$$

$$
\begin{aligned}
& M_{x}=9.045 \mathrm{kN}-\mathrm{m}, \text { clockwise } \\
& M_{y}=18.090 \mathrm{kN}-\mathrm{m}, \text { anticlockwise } \\
& M_{z}=9.045 \mathrm{kN}-\mathrm{m}, \text { clockwise }
\end{aligned}
$$

(ii) Moment about the axis $D E$

$$
\begin{aligned}
\mathbf{r}_{\mathbf{D E}} & =(4-2.5) \mathbf{i}+(3-1.5) \mathbf{j}+(5.0-4) \mathbf{k} \\
& =1.5 \mathbf{i}+1.5 \mathbf{j}+1 \mathbf{k} \\
r_{D E} & =\sqrt{1.5^{2}+1.5^{2}+1^{2}}=2.3452 \\
\therefore \quad \mathbf{n}_{D E} & =\frac{1.5}{2.3452} \mathbf{i}+\frac{1.5}{2.3452} \mathbf{j}+\frac{1}{2.3452} \mathbf{k} \\
& =0.6396 \mathbf{i}+0.6396 \mathbf{j}+0.4264 \mathbf{k}
\end{aligned}
$$

First we find moment about $D$

$$
\begin{aligned}
\mathbf{r}_{\mathbf{D A}}= & (1-2.5) \mathbf{i}+(2-1.5 \mathbf{j}+(3-4) \mathbf{k} \\
= & -1.5 \mathbf{i}+0.5 \mathbf{j}-\mathbf{k} \\
\mathbf{M}_{\mathbf{D}}= & r_{\mathrm{DA}} \times \mathbf{F} \\
= & \left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1.5 & 0.5 & -1 \\
9.045 & 3.015 & -3.015
\end{array}\right| \\
= & \mathbf{i}(-0.5 \times 3.015+3.015)-\mathbf{j}(1.5 \times 3.015+9.045) \\
& +\mathbf{k}(-1.5 \times 3.015-0.5 \times 9.045) \\
= & 0.1508 \mathbf{i}-13.5675 \mathbf{j}-9.045 \mathbf{k} \\
\therefore \quad \mathbf{M}_{\mathbf{D E}}= & \mathbf{M}_{\mathbf{D}} \cdot \mathbf{n}_{\mathbf{D E}} \\
= & (0.1508 \mathbf{i}-13.5675 \mathbf{j}-9.045 \mathbf{k}) .(0.6396 \mathbf{i}+0.6396 \mathbf{j} \\
& +0.4264 \mathbf{k}) \\
= & 0.1508 \times 0.6396-13.5675 \times 0.6396-9.045 \times 0.4264 \\
= & -12.438 \mathrm{kN}-\mathrm{m}
\end{aligned}
$$

### 3.5 COUPLE-MOMENT

Two parallel forces that are equal in magnitude but opposite in direction and separated by a definite distance are said to form a couple. The rotational effect of the couple is called 'Couple Moment' and this moment is having interesting property.

Consider the couple shown in Fig. 3.7. The forces $F_{1}=F$ and $F_{2}=-F$ form a couple. Let ' $O$ ' be a moment centre and $A$ be an arbitrary point on $F_{1}$ and $B$ be an arbitrary point on $F_{2}$.

$$
\begin{aligned}
\mathbf{M}_{\mathbf{o}} & =\mathbf{r}_{O A} \times \mathbf{F}_{1}+\mathbf{r}_{O B} \times \mathbf{F}_{2} \\
& =\mathbf{r}_{O A} \times \mathbf{F}+\mathbf{r}_{O B} \times(-\mathbf{F}) \\
& =\left(\mathbf{r}_{O A}-\mathbf{r}_{O B}\right) \times \mathbf{F} \\
& =\mathbf{r}_{B A} \times \mathbf{F} \\
& =-\mathbf{r}_{A B} \times \mathbf{F}
\end{aligned}
$$



Fig. 3.7

Hence magnitude of $M_{o}=r_{A B} F \sin \theta=F d$, where $d$ is perpendicular distance between the two parallel forces.

The position vector $\mathbf{r}_{\mathbf{A B}}$ does not contain any reference to the moment centre. Hence for any moment centre the moment of the system is the same. The direction of couple moment is normal to the plane containing the two forces forming the couple and its sense is established by the right hand screw rule.

The characteristics of the couple are listed below:

- A couple consists of a pair of equal and opposite parallel forces which are separated by a definite distance.
- The translatory effect of a couple on the body is zero.
- The rotational effect (moment) of a couple about any point is a constant and it is equal to the product of the displacement vector of any two points on the two forces forming the couple and the magnitude of the forces.
Since the only effect of a couple is a moment and this moment is the same about any moment centre, the effect of the couple is unchanged if,
- the couple is rotated through any angle
- the couple is shifted to any other position
- the couple is replaced by any other pair of forces whose rotational effect is the same.

Example 3.5 Determine the moment-couple and the distance between the line of action of the forces $\mathbf{F}_{1}=25 \mathbf{i}+20 \mathbf{j}+30 \mathbf{k}$ acting through point $A(1.0,1.2,1.5)$ and $\mathbf{F}_{2}=-25 \mathbf{i}-20 \mathbf{j}-30 \mathbf{k}$ acting through point $B(1.5,1.8,-1.0)$. The forces are in kN units and the coordinates in metre units.
Solution. $\mathbf{r}_{\mathrm{AB}}=(1.5-1.0) \mathbf{i}+(1.8-1.2) \mathbf{j}+(-1.0-1.5) \mathbf{k}$

$$
=0.5 \mathbf{i}+0.6 \mathbf{j}-2.5 \mathbf{k}
$$

$$
\left.\begin{array}{rl}
\mathbf{M}=\mathbf{M}_{\mathbf{A}} & =\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0.5 & 0.6 & -2.5 \\
25.0 & 20.0 & 30.0
\end{array}\right| \\
& =\mathbf{i}(18+50)-\mathbf{j}(15+62.5)+\mathbf{k}(10-15) \\
& =68 \mathbf{i}-77.5 \mathbf{j}-5 \mathbf{k} \\
\therefore \quad & M
\end{array}\right)=\sqrt{68^{2}+77.5^{2}+5^{2}}=103.224 \mathrm{kN}-\mathrm{m} \quad .
$$

$$
\therefore \quad \mathbf{M}=103.224\left(\frac{68}{103.224} \mathbf{i}-\frac{77.5}{103.224} \mathbf{j}-\frac{5}{103.224} \mathbf{k}\right)
$$

$$
\text { i.e., } \quad \mathbf{M}=103.224(0.6588 \mathbf{i}-0.7508 \mathbf{j}-0.0484 \mathbf{k})
$$

Ans.

Ans.

Now, $\quad \mathbf{F}=25 \mathbf{i}+20 \mathbf{j}+30 \mathbf{k}$

$$
\begin{aligned}
\therefore \quad F & =\sqrt{25^{2}+20^{2}+30^{2}}=43.875 \mathrm{kN} \\
F & =43.875\left(\frac{25}{43.875} \mathbf{i}+\frac{20}{43.875} \mathbf{j}+\frac{30}{43.875} \mathbf{k}\right) \\
& =43.875(0.5698 i+0.4558 j+0.6838 k)
\end{aligned}
$$

Now from Eqn. 3.7,

$$
\begin{aligned}
M & =F d \\
\therefore \quad d & =\frac{M}{F}=\frac{103.224}{43.875}=2.353 \mathrm{~m}
\end{aligned}
$$

Ans.

### 3.6 TYPES OF VECTORS

The various vectors that we have discussed so far may be grouped into three types viz. free, sliding and fixed vectors.

## Free Vector

This is one which can be moved any where in the space maintaining magnitude, direction and sense. In other words, this vector does not have a specified point of application. The couple-moment is a free vector. The movement of a point in a rigid body without rotation is another example of free vector, since it represents the movement of any other point on the body also (Ref. Fig. 3.8).


Fig. 3.8 Free Vectors

## Sliding Vector

A vector which may be applied any where along its line of action maintaining magnitude, direction and sense is termed as sliding vector. The force acting on a rigid body is an example of a sliding vector, since for it the law of transmissibility holds good. Moment of a force is also sliding vector since position vector of any point on the line of action of force can be used to find the moment (Fig. 3.9).


Fig. 3.9 Sliding Vector

## Fixed Vector

This is also termed as bound vector. This type of vector is having a specified point of application. It has magnitude, line of action and direction. A typical fixed vector is shown in Fig. 3.10.


Fig. 3.10 Fixed Vector

### 3.7 RESOLUTION OF A FORCE INTO A FORCE AND A COUPLE

In many engineering mechanics problems, it will be advantageous to resolve a force acting on a body at a particular point into a force acting at some other suitable point and a couple moment. Let $F$ be a force acting at point $A$ on a body as shown in Fig. 3.11(a). Imagine we apply two equal and opposite forces $F$ at $B$. By doing so the system of forces is not disturbed i.e., the system of forces in Fig. 3.11(b), is the same as the system shown in Fig. 3.11(a). Now, the given force $F$ at $A$ and the opposite force $F$ at $B$ form a couple of magnitude $\mathbf{r}_{A B} \times F$ (or $F d$ ). Hence the system shown in Fig. 3.11(b) is the same as that shown in Fig. 3.11(c). Thus the given force $F$ at $A$ is replaced by a force $\mathbf{F}$ at $B$ and a couple-moment of magnitude $\mathbf{r}_{A B} \times F$ (or $F d$ ).


Fig. 3.11

### 3.8 RESULTANT OF NON-CONCURRENT FORCE SYSTEM

The resultant of a force system is the one which will have the same rotational and translatory effect as the given system of forces. It may be a single force, a pure moment or a force and a moment.

Consider the system of forces shown in Fig. 3.12(a).

(a)

(b)

(c)

Fig. 3.12
Each force can be replaced by a force at the same magnitude and direction acting at point ' $O$ ' and its moment about ' $O$ '. Thus the given system is equal to the system shown in Fig. 3.12(b), where $\Sigma \mathbf{M}_{o}$ is the algebraic sum of the moments of the given forces about $O$.

At $O$, the concurrent forces $\mathbf{F}_{1}, \mathbf{F}_{2}, \ldots, \mathbf{F}_{n}$ can be combined as usual to get the resultant force $\mathbf{R}\left(\mathbf{R}=\mathbf{F}_{1}+\mathbf{F}_{2}+\ldots+\mathbf{F}_{n}\right)$. Now the resultant of the given system is equal to a force $\mathbf{R}$ at $O$ and a moment $\Sigma \mathbf{M}_{o}$, as shown in Fig. 3.12(c). Thus the resultant of the non-concurrent forces is

$$
\begin{equation*}
\mathbf{R}=\sum_{i=1}^{n} \mathbf{F}_{i} \quad \text { and } \quad \mathbf{M}_{\mathrm{o}}=\sum_{i=1}^{n} \mathbf{M}_{o i} \tag{3.8}
\end{equation*}
$$

The above equations are equivalent to the following scalar expressions

$$
\begin{aligned}
\mathbf{R}_{x} & =\Sigma \mathbf{F}_{x i} ; \mathbf{R}_{y}=\Sigma \mathbf{F}_{y i} ; \mathbf{R}_{z}=\Sigma \mathbf{F}_{z i} \\
\mathbf{M}_{x} & =\Sigma \mathbf{M}_{x i} ; \mathbf{M}_{y}=\Sigma \mathbf{M}_{y i} ; \mathbf{M}_{z}=\Sigma \mathrm{M}_{z i} \\
R & =\sqrt{\left(\Sigma F_{x i}\right)^{2}+\left(\Sigma F_{y i}\right)^{2}+\left(\Sigma F_{z i}\right)^{2}} \\
M & =\sqrt{\left(\Sigma F_{x i}\right)^{2}+\left(\Sigma M_{y i}\right)^{2}+\left(\Sigma F_{z i}\right)^{2}}
\end{aligned}
$$

## $x$ and $y$ Intercepts of the Resultant

In some two dimensional problems we may be interested in finding the intersection point of resultant $\mathbf{R}$ along $x$ and $y$ axis i.e., $x$ and $y$ intercepts. Let $\mathbf{R}$ be the resultant intersecting $x, y$ coordinates through point $O$ at $A$ and $B$ as shown in Fig. 3.13. Let moment about the moment centre $O$ be $\mathbf{M}_{O}$. The position vector of $A$ is $x \mathbf{i}$ and the position vector of $B$ is $y \mathbf{j}$.


Fig. 3.13

Hence

$$
\begin{aligned}
\mathbf{M}_{O} & =\mathbf{r}_{O A} \times \mathbf{R}=\mathbf{r}_{O B} \mathbf{R} \\
& =x_{i} \times \mathbf{R}=y_{j} \times \mathbf{R}
\end{aligned}
$$

Example 3.6 The system of loads acting on a beam is as shown in Fig. 3.14. Determine the resultant of the loads.
Solution. Selecting the coordinate directions as shown in Fig. 3.14,


Fig. 3.14

$$
\begin{aligned}
\mathbf{F}_{1} & =-30 \mathbf{j} \\
\mathbf{F}_{2} & =-40 \mathbf{j} \\
\mathbf{F}_{3} & =\left(-20 \cos 60^{\circ} \mathbf{i}-20 \sin 60^{\circ} \mathbf{j}\right) \\
& =-10 \mathbf{i}-17.321 \mathbf{j}
\end{aligned}
$$

Taking $A$ as origin, the position vectors of the intersection points of the forces with beam are:

$$
\begin{align*}
\mathbf{r}_{A 1} & =1.5 \mathbf{i} \quad \mathbf{r}_{A 2}=3.0 \mathbf{i} \quad \text { and } \quad \mathbf{r}_{\mathrm{A} 3}=6 \mathbf{i} \\
\therefore \quad \mathrm{M}_{\mathrm{A}} & =\mathbf{r}_{A 1} \times \mathbf{F}_{1}+\mathbf{r}_{A 2} \times \mathbf{F}_{2}+\mathbf{r}_{A 3} \times \mathbf{F}_{3} \\
& =1.5 \mathbf{i} \times(-30 \mathbf{j})+3.0 \mathbf{i} \times(-40 \mathbf{j})+6 \mathbf{i} \times(-10 \mathbf{i}-17.321 \mathbf{j}) \\
& =-45 \mathbf{k}-120 \mathbf{k}-103.926 \mathbf{k} \\
& =-268.926 \mathbf{k} \\
\text { i.e., } \quad & \\
M_{A} & =268.926 \mathrm{kN}-\mathrm{m}, \text { clockwise } \\
\mathbf{R} & =\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3} \\
\mathbf{M}_{\mathbf{O}} & =\mathbf{r}_{\mathbf{O A}} \times \mathbf{F}=\mathbf{r}_{\mathbf{O B}} \times \mathbf{F}  \tag{3.9}\\
& =x \mathbf{i} \times \mathbf{F}=y \mathbf{j} \times \mathbf{F}
\end{align*}
$$

Ans.

Hence $x$ and $y$ intercepts can be found.

$$
\begin{aligned}
\mathbf{R} & =-30 \mathbf{j}-40 \mathbf{j}-10 \mathbf{i}-17.321 \mathbf{j} \\
& =-10 \mathbf{i}-87.321 \mathbf{j} \\
\therefore \quad \quad \mathbf{R} & =\sqrt{10^{2}+87.321^{2}}=87.892 \mathrm{kN}
\end{aligned}
$$

Let $\alpha$ be the inclination to the $x$-axis. Then
on $\quad \alpha_{x}=96.533$ as shown in figure

$$
\alpha_{y}=\cos ^{-1} \frac{-87.321}{87.892}=173.466^{\circ}
$$

Let the resultant intersect $x$-axis at $D$. Let $A D$ be $x$ metres. Then

$$
\begin{array}{rlrl}
\mathbf{r}_{\mathrm{AD}} & =x \mathbf{i} \\
\therefore & \mathbf{M} & =\mathbf{r}_{\mathrm{AD}} \times \mathbf{R}
\end{array}
$$

$$
\begin{aligned}
-268.926 \mathbf{k} & =x \mathbf{i} \times(-10 \mathbf{i}-87.321 \mathbf{j}) \\
& =-87.321 \mathbf{k} \\
\therefore \quad x & =3.08 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 3.7 Find the resultant of the force system shown in Fig. 3.15 acting on a lamina of equilateral triangle of sides 120 mm .


Fig. 3.15
Solution. Taking $x, y$-axes in the plane of lamina and $z$-axis normal to it, 60 kN force:

$$
\mathbf{F}_{1}=-60 \cos 60^{\circ} \mathbf{i}+60 \sin 60^{\circ} \mathbf{j}=-30 \mathbf{i}+51.96 \mathbf{j}
$$

80 N vertical force, $\quad \mathbf{F}_{2}=80 \mathbf{j}$
80 N horizontal force, $\mathbf{F}_{3}=80 \mathbf{i}$
120 N force, $\mathbf{F}_{4}=-120 \cos 30^{\circ} \mathbf{i}-120 \sin 30^{\circ} \mathbf{j}=-103.923 \mathbf{i}-60 \mathbf{j}$

$$
\begin{aligned}
\therefore \quad \mathbf{R} & =\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4} \\
& =(-30+80-103.923) \mathbf{i}+(51.96+80-60) \mathbf{j} \\
& =-53.923 \mathbf{i}+71.96 \mathbf{j}
\end{aligned}
$$

Ans.
$\therefore \quad R=\sqrt{(53.923)^{2}+(71.96)^{2}}=89.922 \mathrm{~N}$

$$
\begin{aligned}
\cos \alpha_{x} & =\frac{-53.923}{89.922}=-0.5997 \\
\alpha_{x} & =126.846^{\circ} \text { as shown in Fig. 3.15. } \\
\alpha_{y} & =\cos ^{-1} \frac{71.96}{89.722}=36.846^{\circ}
\end{aligned}
$$

Taking point $A$ as origin, the position vector of force $F_{1}$ to $F_{4}$ are

$$
\begin{aligned}
& \mathbf{r}_{A 1}=0 \quad \mathbf{r}_{A 2}=60 \mathbf{i}+120 \sin 60^{\circ} \mathbf{j} \\
& \mathbf{r}_{\mathrm{A} 3}=60 \mathbf{i}+120 \sin 60^{\circ} \mathbf{j} \quad \text { and } \quad \mathbf{r}_{A 4}=120 \mathbf{i}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad \mathbf{M}_{4}= & 0+\left(60 \mathbf{i}+120 \sin 60^{\circ} \mathbf{j}\right) \times(80 \mathbf{j}) \\
& +\left(60 \mathbf{i}+120 \sin 60^{\circ} \mathbf{j}\right) \times 80 \mathbf{i}+120 \mathbf{i} \times(-103.923 \mathbf{i}-60 \mathbf{j}) \\
= & 4800 \mathbf{k}+8313.8(-\mathbf{k})-7200 \mathbf{k}=107138.6 \mathbf{k}
\end{aligned}
$$

## Note:

$$
\mathbf{i} \times \mathbf{i}=\mathbf{j} \times \mathbf{j}=0 ; \mathbf{i} \times \mathbf{j}=\mathbf{k} ; \mathbf{j} \times \mathbf{i}=-\mathbf{k}
$$

Let the resultant interest $x$-axis at $D$.
Then

$$
\mathbf{r}_{\mathrm{AD}}=x \mathbf{i}
$$

$$
\therefore \quad \mathbf{r}_{\mathrm{AD}} \times \mathbf{R}=\mathbf{M}_{\mathbf{A}}
$$

$$
x \mathbf{i} \times(-53.923 \mathbf{i}+71.96 \mathbf{j})=-107138.6 \mathbf{k}
$$

$$
x(71.96 \mathbf{k})=107138.6 \mathbf{k}
$$

$$
x=-148.883 \mathrm{~mm} \text { as shown in Figure }
$$

Ans.
Example 3.8 Find the resultant of a set of coplanar forces acting on a lamina as shown in Fig. 3.16. Each square has a side of 10 mm .


Fig. 3.16
Solution. Let $\theta_{1}, \theta_{2}$ and $\theta_{3}$ be the inclination of the forces $F_{1}=5 \mathrm{kN}, F_{2}=10 \mathrm{kN}$ and $F_{3}=3 \mathrm{kN}$. Then,

$$
\begin{aligned}
& \theta_{1}=\tan ^{-1} \frac{10}{10}=45^{\circ} \\
& \theta_{2}=\tan ^{-1} \frac{30}{40}=36.87^{\circ} \\
& \theta_{3}=\tan ^{-1} \frac{10}{20}=26.565^{\circ}
\end{aligned}
$$

and

The vector form of the given forces are:

$$
\begin{aligned}
\mathbf{F}_{1} & =5 \cos \theta_{1} \mathbf{i}+5 \sin \theta_{1} \mathbf{j}=3.536 \mathbf{i}+3.536 \mathbf{j} \\
\mathbf{F}_{2} & =10 \cos \theta_{2} \mathbf{i}-10 \sin \theta_{2} \mathbf{j} \\
& =8 \mathbf{i}-6 \mathbf{j} \\
\mathbf{F}_{3} & =-3 \cos \theta_{3} \mathbf{i}-3 \sin \theta_{3} \mathbf{j} \\
& =-2.683 \mathbf{i}-1.342 \mathbf{j} \\
\therefore \quad \mathbf{R} & =\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3} \\
& =(3.536+8-2.683) \mathbf{i}+(3.536-6-1.342) \mathbf{j} \\
& =8.853 \mathbf{i}-3.806 \mathbf{j} \\
R & =\sqrt{(8.853)^{2}+(3.806)^{2}}=9.636 \mathrm{kN} \\
\cos \alpha_{x} & =\frac{-8.853}{9.636} \text { or } \alpha_{x}=23.257^{\circ} \\
\cos \alpha_{y} & =\frac{-3.806}{9.636} \text { or } \alpha_{y}=113.265^{\circ}
\end{aligned}
$$

Ans.
$\therefore \mathbf{R}$ is as shown in Fig. 3.16.

$$
\text { Now, } \quad \begin{aligned}
\mathbf{r}_{01} & =0 \mathbf{i}+30 \mathbf{j} \\
\mathbf{r}_{02} & =50 \mathbf{i}+0 \mathbf{j} \\
\mathbf{r}_{03} & =10 \mathbf{i}+0 \mathbf{j} \\
\therefore \quad \mathbf{M}_{\mathbf{O}}= & \mathbf{r}_{01} \times \mathbf{F}_{1}+\mathbf{r}_{02} \times \mathbf{F}_{2}+\mathbf{r}_{03} \times \mathbf{F}_{3} \\
= & 30 \mathbf{j} \times(3.536 \mathbf{i}+3.536 \mathbf{j})+50 \mathbf{i} \times(8 \mathbf{i}-6 \mathbf{j}) \\
& +10 \mathbf{i} \times(-2.683 \mathbf{i}-1.342 \mathbf{j}) \\
= & 106.08(-\mathbf{k})-300 \mathbf{k}-13.42 \mathbf{k} \\
= & -419.5 \mathbf{k}
\end{aligned}
$$

If $y$ is the intercept of resultant with $y$-axis, $D$ being the intersection point, then

$$
\begin{aligned}
\mathbf{r}_{\mathbf{O D}} & =y \mathbf{j} \\
\therefore \quad \mathbf{r}_{\mathbf{O D}} \times \mathbf{R} & =\mathbf{M}_{\mathbf{O}} \text {, gives } \\
y \mathbf{j} \times(8.853 \mathbf{i}-3.806 \mathbf{j}) & =-419.5 \mathbf{k} \\
\text { i.e., } 8.853 y(-\mathbf{k}) & =-419.5 \mathbf{k} \\
y & =47.385 \mathrm{~mm}
\end{aligned}
$$

Ans.
Hence the resultant is as shown in Fig. 3.16.
Example 3.9 The system of forces acting on a bell crank is as shown in Fig. 3.17.
Determine the magnitude, direction and the point of application of the resultant.
Solution. The vector form of the forces are
600 N force: $\mathbf{F}_{1}=600 \cos 60^{\circ} \mathbf{i}-600 \sin 60^{\circ} \mathbf{j}$

$$
=300 \mathbf{i}-519.615 \mathbf{j}
$$

1000 N force: $\mathbf{F}_{2}=-1000 \mathbf{j}$
1200 N force: $\mathbf{F}_{3}=-1200 \mathbf{j}$
800 N force: $\mathbf{F}_{4}=-800 \mathbf{i}$


Fig. 3.17
Taking ' $O$ ' as the origin,
The position vectors of these forces are:

$$
\begin{aligned}
& \mathbf{r}_{01}=-400 \mathbf{i}, r_{02}=-200 \mathbf{i} \\
& \mathbf{r}_{03}=200 \cos 60^{\circ} \mathbf{i}+200 \sin 60^{\circ} \mathbf{j}=100 \mathbf{i}+173.205 \mathbf{j} \\
& \mathbf{r}_{04}= 400 \cos 60^{\circ} \mathbf{i}+400 \sin 60^{\circ} \mathbf{j}=200 \mathbf{i}+346.41 \mathbf{j} \\
& \mathbf{R}= \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4} \\
&=(300-800) \mathbf{i}+(-519.615-1000-1200) \mathbf{j} \\
&=-500 \mathbf{i}-2719.615 \mathbf{j} \\
& R= \sqrt{(500)^{2}+(2719.615)^{2}}=2765.195 \mathbf{N} \\
& \therefore \quad \text { Ans. } \\
& \alpha_{x}= \cos ^{-1} \frac{-500}{2765.195}=100.42^{\circ} \\
& \alpha_{y}= \cos ^{-1} \frac{-2719.615}{2765.261}=169.58^{\circ} \\
& \mathbf{M}_{\mathbf{O}}=-400 \mathbf{i} \times(300 \mathbf{i}-519.615 \mathbf{j})-200 \mathbf{i} \times(-1000 \mathbf{j}) \\
&+(100 \mathbf{i}+173.205 \mathbf{j}) \times(-1200 \mathbf{j}) \\
&+(200 \mathbf{i}+346.41 \mathbf{j}) \times(-800 \mathbf{i}) \\
&= 207846 \mathbf{k}+200000 \mathbf{k}-120000 \mathbf{k}-277128(-\mathbf{k}) \\
&= 564974 \mathbf{k}
\end{aligned}
$$

Let the point of application ' $D$ ' of the resultant be at a distance $x$ along $x$-axis. Then

$$
\begin{array}{rlrl}
\mathbf{r}_{\mathrm{OD}} & =x \mathbf{i} \\
\therefore & \mathbf{r}_{\mathbf{O D}} \times \mathbf{R} & =\mathbf{M}_{\mathbf{o}} \\
\therefore & x \mathbf{i} \times(-500 \mathbf{i}-2719.615 \mathbf{j}) & =564974 \mathbf{k} \\
& -2719.615 x \mathbf{k} & =564994 \mathbf{k} \\
& x=-207.74 \mathrm{~mm}
\end{array}
$$

Ans.
Hence the resultant acts as shown in Fig. 3.17.

Example 3.10 Various forces to be considered for the stability analysis of a dam are as shown in Fig. 3.18. The dam is safe, if the resultant force passes through middle third of the base. Verify whether the dam is safe.
Solution. The vector form of the forces are,
500 kN force : $\quad \mathbf{F}_{1}=500 \mathbf{i}$
1120 kN force : $\quad \mathbf{F}_{2}=-1120 \mathbf{j}$
120 kN force: $\quad \mathbf{F}_{3}=120 \mathbf{j}$
420 kN force: $\quad \mathbf{F}_{4}=-420 \mathbf{j}$
$\therefore \quad \mathbf{R}=\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}$

$$
=500 \mathbf{i}+(-1120+120-420) \mathbf{j}
$$

$$
R=\sqrt{(500)^{2}+(1420)^{2}}=1505.457 \mathrm{kN}
$$

Fig. 3.18

$$
=500 \mathbf{i}-1420 \mathbf{j}
$$

Let the resultant pass through point $D$, which is at a distance $x$ from point $O$.
Then

Now

$$
\mathbf{r}_{\mathrm{OD}}=x \mathbf{i}
$$

Then

$$
\begin{aligned}
& \mathbf{r}_{\mathrm{OD}} \times \mathbf{R}= \Sigma \mathbf{M}_{\mathbf{o}} \\
& x \mathbf{i} \times(500 \mathbf{i}-1420 \mathbf{j})= 4 \mathbf{j} \times 500 \mathbf{i}+2 \mathbf{i}(-1120 \mathbf{j}) \\
&+4 \mathbf{i} \times(120 \mathbf{j})+5 \mathbf{i} \times(-420 \mathbf{j}) \\
&-1420 \mathbf{k}= 2000(-\mathbf{k})-2240 \mathbf{k}+480 \mathbf{k}-2100 \mathbf{k} \\
&=-5860 \mathbf{k} \\
& \therefore \quad x=\frac{5860}{1420}=4.126 \mathrm{~m} \\
& \text { Thus } \quad \frac{1}{3} \times 7<x<\frac{2}{3} 7
\end{aligned}
$$

Hence the resultant passes through point $D$ which is within the middle third

$$
\left(\frac{1}{3} \times 7 \text { to } \frac{2}{3} \times 7\right) \text { of the base. Therefore the dam is safe. }
$$

Ans.
Example 3.11 Determine the magnitude, direction and line of action of the equilibriant of the given set of coplanar forces acting on a truess shown in Fig. 3.19 .

Solution. The two 40 kN forces acting on the smooth pulley may be replaced by a pair of 40 kN forces acting at centre of pulley $C$ and parallel to the given forces, since the sum of moments of the two given forces about $C$ is zero.


Fig. 3.19
Vector form of the forces are:
40 kN force at $30^{\circ}$ to vertical:

$$
\mathbf{F}_{1}=-40 \sin 30^{\circ} \mathbf{i}-40 \cos 30^{\circ} \mathbf{j}=-20 \mathbf{i}-34.641 \mathbf{j}
$$

40 kN force $20^{\circ}$ to horizontal:

$$
\mathbf{F}_{2}=40 \cos 20^{\circ} \mathbf{i}-40 \sin 20^{\circ} \mathbf{j}=37.588 \mathbf{i}-13.681 \mathbf{j}
$$

50 kN force:

$$
\mathbf{F}_{3}=-50 \cos 30^{\circ} \mathbf{i}-50 \sin 30^{\circ} \mathbf{j}=-43.301 \mathbf{i}-25 \mathbf{j}
$$

20 kN upward load:

$$
\mathbf{F}_{4}=20 \mathbf{j}
$$

30 kN load:

$$
\mathbf{F}_{5}=-30 \cos 60^{\circ} \mathbf{i}-30 \sin 60^{\circ} \mathbf{j}=-15 \mathbf{i}-25.981 \mathbf{j}
$$

20 kN downward load:

$$
\mathbf{F}_{6}=-20 \mathbf{j}
$$

20 kN inclined load:

$$
\mathbf{F}_{7}=20 \cos 45^{\circ} \mathbf{i}-20 \sin 45^{\circ} \mathbf{j}=14.142 \mathbf{i}-14.142 \mathbf{j}
$$

Taking $A$ as origin,

$$
\begin{aligned}
\mathbf{r}_{A 1}= & 3 \mathbf{j} \quad \mathbf{r}_{A 2}=3 \mathbf{j} \quad \mathbf{r}_{A 3}=2 \mathbf{i}+2 \mathbf{j} \\
\mathbf{r}_{A 4}= & 4 \mathbf{i}+\mathbf{j} \quad \mathbf{r}_{A 5}=6 \mathbf{i} \quad \mathbf{r}_{A 6}=4 \mathbf{i} \quad \mathbf{r}_{A 7}=0 \\
\therefore \quad \mathbf{R}= & \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}+\mathbf{F}_{5}+\mathbf{F}_{6}+\mathbf{F}_{7} \\
= & (-20+37.588-43.301-15+14.142) \mathbf{i} \\
& +(-34.641-13.681-25+20-25.981-20-14.142) \mathbf{j} \\
= & -26.571 \mathbf{i}-113.445 \mathbf{j} \\
R= & \sqrt{(26.571)^{2}+(113.445)^{2}}=116.515 \mathrm{kN} \\
\alpha_{x}= & \cos ^{-1} \frac{-26.571}{116.515}=103.182^{\circ}
\end{aligned}
$$

$$
\alpha_{y}=\cos ^{-1} \frac{-113.445}{116.515}=166.818^{\circ}
$$

Let $R$ intersect $A B$ at a distance $x$ from $A$.
Then $\mathbf{r}_{\mathrm{AD}}=x \mathbf{i}$

$$
\begin{aligned}
\therefore \quad \mathbf{r}_{\mathrm{AD}} \times \mathbf{R}=\mathbf{r}_{A 1} & \times \mathbf{F}_{1}+\mathbf{r}_{A 2} \times \mathbf{F}_{2}+\mathbf{r}_{A 3} \times \mathbf{F}_{3}+\mathbf{r}_{A 4} \times \mathbf{F}_{4} \\
& +\mathbf{r}_{A 5} \times \mathbf{F}_{5}+\mathbf{r}_{A 6} \times \mathbf{F}_{6}+\mathbf{r}_{A 7} \times \mathbf{F}_{7} \\
& x \mathbf{i} \times(-26.571 \mathbf{i}-113.445 \mathbf{j})=3 \mathbf{j} \times(-20 \mathbf{i}-34.641 \mathbf{j})+3 \mathbf{j} \\
& \times(37.588 \mathbf{i}-13.681 \mathbf{j}) \\
& +(2 \mathbf{i}+2 \mathbf{j}) \times(-43.301 \mathbf{i}-25 \mathbf{j})+(4 \mathbf{i}+\mathbf{j}) \times 20 \mathbf{j} \\
& +6 \mathbf{i} \times(-15 \mathbf{i}-25.981 \mathbf{j})+(4 \mathbf{i})(-20 \mathbf{j})+0 \\
\text { i.e., }-113.445 x \mathbf{k}= & -60(-\mathbf{k})+112.764(-\mathbf{k})-50 \mathbf{k}-86.602(-\mathbf{k}) \\
& +80 \mathbf{k}-155.886 \mathbf{k}-80 \mathbf{k}=-172.048 \mathbf{k} \\
x= & 1.516 \mathrm{~m}
\end{aligned}
$$

Equilibriant is opposite the resultant. Hence it is as shown in Fig. 3.19.
Example 3.12 The loads transferred by four columns to concrete mat foundation are as shown in Fig. 3.20. Determine the resultant and the point of application.
Solution. Taking $A$ as the origin and the coordinate system as shown in Figure 3.20,

$$
\begin{aligned}
& \mathbf{F}_{\mathbf{A}}=-160 \mathbf{j}, \mathbf{F}_{\mathbf{B}}=-140 \mathbf{j} ; \\
& \mathbf{F}_{\mathbf{C}}=-120 \mathbf{j} \text { and } \mathbf{F}_{\mathbf{D}}=-100 \mathbf{j}
\end{aligned}
$$



Fig. 3.20
$A(0,0,0) ; B(6,0,0)$,
$C(6,0,4), D(0,0,4)$

$$
\begin{aligned}
\mathbf{r}_{\mathrm{AA}}= & 0 ; \mathbf{r}_{\mathrm{AB}}=6 \mathbf{i} ; \mathbf{r}_{\mathrm{AC}}=6 \mathbf{i}+4 \mathbf{k} ; \mathbf{r}_{\mathrm{AD}}=4 \mathbf{k} \\
\therefore \quad \mathbf{R}= & \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4} \\
= & (-160-140-120-100) \mathbf{j}=-520 \mathbf{j} \\
\mathbf{M}_{\mathbf{A}}= & 0 \times(-160 \mathbf{j})+6 \mathbf{i} \times(-140 \mathbf{j})+(6 \mathbf{i}+4 \mathbf{k})(-120 \mathbf{j}) \\
& +4 \mathbf{k} \times(-100 \mathbf{j}) \\
= & -840 \mathbf{k}-720 \mathbf{k}-480(-\mathbf{i})-400(-\mathbf{i}) \\
= & -1560 \mathbf{k}+880 \mathbf{i}
\end{aligned}
$$

Let the resultant pass through point $E$ on the slab $E(x, 0, z)$. Then

$$
\begin{array}{rlrl} 
& & \mathbf{r}_{\mathrm{AE}} & =x \mathbf{i}+z \mathbf{k} \\
\therefore & \mathbf{M} & =\mathbf{r}_{\mathrm{AE}} \times \mathbf{R} \\
\text { i.e., } & -1560 \mathbf{k}+880 \mathbf{i} & =(x \mathbf{i}+z \mathbf{k})(-520 \mathbf{j}) \\
& & =-520 x \mathbf{k}-520 z(-\mathbf{i}) \\
& & & =-520 x \mathbf{k}+520 z \mathbf{i} \\
\text { i.e., } & 520 x & =1560 \text { and } 520 z=880 \\
\therefore & x & =3.0 \mathrm{~m} \text { and } z=1.692 \mathrm{~m}
\end{array}
$$

Ans.

Ans.

Example 3.13 $a b c d a^{\prime} b^{\prime} c^{\prime} d^{\prime}$ is a parallelepiped as shown in Fig. 3.21. A force of 30 kN acts in the direction of $a c^{\prime}$ and a force of 60 kN acts in the direction of $d b^{\prime}$. A force of 40 kN acts along $a^{\prime} d^{\prime}$. Determine the resultant force-couple.
Solution. Taking ' $a$ ' as origin and the coordinate system as shown in Fig. 3.21.

$$
\begin{aligned}
& a(0,0,0), c^{\prime}(6,4,3) ; d(0,0,3), b^{\prime}(6,4,0) \\
& \therefore \mathbf{r}_{\mathrm{ac}^{\prime}}
\end{aligned}=6 \mathbf{i}+4 \mathbf{j}+3 \mathbf{k}
$$

$$
=46.08 \mathbf{i}+30.72 \mathbf{j}-23.04 \mathbf{k}
$$

$$
\mathbf{F}_{3}=40 \mathbf{k}
$$

$$
\therefore \quad \mathbf{R}=\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}
$$

$$
=(23.047+46.08) \mathbf{i}+(15.36+30.72) \mathbf{j}+(11.523-23.04+40) \mathbf{k}
$$

$$
=69.127 \mathbf{i}+46.08 \mathbf{j}+28.483 \mathbf{k}
$$

$$
\begin{aligned}
R & =\sqrt{(69.127)^{2}+(46.08)^{2}+(28.483)^{2}}=87.825 \mathrm{kN} \\
\alpha_{x} & =\cos ^{-1} \frac{69.127}{87.825}=38.08^{\circ} \\
\alpha_{y} & =\cos ^{-1} \frac{46.08}{87.825}=58.353^{\circ} \\
\alpha_{z} & =\cos ^{-1} \frac{28.483}{87.825}=71.08^{\circ}
\end{aligned}
$$

Ans.

Taking displacement vectors of convenient points on $F_{1}, F_{2}, F_{3}$

$$
\begin{aligned}
\mathbf{r}_{a 1} & =\mathbf{r}_{\mathrm{ac}}=0 \\
\mathbf{r}_{a 2} & =\mathbf{r}_{\mathrm{ad}}=3 \mathbf{k} \\
\mathbf{r}_{a 3} & =\mathbf{r}_{\mathrm{a}}{ }^{\prime}=4 \mathbf{j} \\
\mathbf{M}_{\mathrm{a}} & =\mathbf{r}_{a 1} \times \mathbf{F}_{1}+\mathbf{r}_{a 2} \times \mathbf{F}_{2}+\mathbf{r}_{a 3} \times \mathbf{F}_{3} \\
& =0+3 \mathbf{k} \times(46.08 \mathbf{i}+30.72 \mathbf{j}-23.04 \mathbf{k})+4 \mathbf{j} \times(40 \mathbf{k}) \\
& =138.24(\mathbf{j})+92.16(-\mathbf{i})+160 \mathbf{i} \\
& =67.84 \mathbf{i}+138.24 \mathbf{j}
\end{aligned}
$$

Thus the resultant is a force $R=69.127 \mathbf{i}+46.08 \mathbf{j}+28.483 \mathbf{k}$ and a couple moment $67.84 \mathbf{i}+138.24 \mathbf{j}$.

Example 3.14 A rectangular block of size $6 \mathrm{~m} \times 4 \mathrm{~m} \times 3 \mathrm{~m}$ is divided into $6 \times 4 \times 3$ cubes of side 1 m as shown in Fig. 3.22. It is subjected to forces as given below:

$$
\begin{aligned}
& F_{1}=400 \mathrm{~N} \text { from } A \text { to } B \\
& F_{2}=300 \mathrm{~N} \text { from } C \text { to } B \\
& \text { and } \quad F_{3}=200 \mathrm{~N} \text { from } E \text { to } F .
\end{aligned}
$$

Find the resultant force-couple system acting at the origin $O$ and at $G$.

Solution. The coordinates of various points are

Fig. 3.22


$$
\begin{aligned}
& O(0,0,0) ; A(0,4,2) \\
& B(6,1,0) ; C(2,3,3), D(6,0,1) \\
& E(2,4,0), F(6,2,2) \text { and } G(6,4,0)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{r}_{\mathrm{AB}} & =6 \mathbf{i}-3 \mathbf{j}-2 \mathbf{k} \\
r_{A B} & =\sqrt{6^{2}+3^{2}+2^{2}}=7 \mathrm{~m} \\
\mathbf{r}_{\mathrm{AB}} & =0.857 \mathbf{i}-0.429 \mathbf{j}-0.286 \mathbf{k} \\
\mathbf{F}_{\mathrm{AB}} & =\mathbf{F}_{1}=400(0.857 \mathbf{i}-0.429 \mathbf{j}-0.286 \mathbf{k}) \\
& =342.8 \mathbf{i}-171.6 \mathbf{j}-114.4 \mathbf{k} \\
\mathbf{r}_{\mathbf{C D}} & =4 \mathbf{i}-3 \mathbf{j}-2 \mathbf{k} \\
\therefore \quad & r_{C D}
\end{aligned}=\sqrt{4^{2}+3^{2}+2^{2}}=5.385 .
$$

$$
=222.9 \mathbf{i}-167.1 \mathbf{j}-111.3 \mathbf{k}
$$

$$
r_{E F}=4 \mathbf{i}-2 \mathbf{j}+2 \mathbf{k}
$$

$$
r_{E F}=\sqrt{4^{2}+2^{2}+2^{2}}=4.899
$$

$$
\boldsymbol{n}_{E F}=0.816 \mathbf{i}-0.408 \mathbf{j}+0.408 \mathbf{k}
$$

$$
\mathbf{F}_{\mathbf{E F}}=\mathbf{F}_{3}=200(0.816 \mathbf{i}-0.408 \mathbf{j}+0.408 \mathbf{k})
$$

$$
=163.20 \mathbf{i}-81.60 \mathbf{j}+81.60 \mathbf{k}
$$

$$
\mathbf{R}=\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}
$$

$$
\therefore \quad \mathbf{R}=(342.8+222.9+163.20) \mathbf{i}+(-171.6-167.1-81.60) \mathbf{j}
$$

$$
+(-114.4-111.3+81.60) \mathbf{k}
$$

$$
=728.9 \mathbf{i}-420.3 \mathbf{j}-144.1 \mathbf{k}
$$

Now, $\quad \mathbf{r}_{\mathbf{O A}}=4 \mathbf{j}+2 \mathbf{k}$

$$
\mathbf{r}_{\mathbf{O C}}=2 \mathbf{i}+3 \mathbf{j}+3 \mathbf{k}
$$

$$
\mathbf{r}_{\mathrm{OE}}=2 \mathbf{i}+4 \mathbf{j}
$$

$\therefore$ Moment of all the force about $O$ is given as

$$
\begin{aligned}
\mathbf{M}_{\mathbf{o}}= & \left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 4 & 2 \\
342.8 & -171.6 & -114.4
\end{array}\right|+\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 3 & 3 \\
222.9 & -167.1 & -111.3
\end{array}\right| \\
& +\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 4 & 0 \\
163.2 & -81.60 & 81.60
\end{array}\right| \\
= & \mathbf{i}(-457.6+343.2)-\mathbf{j}(0-685.6)+\mathbf{k}(0-1371.2) \\
& +\mathbf{i}(-333.9+501.3)-\mathbf{j}(-222.6-668.7)+\mathbf{k}(-334.2-668.7) \\
& +\mathbf{i}(326.4-0)-\mathbf{j}(163.2-0)+\mathbf{k}(-163.2-652.8) \\
= & 379.4 \mathbf{i}+1413.7 \mathbf{j}-3190.1 \mathbf{k}
\end{aligned}
$$

Hence force couple system at $O$ is

$$
\begin{aligned}
\mathbf{R} & =728.9 \mathbf{i}-420.3 \mathbf{j}-144.1 \mathbf{k} \\
\mathbf{M}_{\mathbf{o}} & =379.4 \mathbf{i}+1413.7 \mathbf{j}-3190.1 \mathbf{k}
\end{aligned}
$$

Force couple system at $G$ is given by

$$
\begin{aligned}
\mathbf{M}_{\mathbf{G}} & =\mathbf{M}_{\mathbf{0}}+\mathbf{r}_{\mathbf{G o}} \times \mathbf{R} \\
\mathbf{r}_{\mathbf{G o}} & =-6 \mathbf{i}-4 \mathbf{j}
\end{aligned}
$$

$$
\mathbf{M}_{\mathbf{G}}=379.4 \mathbf{i}+1413.7 \mathbf{j}-3190.1 \mathbf{k}+\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-6 & -4 & 0 \\
728.9 & -420.3 & -144.1
\end{array}\right|
$$

$$
=379.4 \mathbf{i}+1413.7 \mathbf{j}-3190.1 \mathbf{k}+\mathbf{i}(576.4)-\mathbf{j}(864.6)
$$

$$
+\mathbf{k}(2521.8+2915.6)
$$

$$
=955.8 \mathbf{i}+549.1 \mathbf{j}+2247.3 \mathbf{k}
$$

Ans.

### 3.9 WRENCH RESULTANT

In a two dimensional problem, the resultant $R$ and the moment $M$ at any point are at right angles to each other, the force being in the plane ( $x-y$ plane) and the moment at right angles to the plane (about $z$-axis). Hence it was possible to replace the resultant force-couple system into a single force acting at a distance $d=\frac{M}{F}$ or having $x$ or $y$ intercepts given by $x \mathbf{i} \times \mathbf{F}=\mathbf{M}_{\mathbf{0}}$ and $y \mathbf{j} \times \mathbf{F}=\mathbf{M}_{\mathbf{0}}$. However in case of a three dimensional problem it is not possible to replace the resultant force-couple system by a single force. Hence we presented the resultant in the form of a force and moment about a specified point.

The simplest equivalent form of the resultant system is a force and a moment about the axis of the force as shown in Fig. 3.23. This form of the resultant system is called a wrench. The word wrench is used because the resulting
combination is a push and twist similar to that exerted by a wrench. A wrench is said to be positive if the resultant force and moment have the same sense of direction [Fig. 3.23(a)], and is said to be negative if the two vectors have the opposite directions [Fig. 3.23(b)]


Fig. 3.23 Wrench System of Resultant
In examples 3.13 and 3.14 we have seen how a given force system can be reduced to a resultant force $\mathbf{R}$ and moment $\mathbf{M}_{O}$. $\mathbf{R}$ and $\mathbf{M}_{O}$ may be in different directions as shown in Fig. 3.24(a). Let $N$ be the plane normal to the plane $\mathbf{R}$ and $\mathbf{M}_{O}$. Let $\mathbf{M}_{1}$ and $\mathbf{M}_{2}$ be the component of $\mathbf{M}_{O}$ in the direction of $\mathbf{R}$ and normal to $\mathbf{R}$ respectively.

Now the resultant force $\boldsymbol{R}$ may be shifted to point $O^{\prime}$ in the plane such that $F d=\mathbf{M}_{2}$ as shown in Fig. 3.24(c). Thus the resultant $\mathbf{R}$ at $O^{\prime}$ replaces the resultant force $\mathbf{R}$ and moment $\mathbf{M}_{2}$ at point $\boldsymbol{O}$. Hence the resultant force-moment system at $O^{\prime}$ is a resultant force $\mathbf{R}$ and moment $\mathbf{M}_{1}$ about the axis of the force, i.e., the system of resultant is a wrench.

(a)

(b)

(c)

Fig. 3.24
Example 3.15 Determine the wrench resultant of the system of forces shown in Fig. 3.25, indicating its intersection $x-y$ plane.
Solution. $a(0,0,0), c^{\prime}(8,4,4)$

$$
\begin{aligned}
\therefore \quad \mathbf{r a c}_{\mathrm{ac}^{\prime}} & =8 \mathbf{i}+4 \mathbf{j}+4 \mathbf{k} \\
r_{a c^{\prime}} & =\sqrt{8^{2}+4^{2}+4^{2}}=9.798 \mathrm{~m} \\
\mathbf{n}_{\mathrm{ac}^{\prime}} & =\frac{8}{9.798} \mathbf{i}+\frac{4}{9.798} \mathbf{j}+\frac{4}{9.798} \mathbf{k} \\
& =0.816 \mathbf{i}+0.408 \mathbf{j}+0.408 \mathbf{k}
\end{aligned}
$$



Fig. 3.25

$$
\begin{aligned}
\mathbf{F}_{1} & =40(0.816 \mathbf{i}+0.408 \mathbf{j}+0.408 \mathbf{k}) \\
& =32.64 \mathbf{i}+16.32 \mathbf{j}+16.32 \mathbf{k} \\
\therefore \quad \mathbf{F}_{2} & =20 \mathbf{j} \text { and } \mathbf{F}_{3}=60 \mathbf{k} \\
\mathbf{R} & =\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3} \\
& =32.64 \mathbf{i}+36.32 \mathbf{j}+76.32 \mathbf{k} \\
R & =\sqrt{(32.64)^{2}+(36.32)^{2}+(76.32)^{2}}=90.605 \mathrm{kN} \\
\mathbf{n} & =\frac{32.64}{90.605} \mathbf{i}+\frac{36.32}{90.605} \mathbf{j}+\frac{76.32}{90.605} \mathbf{k} \\
& =0.360 \mathbf{i}+0.401 \mathbf{j}+0.842 \mathbf{k}
\end{aligned}
$$

Let $D(x, y, 0)$ be a point on $x-y$ plane through which wrench resultant $F$ passes.
$a(0,0,0), b(8,0,0)$ and $a^{\prime}(0,4,0)$

$$
\begin{array}{ll}
\therefore & \mathbf{r}_{\mathbf{D a}}=-x \mathbf{i}-y \mathbf{j} \quad \mathbf{r}_{\mathbf{D b}}=(8-x) \mathbf{i}-y \mathbf{j} \\
& \mathbf{r}_{\mathbf{D a ^ { \prime }}}=-x \mathbf{i}+(4-y) \mathbf{j} \\
\therefore & \mathbf{M}_{\mathbf{D}}=\mathbf{r}_{\mathbf{D a}} \times \mathbf{F}_{1}+\mathbf{r}_{\mathbf{D b}} \times \mathbf{F}_{2}+\mathbf{r}_{\mathbf{D a}^{\prime}} \times \mathbf{F}_{3}
\end{array}
$$

$$
\begin{aligned}
= & \left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-x & -y & 0 \\
32.64 & 16.32 & 16.32
\end{array}\right|+[(8-x) \mathbf{i}-y \mathbf{j}] 20 \mathbf{j} \\
& +[-x \mathbf{i}+(4-y) \mathbf{j}] \times 60 \mathbf{k} \\
= & \mathbf{i}(-16.32 y)-\mathbf{j}(-16.32 x)+\mathbf{k}(-16.32 x+32.64 y) \\
& +20(8-x) \mathbf{k}-60 x(-\mathbf{j})+60(4-y) \mathbf{i} \\
= & (-16.32 y+240-60 y) \mathbf{i}+\mathbf{j}(16.32 x+60 x) \\
& +\mathbf{k}(-16.32 x+32.64 y+160-20 x) \\
= & (240-76.32 y) \mathbf{i}+76.32 x \mathbf{j}+(160-36.32 x+32.64 y) \mathbf{k} \\
\mathbf{n}^{\prime}= & \frac{240+76.32 y}{M} \mathbf{i}+\frac{76.32 x}{M} \mathbf{j}+\frac{160-36.32 x+32.64 y}{M} \mathbf{k}
\end{aligned}
$$

where $M$ is the moment. Then the direction cosine of $M$ and $F$ are the same in a wrench system.

$$
\begin{equation*}
\therefore \quad \frac{240-76.32 y}{M}=0.360 \tag{i}
\end{equation*}
$$

$$
\begin{equation*}
\frac{76.32 x}{M}=0.401 \tag{ii}
\end{equation*}
$$

and $\quad \frac{160-36.32 x+32.64 y}{M}=0.842$
From Eqn. $(i), \quad y=\frac{240-0.36 M}{76.32}$

From Eqn. (ii), $\quad x=\frac{0.401}{76.32} M$
From Eqn. (iii),

$$
\begin{aligned}
& \qquad 160-36.32 \frac{0.401}{76.32} M+\frac{32.64(240-0.36 M)}{76.32}=0.842 M \\
& \text { i.e., } \quad(0.842+0.1909+0.1540) M=160+102.642 \\
& \therefore \quad M
\end{aligned}
$$

Ans.

Ans.

Ans.

Example 3.16 A system of forces has resultant force couple as :

$$
\begin{aligned}
\mathbf{R} & =700 \mathbf{i}-400 \mathbf{j}-150 \mathbf{k} \text { and } \\
\mathbf{M}_{\mathbf{o}} & =400 \mathbf{i}+350 \mathbf{j}-2500 \mathbf{k}
\end{aligned}
$$

Find the wrench system of resultant, indicating its intersection with $x, y$ plane.

## Solution.

$$
\mathbf{R}=700 \mathbf{i}-400 \mathbf{j}-150 \mathbf{k}
$$

$$
\begin{aligned}
R & =\sqrt{700^{2}+400^{2}+150^{2}}=820.06 \\
\mathbf{n}_{\mathbf{R}} & =\frac{700}{820.06} \mathbf{i}-\frac{400}{820.06} \mathbf{j}-\frac{150}{820.06} \mathbf{k} \\
& =0.854 \mathbf{i}-0.488 \mathbf{j}-0.183 \mathbf{k} \\
\mathbf{M}_{\mathbf{o}} & =400 \mathbf{i}+350 \mathbf{j}-2500 \mathbf{k}
\end{aligned}
$$

Let $O^{\prime}$ be the point of intersection of wrench resultant with $x, y$ plane. Its coordinates be $(x, y, 0)$.

Then, $\quad \mathbf{M}_{\mathbf{o}^{\prime}}=\mathbf{M}_{\mathbf{0}}+\mathbf{r}_{\mathbf{o}^{\prime} \mathbf{0}} \times \mathbf{R}$

$$
\begin{aligned}
& =400 \mathbf{i}+350 \mathbf{j}-2500 \mathbf{k}+\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
x & y & 0 \\
700 & -400 & -150
\end{array}\right| \\
& =400 \mathbf{i}+350 \mathbf{j}-2500 \mathbf{k}+\mathbf{i}(-150 y)-\mathbf{j}(-150 x)+\mathbf{k}(-400 x-700 y) \\
& =(400-150 y) \mathbf{i}+(350+150 x) \mathbf{j}+(-2500-400 x-700 y) \mathbf{k}
\end{aligned}
$$

If $M^{\prime}$ is the magnitude of the wrench moment, then

$$
\mathbf{M}_{\mathbf{o}^{\prime}}=\frac{400-150 y}{M^{\prime}} \mathbf{i}+\frac{350-150 x}{M^{\prime}} \mathbf{j}-\frac{2500+400 x+700 y}{M^{\prime}} \mathbf{k}
$$

Since $\mathbf{M}_{\mathbf{o}}{ }^{\prime}$ has to be the same direction cosines as $\mathbf{R}$, we get

$$
\begin{aligned}
& \frac{400-150 y}{M^{\prime}}=0.854 \\
& \frac{350+150 x}{M^{\prime}}=-0.488 \\
& -\frac{2500+400 x+700 y}{M^{\prime}}=-0.183 \\
& \text { From Eqn. }(i), \quad y=\frac{400-0.854 M^{\prime}}{150} \\
& \text { From Eqn. (ii), } \quad x=\frac{-0.488 M^{\prime}-350}{150}
\end{aligned}
$$

Hence from Eqn. (iii),

$$
\begin{aligned}
& 2500+400\left(\frac{-0.488 M^{\prime}-350}{150}\right)+\frac{700\left(400-0.854 M^{\prime}\right)}{150}=0.183 M^{\prime} \\
& 2500-1.301 M^{\prime}-933.33+1886.67-3.985 M^{\prime}=0.183 M^{\prime} \\
& \therefore \quad(0.183+1.301+3.985) M^{\prime}=2500-933.33+1886.67 \\
& M^{\prime}=627.71 \text { units } \\
& \therefore x=\frac{-0.488(627.71)-350}{150}=-4.375 \text { units } \\
& \therefore \\
& y=\frac{400-0.854(627.71)}{150}=-0.907 \text { units } \\
& z=0 \\
& \mathbf{R}=700 \mathbf{i}-400 \mathbf{j}-150 \mathbf{k}
\end{aligned}
$$

Ans.

### 3.10 EQUILIBRIUM OF RIGID BODIES (EQUILIBRIUM OF NONCONCURRENT FORCE SYSTEM)

Under this heading, we consider the analysis of rigid bodies subjected to nonconcurrent force system. Such bodies are in equilibrium, when there is no translation and no rotation. The various forces to be considered in these cases are similar to those considered for the equilibrium conditions of particle, i.e., applied forces and non-applied forces.

## Applied Force

The commonly encountered applied forces on rigid bodies are discussed below:

## Point Force

If a load is acting on the body over a very small area compared to the sizes of the body, without much loss of accuracy in the analysis, it may be approximated as a force acting at a point. Such load is also known as concentrated load.

## Linear Force

A force acting along a line of the body is known as the linear force. Figure 3.26 shows a linear loading on a beam. In this the ordinate represents, the intensity of the force and the abscissa represents the position of the force on the beam. The force $d F$ on an elemental length $d x$ is given by


Fig. 3.26 Linear Force

$$
\begin{equation*}
d F=w d x \tag{3.10}
\end{equation*}
$$

where $w$ is the intensity of the force.
Uniformly distributed loads (Fig. 3.27) and uniformly varying loads (Fig. 3.28) are the particular cases of linear forces.

From Eqn. 3.10 it is obvious that area of load diagram gives total load and centroid of this load is the centroid of the total load. Thus, in Fig. 3.27,


Fig. 3.27 Uniformly Distributed Load (UDL)
total load

$$
W=40 \times 3=120 \mathrm{kN}
$$

and its centroid is at $=1+\frac{3}{2}=2.5 \mathrm{~m}$ from A. In Fig. 3.28(a),
Total load $=\frac{1}{2} \times 80 \times 3=120 \mathrm{kN}$


Fig. 3.28 (a) Uniformly Varying Loads (UVL)


Fig. 3.28 (b) Another method of showing (UDL)
and its centroid is at $1+\frac{2}{3} \times 3=3 \mathrm{~m}$ from $A$.
Many times, for simplicity uniformly distributed load is indicated as shown in Fig. 3.28(b).

## Surface Force

A force acting on the surface of a body is termed as surface force. Common example of such force is the hydrostatic pressure acting on a body immerced in water (Ref. Fig. 3.29). If ' $p$ ' is the intensity of force acting on an elemental area $d A$, then the force acting on the element is


Fig. 3.29 Hydrostatic Force (a surface force)

$$
\begin{equation*}
d F=p . d A \tag{3.11}
\end{equation*}
$$

## Body Force

A body force is the force acting on each and every particle contained in the mass of the body. Self weight of the body and the inertia force on the body when it is rotated (Fig. 3.30) are the common example of such forces.


Fig. 3.30 Inertia Force (a body force)

## External Moment

Some times a couple may act on a body as shown in Fig. 3.31(a). It is equivalent to a external moment $M=F \times a$ as shown in Fig. 3.31(b).

(a)

(b)

Fig. 3.31 External Moment

## Non-applied Forces

Self weight and reaction are the non applied forces acting on the bodies. As explained earlier self weight $W=m g$, where $m$ is mass of the body and ' $g$ ' is the acceleration due to gravity. It acts always vertically downward. Reactions are the self adjusting forces exerted by other bodies in contact or by the supporting bodies. Tables 3.1 and 3.2 show the different types of contacts between the body and other bodies and also the reactive force on the body.

Table 3.1 Types of Reactive Forces in Two-Dimensional Problem

| Sl. no. Types of contact | Reactive forces <br> on the body |
| :--- | :--- |
| 1. Spring Action <br> 2. Flexible cable or rope | $F=k \delta$, where $k$ is spring <br> constant. Away from the <br> body if spring extends, <br> towards the body if spring <br> contracts. <br> Tensile force in cable. <br> Always away from the <br> body. |
| 3. Rmooth Surface |  |

Table 3.1 (Contd.)

| Sl. no. Types of contact | Reactive forces on the body | Remarks |
| :---: | :---: | :---: |
| 5. Roller support/simple support |  | Always normal to support. |
| Support | $\uparrow$ |  |
| 6. Pinned/hinged support |  | No moment but reactive force in any direction, which may be conveniently represented by its components in two mutually perpendicular direction. |
| 7. Fixed or built-in support | $R_{x} \underset{R_{y}^{+}}{\stackrel{\leftrightarrows}{4}}$ | Moment and force in any directions |

Table 3.2 Types of Reactive Force in Three Dimensional Problems


Table 3.2 (Contd.)

| $\begin{aligned} & \text { Sl. } \\ & \text { no. } \end{aligned}$ | Types of contact | Reactive forces in the body | Remarks |
| :---: | :---: | :---: | :---: |
| 5. | Fixed Joint |  | Resultant force and moment in any direction, which may be represented by reaction components, $R_{x}, R_{y}, R_{z}, M_{x}, M_{y}$ and $M_{z}$ |

## Equations of Equilibrium

If a body is said to be in equilibrium, it means that the resultant force-moment system acting on the body is zero. Mathematically,

$$
\begin{equation*}
\mathbf{R}=0 \quad \mathbf{M}=0 \tag{3.12}
\end{equation*}
$$

consider the equation

$$
\begin{gather*}
\mathbf{R}=0 \\
\mathbf{F}_{1}+\mathbf{F _ { 2 }}+\ldots+\mathbf{F}_{n}=0 \\
\text { i.e., } \begin{array}{c}
\mathbf{i} F_{1 x}+\mathbf{j} F_{1 y}+\mathbf{k} F_{1 z}+\mathbf{i} F_{2 x}+\mathbf{j} F_{2 y}+\mathbf{k} F_{2 z}+\ldots+\mathbf{i} F_{n x}+\mathbf{j} F_{n y} \\
+\mathbf{k} F_{n z}=0
\end{array} \\
\text { i.e., } \quad \mathbf{i} \Sigma F_{x}+\mathbf{j} \Sigma F_{y}+\mathbf{k} \Sigma F_{z}=0 \\
\text { Similarly, } \mathbf{M}=0 \text { means } \\
\\
\mathbf{i} \Sigma M_{x}+\mathbf{j} \Sigma M_{y}+\mathbf{k} \Sigma M_{z}=0 \tag{i}
\end{gather*}
$$

In scalar form, Eqns. (i) and (ii) mean

$$
\begin{array}{r}
\Sigma F_{x}=0 ; \Sigma F_{y}=0 ; \Sigma F_{z}=0 \\
\Sigma M_{x}=0 ; \Sigma M_{y}=0 ; \Sigma M_{z}=0 \tag{3.13}
\end{array}
$$

For two dimensional problems in $x, y$ planes, the forces do not exist in $z$-direction and the moment exists only in the $z$-direction. In other words, $\Sigma F_{z}=0, \Sigma M_{x}=0, \Sigma M_{y}=0$, Noting $M_{z}$ as $M$ only, which is normally looked as a moment about a point in $x-y$ plane, the equations of equilibrium reduce to,

$$
\begin{equation*}
\Sigma F_{x}=0 \Sigma F_{y}=0 \text { and } \Sigma M=0 \tag{3.14}
\end{equation*}
$$

For the system of forces in a plane scalar equivalent equations [Eqn. (3.14)] are convenient to use while for the system in space vector form of equations [Eqn. (3.12)] are more useful.

While dealing with the equilibrium conditions of bodies under concurrent force system (particle), the statement was made that if a system is in equilibrium under the action of only three forces, these three forces must be concurrent. Now we are in a position to prove the statement.

Let $\mathbf{F}_{1}, \mathbf{F}_{2}$ and $\mathbf{F}_{3}$ be the three forces acting on a body as shown in Fig. 3.32. Let $\mathbf{F}_{1}$ and $\mathbf{F}_{2}$ intersect at $A$. Then from moment equilibrium condition we get,

$$
\Sigma M_{A}=0
$$



Fig. 3.32

$$
\text { i.e., } \quad F_{3} d=0
$$

where $d$ is the distance of line of action of $F_{3}$ from $A$. Since $F_{3}$ is not zero ' $d$ ' must be zero, i.e., $F_{3}$ also pass through A. Hence the proposition is proved.

### 3.11 BEAMS

Beams are common rigid bodies encountered in engineering which are subjected to non-concurrent system of forces. A beam may be defined as a structural element which has one dimension considerably large compared to the other two dimensions and having supports at few points. A beam is normally subjected to transverse loads and that load is transferred to supports. In this process the beam bends. The types of beams and the reactions developed in such beams is discussed below:

## Simply Supported Beams

When both ends simply rest on supports, the beam is called simply supported beam. In this case at both ends reactions are at right angles to the support (Fig. 3.33). Such beams can support loads in the direction normal to its axis only.


Fig. 3.33 Simply Supported Beam

## Beam with One End Hinged and the Other on Rollers

If one end of the beam is hinged (pinned) and the other end is on rollers, the beams can resist the load in any direction (Fig. 3.34)


Fig. 3.34 Beam with One End Hinged Other on Roller

## Over-hanging Beams

If beams are having projection beyond the support they are termed as overhanging beams. The overhang may be on only one side [Fig. 3.35(a)] or may be on both sides [Ref. Fig. 3.35(b)].


Fig. 3.35 Overhanging Beams

## Cantilever Beams

It the beam has fixed end at one side and a free end at the other side, it is called cantilever beam (Fig. 3.36). At fixed end there are three reaction components and no reactions at free end.


Fig. 3.36 Cantilever Beam

## Propped Cantilever

It is a beam with one end fixed and the other end simply supported. In this case there are three reaction components at fixed end and one at simply supported end (Fig. 3.37).


Fig. 3.37 Propped Cantilever

## Both Ends Hinged

In these beams both ends are having hinged support. Hence total number of reaction components are four, two at each end (Fig. 3.38).


Fig. 3.38 Both Ends Hinged Beam

## Continuous Beam

A beam is said to be continuous, if it is supported at more than two points (Fig. 3.39).


Fig. 3.39 Continuous Beams
In the cases of simply supported beams, beams with one end hinged and the other on rollers, cantilever and overhanging beams, it is possible to determine the reactions for given loadings by using the equations of equilibrium 3.14 only. In other cases, the number of equilibrium equations are less than the number of unknown reactions and hence it is not possible to analyse them by using equations of equilibrium only. The beams which can be analysed using equations of equilibrium only are known as Statically Determinate beams and those which cannot be analysed are known as Statically Indeterminate beams. The anlaysis of statically indeterminate beams is beyond the scope of this book.

Example 3.17 The beam $A B$ of span 12 m shown in Fig. 3.40(a) is hinged at $A$ and is on rollers at $B$. Determine the reactions of $A$ and $B$ for the loading shown in the figure.

(a)


Fig. 3.40
Solution. At support $A$, which is hinged, the reaction can be in any direction. Let this reaction be represented by its components $R_{A x}$ and $R_{A y}$ as shown in Fig. 3.40(b). At $B$ the reaction is at right angle to roller support, i.e., the reaction is vertical. Let it be $R_{B}$. The freebody diagram of the beam is as shown in Fig. 3.40(b). Since this is a non-concurrent force system the moment equilibrium condition should also be considered. The vector form of various forces are

$$
\mathbf{F}_{1}=R_{A x} \mathbf{i} \quad \mathbf{F}_{2}=R_{A y} \mathbf{j}
$$

$$
\begin{aligned}
& \mathbf{F}_{3}=-10 j \mathbf{F}_{4}=-15 \cos 30^{\circ} \mathbf{i}-15 \sin 30^{\circ} \mathbf{j} \\
& \mathbf{F}_{5}=-20 \cos 45^{\circ} \mathbf{i}-20 \sin 45^{\circ} \mathbf{j} ; F_{6}=R_{B} \mathbf{j}
\end{aligned}
$$

The position vector of these forces from $A$ are

$$
\begin{aligned}
& \mathbf{r}_{A 1}=0, \quad \mathbf{r}_{A 2}=0 \quad \mathbf{r}_{a 3}=4 \mathbf{i} \quad \mathbf{r}_{a 4}=6 \mathbf{i} \\
& \mathbf{r}_{A 5}=10 \mathbf{i} \quad \mathbf{r}_{A 6}=12 \mathbf{i} \\
& \therefore \quad \quad \quad \mathrm{M}_{A}=0 \text {, gives } \\
& 0 \times R_{A x} \mathbf{i}+0 \times R_{A y} \mathbf{j}+4 \mathbf{i} \times(-10 \mathbf{j})+6 \mathbf{i} \times\left(-15 \cos 30^{\circ} \mathbf{i}-15 \sin 30^{\circ} \mathbf{j}\right) \\
& +10 \mathbf{i}\left(-20 \cos 45^{\circ} \mathbf{i}-20 \sin 45^{\circ} \mathbf{j}\right)+12 \mathbf{i} \times R_{B} \mathbf{j}=0 \\
& \text { i.e., } \quad-40 \mathbf{k}-90 \sin 30^{\circ} \mathbf{k}-200 \sin 45^{\circ} \mathbf{k}+12 R_{B} \mathbf{k}=0 \\
& \therefore \quad R_{B}=\frac{40+90 \sin 30^{\circ}+200 \sin 45^{\circ}}{12}=18.868 \mathrm{kN} \quad \text { Ans. }
\end{aligned}
$$

Force equilibrium condition $\mathbf{R}=0$, gives,
$\boldsymbol{R}_{A x} \mathbf{i}+\boldsymbol{R}_{A \mathbf{y}} \mathbf{j}-10 \mathbf{j}-15 \cos 30^{\circ} \mathbf{i}-15 \sin 30^{\circ} \mathbf{j}-20 \cos 45^{\circ} \mathbf{i}$
$-20 \sin 45^{\circ} \mathbf{j}+R_{B} \mathbf{j}=0$
i.e., $\quad\left(\boldsymbol{R}_{A x}-15 \cos 30^{\circ}-20 \cos 45^{\circ}\right) \mathbf{i}+\left(R_{A y}-10-15 \sin 30^{\circ}-20\right.$
$\left.\sin 45^{\circ}+R_{B}\right) \mathbf{j}=0$
i.e., $\quad R_{A x}-15 \cos 30^{\circ}-20 \cos 45^{\circ}=0$
and $\quad R_{A y}-10-15 \sin 30^{\circ}-20 \sin 45^{\circ}+R_{B}=0$
From Eqn. (i), $R_{A x}=15 \cos 30^{\circ}+20 \cos 45^{\circ}=27.133 \mathrm{kN}$
Ans.
From Eqn. (ii),

$$
\begin{aligned}
R_{A y} & =10+15 \sin 30^{\circ}+20 \sin 45^{\circ}-R_{B} \\
& =10+15 \sin 30^{\circ}+20 \sin 45^{\circ}-18.868 \\
& =12.774 \mathrm{kN} \\
\therefore \quad \mathbf{R}_{\mathbf{A}} & =\mathbf{R}_{\mathbf{A x}}+\mathbf{R}_{\mathbf{A y}}=27.133 \mathbf{i}+12.774 \mathbf{j} \\
R_{A} & =\sqrt{27.133^{2}+12.774^{2}}=29.989 \mathrm{kN} \\
\alpha_{x} & =\cos ^{-1} \frac{27.133}{29.984}=25.188^{\circ} \\
\alpha_{y} & =\cos ^{-1} \frac{12.774}{29.984}=64.784^{\circ}
\end{aligned}
$$

Ans.

Ans.

Example 3.18 Determine the reactions developed at supports $A$ and $B$ in the beam shown in Fig. 3.41(a).
Solution. The reaction at $B$ will be at right angles to the support i.e., at $30^{\circ}$ to the vertical as shown in Fig. 3.41(b). Let the component of reaction at hinged end $A$ be $R_{A x}$ and $R_{A y}$. FBD of beam is as shown in Fig. 3.41(b).

$$
\text { Now } \begin{aligned}
\mathbf{F}_{1} & =R_{A x} \mathbf{i}, \mathbf{F}_{2}=\boldsymbol{R}_{A y} \mathbf{j}, \mathbf{F}_{3}=-60 \mathbf{j} \\
\mathbf{F}_{4} & =-80 \cos 75^{\circ} \mathbf{i}-80 \sin 75^{\circ} \mathbf{j}, \mathbf{F}_{5}=40 \cos 60^{\circ} \mathbf{i}-40 \sin 60^{\circ} \mathbf{j} \\
\mathbf{F}_{6} & =-R_{B} \sin 30^{\circ} \mathbf{i}+R_{B} \cos 30^{\circ} \mathbf{j} \\
\mathbf{r}_{A 1} & =0 ; \mathbf{r}_{A 2}=0, \boldsymbol{r}_{A 3}=\mathbf{i}, \mathbf{r}_{A 4}=3 \mathbf{i} \\
\mathbf{r}_{A 5} & =5.5 \mathbf{i}, \mathbf{r}_{A 6}=7 \mathbf{i}
\end{aligned}
$$


(a)

(b)

Fig. 3.41

$$
\begin{align*}
\Sigma M_{A} & =0 \rightarrow \\
& 0+0+\mathbf{i} \times(-60 \mathbf{j})+3 \mathbf{i} \times\left(-80 \cos 75^{\circ} \mathbf{i}-80 \sin 75^{\circ} \mathbf{j}\right) \\
& +5.5 \mathbf{i} \times\left(40 \cos 60^{\circ} \mathbf{i}-40 \sin 60^{\circ} \mathbf{j}\right)+7 \mathbf{i} \\
& \times\left(-R_{B} \sin 30^{\circ} \mathbf{i}+R_{B} \cos 30^{\circ} \mathbf{j}\right)=0 \\
& -60 \mathbf{k}-240 \sin 75^{\circ} \mathbf{k}-220 \sin 60^{\circ} \mathbf{k}+7 R_{B} \cos 30^{\circ} \mathbf{k}=0 \\
& \\
\text { i.e., } \quad & R_{B}=\frac{60+240 \sin 75^{\circ}+220+\sin 60^{\circ}}{7 \cos 30^{\circ}}=79.567 \mathrm{kN} \\
& \mathbf{R}=0 \rightarrow \\
& R_{A x} \mathbf{i}+R_{A y} \mathbf{j}-60 \mathbf{j}-80 \cos 75^{\circ} \mathbf{i}-80 \sin 75^{\circ} \mathbf{j} \\
& +40 \cos 60^{\circ} \mathbf{i}-40 \sin 60^{\circ} \mathbf{j} \\
& -R_{B} \sin 30^{\circ} \mathbf{i}+R_{B} \cos 30^{\circ} \mathbf{j}=0 \\
& \left(R_{A x}-80 \cos 75^{\circ}+40 \cos 60^{\circ}-R_{B} \sin 30^{\circ}\right) \mathbf{i} \\
& +\left(R_{A y}-60-80 \sin 75^{\circ}-40 \sin 60^{\circ}+R_{B} \cos 30^{\circ}\right) \mathbf{j}=0 \\
& R_{A x}-80 \cos 75^{\circ}+40 \cos 60^{\circ}-R_{B} \sin 30^{\circ}=0  \tag{i}\\
\text { i.e., } \quad & R_{A y}-60-80 \sin 75^{\circ}-40 \sin 60^{\circ}+R_{B} \cos 30^{\circ}=0 \tag{ii}
\end{align*} . .
$$

Ans.

Substituting the value of $R_{B}$ in Eqn. (i), we get

$$
\begin{aligned}
R_{A x} & =80 \cos 75^{\circ}-40 \cos 60^{\circ}+79.567 \sin 30^{\circ} \\
& =40.489 \mathrm{kN}
\end{aligned}
$$

Ans.
From Eqn. (ii),

$$
\begin{aligned}
R_{A y} & =60+80 \sin 75^{\circ}+40 \sin 60^{\circ}-79.567 \cos 30^{\circ} \\
& =103.008 \mathrm{kN}
\end{aligned}
$$

Ans.
Example 3.19 Find the reactions at supports $A$ and $B$ of the beam shown in Fig. 3.42(a).

Solution. The reaction at $A$ is vertical. Let $R_{B x}$ and $R_{B y}$ be the horizontal and vertical components of the reactions at $B$.

(a)


Fig. 3.42
Noting that in a rigid body, the UDL can be replaced by total load at its centroid, we replace the given udl in this problem by

$$
30 \times 3=90 \mathrm{kN} \text { load acting at } 2+\frac{3}{2}=3.5 \mathrm{~m} \text { from } A .
$$

The various forces in this system are shown in Fig. 3.42 (b).

$$
\begin{aligned}
& \mathbf{F}_{1}=R_{A} \mathbf{j}, \mathbf{F}_{2}=-20 \mathbf{j}, \mathbf{F}_{3}=-30 \times 3 \mathbf{j}=-90 \mathbf{j} \\
& \mathbf{F}_{4}=40 \cos 60^{\circ} \mathbf{i}-40 \sin 60^{\circ} \mathbf{j} \\
& \mathbf{F}_{5}=R_{B x} \mathbf{i} \text { and } \mathbf{F}_{6}=R_{B y} \mathbf{j} \\
& \mathbf{r}_{B 1}=-8 \mathbf{i} ; \mathbf{r}_{B 2}=-6 \mathbf{i} ; \mathbf{r}_{B 3}=(8-3.5) \mathbf{i}=-4.5 \mathbf{i} \\
& \mathbf{r}_{B 4}=-\mathbf{i} ; \mathbf{r}_{B 5}=0 ; \mathbf{r}_{B 6}=0 \\
& \therefore \quad \Sigma M_{B}=0 \text {, gives } \\
& -8 \mathbf{i} \times R_{A} \mathbf{j}-6 \mathbf{i} \times(-20 \mathbf{j})-4.5 \mathbf{i} \times(-90 \mathbf{j}) \\
& -\mathbf{i} \times\left(40 \cos 60^{\circ} \mathbf{i}-40 \sin 60^{\circ} \mathbf{j}\right)+0+0=0 \\
& \text { i.e. }, \quad-8 R_{A} \mathbf{k}+120 \mathbf{k}+405 \mathbf{k}+40 \sin 60 \mathbf{k}=0 \\
& \therefore \quad R_{A}=\frac{120+405+40 \sin 60^{\circ}}{8}=69.655 \mathrm{kN}
\end{aligned}
$$

Resultant force of the system $=0$, gives

$$
\mathbf{R}=0
$$

$R_{A} \mathbf{j}-20 \mathbf{j}-90 \mathbf{j}+40 \cos 60^{\circ} \mathbf{i}-40 \sin 60^{\circ} \mathbf{j}+R_{B x} \mathbf{i}+R_{B y} \mathbf{j}=0$
$\left(40 \cos 60^{\circ}+R_{B x}\right) \mathbf{i}+\left(R_{A}-20-90-40 \sin 60^{\circ}+R_{B y}\right) \mathbf{j}=0$
i.e., $\quad 40 \cos 60^{\circ}+R_{B x}=0$
and $\quad R_{A}-20-90-40 \sin 60^{\circ}+R_{B y}=0$
From Eqn. (i), we get $R_{B x}=-40 \cos 60^{\circ}=-20 \mathrm{kN}$.
Ans.
From Eqn. (ii),

$$
\begin{aligned}
R_{B y} & =-R_{A}+20+90+40 \sin 60^{\circ} \\
& =-69.955+20+90+40 \sin 60^{\circ} \\
& =74.686 \mathrm{kN} \quad \text { Ans. }
\end{aligned}
$$

Example 3.20 Determine the reactions developed in the cantilever beam shown in Fig. 3.43 (a).

(a)


Fig. 3.43
Solution. Let the reactions at fixed support $A$ be $R_{A x}, R_{A y}$ and $M_{A}$ as shown in Fig. 3.43 (b), which is the free body diagram of the beam. The various forces in the system are

$$
\left.\begin{array}{rl}
\mathbf{F}_{1} & =R_{A x} \mathbf{i} \mathbf{F}_{2}=R_{A y} \mathbf{j} \mathbf{F}_{3}=-20 \times 2 \mathbf{j}=-40 \mathbf{j} \text { acting at } 1 \mathrm{~m} \text { from } A . \\
\mathbf{F}_{4} & =-25 \mathbf{j} \quad \mathbf{F}_{5}=-15 \mathbf{j} \quad \mathbf{F}_{6}=-10 \mathbf{j} \\
\mathbf{r}_{A 1} & =0 ; \mathbf{r}_{A 2}=0 \mathbf{r}_{A 3}=1 \mathbf{i} \mathbf{r}_{A 4}=2 \mathbf{i} \\
\mathbf{r}_{A 5} & =3 \mathbf{i} \text { and } \mathbf{r}_{\mathrm{A} 6}=4 \mathbf{i} \\
\Sigma M_{A} & =0 \rightarrow \\
M_{A} \mathbf{k}+0+0+1 \mathbf{i} \times(-40 \mathbf{j})+2 \mathbf{i} \times(-25 \mathbf{j}) \\
+3 \mathbf{i} \times(-15 \mathbf{j})+4 \mathbf{i} \times(-10 \mathbf{j})=0 \\
M_{A} \mathbf{k}-40 \mathbf{k}-50 \mathbf{k}-45 \mathbf{k}-40 \mathbf{k}=0 \\
& \text { or } M_{A}
\end{array}\right)=175 \mathrm{kN}-\mathrm{m} \quad \text { Ans. } \quad .
$$

Resultant force $=0(\mathbf{R}=0)$, gives

$$
\begin{aligned}
& R_{A x} \mathbf{i}+R_{A y} \mathbf{j}-40 \mathbf{j}-25 \mathbf{j}-15 \mathbf{j}-10 \mathbf{j}=0 \\
& R_{A x} \mathbf{i}+\left(R_{A y}-40-25-15-10\right) \mathbf{j}=0
\end{aligned}
$$

i.e., $\quad R_{A x}=0$

Ans.
and $R_{A y}-40-25-15-10=0$
i.e., $\quad R_{A y}=90 \mathrm{kN}$

Ans.
Example 3.21 Determine the reactions at supports $A$ and $B$ of the overhanging beam shown in Fig. 3.44(a).
Solution. The freebody diagram of the beam is shown in Fig. 3.44(b). The 20 kN udl may be replaced by $20 \times 3=60 \mathrm{kN}$ load acting at $3+\frac{3}{2}=4.5 \mathrm{~m}$ from $A$ and uniformly varying load acting in the overhanging portion by its equivalent of $\frac{1}{2} \times 40 \times 1.5=30 \mathrm{kN}$ acting at the centre of gravity of the load triangle, i.e., at 0.5 m from $B$. The FBD of the beam is as shown in Fig. 3.44(b).

(a)


Fig. 3.44
The moment equilibrium condition at $A$ is

$$
\begin{array}{rlrl} 
& \Sigma M_{A}=0 \rightarrow \\
& R_{B} \times 6-30 & \times 2-20 \times 3 \times 4.5-\frac{1}{2} \times 1.5 \times 40 \times(6+0.5)=0 \\
\therefore & R_{B}=87.5 \mathrm{kN} \\
& \Sigma F_{x}=0 \rightarrow R_{A x}=0 \\
& \Sigma F_{y}=0 \rightarrow R_{A y}+R_{B}-30-20 \times 3-\frac{1}{2} \times 1.5 \times 40=0 \\
& & R_{A y}+87.5-30-60-30 & =0 \\
& & R_{A y}=32.5 \mathrm{kN}
\end{array}
$$

Ans.

Ans.

Example 3.22 Determine the reactions in the overhanging beam shown in Fig. 3.45(a).

Solution. The $F B D$ of the beam is as shown in Fig. 3.45 (b).

$$
\begin{aligned}
& \Sigma M_{A}=0 \rightarrow \\
&-60-40 \sin 60^{\circ} \times 5-30 \times 2 \times(6+1)+R_{B} \times 6=0 \\
& \therefore \quad R_{B}=\frac{60+200 \sin 60^{\circ}+420}{6}=108.868 \mathrm{kN}
\end{aligned}
$$

## Note

Inclined force of 40 kN is split into its vertical and horizontal components at its intersection point with beam. Vertical component $40 \sin 60^{\circ}$ gives clockwise moment $40 \sin 60^{\circ} \times 5$ whereas horizontal component $40 \cos 60^{\circ}$ do not give any moment about $A$, since it passes through $A$, i.e., its distance from $A$ is zero.

$$
\begin{aligned}
\Sigma F_{x} & =0 \rightarrow \\
R_{A x}-40 \cos 60^{\circ} & =0 \\
R_{A x}=40 \cos 60^{\circ} & =20 \mathrm{kN}
\end{aligned}
$$

Ans.

(a)

(b)

Fig. 3.45

$$
\begin{array}{rlrl}
\Sigma F_{y} & =0 \rightarrow \\
& & & =0 \\
\therefore & R_{A y}+R_{B}-40 \sin 60^{\circ}-30 \times 2 & =0 \\
\text { or } & R_{A y}+108.8-40 \sin 60^{\circ}-60 & =0 \\
& R_{A y} & =-14.265 \mathrm{kN} \\
& & \text { (i.e., } R_{A y}=14.265 \mathrm{kN}, \text { downward). }
\end{array}
$$

Ans.

Example 3.23 Determine the reactions at supports $A$ and $B$ of the loaded beam shown in Fig. 3.46(a).

(a)

(b)

Fig. 3.46

Solution. The triangular load in portion $C A$ is equivalent to a force of $\frac{1}{2} \times 2 \times 20$ $=20 \mathrm{kN}$ acting downward at the centroid of the triangle i.e., at $\frac{2}{3} m$ from $A$. The trapezoidal load in portion $A B$ is divided into udl of $20 \mathrm{kN} / \mathrm{m}$ acting at 2 m from $A$ and a uniformly varying load of intensity zero at $A$ and intensity $30-20=$ $10 \mathrm{kN} / \mathrm{m}$ at $B$, acting at $\frac{2}{3} \times 4=\frac{8}{3} \mathrm{~m}$ from $B$. $F B D$ of the beam is shown in Fig. 3.46(b).

$$
\begin{array}{rlr}
\Sigma M_{A} & =0 \rightarrow \\
\frac{1}{2} \times 2 \times 20 \times \frac{2}{3}-20 \times 4 \times 2 & -\frac{1}{2} \times 4 \times 10 \times \frac{8}{3}+R_{B} \times 4=0 \\
R_{B} & =50 \mathrm{kN} \\
\Sigma F_{x} & =0 \rightarrow \\
R_{A x} & =0 & \\
\Sigma F_{y} & =0 \rightarrow & \text { Ans. } \\
R_{A y}+R_{B}-\frac{1}{2} \times 2 \times 20-4 \times 20-\frac{1}{2} \times 4 \times 10 & =0 \\
R_{A y}+50-20-80-20 & =0 \\
R_{A y} & =70 \mathrm{kN} & \\
\therefore \quad \text { Ans. }
\end{array}
$$

Ans.

Ans.

Ans.

Example 3.24 Determine the support reactions on the beam shown in Fig. 3.47. Solution. The freebody diagram of the beam is as shown in Fig. 3.47(b).


Fig. 3.47

$$
\begin{aligned}
\Sigma M_{B} & =0 \text { gives, } \\
30 \times 6-R_{A} \times 4+40 \times 4 \times 2-30 & =0 \\
\therefore \quad R_{A}=\frac{30 \times 6+40 \times 8-30}{4} & =117.5 \mathrm{kN} \\
\Sigma F_{x} & =0 \rightarrow \\
R_{B x} & =0 \\
\Sigma F_{y} & =0 \rightarrow \\
\therefore \quad & \quad R_{B y}=30-R_{A}+40 \times 4 \\
\therefore \quad & =30-117.5+160 \\
& =72.5 \mathrm{kN}
\end{aligned}
$$

Ans.

Ans.

Ans.
Example 3.25 Beam $A B 20 \mathrm{~m}$ long, supported on two intermediate supports $C$ and $D, 12 \mathrm{~m}$ apart, carries the loads as shown in Fig. 3.48. How far away should the first support $C$ be located from end $A$, so that the reactions at both the supports are equal?


Fig. 3.48
Solution. Let the support $C$ be at a distance $x$ metres from the end $A$.
Now it is given that $R_{C}=R_{D}$

$$
\begin{aligned}
& \Sigma F_{y}=0 \rightarrow \\
& -30-6 \times 20+R_{C}+R_{D}-50=0 \\
& \therefore \quad R_{C}+R_{D}=200 \\
& \text { Since } \\
& \text { or } \\
& \text { or } \quad R_{C}=100 \mathrm{kN}=R_{D} \\
& \Sigma M_{A}=0 \rightarrow \\
& R_{C} x+R_{D}(12+x)-6 \times 20 \times 10-50 \times 20=0 \\
& 100 x+100(12+x)=1200+1000 \\
& 200 x=1000 \\
& x=5 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 3.26 Determine the reactions developed at $A, B$ and $D$ in the compound beam shown in Fig. 3.49(a).
Solution. The freebody diagrams of beam $A B$ and $C D$ are as shown in Fig. $3.49(\mathrm{~b})$. The given trapezoidal load on $C D$ is divided into udl of intensity $20 \mathrm{kN} / \mathrm{m}$ and a triangular load of intensity zero at $C$ and $10 \mathrm{kN} / \mathrm{m}$ at $D$.


Fig. 3.49
Consider the equilibrium of beam $C D$.

$$
\begin{array}{rlrl} 
& \Sigma F_{x} & =0, \text { gives } \\
R_{D x} & =0 \\
\Sigma M_{D} & =0 \rightarrow \\
& \\
R_{C} 7-20 \times 5 \times(7-2.5) & -\frac{1}{2} \times 5 \times 10(2+5 / 3)=0 \\
& & \\
\therefore & R_{C} & =\frac{20 \times 5 \times 4.5+\frac{1}{2} \times 5 \times 10 \times 3.667}{7} \\
& =77.382 \mathrm{kN} \\
& R_{C}+R_{D}-5 \times 20 & -\frac{1}{2} \times 5 \times 10 & =0 \\
& & 77.382+R_{D}-100-25 & =0 \\
\text { or } & R_{D} & =47.618 \mathrm{kN}
\end{array}
$$

Now consider the equilibrium of beam $A B$

$$
\begin{array}{rlrl}
\Sigma M_{A} & =0 \\
& & \\
& R_{B} \times 5-R_{C} \times 2 & =0 \\
\therefore & R_{B} & =\frac{2}{5} R_{C}=\frac{2}{5} \times 77.382 \\
\text { i.e., } & R_{B} & =30.953 \mathrm{kN}
\end{array}
$$

i.e.,

Ans.

$$
\begin{aligned}
\Sigma F_{y} & =0 \rightarrow \\
R_{A y}+R_{B}-R_{C} & =0 \\
R_{A y} & =R_{C}-R_{B}=77.382-30.952 \\
& =46.43 \mathrm{kN} \\
\Sigma F_{x} & =0 \rightarrow \\
R_{A x} & =0
\end{aligned}
$$

Ans.

Ans.
Example 3.27 The compound bar $A B C D$ is loaded as shown in Fig. 3.50(a). Determine the reactions developed at $A, C$ and $D$.

(b)

Fig. 3.50
Solution. The free body diagrams of beams $A B$ and $C D$ are as shown in Fig. 3.50 (b). The two equal and opposite forces of 20 kN may be replaced by a couple moment of $20 \times 1=20 \mathrm{kN}-\mathrm{m}$ acting at $F$.

Consider the overhanging beam $A E B$.

$$
\Sigma M_{A}=0 \rightarrow
$$

$R_{E} \times 3-20 \times 2-60 \sin 60^{\circ} \times 4=0$
(Note: the component $R_{E} \cos 60^{\circ}$ do not give any moment about $A$ ).

$$
\begin{aligned}
\therefore \quad R_{E} & =\frac{20 \times 2+60 \sin 60^{\circ} \times 4}{3}=82.615 \mathrm{kN} \\
\Sigma F_{x} & =0 \rightarrow \\
R_{A x}+60 \cos 60^{\circ} & =0
\end{aligned}
$$

$$
\text { or } \quad \begin{aligned}
& R_{A x}=-30 \mathrm{kN} \\
& \Sigma F_{y}=0 \rightarrow
\end{aligned}
$$

$R_{A y}+R_{E}-20-60 \sin 60^{\circ}=0$
$R_{A y}+82.615-20-60 \sin 60^{\circ}=0$
or

$$
R_{A y}=-10.654 \mathrm{kN}
$$

## Ans.

Now consider the equilibrium of beam $C D F$.

$$
\begin{aligned}
\Sigma M_{c} & =0 \rightarrow \\
R_{D} \times 3+20-R_{E} \times 2 & =0 \\
R_{D} & =\frac{-20+82.615 \times 2}{3} ; \text { since } R_{E}=82.615 \mathrm{kN} \\
& =48.410 \mathrm{kN} \\
\Sigma F_{x} & =0 \rightarrow \\
R_{C x} & =0 \\
\Sigma F_{y} & =0 \rightarrow \\
R_{c y}+R_{D}-82.625 & =0 \\
\therefore \quad R_{c y} & =82.615-48.410 \\
& =34.205 \mathrm{kN}
\end{aligned}
$$

Ans.
Ans.
Ans.
Ans.

Example 3.28 Determine the reactions developed at the support in rigid cantilever frame shown in Fig. 3.51.

(a)

(b)

Fig. 3.51
Solution. The free body diagram of the rigid frame is as shown in Fig. 3.51(b).

$$
\begin{array}{cc} 
& \Sigma M_{A}=0 \rightarrow \\
& M_{A}-40 \sin 60^{\circ} \times 2+40 \cos 60^{\circ} \times 4-60 \times 1-15 \times 4-20 \times 2=0 \\
M_{A}=149.282 \mathrm{kN}-\mathrm{m} \\
& \Sigma F_{x}=0 \rightarrow
\end{array}
$$

Ans.
$R_{A x}-40 \cos 60^{\circ}+15+20=0$
or

$$
\begin{aligned}
R_{A x} & =-15 \mathrm{kN} \\
& =15 \mathrm{kN}(\leftarrow) \\
\Sigma F_{y} & =0 \rightarrow \\
R_{A y}-40 \sin 60^{\circ}-60 & =0 \\
R_{A y} & =94.641 \mathrm{kN}
\end{aligned}
$$

Ans.

Ans.
Example 3.29. Determine the support reactions of portal frame shown in Fig. 3.52.


Fig. 3.52
Solution. Freebody diagram of the frame is shown in Fig. 3.52(b).

$$
\begin{array}{rlrl}
\Sigma M_{A} & =0 & \\
& & \\
\therefore & R_{D} \times 6-30 \times 2-20 \times 6 \times 3-40 & =0 & \\
R_{D} & =76.667 \mathrm{kN} \\
\Sigma F_{x} & =0 \rightarrow & & \\
& R_{A x}+30 & =0 & \text { Ans. } \\
& R_{A x} & =-30 \mathrm{kN} \\
& \Sigma F_{y} & =0 \rightarrow & \\
& R_{A y} & +R_{D}-20 \times 6=0 & \\
& R_{A y} & =20 \times 6-R_{D}=120-76.667 & \\
\therefore & R_{A y} & =43.333 \mathrm{kN} & \text { Ans. } \\
\text { i.e., } & & &
\end{array}
$$

Example 3.30 The 12 m boom $A B$ weighs 5 kN , the distance of the centre of gravity $G$ being 6 m from $A$. For the position shown, determine the tension $T$ in the cable and the reaction at $B$, [Ref. Fig. 3.53]
Solution. The free body diagram of the beam is shown in Fig. 3.53(b).

$$
\Sigma M_{A}=0 \rightarrow
$$

[Moments are worked after resolving forces normal to and parallel to boom]


Fig. 3.53
$T \sin 15^{\circ} \times 12-20 \sin 60^{\circ} \times 12-5 \sin 60^{\circ} \times 6=0$

$$
\begin{aligned}
T & =75.286 \mathrm{kN} \\
\Sigma F_{x} & =0 \rightarrow \\
R_{A x}-T \cos 15^{\circ} & =0\left(\text { since } T \text { makes } 15^{\circ} \text { with } x \text {-axis }\right] \\
R_{A x} & =75.286 \cos 15^{\circ} \\
& =72.721 \mathrm{kN} \\
\Sigma F_{y} & =0 \rightarrow \\
R_{A y}-5-20-T \sin 15^{\circ} & =0 \\
R_{A y} & =5+20+75.286 \sin 15^{\circ}
\end{aligned}
$$

$$
=44.485 \mathrm{kN}
$$

Example 3.31 A hollow right circular cylinder of radius 800 mm is open at both ends and rests on a smooth horizontal plane as shown in Fig. 3.54(a). Inside the cylinder, there are two spheres weighing 1 kN and 3 kN and with radii 400 mm and 600 mm respectively. The lower sphere rests on the horizontal plane. Neglecting friction, find the minimum weight $W$ of the cylinder for which it will not tip over.
Solution. Join the centres $O_{1}$ and $O_{2}$ and drop $O_{1} D$ perpendicular to horizontal through $O_{2}$.

Now $O_{2} D=1600-r_{1}-r_{2} \quad$ where $r_{1}$ and $r_{2}$ are radii of the spheres.

$$
=1600-400-600=600 \mathrm{~mm}
$$

If $\alpha$ is the inclination to $O_{2} O_{1}$ to horizontal,

$$
\begin{aligned}
\cos \alpha & =\frac{O_{2} D}{O_{1} O_{2}}=\frac{600}{400+600}=0.6 \\
\alpha & =53.130
\end{aligned}
$$

Free body diagrams of cylinder and spheres are shown in Figs. 3.54(b) and 3.54(c).


Fig. 3.54
Consider the equilibrium of the two spheres.

$$
\begin{aligned}
\Sigma M_{O 2} & =0 \rightarrow \\
R_{1} \times O_{1} O_{2} \sin \alpha-1 \times O_{2} D & =0 \\
R_{1} 1000 \sin 53.130^{\circ}-1 \times 600 & =0 \\
R_{1} & =0.75 \mathrm{kN} \\
\Sigma F_{x} & =0 \rightarrow \\
R_{2} & =R_{1}=0.75 \mathrm{kN} \\
\Sigma F_{y} & =0 \rightarrow \\
R_{3}-1-3 & =0 \\
R_{3} & =4 \mathrm{kN}
\end{aligned}
$$

Now consider the equilibrium of the cylinder. When it is about to tip over on point $A$, there is no reaction from ground at $B$. The reaction will be only at $A$.

$$
\begin{aligned}
& \Sigma M_{A}=0 \rightarrow \\
& R_{1} h_{1}-R_{2} h_{2}-W \times 800=0 \\
& 0.75 h_{1}-0.75 h_{2}=800 \mathrm{~W}, \text { since } R_{1}=R_{2}=0.75 \mathrm{kN} \\
& 0.75\left(h_{1}-h_{2}\right)=800 \mathrm{~W} \\
& 0.75 O_{1} D=800 \mathrm{~W} \\
& O_{1} D=1000 \sin \alpha=1000 \sin 53.130^{\circ} \\
& \text { But } \quad \begin{aligned}
\text { or }
\end{aligned} \quad 0.75 \times 1000 \sin 53.130^{\circ}=800 \mathrm{~W} \\
& \quad W
\end{aligned}
$$

Ans.
Example 3.32 A 500 N cylinder, 1 m in diameter is placed between the two cross pieces which make an angle of $60^{\circ}$ with each other. The two cross pieces are pinned at $C$ as shown in Fig. 3.55 (a). Determine the tension in the horizontal rope $D E$ assuming floor is smooth.
Solution. Consider the equilibrium of entire system
Due to symmetry,

$$
\begin{aligned}
R_{A} & =R_{B} \\
\Sigma F_{y} & =0 \rightarrow \\
R_{A}+R_{B} & =500
\end{aligned}
$$


(a)

(c)

Fig. 3.55
or

$$
2 R_{A}=500
$$

i.e.,
$R_{A}=250 \mathrm{~N}=R_{B}$.
Now consider the equilibrium of the cylinder. [Ref Fig. 3.55(b)]. The members $A E$ and $B D$ make $60^{\circ}$ with horizontal. Hence the reactions $R_{2}$ and $R_{1}$ make $60^{\circ}$ with the vertical

$$
\text { i.e., } \begin{aligned}
\Sigma F_{x} & =0 \rightarrow \\
R_{1} \sin 60^{\circ}-R_{2} \sin 60^{\circ} & =0 \\
R_{1} & =R_{2} \\
\Sigma F_{y} & =0 \rightarrow \\
R_{1} \cos 60^{\circ}+R_{2} \cos 60^{\circ}-500 & =0 \\
0.5 R_{1}+0.5 R_{2} & =500 \\
R_{1} & =500 \mathrm{~N}, \text { since } R_{1}=R_{2} .
\end{aligned}
$$

Now consider the equilibrium of the member $A E$. The free body diagram is shown in Fig. 5.55 (c).

$$
\Sigma M_{c}=0 \rightarrow
$$

$T \sin 60^{\circ} \times 1.8-R_{2} C F-R_{A} \sin 30^{\circ} \times 1.2=0$
Now $\quad C F=O F \cot 30^{\circ}=0.5 \cot 30^{\circ}=0.866 \mathrm{~m}$
$\therefore \quad T=\frac{500 \times 0.866+250 \sin 30^{\circ} \times 1.2}{1.8 \times \sin 60^{\circ}}$
i.e., $\quad T=374.00 \mathrm{~N}$

Ans.
Note:
If the reactions on pin at $C$ are required, the other two equilibrium equations namely $\Sigma F_{x}=0$ and $\Sigma F_{y}=0$ are to be considered.

Example 3.33 The frame shown in Fig. 3.56(a) is supported by a hinge at $A$ and a roller at $E$. Compute the horizontal and vertical components of the hinge forces at $B$ and $C$ as experienced by member $A C$.


Fig. 3.56
Solution. Consider the equilibrium of the entire frame.

$$
\begin{aligned}
\Sigma M_{A} & =0 \rightarrow \\
R_{E} \times 5-1200(1+3+1.5) & =0 \\
R_{E} & =1320 \mathrm{~N} \\
\Sigma F_{x} & =0 \rightarrow R_{A x}=0 \\
\Sigma F_{y} & =0 \rightarrow R_{A y}+R_{E}-1200=0 \\
\therefore \quad R_{A y} & =1200-1320, \text { since } R_{E}=1320 \mathrm{~N} \\
& =-120 \mathrm{~N}
\end{aligned}
$$

Consider the equilibrium of member $B D F$ [Fig. 3.56(b)]

$$
\begin{aligned}
\Sigma M_{D} & =0 \rightarrow \\
R_{B y} \times 3-1200 \times 1.5 & =0 \\
R_{B y} & =600(\text { downward })
\end{aligned}
$$

## Note:

The directions of reaction at $B$ on $A B C$ is opposite to that of $A B C$ on member $B D F$. (action and reactions are opposite to each other)

Consider the equilibrium of the member $A C$. [Ref. Fig. 3.56(c)].

$$
\begin{aligned}
\Sigma F_{y} & =0 \rightarrow \\
R_{c y}+(-120)+600 & =0 \\
R_{c y} & =-480 \mathrm{~N}=480 \mathrm{~N}, \text { downward. } \\
\Sigma F_{x} & =0 \rightarrow \\
R_{C x} & =R_{B x} \\
\Sigma M_{C} & =0 \rightarrow
\end{aligned}
$$

Ans.
$R_{B x} \times 3-R_{B y} \times 1.5-(-120) \times 2.5=0$

$$
R_{B x}=\frac{600 \times 1.5-120 \times 2.5}{3}=200 \mathrm{~N}
$$

From Eqn. (i),

$$
R_{C x}=R_{B x}=200 \mathrm{~N} .
$$

Ans.
Example 3.34 A triangular plate weighing 36 kN is suspended by lifting machine by vertical cables at the corners $A, B$, and $C$ as shown in Fig. 3.57. Determine the forces developed in the cables.


Fig. 3.57
Solution. [Note: This is problem of system of forces in space. Vector approach is ideally suited].

Selecting $A$ as origin, the coordinates of various points are
$A(0,0,0), B(4.5,0,0), C(0,0,3.0)$ and $G(1.5,0,1.0)$.
Let $\mathbf{F}_{1}, \mathbf{F}_{2}$ and $\mathbf{F}_{3}$ be the tensile forces in cables at $A, B$ and $C$. Then

$$
\mathbf{F}_{1}=F_{1} \mathbf{j} ; \mathbf{F}_{2}=F_{2} \mathbf{j} \text { and } \mathbf{F}_{3}=F_{3} \mathbf{j}
$$

The load is

$$
\mathbf{F}_{4}=-36 \mathbf{j}
$$

$$
\mathbf{r}_{A 1}=0, \mathbf{r}_{A 2}=4.5 \mathbf{i}, \mathbf{r}_{A 3}=3 \mathbf{k} \text { and } \mathbf{r}_{A 4}=1.5 \mathbf{i}+\mathbf{k}
$$

The moment equilibrium equation about $A$, gives

$$
\begin{array}{cr} 
& O+4.5 \mathbf{i} \times \mathrm{F}_{2} \mathbf{j}+3 \mathbf{k} \times F_{3} \mathbf{j}+(1.5 \mathbf{i}+\mathbf{k}) \times(-36 \mathbf{j})=0 \\
\text { i.e., } & 4.5 F_{2} \mathbf{k}+3 F_{3}(-\mathbf{i})-54 \mathbf{k}-36(-\mathbf{i})=0 \\
\left(-3 F_{3}+36\right) \mathbf{i}+\left(4.5 F_{2}-54\right) \mathbf{k}=0 \\
\text { i.e., } & \begin{aligned}
&-3 F_{3}+36=0 \quad \therefore F_{3}=12 \mathrm{kN} \\
& \text { and } 4.5 F_{2}-54=0 \quad F_{2}=12 \mathrm{kN}
\end{aligned}
\end{array}
$$

Ans.
Ans.
From force equilibrium condition, we have
i.e.,

$$
F_{1} \mathbf{j}+F_{2} \mathbf{j}+\mathrm{F}_{3} \mathbf{j}-36 \mathbf{j}=0
$$

i.e.,

$$
\left(F_{1}+F_{2}+F_{3}-36\right) \mathbf{j}=0
$$

$$
F_{3}=36-F_{1}-F_{2}
$$

$$
=36-12-12=12 \mathrm{kN}
$$

Ans.

Example 3.35 A rectangular slab of size $3 \mathrm{~m} \times 6 \mathrm{~m}$ is to be lifted with the help of three vertical chains hooked to it at $B, C$ and $E$ as shown in Fig. 3.58. If the slab is having uniform thickness and weighs 54 kN determine the forces developed in the chains.


Fig. 3.58
Solution. Let $\mathbf{F}_{1}, \mathbf{F}_{2}$ and $\mathbf{F}_{3}$ be the tensile forces developed in the chains at $B, C$ and $E$. Let $G$ be the centre of gravity of the slab. Then selecting $A$ as origin and the coordinate system as shown in the Fig. 3.58
$A(0,0,0), B(3,0,0), C(3,0,6), D(0,0,6)$
$E(0,0,3)$ and $G(1.5,0,3)$.
Vector form of the forces to be considered are

\[

\]

From Eqn. (ii), $\quad F_{3}=\frac{81}{3}=27 \mathrm{kN}$
Ans.
From Eqn. (i)

$$
\begin{aligned}
6 F_{2}+3 F_{3} & =162 \\
F_{2} & =\frac{162-3 F_{3}}{6}=\frac{162-3 \times 27}{6}=13.5 \mathrm{kN}
\end{aligned}
$$

Ans.

From force equilibrium condition, we have

$$
\begin{array}{r}
\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}=0 \\
\text { i.e. }\left(F_{1}+13.5+27-54\right) \mathbf{i}=0
\end{array}
$$

$$
\therefore \quad F_{1}=13.5 \mathrm{kN} \quad \text { Ans. }
$$

Example 3.36 A bar of uniform cross section, weighing 2000 N is having a ball and socket support at $A$ and its end $B$ is resting at the corner of smooth walls as shown in Fig. 3.59(a). Determine the reactions developed at the ends $A$ and $B$.

(a)

(b)

Fig. 3.59
Solution. Taking the coordinate system with origin at $O$ as shown in Fig. 3.59(a).
$A(6,0,3), B(0,4,0), G(3,2.0,1.5)$
The free body diagram of the bar is as shown in Fig. 3.59(b).
The vector form of the forces are,
and

$$
\begin{aligned}
& \mathbf{F}_{1}=F_{1 x} \mathbf{i}+\mathbf{F}_{1 y} \mathbf{j}+F_{1 z} \mathbf{k} \\
& \mathbf{F}_{2}=F_{2 x} \mathbf{i}+\mathbf{F}_{2 z} \mathbf{k}
\end{aligned}
$$

$$
\mathbf{F}_{3}=-2000 j
$$

$$
\mathbf{r}_{A B}=-6 \mathbf{i}+4 \mathbf{j}-3 \mathbf{k} \quad \text { and } \quad \mathbf{r}_{A G}=-3 \mathbf{i}+2.0 \mathbf{j}-1.5 \mathbf{k}
$$

$$
\Sigma M_{A}=0 \rightarrow
$$

$$
\mathbf{r}_{A B} \times \mathbf{F}_{2}+\mathbf{r}_{A G} \times \mathbf{F}_{3}=0
$$

i.e., $\quad(-6 \mathbf{i}+4 \mathbf{j}-3 \mathbf{k}) \times\left(F_{2 x} \mathbf{i}+F_{2 z} \mathbf{k}\right)+(-3 \mathbf{i}+2.0 \mathbf{j}-1.5 \mathbf{k}) \times(-2000 \mathbf{j})=0$

$$
\left.\begin{array}{l}
\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-6 & 4 & -3 \\
F_{2 x} & 0 & F_{2 z}
\end{array}\right|+6000 \mathbf{k}+3000(-\mathbf{i})=0 \\
\text { i.e., } \\
\mathbf{i}\left(4 F_{2 z}\right)-\mathbf{j}\left(-6 F_{2 z}+3 F_{2 x}\right)+\mathbf{k}\left(0-4 F_{2 x}\right)+6000 \mathbf{k}-3000 \mathbf{i}=0 \\
\text { i.e., }  \tag{i}\\
\text { i.e., }
\end{array} 4 F_{2 z}-3000\right) \mathbf{i}+\mathbf{j}\left(6 F_{2 z}-3 F_{2 x}\right)+\mathbf{k}\left(-4 F_{2 x}+6000\right)=0 \quad 4 F_{2 z}-3000=0 .
$$

and $\quad \begin{aligned} 6 F_{2 z}-3 F_{2 x} & =0 \\ -4 F_{2 x}+6000 & =0\end{aligned}$
From Eqn. (i), $\quad F_{2 z}=750 N$
Ans.
From Eqn. (ii), $\quad F_{2 x}=2 F_{2 z}=1500 \mathrm{~N}$
Ans.
From Eqn. (iii), $\quad F_{2 x}=1500$, same as from Eqn. (ii).
Now consider the force equilibrium equation.

$$
\begin{gather*}
\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}=0 \\
\left(F_{1 x} \mathbf{i}+F_{1 y} \mathbf{j}+F_{1 z} \mathbf{k}\right)+F_{2 x} \mathbf{i}+F_{2 z} \mathbf{k}-2000 \mathbf{j}=0 \\
\text { i.e., } \\
\left(F_{1 x}+F_{2 x}\right) \mathbf{i}+\left(F_{1 y}-2000\right) \mathbf{j}+\left(F_{1 z}+F_{2 z}\right) \mathbf{k}=0  \tag{iv}\\
\text { i.e., }  \tag{v}\\
F_{1 x}+F_{2 x}=0  \tag{vi}\\
F_{1 y}-2000=0 \\
F_{1 z}+F_{2 z}=0
\end{gather*}
$$

From Eqn. (iv), $F_{1 x}=-F_{2 x}=-1500 \mathrm{~N}$
Ans.
From Eqn. (v), $F_{1 y}=2000=2000 \mathrm{~N}$
Ans.
From Eqn. (vi) $F_{1 z}=-F_{2 z}=-750 \mathrm{~N}$
Ans.
Example 3.37 A block of size $4 \mathrm{~m} \times 3 \mathrm{~m} \times 2 \mathrm{~m}$ is having a uniform mass and is weighing 20 kN . It is supported by a ball and socket at $d^{\prime}$ and by a roller support on the horizontal plane at $A$ and by two cables $B a$ and $C a$, which are in the directions of the $a c^{\prime}$ and $a^{\prime} c$ as shown in Fig. 3.60(a). Determine the support reactions.


Fig. 3.60
Solution. Selecting $d$ as the origin, the coordinates of various points are,

$$
\begin{aligned}
& a(0,3,0), b(4,3,0), c(4,0,0), d(0,0,0) \\
& a^{\prime}(0,3,2), b^{\prime}(4,3,2), c^{\prime}(4,0,2), d^{\prime}(0,0,2)
\end{aligned}
$$

$G(2,1.5,1.0) A(4,0,1)$.
Vector form of reactions:

$$
\begin{aligned}
& \mathbf{F}_{1}=F_{1 x} \mathbf{i}+F_{1 y} \mathbf{j}+F_{1 z} \mathbf{k} \\
& \mathbf{F}_{2}=F_{2} \mathbf{j} \\
& \mathbf{r}_{c^{\prime} a}=-4 \mathbf{i}+3 \mathbf{j}-2 \mathbf{k} \\
& \mathbf{r}_{c^{\prime} a}=\sqrt{4^{2}+3^{2}+2^{2}}=5.385 \mathrm{~m} \\
& \mathbf{n}_{c^{\prime} a}=-0.743 \mathbf{i}+0.557 \mathbf{j}-0.371 \mathbf{k} \\
& \mathbf{F}_{3}=F_{3}(-0.743 \mathbf{i}+0.557 \mathbf{j}-0.371 \mathbf{k}) \\
& \mathbf{r}_{c a^{\prime}}=-4 \mathbf{i}+3 \mathbf{j}+2 \mathbf{k} \\
& \mathbf{r}_{c a^{\prime}}=\sqrt{4^{2}+3^{2}+2^{2}}=5.385 \mathrm{~m} \\
& \therefore \quad \mathbf{n}_{c a^{\prime}}=-0.743 \mathbf{i}+0.557 \mathbf{j}-0.371 \mathbf{k} \\
& \therefore \quad \mathbf{F}_{4}=F_{4}(-0.743 \mathbf{i}+0.557 \mathbf{j}-0.371 \mathbf{k}) \\
& \mathbf{F}_{5}
\end{aligned}=\operatorname{self} \mathrm{wt}=-12 \mathbf{j} \mathrm{l} .
$$

Moment equilibrium condition about $d^{\prime}$ is to be considered.

$$
\begin{aligned}
& \mathbf{r}_{d^{\prime} 1}=\mathbf{r}_{d^{\prime} a^{\prime}}=0, \mathbf{r}_{d^{\prime} 2}=\mathbf{r}_{d^{\prime} A}=4 \mathbf{i}-\mathbf{k} \\
& \mathbf{r}_{d^{\prime} 3}=\mathbf{r}_{d^{\prime} c^{\prime}}=4 \mathbf{i}, \mathbf{r}_{d^{\prime} 4}=\mathbf{r}_{d^{\prime} a^{\prime}}=3 \mathbf{j} \\
& \mathbf{r}_{d^{\prime} 4}=\mathbf{r}_{d^{\prime} G}=2 \mathbf{i}+1.5 \mathbf{j}-\mathbf{k}
\end{aligned}
$$

$\therefore$ The moment equilibrium condition about $d^{\prime}$ gives

$$
\begin{aligned}
& 0+(4 \mathbf{i}-\mathbf{k}) \times F_{2} \mathbf{j}+4 \mathbf{i} \times(-0.743 \mathbf{i}+0.557 \mathbf{j}-0.371 \mathbf{k}) F_{3} \\
& +3 \mathbf{j} \times(-0.743 \mathbf{i}+0.557 \mathbf{j}+0.371 \mathbf{k}) F_{4} \\
& +(2 \mathbf{i}+1.5 \mathbf{j}-\mathbf{k}) \times(-12 \mathbf{j})=0
\end{aligned}
$$

i.e., $\quad 4 \mathbf{k} F_{2}-F_{2}(-\mathbf{i})+2.228 F_{3} \mathbf{k}-1.484 F_{3}(-\mathbf{j})$

$$
-2.229(-\mathbf{k}) F_{4}+1.113 F_{4} \mathbf{i}-24 \mathbf{k}+12(-\mathbf{i})=0
$$

$$
\left(F_{2}+1.113 F_{4}-12\right) \mathbf{i}+1.484 F_{3} \mathbf{j}
$$

$$
+\left(4 F_{2}+2.228 F_{3}+2.229 F_{4}-24\right) \mathbf{k}=0
$$

i.e.,

$$
\begin{align*}
F_{2}+1.113 F_{4}-12 & =0  \tag{i}\\
F_{3} & =0 \tag{ii}
\end{align*}
$$

and

$$
\begin{equation*}
4 F_{2}+2.228 F_{3}+2.229 F_{4}-24=0 \tag{iii}
\end{equation*}
$$

Since $F_{3}$ is zero, Eqn. (iii) reduces to

$$
\begin{equation*}
4 F_{2}+2.229 F_{4}=24 \tag{iv}
\end{equation*}
$$

Subtracting 4 times Eqn. (i) from Eqn. (iv), we get

$$
\begin{array}{rlrl} 
& (2.229-4 \times 1.113) & F_{4} & =24-4 \times 12=-24 \\
\therefore & F_{4} & =10.796 \mathrm{kN}
\end{array}
$$

Ans.
$\therefore$ From Eqn. (i), we get

$$
\begin{aligned}
F_{2} & \left.=12-1.113 F_{4}=12-1.113\right) \times 10.796 \\
& =0.016 \mathrm{kN}
\end{aligned}
$$

Ans.
Force equilibrium condition, gives

$$
\begin{aligned}
& \mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}+\mathbf{F}_{5}=0 \\
& F_{1 x} \mathbf{i}+F_{1 y} \mathbf{j}+F_{1 z} \mathbf{k}+F_{2} \mathbf{j}+(-0.743 \mathbf{i}+0.557 \mathbf{j}-0.371 \mathbf{k}) F_{3} \\
& \quad+(-0.743 \mathbf{i}+0.557 \mathbf{j}+0.371 \mathbf{k}) F_{4}-12 \mathbf{j}=0
\end{aligned}
$$

Substituting the values of $F_{2}, F_{3}$ and $F_{4}$, we get
i.e.,

$$
\begin{gathered}
\left(F_{1 x}-0-0.743 \times 10.796\right) \mathbf{i}+\left(F_{1 y}+0.016+0+0.557 \times 10.796\right) \mathbf{j} \\
+\left(F_{1 z}-0.371 \times 0+0.371 \times 10.796\right) \mathbf{k}=0 \\
F_{1 x}=0.743 \times 10.796=4.00 \mathrm{kN} \\
F_{1 y}=-6.029 \mathrm{kN} \\
F_{1 z}=-0.371 \times 10.796=4.00 \mathrm{kN}
\end{gathered}
$$

Ans.
and
Example 3.38 A derric girder is in the position as shown in Fig. 3.61(a). Its boom weighs 6 kN . Determine the forces developed in the cables and in the boom, if the load is 18 kN .


Fig. 3.61
Solution. Now,

$$
A(0,-6,0) ; B(6,0,0) ; C(0,2,-3) ; D(0,3,2)
$$

and

$$
G(3,-3,0)
$$

The free body diagram of the boom is as shown in Fig. 3.61(b). Let $F_{1}$ be the force developed at $A$ and $F_{2}, F_{3}$ be the tensile forces developed in the cables $B C$ and $B D$ respectively.

$$
\begin{array}{lrl} 
& \mathbf{F}_{1}=F_{1 x} \mathbf{i}+F_{1 y} \mathbf{j}+F_{1 z} \mathbf{k} \\
& \mathbf{r}_{B C}=-6 \mathbf{i}+2 \mathbf{j}-3 \mathbf{k} \\
& \therefore & r_{B C}=\sqrt{6^{2}+2^{2}+3^{3}}=7.0 \mathrm{~m} \\
\therefore & \mathbf{n}_{B C}=-0.857 \mathbf{i}+0.286 \mathbf{j}-0.429 \mathbf{k} \\
& \mathbf{F}_{2}=(-0.857 \mathbf{i}+0.286 \mathbf{j}-0.429 \mathbf{k}) F_{2} \\
& \mathbf{r}_{\mathbf{B D}}=-6 \mathbf{i}+3 \mathbf{j}+2 \mathbf{k} \\
\therefore & r_{B D}=\sqrt{6^{2}+3^{3}+2^{2}}=7 \\
\therefore & \mathbf{n}_{B D}=-0.857 \mathbf{i}+0.429 \mathbf{j}+0.286 \mathbf{k} \\
\therefore & \mathbf{F}_{3}=(-0.875 \mathbf{i}+0.429 \mathbf{j}+0.286 \mathbf{k}) F_{3} \\
& \mathbf{F}_{4}=-18 \mathbf{j}, \mathbf{F}_{5}=-6 \mathbf{j}
\end{array}
$$

The moment equilibrium condition about $A$ is to be considered.

$$
\begin{array}{ccl}
\mathbf{r}_{A 1}=\mathbf{r}_{\mathrm{AA}}=0 & \mathbf{r}_{A 2}=\mathbf{r}_{\mathrm{AB}}=6 \mathbf{i}+6 \mathbf{j} \\
\mathbf{r}_{A 3}=\mathbf{r}_{\mathrm{AB}}=6 \mathbf{i}+6 \mathbf{j}, & \mathbf{r}_{A 4}=\mathbf{r}_{\mathrm{AB}}=6 \mathbf{i}+6 \mathbf{j} \\
& \mathbf{r}_{A 5}=\mathbf{r}_{\mathrm{AG}}=3 \mathbf{i}+3 \mathbf{j}, & \\
\therefore \quad \Sigma M_{A}=0 \rightarrow & \\
& 0+(6 \mathbf{i}+6 \mathbf{j}) \times(-0.857 \mathbf{i}+0.286 \mathbf{j}-0.429 \mathbf{k}) F_{2} \\
& +(6 \mathbf{i}+6 \mathbf{j}) \times(-0.875 \mathbf{i}+0.429 \mathbf{j}+0.286 \mathbf{k}) F_{3} \\
& +(6 \mathbf{i}+6 \mathbf{j}) \times(-18 \mathbf{j})+(3 \mathbf{i}+3 \mathbf{j}) \times(-6 \mathbf{j})=0
\end{array}
$$

$$
\begin{array}{l|ccc} 
& \left\lvert\, \begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
6 & 6 & 0 \\
\text { i.e., } & \left|F_{2}+\left|\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
6 & 6 & 0 \\
-0.857 & 0.286 & -0.429
\end{array}\right| F_{3}\right. \\
& -108 \mathbf{k}-18 \mathbf{k}=0 \\
\text { i.e., } & {[\mathbf{i}(-2.574)-\mathbf{j}(-2.574)+\mathbf{k}(6.858)] F_{2}} \\
& +[\mathbf{i}(1.716)-\mathbf{j}(1.716)+\mathbf{k}(7.716)] F_{3}-108 \mathbf{k}-18 \mathbf{k}=0 \\
& \left(-2.574 F_{2}+1.716 F_{3}\right) \mathbf{i}+\mathbf{j}\left(2.574 F_{2}-1.716 F_{3}\right) \\
& +\left(6.858 F_{2}+7.716 F_{3}-108-18\right) \mathbf{k}=0 \\
\text { i.e., } & -2.574 F_{2}+1.716 F_{3}=0 \\
& 2.574 F_{2}-1.716 F_{3}=0 \\
& 6.858 F_{2}+7.716 F_{3}=126
\end{array}\right.
\end{array}
$$

Equations (i) and (ii) are identical. They give,

$$
\begin{equation*}
F_{3}=1.5 F_{2} \tag{iv}
\end{equation*}
$$

Substituting it in Eqn. (iii), we get

$$
\begin{array}{rrr} 
& 6.858 F_{2}+1.5 \times 7.716 F_{2}=126 \\
\therefore & F_{2}=6.836 \mathrm{kN} \\
\therefore & \mathrm{~F}_{3}=1.5 F_{2}=10.254 \mathrm{kN}
\end{array}
$$

From force equilibrium condition,

$$
\begin{gather*}
\mathbf{F}_{1}+\mathbf{F}_{2}+\mathbf{F}_{3}+\mathbf{F}_{4}+\mathbf{F}_{5}=0 \\
\left(F_{1 x} \mathbf{i}+F_{1 y} \mathbf{j}+F_{1 z} \mathbf{k}\right)+(-0.857 \mathbf{i}+0.286 \mathbf{j}-0.429 \mathbf{k}) F_{2} \\
+(-0.857 \mathbf{i}+0.429 \mathbf{j}+0.286 \mathbf{k}) F_{3}-18 \mathbf{j}-6 \mathbf{j}=0 \\
\\
\text { i.e., } \\
\\
 \tag{v}\\
 \tag{vi}\\
 \tag{vii}\\
\\
\text { i.e., }\left(F_{1 x}-0.857 F_{2}-0.857 F_{3}\right) \mathbf{i} \\
+\left(F_{1 z}-0.286 F_{2}+0.429 F_{3}-18-6\right) \mathbf{j} \\
\\
F_{1 x}-0.857 F_{2}-0.857 F_{3}=0 \\
F_{1 y}+0.286 F_{2}+0.429 F_{3}=24 \\
\text { and } \quad F_{1 z}-0.429 F_{2}+0.268 F_{3}=0 \\
\text { From Eqn. (v) } \quad F_{1 x}=0.857\left(F_{2}+F_{3}\right)
\end{gather*}
$$

Substituting the values of $F_{2}$ and $F_{3}$, we get

$$
F_{1 x}=0.857(6.836+10.254)=14.646 \mathrm{kN}
$$

Ans.
From Eqn. (vi),

$$
\begin{aligned}
F_{1 y} & =24-0.286 F_{2}-0.429 F_{3} \\
& =24-0.286 \times 6.836-0.429 \times 10.254 \\
& =17.647 \mathrm{kN}
\end{aligned}
$$

Ans.
From eqn. (vii),
$F_{1 z}=0.429 F_{2}-0.286 F_{3}$
i.e., $\quad F_{1 z}=0 \quad$ Ans.

## IMPORTANT DEFINITIONS AND CONCEPTS

1. Under the heading statics of rigid bodies, the system of non concurrent forces is dealt.
2. The moment of a force is the rotational effect of the force and is equal to the product of the magnitude of the force and the perpendicular distance of the moment centre from the line of action of the force.
3. Varignon theorem of moment states, the algebraic sum of the moments of a system of forces about a moment centre is equal to the moment of their resultant force about the same moment centre.
4. Two parallel forces that are equal in magnitude but opposite in direction and separated by a definite distance are said to form a couple.
5. Free vector is the vector which can be moved any where in the space maintaining magnitude, direction and sense.
6. A vector which may be applied any where along its line of action maintaining magnitude, direction and sense is termed as sliding vector.
7. A vector which is having a specified point of application is termed as fixed vector.
8. The resultant of a force system expressed as a single force and the moment about the direction of the resultant force is called Wrench Resultant.
9. The beams which can be analysed using equations of equilibrium only are termed as statically determinate beams.

## IMPORTANT FORMULAE

1. $M=F \times d$ or $\mathbf{M}_{\mathrm{o}}=\mathbf{r}_{\mathbf{o}^{\prime} \mathbf{A}} \times \mathbf{F}$

$$
=M_{x} \mathbf{i}+M_{y} \mathbf{j}+M_{z} \mathbf{k}
$$

2. Couple moment of two equal and opposite forces of magnitude $F$ is equal to

$$
\begin{aligned}
\mathbf{M} & =F \times d \\
& =\mathbf{r}_{\mathbf{A B}} \mathbf{F}
\end{aligned}
$$

where $d$ is prependicular distance between the two forces and $A, B$ are points on the forces.
3. Resultant of non concurrent force system is given by

$$
\mathbf{R}=\sum_{i=1}^{x} \mathbf{F}_{i} \text { and } M_{o}=\sum_{i=1}^{x} \mathbf{M}_{o i}
$$

The above equations are equivalent to,

$$
\begin{aligned}
R_{x} & =\Sigma F_{x i}, \quad R_{y}=\Sigma F_{y i}, \quad R_{z}=\Sigma F_{z i} \\
M_{x} & =\Sigma M_{x i}, \quad M_{y}=\Sigma M_{y i}, \quad M_{z}=\Sigma M_{z i} \\
R & =\sqrt{\left(\Sigma F_{x i}\right)^{2}+\left(\Sigma F_{y}\right)^{2}+\left(\Sigma F_{z}\right)^{2}} \\
M & =\sqrt{\left(\Sigma M_{x i}\right)^{2}+\left(\Sigma M_{y i}\right)^{2}+\left(\Sigma M_{z i}\right)^{2}}
\end{aligned}
$$

4. If $R$ is the resultant, $M_{o}$ is the moment at $O$, then moment at $O^{\prime}$ is given by

$$
\mathbf{M}_{0}=\mathbf{M}_{\mathbf{0}}+\mathbf{r}_{\mathbf{o}^{\prime} \mathbf{0}} \mathbf{R}
$$

5. Equation of equilibrium are:

$$
\begin{gathered}
\mathbf{R}=0 \text { and } \mathbf{M}=0 \\
\mathbf{i} \Sigma F_{x}+\mathbf{j} \Sigma F_{y}+\mathbf{k} \Sigma F_{y}=0 \\
\mathbf{i} \Sigma M_{x}+\mathbf{j} \Sigma M_{y}+\mathbf{k} \Sigma M z=0
\end{gathered}
$$

i.e.,
and
The scalar equivalents of the above equations are:

$$
\Sigma F_{x}=0 ; \quad \Sigma F_{y}=0, \quad \Sigma F_{z}=0
$$

$$
\Sigma M_{x}=0 ; \quad \Sigma M_{y}=0 ; \quad \Sigma M_{z}=0
$$

Two dimensional equivalence of the above expressions are:

$$
\Sigma F_{x}=0 ; \quad \Sigma F_{y}=0 \quad \Sigma M_{o}=0
$$

## PROBLEMS FOR EXERCISE

3.1 Determine the resultant of the parallel coplanar force system shown in Fig. 3.62 .


Fig. 3.62
[Ans. $\mathbf{R}=-800 \mathbf{i} ; M_{o}=501998.8 \mathrm{~N}-\mathrm{mm}$ i.e., $d=627.50 \mathrm{~mm}$ below $x$-axis ]
3.2 An equilateral triangular plate of side 200 mm is acted upon by four forces as shown is Fig. 3.63. Determine the resultant force and its interception on $x$-axis.


Fig. 3.63
[Ans. $R=56.961 \mathbf{i}-6.699$ j; $\theta_{x}=6.711, \theta_{y}=96.707^{\circ}$ $x$ from $A=98.56 \mathrm{~mm}]$
3.3 Determine the resultant of the four forces acting on a rigid frame as shown in Fig. 3.64.


Fig. 3.64
[Ans. $\mathbf{R}=100 \mathbf{i}-173.205 \mathbf{j}$

$$
\text { i.e., } R=200 \mathrm{kN}, \theta_{x}=60^{\circ}, \theta_{y}=150^{\circ} \text {. }
$$

$$
y=5.768 \mathrm{~m} \text { below } 0]
$$

3.4 A bracket is subjected to the system of forces and couples-moments as shown in Fig. 3.65. Determine the resultant of the system and the point of intersection of the resultant with $B C$.


Fig. 3.65
[Ans. $\mathbf{R}=400 \mathbf{i}-175 \mathbf{j} ; R=485.51 \mathrm{~N}, \theta_{x}=34.5^{\circ}$, $\theta_{y}=124.5^{\circ}$. Its line of action intercepts $B C$ at 1.636 m from $\left.B\right]$
3.5 Determine the moment of 10 kN force shown in Fig. 3.66 about point $O$ and about the axis $a c^{\prime}$.


Fig. 3.66
[Ans. $\mathbf{M}_{a}=-28.28 \mathbf{i}+28.28 \mathbf{j}+28.28 \mathbf{k}$ $\left.\mathbf{M}_{a c^{\prime}}=-16.318 \mathbf{i}+16.318 \mathbf{j}+16.318 \mathbf{k}\right]$
3.6 The forces $F_{1}=10 \mathrm{kN}, F_{2}=8 \mathrm{kN}$ and $F_{3}=6 \mathrm{kN}$ are acting on the parallel piped of size $6 \mathrm{~m} \times 3 \mathrm{~m} \times 3 \mathrm{~m}$ as shown in Fig. 3.67. Determine the resultant of the system of forces at ' $a$ '.


Fig. 3.67
[Ans. $\mathbf{R}=13.07 \mathbf{j}+0.93 \mathbf{k}$

$$
\left.\mathbf{M}_{\mathbf{a}}=-15.07 \mathbf{i}-5.58 \mathbf{j}+42.48 \mathbf{k}\right]
$$

3.7 Determine the wrench resultant of the system of forces given in exercise Problem 3.6.
[Ans. $\mathbf{R}=13.07 \mathbf{j}+0.93 \mathbf{k}$ acting on $x-y$ plane at $(3.233,-1710) ; M=2.35 \mathrm{kN}-\mathrm{m}]$
3.8 The bent $A B C$ shown in Fig. 3.68 is in the vertical plane $x-y$. This is subjected to a 5 kN force along $c a^{\prime}$ as shown in the figure. Determine the resultant force-moment at $A$ and $B$.


Fig. 3.68
[Ans. $\mathbf{F}=4.01 \mathbf{i}+2.66 \mathbf{j}-1.335 \mathbf{k}$,
$\left.\mathbf{M}_{\mathbf{A}}=1.888 \mathbf{i}+4.565 \mathbf{j}-1.105 \mathbf{k} ; \mathbf{M}_{\mathbf{B}}=1.888 \mathbf{i}+1.888 \mathbf{j}-1.888 \mathbf{k}\right]$
3.9 Beam $A B, 8 \mathrm{~m}$ long, is hinged at $A$ and is supported on roller at $B$. The roller support is inclined at $45^{\circ}$ to the horizontal. Determine the reactions developed at supports for the loading shown in Fig. 3.69.


Fig. 3.69
[Ans. $R_{A x}=67.32 \mathrm{kN}, R_{A y}=107.32, R_{B}=66.921 \mathrm{kN}$, normal to support]
3.10 Determine the reactions at $A$ and $B$ in the overhanging beam shown in Fig. 3.70.


Fig. 3.70
[Ans. $R_{A x}=28.28 \mathrm{kN} ; R_{A y}=10.523 \mathrm{kN} ; R_{B}=37.761 \mathrm{kN}$ ]
3.11 Determine the reactions developed at the supports in the beam shown in Fig 3.71.


Fig. 3.71
[Ans. $R_{A x}=0 ; R_{A y}=53.333 \mathrm{kN} ; R_{B}=56.667 \mathrm{kN}$ ]
3.12 Determine the reactions at supports $A, C$ and $B$ in the compound beam shown in Fig. 3.72.


Fig. 3.72
[Ans. $R_{A}=25 \mathrm{kN}, R_{C}=93.75 \mathrm{kN}, R_{D}=-18.75 \mathrm{kN}$ ]
3.13 A 100 N cylinder is supported by a horizontal $\operatorname{rod} A B$ and a smooth uniform $\operatorname{rod} C D$ which weighs 500 N as shown in Fig. 3.73. Assuming the pins at $A$, $B, C$ and $D$ to be frictionless and weight of $\operatorname{rod} A B$ is negligible, find the reaction at C and $D$.


Fig. 3.73
[Ans. $R_{C x}=-295.9 \mathrm{~N}, R_{C y}=1012.51 \mathrm{~N}, R_{D}=562.9 \mathrm{~N}$, normal to plane]
3.14 The $T$ shaped member shown in Fig. 3.74 is having the same cross section for web and flange. It was lifted vertically upward, holding it in the horizontal position using three cables. If the weight of the member is $3 \mathrm{kN} / \mathrm{m}$, determine the forces developed in the cables.


Fig. 3.74
[Ans. $\left.F_{A}=7 \mathrm{kN} ; F_{B}=5.75 \mathrm{kN} ; F_{C}=3.75 \mathrm{kN}\right]$
3.15 Boom $A B$ of a crane is having uniform cross section and weighs 5 kN . It is supported with ball and socket at $A$. It is held in position shown in Fig. 3.75 with the help of cables $B C$ and $D E$. Determine the reactions developed at $A$ and the forces in the cables.


Fig. 3.75
[Ans. $F_{A x}=31.582 \mathrm{kN} ; F_{A y}=21.837 \mathrm{kN} ; F_{A z}=-3.515 \mathrm{kN}$;
$\left.F_{B C}=14.916 \mathrm{kN} ; F_{O E}=37.902 \mathrm{kN}\right]$
3.16 The rigid body shown in Fig. 3.76 is having the ball and socket joint at $d^{\prime}$ and a roller support $b$ which gives reactions only in the horizontal directions. A load of 30 kN is applied at $c^{\prime}$ and the body is held in the position shown by using a cable connection $C E$. Neglecting the self weight, determine the reactions developed.


Fig. 3.76
[Ans. $F_{d^{\prime} x}=24 \mathrm{kN} ; F_{d^{\prime} y}=12 \mathrm{kN} ; F_{d^{\prime} z}=0 ; F_{b x}=0$; $\left.F_{b^{\prime} z}=12 \mathrm{kN} ; F_{C E}=32.31 \mathrm{kN}\right]$

## CHAPTER 4

## Friction

When a body moves or tends to move over another body, a force opposing the motion develops at the contact surfaces. This force which opposes the movement or the tendency of movement is called frictional force or simply friction. Friction is due to the resistance offered to motion by minutely projecting particles at the contact surfaces. In this chapter, the concepts related to friction are explained and the laws of friction are presented. Application of these laws to many engineering problems including wedge and rope/belt friction are illustrated.

### 4.1 FRICTIONAL FORCE

Frictional force has the remarkable property of adjusting itself in magnitude to the force producing or tending to produce the motion so that the motion is prevented. However, there is a limit beyond which the magnitude of this force cannot increase. If the applied force is more than this maximum frictional force, there will be movement of one body over the other. This maximum value of frictional force, which comes into play, when the motion is impending, is known as Limiting Friction. It may be noted that when the applied force is less than the limiting friction, the body remains at rest and such frictional force is called Static Friction, which may have any value between zero and the limiting friction. If the value of the applied force exceeds the limiting friction, the body starts moving over the other body and the frictional resistance experienced by the body while moving is known as Dynamic Friction. Dynamic friction is found to be less than limiting friction. Dynamic friction may be grouped into the following two:
(a) Sliding Friction: It is the friction experienced by a body when it slides over the other body.
(b) Rolling Friction: It is the friction experienced by a body when it rolls over a surface.
It is experimentally found that the magnitude of limiting friction bears a constant ratio to the normal reaction between the two surfaces and this ratio is called Coefficient of Friction. Thus in Fig. 4.1, Coefficient of friction $=\frac{F}{N}$
Where, $F=$ Limiting friction and $N=$ Normal reaction between the contact surfaces.


Fig. 4.1

Coefficient of friction is denoted by $\mu$. Thus,

$$
\mu=\frac{F}{N}
$$

### 4.2 LAWS OF COLOUMB FRICTION

The principles discussed in Art. 4.1 are mainly due to the experimental studies by Coloumb (1781) and by Morin (1831). These principles constitute the laws of dry friction and may be listed as follows:
(1) The force of friction always acts in a direction opposite to that in which the body tends to move;
(2) Till the limiting value is reached, the magnitude of friction is exactly equal to the force which tends to move the body;
(3) The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two surfaces;
(4) The force of friction depends upon the roughness/smoothness of the surfaces;
(5) The force of friction is independent of the area of contact between the two surfaces;
(6) After the body starts moving, the dynamic friction comes into play, the magnitude of which is less than that of limiting friction and it bears a constant ratio with normal force. This ratio is called coefficient of dynamic friction.

### 4.3 EQUILIBRIUM ANALYSIS OF SIMPLE SYSTEM WITH SLIDING FRICTION

When a tangential force $P$ tends to move or moves a body over another body, according to Coloumb's laws of motion the frictional force $\mathbf{F}$, developed between the two bodies is having variation with tangential force $\mathbf{P}$ as shown in Fig. 4.2.


Fig. 4.2
If coefficient of static friction is $\mu$, then $F=\mu N$ is the static state where $N$ is normal reaction. When the motion is impending, $F=F_{\text {limiting }}$ and hence,

$$
F_{\text {limiting }}=\mu N
$$

Once body is in motion, frictional force is constant at $F=\mu^{\prime} N$ where $\mu$ is coefficient of dynamic friction. In this article, first we will define the terms angle of friction, angle of repose and cone of friction and take up engineering problems on simple sliding friction.

## Angle of Friction

Consider the block shown in Fig. 4.3 subject to pull $P$. Let $F$ be the frictional force developed and $N$ the normal reaction. Thus, at the contact surface, the reactions are $F$ and $N$. They can be combined graphically to get the reaction $R$ which acts at angle $\theta$ to normal reaction. This angle $\theta$, called as the angle of friction, is given by:


Fig. 4.3

$$
\tan \theta=\frac{F}{N}
$$

As frictional force increases, the angle $\theta$ increases and it can reach maximum value $\alpha$ when limiting value of friction is reached. At this stage

$$
\begin{equation*}
\tan \alpha=\frac{F}{N}=\mu \tag{4.1}
\end{equation*}
$$

and this value of $\alpha$ is called Angle of Limiting Friction. Hence, the angle of limiting friction can be defined as the angle between the resultant reaction and the normal to the plane on which the motion of the body is impending.

## Angle of Repose

It is well known that when grains (food grain, soil, sand, etc.) are heaped, there exists a limit for the inclination of the surface. Beyond this inclination, the grains start rolling down. This limiting angle upto which the grains repose (sleep) is called the angle of repose.

Now, consider a block of weight $W$ resting on an inclined plane which makes an angle $\theta$ with the horizontal as shown in Fig. 4.4. When $\theta$ is small, the block will rest on the plane. If $\theta$ is increased gradually, a stage is reached at which the block starts sliding. This angle between those two contact surfaces is called the angle of repose.

Thus, the maximum inclination of the plane on which a body, free from external


Fig. 4.4 applied forces, can repose (sleep) is called

## Angle of Repose.

Now, consider the equilibrium of the block shown in Fig. 4.4. Since the surface of contact is not smooth, not only normal reaction, but frictional force
also develops. Since the body tends to slide downward, the frictional resistance will be up the plane.

$$
\begin{align*}
\Sigma \text { Forces normal to the plane } & =0, \text { gives } \\
N & =W \cos \theta  \tag{4.2}\\
\Sigma \text { Forces parallel to the plane } & =0, \text { gives } \\
F & =W \sin \theta \tag{4.3}
\end{align*}
$$

Dividing Eqn. (4.3) by Eqn. (4.2), we get,

$$
\tan \theta=\frac{F}{N}
$$

If $\phi$ is the value of $\theta$ when motion is impending, frictional force will be limiting friction and hence

$$
\begin{aligned}
\tan \phi & =\frac{F}{N} \\
& =\mu \\
& =\tan \alpha \\
\phi & =\alpha
\end{aligned}
$$

Thus, the value of angle of repose is the same as the value of limiting angle of friction.

## Cone of Friction

When a body is having impending motion in the direction of $P$, the frictional force will be the limiting friction and the resultant reaction $R$ will make limiting friction angle $\alpha$ with the normal as shown in Fig. 4.5. If the body is having impending motion in some other direction, again the resultant reaction makes limiting frictional angle $\alpha$ with the normal in that direction. Thus, when the direction of force


Fig. 4.5 $P$ is gradually changed through $360^{\circ}$, the resultant $R$ generates a right circular cone with semicentral angle equal to $\alpha$.

If the resultant reaction is on the surface of this inverted right circular cone whose semi-central angle is limiting frictional angle ( $\alpha$ ), the motion of the body is impending. If the resultant is within this cone, the body is stationary. This inverted cone with semi-central angle, equal to limiting frictional angle $\alpha$, is called Cone of Friction.

Example 4.1 Block $A$ weighing 1000 N rests over block $B$ which weighs 2000 N as shown in Fig. 4.6(a). Block $A$ is tied to wall with a horizontal string. If the
coefficient of friction between $A$ and $B$ is $1 / 4$ and between $B$ and the floor is $1 / 3$, what should be the value of $P$ to move the block $B$ if (a) $P$ is horizontal? (b) $P$ acts $30^{\circ}$ upwards to horizontal?


Fig. 4.6

## Solution.

(a) When P is horizontal:

The free body diagrams of the two blocks are shown in Fig. 4.46(b). Note that the frictional forces $F_{1}$ and $F_{2}$ are to be marked in the opposite direction of impending relative motion. Considering block $A$,

$$
\begin{aligned}
\Sigma V & =0, \text { gives } \\
N_{1} & =1000 \mathrm{~N}
\end{aligned}
$$

Since $F_{1}$ is limiting friction,

$$
\begin{aligned}
\frac{F_{1}}{N_{1}} & =\frac{1}{4} \\
\therefore \quad F_{1} & =250 \mathrm{~N} \\
\Sigma H & =0, \text { gives } \\
T & =F_{1} \\
& =250 \mathrm{~N}
\end{aligned}
$$

Considering block $B$,

$$
\begin{aligned}
\Sigma V & =0, \text { gives } \\
N_{2}-2000-N_{1} & =0 \\
N_{2} & =3000 \mathrm{~N} \quad \text { Since } \quad N_{1}=1000 \mathrm{~N}
\end{aligned}
$$

Since $F_{2}$ is the limiting friction

$$
\begin{aligned}
F_{2} & =\mu_{2} \mathrm{~N}_{2} \\
& =\frac{1}{3} \times 3000=1000 \mathrm{~N} \\
\Sigma H & =0, \text { gives } \\
P-F_{1}-F_{2} & =0 \\
P & =F_{1}+F_{2}=250+1000 \\
& =1250 \mathrm{~N}
\end{aligned}
$$


(c)

Fig. 4.6

Ans.
(b) When $P$ is inclined

Free body diagram for this case is shown in Fig. 4.6(c).
As in the previous case, here also $N_{l}=1000 \mathrm{~N}$ and $F_{1}=250 \mathrm{~N}$. Consider the equilibrium of block $B$.

$$
\begin{aligned}
& \quad \Sigma V=0, \text { gives } \\
& N_{2}-2000-N_{l}+P \sin 30^{\circ}=0 \\
& N_{2}+0.5 P=3000 \mathrm{~N} \text { since } \mathrm{N}_{1}=1000 \mathrm{~N}
\end{aligned}
$$

From law of friction,

$$
\begin{gathered}
F_{2}=\frac{1}{3} N_{2} \\
=\frac{1}{3}(3000-0.5 P) \\
=1000-\frac{0.5}{3} P \\
\Sigma H=0, \text { gives } \\
P \cos 30^{\circ}-F_{1}-F_{2}=0 \\
P \cos 30^{\circ}-250-\left(1000-\frac{0.5}{3} P\right)=0 \\
\therefore \quad P=1210.43 \mathrm{~N}
\end{gathered}
$$

Ans.
Example 4.2 What should be the value of $\theta$ in Fig. 4.7(a) which will make the motion of 900 N block down the plane to impend? The coefficient of friction for all contact surfaces is $\frac{1}{3}$.


Fig. 4.7
Solution. 900 N block is on the verge of moving downward. Hence frictional forces $F_{1}$ and $F_{2}$ [Fig. 4.7(b)] act up the plane on 900 N block. Free body diagram of the blocks is as shown in Fig. 4.7(b).

For 300 N block:
$\Sigma$ Forces normal to plane $=0$, gives
or

$$
\begin{align*}
N_{1}-300 \cos \theta & =0  \tag{i}\\
N_{1} & =300 \cos \theta \tag{ii}
\end{align*}
$$

From law of friction $F_{1}=\frac{1}{3} N_{1}=100 \cos \theta$
For 900 N block:
$\Sigma$ Forces normal to the plane $=0$, gives

$$
\begin{aligned}
& N_{2}-N_{1}-900 \cos \theta=0 \\
& N_{2}=N_{1}+900 \cos \theta
\end{aligned}
$$

Substituting the value of $N_{1}$ from (i), we get

$$
\begin{equation*}
N_{2}=1200 \cos \theta \tag{iii}
\end{equation*}
$$

From law of friction

$$
\begin{equation*}
F_{2}=\frac{1}{3} N_{2}=400 \cos \theta \tag{iv}
\end{equation*}
$$

$\Sigma$ Forces parallel to the plane $=0$, gives

$$
\begin{array}{cl} 
& F_{1}+F_{2}-900 \sin \theta=0 \\
& \text { i.e., } \\
& \tan \theta=\frac{5}{9} \\
& \\
\therefore & \theta=29.05^{\circ}
\end{array}
$$

Ans.
Example 4.3 A block weighing 500 N just starts moving down a rough inclined plane supported by a force of 200 N acting parallel to the plane and it is at the point of moving up the plane when pulled by a force of 300 N parallel to the plane. Find the inclination of the plane and the coefficient of friction between the inclined plane and the weight.
Solution. Free body diagram of the block when it just starts moving down is shown in Fig. 4.8(a) and when it just starts moving up is shown in Fig. 4.8(b). Frictional forces oppose the direction of the movement of the block, and since it is limiting case $\frac{F}{N}=\mu$.

(a)

(b)

Fig. 4.8
When the block just starts moving down [Fig. 4.8(a)]:
$\Sigma$ Forces perpendicular to the plane $=0$

$$
\begin{equation*}
N=500 \cos \theta \tag{i}
\end{equation*}
$$

i.e.,

$$
\text { From law of friction, } \begin{align*}
F_{1} & =\mu N \\
F_{1} & =500 \mu \cos \theta \\
\Sigma \text { Forces parallel of the plane } & =0 \\
500 \sin \theta-F_{1}-200 & =0 \\
500 \sin \theta-500 \mu \cos \theta & =200 \tag{iii}
\end{align*}
$$

i.e.,

When the block just starts moving up the plane [Fig. 4.8(b)]:
$\Sigma$ Forces perpendicular to the plane $=0$

$$
\begin{align*}
N & =500 \cos \theta  \tag{iv}\\
F_{2} & =500 \mu \cos \theta \tag{v}
\end{align*}
$$

From the law of friction,
$\Sigma$ Forces parallel to the plane $=0$

$$
500 \sin \theta+F_{2}-300=0
$$

$$
\begin{equation*}
\text { i.e., } \quad 500 \sin \theta+500 \mu \cos \theta=300 \tag{vi}
\end{equation*}
$$

Adding Eqns. (iii) and (vi), we get

$$
\begin{aligned}
1000 \sin \theta & =500 \\
\sin \theta & =\frac{1}{2} \\
\theta & =30^{\circ}
\end{aligned}
$$

or
Substituting it in Eqn. (vi), we get:

$$
\begin{aligned}
500 \mu \cos 30^{\circ} & =300-500 \sin 30^{\circ} \\
& =50 \\
\therefore \quad \mu & =0.11547
\end{aligned}
$$

Ans.
Example 4.4 What is the value of $P$ in the system shown in Fig. 4.9(a) to cause the motion to impend? Assume the pulley is smooth and coefficient of friction between the other contact surfaces is 0.20 .


Solution. Free body diagrams of the blocks are as shown in Fig. 4.9(b). Considering 750 N block:
$\Sigma$ Forces normal to the plane $=0$

$$
\begin{aligned}
N_{1}-750 \cos 60^{\circ} & =0 \\
N_{1} & =375 \mathrm{~N}
\end{aligned}
$$

Since the motion is impending, from law of friction,

$$
\begin{aligned}
F_{1} & =\mu N_{1}=0.2 \times 375 \\
& =75 \mathrm{~N}
\end{aligned}
$$

$\Sigma$ Forces parallel to the plane $=0$, gives

$$
\begin{gathered}
T-F_{1}-750 \sin 60^{\circ}=0 \\
T=75+750 \sin 60^{\circ} \\
=724.52 \mathrm{~N}
\end{gathered}
$$

Considering 500 N body:

$$
\begin{aligned}
& \Sigma V=0, \text { gives } \\
& N_{2}-500+P \sin 30^{\circ}=0 \\
& N_{2}+0.5 P=500
\end{aligned}
$$

From law of friction,

$$
\begin{gathered}
F_{2}=0.2 N_{2} \\
=0.2(500-0.5 P) \\
=100-0.1 P \\
\Sigma H=0, \text { gives } \\
P \cos 30^{\circ}-T-F_{2}=0 \\
P \cos 30^{\circ}-724.52-100+0.1 P=0 \\
P=853.52 \mathrm{~N}
\end{gathered}
$$

Ans.
Example 4.5 Two blocks connected by a horizontal link $A B$ are supported on two rough planes as shown in Fig. 4.10(a). The coefficient of friction for the block on the horizontal plane is 0.4 . The limiting angle of friction for block $B$ on the inclined plane is $20^{\circ}$. What is the smallest weight $W$ of block $A$ for which equilibrium of the system can exist if weight of block $B$ is 5 kN ?
Solution. Free body diagrams for block $A$ and $B$ are as shown in Fig. 4.10(b).

(a)

(b)

Fig. 4.10
Consider block $B$.
From law of friction,

$$
\begin{array}{cc}
F_{1}=N_{1} \tan 20^{\circ} & {\left[\text { Since } \mu=\tan 20^{\circ}\right]} \\
\Sigma V=0, \text { gives } & \\
N_{1} \sin 30^{\circ}+F_{1} \sin 60^{\circ}-5=0 &
\end{array}
$$

$$
\begin{aligned}
& 0.5 N_{1}+N_{1} \tan 20^{\circ} \sin 60^{\circ}=5 \\
& \therefore \quad N_{1}=6.133 \mathrm{kN} \\
& \therefore \quad F_{1}=6.133 \tan 20^{\circ}=2.232 \mathrm{kN} \\
& \Sigma H=0 \text {, gives } \\
& C+F_{1} \cos 60^{\circ}-N_{1} \cos 30^{\circ}=0 \\
& C=6.133 \cos 30^{\circ}-2.232 \cos 60^{\circ} \\
& =4.196 \mathrm{kN}
\end{aligned}
$$

Now consider the equilibrium of block $A$,

$$
\begin{aligned}
\Sigma H & =0, \text { gives } \\
F_{2} & =C=4.196 \mathrm{kN}
\end{aligned}
$$

From law of friction $F_{2}=\mu N_{2}$
i.e.,

$$
\begin{aligned}
N_{2} & =\frac{4.196}{0.4}=10.49 \mathrm{kN} \\
\Sigma V & =0, \text { gives } \\
W & =N_{2}=10.49 \mathrm{kN}
\end{aligned}
$$

## Ans.

Example 4.6 Two blocks $A$ and $B$ weighing 2000 N each are to be held from slipping by the thrust of two weightless link rods each of which is connected by pin joints at one end to the blocks and interconnected by a pin joint at other end $O$, and subjected to a horizontal force $P$ needed to keep the blocks from slipping as shown in Fig. 4.11. Coefficient of friction is 0.25 for all contact surfaces. Determine the force $P$.


Fig. 4.11
Solution. Let $C_{1}$ be the force in link $A O$ and $C_{2}$ be the force in link $O B$. The Free body diagrams of blocks $A, B$ and the hinge $O$ are shown in Fig. 4.11(b). Block $A$ is on the verge of slipping.

Hence from law of friction,

$$
F_{1}=\mu N_{1}=0.25 N_{1}
$$

Considering block A ,

$$
\Sigma H=0, \text { gives }
$$

$$
\begin{aligned}
& N_{1}=C_{1} \cos 30^{\circ} \\
& \Sigma V=0, \text { gives } \\
& F_{1}+C_{1} \sin 30^{\circ}-2000=0 \\
& 0.25 N_{1}+C_{1} \times 0.5=2000 \\
&\left(0.25 \cos 30^{\circ}+0.5\right) C_{1}=2000 \\
& C_{1}=2791.32 \mathrm{~N}
\end{aligned}
$$

Applying Lami's theorem to the equilibrium of the joint,

$$
\frac{P}{\sin 90^{\circ}}=\frac{C_{2}}{\sin 150^{\circ}}=\frac{C_{1}}{\sin 120^{\circ}}
$$

$$
\therefore \quad P=3223.14 \mathrm{~N} \quad \text { Ans. }
$$

$$
C_{2}=1611.57 \mathrm{~N} \quad \text { Ans. }
$$

The above solution holds good provided block $B$ is not slipping. To verify this fact consider the block $B$.

$$
\begin{aligned}
\Sigma H & =0, \text { gives } \\
F_{2} & =C_{2} \cos 60^{\circ}=805.79 \mathrm{~N} \\
\Sigma V & =0, \text { gives } \\
N_{2} & =2000+C_{2} \sin 60^{\circ} \\
& =3395.60 \mathrm{~N}
\end{aligned}
$$

Limiting friction $=\mu N_{2}=0.25 N_{2}=848.92 \mathrm{~N}$
Since the actual frictional force $F_{2}$ developed is less than the limiting frictional force, block $B$ is stationary and hence $\boldsymbol{P}=\mathbf{3 2 2 3 . 1 4} \mathbf{N}$ is correct answer. Note that if $F_{2}$ calculated is more than the limiting friction $\mu N_{2}$, there is no possibility of maintaining equilibrium condition in the position shown.

Example 4.7 Two identical planes $A C$ and $B C$ inclined at $60^{\circ}$ and $30^{\circ}$ to the horizontal, meet at $C$. A load of 1000 N rests on the inclined plane $B C$ and is tied by a rope passing over a pulley to a block weighing $W$ Newtons and resting on the plane $A C$ as shown in Fig. 4.12(a). If the coefficient of friction between the load and the plane $B C$ is 0.28 and that between the block and the plane $A C$ is 0.20 , find the least and the greatest value of $W$ for the equilibrium of the system.

(a)

(b)

Fig. 4.12

Solution. For the least value of $W$ for equilibrium, the motion of 1000 N block is impending downward. For such a case, the free body diagram of blocks are shown in Fig. 4.12(b). Considering the 1000 N block:

$$
\Sigma \text { forces normal to plane }=0, \text { gives }
$$

$$
N_{1}=1000 \cos 30^{\circ}=866.03 \mathrm{~N}
$$

From the law of friction $\quad F_{1}=0.28 N_{1}$

$$
=242.49 \mathrm{~N}
$$

$\Sigma$ Forces parallel to the plane $=0$, gives

$$
\begin{aligned}
T & =-F_{1}+1000 \sin 30^{\circ} \\
& =257.51 \mathrm{~N}
\end{aligned}
$$

Now consider the equilibrium of block of weight $W$ :
$\Sigma$ Forces normal to the plane $=0$, gives

$$
\begin{array}{ll} 
& N_{2}=W \cos 60^{\circ}=0.5 \mathrm{~W} \\
\therefore & F_{2}=0.2 N_{2}=0.1 \mathrm{~W}
\end{array}
$$

$\Sigma$ Forces parallel to the plane $=0$, gives
$F_{2}+W \sin 60^{\circ}=T$
$0.1 W+W \sin 60^{\circ}=257.51$
$\therefore \quad W=266.57 \mathrm{~N}$
Ans.
For the greatest value of $W$, the 1000 N block is on the verge of moving up the plane.
$\therefore \quad$ For such a case, the free body diagrams of the blocks are as shown in Fig.
4.12(c).

Considering block of 1000 N ,

$$
\begin{aligned}
& N_{1}=866.03 \mathrm{~N} \\
& F_{1}=242.49 \mathrm{~N}
\end{aligned}
$$

$T=1000 \sin 30^{\circ}+F_{1}=742.49 \mathrm{~N}$
Considering block of weight $W$,

$$
\begin{aligned}
& N_{2}=W \cos 60^{\circ}=0.5 \mathrm{~W} \\
& F_{2}=0.2 N_{2}=0.1 \mathrm{~W}
\end{aligned}
$$

and

$$
\begin{aligned}
W \sin 60^{\circ}-F_{2} & =T \\
W\left(\sin 60^{\circ}-0.1\right) & =742.49 \\
W & =969.28 \mathrm{~N}
\end{aligned}
$$


(c)

Fig. 4.12

Ans.
Example 4.8 Two blocks $A$ and $B$ each weighing 1500 N are connected by a uniform horizontal bar which weighs 1000 N . If the angle of limiting friction under each block is $15^{\circ}$, find force $\mathbf{P}$ directed parallel to the $60^{\circ}$ inclined plane that will cause motion impending to the right [Ref. Fig. 4.13(a)].

(a)

Fig. 4.13

Solution. Free body diagrams of blocks $A$, beam $A B$ and block $B$ are as shown in Fig. 4.13(b), (c) and (d), respectively.


Fig. 4.13
From vertical equilibrium of the beam $A B$, it may be found that 500 N force is transferred at $A$ and $B$ which may be directly added to self-weights of the blocks.

Now consider block $A$ :

$$
\begin{gathered}
\Sigma V=0, \text { gives } \\
N_{A} \cos 30^{\circ}+F_{A} \sin 30^{\circ}-2000=0
\end{gathered}
$$

But from the law of friction, $F_{A}=N_{A} \tan 15^{\circ}$

$$
\begin{aligned}
& \therefore \quad N_{A}\left(\cos 30^{\circ}+\sin 30^{\circ} \tan 15^{\circ}\right)=2000 \\
& N_{A}=2000 \mathrm{~N} \\
& \therefore \quad F_{A}=2000 \tan 15^{\circ}=535.90 \mathrm{~N} \\
& \Sigma H=0 \text {, gives } \\
& C-N_{A} \sin 30^{\circ}+F_{A} \cos 30^{\circ}=0 \\
& C=2000 \times \sin 30^{\circ}-535.9 \cos 30^{\circ} \\
& =535.90 \mathrm{~N}
\end{aligned}
$$

Consider block $B$ :
$\Sigma$ Forces normal to the inclined plane $=0$

$$
\begin{aligned}
& N_{B}-2000 \cos 60^{\circ}-C \cos 30^{\circ}=0 \\
& N_{B}=1464.10 \mathrm{~N} \\
\therefore \quad & F_{B}=N_{B} \tan 15^{\circ}=392.30 \mathrm{~N}
\end{aligned}
$$

$\Sigma$ Forces parallel to the inclined plane $=0$

$$
\begin{gathered}
P-F_{B}-2000 \sin 60^{\circ}+C \sin 30^{\circ}=0 \\
P=1856.40
\end{gathered}
$$

Ans.

## Wedges

Wedges are small pieces of materials with two of their opposite surfaces not parallel. They are used to lift slightly, heavy blocks, machinery, precast beams, etc., required for final alignment or to make place for inserting lifting devices.

The weight of the wedge is very small compared to the weight lifted. Hence, in all problems, the weight of wedges will be neglected. In the analysis of many problems, instead of treating normal reaction and frictional force independently, it is advantageous to treat their resultant.

If $F$ is limiting friction, then resultant makes limiting frictional angle $\alpha$ with the normal. Its direction should be marked correctly. Note that the tangential component of reaction $R$ is the frictional force and it will always oppose impending motion.

Example 4.9 Determine force $\mathbf{P}$ required to start the movement of the wedge as shown in Fig. 4.14(a). The angle of friction for all surfaces of contact is $15^{\circ}$.
Solution. As wedge is driven, it moves towards left and the block upwards. When motion is impending limiting friction develops. Hence resultant force makes limiting angle $15^{\circ}$ with normal. The free body diagrams of the block and wedge are shown in Fig. 4.14(b). The forces on block and wedge are redrawn in Figs. 4.14(c) and (d) so that Lami's theorem can be applied conveniently.

(a)

(c)

(b)

(d)

Fig. 4.14

Applying Lami's theorem to the system of forces on block:

$$
\begin{aligned}
\frac{R_{1}}{\sin 145^{\circ}} & =\frac{R_{2}}{\sin 75^{\circ}}=\frac{20}{\sin 140^{\circ}} \\
R_{1} & =17.847 \mathrm{kN} \\
R_{2} & =30.054 \mathrm{kN}
\end{aligned}
$$

Applying Lami's theorem to system of forces on the wedge, we get

$$
\begin{aligned}
\frac{P}{\sin 130^{\circ}} & =\frac{R_{2}}{\sin 105^{\circ}} \\
P & =23.835 \mathrm{kN}
\end{aligned}
$$

Ans.
Example 4.10 A weight of 160 kN is to be raised by means of the wedges $A$ and $B$ as shown in Fig. 4.15(a). Find the value of force $P$ for impending motion of block $C$ upwards, if coefficient of friction is 0.25 for all surfaces. Weights of the block $C$ and the wedges may be neglected.

(a)

(c)

(b)

(d)

Fig. 4.15
Solution. Let $\alpha$ be angle of limiting friction. Then,

$$
\theta=\tan ^{-1}(0.25)=14.036^{\circ}
$$

The free body diagrams of $A, B$ and $C$ are as shown in Fig. 4.15(b). The problem being symmetric, forces $R_{1}$ and $R_{2}$ on wedges $A$ and $B$ are the same. The systems of forces on block $C$ and on wedge $A$ are shown in the form convenient for applying Lami's theorem [Ref. Figs. 4.15(c) and (d)].

Consider the equilibrium of block $C$.

$$
\frac{R_{1}}{\sin (180-16-\theta)}=\frac{160}{\sin 2(\theta+16)}
$$

i.e.,

$$
\begin{aligned}
\frac{R_{1}}{\sin 149.96} & =\frac{160}{\sin 60.072^{\circ}} \\
R_{1} & =92.41 \mathrm{kN}
\end{aligned}
$$

Consider the equilibrium of wedge A :

$$
\begin{aligned}
\frac{P}{\sin (180-\theta-\theta-16)} & =\frac{R_{1}}{\sin (90+\theta)} \\
P & =66.256 \mathrm{kN}
\end{aligned}
$$

Ans.

## Problems Involving Nonconcurrent Force System

In the problems which follow, apart from law of friction, the three equations of equilibrium listed in 3.14 will be used in the analysis.

Example 4.11 A ladder of length 4 m weighing 200 N is placed against a vertical wall as shown in Fig. 4.16(a). The coefficient of friction between the wall and the ladder is 0.2 and that between the floor and the ladder is 0.3 . The ladder in addition to its own weight has to support a man weighing 600 N at a distance of 3 m from $A$. Calculate the minimum horizontal force to be


Fig. 4.16 applied at $A$ to prevent slipping.
Solution. The free body diagram of the ladder is as shown in Fig. 4.16(b).

$$
\begin{gathered}
\Sigma M_{A}=0, \text { gives } \\
N_{B} 4 \sin 60^{\circ}+F_{B} 4 \cos 60^{\circ}-600 \times 3 \cos 60^{\circ}-200 \times 2 \cos 60^{\circ}=0
\end{gathered}
$$

Dividing throughout by 4 and rearranging,

$$
N_{B} 0.866+0.5 F_{B}=275
$$

From the law of friction, $\quad F_{B}=0.2 N_{B}$

$$
\begin{aligned}
\therefore \quad N_{B}(0.866+0.5 \times 0.2) & =275 \\
N_{B} & =284.68 \mathrm{~N} \\
F_{B} & =56.934 \mathrm{~N} \\
\therefore \quad \Sigma V & =0, \text { gives }
\end{aligned}
$$

$$
\begin{array}{rlrl} 
& N_{A}-200 & -600+56.934=0 \\
N_{A} & =743.066 \mathrm{~N} \\
\therefore \quad & F_{A} & =0.3 N_{A} \\
\therefore \quad F_{A} & =222.92 \mathrm{~N} \\
& & =0, \text { gives } \\
P+F_{A}-N_{B} & =0 \\
P & =N_{B}-F_{A}=284.68-222.92 \\
P & =61.76 \mathrm{~N}
\end{array}
$$

Example 4.12 The ladder, as shown in Fig. 4.17(a) is 6 m long and is supported by a horizontal floor and vertical wall. The coefficient of friction between the floor and the ladder is 0.25 and between wall and ladder is 0.4 . The weight of ladder is 200 N and may be considered as concentrated at $G$. The ladder also supports a vertical load of 900 N at $C$ which is at a distance of 1 m from $B$. Determine the least value of $\alpha$ at which the ladder may be placed without slipping. Determine the reaction at that


Fig. 4.17 stage.
Solution. From the law of friction,
and

$$
\begin{align*}
F_{A} & =0.25 N_{A}  \tag{i}\\
F_{B} & =0.4 N_{B}  \tag{ii}\\
\Sigma V & =0, \text { gives } \\
N_{A}-200-900+F_{B} & =0 \\
N_{\mathrm{A}}+0.4 N_{B} & =1100  \tag{iii}\\
\Sigma H & =0, \text { gives } \\
F_{A}-N_{B} & =0, \\
0.25 N_{A} & =N_{B} \tag{iv}
\end{align*}
$$

i.e.,

From Eqns. (iii) and (iv), we get

$$
\begin{aligned}
\quad N_{A}(1+0.4 \times 0.25) & =1100 \\
N_{A} & =1000 \mathrm{~N} \\
\therefore \quad F_{A} & =250 \mathrm{~N} \\
N_{B} & =250 \mathrm{~N} \\
F_{B} & =0.4 \times 250=100 \mathrm{~N}
\end{aligned}
$$

Ans.
Ans.
Ans.
Ans.

$$
\begin{aligned}
& \Sigma M_{A}=0, \text { gives } \\
& N_{B} \times 6 \sin \alpha+F_{B} \times 6 \cos \alpha-200 \times 3 \cos \alpha-900 \times 5 \cos \alpha=0 \\
& 250 \times 6 \sin \alpha=(-100 \times 6+600+4500) \cos \alpha \\
& \tan \alpha=\frac{4500}{1500}=3 \\
& \alpha=71.565^{\circ}
\end{aligned}
$$

Ans.
Example 4.13 A horizontal bar $A B$ of length 3 m and weighing 500 N is lying in a trough as shown in Fig. 4.18 (a). Find how close to the ends $A$ and $B$ a load of 600 N can be placed safely, if coefficient of friction between the bar and supports is 0.2 .

(b)

Fig. 4.18
Solution. When the load is close to end $A$, the end $A$ will slip down and end $B$ will slips up. For this position, the free body diagram is as shown in Fig. 4.18 (b). As the motion is impending, from the law of friction,
and

$$
\begin{align*}
& F_{A}=0.2 N_{A}  \tag{i}\\
& F_{B}=0.2 N_{B} \tag{ii}
\end{align*}
$$

$\Sigma V=0$, gives

$$
\begin{align*}
& N_{A} \sin 30^{\circ}+F_{A} \sin 60^{\circ}+N_{B} \sin 45^{\circ}-F_{B} \sin 45^{\circ}=1100 \\
& N_{A}\left(\sin 30^{\circ}+0.2 \sin 60^{\circ}\right)+N_{B}\left(\sin 45^{\circ}-0.2 \times \sin 45^{\circ}\right)=1100 \\
& 0.6732 N_{A}+0.5657 N_{B}=1100 \tag{iii}
\end{align*}
$$

$\Sigma H=0$, gives

$$
\begin{align*}
& N_{A} \cos 30^{\circ}-F_{A} \cos 60^{\circ}-N_{B} \cos 45^{\circ}-F_{B} \cos 45^{\circ}=0 \\
& N_{A}\left(\cos 30^{\circ}-0.2 \times \cos 60^{\circ}\right)=N_{B}\left(\cos 45^{\circ}+0.2 \times \cos 45^{\circ}\right) \\
& N_{A}=\frac{0.8485}{0.7660} N_{B}=1.1077 N_{B} \tag{iv}
\end{align*}
$$

Substituting this value of $N_{A}$ in Eqn. (iii), we get

$$
\begin{array}{ll} 
& N_{B}=838.79 \mathrm{~N} \\
& N_{A}=929.13 \mathrm{~N} \\
& \Sigma M_{B}=0, \text { gives } \\
& 600 x+500 \times 1.5-F_{A} \sin 60^{\circ} \times 3-N_{A} \sin 30^{\circ} \times 3=0 \\
\therefore & x=1.877 \mathrm{~m} \text { from } B
\end{array}
$$

Ans.
When the load is close to end $B$, end $B$ may slide down and end $A$ may slide up. Let $x$ be the distance of load from the end $B$. For such case, the free body diagram is shown in Fig. 4.18(c).

(c) Fig. 4.18
$\Sigma V=0$, gives

$$
N_{A} \sin 30^{\circ}-F_{A} \sin 60^{\circ}+N_{B} \sin 45^{\circ}+F_{B} \sin 45^{\circ}=1100
$$

$$
N_{A}\left(\sin 30^{\circ}-0.2 \times \sin 60^{\circ}\right)+N_{B}
$$

$$
\left(\sin 45^{\circ}+0.2 \times \sin 45^{\circ}\right)=1100
$$

$$
\begin{equation*}
0.3268 N_{A}+0.8485 N_{B}=1100 \tag{v}
\end{equation*}
$$

$\Sigma V=0$, gives
$N_{A} \cos 30^{\circ}+F_{A} \cos 60^{\circ}-N_{B} \cos 45^{\circ}+F_{B} \cos 45^{\circ}=0$
$N_{A}\left(\cos 30^{\circ}+0.2 \times \cos 60^{\circ}\right)=N_{B}\left(\cos 45^{\circ}-0.2 \times \cos 45^{\circ}\right)$

$$
\begin{equation*}
N_{A}=\frac{0.5657}{0.9660} N_{B}=0.5856 N_{B} \tag{vi}
\end{equation*}
$$

Substituting it in (v), we get

$$
\text { and hence } \quad \begin{aligned}
N_{A} & =0.5856 \times 1057.82 \\
& =619.67 \mathrm{~N} \\
\Sigma M_{B} & =0, \text { gives } \\
600 x+500 \times 1.5- & N_{A} \sin 30^{\circ} \times 3+F_{A} \sin 60^{\circ} \times 3=0 \\
\therefore \quad x & =-0.237 \mathrm{~m}
\end{aligned}
$$

$$
N_{B}=1057.82 \mathrm{~N}
$$

It means that the motion will be impending when the load is at 0.237 m to the right of $B$, which is not a possible case of loading. Hence load can be placed even on point $B$ safely. Thus, 600 N load can be placed anywhere between point $B$ and a point 1.877 m from $B$.

### 4.4 ROLLING RESISTANCE

The cylinder or sphere may slide on the surface as blocks do. In such case [Ref. Fig. 4.19(a)], the normal reaction is equal to $W$ and frictional force is equal to
$\mu W$, same as in case of a block. However, the cylinder or sphere may roll over the surface. In this case we resistance to motion by friction is much less, hence a person prefers to roll them rather than drag them. Figure 4.19(b) shows the rolling resistance to a roller.


Fig. 4.19
Rolling resistance occurs due to small deformations of the surface, which is as shown in Fig. 4.19(b) to an exaggerated scale. The resistance $R$ from the surface, $W$ the load on the wheel and $P$ the effort applied, form the system of forces acting on the wheel. Since they are only three in number, they must form the system of concurrent forces. Hence the resistance $R$ also passes through the centre of roller. If $R$ makes angle $\phi$ with the vertical, then

$$
\begin{align*}
& \begin{aligned}
\Sigma V & =0, \text { gives } W=R \cos \phi, \quad \text { and } \\
\Sigma H & =0, \text { gives } P=R \sin \phi \\
\therefore \quad & \frac{P}{W}
\end{aligned} \\
&=\tan \phi \\
&=\frac{a}{r}, \text { when } r \text { is radius of the roller } \\
& \text { Thus, } \quad P=\frac{W a}{r} \\
& \text { But } \quad P=F \quad \therefore \quad F=\frac{W a}{r}
\end{align*}
$$

The distance ' $a$ ' in Eqn. (4.5) is termed as the coefficient of rolling resistance. This coefficient had the unit of length.

The coefficient of rolling resistance for some of the materials is shown in Table 4.1.

Table 4.1 Coefficient of Rolling Resistance

| Contact Surfaces | Range of a in (mm) |
| :--- | :---: |
| Steel on steel | $0.18-0.40$ |
| Steel on wood | $1.5-2.50$ |
| Tyres on smooth roads | $0.5-0.75$ |
| Tyres on mud roads | $1.0-1.5$ |
| Hardened steel on hardened steel | $0.005-0.015$ |

From Eqn. (4.5), it may be observed that less force is required to roll a wheel of bigger radius.

Example 4.14 Determine the rolling resistance of a railroad car and a truck with a trailer. Given,

Total weight in each case 800 kN
In case of railroad car,

$$
\begin{aligned}
\text { Diameter of the wheel } & =750 \mathrm{~mm} \\
\text { Rolling reistance } & =0.025 \mathrm{~mm}
\end{aligned}
$$

In case of truck and trailer,
Diameter of the wheel $=1200 \mathrm{~mm}$
Rolling resistance $=0.65 \mathrm{~mm}$
Solution. From Eqn. 4.5, we have

$$
F=\frac{W a}{r}
$$

$\therefore$ For railroad

$$
F=\frac{800 \times 0.025}{(750 / 2)}=0.0533 \mathrm{kN}=53.3 \mathrm{~N}
$$

In case of truck and trailer,

$$
F=\frac{800 \times 0.65}{(1200 / 2)}=0.8667 \mathrm{kN}=866.7 \mathrm{~N}
$$

Note: The number of wheels $n$ plays no role here, since we have to divide the load by number of wheels to get the load per wheel and then multiply by the number of wheels to get the total resistance.

Example 4.15 The coefficient to rolling resistance of a cylinder on a flat surface is 1.3 mm . At what inclination of the surface, the cylinder of radius $r=300 \mathrm{~mm}$ will start rolling down.
Solution. Rolling resistance $=F=\frac{W a}{r}$

$$
\begin{equation*}
=W \frac{1.3}{300} \tag{i}
\end{equation*}
$$

Referring to FBD of roller shown in Fig 4.20,

$$
\text { we find } \quad F=W \sin \theta
$$

From Eqns. (i) and (ii), we get

$$
\begin{array}{rlrl}
W \sin \theta & =W \frac{1.3}{300} \\
\therefore \quad & \theta & =\sin ^{-1} \frac{1.3}{300}=0.248^{\circ}
\end{array}
$$



Ans.
Fig. 4.20

Example 4.16 A circular cylinder of radius 0.6 m and weighing 3000 N is in contact with a rectangular block of 2000 N on an incline at $30^{\circ}$ to horizontal as shown in Fig. 4.21(a). If the coefficient of static friction is 0.5 , determine the minimum force $F$ to be applied up the plane on the block to prevent the bodies from sliding down.

(a)

(c)

Fig. 4.21
Solution. Figure 4.20(b) shows FBD of cylinder and Fig. 4.20(c) shows that of the block.

Consider the equilibrium of cylinder. It may slide at $A$ or it may roll. If it slides $F_{A}=\mu N_{A}=0.5 N_{A}$. If it rolls, its value is less than $0.5 N_{A}$. Let us assume the second possibility and try the analysis
$\Sigma$ Forces normal to plane $=0$, gives

$$
\begin{equation*}
N_{A}-3000 \cos 30^{\circ}+F_{B}=0 \tag{i}
\end{equation*}
$$

$\Sigma$ Forces parallel to plane $=0$, gives

$$
\begin{align*}
& N_{B}+F_{A}-3000 \sin 30^{\circ}=0  \tag{ii}\\
& \quad \Sigma M_{o}=0, \text { gives } F_{B} \times 0.6-F_{A} \times 0.6=0
\end{align*}
$$

or

$$
\begin{equation*}
F_{A}=F_{B} \tag{iii}
\end{equation*}
$$

$$
F_{B}=\mu N_{B}=0.5 N_{B}
$$

$$
\begin{equation*}
\therefore \quad F_{A}=\mu N_{B}=0.5 N_{B} \tag{iv}
\end{equation*}
$$

$\therefore$ From Eqn. (ii)

$$
\begin{aligned}
N_{B}+0.5 N_{B} & =3000 \sin 30 \\
N_{B} & =1000 \mathrm{~N} \\
F_{A} & =F_{B}=0.5 \times 1000=500 \mathrm{~N}
\end{aligned}
$$

From Eqn. (i)

$$
\begin{aligned}
N_{A} & =3000 \cos 30^{\circ}-500 \\
& =2098.08 \mathrm{~N}
\end{aligned}
$$

Now check for $F_{A}$. If sliding is to take place, $N_{A}$ would have been $0.5 \times 2098.08=1049.04 \mathrm{~N}$. Thus $F_{A}$ is between 0 and 1049.04 , which means rolling is taking place at $A$ as observed.

Now, consider the equilibrium of the block.

$$
\Sigma \text { Forces normal to plane }=0 \text {, gives }
$$

$$
N_{C}-2000 \cos 30^{\circ}-F_{B}=0
$$

i.e.,

$$
\begin{equation*}
N_{C}=2000 \cos 30^{\circ}+500=2232.05 \mathrm{~N} \tag{v}
\end{equation*}
$$

$\Sigma$ Forces parallel to plane $=0$, gives

$$
\begin{gather*}
F+F_{C}-N_{B}-2000 \sin 30^{\circ}=0  \tag{vi}\\
F_{C}=\mu N_{C}=0.5 \times 2232.05=1116.03 \mathrm{~N}
\end{gather*}
$$

But
Substituting the values of $N_{B}$ and $F_{C}$ in Eqn. (vi), we get,

$$
\begin{aligned}
& F=-1116.03+1000+2000 \sin 30^{\circ} \\
& F=883.97 \mathrm{~N}
\end{aligned}
$$

Ans.

### 4.5 BELT FRICTION

The transmission of power by means of belt or rope drives is possible because of friction which exists between the wheels and the belt. Similarly, band brakes stop the rotating discs because of friction between the belt and the disc. All along the contact surface, the frictional resistance develops. Hence, the tension in the belt is more on the side it is pulled and is less on the other side. Accordingly, the two sides of the belt may be called as tight side and slack side.

## Relationship Between Tight Side and Slack Side Forces in a Rope

Figure 4.22(a) shows a load $W$ being pulled by a force $P$ over a fixed drum. Let the forces on slack side be $T_{1}$ and on tight side be $T_{2}$ [Fig. 4.22(b)]. $T_{2}$ is more


Fig. 4.22
than $T_{1}$ because frictional force develops between drum and the rope [Fig. 4.21 (c)]. Let $\theta$ be the angle of contact between rope and the drum. Now, consider an elemental length of rope as shown in Fig. 4.22(d). Let $T$ be the force on slack side and $T+d T$ on tight side. There will be normal reaction $N$ on the rope in the radial direction and frictional force $F=\mu N$ in the tangential direction. Then,
$\Sigma$ Forces in radial direction $=0$, gives

$$
N-T \sin \frac{d \theta}{2}-(T+d T) \sin \frac{d \theta}{2}=0
$$

Since $d \theta$ is small, $\quad \sin \frac{d \theta}{2}=\frac{d \theta}{2}$
$\therefore \quad N-T \frac{d \theta}{2}-(T+d T) \frac{d \theta}{2}=0$
i.e.,

$$
\begin{equation*}
N=\left(T+\frac{d T}{2}\right) d \theta \tag{i}
\end{equation*}
$$

From the law of friction,

$$
\begin{align*}
F & =\mu N \\
& =\mu\left(T+\frac{d T}{2}\right) d \theta \tag{ii}
\end{align*}
$$

where $\mu$ is coefficient of friction.
Now, $\quad \Sigma$ Forces in tangential direction $=0$, gives

$$
(T+d T) \cos \frac{d \theta}{2}=F+T \cos \frac{d \theta}{2}
$$

Since $\quad \frac{d \theta}{2}$ is small, $\cos \frac{d \theta}{2}=1$
$\therefore \quad T+d T=F+T$
or $\quad d T=F$
From Eqns. (ii) and (iii) $\quad d T=\mu\left(T+\frac{d T}{2}\right) d \theta$
Neglecting small quantity of higher order, we get

$$
\begin{aligned}
d T & =\mu T d \theta \\
\frac{d T}{T} & =\mu d \theta
\end{aligned}
$$

Integrating both sides over 0 to $\theta$,

$$
\int_{T_{1}}^{T_{2}} \frac{d T}{T}=\int_{0}^{\theta} \mu d \theta
$$

$$
\begin{array}{ll}
\therefore \quad[\log T]_{T_{1}}^{T_{2}} & =\mu[\theta]_{0}^{\theta} \\
\log \frac{T_{2}}{T_{1}} & =\mu \theta \\
\frac{T_{2}}{T_{1}} & =e^{\mu \theta} \\
\text { i.e., } \quad T_{2} & =T_{1} e^{\mu \theta}
\end{array}
$$

Note: $\theta$ should be in radians.

Example 4.17 A rope making $1 \frac{1}{4}$ turns around a stationary horizontal drum is used to support a weight $W$ (Fig. 4.23). If the coefficient of friciton is 0.3 , what range of weight can be supported by exerting a 600 N force at the other end of the rope?
Solution. Angle of contact $=1.25 \times 2 \pi=2.5 \pi$


Fig. 4.23
(1) Let the impending motion of the weight be downward.

Then,

$$
\begin{aligned}
T_{1} & =600 \mathrm{~N} ; T_{2}=W \\
\frac{W}{600} & =e^{\mu 2.5 \pi}=e^{0.3 \times 2.5 \pi}=e^{0.75 \pi} \\
W & =6330.43 \mathrm{~N}
\end{aligned}
$$

(2) Let the impending motion of weight be upwards. Then,

$$
\begin{aligned}
T_{1} & =W ; T_{2}=600 \mathrm{~N} \\
T_{2} & =T_{1} e^{\mu \theta} \\
600 & =W e^{0.75 \pi} \\
W & =56.87 \mathrm{~N}
\end{aligned}
$$

Thus, a 600 N force can support a range of loads between 56.87 N to 6330.43 N weight on the other side of drum.

Example 4.18 In Fig. 4.24(a), the coefficient of friction is 0.20 between the rope and the fixed drum, and between other surfaces of contact $\mu=0.3$. Determine the minimum weight $W$ to prevent downward motion of the 1000 N body.
Solution. Since 1000 N weight is on the verge of sliding downwards, the rope connecting it is the tight side and the rope connecting $W$ is the slack side. Free body diagrams for $W$ and 1000 N body are shown in Fig. 4.24(b).

Now, $\quad \begin{aligned} \cos \alpha & =0.8 \\ \sin \alpha & =0.6\end{aligned}$
Consider the equilibrium of weight $W$.

(a)

(b)

Fig. 4.24
$\Sigma$ Forces perpendicular to the plane $=0$, gives

$$
\begin{array}{ll} 
& N_{1}=W \cos \alpha \\
& N_{1}=0.8 \mathrm{~W} \\
\therefore \quad & F_{1}=\mu N_{1}=0.3 \times 0.8 \mathrm{~W} \\
& F_{1}=0.24 \mathrm{~W} \tag{ii}
\end{array}
$$

$\Sigma$ Forces parallel to the plane $=0$, gives

$$
\begin{align*}
T_{1} & =F_{1}+W \sin \alpha=0.24 W+0.6 \mathrm{~W} \\
& =0.84 \mathrm{~W} \tag{iii}
\end{align*}
$$

Angle of contact of rope with the pulley $=180^{\circ}=\pi$ radians
Applying friction equation, we get

$$
\begin{aligned}
& T_{2}=T_{1} e^{\mu \theta}=T_{1} e^{0.3 \pi} \\
& T_{2}=2.566 T_{1}
\end{aligned}
$$

Substituting the value of $T_{1}$ from Eqn. (iii)

$$
\begin{equation*}
T_{2}=2.156 \mathrm{~W} \tag{iv}
\end{equation*}
$$

Now, consider 1000 N body,
$\Sigma$ forces perpendicular to the plane $=0$, gives

$$
N_{2}-N_{1}-1000 \cos \alpha=0
$$

Substituting the value of $N_{1}$ from Eqn. (i),

$$
\begin{array}{ll} 
& N_{2}=0.8 W+1000 \times 0.8=0.8 \mathrm{~W}+800 \\
\therefore & F_{2}=0.3 N_{2}=0.24 W+240 \tag{v}
\end{array}
$$

$\Sigma$ forces parallel to the plane $=0$, gives

$$
F_{1}+F_{2}-1000 \sin \alpha+T_{2}=0
$$

Substituting the values from Eqns. (ii), (iv) and (v)

$$
\begin{aligned}
& 0.24 W+0.24 W+240-1000 \times 0.6+2.156 W=0 \\
& W=136.56 \mathrm{~N}
\end{aligned}
$$

## Ans.

Example 4.19 A torque of $300 \mathrm{~N}=\mathrm{m}$ acts on the brake drum shown in Fig. 4.25(a). If the brake band is in contact with the brake drum through $250^{\circ}$ and the coefficient of friction is 0.3 , determine force $P$ applied at the end of the brake lever for the position shown in the figure.


Fig. 4.25

## Solution.

Figure 4.25(b) shows free body diagrams of brake drum and the lever arm.

$$
\begin{array}{lc}
\text { Now } & T_{2}=T_{1} e^{\mu \theta} \\
& \theta=\frac{250 \pi}{180} \text { radians and } \mu=0.3 \\
\therefore & \mu \Theta=1.309 \\
\therefore & T_{2}=T_{1} e^{1.309}=3.7025 T_{1} \\
\text { Now, } & \left(T_{2}-T_{1}\right) r=M \\
& (3.7025-1) T_{1} \times 250=300 \times 10^{3} \\
\therefore & T_{1}=444.04 \mathrm{~N} \\
\therefore & T_{2}=1644.058 \mathrm{~N}
\end{array}
$$

Consider the lever arm. Taking moment about the hinge, we get

$$
\begin{aligned}
T_{2} \times 50 & =P \times 300 \\
P & =274.0 \mathrm{~N}
\end{aligned}
$$

Ans.

## IMPORTANT DEFINITIONS AND CONCEPTS

1. The magnitude of limiting friction bears a constant ratio to the normal reaction between the two surfaces of contact and this ratio is called Coefficient of Friction.
2. Angle of Limiting Friction can be defined as the angle between the resultant reaction and the normal to the plane on which the motion of the body is impending.
3. The maximum inclination of the plane on which a body free from external applied forces, can repose (sleep) is called Angle of Repose.
4. The inverted cone with semi-central angle equal to limiting frictional angle $\alpha$ is called Cone of Friction.

## IMPORTANT FORMULAE

1. $\mu=\frac{F}{N}$, where $F$ is limiting friction
$N$ is normal reaction.
2. $\tan \alpha=\frac{F}{N}=\mu$
where $\alpha=$ Angle of friction $=$ Angle of repose
3. $F=\frac{W a}{r}$
where $a$ is coefficient of rolling resistance
$r=$ Radius of roller
$W=$ Weight of roller
$F=$ Rolling friction
4. $T_{2}=T_{1} e^{\mu \theta}$
where $T_{1}=$ Tension on slack side
$T_{2}=$ Tension on tight side
$\theta=$ Angle of contact

## PROBLEMS FOR EXERCISE

4.1 A pull of 180 N applied upward at $30^{\circ}$ to a rough horizontal plane was required to just move a body resting on the plane while a push of 220 N applied along the same line of action was required to just move the same body downwards. Determine the weight of the body and the coefficient of friction.
[Ans. $W=990 \mathrm{~N} ; \mu=0.1722$ ]
4.2 The block $A$ as shown in Fig. 4.26, weighs 2000 N. The cord attached to $A$ passes over a frictionless pulley and supports a weight equal to 800 N . The value of coefficient of friction between $A$ and the horizontal plane is 0.35 . Solve for horizontal force $P$ : (1) if motion is


Fig. 4.26 impending towards the left, and (2) if the motion is impending towards the right.
[Ans. (1) 1252.82 N ; (2) 132.82 N ]
4.3 A 3000 N block is placed on an inclined plane as shown in Fig. 4.27. Find the maximum value of $W$ for equilibrium if tipping does not occur. Assume coefficient of friction as 0.2 .
[Ans. 1014.96 N]


Fig. 4.27


Fig. 4.28
4.4 Find whether block $A$ is moving up or down the plane in Fig. 4.28 for the data given below. Weight of block $A=300 \mathrm{~N}$. Weight of block $B=600 \mathrm{~N}$. Coefficient of limiting friction between plane $A B$ and block $A$ is 0.2 . Coefficient of limiting friction between plane $B C$ and block $B$ is 0.25 . Assume pulley as smooth.
[Ans. The block $A$ moves up]
4.5 Two identical blocks $A$ and $B$ are connected by a rod and they rest against vertical and horizontal planes respectively as shown in Fig. 4.29. If sliding impends when $\theta=45^{\circ}$, determine the coefficient of friction, assuming it to be the same for both floor and wall.
[Ans. 0.414]


Fig. 4.29


Fig. 4.30
4.6 Determine force $P$ required to start the wedge as shown in Fig. 4.30. The angle of friction for all surfaces of contact is $15^{\circ}$. [Ans. 26.6784 kN ]
4.7 Two blocks $A$ and $B$ weighing 3 kN and 15 kN , respectively, are held in position against an inclined plane by applying force $P$ as shown in Fig. 4.31. Find the least value of $P$ which will induce motion of the block $A$ upwards. Angle of friction for all contact surfaces is $12^{\circ}$.

[Ans. 14.025 kN ]

Fig. 4.31
4.8 In Fig. 4.32, $C$ is a stone block weighing 6 kN . It is being raised slightly by means of two wooden wedges $A$ and $B$ with a force $P$ on wedge $B$. The angle between the contacting surfaces of the wedge is $5^{\circ}$. If coefficient of friction is 0.3 for all contacting surfaces, compute the value of $P$ required to impend upward motion of the block $C$. Neglect weight of the wedges.
[Ans. 2.344 kN ]


Fig. 4.32


Fig. 4.33
4.9 Find the horizontal force $P$ required to push block $A$ of weight 150 N which carries block $B$ of weight 1280 N as shown in Fig. 4.33. Take angle of limiting friction between floor and block $A$ as $14^{\circ}$ and that between vertical wall and block $B$ as $13^{\circ}$ and coefficient of limiting friction between the blocks as 0.3 .
[Ans. $P=1294.2 \mathrm{~N}]$
4.10 The level of a precast beam weighing $20,000 \mathrm{~N}$ is to be adjusted by driving a wedge as shown in Fig. 4.34. If coefficient of friction between the wedge and pier is 0.35 and that between beam and the wedge is 0.25 ,


Fig. 4.34
determine the minimum force $P$ required on the wedge to make adjustment of the beam. Angle of the wedge is $15^{\circ}$. (Hint: Vertical component of reaction on wedge at contact with beam $=1 / 2$ vertical load on beam $=$ $10,000 \mathrm{kN}$ )
[Ans. 9057.4 N ]
4.11 A ladder 5 m long rests on a horizontal ground and leans against a smooth vertical wall at an angle of $70^{\circ}$ with the horizontal. The weight of the ladder is 300 N . The ladder is on the verge of sliding when a man weighing 750 N stands on a rung 1.5 m high. Calculate the coefficient of friction between the ladder and the floor.
[Ans. $\mu=0.1837$ ]
4.12 A 4 m ladder weighing 200 N is placed against a vertical wall as shown in Fig. 4.35. As a man weighing 800 N , reaches a point 2.7 m from $A$, the ladder is about to slip. Assuming that the coefficient of friction between the ladder and the wall is 0.2 , determine the coefficient of friction between the ladder and the floor.
[Ans. 0.3548]


Fig. 4.35
4.13 Determine the maximum weight that can be lowered by a person who can exert a 300 N pull on rope if the rope is wrapped $2 \frac{1}{2}$ turns round a horizontal spur as shown in Fig. 4.36. Coefficient of friction between spur and the rope is 0.3 .
[Ans. 33395.33 N]
4.14 Determine the minimum value of $W$ required to cause motion of blocks $A$ and $B$ towards right (Ref. Fig. 4.37. Each block weighs 3000 N and coefficient of friction between blocks and inclined planes is 0.2 . Coefficient of friction between the drum and the rope is 0.1 . Angle of wrap over the drum is $90^{\circ}$.
[Ans. 3065.18 N ]


Fig. 4.36


Fig. 4.37
4.15 Block $A$ as shown in Fig. 4.38 weighs 2000 N. The cord attached to $A$ passes over a fixed drum and supports a weight equal to 800 N . The value of coefficient of friction between $A$ and the horizontal plane is 0.25 and between the rope and the fixed drum is 0.1 . Solve for $P$ : (1) if motion is impending towards the left, (2) if the motion is impending towards the right.
[Ans. (1) 1230.94 N ; (2) 143.0 N ]
4.16 The dimension of a brake drum is as shown in Fig. 4.39. Determine torque $M$ exerted on the drum if load $P=50 \mathrm{~N}$. Assume coefficient of kinetic friction between rope and drum to be 0.15 .
[Ans. 747.685 N-m]
4.17 Determine force $F$ required to move up the cylinder and the block shown in Fig. 4.21(a).
[Ans. $F=3116.025 \mathrm{~N}$ ]


Fig. 4.38


Fig. 4.39

## Chapter 5

## Properties of Surfaces and Solids

Under this topic, first we will see how to find the areas of given figures and the volumes of given solids. Then the terms centre of gravity and centroids are explained. Though the title of this topic does not indicate the centroid of line segment, that term is also explained, since the centroid of line segment will be useful in finding the surface area and volume of solids using theorems of Pappus and Guldinus. Then the term first moment of area is explained and the method of finding centroid of plane areas and volumes is illustrated. After explaining the term second moment of area, the method of finding moment of inertia of plane figures about $x-x$ or $y-y$ axis is illustrated. The term product moment of inertia is defined and the method of finding principal moment of inertia is presented. At the end, the method of finding mass moment of inertia is presented.

### 5.1 DETERMINATION OF AREAS AND VOLUMES

In the school education, methods of finding areas and volumes of simple cases are taught by many methods. Here we will see the general approach which is common to all cases, i.e., by the method of integration. In this method, the expression for an elemental area will be written then suitable integrations are carried out so as to take care of entire surface/volume. This method is illustrated with standard cases below, first for finding the areas and later for finding the volumes:

## A: Area of Standard Figures

(i) Area of a rectangle:

Let the size of rectangle be $b \times d$ as shown in Fig. 5.1. $d A$ is an elemental area of side $d x \times d y$.

Area of rectangle,

$$
\begin{aligned}
A & =\oint d A=\int_{-b / 2}^{b / 2} \int_{-d / 2}^{d / 2} d x d y \\
& =[x]_{-b / 2}^{b / 2}[y]_{-d / 2}^{d / 2} \\
& =b d
\end{aligned}
$$



Fig. 5.1

If we take element as shown in Fig. 5.2,

$$
\begin{aligned}
A & =\int_{-d / 2}^{d / 2} d A=\int_{-d / 2}^{d / 2} b \cdot d y=b[y]_{-d / 2}^{d / 2} \\
& =b d
\end{aligned}
$$

(ii) Area of a triangle of base width ' $b$ ' height ' $h$ '.
Referring to Fig. 5.3, let the element be selected as shown by hatched lines.

Then

$$
\begin{aligned}
d A & =b^{\prime} d y=b \frac{y}{h} d y \\
A & =\int_{0}^{h} d A=\int_{0}^{h} b \frac{y}{h} d y \\
& =\frac{b}{h}\left[\frac{y^{2}}{2}\right]_{0}^{h}=\frac{b h}{2}
\end{aligned}
$$



Fig. 5.2
hen


Fig. 5.3
(iii) Area of a circle

Consider the elemental area $d A=r d \theta d r$ as shown in Fig. 5.4. Now,

$$
d A=r d \theta d r
$$

$r$ varies from 0 to $R$ and $\theta$ varies from 0 to $2 \pi$

$$
\begin{aligned}
\therefore & =\int_{0}^{2 \pi} \int_{0}^{R} r d \theta d r \\
& =\int_{0}^{2 \pi}\left[\frac{r^{2}}{2}\right]_{0}^{R} d \theta \\
& =\int_{0}^{2 \pi} \frac{R^{2}}{2} d \theta \\
& =\frac{R^{2}}{2}[\theta]_{0}^{2 \pi} \\
& =\frac{R^{2}}{2} \cdot 2 \pi=\pi R^{2}
\end{aligned}
$$



Fig. 5.4

In the above derivation, if we take variation of $\theta$ from 0 to $\pi$, we get the area of semicircle as $\frac{\pi R^{2}}{2}$ and if the limit is from 0 to $\pi / 2$ the area of quarter of a circle is obtained as $\frac{\pi R^{2}}{4}$.
(iv) Area of a sector of a circle

Area of a sector of a circle with included angle $2 \alpha$, as shown in Fig. 5.5 is to be determined. The elemental area is as shown in the figure:

$$
d A=r d \theta \cdot d r
$$

$\theta$ varies from $-\alpha$ to $\alpha$ and $r$ varies from 0 to $R$

$$
\begin{aligned}
\therefore \quad & =\oint d A=\int_{-\alpha}^{\alpha} \int_{0}^{R} r d \theta d r \\
& =\int_{-\alpha}^{\alpha}\left[\frac{r^{2}}{2}\right]_{0}^{R} d \theta=\int_{-\alpha}^{\alpha} \frac{R^{2}}{2} d \theta \\
& =\left[\frac{R^{2}}{2} \theta\right]_{-\alpha}^{\alpha} \\
& =\frac{R^{2}}{2}(2 \alpha)=R^{2} \alpha
\end{aligned}
$$



Fig. 5.5
(v) Area of a parabolic spandrel

Two types of parabolic curves are possible:
(a) $y=k x^{2}$
(b) $y^{2}=k x$

Case $a$ : This curve is shown in Fig. 5.6.
The area of the element

$$
\begin{aligned}
& d A=y d x \\
& =k x^{2} d x \\
& \therefore \quad A=\int_{0}^{a} d A=\int_{0}^{a} k x^{2} a x \\
& =k\left[\frac{x^{3}}{3}\right]_{0}^{a}=\frac{k a^{3}}{3} \\
& \text { Fig. } 5.6
\end{aligned}
$$

We know, when

$$
x=a, y=h
$$

i.e., $\quad h=k a^{2}$ or $k=\frac{h}{a^{2}}$
$\therefore \quad A=\frac{k a^{3}}{3}=\frac{h}{a^{2}} \frac{a^{3}}{3}=\frac{1}{3} h a=\frac{1}{3}$ rd the area of rectangle
of size $a \times h$

Case $b$ : In this case, $y^{2}=k x$
Referring to Fig. 5.7

$$
\begin{aligned}
d A & =y d x=\sqrt{k x} d x \\
A & =\int_{0}^{a} y d x=\int_{0}^{a} \sqrt{k x} d x \\
& =\sqrt{k}\left[x^{3 / 2} \frac{2}{3}\right]_{0}^{a} \\
& =\sqrt{k} \frac{2}{3} a^{3 / 2}
\end{aligned}
$$



Fig. 5.7

We know that, when $x=a, y=h$

$$
\therefore \quad h^{2}=k a \text { or } k=\frac{h^{2}}{a}
$$

Hence

$$
A=\frac{h}{\sqrt{a}} \cdot \frac{2}{3} \cdot a^{3 / 2}
$$

i.e., $\quad A=\frac{2}{3} h a=\frac{2}{3} \mathrm{rd}$ the area of rectangle of size $a \times h$
(vi) Surface area of a cone

Consider the cone shown in Fig. 5.8. Now,

$$
y=\frac{x}{h} R
$$

Surface area of the element,

$$
\begin{aligned}
d A & =2 \pi y d l=2 \pi \frac{x}{h} R d l \\
& =2 \pi \frac{x}{h} R \frac{d x}{\sin \alpha} \\
\therefore \quad A & =\frac{2 \pi R}{h \sin \alpha}\left[\frac{x^{2}}{2}\right]_{0}^{h} \\
& =\frac{\pi R h}{\sin \alpha} \\
& =\pi R l
\end{aligned}
$$



Fig. 5.8
(vii) Surface area of a sphere

Consider the sphere of radius $R$ as shown in Fig. 5.9. The element considered is the parallel circle at distance $y$ from the diametral axis of sphere.


Fig. 5.9

$$
\begin{aligned}
d S & =2 \pi x R d \theta \\
& =2 \pi R \cos \theta R d \theta, \text { since } x=R \cos \theta \\
\therefore \quad S & =2 \pi R^{2} \int_{-\pi / 2}^{\pi / 2} \cos \theta d \theta \\
& =2 \pi R^{2}[\sin \theta]_{-\pi / 2}^{\pi / 2} \\
& =4 \pi R^{2}
\end{aligned}
$$

## B. Volume of Standard Solids

(i) Volume of a parallelepiped

Let the size of the parallelepiped be $a \times b \times c$ as shown in Fig. 5.9. The volume of the element is

$$
d V=d x d y d z
$$

$$
\begin{aligned}
V & =\int_{0}^{a} \int_{0}^{b} \int_{0}^{c} d x d y d z \\
& =[x]_{0}^{a}[y]_{0}^{b}[z]_{0}^{c} \\
& =a b c
\end{aligned}
$$

(ii) Volume of a cone

Referring to Fig. 5.8

$$
d V=\pi y^{2} \cdot d x=\pi \frac{x^{2}}{h^{2}} R^{2} d x, \text { since } y=\frac{x}{h} R
$$

$$
\begin{aligned}
V & =\frac{\pi}{h^{2}} R^{2} \int_{0}^{h} x^{2} d x=\frac{\pi}{h^{2}} R^{2}\left[\frac{x^{3}}{3}\right]_{0}^{h} \\
& =\frac{\pi}{h^{2}} R^{2} \frac{h^{3}}{3}=\frac{\pi R^{2} h}{3}
\end{aligned}
$$

(iii) Volume of a sphere

Referring to Fig. 5.9

$$
\begin{aligned}
& d V & =\pi x^{2} d y \\
\text { but } & x^{2}+y^{2} & =R^{2} \\
\text { i.e., } & x^{2} & =R^{2}-y^{2}
\end{aligned}
$$

$$
\therefore \quad d V=\pi\left(R^{2}-y^{2}\right) d y
$$

$$
V=\int_{-R}^{R} \pi\left(R^{2}-y^{2}\right) d y
$$

$$
=\pi\left[R^{2} y-\frac{y^{3}}{3}\right]_{-R}^{R}
$$

$$
=\pi\left[R^{2} \cdot R-\frac{R^{3}}{3}-\left\{-R^{3}-\frac{(-R)^{3}}{3}\right\}\right]
$$

$$
=\pi R^{3}\left[1-\frac{1}{3}+1-\frac{1}{3}\right]
$$

$$
=\frac{4}{3} \pi R^{3}
$$

The surface areas and volumes of solids of revolutions like cone, spheres may be easily found using theorems of Pappus and Guldinus. This will be taken up latter in this chapter, since it needs the term centroid of generating lines.

### 5.2 CENTRE OF GRAVITY AND CENTROIDS

Consider the suspended body shown in Fig. 5.10(a). The self weight of various parts of this body are acting vertically downward. The only upward force is the force $T$ in the string. To satisfy the equilibrium condition the resultant weight of the body, $W$ must act along the line of string $1-1$. Now, if the position is changed and the body is suspended again [Fig. 5.10(b)], it will reach equilibrium condition in a particular position. Let the line of action of the resultant weight be 2-2 intersecting $1-1$ at $G$. It is obvious that if the body is suspended in any other position, the line of action of resultant weight $W$ passes through $G$. This point is called the centre of gravity of the body. Thus centre of gravity can be defined as the point through which the resultant of force of gravity of the body acts.


Fig. 5.10
The above method of locating centre of gravity is the practical method. If one desires to locate centre of gravity of a body analytically, it is to be noted that the resultant of weight of various portions of the body is to be determined. For this Varignon's theorem, which states the moment of resultant force is equal to the sum of moments of component forces, can be used.

Referring to Fig. 5.11, let $W_{i}$ be the weight of an element in the given body. $W$ be the total weight of the body. Let the coordinates of the element be $x_{i}, \mathrm{y}_{i}, z_{i}$ and that of centroid $G$ be $x_{2}, y_{2}, z_{2}$. Since $W$ is the resultant of $W_{i}$ forces,


Fig. 5.11

$$
\begin{array}{lrl} 
& & W \\
& =W_{1}+W_{2}+W_{3} \cdots \\
& =\Sigma W_{i} \\
\text { and } & & W x_{c} \\
& =W_{1} x_{1}+W_{2} x_{2}+W_{3} x_{3}+\ldots \\
\text { Similarly, } & & W x_{c} \tag{5.1}
\end{array}=\Sigma W_{i} x_{i}=\oint x d w,
$$

If $M$ is the mass of the body and $m_{i}$ that of the element, then

$$
M=\frac{W}{g} \text { and } m_{i}=\frac{W_{i}}{g} \text {, hence we get }
$$

and

$$
\left.\begin{array}{l}
M x_{c}=\Sigma m_{i} x_{i}=\oint x d m \\
M y_{c}=\Sigma m_{i} y_{i}=\oint y d m \\
M z_{c}=\Sigma m_{i} z_{i}=\oint z d m
\end{array}\right\}
$$

If the body is made up of uniform material of unit weight $\gamma$, then we know $W_{i}=V_{i} \gamma$, where $V$ represents volume, then Eqn. (5.1) reduces to

$$
\left.\begin{array}{l}
V x_{c}=\Sigma V_{i} x_{i}=\oint x d V \\
V y_{c}=\Sigma V_{i} y_{i}=\oint y d V \\
V z_{c}=\Sigma V_{i} z_{i}=\oint z d V
\end{array}\right\}
$$

If the body is a flat plate of uniform thickness, in $x-y$ plane, $W_{i}=\gamma A_{i} t$ (Ref. Fig. 5.12). Hence Eqn. (5.1) reduces to

$$
\left.\begin{array}{l}
A x_{c}=\Sigma A_{i} x_{i}=\oint x d A \\
A y_{c}=\Sigma A_{i} y_{i}=\oint y d A
\end{array}\right\}
$$



Fig. 5.12
If the body is a wire of uniform cross-section in plane $x, y$ (Ref. Fig. 5.13) the Eqn. (5.1) reduces to

$$
\left.\begin{array}{l}
L x_{c}=\Sigma L_{i} x_{i}=\oint x d L  \tag{5.5}\\
L y_{c}=\Sigma L_{i} y_{i}=\oint y d L
\end{array}\right\}
$$



Fig. 5.13

The term centre of gravity is used only when the gravitational forces (weights) are considered. This term is applicable to solids. Equations (5.2) in which only masses are used, the point obtained is termed as centre of mass. The central points obtained for volumes, surfaces and line segments [obtained by Eqns. (5.3), (5.4) and (5.5)] are termed as centroids.

### 5.3 CENTROID OF A LINE

Centroid of a line can be determined using Eqn. (5.5). Method of finding the centroid of a line for some standard cases is illustrated below:
(i) Centroid of a straight line:

Selecting the $x$-coordinate along the line (Fig. 5.14), we have

$$
\begin{aligned}
L x_{c} & =\int_{0}^{L} x d x=\left[\frac{x^{2}}{2}\right]_{0}^{L}=\frac{L^{2}}{2} \\
\therefore \quad x_{c} & =\frac{L}{2}
\end{aligned}
$$



Fig. 5.14

Thus the centroid lies at midpoint of a straight line, whatever be the orientation of line (Ref. Fig. 5.15).


Fig. 5.15
(ii) Centroid of a arc of a circle : Referring to Fig. 5.16,

$$
\begin{aligned}
L & =\text { Length of } \operatorname{arc}=R 2 \alpha \\
d L & =R d \theta
\end{aligned}
$$

Hence from Eqn. (5.5), we have

$$
\begin{aligned}
x_{c} L & =\int_{-\alpha}^{\alpha} x d L \\
\text { i.e., } \quad x_{c} R 2 \alpha & =\int_{-\alpha}^{\alpha} R \cos \theta \cdot R d \theta \\
& =R^{2}[\sin \theta]_{-\alpha}^{\alpha} \quad . .
\end{aligned}
$$



Fig. 5.16

$$
\begin{align*}
& \therefore \quad x_{c}=\frac{R^{2} \times 2 \sin \alpha}{2 R \alpha}=\frac{R \sin \alpha}{\alpha} \\
& \text { and } \quad y_{c} L 2 \alpha \int_{-\alpha}^{\alpha} y d L=\int_{-\alpha}^{\alpha} R \sin \theta \cdot R d \theta \\
& =R^{2}[-\cos \theta]_{-\alpha}^{\alpha}  \tag{ii}\\
& =0 \\
& \therefore \quad y_{c}=0
\end{align*}
$$

From Eqns. (i) and (ii) we can get the centroid of semicircle, as shown in Fig. 5.17, by putting $\alpha=\pi / 2$ and for quarter of a circle, as shown in Fig. 5.18, by putting $\alpha$ varying from zero to $\pi / 2$.


Fig. 5.17


Fig. 5.18

For semi circle, $\quad x_{c}=\frac{2 R}{\pi}$

$$
y_{c}=0
$$

For quarter of a circle,

$$
\begin{aligned}
& x_{c}=\frac{2 R}{\pi} \\
& y_{c}=\frac{2 R}{\pi}
\end{aligned}
$$

(iii) Centroid of composite line segments:

The results obtained for standard cases may be used for various segments and then Eqns. (5.5) in the form

$$
\begin{aligned}
& x_{c} L=\Sigma L_{i} x_{i} \\
& y_{c} L=\Sigma L_{i} y_{i}
\end{aligned}
$$

may be used to get centroid $x_{c}$ and $y_{c}$. If the line segments are in space, the expression $z_{c} L=\Sigma L_{i} z_{i}$ may also be used. The method is illustrated with few examples below:

Example 5.1 Determine the centroid of the wire shown in Fig. 5.19.


Fig. 5.19
Solution. The wire is divided into three segments $A B, B C$ on $C D$. Taking $A$ as origin the coordinates of the centroids of $A B, B C$ and $C D$ are
$G_{1}(300,0) ; G_{2}(600,100)$ and $G_{3}\left(600-150 \cos 45^{\circ} ; 200+150 \sin 45^{\circ}\right)$ i.e., $\quad G_{3}(493.93,306.07)$

$$
L_{1}=600 \mathrm{~mm}, L_{2}=200 \mathrm{~mm}, L_{3}=300 \mathrm{~mm}
$$

$\therefore$ Total length $L=600+200+300=1100 \mathrm{~mm}$
$\therefore$ From the Eqn. $L x_{c}=\Sigma L_{i} x_{i}$, we get

$$
\begin{aligned}
1100 x_{c} & =L_{1} x_{1}+L_{2} x_{2}+L_{3} x_{3} \\
& =600 \times 300+200 \times 600+300 \times 493.93 \\
\therefore \quad x_{c} & =407.44 \mathrm{~mm} \\
\text { Now, } \quad L_{c} & =\Sigma L_{i} y_{i} \\
& 1100 y_{c}
\end{aligned}=600 \times 0+200 \times 100+300 \times 306.07
$$

Example 5.2 Locate the centroid of the uniform wire bent as shown in Fig. 5.20.


Fig. 5.20
Solution. The composite figure is divided into 3 simple figures and taking $A$ as origin, coordinates of their centroids noted down as shown below:
$A B$-a straight line

$$
L_{1}=400 \mathrm{~mm}, \quad G_{1}(200,0)
$$

$B C$-a semicircle

$$
\begin{array}{rr}
L_{2}=150 \pi=471.24, & G_{2}\left(550, \frac{2 \times 150}{\pi}\right) \\
\text { i.e., } G_{2}(550,95.49)
\end{array}
$$

$C D$-a straight line

$$
\begin{aligned}
& L_{3}=250 ; x_{3}=400+300+\frac{250}{2} \cos 30^{\circ}=808.25 \mathrm{~mm} \\
& y_{3}=125 \sin 30^{\circ}=62.5 \mathrm{~mm} \\
& L=L_{1}+L_{2}+\mathrm{L}_{3}=1121.24 \mathrm{~mm} \\
& \therefore \quad \text { Total length } \quad \begin{aligned}
& \\
& L x_{c}=\Sigma L_{i} x_{i} \text { gives } \\
& 1121.24 x_{c}=400 \times 200+471.24 \times 550+250 \times 808.25 \\
& x_{c}=482.72 \mathrm{~mm} \\
& L y_{c}=\Sigma L_{i} y_{i} \text { gives } \\
& \text { Ans. } \\
& 1121.24 y_{c}=400 \times 0+471.24 \times 95.49+250 \times 62.5 \\
& y_{c}=54.07 \mathrm{~mm}
\end{aligned} \quad \text { Ans. }
\end{aligned}
$$

Example 5.3 Locate the centroid of uniform wire shown in Fig. 5.21. Note: portion $A B$ is in $x-z$ plane, $B C$ in $y-z$ plane and $C D$ in $x-y$ plane. $A B$ and $B C$ are semicircular in shape.


Fig. 5.21
Solution. The length and the centroid of portions $A B, B C$ and $C D$ are as shown in table below:

Table 5.1

| Portion | $L_{i}$ | $x_{i}$ | $y_{i}$ | $z_{i}$ |
| :---: | :--- | :--- | :--- | :--- |
| $A B$ | $100 \pi$ | 100 | 0 | $\frac{2 \times 100}{\pi}$ |
| $B C$ | $140 \pi$ | 0 | 140 | $\frac{2 \times 140}{\pi}$ |
| $C D$ | 300 | $300 \sin 45^{\circ}$ | $280+300 \cos 45^{\circ}$ <br> $=492.13$ | 0 |

$$
\begin{array}{lrl}
\therefore & L & =100 \pi+140 \pi+300=1053.98 \mathrm{~mm} \\
& \text { From equation } \quad L x_{c} & =\Sigma L_{i} x_{i}, \text { we get } \\
1053.98 x_{c} & =100 \pi \times 100+140 \pi \times 0+300 \times 300 \sin 45^{\circ} \\
x_{c} & =90.19 \mathrm{~mm} & \text { Ans. } \\
\text { Similarly, } \quad 1053.98 y_{c} & =100 \pi \times 0+140 \pi \times 140+300 \times 492.13 \\
y_{c} & =198.50 \mathrm{~mm} \\
\text { and } & & \text { Ans. } \\
1053.98 z_{c} & =100 \pi \times \frac{200}{\pi}+140 \pi \times \frac{2 \times 140}{\pi}+300 \times 0 \\
z_{c} & =56.17 \mathrm{~mm} & \text { Ans. }
\end{array}
$$

### 5.4 FIRST MOMENT OF AREA AND CENTROID

From Eqn. (5.1), we have

$$
x_{c}=\frac{\Sigma W_{i} x_{i}}{W}, y_{c}=\frac{\Sigma W_{i} y_{i}}{W} \text { and } z_{c}=\frac{\Sigma W_{i} z_{i}}{W}
$$

From the above equation we can make the statement that distance of centre of gravity of a body from an axis is obtained by dividing moment of the gravitational forces acting on the body, about the axis, by the total weight of the body. Similarly from Eqn. (5.4), we have,

$$
x_{c}=\frac{\Sigma A_{i} x_{i}}{A}, y_{c}=\frac{\Sigma A_{i} y_{i}}{A}
$$

By terming $\Sigma A_{i} x_{i}$ as the moment of area about the axis, we can say centroid of plane area from any axis is equal to moment of area about the axis divided by the total area. The moment of area $\Sigma A_{i} x_{i}$ is termed as first moment of area also just to differentiate this from the term $\Sigma A_{i} x_{i}^{2}$, which will be dealt latter. It may be noted that since the moment of area about an axis divided by total area gives the distance of the centroid from that axis, the moment of area is zero about any centroidal axis.

## Difference Between Centre of Gravity and Centroid

From the above discussion, we can draw the following differences between centre of gravity and centroid:
(i) The term centre of gravity applies to bodies with weight, and centroid applies to lines, plane areas and volumes.
(ii) Centre of gravity of a body is a point through which the resultant gravitational force (weight) acts for any orientation of the body, whereas centroid is a point in a line, plane area volume such that the moment of area about any axis through that point is zero.

## Use of Axis of Symmetry

Centroid of an area lies on the axis of symmetry if it exits. This is useful theorem to locate the centroid of an area. This theorem can be proved as follows: Consider the area shown in Fig. 5.22. In this figure, $y-y$ is the axis of symmetry. From Eqn. (5.4), the distance of centroid from this axis is given by:

$$
\frac{\Sigma A_{i} x_{i}}{A}
$$

Consider the two elemental areas shown in Fig. 5.22, which are equal in size and are equidistant from the axis, but on either side. Now the sum of moments of these areas cancel each other since the areas and distances are the same, but signs of distances are opposite.


Fig. 5.22 Similarly, we can go on considering an area on one side of symmetric axis and corresponding image area on the other side, and prove that total moments of area ( $\Sigma A_{i} x_{i}$ ) about the symmetric axis is zero. Hence the distance of centroid from the symmetric axis is zero, i.e., centroid always lies on symmetric axis.

Making use of the symmetry, we can conclude that:
(i) Centroid of a circle is its centre (Fig. 5.23);
(ii) Centroid of a rectangle of sides $b$ and $d$ is at distance $\frac{b}{2}$ and $\frac{d}{2}$ from the corner as shown in Fig. 5.24.


Fig. 5.23


Fig. 5.24

## Determination of Centroid of Simple Figures From First Principles

For simple figures like triangle and semicircle, we can write general expression for the elemental area and its distance from an axis. Then Eqn. (5.4)

$$
\begin{aligned}
& \bar{y}=\frac{\int y d A}{A} \\
& \bar{x}=\frac{\int x d A}{A}
\end{aligned}
$$

The location of the centroid using the above equations may be considered as finding centroid from first principle. Now, let us find centroid of some standard figures from first principles.

## Centroid of a Triangle

Consider the triangle $A B C$ of base width $b$ and height $h$ as shown in Fig. 5.25. Let us locate the distance of centroid from the base. Let $b_{1}$ be the width of elemental strip of thickness $d y$ at a distance $y$ from the base. Since $\triangle A E F$ and $\triangle A B C$ are similar triangles, we can write:

$$
\begin{aligned}
& \frac{b_{1}}{b}=\frac{h-y}{h} \\
& b_{1}=\left(\frac{h-y}{h}\right) b=\left(1-\frac{y}{h}\right) b \text { we can write: }
\end{aligned}
$$

$\therefore$ Area of the element $=d A=b_{1} d y$

$$
=\left(1-\frac{y}{h}\right) b d y
$$

Area of the triangle $A=\frac{1}{2} b h$
$\therefore$ From Eqn. (5.4), we have

$$
\bar{y}=\frac{\text { Moment of area }}{\text { Total area }}=\frac{\int y d A}{A}
$$

Now,

$$
\begin{aligned}
\int y d A & =\int_{0}^{h} y\left(1-\frac{y}{h}\right) b d y \\
& =\int_{0}^{h}\left(y-\frac{y^{2}}{h}\right) b d y \\
& =b\left[\frac{y^{2}}{2}-\frac{y^{3}}{3 h}\right]_{0}^{h} \\
& =\frac{b h^{2}}{6}
\end{aligned}
$$

$$
\begin{array}{ll}
\therefore & \bar{y}=\int \frac{y d A}{A}=\frac{b h^{2}}{6} \times \frac{1}{\frac{1}{2} b h} \\
\therefore & \bar{y}=\frac{h}{3}
\end{array}
$$

Thus the centroid of a triangle is at a distance $\frac{h}{3}$ from the base (or $\frac{2 h}{3}$ from the apex) of the triangle where $h$ is the height of the triangle.

## Centroid of a Semicircle

Consider the semicircle of radius $R$ as shown in Fig. 5.26. Due to symmetry, centroid must lie on $y$-axis. Let its distance from diametral axis be $\bar{y}$. To find $\bar{y}$, consider an element at a distance $r$ from the centre $O$ of the semicircle, radial width being $d r$ and bound by radii at $\theta$ and $\theta+d \theta$.

$$
\text { Area of element }=r d \theta d r
$$

Its moment about diametral axis $x$ is given by:

$$
r d \theta \times d r \times r \sin \theta=r^{2} \sin \theta d r d \theta
$$

$\therefore$ Total moment of area about diametral axis,

$$
\begin{aligned}
\int_{0}^{\pi} \int_{0}^{R} r^{2} \sin \theta d r d \theta & =\int_{0}^{\pi}\left[\frac{r^{3}}{3}\right]_{0}^{R} \sin \theta d \theta \\
& =\frac{R^{3}}{3}[-\cos \theta]_{0}^{\pi}
\end{aligned}
$$

$$
=\frac{R^{3}}{3}[1+1]=\frac{2 R^{3}}{3}
$$

Area of semicircle $\quad A=\frac{1}{2} \pi R^{2}$

$$
\begin{aligned}
\therefore \quad \bar{y} & =\frac{\text { Moment of area }}{\text { Total area }}=\frac{\frac{2 R^{3}}{3}}{\frac{1}{2} \pi R^{2}} \\
& =\frac{4 R}{3 \pi}
\end{aligned}
$$

Thus, the centroid of the circle is at a distance $\frac{4 R}{3 \pi}$ from the diametral axis.

## Centroid of Sector of a Circle

Consider the sector of a circle of angle $2 \alpha$ as shown in Fig. 5.27. Due to symmetry, centroid lies on $x$-axis. To find its distance from the centre $O$, consider the elemental area shown.

Area of the element $=r d \theta d r$
Its moment about $y$ axis

$$
\begin{aligned}
& =r d \theta \times d r \times r \cos \theta \\
& =r^{2} \cos \theta d r d \theta
\end{aligned}
$$

$\therefore$ Total moment of area about $y$ axis

$$
\begin{aligned}
& =\int_{-\alpha}^{\alpha} \int_{0}^{R} r^{2} \cos \theta d r d \theta \\
& =\left[\frac{r^{3}}{3}\right]_{0}^{R}[\sin \theta]_{-\alpha}^{\alpha} \\
& =\frac{R^{3}}{3} 2 \sin \alpha
\end{aligned}
$$



Fig. 5.27

Total area of the sector

$$
\begin{aligned}
& =\int_{-\alpha}^{\alpha} \int_{0}^{R} r d r d \theta \\
& =\int_{-\alpha}^{\alpha}\left[\frac{r^{2}}{2}\right]_{0}^{R} d \theta \\
& =\frac{R^{2}}{2}[\theta]_{-\alpha}^{\alpha} \\
& =R^{2} \alpha
\end{aligned}
$$

$\therefore$ The distance of centroid from centre $O$

$$
\begin{aligned}
& =\frac{\text { Moment of area about } y \text {-axis }}{\text { Area of the figure }} \\
& =\frac{\frac{2 R^{3}}{3} \sin \alpha}{R^{2} \alpha}=\frac{2 R}{3 \alpha} \sin \alpha
\end{aligned}
$$

## Centroid of Parabolic Spandrel

Consider the parabolic spandrel shown in Fig. 5.28. Height of the element at a distance $x$ from $O$ is $y=k x^{2}$.

Width of element $=d x$
$\therefore$ Area of the element $=k x^{2} d x$
$\therefore$ Total area of spandrel $=\int_{0}^{a} k x^{2} d x$

$$
\left[\frac{k x^{3}}{3}\right]_{0}^{a}=\frac{k a^{3}}{3}
$$



Fig. 5.28

Moment of area about $y$-axis

$$
\begin{aligned}
& =\int_{0}^{a} k x^{2} d x \times x \\
& =\int_{0}^{a} k x^{3} d x \\
& =\frac{k a^{4}}{4}
\end{aligned}
$$

Moment of area about $x$-axis

$$
\begin{aligned}
& =\int_{0}^{a} k x^{2} d x \frac{k x^{2}}{2}=\int_{0}^{a} \frac{k^{2} x^{4}}{2} d x \\
& =\frac{k^{2} a^{5}}{10} \\
\therefore \quad \bar{x} & =\frac{k a^{4}}{4} \div \frac{k a^{3}}{3}=\frac{3 a}{4} \\
\bar{y} & =\frac{k^{2} a^{5}}{10} \div \frac{k a^{3}}{3}=\frac{3}{10} k a^{2}
\end{aligned}
$$

From Fig. 5.28, at $x=a, y=h$

$$
\begin{array}{ll}
\therefore & h=k a^{2} \text { or } k=\frac{h}{a^{2}} \\
\therefore & \bar{y}=\frac{3}{10} \times \frac{h}{a^{2}} a^{2}=\frac{3 h}{10}
\end{array}
$$

Thus, centroid of spandrel is $\left(\frac{3 a}{4}, \frac{3 h}{10}\right)$
Centroids of some common figures are shown in Table 5.2.

Table 5.2 Centroid of Some Common Figures

| Shape | Figure | $\bar{x}$ | $\bar{y}$ | Area |
| :---: | :---: | :---: | :---: | :---: |
| Triangle |  | - | $\frac{h}{3}$ | $\frac{b h}{2}$ |
| Semicircle |  | 0 | $\frac{4 R}{3 \pi}$ | $\frac{\pi R^{2}}{2}$ |
| Quarter circle |  | $\frac{4 R}{3 \pi}$ | $\frac{4 R}{3 \pi}$ | $\frac{\pi R^{2}}{4}$ |
| Sector of a circle | $\frac{t^{y}}{2 \alpha}-\theta_{x}$ | $\frac{2 R}{3 \alpha} \sin a$ | 0 | $\alpha R^{2}$ |
| Parabola |  | 0 | $\frac{3 h}{5}$ | $\frac{4 a h}{3}$ |
| Semi Parabola |  | $\frac{3}{8} a$ | $\frac{3 h}{5}$ | $\frac{2 a h}{3}$ |
| Parabolic Spandrel |  | $\frac{3 a}{4}$ | $\frac{3 h}{10}$ | $\frac{a h}{3}$ |

## Centroid of Composite Sections

So far, the discussion was confined to locating the centroid of simple figures such as rectangle, triangle, circle, semicircle, etc. In engineering practice, use of sections which are built up of many simple sections is very common. Such sections may be called as built-up sections or composite sections. To locate the centroid of composite sections, one need not go for the first principles (method of integration). The given composite section can be split into suitable simple figures and then the centroid of each simple figure can be found by inspection or using the standard formulae listed in Table 5.2. Assuming the area of the simple figure as concentrated at its centroid, its moment about an axis can be found by multiplying the area with distance of its centroid from the reference axis. After determining moment of each area about reference axis, the distance of centroid from the axis is obtained by dividing total moment of area by total area of the composite section.

Example 5.4 Locate the centroid of the T-section shown in the Fig. 5.29.
Solution. Selecting the axis as shown in Fig. 5.29, we can say that due to symmetry, centroid lies on $y$-axis, i.e., $\bar{x}=0$

Now the given T-section may be divided into two rectangles $A_{1}$ and $A_{2}$ each of size 100 $\times 20$ and $20 \times 100$. The centroid of $A_{1}$ and $A_{2}$ are $g_{1}(0,10)$ and $g_{2}(0,70)$ respectively.
$\therefore$ The distance of centroid from top is given by:

$$
\begin{aligned}
\bar{y} & =\frac{100 \times 20 \times 10+20 \times 100 \times 70}{100 \times 20+20 \times 100} \\
& =40 \mathrm{~mm}
\end{aligned}
$$

Hence, centroid of T-section is on the symmetric axis at a distance 40 mm from the


Fig. 5.29 top.

Ans.
Example 5.5 Find the centroid of the unequal angle $200 \times 150 \times 12 \mathrm{~mm}$, shown in Fig. 5.30.

Solution. The given composite figure can be divided into two rectangles:

$$
\begin{aligned}
A_{1} & =150 \times 12=1800 \mathrm{~mm}^{2} \\
\text { Total area } \quad A_{2} & =(200-12) \times 12=2256 \mathrm{~mm}^{2} \\
A & =A_{1}+A_{2}=4056 \mathrm{~mm}^{2}
\end{aligned}
$$

Selecting the reference axes $x$ and $y$ as shown in Fig. 5.30. The centroid of $A_{1}$ is $g_{1}(75,6)$ and that of $A_{2}$ is:

$$
\begin{array}{ll} 
& g_{2}\left[6,12+\frac{1}{2}(200-12)\right] \\
\text { i.e., } \quad & \\
\therefore \quad & g_{2}(6,106) \\
\therefore & =\frac{\text { Moment about } y \text {-axis }}{\text { Total area }} \\
& =\frac{A_{1} x_{1}+A_{2} x_{2}}{A} \\
& =\frac{1800 \times 75+2256 \times 6}{4056} \\
& =36.62 \mathrm{~mm} \\
\bar{y} & =\frac{\text { Moment about } x \text {-axis }}{\text { Total area }} \\
& =\frac{A_{1} y_{1}+A_{2} x_{2}}{A}
\end{array}
$$



Fig. 5.30

$$
\begin{aligned}
& =\frac{1800 \times 6+2256 \times 106}{4056} \\
& =61.62 \mathrm{~mm}
\end{aligned}
$$

Thus, the centroid is at $\bar{x}=36.62 \mathrm{~mm}$ and $\bar{y}=61.62 \mathrm{~mm}$ as shown in the figure.

Example 5.6 Locate the centroid of the I-section shown in Fig. 5.31.


Fig. 5.31
Solution. Selecting the co-ordinate system as shown in Fig. 5.31, due to symmetry centroid must lie on $y$-axis,
i.e.,

$$
\bar{x}=0
$$

Now, the composite section may be split into three rectangles

$$
A_{1}=100 \times 20=2000 \mathrm{~mm}^{2}
$$

Centroid of $A_{1}$ from the origin is:

$$
\begin{array}{ll} 
& y_{1}=30+100+\frac{20}{2}=140 \mathrm{~mm} \\
\text { Similarly, } & A_{2}=100 \times 20=2000 \mathrm{~mm}^{2} \\
y_{2}=30+\frac{100}{2}=80 \mathrm{~mm} \\
A_{3}=150 \times 30=4500 \mathrm{~mm}^{2}, \text { and } \\
& y_{3}=\frac{30}{2}=15 \mathrm{~mm}
\end{array}
$$

$$
\begin{aligned}
\therefore \quad \bar{Y} & =\frac{A_{1} Y_{1}+A_{2} Y_{2}+A_{3} Y_{3}}{A} \\
& =\frac{2000 \times 140+2000 \times 80+4500 \times 15}{2000+2000+4500} \\
& =59.71 \mathrm{~mm}
\end{aligned}
$$

Thus, the centroid is on the symmetric axis at a distance 59.71 mm from the bottom as shown in Fig. 5.31.

Example 5.7 Determine the centroid of the section of the concrete dam shown in Fig. 5.32.


Fig. 5.32
Solution. Let the axis be selected as shown in Fig. 5.32. Note that it is convenient to take axis in such a way that the centroids of all simple figures are having positive coordinates. If coordinate of any simple figure comes out to be negative, one should be careful in assigning the sign of moment of area of that figure.

The composite figure can be conveniently divided into two triangles and two rectangles, as shown in Fig. 5.32.

$$
\text { Now, } \quad \begin{aligned}
A_{1} & =\frac{1}{2} \times 2 \times 6=6 \mathrm{~m}^{2} \\
A_{2} & =2 \times 7.5=15 \mathrm{~m}^{2} \\
A_{3} & =\frac{1}{2} \times 3 \times 5=7.5 \mathrm{~m}^{2} \\
A_{4} & =1 \times 4=4 \mathrm{~m}^{2} \\
A & =\text { Total area }=32.5 \mathrm{~m}^{2}
\end{aligned}
$$

Centroids of simple figures are:

$$
\begin{aligned}
x_{1} & =\frac{2}{3} \times 2=\frac{4}{3} \mathrm{~m} \\
y_{1} & =\frac{1}{3} \times 6=2 \mathrm{~m} \\
x_{2} & =2+1=3 \mathrm{~m} \\
y_{2} & =\frac{7.5}{2}=3.75 \mathrm{~m} \\
x_{3} & =2+2+\frac{1}{3} \times 3=5 \mathrm{~m} \\
y_{3} & =1+\frac{1}{3} \times 5=\frac{8}{3} \mathrm{~m} \\
x_{4} & =4+\frac{4}{2}=6 \mathrm{~m} \\
y_{4} & =0.5 \mathrm{~m} \\
\bar{x} & =\frac{A_{1} x_{1}+A_{2} x_{2}+A_{3} x_{3}+A_{4} x_{4}}{A} \\
& =\frac{6 \times \frac{4}{3}+15 \times 3+7.5 \times 5+4 \times 6}{32.5} \\
& =3.523 \mathrm{~m} \\
\bar{y} & =\frac{A_{1} y_{1}+A_{2} y_{2}+A_{3} y_{3}+A_{4} y_{4}}{A} \\
\bar{x} & =3.523 \mathrm{~m} \\
\bar{y} & =2.777 \mathrm{~m} \\
& =\frac{6 \times 2+15 \times 3.75+7.5 \times \frac{8}{3}+4 \times 0.5}{32.5} \\
& =2.777 \mathrm{~m} \\
& =2
\end{aligned}
$$

The centroid is at
Ans.
and

Ans.
Example 5.8 Determine the centroid of the area shown in Fig. 5.33 with respect to the axis shown.


Fig. 5.33

Solution. The composite section is divided into three simple figures, a triangle, a rectangle and a semicircle.

| Now, area of triangle | $A_{1}=\frac{1}{2} \times 3 \times 4=6 \mathrm{~m}^{2}$ |
| :--- | :--- |
| Area of rectangle | $A_{2}=6 \times 4=24 \mathrm{~m}^{2}$ |
| Area of semicircle | $A_{3}=\frac{1}{2} \times \pi \times 2^{2}=6.2832 \mathrm{~m}^{2}$ |
| $\therefore$ Total area | $A=36.2832 \mathrm{~m}^{2}$ |

The coordinates of centroids of these three simple figures are:

$$
\begin{aligned}
x_{1} & =6+\frac{1}{3} \times 3=7 \mathrm{~m} \\
y_{1} & =\frac{4}{3} \mathrm{~m} \\
x_{2} & =3 \mathrm{~m} \\
y_{2} & =2 \mathrm{~m} \\
x_{3} & =\frac{-4 R}{3 \pi}=-\frac{4 \times 2}{3 \pi}=-0.8488 \mathrm{~m} \\
y_{3} & \left.=2 \mathrm{~m} \quad \text { (Note carefully the sign of } x_{3}\right) . \\
\bar{x} & =\frac{A_{1} x_{1}+A_{2} x_{2}+A_{3} x_{3}}{A} \\
& =\frac{6 \times 7+24 \times 3+6.2832 \times(-0.8488)}{36.2832}
\end{aligned}
$$

i.e.,

$$
\bar{x}=2.995 \mathrm{~m}
$$

$$
\bar{y}=\frac{A_{1} y_{1}+A_{2} y_{2}+A_{3} y_{3}}{A}
$$

$$
=\frac{\frac{6 \times 4}{3}+24 \times 2+6.2832 \times 2}{36.2832}
$$

Ans.

Ans.

Example 5.9 In a gusset plate, there are six rivet holes of 21.5 mm diameter as shown in Fig. 5.34. Find the position of the centroid of the gusset plate.


Fig. 5.34

Solution. The composite area is equal to a rectangle of size $160 \times 280 \mathrm{~mm}$ plus a triangle of size 280 mm base width and 40 mm height and minus areas of six holes. In this case also, Eqn. (5.4) can be used for locating centroid by treating area of holes as negative. The area of simple figures and their centroids are as shown in Table 5.3.

Table 5.3

| Figure | Area in $\mathrm{mm}^{2}$ | $x_{i}$ in mm | $y_{i}$ in mm |
| :--- | :--- | :--- | :--- |
| Rectangle | $160 \times 280=44,800$ | 140 | 80 |
| Triangle | $\frac{1}{2} \times 280 \times 40=5600$ | $\frac{560}{3}$ | $160+\frac{40}{3}=173.33$ |
|  |  |  |  |
| 1st hole | $\frac{-\pi \times 21.5^{2}}{4}=-363.05$ | 70 | 50 |
| 2nd hole | -363.05 | 140 | 50 |
| 3rd hole | -363.05 | 210 | 50 |
| 4th hole | -363.05 | 70 | 120 |
| 5th hole | -363.05 | 140 | 130 |
| 6th hole | -363.05 | 210 | 140 |
| $\therefore$ | $A=\Sigma A_{i}=48,221.70$ |  |  |
|  |  |  |  |

$$
\begin{aligned}
\therefore \quad A & =\Sigma A_{i}=48,221.70 \\
\therefore \quad \Sigma A_{i} x_{i} & =44,800 \times 140+5600 \times \frac{560}{3} \\
& -363.05(70+140+210+70+140+210) \\
& =70,12,371.3 \mathrm{~mm}^{3} \\
\bar{x} & =\frac{\Sigma A_{i} x_{i}}{A}=145.42 \mathrm{~mm} \\
\Sigma A_{i} y_{i} & =44,800 \times 80+5600 \times 173.33 \\
& -363.05(50 \times 3+120+130+140) \\
& =43,58,601 \mathrm{~mm}^{3} \\
\bar{y} & =\frac{\Sigma A_{i} y_{i}}{A}=\frac{43,58,601}{48221.70} \\
& =90.39 \mathrm{~mm}
\end{aligned}
$$

Thus, the coordinates of centroid of composite figure is given by:

$$
\begin{aligned}
& \bar{x}=145.42 \mathrm{~mm} \\
& \bar{y}=90.39 \mathrm{~mm}
\end{aligned}
$$

## Ans.

Ans.
Example 5.10 Determine the coordinates $x_{c}$ and $y_{c}$ of the centre of a 100 mm diameter circular hole cut in a thin plate so that this point will be the centroid of the remaining shaded area shown in Fig. 5.35 (All dimensions are in mm ).

Solution. If $x_{c}$ and $y_{c}$ are the coordinates of the centre of the circle, centroid also must have the coordinates $x_{c}$ and $y_{c}$ as per the condition laid down in the problem. The shaded area may be considered as a rectangle of size $200 \mathrm{~mm} \times 150 \mathrm{~mm}$ minus a triangle of sides $100 \mathrm{~mm} \times 75 \mathrm{~mm}$ and a circle of diameter 100 mm .

$$
\begin{aligned}
& \begin{aligned}
\therefore \text { Total area } & =200 \times 150-\frac{1}{2} \times 100 \times 75-\left(\frac{\pi}{4}\right) 100^{2} \\
& =18,396 \mathrm{~mm}^{2}
\end{aligned} \\
& \begin{aligned}
\bar{x} & =x_{c}
\end{aligned} \\
& \text { Hence, } x_{c}=\frac{200 \times 150 \times 100-\frac{1}{2} \times 100 \times 75 \times\left[200-\left(\frac{100}{3}\right)\right]-\frac{\pi}{4} \times 100^{2} \times x_{c}}{18,396} \\
& \therefore x_{c}(18,396)=200 \times 150 \times 100-\frac{1}{2} \times 100 \times 75 \times 166.67-\frac{\pi}{4} \times 100^{2} x_{c} \\
& \qquad x_{c}=\frac{23,75000}{26,250}=90.48 \mathrm{~mm}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& 18,396 y_{c}=200 \times 150 \times 75-\frac{1}{2} \times 100 \times 75 \times(150-25)-\frac{\pi}{4} \times 100^{2} y_{c} \\
& \therefore
\end{aligned} y_{c}=\frac{17,81250.0}{26,250}=67.86 \mathrm{~mm} \quad \text { Ans. }
$$

Centre of the circle should be located at $(90.48,67.86)$ so that this point will be the centroid of the remaining shaded area shown in Fig. 5.35.

Note: The centroid of the given figure will coincide with the centroid of the figure without circular hole. Hence, the centroid of the given figure may be obtained by determining the centroid of the figure without the circular hole also.

Example 5.11 Determine the coordinates of the centroid of the plane area shown in Fig. 5.36 with reference to the axis shown. Take $x=40 \mathrm{~mm}$
Solution. The composite figure is divided into the following simple figures:
(i) A rectangle

$$
\begin{aligned}
A_{1} & =(14 x) \times(12 x)=168 x^{2} \\
x_{1} & =7 x ; y_{1}=6 x:
\end{aligned}
$$

(ii) A triangle

$$
\begin{aligned}
& A_{2}=\frac{1}{2}(6 x) \times(4 x)=12 x^{2} \\
& x_{2}=14 x+2 x=16 x \\
& y_{2}=\frac{4 x}{3}
\end{aligned}
$$



Fig. 5.36
(iii) A rectangle to be subtracted

$$
\begin{aligned}
A_{3} & =(-4 x) \times(4 x)=-16 x^{2} \\
\mathrm{x}_{3} & =2 x ; y_{3}=8 x+2 x=10 x
\end{aligned}
$$

(iv) A semicircle to be subtracted

$$
\begin{aligned}
& A_{4}=-\frac{1}{2} \pi(4 x)^{2}=-8 \pi x^{2} \\
& x_{4}=6 x \\
& y_{4}=\frac{4 R}{3 \pi}=4 \times \frac{(4 x)}{3 \pi}=\frac{16 x}{3 \pi}
\end{aligned}
$$

(v) A quarter of a circle to be subtracted

$$
\begin{aligned}
A_{5}= & -\frac{1}{4} \times \pi(4 x)^{2}=-4 \pi x^{2} \\
x_{5}= & 14 x-\frac{4 R}{3 \pi}=14 x-(4)\left(\frac{4 x}{3 \pi}\right)=12.3023 x \\
y_{5}= & 12 x-4 \times\left(\frac{4 x}{3 \pi}\right)=10.3023 x \\
A= & 168 x^{2}+12 x^{2}-16 x^{2}-8 \pi x^{2}-4 \pi x^{2} \\
= & 126.3009 x^{2} \\
\bar{x}= & \frac{\Sigma A_{i} x_{i}}{A} \\
\Sigma A_{i} x_{i}= & 168 x^{2} \times 7 x+12 x^{2} \times 16 x-16 x^{2} \times 2 x \\
& -8 \pi x^{2} \times 6 x-4 \pi x^{2} \times 12.3023 x \\
= & 1030.6083 x
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad \bar{x} & =\frac{1030.6083 x^{3}}{126.3009 x^{2}} \\
& =8.1599 x=8.1599 \times 40 \quad(\text { since } x=40 \mathrm{~mm}) \\
& =326.40 \mathrm{~mm} \\
\bar{y} & =\frac{\Sigma A_{i} y_{i}}{A} \\
\Sigma A_{i} x_{i} & =168 x^{2} \times 6 x+12 x^{2} \times \frac{4 x}{3}-16 x^{2} \times 10 x \\
& \\
& =691.8708 x^{3} \\
\therefore \quad \bar{y} & =\frac{691.8708}{126.3009 x^{2}} \\
& =5.4780 x \\
& =219.12 \mathrm{~mm}
\end{aligned} \quad \begin{aligned}
& \text { (since } x=40 \mathrm{~mm})
\end{aligned}
$$

Centroid is at $(326.40,219.12)$

### 5.5 SECOND MOMENTS OF PLANE AREA

Consider the area shown in Fig. 5.37(a). $d A$ is an elemental area with coordinates as $x$ and $y$. The term $\Sigma y_{i}^{2} d A_{i}$ is called moment of inertia of the area about $x$-axis and is denoted as $I_{x x}$. Similarly, the moment of inertia about $y$-axis is

$$
I_{y y}=\Sigma x^{2}{ }_{i} d A_{i}
$$

In general, if $r$ is the distance of elemental area $d A$ from the axis $A B$ [Fig. 5.37(b)], the sum of the terms $\Sigma r^{2} d A$ to cover the entire area is called moment of inertia of the area about the axis $A B$. If $r$ and $d A$ can be expressed in general terms, for any element, then the sum becomes an integral. Thus,

$$
\begin{equation*}
I_{A B}=\Sigma r_{i}^{2} d A_{i}=\int r^{2} d A \tag{5.6}
\end{equation*}
$$



Fig. 5.37

The term $r d A$ may be called as moment of area, similar to moment of a force, and hence $r^{2} d A$ may be called as moment of moment of area or the second moment of area. Thus, the moment of inertia of area is nothing but second moment of area. In fact, the term 'second moment of area' appears to correctly signify the meaning of the expression $\Sigma r^{2} d A$. The term 'moment of inertia' is rather a misnomer. However, the term moment of inertia has come to stay for long time, and hence it will be used in this book also.

Though moment of inertia of plane area is purely a mathematical term, it is one of the important properties of areas. The strength of members subject to bending depends on the moment of inertia of its cross-sectional area. Students will find this property of area very useful when they study subjects such as strength of materials, structural design and machine design.

The moment of inertia is a fourth dimensional term, since it is a term obtained by multiplying area by the square of the distance. Hence, in SI units, if metre (m) is the unit for linear measurements used then $m^{4}$ is the unit of moment of inertia. If millimetre ( mm ) is the unit used for linear measurements, then $\mathrm{mm}^{4}$ is the unit of moment of inertia. In MKS system $\mathrm{m}^{4} \mathrm{or} \mathrm{cm}^{4}$ and in FPS system $\mathrm{ft}^{4}$ or in ${ }^{4}$ are commonly used as units for moment of inertia.

## Polar Moment of Inertia

Moment of inertia about an axis perpendicular to the plane of an area is known as polar moment of inertia. It may be denoted as $J$ or $I_{z z}$. Thus, the moment of inertia about an axis perpendicular to the plane of the area at $O$ in


Fig. 5.38 Fig. 5.38 is called polar moment of inertia at point $O$, and is given by

$$
\begin{equation*}
I_{z z}=\Sigma r^{2} d A \tag{5.7}
\end{equation*}
$$

## Radius of Gyration

Radius of gyration is a mathematical term defined by the relation

$$
\begin{equation*}
k=\sqrt{\frac{I}{A}} \tag{5.8}
\end{equation*}
$$

where $\quad k=$ radius of gyration
and $\quad A=$ the cross-sectional area
Suffixes with moment of inertia $I$ also accompany the term radius of gyration $k$. Thus, we can have

$$
k_{x x}=\sqrt{\frac{I_{x x}}{A}}
$$

$$
\begin{aligned}
k_{y y} & =\sqrt{\frac{I_{y y}}{A}} \\
k_{A B} & =\sqrt{\frac{I_{A B}}{A}}
\end{aligned}
$$

and so on.
The relation between radius of gyration and moment of inertia can be put in the form:

$$
I=A k^{2}
$$

From the above relation, a geometric meaning can be assigned to the term 'radius of gyration.' We can consider $k$ as the distance at which the complete area is squeezed and kept as a strip of negligible width (Fig. 5.39) such that there is no change in the moment of inertia.

## Theorems of Moments of Inertia

There are two theorems of moment of inertia:
(i) Perpendicular axis theorem, and
(ii) Parallel axis theorem.


Fig. 5.39

These are explained and proved below :

## Perpendicular Axis Theorem

The moment of inertia of an area about an axis pependicular to its plane (polar moment of inertia) at any point $O$ is equal to the sum of moments of inertia about any two mutually perpendicular axis through the same point $O$ and lying in the plane of the area.

Referring to Fig. 5.40, if $z-z$ is the axis normal to the plane of paper passing through point $O$, as per this theorem,

$$
\begin{equation*}
I_{z z}=I_{x x}+I_{y y} \tag{5.10}
\end{equation*}
$$

The above theorem can be easily proved. Let us consider an elemental area $d A$ at a distance $r$ from $O$.


Fig. 5.40

Let the coordinates of $d A$ be $x$ and $y$. Then from definition:

$$
\begin{aligned}
I_{z z} & =\Sigma r^{2} d A \\
& =\Sigma\left(x^{2}+y^{2}\right) d A \\
& =\Sigma x^{2} d A+\Sigma y^{2} d A \\
I_{z z} & =I_{x x}+I_{y y}
\end{aligned}
$$

## Parallel Axis Theorem

Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axis. Referring to Fig. 5.41, the above theorem means:

$$
\begin{equation*}
I_{A B}=I_{G G}+A y_{c}^{2} \tag{5.11}
\end{equation*}
$$

where

$$
\begin{aligned}
& I_{A B}=\text { Moment of inertia about axis } A B \\
& I_{G G^{\prime}}=\text { Moment of inertia about centroidal axis } G G^{\prime} \text { parallel to } A B . \\
& A=\text { The area of the plane figure given and } \\
& y_{c}=\text { The distance between the axis } A B \text { and the parallel centroidal } \\
& \text { axis } G G^{\prime} .
\end{aligned}
$$

Proof: Consider an elemental parallel strip $d A$ at a distance $y$ from the centroidal axis (Fig 5.41).

Then, $\quad I_{A B}=\Sigma\left(y+y_{c}\right)^{2} d A$

$$
=\Sigma\left(y^{2}+2 y y_{c}+y_{c}^{2}\right) d A
$$

$$
=\Sigma y^{2} d A+\Sigma 2 y y_{c} d A+\Sigma y_{c}^{2} d A
$$

Now, $\Sigma y^{2} d A=$ Moment of inertia about the axis $G G^{\prime}$

$$
\begin{aligned}
& =I_{G G^{\prime}} \\
\Sigma 2 y y_{c} d A & =2 y_{c} \Sigma y d A \\
& =2 y_{c} A \frac{\Sigma y d A}{A}
\end{aligned}
$$



Fig. 5.41

In the above, term $2 y_{c} A$ is constant and $\frac{\Sigma y d A}{A}$ is the distance of centroid from the reference axis $G G^{\prime}$. Since $G G^{\prime}$ is passing through the centroid itself $\frac{y d A}{A}$ is zero and hence the term $\Sigma 2 y y_{c} d A$ is zero.

Now, the third term,

$$
\begin{aligned}
\Sigma y_{c}^{2} d A & =y_{c}^{2} \Sigma d A \\
& =A y_{c}^{2} \\
\therefore \quad I_{A B} & =I_{G G^{\prime}}+A y_{c}^{2}
\end{aligned}
$$

## Note:

The above equation cannot be applied to any two parallel axis. One of the axis $\left(G G^{\prime}\right)$ must be centroidal axis only.

### 5.6 MOMENT OF INERTIA FROM FIRST PRINCIPLES

For simple figures, moment of inertia can be obtained by writing the general expression for an element and then carrying out integration so as to cover the entire area. This procedure is illustrated with the following three cases:
(i) Moment of inertia of a rectangle about the centroidal axis
(ii) Moment of inertia of a triangle about the base
(iii) Moment of inertia of a circle about a diametral axis

Moment of Inertia of a Rectangle about the Centroidal Axis: Consider a rectangle of width $b$ and depth $d$ (Fig. 5.42). Moment of inertia about the centroial axis $x-x$ parallel to the short side is required.

Consider an elemental strip of width $d y$ at a distance $y$ from the axis. Moment of inertia of the elemental strip about the centroidal axis $x x$ is:

$$
\begin{aligned}
& =y^{2} d A \\
& =y^{2} b d y \\
\therefore \quad I_{x x} & =\int_{-d / 2}^{d / 2} y^{2} b d y \\
& =b\left[\frac{y^{3}}{3}\right]_{-d / 2}^{d / 2} \\
& =b\left[\frac{d^{3}}{24}+\frac{d^{3}}{24}\right] \\
I_{x x} & =\frac{b d^{3}}{12}
\end{aligned}
$$



Fig. 5.42

Moment of Inertia of a Triangle About its Base: Moment of inertia of a triangle with base width $b$ and height $h$ is to be determined about the base $A B$ (Fig. 5.43).


Fig. 5.43
Consider an elemental strip at a distance $y$ from the base $A B$. Let $d y$ be the thickness of the strip and $d A$ its area. Width of this strip is given by:

$$
b_{1}=\frac{(h-y)}{h} \times b
$$

Moment of inertia of this strip about $A B$

$$
\begin{aligned}
& =y^{2} d A \\
& =y^{2} b_{1} d y \\
& =y^{2} \frac{(h-y)}{h} \times b \times d y
\end{aligned}
$$

$\therefore$ Moment of inertia of the triangle about $A B$,

$$
\begin{aligned}
I_{A B} & =\int_{0}^{h} \frac{y^{2}(h-y) b d y}{h} \\
& =\int_{0}^{h}\left(y^{2}-\frac{y^{3}}{h}\right) b d y \\
& =b\left[\frac{y^{3}}{3}-\frac{y^{4}}{4 h}\right]_{0}^{h} \\
& =b\left[\frac{h^{3}}{3}-\frac{h^{4}}{4 h}\right] \\
I_{A B} & =\frac{b h^{3}}{12}
\end{aligned}
$$

Moment of Inertia of Circle about its Diametral Axis: Moment of inertia of a circle of radius $R$ is required about it's diametral axis as shown in Fig. 5.44.

Consider an element of sides $r d \theta$ and $d r$ as shown in the figure. It's moment of inertia about the diametral axis $x-x$ :

$$
\begin{aligned}
& =y^{2} d A \\
& =(r \sin \theta)^{2} r d \theta d r \\
& =r^{3} \sin ^{2} \theta d \theta d r
\end{aligned}
$$

$\therefore$ Moment of inertia of the circle about $x-x$ is given by

$$
\begin{aligned}
\mathrm{I}_{x x} & =\int_{0}^{R} \int_{0}^{2 \pi} r^{3} \sin ^{2} \theta d \theta d r \\
& =\int_{0}^{R} \int_{0}^{2 \pi} r^{3} \frac{(1-\cos 2 \theta)}{2} d \theta d r \\
& =\int_{0}^{R} \frac{r^{3}}{2}\left[\theta-\frac{\sin 2 \theta}{2}\right]_{0}^{2 \pi} d r \\
& =\left[\frac{r^{4}}{8}\right]_{0}^{R}[2 \pi-0+0-0]
\end{aligned}
$$



Fig. 5.44

$$
\begin{aligned}
& =\frac{2 \pi}{8} R^{4} \\
I_{x x} & =\frac{\pi R^{4}}{4}
\end{aligned}
$$

If $d$ is the diameter of the circle, then

$$
\begin{aligned}
R & =\frac{d}{2} \\
\therefore \quad I_{x x} & =\frac{\pi}{4}\left(\frac{d}{2}\right)^{4} \\
I_{x x} & =\frac{\pi d^{4}}{64}
\end{aligned}
$$

## Moment of Inertia of Standard Sections

Rectangle: Referring to Fig. 5.45.
(a) $I_{x x}=\frac{b d^{3}}{12}$ as derived from first principle
(b) $I_{y y}=\frac{d b^{3}}{12}$ can be derived on the same lines.
(c) About the base $A B$, from parallel axis theorem,


Fig. 5.45

$$
\begin{aligned}
I_{A B} & =I_{x x}+A y_{c}^{2} \\
& =\frac{b d^{3}}{12}+b d\left(\frac{d}{2}\right)^{2}, \text { since } y_{c}=\frac{d}{2} \\
& =\frac{b d^{3}}{12}+\frac{b d^{3}}{4} \\
& =\frac{b d^{3}}{3}
\end{aligned}
$$

Symmetric Hollow Rectangular Section: Referring to Fig. 5.46, Moment of inertia $I_{x x}=$ Moment of inertia of larger rectangle-Moment of inertia of hollow portion.That is,

$$
\begin{aligned}
& =\frac{B D^{3}}{12}-\frac{b d^{3}}{12} \\
& =\frac{1}{12}\left(B D^{3}-b d^{3}\right)
\end{aligned}
$$



Fig. 5.46

Triangle—Referring to Fig. 5.47.
(a) About the base:

As found from first principles

$$
I_{A B}=\frac{b h^{3}}{12}
$$

(b) About centroidal axis, $x-x$ parallel to base:


Fig. 5.47

From parallel axis theorem,

$$
I_{A B}=I_{x x}+A y_{c}^{2}
$$

Now, $y_{c}$, the distance between the non-centroidal axis $A B$ and centroidal axis $x-x$, is equal to $\frac{h}{3}$.

$$
\begin{aligned}
\therefore \quad \frac{b h^{3}}{12} & =I_{x x}+\frac{1}{2} b h\left(\frac{h}{3}\right)^{2} \\
& =I_{x x}+\frac{b h^{3}}{18} \\
\therefore \quad I_{x x} & =\frac{b h^{3}}{12}-\frac{b h^{3}}{18} \\
& =\frac{b h^{3}}{36}
\end{aligned}
$$

Moment of Inertia of a Circle about any diametral axis

$$
=\frac{\pi d^{4}}{64}
$$

(as found from first principles)
Moment of Inertia of a Concentric Hollow Circle: Referring to Fig. 5.48. $I_{A B}=$ Moment of inertia of solid circle of diameter $D$ about $A B$ minus Moment of inertia of circle of diameter $d$ about $A B$. That is,

$$
\begin{aligned}
& =\frac{\pi D^{4}}{64}-\frac{\pi d^{4}}{64} \\
& =\frac{\pi}{64}\left(D^{4}-d^{4}\right)
\end{aligned}
$$



Fig. 5.48

Moment of Inertia of a Semicircle: (a) About Diametral Axis:
If the limit of integration is put as 0 to $\pi$ instead of 0 to $2 \pi$ in the derivation for the moment of inertia of a circle about diametral axis the moment of inertia of a semicircle is obtained. It can be observed that the moment of inertia of a semicircle (Fig. 5.49) about the diametral axis $A B$ :

$$
=\frac{1}{2} \times \frac{\pi d^{4}}{64}=\frac{\pi d^{4}}{128}
$$

(b) About centroidal axis $x-x$ ': Now, the distance of centroidal axis $y_{c}$ from the diametral axis is given by:


Fig. 5.49

$$
y_{c}=\frac{4 R}{3 \pi}=\frac{2 d}{3 \pi}
$$

and, $\quad$ Area $A=\frac{1}{2} \times \frac{\pi d^{2}}{4}=\frac{\pi d^{2}}{8}$
From parallel axis theorem,

$$
\begin{aligned}
I_{A B} & =I_{x x}+A y_{c}^{2} \\
\frac{\pi d^{4}}{128} & =I_{x x}+\frac{\pi d^{2}}{8} \times\left(\frac{2 d}{3 \pi}\right)^{2} \\
I_{x x} & =\frac{\pi d^{4}}{128}-\frac{d^{4}}{18 \pi} \\
& =0.0068598 d^{4}
\end{aligned}
$$

Moment of Inertia of a Quarter of a Circle: (a) About the base:
If the limit of integration is put as 0 to $\frac{\pi}{2}$ instead of 0 to $2 \pi$ in the derivation for moment of inertia of a circle, the moment of inertia of a quarter of a circle is obtained. It can be observed that moment of inertia of the quarter of a circle about the base $A B$.

$$
=\frac{1}{4} \times \frac{\pi d^{4}}{64}=\frac{\pi d^{4}}{256}
$$

(b) About centroidal axis $x-x$

Now, the distance of centroidal axis $y_{c}$ from the base is given by:

$$
y_{c}=\frac{4 R}{3 \pi}=\frac{2 d}{3 \pi}
$$

and the area

$$
A=\frac{1}{4} \times \frac{\pi d^{2}}{4}=\frac{\pi d^{2}}{16}
$$



Fig. 5.50

From parallel axis theorem,

$$
\begin{aligned}
I_{A B} & =I_{x x}+A y_{c}^{2} \\
\frac{\pi d^{4}}{256} & =I_{x x}+\frac{\pi d^{2}}{16}\left(\frac{2 d}{3 \pi}\right)^{2} \\
I_{x x} & =\frac{\pi d^{4}}{256}-\frac{d^{4}}{36 \pi} \\
& =0.00343 d^{4}
\end{aligned}
$$

The moment of inertia of common standard sections are presented in Table 5.4.

Table 5.4 Moment of Inertia of Standard Sections

(Contd...)


### 5.7 MOMENT OF INERTIA OF COMPOSITE SECTIONS

Beams and columns having composite sections are commonly used in structures. Moment of inertia of these sections about an axis can be found by the following steps:
(i) Divide the given figure into a number of simple figures.
(ii) Locate the centroid of each simple figure by inspection or using standard expressions.
(iii) Find the moment of inertia of each simple figure about its centroidal axis. Add the term $A y^{2}$ where $A$ is the area of the simple figure and $y$ is the distance of the centroid of the simple figure from the reference axis. This gives moment of inertia of the simple figure about the reference axis.
(iv) Sum up moments of inertia of all simple figures to get the moment of inertia of the composite section.
The procedure given above is illustrated below. Referring to the Fig. 5.51, it is required to find out the moment of inertia of the section about axis $A-B$.
(i) The section in the figure is divided into a rectangle, a triangle and a semicircle. The areas of the simple figures $A_{1}, A_{2}$ and $A_{3}$ are calculated.


Fig. 5.51
(ii) The centroids of the rectangle $\left(g_{1}\right)$, triangle $\left(g_{2}\right)$ and semicircle $\left(g_{3}\right)$ are located. The distances $y_{1}, y_{2}$ and $y_{3}$ are found from the axis $A B$.
(iii) The moment of inertia of the rectangle about it's centroid $\left(I_{g 1}\right)$ is calculated using standard expression. To this, the term $A_{1} y_{1}^{2}$ is added to get the moment of inertia about the axis $A B$ as:

$$
I^{2}=I_{g_{1}}+A_{1} y_{1}^{2}
$$

Similarly, the moment of inertia of the triangle $\left(I_{2}=I_{g_{2}}+A_{2} y_{2}^{2}\right)$ and of semicircle $\left(I_{3}=I_{g_{3}}+A_{3} y_{3}^{2}\right)$ about axis $A B$ are calculated.
(iv) Moment of inertia of the composite section about $A B$ is given by:

$$
\begin{align*}
I_{A B} & =I_{1}+I_{2}+I_{3} \\
& =I_{g_{1}}+A_{1} y_{1}^{2}+I_{g_{2}}+A_{2} y_{2}^{2}+I_{g_{3}}+A_{3} y_{3}^{2} \tag{5.12}
\end{align*}
$$

In most engineering problems, moment of inertia about the centroidal axis is required. In such cases, first locate the centroidal axis as discussed in Section 5.4 and then find the moment of inertia about this axis.

Referring to Fig. 5.52, first the moment of area about any reference axis, say $A B$ is taken and is divided by the total area of section to locate centroidal axis $x-x$. Then the distances of centroid of individual figures $y_{c_{1}}, y_{c_{2}}$ and $y_{c_{3}}$ from the axis $x-x$ are determined. The moment of inertia of the composite section about the centroidal axis $x-x$ is calculated using the expression:

$$
\begin{equation*}
I_{x x}=I_{g_{1}}+A_{1} y_{c_{1}}^{2}+I_{g_{2}}+A_{2} y_{c_{2}}^{2}+I_{g_{3}}+A_{3} y_{c_{3}}^{2} \tag{5.13}
\end{equation*}
$$



Fig. 5.52

Sometimes the moment of inertia is found about a convenient axis and then using parallel axis theorem, the moment of inertia about centroidal axis is found.

In the above example, the moment of inertia $I_{A B}$ is found and $y_{C}$, the distance of $C G$ from axis $A B$ is calculated. Then from parallel axis theorem,

$$
\begin{aligned}
I_{A B} & =I_{x x}+A y_{c}^{2} \\
I_{x x} & =I_{A B}-A y_{c}^{2}
\end{aligned}
$$

where $A$ is the area of composite section.
Example 5.12 Determine the moment of inertia of the section shown in Fig. 5.53 about an axis passing through the centroid and parallel to the top most fibre of the section.

Also determine moment of inertia about the axis of symmetry. Hence find out radii of gyration.

Solution. The given composite section can be divided into two rectangles as follows:

Area

$$
\begin{aligned}
A_{1} & =150 \times 10=1500 \mathrm{~mm}^{2} \\
A_{2} & =140 \times 10=1400 \mathrm{~mm}^{2} \\
A & =A_{1}+A_{2}=2900 \mathrm{~mm}^{2}
\end{aligned}
$$

Area


Fig. 5.53

Total Area $\quad A=A_{1}+A_{2}=2900 \mathrm{~mm}^{2}$
Due to symmetry, centroid lies on the symmetric axis $y-y$.
The distance of the centroid from the top most fibre is given by:

$$
\begin{aligned}
y_{c} & =\frac{\text { Sum of moment of the areas about the top most fibre }}{\text { Total area }} \\
& =\frac{1500 \times 5+1400(10+70)}{2900} \\
& =41.21 \mathrm{~mm}
\end{aligned}
$$

Referring to the centroidal axis $x-x$ and $y-y$, the centroid of $A_{1}$ is $g_{1}$ $(0.0,36.21)$ and that of $A_{2}$ is $g_{2}(0.0,38.79)$.

Moment of inertia of the section about $x-x$ axis
$I_{x x}=$ Moment of inertia of $A_{1}$ about $x-x$ axis + Moment of inertia of $A_{2}$ about $x-x$ axis.
$\therefore \quad I_{x x}=\frac{150 \times 10^{3}}{12}+1500(36.21)^{2}+\frac{10 \times 140^{3}}{12}+1400(38.79)^{2}$
i.e., $\quad I_{x x}=63,72442.5 \mathrm{~mm}^{4}$

Ans.
Similarly,

$$
I_{y y}=\frac{10 \times 150^{3}}{12}+\frac{140 \times 10^{3}}{12}=28,24166.7 \mathrm{~mm}^{4}
$$

Ans.
Hence, the moment of inertia of the section about an axis passing through the centroid and parallel to the top most fibre is $63,72442.5 \mathrm{~mm}^{4}$ and moment of inertia of the section about the axis of symmetry is $28,24166.66 \mathrm{~mm}^{4}$.

The radius of gyration is given by:

$$
\begin{aligned}
k & =\sqrt{\frac{I}{A}} \\
\therefore \quad k_{x x} & =\sqrt{\frac{I_{x x}}{A}} \\
& =\sqrt{\frac{63,72442.5}{2900}} \\
\text { Similarly, } & k_{x x x}
\end{aligned}=46.88 \mathrm{~mm},
$$

Ans.

## Ans.

Example 5.13 Determine the moment of inertia of the L-section shown in Fig. 5.54 about its centroidal axis parallel to the legs. Also find out the polar moment of inertia.

Solution. The given section is divided into two rectangles $A_{1}$ and $A_{2}$.
Area

$$
\begin{aligned}
A_{1} & =125 \times 10=1250 \mathrm{~mm}^{2} \\
A_{2} & =75 \times 10=750 \mathrm{~mm}^{2} \\
& =2000 \mathrm{~mm}^{2}
\end{aligned}
$$

Area
Total Area
First, the centroid of the given section is to be located.
Two reference axis (1)-(1) and (2)-(2) are chosen as shown in Fig. 5.54.
The distance of centroid from the axis (1)-(1)

$$
\begin{aligned}
& =\frac{\text { Sum of moment of areas } A}{\text { Total }} \\
& =\frac{1250 \times 5+750\left(10+\frac{75}{2}\right)}{2000} \\
& =20.94 \mathrm{~mm}
\end{aligned}
$$

Similarly, the distance of the centroid from the axis (2)-(2)

$$
\begin{aligned}
& =\bar{y}=\frac{1250 \times \frac{125}{2}+750 \times 5}{2000} \\
& =40.94 \mathrm{~mm}
\end{aligned}
$$



Fig. 5.54

With respect to the centroidal axis $x-x$ and $y-y$, the centroid of $A_{1}$ is $g_{1}$ $(15.94,21.56)$ and that of $A_{2}$ is $g_{2}(26.56,35.94)$.
$\therefore \quad I_{x x}=$ Moment of inertia of $A_{1}$ about $x-x$ axis + Moment of inertia of $A_{2}$ about $x-x$ axis
$\therefore \quad I_{x x}=\frac{10 \times 125^{3}}{12}+1250 \times 21.56^{2}+\frac{75 \times 10^{3}}{12}+750 \times 39.94^{2}$
i.e., $\quad I_{x x}=34,11298.9 \mathrm{~mm}^{4}$

Ans.
Similarly,

$$
I_{y y}=\frac{125 \times 10^{3}}{12}+1250 \times 15.94^{2}+\frac{10 \times 75^{3}}{12}+750 \times 26.56^{2}
$$

i.e., $\quad I_{y y}=12,08658.9 \mathrm{~mm}^{4}$

Ans.
Polar moment of inertia $=I_{x x}+I_{y y}$

$$
\begin{aligned}
& =34,11298.9+12,08658.9 \\
I_{z z} & =46,19957.8 \mathrm{~mm}^{4}
\end{aligned}
$$

Ans.
Example 5.14 Determine the moment of inertia of the symmetric I-section shown in Fig. 5.55 about its centroidal axes $x-x$ and $y-y$.


Fig. 5.55
Also, determine moment of inertia of the section about a centroidal axis perpendicular to $x-x$ axis and $y-y$ axes.
Solution. The section is divided into three rectangles $A_{1}, A_{2}, A_{3}$.
Area

$$
A_{1}=200 \times 9=1800 \mathrm{~mm}^{2}
$$

Area

$$
A_{2}=(250-9 \times 2) \times 6.7=1554.4 \mathrm{~mm}^{2}
$$

Area

$$
A_{3}=200 \times 9=1800 \mathrm{~mm}^{2}
$$

Total Area

$$
A=5154.4 \mathrm{~mm}^{2}
$$

The section is symmetrical about both $x-x$ and $y-y$ axis. Therefore, its centroid will coincide with the centroid of rectangle $A_{2}$.

With respect to the centroidal axis $x-x$ and $y-y$, the centroid of rectangle $A_{1}$ is $g_{1}(0.0,120.5)$, that of $A_{2}$ is $g_{2}(0.0,0.0)$ and that of $A_{3}$ is $g_{3}(0.0,-120.5)$.

$$
\begin{aligned}
I_{x x}= & \text { Moment of inertia of } A_{1}+\text { Moment of inertia of } A_{2}+\text { Moment } \\
& \text { of inertia of } A_{3} \text { about } x-x \text { axis }
\end{aligned}
$$

$$
\begin{aligned}
I_{x x}= & \frac{200 \times 9^{3}}{12}+1800 \times 120.5^{2}+\frac{6.7 \times 232^{3}}{12}+0 \\
& +\frac{200 \times 9^{3}}{12}+1800 \times(-120.5)^{2}
\end{aligned}
$$

$$
I_{x x}=5,92,69202 \mathrm{~mm}^{4}
$$

Ans.
Similarly,

$$
\begin{aligned}
& I_{y y}=\frac{9 \times 200^{3}}{12}+\frac{232 \times 6.7^{3}}{12}+\frac{9 \times 200^{3}}{12} \\
& I_{y y}=1,20,05815 \mathrm{~mm}^{4}
\end{aligned}
$$

Ans.
Moment of inertia of the section about a centroidal axis perpendicular to $x-x$ and $y-y$ axes is nothing but polar moment of inertia, and is given by:

$$
\begin{aligned}
I_{z z} & =I_{x x}+I_{y y} \\
& =5,92,69202+1,20,05815 \\
I_{z z} & =7,12,75017 \mathrm{~mm}^{4}
\end{aligned}
$$

Ans.
Example 5.15 Compute the second moment of area of the channel section shown in Fig. 5.56 about centroidal axes $x-x$ and $y-y$.
Solution. The section is divided into three rectangles $A_{1}, A_{2}$ and $A_{3}$.
Area $\quad A_{1}=100 \times 13.5=1350.0 \mathrm{~mm}^{2}$
Area $\quad A_{2}=(400-27) \times 8.1=3021.3 \mathrm{~mm}^{2}$
Area $\quad A_{3}=100 \times 13.5=1350.0 \mathrm{~mm}^{2}$
Total Area $A=5721.3 \mathrm{~mm}^{2}$
The given section is symmetric about horizontal axis passing through the centroid $g_{2}$ of the rectangle $A_{2}$. A reference axis (1)-(1) is chosen as shown in Fig. 5.56. The distance of the centroid of the section from (1)-(1)

$$
\begin{aligned}
& =\frac{1350 \times 50+3021.3 \times \frac{8.1}{2}+1350 \times 50}{5721.3} \\
& =25.73 \mathrm{~mm}
\end{aligned}
$$



Fig. 5.56

With reference to the centroidal axes $x-x$ and $y-y$, the centroid of the rectangle $A_{1}$ is $g_{1}(24.27,193.25)$ that of $A_{2}$, is $g_{2}(21.68,0.0)$ and that of $A_{3}$ is $g_{3}(24.27$, 193.25).
$\therefore \quad I_{x x}=$ Moment of inertia of $A_{1}, A_{2}$ and $A_{3}$ about $x-x$

$$
\begin{aligned}
= & \frac{100 \times 13.5^{3}}{12}+1350 \times 193.25^{2}+\frac{8.1 \times 373^{3}}{12}+\frac{100 \times 13.5^{3}}{12} \\
& +1350 \times 193.25^{2} \\
I_{x x}= & 1.359 \times 10^{8} \mathrm{~mm}^{4}
\end{aligned}
$$

Ans.

Similarly,

$$
\begin{aligned}
I_{y y}= & \frac{13.5 \times 100^{3}}{12}+1350 \times 24.27^{2}+\frac{273 \times 8.1^{3}}{12}+3021.13 \\
& \times 21.68^{2}+\frac{13.5 \times 100^{3}}{12}+1350 \times 24.27^{2} \\
I_{y y}= & 52,72557.6 \mathrm{~mm}^{4}
\end{aligned}
$$

Ans.
Example 5.16 Determine the polar moment of inertia of the I-section shown in the Fig. 5.57. Also determine the radii of gyration with respect to $x-x$ axis and $y-y^{\prime}$ axis.


Fig. 5.57
Solution. The section is divided into three rectangles as shown in Fig. 5.57.

Area
Area

$$
A_{1}=80 \times 12=960 \mathrm{~mm}^{2}
$$

$$
A_{2}=(150-22) \times 12=1536 \mathrm{~mm}^{2}
$$

Area

$$
A_{3}=120 \times 10=1200 \mathrm{~mm}^{2}
$$

$$
A=3696 \mathrm{~mm}^{2}
$$

Due to symmetry, centroid lies on axis $y-y$. The bottom fibre (1)-(1) is chosen as reference axis to locate the centroid.

The distance of the centroid from (1)-(1)

$$
\begin{aligned}
& =\frac{\text { Sum of moments of the areas of the rectangles about }(1)-(1)}{\text { Total area of section }} \\
& =\frac{960 \times(150-6)+1536 \times\left(\frac{128}{2}+10\right)+1200 \times 5}{3696} \\
& =69.78 \mathrm{~mm}
\end{aligned}
$$

With reference to the centroidal axes $x-x$ and $y-y$, the centroid of the rectangles $A_{1}$ is $g_{1}(0.0,74,22)$, that of $A_{2}$ is $g_{2}(0.0,4.22)$ and that of $A_{3}$ is $g_{3}$ (0.0, 64.78).

$$
\begin{aligned}
I_{x x}= & \frac{80 \times 12^{3}}{12}+960 \times 74.22^{2}+\frac{12 \times 128^{3}}{12}+1536 \times 4.22^{2} \\
& +\frac{120 \times 10^{3}}{12}+1200 \times 64.78^{2} \\
I_{x x}= & 1,24,70028 \mathrm{~mm}^{4} \\
I_{y y}= & \frac{12 \times 80^{3}}{12}+\frac{128 \times 12^{3}}{12}+\frac{10 \times 120^{3}}{12} \\
= & 19,70432 \mathrm{~mm}^{4}
\end{aligned}
$$

Polar Moment of Inertia $=I_{x x}+I_{y y}$

$$
\begin{aligned}
& =1,24,70027+19,70432 \\
& =1,44,40459 \mathrm{~mm}^{4} \\
\therefore \quad k_{x x} & =\sqrt{\frac{I_{x x}}{A}}=\sqrt{\frac{1,24,70027}{3696}} \\
& =58.09 \mathrm{~mm} \\
k_{y y} & =\sqrt{\frac{I_{y y}}{A}}=\sqrt{\frac{19,70432}{3696}} \\
& =23.09 \mathrm{~mm} .
\end{aligned}
$$

Ans.

Ans.

## Ans.

Example 5.17 Determine the moment of inertia of the built-up section shown in Fig. 5.58 about its centroidal axis $x-x$ and $y-y$.


Fig. 5.58

Solution. The given composite section may be divided into simple rectangles and triangles as shown in Fig. 5.58.

| Area | $A_{1}=100 \times 30=3000 \mathrm{~mm}^{2}$ |
| :--- | :--- |
| Area | $A_{2}=100 \times 25=2500 \mathrm{~mm}^{2}$ |
| Area | $A_{3}=200 \times 20=4000 \mathrm{~mm}^{2}$ |
| Area | $A_{4}=\frac{1}{2} \times 87.5 \times 20=875 \mathrm{~mm}^{2}$ |
| Area | $A_{5}=\frac{1}{2} 87.5 \times 20=875 \mathrm{~mm}^{2}$ |
| Total area | $A=11250 \mathrm{~mm}^{2}$ |

Due to symmetry, centroid lies on the axis $y-y$.
A reference axis (l)-(l) is choosen as shown in the figure.
The distance of the centroidal axis from (l)-(1)

$$
\begin{aligned}
& =\frac{\text { Sum of moment of areas about }(1)-(1)}{\text { Total area }} \\
\overline{\mathrm{y}} & =\frac{3000 \times 135+2500 \times 70+4000 \times 10+875\left(\frac{1}{3} \times 20+20\right) \times 2}{11250} \\
& =59.26 \mathrm{~mm}
\end{aligned}
$$

With reference to the centroidal axis $x-x$ and $y-y$, the centroid of the rectangle $A_{1}$ is $(0.0,75.74)$, that of $A_{2}$ is $g_{2}(0.0,10.74)$, that of $A_{3}$ is $g_{3}(0.0,49.26)$, the centroid of triangle $A_{4}$ is $g_{4}(41.66,32.59)$ and that of $A_{5}$ is $g_{5}(41.66,32.59)$.

$$
\begin{aligned}
I_{x x}= & \frac{100 \times 30^{3}}{12}+3000 \times 75.74^{2}+\frac{25 \times 100^{3}}{12}+2500 \times 10.74^{2} \\
& +\frac{200 \times 20^{3}}{12}+4000 \times 49.26^{2}+\frac{87.5 \times 20^{3}}{36}+875 \times 32.59^{2} \\
& +\frac{87.5 \times 20^{3}}{36}+875 \times 32.59^{2} \\
I_{x x}= & 3,15,43447 \mathrm{~mm}^{4} \\
I_{y y}= & \frac{30 \times 100^{3}}{12}+\frac{100 \times 25^{3}}{12}+\frac{20 \times 200^{3}}{12}+\frac{20 \times 87.5^{3}}{36}+875 \times 41.66^{2} \\
& +\frac{20 \times 87.5^{3}}{36}+875 \times 41.66^{2} \\
I_{y y}= & 1,97,45122 \mathrm{~mm}^{4} .
\end{aligned}
$$

Example 5.18 Determine the moment of inertia of the built-up section shown in Fig. 5.59 about an axis $A B$ passing through the top most fibre of the section as shown.


Fig. 5.59
Solution. In this problem, it is required to find out the moment of inertia of the section about an axis $A B$. So there is no need to find out the position of the centroid.

The given section is split up into simple rectangles as shown in Fig. 5.59.
Now,
Moment of inertia about $A B=$ Sum of moments of inertia of the rectangles about $A B$

$$
\begin{aligned}
= & \frac{400 \times 20^{3}}{12}+400 \times 20 \times 10^{2}+\left[\frac{100 \times 10^{3}}{12}+100 \times 10 \times(20+5)^{5}\right] \times 2 \\
& +\left[\frac{10 \times 380^{3}}{12}+10 \times 380 \times(30+190)^{2}\right] \times 2 \\
& +\left[\frac{100 \times 10^{3}}{12}+100 \times 10 \times(20+10+380+5)^{5}\right] \times 2 \\
I_{A B}= & 8.06093 \times 10^{8} \mathrm{~mm}^{4} . \quad \text { Ans. }
\end{aligned}
$$

Example 5.19 Calculate the moment of inertia of the built-up section shown in Fig. 5.60 about a centroidal axis parallel to $A B$. All members are 10 mm thick. Solution. The built-up section is divided into six simple rectangles as shown in the figure.

The distance of centroidal axis from $A B$

$$
=\frac{\text { Sum of the moment of areas about } A B}{\text { Total area }}
$$



Fig. 5.60

$$
\begin{aligned}
= & \frac{\Sigma A_{i} y_{i}}{A} \\
\text { Now, } \quad \Sigma A_{i} y_{i}= & 250 \times 10 \times 5+2 \times 40 \times 10 \times(10+20)+40 \times 10 \\
& \times(10+5)+40 \times 10 \times 255+250 \times 10 \times(10+125) \\
= & 4,82000 \mathrm{~mm}^{3} \\
A== & 2 \times 250 \times 10+40 \times 10 \times 4 \\
= & 6600 \mathrm{~mm}^{2} \\
\therefore \quad \bar{y}= & \frac{\Sigma A_{i} y_{i}}{A}=\frac{4,82000}{6600} \\
= & 73.03 \mathrm{~mm}
\end{aligned}
$$

Now,
$\left.\begin{array}{l}\text { Moment of inertia about } \\ \text { the centroidal axis }\end{array}\right\}=\left\{\begin{array}{l}\text { Sum of the moment of inertia } \\ \text { of the individual rectangles }\end{array}\right.$

$$
\begin{aligned}
= & \frac{250 \times 10^{3}}{12}+250 \times 10 \times(73.03-5)^{2} \\
& +\left[\frac{10 \times 40^{3}}{12}+40 \times 10(73.03-30)^{2}\right] \times 2 \\
& +\frac{40 \times 10^{3}}{12}+40 \times 10(73.03-15)^{2}+\frac{10 \times 250^{3}}{12}+250 \\
& \times 10(73.03-135)^{2}+\frac{40 \times 10^{3}}{12}+40 \times 10(73.03-255)^{2} \\
I_{x x}= & 5,03,99395 \mathrm{~mm}^{5} .
\end{aligned}
$$

Ans.
Example 5.20 A built-up section of structural steel consists of a flange plate 400 $\mathrm{mm} \times 20 \mathrm{~mm}$, a web plate $600 \mathrm{~mm} \times 15 \mathrm{~mm}$ and two angles $150 \mathrm{~mm} \times 150 \mathrm{~mm}$ $\times 10 \mathrm{~mm}$ assembled to form a section as shown in Fig. 5.61. Determine the moment of inertia of the section about the horizontal centroidal axis.


Fig. 5.61
Solution. Each angle is divided into two rectangles as shown in Fig. 5.61. The distance of the centroidal axis from the bottom fibres of section

$$
\begin{aligned}
& =\frac{\text { Sum of the moment of the areas about bottom fibres }}{\text { Total area of the section }} \\
& =\frac{\Sigma A_{i} y_{i}}{A}
\end{aligned}
$$

$$
\text { Now, } \Sigma A_{i} y_{i}=600 \times 15 \times\left(\frac{600}{2}+20\right)+140 \times 10 \times(70+30) \times 2
$$

$$
+150 \times 10 \times(5+20) \times 2+400 \times 20 \times 10
$$

$$
=33,15000 \mathrm{~mm}^{3}
$$

$$
A=600 \times 15+140 \times 10 \times 2+150 \times 10 \times 2+400 \times 20
$$

$$
=22,800 \mathrm{~mm}^{2}
$$

$$
\therefore \quad \bar{y}=\frac{\Sigma A_{i} y_{i}}{A}=\frac{13,15000}{22,800}
$$

$$
=145.39 \mathrm{~mm}
$$

$\left.\begin{array}{l}\text { Moment inertia of section } \\ \text { about centroidal axis }\end{array}\right\}=\left\{\begin{array}{l}\text { Sum of the moment of inertia of the all } \\ \text { simple figures about centroidal axis }\end{array}\right.$

$$
\begin{aligned}
= & \frac{15 \times 600^{3}}{12}+600 \times 15(145.39-320)^{2} \\
& +\left[\frac{10 \times 140^{3}}{12}+1400(145.39-100)^{2}\right] \times 2 \\
& +\left[\frac{150 \times 10^{3}}{12}+1500 \times(145.39-15)^{2}\right] \times 2 \\
& +\frac{400 \times 20^{3}}{12}+400 \times 20 \times(145.39-10)^{2} \\
I_{x x}= & 7.45156 \times 10^{8} \mathrm{~mm}^{4} .
\end{aligned}
$$

Ans.

Example 5.21 Compute the moment of inertia of the $100 \mathrm{~mm} \times 150 \mathrm{~mm}$ rectangle shown in Fig. 5.62 about $x-x$ axis to which it is inclined at an angle

$$
\theta=\sin ^{-1}\left(\frac{4}{5}\right)
$$

Solution. The rectangle is divided into four triangles as shown in the figure. [The dividing line between triangles $A_{1}$ and $A_{2}$ is parallel to $x-x$ axis].

Now

$$
=\sin ^{-1}\left(\frac{4}{5}\right)=53.13^{\circ}
$$

From the geometry of Fig. 5.62,

$$
\begin{aligned}
B K & =A B \sin \left(90^{\circ}-\theta\right) \\
& =100 \sin \left(90^{\circ}-53.13^{\circ}\right) \\
& =60 \mathrm{~mm} \\
N D & =B K=60 \mathrm{~mm} \\
\therefore \quad F D & =\frac{60}{\sin \theta}=\frac{60}{\sin 53.13}=75 \mathrm{~mm} \\
\therefore \quad A F & =150-F D=75 \mathrm{~mm} \\
\text { Hence } \quad F L & =M E=75 \sin \theta=60 \mathrm{~mm} \\
& \quad A E
\end{aligned}
$$

$\left.\begin{array}{l}\text { Moment of inertia of the } \\ \text { section about } x-x \text { axis }\end{array}\right\}=\left\{\begin{array}{l}\text { Sum of the moments of inertia of individual } \\ \text { triangular areas about } x-x \text { axis }\end{array}\right.$

$$
\begin{aligned}
= & I_{D F C}+I_{F C E}+I_{F E A}+I_{A E B} \\
= & \frac{125 \times 60^{3}}{36}+\frac{1}{2} \times 125 \times 60 \times\left(60+\frac{1}{3} \times 60\right)^{2} \\
& +\frac{125 \times 60^{3}}{36}+\frac{1}{2} \times 125 \times 60 \times\left(\frac{2}{3} \times 60\right)^{2}+\frac{125 \times 60^{3}}{36}+\frac{1}{2} \times 125 \\
& \times 60 \times\left(\frac{1}{3} \times 60\right)^{2}+\frac{125 \times 60^{3}}{36}+\frac{1}{2} \times 125 \times 60 \times\left(\frac{1}{3} \times 60\right)^{2} \\
I_{x x}= & 3,60,00000 \mathrm{~mm}^{4}
\end{aligned}
$$

Example 5.22 Find moment of inertia of the shaded area shown in Fig. 5.63 about axis $A B$.
Solution. The section is divided into a triangle $P Q R$, a semicircle $P S Q$ having base on axis $A B$ and a circle having its centre on axis $A B$.


Fig. 5.63
Now,
$\left.\begin{array}{l}\text { Moment of inertia of the } \\ \text { section about axis } A B\end{array}\right\}=\left\{\begin{array}{l}\text { Moment of inertia of triangle } P Q R \text { about } \\ A B+\text { Moment of inertia of semicircle } \\ P S Q \text { about } A B-\text { Moment of inertia of } \\ \text { circle about } A B\end{array}\right.$

$$
\begin{aligned}
& =\frac{80 \times 80^{3}}{12}+\frac{\pi}{128} \times 80^{4}-\frac{\pi}{64} \times 40^{4} \\
I_{A B} & =42,92979 \mathrm{~mm}^{4} .
\end{aligned}
$$

Ans.
Example 5.23 Find the second moment of the shaded portion shown in Fig. 5.64 about its centroidal axis.


Fig. 5.64

Solution. The section is divided into three simple figures viz., a triangle $A B C$, a rectangle $A C D E$ and a semicircle.

Total Area $=$ Area of triangle $A B C+$ Area of rectangle $A C D E$ - Area of semi circle

$$
\begin{aligned}
A= & \frac{1}{2} \times 80 \times 20+40 \times 80-\frac{1}{2} \times \pi \times 20^{2} \\
= & 3371.68 \\
A \bar{y}= & \frac{1}{2} \times 80 \times 20\left(\frac{1}{3} \times 20+40\right)+40 \times 80 \times 20-\frac{1}{2} \times \pi 20^{2} \times \frac{4 \times 20}{3 \pi} \\
= & 95991.77 \\
\bar{y}= & \frac{95991.77}{3371.6} \\
= & 28.47 \mathrm{~mm} \\
A \bar{x}= & \frac{1}{2} \times 30 \times 20 \times \frac{2}{3}+30+\frac{1}{2} \times 50 \times 20 \times\left(\frac{1}{3} \times 50 \times 30\right) \\
& +40 \times 80 \times 40-\frac{1}{2} \times \pi \times 20^{2} \times 40 \\
= & 132203.6 \\
\therefore \quad \bar{x}= & \frac{A \bar{x}}{A}=\frac{132203.6}{3371.68}=37.21 \mathrm{~mm}
\end{aligned}
$$

$$
\left.\begin{array}{l}
\text { Moment of inertia about } \\
\text { centroidal } x-x \text { axis }
\end{array}\right\}=\left\{\begin{array}{l}
\text { Moment of inertia of triangle } A B C \\
\text { about } x-x \text { axis }+ \text { Moment of inertia } \\
\text { of rectangle about } x-x \text { axis }- \text { Moment } \\
\text { of semicircle about } x-x \text { axis }
\end{array}\right.
$$

$$
\therefore \quad I_{x x}=\frac{80 \times 20^{3}}{36}+\frac{1}{2} \times 80 \times 20\left(60-\frac{2}{3} \times 20-28.47\right)^{2}+\frac{80 \times 40^{3}}{12}
$$

$$
+80 \times 40 \times(28.47-20)^{2}-\left[0.0068598 \times 20^{4}+\frac{1}{2} \pi \times 20^{2}\right.
$$

$$
\left.\left(28.47-\frac{4 \times 20^{2}}{3 \pi}\right)\right]
$$

$$
I_{x x}=6,86944 \mathrm{~mm}^{4}
$$

Ans.

Similarly,

$$
\begin{aligned}
I_{y y}= & \frac{20 \times 30^{3}}{36}+\frac{1}{2} \times 20 \times 30\left(39.21-\frac{2}{3} \times 30\right)^{2}+\frac{20 \times 50^{3}}{36}+\frac{1}{2} \\
& \times 20 \times 50 \times\left[39.21-\left(30+\frac{1}{3} \times 50\right)\right]^{2}+\frac{40 \times 80^{3}}{12} \\
& +40 \times 80(39.21-40)^{2}-\frac{1}{2} \times \frac{\pi}{64} \times 40^{4}-\frac{1}{2} \times \frac{\pi}{4} \\
& \times 40^{2}(40-39.21)^{2} \\
= & 1868392 \mathrm{~mm}^{4} .
\end{aligned}
$$

Ans.

### 5.8 PRODUCT MOMENT OF PLANE AREA

Consider the plane area shown in Fig. 5.65. $x$ and $y$ are the cartesian coordinate axes selected and $\delta A_{i}$ is an element in the area with coordinates $x_{i}$ and $y_{i}$. The term $\Sigma x_{i} y_{i} d A_{i}$ to cover the entire area is termed product moment of the plane area or product moment of inertia.


Fig. 5.65
If the term $x_{i} y_{i} d A_{i}$ can be expressed in general term for any element in the area and appropriate limits can be found for $x$ and $y$ to cover entire area then $\Sigma x_{i} y_{i} d A_{i}$ can be replaced by $\oint x y d A$. Denoting the product moment of inertia by $I_{x y}$, we have

$$
\begin{equation*}
\text { Product moment of area }=I_{x y}=\Sigma x y d A=\oint x y d A \tag{5.14}
\end{equation*}
$$

For this term also, the unit is $\mathrm{mm}^{4}$ or $\mathrm{m}^{4}$. We noticed $I_{x x}$ and $I_{y y}$ terms were always positive, since they involved the terms $x^{2}$ and $y^{2}$. But the product moment of area $I_{x y}$ can be negative also since one of the coordinates may be positive and the other may be negative.

For composite sections product moment of inertia found by $I_{x y}=A_{1} x_{1} y_{1}+$ $A_{2} x_{2} y_{2}+\ldots$ where $x_{i} y_{i}$ are the coordinates of centroidal axis of $A_{i}$.

The term product moment of inertia will be very useful when the bending stresses in a structure with unsymmetric cross-sections are to be determined.

### 5.9 PRINCIPAL MOMENT OF INERTIA

The axis about which product moment of inertia are zero are termed as principal axis. Incidently the moment of inertia are maximum and minimum about these axes, and they are known as principal moment of inertia. If the axes considered are through the centroid of the area, then the axes are known as centroidal principal axes.

Consider the product moment of inertia about a symmetric axis as shown in Fig. 5.66. If we consider the two elemental areas at a time, which are equal but one having the coordinates $(x, y)$ and the other $(-x, y)$, we find their contribution to the term $I_{x y}$ is zero. Since $y$ axis is symmetric axis, we can find one to one match for $x y d A$ and $-x y d A$ terms over entire area and hence $I_{x y}$ is zero. Hence a symmetric axis is a principal axis. If we consider $I_{x y}$ term about an axis at right angles to the symmetric axis, then also we find $I_{x y}$ term will be zero. Hence the


Fig. 5.66 symmetric axis and axis at right angle to symmetric axis are principal axes.

If there is no axis of symmetry in the plane area considered, any convenient cartesian axis $x-y$ is selected and $I_{x x}, I_{y y}$ and $I_{x y}$ are computed. If $u, v$ are the principal axes, making angle $\theta$ with $x, y$ axes, (Ref. Fig. 5.67) then it is possible to find the relations between $I_{u v}$ and $I_{x y}, I_{x x}, I_{y y}$. Then make use of the definition that $I_{u v}$ should be zero since $u-v$ are principal axes to get angle $\theta$. Then the principal moment of inertia $I_{u u}$ and $I_{\nu v}$ can be found.


Fig. 5.67

Consider the elemental area $d A$ at point $P$. Its coordinates are $x$ and $y$ with respect to $x-y$ axes and $u$ and $v$ with respect to $u-v$ axes. $P M$ and $P N$ are perpendicular to $x$-axis and $u$-axes. Let $M^{\prime}$ be the perpendicular to $u$-axis. $N^{\prime}$ is the intersection of $P N$ with the line $M N^{\prime}$ which is parallel to $x$-axis.

Now,

$$
\begin{align*}
u & =G N=G M^{\prime}+M^{\prime} N=G M^{\prime}+M N^{\prime} \\
& =G M \cos \theta+P M \sin \theta \\
& =x \cos \theta+y \sin \theta  \tag{5.15}\\
v & =P N=P N^{\prime}-N N^{\prime}=P N^{\prime}-M M^{\prime} \\
& =P M \cos \theta-G M \sin \theta \\
& =y \cos \theta-x \sin \theta \tag{5.16}
\end{align*}
$$

According to definitions
and

$$
I_{y x}=\Sigma y^{2} d A, I_{y y}=\Sigma x^{2} d A, I_{x y}=\Sigma x y d A
$$

$$
I_{u u}=\Sigma v^{2} d A, I_{v v}=\Sigma u^{2} d A, I_{u v}=\Sigma u v d A
$$

Consider the term $I_{u v}$

$$
\begin{aligned}
I_{u v}= & \Sigma u v d A \\
= & \Sigma(x \cos \theta+y \sin \theta)(y \cos \theta-x \sin \theta) d A \\
= & \Sigma\left(x y \cos ^{2} \theta-x^{2} \sin \theta \cos \theta+y^{2} \sin \theta \cos \theta-x y \sin ^{2} \theta\right) d A \\
= & \cos ^{2} \theta \Sigma x y d A-\sin \theta \cos \theta \Sigma x^{2} d A+\sin \theta \cos \theta \Sigma y^{2} d A \\
& -\sin ^{2} \theta \Sigma x y d A \\
= & I_{x y} \cos ^{2} \theta-I_{y y} \sin \theta \cos \theta+I_{x x} \sin \theta \cos \theta-I_{x y} \sin ^{2} \theta \\
= & I_{x y}\left(\cos ^{2} \theta-\sin ^{2} \theta\right)-\sin \theta \cos \theta\left(I_{y y}-I_{x x}\right) \\
= & I_{x y} \cos 2 \theta-\frac{\left(I_{y y}-I_{x x}\right)}{2} \sin 2 \theta
\end{aligned}
$$

Since $u-v$ are principal axes,

$$
\begin{align*}
I_{u v} & =0 \\
\therefore \text { we get } \quad \tan 2 \theta & =\frac{2 I_{x y}}{I_{y y}-I_{x x}} \tag{5.17}
\end{align*}
$$

Using $I_{x x}, I_{y y}$ and $I_{x y}$ values, $\theta$ can be found from eqn. (5.17), in other words, the principal axis can be located. Equation (5.17) gives two values for $\theta$ which differ by $90^{\circ}$, one being minimum principal axis and the other maximum principal axis.

$$
\text { Now, } \quad \begin{aligned}
I_{u u} & =\Sigma v^{2} d A \\
& =\Sigma(y \cos \theta-x \sin \theta)^{2} d A \\
& =\Sigma\left(y^{2} \cos ^{2} \theta-2 x y \sin \theta \cos \theta+x^{2} \sin ^{2} \theta\right) d A \\
& =\Sigma y^{2} \cos ^{2} \theta d A-\Sigma 2 x y \sin \theta \cos \theta d A+\Sigma x^{2} \sin ^{2} \theta d A
\end{aligned}
$$

$$
\begin{aligned}
& =I_{x x} \cos ^{2} \theta-I_{x y} \sin 2 \theta+I_{y y} \sin ^{2} \theta \\
& =I_{x x} \frac{1+\cos 2 \theta}{2}+I_{y y} \frac{1-\cos 2 \theta}{2}-I_{x y} \sin 2 \theta \\
& =\frac{1}{2}\left(I_{x x}+I_{y y}\right)+\frac{\left(I_{x x}-I_{y y}\right)}{2} \cos 2 \theta-I_{x y} \sin 2 \theta
\end{aligned}
$$

Representing Eqn. (5.17) for the direction of principal axis ' $2 \theta$ ' by the sides of the triangles as shown in Fig. 5.68, we get

$$
\sin 2 \theta=\frac{I_{x y}}{\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}}} \text { and } \cos 2 \theta=\frac{\frac{I_{y y}-I_{x x}}{2}}{\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}}}
$$



Fig. 5.68
and $I_{u u}=\frac{1}{2}\left(I_{x x}+I_{y y}\right)+\frac{I_{x x}-I_{y y}}{2} \frac{\frac{I_{y y}-I_{x x}}{2}}{\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}}}$

$$
\begin{gather*}
-I_{x y} \frac{I_{x y}}{\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}}} \\
=\frac{1}{2}\left(I_{x x}+I_{y y}\right)-\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}} \tag{5.18}
\end{gather*}
$$

Similarly,

$$
\begin{aligned}
I_{v v} & =\Sigma u^{2} d A=\Sigma(x \cos \theta+y \sin \theta)^{2} d A \\
& =\Sigma x^{2} \cos ^{2} \theta d A+\Sigma 2 x y \sin \theta \cos \theta d A+\Sigma y^{2} \sin ^{2} \theta d A \\
& =I_{y y} \cos ^{2} \theta+I_{x y} 2 \sin \theta \cos \theta+I_{x x} \sin ^{2} \theta
\end{aligned}
$$

$$
\begin{align*}
&=I_{y y} \frac{1+\cos 2 \theta}{2}+I_{x x} \frac{1-\cos 2 \theta}{2}+I_{x y} \sin 2 \theta \\
&=\frac{I_{y y}+I_{x x}}{2}+\frac{I_{y y}-I_{x x}}{2} \cos 2 \theta+I_{x y} \sin 2 \theta \\
&=\frac{I_{x x}}{2}+I_{y y}+\frac{I_{y y}-I_{x x}}{2} \frac{\frac{I_{y y}-I_{x x}}{2}}{\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}}}+I_{x y} \frac{I_{x y}}{\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}}} \\
&=\frac{I_{x x}+I_{y y}}{2}+\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}} \tag{5.19}
\end{align*}
$$

Thus Eqn. (5.17) gives the direction of principal axis and Eqns. (5.18) and (5.19) give the principal moment of inertia.

Example 5.24 Determine the centroidal principal moment of inertia of the equal angle section $50 \mathrm{~mm} \times 50 \mathrm{~mm} \times 10 \mathrm{~mm}$ shown in Fig. 5.69.


Fig. 5.69
Solution. Let $C_{x x}$ and $C_{y y}$ be the distances of centroid from $x$ and $y$ axes as shown in Fig. 5.69.

$$
\therefore \quad C_{x x}=\frac{50 \times 10 \times 5+40 \times 10 \times(10+20)}{50 \times 10+40 \times 10}=16.11 \mathrm{~mm}
$$

Similarly $C_{y y}=16.11 \mathrm{~mm}$

$$
\begin{aligned}
\therefore \quad I_{x x}= & \frac{1}{12} \times 50 \times 10^{3}+50 \times 10 \times(16.11-5.0)^{2}+\frac{1}{12} \\
& \times 40^{3} \times 10+40 \times 10 \times(30-16.11)^{2} \\
= & 196389 \mathrm{~mm}^{4}
\end{aligned}
$$

Similarly $I_{y y}=196389 \mathrm{~mm}^{4}$ [since it is equal angle].
Coordinates of the two simple rectangles w.r.t. centroidal axes are,

$$
\begin{array}{ll} 
& g_{1}[25-16.11,-(16.11-5)], g_{2}[-(16.11-5), 30-16.11] \\
\text { i.e., } & g_{1}(8.89,-11.11) \text { and } g_{2}(-11.11,13.89) \\
\therefore \quad & I_{x y} \\
& =A_{1} x_{1} y_{1}+A_{2} x_{2} y_{2} \\
& =50 \times 10 \times 8.89 \times(-11.11)+40 \times 10 \times(-11.11) \times 13.89 \\
& =-111111.11 \mathrm{~mm}^{4}
\end{array}
$$

Direction of principal axis is given by $\theta$, where

$$
\begin{aligned}
& \tan 2 \theta=\frac{2 I_{x y}}{I_{y y}-I_{x x}}=\infty \text { since } I_{y y}=I_{x x} \\
& \therefore \quad 2 \theta=90^{\circ} \text { or } \theta_{1}=45^{\circ} \text { and } \theta_{2}=135^{\circ} \\
& \therefore \quad \begin{aligned}
I_{u u} & =\frac{1}{2}\left(I_{x x}+I_{y y}\right)-\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}} \\
& =\frac{1}{2}(196389+196389)-\sqrt{0+111111.11^{2}} \\
& =196389-111111.11 \\
I_{u u} & =85277.89 \mathrm{~mm}^{4}
\end{aligned} \\
& \text { Similarly } I_{v v}=196389+11111.11=307500.11 \mathrm{~mm}^{4}
\end{aligned}
$$

Ans.

Thus in case of equal angles, principal axes are at $45^{\circ}$ to $x-y$ axes.
Example 5.25 Determine the principal moment of inertia of the unequal angle section $125 \times 85 \times 10 \mathrm{~mm}$ shown in Fig. 5.70.

## Solution.

$$
\begin{aligned}
& C_{y y}=\frac{85 \times 10 \times \frac{85}{2}+115 \times 10 \times 5}{85 \times 10+115 \times 10}=20.94 \mathrm{~mm} \\
& C_{x x}=\frac{85 \times 10 \times 5+115 \times 10 \times\left(10+\frac{115}{2}\right)}{85 \times 10+115 \times 10}=40.94 \mathrm{~mm}
\end{aligned}
$$

The coordinates $g_{1}$ and $g_{2}$ w.r.t. to centroidal $x, y$ axes are

$$
\left.g_{1}\left[\frac{85}{2}-20.04,-(40.94-5)\right] ; g_{2}[-20.94-5), 10+\frac{115}{2}-40.94\right]
$$



Fig. 5.70
i.e., $\quad g_{1}[22.466,-35.94] ; g_{2}[-15.94,26.56]$

$$
I_{x x}=\frac{1}{12} \times 85 \times 10^{3}+85 \times 10 \times(-35.94)^{2}
$$

$$
+\frac{1}{12} \times 10 \times 115^{3}+10 \times 115(26.56)^{2}
$$

$$
=3183659 \mathrm{~mm}^{4}
$$

$$
I_{y y}=\frac{1}{12} \times 10 \times 85^{3}+10 \times 85 \times 22.46^{2}+\frac{1}{12} \times 115 \times 10^{3}
$$

$$
+115 \times 10(-15.95)^{2}
$$

$$
=1242700 \mathrm{~mm}^{4}
$$

$$
I_{x y}=A_{1} x_{1} y_{1}+A_{2} y_{2} x_{2}
$$

$$
=85 \times 10 \times 22.46(-35.94)+115 \times 10 \times(-15.94) \times 26.54
$$

$$
=-1172635 \mathrm{~mm}^{4}
$$

$$
\tan 2 \theta=\frac{-1172635}{\frac{1242700-3183659}{2}}=1.2083
$$

$$
\therefore \quad 2 \theta=50.39
$$

$$
\therefore \quad \theta_{1}=25.19^{\circ} ; \quad \theta_{2}=115.19^{\circ}
$$

$$
\begin{aligned}
& I_{u u}=\frac{I_{x x}+I_{y y}}{2}-\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}} \\
& =\frac{3183659+1242700}{2}-\sqrt{\left(\frac{1242700-3183659}{2}\right)^{2}+(1172635)^{2}}
\end{aligned}
$$

$$
=2213179-1522138
$$

$$
\begin{aligned}
\therefore \quad I_{u u} & =691041 \mathrm{~mm}^{4} \\
I_{v v} & =2213179+1522138
\end{aligned}
$$

$$
I_{v v}=3735317 \mathrm{~mm}^{4}
$$

Ans.
Ans.

### 5.10 THEOREMS OF PAPPUS-GULDINUS

There are two important theorems, first proposed by Greek scientist (about 340 AD) and then restated by Swiss mathematician Paul Guldinus (1640) for determining the surface area and volumes generated by rotating a curve and a plane area about a non-intersecting axis, some of which are shown in Fig. 5.71. These theorems are known as Pappus-Guldinus theorems.


Fig. 5.71

## Theorem I

The area of surface generated by revolving a plane curve about a non-intersecting axis in the plane of the curve is equal to the length of the generating curve times the distance travelled by the centroid of the curve in the revolution.

Proof: Figure 5.72 shows the isometric view of the plane curve rotated about $x$ axis by angle $\theta$. We are interested in finding the surface area generated by rotating the curve $A B$. Let $d L$ be the elemental length on the curve at $D$. Its coordinate be y. Then the elemental surface area generated by this element at $D$

$$
\begin{aligned}
d A & =d L(y \theta) \\
\therefore \quad A & =\int d L(y \theta) \\
& =\theta \int y d L \\
& =\theta L y_{c} \\
& =L\left(y_{c} \theta\right)
\end{aligned}
$$



Fig. 5.72
Thus we get area of the surface generated as length of the generating curve times the distance travelled by the centroid.

## Theorem II

The volume of the solid generated by revolving a plane area about a non-intersecting axis in the plane is equal to the area of the generating plane times the distance travelled by the centroid of the plane area during the revolution.

Proof: Consider the plane area $A B C$, which is rotated through an angle $\theta$ about $x$-axis as shown in Fig. 5.73.


Fig. 5.73

Let $d A$ be the elemental area of distance $y$ from $x$-axis. Then the volume generated by this area during rotation is given by

$$
\begin{aligned}
d V & =d A y \theta \\
\therefore \quad V & =\int d A y \theta \\
& =\theta \int y d A \\
& =\theta A y_{c} \\
& =A\left(y_{c} \theta\right)
\end{aligned}
$$

Thus the volume of the solid generated is area times the distance travelled by its centroid during the rotation. Using Pappus-Guldinus theorems, surface area and volumes of cones and spheres can be calculated as shown below:
(i) Surface area of a cone: Referring to Fig. 5.74 (a),

Length of the line generating cone $=L$
Distance of centroid of the line from the axis of rotation $=y=\frac{R}{2}$
In one revolution centroid moves by distance $=2 \pi y=\pi R$
$\therefore$ Surface area $=L \times(\pi R)=\pi R L$
(ii) Volume of a cone: Referring to Fig. 5.74(b),

Area generating solid cone $=\frac{1}{2} h R$
Centroid $G$ is at a distance
$y=\frac{R}{3}$

(a)

(b)

Fig. 5.74
$\therefore$ The distance moved by the centroid in one revolution $=2 \pi y=2 \pi \frac{R}{3}$
$\therefore$ Volume of solid cone $=\frac{1}{2} h R \times \frac{2 \pi R}{3}$

$$
=\frac{\pi R^{2} h}{3}
$$

(iii) Surface area of sphere: Sphere of radius $R$ is obtained by rotating a semi circular arc of radius $R$ about its diametral axis. Referring to Fig. 5.75(a), Length of the arc $=\pi R$
Centroid of the arc is at $y=\frac{2 R}{\pi}$ from the diametral axis (i.e., axis of rotation)
$\therefore$ Distance travelled by centroid of the arc in one revolution

$$
=2 \pi y=2 \pi \frac{2 R}{\pi}=4 R
$$

$\therefore$ Surface area of sphere $=\pi R \times 4 R$

$$
=4 \pi R^{2}
$$

(iv) Volume of sphere: Solid sphere of radius $R$ is obtained by rotating a semi-circular area about its diametral axis. Referring to Fig. 5.75(b).

$$
\text { Area of semi-circle }=\frac{\pi R^{2}}{2}
$$

Distance of centroid of semi-circular area from its centroidal axis

$$
=y=\frac{4 R}{3 \pi}
$$

$\therefore$ The distance travelled by the centroid in one revolution

$$
=2 \pi y=2 \pi \frac{4 R}{3 \pi}=\frac{8 \pi R}{3 \pi}
$$

$\therefore \quad$ Volume of sphere $=\frac{\pi R^{2}}{2} \times \frac{8 \pi R}{3 \pi}$

$$
=\frac{4 \pi R^{3}}{3}
$$


(a)

(b)

Fig. 5.75

### 5.11 CENTRE OF GRAVITY OF SOLIDS

Centre of gravity of solids may be found using eqn. (5.1) which will be same as those found from Eqns. (5.2) and (5.3) if the mass is uniform. Hence centre of gravity of solids, centre of gravity of mass or centroid of volumes is the same for all solids with uniform mass. For standard solids, the centre of gravity may be
found from first principles and the results obtained for standard solids may be used to find centre of gravity of composite solids. The procedure is illustrated with Examples 5.26 to 5.29.

Example 5.26 Locate the centre of gravity of the right circular cone of base radius $r$ and height $h$ shown in Fig. 5.76.


Fig. 5.76
Solution. Taking origin at the vertex of the cone and selecting the axis as shown in Fig. 5.76, it can be observed that due to symmetry the coordinates of centre of gravity $\bar{y}$ and $\bar{z}$ are equal to zero, i.e., the centre of gravity lies on the axis of rotation of the cone. To find its distance $\bar{x}$ from the vertex, consider an elemental plate at a distance $x$. Let the thickness of the elemental plate be $d x$. From the similar triangles $O A B$ and $O C D$, the radius of elemental plate $z$ is given by

$$
z=\frac{x}{h} r
$$

$\therefore$ Volume of the elemental plate $d V$

$$
d V=\pi z^{2} d x=\pi x^{2} \frac{r^{2}}{h^{2}} d x
$$

If $\gamma$ is the unit weight of the material of the cone, then weight of the elemental plate is given by:

$$
\begin{align*}
d W & =\gamma \pi x^{2} \frac{r^{2}}{h^{2}} d x  \tag{i}\\
W & =\int_{0}^{h} \gamma \frac{\pi r^{2}}{h^{2}} x^{2} d x \\
& =\gamma \frac{\pi r^{2}}{h^{2}}\left[\frac{x^{3}}{3}\right]_{0}^{h} \\
& =\gamma \pi \frac{r^{2} h}{3} \tag{ii}
\end{align*}
$$

$\left[\right.$ Note: $\frac{\pi r^{2} h}{3}$ is volume of cone $]$

Now, substituting the value of $d W$ in (i), above we get:

$$
\begin{align*}
\int x \cdot d W & =\int_{0}^{h} \gamma \frac{\pi r^{2}}{h^{2}} x^{2} \cdot x \cdot d x \\
& =\gamma \frac{\pi r^{2}}{h^{2}}\left[\frac{x^{4}}{4}\right]_{0}^{h} \\
& =\gamma \frac{\pi r^{2} h^{2}}{4} \tag{iii}
\end{align*}
$$

From Eqn. 5.1, we have

$$
\begin{array}{rlrl} 
& W \bar{x} & =\int x d W \\
& & & \\
\text { i.e., } & \frac{\pi r^{2} h}{3} \bar{x} & =\frac{\gamma \pi r^{2} h^{2}}{4} \\
\therefore & \bar{x} & =\frac{3}{4} h
\end{array}
$$

Thus, in a right circular cone, centre of gravity lies at a distance $\frac{3}{4} h$ from vertex along the axis of rotation, i.e., at a distance $\frac{h}{4}$ from the base.

Example 5.27 Determine the centre of gravity of a solid hemisphere of radius $r$ from its diametral axis.
Solution. Due to symmetry, centre of gravity lies on the axis of rotation. To find its distance $\bar{x}$ from the base along the axis of rotation, consider an elemental plate at a distance $x$ as shown in Fig. 5.77.

Now,

$$
\begin{aligned}
x^{2}+z^{2} & =r^{2} \\
z^{2} & =r^{2}-x^{2}
\end{aligned}
$$

Volume of elemental plate

$$
\begin{equation*}
d V=\pi z^{2} d x=\pi\left(r^{2}-x^{2}\right) d x \tag{ii}
\end{equation*}
$$

$\therefore$ Weight of elemental plate

$$
\begin{equation*}
d W=\gamma d v=\gamma \pi\left(r^{2}-x^{2}\right) d x \tag{iii}
\end{equation*}
$$

$\therefore$ Weight of hemisphere

$$
\begin{align*}
W & =\int d W=\int_{0}^{r} \gamma \pi\left(r^{2}-x^{2}\right) d x \\
& =\gamma \pi\left[r^{2} x-\frac{x^{2}}{3}\right]_{0}^{r} \\
& =\frac{2 \gamma \pi r^{3}}{3} \tag{iv}
\end{align*}
$$



Fig. 5.77

Moment of weight about $z$-axis

$$
\begin{align*}
& =\int_{0}^{r} x d W \\
& =\int_{0}^{r} x \pi\left(r^{2}-x^{2}\right) d x \\
& =\pi\left[r^{2} \frac{x^{2}}{2}-\frac{x^{4}}{4}\right]_{0}^{r} \\
& =\frac{\pi r^{4}}{4} \tag{v}
\end{align*}
$$

$\therefore \quad \bar{x}$, the distance of centre of gravity from base is given by:

$$
W \bar{x}=\int_{0}^{r} x d W
$$

i.e., From (iv) and (v) above, we get

$$
\frac{2 \gamma \pi r^{3}}{3} \bar{x}=\frac{\gamma \pi r^{4}}{4} \quad \bar{x}=\frac{3}{8} r
$$

Thus, the centre of gravity of a solid hemisphere of radius $r$ is at a distance $\frac{3}{8} r$ from its diametral axis.
Example 5.28 Determine the maximum height $h$ of the cylindrical portion of the body with hemispherical base shown in Fig. 5.78 so that it is in stable equilibrium on its base.
Solution. The body will be stable on its base as long as its centre of gravity is in hemispherical base. The limiting case is when it is on the plane $x-x$ shown in the figure.

Centroid lies on the axis of rotation. Mass of cylindrical portion
$m_{1}=\pi r^{2} h \rho$, where $\rho$ is unit mass of material. Its centre of gravity $g_{1}$ is at a height

$$
z_{1}=\frac{h}{2} \text { from } x \text {-axis. }
$$



Fig. 5.78

Mass of hemispherical portion

$$
m_{2}=\rho \frac{2 \pi r^{2}}{3}
$$

and its $C G$ is at a distance

$$
z_{2}=\frac{3 r}{8} \text { from } x-x \text { plane }
$$

Since centroid is to be on $x-x$ plane $\bar{z}=0$

$$
\begin{array}{rlrl}
\text { i.e., } & \Sigma m_{i} z_{i} & =0 \\
& \therefore & \frac{m_{1} h}{2}-m_{2} \frac{3}{8} r & =0 \\
& \therefore & \pi r^{2} h \rho \frac{h}{2} & =\rho \frac{2 \pi r^{3}}{3} \frac{3}{8} r \\
& \therefore & h^{2} & =\frac{1}{2} r^{2} \\
& \text { or } & h & =\frac{r}{\sqrt{2}}=0.707 r
\end{array}
$$

Ans.

Example 5.29 A concrete block of size $0.60 \mathrm{~m} \times 0.75 \mathrm{~m} \times 0.5 \mathrm{~m}$ is cast with a hole of diameter 0.2 m and depth 0.3 m as shown in Fig. 5.79. The hole is completely filled with steel balls weighing 2500 N . Locate the centre of gravity of the body. Take the weight of concrete $=25000 \mathrm{~N} / \mathrm{m}^{3}$.


Fig. 5.79
Solution. Weight of solid concrete block:

$$
W_{1}=0.6 \times 0.75 \times 0.5 \times 25000=5625 \mathrm{~N}
$$

Weight of concrete $\left(W_{2}\right)$ removed for making hole:

$$
W_{2}=\frac{\pi}{4} \times 0.2^{2} \times 0.3 \times 25000=235.62 \mathrm{~N}
$$

Taking origin as shown in the figure, the centre of gravity of solid block is $(0.375,0.3,0.25)$ and that of hollow portion is $(0.5,0.4,0.15)$. The following Table 5.5 may be prepared now.

Table 5.5

| Simple body | $W_{i}$ | $x_{i}$ | $W_{i} x_{i}$ | $y_{i}$ | $W_{i} y_{i}$ | $z_{i}$ | $W_{i} z_{i}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1. Solid block | 5625 | 0.375 | 2109.38 | 0.3 | 1687.5 | 0.25 | 1406.25 |
| 2. Hole in con- | -235.62 | 0.5 | -117.81 | 0.4 | -94.25 | 0.15 | -35.34 |
| crete block |  |  |  |  |  |  |  |
| 3. Steel balls | 2500 | 0.5 | 1250.0 | 0.4 | 1000.0 | 0.15 | 375.0 |

$$
\begin{aligned}
& \Sigma W_{i}=7889.38 \quad \Sigma W_{i} x_{i}=3241.57 \quad \Sigma W_{i} y_{i}=2593.25 \quad \Sigma W_{i} z_{i}=1745.91 \\
& \therefore \quad \bar{x}=\frac{\Sigma W_{i} x_{i}}{W}=\frac{\Sigma W_{i} x_{i}}{W_{i}} \text { or } \bar{x}=\frac{3241.57}{7889.38}=0.411 \mathrm{~m}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \bar{y}=\frac{2593.25}{7887.38}=0.329 \mathrm{~m} \\
& \bar{z}=\frac{1745.91}{7889.38}=0.221 \mathrm{~m}
\end{aligned}
$$

Ans.

Ans.

### 5.12 MASS MOMENT OF INERTIA

Mass moment of inertia of a body about an axis is defined as the sum total of product of its elemental masses and square of their distances from the axis.

Thus for the $i$ th element of mass $d m_{i}$ located at coordinates $\left(x_{i}, y_{i}, z_{i}\right)$ (Ref. Fig. 5.80).
and

$$
\begin{aligned}
& d I_{x x}=\left(y_{i}^{2}+z_{i}^{2}\right) d m_{i} \\
& d I_{y y}=\left(x_{i}^{2}+z_{i}^{2}\right) d m_{i} \\
& d I_{z z}=\left(x_{i}^{2}+y_{i}^{2}\right) d m_{i}
\end{aligned}
$$

If $\rho$ is the mass density of the material, we know that $d m=\rho d V$. Hence

$$
\begin{align*}
I_{x x} & =\Sigma\left(y_{i}^{2}+z_{i}^{2}\right) d m_{i}=\oint\left(y^{2}+z^{2}\right) d m \\
& =\oint\left(y^{2}+z^{2}\right) \rho d V \\
I_{y y} & =\Sigma\left(x_{i}^{2}+z_{i}^{2}\right) d m_{i}=\oint\left(x^{2}+z^{2}\right) d m \\
& =\oint\left(x^{2}+z^{2}\right) \rho d V \\
I_{z z} & =\Sigma\left(x_{i}^{2}+y_{i}^{2}\right) d m_{i}=\oint\left(x^{2}+y^{2}\right) d m \\
& =\oint\left(x^{2}+y^{2}\right) \rho d V \tag{5.20}
\end{align*}
$$



Fig. 5.80 Mass Moment of Inertia

The product inertia of mass of the body are similarly defined as,
and

$$
\begin{aligned}
& I_{x y}=I_{y x}=\oint x y \rho d V \\
& I_{x z}=I_{z x}=\oint x z \rho d V
\end{aligned}
$$

$$
I_{y z}=I_{z y}=\oint y z \rho d V
$$

Thus for a body there are nine components of inertia matrix as shown below:

$$
I=\left[\begin{array}{lll}
I_{x x} & I_{x y} & I_{x z} \\
I_{y x} & I_{y y} & I_{y z} \\
I_{z x} & I_{z y} & I_{z z}
\end{array}\right]
$$

In the above matrix as already shown,

$$
I_{x y}=I_{y x} ; \quad I_{x z}=I_{z x} \text { and } I_{y z}=I_{z y}
$$

We can also see that,

$$
\begin{aligned}
I_{x x}+I_{y y}+I_{z z} & =\oint 2\left(x^{2}+y^{2}+z^{2}\right) \rho d V \\
& =\oint 2 r^{2} \rho d V
\end{aligned}
$$

where $r$ is the position vector $x i+y j+z k$. Since whatever be the orientation of the coordinate system $x, y, z$ at the origin, sum of the inertia at that origin is constant.

## Radius of Gyration

Radius of Gyration is that distance which when squared and multiplied with total mass of the body gives the mass moment of inertia of the body. Thus if $I$ is moment of inertia of a body of mass $M$ about an axis, then its radius of gyration about that axis is given by the relation

$$
\begin{equation*}
I=M k^{2} \text { or } k=\sqrt{\frac{I}{M}} \tag{5.21}
\end{equation*}
$$

A physical meaning may be assigned to the distance at which the entire mass can be assumed to be concentrated such that the moment of inertia of the actual body and the concentrated mass is the same.

### 5.13 DETERMINATION OF MASS MOMENT OF INERTIA FROM FIRST PRINCIPLES

Mass moment of inertia of simple bodies can be determined from its definition. The following steps may be followed:
(a) Take a general element.
(b) Write the expression for mass of the element, $d m$, and its distance, $r$, from the axis.
(c) Integrate the term $r^{2} d m$ between suitable limits such that the entire mass of the body is covered. The procedure is illustrated with the examples given below.

Example 5.30 Determine the mass moment of inertia of a uniform rod of length L about its: (a) centroidal axis normal to rod, and (b) axis at the end of the rod and normal to it.

## Solution.

(a) About Centroidal Axis Normal to Rod

Consider an elemental length $d x$ at a distance $x$ from centroidal axis $y-y$ as shown in Fig. 5.81. Let the mass of rod be $m$ per unit length. Then mass of the element $d m=m d x$.

$$
\begin{aligned}
\therefore \quad I & =\int_{-\frac{L}{2}}^{\frac{L}{2}} x^{2} d m=\int_{-\frac{L}{2}}^{\frac{L}{2}} x^{2} \times m d x \\
& =m\left[\frac{x^{3}}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}} \\
& =\frac{m L^{3}}{12} \\
\text { i.e., } \quad I & =\frac{M L^{2}}{12}
\end{aligned}
$$

where $M=m L$ is total mass of the rod.
(b) About Axis at the End of the Rod and Normal to it:

Consider an element of length $d x$ at a distance $x$ from end as shown in Fig. 5.82. Moment of inertia of the rod about $y$-axis:

$$
\begin{aligned}
I & =\int_{0}^{L} x^{2} m d x \\
& =m\left[\frac{x^{3}}{3}\right]_{0}^{L} \\
& =\frac{m L^{3}}{3} \\
I & =\frac{M L^{2}}{3}
\end{aligned}
$$

Example 5.31 Determine the moment of inertia of a rectangular plate of size $a \times b$ and thickness $t$ about its centroidal axis.
Solution. To find $I_{x x}$
Consider an elemental strip of width $d y$ at a distance $y$ from $x$-axis as shown in Fig. 5.83 (a). Mass of the element:

$$
d m=\rho b \times t \times d y
$$

$$
\text { ( } \rho \text {-Unit mass of the material) }
$$


(a)

(b)

Fig. 5.83

$$
\left.\begin{array}{rl}
\therefore & I_{x x}
\end{array}=\int_{-a / 2}^{a / 2} y^{2} d m=\int_{-a / 2}^{a / 2} y^{2} \rho b t d y\right]
$$

But mass of the plate $M=\rho$ bta

$$
\therefore \quad I_{x x}=\frac{M a^{2}}{12}
$$

To find $I_{y y}$
Taking an elemental strip parallel to $y$-axis, it can be easily shown that:

$$
\therefore \quad I_{y y}=\frac{M b^{2}}{12}
$$

To find $I_{z z}$
Consider an element of size $d x d y$ and thickness $t$ as shown in Fig. 5.83(b).
Now

$$
\begin{aligned}
r^{2} & =x^{2}+y^{2} \\
I_{z z} & =\int r^{2} d m \\
& =\int\left(x^{2}+y^{2}\right) d m \\
& =\int x^{2} d m+\int y^{2} d m \\
& =I_{x x}+I_{y y}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{M a^{2}}{12}+\frac{M b^{2}}{12} \\
I_{z z} & =\frac{1}{12} M\left(a^{2}+b^{2}\right)
\end{aligned}
$$

Example 5.32 Find the moment of inertia of circular plate of radius $R$ and thickness $t$ about its centroidal axis.

Solution. Consider an elemental area $r d \theta d r$ and thickness $t$ as shown in Fig. 5.84. $d m=$ Mass of the element $=\rho r d \theta d r t=\rho t r d \theta d r$

Its distance from $x$-axis $=r \sin \theta$

$$
\begin{aligned}
I_{x x} & =\oint(r \sin \theta)^{2} d m \\
& =\int_{0}^{R 2 \pi} \int_{0}^{2} r^{2} \sin ^{2} \theta \rho t r d \theta d r \\
& =\rho t \int_{0}^{R 2 \pi} \int_{0}^{2} r^{3}\left(\frac{1-\cos 2 \theta}{2}\right) d r d \theta \\
& =\rho t \int_{0}^{R} \frac{r^{3}}{2}\left[\theta-\frac{\sin 2 \theta}{2}\right]_{0}^{2 \pi} d r \\
& =\rho t \int_{0}^{R} \frac{r^{3}}{2} \times 2 \pi d r \\
& =\rho t \pi\left[\frac{r^{4}}{4}\right]_{0}^{R} \\
& =\rho t \frac{\pi R^{4}}{4}
\end{aligned}
$$

Mass of the plate $M=\rho \times \pi R^{2} t$

$$
\begin{array}{ll}
\therefore & I_{x x}=\frac{M R^{2}}{4} \\
\text { Similarly, } & I_{y y}=\frac{M R^{2}}{4}
\end{array}
$$

Actually $I=\frac{M R^{2}}{4}$ is moment of inertia of circular plate about any diametral axis in the plate.

To find $I_{z z}$, consider the same element.

$$
\begin{aligned}
I_{z z} & =\oint r^{2} d m=\int_{0}^{R} \int_{0}^{2 \pi} r^{2} \rho t r d r d \theta \\
& =\rho t \int_{0}^{R} r^{3}[\theta]_{0}^{2 \pi} d r \\
& =\rho t \int_{0}^{R} 2 \pi r^{3} d r \\
& =\rho t 2 \pi\left[\frac{r^{4}}{4}\right]_{0}^{R} \\
& =\rho t 2 \pi \frac{R^{4}}{4}=\rho t \frac{\pi R^{4}}{2}
\end{aligned}
$$

But total mass $\quad M=\rho t \pi R^{2}$

$$
\therefore \quad I_{z z}=\frac{M R^{2}}{2}
$$

Example 5.33 Determine the mass moment of inertia of a circular ring of uniform cross-section.
Solution. Consider uniform ring of radius $R$ as shown in Fig. 5.85. Let its mass per unit length be $m$.

Hence, Total mass $M=2 \pi R m$
Consider an elemental length $d s=R d \theta$ at an angle $\theta$ to the diametral axis $x-x$. The distance of the element from $x$-axis is $R \sin \theta$ and mass of element is $m R d \theta$

$$
\begin{aligned}
\therefore & =\int_{0}^{2 \pi}(R \sin \theta)^{2} m R d \theta \\
& =R^{3} m \int_{0}^{2 \pi} \sin ^{2} \theta d \theta \\
& =m R^{3} \int_{0}^{2 \pi}\left(\frac{1-\cos 2 \theta}{2}\right) d \theta \\
& =\frac{m R^{3}}{2}\left[\theta-\frac{\sin 2 \theta}{2}\right]_{0}^{2 \pi}=m R^{3} \pi
\end{aligned}
$$



Fig. 5.85
$\begin{array}{lr}\text { But } & M=2 \pi R m \\ \therefore & I=\frac{M R^{2}}{2}\end{array}$
Example 5.34 Find the mass moment of inertia of the solid cone of height $h$ and base radius $R$ about:
(i) its axis of rotation and
(ii) an axis through vertex normal to the axis of rotation.


Fig. 5.86
Solution. Moment of Inertia about its Axis of Rotation:
Consider an elemental plate at distance $x$. Let its radius be $r$ and thickness $d x$.
Mass of the elemental plate $=\rho \pi r^{2} d x$
But, the moment of inertia of circular plate about normal axis through its centre is

$$
\begin{align*}
& =\frac{1}{2} \times \text { Mass } \times \text { Square of radius } \\
& =\frac{1}{2} \times \rho \pi r^{2} d x \cdot r^{2} \\
& =\rho \frac{\pi r^{4} d x}{2} \tag{i}
\end{align*}
$$

But now, $\quad r=\left(\frac{x}{h}\right) R$
$\therefore$ Moment of inertia of the elemental plate about

$$
\begin{equation*}
x \text {-axis }=\rho \frac{\pi}{2} R^{4} \frac{x^{4}}{h^{4}} d x \tag{ii}
\end{equation*}
$$

$\therefore$ Moment of inertia of the cone about $x$-axis,

$$
I_{x x}=\int_{0}^{h} \frac{\rho \pi}{2} R^{4} \frac{x^{4}}{h^{4}} d x
$$

$$
\begin{align*}
& =\frac{\rho \pi}{2} \frac{R^{4}}{h^{4}}\left[\frac{x^{5}}{5}\right]_{0}^{h} \\
I_{x x} & =\frac{\pi R^{4} h}{10} \tag{iii}
\end{align*}
$$

But mass of the cone

$$
\begin{align*}
M & =\int_{0}^{h} \pi r^{2} d x=\int_{0}^{h} \pi R^{2} \frac{x^{2}}{h^{2}} d x \\
& =\frac{\pi R^{2}}{h^{2}}\left[\frac{x^{3}}{3}\right]_{0}^{h}  \tag{iv}\\
& =\frac{\pi R^{2} h}{3}
\end{align*}
$$

From (iii) and (iv), we get

$$
I_{x x}=\frac{3}{10} M R^{2}
$$

Moment of Inertia about an Axis through Vertex and Normal to the Axis of Rotation:

Consider an element of size $r d \theta \times d r \times d x$ as shown in Fig. 5.87. Let its coordinates be $x, y, z$ and $z_{1}$ be the radius of the plate at distance $x$ from vertex.

Now, the mass of this element

$$
d m=\rho r d \theta d r d x
$$



Fig. 5.87
and its distance from $y$-axis is given by:

$$
l=\sqrt{x^{2}+z^{2}}
$$

$\therefore$ Moment of inertia of the cone about $y$-axis:

$$
I_{y}=\oint l^{2} d m
$$

$$
\begin{aligned}
& =\int_{0}^{h} \int_{0}^{z_{1}} \int_{0}^{2 \pi}\left(x^{2}+z^{2}\right) \rho r d \theta d r d x \\
& \text { but } \quad z=r \sin \theta \\
& \therefore \quad I_{y y}=\int_{0}^{h} \int_{0}^{z_{1}} \int_{\theta}^{2 \pi}\left(x^{2}+r^{2} \sin ^{2} \theta\right) \rho r d \theta d r d x \\
& =\int_{0}^{h} \int_{0}^{z_{1}} \int_{0}^{2 \pi} \rho\left(x^{2} r+r^{3} \frac{1-\cos 2 \theta}{2}\right) d \theta d r d x \\
& =\int_{0}^{h} \int_{0}^{z_{1}} \rho\left[x^{2} r \theta+\frac{r^{3}}{2}\left(\theta-\frac{\sin 2 \theta}{2}\right)\right]_{0}^{2 \pi} d r d x \\
& =\int_{0}^{h} \int_{0}^{z_{1}} \rho\left[2 \pi x^{2} r+\frac{r^{3}}{2} 2 \pi\right] d r d x \\
& =\int_{0}^{h} \rho\left[\pi x^{2} r^{2}+\pi \frac{r^{4}}{4}\right]_{0}^{z_{1}} d x \\
& =\int_{0}^{h} \rho\left[\pi x^{2} z_{l}^{2}+\frac{\pi z_{l}^{4}}{4}\right] d x \\
& \text { but } \\
& z_{l}=\frac{x}{h} R \\
& \therefore \quad I_{y y}=\int_{0}^{h} \rho\left[x^{2} \frac{x^{2}}{h^{2}} R^{2}+\frac{\pi}{4} \frac{x^{4}}{h^{4}} R^{4}\right] d x \\
& =\rho\left[\pi \frac{x^{5}}{5 h^{2}} R^{2}+\frac{\pi x^{5}}{20 h^{4}} R^{4}\right]_{0}^{h} \\
& =\rho\left[\frac{\pi}{5} h^{3} R^{2}+\frac{\pi}{20} h R^{4}\right]=\frac{\pi \rho R^{2} h}{5}\left(h^{2}+\frac{R^{2}}{4}\right) \\
& \text { but Mass of cone } M=\frac{1}{3} \rho \pi R^{2} h \\
& \therefore \quad I_{y y}=\frac{3 M}{5}\left(h^{2}+\frac{R^{2}}{4}\right)
\end{aligned}
$$

Example 5.35 Determine the moment of inertia of a solid sphere of radius $R$ about its diametral axis.
Solution. Consider an elemental plate of thickness $d y$ at distance $y$ from the diametral axis as shown in Fig. 5.88. Radius of this elemental circular plate $x$ is given by the relation:

$$
\begin{equation*}
x^{2}=R^{2}-y^{2} \tag{i}
\end{equation*}
$$

$\therefore$ Mass of the elemental plate $d m=\rho \pi x^{2} d y$

$$
\begin{equation*}
=\rho \pi\left(R^{2}-y^{2}\right) d y \tag{ii}
\end{equation*}
$$

Moment of inertia of this circular plate element about $y$-axis is:

$$
\begin{align*}
& =\frac{1}{2} \times \text { Mass } \times \text { Square of radius } \\
& =\frac{1}{2} \times \rho \pi x^{2} d y \times x^{2} \\
& =\rho \frac{\pi}{2} x^{4} d y \\
& =\rho \frac{\pi}{2}\left(R^{2}-y^{2}\right)^{2} d y \\
& =\rho \frac{\pi}{2}\left(R^{4}-2 R^{2} y^{2}+y^{4}\right) d y \\
\therefore \quad I_{y y} & =2 \int_{0}^{R} \rho \frac{\pi}{2}\left(R^{4}-2 R^{2} y^{2}+y^{4}\right) d y \\
& =\rho \pi\left[R^{4} y-\frac{2 R^{2} y^{3}}{3}+\frac{y^{5}}{5}\right]_{0}^{R} \\
& =\rho \pi R^{5}\left[1-\frac{2}{3}+\frac{1}{5}\right] \\
& =\frac{8}{15} \rho \pi R^{5} \tag{iii}
\end{align*}
$$



Fig. 5.88
but Mass of sphere, $M=2 \int_{0}^{R} d m=2 \int_{0}^{R} \rho \pi x^{2} d y$

$$
\begin{aligned}
& =2 \int_{0}^{R} \rho \pi\left(R^{2}-y^{2}\right) d y \\
& =2 \rho \pi\left(R^{2} y-\frac{y^{3}}{3}\right)_{0}^{R}
\end{aligned}
$$

$$
\begin{align*}
& =2 \rho \pi\left[R^{3}-\frac{R^{3}}{3}\right] \\
M & =\frac{4 \pi R^{3}}{3} \tag{iv}
\end{align*}
$$

From (iii) and (iv) above, we get

$$
I_{y y}=\frac{2}{5} M R^{2}
$$

Example 5.36 Using the moment of inertia expression for plates, find the expressions for moment of inertia of
(a) Parallelepiped and
(b) Circular cylinder about $z$-axis as shown in Fig. 5.89.
Solution. Parallelepiped can be looked upon as a rectangular plate of thickness $t=l$, and similarly a solid cylinder as a circular plate of thickness $l$. Hence the expressions for moment of inertia are:

$$
\begin{aligned}
& \frac{1}{12} M\left(a^{2}+b^{2}\right) \text { for parallelepiped } \\
& \frac{M R^{2}}{2} \text { for cylinder }
\end{aligned}
$$

where
Mass of parallelepiped $\quad=a b l \rho$
and that of cylinder $\quad=\pi R^{2} l \rho$

## Parallel Axis Theorem

The parallel axis theorem for the mass moment of inertia may be stated as given below:

The mass moment of inertia of a body about an axis at a distance $d$ and parallel to a centroidal axis is equal to the sum of moment of inertia about centroidal axis and the product of mass and square of distance between the two parallel axes. Thus if $I_{G G}$ is the moment of inertia of a body of mass $M$ about a centroidal axes and $I_{A A}$ is the moment of inertia about a parallel axis through $A$ which is at a distance $d$ from the centroidal axis, then

$$
\begin{equation*}
I_{A A}=I_{G G}+M d^{2} \tag{5.22}
\end{equation*}
$$

Proof: Consider the solid shown in Fig. 5.90. (a) shows the elevation of the solid and Fig. 5.90(b) shows its cross-sectional view.
$A A$ is an axis parallel to the centroidal $G-G^{\prime}$ axis at a distance ' $d$ ' from the centroid.

Since,

$$
\int r_{G^{2}} d m=I_{G G}
$$

$$
\int d^{2} d m=d^{2} \quad \int d m=M
$$

$$
\text { and } \quad 2 \int r_{\mathbf{G}} \mathbf{d} d m=2 \mathbf{d} \int r_{\mathbf{G}} d m
$$


(a)
(b)

$$
\begin{aligned}
& =2 \mathbf{d} M\left(\int \frac{r_{\mathbf{G}}}{m} d m\right) \\
& =0
\end{aligned}
$$

Fig. 5.90

Since $\frac{1}{M} \int r_{\mathbf{G}} d m=$ Distance of centroid from the centroidal axis.
Thus, we get

$$
I_{A A}=I_{G G}+M d^{2}
$$

## Moment of Inertia of Composite Bodies

In order to determine the moment of inertia, the composite body is divided into a set of simple bodies. The centre of gravity and moment of inertia expressions for such simple bodies are known. Moment of inertia of simple bodies about their centroidal axis are calculated and then using parallel axis theorem, moment of inertia of each simple body is found about the required axis. Summing up of the moment of inertia of each simple body about the required axis, gives the moment of inertia of the composite body. The procedure is illustrated with Examples 5.37 and 5.38.

Example 5.37 Determine the radius of gyration of the body shown in Fig. 5.91 about the centroidal $x$-axis. The grooves are semicircular with radius 40 mm . All dimensions shown are in mm .
Solution. The composite body may be divided into
(i) A solid block of size $80 \times 120 \times 100 \mathrm{~mm}$ and,
(ii) Two semicircular grooves each of radius 40 mm and length 80 mm .

$$
\begin{aligned}
& I_{A A}=\int \mathbf{r}^{2} d m \\
& \text { but } \quad \mathbf{r}=r_{\mathbf{G}}+\mathbf{d} \\
& \therefore \quad \mathbf{r}^{2}=\left(r_{\mathbf{G}}+\mathbf{d}\right) \times\left(r_{\mathbf{G}}+\mathbf{d}\right) \\
& =r_{G^{2}}+2 \mathbf{r}_{G} \mathbf{d}+\mathbf{d}^{2} \\
& \therefore \quad I_{A A}=\int\left(r_{\boldsymbol{G}}^{2}+2 \mathbf{r}_{G} \mathbf{d}+\mathbf{d}^{2}\right) d m \\
& =\int\left(r_{\boldsymbol{G}}^{2} d m+2 \int r_{\mathbf{G}} d d m+\int \mathbf{d}^{2} d m\right. \\
& =I_{G G}+0+M d^{2}
\end{aligned}
$$

Mass of solid block,

$$
\begin{aligned}
M_{1} & =80 \times 120 \times 100 \rho \\
& =960000 \rho
\end{aligned}
$$

where, $\rho$ is mass of $1 \mathrm{~mm}^{3}$ of material.
Its moment of inertia about $x$ axis

$$
\begin{aligned}
I_{x_{1}} & =M_{1}\left(\frac{\left(100^{2}+120^{2}\right)}{12}\right) \\
& =960 \times 10^{3} \rho \frac{\left(100^{2}+120^{2}\right)}{12}=1.952 \times 10^{9} \rho
\end{aligned}
$$

Semicircular groove:
Mass, $\quad M=\frac{1}{2} \pi r^{2} l \rho$

$$
=\frac{1}{2} \pi 40^{2} \times 80 \rho=201061.93 \rho
$$

Moment of inertia about the axis parallel to $x$-axis through centre of semicircle:

$$
\begin{aligned}
& =\frac{1}{2} \text { of that of cylinder } \\
& =\frac{1}{2} \times 2 M_{2} \times \frac{r^{2}}{2}=\frac{M_{2} r^{2}}{2}
\end{aligned}
$$

Centre of gravity of semicircular groove from this axis is at a distance

$$
\begin{aligned}
d & =\frac{4 r}{3 \pi} \\
& =4 \times \frac{40}{3 \pi}=16.9765 \mathrm{~mm}
\end{aligned}
$$

$\therefore$ Moment of inertia about the axis through centre of gravity $I_{G G}$ is given by:

$$
\begin{aligned}
\frac{M_{2} r^{2}}{2} & =I_{G G}+M_{2} d^{2} \\
\therefore \quad I_{G G} & =M_{2}\left(\frac{r^{2}}{2}-d^{2}\right) \\
& =201061.93 \rho\left(\frac{40^{2}}{2}-16.9765^{2}\right) \\
& =1.029 \times 10^{8} \rho
\end{aligned}
$$

The distance of this centroid from $x$-axis is

$$
\begin{aligned}
d^{\prime} & =60-16.9765=43.0235 \mathrm{~mm} \\
I_{x_{2}} & =I_{G G}+M_{2} d^{\prime 2} \\
& =1.029 \times 10^{8} \rho+201061.93 \rho \times 43.0235^{2} \\
& =4.7509 \times 10^{8} \rho
\end{aligned}
$$

$\therefore I_{x x}$ of composite body

$$
=I_{x_{1}}-2 I_{x_{2}}
$$

(Since there are two semicircular grooves placed symmetrically with respect to $x-x$ axis)

$$
\begin{aligned}
I_{x x} & =1.952 \times 10^{9} \rho-2 \times 4.7507 \times 10^{8} \rho \\
& =10.0816 \times 10^{8} \rho \\
\text { Total mass } \quad M & =M_{1}-2 M_{2} \\
& =960000 \rho-2 \times 201061.93 \rho \\
& =557876.14 \rho \text { units } \\
\text { i.e., } \quad k & =\sqrt{\frac{I}{M}}=\sqrt{\frac{10.0816 \times 10^{8}}{552876.14}} \\
& k
\end{aligned}
$$

Example 5.38 A cast iron flywheel has the following dimensions:
Diameter $=1.5 \mathrm{~m}$
Rim width $=300 \mathrm{~mm}$
Thickness of rim $=50 \mathrm{~mm}$
Hub length $=200 \mathrm{~mm}$
Outer diameter of hub $=250 \mathrm{~mm}$
Inner diameter of hub $=100 \mathrm{~mm}$
Arms: 6 equally spaced uniform slender rods of length 0.575 m Cross-sectional area of each arm $=8000 \mathrm{~mm}^{2}$

Determine the moment of inertia of the wheel about the axis of rotation. Take mass of cast iron as $7200 \mathrm{~kg} / \mathrm{m}^{3}$.


Fig. 5.92

Solution. Moment of inertia of rim:
Outer diameter $=1.5 \mathrm{~m}$
Thickness $\quad=0.05 \mathrm{~m}$
$\therefore$ Inner diameter $=1.5-2 \times 0.05=1.4 \mathrm{~m}$
Treating it as a solid circular plate of 1.5 m diameter in which a circular plate of diameter 1.4 m is cut and noting that for a plate moment of inertia about the axis is:

$$
\frac{\text { Mass } \times \text { Square of radius }}{2}
$$

The moment of inertia of rim is

$$
\begin{aligned}
I_{1} & =\frac{M_{0} R_{0}^{2}-M_{i} R_{i}^{2}}{2} \\
& =\frac{1}{2}\left[\left(\pi R_{0}^{2} t \rho R_{0}^{2}-\pi R_{i}^{2} t \rho R_{i}^{2}\right)\right]
\end{aligned}
$$

where, $\quad R_{0}=$ Outer radius $=\frac{1.5}{2}=0.75$

$$
R_{i}=\text { Inner radius }=\frac{1.4}{2}=0.7
$$

$M_{0}$ and $M_{1}$ are the masses of circular plates of radii $R_{0}$ and $R_{i}$, respectively.

$$
\begin{array}{lrl}
\therefore & t & =\text { Width }=0.3 \\
p & =\text { Mass per cubic meter }=7200 \mathrm{~kg} / \mathrm{m}^{3} \\
\text { i.e., } & I_{1} & =\left(\frac{1}{2}\right) \pi t\left[R_{0}^{4}-R_{i}^{4}\right] \\
\therefore & & \mathrm{I}_{1}
\end{array}
$$

## Moment of inertia of hub:

This hollow cylinder may be considered as a circular plate with circular cut out and thickness equal to length of cylinder.

$$
\begin{array}{lr}
\text { In this case, Outer radius } & R_{0}=\frac{0.25}{2}=0.125 \mathrm{~m}, \text { and } \\
\text { Inner radius } & R_{i}=\frac{0.1}{2}=0.05 \mathrm{~m} \\
& t=\text { Length of cylinder } 0.2
\end{array}
$$

As in the above case, the moment of inertia can be worked out as the moment of inertia of solid plate minus the moment of inertia of hollow plate.

$$
\begin{aligned}
I_{2} & =\frac{1}{2} \times \pi \times 0.2 \times 7200\left(0.125^{4}-0.05^{4}\right) \\
& =0.5381 \text { units }
\end{aligned}
$$

Moment of Inertia of Arms:
Moment of inertia of arm about its centre of gravity is $=\frac{M l^{2}}{12}$ and when it is shifted to axis of rotation it will be equal to $\frac{M l^{2}}{12}+M d^{2}$

$$
\text { Now, } \quad \begin{aligned}
A & =8000 \mathrm{~mm}^{2}=8000 \times 10^{-9} \mathrm{~m}^{2} \\
l & =0.575 \mathrm{~m} \\
d & =\frac{0.575}{2}+0.125=0.4125 \mathrm{~m} \\
M & =l A \rho=0.575 \times 8000 \times 10^{-9} \times 7200 \\
& =0.03312 \mathrm{~kg}
\end{aligned}
$$

As there are six such arms,

$$
\begin{aligned}
I_{3} & =6 \times \frac{M l^{2}}{2} \times d^{2} \\
& =6 \times 0.03312\left(\frac{0.575^{2}}{12}\right) \times 0.4125^{2} \\
& =0.0393 \text { units }
\end{aligned}
$$

$\therefore$ Moment of inertia of flywheel,

$$
\begin{aligned}
I & =I_{1}+I_{2}+I_{3} \\
& =258.90+0.5381+0.0393 \\
& =259.4774 \text { units }
\end{aligned}
$$

Note: Moment of inertia is contributed mainly by rim.

### 5.14 RELATION BETWEEN MASS MOMENT OF INERTIA AND AREA MOMENT OF INERTIA

Referring back to the flat plate of uniform thickness ' $t$ ' shown in Fig. 5.93, we can write

$$
\begin{aligned}
m_{i} & =\rho d v=\rho t A \\
\therefore \quad I_{M x x} & =\Sigma\left(y_{i}^{2}+z_{i}^{2}\right) d m_{i} \\
& =\Sigma y_{i}^{2} d m_{i}
\end{aligned}
$$

Since $z=0$ for all elements when $z$ is normal to the plane of plate

$$
\begin{aligned}
\therefore \quad I_{M x x} & =\Sigma y_{i}^{2} \rho t A_{i} \\
& =\rho t \Sigma y_{i}^{2} d A_{i} \\
& =\rho t I_{x x}
\end{aligned}
$$

Similarly,

$$
\begin{align*}
& I_{M y y}=\rho t I_{y y} \\
& I_{M x y}=\rho t I_{x y} \tag{5.23}
\end{align*}
$$

## IMPORTANT DEFINITIONS AND CONCEPTS

1. Centre of gravity of a body may be defined as the point through which the resultant of force of gravity of the body acts.
2. The centre point obtained by dividing first moment of masses about reference axis by the total mass is termed as centre of mass.
3. The centre point obtained by dividing the first moment of line segment by total length of line or first moment of area about reference axis with total area or first moment of volume about reference axis by total volume is termed as centroid.
4. To find centroid of simple figures's first principles is used, i.e., the first moment is obtained by integrating the value for an element over suitable limits. To find centroid of composite figures, the expression used for area is

$$
x_{i}=\frac{\Sigma a_{i} x_{i}}{A}, \quad y_{i}=\frac{\Sigma a_{i} y_{i}}{A}
$$

where $a_{i}$ is standard areas and $x_{i}, y_{i}$ the coordinates for the centroid of the standard area. Similarly, the equations for lines and volumes may be used.
5. If there exists an axis of symmetry the centroid lies on that axis.
6. Table 5.1 shows centroid of standard areas.
7. Second moment of area is termed as sum of area times the square of the distance from the axis considered, which is in the plane of the area. This is also termed as moment of inertia of area.
8. Second moment of area about an axis normal to the area is termed as polar moment of area.
9. The moment of inertia of an area about an axis perpendicular to its plane (i.e., polar moment of inertia) at any point $O$ is equal to the sum of moments of inertia about any two mutually perpendicular axes through the same point $O$ and lying in the plane of the area.
10. Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axes.
11. Moment of inertia for standard sections is shown in Table 5.4.
12. The term $\Sigma x y d A$ covering the entire area is termed as product moment of inertia of the area.
13. The axes about which product moment of inertia are zero are termed as principal axes.
14. The area of surface generated by revolving a plane curve about a nonintersecting axis in the plane of the curve is equal to the length of the generating curve times the distance travelled by the centroid of the curve in the revolution. This is known as Pappus-Guldinus theorem I.
15. Pappus-Guldinus second theorem states that the volume of the solid generated by revolving a plane area about a non-intersecting axis in the plane is equal to the area of the generating plane times the distance travelled by the centroid of the plane area during the rotation.

## IMPORTANT FORMULAE

1. $W x_{c}=\Sigma W_{i} x_{i}=\oint x d w$
$W y_{c}=\Sigma W_{i} y_{i}=\oint y d w$
$W z_{c}=\Sigma W_{i} z_{i}=\oint z d w$
$M x_{c}=\Sigma M_{i} x_{i}=\oint x M d m$
$M y_{c}=\Sigma M_{i} y_{i}=\oint y M d m$
$M z_{c}=\Sigma M_{i} z_{i}=\oint z d m$
$V x_{c}=\Sigma V_{i} x_{\mathrm{i}}=\oint x d V$
$V y_{c}=\Sigma V_{i} y_{\mathrm{i}}=\oint y d V$
$V z_{c}=\Sigma V_{i} z_{\mathrm{i}}=\oint z d V$
$A x_{c}=\Sigma A_{i} x_{i}=\oint x d A$
$A y_{c}=\Sigma A_{i} y_{i}=\oint y d A$
$L x_{c}=\Sigma L_{i} y_{i}=\oint x d L$
$L y_{c}=\Sigma L_{i} y_{i}=\oint y d L$
2. $I_{x x}=\Sigma y_{i}^{2} d A_{i}=\oint y^{2} d A$

$$
\begin{aligned}
& I_{y y}=\Sigma x_{i}^{2} d A_{i}=\oint x^{2} d A \\
& I_{z z}=\Sigma r_{i}^{2} d A_{i}=\oint r^{2} d A
\end{aligned}
$$

3. Radius of gyration $k=\sqrt{\frac{I}{A}}$, i.e., $A k^{2}=I$
4. $I_{z z}=I_{x x}+I_{y y}$
5. $I_{A B}=I_{G G}+A y_{c}^{2}$
6. $I_{x y}=\Sigma x_{i} y_{i} d A_{i}=\oint x y d A$
7. If $u-v$ axes make an anticlockwise angle $\theta$ with $x-y$ axes, then for any point,

$$
\begin{aligned}
& u=x \cos \theta+y \sin \theta \\
& v=y \cos \theta-x \sin \theta
\end{aligned}
$$

8. Principal axis $\theta$ is given by

$$
\begin{aligned}
\tan 2 \theta & =\frac{2 I_{x y}}{I_{y y}-I_{x x}} \\
I_{u u} & =\frac{I_{x x}+I_{y y}}{2}-\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}} \\
I_{v v} & =\frac{I_{x x}+I_{y y}}{2}+\sqrt{\left(\frac{I_{y y}-I_{x x}}{2}\right)^{2}+I_{x y}^{2}}
\end{aligned}
$$

9. Mass moment of inertia,

$$
\begin{array}{r}
\qquad I_{M}=\Sigma r_{i}^{2} d m_{i}=\oint r^{2} d m \\
\text { Radius of gyration, } k_{m}=\sqrt{\frac{I_{M}}{M}}
\end{array}
$$

## PROBLEMS FOR EXERCISE

5.1 Determine the centroid of the built-up section in Fig. 5.93. Express the coodinates of centroid with respect to $x$ and $y$-axes shown.
[Ans. $\bar{x}=48.91 \mathrm{~mm} ; \bar{y}=61.30 \mathrm{~mm}$ ]
5.2 Determine the centroid of the reinforced concrete retaining wall section shown in Fig. 5.94.
[Ans. $\bar{x}=1.848 \mathrm{~m} ; \bar{y}=1.825 \mathrm{~m}$ ]


Fig. 5.93
5.3 Find the coordinates of the centroid of the shaded area with respect to the axis shown in Fig. 5.95.
[Ans. $x=43.98 \mathrm{~mm} ; y=70.15 \mathrm{~mm}$ ]
5.4 A circular plate of uniform thickness and of diameter 500 mm as shown in Fig. 5.96 has two circular holes of 40 mm diameter each. Where should a 80 mm diameter hole be drilled so that the centre of gravity of the plate will be at the geometric centre?
[Ans. $x=50 \mathrm{~mm} ; y=37.5 \mathrm{~mm}$ ]


Fig. 5.95


Fig. 5.96
5.5 With respect to the coordinate axes $x$ and $y$, locate the centroid of the shaded area shown in Fig. 5.97.
[Ans. $\bar{x}=97.47 \mathrm{~mm} ; \bar{y}=70.69 \mathrm{~mm}$ ]
5.6 Locate the centroid of the plane area shown in Fig. 5.98.
[Ans. $\bar{x}=104.10 \mathrm{~mm} ; \bar{y}=44.30 \mathrm{~mm}$ ]


Fig. 5.97


Fig. 5.98
5.7 Determine the coordinates of the centroid of shaded area shown in Fig. 5.99 with respect to the corner point $O$. Take $x=40 \mathrm{~mm}$.
[Ans. $\bar{x}=260.07 \mathrm{~mm} ; \bar{y}=113.95 \mathrm{~mm}$ ]
5.8 $A B C D$ is a square section of sides 100 mm . Determine the ratio of moment of inertia of the section about centroidal axis parallel to a side to that about diagonal $A C$.
[Ans.1]


Fig. 5.99
5.9 The cross-section of a rectangular hollow beam is as shown in Fig. 5.100. Determine the polar moment of inertia of the section about centroidal axes. [Ans. $I_{x x}=10538667 \mathrm{~mm}^{4} ; I_{y y}=4906667 \mathrm{~mm}^{4} ; I_{z z}=15445334 \mathrm{~mm}^{4}$ ]
5.10 The cross-section of a prestressed concrete beam is shown in Fig. 5.101. Calculate the moment of inertia of this section about the centroidal axes parallel to and perpendicular to top edge. Also determine the radii of gyration.
[Ans. $I_{x x}=1.15668 \times 10^{10} \mathrm{~mm}^{4} ; k_{x x}=231.95 \mathrm{~mm}$; $\left.I_{y y}=8.75729 \times 10^{9} \mathrm{~mm}^{4} ; k_{y y}=201.82 \mathrm{~mm}\right]$

5.11 The strength of a 400 mm deep and 200 mm wide I-beam of uniform thickness 10 mm , is increased by welding a 250 mm wide and 20 mm thick plate to its upper flange as shown in Fig. 5.102. Determine the moment of inertia and the radii of gyration of the composite section with respect to centroidal axes parallel to and perpendicular to the bottom edge $A B$.
[Ans. $\begin{array}{r}I_{x x}=3.32393 \times 10^{8} \mathrm{~mm}^{4} ; k_{x x}=161.15 \mathrm{~mm} ; \\ \left.I_{y y}=3,9406667 \mathrm{~mm}^{4} ; k_{y y}=55.49 \mathrm{~mm}\right]\end{array}$
5.12 The cross-section of a gantry girder is as shown in Fig. 5.103. It is made up of an I-section of depth 450 mm , flange width 200 mm and a channel of size $400 \mathrm{~mm} \times 150 \mathrm{~mm}$. Thickness of all members is 10 mm . Find the moment of inertia of the section about the horizontal centroid axis.
[Ans. $I_{x x}=4.2198 \times 10^{8} \mathrm{~mm}^{4}$ ]


Fig. 5.102


Fig. 5.103
5.13 A plate girder is made up of a web plate of size $400 \mathrm{~mm} \times 10 \mathrm{~mm}$, four angles of size $100 \mathrm{~mm} \times 100 \mathrm{~mm} \times 10 \mathrm{~mm}$ and cover plates of size 300 mm $\times 10 \mathrm{~mm}$ as shown in Fig. 5.104. Determine the moment of inertia about horizontal and vertical centroidal axes.
[Ans. $I_{x x}=5.35786 \times 10^{8} \mathrm{~mm}^{4} ; I_{y y}=60850,667 \mathrm{~mm}^{4}$ ]
5.14 Determine the moment of inertia and radii of gyration of the area shown in Fig. 5.105 about the base $A-B$ and the centroidal axis parallel to $A B$.
[Ans. $I_{A B}=4815000 \mathrm{~mm}^{4} ; I_{x x}=1824231 \mathrm{~mm}^{4}$ ]


Fig. 5.104


Fig. 5.105
5.15 Determine the moment of inertia of section shown in Fig. 5.106 about the vertical centroidal axis.
[Ans. $I_{y y}=50382857 \mathrm{~mm}^{4}$ ]
5.16 A semi-circular cut is made in rectangular wooden beam as shown in Fig. 5.107. Determine the polar moment of inertia of the section about the centroidal axes.
[Ans. $I_{x x}=33581456 \mathrm{~mm}^{4} ; I_{y y}=10045631 \mathrm{~mm}^{4}$;
$\left.I_{z z}=22098980 \mathrm{~mm}^{4}\right]$

5.17 Determine the moment of inertia of the section shown in Fig. 5.108 about the horizontal centroidal axis. Also find the moment of inertia of the section about the symmetrical axis. Hence find the polar moment of inertia.
[Ans. $I_{x x}=5409046 \mathrm{~mm}^{4} ; I_{y y}=1455310 \mathrm{~mm}^{4} ; I_{z z}=6864356 \mathrm{~mm}^{4}$ ]
5.18 The cross-section of a machine part is as shown in Fig. 5.109. Determine its moment of inertia and radius of gyration about the horizontal centroidal axis.
[Ans. $I_{x x}=5249090.85 \mathrm{~mm}^{4} ; k_{x x}=27.05 \mathrm{~mm}$ ]


Fig. 5.108


Fig. 5.109
5.19 The cross-section of a plain concrete culvert is as shown in Fig. 5.110. Determine the moment of inertia about the horizontal centroidal axes.

$$
\text { [Ans. } \left.I_{x x}=5.45865 \times 10^{10} \mathrm{~mm}^{4}\right]
$$

5.20 Determine the centroid of the built-up section shown in Fig. 5.111 and find the moment of inertia and radius of gyration about the horizontal centroidal axis
[Ans. $I_{x x}=1267942 \mathrm{~mm}^{4} ; k_{x x} 18.55 \mathrm{~mm}$ ]

5.21 Determine the centre of gravity of the pyramid shown in Fig. 5.112.
$\left[\right.$ Ans. $\left.x=\frac{3}{4} h\right]$


Fig. 5.112
5.22 A steel ball of diameter 150 mm rests centrally over a concrete cube of size 150 mm . Determine the centre of gravity of the system, taking weight of concrete $=25000 \mathrm{~N} / \mathrm{m}^{3}$ and that of steel $80000 \mathrm{~N} / \mathrm{m}^{3}$.
[Ans. 147.78 mm from base]
5.23 Locate the centre of gravity of the wire shown in Fig. 5.113. Portion $B C$ is in $x-y$ plane and semicircle $C D$ is in $x-z$ plane.
[Ans. $\bar{x}=43.42 \mathrm{~mm} ; \bar{y}=29.63 \mathrm{~mm} ; \bar{z}=25.79 \mathrm{~mm}$ ]


Fig. 5.113


Fig. 5.114
5.24 Determine the moment of inertia of the link shown in Fig. 5.114 about $x$-axis.
[Ans. $23.9 \times 10^{8} \rho$ units]

## chapter 6

## Dynamics of Particles-Kinematics

Dynamics is the branch of mechanics that deals with the bodies in motion. It branches into two streams-kinematics and kinetics. Kinematics is the branch of dynamics that deals with the motion of bodies without referring the forces causing the motion and kinetics is the branch of dynamics that deals with the motion of the bodies referring to the forces causing the motion. As already defined in Chapter 1, if the size of the body is negligible in the study, the body is referred as a particle. In this chapter, we will deal with kinematics of the particle.

In this chapter, the terminologies like displacement, velocity and acceleration are explained in details. Then the kinematics of the particles is dealt under the sub-headings rectilinear motion, plane curvilinear motion, motion in space and relative motion.

### 6.1 TERMINOLOGIES IN DYNAMICS

A body is said to be in motion, if it is changing its position with reference to a point. A person on a scooter is in motion when referred to a fixed point on road side but is at rest when referred to the scooter itself. In most of the problems, the reference point is implied. For all engineering problems, a convenient fixed point on earth is an implied reference point. Other examples of implied reference points are the centre of earth for the study of satellite motion, centre of the sun for the study of motion of a solar system and mass centre of the solar system for the study of interplanatory motion.

In a general motion, we need three coordinates to describe the motion of the particle and hence the particle is termed to possess three degrees of motion. We can call such motion as space motion also. Figure 6.1 shows a particle $P$ which is in general motion. We need $x, y, z$ coordinates to study its motion. If the particle is in a plane motion, only two coordinates are enough to describe the motion. Figure 6.2 shows this case.


Fig. 6.1 General Motion/Motion in Space


Fig. 6.2 Plane Motion
Fig. 6.3 Rectilinear Motion
The simplest case is the motion along a straight line, which is termed as rectilinear motion. A car moving in a straight line is an example of a body in rectilinear motion (Ref. Fig. 6.3). To describe this motion, even a single coordinate is enough.

## Displacement

Displacement of the body in a time interval may be defined as the linear distance between the two positions of the body in the beginning and at the end of the time interval. Referring to Fig. 6.1, at the beginning of the time interval if the body is at $P$ and after a time interval $\Delta t$, if it moves to $\mathrm{P}^{\prime}$, the displacement of the body in the time interval $\Delta t$ is,
but

$$
\begin{aligned}
(r+\Delta r)-r & =\Delta r \\
r & =x i+y j+z k
\end{aligned}
$$

and

$$
\therefore \quad \Delta r=(r+\Delta r)-r ~ 子 ~=\Delta x i+\Delta y j+\Delta z k
$$

The unit of displacement in SI is $\mathrm{mm}, \mathrm{m}, \mathrm{km}$, etc.

## Velocity

The rate of change of displacement with respect to time is called velocity. Thus,

$$
\begin{aligned}
v & =\lim _{\Delta t \rightarrow 0} \frac{\Delta r}{\Delta t} \\
& =\frac{d r}{d t} \\
& =\lim _{\Delta t \rightarrow 0}\left(\frac{\Delta x}{\Delta t} i+\frac{\Delta y}{\Delta t} j+\frac{\Delta z}{\Delta t} k\right) \\
& =\frac{d x}{d t} i+\frac{d y}{d t} j+\frac{d z}{d t} k
\end{aligned}
$$

$$
\begin{equation*}
=v_{x} i+v_{y} j+v_{z} k \tag{6.2}
\end{equation*}
$$

where

$$
v_{x}=\frac{d x}{d t} \text { is } x \text {-component of the velocity }
$$

$$
v_{y}=\frac{d y}{d t} \text { is } y \text {-component of the velocity }
$$

and

$$
v_{z}=\frac{d z}{d t} \text { is } z \text {-component of the velocity }
$$

$$
\therefore \quad v=\sqrt{v_{x}^{2}+v_{y}^{2}+v_{z}^{2}}
$$

In SI units, usually metre is the unit for displacement and second is the unit of time. Hence from Eqn. (6.2), it may be observed that unit of velocity is $\mathrm{m} / \mathrm{s}$. Many times kilometer per hour ( $\mathrm{km} / \mathrm{h}$ or kmph ) is also used as unit of velocity. The relation between the two units can be easily derived as

$$
1 \mathrm{kmph}=\frac{1 \times 1000}{60 \times 60} \mathrm{~m} / \mathrm{s}
$$

## Acceleration

The rate of change of velocity with respect to time is termed as acceleration.
Thus,

$$
\begin{align*}
a & =\lim _{\Delta t \rightarrow 0} \frac{\Delta v}{d t} \\
& =\frac{d v}{d t} \\
& =\frac{d v_{x}}{d t} i+\frac{d v_{y}}{d t} j+\frac{d v_{z}}{d t} k \\
& =a_{x} i+a_{y} j+a_{z} k \tag{6.3}
\end{align*}
$$

where

$$
\begin{aligned}
& a_{x}=\frac{d v_{x}}{d t}=\frac{d}{d t}\left(\frac{d x}{d t}\right)=\frac{d^{2} x}{d t^{2}} \\
& a_{y}=\frac{d v_{y}}{d t}=\frac{d}{d t}\left(\frac{d y}{d t}\right)=\frac{d^{2} y}{d t^{2}} \\
& a_{z}=\frac{d v_{z}}{d t}=\frac{d}{d t}\left(\frac{d z}{d t}\right)=\frac{d^{2} z}{d t^{2}}
\end{aligned}
$$

Thus, $a_{x}, a_{y}$ and $a_{z}$ are the components of accelerations. Hence

$$
a=\sqrt{a_{x}^{2}+a_{y}^{2}+a_{z}^{2}}
$$

The unit of acceleration is $\mathrm{m} / \mathrm{s}^{2}$ since the unit of velocity is $\mathrm{m} / \mathrm{s}$ and acceleration is the rate of change of velocity with respect to time.

### 6.2 RECTILINEAR MOTION

The motion of a particle along a straight line is called rectilinear motion. Let a body move along direction $s$. If we take this direction as $x$-axis, then

$$
\begin{aligned}
& r=x i=s i \\
\therefore \quad & V=\frac{d s}{d t} i \quad \text { and } \quad a=\frac{d v}{d t} i=\frac{d^{2} s}{d t^{2}} i
\end{aligned}
$$

Since all the vectors in this study are along only one axis, the vector notation may be dropped in writing and kept in mind that all vector quantities are in the direction of motion. Hence we can say
$s$-is displacement along the line of motion,
$v$-is the velocity in the direction of motion, and
and $a$-is the acceleration in the direction of motion.
Thus, $\quad v=\frac{d s}{d t}$

$$
\begin{equation*}
a=\frac{d v}{d t}=\frac{d^{2} s}{d t^{2}} \tag{6.4}
\end{equation*}
$$

The readers must clearly distinguish between the terms
(i) Distance and displacement,
(ii) Speed and velocity, and
(iii) Acceleration and retardation.

## (i) Distance and Displacement

Let a car move along a road, the longitudinal section of which is as shown in Fig. 6.4. Let the time interval to move from the position $A$ to position $B$ be $t$.


Fig. 6.4

Then the distance moved by the body in the time interval is the distance measured along the hatched line, whereas the displacement is the linear distance from $A$ to $B$ measured along the direction of interest in the study. Thus the distance is the scalar quantity while displacement is the vector.

## (ii) Speed and Velocity

The rate of change of distance with respect to time is defined as speed, where as the rate of change of displacement is called velocity. Speed has only magnitude, whereas velocity has both magnitude and direction; in other words, speed is scalar while velocity is a vector.

## (iii) Acceleration and Retardation

Rate of change of velocity with respect to time has been defined as acceleration. This can be positive or negative. In practice, negative acceleration is called as retardation or deceleration.

### 6.3 MOTION CURVES

Motion curves are the graphical representation of the displacement, velocity and acceleration with time.

## Displacement-Time Curve (s-t curve)

Displacement-Time curve is a curve with time as abscissa and displacement as ordinate (Fig. 6.5). At any instant of time $t$, velocity $v$ is given by

$$
v=\frac{d s}{d t}
$$

If a body is having non-uniform motion, its displacement at various time interval may be observed and $s-t$ curve plotted. Velocity at any time may be found from the slope of $s-t$ curve.

## Velocity-time Curve (v-t Curve)



Fig. 6.5

In velocity-time curve diagram, the abscissa represents time and ordinate, the velocity of the motion. Such a curve is shown in Fig. 6.6. Acceleration $a$ is given by the slope of the $v-t$ curve.

$$
\text { i.e., } \quad a=\frac{d v}{d t}
$$



Fig. 6.6
Thus, acceleration at any time is the slope of $v-t$ curve at the time, as shown in Fig. 6.6.

| Now, |  | $\frac{d s}{d t}$ | $=v$ |
| ---: | :--- | ---: | :--- |
|  | $\therefore$ | $d s$ | $=v d t$ |
|  | or | $s$ | $=\int v d t$ |

Referring to Fig. 6.6, $v d t$ is the elemental area under the curve at time $t$ in the interval $d t$. Hence the shaded area under the curve between $t_{1}$ and $t_{2}$ shown in Fig. 6.6 represents displacement $s$ of the moving body in the time interval between $t_{1}$ and $t_{2}$. Thus in $v-t$ curve;
(i) Slope of the curve represents acceleration, and
(ii) Area under the curve represents displacement.

## Acceleration-Time Curve (a-tcurve)

If a body is moving with varying acceleration, its motion can be studied more conveniently by drawing a curve with time as abscissa and acceleration as ordinate. Such a curve is called acceleration-time curve (Ref. Fig. 6.7).
or

$$
\begin{aligned}
\frac{d v}{d t} & =a \\
d v & =a d t \\
v & =\int a d t
\end{aligned}
$$

or
Hence the area under the curve represents velocity.


Fig. 6.7

### 6.4 MOTION WITH UNIFORM VELOCITY

Consider the motion of a body moving with uniform velocity $v$. Now,
or

$$
\begin{aligned}
\frac{d s}{d t} & =v \\
s & =\int v d t
\end{aligned}
$$

$=v t($ since $v$ is constant $) \quad \ldots$ (6.5)
$v-t$ curve for such a motion is shown in Fig. 6.8. It can be easily seen that the distance travelled $s$, from starting point, in time $t$ is given by the shaded area, which is a rectangle, or $s$ $=v t$, which is same as Eqn. (6.5).


Fig. 6.8

### 6.5 MOTION WITH UNIFORM ACCELERATION

Consider the motion of a body with uniform acceleration $a$.
Let $u$ - Initial velocity,
$v$ - Final velocity,
and $\quad t$ - Time taken for change of velocity from $u$ to $v$.
Acceleration is defined as the rate of change of velocity. Since it is uniform, we can write
or

$$
\begin{align*}
& a=\frac{v-u}{t} \\
& v=u+a t
\end{align*}
$$

Displacement $s$ is given by,

$$
s=\text { Average velocity } \times \text { Time }
$$

$$
=\frac{u-v}{2} t
$$

Substituting the value of $v$ from Eqn. (6.6) into Eqn. (6.7), we get

$$
\begin{align*}
s & =\frac{u+u+a t}{2} t \\
& =u t+\frac{1}{2} a t^{2}
\end{align*}
$$

From Eqn. (6.6),

$$
t=\frac{v-u}{a}
$$

Substituting it into eqn. 6.7

$$
s=\frac{u+v}{2} \frac{v-u}{a}
$$

$$
\begin{align*}
& =\frac{v^{2}-u^{2}}{2 a} \\
\text { i.e., } \quad v^{2}-u^{2} & =2 a s \tag{6.9}
\end{align*}
$$

Thus equations of motion of a body moving with constant acceleration are
and

$$
\left.\begin{array}{rlr}
v & =u+a t \\
s & =u t+\frac{1}{2} a t^{2} & \\
(b)  \tag{6.10}\\
v^{2}-u^{2} & =2 a s & \quad(c)
\end{array}\right\}
$$

Equation (6.10) can be derived by integration technique also as given below: From definition of acceleration

$$
\begin{aligned}
\frac{d v}{d t} & =a \\
d v & =a d t
\end{aligned}
$$

Since ' $a$ ' is constant,

$$
\begin{equation*}
v=a t+c \tag{6.10a}
\end{equation*}
$$

where $c$ is constant of integration.
when

$$
t=0, \quad \text { Velocity }=\text { Initial velocity }, u
$$

Substituting these values in (6.10a), we get
$\therefore \quad u=0+c$
or

$$
\begin{align*}
& c=u \\
& v=u+a t \tag{a}
\end{align*}
$$

Thus,
From the definition of velocity,

$$
\begin{array}{rlrl}
\frac{d s}{d t} & =v \\
& =u+a t \\
& \text { or } & & \\
\therefore & d s & =(u+a t) d t \\
\therefore & s & =u t+\frac{1}{2} a t^{2}+C_{2}
\end{array}
$$

where, $C_{2}$ is constant of integration.

$$
\begin{equation*}
\text { When } \quad t=0, \quad s=0 \text {. } \tag{b}
\end{equation*}
$$

$\therefore \quad C_{2}=0 \quad$ and hence $s=u t+\frac{1}{2} a t^{2}$
From definition of acceleration,

$$
\begin{aligned}
a & =\frac{d v}{d t} \\
& =\frac{d v}{d s} \cdot \frac{d s}{d t}
\end{aligned}
$$

$$
\begin{equation*}
=\frac{d v}{d s} v \quad[\text { since } d s / d t=v] \tag{6.11}
\end{equation*}
$$

$\therefore \quad a d s=v d v$
By integrating,

$$
\begin{aligned}
a \int d s & =\int_{u}^{v} v d v \\
a s & =\left[v^{2} / 2\right]_{u}^{v} \\
& =\frac{v^{2}}{2}-\frac{u^{2}}{2}
\end{aligned}
$$

or

$$
\begin{equation*}
v^{2}-u^{2}=2 a s \tag{c}
\end{equation*}
$$

The equations of linear motions can be found conveniently referring to $v-t$ diagram. Since acceleration is uniform, the slope of the curve is constant, i.e., it is a straight line, as shown in Fig. 6.9.

$$
\text { Now, } \quad \begin{align*}
a & =\text { Slope of the diagram } \\
& =\tan \theta \\
& =\frac{B D}{A D}=\frac{B C-C D}{O C}=\frac{B C-O A}{O C} \\
& =\frac{v-u}{t} \\
\therefore & =u+a t  \tag{d}\\
s & =\text { Area } A O C B \\
& =\text { Area of rectangle } A O C D+\text { Area of } \triangle A B D \\
& =A O \times O C+\frac{1}{2} \times A D \times B D \\
& =u t+\frac{1}{2} A D \times A D \tan \theta \\
& =u t+\frac{1}{2} \times t \times t \times a \\
s & =u t+\frac{1}{2} a t^{2} \tag{e}
\end{align*}
$$

We can also write,

$$
\begin{aligned}
s & =\text { Area of parallelogram } A O C B \\
& =\frac{1}{2}(A O+B C) O C \\
& =\frac{1}{2}(u+v) t
\end{aligned}
$$

### 6.6 ACCELERATION DUE TO GRAVITY

In Chapter 1, it has been shown that the acceleration due to gravity is constant for all practical purposes when we treat the motion of the bodies near earth's surface. Its value is found to be $9.81 \mathrm{~m} / \mathrm{s}^{2}$ and is always directed towards centre of the earth, i.e., vertically downwards. Hence, if vertically downward motion of a body is considered, the value of acceleration $a$ in Eqn. (6.10) is $9.81 \mathrm{~m} / \mathrm{s}^{2}$ and if vertically upward motion is considered, then

$$
a=-g=-9.81 \mathrm{~m} / \mathrm{s}^{2}
$$

Example 6.1 A particle is projected vertically upwards from the ground with an initial velocity of $u \mathrm{~m} / \mathrm{s}$.

Find:
(i) The time taken to reach the maximum height;
(ii) The maximum height reached;
(iii) Time required for descending; and
(iv) Velocity when it strikes the ground.

Solution. Consider the upward motion of the particle.
Initial velocity $=u$
Since vertically upward motion is considered,

$$
a=-g=-9.81 \mathrm{~m} / \mathrm{s}^{2}
$$

When maximum height is reached, final velocity $v=0$.
From equation of motion,
we get

$$
v=u+a t
$$

$$
\begin{equation*}
\therefore \quad t=\frac{u}{g} \tag{1}
\end{equation*}
$$

Let the maximum height reached (displacement) $s$ be $h$.
From equation of motion,

$$
\begin{aligned}
v^{2}-u^{2} & =2 a s, \text { we get } \\
0-u^{2} & =-2 g h
\end{aligned}
$$

$$
\begin{equation*}
h=\frac{u^{2}}{2 g} \tag{2}
\end{equation*}
$$

$$
\begin{align*}
& \text { Substituting } \quad t=\frac{v-u}{a} \text { from Eqn. (d), we get } \\
& s=\frac{1}{2}(u+v) \frac{(v-u)}{a} \\
& \text { i.e., } \\
& 2 a s=v^{2}-u^{2} \tag{f}
\end{align*}
$$

Now, consider the downward motion of the particle. It starts with zero velocity $\left(u_{2}=0\right)$ from a height $h(=s)$. Let it strike the ground with final velocity $v_{2}$. Acceleration due to gravity, $g=9.81 \mathrm{~m} / \mathrm{s}^{2}$.

From the equation of motion, $v^{2}-u^{2}=2 a s$, we get

$$
\begin{align*}
v_{2}^{2}-0 & =2 g h \\
v_{2}^{2} & =2 g \frac{u^{2}}{2 g} \quad\left[\text { From Eqn. (2), } h=\frac{u^{2}}{2 g}\right] \\
& =u^{2} \\
v_{2} & =u \tag{3}
\end{align*}
$$

When a particle is freely projected, the magnitude of its velocity at any given elevation is the same during both upward and downward motion.

From the relation, $v=u+a t$, we get

$$
\begin{align*}
v_{2} & =0+g t \\
t & =\frac{v}{g}=\frac{u}{g} \tag{4}
\end{align*}
$$

Thus, the time taken for the upward motion is same as the time taken for the downward motion.

Example 6.2 A small steel ball is shot vertically upwards from the top of a building 25 m above the ground with an initial velocity of $18 \mathrm{~m} / \mathrm{s}$.
(a) In what time, will it reach the maximum height?
(b) How high above the building will the ball rise?
(c) Compute the velocity with which it will strike the ground and the total time it is in motion.
Solution. For upward motion:

$$
\begin{aligned}
u & =18 \mathrm{~m} / \mathrm{s} \\
v & =0 \\
a & =-9.81 \mathrm{~m} / \mathrm{s}^{2} \\
s & =h
\end{aligned}
$$

and
Let $t_{1}$ be time taken to reach the maximum height.
From equation of motion,


Fig. 6.10

$$
v=u+a t
$$

we get

$$
0=18-9.81 t_{1}
$$

$$
t_{1}=1.83 \mathrm{sec}
$$

Ans.
From the relation, $v^{2}-u^{2}=2 a \mathrm{~s}$, we get

$$
0-18^{2}=2(-9.81) h
$$

$$
\therefore \quad h=\frac{18^{2}}{2 \times 9.81}=16.51 \mathrm{~m}
$$

Ans.
$\therefore \quad$ Total height from the ground

$$
\begin{aligned}
& =25+h=25+16.51 \\
& =41.51 \mathrm{~m}
\end{aligned}
$$

Downward motion:
With usual notations:
$u=0, v=v_{2}, \quad s=41.51 \mathrm{~m}, \quad a=+9.81 \mathrm{~m} / \mathrm{s}^{2}$
$t=t_{2}$
From the relation $v^{2}-u^{2}=2$ as, we get

$$
\begin{aligned}
v_{2}^{2}-0 & =2 \times 9.81 \times 41.51 \\
v_{2} & =28.54 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

## Ans.

From the relation $v=u+a t$, we get

$$
\begin{array}{rlrl} 
& & 28.54 & =0+9.81 t_{2} \\
\therefore & t_{2} & =2.91 \mathrm{~s}
\end{array}
$$

$\therefore \quad$ Total time during which the body is in motion

$$
\begin{aligned}
& =t_{1}+t_{2}=1.83+2.91 \\
& =4.74 \mathrm{~s}
\end{aligned}
$$

Ans.


Fig. 6.11

For motion from $A$ to $C$,

$$
u=0, s=h+2.45, a=g=9.81 \mathrm{~m} / \mathrm{s}^{2}
$$

$$
\text { Time }=t+0.5
$$

$\therefore \quad h+2.45=0 \times t+\frac{1}{2} \mathrm{~g}(t+0.5)^{2}$

$$
\begin{equation*}
=\frac{1}{2} g\left(t^{2}+t+0.25\right) \tag{2}
\end{equation*}
$$

Subtracting (1) from (2), we get

$$
\begin{aligned}
2.45 & =\frac{1}{2} g(t+0.25) \\
& =\frac{1}{2} \times 9.81(t+0.25) \\
t & =0.2495 \mathrm{~s}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad h & =\frac{1}{2} g t^{2} \\
& =\frac{1}{2} \times 9.81 \times(0.2495)^{2} \\
h & =0.305 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 6.4 A ball is dropped from the top of a tower 30 m high. At the same instant, a second ball is thrown upward from the ground with an initial velocity of $15 \mathrm{~m} / \mathrm{s}$. When and where do they cross and with what relative velocity?
Solution. Let the two balls cross each other at a height $h$ from the ground (Fig. 6.12) after $t$ seconds.

For the motion of first ball,
$u=0, \quad s=30-h$ and $a=9.81 \mathrm{~m} / \mathrm{s}^{2}$.
Using the equation

$$
\begin{align*}
s & =u t+\frac{1}{2} a^{2}, \text { we get } \\
30-h & =0 \times t+\frac{1}{2} \times 9.81 \times t^{2} \tag{1}
\end{align*}
$$



Fig. 6.12

For the motion of second ball,

$$
\begin{array}{ll} 
& u=15 \mathrm{~m} / \mathrm{s}, \quad s=h \text { and } a=-9.81 \mathrm{~m} / \mathrm{s}^{2} \\
\therefore & h=15 t-\frac{1}{2} \times 9.81 t^{2} \tag{2}
\end{array}
$$

Adding (1) and (2), we get

$$
\begin{aligned}
30 & =15 t \\
t & =2 \mathrm{~s}
\end{aligned}
$$

or
Ans.

Ans.
At

$$
h=10.38 \mathrm{~m}
$$

$$
t=2 \text { seconds }
$$

(i) Downward velocity of first ball,

$$
v_{1}=0+9.81 \times 2=19.62 \mathrm{~m} / \mathrm{s}
$$

(ii) Upward velocity of second ball,

$$
\begin{aligned}
v_{2} & =15-9.81 \times 2 \\
& =-4.62 \mathrm{~m} / \mathrm{s} \\
v_{2} & =4.62 \mathrm{~m} / \mathrm{s} \quad \text { downward }
\end{aligned}
$$

$\therefore$ Relative velocity $=19.62-4.62$

$$
=15 \mathrm{~m} / \mathrm{s}
$$

Ans.

Example 6.5 A stone dropped into a well is heard to strike the water in 4 seconds. Find the depth of the well, assuming the velocity of sound to be $335 \mathrm{~m} / \mathrm{s}$.
Solution. Let $h$-Depth of the well,
$t_{1}$-Time taken by stone to strike water, and
$t_{2}$-Time taken by sound to travel the height $h$.
Then,

$$
\begin{equation*}
t_{1}+t_{2}=4 \tag{1}
\end{equation*}
$$

For the downward motion of the stone,

$$
\begin{align*}
& h=0 \times t_{1}+\frac{1}{2} g t_{1}^{2} \\
& \text { i.e., } \quad h=\frac{1}{2} g t_{1}^{2} \tag{2}
\end{align*}
$$

Since sound moves with uniform velocity, for the upward motion of sound, we have

$$
\begin{equation*}
h=335 t_{2} \tag{3}
\end{equation*}
$$

From Eqns. (2) and (3), we have

$$
\frac{1}{2} g t_{1}^{2}=335 t_{2}
$$

But from Eqn. (1), $t_{2}=4-t_{1}$
Hence $\quad \frac{1}{2} g t_{1}^{2}=335\left(4-t_{1}\right)$
Substituting the value of $g=9.81 \mathrm{~m} / \mathrm{s}^{2}$, we get

$$
\frac{9.81}{2} t_{1}^{2}=335\left(4-t_{1}\right)
$$

$$
t_{1}^{2}+68.30 t_{1}-273.19=0
$$

i.e.,

$$
\begin{aligned}
t_{1} & =\frac{-68.3+\sqrt{68.3^{2}-4 \times 1 \times(-273.19)}}{2} \\
& =3.79 \mathrm{~s} \\
\therefore \quad h & =\frac{1}{2} g t_{1}^{2}=\frac{1}{2} \times 9.81 \times(3.79)^{2} \\
& h
\end{aligned} \quad=70.44 \mathrm{~m}
$$

Example 6.6 A particle under a constant deceleration is moving in a straight line and covers a distance of 20 m in first two seconds and 40 m in the next 5 seconds. Calculate the distance it covers in the subsequent 3 seconds and the total distance covered, before it comes to rest.
Solution. Let the particle start from $A$ and come to halt at $E$ as shown in Fig. 6.13. Let the acceleration of the particle be $a$.

Note that if the particle is having deceleration the value of $a$ will be negative.
Let initial velocity be $u \mathrm{~m} / \mathrm{s}$.
Considering the motion between $A$ and $B$, we get
or

$$
\begin{aligned}
& 20=u \times 2+\frac{1}{2} a \times 2^{2} \\
& 20=2 u+2 a \\
& 10=u+a
\end{aligned}
$$



Considering the motion between $A$ and $C$, we get

$$
\begin{align*}
& 60=7 u+\frac{1}{2} \times a \times 7^{2} \\
& 60=7 u+24.5 a  \tag{2}\\
& 70=7 u+7 a \tag{3}
\end{align*}
$$

From (1),
$\therefore$ Subtracting (2) from (3), we get
or $\quad a=-.571 \mathrm{~m} / \mathrm{s}^{2} \quad$ (i.e., the body is decelerating)
From (1), $\quad u=10-a=10-(-0.571)$

$$
=10.571 \mathrm{~m} / \mathrm{s}
$$

Let distance $A D$ be equal to $s_{1}$. Then

$$
\begin{aligned}
s_{1} & =10.571 \times 10+\frac{1}{2}(-0.571) \times 10^{2} \\
& =77.16 \mathrm{~m}
\end{aligned}
$$

$\therefore$ Distance covered in the interval 7 seconds to 10 seconds is

$$
C D=77.16-60=17.16 \mathrm{~m}
$$

Ans.
Let the particle come to rest at a distance $s$ from the starting point, i.e., $A E=s$. Then, using the relation $v^{2}-u^{2}=2 a s$, we get

$$
\begin{aligned}
& 0-(-10.571)^{2} & =2(-0.571) s \\
\therefore & s & =\frac{10.571^{2}}{2 \times 571}=97.85 \mathrm{~m}=97.85 \mathrm{~m}
\end{aligned}
$$

Ans.

Example 6.7 A motorist is travelling at 80 kmph , when he observes a traffic light 200 m ahead of him turns red. The traffic light is timed to stay red for 10 seconds. If the motorist wished to pass the light without stopping, just as it turns green, determine: (a) the required uniform deceleration of the motor, and $(b)$ the speed of the motor as it passes the light.
Solution. Initial velocity $u=80 \mathrm{kmph}$

$$
\begin{aligned}
& =\frac{80 \times 1000}{60 \times 60}=22.22 \mathrm{~m} / \mathrm{s} \\
t & =10 \text { seconds, } s=200 \mathrm{~m}
\end{aligned}
$$

Let ' $a$ ' be acceleration.
Using the equation of motion,
we get

$$
s=u t+\frac{1}{2} a t^{2}
$$

$$
200=22.22 \times 10+\frac{1}{2} a \times 10^{2}
$$

$\therefore \quad a=-0.444 \mathrm{~m} / \mathrm{s}^{2} \quad$ (i.e., it is decelerating) Ans.
Final velocity (speed when it passed the signal)

$$
\begin{aligned}
v & =u+a t \\
& =22.22-0.444 \times 10 \\
v & =17.78 \mathrm{~m} / \mathrm{s} \\
& =17.78 \times \frac{60 \times 60}{1000} \\
\text { i.e., } \quad v & =64 \mathrm{kmph} .
\end{aligned}
$$

Ans.
Example 6.8 The greatest possible acceleration or deceleration that a train may have is $a$ and its maximum speed is $v$. Find the minimum time in which the train can get from one station to the next, if the total distance is $s$.
Solution. To have the minimum travel time, the train must accelerate at maximum acceleration to reach maximum velocity, then run at that velocity and finally decelerate at maximum rate so that this time is kept the least. Velocity-time diagram for this motion is shown in Fig. 6.14.

Let $\quad t_{1}$-Time for acceleration


Fig. 6.14
$t_{2}$-Time of uniform motion
and
$t_{3}$-Time for deceleration
$\therefore$ Total time of travel, $t=t_{1}+t_{2}+t_{3}$
Now

$$
\begin{equation*}
v=t_{1} \tan \theta \tag{1}
\end{equation*}
$$

$$
=a t_{1} \quad\left[\text { Since } \tan \theta=\text { Acceleration }=a \mathrm{~m} / \mathrm{s}^{2}\right]
$$

and also

$$
\begin{equation*}
\mathrm{v}=t_{3} \tan \theta=a t_{3} \tag{2}
\end{equation*}
$$

$\therefore$

$$
\begin{equation*}
t_{1}=t_{3} \tag{3}
\end{equation*}
$$

Let $\quad s_{1}$ —Distance travelled while accelerating
$s_{2}$-Distance travelled with uniform velocity
and
$s_{3}$ —Distance travelled while decelerating
Then

$$
\begin{aligned}
s & =s_{1}+s_{2}+s_{3} \\
& =\frac{1}{2} v t_{1}+v t_{2}+\frac{1}{2} v t_{3} \\
& =v t_{1}+v t_{2} \quad \quad\left(\text { Since } t_{1}=t_{3}\right)
\end{aligned}
$$

or

$$
\begin{equation*}
\frac{s}{v}=t_{1}+t_{2} \tag{4}
\end{equation*}
$$

From (1), (4) and (2), we have

$$
\begin{aligned}
& t=\left(t_{1}+t_{2}\right)+t_{3} \\
& t=\frac{s}{v}+\frac{v}{a}
\end{aligned}
$$

Ans.

Example 6.9 Two stations $P$ and $Q$ are 5.2 km apart. An automobile starts from rest from station $P$ and accelerates uniformly to attain a speed of 48 kmph in 30 seconds. This speed is maintained until the brakes are applied. The automobile comes to rest at station $Q$ with a uniform retardation of one metre per second. Determine the total time required to cover the distance between these two stations.
Solution. Now, $\quad v=48 \mathrm{kmph}$

$$
=\frac{48 \times 1000}{60 \times 60}=13.33 \mathrm{~m} / \mathrm{s}
$$

The time-velocity diagram is shown in the Fig. 6.15.


Fig. 6.15
When brakes are applied, the automobile retards from a velocity of 13.33 $\mathrm{m} / \mathrm{s}$ to zero in $t_{3}$ seconds at the rate $1 \mathrm{~m} / \mathrm{s}^{2}$. Then

$$
\begin{aligned}
t_{3} \tan \theta & =13.33 \\
t_{3} \times 1 & =13.33 \\
t_{3} & =13.33 \mathrm{~s}
\end{aligned}
$$

Let $t_{2}$ be the time during which the automobile travels at uniform velocity.
Now,

$$
s=s_{1}+s_{2}+s_{3}
$$

where,
$s=$ Total distance covered $=52000 \mathrm{~m}$.
$s_{1}=$ Distance covered while accelerating
$s_{2}=$ Distance covered with uniform velocity
and

$$
s_{3}=\text { Distance covered while retarding. }
$$

$$
\begin{aligned}
\therefore \quad 5200 & =\frac{1}{2} \times 30 \times 13.33+13.33 \times t_{2}+\frac{1}{2} \times 13.33 \times 13.33 \\
t_{2} & =368.33 \mathrm{~s} .
\end{aligned}
$$

$\therefore$ Total time to cover the distance between the two stations

$$
\begin{aligned}
& =30+368.33+13.333 \\
& =411.66 \mathrm{~s} .
\end{aligned}
$$

Ans.
Example 6.10 A cage descends a mine shaft with an acceleration of $0.6 \mathrm{~m} / \mathrm{s}^{2}$. After the cage has travelled 30 m a stone is dropped from the top of the shaft. Determine; (1) the time taken by the stone to hit the cage and (2) distance travelled by the cage before impact.
Solution. Let $t$ be the time in seconds during which stone is in motion. Let $s$ be the distance from the top of shaft where impact takes place.

From motion of stone, we get

$$
\begin{equation*}
s=\frac{1}{2} g t^{2}=\frac{1}{2} \times 9.81 t^{2} \tag{1}
\end{equation*}
$$

Consider the motion of the cage:
Let $t_{1}$ be the time taken to travel first 30 m .

$$
\begin{aligned}
a & =0.6 \mathrm{~m} / \mathrm{s}^{2} ; u=0 \\
\therefore \quad 30 & =0+\frac{1}{2} \times 0.6 t_{1}^{2} \\
t_{1} & =10 \mathrm{~s}
\end{aligned}
$$

$\therefore$ When the stone strikes, the cage has moved for $(t+10)$ seconds

$$
\begin{equation*}
\therefore \quad s=0+\frac{1}{2} \times 0.6 \times(t+10)^{2} \tag{2}
\end{equation*}
$$

From (1) and (2), we have

$$
\frac{1}{2} \times 9.81 t^{2}=\frac{1}{2} \times 0.6(t+10)^{2}
$$

Solving the quadratic equation in $t$, we get

$$
\begin{aligned}
& t=3.286 \mathrm{~s} \\
\therefore \quad s & =\frac{1}{2} g t^{2} \\
& =\frac{1}{2} \times 9.81 \times(3.286)^{2} \\
\text { i.e., } & s
\end{aligned}
$$

Ans.

Ans.
Example 6.11 Two trains $A$ and $B$ leave the same station in parallel lines. Train $A$ starts with a uniform acceleration of $0.15 \mathrm{~m} / \mathrm{s}^{2}$ and attains a speed of 27 kmph when the steam is reduced to keep speed constant. Train $B$ leaves 40 seconds later with uniform acceleration of $0.3 \mathrm{~m} / \mathrm{s}^{2}$ to attain a maximum speed of 54 kmph . When and where will $B$ overtake $A$ ?
Solution. Constant speed of

$$
\begin{aligned}
A & =27 \mathrm{kmph} \\
& =27 \times \frac{1000}{60 \times 60} \\
& =7.5 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Constant speed of $B$

$$
\begin{aligned}
& =54 \mathrm{kmph} \\
& =15 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

The velocity-time diagram for the trains $A$ an $B$ are shown in Figs. 6.16 (a) and (b) respectively.


Fig. 6.16
Let the train $B$ overtake train $A$ after $t$ seconds at a distance $s$ from the station. Motion of train $A$ :

Time taken to attain uniform acceleration be $t_{1}$.
From the relation $\mathrm{v}=u+a t$, we get

$$
\begin{aligned}
& & 7.5 & =0+0.15 t_{1} \\
\therefore & & t_{1} & =50 \mathrm{~s}
\end{aligned}
$$

Distance travelled in time $t$

$$
\begin{align*}
s & =\text { Shaded area in Fig. } 6.12 \text { (a) } \\
& =\frac{1}{2} t_{1} \times 7.5+7.5\left(t-t_{1}\right) \\
& =\frac{1}{2} \times 50 \times 7.5+7.5(t-50) \\
& =7.5 t-187.5 \tag{1}
\end{align*}
$$

Motion of train $B$ :
From the relation $v=u+a t$, we get

$$
\begin{array}{rlrl} 
& & 15 & =0+0.3 t_{2} \\
\therefore & t_{2} & =50 \mathrm{~s}
\end{array}
$$

Distance travelled in ' $t$ ' seconds after train $A$ started:

$$
\begin{align*}
s & =\frac{1}{2} \times t_{2} \times 15+15\left(t-t_{2}-40\right) \\
& =\frac{1}{2} \times 50 \times 15+15(t-50-40) \\
& =15 t-975 \tag{2}
\end{align*}
$$

From (1) and (2), we get

$$
\begin{aligned}
7.5 t-187.5 & =15 t-975 \\
t & =105 \mathrm{~s}
\end{aligned}
$$

Ans.
Train $B$ overtakes train A 105 seconds after train $A$ has started.
Distance ' $s$ ' is given by

$$
\begin{aligned}
s & =15 t-975 \\
& =15 \times 105-975=600 \mathrm{~m} .
\end{aligned}
$$

Ans.
Example 6.12 Two cars are travelling towards each other on a single lane road at the velocities $12 \mathrm{~m} / \mathrm{s}$ and $9 \mathrm{~m} / \mathrm{s}$, respectively. When 100 m apart, both drivers realise the situation and apply their brakes. They succeeds in stopping simultaneously and just short of colliding. Assume constant deceleration for each case determine: (a) time required for cars to stop, $(b)$ deceleration of each car, and $(c)$ the distance travelled by each car while slowing down.
Solution. Let ' $A$ ' be the position of one car and ' $B$ ' that of other car when drivers see each other and apply brakes (Fig. 6.17).


Fig. 6.17
After applying brakes, let them meet at $C$. Let $A C=x$ and time taken be $t$ seconds. Considering the motion of the first car:

$$
u=12 \mathrm{~m} / \mathrm{s}, \quad v=0, \quad s=x .
$$

Let $a_{1}$ be aceleration ( $a_{1}$ as it will automatically come out negative since it is retardation case).

From the equation of motion $v=u+a t$, we get

$$
\begin{equation*}
0=12+a_{1} t \tag{1}
\end{equation*}
$$

i.e., $\quad a_{1}=-\frac{12}{t}$

From the equation $v^{2}-u^{2}=2 a s$, we get

$$
\begin{align*}
0-12^{2} & =2 a_{1} x \\
-144 & =2 \times\left(-\frac{12}{t}\right) x \\
x & =6 t . \tag{2}
\end{align*}
$$

or
Consider the motion of second car:

$$
u=9 \mathrm{~m} / \mathrm{s} . v=0, a=a_{2}, \text { time }=t \text { and } s=100-x
$$

From the equation of motion $v=u+a t$, we get

$$
\begin{aligned}
& 0 & =9+a_{2} t \\
\therefore & a_{2} & =-9 / t
\end{aligned}
$$

and from the equation of motion, $v^{2}-u^{2}=2 a s$, we get

$$
\begin{align*}
0-9^{2} & =2 a_{2}(100-x) \\
-81 & =2\left(-\frac{9}{t}\right)(100-x) \\
100-x & =4.5 t \tag{4}
\end{align*}
$$

i.e.,

Adding (2) and (4), we get
or

$$
100=10.5 t
$$

$$
t=9.524 \mathrm{~s}
$$

Ans.
From (1)

$$
a_{1}=-12 / t=-1.26 \mathrm{~m} / \mathrm{s}^{2}
$$

Ans.
From (3) $\quad a_{2}=-9 / t=-0.945 \mathrm{~m} / \mathrm{s}^{2}$
Ans.
From (2)

$$
x=57.14 \mathrm{~m}
$$

Ans.
Distance travelled by the second car is

$$
\begin{aligned}
& =100-x \\
& =42.86 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 6.13 A car and a truck are both travelling at the constant speed of 45 kmph . The car is 10 m behind the truck. The truck driver suddenly applies his brakes, causing the truck to decelerate at the constant rate of $2 \mathrm{~m} / \mathrm{s}^{2}$. Two seconds later, the driver of the car applies his brakes and just manages to avoid a rear end collision. Determine the constant rate at which the car decelerated.
Solution. Velocity of truck and car $=45 \mathrm{kmph}$

$$
\text { i.e., } \quad u=12.5 \mathrm{~m} / \mathrm{s} \text {. }
$$

Taking $A$ as the reference point (Fig. 6.18), the distance moved by the truck $s_{T}$ is given by
i.e., $\quad s_{T}=10+12.5 t+\frac{1}{2} a_{T} t^{2}$

$\leftarrow 10 \mathrm{~m} \rightarrow$
Fig. 6.18
where, $a_{T}$ is the acceleration of truck $=-2 \mathrm{~m} / \mathrm{s}^{2}$
and $\quad t$ is the time at any instant after the brakes are applied.

$$
\begin{equation*}
\therefore \quad s_{T}=10+12.5 t-t^{2} \tag{1}
\end{equation*}
$$

The distance moved by the car $s_{C}$ during this time is given by

$$
s_{C}=12.5 \times 2+12.5(t-1)+\frac{1}{2} a_{C}(t-2)^{2}
$$

where, $a_{C}$ is acceleration of the car.
Note that $u \times 2$ gives the distance moved by the car before applying the brakes to it.

$$
\begin{align*}
s_{C} & =12.5 \times 2+12.5(t-2)+\frac{1}{2} a_{C}(t-2)^{2} \\
\therefore \quad & =12.5 t+\frac{1}{2} a_{C}(t-2)^{2} \tag{2}
\end{align*}
$$

The $s-t$ curve for the motion of car and truck is given in Fig. 6.19.
Since there is no possibility of car jumping over the truck, there is only one point of contact between the two vehicles which is shown by the point $C$ in Fig. 6.19. At this time, the two vehicles need not have zero velocity. They will have same velocity which will be equal to slope of the curves at $C$. Beyond point $C$, the car will be having velocity lesser than that of the truck. Finally, the car will stop at distance $s_{1}$ and truck will stop at distance $s_{2}$ as shown in the figure.


Fig. 6.19
At point $C, s_{T}=s_{C}$
Substituting the values of $s_{T}$ and $s_{C}$ from (1) and (2), we get

$$
\begin{aligned}
10+12.5 t-t^{2} & =12.5 t+\frac{1}{2} a_{C}(t-2)^{2} \\
& =12.5 t+\frac{1}{2} a_{C} t^{2}-2 a_{C} t+2 a_{C}
\end{aligned}
$$

i.e., $\quad t^{2}\left(\frac{1}{2} a_{C}+1\right)-2 a_{C} t+\left(2 a_{C}-10\right)=0$

This is a quadratic equation in $t$.
$\therefore \quad t=\frac{2 a_{C}}{2\left(1 / 2 a_{C}+1\right)} \pm \frac{\sqrt{\left(2 a_{C}\right)^{2}-4 \times\left(\frac{1}{2} a_{C}+1\right)\left(2 a_{C}-10\right)}}{2\left(\frac{1}{2} a_{C}+1\right)}$
Since there is only one contact point $C$, there should be only one value for $t$.
Hence in (3) the term under the root should be zero, i.e.,

$$
\left(2 a_{C}\right)^{2}-4\left(1 / 2 a_{C}+1\right)\left(2 a_{C}-10\right)=0
$$

i.e.,

$$
4 a_{C}^{2}-\left(2 a_{C}+4\right)\left(2 a_{C}-10\right)=10
$$

$\begin{aligned} \text { i.e., } & 12 a_{C}+40 & =0 \\ \text { or } & a_{C} & =-\frac{10}{3}\end{aligned}$
i.e., the deceleration of the car is $\frac{10}{3} \mathrm{~m} / \mathrm{s}^{2}$.

Ans.

### 6.7 MOTION WITH VARYING ACCELERATION

A vehicle is normally not accelerated uniformly. Initially it starts with zero acceleration, then the rate of acceleration is increased and when the desired speed is nearing, the rate of acceleration is reduced. By the time desired speed is picked up, acceleration is brought to zero. Thus there are situations with varying acceleration. If the variation of acceleration or velocity or displacement with respect to time is known, such problems can be solved using the differential equations:

$$
\begin{aligned}
& v=\frac{d s}{d t} \\
& a=\frac{d v}{d t}=\frac{d^{2} s}{d t^{2}}=v \frac{d v}{d s}
\end{aligned}
$$

Example 6.14 The motion of a particle moving in a straight line is given by the expression

$$
s=t^{3}-3 t^{2}+2 t+5
$$

where, $s$ is the displacement in metres and $t$ is the time in seconds. Determine: (1) velocity and acceleration after 4 seconds; (2) maximum or minimum velocity and corresponding displacement; (3) time at which velocity is zero.

## Solution.

$$
\begin{array}{ll} 
& s=t^{3}-3 t^{2}+2 t+5 \\
\therefore & v=\frac{d s}{d t}=3 t^{2}-6 t+2 \\
\text { and } & a=\frac{d^{2} s}{d t^{2}}=6 t-6
\end{array}
$$

Hence, after 4 seconds,
and

$$
\begin{aligned}
& \mathrm{v}=3 \times 4^{2}-6 \times 4+2=26 \mathrm{~m} / \mathrm{s} \\
& a=6 \times 4-6=18 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Ans.
Ans.
The velocity is maximum or minimum when $\frac{d v}{d t}=a=0$. From (3), we get

$$
0=6 t-6 \text { or } t=1
$$

$\therefore$ Corresponding velocity $=3 \times 1^{2}-6 \times 1+2=-1 \mathrm{~m} / \mathrm{s}$.
Hence it should be minimum velocity.
(or $\frac{d^{2} v}{d t^{2}}$ is $+v e$, hence it is minimum velocity).
Minimum velocity $=-1 \mathrm{~m} / \mathrm{s}$.
Ans.
Let at time $t$, the velocity be zero. Then

$$
\begin{array}{rl} 
& 0=3 t^{2}-6 t+2 \\
\therefore \quad t & t=\frac{6 \pm \sqrt{36-4 \times 3 \times 2}}{2 \times 3}=1.577 \text { and } 0.423 \mathrm{~s} .
\end{array}
$$

Velocity is zero at $t=0.423 \mathrm{~s}$ and 1.577 s .

## Ans.

Example 6.15 The velocity of a particle moving in a straight line is given by the expression

$$
v=t^{3}-t^{2}-2 t+2
$$

The particle is found to be at a distance 4 m from station A after 2 seconds. Determine: (a) acceleration and displacement after 4 seconds; and (b) maximum/ minimum acceleraton.
Solution. (a) Know, $v=t^{3}-t^{2}-2 t+2$

$$
\begin{equation*}
a=\frac{d v}{d t}=3 t^{2}-2 t-2 \tag{1}
\end{equation*}
$$

Hence, acceleration after 4 seconds

$$
=3 \times 4^{2}-2 \times 4-2=38 \mathrm{~m} / \mathrm{s}^{2}
$$

Ans.
Now

$$
\frac{d s}{d t}=v=t^{3}-t^{2}-2 t+2
$$

$$
\therefore \quad s=\frac{t^{4}}{4}-\frac{t^{3}}{3}-t^{2}+2 t+C
$$

where $C$ is constant of integration.
From the given condition, $s=4 \mathrm{~m}$ when $t=2 \mathrm{~s}$, we get

Ans.

$$
\begin{aligned}
& 4=\frac{2^{4}}{4}-\frac{2^{3}}{3} 2^{2}+2 \times 2+C \\
& \text { i.e., } \quad C=\frac{4}{3} \\
& \therefore \quad s=\frac{t^{4}}{4}-\frac{t^{3}}{3}-t^{2}+2 t+\frac{4}{3} \\
& \text { when } \\
& t=4 \mathrm{~s} \text {, } \\
& s=\frac{4^{4}}{4}-\frac{4^{3}}{3}-4^{2}+2 \times 4+\frac{4}{3} \\
& =36 \mathrm{~m}
\end{aligned}
$$

(b) Acceleration $a$ is maximum or minimum, when

$$
\left.\begin{array}{rl}
\frac{d a}{d t} & =0 \\
\text { i.e., } & 6 t-2
\end{array}\right)=0
$$

Since $\frac{d^{2} a}{d t^{2}}$ is a positive quantity, the above condition is for the minimum value.
$\therefore$ Minimum value of acceleration $=3 \times(1.3)^{2}-2 \times 1 / 3-2$

$$
=-2.333 \mathrm{~m} / \mathrm{s}^{2}
$$

Ans.
Example 6.16 A body moves along a straight line and its acceleration $a$ which varies with time is given by $a=2-3 t$. Five seconds after start of the observations, its velocity is found to be $20 \mathrm{~m} / \mathrm{s}$. Ten seconds after start of the observation, the body is at 85 m from the origin. Determine: (a) its acceleration, velocity and distance from the origin, and $(b)$ Determine the time in which the velocity becomes zero and the corresponding distance from the origin; and (c) Describe the motion diagramatically.
Solution. In this problem,

$$
\begin{array}{rlrl} 
& & a & =2-3 t \\
\text { i.e., } & \frac{d v}{d t} & =2-3 t \\
\therefore & v & =2 t-\frac{3 t^{2}}{2}+C_{1} \tag{1}
\end{array}
$$

where $C_{1}$ is constant of integration.
Now, $v=20 \mathrm{~m} / \mathrm{s}$, when $t=5 \mathrm{~s}$
Hence from (1),

$$
\begin{array}{ll}
20 & =2 \times 5-\frac{3}{2} \times 5^{2}+C_{1} \\
\therefore \quad & C_{1}=47.5
\end{array}
$$

Hence, (1) becomes

$$
\begin{equation*}
v=47.5+2 t-1.5 t^{2} \tag{2}
\end{equation*}
$$

i.e., $\quad \frac{d s}{d t}=47.5+2 t-1.5 t^{2}$
$\therefore \quad s=47.5 t+t^{2}-0.5 t^{3}+C_{2}$
where, $C_{2}$ is constant of integration.

It is given that,

$$
s=85 \mathrm{~m} \quad \text { when } \quad t=10 \mathrm{~s}
$$

Hence, we get

$$
\begin{equation*}
85=47.5 \times 10+10^{2}-0.5 \times 10^{3}+C_{2} \tag{4}
\end{equation*}
$$

$\therefore \quad C_{2}=10$
Hence, $\quad s=10+47.5 t+t^{2}-0.5 t^{3}$
(a) When $t=0$,
and

$$
\begin{aligned}
a & =2-3 \times 0=2 \mathrm{~m} / \mathrm{s}^{2} \\
v & =47.5+0-0=47.5 \mathrm{~m} / \mathrm{s} \\
s & =10-0+0+0-0=10 \mathrm{~m}
\end{aligned}
$$

(b) Let the time when velocity becomes zero be $t$.

Then from (2), we have

$$
\begin{aligned}
\therefore \quad 0 & =47.5+2 t-1.5 t^{2} \\
t & =6.33 \mathrm{~s}
\end{aligned}
$$

Ans.
Corresponding distance from origin

$$
\begin{aligned}
s & =10+47.5 \times 6.33+6.33^{2}-0.5 \times 6.33^{3} \\
& =223.926 \mathrm{~m}
\end{aligned}
$$

## Ans.

(c) The values of displacement, velocity and acceleration for various values of $t$ are tabulated below and then $s-t, v-t$ and $a-t$ diagrams are drawn (Fig. 6.20).

| $t$ | $s$ | $v$ | $a$ |
| :---: | ---: | ---: | :---: |
| 0 | 10.0 | 47.5 | 2 |
| 1 | 58.0 | 47.0 | -1.0 |
| 2 | 105.0 | 45.5 | -4.0 |
| 3 | 148.0 | 40.0 | -7.0 |
| 4 | 184.0 | 31.5 | -10.0 |
| 5 | 210.0 | 20.0 | -13.0 |
| 6 | 223.0 | 5.5 | -16.0 |
| 7 | 220.0 | -12.0 | -19.0 |
| 8 | 198.0 | -32.5 | -22.0 |

Example 6.17 A particle starts with an initial velocity of $8 \mathrm{~m} / \mathrm{s}$ and moves along a straight line. Its acceleration at any time $t$ after start, is given by the expression $\lambda-\mu t$, where $\lambda$ and $\mu$ are constants. Determine the equation for displacement, if the particle covers a distance of 40 m in 5 seconds and stops.

## Solution.

$$
a=\lambda-\mu t
$$

i.e.,

$$
\frac{d v}{d t}=\lambda-\mu t
$$

$\therefore \quad v=\lambda t-\frac{\mu t^{2}}{2}+C_{1}$
when,

$$
t=0, v=8 \mathrm{~m} / \mathrm{s}
$$

$$
\therefore \quad 8=0-0+C_{1} \quad \text { or } \quad C_{1}=8
$$



Fig. 6.20

$$
\therefore \quad v=8+\lambda t-\frac{\mu t^{2}}{2}
$$

$$
\text { i.e., } \quad \frac{d s}{d t}=8+\lambda t-\frac{\mu t^{2}}{2}
$$

$$
\therefore \quad s=C_{2}+8 t+\frac{\lambda t^{2}}{2}-\frac{\mu t^{3}}{6}
$$

when, $\quad t=0, s=0$

$$
\therefore \quad C_{2}=0
$$

or

$$
s=8 t+\frac{\lambda t^{2}}{2}-\frac{\mu t^{3}}{6}
$$

After 5 seconds, $s=40 \mathrm{~m}$,
$\therefore \quad 40=8 \times 5+\frac{25}{2} \lambda-\frac{\mu}{6} \times 125$
$\therefore \quad \mu=0.6 \lambda$
when $t=5 \mathrm{~s}, v=0$,

$$
\begin{equation*}
\therefore \quad 0=8+5 \lambda-\mu \frac{25}{2} \tag{2}
\end{equation*}
$$

From (1) and (2),

$$
\begin{array}{ll} 
& 0=8+5 \lambda-0.6 \lambda \frac{25}{2} \\
\therefore \quad \text { Hence, } \quad & \lambda=3.2 \\
& \mu=0.6 \times 3.2=1.92
\end{array}
$$

$\therefore$ Equation of displacement is

$$
\begin{array}{ll}
s & =8 t+\frac{3.2 t^{2}}{2}-\frac{1.92 t^{3}}{6} \\
\text { i.e., } \quad s=8 t+1.6 t^{2}-0.32 t^{3}
\end{array}
$$

## Ans.

Example 6.18 A car is moving with a velocity of 72 kmph . After seeing a child on the road, the brakes are applied and the vehicle is stopped in a distance of 15 m . If the retardation produced is proportional to distance from the point where brakes are applied, find the expression for retardation.
Solution. Let the expression for retardation be

$$
\begin{array}{rlrl} 
& a & =-k s, \quad \begin{array}{l}
\text { where } s \\
\text { and } \quad k
\end{array} & =\text { Distance travelled } \\
\therefore & v \frac{d v}{d s} & =-k s \\
v d v & =-k s d s \\
\therefore & \frac{v^{2}}{2} & =-\frac{k s^{2}}{2}+C_{1}
\end{array}
$$

At the time, the brakes are applied, $s=0$ and $v=72 \mathrm{kmph}$
i.e., $\quad v=\frac{72 \times 1000}{36 \times 36}=20 \mathrm{~m} / \mathrm{s}$
$\therefore \quad \frac{20 \times 20}{2}=-0+C_{1}$
or

$$
C_{1}=200
$$

$$
\therefore \quad \frac{v^{2}}{2}=\frac{-k s^{2}}{2}+200
$$

When vehicle stops, $v=0$ and $s=15 \mathrm{~m}$

$$
\therefore \quad 0=-k \times \frac{15^{2}}{2}+200
$$

$$
\therefore \quad k=400 / 225=1.778
$$

Hence the expression for retardation is

$$
a=-1.778 \mathrm{~s} .
$$

Ans.

### 6.8 PLANE CURVILINEAR MOTION REFERRED TO RECTANGULAR COORDINATES

Such a motion can be studied referring to only two rectangular coordinates, say $x$ and $y$. Referring to Fig. 6.21.

$$
\mathbf{r}=x \mathbf{i}+y \mathbf{j}
$$



Fig. 6.21

$$
\begin{aligned}
\mathbf{v} & =\frac{d \mathbf{r}}{d t}=\frac{d x}{d t} \mathbf{i}+\frac{d y}{d t} \mathbf{j} \\
& =v_{x} \mathbf{i}+v_{y} \mathbf{j}
\end{aligned}
$$

and

$$
\begin{align*}
\mathbf{a} & =\frac{d v}{d t}=\frac{d v_{x}}{d t} \mathbf{i}+\frac{d v_{y}}{d t} \mathbf{j}  \tag{6.12}\\
& =\frac{d^{2} x}{d t^{2}} \mathbf{i}+\frac{d^{2} y}{d t^{2}} \mathbf{j} \\
& =a_{x} \mathbf{i}+a_{y} \mathbf{j}
\end{align*}
$$

$$
\text { Hence } \begin{aligned}
v=\sqrt{v_{x}^{2}+v_{y}^{2}}, & \tan \theta=\frac{v_{y}}{v_{x}} \\
a=\sqrt{a_{x}^{2}+a_{y}^{2}}, & \tan \theta=\frac{a_{y}}{a_{x}}
\end{aligned}
$$

Example 6.19 The motion of a particle in a plane is defined by $x=t^{3}+3 t-4$ and $y=t^{2}-2 t+6$, the units being metre and second. Determine the velocity and acceleration after 3 seconds.

## Solution.

$$
\begin{array}{lll}
\text { Solution. } & x=t^{3}+3 t-4 & y=t^{2}-2 t+6 \\
\therefore & v_{x}=\frac{d x}{d t}=3 t^{2}+3 & v_{y}=\frac{d y}{d t}=2 t-2 \\
& a_{x}=\frac{d v_{x}}{d t}=6 t & a_{y}=\frac{d v_{y}}{d t}=2
\end{array}
$$

$\therefore$ After 3 seconds,

$$
\begin{array}{ll}
v_{x}=3 \times 3^{2}+3=30 \mathrm{~m} / \mathrm{s} & v_{y}=2 \times 3-2=4 \mathrm{~m} / \mathrm{s} \\
a_{x}=6 \times 3=18 \mathrm{~m} / \mathrm{s}^{2} & a_{y}=2 \mathrm{~m} / \mathrm{s}^{2}
\end{array}
$$

Hence

$$
v=\sqrt{v_{x}^{2}+v_{y}^{2}}=\sqrt{30^{2}+4^{2}}
$$

$$
v=30.265 \mathrm{~m} / \mathrm{s}
$$

Ans.

$$
\tan \theta=\frac{v_{y}}{v_{x}}=\frac{4}{30}
$$

$$
\theta=7.595^{\circ}
$$

$$
a=\sqrt{a_{x}^{2}+a_{y}^{2}}=\sqrt{18^{2}+2^{2}}
$$

$$
a=18.111 \mathrm{~m} / \mathrm{s}^{2} .
$$

$$
\tan \phi=\frac{a_{y}}{a_{x}}=\frac{2}{18}
$$

$$
\therefore \quad \phi=6.340^{\circ} .
$$

Example 6.20 A particle moves in $x-y$ plane such that $\mathbf{a}=(2-3 t) \mathbf{i}+(2 t-1) \mathbf{j}$. Five seconds after start of the observation, its velocity vector is found to be $20 \mathbf{i}+10 \mathbf{j}$. Ten seconds from the start of the observation, the body is at point $P(85 \mathrm{~m}, 200 \mathrm{~m})$. Determine the expression for the position vector of the particle at any time ' $t$ '.

## Solution.

$$
\mathbf{a}=(2-3 t) \mathbf{i}+(2 t-1) \mathbf{j}
$$

i.e.,

$$
a_{x}=(2-3 t)
$$

$$
a_{y}=2 t-1
$$

i.e.,

$$
\frac{d v_{x}}{d t}=2-3 t \quad \frac{d v_{y}}{d t}=2 t-1
$$

$$
\begin{array}{lll}
\text { i.e., } & v_{x}=2 t-\frac{3 t^{2}}{2}+C_{1} & v_{y}=t^{2}-t+C_{2} \\
\text { After } 5 \text { seconds, } & v_{x}=20 \mathrm{~m} / \mathrm{s} & v_{y}=10 \mathrm{~m} / \mathrm{s} \\
\therefore & 20=2 \times 5-\frac{3 \times 5^{2}}{2}+C_{1} & 10=5^{2}-5+C_{2} \\
\therefore & C_{1}=47.5 & C_{2}=-10 \\
\therefore & v_{x}=2 t-1.5 t^{2}+47.5 & v_{y}=t^{2}-t-10 \\
\text { i.e., } & \frac{d x}{d t}=2 t-1.5 t^{2}+47.5 & \frac{d y}{d t}=t^{2}-t-10 \\
\therefore & x=t^{2}-0.5 t^{3}+47.5 t+C_{3} \quad y=\frac{t^{3}}{3}-\frac{t^{2}}{2}-10 t+C_{4}
\end{array}
$$

After 10 seconds, $x=85 \mathrm{~m}$ and $y=200 \mathrm{~m}$

$$
\begin{array}{llrl}
\therefore & 85=1 \times 10^{2}-0.5 \times 10^{3}+47.5 \times 10+C_{3} ; & 200 & =\frac{10^{3}}{3}-\frac{10^{2}}{2}-10 \times 10+C_{4} \\
& C_{4}=16.667 \\
\therefore & x=t^{2}-0.5 t^{3}+47.5 t+10 ; & y=\frac{t^{3}}{3}-\frac{t^{2}}{2}-10 t+16.667 \\
\therefore & \mathbf{r} & \\
& & \left(10+47.5 t+t^{2}-0.5 t^{3}\right) \mathbf{i}+\left(16.667-10 t-\frac{t^{2}}{2}+\frac{t^{3}}{3}\right) \mathbf{j}
\end{array}
$$

### 6.9 MOTION OF PROJECTILES

If a particle is freely projected in the air in the direction other than vertical, we observe that it moves in a curved path. Such particles (objects) are called projectiles. The motion of the component has a vertical component and a horizontal component. The vertical component of the motion is subjected to gravitational retardation, while the horizontal component remains constant if air resistance is neglected. Referring to Fig. 6.22,

$$
\mathbf{r}=x \mathbf{i}+y \mathbf{j}
$$

If $u$ is the initial velocity at angle $\alpha$ to the horizontal direction ( $x$-direction), then initially

$$
\begin{aligned}
& v_{x}=\frac{d x}{d t}=u \cos \alpha \\
& v_{y}=\frac{d y}{d t}=u \sin \alpha \\
& a_{x}=0 \quad \text { and } \quad a_{y}=-g \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$



Fig. 6.22
Near earth surface $g=9.81 \mathrm{~m} / \mathrm{s}^{2}$. Hence in all engineering problems near earth surface, $g=9.81 \mathrm{~m} / \mathrm{s}^{2}$ is taken.

Some of the terminologies used in connection with the motion of projectiles are explained below:

Velocity of Projection: The velocity with which the particle is projected is called velocity of projection ( $u \mathrm{~m} / \mathrm{s}$ ).

Angle of Projection: The angle made by the direction of projection with horizontal axis is termed as angle of projection ( $\alpha$ in Fig. 6.22).

Trajectory: The path traced by the projectile is termed or the trajectory.
Horizontal Range: The horizontal distance through which the projectile travels in its flight is termed the horizontal range or simply range of projection.

Time of Flight: The time interval during which the projectile is in motion is called as time of flight.

The study of motion of projectile is discussed under the following four subheadings:

1. Horizontal projection
2. Inclined projection on level ground
3. Inclined projection with point of projection and point of strike at different levels
4. Inclined projection on inclined plane.

## Horizontal Projection

Consider a particle thrown horizontally from point $A$ with a velocity $v_{0} \mathrm{~m} / \mathrm{s}$ as shown in Fig. 6.23. At any instant, the particle is subjected to:
(a) Horizontal motion with constant velocity $v_{0} \mathrm{~m} / \mathrm{s}$, i.e., $v_{x}=v_{x 0}=v_{0}$
(b) Vertical motion with initial velocity zero and moving with acceleration due to gravity g, i.e.,

$$
v_{y}=v_{y 0}+a t=-g t \quad \text { since } \quad v_{y 0}=0 \quad \text { and } \quad a=-g
$$



Let $h$ be the height of $A$ from the ground.
Considering vertical motion,

$$
\begin{align*}
& y_{B}=-h \\
& \therefore \quad-h=0 \times t-\frac{1}{2} g t^{2} \\
& \therefore \quad h=\frac{1}{2} g t^{2} \tag{6.13}
\end{align*}
$$

This expression gives the time of flight. During this period, the particle moves horizontally with uniform velocity, $v_{0} \mathrm{~m} / \mathrm{s}$.
$\therefore \quad$ Range $=v_{0} t$
Example 6.21 A pilot flying his bomber at a height of 2000 m with a uniform horizontal velocity of 600 kmph wants to strike a target (Ref. Fig. 6.24). At what distance from the target, he should release the bomb?
Solution.

$$
\begin{aligned}
y_{B} & =-h=-2000 \mathrm{~m} \\
v_{x_{0}} & =600 \mathrm{kmph} \\
v_{x_{0}} & =600 \mathrm{kmph} \\
& =\frac{600 \times 1000}{60 \times 60} \\
& =166.67 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Initial velocity in vertical direction $=v_{y_{0}}$ $=0$ and gravitational acceleration $=9.81$


Fig. 6.24 $\mathrm{m} / \mathrm{s}^{2}$. If $t$ is the time of flight, considering vertical motion, we get

$$
\begin{aligned}
& & -2000 & =0 \times t-\frac{1}{2} \times 9.81 t^{2} \\
\therefore & & t & =20.19 \mathrm{~s}
\end{aligned}
$$

During this period horizontal distance travelled by the bomb

$$
\begin{aligned}
& =v_{0} t, \quad \text { since } v_{x}=v_{x_{0}}=v_{0} \\
& =166.67 \times 20.19 \\
& =3365.46 \mathrm{~m}
\end{aligned}
$$

Bomb should be released at 3365.46 m from the target.
Ans.

Example 6.22 A person wants to jump over a ditch as shown in Fig. 6.25. Find the minimum velocity with which he should jump.
Solution. Taking $A$ as origin,

$$
y_{B}=-h=-2 \mathrm{~m}
$$

and

$$
\text { Range }=3 \mathrm{~m}
$$

Let $t$ be the time of flight and $v_{x 0}$ the minimum horizontal velocity required. Considering the vertical motion:

$$
\begin{aligned}
-h & =-\frac{1}{2} g t^{2} \\
\therefore \quad 2 & =\frac{1}{2} 9.81 t^{2} \\
\therefore \quad t & =0.6386 \mathrm{~s}
\end{aligned}
$$



Fig. 6.25

Considering horizontal motion of uniform velocity, we get

$$
\begin{array}{rlrl} 
& & 3 & =v_{0} \times 0.6386 \\
v_{0} & =4.698 \mathrm{~m} / \mathrm{s} . & \left(\text { Since } v_{x}=v_{x_{0}}=v_{0}\right) \\
\therefore &
\end{array}
$$

Example 6.23 A pressure tank issues water at $A$ with a horizontal velocity $v_{0}$ as shown in Fig. 6.26. For what range of values of $v_{0}$, will water enter the opening $B C$ ?
Solution. Required velocity to enter at B:

$$
y_{B}=-h=-1 \mathrm{~m}
$$

If $t_{l}$ is the time of flight, considering vertical motion,

$$
-1=-\frac{1}{2} \times 9.81 t_{1}^{2}
$$

or $\quad t_{1}=0.4515 \mathrm{~s}$
Considering horizontal motion,

$$
\begin{aligned}
v_{x} & =v_{x 0}=v_{0} \\
v_{01} t_{1} & =3 \\
\therefore \quad v_{01} & =\frac{3}{0.4515}=6.44 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$



Required velocity to enter at $C$ :
Let $t_{2}$ be time required for the flight from $A$ to $C$.

$$
y_{C}=-2.5 \mathrm{~m} ; \text { Range }=3 \mathrm{~m} .
$$

Considering vertical motion,

$$
\begin{aligned}
-2.5 & =-\frac{1}{2} \times 981 t_{2}^{2} \\
t_{2} & =0.714 \mathrm{~s}
\end{aligned}
$$

Considering the horizontal motion,

$$
\begin{array}{ll} 
& v_{02} t_{2}=3 \\
\therefore & v_{02}=\frac{3}{0.714}=4.202 \mathrm{~m} / \mathrm{s}
\end{array}
$$

Therefore the range of velocity for which the jet can enter the opening BC is $4.202 \mathrm{~m} / \mathrm{s}$ to $6.644 \mathrm{~m} / \mathrm{s}$.

Ans.
Example 6.24 A rocket is released from a jet fighter flying horizontally at 1200 kmph at an altitude of 3000 m above its target. The rocket thrust gives it a constant horizontal acceleration of $6 \mathrm{~m} / \mathrm{s}^{2}$. At what angle below the horizontal should pilot see the target at the instant of releasing the rocket in order to score a hit?
Solution. Referring to Fig. 6.27, $h=-3000 \mathrm{~m}$.


Fig. 6.27
In vertical direction, the rocket has initial velocity equal to zero and moves with gravitational acceleration $9.81 \mathrm{~m} / \mathrm{s}^{2}$.

Hence if $t$ is the time of flight,
i.e.,

$$
y_{B}=-3000=0-\frac{1}{2} \times 9.81 t^{2}
$$

In the horizontal direction, rocket has initial velocity $=1200 \mathrm{kmph}$ and acceleration $=6 \mathrm{~m} / \mathrm{s}^{2}$.

$$
\text { Now, } \quad \begin{aligned}
v_{0} & =1200 \mathrm{kmph} \\
& =\frac{1200 \times 1000}{60 \times 60}=333.33 \mathrm{~m} / \mathrm{s} \\
v_{x} & =v_{x_{0}}+6 t \\
& =v_{0}+6 t
\end{aligned}
$$

$\therefore$ Horizontal distance covered during the time of flight $=$ Range

$$
\begin{aligned}
& =v_{0} t+\frac{1}{2} a t^{2} \\
& =333.33 \times 24.73+\frac{1}{2} \times 6 \times 24.73^{2} \\
& =10,078.5 \mathrm{~m}
\end{aligned}
$$

$\therefore$ The angle $\theta$ below the horizontal at which the pilot must see the target while releasing the rocket, is given by

$$
\begin{aligned}
\tan \theta & =\frac{3000}{10,078.5} \\
\therefore \quad \theta & =16.576^{\circ} .
\end{aligned}
$$

Ans.

## Inclined Projection on Level Ground

Consider the motion of a projectile, projected from point $A$ with velocity of projection $v_{0}$ and angle of projection $\alpha$, as shown in Fig. 6.28. Let the ground be a horizontal surface.


Fig. 6.28
The particle has motion in vertical as well as horizontal directions.

## Vertical Motion

Initial velocity $v_{y_{0}}=v_{0} \sin \alpha$, upward
Gravitational acceleration $=g=9.81 \mathrm{~m} / \mathrm{s}^{2}$, downward

$$
\text { i.e., } \quad a=-g=-9.81 \mathrm{~m} / \mathrm{s}^{2}
$$

$$
\therefore \quad v_{y}=v_{y_{0}}-g t=v_{0} \sin \alpha-g t
$$

Hence initially the particle moves upward with velocity $v_{y_{0}} \sin \alpha$ and

$$
a=-9.81 \mathrm{~m} / \mathrm{s}^{2}
$$

Velocity beomes zero after some time (at $C$ ) and then the particle starts moving downward with gravitational acceleration.

## Horizontal Motion

Horizontal component of initial velocity

$$
v_{x_{0}}=v \cos \alpha
$$

Neglecting air friction, we can say that the projectile is having uniform velocity $v \cos \alpha$ during its entire flight, i.e.,

$$
v_{x}=v \cos \alpha
$$

## Equation of the Trajectory

Let $P(x, y)$ represent the position of projectile after $t$ seconds. Considering the vertical motion,

$$
\begin{equation*}
y=\left(v_{0} \sin \alpha\right) t-\frac{1}{2} g t^{2} \tag{6.15}
\end{equation*}
$$

Considering horizontal motion,

$$
\begin{align*}
& x & =\left(v_{0} \cos \alpha\right) t  \tag{6.16}\\
\therefore & t & =\frac{x}{v_{0} \cos \alpha}
\end{align*}
$$

Substituting this value in Eqn. (6.15), we get

$$
\begin{array}{ll}
\left.\qquad \begin{array}{rl}
y & =v_{0} \sin \alpha \frac{x}{v_{0} \cos \alpha}-\frac{1}{2} g\left(\frac{x}{v_{0} \cos \alpha}\right)^{2} \\
\text { i.e., } & y
\end{array}\right)=x \tan \alpha-\frac{1}{2} \frac{g x^{2}}{v_{0}^{2} \cos ^{2} \alpha} \\
\text { But } \quad \frac{1}{\cos ^{2} \alpha} & =\sec ^{2} \alpha=\left(1+\tan ^{2} \alpha\right) \tag{6.17}
\end{array}
$$

Hence Eqn. (6.17) reduces to the form

$$
\begin{equation*}
y=x \tan \alpha-\frac{1 g x^{2}}{2 v_{0}^{2}}\left(1+\tan ^{2} \alpha\right) \tag{6.18}
\end{equation*}
$$

This is an equation of a parabola. Hence the equation of the trajectory is a parabola.

## Maximum Height

When the particle reaches maximum height, the vertical component of the velocity will be zero. Considering vertical motion,

$$
\begin{aligned}
\text { Iinitial velocity } & =v_{0} \sin \alpha \\
\text { Final velocity } & =0 \\
\text { Acceleration } & =-g
\end{aligned}
$$

Using the equation of rectilinear motion $v^{2}-u^{2}=2 a s$, we get

$$
\begin{align*}
0-\left(v_{0} \sin \alpha\right)^{2} & =-2 g h \\
h & =\frac{v_{0}^{2} \sin ^{2} \alpha}{2 g} \tag{6.19}
\end{align*}
$$

## Time Required to Reach Maximum Height

Using first equation of motion $(v=u+a t)$, when projectile reaches maximum height,

$$
\begin{array}{ll} 
& 0=v_{0} \sin \alpha-g t \\
\therefore & t=\frac{v_{0} \sin \alpha}{g} \tag{6.20}
\end{array}
$$

## Time of Flight of Projectile

Motion of the projectile in vertical direction is given by Eqn. (6.15) as

$$
y=\left(v_{0} \sin \alpha\right) t-\frac{1}{2} g t^{2}
$$

At the end of flight, $\quad y=0$

$$
\begin{array}{ll}
\therefore & 0=\left(v_{0} t \sin \alpha\right)-\frac{1}{2} g t^{2} \\
& =t\left(v_{0} \sin \alpha-\frac{1}{2} g t\right) \\
\text { or } & t=0 \\
& t=\frac{2 v_{0} \sin \alpha}{g}
\end{array}
$$

$t=0$, gives initial position ' 0 ' of the projectile. Hence time of flight is given by

$$
\begin{equation*}
t=\frac{2 v_{0} \sin \alpha}{g} \tag{6.21}
\end{equation*}
$$

## Horizontal Range

During the time of the flight, projectile moves in horizontal direction with uniform velocity $v_{0} \cos \alpha$. Hence the horizontal distance traced by the projectile in this time is given by:

$$
\begin{align*}
R & =\left(v_{0} \cos \alpha\right) t \\
& =v_{0} \cos \alpha \frac{2 v_{0} \sin \alpha}{g} \\
R & =\frac{v_{0}^{2} \sin 2 \alpha}{g} \tag{6.22}
\end{align*}
$$

## Maximum Range

In Eqn. (6.22), $\sin 2 \alpha$ can have maximum value of 1 .

$$
\begin{equation*}
\text { Hence the maximum range }=\frac{v_{0}^{2}}{g} \tag{6.23}
\end{equation*}
$$

and the angle of projection for this is when
or

$$
\begin{align*}
\sin 2 \alpha & =1 \\
2 \alpha & =90^{\circ}, \quad \text { i.e., } \quad \alpha=45^{\circ} \tag{6.24}
\end{align*}
$$

## Angle of Projection for the Required Range

In Eqn. (6.22), we have the expression for range as

$$
\begin{aligned}
& R=\frac{v_{0}^{2} \sin 2 \alpha}{g} \\
& \therefore \quad \sin 2 \alpha=\frac{g R}{v_{0}^{2}} \\
& \text { Since } \quad \sin (2 \alpha)=\sin (180-2 \alpha)
\end{aligned}
$$

There are two values of $\alpha$ which give the same result.
and another is

$$
2 \alpha_{1}=2 \alpha, \quad \text { i.e., } \quad \alpha_{1}=\alpha
$$

$\therefore \quad \alpha_{1}+\alpha_{2}=90^{\circ}$
Hence if
$\alpha_{1}=45^{\circ}+\theta$
$\alpha_{2}=45-\theta$
Thus there are two angles of projection for the required range as shown in Fig. 6.29.


Fig. 6.29
Example 6.25 A body is projected at an angle such that its horizontal range is 3 times the maximum height. Find the angle of projection.
Solution. Let $v_{0}$ be the velocity of projection and $\alpha$ the angle of projection. Then

$$
\text { Maximum height reached }=\frac{v_{0}^{2} \sin ^{2} \alpha}{2 g}
$$

and $\quad$ Range $=\frac{v_{0}^{2} \sin 2 \alpha}{g}$
$\therefore \quad$ In this case

$$
\begin{array}{rlrl} 
& & \frac{v_{0}^{2} \sin 2 \alpha}{g} & =3 \times \frac{v_{0}^{2} \sin ^{2} \alpha}{2 g} \\
& \therefore & \sin 2 \alpha & =\frac{3}{2} \sin ^{2} \alpha \\
\text { i.e., } & 2 \sin \alpha \cos \alpha & =\frac{3}{2} \sin ^{2} \alpha \\
\text { or } & \tan \alpha & =\frac{4}{3} \\
\text { i.e., } & \alpha & =53.13^{\circ} .
\end{array}
$$

Ans.
Example 6.26 A projectile is aimed at a target on the horizontal plane and falls 12 m short when the angle of projection is $15^{\circ}$, while it overshoots by 24 m when the angle is $45^{\circ}$. Find the angle of projection to hit the target.
Solution. Let $s$ be the distance of the target from the point of projection (Ref. Fig. 6.30) and $v_{0}$ be the velocity of projection.

Range of projection is given by the expression

$$
R=\frac{v_{0}^{2} \sin 2 \alpha}{g}
$$

Applying it to first case, we get


Fig. 6.30

$$
\begin{align*}
s-12 & =\frac{v_{0}^{2}}{g} \sin \left(2 \times 15^{\circ}\right) \\
& =\frac{v_{0}^{2}}{g} \times \frac{1}{2}=\frac{v_{0}^{2}}{2 g} \tag{1}
\end{align*}
$$

From the second case, we have

$$
\begin{align*}
s+24 & =\frac{v_{0}^{2}}{g} \sin \left(2 \times 45^{\circ}\right) \\
& =\frac{v_{0}^{2}}{g} \tag{2}
\end{align*}
$$

From (1) and (2), we get

$$
\begin{array}{rlrl} 
& & s+24 & =2(s-12) \\
\therefore & s & =48 \mathrm{~m}
\end{array}
$$

Let the correct angle of projection be $\alpha$. Then

$$
\begin{equation*}
48=\frac{v_{0}^{2}}{g} \sin 2 \alpha \tag{3}
\end{equation*}
$$

From Eqn. (2), $\frac{v_{0}^{2}}{g}=s+24=48+24$

$$
=72 \mathrm{~m}
$$

$\therefore \quad$ From Eqn. (3), $48=72 \sin 2 \alpha$

$$
\begin{aligned}
2 \alpha & =41.81^{\circ} \\
\alpha & =20.905^{\circ} .
\end{aligned}
$$

Ans.
Example 6.27 The horizontal component of the velocity of a projectile is twice its initial vertical component. Find the range on the horizontal plane, if the projectile passes through a point 18 m horizontally and 3 m vertically above the point of projection.
Solution. Let $v_{0}$ be the initial velocity and $\alpha$ its angle of projection.
Vertical component of velocity $=v_{0} \sin \alpha$
Horizontal component of velocity $=v_{0} \cos \alpha$
In this problem,

$$
\begin{aligned}
& v_{0} \cos \alpha & =2 v_{0} \sin \alpha \\
\therefore & \tan \alpha & =\frac{1}{2} \\
\therefore & \alpha & =26.565^{\circ}
\end{aligned}
$$

It is given that when $x=18 m, y=3 \mathrm{~m}$. Using the equation of trajectory,

$$
y=x \tan \alpha-\frac{1}{2} \times \frac{g x^{2}}{v_{0}^{2} \cos ^{2} \alpha}
$$

we get

$$
3=18 \times \frac{1}{2}-\frac{1}{2} \times \frac{9.81 \times 18^{2}}{v_{0}^{2} \cos ^{2} 26.565^{\circ}} \quad[\text { since } \tan \alpha=1 / 2]
$$

i.e., $\quad v_{0}^{2}=\frac{9.81 \times 18^{2}}{6 \times 2 \times \cos ^{2} 26.565^{\circ}}$
i.e., $\quad v_{0}=18.196 \mathrm{~m} / \mathrm{s}$
$\therefore$ Range on the horizontal plane

$$
\begin{aligned}
& =\frac{v_{0}^{2} \sin 2 \alpha}{g} \\
& =\frac{18.196^{2} \times \sin \left(2 \times 26.565^{\circ}\right)}{9.81} \\
& =27.00 \mathrm{~m} .
\end{aligned}
$$

Ans.

Example 6.28 Find the least initial velocity with which a projectile is to be projected so that it clears a wall 4 m height located at a distance of 5 m , and strikes the ground at a distance of 4 m beyond the wall as shown in Fig. 6.31. The point of projection is at the same level as the foot of the wall.
Solution. Let $v_{0}$ be the initial velocity required and $\alpha$ the angle of projection. In this problem, Range $=9 \mathrm{~m}$ and at $P$, the top of wall, $x=5 \mathrm{~m}, y=4 \mathrm{~m}$.

$$
\begin{array}{ll}
\therefore & 9=\frac{v_{0}^{2} \sin 2 \alpha}{g} \\
\therefore & v_{0}^{2}=\frac{9 g}{\sin 2 \alpha}
\end{array}
$$

From the equation of trajectory, we have
(i)


Fig. 6.31

$$
\begin{aligned}
& y=x \tan \alpha-\frac{1}{2} \frac{g x^{2}}{v_{0}^{2} \cos ^{2} \alpha} \\
& 4=5 \tan \alpha-\frac{1}{2} \frac{g \times 5^{2}}{\frac{9 g}{\sin 2 \alpha} \cos ^{2} \alpha}
\end{aligned}
$$

Substituting $2 \sin \alpha \cos \alpha$ for $\sin 2 \alpha$, we get

\[

\]

## Ans.

## Inclined Projection with Point of Projection and Point of Strike at Different Levels

Equations (6.20) and (6.21) are to be used only when the point of projection and the point of striking the ground are at the same level. Now let us consider the case, when the point of projection is at a height $y_{0}$ above the point of strike as shown in Fig. 6.32.


Fig. 6.32
From equation of motion in vertical direction, we have

$$
y=v_{0} \sin \alpha \times t-\frac{1}{2} g t^{2}
$$

By putting $y=-y_{0}$ in the above equation, the time required to reach $B$ (time of flight) is obtained.

$$
-y_{0}=v_{0} \sin \alpha \times t-\frac{1}{2} g t^{2}
$$

Once the time of flight is known, horizontal range can be found from the relation:

$$
R=v_{0} \cos \alpha \times t
$$

Maximum height above the point of projection and time required to reach it can be found as usual from Eqns. (6.19) and (6.20).

Example 6.29 A bullet, is fired from a height of 120 m at a velocity of 360 kmph at an angle of $30^{\circ}$ upwards. Neglecting air resistance, find
(a) total time of flight,
(b) horizontal range of the bullet,
(c) maximum height reached by the bullet, and
(d) final velocity of the bullet just before touching the ground.


Fig. 6.33

Solution. Velocity of projection

$$
\begin{aligned}
v_{0} & =360 \mathrm{kmph} \\
& =\frac{360 \times 1000}{60 \times 60}=100 \mathrm{~m} / \mathrm{s} .
\end{aligned}
$$

(a) Total time of flight

$$
y_{0}=-120 \mathrm{~m} .
$$

Considering vertical motion

$$
\begin{aligned}
& \qquad \begin{aligned}
y & =v_{0} \sin \alpha \times t-\frac{1}{2} g t^{2}, \\
-120 & =100 \sin 30^{\circ} \times t-\frac{1}{2} \times 9.81 t^{2} \\
t^{2}-10.194 t-24.465 & =0 \\
\text { or } \quad t & =\frac{10.194 \pm \sqrt{10.194^{2}+4 \times 1 \times 24.465}}{2} \\
\text { or } \quad t & =12.20 \mathrm{~s} .
\end{aligned}
\end{aligned}
$$

(The negative value is neglected since it does not give any practical meaning).
(b) Maximum height reached by the bullet

$$
\begin{aligned}
h & =\frac{v_{0}^{2} \sin ^{2} \alpha}{2 g}=\frac{100^{2} \sin ^{2} 30^{\circ}}{2 \times 9.81} \\
& =127.42 \mathrm{~m} \text { above point } A
\end{aligned}
$$

i.e., $\quad 127.42+120=247.42 \mathrm{~m}$ above the ground.

## Ans.

(c) Horizontal range

$$
\begin{aligned}
& =v_{0} \cos \alpha \times t \\
& =100 \cos 30^{\circ} \times 12.2 \\
& =1056.55 \mathrm{~m} .
\end{aligned}
$$

Ans.
(d) Velocity of the bullet just before striking the ground:

Vertical component of velocity

$$
\begin{aligned}
v_{y} & =v_{0} \sin \alpha-g t \\
& =100 \sin 30^{\circ}-9.81 \times 12.2 \\
& =-69.682 \mathrm{~m} / \mathrm{s} \\
& =69.682 \mathrm{~m} / \mathrm{s} \text { downward }
\end{aligned}
$$

Horizontal component of velocity,

$$
v_{x}=100 \cos \alpha=86.603 \mathrm{~m} / \mathrm{s}
$$



Fig. 6.34

Thus,

$$
\mathbf{v}=v_{x} \mathbf{i}+v_{y} \mathbf{j}=86.603 \mathbf{i}+69.682 \mathbf{j}
$$

Referring to Fig. 6.34, the velocity at strike

$$
\begin{aligned}
& v=\sqrt{69.682^{2}+86.603^{2}} \\
& v=111.16 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Ans.

$$
\begin{aligned}
\theta & =\tan ^{-1} \frac{69.682}{86.603} \\
& =38.82^{\circ} \text { as shown in Fig. 6.34. }
\end{aligned}
$$

Ans.
Example 6.30 A cricket ball thrown by a fielder from a height of 2 m , at an angle of $30^{\circ}$ to the horizontal, with an initial velocity of $20 \mathrm{~m} / \mathrm{s}$, hits the wickets at a height of 0.5 m from the ground. How far was the fielder from the wickets?


Fig. 6.35
Solution. Initial velocity $v_{0}=20 \mathrm{~m} / \mathrm{s}$
Angle of projection $\alpha=30^{\circ}$

$$
y_{0}=-(2.0-0.5)=-1.5 \mathrm{~m}
$$

Time of flight $t$ is given by the expression

$$
\begin{aligned}
-1.5 & =20 \sin 30^{\circ} \times t-\frac{1}{2} \times 9.81 t^{2} \\
t^{2}-2.0387 t-0.3058 & =0 \\
\text { i.e., } \quad t & =2.179 \mathrm{~s}
\end{aligned}
$$

$\therefore$ The distance of the fielder from the wickets $=$ Range $=v_{0} \cos \alpha \times t$

$$
\begin{aligned}
& =20 \cos 30^{\circ} \times 2.197 \\
& =37.742 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 6.31 Gravel is thrown into a bin from the top of a conveyor (Ref. Fig. 6.36 ) with a velocity of $5 \mathrm{~m} / \mathrm{s}$. Determine:
(a) time it takes the gravel to hit the bottom of the bin;


Fig. 6.36
(b) the horizontal distance from the end of the conveyor to the bin, where the gravel strikes the bin, and
(c) the velocity at which the gravel strikes the bin.

Solution. (a) Taking $A$ as origin of the coordinate system as shown in the figure, vertical motion is represented by:
where.

$$
y=v_{0} \sin \alpha \times t-\frac{1}{2} g t^{2}
$$

and

$$
\begin{aligned}
& v_{0}=5 \mathrm{~m} / \mathrm{s} \\
& \alpha=50^{\circ}
\end{aligned}
$$

For the point $B$;
$\therefore \quad-10=5 \sin 50^{\circ} \times t-\frac{1}{2} 9.81 t^{2}$
i.e., $\quad t=1.871 \mathrm{~s}$

Ans.
(b) Horizontal distance travelled in this time

$$
\begin{aligned}
& =v_{x} t \\
& =v_{0} \cos \alpha \times t \\
& =5 \cos 50^{\circ} \times 1.871 \\
& =6.013 \mathrm{~m} .
\end{aligned}
$$

Ans.
(c) Vertical component of velocity of gravel at the time of striking the bin is given by.

$$
\begin{aligned}
v_{y} & =v_{0} \sin \alpha-g t \\
& =5 \sin 50^{\circ}-9.81 \times 1.871 \\
& =-14.524 \\
& =14.524 \mathrm{~m} / \mathrm{s} \text { (downward). }
\end{aligned}
$$

Horizontal component of velocity

$$
\begin{aligned}
v_{x} & =\mathrm{v}_{0} \cos \alpha \\
& =5 \cos 50^{\circ}=3.214 \mathrm{~m} / \mathrm{s} \\
\therefore \quad & \mathbf{v}
\end{aligned}=v_{x} \mathbf{i}+v_{y} \mathbf{j}=3.214 \mathbf{i}-14.524 \mathbf{j} \mathbf{j}
$$



Fig. 6.37
$\therefore$ Velocity of strike

$$
\begin{aligned}
v & =\sqrt{14.524^{2}+3.214^{2}} \\
& =14.875 \mathrm{~m} / \mathrm{s} \\
\theta & =\tan ^{-1} \frac{14.524}{3.214} \\
& =77.52^{\circ} \text { to horizontal as shown in Fig. 6.37. }
\end{aligned}
$$

Ans.

Example 6.32 A soldier fires a bullet with a velocity of $31.32 \mathrm{~m} / \mathrm{s}$ at an angle $\alpha$ upwards from the horizontal from his position on a hill to strike a target which is 100 m away and 50 m below his position. Find the angle of projection $\alpha$. Find also the velocity with which the bullet strikes the object.


Fig. 6.38
Solution. The equation of the trajectory of bullet is [from Eqn. (6.18)]

$$
y=x \tan \alpha-\frac{1}{2} \frac{g x^{2}}{v_{0}^{2}}\left(\tan ^{2} \alpha+1\right)
$$

Now, for the point on the ground where bullet strikes,
$y=-50 \mathrm{~m} ; x=100 \mathrm{~m}$
and

$$
v_{0}=31.32 \mathrm{~m} / \mathrm{s}
$$

$$
\therefore \quad-50=100 \tan \alpha-\frac{1}{2} \times \frac{9.81 \times 100^{2}}{31.32^{2}}\left(\tan ^{2} \alpha+1\right)
$$

$$
=100 \tan \alpha-50\left(\tan ^{2} \alpha+1\right)
$$

i.e., $\quad \tan ^{2} \alpha-2 \tan \alpha=0$
$\tan \alpha(\tan \alpha-2)=0$
$\therefore \quad \alpha=0$,
or $\quad \alpha=\tan ^{-1} 2=63.435^{\circ} \quad$ Ans.
$\alpha_{0}=0$ is neglected since $\alpha$ is upward angle as given in the problem.
Horizontal component of velocity

$$
\begin{aligned}
v_{x} & =31.32 \times \cos 63.435^{\circ} \\
& =14.007 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Vertical component of velocity of strike $v_{y}$ will be downward. The vertical component of initial velocity was $31.32 \sin 63.435^{\circ}=28.013 \mathrm{~m} / \mathrm{s}$ upward.

$$
\begin{aligned}
v_{y}^{2}-(-28.013)^{2} & =2 \times 9.81 \times 50 \\
v_{y} & =42.02 \mathrm{~m} / \mathrm{s} \text { downward } \\
v & =\sqrt{v_{x}^{2}+v_{y}^{2}}=\sqrt{14.007^{2}+42.02^{2}} \\
& =44.294 \mathrm{~m} / \mathrm{s} \quad \text { Ans. } \\
\theta & =\tan ^{-1}\left(\frac{v_{y}}{v_{x}}\right)=\frac{42.02}{14.007} \quad \\
& =71.565^{\circ} \text { to horizontal, as shown in Fig. 6.39. Ans. }
\end{aligned}
$$

Example 6.33 A ball rebounds at $A$ and strikes the inclined plane at point $B$ at a distance $s=76 \mathrm{~m}$ as shown in Fig. 6.40. If the ball rises to a maximum height
$h=19 \mathrm{~m}$ above the point of projection, compute the initial velocity and the angle of projection $\alpha$.


Fig. 6.40
Solution. At $A$ the vertical component of velocity $=v_{0} \sin \alpha$
When maximum height $h=19 \mathrm{~m}$ is reached, vertical component of velocity is zero.

$$
\therefore \begin{aligned}
0-\left(v_{0} \sin \alpha\right)^{2} & =2(-g) 19 \\
\text { Substituting } \mathrm{g} & =9.81 \mathrm{~m} / \mathrm{s}^{2}, \\
v_{0} \sin \alpha & =19.308 \\
y \text { co-ordinate of } B & =-A B \times \sin \theta \\
& =-76 \times \frac{1}{\sqrt{3^{2}+1^{2}}}=-24.033 \mathrm{~m}
\end{aligned}
$$

Considering the motion in vertical upward direction:

$$
\begin{aligned}
-24.033 & =\left(v_{0} \sin \alpha\right) t-\frac{1}{2} 9.81 t^{2} \\
& =19.308 t-\frac{1}{2} 9.81 t^{2}
\end{aligned}
$$

$$
\begin{aligned}
t^{2}-3.936 t-4.9 & =0 \\
\therefore \quad t & =4.93 \mathrm{~s} \\
x \text { co-ordinate of } B & =A B \times \cos \theta \\
& =76 \times \frac{3}{\sqrt{10}}=72.1 \mathrm{~m}
\end{aligned}
$$

Considering the horizontal motion of the ball:

$$
\begin{align*}
v_{0} \cos \alpha \times t & =72.1 \\
v_{0} \cos \alpha \times 4.93 & =72.1 \\
v_{0} \cos \alpha & =14.625 \tag{2}
\end{align*}
$$

or

From Eqn. (1) and (2)

$$
\tan \alpha=\frac{19.308}{14.625}=1.32
$$

$$
\therefore \quad \alpha=52.86^{\circ}
$$

Ans.
From Eqn. (2),

$$
v_{0}=\frac{14.625}{\cos 52.86^{\circ}}
$$

i.e.,

$$
v_{0}=24.222 \mathrm{~m} / \mathrm{s}
$$

Ans.
Note: Though this example is about projection on inclined plane, to be discussed in next article, it could be solved now only as $x$ and $y$-coordinates of the point of strike $B$ are known.

## Projection on Inclined Plane

Let $A B$ be a plane inclined at an angle $\beta$ to the horizontal as shown in the Fig. 6.41. A projectile is fired up the plane from point $A$ with initial velocity $v_{0} \mathrm{~m} / \mathrm{s}$ and an angle $\alpha$. Now, the range on the inclined plane $A B$ and the time of flight are to be determined.


Fig. 6.41
Let the inclined range $A B$ be denoted by $R . A D$ be the corresponding horizontal range.

$$
\therefore \quad A D=R \cos \beta \text { and } \quad D B=R \sin \beta
$$

The equation of trajectory of the projectile is given by

$$
y=x \tan \alpha-\frac{1}{2} \frac{g x^{2}}{v_{0}^{2} \cos ^{2} \alpha}
$$

Applying this equation to point $B$, we get

$$
R \sin \beta=R \cos \beta \tan \alpha-\frac{1}{2} \frac{g R^{2} \cos ^{2} \beta}{v_{0}^{2} \cos ^{2} \alpha}
$$

$$
\text { i.e., } \quad R\left(\frac{1}{2} \frac{g \cos ^{2} \beta}{v_{0}^{2} \cos ^{2} \alpha}\right)=\cos \beta \tan \alpha-\sin \beta
$$

$$
\therefore \quad R=\frac{2 v_{0}^{2} \cos ^{2} \alpha}{g \cos ^{2} \beta}(\cos \beta \tan \alpha-\sin \beta)
$$

$$
\begin{align*}
&=\frac{2 v_{0}^{2} \cos \alpha}{g \cos ^{2} \beta}(\cos \beta \cdot \sin \alpha-\sin \beta \cdot \cos \alpha) \\
&=\frac{2 v_{0}^{2} \cos \alpha}{g \cos ^{2} \beta} \sin (\alpha-\beta)  \tag{6.25a}\\
& \text { i.e., } \quad \begin{aligned}
R & =\frac{v_{0}^{2}}{g \cos ^{2} \beta}[\sin (2 \alpha-\beta)-\sin \beta] \\
& {[\text { Since } 2 \cos A \sin B=\sin (A+B)-\sin (A-B)}
\end{aligned} \tag{6.25b}
\end{align*}
$$

## Time of flight:

Let $t$ be the time of flight. The horizontal distance covered during the flight

$$
\begin{align*}
& =A D=v_{0} \cos \alpha \times t \\
& \therefore \quad t=\frac{A D}{v_{0} \cos \alpha}=\frac{R \cos \beta}{v_{0} \cos \alpha} \\
& =\frac{2 v_{0}^{2} \cos \alpha}{g \cos ^{2} \beta} \times \frac{\sin (\alpha-\beta)}{v_{0} \cos \alpha} \cos \beta \\
& =\frac{2 v_{0} \sin (\alpha-\beta)}{g \cos \beta} \tag{6.26}
\end{align*}
$$

For the given values of $\alpha$ and $\beta$, the range is maximum when:

$$
\sin (2 \alpha-\beta)=1
$$

i.e.,

$$
2 \alpha-\beta=\frac{\pi}{2}
$$

or

$$
\begin{equation*}
\alpha=\frac{\pi}{4}+\frac{\beta}{2} \tag{6.27}
\end{equation*}
$$

Referring to Fig. 6.42,

$$
\begin{aligned}
\theta_{1} & =\frac{\pi}{4}+\frac{\beta}{2}-\beta \\
& =\frac{\pi}{4}-\frac{\beta}{2}
\end{aligned}
$$

and

$$
\theta_{2}=\frac{\pi}{2}-\beta
$$



Fig. 6.42

Thus,

$$
\theta_{2}=2 \theta_{1}
$$

i.e., the range on the given plane is maximum, when the angle of projection bisects the angle between the vertical and inclined planes.

If the projection is down the plane, Eqns. (6.25) to (6.27) can be still used, but the value of $\beta$ should be taken negative.

Example 6.34 A plane has a slope of 5 in 12 . A shot is projected with a velocity of $200 \mathrm{~m} / \mathrm{s}$ at an upward angle of $30^{\circ}$ to horizontal. Find the range on the plane if:
(a) the shot is fired up the plane;
(b) the shot is fired down the plane.

Solution. Initial velocity $v_{0}=200 \mathrm{~m} / \mathrm{s}$.
Angle of projection $\alpha=30^{\circ}$
Inclination of the plane $=\tan ^{-1}\left(\frac{5}{12}\right)=22.62^{\circ}$
(a) When the shot is fired up the plane:

$$
\beta=22.62^{\circ}
$$

$$
\begin{aligned}
\text { Range } & =\frac{v_{0}^{2}}{g \cos ^{2} \beta}[\sin (2 \alpha-\beta)-\sin \beta] \\
& =\frac{200^{2}}{9.81 \times \cos ^{2} 22.62^{\circ}}\left[\sin \left(2 \times 30-22.62^{\circ}\right)-\sin 22.62^{\circ}\right]
\end{aligned}
$$

i.e., $\quad$ Range $=1064.65 \mathrm{~m}$

Ans.
(b) When the shot is fired down the plane:

$$
\begin{gathered}
\beta=-22.62^{\circ} \\
\text { Range }=\frac{200 \times 200}{9.81 \cos ^{2}(22.62)^{\circ}}\left[\sin \left(2 \times 30+22.62^{\circ}\right)-\sin \left(-22.62^{\circ}\right)\right] \\
=\frac{200 \times 200}{9.81 \cos ^{2} 22.62^{\circ}}\left[\sin 82.62^{\circ}+\sin 22.62^{\circ}\right] \\
\text { Range }=6586.27 \mathrm{~m}
\end{gathered}
$$

Ans.
Example 6.35 A person can throw a ball at a maximum velocity of $30 \mathrm{~m} / \mathrm{s}$. If he wants to get maximum range on the plane inclined at $20^{\circ}$ to horizontal, at what angle should the ball be projected and what would be the maximum range (a) up the plane (b) down the plane?
Solution. (a) Up the plane:
For getting maximum range, the direction of projection must bisect the angle between the plane and the vertical. Referring to Fig. 6.43.

$$
\begin{array}{ll} 
& \theta=\frac{90-20}{2}=35^{\circ} \\
\therefore \quad \alpha=\theta+20^{\circ}=55^{\circ}
\end{array}
$$



Fig. 6.43

Ans.

$$
\text { Max. Range }=\frac{v_{0}^{2}}{g \cos ^{2} \beta}[\sin (2 \alpha-\beta)-\sin \beta]
$$

$$
\begin{aligned}
& =\frac{30 \times 30}{9.81 \cos ^{2} 20^{\circ}}\left[\sin \left(2 \times 55-20^{\circ}\right)-\sin 20^{\circ}\right] \\
& =68.362 \mathrm{~m} .
\end{aligned}
$$

Ans.
(b) Down the plane:

For maximum projection, direction of the projection bisects the angle between the plane and vertical.
Referring to Fig. 6.44, we have


Fig. 6.44

$$
\begin{aligned}
\theta & =\frac{1}{2}(90+20)=55^{\circ} \\
\alpha & =\theta-\beta=55-20 \\
& =35^{\circ}
\end{aligned}
$$

$\therefore \quad$ Maximum Range down the plane

$$
\begin{aligned}
& =\frac{30 \times 30}{9.81 \cos ^{2}\left(-20^{\circ}\right)}\left[\sin \left(2 \times 35+20^{\circ}\right)-\sin \left(-20^{\circ}\right)\right] \\
& =139.432 \mathrm{~m}
\end{aligned}
$$

Ans.

### 6.10 TANGENTIAL AND NORMAL ACCELERATIONS

Consider the motion of a particle moving along the path shown by dotted lines in Fig. 6.45 (a). Let its velocity at $A$ be $v$ in the tangential direction. Let $e_{t}$ be the unit tangential direction and $e_{n}$, the unit normal direction. Let the particle move from the position $A$ to $A^{\prime}$ in time interval $d t$. $\rho$ is the radius of curvature at $A$ and $\theta$ is the angle $A O A^{\prime}$. Then,

$$
\begin{align*}
v_{t} & =v e_{t} \\
d s & =\rho \theta \\
\therefore \quad \frac{d s}{d t} & =\rho \frac{d \theta}{d t} \\
\text { i.e., } \quad v_{t} & =\rho \frac{d \theta}{d t} \tag{1}
\end{align*}
$$



Fig. 6.45
and hence $\quad \mathbf{v}=v \mathbf{e}_{t}=\rho \frac{d \theta}{d t} . \mathbf{e}_{t}$

Now

$$
\begin{align*}
\mathbf{a} & =\frac{d \mathbf{v}}{d t}=\frac{d}{d t}\left(v \mathbf{e}_{t}\right) \\
& =v \frac{d \mathbf{e}_{t}}{d t}+\mathbf{e}_{t} \frac{d \mathbf{v}}{d t} \\
& =v \frac{d \mathbf{e}_{t}}{d t}+\mathbf{e}_{\mathbf{t}} a_{t} \tag{2}
\end{align*}
$$

where $a_{t}$ is acceleration in tangential direction.
Referring to Fig. 6.45(b), we find

$$
d \mathbf{e}_{t}=\mathbf{e}_{t} d \theta
$$

and the direction of $d e_{t}$ is normal to the direction of $e_{t}$, i.e., it is in the direction $e_{n}$.
$\therefore \quad d \mathbf{e}_{t}=e_{n} d \theta$
or

$$
\frac{d \mathbf{e}_{\mathbf{t}}}{d t}=\mathbf{e}_{\mathbf{n}} \frac{d \theta}{d t}
$$

$\therefore$ From Eqn. (2), we get

$$
\mathbf{a}=v \mathbf{e}_{n} \frac{d \theta}{d t}+\mathbf{e}_{t} a_{t}
$$

But from Eqn. (1), $\frac{d \theta}{d t}=\frac{v}{\rho}$

Hence

$$
\begin{align*}
& \mathbf{a}=v \mathbf{e}_{n} \frac{v}{\rho}+\mathbf{e}_{t} a_{t} \\
& \mathbf{a}=\frac{v^{2}}{\rho} \mathbf{e}_{n}+\mathbf{e}_{t} a_{t} \tag{6.28}
\end{align*}
$$

Thus acceleration in the normal (radially inward) direction is

$$
\begin{equation*}
a_{n}=\frac{v^{2}}{\rho} \tag{6.28a}
\end{equation*}
$$

and in the tangential direction,

$$
\begin{array}{rlrl} 
& a_{t} & =\frac{d v}{d t}=\frac{d^{2} s}{d t^{2}} \\
\therefore \quad a & =\sqrt{a_{n}^{2}+a_{t}^{2}} \tag{6.28c}
\end{array}
$$

If the equation of motion is expressed as $y=f(x)$, we can find the expression for radius of curvature. Referring to Fig. 6.46,


Fig. 6.46
It may be easily shown that
and

$$
\angle D M E=d \theta
$$

$$
\angle A C B=\angle \mathrm{DME}
$$

Hence $\quad \angle A C B=d \theta$
$\therefore \quad d s=\rho d \theta$
$\therefore \quad \frac{1}{\rho}=\frac{d \theta}{d s}$
Since $d s$ is an elemental length, treating $A B F$ as a triangle,

$$
\begin{align*}
\frac{d s}{d x} & =\sec \theta  \tag{2}\\
\frac{d y}{d x} & =\tan \theta  \tag{3}\\
\frac{d^{2} y}{d x^{2}} & =\sec ^{2} \theta \frac{d \theta}{d x} \\
& =\sec ^{2} \theta \cdot \frac{d \theta}{d s} \cdot \frac{d s}{d x} \\
& =\sec ^{2} \theta \frac{1}{\rho} \sec \theta\left[\text { Since } \frac{1}{\rho}=\frac{d \theta}{d s}(1) \text { and } \frac{d s}{d x}=\sec \theta(2)\right] \\
& =\frac{1}{\rho} \sec ^{3} \theta
\end{align*}
$$

$$
\therefore \quad \rho=\frac{\sec ^{3} \theta}{\frac{d^{2} y}{d x^{2}}}=\frac{\left(1+\tan ^{2} \theta\right)^{3 / 2}}{\frac{d^{2} y}{d x^{2}}}
$$

$$
\begin{equation*}
=\frac{\left(1+\left(\frac{d y}{d x}\right)^{2}\right)^{3 / 2}}{\frac{d^{2} y}{d x^{2}}} \tag{6.29}
\end{equation*}
$$

Example 6.36 A car moving at 90 kmph was slowed down from the beginning of the curve at $A$, shown in Fig. 6.47 (a). After travelling a distance of 100 m , the speed reduced to 45 kmph at $B$. If radius of the curve is 300 m , determine the net acceleration of the car at point $A$ and at point $B$. Assume retardation is constant.

(a)

(b)

(c)

Fig. 6.47

## Solution.

$$
\begin{aligned}
& \text { Initial velocity }=90 \mathrm{kmph}=\frac{90 \times 1000}{60 \times 60}=25 \mathrm{~m} / \mathrm{s} \\
& \text { Final velocity }=45 \mathrm{kmph}=\frac{45 \times 1000}{60 \times 60}=12.5 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

$\therefore \quad$ From the Eqn., $v^{2}-u^{2}=2$ as, we get

$$
\begin{aligned}
12.5^{2}-25^{2} & =2 a_{t} 100 \\
a_{t} & =-2.344 \mathrm{~m} / \mathrm{s}^{2}, \text { constant throughout }
\end{aligned}
$$

At $A$

$$
a_{n}=\frac{v^{2}}{\rho}=\frac{25^{2}}{300}=2.083
$$

$$
\therefore \quad a_{A}=\sqrt{a_{t}^{2}+a_{n}^{2}}=\sqrt{2.344^{2}+2.083^{2}}
$$

$$
=3.136 \mathrm{~m} / \mathrm{s}^{2}
$$

$$
\tan \theta_{A}=\frac{a_{n}}{a_{t}}=\frac{2.083}{2.344}
$$

$\therefore \quad \theta_{A}=41.626^{\circ}$ as shown in Fig. 6.47(b)
At $B$

$$
a_{n}=\frac{v^{2}}{\rho}=\frac{12.5}{300}=0.521 \mathrm{~m} / \mathrm{s}^{2}
$$

$$
\therefore \quad a_{B}=\sqrt{2.344^{2}+0.521^{2}}
$$

$$
=2.401
$$

$$
\tan \theta_{B}=\frac{a_{n}}{a_{t}}=\frac{0.521}{2.344}
$$

$$
\theta_{B}=12.531^{\circ}, \text { as shown in Fig. 6.47(c). }
$$

Example 6.37 A car is moving down a sloping ground represented by the curve $x^{2}=240 y$, where $x$ and $y$ are in metres. When the car is at position $A$ as shown in Fig. 6.48 (a), its velocity is 72 kmph and the retardation is $2.4 \mathrm{~m} / \mathrm{s}^{2}$. Determine the total acceleration at $A$.


Fig. 6.48

## Solution.

Now

$$
y=\frac{x^{2}}{240}
$$

$$
\frac{d y}{d x}=\frac{x}{120}
$$

and

$$
\frac{d^{2} y}{d x^{2}}=\frac{1}{120}
$$

$$
\therefore \quad \rho=\frac{\left(1+\left(\frac{d y}{d x}\right)^{2}\right)^{3 / 2}}{\frac{d^{2} y}{d x^{2}}}=\frac{\left[1+\left(\frac{x}{120}\right)^{2}\right]^{3 / 2}}{\frac{1}{120}}
$$

At A,

$$
y=15 \mathrm{~m}
$$

$$
\begin{aligned}
& \therefore \quad \begin{aligned}
& x=\sqrt{240 y}=\sqrt{240 \times 15}=60 \mathrm{~m} . \\
& \therefore \quad \text { At } A, \rho \\
& \therefore \quad \text { At } A, \\
& \therefore \quad \text { At } A,
\end{aligned} \\
& \quad \begin{aligned}
v & =72 \mathrm{kmph}=\frac{72 \times 1000}{60 \times 60}=20 \mathrm{~m} / \mathrm{s} \\
a_{n} & =\frac{v^{2}}{\rho}=\frac{(20)^{2}}{167.705}=2.385 \mathrm{~m} / \mathrm{s}^{2} \\
a_{t} & =-2.4 \mathrm{~m} / \mathrm{s} \\
a_{t} & =\sqrt{a_{t}^{2}+a_{n}^{2}} \\
& =\sqrt{(2.4)^{2}+(2.385)^{2}} \\
& =3.384 \mathrm{~m} / \mathrm{s}^{2} \\
\tan \theta & =\frac{a_{n}}{a_{t}}=\frac{2.385}{2.4}=44.82^{\circ} \text { as shown in Fig. } 6.48 \text { (b). }
\end{aligned} \text { (giv }
\end{aligned}
$$

Example 6.38 The motion of a particle in $x-y$ plane is given by $\mathbf{r}=t \mathbf{i}+\left(3 t^{2}-4 t\right) \mathbf{j}$, where the distances are in metres and time is in seconds. Determine the radius of curvature and the normal and tangential acceleration when the particle crosses the $z$-axis after the start of motion.

## Solution.

Let

$$
\mathbf{r}=x \mathbf{i}+y \mathbf{j}=t \mathbf{i}+\left(3 t^{2}-4 t\right) \mathbf{j}
$$

$\therefore \quad x=t$ and $y=3 x^{2}-4 x$ $=x(3 x-4)$
$y=0 \quad$ at $\quad x=0$ and at $x=\frac{4}{3} \mathrm{~m}$.
Hence the point of interest $A$ is at $x=\frac{4}{3} \mathrm{~m}$.


Fig. 6.49

$$
y=3 x^{2}-4 x
$$

$$
\therefore \quad \frac{d y}{d x}=6 x-4
$$

$$
\frac{d^{2} y}{d x^{2}}=6
$$

$$
\begin{aligned}
\therefore \frac{d y}{d x} /_{A}= & 6 \times \frac{4}{3}-4=4 \\
& \therefore \frac{d^{2} y}{d x^{2}} /_{A}=6
\end{aligned}
$$

At $A$,

$$
\begin{aligned}
& \therefore \quad \rho=\frac{\left[1+\left(\frac{d y}{d x}\right)^{2}\right]^{3 / 2}}{\frac{d^{2} y}{d x^{2}}}=\frac{\left[1+4^{2}\right]^{3 / 2}}{6} \\
& \text { Now, } \quad \begin{aligned}
& =11.682 \mathrm{~m} \\
x & =t
\end{aligned} \quad \text { and } \quad y=3 t^{2}-4 t \\
& \therefore \quad v_{x}=\frac{d x}{d t}=1 \quad v_{y}=\frac{d y}{d t}=6 t-4 \\
& a_{x}=\frac{d^{2} x}{d t^{2}}=0 \quad a_{y}=\frac{d^{2} y}{d t^{2}}=6 \\
& \text { At } A, \quad x=\frac{4}{3}, t=x=\frac{4}{3} \\
& \therefore \quad v_{x}=1 \mathrm{~m} / \mathrm{s} \quad v_{y}=6 \times \frac{4}{3}-4=4 \mathrm{~m} / \mathrm{s} \text {. } \\
& v=\sqrt{1^{2}+4^{2}}=4.123 \mathrm{~m} / \mathrm{s} \\
& a_{x}=0 \quad a_{y}=6 \mathrm{~m} / \mathrm{s}^{2} \\
& \therefore \quad a=\sqrt{a_{x}^{2}+a_{y}^{2}}=6 \mathrm{~m} / \mathrm{s}^{2} \\
& \text { At } A, \quad a_{n}=\frac{v^{2}}{\rho}=\frac{(4.123)^{2}}{11.682}=1.455 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

From the relation,
we get

$$
a=\sqrt{a_{n}^{2}+a_{t}^{2}}
$$

$\therefore \quad a_{1}=1.970 \mathrm{~m} / \mathrm{s}^{2}$
Thus at $A$,
and

$$
\left.\begin{array}{rl}
\rho & =11.682 \mathrm{~m} \\
a_{n} & =1.455 \mathrm{~m} / \mathrm{s}^{2} \\
a_{t} & =1.970 \mathrm{~m} / \mathrm{s}^{2}
\end{array}\right\}
$$

Ans.

### 6.11 ROTATIONAL MOTION

It is an important special case of plane curvilinear motion where the radius of curvature $\rho$ becomes the constant radius $r$ of the circle.

The displacement of the particle in rotation is measured in terms of angular displacement $\theta$, where $\theta$ is in radians. Thus when a particle moves from position $A$ to $B$, the displacement is $\theta$ as shown in Fig. 6.50. This displacement has the direction-clockwise or counter-clockwise.


Fig. 6.50
The rate of change of displacement with time is called angular velocity and is denoted by $\omega$.

Thus,

$$
\begin{equation*}
\omega=\frac{d \theta}{d t} \tag{6.29a}
\end{equation*}
$$

The rate of change of angular velocity with time is termed as angular acceleration and is denoted by $\alpha$. Thus,

$$
\begin{equation*}
\alpha=\frac{d \omega}{d t}=\frac{d^{2} \theta}{d t^{2}} \tag{6.30}
\end{equation*}
$$

The angular acceleration may be expressed in another useful form as shown below:

$$
\begin{align*}
\alpha & =\frac{d \omega}{d t}=\frac{d \omega}{d \theta} \cdot \frac{d \theta}{d t} \\
& =\omega \frac{d \omega}{d \theta} \tag{6.31}
\end{align*}
$$

Noting that when the particle moves from $A$ to $B$ (Ref. Fig. 6.50), the distance travelled by it is

$$
\begin{aligned}
& s=r \theta \\
\therefore \quad v & =\frac{d s}{d t}=r \frac{d \theta}{d t}=r \omega
\end{aligned}
$$

and

$$
\begin{equation*}
a_{t}=\frac{d v}{d t}=r \frac{d^{2} \theta}{d t^{2}}=r \alpha \tag{6.32}
\end{equation*}
$$

$$
a_{n}=\frac{v^{2}}{r}=\frac{(r \omega)^{2}}{r}=r \omega^{2}
$$

### 6.12 PLANE CURVILINEAR MOTION IN POLAR COORDINATES

In this system, position of a particle is described by radial distance $r$ from the fixed position and angular distance $\theta$ from a reference axis. Thus the coordinates of point $A$ shown in Fig. 6.51 (a) are ( $r, \theta$ ). Let $e_{r}$ and $e_{\theta}$ be the unit directions in radial and angular directions. Hence the position vector

$$
\begin{equation*}
\mathbf{r}=r e_{\mathbf{r}} \tag{1}
\end{equation*}
$$



Fig. 6.51

$$
\begin{align*}
\therefore \quad \mathbf{v} & =\frac{d r}{d t}=\frac{d}{d t}\left(r \mathbf{e}_{\mathbf{r}}\right) \\
& =\frac{d r}{d t} \mathbf{e}_{\mathbf{r}}+r \frac{d \mathbf{e}_{\mathbf{r}}}{d \theta} \tag{2}
\end{align*}
$$

When the particle moves from $A$ to $A^{\prime}$ at an elemental distance, the changes in $\mathbf{e}_{\mathbf{r}}$ and $\mathbf{e}_{\theta}$ take place as shown in Fig. 6.51(b).

Now, $\quad d \mathbf{e}_{\mathbf{r}}=1 \times d \theta$ in magnitude since magnitude of $\mathbf{e}_{\mathbf{r}}=1$.
It is directed towards $\mathbf{e}_{\theta}$. Hence

$$
\begin{array}{r}
d \mathbf{e}_{\mathbf{r}}=d \theta \cdot \mathbf{e}_{\theta} \\
\text { Similarly, }  \tag{4}\\
d \mathbf{e}_{\theta}=-d \theta \cdot \mathbf{e}_{\mathbf{r}}
\end{array}
$$

Negative sign since it is in the opposite direction of increasing $\mathbf{e}_{\mathbf{r}}$.
From equations (2) and (3), we get

$$
\begin{aligned}
\mathbf{v} & =\frac{d r}{d t} \mathbf{e}_{\mathbf{r}}+\mathbf{r} \frac{d \theta}{d t} \mathbf{e}_{\theta} \\
& =\dot{r} \mathbf{e}_{\mathbf{r}}+\mathrm{r} \dot{\theta} \mathbf{e}_{\theta} \\
& =\boldsymbol{v}_{\boldsymbol{r}} \mathbf{e}_{\mathbf{r}}+\boldsymbol{v}_{\theta} \mathbf{e}_{\theta}
\end{aligned}
$$

where

$$
\begin{equation*}
v_{r}=\dot{r}=\frac{d r}{d t} \tag{6.33}
\end{equation*}
$$

and

$$
v_{\theta}=r \dot{\theta}=r \frac{d \theta}{d t}
$$

$$
v=\sqrt{v_{r}^{2}+v_{\theta}^{2}}
$$

Now consider the acceleration of the particle.

$$
\begin{aligned}
a & =\frac{d v}{d t} \\
& =\frac{d v}{d t}\left(\dot{r} \mathbf{e}_{\mathbf{r}}+r \dot{\theta} \mathbf{e}_{\theta}\right) \\
& =\frac{d \dot{r}}{d t} \cdot \mathbf{e}_{\mathbf{r}}+\dot{r} \frac{d \mathbf{e}_{\mathbf{r}}}{d t}+\frac{d r}{d t} \dot{\theta} \mathbf{e}_{\theta}+r \frac{d \dot{\theta}}{d t} \mathbf{e}_{\theta}+r \dot{\theta} \frac{d \mathbf{e}_{\theta}}{d t}
\end{aligned}
$$

Substituting for $d e_{r}$ and $d e_{\theta}$ from Eqns. (3) and (4), we get
where

$$
\begin{aligned}
a & =\ddot{r} \mathbf{e}_{\mathbf{r}}+\dot{r} \frac{d \theta}{d t} \mathbf{e}_{\theta}+\dot{r} \dot{\theta} \mathbf{e}_{\theta}+r \ddot{\theta} \mathbf{e}_{\theta}+r \dot{\theta}\left(-\frac{d \theta}{d t} \mathbf{e}_{r}\right) \\
& =\ddot{r} \mathbf{e}_{\mathbf{r}}+\dot{r} \dot{\theta} \mathbf{e}_{\theta}+\dot{r} \dot{\theta} \mathbf{e}_{\theta}+r \ddot{\theta} \mathbf{e}_{\theta}-r \dot{\theta}^{2} \mathbf{e}_{\mathbf{r}} \\
& =\left(\dot{r}-r \dot{\theta}^{2}\right) \mathbf{e}_{\mathbf{r}}+(2 \dot{r} \dot{\theta}+r \ddot{\theta}) \mathbf{e}_{\theta} \\
\mathbf{a} & =a_{r} \mathbf{e}_{\mathbf{r}}+a_{\theta} \mathbf{e}_{\theta}
\end{aligned}
$$

and

$$
\begin{align*}
a_{r} & =\ddot{r}-r \dot{\theta}^{2}  \tag{6.34}\\
a_{\theta} & =2 \dot{r} \dot{\theta}+r \ddot{\theta} \\
a & =\sqrt{a_{r}^{2}+a_{\theta}^{2}}
\end{align*}
$$

Example: 6.39 A radar locates a bomber plane at $60^{\circ}$ to horizontal. At that instant the tracking data gives $r=60 \mathrm{~km}, \dot{r}=1000 \mathrm{~m} / \mathrm{s}$ and $\dot{\theta}=-1.0$ degrees/ second. The acceleration of the bomber is only due to gravitational attraction which is given by $g=9.4 \mathrm{~m} / \mathrm{s}^{2}$. Determine the values of velocity, radial and angular acceleration of the bomber plane.
Solution. Figure 6.52 shows the position of radar and the bomber plane.
$r=60 \mathrm{~km}=60000 \mathrm{~m}$

(a)

(b)

(c)

Fig. 6.52

$$
v=1447.972 \mathrm{~m} / \mathrm{s} \text { as shown in Fig. 6.52(b). Ans. }
$$

From the relation,

$$
a_{r}=\ddot{r}-r \dot{\theta}^{2} \text {, we get }
$$

$$
-8.14=\ddot{r}-60000\left(1.0 \times \frac{\pi}{180}\right)^{2}
$$

$\therefore \quad \ddot{r}=10.136 \mathrm{~m} / \mathrm{s}^{2}$
From the relation,

$$
\begin{aligned}
a_{\theta} & =r \ddot{\theta}+2 \dot{r} \dot{\theta}, \text { we get } \\
-4.7 & =60000 \ddot{\theta}+2 \times 1000 \times\left(-1 \times \frac{\pi}{180}\right) \\
\ddot{\theta} & =5.034 \times 10^{-4} \mathrm{rad} / \mathrm{s}
\end{aligned}
$$

Ans.
Example 6.40 A particle moves in counter-clockwise direction on a circular path of radius 25 m . Its movement is found to be $s=t^{3}+8$, where $s$ is in metres and $t$ in seconds. When $t=2 \mathrm{~s}$,
(i) Using rectangular coordinates determine, $v_{x}, v_{y}, v$ and $a_{x}, a_{y}, a$.
(ii) Using polar coordinates, determine

$$
v_{r}, v_{\theta}, v \text { and } a_{r} a_{\theta} \text { and } a .
$$

(iii) Using tangential and normal components, determine $v_{n}, v_{t}, v$ and $a_{n}, a_{t}$ and $a$.
Solution. Let the position of the particle be at $P$ after $t$ seconds, the corresponding angular displacement being $\theta$ as shown in Fig. 6.53.
Now,

$$
A P=s=t^{3}+8 \quad \text { and } \quad r=25 \mathrm{~m}
$$

We know,

$$
r \theta=s
$$

$$
\begin{aligned}
& v_{r}=\dot{r}=\frac{d r}{d t}=1000 \mathrm{~m} / \mathrm{s} \\
& \theta=-1.0 \mathrm{deg} / \mathrm{s}=-1.0 \times \frac{\pi}{180} \mathrm{rad} / \mathrm{s} \\
& \therefore \quad v_{\theta}=r \theta=-60000 \times 1.0 \times \frac{\pi}{180} \\
& =-1047.198 \mathrm{~m} / \mathrm{s} \\
& \therefore \quad v=\sqrt{v_{r}^{2}+v_{\theta}^{2}}=\sqrt{(1000)^{2}+1047.198^{2}} \\
& \text { Now the total acceleration is } g=9.40 \mathrm{~m} / \mathrm{s}^{2} \text { downward as shown in } \\
& \text { Fig. 6.52(c). Its radial and angular acceleration are } \\
& \text { and } \\
& a_{r}=-9.40 \sin 60^{\circ}=-8.141 \mathrm{~m} / \mathrm{s}^{2} \\
& a_{\theta}=-9.40 \cos 60^{\circ}=-4.7 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$



Fig. 6.53

$$
\begin{array}{ll}
\therefore & \theta=\frac{s}{r}=\frac{1}{25}\left(t^{3}+8\right) \\
\therefore & \frac{d \theta}{d t}=\frac{3 t^{2}}{25} \tag{2}
\end{array}
$$

$$
\begin{equation*}
\frac{d^{2} \theta}{d t^{2}}=\frac{6 t}{25} \tag{3}
\end{equation*}
$$

(i) Using rectangular coordinate $x, y$

From Fig. 6.53,

$$
\begin{aligned}
& x=25 \cos \theta \quad y \\
& \therefore \quad v_{x}=\frac{d x}{d t}=-25 \sin \theta \\
& \therefore \quad \sin \theta \frac{d \theta}{d t}=-25 \sin \theta \frac{3 t^{2}}{25}=-3 t^{2} \sin \theta
\end{aligned}
$$

when $t=2 \mathrm{~s}$ from Eqn. (1),

$$
\begin{aligned}
\theta & =\frac{1}{25}\left(2^{3}+8\right)=0.64 \mathrm{rad} . \\
\therefore \quad v_{x} & =-3 \times 2^{2} \sin (0.64)=-7.166 \mathrm{~m} / \mathrm{s} \\
\text { Similarly, } \quad v_{y} & =\frac{d y}{d t}=25 \cos \theta \cdot \frac{d \theta}{d t}=25 \cos \theta \frac{3 t^{2}}{25} \\
& =3 t^{2} \cos \theta \\
& =3 \times 2^{2} \times \cos (0.64) \\
& =9.625 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

$$
\begin{aligned}
v & =\sqrt{v_{x}^{2}+v_{y}^{2}}=\sqrt{(-7.166)^{2}+9.625^{2}}=12.0 \mathrm{~m} / \mathrm{s} \\
a_{x} & =\frac{d v_{x}}{d t}=\frac{d}{d t}\left(-3 t^{2} \sin \theta\right) \\
& =-6 t \sin \theta-3 t^{2} \cos \theta \frac{d \theta}{d t} \\
& =-6 t \sin \theta-3 t^{2} \cos \theta \frac{3 t^{2}}{25}
\end{aligned}
$$

Since

$$
\frac{d \theta}{d t}=\frac{3 t^{2}}{25} ; \text { Eqn (2). }
$$

when $t=2 \mathrm{~s}$

$$
\begin{aligned}
a_{x} & =-6 \times 2 \sin (0.64)-3 \times 2^{2} \cos (0.64) \times \frac{3 \times 2^{2}}{25} \\
& =-11.786 \mathrm{~m} / \mathrm{s}^{2} \\
a_{y} & =\frac{d v_{y}}{d t}=\frac{d}{d t}\left(3 t^{2} \cos \theta\right) \\
& =6 t \cos \theta+3 t^{2}(-\sin \theta) \frac{d \theta}{d t} \\
& =6 t \cos \theta-3 t^{2} \sin \theta \frac{3 t^{2}}{25}
\end{aligned}
$$

when $t=2 \mathrm{~s}$

$$
\begin{aligned}
a_{y} & =6 \times 2 \cos (0.64)-3 \times 2^{2} \sin (0.64) \times \frac{3 \times 2^{2}}{25} \\
& =6.185 \mathrm{~m} / \mathrm{s}^{2} \\
\therefore \quad a & =\sqrt{a_{x}^{2}+a_{y}^{2}} \\
& =\sqrt{(-11.786)^{2}+(6.185)^{2}} \\
& =13.311 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

(ii) Using polar coordinates

$$
\begin{array}{llr}
r=20 \mathrm{~m} & \theta=\frac{t^{3}+8}{25} \\
\therefore & v_{r}=\frac{d r}{d t}=0 ; & v_{\theta}=\frac{d \theta}{d t}=3 t^{2}
\end{array}
$$

$\therefore$ when $t=2 \mathrm{~s}$

$$
\begin{aligned}
v_{r} & =\frac{d r}{d t}=0 \quad \text { and } \quad v_{0}=3 \times 2^{2}=12 \mathrm{~m} / \mathrm{s} \\
\therefore & =\sqrt{v_{r}^{2}+v_{\theta}^{2}}=12 \mathrm{~m} / \mathrm{s} \\
a_{r} & =\ddot{r}-r \dot{\theta}^{2} \\
& =\frac{d^{2} r}{d t^{2}}-r\left(\frac{d \theta}{d t}\right)^{2} \\
& =0-25\left(\frac{3 t^{2}}{25}\right)^{2}=-\frac{9 t^{4}}{25}
\end{aligned}
$$

Ans.

Ans.
$\therefore$ when $t=2 \mathrm{~s}$

$$
\begin{aligned}
a_{r} & =\frac{-9 \times 2^{4}}{25}=-5.76 \mathrm{~m} / \mathrm{s}^{2} \\
a_{\theta} & =r \ddot{\theta}+2 \dot{r} \dot{\theta} \\
& =r\left(\frac{d^{2} \theta}{d t^{2}}\right)+2 \cdot \frac{d r}{d t} \cdot \frac{d \theta}{d t} \\
& =25\left(\frac{6 t}{25}\right)+2 \times 0 \times \frac{3 t^{2}}{25}
\end{aligned}
$$

Ans.

Ans.

Ans.
(iii) Using normal and tangential directions

$$
\begin{aligned}
& s=t^{3}+8 \\
\therefore & v \\
\therefore & =\frac{d s}{d t}=3 t^{2}
\end{aligned}
$$

$\therefore$ when $t=2 \mathrm{~s}$

$$
\begin{aligned}
& v \\
& =12 \mathrm{~m} / \mathrm{s} \\
& v_{n}
\end{aligned}=0
$$

Ans.

Ans.
and
$\therefore$ when $t=2 \mathrm{~s}$

$$
\begin{aligned}
a_{t} & =12 \mathrm{~m} / \mathrm{s} \\
a_{n} & =\frac{v^{2}}{r}=\frac{12^{2}}{25}=5.76 \mathrm{~m} / \mathrm{s}^{2} \\
\therefore \quad a & =\sqrt{a_{t}^{2}+a_{n}^{2}} \\
& =\sqrt{12^{2}+(5.26)^{2}} \\
a & =13.311 \mathrm{~m} / \mathrm{s}^{2} .
\end{aligned}
$$

Ans.

Ans.

### 6.13 CURVILINEAR MOTION IN SPACE

Consider the right hand Cartesian coordinate system $x, y, z$ as shown in Fig. 6.53. Let $P$ be the position of particle at the instant under consideration. The following three types of coordinate systems are commonly used in the study of the motion of particles in space:
(i) rectangular coordinates $(x, y, z)$
(ii) cylindrical coordinates ( $r, \theta, z$ )
(iii) spherical coordinates ( $R, \theta, \phi$ )

All these coordinate systems are shown in Fig. 6.53.
(i) Rectangular Coordinate System ( $x, y, z$ )

The position vector $R$ is defined by

$$
\begin{array}{ll} 
& \mathbf{R}=x \mathbf{i}+y \mathbf{j}+2 \mathbf{k} \\
\therefore & \mathbf{v}=\dot{\mathbf{R}}=\ddot{x} \mathbf{i}+\ddot{y} \mathbf{j}+z \dot{z}  \tag{6.35}\\
\text { and } & \mathbf{a}=\ddot{\mathbf{R}}=\ddot{x} \mathbf{i}+\ddot{y} \mathbf{j}+\ddot{z} \mathbf{k}
\end{array}
$$

(ii) Cylindrical Coordinate System (r, $\theta, z$ )
$r$ and $\theta$ are in $x-y$ plane and $z$ is normal to $x-y$ plane as shown in Fig. 6.53. In this system it may be observed that,

$$
\begin{array}{ll}
\therefore \quad & \begin{aligned}
\mathbf{R} & =r \mathbf{e}_{\mathbf{r}}+z \mathbf{k} \\
\mathbf{v} & =\dot{\mathbf{R}}=\dot{r} \mathbf{e}_{\mathbf{r}}+r \dot{\theta} \mathbf{e}_{\theta}+\dot{z} \mathbf{k} \\
& =v_{r} \mathbf{e}_{\mathbf{r}}+v_{\theta} \mathbf{e}_{\theta}+v_{z} \mathbf{k} \\
\text { i.e., } \quad v_{r} & =\dot{r}, v_{\theta}=r \dot{\theta}, v_{z}=\dot{z} \\
& \mathbf{a}
\end{aligned}=\dot{\mathbf{v}}=\left(\ddot{r}-r \dot{\theta}^{2}\right) \mathbf{e}_{\mathbf{r}}+(r \ddot{\theta}+2 \dot{r} \dot{\theta}) \mathbf{e}_{\theta}+\ddot{z} \mathbf{k} \\
& =a_{r} \mathbf{e}_{\mathbf{r}}+a_{\theta} \mathbf{e}_{\theta}+\ddot{z} \mathbf{k}
\end{array}
$$

i.e.,
$a_{r}=\ddot{r}-r \dot{\theta}^{2}, a_{\theta}=r \ddot{\theta}+2 \dot{r} \dot{\theta}$ and $a_{z}=\ddot{z}$
(iii) Spherical Coordinates $(R, \theta, \phi)$

In this system, $R$ is the position vector of $P, \theta$ is the angle made by the projection of $O R$ in $x-y$ plane with $x$-axis and $\phi$ is the angle made by $O P^{\prime}$ with $z$-plane. The velocity and the acceleration in such system are

Thus, $\quad v_{r}=\dot{R}, v_{\theta}=R \dot{\theta} \cos \phi, v_{\phi}=R \dot{\phi}$
and

$$
\begin{aligned}
v & =\dot{R} \mathbf{e}_{\mathbf{r}}+R \dot{\theta} \cos \phi \mathbf{e}_{\theta}+R \phi \mathbf{e}_{\phi} \\
& =v_{r} \mathbf{e}_{\mathbf{r}}+v_{\theta} \mathbf{e}_{\theta}+v_{\phi} \mathbf{e}_{\phi}
\end{aligned}
$$

and

$$
\begin{equation*}
\mathbf{a}=a_{R} \mathbf{e}_{\mathbf{R}}+a_{\theta} \mathbf{e}_{\theta}+a_{\phi} \mathbf{e}_{\phi} \tag{6.37}
\end{equation*}
$$

where

$$
\begin{align*}
& a_{R}=\ddot{R}-R \dot{\phi}^{2}-R \dot{\theta}^{2} \cos ^{2} \phi \\
& a_{\theta}=\frac{\cos \phi}{R} \cdot \frac{d}{d t}\left(R^{2} \dot{\theta}\right)-2 R \dot{\theta} \dot{\phi} \sin \phi  \tag{6.38}\\
& a_{\phi}=\frac{1}{R} \cdot \frac{d}{d t}\left(R^{2} \dot{\phi}\right)+R \dot{\theta}^{2} \sin \phi \cos \phi
\end{align*}
$$

As the study of curvilinear motion is treated out of the scope of this book, only the final results are presented and no derivation or solution of problems are attempted here.

### 6.14 RELATIVE MOTION

So far we dealt with the motion of the particles with reference to a fixed point. Now we discuss the motion of the particle with reference to a moving point, in other words, motion of a particle as observed by a moving person. When two trains move in parallel lines in the same direction with the same speed, an observer in one of the train does not feel the motion of other train. If one train moves a little faster, the observer feels a slight motion. If the two trains are moving in opposite direction, the observer feels that speed is very high. These are some common examples of relative motion. The motion of two particles need not be always in parallel paths. They may be in any direction and in any plane.

Figure 6.54 shows two points $A$ and $B$ moving with reference to the fixed frame $X Y Z$ at fixed point $O$. If the motion of $B$ w.r.t. the reference frame $x y z$ at


Fig. 6.54 Translation of $x y z$ Frame w.r.t. Fixed Frame $X Y Z$
$A$ is required, then, we call it as relative motion of $B$ with respect to $A$. This is given by the relation,

$$
\begin{align*}
\mathbf{r}_{B / A} & =\mathbf{r}_{B}-\mathbf{r}_{A} \\
\mathbf{v}_{B / A} & =\mathbf{v}_{B}-\mathbf{v}_{A}  \tag{6.39}\\
\mathbf{a}_{B / A} & =\mathbf{a}_{B}-\mathbf{a}_{A}
\end{align*}
$$

Similarly, the relative motion of $A$ with respect to $B$ is given by

$$
\text { and } \quad \begin{align*}
\mathbf{r}_{A / B} & =\mathbf{r}_{A}-\mathbf{r}_{B} \\
\mathbf{v}_{A / B} & =\mathbf{v}_{A}-\mathbf{v}_{B} \\
\mathbf{a}_{A / B} & =\mathbf{a}_{A}-\mathbf{a}_{B} \tag{6.40}
\end{align*}
$$

The moving reference frame $x y z$ may have only translation with respect the fixed frame $X Y Z$ or it can have rotation also. In this book discussion is limited to only the translation of the moving frame.

Example 6.41 The motion of two particles as referred to a fixed frame $X, Y$ is given below:

$$
\begin{aligned}
& \mathbf{r}_{A}=t^{3} \mathbf{i}+2 t \mathbf{j} \\
& \mathbf{r}_{B}=t^{2} \mathbf{i}+3 t \mathbf{j}
\end{aligned}
$$

where $r$ is in metres and $t$ in seconds. Determine the relative position, velocity and acceleration of $B$ w.r.t. $A$ after 2 seconds.

## Solution.

$$
\begin{aligned}
\mathbf{r}_{B / A} & =\mathbf{r}_{B}-\mathbf{r}_{A} \\
& =t^{2} \mathbf{i}+3 t \mathbf{j}-\left(t^{3} \mathbf{i}+\mathbf{2} t \mathbf{j}\right) \\
& =\left(t^{2}-t^{3}\right) \mathbf{i}+(3 t-2 t) \mathbf{j} \\
\therefore \quad \mathbf{v}_{B / A} & =\frac{d}{d t}\left(r_{B / A}\right)=\left(2 t-3 t^{2}\right) \mathbf{i}+(3-2) \mathbf{j} \\
\mathbf{a}_{B / A} & =\frac{d}{d t}\left(V_{B / A}\right)=(2-6 t) \mathbf{i}+0 \mathbf{j}
\end{aligned}
$$

$\therefore$ After 2 seconds

$$
\begin{aligned}
\mathbf{r}_{B / A} & =\left(2^{2}-2^{3}\right) \mathbf{i}+(3 \times 2-2 \times 2) \mathbf{j}=-4 \mathbf{i}+2 \mathbf{j} \\
\mathbf{r}_{B / A} & =\sqrt{4^{2}+2^{2}}=4.472 \mathrm{~m} \\
\mathbf{v}_{B / A} & =\left(2 \times 2-3 \times 2^{2}\right) \mathbf{i}+\mathbf{j} \\
& =-8 \mathbf{i}+\mathbf{j} \\
\therefore \quad v_{B / A} & =\sqrt{8^{2}+1^{2}}=8.062 \mathrm{~m} / \mathrm{s} \\
& \mathbf{a}_{B / A}
\end{aligned}=(2-6 \times 2) \mathbf{i}+0 \mathbf{j},
$$

Example 6.42 A passenger train 250 m long, moving with a velocity of 72 kmph , overtakes a goods train, moving on a parallel path in the same direction, completely in 45 seconds. If the length of the goods train is 200 m , determine the speed of the goods train.


Fig. 6.55

## Solution.

Let
Velocity of passenger train
$V_{B}=$ velocity of goods train.
$V_{A}=72 \mathrm{kmph}$

$$
=72 \times \frac{1000}{60 \times 60}=20 \mathrm{~m} / \mathrm{s}
$$

Relative velocity of $B$ with respect to $A$,

$$
V_{A / B}=V_{r}=V_{A}-V_{B}=20-V_{B}
$$

Relative distance moved to overtake the goods train

$$
=250+200=450 \mathrm{~m} .
$$

Now, 45 seconds are required to cover this relative distance

$$
\begin{aligned}
\left(20-V_{B}\right) 45 & =450 \\
V_{B} & =10 \mathrm{~m} / \mathrm{s} \\
\text { i.e., } \quad V_{B} & =\frac{10 \times 60 \times 60}{1000}=36 \mathrm{kmph} .
\end{aligned}
$$

Ans.

Example 6.43 A passenger train is 240 m long and is moving with a constant velocity of 72 kmph . At a particular time, its engine approaches last compartment of a goods train moving on a parallel track in the same direction. 25 seconds later its engine starts overtaking the engine of goods train. It took 30 seconds more to completely overtake the goods train. Determine the length and speed of the goods train.
Solution. Velocity of passenger train

$$
\begin{aligned}
V_{A} & =72 \mathrm{kmph} \\
& =20 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Let velocity of goods train be $V_{B} \mathrm{~m} / \mathrm{s}$ and its length $x$ metres.
Relative velocity $=20-V_{B} \mathrm{~m} / \mathrm{s}$
When $t=25 \mathrm{~s}$, relative distance moved is $x$ metres.

$$
\begin{equation*}
\therefore \quad\left(20-V_{B}\right) 25=x \tag{1}
\end{equation*}
$$

In the next, $t=30$ seconds,
Relative distance moved = Length of passenger train

$$
\begin{equation*}
=240 \mathrm{~m} \tag{2}
\end{equation*}
$$

$\therefore \quad\left(20-V_{B}\right) 30=240$
$\therefore \quad V_{B}=12 \mathrm{~m} / \mathrm{s}$

$$
=\frac{12 \times 60 \times 60}{1000}=43.2 \mathrm{~km} / \mathrm{h}
$$

Ans.
Substituting the value of $V_{B}$ in (1)

$$
\begin{array}{rlrl} 
& & (20-12) 25 & =x \\
\therefore & x & =200 \mathrm{~m} .
\end{array}
$$

Ans.
Example 6.44 Two trains $A$ and $B$ are moving on parallel tracks in opposite direction. Velocity of $A$ is twice that of $B$. They take 18 seconds to pass each other. Determine their velocities, given, the length of train $A$ is 240 m and that of $B$ is 300 m .
Solution. Taking the direction of motion of train $A$ as positive.
Let velocity of A be $v V \mathrm{~m} / \mathrm{s}$
$\therefore \quad$ Velocity of $B=-V / 2 \mathrm{~m} / \mathrm{s}$

$$
\text { Relative velocity }=V-\left(-\frac{V}{2}\right)=1.5 \mathrm{~V}
$$

Relative distance moved in 18 seconds

$$
\begin{array}{rlrl} 
& =240+300=540 \mathrm{~m} \\
\therefore & & 1.5 \mathrm{~V} \times 18 & =540 \\
& & \mathrm{~V} & =20 \mathrm{~m} / \mathrm{s}=72 \mathrm{kmph} \\
& \therefore & \text { Velocity of } B & =-10 \mathrm{~m} / \mathrm{s}=-36 \mathrm{kmph}, \\
\text { i.e., } & & =36 \mathrm{kmph} .
\end{array}
$$

Ans.

Example 6.45 Two ships move from a port at the same time. Ship $A$ has velocity of 30 kmph and is moving in $\mathrm{N} 30^{\circ} \mathrm{W}$ while ship $B$ is moving in south-west direction with a velocity of 40 kmph . Determine the relative velocity of $A$ with respect to $B$ and the distance between them after half an hour.
Solution. Taking west direction as $x$-axis and north direction as $y$-axis,


$$
\begin{aligned}
& \qquad \tan \theta=\frac{13.284}{54.264} \\
& \therefore \quad \theta=13.76^{\circ} \\
& \therefore \quad \text { i.e., from B, ship A appears to move with a } \\
& \text { velocity of } 55.866 \mathrm{kmph} \text { in } N 13.76^{\circ} \text { E direction. }
\end{aligned}
$$

> Ans.
$\therefore$ Relative distance after half an hour

$$
\begin{aligned}
& =\text { Relative velocity } \times \text { time } \\
& =55.866 \times 1 / 2=27.933 \mathrm{~km}
\end{aligned}
$$



Fig. 6.57

Ans.
Example 6.46 A enemy ship was located at a distance of 25 km in north-west direction by a warship. If the enemy ship is moving with a velocity of 18 kmph $\mathrm{N} 30^{\circ} \mathrm{E}$, in which direction the warship must move with a velocity of 36 kmph , to strike at its earliest? Assume the fire range of warship is 5 km . When is the shell to be fired?
Solution. Let the enemy ship $B$ be at $M$ and the warship ( $A$ ) be at N as shown in Fig. 6.58. The required direction of movement of warship be $\theta$ to north direction. Taking north as $y$ direction and west as $x$ direction, it's components of velocity are

$$
V_{A y}=36 \cos \theta \text { and } V_{A x}=36 \sin \theta
$$

Components of velocity of enemy ship are

$$
\begin{aligned}
V_{B y} & =18 \cos 30^{\circ} \text { and } V_{B x}=-18 \sin 30^{\circ} \\
& =-15.588 \mathrm{kmph}=-9 \mathrm{kmph}
\end{aligned}
$$

The relative velocity of $A$ with respect to $B$ is represented by

$$
\begin{aligned}
& V_{r x}=36 \sin \theta-(-9)=36 \sin \theta+9 \\
& V_{r y}=36 \cos \theta-15.588
\end{aligned}
$$



Fig. 6.58
$\therefore$ The direction of relative velocity $\alpha$ with north is given by

$$
\tan \alpha=\frac{V_{r x}}{V_{r y}}=\frac{36 \sin \theta+9}{36 \cos \theta-15.588}
$$

To approach the enemy ship at earliest, the direction of relative velocity $\alpha$ should be $M N$ i.e., $45^{\circ}$ to north.

$$
\therefore \quad \tan 45^{\circ}=\frac{36 \sin \theta+9}{36 \cos \theta-15.588}
$$

or $36 \cos \theta-15.588=36 \sin \theta+9$
(Since $\tan 45^{\circ}=1$ )

$$
\begin{equation*}
\cos \theta-\sin \theta=0.683 \tag{1}
\end{equation*}
$$

Let $x=\sin \theta$, then $\cos \theta=\sqrt{1-x^{2}}$
$\therefore$ From (1)

$$
\begin{array}{rlrl} 
& \sqrt{1-x^{2}}-x & =0.683 \\
& & \sqrt{1-x^{2}} & =0.683+x \\
& 1-x^{2} & =(0.683+x)^{2} \\
\text { i.e., } & 2 x^{2}+1.366 x-0.5335 & =0 \\
\text { i.e., } & x & =0.2777 \\
\therefore & \theta & =16.12^{\circ}
\end{array}
$$

Warship must move in $N$ 16.12 W direction.
Ans.
$\therefore \quad V_{r x}=36 \sin \theta+9=19.0$

$$
V_{r y}=36 \cos \theta-15.588=19.0 \mathrm{kmph}
$$

$\therefore$ Relative velocity $V_{A / B}=V_{r}=\sqrt{19^{2}+19^{2}}$

$$
=26.870 \mathrm{kmph}
$$

Relative distance to be moved before firing

$$
\begin{aligned}
& =\text { Distance between } M N-\text { Range of gun } \\
& =25-5=20 \mathrm{~km}
\end{aligned}
$$

$\therefore$ Time interval $t$ is given by

$$
\begin{aligned}
V_{r} t & =20 \\
t & =\frac{20}{26.870}=0.744 \text { hour }
\end{aligned}
$$

i.e., 4 min and 40 sec after sighting the enemy ship the shell is to be fired. Ans.

Example 6.47 A motor boat has to cross a river 100 m wide and flow of 5 kmph . Assumimg the boat moves with uniform velocity of 15 kmph throughout, find the direction in which it should move to reach the other bank in the minimum time. Where and when it will reach the other bank.

If the boat has to touch exactly the opposite bank, in which direction should it be set and how much time is required for it to reach opposite bank?
Solution. Let the motor boat start from point $A$ as shown in Fig. 6.59. The boat moves in the direction of the resultant velocity $V$ and reaches point $C$ on the other bank. This resultant velocity


Fig. 6.59 is due to the velocity of boat and velocity of the flow of river.

To reach opposite bank in minimum time, the component of resultant velocity must have the maximum value in $x$-direction. Only driving force of the boat gives component in $x$-direction. Hence boat should be set in straight across the river.

$$
V_{r x}=15 \mathrm{kmph}
$$

Distance to be moved in $x$ direction $=1 \mathrm{~km}$
$\therefore \quad$ The time interval $t$ required is

$$
\begin{aligned}
15 t & =1 \quad \text { or } \quad t=\frac{1}{15} \text { hour } \\
& =4 \mathrm{~min} .
\end{aligned}
$$

Ans.
During this period the boat will move down the stream due to the component of velocity $V_{y}$, where

$$
V_{y}=5 \mathrm{kmph}
$$

$\therefore \quad$ Distance moved in downstream direction

$$
\begin{aligned}
& =5 \times \frac{1}{15}=\frac{1}{3} \mathrm{~km} \\
& =333.33 \mathrm{~m}
\end{aligned}
$$

Ans.
If the boat has to touch the bank straight across, the resultant velocity of the boat should be directed in $x$ direction, i.e., resultant velocity of the boat with respect to stream in $y$ direction $=0$. Let the direction of boat be set at $\theta$ to $x$ direction as shown in Fig. 6.60.
$\therefore \quad 15 \sin \theta=5$
or

$$
\sin \theta=\frac{1}{3}
$$

$$
\theta=19.47^{\circ}
$$

Ans.
Example 6.48 Ship $A$ is approaching a port in due East direction with a velocity of 15 kmph . When this ship was 50 km from port, ship $B$ sails in $\mathrm{N} 45^{\circ} \mathrm{W}$ direction with a velocity of 25 kmph from the port. After what time the two ships are at minimum distance and how for each has travelled.
Solution. Let west be $x$ and north be $y$-axis
and

$$
\begin{aligned}
& V_{B x}=25 \sin 45^{\circ}=17.678 \mathrm{kmph} \\
& V_{B y}=25 \cos 45^{\circ}=17.678 \mathrm{kmph} \\
& V_{A x}=-15 \mathrm{kmph} \\
& V_{A y}=0.0
\end{aligned}
$$

Let $V_{r}$ be the relative velocity of $B$ with respect to $A\left(V_{B / A}\right)$.


Fig. 6.61

$$
\begin{array}{ll}
\therefore \quad V_{r x} & =17.678-(-15)=32.678 \mathrm{kmph} \\
V_{r y} & =17.678 \mathrm{kmph} \\
\therefore \quad V_{r} & =\sqrt{32.678^{2}+17.678^{2}} \\
& =37.153 \mathrm{kmph}
\end{array}
$$



Fig. 6.60
or $\quad \theta=19.47$

$$
\begin{aligned}
& \tan \alpha & =\frac{V_{r y}}{V_{r x}}=\frac{17.678}{32.678} \\
\therefore \quad & & =28.41^{\circ} .
\end{aligned}
$$

Holding $A$ as stationary at $M$ and allowing $B$ to move with relative velocity $V_{B / A}=V_{r}$ from N , the two ships will be at minimum distance at point $P$ as shown in Fig. 6.62.
$\therefore$

$$
\begin{aligned}
V_{r} t & =50 \cos \alpha \\
37.153 t & =50 \cos 28.41^{\circ} \\
t & =1.1837 \text { hours }
\end{aligned}
$$

Ans.


Fig. 6.62
During this period $A$ has moved in due East direction a distance of

$$
15 \times 1.1837=17.756 \mathrm{~km}
$$

and $B$ has moved in $\mathrm{N} 45^{\circ} \mathrm{W}$ a distance of $25 \times 1.1837=29.593 \mathrm{~km}$
Example 6.49 Ship $B$ is approaching a port at 18 kmph speed moving in East direction. When it was 60 km away, ship $A$ left the port with a speed of 24 kmph in $\mathrm{S} 60^{\circ} \mathrm{W}$. If the two ships can exchange the signal when they are within a range of 25 km , find when and how long they can exchange the signal.
Solution. Considering west as $x$-axis and south as $y$-axis;

$$
\begin{aligned}
V_{A x} & =24 \cos 30^{\circ}=20.785 \\
V_{A y} & =24 \sin 30^{\circ}=12 \\
\mathrm{~V}_{B x} & =-18 \\
\mathrm{~V}_{B y} & =0
\end{aligned}
$$

Let relative velocity of $A$ with respect to $B\left(V_{A / B}\right)$ be $V_{r}$ at an angle $\alpha$ to west direction.

$$
\begin{aligned}
V_{r x} & =V_{A x}-V_{B x} \\
& =38.785 \mathrm{kmph} \\
V_{r y} & =V_{A y}-V_{B y}=12 \\
V & =\sqrt{V_{r x}^{2}+V_{r y}^{2}}=\sqrt{38.785^{2}+12^{2}} \\
& =40.599 \mathrm{kmph}
\end{aligned}
$$

$$
\begin{aligned}
\tan \alpha & =\frac{V_{r y}}{V_{r x}}=\frac{12}{38.785} \\
\therefore \quad \alpha & =17.192 .
\end{aligned}
$$

Holding $B$ stationary and allowing $A$ to move with relative velocity, the minimum distance between the two ships $B C$ (Fig. 6.64) is given by

$$
B C=60 \sin \alpha=17.735 \mathrm{~km}
$$



Fig. 6.64
The two ships can exchange the signals from point $D$ to $E$ when

$$
B D=B E=25 \mathrm{~km}
$$

From the triangle $B C D$,

$$
D C=\sqrt{25^{2}-17.735^{2}}=17.62 \mathrm{~km}
$$

Since $B C$ is the foot of isoceles triangle $B E D$,

$$
C E=D C=17.62 \mathrm{~km}
$$

From triangle $A B C$,

$$
\begin{aligned}
A C & =60 \cos \alpha \\
& =57.319 \mathrm{~km} \\
\therefore \quad & A D \\
\text { and } \quad & A C-D C=57.319-17.62 \\
& =39.699 \mathrm{~km} \\
& A E
\end{aligned}
$$

Time taken to reach $D$,

$$
\begin{aligned}
t_{1} & =\frac{A D}{V_{r}}=\frac{39.699}{40.899} \\
& =0.9778 \mathrm{hr} \\
& =58 \mathrm{~min} 40 \mathrm{~s}
\end{aligned}
$$

Time taken to reach $E$,

$$
\begin{aligned}
t_{2} & =\frac{A E}{V_{r}}=\frac{74.939}{40.599} \\
& =1.84548 \mathrm{hr} \\
& =1 \mathrm{hr} .50 \mathrm{~min} .45 \mathrm{~s}
\end{aligned}
$$

Hence the two ships can start exchanging signals, 58 min 40 s after ship $A$ leaves the port and continue to do so for 52 min and 5 s .

Example 6.50 When a train is moving with a velocity of 36 kmph , a passenger observes rain drops at $30^{\circ}$ to vertical. When the velocity of train increases to 54 kmph , he observes rain at $45^{\circ}$ to vertical. Determine the true velocity and direction of rain drops. Assume rain drops are in the parallel vertical plane to that of train's vertical plane.
Solution. Let the true velocity of rain be $V$ kmph at a true angle $\theta$ with vertical as shown in Fig. 6.65.

Taking the direction of train as $x$ and that of vertical downward as $y$. Velocity components of rain are:

$$
\begin{aligned}
V_{1 x} & =V \sin \theta \\
V_{1} y & =V \cos \theta
\end{aligned}
$$



Fig. 6.65

When the velocity of train was 36 kmph ,

$$
V_{2 x}=36 \text { and } V_{2 y}=0
$$

$\therefore \quad$ The relative velocity components of rain with respect to train are:

$$
\begin{aligned}
V_{r x} & =\mathrm{V} \sin \theta-36 \\
V_{r y} & =V \cos \theta \\
\therefore \quad \tan \alpha & =\frac{V \sin \theta-36}{V \cos \theta}
\end{aligned}
$$

Now $\alpha$ the direction of relative velocity is given as $30^{\circ}$.

$$
\begin{equation*}
\therefore \quad \tan 30^{\circ}=\frac{V \sin \theta-36}{V \cos \theta} \tag{1}
\end{equation*}
$$

when the velocity of train is $54 \mathrm{kmph}, \alpha=45^{\circ}$

$$
\therefore \quad \begin{align*}
\tan 45^{\circ} & =\frac{V \sin \theta-54}{V \cos \theta} \\
1 & =\frac{V \sin \theta-54}{V \cos \theta} \\
V \cos \theta & =V \sin \theta-54 \tag{2}
\end{align*}
$$

Substituting this value in (1), we get

$$
\begin{align*}
0.577 & =\frac{V \sin \theta-36}{V \sin \theta-54} \\
\therefore \quad V \sin \theta & =-11.402 \tag{3}
\end{align*}
$$

Substituting it in (2), we get

$$
\begin{equation*}
V \cos \theta=-65.402 \tag{4}
\end{equation*}
$$

Squaring and adding (3) and (4), we get

$$
\begin{aligned}
V^{2} & =4407.43 \\
V & =66.388 \mathrm{kmph} \\
\sin \theta & =-\frac{11.402}{66.388}=-0.1717 \\
\therefore \quad \theta & =-9.89^{\circ} .
\end{aligned}
$$

Ans.

Ans.
Example 6.51 A jet of water is discharged from a tank at $A$ with a velocity of $20 \mathrm{~m} / \mathrm{s}$ to strike a moving plate at $B$ as shown in Fig. 6.66. If the plate is moving in vertically downward direction with a velocity of $1 \mathrm{~m} / \mathrm{s}$, determine the relative velocity of water striking the plate.


Fig. 6.66
Solution. The water is moving with initial velocity of $20 \mathrm{~m} / \mathrm{s}$ horizontally and is subjected to gravitational force, which causes vertical motion also. Horizontal component of the velocity is not altered if we neglect air resistance.
$\therefore$ Time taken to move a horizontal distance of $5 \mathrm{~m}, t=\frac{5}{20}=\frac{1}{4}$ second.
During this period vertical downward velocity gained by the water

$$
=0+9.81 \times \frac{1}{4}=2.453 \mathrm{~m} / \mathrm{s}
$$

Velocity of plate:
Horizontal component $=0$
Vertical downward component $=1 \mathrm{~m} / \mathrm{s}$
$\therefore$ Relative velocity of water with respect to plate has the components

$$
\begin{aligned}
V_{r y} & =2.453-1.0=1.453 \mathrm{~m} / \mathrm{s} \\
V_{r x} & =20 \mathrm{~m} / \mathrm{s} \\
V_{r} & =\sqrt{20^{2}+1.4535^{2}}=20.052 \mathrm{~m} / \mathrm{s} . \text { Ans. } \\
\tan \alpha & =1.453 / 20 \\
\alpha & =4.16^{\circ} \text { as shown in Fig. } 6.67
\end{aligned}
$$



Fig. 6.67

Ans.
Example 6.52 A railway carriage 4.8 m long and 1.8 m wide is moving at 96 kmph , when a pistol shot hits the carriage in a direction making an angle of $10^{\circ}$ to
the direction of the track. The shot enters the carriage in one corner and passes out diagonally at the opposite corner. Find the speed of the shot and the time taken for the shot to traverse the carriage.
Solution. Let $V$ be the velocity of shot, $x$ be the direction in which train is moving and $y$ be the direction at right angles to it as shown in Fig. 6.68.


Fig. 6.68
Velocity components of train are:

$$
\begin{aligned}
& V_{1 x}=96 \mathrm{kmph} \\
& V_{1 y}=0
\end{aligned}
$$

Velocity components of bullet are:

$$
\begin{aligned}
& V_{2 x}=V \cos 10^{\circ}=0.948 V \\
& V_{2 y}=V \sin 10^{\circ}=0.1736 V
\end{aligned}
$$

Components of relative velocity of $B$ with respect to $A$ are:

$$
V_{r x}=0.9848 V-96
$$

and

$$
V_{r y}=0.1736 \mathrm{~V} .
$$

Direction of relative velocity $\alpha$ with $x$-axis is given by

$$
\tan \alpha=\frac{V_{r y}}{V_{r x}}=\frac{0.1736 \mathrm{~V}}{0.9848 \mathrm{~V}-96}
$$

It is given that relative velocity of bullet with respect to train makes an angle $\alpha$ such that

$$
\begin{aligned}
& \tan \alpha & =\frac{1.8}{4.8}=-0.375 \\
\therefore & 0.375 & =\frac{0.1736 V}{0.9848 V-96}
\end{aligned}
$$

$$
V=183.96 \mathrm{kmph}=51.1 \mathrm{~m} / \mathrm{s}
$$

Ans.
Considering the motion in $y$ direction,

$$
V_{r y}=0.1736 \times 51.1 \mathrm{~m} / \mathrm{s}
$$

Relative distance travelled $=1.8 \mathrm{~m}$
$\therefore \quad$ Time required $=t=\frac{1.8}{0.1736 \times 51.1}=0.203 \mathrm{~s}$.
Ans.

## IMPORTANT DEFINITIONS AND CONCEPTS

1. Dynamics is the branch of mechanics that deals with the bodies in motion.
2. Kinematics is the branch of dynamics that deals with the motion of bodies without referring the forces causing the motion.
3. Kinetics is the branch of dynamics that deals with, the motion of the bodies referring to the forces causing the motion.
4. In a general motion also known as motion in space we need three coordinates to describe the motion. If the particle is in a plane motion only two coordinates are enough to describe the motion. The motion along a straight line is termed as the rectilinear motion.
5. Displacement of the body in a time interval may be defined as the linear distance between the two positions of the body in the beginning and at the end of the time interval.
6. The rate of change of displacement with respect to time is termed as velocity.
7. The rate of change of velocity with respect to time is termed as acceleration.
8. Motion of a body with reference to a moving reference point is called relative motion.

## IMPORTANT FORMULAE

1. 

$$
\Delta \mathbf{r}=\Delta x \mathbf{i}+\Delta y \mathbf{j}+\Delta z \mathbf{k}
$$

$$
\mathbf{v}=\frac{d r}{d t}=v_{x} \mathbf{i}+v_{y} \mathbf{j}+v_{z} \mathbf{k}
$$

where

$$
v_{x}=\frac{d x}{d t}, v_{y}=\frac{d y}{d t} \text { and } v_{z}=\frac{d z}{d t}
$$

$$
\mathbf{a}=\frac{d \mathbf{v}}{d t}=a_{x} \mathbf{i}+a_{y} \mathbf{j}+a_{z} \mathbf{k}
$$

where

$$
a_{x}=\frac{d v_{x}}{d t}, a_{y}=\frac{d v_{y}}{d t} \text { and } a_{z}=\frac{d v_{z}}{d t}
$$

In rectilinear motion
$v=\frac{d s}{d t}$ velocity is slope of $s-t$ curve
$a=\frac{d v}{d t}$ slope of $v-t$ curve represents acceleration. Area under the curve represents displacement. Area under $a-t$ curve represents change in velocity.

If in rectilinear motion, velocity is uniform,

$$
s=v t
$$

If motion is with uniform acceleration,

$$
\begin{aligned}
v & =u+a t \\
s & =u t+\frac{1}{2} a t^{2} \\
v^{2}-u^{2} & =2 a s
\end{aligned}
$$

In plane curvilinear motion

$$
\rho=\frac{\left(1+\left(\frac{d y}{d x}\right)^{2}\right)^{3 / 2}}{\frac{d^{2} y}{d x^{2}}}
$$

Rectangular coordinates

$$
\begin{aligned}
& \mathbf{r}=x \mathbf{i}+y \mathbf{j} \\
& \mathbf{v}=v_{x} \mathbf{i}+v_{y} \mathbf{j} \\
& \mathbf{a}=a_{x} \mathbf{i}+a_{y} \mathbf{j}
\end{aligned}
$$

In case of projectile

$$
\begin{aligned}
v_{x} & =u \cos \alpha, v_{y}=u \sin \alpha, a_{y}=-g \\
y & =v_{0} \sin \alpha t-\frac{1}{2} g t^{2} \\
& =x \tan \alpha-\frac{1}{2} \frac{g x^{2}}{v_{0}^{2} \cos ^{2} \alpha} \\
& =x \tan \alpha-\frac{1}{2} \frac{x^{2}}{v_{0}^{2}}\left(1+\tan ^{2} \alpha\right)
\end{aligned}
$$

Time required to reach maximum height $t=\frac{v_{0} \sin \alpha}{g}$

$$
\text { Time of flight }=\frac{x}{v_{0} \cos \alpha}
$$

2. Tangential and normal coordinates

$$
\begin{aligned}
& v_{t}=v=\rho \frac{d \theta}{d t}, \quad a_{t}=\frac{d v_{t}}{d t}=\frac{d v}{d t}, \quad a_{n}=\frac{v^{2}}{\rho} \\
& \mathbf{a}=a_{n} \mathbf{e}_{\mathbf{n}}+a_{t} \mathbf{e}_{\mathbf{t}}
\end{aligned}
$$

3. In polar coordinates
where

$$
\mathbf{v}=v_{r} \mathbf{e}_{\mathbf{r}}+v_{\theta} \mathbf{e}_{\theta}
$$

$$
v_{r}=\frac{d r}{d t} \text { and } \quad v_{\theta}=r \dot{\theta}
$$

$$
\mathbf{a}=a_{r} \mathbf{e}_{\mathbf{r}}+a_{\theta} \mathbf{e}_{\theta}
$$

where

$$
\begin{aligned}
& a_{r}=\ddot{r}-r \dot{\theta}^{2} \\
& a_{\theta}=2 \dot{r} \dot{\theta}+r \ddot{\theta}
\end{aligned}
$$

Relative motion of $B$ w.r.t. $A$ is given by

$$
\begin{aligned}
\mathbf{r}_{B / A} & =\mathbf{r}_{B}-\mathbf{r}_{A} \\
\mathbf{V}_{B / A} & =\mathbf{V}_{B}-\mathbf{V}_{A} \\
\mathbf{a}_{B / A} & =\mathbf{a}_{B}-\mathbf{a}_{A}
\end{aligned}
$$

## PROBLEMS FOR EXERCISES

6.1 A ball is thrown vertically upwards with an initial velocity of $36 \mathrm{~m} / \mathrm{s}$. After 2 seconds, another ball is thrown vertically upwards. What should be its initial velocity so that it crosses the first ball at a height of 30 m ?
[Ans. $28.34 \mathrm{~m} / \mathrm{s}$ ]
6.2 A stone is dropped from the top of a tower. During the last second of its flight, it is found to fall $1 / 4$ th of the whole height of tower. Find the height of the tower. What is the velocity with which the stone hits the ground?

$$
\text { [Ans. } h=273.27 \mathrm{~m} ; v=73.22 \mathrm{~m} / \mathrm{s} \text {.] }
$$

6.3 A stone is dropped into a well without initial velocity. Its splash is heard after 3.5 seconds. Another stone is dropped with some initial velocity and its splash is heard after 3 seconds. Determine the initial velocity of the second stone if velocity of sound is $335 \mathrm{~m} / \mathrm{s}$.
[Ans. $u=5.34 \mathrm{~m} / \mathrm{s}$ ]
6.4 A ship, while being launched, slips down the skid with uniform acceleration. If 10 seconds are required to traverse the first 5 metres, what time will be required to slide the total distance of 120 m ? With what velocity the ship strikes the water?
[Ans. $48.99 \mathrm{~s} ; 4.899 \mathrm{~m} / \mathrm{s}$ ]
6.5 A train covers a distance of 1.6 km between two stations $A$ and $B$ in 2 minutes, starting from rest. In the first minute of its motion, it accelerates uniformly and in the last 30 seconds it retards uniformly and comes to rest. It moves with uniform velocity during the rest of the period. Find:
(a) Its acceleration in the first minute;
(b) Its retardation in the last 30 seconds, and
(c) Constant velocity reached by the train.
[Ans. (a) $0.3555 \mathrm{~m} / \mathrm{s}^{2}$; (b) $0.711 \mathrm{~m} / \mathrm{s}^{2}$; (c) $21.3333 \mathrm{~m} / \mathrm{s}$ ]
6.6 The elevator in an office building starts from ground floor with an acceleration of $0.6 \mathrm{~m} / \mathrm{s}^{2}$ for 4 seconds. During the next 8 seconds, it travels with uniform velocity. Then suddenly power fails and elevator stops after 3 seconds. If floors are 3.5 m apart, find the floor near which the elevator stops. Assume the retardation is uniform.
[Ans. near eighth floor; $h=27.6 \mathrm{~m}$ ]
6.7 A train, starting from rest, is uniformly accelerated during the first 250 m of its run and runs next 750 m at uniform speed. It is then brought to rest in 50 seconds under uniform retardation. If the time taken for the entire journey is 5 minutes, find the acceleration with which the train started.
[Ans. $0.05 \mathrm{~m} / \mathrm{s}^{2}$ ]
6.8 Three marks $A, B, C$ spaced at a distance of 100 m are made along a straight road. A car starting from rest and accelerating uniformly passes the mark $A$ and takes 10 seconds to reach the mark $B$ and further 8 seconds to reach mark C. Calculate:
(a) the magnitude of the acceleration of the car;
(b) the velocity of the car at $A$;
(c) the velocity of the car at $B$; and
(d) the distance of the mark $A$ from the starting point.
[Ans. (a) $0.2778 \mathrm{~m} / \mathrm{s}^{2}$; (b) $8.61 \mathrm{~m} / \mathrm{s}$;
(c) $10.833 \mathrm{~m} / \mathrm{s}$; and (d) 133.47 m$]$
6.9 Two automobiles travelling in the same direction in adjacent lanes are stopped at a highway traffic signal. As the signal turns green, automobile $A$ accelerates at a constant rate of $1.0 \mathrm{~m} / \mathrm{s}^{2}$. Two seconds later automobile $B$ starts and accelerates at a constant rate of $1.3 \mathrm{~m} / \mathrm{s}^{2}$. Determine,
(a) when and where $B$ will overtake $A$, and
(b) the speed of each automobile at that time.
[Ans. (a) $t=16.268 \mathrm{~s} ; s=132.321 \mathrm{~m}$
(b) $\left.v_{1}=16.268 \mathrm{~m} / \mathrm{s} ; v_{2}=18.548 \mathrm{~m} / \mathrm{s}\right]$
6.10 A particle moves along a straight line. Its motion is represented by the equation

$$
s=16 t+4 t^{2}-3 t^{3}
$$

where, $s$ is in metres and $t$, in seconds. Determine,
(a) displacement, velocity and acceleration 2 seconds after start;
(b) displacement and acceleration when velocity is zero; and
(c) displacement and velocity when acceleration is zero
[Ans. (a) $24 \mathrm{~m},-4 \mathrm{~m} / \mathrm{s},-28 \mathrm{~m} / \mathrm{s}^{2}$;
(b) $24.3 \mathrm{~m},-25.3 \mathrm{~m} / \mathrm{s}^{2}$; (c) $7.64 \mathrm{~m}, 17.78 \mathrm{~m} / \mathrm{s}$ ]
6.11 The acceleration of a body starting from rest is given by the equation:

$$
a=12-0.1 \mathrm{~s}
$$

where $a$, is the acceleration in $\mathrm{m} / \mathrm{s}^{2}$ and $s$ is the displacement in metres. Determine the velocity of the body when a distance of 100 m is covered and the distance at which velocity will be zero again.

$$
\text { [Ans. } s=100, v=37.42 ; v=0, s=240]
$$

6.12 A particle moving in a straight line with an initial velocity $u \mathrm{~m} / \mathrm{s}$ is subjected at any instant to a retardation of $\lambda v^{3}$ where $\lambda$ is a constant and $v$ is the velocity at that instant in $\mathrm{m} / \mathrm{s}$. Calculate the velocity when the distance travelled is $s$ metres and the time for doing so.

$$
\left[\text { Ans. } v=1 \frac{u}{\lambda u s+1} ; t=s\left(\frac{\lambda s}{2}+\frac{1}{u_{1}}\right)\right]
$$

6.13 The motion of a particle in a plane is defined by $x=t^{3}+2 t^{2}$ and $y=t^{3}-2 t$, the units being metres and seconds. Determine the velocity and acceleration after 2 seconds.
[Ans. $v_{x}=20 \mathrm{~m} / \mathrm{s}, v_{y}=10 \mathrm{~m} / \mathrm{s}, v=22.36 \mathrm{~m} / \mathrm{s}$
$\left.a_{x}=16 \mathrm{~m} / \mathrm{s}, a_{y}=12 \mathrm{~m} / \mathrm{s}, a=20 \mathrm{~m} / \mathrm{s}\right]$
6.14 A particle is found to possess the velocities as $v_{x}=t^{2}+2$ and $v_{y}=t^{2}-2 t$. After one second the particle passed through the point $P(4,6)$. Determine the expression for its position vector.

$$
\left[\text { Ans. } \mathbf{r}=\left(\frac{t^{3}}{3}+2 t+1.667\right) \mathbf{i}+\left(\frac{t^{3}}{3}-t^{2}+6.667\right) \mathbf{j}\right]
$$

6.15 Two particles are dropped simultaneously from a height 100 m above the ground. One of them, in its mid path hits a fixed plane, inclined at an angle as shown in Fig. 6.69. As a result of this impact, the direction of its velocity becomes horizontal. Compute the time taken by the two particles to reach the ground.
[Ans. $4.515 \mathrm{~s} ; 6.385 \mathrm{~s}$.]


Fig. 6.69
6.16 (a) A projectile is fired from a gun with an initial velocity $u$ in a direction which makes an angle $\alpha$ with the horizontal. Neglecting the resistance of air find:
(1) the time of flight of the projectile to reach the level from which it started.
(2) its range on the horizontal plane, through the point of projection;
(3) the greatest height reached by it
(4) the equation of its path
(b) A Projectile is fired at a velocity of $800 \mathrm{~m} / \mathrm{s}$ at an angle of $40^{\circ}$ measured from the horizontal.
Determine the time of flight, range, and the greatest height reached.
[Ans. $t=104.838 \mathrm{~s} ; R=64248.42 \mathrm{~m}$; and $h=13477.71 \mathrm{~m}$ ]
6.17 A stone is projected upwards from the ground with velocity of $16 \mathrm{~m} / \mathrm{s}$ at an angle of $60^{\circ}$ to the horizontal. With what velocity must another stone be projected at an angle of $45^{\circ}$ to the horizontal from the same point in order:
(1) to have the same horizontal range?
(2) to attain the same maximum height?
[Ans. (1) $14.89 \mathrm{~m} / \mathrm{s}$; (2) $19.6 \mathrm{~m} / \mathrm{s}$ ]
6.18 A fire brigade man wants to extinguish a fire at a height of 6 m above the nozzle standing at a distance 5 m away from the fire. Find (1) the minimum velocity of the nozzle discharge required, (2) velocity of discharge if he could extinguish with angle of projection of $60^{\circ}$.
[Ans. (1) $u=11.754 \mathrm{~m} / \mathrm{s}$, (2) $u=13.58 \mathrm{~m} / \mathrm{s}$ ]
6.19 Maximum range of a field gun is 2000 m . If a target at a distance of 1200 m is to be hit, what should be the angle of projection?
[Ans. $\theta=18.435^{\circ}$ or $71.565^{\circ}$ ]
6.20 A projectile is fired with a velocity of $300 \mathrm{~m} / \mathrm{s}$ at an upward angle of $60^{\circ}$ to horizontal. Neglecting air resistance determine: (1) time of flight, (2) range and (3) its position after 35 seconds and (4) its velocity after 35 seconds. $\quad[$ Ans. (1) $t=52.97 \mathrm{~s}$; (2) $R=7945.19 \mathrm{~m}$; (3) $x=5250 \mathrm{~m}$;

$$
\left.y=3084.64 \mathrm{~m} \text {; (4) } 171.69 \mathrm{~m} / \mathrm{s} \text { at } \theta=29.116^{\circ} \text { to horizontal }\right]
$$

6.21 A rocket is projected vertically upward until it is 50 km above the launching site. At this instant it is turned so that its velocity is directed 3 upward and 4 horizontal and the power is shut off. The velocity, when the power is shut off is $1680 \mathrm{~m} / \mathrm{s}$. The rocket strikes the ground at the same elevation as the launching site. Determine: (1) the horizontal distance travelled by the rocket; and (2) the velocity with which the rocket strikes the ground.
[Ans. (1) $3,31708.4 \mathrm{~m}$; (2) $1950.23 \mathrm{~m} / \mathrm{s} ; \theta=46.434^{\circ}$ to horizontal]
6.22 A projectile is fired from a point 0 with the same velocity as it would be due to a fall of a 100 m from rest. The projectile hits a mark at a depth of 50 m below 0 to at a horizontal distance of 100 m from the vertical line through 0 . Determine the two possible directions of projections and show that they are at right angles to each other. [Ans. $\alpha_{1}=76.72^{\circ}$ above horizontal; and $\alpha_{2}=13.28^{\circ}$ below horizontal]
6.23 A stunt man wants to drive his motorbyke across a gap as shown in Fig. 6.70. What should be the minimum velocity of take-off. For smooth landing at this velocity what should be the inclination of the landing ramp?
[Ans. $u=6.56 \mathrm{~m} / \mathrm{s} ; \theta=51.207^{\circ}$ ]


Fig. 6.70
6.24 A gun is fired from the top of a hill 100 m above the sea level. Ten seconds later the shell is found to hit the warship 1000 m away from the position of the gun. Determine the velocity and angle of projection of the shell. With what velocity shell strikes the ship?

$$
\begin{array}{r}
{\left[\text { Ans. } u=107.354 \mathrm{~m} / \mathrm{s} ; \alpha=21.331^{\circ}\right.} \\
v=139.868 \mathrm{~m} / \mathrm{s} \text { and } \theta=45^{\circ} \text { ] }
\end{array}
$$

6.25 A missile projected from a rocket travelled 20 km vertically and 10 km horizontally from the launching point, when fuel of the rocket is exhausted. At this stage missile has acquired a velocity of $1600 \mathrm{~m} / \mathrm{s}$ at an angle of $35^{\circ}$ above the horizontal. Assuming that the rest of the flight is under the influence of the gravity and neglecting air resistance and curvature of the earth, calculate: (1) the total horizontal range from the launching point; and (2) the time of flight after the fuel has completely burnt.
[Ans. $R=271.061 \mathrm{~km} ; t_{1}=206.81 \mathrm{~s}$ ]
6.26 A particle is projected down a plane with an initial velocity of $u \mathrm{~m} / \mathrm{s}$.

Show that its maximum range is equal to $\frac{u^{2}}{g} \sec \beta(\sec \beta+\tan \beta)$ where, $\beta$ is the inclination of the plane to the horizontal.
6.27 A particle is projected from a point on an incline with a velocity of $30 \mathrm{~m} / \mathrm{s}$. The angle of projection and the angle of the plane are $55^{\circ}$ and $20^{\circ}$ to the horizontal respectively. Show that the range up the plane is the maximum one for the given plane. Find the range and time of flight of the particle.

$$
\text { [Ans. } R=68.36 \mathrm{~m} ; t=3.73 \mathrm{~s} \text {.] }
$$

6.28 While negotiating a curve of radius 800 m the driver of a train slowed it down at the rate of $0.3 \mathrm{~m} / \mathrm{s}^{2}$ in its tangential direction. If at point $A$ speed is 90 kmph , what will be the speed and total acceleration of the train after travelling a distance of 400 m .

$$
\begin{aligned}
{\left[\text { Ans. } v=19.621 \mathrm{~m} / \mathrm{s} ; a_{n}\right.} & =0.481 \mathrm{~m} / \mathrm{s}^{2}, \theta_{t}=3 \mathrm{~m} / \mathrm{s}^{2} \\
\therefore a & \left.=3.038 \mathrm{~m} / \mathrm{s}^{2} \theta=9.108^{\circ}\right]
\end{aligned}
$$

6.29 A ball is attached to the end of a string of length 0.6 m and swung in the vertical plane. The speed of the ball is $4 \mathrm{~m} / \mathrm{s}$ at the highest point $A$ and $6.4 \mathrm{~m} / \mathrm{s}$ at the lowest point $B$. Determine the total acceleration of the ball at the $A$ and $B$.
[Hints: $s=\pi r=0.6 \pi, V_{B}^{2}=V_{A}^{2}+2 a_{t} s$ Hence $a_{t}=6.621 \mathrm{~m} / \mathrm{s}^{2}$ ]

$$
\begin{aligned}
& \text { [Ans. } a_{A}=27.487 \mathrm{~m} / \mathrm{s}^{2} \theta=76.056^{\circ} \\
& \qquad a_{B}=68.587 \mathrm{~m} / \mathrm{s}^{2} \theta=84.460^{\circ} \text { ] }
\end{aligned}
$$

6.30 A collar is sliding along the arm such that its distance from the hinge is given by $r=2.4-0.2 t^{2}$ where $r$ is in metres and $t$ is in second. [Ref Fig. 6.71]. If the arm itself is rotating in a plane such that $\theta=0.12 t^{2}$ with a reference axis $x$, determine the velocity and the direction of the slider when $\theta=60^{\circ}$.


Fig. 6.71
[Hint: $\theta=\pi / 3=0.12 t^{2}$; find $t$. Then using the equation $r=2.4-0.2 t^{2}$ and $\theta=0.12 t^{2}$ assemble $\dot{r}, \ddot{r}, \dot{\theta}, \ddot{\theta}$ and find $v_{r}, v_{\theta}, a_{r}$ and $a_{\theta}$. Then find $v$ and $a$.
[Ans. $v=1.269 \mathrm{~m} / \mathrm{s}$ at $21.443^{\circ}$ to radial direction $a=2.147 \mathrm{~m} / \mathrm{s}^{2}$ at $64.438^{\circ}$ to radial direction]
6.31 Two trains $A$ and $B$ are moving on a parallel tracks with velocities 72 kmph and 36 kmph in the same direction. The driver of train $A$ finds that it took 42 seconds to overtake train $B$ while driver of train $B$ finds that $\operatorname{train} A$ took 30 seconds to overtake him.
Determine:
(a) length of each train
(b) time taken for complete overtaking

If two trains move in opposite direction with the same velocities, how much time is taken for complete crossing.
[Ans. (a) $L_{A}=300 \mathrm{~m}, L_{B}=420 \mathrm{~m}$; (b) $t=72 \mathrm{~s}, t=24 \mathrm{~s}$ ]
6.32 An aeroplane is flying horizontally at a height of 2000 m at a speed of 800 kmph towards a battle tank. The muzzle velocity of the gun of the tank is 600 metres per second. At what angle the shell should be fired when the plane is at 3000 m from the tank? Neglect air resistance and effect of gravity on the shell.
[Ans. $\theta=81.387^{\circ}$ ]
6.33 Two ships move simultaneously from a port, one moving in $\mathrm{N} 45^{\circ} \mathrm{W}$ and at 25 kmph another at $\mathrm{S} 60^{\circ} \mathrm{W}$, at 15 kmph . Determine the relative velocity of first ship with respect to second ship. When are they going to be 15 km apart?
[Ans. $V_{r}=25.61 \mathrm{kmph}, t=35.14 \mathrm{~min}$.]
6.34 Ship $A$ is approaching a port from $\mathrm{S} 40^{\circ} \mathrm{W}$ direction at 20 kmph . When ship $A$ was 20 km from the port, ship $B$ leaves the port at N 60 W with a velocity of 25 kmph . Determine the relative velocity of $A$ with respect to $B$. When are they at least distance?
[Ans. $V_{A / B}=34.62 \mathrm{kmph}, \theta=\mathrm{S} 85^{\circ} .326 \mathrm{~W}$ ]
6.35 Two coastal guard ships are 40 km apart, ship $A$ being due north of $B$. Ship $A$ is moving $\mathrm{S} 45^{\circ} \mathrm{W}$ with a velocity of 18 kmph while ship $B$ is moving west with a velocity of 24 kmph . If the two ships can exchange signals when they are 30 km apart, when they can begin and how long they can exchange the signals?
[Ans. They can begin after 56.18 m and continue for 1 hr .39 min .]
6.36 To an observer on a ship moving due south with a velocity of 18 kmph , wind appears to blow from due west. After the ship changes the course and moves in due west with the same velocity, wind appears to blow from $\mathrm{S} 45^{\circ}$ W. Assuming the wind has not changed the direction during this period, find the true direction and velocity of the wind.
[Ans. $v=40.25 \mathrm{kmph}, \theta=26.565^{\circ}$ ]
6.37 A rifleman on a train, moving with a speed of 60 kmph wants to shoot at a stationary object on the ground when it was sighted at $30^{\circ}$ to the train. At what angle should he aim if the bullet velocity is 700 kmph ?

If the object on the ground is not stationary, but moves away at a velocity of 40 kmph at right angles to the train, what should be the angle of the rifle at the time of shooting? Assume object remains in firing range and neglect air resistance and gravitational effect on the bullet.
[Ans. (1) $\phi=32.46^{\circ}$, (2) $\phi=33.40^{\circ}$ ]
6.38 When a cyclist is riding west at 20 kmph , he feels it is raining at an angle of $45^{\circ}$ with the vertical. When he rides at 15 kmph , he feels it is raining at an angle of $30^{\circ}$ with the vertical. What is the velocity of the rain?
[Ans. $V=14.377 \mathrm{kmph}$ ]

## CHAPTER 7

## Dynamics of Particles-Kinetics

As already defined in Chapter 6, the kinetics is the branch of dynamics that deals with the motion of the bodies referring to the forces causing the motion. In this chapter, we will deal the kinetics of particles under the following sub-headings:

1. Newton's Law-D'Alembert's principle,
2. Work-energy equations,
3. Impulse momentum, and
4. Impact of elastic bodies.

### 7.1 NEWTON'S LAW—D'ALEMBERT'S PRINCIPLES

## Newton's Second Law of Motion

D'Alembert's principle is, in fact, the application of Newton's second law to a moving body and looking at it from a different angle. Observing the motion of falling bodies, Galileo discovered first two laws of motion which are now commonly known as Newton's laws of motion. However, Newton generalised the laws and demonstrated their validity by astronomical predictions. These laws are presented in the first chapter. According to Newton's second law, the rate of change of momentum is directly proportional to the impressed force and takes place in the direction, in which the force acts. The above definition leads to the statement that force is directly proportional to product of mass and acceleration (Eqn. 1.3) and unit of force is so selected that constant of proportionality reduces to unity (Eqn. 1.5). Hence, finally, Newton's second law reduceds to the statement,

$$
\text { Force }=\text { Mass } \times \text { Acceleration } .
$$

Instead of single force, if a system of forces acts on a particle, the above statement reduces to the statement that resultant of forces is equal to the product of mass and acceleration in the direction of the resultant force. Mathematically:

$$
\begin{equation*}
R=m a \tag{7.1}
\end{equation*}
$$

where, $R$ is the resultant of forces acting on the particle. Hence many times Newton's second law is stated as: a particle acted upon by an unbalanced system of forces has an acceleration directly proportional and in line with the resultant force.

## D'Alembert's Principle

French mathematician Jeanle Rond d'Alembert proved (1743) that the Newton's second law of motion is applicable not only to the motion of a particle but also to the motion of a body and looked at Eqn. 7.1 from different angle. The equation $R=m a$ may be written as:

$$
\begin{equation*}
R-m a=0 \tag{7.2}
\end{equation*}
$$

The term ' $-m a$ ' may be looked as a force of magnitude $m \times a$, applied in the opposite direction of motion and is termed as the inertia force or reverse effective force. D'Alembert looks at Eqn. 7.2 as an equation of equilibrium and states that the system of forces acting on a body in motion is in dynamics equilibrium with the inertia force of the body. This is known as D'Alembert's principle.

Let the body, as shown in Fig. 7.1, be subjected to a system of forces causing the body to move with an acceleration $a$ in the direction of the resultant. Then let us apply a force equal to $m a$ in the reversed direction of acceleration as shown in Fig. 7.1. Now according to D'Alembert's principle, the equations of equilibrium $\Sigma F_{x}=0$ and $\Sigma F_{y}=0$ may be used for the system of forces shown in Fig. 7.2.


Fig. 7.1


Fig. 7.2

Equation (7.2) is applicable to any direction referred. The direction may be $x, y$, tangential, normal, radial, etc.

The inertia force $-m a$ has a physical meaning. According to Newton's first law of motion, a body continues to be in the state of rest or of uniform motion in a straight line unless acted by an external force. That means everybody has a tendency to continue in its state of rest or of uniform motion. This tendency is called inertia. Hence inertia force is the resistance offered by a body to the change in its state of rest or of uniform motion.

Many scientists are critical of D'Alembert's principle. Usually, equilibrium equations are applied to a system of forces acting on a body. Inertia force is not acting on the moving body. Actually, this is the force exerted by the moving body to resist the change in its stage. Hence D'Alembert is criticized for messing-up the concept of equations of equilibrium.

However, many engineers prefer to use D'Alembert's principle, since just by applying a reverse effective force, the moving body can be treated as a body in equilibrium and can be analysed using equations of static equilibrium.

Example 7.1 A man weighing $W$ Newton entered a lift which moved with an acceleration of $a \mathrm{~m} / \mathrm{s}^{2}$. Find the force exerted by the man on the floor of lift when
(a) lift is moving downward
(b) lift is moving upward.

## Solution.

(a) When the lift is moving downward, the inertia force $m a=\frac{W}{g} \times a$ should be applied in upward direction as shown in Fig. 7.3 (a). Sum of forces in vertical direction $=0$ for the lift, gives

(a)

Lift moving Lift moving downward

(b) upward

$$
\begin{align*}
R_{1}-W+\frac{W}{g} a & =0 \\
R_{1} & =W\left(1-\frac{a}{g}\right) \tag{1}
\end{align*}
$$

(b) When lift is moving upward, inertia force $\frac{W}{g} a$ should be applied in the downward direction as shown in Fig. 7.3(b). Applying the equation of equilibrium, we get

$$
\begin{align*}
R_{2}-W-\frac{W}{g} a & =0 \\
R_{2} & =W\left(1+\frac{a}{g}\right) \tag{2}
\end{align*}
$$

Thus when lift is moving with acceleration downward, the man exerts less force on the floor of the lift, and while moving upward, he exerts more force. From Eqns. (1) and (2), it may be observed that if the lift moves with uniform velocity then the acceleration $a$ is zero and hence the man exerts force equal to his own weight on the lift.

Example 7.2 An elevator cage of a mine shaft weighing 8 kN , when empty, is lifted or lowered by means of a wire rope. Once a man weighting 600 N , entered it and lowered with uniform acceleration such that when a distance of 187.5 m was covered, the velocity of the cage was $25 \mathrm{~m} / \mathrm{s}$. Determine the tension in the rope and the force exerted by the man on the floor of the cage.

Solution. In this problem

| Initial velocity | $u=0$ |
| :--- | :--- |
| Final velocity | $v=25 \mathrm{~m} / \mathrm{s}$ |
| Distance covered | $s=187.5 \mathrm{~m}$ |

Using the equation of motion,
we get

$$
v^{2}-u^{2}=2 a s
$$

or

$$
\begin{aligned}
25^{2}-0 & =2 a 187.5 \\
a & =1.667 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Figure 7.4 (a) shows free body diagram of elevator cage and the man with inertia force $\frac{W}{g} a=\frac{8000+600}{9.81} a$ applied in upward direction (since the motion is downward). Summing up the forces in vertical direction, we have

$$
T+\frac{8600}{9.81} 1.667-8600=0
$$

$$
T=7138.90 \mathrm{~N}
$$


(a)

(b)

Ans.

Fig. 7.4
Figure. 7.4 (b) shows the free body diagram of the man along with inertia force $\frac{600}{9.81} a$ applied in upward direction. Sum of the vertical forces should be zero. That is,

$$
\begin{array}{r}
\Sigma V=0, \text { gives } \\
R+\frac{600}{9.81} a-600=0 \\
\text { i.e., } \quad R+\frac{600}{9.81} \times 1.667-600=0
\end{array}
$$

or

$$
R=498.06 \mathrm{~N}
$$

Ans.
Example 7.3 A motorist travelling at a speed of 70 kmph , suddenly applies brakes and halts after skidding 50 m . Determine:
(a) the time required to stop the car,
(b) the coefficient of friction between the tyres and the road.

Solution. (a) Initial velocity $u=70 \mathrm{kmph}$

$$
=\frac{70 \times 1000}{60 \times 60}=19.44 \mathrm{~m} / \mathrm{s}
$$

Final velocity

$$
v=0
$$

Displacement

$$
s=50 \mathrm{~m}
$$

Using the equation of linear motion $v^{2}=u^{2}+2 a s$, we get

$$
\begin{aligned}
& 0=19.44^{2}+2 a \times 50 \\
& a=-3.78 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

i.e., the retardation is $3.78 \mathrm{~m} / \mathrm{s}^{2}$.

$$
\left.\begin{array}{rl}
\text { Using the relation } & v \\
\therefore \quad 0 & =u+a t, \text { we get } \\
\therefore & t
\end{array}\right)=5.14 \mathrm{sec} .
$$

(b) Inertia force must be applied in the opposite direction of acceleration, which means, it should be applied in the direction of motion while retarding. Figure 7.5 shows the free body of the motor along with inertia force.

Sum of force normal to road $=0$, gives

$$
N=W
$$

From the law of friction, $\quad F=\mu N=\mu W$
Sum of forces in the direction of motion $=0$, gives

$$
\begin{aligned}
F & =\frac{W}{9.81} 3.78 & \begin{array}{|l}
W \\
\mu W
\end{array} & =\frac{W \times 3.78}{9.81}
\end{aligned} \quad F \frac{W}{9.81} \times 3.78
$$

$\therefore \quad \mu=0.385$.
Ans.
Example 7.4 A block weighing 1 kN rests on a horizontal plane as shown in Fig. 7.6(a). Find the magnitude of the force $P$ required to give the block an acceleration of $3 \mathrm{~m} / \mathrm{s}^{2}$ to the right. The coefficient of friction between the block and the plane is 0.25 .

(a)

(b)

Fig. 7.6

Solution. Free body diagram of the block along with inertia force $m a=\frac{1}{g} \times 3=$ $\frac{3}{9.81} \mathrm{kN}$ is shown in Fig. 7.6(b). In the figure, $N$ is the normal reaction and $F$ is the frictional forces.

$$
\begin{align*}
\Sigma V & =0, \text { gives } \\
N-1-P \sin 30^{\circ} & =0 \\
N & =1+\frac{P}{2} \tag{1}
\end{align*}
$$

or
From the law of friction,

$$
\begin{align*}
& F=\mu N=0.25\left(1+\frac{P}{2}\right)  \tag{2}\\
& \Sigma H=0, \text { gives } \\
& P \cos 30^{\circ}-F-\frac{3}{9.81}=0 \\
& P \cos 30^{\circ}-0.25\left(1+\frac{P}{2}\right)-\frac{3}{9.81}=0 \\
& P=0.561 \mathrm{kN} .
\end{align*}
$$

Ans.
Example 7.5 A 750 N crate rests on a 500 N cart. The coefficient of friction between the crate and the cart is 0.3 and between cart and the road is 0.2 . If the cart is to be pulled by a force $P$ [Ref. Fig. 7.7 (a)] such that the crate does not slip, determine: (a) the maximum allowable magnitude of $P$, and (b) the corresponding acceleration of the cart.


Fig. 7.7

Solution. Let the maximum acceleration be $a$, at which 750 N crate is about to slip. Hence frictional force will have limiting value $(=\mu N)$. Consider the free body diagram of crate along with inertia force, as shown in Fig. 7.7 (b).
$\therefore$ Frictional force

$$
\begin{aligned}
\Sigma V & =0, \text { gives } \\
N & =W=750 \text { Newton } \\
F & =\mu N=0.3 \times 750 \\
& =225 \mathrm{~N} \\
\Sigma H & =0, \text { gives } \\
225 & =\frac{750}{9.81} a
\end{aligned}
$$

$$
\therefore \quad a=2.943 \mathrm{~m} / \mathrm{s}^{2}
$$

Ans.
Consider now dynamic equilibrium of crate and the cart shown as a single body in Fig. 7.7 (c).

$$
\begin{aligned}
\Sigma V & =0, \text { gives } \\
N & =1250 \text { Newton }
\end{aligned}
$$

$$
\text { Frictional force }=\mu N=0.2 \times 1250=250 \text { Newton }
$$

$$
\Sigma H=0, \text { gives }
$$

$$
P-250-\frac{1250}{9.81} \times 2.943=0
$$

$$
P=625 \text { Newton }
$$

Ans.
Example 7.6 A body weighing 1200 N rests on a rough plane inclined at $12^{\circ}$ to the horizontal. It is pulled up the plane by means of a light flexible rope running parallel to the plane and passing over a light frictionless pulley at the top of the plane as shown in Fig. 7.8 (a). The portion of the rope beyond the pulley hangs vertically down and carries a weight of 800 N at its end. If the coefficient of friction for the plane and the body is 0.2 , find:
(a) tension in the rope,
(b) acceleration with which the body moves up the plane, and
(c) the distance moved by the body in 3 seconds after starting from rest.


(b)

(c)

Fig. 7.8

Solution. Let $a$ be the acceleration of the system.
Free body diagrams of 1200 N block and 800 N block are shown in Figs. 7.8 (b) and (c) along with inertia forces. According to D'Alembert, we can treat these bodies as in static equilibrium.

Consider 1200 N block.
Sum of forces normal to the plane $=0$, gives

$$
\begin{aligned}
& N=1200 \cos 12^{\circ}=0 \\
& N=1173.77 \text { Newton }
\end{aligned}
$$

From the law of friction

$$
F=\mu N=0.2 \times 1173.77=234.76 \mathrm{~N}
$$

Sum of forces parallel to the inclined plane $=0$, gives

$$
\frac{1200}{9.81} a+1200 \sin 12^{\circ}+F-T=0
$$

i.e.,

$$
\begin{equation*}
122.32 a-T=-484.25 \tag{1}
\end{equation*}
$$

Consider the free body diagram of 800 N block as shown in Fig. 7.8 (c)

$$
\begin{equation*}
T+\frac{800}{9.81} a=800 \tag{2}
\end{equation*}
$$

Adding Eqns. (1) and (2), we get

$$
\left(122.32+\frac{800}{9.81}\right) a=800-484.25
$$

$$
\therefore \quad a=1.549 \mathrm{~m} / \mathrm{s}^{2}
$$

Ans.
Substituting it in (2), we get

$$
T=800-\frac{800}{9.81} \times 1.549
$$

i.e.,

$$
T=673.68 \mathrm{~N}
$$

Initial velocity $=0$

$$
\begin{aligned}
a & =1.549 \mathrm{~m} / \mathrm{s}^{2} \\
t & =3 \mathrm{~s}
\end{aligned}
$$

Using the equation $s=u t+\frac{1}{2} a t^{2}$, we get distance moved in 3 seconds as

$$
\begin{aligned}
& =0 \times 3+\frac{1}{2} \times 1.549 \times 3^{2} \\
& =6.971 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 7.7 Two weights 800 N and 200 N are connected by a thread and they move along a rough horizontal plane under the action of a force of 400 N applied to the 800 N weight as shown in Fig. 7.9 (a). The coefficient of friction between the sliding surface of the weights and the plane is 0.3 .


Fig. 7.9
D' Alembert's principle to be used to determine the acceleration of the weight and tension in the thread.
Solution. Free body diagrams of 200 N and 800 N blocks along with inertia forces are shown in Fig. 7.9 (b), in which $a$ is alteration of the system. Consider the dynamic equilibrium of 200 N weight.

$$
\begin{align*}
\Sigma V & =0, \text { gives } \\
N_{1} & =200 \mathrm{~N}  \tag{1}\\
F_{1} & =\mu N_{1}=0.3 \times 200=60 \mathrm{~N}  \tag{2}\\
\Sigma H & =0, \text { gives }
\end{align*}
$$

$$
T-F_{1}-\frac{200}{9.81} a=0
$$

i.e.,

$$
\begin{equation*}
T-\frac{200}{9.81} a=60 \tag{3}
\end{equation*}
$$

Since

$$
F_{1}=60 \mathrm{~N}
$$

Consider 800 N body,

$$
\begin{align*}
& \Sigma V=0, \text { gives } \\
& N_{2}=800 \mathrm{~N}  \tag{4}\\
& F_{2}=\mu N_{2}=0.3 \times 800 \\
&=240 \mathrm{~N}  \tag{5}\\
& \text { From law of friction, }  \tag{6}\\
& \Sigma H=0, \text { gives } \\
&-T-\frac{800}{9.81} a-F_{2}+400=0 \\
& \text { or } \\
& \text { Since } \quad T+\frac{800}{9.81} a=160 \\
& F_{2}=240 \mathrm{~N}
\end{align*}
$$

Subtracting Eqns. (3) from (6), we get

$$
\begin{array}{rlrl}
\therefore \quad\left(\frac{200}{9.81}+\frac{800}{9.81}\right) a & =160-60 \\
\therefore & a & =0.981 \mathrm{~m} / \mathrm{s}^{2}
\end{array}
$$

## Ans.

Substituting it in Eqn. (6), we get

$$
\begin{array}{ll} 
& T=160-\frac{800}{9.81} \times 0.981 \\
\text { i.e., } \quad & T=80 \mathrm{~N} .
\end{array}
$$

## Ans.

Example 7.8 Two rough planes inclined at $30^{\circ}$ and $60^{\circ}$ to horizontal are placed back to back as shown in Fig. 7.10 (a). The blocks of weights 50 N and 100 N are placed on the faces and are connected by a string running parallel to planes and passing over a frictionless pulley. If the coefficient of friction between planes and blocks is 1.3 , find the resulting accleration and tension in the string.
Solution. Let the assembly move down the $60^{\circ}$ plane by an acceleration ' $a$ ' $\mathrm{m} / \mathrm{s}^{2}$. Free body diagrams of 100 N and 50 N blocks along with inertia forces are shown in Figs. 7.10 (b) and 7.10 (c) respectively.

(a)

(b)

(c)

Fig. 7.10
Consider the block weighing 100 N :
Sum of forces normal to the plane $=0$, gives

$$
\begin{equation*}
N_{1}=100 \cos 60^{\circ}=50 \mathrm{~N} \tag{1}
\end{equation*}
$$

From the law of friction,

$$
\begin{equation*}
F_{1}=\mu N=\frac{1}{3} \times 50=16.67 \mathrm{~N} \tag{2}
\end{equation*}
$$

Sum of forces parallel to the plane $=0$, gives

$$
\begin{align*}
T+\frac{100}{9.81} a-100 \sin 60^{\circ}+F_{1} & =0 \\
T+\frac{100}{9.81} a & =69.93  \tag{3}\\
F_{1} & =16.67
\end{align*}
$$

Since
Now consider 50 N block:
Sum of forces normal to plane $=0$, gives

$$
\begin{equation*}
N_{2}=50 \cos 30^{\circ}=43.30 \mathrm{~N} \tag{4}
\end{equation*}
$$

From the law of friction, $F_{2}=\mu N_{2}$

$$
=\frac{1}{3} \times 43.3=14.43 \mathrm{~N}
$$

Sum of forces parallel to $30^{\circ}$ plane $=0$, gives

$$
\begin{align*}
\frac{50}{9.81} a+F_{2}+50 \sin 30^{\circ}-T & =0 \\
\frac{50}{9.81} a-T & =-39.43  \tag{6}\\
F_{2} & =14.43
\end{align*}
$$

Since
Adding Eqns. (3) and (6), we get

$$
\left(\frac{100}{9.81}+\frac{50}{9.81}\right) a=69.93-39.43
$$

or

$$
a=1.9947 \mathrm{~m} / \mathrm{s}^{2}
$$

Ans.
From Eqn. (3),

$$
T=69.93-\frac{100}{9.81} \times 1.9947=49.6 \mathrm{~N}
$$

Ans.
Example 7.9 When they are 18 m apart, two blocks $A$ and $B$ are released from rest on a $30^{\circ}$ incline. The coefficient of friction under the upper block $A$ is 0.2 and that under the lower block $B$ is 0.4 [Ref. Fig. 7.11 (a)]. In what time does block $A$ reach the block $B$ ? After they touch and move as a single unit, what will be the contact force between them? Weights of the block $A$ and $B$ are 100 N , and 80 N respectively.

(a)

(b)

Fig. 7.11

Solution. Let block $A$ move with an acceleration $\mathrm{a}_{1}$, and block $B$ with an acceleration $a_{2}$. The free body diagrams of the blocks $A$ and $B$ along with inertia forces are shown in Figs. 7.11 (b) and (c) respectively.

Consider block A.
Sum of forces normal to the plane $=0$, gives

$$
\begin{equation*}
N_{1}=W_{A} \cos \theta=W_{A} \cos 30^{\circ} \tag{1}
\end{equation*}
$$

From the law of friction, $F_{A}=\mu N_{A}$

$$
\begin{equation*}
=0.2 W_{A} \cos 30^{\circ} \tag{2}
\end{equation*}
$$

Sum of forces parallel to the plane $=0$, gives

$$
\begin{gather*}
\frac{W_{A}}{9.81} a_{1}+F_{A}-W_{A} \sin 30^{\circ}=0 \\
\therefore \quad \frac{W_{A}}{9.81} a_{1}+0.2 W_{A} \cos 30^{\circ}-W_{A} \sin 30^{\circ}=0 \\
\frac{a_{1}}{9.81}+0.2 \cos 30^{\circ}-\sin 30^{\circ}=0 \\
a_{1}=3.2058 \mathrm{in} / \mathrm{s}^{2} \tag{3}
\end{gather*}
$$

## Consider block $B$

Sum of forces normal to the plane $=0$, gives

$$
\begin{equation*}
N_{B}=W_{B} \cos 30^{\circ} \tag{4}
\end{equation*}
$$

From the law of friction, $F_{B}=\mu N_{B}$
i.e.,

$$
\begin{equation*}
F_{B}=0.4 W_{B} \cos 30^{\circ} \tag{5}
\end{equation*}
$$

Sum of forces parallel to the plane $=0$, gives

$$
\begin{array}{rlrl} 
& \frac{W_{B}}{9.81} a_{2}+F_{B}-W_{B} \sin 30^{\circ} & =0 \\
\therefore & \frac{W_{B}}{9.81} a_{2}+0.4 W_{B} \cos 30^{\circ}-W_{B} \sin 30^{\circ}=0 \\
a_{2} & =1.5067 \mathrm{~m} / \mathrm{s}^{2} .
\end{array}
$$

Let $t$ be the time elapsed until the blocks touch each other.
Displacement of block $A$ in this period is given as

$$
s_{1}=u_{1} t+\frac{1}{2} a_{1} t^{2}=\frac{1}{2} \times 3.2058 t^{2}
$$

Since initial velocity $u_{1}=0$
Displacement of block $B$ in this time is give as

$$
s_{2}=u_{2} t+\frac{1}{2} a_{2} t^{2}=\frac{1}{2} \times 1.5067 t^{2}
$$

When the two blocks touch each other,

$$
\begin{aligned}
s_{1} & =s_{2}+18 \\
\therefore \quad \frac{1}{2} \times 3.2058 t^{2} & =\frac{1}{2} \times 1.5067 t^{2}+18 \\
\therefore \quad t & =4.60 \mathrm{~s} .
\end{aligned}
$$

Ans.
After the blocks touch each other, let the common acceleration be $a$. Summing up the forces including inertia forces along the inclined plane, we have

$$
\begin{gathered}
\frac{100}{9.81} a+0.2 \times 100 \cos 30^{\circ}-100 \sin 30^{\circ}+\frac{80}{9.81} a \\
+0.4 \times 80 \cos 30^{\circ}-80 \sin 30^{\circ}=0 \\
a=2.45 \mathrm{~m} / \mathrm{s}^{2}
\end{gathered}
$$

Considering the free body diagram of any one of the blocks, contact force $P$ can be obtained. Free body diagram of block $A$ along with inertia force is shown in Fig. 7.11 (d).

Now sum of forces parallel to plane $=0$, gives

$$
\begin{gathered}
P-100 \sin 30^{\circ}+F+\frac{100}{9.81} a=0 \\
P-100 \sin 30^{\circ}+0.2 \times 100 \cos 30^{\circ}+\frac{100}{9.81} \times 2.45=0
\end{gathered}
$$

$$
P=7.7 \mathrm{~N}
$$

Ans.
Example 7.10 Two bodies weighing 300 N and 450 N are hung to the ends of a rope passing over an ideal pulley as shown in Fig. 7.12 (a). With what acceleration does the heavier body come down? What is the tension in the string?
Solution. Let $a$ be the acceleration with which the system moves and $T$ be the tension in the string.

Free body diagrams of 300 N block and 450 N block along with inertia forces are shown in Figs. 7.12(b) and 7.12 (c).


Fig. 7.12

Consider the body weighing 300 N

$$
\Sigma V=0, \text { gives }
$$

$$
T-\frac{300}{9.81} a-300=0
$$

i.e., $\quad T-\frac{300}{9.81} a=300$

Considering the body weighing 450 N , we get

$$
\begin{equation*}
T+\frac{450}{9.81} a=450 \tag{2}
\end{equation*}
$$

Subtracting Eqn. (1) from Eqn. (2), we get

$$
\begin{aligned}
\frac{450}{9.81} a+\frac{300}{9.81} a & =450-300 \\
a & =1.962 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Ans.
Substituting this value in Eqn. (1), we get

$$
T=300+\frac{300}{9.81} \times 1.962
$$

i.e.,

$$
T=360 \mathrm{~N}
$$

Ans.
Example 7.11 Determine the tension in the string and accelerations of blocks $A$ and $B$ weighing 1500 N and 500 N connected by an inextensible string as shown in Fig. 7.13 (a). Assume that the pulleys are frictionless and weightless.


Fig. 7.13
Solution. In this pulley system, it may be observed that if 1500 N block moves downward by distance $x, 500 \mathrm{~N}$ block moves up by $2 x$. Hence if acceleration of 1500 N block is $a$ then acceleration of 500 N block is $2 a$. The free body diagrams of 1500 N block and 500 N block are shown in Figs. 7.13 (b) and (c), along with inertia forces. According to D'Alembert's principle, the system of forces shown in Figs. 7.13 (b) and (c) may be treated to be in equilibrium.

Considering 1500 N block, we get

$$
\begin{equation*}
2 T+\frac{1500}{9.81} a=1500 \tag{1}
\end{equation*}
$$

Considering 500 N block, we get

$$
\begin{equation*}
T-\frac{500}{9.81}(2 a)=500 \tag{2}
\end{equation*}
$$

From Eqns. (1) and (2), we get

$$
\begin{aligned}
\therefore \quad\left(\frac{1500}{9.81}+\frac{2000}{9.81}\right) a & =500 \\
\therefore \quad a & =1.401 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Ans.
Substituting it in Eqn. (1), we get

$$
\begin{aligned}
2 T & =1500-\frac{1500}{9.81} \times 1.401 \\
T & =642.89 \mathrm{~N}
\end{aligned}
$$

Ans.
Example 7.12 An engine weighing 500 kN pulls a carriage weighing 1500 kN up an incline of 1 in 100 . The train starts from rest and moves with a constant acceleration against a resistance of $5 \mathrm{~N} / \mathrm{kN}$. It attains a maximum speed of 36 kmph in 1 km distance. Determine the tension in the coupling between the carriage and the engine and the tractive force developed by the engine.
Solution. Initial velocity $u=0$
Final velocity

$$
v=36 \mathrm{kmph}=\frac{36 \times 1000}{60 \times 60}=10 \mathrm{~m} / \mathrm{s}^{2}
$$

Displacement

$$
s=1 \mathrm{~km}=1000 \mathrm{~m} .
$$

From the kinematic equation $v^{2}=u^{2}+2 a s$, we get

$$
\begin{aligned}
10^{2} & =0+2 \times a \times 1000 \\
a & =0.05 \mathrm{~m} / \mathrm{s}^{2} .
\end{aligned}
$$

To find the tension in the coupling between engine and train, dynamic equilibrium of the coaches only may be considered. Let $T$ be the tension in coupling. Various forces acting parallel to the track are shown in Fig. 7.14.


Fig. 7.14

$$
\begin{aligned}
\text { Tractive resistance } & =5 \times 1500 \\
& =7500 \mathrm{~N} \\
& =7.5 \mathrm{kN} \text { down the plane. }
\end{aligned}
$$

Component of weight of train

$$
=1500 \times \frac{1}{100}=15 \mathrm{kN} \text { down the plane. }
$$

Note: Since slope of track is very small, $\sin \theta=\tan \theta=\frac{1}{100}$.

$$
\begin{aligned}
& \text { Inertia force }=\frac{W}{g} a \\
& \frac{1500}{9.81} \times 0.05=7.645 \mathrm{kN} \quad(\text { down the plane })
\end{aligned}
$$

Dynamic equilibrium equation is

$$
T-7.5-15-7.645=0
$$

or

$$
T=30.145 \mathrm{kN} .
$$

Ans.
To find tractive force developed by the engine, consider dynamic equilibrium of entire train.

Figure 7.15 shows free body diagram of the entire train along with inertia force. Let $P$ be tractive force developed.

Total tractive resistance

$$
\begin{aligned}
& =5 \times 2000=10,000 \mathrm{~N} \\
& =10 \mathrm{kN} \text {, down the plane. }
\end{aligned}
$$

$$
\text { Inertia force }=\frac{W}{g} a
$$



Fig. 7.15
$=\frac{2000}{9.81} \times 0.05$
$=10.194 \mathrm{kN}$, down the plane.
Component of weight down the plane $=2000 \times \frac{1}{100}=20 \mathrm{kN}$
From the dynamic equilibrium equation along the track, we have

$$
P-10-10.194-20=0
$$

Therefore,

$$
P=40.194 \mathrm{kN}
$$

Ans.

### 7.2 WORK-ENERGY METHOD

In the last article, kinetic problems were solved using D'Alembert's dynamic equilibrium condition. In this article, another approach, called work-energy approach, is used to solve kinetic problems. This method is advantageous over

D'Alembert's method when the problem involves velocities, rather than acceleration. The terms work, energy and power are explained first, then the work-energy equation is derived. Using this equation, a number of kinetic problems are solved.

## Work

The work done by a force on a moving body is defined as the product of the force and the distance moved in the direction of the force. In Fig. 7.16, various forces acting on a particle are shown. If the particle moves a distance $s$ in $x$ direction, from $A$ to $B$, then the work done by various forces are as given below:

| Force | Work done |
| :---: | :--- |
| $F_{1}$ | $F_{1} s$ |
| $F_{2}$ | $+F_{2}(-s)=-F_{2} s$ |
| $F_{3}$ | $F_{3} \times 0=0$ |
| $F_{4}$ | $F_{4} s \cos \theta$ |



Fig. 7.16

The general expression for work done by a force $F$ is $F s \cos \theta$, where $\theta$ is the angle between the force and the direction of motion. This expression may be rearranged as $F \cos \theta \times s$, giving the new definition of work done. Thus work done by a force may be defined as the product of component of force in the direction of motion and the distance moved.

From the definition of work, it is obvious that unit of work is obtained by multiplying unit of force by unit of length. Hence, if Newton is unit of force and metre is unit of displacement, unit of work will be $\mathrm{N}-\mathrm{m}$. One $\mathrm{N}-\mathrm{m}$ of work is denoted by the term Joule $(J)$. Hence one Joule may be defined as the amount of work done by one Newton force when the particle moves 1 metre in the direction of that force.

The other commonly used units are: kilo joules $k J$ (i.e., kN-m) or milli Joules $m J$ ( $\mathrm{N}-\mathrm{mm}$ ), etc.

## Work Done by a Varying Force

Let the varying force acting at any instance on the particle be $F$. Now if the particle moves a small distance $\delta s$, then the work done by the force is $F \times \delta s$. Work done by the force in moving the body by a distance $s$ is $\Sigma F s$. Thus if a force versus displacement curve is drawn (Fig. 7.17), the area under the curve gives the work done by the force. If the variation of $F$ is in a regular fashion, then

$$
\begin{equation*}
\Sigma F s=\int F d s \tag{7.3}
\end{equation*}
$$



Fig. 7.17

## Energy

Energy is defined as the capacity to do work. There are many forms of energies such as heat energy, mechanical energy, electrical energy and chemical energy. In engineering mechanics, we are interested in mechanical energy. This energy may be classified into potential energy and kinetic energy.

Potential energy is the capacity to do work due to the position of the body. A body of weight ' $W$ ' held at a height $h$ possesses an energy $W h$.

Kinetic energy is the capacity to do work due to motion of the body. Consider a car moving with a velocity $v \mathrm{~m} / \mathrm{s}$. (Fig. 7.18). If the engine is stopped, it still moves forward, doing work against frictional resistance and stops at a certain distance $s$. From the kinematic of the motion, we have

$$
\begin{aligned}
0-u^{2} & =2 a s \\
a & =-\frac{u^{2}}{2 s}
\end{aligned}
$$

From D'Alembert's principle, we have

$$
F+\frac{W}{g} a=0
$$



Fig. 7.18
i.e., $\quad F-\frac{W}{g} \times \frac{u^{2}}{2 s}=0$
or

$$
F=\frac{W u^{2}}{2 g s}
$$

$\therefore \quad$ Work done $=F \times s=\frac{W u^{2}}{2 g}$
This work is done by the energy, stored initially in the body.
$\therefore \quad$ Kinetic energy $=\frac{1}{2} \times \frac{W}{g} v^{2}$, where $v$ is the velocity of the body.

Unit of energy is same as that of work, since it is nothing but capacity to do work. It is measured in Joules $\mathbf{J}(\mathrm{N}-\mathrm{m})$ or kilo Joules kJ (i.e., kN-m).

## Power

Power is defined as time rate of doing work. Unit of power is Watt $(W)$ and is defined as one Joule of work done in one second. In practice, kilowatt is the commonly used unit which is equal to 1000 watts. Horse power is the unit used in MKS and FPS systems.

$$
\begin{aligned}
& 1 \text { metric H.P. }=735.75 \text { watts } \\
& 1 \text { British H.P. }=745.8 \text { watts. }
\end{aligned}
$$

Example 7.13 A pump lifts $40 \mathrm{~m}^{3}$ of water to a height of 50 m and delivers it with a velocity of $5 \mathrm{~m} / \mathrm{s}$. What is the amount of energy spent during this process? If the job is done in half an hour, what is the input power of the pump which has an overall efficiency of $70 \%$ ?
Solution. Output energy of the pump is spent in lifting $40 \mathrm{~m}^{3}$ of water to a height of 50 m and delivering it with the given kinetic energy of delivery.

Work done in lifting $40 \mathrm{~m}^{3}$ of water to a height of 50 m is

$$
=W h
$$

where $W$ is weight of $40 \mathrm{~m}^{3}$ of water

$$
\begin{aligned}
& =40 \times 9810 \times 50 \\
& =1,96,20000 \mathrm{~N}-\mathrm{m}
\end{aligned}
$$

Note: $1 \mathrm{~m}^{3}$ of water weighs $=9810$ Newtons
Kinetic energy at delivery

$$
\begin{aligned}
& =\frac{1}{2} \times \frac{W}{g} v^{2} \\
& =\frac{1}{2} \times \frac{40 \times 9810}{9.81} \times 5^{2} \\
& =5,00000 \mathrm{~N}-\mathrm{m} \\
\text { Total energy spent } & =1,96,20000+5,00000 \\
\text { Energy spent } & =2,01,20000 \mathrm{~N}-\mathrm{m}
\end{aligned}
$$

This energy is spent by the pump in half an hour, i.e., in $30 \times 60=1800 \mathrm{~s}$.
$\therefore \quad$ Output power of pump $=$ Output energy spent per second

$$
\begin{aligned}
& =\frac{2,01,20000}{1800} \\
& =11177.8 \mathrm{watts} \\
& =11.1778 \mathrm{~kW} .
\end{aligned}
$$

$$
\begin{aligned}
\text { Input power } & =\frac{\text { Output power }}{\text { Efficiency }} \\
& =\frac{11.1778}{0.7} \\
& =15.9683 \mathrm{~kW} .
\end{aligned}
$$

Ans.
Example 7.14 A man wishes to move wooden box of 1 metre cube to a distance of 5 m with the least amount of work. If the block weighs 1 kN and the coefficient of friction is 0.3 , find whether he should tip it or slide it.


Fig. 7.19

## Solution.

Work to be done in sliding
Normal reaction

$$
N=W=1 \mathrm{kN}
$$

$\therefore \quad$ Frictional force $=F=\mu N=0.3 \mathrm{kN}$
Applied force $\quad P=F=0.3 \mathrm{kN}$
Work to be done in sliding to a distance of $5 \mathrm{~m}=P \times 5$

$$
\begin{align*}
& =0.3 \times 5=1.5 \mathrm{kN}-\mathrm{m} \\
& =1.5 \mathrm{~kJ} \tag{1}
\end{align*}
$$

Work to be done in tipping
In one tipping, the centre of gravity of box is to be raised to a height (Ref. Fig. 7.19)

$$
=\frac{1}{\sqrt{2}}-0.5=0.207 \mathrm{~m}
$$

$\therefore \quad$ Work done in one tipping $=W h$

$$
\begin{aligned}
& =1 \times 0.207 \\
& =0.207 \mathrm{~kJ}
\end{aligned}
$$

To move a distance of 5 m , five tippings are required. Hence work to be done in moving it by 5 metres, by tipping

$$
\begin{equation*}
=5 \times 0.207=1.035 \mathrm{~kJ} \tag{2}
\end{equation*}
$$

Since the man needs to spend only 1.035 kJ when tipping and it is less than 1.5 kJ to be spent in sliding, the man should move the box by tipping.

## Work-Energy Equation for Translation

Consider the body as shown in Fig. 7.20, subjected to a system of forces $F_{1}, F_{2}$. $\ldots$ and moving with an acceleration $a$ in $x$-direction. Let its initial velocity at $A$ be $u$ and final velocity when it moves distance $A B=s$ be $v$. Then the resultant of system of the forces must be in $x$-direction. Let

$$
\begin{equation*}
R=\Sigma F_{x} \tag{7.6}
\end{equation*}
$$



Fig. 7.20
From Newton's second law of motion,

$$
R=\frac{W}{g} a
$$

Multiplying both sides by elementary distance $d s$, we get

$$
\begin{aligned}
R d s & =\frac{W}{g} a d s \\
& =\frac{W}{g} v \frac{d v}{d s} d s \quad\left[\text { Since } a=v \frac{d v}{d s}\right] \\
& =\frac{W}{g} v d v
\end{aligned}
$$

Integrating both sides for the motion from $A$ to $B$, we get

$$
\begin{align*}
\int_{0}^{S} R d s & =\int_{u}^{v} \frac{W}{g} v d v \\
R s & =\frac{W}{g}\left[v^{2} / 2\right]_{u}^{v} \\
& =\frac{W}{2 g}\left(v^{2}-u^{2}\right) \tag{7.7}
\end{align*}
$$

Now $R s$ is the work done by the forces acting on the body. $\frac{W}{2 g} v^{2}$ is final kinetic energy and $\frac{W u^{2}}{2 g}$ is initial kinetic energy. Hence we can say, work done in a motion is equal to change in kinetic energy. That is,

Work done $=$ Final kinetic energy - Initial kinetic energy and it is called Work-Energy Equation.

This work energy principle may be stated as the work done by a system of forces acting on a body during a displacement is equal to the change in kinetic energy of the body during the same displacement.

Using this work-energy equation a number of kinetic problems can be solved. This will be found more useful than D'Alembert's principle, when we are not interested in finding acceleration in the problem, but mainly interested in velocity and distance.

Example 7.15 A body weighing 300 N is pushed up a $30^{\circ}$ plane by a 400 N force acting parallel to the plane. If the initial velocity of the body is $1.5 \mathrm{~m} / \mathrm{s}$ and coefficient of kinetic friction is $\mu=0.2$, what velocity will the body have after moving 6 m ?
Solution. Consider the free body diagram of the body as shown in Fig. 7.21.


Fig. 7.21
Sum of forces normal to plane $=0$, gives

| $N$ | $=300 \times \cos 30^{\circ}$ |
| ---: | :--- |
|  | $=259.81$ Newton |
| $\therefore$ Frictional force | $F$ |
|  | $=\mu N$ |
|  | $=0.2 \times 259.81=51.96 \mathrm{~N}$ |
|  | Initial velocity |
| Displacement | $u$ |

Equating the work done by forces along the plane to change in kinetic energy, we get

$$
(400-F-W \sin \theta) s=\left(\frac{1}{2}\right) \frac{W}{g}\left(v^{2}-u^{2}\right)
$$

$$
\begin{array}{rlrl}
\left(400-51.96-300 \times \frac{1}{2}\right) 6 & =\frac{1}{2} \times\left(\frac{300}{9.81}\right)\left(v^{2}-1.5^{2}\right) \\
\therefore \quad 77.71 & =v^{2}-2.25 \\
\therefore & v & =8.942 \mathrm{~m} / \mathrm{s} .
\end{array}
$$

Ans.
Example 7.16 Find the power of a locomotive, drawing a train whose weight including that of engine is 420 kN up an incline 1 in 120 at a steady speed of 56 kmph, the frictional resistance being $5 \mathrm{~N} / \mathrm{kN}$.

While the train is ascending the incline, the steam is shut off. Find how far it will move before coming to rest, assuming that the resistance to motion remains the same.
Solution. Figure 7.22 (a) shows the system of forces acting on the locomotive while moving up the incline with steady speed.


Fig. 7.22

$$
\begin{aligned}
v & =56 \mathrm{kmph} \\
& =\frac{56 \times 1000}{60 \times 60}=15.556 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

$$
F=5 \times 420=2100 \text { Newton }
$$

$$
=2.1 \mathrm{kN}
$$

$$
P=F+W \sin \theta
$$

$$
=2.1+420 \times \frac{1}{120}
$$

$$
=5.6 \mathrm{kN}
$$

$\therefore \quad$ Power of locomotive $=$ Work done by $P$ per second

$$
=P \times \text { Distance moved per second }
$$

$$
=P \times v
$$

$$
=5.6 \times 15.556
$$

$$
=87.11 \mathrm{kN}-\mathrm{m} / \mathrm{s}
$$

Ans.
When steam is put off, let it move a distance $s$ before coming to rest.
Initial velocity

$$
u=15.556 \mathrm{~m} / \mathrm{s}
$$

Final velocity
$v=0$

Figure 7.21 (b) shows the system of forces acting in this motion. Resultant force parallel to the plane is

$$
\begin{aligned}
& =F+W \sin \theta \\
& =2.1+420 \times\left(\frac{1}{120}\right)=5.6 \mathrm{kN}(\text { down the plane })
\end{aligned}
$$

Writing work-energy equation for motion up the plane, we have

$$
\begin{aligned}
-5.6 \times s & =\frac{1}{2} \times \frac{420}{9.81}\left(0-15.556^{2}\right) \\
s & =924.98 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 7.17 A tram car weight 120 kN , the tractive resistance being $5 \mathrm{~N} / \mathrm{kN}$. What power will be required to propel the car at a uniform speed of 20 kmph ?
(a) on level surface,
(b) up an incline of 1 in 300, and
(c) down an inclination of 1 in 300 ?

Take efficiency of motor as $80 \%$.


Fig. 7.23
Solution. Figures 7.23 (a), (b) and (c) show the free body diagram of the locomotive in the three cases given.

In all the cases, Frictional resistance

$$
F=5 \mathrm{~N} / \mathrm{kN}=5 \times 120=600 \mathrm{~N}=0.6 \mathrm{kN}
$$

The locomotive is moving with uniform velocity. Hence it is in equilibrium.

Now,

$$
v=20 \mathrm{kmph}=\frac{20 \times 1000}{60 \times 60}=5.556 \mathrm{~m} / \mathrm{s}
$$

(a) On level track

$$
\begin{aligned}
P & =F=0.6 \mathrm{kN} \\
\therefore \quad \text { Output power } & =P v \\
& =0.6 \times 5.556 \\
& =3.333 \mathrm{~kW} \\
\eta & =80 \%=0.8
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad \text { Input power } & =\frac{\text { Output power }}{\eta} \\
& =\frac{3.3333}{0.8} \\
& =4.167 \mathrm{~kW} .
\end{aligned}
$$

(b) Up the plane

The component of weight $W \sin \theta$ acts down the plane and

$$
\sin \theta \approx \tan \theta=\frac{1}{300}
$$

$$
\therefore \quad P=F+W \sin \theta=0.6+120 \times \frac{1}{300}
$$

$$
=1 \mathrm{kN}
$$

Output power required $=P \times v$

$$
=1 \times 5.5556
$$

$$
=5.5556 \mathrm{~kW}
$$

$\therefore \quad$ Input power of engine $=\frac{\text { Output power }}{\eta}$

$$
\begin{aligned}
& =\frac{5.5556}{0.8} \\
& =6.94 \mathrm{~kW} .
\end{aligned}
$$

Ans.
(c) Down the incline plane

Referring to Fig. 7.23 (c), we have

$$
\begin{aligned}
P & =F-W \sin \theta=0.6-120 \times \frac{1}{300}=0.2 \\
\therefore \quad \text { Output power } & =0.2 \times 5.5556 \\
& =1.1111 \mathrm{~kW} \\
\text { Input power } & =\frac{1.1111}{0.8}=1.389 \mathrm{~kW} .
\end{aligned}
$$

Ans.

Example 7.18 In a police investigation of tyre marks, it was concluded that a car while in motion along a straight level road skidded for a total of 60 metres after the brakes were applied. If the coefficient of friction between the tyres and the pavement is estimated as 0.5 , what was the probable speed of the car just before the brakes were applied?
Solution. Let the probable speed of the car just before brakes were applied be $u$ $\mathrm{m} / \mathrm{s}$. Free body diagram of the car is shown in Fig. 7.24.


Fig. 7.24

$$
\begin{array}{rlrl} 
& \text { Now, } & \Sigma V & =0, \text { gives } \\
N & =W \\
\therefore \quad & F & =\mu N=\mu W=0.5 \mathrm{~W} \tag{1}
\end{array}
$$

Only force in the direction of motion is $F$.
Now, $\quad$ Final velocity $=0$
Displacement, $s=60 \mathrm{~m}$
Applying work-energy equation, we have

$$
\begin{aligned}
-F \times s & =\frac{W}{2 g}\left(v^{2}-u^{2}\right) \\
-0.5 W \times 60 & =\frac{W}{2 \times 9.81}\left(0-u^{2}\right) \\
\therefore \quad u & =24.261 \mathrm{~m} / \mathrm{s} \\
& =\frac{24.261 \times 60 \times 60}{1000}=87.34 \mathrm{kmph} . \quad \text { Ans. }
\end{aligned}
$$

Example 7.19 A block weighing 2500 N rests on a level horizontal plane, for which coefficient of friction is 0.20 . This block is pulled by a force of 1000 N acting at an angle of $30^{\circ}$ to the horizontal. Find the velocity of the block after it moves 30 m starting from rest. If the force of 1000 N is then removed, how much further will it move? Use work-energy method.
Solution. Free body diagrams of the block for the two cases are shown in Figs. 7.25 (a) and (b).


Fig. 7.25

When pull $P$ is acting, we have

$$
\begin{aligned}
N & =W-P \sin 30^{\circ} \\
& =2500-1000 \sin 30^{\circ}=2000 \text { Newton } \\
F & =\mu N=0.2 \times 2000 \\
& =400 \text { Newton } \\
\text { Initial velocity } & =0
\end{aligned}
$$

Let final velocity be $v$.
Displacement $=s=30 \mathrm{~m}$
Applying work-energy equation for the horizontal motion, we have

$$
\begin{aligned}
\left(P \cos 30^{\circ}-F\right) s & =\frac{W}{2 g}\left(v^{2}-u^{2}\right) \\
(0.866 \times 1000-400) 30 & =\frac{2500}{2 \times 9.81}\left(v^{2}-0\right) \\
v & =10.4745 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Ans.
Now, if the force 1000 N is removed, let the distance moved be ' $s$ ', before the body comes to rest.
$\therefore \quad$ Initial velocity $=10.4745 \mathrm{~m} / \mathrm{s}$.
Final velocity $=0$
Applying work-energy equation for the motion in horizontal direction, we get

$$
\begin{aligned}
-F \times s & =\frac{W}{2 g}\left(v^{2}-u^{2}\right) \\
-400 \times s & =\frac{2500}{2 \times 9.81}\left(0-10.4745^{2}\right) \\
s & =22.37 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 7.20 A small block starts from rest at point A and slides down the inclined plane as shown in Fig. 7.26 (a). What distance along the horizontal plane will it travel before coming to rest? The coefficient of kinetic friction between the block and either plane is 0.3 . Assume that the initial velocity with which it starts to move along $B C$ is of the same magnitude as that gained in sliding from $A$ to $B$.


Fig. 7.26

Length

$$
A B=\sqrt{3^{2}+4^{2}}=5 \mathrm{~m}
$$

$$
\therefore \quad \sin \theta=0.6 \text { and } \cos \theta=0.8
$$

Consider the free body diagram of the block on inclined plane at $A$ [Fig. 7.26 (b)]. It moves down the plane. Hence

Sum of forces normal to plane $=0$, gives

$$
\begin{aligned}
N_{1} & =W \cos \theta \\
& =W \times 0.8
\end{aligned}
$$

$$
\therefore \quad \quad F_{1}=\mu N_{1}=0.3 \mathrm{~W} \times 0.8=0.24 \mathrm{~W} .
$$

Applying work-energy equation for the motion from $A$ to $B$, velocity at $B$, $v_{B}$ is given by

$$
\begin{aligned}
\left(W \sin \theta-F_{1}\right) s & =\frac{W}{2 g}\left(v_{B}^{2}-0\right) \\
(0.6 W-0.24 W) \times 5 & =\frac{W}{2 \times 9.81}\left(v_{B}^{2}\right) \\
v_{B} & =5.943 \mathrm{~m} / \mathrm{s} .
\end{aligned}
$$

For the motion on horizontal plane, free body diagram of the block is shown in Fig. 7.26 (c).

$$
\begin{aligned}
\text { Initial velocity } & =v_{B}=5.943 \mathrm{~m} / \mathrm{s} \\
\text { Final velocity } & =0
\end{aligned}
$$

Writing work-energy equation for the motion along $B C$, we have

$$
-F_{2} s=\frac{W}{2 g}\left(0-v_{B}^{2}\right)
$$

but

$$
F_{2}=\mu N=\mu W=0.3 \mathrm{~W}
$$

or

$$
-0.3 W s=-\frac{W}{2 \times 9.81}(5.943)^{2}
$$

Hence,

$$
s=6 \mathrm{~m} .
$$

Ans.

## Motion of Connected Bodies

Work-energy equation may be applied to the connected bodies also. There is no need to separate connected bodies and work out for forces in the connecting member. Note down the various forces acting on connected bodies. Equate summation of work done by forces acting on bodies to the summation of change in kinetic energy of the bodies. While writing work done, note that force components in the direction of motion are to be multiplied by the distance moved.

Consider the two connected bodies as shown in Fig. 7.27.


Fig. 7.27
Under the pull $P$, both bodies move the same distance and with the same velocity. Hence, initial velocity, final velocity and displacement are the same for the two bodies. Out of the three forces $W_{1}, N_{1}$ and $F_{1}$ acting on first body, only the frictional force $F_{1}$ will do the work. Among the various forces acting on the second body, the applied force $P$, frictional force $F_{2}$ and the down the plane component of weight, $W_{2} \sin \theta$ will do the work. Hence the work-energy equation for that system will be given as

$$
\begin{aligned}
& \quad-F_{1} s+\left(P-F_{2}-W_{2} \sin \theta\right) s=\frac{W_{1}}{2 g}\left(v^{2}-u^{2}\right)+\frac{W_{2}}{2 g}\left(v^{2}-u^{2}\right) \\
& \text { i.e., } \quad\left(-F_{1}+P-F_{2}-W_{2} \sin \theta\right) s=\frac{W_{1}+W_{2}}{2 g}\left(v^{2}-u^{2}\right) .
\end{aligned}
$$

Example 7.21 Determine the constant force $P$ that will give the system of bodies shown in Fig. 7.28(a), a velocity of $3 \mathrm{~m} / \mathrm{s}$ after moving 4.5 m from rest. Coefficient of friction between the blocks and the plane is 0.3 . Pulleys are smooth.


Fig. 7.28

Solution. The system of forces acting on connected bodies is shown in Fig. 7.28 (b).

$$
\begin{array}{ll}
\therefore \quad & N_{1}=250 \mathrm{~N} \\
F_{1} & =\mu N_{1}=0.3 \times 250=75 \mathrm{~N} \\
& N_{2}=1000 \cos \theta=1000 \times \frac{3}{5}=600 \mathrm{~N} \\
\therefore \quad & F_{2}=0.3 N_{2}=0.3 \times 600 \\
& =180 \mathrm{~N} \\
& N_{3}=500 \mathrm{~N} \\
\therefore \quad & F_{3}=\mu N_{3}=0.3 \times 500=150 \mathrm{~N}
\end{array}
$$

Let the constant force be $P$. Writing work-energy equation, we have

$$
\begin{aligned}
& \left(P-F_{1}-F_{2}-1000 \sin \theta-F_{3}\right) s=\frac{W_{1}+W_{2}+W_{3}}{2 g}\left(v^{2}-u^{2}\right) \\
& (P-75-180-1000 \times 0.8-150) 4.5=\frac{250+1000+500}{2 \times 9.81}\left(3^{2}-0\right)
\end{aligned}
$$

$$
P=1383.39 \mathrm{~N}
$$

Ans.
Note: Work done is force $\times$ distance moved in the direction of force. Hence, $N_{1}, N_{2}, N_{3}, 250,1000 \cos \theta$ and 500 forces are not contributing to the work done.

Example 7.22 In what distance will body $A$ of Fig. 7.29 attain a velocity of $3 \mathrm{~m} / \mathrm{s}$, starting from rest? Take $\mu=0.2$. Pulleys are frictionless and weightless. Solution. Let $\theta_{1}$ and $\theta_{2}$ be the slopes of inclined planes.

Then

$$
\begin{aligned}
& \sin \theta_{1}=\frac{4}{5}=0.8, \cos \theta_{1}=0.6 \\
& \sin \theta_{2}=\frac{3}{5}=0.6, \cos \theta_{2}=0.8
\end{aligned}
$$



Fig. 7.29
By observing pulley system, it may be concluded that if 1500 N body moves a distance ' $s$ ', 2000 N body moves a distance $0.5 s$, and if velocity of 1500 N
block is $v$, then that of 2000 N block will be $0.5 v$. Assuming 2000 N body moves up the plane and 1500 N body moves down the plane, the forces acting on 1500 N that will do work are:

$$
1500 \sin \theta_{1}=1200 \mathrm{~N} \text { down the plane }
$$

and $\quad F_{1}=\mu \times 1500 \cos \theta_{1}=0.2 \times 1500 \times 0.6=180 \mathrm{~N}$ up the plane
The forces acting on 2000 N block that do work when it slides up are:
$2000 \sin \theta_{2}=2000 \times 0.6=1200 \mathrm{~N}$ down the plane
and $\quad F_{2}=0.2 \times 2000 \cos \theta_{2}=0.2 \times 2000 \times 0.8$ $=320 \mathrm{~N}$ down the plane.
Equating work done by various forces to change in the kinetic energy of the system, we get
$(1200-180) s-(1200+320) 0.5 s$

$$
=\left\{\frac{1500}{2 \times 9.81}\right\}\left(v^{2}-0\right)+\frac{2000}{2 \times 9.81}\left\{(0.5 v)^{2}-0\right\}
$$

In the present case,

$$
v=3 \mathrm{~m} / \mathrm{s} .
$$

$$
\begin{aligned}
250 s & =\frac{1500}{2 \times 9.81} \times 3^{2}+\frac{2000}{2 \times 9.81} \times 1.5^{2} \\
s & =3.7 \mathrm{~m}
\end{aligned}
$$

Ans.
Since $s$ is positive, the assumed direction of motion is correct. If it comes out negative, note that recalculations are to be made since the frictional force changes the sign, if motion is reversed.

Example 7.23 Two bodies weighing 300 N and 450 N are hung to the ends of a rope passing over an ideal pulley as shown in Fig. 7.30 (a). How much distance the blocks will move in increasing the velocity of system from $2 \mathrm{~m} / \mathrm{s}$ to $4 \mathrm{~m} / \mathrm{s}$ ? How much is the tension in the string? Use work energy method.


Fig. 7.30
Solution. 450 N block moves down and 300 N block moves up. The arrangement is such that both bodies will be having same velocity and both will move by the
same distance. Let ' $s$ ' be the distance moved. Writing work-energy equation for the system, we get

$$
\begin{aligned}
450 s-300 s & =\left[\frac{450}{2 \times 9.81}\right]\left(v^{2}-u^{2}\right)+\left[\frac{300}{2 \times 9.81}\right]\left(v^{2}-u^{2}\right) \\
150 s & =\left[\frac{450}{2 \times 9.81}\right] \times\left(4^{2}-2^{2}\right)+\left[\frac{300}{2 \times 9.81}\right]\left(4^{2}-2^{2}\right) \\
s & =3.058 \mathrm{~m} .
\end{aligned}
$$

Let $T$ be the tension in the string. Consider work-energy equation for any one body, say 450 N body as shown in Fig. 7.30 (b).

$$
\begin{aligned}
450 s-T s & =\left[\frac{450}{2 \times 9.81}\right]\left(4^{2}-2^{2}\right) \\
(450-T) 3.058 & =\left[\frac{450}{2 \times 9.81}\right] \times 12 \\
T & =360 \mathrm{~N} .
\end{aligned}
$$

Ans.

## Work Done by a Spring

Consider a body attached to a spring as shown in Fig. 7.31. It is obvious that if the body moves out from its undeformed position, tensile force develops in the spring and if it moves towards the supports, compressive force develops. In other words, the force of a spring is always directed towards its normal position. Experimental results have shown that the magnitude of the force developed in the spring is directly proportional to its displacement from the undeformed position. Thus, if $F$ is the force in the spring due to deformation $x$ from its undeformed position
i.e.,

$$
\begin{align*}
& F \propto x \\
& F=k x \tag{7.9}
\end{align*}
$$



Fig. 7.31
where the constant of proportionality $k$ is called spring constant and is defined as a force required for unit deformation of the spring. Hence the unit of spring constant is $\mathrm{N} / \mathrm{m}$ or $\mathrm{kN} / \mathrm{m}$.

At any instant if the displacement is $d x$, the work done by spring force $d U$ is given by

$$
d U=-F d x=-k x d x
$$

Note: The negative sign is used since the force of spring is in the opposite direction of displacement.
$\therefore$ Work done in the displacement of the body from $x_{1}$ to $x_{2}$ is given by

$$
\begin{aligned}
U & =\int_{x_{1}}^{x_{2}}-k x d x=-k\left[\frac{x^{2}}{2}\right]_{x_{1}}^{x_{2}} \\
& =-\frac{1}{2} k\left(x_{2}^{2}-x_{1}^{2}\right)
\end{aligned}
$$

If the work done is to be found while moving from undeformed position to displacement $x$, then

$$
\begin{align*}
U & =-\frac{1}{2} k\left(x^{2}-0^{2}\right) \\
& =-\frac{1}{2} k x^{2} \tag{7.10}
\end{align*}
$$

Note: The negative sign is used with the expression for work done by the spring, since whenever spring is deformed, the force of spring is in the opposite direction of deformation. However, if a deformed spring is allowed to move towards its normal position, work done will be positive, since the movement and the force of spring are in the same direction.

Example 7.24 A 3000 N block starting from rest as shown in Fig. 7.32 slides down a $50^{\circ}$ incline. After moving 2 m , it strikes a spring whose stiffness is 20 $\mathrm{N} / \mathrm{mm}$. If the coefficient of friction between the block and the incline is 0.2 , determine the maximum deformation of the spring and the maximum velocity of the block.

Solution. Normal reaction

$$
\begin{aligned}
N & =3000 \cos 50^{\circ} \\
F & =\mu N=0.2 \times 3000 \cos 50^{\circ} \\
& =385.67 \mathrm{~N}
\end{aligned}
$$

$\therefore \quad$ Frictional force

Let the maximum deformation of spring be $s \mathrm{~mm}$. The body was at rest and is again at rest, when it moves a distance $(2000+s)$ milli metres. Applying work-energy equation, we have

$$
\begin{aligned}
& \left(3000 \sin 50^{\circ}-F\right)(2000+s)-\frac{1}{2} k s^{2}=0 \\
& \left(3000 \sin 50^{\circ}-385.67\right)(2000+s)=\frac{1}{2} \times 20 \times s^{2}
\end{aligned}
$$



Fig. 7.32

$$
191.246(2000+s)=s^{2}
$$

Solving the quadratic equation, we get

$$
s=721.43 \mathrm{~mm} .
$$

Ans.
The velocity will be maximum when acceleration $\frac{d v}{d t}$ is zero. Acceleration is zero, when force is zero. Net force acting on the body is zero when spring force developed balances the force exerted by the body. Let $x$ be the deformation when the net force on the body in the direction of motion is zero. Then referring to Fig. 7.33,

$$
\begin{aligned}
k x & =W \sin \theta-F \\
20 x & =3000 \sin 50^{\circ}-385.67 \\
x & =95.62 \mathrm{~mm}
\end{aligned}
$$

Now applying work-energy equations, we have


Fig. 7.33

$$
\begin{aligned}
&\left(3000 \sin 50^{\circ}-385.67\right)(2000+x)-\frac{1}{2} k x^{2}=\frac{3000}{2 \times 9810}\left(v^{2}-0\right) \\
&\left(3000 \sin 50^{\circ}-385.67\right)(2000+95.62)-\frac{1}{2} \times 20 \times 95.62^{2} \\
&=\left[\frac{3000}{2 \times 9810}\right] v^{2} \\
& \text { i.e., } \quad v=5060.9 \mathrm{~mm} / \mathrm{s} \\
& v \\
&=5.061 \mathrm{~m} / \mathrm{s} .
\end{aligned}
$$

Ans.
Note: Maintain the consistency of units

$$
\begin{aligned}
g & =9.81 \mathrm{~m} / \mathrm{s} \\
& =9810 \mathrm{~mm} / \mathrm{s} .
\end{aligned}
$$

Example 7.25 A wagon weighing 500 kN starts from rest, runs 30 metre down one per cent grade and strikes the bumper post. If the rolling resistance of the track is $5 \mathrm{~N} / \mathrm{kN}$, find the velocity of the wagon when it strikes the post.

If the bumper spring which compresses 1 mm for every 15 kN , determine by how much this spring will be compressed.

Solution. Component of weight down the plane

$$
\begin{aligned}
& =W \sin \theta=500 \times \frac{1}{100}=5 \mathrm{kN} \\
\text { Track resistance } & =5 \mathrm{~N} / \mathrm{kN} \\
& =5 \times 500=2500 \mathrm{~N} \\
& =2.5 \mathrm{kN}
\end{aligned}
$$



Fig. 7.34
The wagon starts from rest $(u=0)$, and moves a distance $s=30 \mathrm{~m}$ before striking the bumper.

Let the velocity of wagon while striking be $v \mathrm{~m} / \mathrm{s}$.
Applying work-energy equation (Ref. Fig. 7.34), we get

$$
\begin{aligned}
(W \sin \theta-F) s & =\frac{W}{2 g}\left(v^{2}-u^{2}\right) \\
(5-2.5) \times 30 & =\frac{500}{2 \times 9.81}\left(v^{2}-0\right) \\
v & =1.716 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Let spring compression be $x$.
The spring constant is

$$
\begin{aligned}
k & =15 \mathrm{kN} / \mathrm{mm} \\
& =15000 \mathrm{kN} / \mathrm{m}
\end{aligned}
$$

and velocity is zero.
Applying work-energy equation, we get
$(5.0-2.5)(30+x)-\frac{1}{2} k x^{2}=\frac{W}{2 g}(0-0)$
$2.5(30+x)-\frac{1}{2} \times 15000 x^{2}=0$

$$
3000 x^{2}-x-30=0
$$

$x=\frac{+1 \pm \sqrt{1+4 \times 3000 \times 30}}{2 \times 3000}$

$$
=0.1002 \mathrm{~m}
$$

i.e.,

$$
x=100.2 \mathrm{~mm} .
$$

Ans.
Note: Compared to work done by wagon in moving 30 m , if the work done in moving through distance $x$ is neglected, then work-energy equation reduces to

$$
\begin{aligned}
(5-2.5) 30-\frac{1}{2} k x^{2} & =0 \\
x & =\sqrt{\frac{2.5 \times 30 \times 2}{15000}}=0.1 \mathrm{~m} \\
\text { i.e., } \quad x & =100 \mathrm{~mm} .
\end{aligned}
$$

Ans.

### 7.3 IMPULSE MOMENTUM

It is clear from the discussion of the previous chapters that for solving kinetic problems, involving force and acceleration, D'Alembert's principle is useful and that for the problems involving force, velocity and displacement, the work-energy method is useful. In this chapter, the impulse momentum method is dealt which is useful for solving the problems involving force, time and velocity.

If $R$ is the resultant force acting on a body of mass $m$, then from Newton's second law,

$$
\begin{array}{lrl} 
& R & =m a \\
\text { but acceleration } & a & =\frac{d v}{d t} \\
\therefore & R & =m \frac{d v}{d t} \\
\text { i.e., } & R d t & =m d v \\
\therefore & \int R d t & =\int m d v
\end{array}
$$

If initial velocity is $u$ and after time interval $t$ it becomes $v$, then

$$
\begin{align*}
\int_{0}^{t} R d t & =m[v]_{u}^{v} \\
& =m v-m u \tag{7.11}
\end{align*}
$$

The term $\int_{0}^{t} R \cdot d t$ is called impulse. If the resultant force is in Newton and time is in second, the unit of impulse will be N -s.

If $R$ is constant during time interval $t$, then impulse is equal to $R \times t$.
The term mass $\times$ velocity is called momentum. Substituting dimensional equivalence, we get

$$
\begin{aligned}
m v & =\frac{W}{g} v=\frac{\mathrm{N}}{\mathrm{~m} / \mathrm{s}^{2}} \mathrm{~m} / \mathrm{s} \\
& =\mathrm{N}-\mathrm{s}
\end{aligned}
$$

Thus the momentum has also unit N -s. Equation (2) satisfies the requirement of dimensional homogenity. Equation (2) can now be expressed as

$$
\begin{equation*}
\text { Impulse }=\text { Final momentum }- \text { Initial momentum } \tag{7.12}
\end{equation*}
$$

Since the velocity is a vector quantity, impulse is also a vector quantity. The impulse momentum equation (Eqn. 7.11 or 7.12) holds good when the directions of $R, u$ and $v$ are the same. Impulse momentum equation can be stated as follows:

The component of the resultant linear impulse along any direction is equal to change in the component of momentum in that direction.

The impulse momentum equation can be applied in any convenient direction and the kinetic problems involving force, velocity and time can be solved easily.

Example 7.26 A glass marble, whose weight is 0.2 N , falls from a height of 10 m and rebounds to a height of 8 metres. Find the impulse and the average force between the marble and the floor if the time during which they are in contact is 1 / 10 of a second.
Solution. Applying kinematic equations, for the freely falling body, the velocity with which marble strikes the floor

$$
\begin{align*}
& =\sqrt{2 g h} \\
& =\sqrt{2 \times 9.81 \times 10} \\
& =14.007 \mathrm{~m} / \mathrm{s} \quad \text { (downward) } \tag{1}
\end{align*}
$$

Similarly applying kinematic equations for the marble moving up, we get the velocity of rebound

$$
\begin{aligned}
& =\sqrt{2 g \times 8}=\sqrt{2 \times 9.81 \times 8} \\
& =12.528 \mathrm{~m} / \mathrm{s} \quad \text { (upward) }
\end{aligned}
$$



Fig. 7.35

Taking upward direction as positive and applying impulse momentum equation, we get

$$
\begin{aligned}
\text { Impulse } & =\frac{W}{g}(v-u) \\
& =\frac{0.2}{9.81}[12.52-(-14.007)] \\
& =0.541 \mathrm{~N}-\mathrm{s} .
\end{aligned}
$$

Ans.
If $F$ is the average force, then

$$
\begin{aligned}
F t & =0.541 \\
F \times 1 / 10 & =0.541 \mathrm{~N} \\
F & =5.41 \mathrm{~N} .
\end{aligned}
$$

Ans.
Example 7.27 A 1 N ball is bowled to a batsman. The velocity of ball was $20 \mathrm{~m} / \mathrm{s}$ horizontally just before batsman hits it. After hitting, it went away with a velocity of $48 \mathrm{~m} / \mathrm{s}$ at an inclination of $30^{\circ}$ to horizontal as shown in Fig. 7.36 (a). Find the average force exerted on the ball by the bat if the impact lasts for 0.02 s .


Fig. 7.36

Solution. Let $P_{x}$ be the horizontal component of the force and $P_{y}$ be the vertical component. Applying impulse momentum condition in horizontal direction, we have

$$
\begin{aligned}
P_{x} \times 0.02 & =\frac{1}{9.81}\left[48 \cos 30^{\circ}-(-20)\right] \\
P_{x} & =313.81 \mathrm{~N}
\end{aligned}
$$

Applying impulse momentum equation in vertical direction, we have

$$
\begin{aligned}
P_{y} \times 0.02 & =\frac{1}{9.81}\left(48 \sin 30^{\circ}-0\right) \\
\therefore \quad P_{y} & =122.32 \mathrm{~N}
\end{aligned}
$$

$\therefore$ Resultant force

$$
\begin{aligned}
& P=\sqrt{P_{x}^{2}+P_{y}^{2}}=\sqrt{313.81^{2}+122.32^{2}} \\
& P=336.81 \mathrm{~N} \\
& \theta=\tan ^{-1}\left(\frac{P_{y}}{P_{x}}\right)=\tan ^{-1}\left(\frac{122.32}{313.81}\right)
\end{aligned}
$$

Ans.

Ans.
i.e., $\theta=21.30^{\circ}$ to horizontal as shown in Fig. 7.36(b).

Example 7.28 A 1500 N block is in contact with a level plane, the coefficient of friction between two contact surfaces being 0.1 . If the block is acted upon by a horizontal force of 300 N , what time will elapse before the block reaches a velocity of $16 \mathrm{~m} / \mathrm{s}$ starting from rest? If 300 N force is then removed, how much longer will the block continue to move? Solve the problem using impulse momentum equation.
Solution. Consider the FBD of the block as shown in Fig. 7.37.

$$
\text { Normal reaction }=1500 \mathrm{~N}
$$

$\therefore \quad$ Frictional force $F=\mu N=0.1 \times 1500$

$$
=150 \mathrm{~N}
$$

Applying impulse momentum equation in the horizontal direction, we have

$$
\begin{aligned}
(300-150) t & =\left[\frac{1500}{9.81}\right](v-u) \\
& =\left[\frac{1500}{9.81}\right](16-0) \\
t & =16.31 \mathrm{~s}
\end{aligned}
$$



Fig. 7.37

Ans.
If force is then removed the only horizontal force is $F=150 \mathrm{~N}$. Applying impulse momentum equation for the motion towards right, we have

$$
\begin{aligned}
-150 t & =\frac{1500}{9.81}(0-16) \\
t & =16.31 \mathrm{~s}
\end{aligned}
$$

The block takes another 16.31 s before it comes to rest.
Ans.

Example 7.29 A 20 kN automobile is moving at a speed of 70 kmph when the brakes are fully applied causing all four wheels to skid. Determine the time required to stop the automobile ( $a$ ) on concrete road for which $\mu=0.75$, (b) on ice for which $\mu=0.08$.
Solution. Initial velocity of the vehicle

$$
\begin{aligned}
u & =70 \mathrm{kmph}=\frac{70 \times 1000}{60 \times 60} \\
& =19.44 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Final velocity $v=0$.
Free body diagram is shown in Fig. 7.38.

$$
\begin{aligned}
F & =\mu N \\
& =\mu W=20 \mu
\end{aligned}
$$

Applying impulse momentum equation, we have


Fig. 7.38

$$
\begin{aligned}
-F t & =\frac{W}{g}(v-u) \\
-20 \mu t & =\left[\frac{20}{9.81}\right](19.44-0) \\
\therefore \quad t & =\frac{1.982}{\mu}
\end{aligned}
$$

(a) On concrete road, $\mu=0.75$

$$
\therefore \quad t=\frac{1.982}{0.75}=2.64 \mathrm{~s}
$$

Ans.
(b) On ice, $\mu=0.08$
$\therefore \quad t=\frac{1.982}{0.08}=24.78 \mathrm{~s}$.
Ans.

Example 7.30 A block weighing 130 N is on an incline, whose slope is 5 vertical to 12 horizontal. Its initial velocity down the incline is $2.4 \mathrm{~m} / \mathrm{s}$. What will be its velocity 5 s . later? Take coefficient of friction at contact surface $=0.3$.

## Solution.

$$
\begin{aligned}
\tan \theta & =\frac{5}{12}, \quad \theta=22.62^{\circ} \\
N & =W \cos \theta=130 \cos 22.62^{\circ}=120 \text { Newton } \\
F & =\mu N=0.3 \times 120=36 \text { Newton }
\end{aligned}
$$



Fig. 7.39
$\Sigma$ Forces down the plane

$$
\begin{aligned}
& =R=W \sin \theta-F=130 \sin 22.62^{\circ}-36 \\
& =14.0 \mathrm{~N} \\
\text { Initial velocity } \quad u & =2.4 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Let final velocity be $v \mathrm{~m} / \mathrm{s}$
Time interval $t=5 \mathrm{~s}$
Applying impulse momentum equation,

$$
\begin{aligned}
R t & =\frac{W}{g}(v-u), \text { we get } \\
14 \times 5 & =\left[\frac{130}{9.81}\right](v-2.4) \\
\therefore \quad v & =7.68 \mathrm{~m} / \mathrm{s} . \quad \text { Ans. }
\end{aligned}
$$

## Connected Bodies

The problems involving connected bodies may be solved by any one of the following two methods:

First Method: Free body diagrams of each body is drawn separately. Impulse momentum equation for each body in the direction of its motion is written, and then the equations are solved to get the required values.

Second Method: If the connected bodies have same displacement in the same time, the impulse of internal tension in connecting chord's will get cancelled. Hence free body diagram of combined bodies may be considered and impulse moment equation is applied in the direction of motion of combined bodies. This method is applicable only if displacement of each body is the same in given time.

Example 7.31 Determine the time required for the weights shown in Fig. 7.40 (a) to attain a velocity of $9.81 \mathrm{~m} / \mathrm{s}$. What is the tension in the chord? Take $\mu=0.2$ for both planes. Assume the pulleys to be frictionless.

## Solution.

First Method: Free body diagram of the two blocks are as shown in Fig. 7.40(b).

For 2000 N block:

$$
\begin{aligned}
N_{1} & =W_{1} \cos 30^{\circ}=2000 \cos 30^{\circ} \\
& =1732.05 \mathrm{~N} \\
F_{1} & =\mu N_{1}=0.2 \times 1732.05 \\
& =346.41 \mathrm{~N}
\end{aligned}
$$

For 1800 N block:

$$
\begin{aligned}
N_{2} & =W_{2} \cos 60^{\circ}=1800 \cos 60^{\circ} \\
& =900 \mathrm{~N} \\
F_{2} & =\mu N_{2}=0.2 \times 900 \\
& =180 \mathrm{~N}
\end{aligned}
$$


(a)

(b)

(c)

Fig. 7.40
Let $T$ be the tension in the chord.

| Initial velocity | $u$ | $=0$ |
| ---: | :--- | ---: | :--- |
| Final velocity | $v$ | $=9.81 \mathrm{~m} / \mathrm{s}$ |

Applying impulse moment equation for the 2000 N block in upward direction parallel to the plane, we get

$$
\begin{align*}
\left(T-2000 \sin 30^{\circ}-F_{1}\right) t & =\left[\frac{2000}{9.81}\right](v-u) \\
\left(T-2000 \sin 30^{\circ}-346.4\right) t & =2000 \\
(T-1346.41) t & =2000 \tag{1}
\end{align*}
$$

Applying impulse momentum equation for 1800 N block in the direction parallel to $60^{\circ}$ incline plane, we get

$$
\begin{align*}
\left(1800 \sin 60^{\circ}-T-F_{2}\right) t & =\left[\frac{1800}{9.81}\right](v-u) \\
\left(1800 \sin 60^{\circ}-T-180\right) t & =\left[\frac{1800}{9.81}\right](9.81-0) \\
(1378.85-T) t & =1800 \tag{2}
\end{align*}
$$

Dividing Eqn. (1) by Eqn. (2), we get

$$
\begin{array}{rlrl} 
& & \frac{(T-1346.41)}{(1378.85-T)} & =\frac{2000}{1800} \\
\therefore & T & =1363.48 \mathrm{~N}
\end{array}
$$

Ans.
Substituting it in Eqn. (2), we get

$$
\begin{aligned}
\therefore & (1378.85-1363.48) t & =1800 \\
\therefore & t & =117.11 \mathrm{~s}
\end{aligned}
$$

Second Method: Since the displacement of both bodies are same in given time, consider combined FBD of the blocks as shown in Fig. 7.40(c). Writing impulse momentum equation in the direction of motion, we have

$$
\begin{aligned}
& \left(1800 \sin 60^{\circ}-F_{2}-2000 \sin 30^{\circ}-F_{1}\right) t=\frac{2000+1800}{9.81}(\mathrm{v}-u) \\
& \left.1800 \sin 60^{\circ}-180-2000 \sin 30^{\circ}-346.41\right) t=\frac{3800}{9.81} \times 9.81
\end{aligned}
$$

To find tension in the chord, consider the impulse momentum equation of any one block, say 2000 N block.

$$
\begin{gathered}
\left(T-2000 \sin 30^{\circ}-F_{1}\right) t=\frac{2000}{9.81}(v-u) \\
\\
\\
\\
\left(T-2000 \sin 30^{\circ}-346.41\right) 117.11=\frac{2000}{9.81} \times 9.81 \\
\therefore \quad
\end{gathered}
$$

Ans.
Example 7.32 Determine the tension in the strings and the velocity of 1500 N block shown in Fig. 7.41(a) 5 seconds after starting from
(a) rest,
(b) starting with a downward velocity of $3 \mathrm{~m} / \mathrm{s}$.

Assume that pulleys are weightless and frictionless.


Fig. 7.41
Solution. When 1500 N block moves a distance $s$, in the same time 500 N block moves a distance $2 s$. Hence if velocity of 1500 N block is $v \mathrm{~m} / \mathrm{s}$ that of 500 N block will be $2 v \mathrm{~m} / \mathrm{s}$. Let $T$ be the tension in the chord connecting 500 N block.

Hence tension in the wire connecting 1500 N block will be 2T. [see Fig. 7.41(b)]. Since the velocities of two blocks are different only first method is to be used.

Case $a$ : Initial velocity $u=0, t=5 \mathrm{~s}$.
Writing impulse momentum equation for 500 N block, we have

$$
\begin{align*}
(T-500) t & =\frac{500}{9.81}(2 v-u) \\
(T-500) 5 & =\frac{500}{9.81} \times(2 v-0) \\
T-500 & =\frac{200 v}{9.81} \tag{1}
\end{align*}
$$

Applying impulse momentum equation to 1500 N block, we have

$$
\begin{align*}
(1500-2 T) t & =\frac{1500}{9.81}(v-0) \\
(1500-2 T) 5 & =\frac{1500}{9.81} v \\
1500-2 T & =\frac{300 v}{9.81} \tag{2}
\end{align*}
$$

Adding Eqn. (2) in 2 times Eqn. (1), we get

$$
1500-2 \times 500=\frac{400 v}{9.81}+\frac{300 v}{9.81}=\frac{700 v}{9.81}
$$

$\therefore \quad v=7.007 \mathrm{~m} / \mathrm{s}$.
Ans.
Substituting it in Eqn. (1), we get

$$
\left.\begin{array}{rl}
T-500 & =\frac{200}{9.81} \times 7.007 \\
\therefore \quad T & =642.86 \mathrm{~N} \\
\therefore \quad \text { Case } b: \text { Initial velocity } u & =3 \mathrm{~m} / \mathrm{s}
\end{array}\right\} \text { Impulse momentum equation for } 500 \mathrm{~N} \text { body will be }
$$

Ans.

$$
\begin{align*}
\qquad(T-500) 5 & =\frac{500}{9.81}(2 v-3) \\
\text { i.e., } \quad T-500 & =\frac{100(2 v-3)}{9.81} \tag{3}
\end{align*}
$$

Impulse momentum equation for 1500 N body will be

$$
\begin{equation*}
(1500-2 T) 5=\frac{1500}{9.81}(v-3) \tag{4}
\end{equation*}
$$

i.e., $\quad 1500-2 T=\frac{300}{9.81}(v-3)$

Adding Eqn. (4) and 2 times Eqn. (1), we get

$$
\begin{aligned}
1500-1000 & =\frac{100}{9.81}(7 v-15) \\
v & =9.15 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Ans.
Substituting it in Eqn. (3), we get

$$
\begin{array}{rlrl} 
& & T-500 & =\frac{100}{9.81}(2 \times 9.15-3) \\
\therefore & T & =655.96 \mathrm{~N} .
\end{array}
$$

Ans.
Example 7.33 The system shown in Fig. 7.42(a) has a rightward velocity of $3 \mathrm{~m} / \mathrm{s}$. Determine its velocity after 5 seconds. Take $\mu=0.2$ for the surfaces in contact. Assume pulleys to be frictionless.
Solution. Since all bodies have same displacement in given time, consider the combined FBD of the system.

$$
\begin{aligned}
& N_{1}=500 \mathrm{~N} \\
& F_{1}=0.2 \times 500=100 \mathrm{~N} \\
& N_{2}=1000 \cos 30^{\circ}=866.03 \\
& F_{2}=0.2 \times N_{2}=173.2 \mathrm{~N}
\end{aligned}
$$



Fig. 7. 42
Writing impulse momentum equation for whole system, we get

$$
\begin{gathered}
\left(2000-F_{1}-1000 \sin 30^{\circ}-F_{2}\right) t=\frac{2000+500+1000}{9.81}(v-u) \\
\left(2000-100-1000 \sin 30^{\circ}-173.2\right) 5=\left[\frac{3500}{9.81}\right](v-3) \\
v=20.19 \mathrm{~m} / \mathrm{s} .
\end{gathered}
$$

Ans.
Example 7.34 Just before a force $P$ is applied, the system, as shown in Fig. 7.43(a), has a rightward velocity of $4 \mathrm{~m} / \mathrm{s}$. Determine the value of $P$ that will give a leftward velocity of $6 \mathrm{~m} / \mathrm{s}$ in a time interval of 20 second. Take coefficient of friction $=0.2$ and assume the pulley to be ideal.
Solution. When the system is moving rightward, frictional force acts leftwards as shown in Fig. 7.42(b). Force $P$ first brings the system to stationary position,

(a)

(b)

(c)

Fig. 7.43
then starts moving leftward. At this stage, the frictional force acts rightward as shown in Fig. 7.43(c). Let $t_{1}$ be the time required to bring the system to the stationary condition.

$$
\begin{aligned}
& N=1000 \text { Newton } \\
& F=0.2 \times 1000=200 \mathrm{~N}
\end{aligned}
$$

Applying impulse momentum equation for the motion upto stationary condition, we have

$$
\begin{align*}
(400-200-P) t_{1} & =\frac{400+1000}{9.81}(0-4) \\
(P-200) t_{1} & =\frac{5600}{9.81} \tag{1}
\end{align*}
$$

Applying impulse momentum equation for the motion from stationary position to leftward motion, after total time of 20 second, we have

$$
\begin{align*}
(P-F-400)\left(20-t_{1}\right) & =\frac{1000+400}{9.81}(v-0) \\
(P-600)\left(20-t_{1}\right) & =\frac{1400}{9.81} \times 6 \tag{2}
\end{align*}
$$

Simultaneous Eqns. (1) and (2) may be solved to get $t_{1}$ and $P$. Trial and error method may be advantageously used here. Looking at Eqn. (2), the value of $P$ should be more than 600 . Let us take a trial value of $P$ as 700 . From Eqn. (1)

$$
t_{1}=1.142 \mathrm{~s}
$$

Substituting it in Eqn. (2), we get

$$
P=645.41 \mathrm{~N}
$$

Substituting this value of $P$ in Eqn. (1), we get

$$
t_{1}=1.282 \mathrm{~s}
$$

Substituting it in Eqn. (2), we get

$$
P=645.74 \mathrm{~N}
$$

This value is almost same as trial value 646.41 N
Hence $\quad P=645.74 \mathrm{~N}$.
Ans.

## Force of Jet on a Vane

In hydroelectric generating stations, a jet of water is made to impinge on the vanes of turbines and get deflected by a certain angle. During this process, a force is exerted by the jet on the vane and that causes rotation of turbine. This mechanical energy is further converted into electrical energy. The force exerted by the jet on the vane, moving or stationary, can be determined by applying impulse momentum equations. This is illustrated with Examples 7.35 and 7.36.

Example 7.35 A nozzle issues a jet of water 50 mm in diameter, with a velocity of $30 \mathrm{~m} / \mathrm{s}$ which impinges tangentially upon a perfectly smooth and stationary vane, and deflects it through an angle of $30^{\circ}$ without any loss of velocity (see Fig. 7.44). What is the total force exerted by the jet upon the vane?


Fig. 7.44
Solution. Weight of water whose momentum is changed in ' $t$ ' seconds

$$
\begin{aligned}
& =(\pi / A)(0.05)^{2} \times 30 \times 9810 \times t \\
& =577.86 t \text { Newtons }
\end{aligned}
$$

Note: 1 cubic metre of water weighs 9810 Newtons.
Let $P_{x}$ and $P_{y}$ be the components of reactive force of vane.
Applying impulse momentum equation in $x$-direction, we get

$$
\begin{aligned}
-P_{x} t & =\frac{577.86 t}{9.81}\left(30 \cos 30^{\circ}-30\right) \\
P_{x} & =236.75 \mathrm{~N}
\end{aligned}
$$

Applying the impulse momentum equation in $y$-direction

$$
\begin{aligned}
P_{y} t & =\frac{5.77 .86 t}{9.81}\left(30 \sin 30^{\circ}-0\right) \\
P_{y} & =883.58 \mathrm{~N} \\
P & =\sqrt{P_{x}^{2}+P_{y}^{2}}=\sqrt{236.75^{2}+883.58^{2}} \\
& =914.75 \mathrm{~N}
\end{aligned}
$$

Inclination with horizontal is given as

$$
\begin{aligned}
& \theta=\tan ^{-1}\left(\frac{P_{y}}{P_{x}}\right)=\tan ^{-1} \frac{883.58}{236.75} \\
& \theta=75.0^{\circ}
\end{aligned}
$$

Force $P$ shown in Fig. 7.44(a) is the reactive force of the vane. The force of jet is equal and opposite to this force as shown in Fig. 7.45(b).


Fig. 7.45
Example 7.36 In the previous example, if the vane is moving with a velocity of $10 \mathrm{~m} / \mathrm{s}$ towards right, what will be the pressure exerted by the jet?
Solution. Velocity of approach $=30-10=20 \mathrm{~m} / \mathrm{s}$
Weight of water impinging in $t$ seconds

$$
\begin{aligned}
& =(\pi / 4) \times(0.05)^{2} \times 20 \times t \times 9810 \\
& =385.24 t \\
\text { Velocity of departure } & =\text { Velocity of approach } \\
& =30-10=20 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Writing impulse moment equation in $x$-direction, we get

$$
\begin{aligned}
-P_{x} t & =\frac{385.24 t}{9.81}\left(20 \sin 30^{\circ}-0\right) \\
P_{x} & =105.22 \mathrm{~N}
\end{aligned}
$$

Applying impulse moment equation in $y$-direction, we get
i.e.,

$$
\begin{aligned}
P_{y} t & =\frac{385.24 t}{9.81}\left(20 \cos 30^{\circ}-20\right) \\
P_{y} & =392.70 \mathrm{~N} \\
P & =\sqrt{P_{x}^{2}+P_{y}^{2}}=\sqrt{105.22^{2}+392.70^{2}}
\end{aligned}
$$

Its inclination to the horizontal

$$
\begin{aligned}
\theta & =\tan ^{-1}\left(\frac{P_{y}}{P_{x}}\right)=\tan ^{-1}\left(\frac{392.70}{105.22}\right) \\
& =75^{\circ}
\end{aligned}
$$

Reaction of vane $P$ is as shown in Fig. 7.46(a). The pressure exerted by the jet is equal and opposite to it and is as shown in Fig. 7.46(b).


Fig. 7.46

## Conservation of Momentum

In a system, if the resultant force $R$ is zero, the impulse momentum Eqn. (7.12) reduces to final momentum equal to initial momentum. Such situations arises in many cases because the force system consists of only action and reaction on the elements of the system. The resultant force is zero, only when entire system is considered, but not when the free body of each element of the system is considered. When a person jumps off a boat, the action of the person is equal and opposite to the reaction of the boat. Hence the resultant force is zero in the system. If $W_{1}$ is the weight of the person and $W_{2}$ that of the boat, $v$ is velocity of the person and the boat before the person jumps out of the boat and $v_{1}, v_{2}$ are the velocities of person and the boat after jumping, then according to principle of conservation of momentum,

$$
\frac{W_{1}+W_{2}}{g} v=\frac{W_{1}}{g} v_{1}+\frac{W_{2}}{g} v_{2}
$$

Similar equation holds good when we consider the system of a gun and shell. The principle of conservation of momentum may be stated as, the momentum is conserved in a system in which resultant force is zero. In other words, in a system if the resultant force is zero, initial momentum will remain equal to final momentum.

It must be noted that conservation of momentum applies to entire system and not to individual elements of the system.

Example 7.37 A 800 N man, moving horizontally with a velocity of $3 \mathrm{~m} / \mathrm{s}$, jumps off the end of a pier into a 3200 N boat. Determine the horizontal velocity of the boat $(a)$ if it had no initial velocity, and $(b)$ if it was approaching the pier with an initial velocity of $0.9 \mathrm{~m} / \mathrm{s}$.
Solution. Weight of man $W_{1}=800 \mathrm{~N}$
Velocity with which man is running $v=3 \mathrm{~m} / \mathrm{s}$
Weight of the system after man jumps into boat $=800+3200=4000 \mathrm{~N}$.
(a) Initial velocity of boat $=0$

Since the action of the man is equal to the reaction of the boat, the principle of conservation of momentum can be applied to the system consisting of the man and the boat.

$$
\begin{aligned}
\text { Initial momentum } & =\text { Final momentum } \\
\frac{800}{9.81} \times 3+\frac{3200}{9.81} \times 0 & =\frac{4000}{9.81} \times V \\
V & =0.6 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Ans.
(b) Initial velocity of boat $=0.9 \mathrm{~m} / \mathrm{s}$ towards the pier

$$
=-0.9 \mathrm{~m} / \mathrm{s}
$$

Applying principle of conservation of momentum, we get

$$
\begin{array}{rlrl} 
& & \frac{800}{9.81} \times 3+\frac{3200}{9.81}(-0.9) & =\frac{4000}{9.81} V \\
\therefore & V & =-0.12 \mathrm{~m} / \mathrm{s} .
\end{array}
$$

i.e., velocity of boat and man will be $0.12 \mathrm{~m} / \mathrm{s}$ towards the pier.

Ans.
Example 7.38 A car weighing $11,000 \mathrm{~N}$ and running at $10 \mathrm{~m} / \mathrm{s}$ holds three men each weighing 700 N . The men jump off from the back end gaining a relative velocity of $5 \mathrm{~m} / \mathrm{s}$ with the car. Find the speed of the car if the three men jump off
(i) in succession,
(ii) all together

Solution. (i) when three men jump off in succession
Initial velocity $u=10 \mathrm{~m} / \mathrm{s}$
Let the velocity of car when
(a) first man jumps be $v_{1} \mathrm{~m} / \mathrm{s}$
(b) second man jumps be $v_{2} \mathrm{~m} / \mathrm{s}$
(c) third man jumps be $v_{3} \mathrm{~m} / \mathrm{s}$

Velocity of the first man w.r.t. fixed point when he jumps $=v_{1}-5$.
Applying principle of conservation of momentum when the first man jumps, we get

$$
\begin{align*}
(11,000+3 \times 700) 10 & =(11,000+2 \times 700) v_{1}+700\left(v_{1}-5\right) \\
& =(11,000+3 \times 700) v_{1}-700 \times 5 \\
v_{1} & =10+\frac{700 \times 5}{11,000+3 \times 700} \tag{1}
\end{align*}
$$

When the second man jumps :

$$
\begin{align*}
(11,000+2 \times 700) v_{1} & =(11,000+700) v_{2}+700\left(v_{2}-5\right) \\
v_{2} & =v_{1}+\frac{700 \times 5}{11,000+2 \times 700} \tag{2}
\end{align*}
$$

When the third man jumps:

$$
\begin{align*}
(11,000+700) v_{2} & =11,000 v_{3}+700\left(v_{3}-5\right) \\
v_{3} & =v_{2}+\frac{700 \times 5}{11,000+700} \tag{3}
\end{align*}
$$

From (1), (2) and (3), we get

$$
\begin{aligned}
\mathrm{v}_{3} & =10+700 \times 5\left(\frac{1}{11,000+3 \times 700}+\frac{1}{11,000+2 \times 700}+\frac{1}{11,000+700}\right) \\
& =10.849 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

(ii) When three men jump together:

Let the velocity of the car be $v$ when three men jump together. Applying principle of conservation of momentum, we get

$$
\begin{aligned}
(11,000+3 \times 700) 10 & =11,000 v+3 \times 700(v-5) \\
& =(11,000+3 \times 700) v-3 \times 700 \times 5
\end{aligned}
$$

where, $(v-5)$ is the relative velocity of the men when they jump,

$$
\begin{array}{ll}
v=10+\frac{3 \times 700 \times 5}{11,000+3 \times 700} \\
\text { i.e., } \quad v=10.802 \mathrm{~m} / \mathrm{s} .
\end{array}
$$

Ans.
Example 7.39 A car weighing 50 kN and moving at 54 kmph along the main road collides with a lorry of weight 100 kN which emerges at 18 kmph from a cross road at right angles to main road. If the two vehicles lock after collision, what will be the magnitude and direction of the resulting velocity?


Fig. 7.47
Solution. Let the velocity of the vehicles after collision be $v_{x}$ in $x$-direction (along main road) and $v_{y}$ in $y$-direction (along cross road) as shown in Fig. 7.47(a). Applying impulse moment equation along $x$-direction, we get

$$
\begin{aligned}
\frac{50 \times 54}{9.81}+0 & =\frac{(50+100)}{9.81} v_{x} \\
v_{x} & =18 \mathrm{kmph}
\end{aligned}
$$

Applying impulse momentum equation in $y$-direction, we get

$$
\begin{aligned}
0+\frac{100 \times 18}{9.81} & =\frac{(50+100)}{9.81} v_{y} \\
v_{y} & =12 \mathrm{kmph} \\
\therefore \quad \text { Resultant velocity } \quad & =\sqrt{v_{x}^{2}+v_{y}^{2}} \\
& =\sqrt{18^{2}+12^{2}} \\
& =21.63 \mathrm{kmph}
\end{aligned}
$$

Ans.
Its inclination to main road

$$
\begin{aligned}
\theta & =\tan ^{-1}\left(\frac{v_{y}}{v_{x}}\right)=\tan ^{-1} 12 / 18 \\
& =33.69^{\circ}
\end{aligned}
$$

Ans.
As shown in Fig. 7.46(b).
Example 7.40 A gun weighing 300 kN fires a 5 kN projectile with a velocity of $300 \mathrm{~m} / \mathrm{s}$. With what velocity will the gun recoil? If the recoil is overcome by an average force of 600 kN , how far will the gun travel? How long will it take?
Solution. Applying principles of conservation of momentum to the system of gun and the projectile, we get

$$
\begin{aligned}
0 & =300 \times v+5 \times 300 \\
v & =-5 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

i.e., Gun will have a velocity of $5 \mathrm{~m} / \mathrm{s}$ in the direction opposite to that of bullet.

Ans.
Let the gun recoil for a distance ' $s$ '.
Using work-energy equation, we have

$$
\begin{aligned}
-600 \times s & =\frac{300}{2 \times 9.81}\left(0-5^{2}\right) \\
s & =0.637 \mathrm{~m}
\end{aligned}
$$

Ans.
Applying impulse momentum equation to gun, we get

$$
\begin{aligned}
-600 t & =(300 / 9.81)(0-5) \\
t & =0.255 \mathrm{~s} .
\end{aligned}
$$

Ans.
Example 7.41 A bullet weighing 0.3 N is fired horizontally into a body weighing 100 N which is suspended by a string 0.8 m long. Due to this impact, the body swings through an angle of $30^{\circ}$. Find the velocity of the bullet and the loss in the energy of the system.
Solution. Let the velocity of the block be $u$ immediately after bullet strikes it. Applying work-energy equation for the block, we get

$$
-W h=\left(\frac{w}{2 g}\right)\left(v^{2}-u^{2}\right)
$$



Fig. 7.48

$$
-100.3 \times 0.8\left(1-\cos 30^{\circ}\right)=\left[\frac{100.3}{2 \times 9.81}\right]\left(0-u^{2}\right)
$$

Note: Work done by the weight is negative since weight is a force acting downwards, whereas body has moved upwards

$$
u=1.025 \mathrm{~m} / \mathrm{s} .
$$

Let $v$ be the velocity of bullet before striking the block. Applying principle of conservation of momentum to the bullet and block system, we get

$$
\begin{aligned}
\frac{0.3}{9.81} v+0 & =\frac{100+0.3}{9.81} u \\
0.3 v & =100.3 \times 1.025 \\
v & =342.82 \mathrm{~m} / \mathrm{s} .
\end{aligned}
$$

$$
\begin{aligned}
\text { Initial energy of bullet } & =\frac{0.3}{2 \times 9.81}(342.82)^{2} \\
& =1797.03 \mathrm{~J}
\end{aligned}
$$

Energy of the block and bullet system

$$
\begin{aligned}
& =\frac{1}{2} \times \frac{100+0.3}{9.81} 1.025^{2} \\
& =5.37 \mathrm{~J} \\
\therefore \quad \text { Loss of energy } & =1797.03-5.37 \\
& =1791.66 \mathrm{~J} .
\end{aligned}
$$

## Ans.

Example 7.42 A bullet weighs 0.5 N and moving with a velocity of $400 \mathrm{~m} / \mathrm{s}$ hits centrally a 30 N block of wood moving away at $15 \mathrm{~m} / \mathrm{s}$ and gets embedded in it. Find the velocity of the bullet after the impact and the amount of kinetic energy lost.
Solution. Initial momentum of the system $=$ Final momentum

$$
\begin{aligned}
\frac{0.5}{9.81} \times 400+\frac{30}{9.81} \times 15 & =\frac{(30+0.5)}{9.81} v \\
v & =21.31 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$



Fig. 7.49
Kinetic energy lost $=$ Initial K.E. - Final K.E.

$$
\begin{aligned}
& =\left(\frac{1}{2} \times \frac{0.5}{9.81} \times 400^{2}+\frac{1}{2} \times \frac{30}{9.81} \times 15^{2}\right)-\frac{1}{2} \times \frac{30.5}{9.81} \times 21.31^{2} \\
& =3715.47 \mathrm{~J} .
\end{aligned}
$$

## Pile and Pile Hammer

If the safe bearing capacity of the soil is too less, a set of reinforced concrete or steel poles are driven in the soil. Such poles are known as piles. Over the group of piles, concrete cap is cast and on it the structure is built.

The piles are driven by pile hammer. It consists of a movable weight called the hammer (see Fig. 7.50). The hammer is raised to a convenient height $h$ and freely dropped. It is guided to fall over the pile. After the hammer strikes the pile, the hammer and the pile move downward together. The kinetic energy of the pile and the hammer is utilised in doing the work against resistance of the ground and pile gets driven by a distance $s$. By repeated hammering, the pile is driven to required depth. If the distance moved per blow is known, earth resistance can be calculated. A general equation


Fig. 7.50 is derived in Example 7.43 and then two specific examples are solved.

Example 7.43 A pile of weight $W$ is driven vertically through a distance $s$ when a hammer of weight $w$ is dropped from a height $h$. Calculate the average resistance of the ground, the loss of kinetic energy during the impact and the time during which the pile is in motion.
Solution. Initial velocity of hammer $=0$
Distance moved before striking pile $=h \mathrm{~m}$
Gravitational acceleration $=g \mathrm{~m} / \mathrm{s}^{2}$
$\therefore$ Velocity of hammer while striking pile $v$ is given by

$$
\begin{align*}
v^{2}-u^{2} & =2 g h \\
v^{2}-0 & =2 g h \\
v & =\sqrt{2 g h} \tag{1}
\end{align*}
$$

After striking, the hammer and the pile move together. Hence the momentum is conserved. Applying the principle of conservation of momentum, we have

$$
\frac{w v}{g}=\frac{(w+W)}{g} V
$$

where, $V$ is the velocity of hammer and the pile immediately after the strike.

$$
\begin{equation*}
V=\frac{w}{w+W} v \tag{2}
\end{equation*}
$$

With this initial velocity, the pile and hammer start moving downwards and they stop moving after a distance $s$. During this process, work is done against the resistance $R$ of the earth. Applying work-energy equation, we have

$$
\begin{align*}
(w+W) s-R \times s & =\frac{1}{2} \frac{w+W}{g}\left(0-V^{2}\right) \\
R s & =(w+W) s+\frac{w+W}{2 g} V^{2} \tag{3}
\end{align*}
$$

From (2)

$$
\left.\begin{array}{rl}
V & =\frac{w}{w+W} v \\
\therefore & R s
\end{array}\right)
$$

$\therefore \quad R s=(w+W) s+\frac{w^{2}}{2 g(w+W)} 2 g h$

$$
=(w+W) s+\frac{w^{2}}{w+W} h
$$

or

$$
R=w+W+\frac{w^{2}}{w+W} \frac{h}{s}
$$

Ans.

## Loss of Kinetic Energy during the Impact :

$$
\begin{aligned}
& =\frac{w}{2 g} v^{2}-\frac{w+W}{2 g} V^{2} \\
& =\frac{w}{2 g} v^{2}-\frac{w+W}{2 g}\left(\frac{w}{w+W}\right)^{2} v^{2}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{v^{2}}{2 g}\left\{w-\frac{w^{2}}{w+W}\right\} \\
& =\frac{2 g h}{2 g} w\left\{1-\frac{w}{w+W}\right\} \\
& =w h \frac{W}{w+W}
\end{aligned}
$$

$$
\text { Loss of KE }=\frac{w W h}{w+W}
$$

Ans.

## Time during which the pile is in motion :

Let $t$ be the time during which the pile is in motion. Applying impulse momentum equation

$$
\begin{aligned}
& {[(w+W)-R] t=\frac{(w+W)}{g}(0-V)} \\
& {[R-(w+W)] t=\frac{w+W}{g} V}
\end{aligned}
$$

Substituting for $R$ and V , we get

$$
\begin{aligned}
\frac{w^{2}}{w+W} \frac{h}{s} t & =\frac{w+W}{g} \frac{w}{w+W} v \\
& =\frac{w}{g} \sqrt{2 g h} \\
t & =\frac{w+W}{w g h} s \sqrt{2 g h}=\frac{w+W}{w} s \sqrt{2 / g h}
\end{aligned}
$$

Ans.

Example 7.44 A pile hammer weighing 20 kN drops from a height of 750 mm on a pile of 10 kN . The pile penetrates 100 mm per blow. Assuming that the motion of the pile is resisted by a constant force, find the resistance of the ground against penetration.
Solution. Initial velocity of hammer $\quad u=0$
Distance moved $=h=750 \mathrm{~mm}=0.75 \mathrm{~m}$
Acceleration $g=9.81 \mathrm{~m} / \mathrm{s}^{2}$
$\therefore \quad$ Velocity at the time of strike $=\sqrt{2 g h}$

$$
=\sqrt{2 \times 9.81 \times 0.75}=3.836 \mathrm{~m} / \mathrm{s}
$$

Applying the principle of conservation of momentum to pile and hammer, we get velocity $V$ of the pile and hammer immediately after the impact,

$$
\frac{20}{9.81} \times 3.836=\frac{20+10}{9.81} V
$$

$$
V=\frac{20}{30} \times 3.836=2.557 \mathrm{~m} / \mathrm{s}
$$

Applying work-energy equation to the motion of the hammer and pile, resistance $R$ of the ground can be obtained.

$$
\begin{aligned}
(20+10-R) s & =\frac{20+10}{2 g}\left(0-V^{2}\right) \\
(30-R) 0.1 & =\frac{30}{2 \times 9.81}\left(-2.557^{2}\right) \\
R & =130 \mathrm{kN} .
\end{aligned}
$$

Ans.
Example 7.45 A pile hammer, weighing 15 kN drops from a height of 600 mm on a pile of 7.5 kN . How deep does a single blow of hammer drive the pile if the resistance of the ground to pile is 140 kN ? Assume that ground resistance is constant.

## Solution.

$$
h=600 \mathrm{~mm}=0.6 \mathrm{~m}
$$

Velocity of hammer at the time of strike

$$
v=\sqrt{2 g h}=\sqrt{2 \times 9.81 \times 0.6}=3.431 \mathrm{~m} / \mathrm{s}
$$

Let $V$ be the velocity of the pile and hammer immediately after impact. Applying principle of conservation of momentum to the system of pile and pile hammer, we get

$$
\begin{aligned}
\frac{15}{9.81} \times 3.431+0 & =\frac{15+7.5}{9.81} V \\
V & =\frac{15 \times 3.431}{22.5}=2.287 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Now applying work-energy equation to the system, we get

$$
\begin{aligned}
(15+7.5-R) s & =\frac{15+7.5}{2 \times 9.81}\left(0-V^{2}\right) \\
(15+7.5-140) s & =\frac{15+7.5}{2 \times 9.81}(-2.287)^{2} \\
s & =\frac{22.5}{2 \times 9.81}\left(+2.287^{2}\right) \frac{1}{117.5} \\
& =0.051 \mathrm{~m} \\
s & =51 \mathrm{~mm}
\end{aligned}
$$

Ans.
Example 7.46 A hammer weighing 5 N is used to drive a nail of weight 0.2 N with a velocity of $5 \mathrm{~m} / \mathrm{s}$ horizontally into a fixed wooden block. If the nail
penetrates by 20 mm per blow, calculate the resistance of the block, which may be assumed uniform.
Solution. Applying impulse momentum equation to the system of hammer and nail, we get

$$
\begin{aligned}
\frac{5}{9.81} \times 5+0 & =\frac{5+0.2}{9.81} V \\
V & =4.808 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$



Fig. 7.51

Applying work-energy equation to the system shown in Fig. 7.51, we get

$$
\begin{aligned}
-R S & =\frac{5.2}{2 \times 9.81}\left(0-4.808^{2}\right) \\
R \times 0.02 & =\frac{5.2}{2 \times 9.81} \times 4.808^{2} \\
R & =306.3 \mathrm{~N}
\end{aligned}
$$

Ans.
Note: Weights of hammer and nail do not do any work, since there is no displacement in the directions of these forces.

### 7.4 IMPACT OF ELASTIC BODIES

A collision between two bodies is said to be impact, if the bodies are in contact for a short interval of time and exert very large force on each other during this short period. On impact, the bodies deform first and then recover due to elastic properties and start moving with different velocities. The velocity with which they separate depends not only on their velocity of approach but also on the shape, size, elastic property and the line of impact. In this book, the velocity of the bodies during the short period of impact is not considered. Only the velocities of colliding bodies before impact and after impact are considered. Some of the new technical terms used in this article are first defined, then the cases of direct impact and oblique impact are analysed. The expression for the loss of kinetic energy on impact is derived.

## Definitions

(i) Line of Impact: Common normal to the colliding surfaces is known as line of impact.
(ii) Direct Impact: If the motion of the two colliding bodies is directed along the line of impact, the impact is said to be direct impact.
(iii) Oblique Impact: If the motion of one or both of the colliding bodies is not directed along the line of impact, the impact is known as oblique impact.
(iv) Central Impact: If the mass centres of colliding bodies are on the line of impact, the impact is called central impact.
(v) Eccentric Impact: Even if mass centre of one of the colliding bodies is not on the line of impact, the impact is called eccentric impact.

These terms are illustrated in Fig. 7.52.

(a) Direct Central Impact

(c) Direct Eccentric Impact

(b) Oblique Central Impact

(d) Oblique Eccentric Impact

Fig. 7.52

## Coefficient of Restitution

During the collision, the colliding bodies initially undergo a deformation for a small time interval and then recover the deformation in a further small time interval. So the period of collision (or time of impact) consists of two time intervals, Period of Deformation and Period of Restitution. "Period of Deformation is the time elapsed between the instant of the initial contact and the instant of maximum deformation of the bodies." Similarly, "Period of Restitution is the time elapsed between the instant of the maximum deformation condition and the instant of separation of the bodies." Therefore,

Impulse during deformation $=F_{D} d t$
where $F_{D}$ refers to the force that acts during the period of deformation.
The magnitude of $F_{D}$ varies from zero at the instant initial contact to the maximum value at the instant of maximum deformation.

Similarly, impulse during restitution $=F_{R} d t$
where $F_{R}$ refer to the force that acts during period of restitution.
The magnitude of $F_{R}$ varies from a maximum value at the instant of maximum deformation condition to zero at the instant of just separation of the bodies.


Fig. 7.53

Let
$m_{1}$ - Mass of the first body
$m_{2}$ - Mass of the second body
$u_{1}$ - Velocity of the first body before impact
$u_{2}$ - Velocity of the second body before impact
$v_{1}$ - Velocity of the first body after impact
and $\quad v_{2}$ — Velocity of the second body after impact
At the instant of maximum deformation, the colliding bodies will have same velocity. Let the velocity of the bodies at the instant of maximum deformation be $U_{D \max }$.

Applying impulse momentum principle for the first body, we have
and

$$
\begin{align*}
& F_{D} d t=m_{1} U_{D \max }-m_{1} u_{1}  \tag{1}\\
& F_{R} d t=m_{1} v_{1}-m_{1} U_{D \max } \tag{2}
\end{align*}
$$

Now, dividing (2) by (1)

$$
\begin{array}{ll}
\frac{F_{R} d t}{F_{D} d t}=\frac{m_{1} v_{1}-m_{1} U_{D \max }}{m_{1} U_{D \max }-m_{1} u_{1}} \\
\text { i.e., } \quad \frac{F_{R} d t}{F_{D} d t}=\frac{v_{1}-U_{D \max }}{U_{D \max }-u_{1}} \tag{3}
\end{array}
$$

Similar analysis for the second body gives,

$$
\begin{equation*}
\frac{F_{R} d t}{F_{D} d t}=\frac{U_{D \max }-v_{2}}{u_{2}-U_{D \max }} \tag{4}
\end{equation*}
$$

From (3) and (4)

$$
\begin{aligned}
\frac{F_{R} d t}{F_{D} d t} & =\frac{v_{1}-U_{D \max }}{U_{D \max }-u_{1}}=\frac{U_{D \max }-v_{2}}{u_{2}-U_{D \max }} \\
& =\frac{v_{1}-U_{D \max }+U_{D \max }-v_{2}}{U_{D \max }-u_{1}+u_{2}-U_{D \max }} \\
& =\frac{v_{1}-v_{2}}{u_{2}-u_{1}}=\frac{v_{2}-v_{1}}{u_{1}-u_{2}} \\
& =\frac{\text { Relative velocity of separation }}{\text { Relative velocity of approach }}
\end{aligned}
$$

Sir Isaac Newton conducted the experiments and observed that when collision of two bodies takes place, relative velocity of separation bears a constant ratio to the relative velocity of approach, the relative velocities being measured along the line of impact. This constant ratio is called as the coefficient of restitution and is denoted by letter $e$. Hence from (5), we have

$$
\begin{equation*}
e=\frac{F_{R} d t}{F_{D} d t}=\frac{v_{2}-v_{1}}{u_{1}-u_{2}} \tag{7.13}
\end{equation*}
$$

The coefficient of restitution of two colliding bodies may be taken as the ratio of impulse during restitution period to the impulse during the deformation period. This is also equal to the ratio of relative velocity of separation to the relative velocity of approach of the colliding bodies, the relative velocities being measured in the line of impact.

## Notes:

(i) The relative velocities are to be considered only along the line of impact,
(ii) Signs of the velocities are to be considered carefully in Eqn. 7.13.

For perfectly elastic bodies, the magnitude of relative velocity after impact will be same as that before impact and hence coefficient of restitution will be 1. Perfectly inelastic bodies cling together and hence the velocity of separation will be zero, i.e., coefficient of restitution will be zero. The coefficient of restitution always lies between 0 and 1 .

The value of coefficient of restitution depends not only on the material property, but it also depends on the shape and size of the body. Hence the coefficient of restitution is the property of two colliding bodies but not merely of material of the colliding bodies.

## Direct Central Impact

Let $u_{1}$ and $u_{2}$ be initial velocities and $v_{1}$ and $v_{2}$ be the velocities after collision of bodies 1 and 2 respectively.

Let $W_{1}$ and $W_{2}$ be the weight of the colliding bodies and $e$ coefficient of restitution, $u_{1}, u_{2}, W_{1}, W_{2}$ and $e$ are known quantities. Velocities after collision, $v_{1}$ and $v_{2}$, are the unknowns. To find these two unknowns, we need two equations. One equation is obtained by the principle of conservation of momentum as,

$$
\begin{align*}
& \frac{W_{1}}{g} u_{1}+\frac{W_{2}}{g} u_{2} & =\frac{W_{1}}{g} v_{1}+\frac{W_{2}}{g} v_{2} \\
\text { i.e., } & W_{1} u_{1}+W_{2} u_{2} & =W_{1} v_{1}+W_{2} v_{2} \tag{7.14}
\end{align*}
$$

Another equation based on the definition of coefficient of restitution may be written as,

$$
\begin{equation*}
e\left(u_{1}-u_{2}\right)=v_{2}-v_{1} \tag{7.15}
\end{equation*}
$$

From Eqns. (7.14) and (7.15) the unknown velocities $v_{1}$ and $v_{2}$ may be found.

Example 7.47 Direct central impact occurs between a 300 N body moving to the right with a velocity of $6 \mathrm{~m} / \mathrm{s}$ and 150 N body moving to the left with a velocity of $10 \mathrm{~m} / \mathrm{s}$. Find the velocity of each body after impact if the coefficient of restitution is 0.8 .
Solution. Referring to Fig. 7.54, initial velocity of 300 N body

$$
u_{1}=6 \mathrm{~m} / \mathrm{s}
$$

Initial velocity of 150 N body, $u_{2}=-10 \mathrm{~m} / \mathrm{s}$


Fig. 7.54
Let final velocity of 300 N body $=v_{1} \mathrm{~m} / \mathrm{s}$ and
Final velocity of 150 N body $=v_{2} \mathrm{~m} / \mathrm{s}$
From principle of conservation of momentum, we have

$$
\begin{align*}
\frac{300}{g} \times 6+\frac{150}{g}(-10) & =\frac{300}{g} v_{1}+\frac{150}{g} v_{2} \\
\text { i.e., } & 2 v_{1}+v_{2}
\end{align*}=2
$$

From the definition of coeff. of restitution, we have
i.e.,

$$
\begin{align*}
e\left(u_{1}-u_{2}\right) & =v_{2}-v_{1} \\
0.8(6+10) & =v_{2}-v_{1} \\
v_{2}-v_{1} & =14.4 \tag{2}
\end{align*}
$$

From (1) and (2), we get

$$
\begin{aligned}
3 v_{1} & =-12.4 \\
v_{1} & =-4.133 \mathrm{~m} / \mathrm{s} \\
v_{2} & =14.4+v_{1}=14.4-4.133 \\
v_{2} & =10.267 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Ans.
and hence
Ans.
Example 7.48 A 80 N body moving to the right at a speed of $3 \mathrm{~m} / \mathrm{s}$ strikes a 10 N body that is moving to the left at a speed of $10 \mathrm{~m} / \mathrm{s}$. The final velocity of 10 N body is $4 \mathrm{~m} / \mathrm{s}$ to the right. Calculate the coefficient of restitution and the final velocity of the 80 N body.
Solution. Here

$$
\begin{array}{ll}
u_{1}=3 \mathrm{~m} / \mathrm{s} & u_{2}=-10 \mathrm{~m} / \mathrm{s} \\
v_{1}=? & v_{2}=4 \mathrm{~m} / \mathrm{s}
\end{array}
$$

Applying the principles of conservation of momentum to the colliding bodies, we get
i.e.,

$$
\begin{aligned}
\frac{80}{g} \times 3+\frac{10}{g}(-10) & =\frac{80}{g} v_{1}+\frac{10}{g} \times 4 \\
v_{1} & =\frac{80 \times 3-100-40}{80}=1.25 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

From the definition of coefficient of restitution, we get

$$
\begin{aligned}
e\left(u_{1}-u_{2}\right) & =v_{2}-v_{1} \\
e(3+10) & =4-1.25 \\
e & =0.212
\end{aligned}
$$

Ans.

Example 7.49 A golf ball is dropped from a height of 10 m on a fixed steel plate. The coefficient of restitution is 0.894 . Find the height to which the ball rebounds on the first, second and third bounces.
Solution. Initial height $h_{0}=10 \mathrm{~m}$
Velocity of golf ball before impact $u_{1}=\sqrt{2 g h_{0}}$
Velocity of steel plate before impact $u_{2}=0$
Velocity of steel plate after impact $v_{2}=0$
Let velocity of golf ball after impact be $v_{1}$.
From the definition of coefficient of restitution, we have

$$
\begin{aligned}
e\left(u_{1}-u_{2}\right) & =v_{2}-v_{1} \\
e\left(\sqrt{2 g h_{0}}-0\right) & =0-v_{1} \\
v_{1} & =-e \sqrt{2 g h_{0}} \\
& =e \sqrt{2 g h_{0}} \quad \text { in upward direction }
\end{aligned}
$$

From kinematic equation, the height $h_{1}$ to which the ball will rise is given by

$$
\begin{aligned}
v_{1}^{2}-0 & =2 g h_{1} \\
h_{1} & =\frac{v_{1}^{2}}{2 g}=\frac{e^{2} \times 2 g h_{0}}{2 g} \\
\text { i.e., } \quad \text { Now, } \quad h_{1} & =e^{2} h_{0} \\
e & =0.894, h_{0}=10 \mathrm{~m} \\
\therefore \quad h_{1} & =0.894^{2} \times 10 \\
& =7.992 \mathrm{~m} .
\end{aligned}
$$

Similarly, after second bounce the height to which the ball will rise is given by

$$
\begin{aligned}
h_{2} & =e^{2} h_{1}=0.894^{2} \times 7.992 \\
& =6.338
\end{aligned}
$$

Ans.
And, after third bounce, the height

$$
\begin{aligned}
& h_{3}=e^{2} h_{2}=0.984^{2} \times 6.388 \\
& h_{3}=5.105 \mathrm{~m}
\end{aligned}
$$

## Ans.

Example 7.50 A ball is dropped from a height of 1 m on a smooth floor. The height of first bounce is 0.810 m .
Determine:
(a) coefficient of the restitution.
(b) expected height of second bounce.

Solution. (a) Velocity of ball before striking the floor

$$
u_{1}=\sqrt{2 g h}=\sqrt{2 \times 9.81 \times 1}=4.429 \mathrm{~m} / \mathrm{s}
$$

Velocity of ball after striking the floor

$$
\begin{aligned}
v_{1} & =-\sqrt{2 g h_{1}}=-\sqrt{2 \times 9.8 \times 0.810} \\
& =-3.987 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

There is no movement of the floor, before and after striking
i.e.,

$$
u_{2}=v_{2}=0
$$

From the definition of coefficient of restitution, we have

$$
\begin{aligned}
e\left(u_{1}-u_{2}\right) & =v_{2}-v_{1} \\
e 4.429 & =0+3.987 \\
e & =\frac{3.987}{4.429}=0.9 .
\end{aligned}
$$

Ans.
(b) Let velocity of the ball after second bounce be $v_{2}$.

Velocity of strike in this case is $v_{1}=3.987 \mathrm{~m} / \mathrm{s}$ downward.
$\therefore \quad e(3.987-0)=0-v_{2}$
$v_{2}=-3.586 \mathrm{~m} / \mathrm{s}$
$=3.586 \mathrm{~m} / \mathrm{s}$ upward
Expected height $h_{2}$ is given by the kinematic equation as

$$
\begin{aligned}
& h_{2}=\frac{v_{2}^{2}}{2 g}=\frac{3.586^{2}}{2 \times 9.81} \\
& h_{2}=0.6561 \mathrm{~m} .
\end{aligned}
$$

Ans.
Example 7.51 A 10 N ball traverses a frictionless tube as shown in Fig. 7.55, falling through a height of 2 m . It then strikes a 20 N ball hung from a rope 1.2 m long. Determine the height to which the hanging ball will rise:
(i) if the collision is perfectly elastic.
(ii) if the coefficient of restitution is 0.7 .

Solution. Velocity of 10 N ball after falling through 2 m height

$$
\begin{aligned}
u_{1} & =\sqrt{2 g h}=\sqrt{2 \times 9.81 \times 2} \\
& =6.264 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$



Fig. 7.55

Velocity of 20 N ball before impact

$$
u_{2}=0
$$

Let the velocities of 10 N and 20 N balls after impact be $v_{1}$ and $v_{2}$, respectively. By principle of conservation of momentum, we have

$$
\begin{align*}
m_{1} u_{1}+m_{2} u_{2} & =m_{1} v_{1}+m_{2} v_{2} \\
\frac{10}{g} 6.264+0 & =\frac{10}{g} v_{1}+\frac{20}{g} v_{2} \\
\text { i.e., } \quad v_{1}+2 v_{2} & =6.264 \tag{1}
\end{align*}
$$

From the definition of coefficient of restitution, we have

$$
\begin{equation*}
e\left(u_{1}-u_{2}\right)=v_{2}-v_{1} \tag{2}
\end{equation*}
$$

Case (i): $e=1$
$\therefore$
i.e.,

$$
\begin{align*}
u_{1}-u_{2} & =v_{2}-v_{1} \\
v_{2}-v_{1} & =6.264 \tag{3}
\end{align*}
$$

From Eqns. (1) and (3), $v_{2}=\frac{2 \times 6.264}{3}=4.176 \mathrm{~m} / \mathrm{s}$.

$$
\begin{aligned}
\therefore \quad v_{1} & =6.264-2 \times 4.176 \\
& =-2.088 \mathrm{~m} / \mathrm{s} .
\end{aligned}
$$

Let $h$ be the height to which hanging ball will rise after impact. Applying work-energy equation, we have

Change in K.E. $=$ Work done

$$
\begin{aligned}
0-\frac{20}{2 g} v_{2}^{2} & =-20 \times h \\
h & =\frac{v_{2}^{2}}{2 g}=\frac{4.176^{2}}{2 \times 9.81} \\
h & =0.889 \mathrm{~m}
\end{aligned}
$$

Ans.
Case (ii): $e=0.7$
From Eqn. (2), we get
i.e.,

$$
\begin{align*}
0.7(6.264) & =v_{2}-v_{1} \\
v_{2}-v_{1} & =4.385 \tag{4}
\end{align*}
$$

From Eqns. (1) and (4), we get

$$
\begin{aligned}
v_{2} & =\frac{6.264+4.385}{3} \\
& =3.55 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

$\therefore$ The height to which ball will rise

$$
\begin{array}{ll} 
& h_{2}=\frac{v_{2}^{2}}{2 g}=\frac{3.55^{2}}{2 \times 9.81} \\
\text { i.e., } & h_{2}=0.642 \mathrm{~m} .
\end{array}
$$

Ans.

## Oblique Impact

Referring to Fig. 7.52 (b) note that, the expression for coefficient of restitution, $e\left(u_{1}-u_{2}\right)=v_{2}-v_{1}$, holds good only for the component of velocities in the line of impact. The component of velocities in a direction at right angles to line of impact remains unaltered.

Example 7.52 The magnitude and direction of the two identical smooth balls before central oblique impact are as shown in Fig. 7.55. Assuming coefficient of restitution $e=0.9$, determine the magnitude and direction of the velocity of each ball after the imapct.
Solution. Component of velocity of A before impact
(i) Normal to line of impact $=9 \sin 30^{\circ}$

$$
u_{A Y}=4.5 \mathrm{~m} / \mathrm{s}
$$

(ii) In the line of impact $=9 \cos 30^{\circ}$

$$
u_{A X}=7.794 \mathrm{~m} / \mathrm{s}
$$

Component of velocity of $B$ before impact:
(a) Normal to line of impact $=12 \sin 60^{\circ}$

$$
u_{B Y}=10.392 \mathrm{~m} / \mathrm{s}
$$




Fig. 7.56
(b) In the line of impact

$$
\begin{aligned}
u_{B X} & =-12 \cos 60^{\circ} \\
& =-6 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Component of velocities only in the line of impact get affected by impact.

$$
\begin{aligned}
v_{A Y} & =u_{A Y}=4.5 \mathrm{~m} / \mathrm{s} \\
v_{B Y} & =u_{B Y}=12 \sin 60^{\circ}=10.392 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Let $v_{A X}$ and $v_{B X}$ be the velocities after impact. Applying principle of conservation of momentum in the line of impact, we get
or

$$
\begin{align*}
7.794-6 & =v_{A X}+v_{B X} \\
v_{A X}+v_{B X} & =1.794 \tag{1}
\end{align*}
$$

From the definition of coefficient of restitution, we obtain

$$
\begin{align*}
0.9\left(u_{A X}-u_{B X}\right) & =v_{B X}-v_{A X} \\
v_{B X}-v_{A X} & =0.9(7.794+6) \\
& =12.415 \tag{2}
\end{align*}
$$

From Eqns. (1) and (2), we get

$$
\begin{aligned}
v_{B X} & =\frac{12.415+1.794}{2}=7.104 \mathrm{~m} / \mathrm{s} \\
v_{A X} & =1.794-7.104 \\
& =-5.310 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$



Fig. 7.57

$$
\begin{aligned}
v_{A} & =\sqrt{5.31^{2}+4.5^{2}}=6.960 \mathrm{~m} / \mathrm{s} \\
\theta_{A} & =\tan ^{-1} \frac{4.5}{5.31}=40.28^{\circ} \text { as shown in Fig. } 7.57 \\
v_{B} & =\sqrt{7.104^{2}+10.392^{2}}=12.588 \mathrm{~m} / \mathrm{s} \\
\text { and } \quad \theta_{B} & =\tan ^{-1}\left(\frac{10.392}{7.104}\right)=55.643 \text { as shown in Fig. } 7.57(\mathrm{~b})
\end{aligned}
$$

Ans.
Ans.

Ans.

Ans.

Example 7.53 A ball is dropped from a height of 3 m upon a $15^{\circ}$ incline. If $e=0.8$, find the resultant velocity of the ball after impact.
Solution. The ball falls freely under gravity from a height of $h_{0}=3 \mathrm{~m}$. Hence its vertical downward velocity at the instance of striking the incline

$$
\begin{aligned}
u_{1} & =\sqrt{2 g h_{0}}=\sqrt{2 \times 9.81 \times 3} \\
& =7.672 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Line of impact is normal to the plane, i.e., at $15^{\circ}$ to vertical. Taking axes $x$ and $y$ as shown in Fig. 7.58, velocity of the ball before striking the plane:
(i) normal to line of impact $u_{1 x}=u_{1} \sin 15^{\circ}=1.986 \mathrm{~m} / \mathrm{s}$
(ii) in the line of impact $u_{1 y}=-u \cos 15^{\circ}=-7.411 \mathrm{~m} / \mathrm{s}$


Fig. 7.58

Let the velocity after impact be $v_{1}$.
In the direction normal to line of impact, the component of velocity is not affected.

$$
v_{1 x}=u_{1 x}=1.986 \mathrm{~m} / \mathrm{s}
$$

Initial and final velocities of floor $=0$
From the definition of coefficient of restitution

$$
\begin{aligned}
0.8(-7.412-0) & =0-v_{1 y} \\
v_{l y} & =5.929 \mathrm{~m} / \mathrm{s} \\
v_{1} & =\sqrt{v_{1 x}^{2}+v_{1 y}^{2}}=\sqrt{1.986^{2}+5.929^{2}} \\
& =6.253 \mathrm{~m} / \mathrm{s} \\
\theta & =\tan ^{-1}\left(\frac{v_{1 x}}{v_{1 y}}\right)=\tan ^{-1}\left(\frac{1.986}{5.929}\right) \\
& =18.52^{\circ} \text { to the line of impact. }
\end{aligned}
$$

Ans.
$\therefore \quad$ Inclination to the plane

$$
=90-18.52=73.48^{\circ} .
$$

Example 7.54 A ball falls vertically for 3 seconds on a plane inclined at $20^{\circ}$ to the horizontal axis. If the coefficient of restitution is 0.8 , when and where will the ball strike the plane again?
Solution. Velocity of ball while striking the plane

$$
=3 \times g=3 g \quad \text { downward }
$$

Component of velocity down the plane

$$
=3 g \sin 20^{\circ}
$$



Fig. 7.59
Component of velocity in the line of impact

$$
u_{y}=-3 g \cos 20^{\circ}
$$

Velocity after the impact $v_{y}=e u_{y}=0.8 \times 3 \mathrm{~g} \cos 20^{\circ}$

$$
=2.4 \mathrm{~g} \cos 20^{\circ}
$$

Acceleration in the line of impact $=-g \cos 20^{\circ}$
Considering the motion normal to the plane and using kinematic equation

$$
s=u t+\frac{1}{2} a t^{2}
$$

we get,

$$
\begin{aligned}
0 & =2.4 \mathrm{~g} \cos 20^{\circ} t-\left(\frac{1}{2}\right) g \cos 20^{\circ} t^{2} \\
t & =4.8 \mathrm{~s}
\end{aligned}
$$

Component of velocity parallel to plane is not affected by the impact, since it is normal to the line of impact.

$$
v_{x}=3 g \sin 20^{\circ}
$$

Acceleration in this direction $=g \sin 20^{\circ}$
$\therefore \quad$ Distance travelled in 4.8 second is given by

$$
\begin{aligned}
s & =3 g \sin 20^{\circ} t-\frac{1}{2} g \sin 20^{\circ} t^{2} \\
& =g \sin 20^{\circ}\left(3 \times 4.8-\frac{1}{2} \times 4.8^{2}\right) \\
& =9.81 \sin 20^{\circ}\left(3 \times 4.8-\frac{1}{2} \times 4.8^{2}\right) \\
s & =86.967 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 7.55 A ball is thrown at an angle $\theta$ with the normal to a smooth wall. It rebounds at an angle $\theta^{\prime}$ with the normal. Show that the coefficient of restitution is expressed by

$$
e=\frac{\tan \theta}{\tan \theta^{\prime}}
$$

Solution. Component normal to line of impact is unaltered after impact. If $u_{1}$ is velocity of approach and $v_{1}$ is the velocity of separation of the ball (Fig. 7.60), we get

$$
u_{1} \sin \theta=v_{1} \sin \theta^{\prime}
$$



Fig. 7.60
or

$$
\begin{equation*}
v_{1}=u_{1} \frac{\sin \theta}{\sin \theta^{\prime}} \tag{1}
\end{equation*}
$$

The velocities of wall before and after impact, i.e., $u_{2}, v_{2}=0$. Hence from the definition of coefficient of restitution,

$$
\begin{equation*}
e\left(u_{1} \cos \theta-0\right)=0-\left(-v_{1} \cos \theta^{\prime}\right)=v_{1} \cos \theta^{\prime} \tag{2}
\end{equation*}
$$

From Eqns. (1) and (2)
or

$$
\begin{aligned}
e u_{1} \cos \theta & =u_{1} \frac{\sin \theta}{\sin \theta^{\prime}} \cos \theta^{\prime} \\
e & =\frac{\tan \theta}{\tan \theta^{\prime}}
\end{aligned}
$$

Example 7.56 A ball is dropped on an inclined plane and is observed to move horizontally after the impact. If the coefficient of restitution is $e$, determine the inclination of the plane and the velocity after impact.
Solution. Let $\theta$ be the inclination of the plane to horizontal. Hence the line of impact is at right angles to the plane. The vertical downward velocity before striking the plane be $u$ and the horizontal velocity after impact be $v$. During impact, component of velocity normal to the line of impact is not altered. Hence

$$
\begin{aligned}
u \sin \theta & =v \cos \theta \\
v & =u \tan \theta
\end{aligned}
$$



Fig. 7.61

Considering the velocities of ball and inclined plane (which is zero) the coefficient of restitution $e$ is given by

$$
\begin{aligned}
e(u \cos \theta-0) & =[0-(-v \sin \theta)] \\
e u \cos \theta & =v \sin \theta
\end{aligned}
$$

Substituting the value of $v$ from (1), we get
or

$$
\begin{aligned}
e u \cos \theta & =u \tan \theta \sin \theta \\
e & =\tan ^{2} \theta
\end{aligned}
$$

or

$$
\tan \theta=\sqrt{e}
$$

## Ans.

From (1) $v=u \tan \theta$
i.e.,

$$
v=u \sqrt{e} .
$$

Ans.
Example 7.57 A ball is dropped from a height $h_{0}=1.2 \mathrm{~m}$ on a smooth floor as shown in Fig. 7.62. Knowing that for the first bounce, $h_{1}=1 \mathrm{~m}$ and $D_{1}=0.4 \mathrm{~m}$, determine:
(a) the coefficient of restitution,
(b) the height and the range of the second bounce.


Fig. 7.62

Solution. Line of impact being vertical, horizontal component of velocity is not affected by impact.

Let this value be $u_{x}$.
The ball is dropped from height of $h_{0}=1.2 \mathrm{~m}$.
Hence vertical component of velocity before first impact

$$
u_{y}=\sqrt{2 g h_{0}} \quad \text { downward }
$$

Let the vertical component of velocity after first impact be $v_{1 y}$.
Since after first bounce, the ball has raised to

$$
\begin{aligned}
h_{1} & =1 \mathrm{~m}, \\
v_{1 y} & =\sqrt{2 g h_{1}} \quad \text { upward. } \\
e & =\frac{\text { Relative velocity of separation }}{\text { Relative velocity of approach }} \\
& =\frac{\sqrt{2 g h_{1}}}{\sqrt{2 g h_{0}}}=\sqrt{h_{1} / h_{0}} \\
\sqrt{1 / 1.2} & =0.913
\end{aligned}
$$

Ans.
Time of flight in first bounce $t_{1}=\frac{2 v_{1 y}}{g}=\frac{2 \sqrt{2 g h_{1}}}{g}=\frac{2 \sqrt{2 \times 9.81 \times 1}}{9.81}$

Range,

$$
\begin{aligned}
& =0.903 \mathrm{~s} \\
D_{1} & =u_{x} t \\
0.4 & =u_{x} \times 0.903 \\
u_{x} & =0.443 \mathrm{~m} / \mathrm{s} .
\end{aligned}
$$

Vertical component of velocity after second bounce,

$$
\left.\begin{array}{rl}
v_{2 y} & =e \sqrt{2 g h_{1}} \\
& =0.903 \sqrt{2 \times 9.81 \times 1} \\
\therefore \quad & =0.4 \mathrm{~m} / \mathrm{s} \\
\therefore \quad & h_{2}
\end{array}\right)=\frac{v_{2 y}^{2}}{2 g}=\frac{4 \times 4}{2 \times 9.81}=0.815 \mathrm{~m}
$$

Ans.

$$
\text { Time of flight, } \quad t_{2}=2 \times \frac{v_{2 y}}{g}=\frac{2 \times 4}{9.81}
$$

$$
\text { Horizontal range } \quad \begin{aligned}
& =0.815 \mathrm{~s} \\
D_{2} & =u_{x} \times t_{2} \\
& =0.443 \times 0.815 \\
& =0.316 \mathrm{~m} .
\end{aligned}
$$

Ans.

## Loss of Kinetic Energy

During collision, the kinetic energy is lost due to imperfect elastic action. Energy is also lost due to
(a) heat generated,
(b) sound generated, and
(c) vibration of colliding bodies.

The loss of kinetic energy can be found by finding kinetic energy before impact and after impact. Let $u_{1}, u_{2}$ be initial velocities and $v_{1}, v_{2}$ be final velocities of two bodies colliding in the line of impact, their weights being $W_{1}$ and $W_{2}$. Then

$$
\begin{aligned}
& \text { Initial K.E. }= \\
& \frac{W_{1}}{2 g} u_{1}^{2}+\frac{W_{2}}{2 g} u_{2}^{2} \\
\therefore \quad \text { Final K.E. } & =\frac{W_{1}}{2 g} v_{1}^{2}+\frac{W_{2}}{2 g} v_{2}^{2} \\
\therefore \quad \text { Loss of K.E. } & =\text { Initial K.E. }- \text { Final K.E. }
\end{aligned}
$$

$$
\begin{aligned}
& =\left(\frac{W_{1}}{2 g} u_{1}^{2}+\frac{W_{2}}{2 g} v_{2}^{2}\right)-\left(\frac{W_{1}}{2 g} v_{1}^{2}+\frac{W_{2}}{2 g} v_{2}^{2}\right) \\
& =\frac{W_{1}}{2 g}\left(u_{1}^{2}-v_{1}^{2}\right)+\frac{W_{2}}{2 g}\left(u_{2}^{2}-v_{2}^{2}\right) .
\end{aligned}
$$

Example 7.58 A sphere weighing 10 N moving at $3 \mathrm{~m} / \mathrm{s}$ collides with another sphere weighing 50 N moving in the same line at $0.6 \mathrm{~m} / \mathrm{s}$. Find the loss of kinetic energy during impact and show that the direction of motion of the first sphere is reversed after the impact. Assume coefficient of restitution as 0.75 .

## Solution.



Fig. 7.63
Let velocities after impact be $v_{1}$ and $v_{2}$.
From principles of conservation of momentum,

$$
\begin{align*}
\frac{10}{g}(3)+\frac{50}{g}(0.6) & =\frac{10}{g} v_{1}+\frac{50}{g} v_{2} \\
v_{1}+5 v_{2} & =6 \tag{1}
\end{align*}
$$

From the definition of coefficient of restitution, we have
or

$$
\begin{align*}
0.75(3-0.6) & =v_{2}-v_{1} \\
v_{2}-v_{1} & =1.8  \tag{2}\\
v_{2} & =1.3 \mathrm{~m} / \mathrm{s} \\
v_{1} & =6-1.3 \times 5=-0.5 \mathrm{~m} / \mathrm{s}
\end{align*}
$$

From (1) and (2),
and
Thus the velocity of first ball is reversed after impact.
Loss of K.E. = Initial K.E. - Final K.E.

$$
\begin{aligned}
& =\frac{10}{g} 3^{2}+\frac{50}{g} 0.6^{2}-\left\{\frac{10}{g} \times(0.5)^{2}+\frac{50}{2 g} \times 1.3^{2}\right\} \\
& =\frac{10}{2 \times 9.81}(9+1.8-0.25-8.45) \\
& =1.07 \text { Joules. }
\end{aligned}
$$

Ans.
Example 7.59 Find the loss of kinetic energy in Example 7.52 if the weight of each ball is 10 N .
Solution. Component of velocities in $y$-direction is not affected. Hence no change in K.E. due to the components of velocities in $y$-direction. In $x$-direction,

$$
\begin{aligned}
u_{A X} & =7.79 \mathrm{~m} / \mathrm{s} . & & u_{B X}=-6 \mathrm{~m} / \mathrm{s} \\
v_{A X} & =-5.31 \mathrm{~m} / \mathrm{s} . & & v_{B X}=7.104 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

$$
\text { Mass of the ball }=\frac{10}{g}=\frac{10}{9.81}
$$

$$
\therefore \quad \text { Loss of K.E. }=\frac{10}{2 \times 9.81}\left[u_{A X}^{2}+u_{B X}^{2}-v_{A X}^{2}-v_{B X}^{2}\right]
$$

$$
=\frac{10}{2 \times 9.81}\left[7.79^{2}+(-6)^{2}-(-5.31)^{2}-7.104^{2}\right]
$$

$$
=18.02 \text { Joules. }
$$

Ans.

## IMPORTANT DEFINITIONS AND CONCEPTS

1. Generalised form of Newton's second law is $R=m a$ and may be stated as a particle acted upon by an unbalanced system of forces has an acceleration directly proportional and in line with the resultant force.
2. D'Alembert looked at Newton's second law of motion as $R-m a=0$ as an equation of equilibrium and stated that the system of forces acting on a body in motion is in dynamic equilibrium with the inertia force ' $m a$ ' of the body.
3. The work done by a force on a moving body is defined as the product of the force and the distance moved in the direction of the force.
4. Joule (J) is the unit of force and may be defined as the amount of work done by one Newton force when the particle moves 1 metre distance in the direction of that force.
5. Energy is defined as the capacity to do the work. Potential energy is the capacity to do the work due to the position of the body and the kinetic energy is the capacity to do the work due to the motion of the body. The unit of energy is same as that of work.
6. Power is defined as the rate of doing work. Unit of power is watt (W) and is defined as one Joule of work done in one second. The work-energy principle may be stated as the work done by a system of forces acting on a body during a displacement is equal to the change in kinetic energy of the body during the same displacement.
7. The impulse momentum principle can be stated as 'the component of the resultant linear impulse along any direction is equal to the component of momentum in that direction.
8. The principle of conservation of momentum may be stated as 'the momentum is conserved in a system in which resultant force is zero. In other words, in a system if the resultant force is zero, initial momentum will remain equal to the final momentum.
9. Coefficient of restitution may be defined as the ratio between relative velocity of separation and relative velocity of approach along the line of impact, when two bodies collide.

## IMPORTANT FORMULAE

1. If a body of mass $m$ is moving with acceleration ' $a$ ', the body may be treated as in equilibrium along with the system of forces and inertia force $m \times a$ applied in the opposite direction of motion.
2. The work-energy equation is

$$
R S=\frac{W}{g}\left(v^{2}-u^{2}\right), \text { with usual notations. }
$$

3. The impulse momentum equation is

$$
\begin{aligned}
R t & =m v-m u \text { in any direction } \\
& =\mathrm{m}(v-u)
\end{aligned}
$$

4. According to conservation of momentum equation, when resultant force $R=0$,

> Initial Momentum = Final Momentum

If two bodies $W_{1}$ and $W_{2}$ moving with different velocities $v_{1}$ and $v_{2}$ join together and start moving, the above equation becomes,

$$
\frac{W_{1}}{g} v_{1}+\frac{W_{2}}{g} v_{2}=\frac{W_{1}+W_{2}}{g} v
$$

where $v$ is the velocity after the two bodies start moving together.
5. Coefficient of restitution $e$ is given by

$$
e=\frac{v_{2}-v_{1}}{u_{1}-u_{2}}
$$

6. Kinetic energy $=\frac{W}{2 g} v^{2}$.
$\therefore$ Loss of kinetic energy due to impact is

$$
=\frac{W_{1}}{2 g}\left(u_{1}^{2}-v_{1}^{2}\right)+\frac{W_{2}}{2 g}\left(u_{2}^{2}-v_{2}^{2}\right)
$$

## PROBLEMS FOR EXERCISE

7.1 A mine cage weighs 12 kN and can carry a maximum load of 20 kN . The average frictional resistance of the slide guys is 500 N . What constant cable tension is required to give a loaded cage an upward velocity of $3 \mathrm{~m} / \mathrm{s}$, from rest in a distance of 3 m ?
[Ans. 37.393 kN ]
7.2 A train weighing 3000 kN is moving up a slope 2 in 100 with an acceleration of $0.04 \mathrm{~m} / \mathrm{s}^{2}$. Tractive resistance is $6 \mathrm{~N} / \mathrm{kN}$. Determine the acceleration of the train if it moves with the same tractive force:
(a) on a level track;
(b) down the plane inclined at 2 in 100 .
[Ans. $0.2362 \mathrm{~m} / \mathrm{s}^{2} ; 0.4324 \mathrm{~m} / \mathrm{s}^{2}$ ]
7.3 The coefficient of friction between the crate and the cart shown in Fig. 7.64 is 0.25 and that between cart and the road is 0.15 . If the cart is pulled by a force of 500 N , determine the acceleration of the cart. Check whether there is slipping of the crate at this stage.
[Ans. $a=1.4138 \mathrm{~m} / \mathrm{s}^{2}$; No slipping]


Fig. 7.64
7.4 A block weighing 2500 N rests on a level horizontal plane which has a coefficient of friction 0.20 . This block is pulled by a force of 1000 N , which is acting at an angle of $30^{\circ}$ to the horizontal. Find the velocity of the block, after it moves 30 m , starting from rest. If the force of 1000 N is then removed, how much further will it move?
[Ans. $v=10.4748 \mathrm{~m} / \mathrm{s} ; s=27.961 \mathrm{~m}$ ]
7.5 In what distance will body 1 shown in Fig. 7.65 attain a velocity of a $3 \mathrm{~m} / \mathrm{s}$ starting from rest? Take coefficient of friction between the blocks and plane as 0.2 . Assume pulley is smooth. What is the tension in the chord?
[Ans. $s=12.953 \mathrm{~m} ; T=159.08 \mathrm{~N}$ ]
7.6 Determine the constant force $P$ that will give the system of bodies, shown in Fig. 7.66, a velocity of $3 \mathrm{~m} / \mathrm{s}$ after moving a distance of 4.5 m from the position of rest. Coefficient of friction at all contact points is 0.2 . Assume pulleys as frictionless.
[Ans. 448.39 N ]


Fig. 7.65


Fig. 7.66
7.7 Two blocks of weight $W_{1}$ and $W_{2}$ are connected by inextensible wire passing over a smooth pulley as shown in Fig. 7.67. If $W_{1}$ is greater than $W_{2}$, find the tension in the string and the acceleration of the system.

$$
\left[\text { Ans. } a=g\left(\frac{W_{1}-W_{2}}{W_{1}+W_{2}}\right) ; T=\frac{2 W_{1} W_{2}}{W_{1}+W_{2}}\right]
$$



Fig. 7.67
7.8 Find the acceleration of the three weights shown in Fig. 7.68. Assume pulleys are frictionless. Find also the tension in the cables.


Fig. 7.68
[Ans. Acceleration of 200 N block $=\frac{13}{37} g$,

$$
40 \mathrm{~N} \text { block }=\frac{23}{37} \mathrm{~g},
$$

$$
40 \mathrm{~N} \text { block }=\frac{3}{37} \mathrm{~g}
$$

Tension in uppper cable $=129.73 \mathrm{~N}$
Tension in lower block $=64.85 \mathrm{~N}$
7.9 The weights of the three blocks shown in Fig. 7.69 are $W_{A}=100 \mathrm{~N}$, $W_{B}=200 \mathrm{~N}$ and $W_{C}=200 \mathrm{~N}$. Coefficient of friction between block $A$ and the floor is 0.2 , that between floor and block $C$ is 0.25 . Assuming pulleys as weightless and smooth, find the acceleration of each block.
$\left[\right.$ Hint: $\left.a_{B}=\left(\frac{a_{A}+a_{C}}{2}\right)\right]$
[Ans. $\left.a_{A}=0.5 \mathrm{~g}, a_{B}=0.3 \mathrm{~g}, a_{C}=0.1 \mathrm{~g}\right]$


Fig. 7.69
7.10 A $10,000 \mathrm{kN}$ train is accelerated at a constant rate up a $2 \%$ grade. The track resistance is constant at $9 \mathrm{~N} / \mathrm{kN}$. The velocity increases from $9 \mathrm{~m} / \mathrm{s}$ to $18 \mathrm{~m} / \mathrm{s}$ in a distance of 600 metres. Determine the maximum power developed by the locomotive.
[Ans. $P=496.422 \mathrm{kN}$, power $=8935.6 \mathrm{~kW}$ ]
7.11 Find the power required to pull a train up an incline of 1 in 200 at a speed of 36 kmph , if the weight of the train is 3000 kN and the track resistance is $5 \mathrm{~N} / \mathrm{kN}$. Also determine the maximum speed with which the train moves up on incline of 1 in 100 with the same power.
[Ans. Power $=300 \mathrm{~kW} ; v=24 \mathrm{kmph}]$
7.12 On seeing a child on the road, the driver of a car applied brakes instantaneously. The car just ran over the child after skidding a total distance of 28.2 metres in the direction of motion before coming to a stop. The traffic police sued the driver on the ground of overspeeding. The speed limit of the section was 40 kmph . Would you justify the police action if the coefficient of friction between the tyres and the road is 0.5 ? The weight of the car is 20 kN and it was travelling on a level surface.
[Ans. $v=59.877 \mathrm{kmph}$, police action is justified]
7.13 A mine cage weighs 12 kN and can carry a maximum load of 20 kN . The average frictional resistance of the slide guides is 500 N . What constant cable tension is required to give a loaded cage an upward velocity of $3 \mathrm{~m} / \mathrm{s}$ from rest in a distance of 3 m ?
[Ans. $T=37.47 \mathrm{kN}$ ]
7.14 A 500 N body moves along the two inclines for which the coefficient of friction is 0.2 (Fig. 7.70). If the body starts from rest at $A$ and slides 60 m down the $30^{\circ}$ incline, how far will it then move along the other incline? What will be its velocity when it returns to $B$ ?
[Ans. $s=25.8 \mathrm{~m} ; v=14.92 \mathrm{~m} / \mathrm{s}$ ]


Fig. 7.70


Fig. 7.71
7.15 In what distance will body $A$ shown in Fig. 7.71 attain a velocity of $3 \mathrm{~m} / \mathrm{s}$ starting from rest? Take coefficient of friction between the blocks and the plane as 0.2 . Assume the pulley is smooth.
[Ans. $s=6.541 \mathrm{~m}$ ]
7.16 Two blocks, weighing 400 N and 500 N are connected by inextensible flexible wire running around a smooth pulley as shown in Fig. 7.72. Find what will be the velocity of the system if the distance moved by the blocks is 3 m starting from rest.
[Ans. $v=2.557 \mathrm{~m} / \mathrm{s}$ ]
7.17 Two blocks are connected by inextensible wires as shown in Fig. 7.73. Find by how much distance block 400 N will move in increasing its velocity to $5 \mathrm{~m} / \mathrm{s}$ from $2 \mathrm{~m} / \mathrm{s}$. Assume pulleys are frictionless and weightless.
[Ans. $s=5.89 \mathrm{~m}$ ]


Fig. 7.72


Fig. 7.73
7.18 By using work-energy equation, calculate the velocity and acceleration of the block $A$ shown in Fig. 7.74 after it has moved 6 m from rest. The coefficient of kinetic friction is 0.3 and the pulleys are considered to be frictionless and weightless. Also calculate the tension in the string attached to $A$.

$$
\text { [Ans. } \left.v=3.544 \mathrm{~m} / \mathrm{s} ; a=1.046 \mathrm{~m} / \mathrm{s}^{2} ; T=189.33 \mathrm{~N}\right]
$$



Fig. 7.74
7.19 How far block $A$ shown in Fig. 7.75 will move when its velocity increases from $3 \mathrm{~m} / \mathrm{s}$ to $8 \mathrm{~m} / \mathrm{s}$. Assume pulleys are weightless and frictionless. Radius of larger pulley is 0.3 m and that of smaller pulley is 0.2 m .
[Ans. $s=15.885 \mathrm{~m}$ ]


Fig. 7.75
7.20 A wagon weighing 600 kN starts from rest, runs 30 m down a $1 \%$ grade and strikes a post. If the rolling resistance of the track is 5 N per kN , find the velocity of the wagon when it strikes the post.
If the impact is to be cushioned by means of one bumper spring, which compresses 1 mm per 25 kN weight, determine how much the bumper spring get compressed.
[Ans. $v=1.716 \mathrm{~m} / \mathrm{s}, s=84.85 \mathrm{~mm}$ ]
7.21 A spring is used to stop a 1000 N package which is moving down a $20^{\circ}$ incline. The spring has a constant $k=150 \mathrm{~N} / \mathrm{mm}$ and is held by the cables so that it is initially compressed by 100 mm . If the velocity of the package is 6 $\mathrm{m} / \mathrm{s}$, when it is 10 m from the spring, determine the maximum additional deformation of the spring in bringing the package to rest. Take coefficient of friction $=0.25$.
[Ans. $s=195.6 \mathrm{~mm}$ ]
7.22 A cricket ball weighing one Newton approaches a batsman with a velocity of $18 \mathrm{~m} / \mathrm{s}$ in the direction shown in Fig. 7.76. After hit by the bat at $B$, it moves out with a velocity of $40 \mathrm{~m} / \mathrm{s}$ at $45^{\circ}$ to horizontal. If the bat and ball were in contact for 0.02 s , determine the impulsive force exerted by the bat.
[Ans. 230.37 N]


Fig. 7.76
7.23 A block weighing 200 N is pulled up a $30^{\circ}$ plane by a force $P$ producing a velocity of $5 \mathrm{~m} / \mathrm{s}$ in 5 seconds. If the coefficient of friction is 0.2 , determine the magnitude of force $P$. At this stage if force $P$ is removed, how much more time will it take to come to rest?
[Ans. $155 \mathrm{~N} ; 0.757 \mathrm{~s}]$
7.24 The initial velocity of 500 N block is $6 \mathrm{~m} / \mathrm{s}$ towards left. At this stage a weight of 250 N is applied as shown in Fig. 7.77. Determine the time at which the block has (a) no velocity, (b) a velocity of $4 \mathrm{~m} / \mathrm{s}$ to the right. Take coefficient of friction 0.2 and assume pulley as ideal.
[Ans. (a) 0.87 s ; (b) 2.23 s$]$
7.25 Determine the tension in the string and the velocity of 2000 N block shown in Fig. 7.78, 6 second after starting with a velocity of $3 \mathrm{~m} / \mathrm{s}$.
[Ans. $V=25.07 \mathrm{~m} / \mathrm{s}, T=1280 \mathrm{~N}$ ]


Fig. 7.77


Fig. 7.78
7.26 Force $P=1900 \mathrm{~N}$ shown in Fig. 7.79 was applied to 200 N block when the block was moving with rightward velocity of $5 \mathrm{~m} / \mathrm{s}$. Determine the time at which the system has (a) no velocity, (b) a velocity of $3 \mathrm{~m} / \mathrm{s}$ towards left. Coefficient of friction between blocks and surface $=0.2$. Assume pulley to be ideal.
[Ans. (a) 0.8676 s ; (b) 1.96 s ]


Fig. 7.79
7.27 An engine weighing 500 kN pulls a train weighing 1500 kN up an incline of 1 in 100. The train starts from rest and moves with a constant acceleration against a resistance of $5 \mathrm{~N} / \mathrm{kN}$. It attains a speed of 18 kmph in 60 seconds. Determine the tension in the draw bar connecting train and the engine. What will be its speed 90 seconds after the start?
[Ans. $P=35.242 \mathrm{kN} ; v=27 \mathrm{kmph}]$
7.28 Determine the force exerted by a 60 mm diameter jet of water flowing at $25 \mathrm{~m} / \mathrm{s}$ on (a) a vertical stationary plate, (b) a cup that turns the water through $120^{\circ}$.
[Ans. (a) $P=1767.146 \mathrm{kN}$;
(b) $P=3006.787 \mathrm{kN}$ at $\theta=30^{\circ}$ to horizontal]
7.29 A jet of water of 50 mm diameter strikes a series of vanes horizontally at a speed of $36 \mathrm{~m} / \mathrm{s}$ and gets deflected through $45^{\circ}$. Determine the force exerted by the jet on vanes, if the vanes are moving away from the jet at a velocity of $6 \mathrm{~m} / \mathrm{s}$. [Ans. $P=1252.516 \mathrm{kN}$ and $\theta=22.5^{\circ}$ to horizontal]
7.30 A shot is fired horizontally from a gun boat towards a target. The total weight of gun boat including men, gun, shells etc. in it is 15 kN . The weight of shell is 15 N and emerges out at a velocity of $300 \mathrm{~m} / \mathrm{s}$. What will be the velocity of the boat when the shell is fired? If the target weighs 1 kN suspended by a rope of length 2 m , by what angle it will swing if the shell gets embedded in it?
[Ans. $v=0.3 \mathrm{~m} / \mathrm{s} ; \theta=60.06^{\circ}$ ]
7.31 A bullet weighing 0.3 N and moving at $600 \mathrm{~m} / \mathrm{s}$ penetrates 40 N body shown in Fig. 7.80 and emerges with a velocity of $180 \mathrm{~m} / \mathrm{s}$. How far and how long will the block move, if the coefficient of friction between the body and the horizontal floor is 0.3 ? [Ans. $s=1.686 \mathrm{~m} ; t=1.07 \mathrm{~s}$ ]


Fig. 7.80
7.32 A cannon weighing 200 kN fires a shell weighing 1 kN with a muzzle velocity of $800 \mathrm{~m} / \mathrm{s}$. Calculate the velocity with which the cannon recoils and the uniform force required to stop it within 400 mm distance. In how much time will it stop? [Ans. $v=4.0 \mathrm{~m} / \mathrm{s} ; F=407.747 \mathrm{kN} ; t=0.20 \mathrm{~s}$ ]
7.33 A man weighing 750 N and a boy weighing 500 N jump from a boat to a pier with a horizontal velocity of $5 \mathrm{~m} / \mathrm{s}$ relative to the boat. The boat weighs 4000 N and was stationary before they jumped. Determine the velocity of the boat if
(a) they jump together;
(b) boy jumps first and the man latter;
(c) man jumps first and the boy latter.
[Ans. (a) $1.563 \mathrm{~m} / \mathrm{s}$; (b) $1.464 \mathrm{~m} / \mathrm{s}$; (c) $1.458 \mathrm{~m} / \mathrm{s}$ ]
7.34 A pile hammer weighing 8 kN falls freely from a height of 1.5 m on a pile weighing 5 kN . For each blow, the pile is driven by 80 mm . Determine
(a) resistance of the ground,
(b) loss of kinetic energy during the impact,
(c) time during which the pile is in motion for each blow.
[Ans. (a) $R=105.31 \mathrm{kN}$; (b) Loss of K.E. $=4.615 \mathrm{kN}-\mathrm{m}$; (c) $t=0.048 \mathrm{~s}$ ]
7.35 Two bodies, one of which is 400 N with a velocity of $8 \mathrm{~m} / \mathrm{s}$ and the other of 250 N with a velocity of $12 \mathrm{~m} / \mathrm{s}$, move towards each other along a straight line and impinge centrally. Find the velocity of each body after impact if the coefficient of restitution is 0.8 .
[Ans. $v_{1}=-5.846 \mathrm{~m} / \mathrm{s} ; v_{2}=10.154 \mathrm{~m} / \mathrm{s}$ ]
7.36 A ball is dropped from a height of $h_{0}$ metres on a floor. Show that after first bounce it will rise to height $h_{1}$ given by $h_{1}=e^{2} h_{0}$, where $e$ is coefficient of restitution. Determine to what height the ball will rise after 3 bounces if dropped from a height of 2 m , if the coefficient of restitution is 0.8 .
[Ans. 0.524 m ]
7.3710 N sphere shown in Fig. 7.81 is released from rest when $\theta_{A}=90^{\circ}$. The coefficient of restitution between the sphere and the block is 0.70 . If the coefficient of friction between the block and the horizontal surface is 0.3 , determine how far the block will move after impact. [Ans. 0.934 m ]
7.38 100 N ball shown in Fig. 7.82 is released from the position shown by continuous line. It strikes a freely suspended 75 N ball. After impact, 75 N ball is raised by an angle $\theta=48^{\circ}$. Determine the coefficient of restitution.
[Hint: $u_{1}=\sqrt{2 g h_{1}}$, where $h_{1}=3\left(1-\cos 60^{\circ}\right)$

$$
\left.v_{2}=\sqrt{2 g h_{2}}, \text { where } h_{2}=2\left(1-\cos 48^{\circ}\right)\right]
$$

[Ans. $e=0.162$ ]


Fig. 7.81


Fig. 7.82
7.39 Two identical balls, moving horizontally collide as shown in Fig. 7.83. Determine their velocities after impact, if the coefficient of restitution is 0.75. [Ans. $v_{1}=8.718 \mathrm{~m} / \mathrm{s} ; \theta_{1}=23.41^{\circ}, v_{2}=8.718 \mathrm{~m} / \mathrm{s} ; \theta_{2}=96.587^{\circ}$ ]


Fig. 7.83
7.40 Find the velocities of the two balls shown in Fig. 7.84 after impact, if the coefficient of restitution $=0.6$.
[Ans. $v_{A}=4.576 \mathrm{~m} / \mathrm{s} ; \theta_{A}=67.99^{\circ}$

$$
\left.v_{B}=6.3932 \mathrm{~m} / \mathrm{s} ; \theta_{B}=38.73^{\circ}\right]
$$

7.41 Central impact takes place between a 40 N ball and a stationary block of 60 N as shown in Fig. 7.85. The block rests on rollers and move freely on horizontal surface. Find the velocity of the block and ball after impact. Take coefficient of restitution as 0.8 . [Ans. Velocity of block $=5.76 \mathrm{~m} / \mathrm{s}$; velocity of ball $=4.04 \mathrm{~m} / \mathrm{s}$ at $67.88^{\circ}$ to horizontal]


Fig. 7.84


Fig. 7.85
7.42 A ball drops on to a smooth horizontal floor and bounces as shown in Fig. 7.86. Derive expressions for coefficient of restitution in terms of (a) two successive heights, (b) two successive ranges. Determine also time of flights in $n$th bounce.
[Ans. (a) $e=\sqrt{h_{n} /\left(h_{n-1}\right)}$, (b) $e=D_{n} / D_{n-1}, t=2 \sqrt{2 h_{n} / g}$ ]


Fig. 7.86
7.43 80 N and 150 N bodies are approaching each other with a velocity of $20 \mathrm{~m} / \mathrm{s}$ and $6 \mathrm{~m} / \mathrm{s}$ respectively. What will be the velocity of each body after impact? How much is the loss of kinetic energy? Take coefficient of restitution $=0.6$. Assume 80 N block is moving from left to right.
[Ans. For 80 N block, $v_{1}=\longleftarrow 7.13 \mathrm{~m} / \mathrm{s}$.
150 N block, $\mathrm{v}_{2}=\xrightarrow[8.47]{ } \mathrm{m} / \mathrm{s}$.
Loss of K.E. $=1150.43$ Joules]

## chapter 8

## Elements of Rigid Body Dynamics

As already discussed in Chapter 1, a body is said to be rigid if the relative position between any two points in a body does not change. To start with different types of rigid body motions are explained, then kinematics of rigid bodies is discussed and finally kinetics is taken up. However, the detailed discussion is restricted to plane motion only.

### 8.1 TYPES OF RIGID BODY MOTION

The meaning of terms translation, rotation, general plane motion and general space motion is presented in this article.

## (a) Translation

A motion is said to be translational, if a straight line drawn on the moving body remains parallel to its original position at any time. During translation if the path traced is a straight line, it is called rectilinear translation and if the path is curved it is called curvilinear translation. Figures 8.1 (a) and (b) show such motion.

(a) Rectilinear Translation

(b) Curvilinear Translation

Fig. 8.1 Translation

## (b) Rotation

A motion is said to be rotational if all the particles of a rigid body move in a concentric circle. The common centre of the circle may be located within or outside the body as shown in Figs. 8.2 (a) and (b). The rotation may be about an axis also as shown in Fig. 8.2 (c).

(a)

(b) Rotation about axis $O O^{\prime}$


Fig. 8.2

## (c) General Plane Motion

A body is said to have general plane motion if it possesses translation and rotation simultaneously. Common examples of such motions are a wheel rolling on straight line and a rod sliding against a wall at one end and the floor at the other end. Figures 8.3 (a) and (b) illustrate how these two cases can be split into translation and rotation.


Fig. 8.3

## (d) Motion in Space

In general, a body can have translation and rotation in space. This motion can be split into three cases of translation and three cases of rotation as shown in Fig. 8.4. This case is considered beyond the scope of this book and hence not discussed further.


Fig. 8.4 Six Modes of Motion in Space

### 8.2 KINEMATICS OF RIGID BODY MOTION

## (a) Translation

Let $A$ and $B$ be two points in a body in rectilinear translation. Let their position be represented by $\mathbf{r}_{\mathbf{A}}$ and $\mathbf{r}_{\mathbf{B}}$ with respect to fixed cartesian axis as shown in Fig. 8.5. Then

$$
\begin{equation*}
r_{B}=r_{A}+r_{B / A} \tag{8.1}
\end{equation*}
$$

From the definition of translation, $\mathbf{r}_{\mathbf{B} / \mathbf{A}}$ is constant.


Fig. 8.5
$\therefore \quad$ Differentiating Eqn. (8.1), w.r.t. time, we get

$$
\begin{equation*}
\mathbf{v}_{B}=\mathbf{v}_{A} \quad \text { and } \quad \frac{d}{d t}\left(\mathbf{r}_{\mathbf{B} / \mathbf{A}}\right)=0 \tag{8.2}
\end{equation*}
$$

Again differentiating Eqn. (8.2), we get

$$
\mathbf{a}_{B}=\mathbf{a}_{A}
$$

Thus all the points in rigid body experience same velocity and acceleration, when the body is in rectilinear translation.

In case of curvilinear translation (Fig. 8.6), the curve traced by any two points is identical. Hence they have same velocity and acceleration at any point. Only difference from rectilinear translation case is the path traced is curve and hence it has not only tangential acceleration but is having radial inward


Fig. 8.6 acceleration $\frac{v^{2}}{\rho}$ at any instant.

## (b) Rotation

If $\theta$ is angular displacement, $\omega$ is angular acceleration and $\alpha$ is the angular acceleration, the following relations used for a particle in Art. 6.11 hold good for
any point in rigid body:

$$
\begin{align*}
& \omega=\frac{d \theta}{d t}  \tag{8.4}\\
& \alpha=\frac{\partial \omega}{\partial t}=\frac{d^{2} \theta}{d t^{2}} \\
& \alpha=\frac{\partial \omega}{\partial t}=\frac{d \omega}{\partial \theta} \cdot \frac{\partial \theta}{\partial t}=\omega \cdot \frac{\partial \omega}{\partial \theta} \\
& v=r \omega \\
& a_{t}=r \frac{d^{2} \theta}{d t^{2}}=r \alpha \\
& a_{n}=\frac{v^{2}}{r}=r \omega^{2}
\end{align*}
$$

The rotation with uniform angular velocity and uniform acceleration has considerable engineering application.

## Uniform Angular Velocity

If the angular velocity is uniform, the angular distance moved in $t$ seconds by a body having angular velocity $\omega$ radians/second is given by

$$
\begin{equation*}
\theta=\omega t \text { radians } \tag{8.5}
\end{equation*}
$$

Note that the uniform angular velocity is characterized by zero angular acceleration.

Many a times, the angular velocity is given in terms of number of revolutions per minute (rpm). Since there are $2 \pi$ radians in one revolution and 60 seconds in one minute, the angular acceleration $\omega$ is given by

$$
\begin{equation*}
\omega=\frac{2 \pi N}{60} \mathrm{rad} / \mathrm{s} \tag{8.6}
\end{equation*}
$$

where, $N$ - is in rpm.
Since angular velocity is uniform, the time taken for one revolution $T$ is given by Eqn. (8.5), as
or

$$
\begin{align*}
2 \pi & =\omega T \\
T & =2 \pi / \omega \tag{8.7}
\end{align*}
$$

## Uniformly Accelerated Rotation

Let us consider the uniformly accelerated motion with angular acceleration $\alpha$. From the definition,

$$
\begin{aligned}
\frac{\partial \omega}{\partial t} & =\alpha \\
\omega & =\alpha t+C_{1}
\end{aligned}
$$

where, $C_{1}$ is constant of integration.
If the initial velocity is $\omega_{0}$, then

$$
\begin{align*}
\omega_{0} & =\alpha \times 0+C_{1} \text { or } C_{1}=\omega_{0} \\
\omega & =\omega_{0}+\alpha t \tag{1}
\end{align*}
$$

Again from the definition of angular velocity, we have

$$
\begin{aligned}
\frac{d \theta}{d t} & =\omega=\omega_{0}+\alpha t \\
\therefore \quad \theta & =\omega_{0} t+\frac{1}{2} \alpha t^{2}+C_{2}
\end{aligned}
$$

where, $C_{2}$ is constant of integration.
Measuring the angular displacement from the instant of reckoning the interval of time, we get

$$
\begin{gather*}
0=0+0+C_{2} \\
\theta=\omega_{0} t+1 / 2 \alpha t^{2} \tag{2}
\end{gather*}
$$

From Eqn. 8.4

$$
\alpha=\omega \frac{d \omega}{d \theta}
$$

or

$$
\alpha d \theta=\omega d \omega
$$

Integrating, we get

$$
a \theta=\frac{\omega^{2}}{2}+C_{3}
$$

where, $C_{3}$ is constant of integration
Initially $\theta=0$ and $\omega=\omega_{0}$
Hence, we get

$$
\alpha \times 0=\frac{\omega_{0}^{2}}{2}+C_{3}
$$

or

$$
C_{3}=\frac{-\omega_{0}^{2}}{2}
$$

$$
\therefore \quad \alpha \theta=\frac{\omega^{2}}{2}+\frac{-\omega_{0}^{2}}{2}
$$

or

$$
\begin{equation*}
\omega^{2}-\omega_{0}^{2}=2 \alpha \theta \tag{3}
\end{equation*}
$$

Thus for uniformly accelerated angular motion, we have
and

$$
\begin{align*}
\omega & =\omega_{0}+\alpha t  \tag{i}\\
\theta & =\omega_{0} t+\frac{1}{2} \alpha t^{2}  \tag{8.8}\\
\omega^{2}-\omega_{0}^{2} & =2 \alpha s
\end{align*}
$$

Carefully note down similarity of Eqn. (8.8) with uniformly accelerated linear motion equations listed below:
and

$$
\begin{align*}
v & =u+a t  \tag{i}\\
s & =u t+\frac{1}{2} a t^{2}  \tag{ii}\\
v^{2}-u^{2} & =2 a s \tag{iii}
\end{align*}
$$

The problems of uniform angular retardation can be handled with Eqn. (8.8) by noting that retardation is negative quantity of acceleration, as has been done in linear motion.

Example 8.1 The rotation of a flywheel is governed by the equation $\omega=3 t^{2}-2 t$ +2 where $\omega$ is in radians per second and $t$ is in seconds. After one second from the start, the angular displacement was 4 radians. Determine the angular displacement, angular velocity and angular acceleration of the flywheel when $t=3$ seconds.
Solution. Here,

|  | $\omega$ |
| ---: | :--- |
|  | $=3 t^{2}-2 t+2$ |
| i.e., | $\frac{d \theta}{d t}$ |$=3 t^{2}-2 t+2, ~=t^{3}-t^{2}+2 t+C$

where, $C$ is constant of integration

| when | $t=1, \theta=A$, |
| :--- | :--- |
| $\therefore$ | $4=1-1+2+C$ |
| i.e., | $C=2$ |
| $\therefore$ | $\theta=t^{3}-t^{2}+2 t+2$ |

When, $t=3$,

$$
\begin{aligned}
\theta & =3^{3}-3^{2}+2 \times 3+2=26 \mathrm{radians} \\
\omega & =3 \times 3^{2}-2 \times 3+2=23 \mathrm{rad} / \mathrm{s}
\end{aligned}
$$

Angular acceleration $\alpha$ is given by

$$
\alpha=\frac{d \omega}{d t}=6 t-2
$$

$\therefore$ When $t=3$,

$$
\alpha=6 \times 3-2=16 \mathrm{rad} / \mathrm{s}^{2}
$$

Ans.
Example 8.2 The angular acceleration of a flywheel is given by $\alpha=12-t$, where, $\alpha$ is in rad $/ \mathrm{s}^{2}$ and $t$ is in seconds. If the angular velocity of the flywheel is $60 \mathrm{rad} / \mathrm{s}$ at the end of 4 seconds, determine the angular velocity at the end of 6 seconds. How many revolutions take place in these 6 seconds?

## Solution.

$$
\alpha=12-t
$$

i.e.,

$$
\begin{aligned}
\frac{d \omega}{d t} & =12-t \\
\omega & =12 t-\frac{t^{2}}{2}+\mathrm{C}
\end{aligned}
$$

or
where, $C$ is constant of integration,
When $t=4 \mathrm{~s}, \omega=60 \mathrm{rad} / \mathrm{s}$
$\therefore \quad 60=12 \times 4-\frac{4^{2}}{2}+C_{1}$
i.e., $\quad C_{1}=20$
$\therefore \quad \omega=12 t-\frac{t^{2}}{2}+20$
when $t=6 \mathrm{~s}$

$$
\omega=12 \times 6-\frac{6^{2}}{2}+20=74 \mathrm{rad} / \mathrm{s}
$$

Ans.
Now,

$$
\frac{d \theta}{d t}=\omega=12 t-\frac{t^{2}}{2}+20
$$

$$
\therefore \quad \theta=6 t^{2}-t^{3} / 6+20 t+C_{2}
$$

where, $C_{2}$ is constant of integration
when $t=0, \theta_{0}=C_{2}$
when $t=6 \mathrm{~s}$,

$$
\begin{aligned}
\theta_{6} & =6 \times 6^{2}-6^{3} / 6+20 \times 6+C_{2} \\
& =180+C_{2}
\end{aligned}
$$

$\therefore$ Angular displacement during 6 seconds

$$
=\theta_{6}-\theta_{0}=180+C_{2}-C_{2}=180 \mathrm{rad}
$$

$\therefore$ Number of revolutions $=\frac{180}{2 \pi}=28.648$

## Ans.

Example 8.3 A wheel rotating about a fixed axis at 20 revolutions per minute is uniformly accelerated for 70 seconds during which it makes 50 revolutions. Find the (i) angular velocity at the end of this interval and (ii) time required for the velocity to reach 100 revolutions per minute.
Solution. (i) Initial velocity $\quad \omega_{0}=20 \mathrm{rpm}$

$$
\begin{aligned}
& =\frac{20 \times 2 \pi}{60} \mathrm{rad} / \mathrm{s} \\
& =2.0944 \mathrm{rad} / \mathrm{s} \\
t & =70 \mathrm{~s}
\end{aligned}
$$

Angular displacement $\quad \theta=50$ revolutions

$$
=50 \times 2 \pi=100 \pi \text { radians }
$$

Using kinematic equation,

$$
\theta=\omega_{0} t+\frac{1}{2} \alpha t^{2}
$$

we get

$$
\begin{aligned}
100 \pi & =2.0944 \times 70+\frac{1}{2} \alpha \times 70^{2} \\
\alpha & =0.06839 \mathrm{rad} / \mathrm{s}^{2}
\end{aligned}
$$

$\therefore$ Angular velocity at the end of 70 seconds interval

$$
\begin{aligned}
\omega & =\omega_{0}+\alpha t \\
& =2.0944+0.06839 \times 70 \\
& =6.8816 \mathrm{rad} / \mathrm{s}
\end{aligned}
$$

Ans.
(ii) Let the time required for the velocity to reach 100 rpm be $t$.

$$
\begin{aligned}
\omega & =100 \mathrm{rpm} \\
& =\frac{100 \times 2 \pi}{60} \mathrm{rad} / \mathrm{s}
\end{aligned}
$$

Using the relation $\omega=\omega_{0}+\alpha t$, we get
i.e.,

$$
t=\frac{\omega-\omega_{0}}{\alpha}=\left(\frac{200 \pi}{60}-2.0944\right) \frac{1}{0.06839}
$$

$$
t=122.497
$$

Ans.
Example 8.4 A flywheel, which accelerates at uniform velocity, is observed to have made 100 revolutions to increase its velocity from 120 rpm to 160 rpm . If the flywheel originally started from rest, determine
(i) the value of acceleration,
(ii) time taken to increase the velocity from 120 rpm to 160 rpm and
(iii) revolution made in reaching a velocity of 160 rpm , starting from rest.

## Solution.

$$
\begin{aligned}
\theta & =100 \text { revolution }=200 \pi \text { radians } \\
\omega_{0} & =120 \mathrm{rpm}=\frac{120 \times 2 \pi}{60}=4 \pi \mathrm{rad} / \mathrm{s} \\
\omega & =160 \mathrm{rpm}=\frac{160 \times 2 \pi}{60}=16.755 \mathrm{rad} / \mathrm{s}
\end{aligned}
$$

Using the kinematic relation $\omega^{2}-\omega_{0}^{2}=2 \alpha \theta$, we get

$$
\begin{aligned}
16.755^{2}-(4 \pi)^{2} & =2 \alpha \times 200 \pi \\
\alpha & =0.0977 \mathrm{rad} / \mathrm{s}^{2}
\end{aligned}
$$

From the kinematic relation $\omega=\omega_{0}+\alpha t$,
we get,
$\therefore$
Let $\theta^{\prime}$ be the total angular displacement in reaching the velocity of 160 rpm . Then,

$$
\begin{aligned}
16.755^{2}-0^{2} & =2 \times 0.0977 \times \theta^{\prime} \\
\theta^{\prime} & =1436.1 \mathrm{rad} \\
\theta^{\prime} & =\frac{1436.1}{2 \pi}=228.57 \text { revolutions }
\end{aligned}
$$

Ans.

Example 8.5 Power supply was cut off to a power driven wheel when it was rotating at a speed of 900 rpm . It was observed to come to rest after making 360 revolutions. Determine its angular retardation and time it took to come to rest after power supply was cut off.

## Solution.

$$
\begin{aligned}
\omega_{0} & =900 \mathrm{rpm}=\frac{900 \times 2 \pi}{60}=30 \pi \\
\omega & =0 \\
\theta & =360 \text { revolutions }=360 \times 2 \pi=720 \pi \mathrm{rad}
\end{aligned}
$$

From the relation $\omega^{2}-\omega_{0}^{2}=2 \alpha \theta$,
we get

$$
\begin{aligned}
0-(30 \pi)^{2} & =2 \alpha \times 720 \pi \\
\alpha & =-1.9635 \mathrm{rad} / \mathrm{s}^{2}
\end{aligned}
$$

i.e., retardation is $1.9635 \mathrm{rad} / \mathrm{s}^{2}$

Ans.
Using the relation $\quad \begin{aligned} \omega & =\omega_{0}+\alpha t, \text { we get } \\ 0 & =30 \pi-1.9635 t \\ t & =\frac{30 \pi}{1.9635}=48 \mathrm{~s} .\end{aligned}$

$$
\begin{aligned}
& 0=30 \pi-1.9635 t \\
& t=\frac{30 \pi}{1.9635}=48 \mathrm{~s}
\end{aligned}
$$

Ans.
Example 8.6 The step pulley shown in Fig. 8.7 starts from rest and accelerates at $2 \mathrm{rad} / \mathrm{s}^{2}$. How much time is required for block $A$ to move 20 m ? Find also the velocities of $A$ and $B$ at that time.
Solution. When $A$ moves by 20 m , the angular displacement $\theta$ of pulley is given by

Now,

$$
\begin{aligned}
r \theta & =s \\
1 \times \theta & =20 \\
\theta & =20 \mathrm{radians} \\
\alpha & =2 \mathrm{rad} / \mathrm{s}^{2} \text { and } \omega_{0}=0
\end{aligned}
$$

From the kinematic relation
we get

$$
\theta=\omega_{0} t+\frac{1}{2} \alpha t^{2}
$$

$$
20=0+\frac{1}{2} \times 2 t^{2}
$$



Fig. 8.7

$$
t=4.472 \mathrm{~s}
$$

Ans.

Velocity of pulley at this time

$$
\begin{aligned}
\omega & =\omega_{0}+\alpha t \\
& =0+2 \times 4.472 \\
& =8.944 \mathrm{rad} / \mathrm{s} \\
\therefore \text { Velocity of } A, \quad v_{A} & =1 \times 8.944 \\
& =8.944 \mathrm{~m} / \mathrm{s} \\
\text { and velocity of } B, \quad v_{B} & =0.6 \times 8.944 \\
& =5.367 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Ans.

Ans.

## (c) Kinematics of General Plane Motion

For the analysis of general plane motion, it is convenient to split the motion into translation and pure rotation cases. The analysis for these two cases is carried out separately and then combined to get the final motion. The kinematic analysis of general plane motion is illustrated with the following four problems in this article.

Example 8.7 Derive the relationship between the linear motion of geometric centre and angular motion of a wheel rolling without slipping.
Solution. Consider a wheel of radius $r$ rolling with angular velocity $\omega$ and angular acceleration $\alpha$. In Fig. 8.8, the dotted line shows the original position and solid line shows the position after rotating through angular distance $\theta$. Since there is no slip, linear distance $B C=s_{A}$ must be equal to angular distance $B C=r \theta$,
i.e.,

$$
\begin{equation*}
s_{A}=r \theta \tag{1}
\end{equation*}
$$



Fig. 8.8

Thus when wheel rotates without slip, the relationship between motion of its geometric centre and its angular motion are
and

$$
\left.\begin{array}{rr}
s_{A}=r \theta & \ldots(i)  \tag{8.10}\\
v_{A}=r \omega & \ldots(i i) \\
a_{A}=r \alpha & \ldots(i i i)
\end{array}\right\}
$$

Example 8.8 A wheel of radius 1 m rolls freely with an angular velocity of $5 \mathrm{rad} / \mathrm{s}$ and with an angular acceleration of $4 \mathrm{rad} / \mathrm{s}^{2}$, both clockwise as shown in Fig. 8.9(a). Determine the velocities and accelerations of points $B$ and $D$ shown in the figure.


Fig. 8.9(a)
Solution. The motion of points $B$ and $D$ may be split into translation of geometric centre $A$ and rotation about $A$. Translation of $A$ is given by

$$
\begin{aligned}
& v_{A}=r \omega=1 \times 5=5 \mathrm{~m} / \mathrm{s} \\
& a_{A}=r \alpha=1 \times 4=4 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Now, consider the rotation of $B$ about $A$. Its relative linear velocity with respect to $A$ is horizontal (normal to $A B$ ) and is given by

$$
\begin{aligned}
& v_{B / A} & =A B \times \omega=1 \times 5=5 \mathrm{~m} / \mathrm{s} \\
\therefore & v_{B} & =v_{A}+v_{B / A}=5+5=10 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Similarly, the tangential acceleration of $B$ with respect to $A$ is, $a_{B / A}=1 \times 4$ $=4 \mathrm{~m} / \mathrm{s}^{2} . B$ has also got radial inward acceleration of magnitude,

$$
a_{n}=\frac{v_{B / A}^{2}}{r}=\frac{5^{2}}{1}=25 \mathrm{~m} / \mathrm{s}^{2}
$$

Hence the acceleration of $B$ is given by the three vectors shown in Fig. 8.9(b).

$$
a_{B}=\sqrt{8^{2}+25^{2}}=26.249 \mathrm{~m} / \mathrm{s}^{2}
$$



Fig. 8.9(b)
and its inclination to the horizontal is given by

$$
\tan \theta=\frac{25}{8}
$$

$$
\therefore \theta=72.26^{\circ}
$$

Ans.

Now consider the rotation of point $D$.
Due to rotation about $A, D$ has a linear velocity of

$$
v_{D / A}=r \omega=0.6 \times 5=3 \mathrm{~m} / \mathrm{s} .
$$

Tangential of $A D$, i.e., at $60^{\circ}$ to vertical

$$
\begin{array}{ll}
\therefore \quad v_{D x} & =v_{A}+v_{D / A} \sin 60^{\circ} \\
& =5+3 \sin 60^{\circ}=7.598 \mathrm{~m} / \mathrm{s} \\
& v_{D y}
\end{array}=v_{D / A} \cos 60^{\circ} .
$$



Fig. 8.9(c)
and its inclination to horizontal, $\theta$, is given by $\tan \theta=\frac{1.5}{7.598}$
$\therefore \quad \theta=11.17^{\circ}$
Ans.
Due to rotation about $A, D$ has a tangential acceleration $=r \alpha=0.6 \times 4$ $=2.4 \mathrm{~m} / \mathrm{s}^{2}$ and radial inward acceleration

$$
\frac{v_{D / A}^{2}}{r}=\frac{3^{2}}{0.6}=15 \mathrm{~m} / \mathrm{s}^{2}
$$

Since

$$
v_{D / A}=0.6 \times 5=3 \mathrm{~m} / \mathrm{s}^{2}
$$



Fig. 8.9(d)

Hence total acceleration of $D$ is vectorial sum of $(i)$ acceleration due to translation $a_{A}=4 \mathrm{~m} / \mathrm{s}^{2}$ (ii) tangential acceleration $r \alpha=2.4 \mathrm{~m} / \mathrm{s}^{2}$ due to rotation and (iii) radial inward acceleration of $15 \mathrm{~m} / \mathrm{s}^{2}$.


Fig. 8.9(e)


Fig. 8.9(f)

Let $a_{D}$ be acceleration of $D$ and its inclination to horizontal be $\theta$ as shown in Fig. 8.9(f).

Then

$$
\begin{aligned}
a_{D} \sin \theta & =Y=2.4 \cos 60^{\circ}+15.0 \cos 30^{\circ} \\
& =14.190 \mathrm{~m} / \mathrm{s}^{2} \downarrow \\
a_{D} \cos \theta & =4+2.4 \sin 60^{\circ}-15 \sin 30^{\circ} \\
& =\overline{-1.422} \mathrm{~m} / \mathrm{s}^{2}=\overleftarrow{1.422} \mathrm{~m} / \mathrm{s}^{2} \\
a_{D} & =\sqrt{14.190^{2}+1.422^{2}}=14.261 \mathrm{~m} / \mathrm{s}^{2} \\
\theta & =\tan ^{-1} \frac{14.190}{1.422}=84.277^{\circ}
\end{aligned}
$$

Ans.
Ans.

Example 8.9 A slender bar $A B$ of length 3 m which remains always in the same vertical plane has its ends $A$ and $B$ constrained to remain in contact with a horizontal floor and a vertical wall respectively as shown in Fig. 8.10(a). Determine the velocity and acceleration of the end $B$ at the position shown in the figure, if the point $A$ has a velocity of $2 \mathrm{~m} / \mathrm{s}$ and an acceleration of $1.6 \mathrm{~m} / \mathrm{s}^{2}$ towards left.


Fig. 8.10 (a), (b), (c)
Solution. The plane motion of $B$ can be split into translation of $A$ and rotation about $A$ as shown in Fig. 8.10(b). Due to rotation about $A, B$ has the relative velocity $v_{B / A}=3 \omega$ where $\omega$ is angular velocity of $B$ about $A$. Since the actual velocity of $B$ is vertically downwards the vector diagram for the velocity of $B$ is as shown in Fig. 8.10(c). From vector diagram.

$$
\begin{aligned}
v_{B} & =v_{A} \cot 60^{\circ}=2 \cot 60^{\circ} \\
& =1.555 \mathrm{~m} / \mathrm{s} \\
v_{B / A} & =\frac{v_{A}}{\sin 60^{\circ}}=\frac{2}{\sin 60^{\circ}}=2.309 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Ans.

Acceleration of point $B$ : There are three components of acceleration as shown in Fig. 8.10(d).
(i) Due to translation of $A=a_{A}=1.6 \mathrm{~m} / \mathrm{s}^{2}$
(ii) Due to rotation about $A$, tangential acceleration $=r \alpha=3 \alpha$ where $\alpha$ is angular acceleration, and
(iii) radial inward acceleration $=\frac{v_{B / A}^{2}}{r}$

$$
=\frac{2.3094^{2}}{3}=1.778 \mathrm{~m} / \mathrm{s}^{2}
$$



Fig. 8.10 (d), (e)
The net resultant acceleration of $B, a_{B}$ is vertically downwards as shown in vector diagram 8.10 (e). Since the resultant acceleration $a_{B}$ is not having any component in the horizontal direction, considering horizontal forces, we get
or

$$
\alpha=0.958 \mathrm{rad} / \mathrm{s}^{2}
$$

Considering vertical components, we get
i.e.,

$$
\begin{aligned}
& a_{B}=1.778 \sin 60^{\circ}+3 \alpha \sin 30^{\circ} \\
& a_{B}=2.9776 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Ans.
Example 8.10 Length of crank $A B$ is 100 mm and that of connecting rod is 250 mm as shown in Fig. 8.11(a). If the crank is rotating in clockwise direction at 1500 rpm , for the position shown in Fig. 8.11(a), determine the angular velocity of the connecting rod and the velocity of the piston.


Fig. 8.11(a)

## Solution.

Angular velocity of the crank

$$
\begin{aligned}
\omega & =1500 \mathrm{rpm} \\
& =\frac{1500 \times 2 \pi}{60}=157.08 \mathrm{rad} / \mathrm{s}
\end{aligned}
$$



Fig. 8.11(b)
$\therefore \quad$ Tangential velocity of end $B$ [Fig. 8.11(b)] is given by

$$
\begin{aligned}
v_{B} & =r \omega=0.100 \times 157.080 \\
& =15.708 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Now consider the motion of connecting rod $B C$. Inclination of $B C$ to horizontal ' $\theta$ ' is given by

$$
\begin{aligned}
\frac{100}{\sin \theta^{\circ}} & =\frac{250}{\sin 30^{\circ}} \\
\sin \theta & =\frac{100 \times \sin 30^{\circ}}{250} \text { or } \theta=11.537^{\circ}
\end{aligned}
$$

The motion of $B C$ can be split into translation of $B$ and rotation of $B C$ about point $B$ as shown in Fig. 8.11(c).


Fig. 8.11(c)
Let $\omega^{\prime}$ be the angular velocity of $B C$. Tangential velocity of point $C$ w.r.t. $B$, due to rotation, $v_{C / B}$ is given by

$$
v_{C / B}=r \omega^{\prime}=0.25 \omega^{\prime}
$$

The resultant velocity of point $C$ is $v_{C}$ and is horizontal. Considering the vertical component of velocities, we get

$$
\begin{aligned}
0 & =v_{B} \sin 60^{\circ}+v_{C / B} \cos \theta \\
& =15.708 \sin 60^{\circ}+0.25 \omega^{\prime} \cos 11.537^{\circ} \\
& =13.6035+0.2449 \omega^{\prime} \\
\text { i.e., } \quad \omega^{\prime} & =55.547 \mathrm{rad} / \mathrm{s}
\end{aligned}
$$

Ans.
Considering the horizontal component of velocities, we get

$$
\begin{aligned}
v_{C} & =v_{B} \cos 60^{\circ}+v_{C / B} \sin \theta \\
& =15.7080 \cos 60^{\circ}+0.25 \times 55.547 \times \sin 11.5378^{\circ} \\
\text { i.e., } \quad v_{C} & =10.631 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Ans.

## Instantaneous Axis of Rotation

In the previous article, it is shown that plane motion of a point $\left(v_{B}\right)$ can be split into translation of another point $\left(v_{A}\right)$ and rotation of about that point $\left(r \omega=v_{B / A}\right)$. At any instant, it is possible to locate a point in the plane which has zero velocity, and hence plane motion of other points may be looked upon as pure rotation about this point. Such point is called Instantaneous Centre and the axis passing through this point and at right angles to the plane of motion is called Instantaneous Axis of Rotation.

Consider the rigid body, as shown in Fig. 8.12, which has plane motion. Let $B$ be a point having velocity $v_{B}$ at the instant considered. Now locate a point $C$ on perpendicular to the direction $v_{B}$ at $B$ at a distance $r_{B}$ from the axis of rotation. Now plane motion of $B$ can be split into translation of $C$ and rotation about $C$.
$\therefore v_{B}=v_{C}+r_{B} \omega$ and the direction of the velocity is at right angles to $C B$. If we write $r_{B}=\frac{v_{B}}{\omega}$, then from the above relation we get


Fig. 8.12

$$
v_{B}=v_{C}+\frac{v_{B}}{\omega} \cdot \omega
$$

$$
\text { i.e., } \quad v_{C}=0
$$

Thus if point ' $C$ ' is selected at a distance $\frac{v_{B}}{\omega}$ along the perpendicular to the direction of velocity at $B$, the plane motion of $B$ reduces to pure rotation about $C$. Hence $C$ is instantaneous centre.

If $D$ is any other point on the rigid body, its velocity will be given by

$$
\begin{aligned}
v_{D} & =v_{C}+r_{D} \omega \\
& =r_{D} \omega, \text { since } v_{C}=0,
\end{aligned}
$$

and it will be at right angles to $C D$.
Thus if the instantaneous centre is located, motion of all other points at that instant can be found by pure rotation about $C$.

## Methods of Locating Instantaneous Centre

Instantaneous centre can be located by any one of the following two methods:
(i) If the angular velocity $\omega$ and linear velocity $v_{B}$ are known, instantaneous centre is located at a distance $\frac{v_{B}}{\omega}$ along the perpendicular to the direction of $v_{B}$ at $B$.
(ii) If the linear velocities of two points of the rigid body are known, say $v_{B}$ and $v_{D}$, drop perpendiculars to them at $B$ and $D$. The intersection point is the instantaneous centre.

## Use of Instantaneous Centre

If instantaneous centre is located, the velocities of all other points on the rigid body at that instant can be determined by treating the plane motion as pure rotation about the instantaneous centre. It may be noted that point $C$ located as instantaneous centre has zero velocity only at that instant. If it has zero velocity at time $t$, at time $t+d t$ it has some velocity. It means, the instantaneous centre is not having zero acceleration. Hence for the acceleration calculations, we can not treat the plane motion as pure rotation about instantaneous centre.

The plane motion can be treated as a case of pure rotation about instantaneous centre only for velocity calculations.

Example 8.11 Determine the velocities of the points $B$ and $D$ given in Example 8.8 by instantaneous centre method.

Solution. Angular velocity $\omega=5 \mathrm{rad} / \mathrm{s}$
Wheel radius $=1 \mathrm{~m}$
$\therefore$ Velocity of geometric centre $A, v_{A}=\omega=1 \times$ $5=5 \mathrm{~m} / \mathrm{s}$ and it is horizontal in the rightward direction. Hence instantaneous centre is in the vertically downward direction at a distance

$$
=v_{A} / \omega=5 / 5=1 \mathrm{~m} \text {, i.e., at point } C \text {, which }
$$ is in contact with floor (see Fig. 8.13).

$$
\begin{aligned}
\therefore \quad v_{B} & =C B \times \omega=2 \times 5 \\
& =10 \mathrm{~m} / \mathrm{s} \\
v_{D} & =C D \times \omega
\end{aligned}
$$



Fig. 8.13

Now $C P=C A+A P$, where $P$ is foot of $\perp^{r}$ of $D$ on $C B$.

$$
\begin{array}{lrl}
\therefore & C P & =1+0.6 \times \sin 60^{\circ}=1.520 \mathrm{~m} \\
& P D & =0.6 \times \cos 60^{\circ}=0.3 \mathrm{~m} \\
\therefore & C D=\sqrt{C P^{2}+P D^{2}} & =\sqrt{1.520^{2}+0.3^{2}} \\
& & =1.549 \mathrm{~m} \\
\therefore & v_{D} & =C D \times \omega=1.549 \times 5.0 \\
\text { i.e., } & v_{D} & =7.745 \mathrm{~m} / \mathrm{s}
\end{array}
$$

Ans.
Its direction is at right angle to $C D$. Its inclination to horizontal, $\theta$ is given by
i.e.,

$$
\begin{aligned}
& \theta=\angle P C D=\tan ^{-1} \frac{0.3}{1.520} \\
& 0=11.17^{\circ}
\end{aligned}
$$

Example 8.12 Find the velocity of $B$ in example 8.9 by instantaneous centre method.

Solution. Since velocity at $A\left(v_{A}\right)$ is horizontal and velocity at $B\left(v_{B}\right)$ is vertical, instantaneous centre is point $C$, which is obtained by dropping perpendicular to the directions $v_{A}$ and $v_{B}$ at points $A$ and $B$ respectively. Now,

$$
\begin{aligned}
v_{A} & =A C \times \omega \\
2 & =3 \sin 60^{\circ} \times \omega \\
\omega & =0.770 \mathrm{rad} / \mathrm{s} \\
v_{B} & =B C \times \omega=3 \cos 60^{\circ} \times 0.770
\end{aligned}
$$

and

$$
\text { i.e., } \quad v_{B}=1.555 \mathrm{~m} / \mathrm{s}
$$



Fig. 8.14

Ans.

### 8.3 KINETICS OF RIGID BODY TRANSLATION

Since in this case all the particles in the body possess same acceleration, from Newton's second law of motion, we have

$$
R=\Sigma m_{i} a_{i}=m a_{c}
$$

where $a_{c}$ is the acceleration of mass centre. Hence rigid body kinetic problems are not different from the kinetic problems of particle dealt in Chapter 7. All the problems dealt under D'Alembert's principle, impulse momentum equation, workenergy principles and impact of elastic body problems may be looked as the problems under kinetics of rigid body translation.

In case of curvilinear translation, we know that there will be radial inward acceleration due to tangential velocity and it is given by

$$
a_{n}=\frac{v^{2}}{\rho}=\frac{v^{2}}{r},
$$

where $r$ is radius of the curve. Due to this, there is radial inward force of magnitude $m \frac{v^{2}}{r}=\frac{w}{g} \frac{v^{2}}{r}$ experienced by the body. This is known as centripetal force. To keep the body in dynamic equilibrium, the inertia force $m \frac{v^{2}}{r}=\frac{w}{g} \frac{v^{2}}{r}$ is to be applied to body in radial outward direction. This force is known as centrifugal force. While writing dynamic equilibrium condition, the inertia force also should be considered. In this article, we consider the problems of vehicles moving in curvilinear path with uniform velocity. The problems involving vehicles moving with acceleration in tangential directions need consideration of inertia force in tangentially opposite direction of motion also. Under this, we will consider:
(a) Motion on level roads
(b) Need for banking of roads and super elevations of rails.
(c) Design speed and
(d) Skidding and overturning on banked roads

## (a) Motion on Level Roads

Consider a body moving with uniform velocity on a circular curve of radius $r$. Let the road be flat. The cross-sectional view is shown in Fig. 8.15. If $W$ is the weight of the body, the inertia force is given by

$$
\begin{equation*}
\frac{W}{g} a=\frac{W}{g} \frac{v^{2}}{r} \tag{8.11}
\end{equation*}
$$

It acts radially outward. Let pressure on inner wheel be $R_{1}$ and on outer wheel be $R_{2}$, total frictional forces be $F$ and centre of gravity be at a height $h$.


Fig. 8.15

## Condition for Skidding

According to D'Alembert's principle, the body is in dynamic equilibrium under the action of forces;

$$
\begin{aligned}
& \quad W \text { - Weight of the vehicle } \\
& R_{1}, R_{2} \text { - The reactions } \\
& F \text { - Frictional force } \\
& \frac{W}{g} \frac{v^{2}}{r} \text { — Inertia force }
\end{aligned}
$$

Skidding takes place when frictional force reaches limiting value, i.e., when $F=\mu W$, where, $\mu$ is the coefficient of friction between road surface and the wheels.

When skidding is about to take place,
Forces in horizontal direction $=0$
i.e.,

$$
\begin{align*}
\frac{W}{g} \frac{v^{2}}{r} & =F=\mu W \\
v^{2} & =\mu g r \\
v & =\sqrt{\mu g r} \tag{8.12}
\end{align*}
$$

This is the limiting value of velocity from the consideration of skidding.

## Condition for Overturning

Let
$B$ - Distance between inner and outer wheels
$h$ - Height of C.G. from ground level
Taking moment about outer wheel, we get

$$
\begin{align*}
R_{1} B+\frac{W}{g} \frac{v^{2}}{r} h-W \frac{B}{2} & =0 \\
R_{1} & =\frac{W}{2}-\frac{W}{g} \frac{v^{2}}{r} \frac{h}{B} \tag{8.13}
\end{align*}
$$

and hence

$$
\begin{align*}
R_{2} & =W-R_{1} \\
& =W-\frac{W}{2}+\frac{W}{g} \frac{v^{2}}{r} \frac{h}{B} \\
& =\frac{W}{2}+\frac{W}{g} \frac{v^{2}}{r} \frac{h}{B} \tag{8.14}
\end{align*}
$$

From Eqn. (8.13), it may be observed that as velocity of the body increases, $R_{1}$ decreases. A stage may reach when $R_{1}$ becomes zero and after that vehicle overturns on outer wheel. To avoid this situation, the maximum speed should be limited to value of $v$ given by Eqn. (8.13) when $R_{1}=0$.

$$
\begin{array}{ll}
\text { i.e., } & 0=\frac{W}{2}-\frac{W}{g} \frac{v^{2}}{r} \frac{h}{B} \\
\text { or } & v^{2}=\frac{g r}{2} \frac{B}{h} \\
\text { i.e., } & v=\sqrt{\frac{g r}{2} \frac{B}{h}}
\end{array}
$$

In case of rail cars, the required lateral force $F$ is offered by bolts holding the rail to the sleepers as shown in Fig. 8.16. The bolts are designed to resist such force and there is no chance of skidding. However, chance of overturning is there. The distance between inner and outer wheel is equal to gauge of the railway track, which is normally represented by letter $G$. Hence Eqn. (8.15) take the form


Fig. 8.16

$$
\begin{equation*}
v=\sqrt{\frac{g r}{2} \frac{G}{h}} \tag{8.16}
\end{equation*}
$$

## (b) Need For Banking of Roads and Super Elevation of Rails

When a vehicle moves around a flat curve, a radial outward force $\frac{W}{g} \frac{v^{2}}{r}$ is experienced which increases the pressure on the outer wheel and decreases the pressure on the inner wheel. It creates discomfort to passengers. Apart from this, the speed is to be reduced considerably on curved path to avoid skidding and overturning. If the road is banked (sloping downward towards centre of curve in the cross-section) or the track is given super elevation (raising the outer rail over inner rail) as shown in Fig. 8.17, the following may be achieved:
(i) Skidding and overturning avoided


Fig. 8.17
(ii) Higher speed permitted
(iii) Lateral pressure $F$ eliminated, giving comfort to passenger
(iv) Excess wear and tear of wheels avoided.

## (c) Design Speed

The speed on a banked curved path for which no lateral pressure develops is called the 'design speed on that curve. If vehicle moves with designed speed, there will be equal pressure on inner and outer wheels and hence passengers will not experience any discomfort.

Let $\alpha$ be the angle of banking as shown in Fig. 8.18. Various forces to be considered for dynamic equilibrium at this stage are:
(i) Self weight ' $W$ ' acting vertically downward,
(ii) Centrifugal force $\frac{W}{g} \frac{v^{2}}{r}$ acting radially outward (horizontal), and
(iii) Reactions $R_{1}$ and $R_{2}$ acting normal to the road surface.
For designed speed, there will not be any lateral (frictional) force and wheel reactions are normal to road.
$\Sigma$ Forces to the inclined plane $=0$, gives


Fig. 8.18

$$
\begin{align*}
\frac{W}{g} \frac{v^{2}}{r} \cos \alpha-W \sin \alpha & =0 \\
\tan \alpha & =\frac{v^{2}}{g r} \tag{8.17}
\end{align*}
$$

This is the relationship between angle of banking and design speed. If designed speed is higher, angle of banking is also higher.

## (d) Skidding and Overturning on Banked Roads

The chances of skidding and overturning on banked roads is not ruled out if the vehicles move at speeds much higher than designed speed. However, this speed
works out to be much more than what it would be [Eqns. (8.12) and (8.16)] on flat roads.

Let the frictional forces developed at inner and outer wheels be $F_{1}$ and $F_{2}$ respectively. Free body diagram of the vehicle moving on a banked curved path along with inertia force
$\frac{W}{g} \frac{v^{2}}{r}$ is shown in Fig. 8.19.
(i) Condition for skidding

Fig. 8.19


Consider the dynamic equilibrium of the body in lateral direction.
Summing up the forces normal to the inclined surface, we get

$$
\begin{equation*}
R_{1}+R_{2}=W \cos \alpha+\frac{W}{g} \frac{v^{2}}{r} \sin \alpha \tag{1}
\end{equation*}
$$

Summing up forces parallel to the inclined surface, we get

$$
\begin{equation*}
F_{1}+F_{2}+W \sin \alpha=\frac{W}{g} \frac{v^{2}}{r} \cos \alpha \tag{2}
\end{equation*}
$$

At the time of skidding, $F_{1}=\mu R_{1}$ and

$$
F_{2}=\mu R_{2}
$$

where, $\mu$ is coefficient of friction
$\therefore \quad$ Equation (2) reduces to

$$
\mu\left(R_{1}+R_{2}\right)+W \sin \alpha=\frac{W}{g} \frac{v^{2}}{r} \cos \alpha
$$

From Eqn. (1), substituting the value of $R_{1}+R_{2}$, we get

$$
\mu\left(W \cos \alpha+\frac{W}{g} \frac{v^{2}}{r} \sin \alpha\right)+W \sin \alpha=\frac{W}{g} \frac{v^{2}}{r} \cos \alpha
$$

or

$$
\mu \cos \alpha+\sin \alpha=\frac{v^{2}}{g r}(\cos \alpha-\mu \sin \alpha)
$$

or

$$
\begin{align*}
\frac{v^{2}}{g r} & =\frac{\mu \cos \alpha+\sin \alpha}{\cos \alpha-\mu \sin \alpha} \\
& =\frac{\mu+\tan \alpha}{1-\mu \tan \alpha} \\
v & =\sqrt{g r \frac{\mu+\tan \alpha}{1-\mu \tan \alpha}} \tag{8.18}
\end{align*}
$$

Substituting $\mu=\tan \phi$, where, $\phi$ is angle of friction, we get

$$
\begin{align*}
\frac{v^{2}}{g r} & =\frac{\tan \phi+\tan \alpha}{1-\tan \phi \tan \alpha} \\
\text { i.e., } \quad v & =\sqrt{\tan (\alpha+\phi)+g r} \tag{8.19}
\end{align*}
$$

(ii) Condition for overturning:

Let $h$ be the height of centre of gravity of vehicle above road surface and $B$ be the distance between inner and outer wheels. At the time of overturning, $R_{1}=$ 0 . Taking moment about contact point of outer wheel with the road, we have

$$
\begin{align*}
& W \cos \alpha \frac{B}{2}+W(\sin \alpha) h+\frac{W v^{2}}{g r} \sin \alpha \frac{B}{2}-\frac{W v^{2}}{g r} \cos \alpha h=0 \\
& \frac{B}{2} \cos \alpha+h \sin \alpha=\frac{v^{2}}{g r}\left[\left(-\frac{B}{2} \sin \alpha+h \cos \alpha\right)\right] \\
& \text { i.e., } \quad \frac{v^{2}}{g r}=\frac{(B / 2) \cos \alpha+h \sin \alpha}{h \cos \alpha-(B / 2) \sin \alpha}=\frac{B+2 h \tan \alpha}{2 h-B \tan \alpha} \\
& v=\sqrt{g r \frac{B+2 h \tan \alpha}{2 h-B \tan \alpha}}
\end{align*}
$$

In case of rail cars, if $e$ is super elevation and $G$ is the gauge, we have
and

$$
\begin{aligned}
\tan \alpha & =\frac{e}{G} \\
B & =G
\end{aligned}
$$

Hence limiting speed from the consideration of overturning is

$$
\begin{align*}
v & =\sqrt{g r \frac{G+2 h e / G}{2 h-(e / G) G}} \\
& =\sqrt{g r \frac{G+(2 h e / G)}{2 h-e}} \tag{8.21}
\end{align*}
$$

Note: There is no question of skidding in rail cars.

Example 8.13 An automobile weighing 25 kN moves on a road, the longitudinal section of which is shown in Fig. 8.20. If it moves with uniform velocity of 50 kmph , what is the vertical reaction experienced at points $A, B$ and C .


Fig. 8.20

## Solution.

$$
\text { Velocity } v=50 \mathrm{kmph}=13.889 \mathrm{~m} / \mathrm{s} .
$$

(i) When automobile is at A: Centrifugal force is vertically upward (radially outward). Its magnitude is

$$
\begin{aligned}
& =\frac{W v^{2}}{g r}=\frac{25}{9.81} \times \frac{13.889^{2}}{80} \\
& =6.145 \mathrm{kN} \\
\therefore \quad \text { Vertical reaction } & =W-6.145 \\
& =25-6.145 \\
& =18.855 \mathrm{kN} .
\end{aligned}
$$

Ans.
(ii) When automobile is at B: Centrifugal force is vertically downward and is equal to

$$
\begin{aligned}
& =\frac{W}{g} \frac{v^{2}}{r}=\frac{25}{9.81} \times \frac{13.889^{2}}{120} \\
& =4.097 \mathrm{kN} \\
\therefore \quad \text { Vertical reaction } & =W+4.007 \\
& =25+4.097 \\
& =29.097 \mathrm{kN}
\end{aligned}
$$

Ans.
(iii) On level track at $C$ :

Vertical reaction $=W=25 \mathrm{kN}$
Ans.
Example 8.14 A car weighing 15 kN goes round a flat curve of 50 m radius. The distance between inner and outer wheel is 1.5 m and the C.G. is 0.75 m above the road level. What is the limiting speed of the car on this curve? Determine the normal reactions developed at the inner and outer wheels if the car negotiates the curve with a speed of 40 kmph . Take coefft.


Fig. 8.21 of friction $\mu=0.4$.
Solution. Consider the dynamic equilibrium of the car, forces acting on which are as shown in Fig. 8.21.

Limiting speed ' $v$ ' from the consideration of avoiding skidding is given by

$$
\begin{aligned}
\mu W & =\frac{W}{g} \frac{v^{2}}{r} \\
v & =\sqrt{\mu g r} \\
& =\sqrt{0.4 \times 9.81 \times 50} \\
& =14.007 \mathrm{~m} / \mathrm{s} \\
& =14.007 \times \frac{60 \times 60}{1000}=50.42 \mathrm{kmph}
\end{aligned}
$$

Limiting speed from the consideration of preventing overturning:
Taking moment about point of contact of outer wheel with road and noting that $R_{1}=0$, when the vehicle is about to overturn, we get

$$
\begin{aligned}
W \times \frac{B}{2} & =\frac{W}{g} \frac{v^{2}}{r} h \\
v & =\sqrt{\frac{g r}{2} \frac{B}{h}}=\sqrt{\frac{9.81 \times 50}{2} \times \frac{1.50}{0.75}} \\
& =22.147 \mathrm{~m} / \mathrm{s}=79.73 \mathrm{kmph}
\end{aligned}
$$

$\therefore \quad$ Limiting speed $v=50.42 \mathrm{kmph}$.
Ans.
If the vehicle moves with a velocity of 40 kmph ,

$$
v=40 \mathrm{kmph}=40 \times \frac{1000}{60 \times 60}=11.111 \mathrm{~m} / \mathrm{s}
$$

Taking moment about outer wheel, we get

$$
\begin{aligned}
R_{1} \times B+\frac{W}{g} \frac{v^{2}}{r} h & =W \times \frac{B}{2} \\
R_{1} & =\frac{W}{2}-\frac{W}{g} \frac{v^{2}}{r} \frac{h}{B} \\
& =\frac{15}{2}-\frac{15}{9.81} \times \frac{11.111^{2}}{50} \times \frac{0.75}{1.5} \\
\therefore \quad R_{1} & =5.612 \mathrm{kN} \\
\text { i.e., } \quad R_{2} & =W-R_{1}=15-5.612 \\
R_{2} & =9.388 \mathrm{kN}
\end{aligned}
$$

Ans.

Example 8.15 Find the angle of banking for a highway curve 200 m radius designed to accommodate cars travelling at 120 kmph if the coefficient of friction between tyres and the road is 0.6 .
Solution. $v=120 \mathrm{kmph}=33.333 \mathrm{~m} / \mathrm{s}$

If $\alpha$ is the angle of banking, then

$$
\begin{aligned}
\tan \alpha & =\frac{33.333^{2}}{9.81 \times 200}=0.566 \\
\alpha & =29.52^{\circ}
\end{aligned}
$$

Ans.
Example 8.16 Find at what speed a vehicle can move round a curve of 40 m radius without side slip (i) on a level road, (ii) on a road banked to an inclination of 1 in 10 .

At what speed can the vehicle travel on banked road without any lateral frictional force? Assume the coefficient of friction between the vehicle and the road is 0.4.
Solution. $r=40 \mathrm{~m}, \quad \mu=0.4$
(i) On level road, limiting speed from the consideration of avoiding skidding is given by

$$
\begin{aligned}
v & =\sqrt{\mu g r}=\sqrt{0.4 \times 9.81 \times 40} \\
& =12.528 \mathrm{~m} / \mathrm{s} \\
\text { i.e., } \quad v & =45.102 \mathrm{kmph}
\end{aligned}
$$

Ans.
(ii) On a road banked to an inclination of 1 in 10:

$$
\tan \alpha=1 / 10=0.1
$$

$$
v=\sqrt{g r \frac{\mu+\tan \alpha}{1-\mu \tan \alpha}}
$$

$$
=\sqrt{9.81 \times 40 \frac{0.4+0.1}{1-0.4 \times 0.1}}
$$

$$
=14.296 \mathrm{~m} / \mathrm{s}
$$

i.e.,

$$
v=51.466 \mathrm{kmph}
$$

Ans.
If lateral forces are not to be experienced, the vehicle should travel with designed speed of curve. It is given by

$$
\begin{aligned}
\tan \alpha & =\frac{v^{2}}{g r} \\
0.1 & =\frac{v^{2}}{9.81 \times 40} \\
\text { i.e., } v & =6.264 \mathrm{~m} / \mathrm{s} \\
v & =22.551 \mathrm{kmph}
\end{aligned}
$$

Ans.

Example 8.17 A car weighing 20 kN , goes around a curve of 60 m radius banked at an angle of $30^{\circ}$. Find the frictional force acting on tyres and normal reaction on outer and inner wheels, when the car is travelling at 96 kmph . The coefficient of friction between the tyres and the road is 0.6 . Take width of wheel base $B=1.6 \mathrm{~m}$ and height of C.G. of the vehicle above road level $h=0.8 \mathrm{~m}$.
Solution. Free body diagram of the vehicle along with inertia force (centrifugal force) $\frac{W}{g} \frac{v^{2}}{r}$ is shown in Fig. 8.22.


Fig. 8.22
Now, the velocity of car $v=96 \mathrm{kmph}=26.667 \mathrm{~m} / \mathrm{s}$
Consider the dynamic equilibrium of the vehicle.
$\Sigma$ Forces parallel to the road surface $=0$, gives

$$
\begin{aligned}
F+W \sin 30^{\circ} & =\frac{W}{g} \frac{v^{2}}{r} \cos 30^{\circ} \\
F & =W\left[\left(v^{2} / g r\right) \cos 30^{\circ}-\sin 30^{\circ}\right] \\
& =20\left[\frac{26.667^{2}}{9.81 \times 60} \cos 30^{\circ}-\sin 30^{\circ}\right] \\
& =10.925 \mathrm{kN}
\end{aligned}
$$

Ans.
Taking moment about point of contact of outer wheel with road surface, we get

$$
\begin{aligned}
& R_{1} \times B-W \sin 30^{\circ} \times h-W \cos 30^{\circ} \times \frac{B}{2}+\frac{W}{g} \frac{v^{2}}{r} \cos 30^{\circ} \times h \\
&-\frac{W}{g} \frac{v^{2}}{r} \sin 30^{\circ} \times \frac{B}{2}=0 \\
& R_{1}=W\left[\frac{h}{B} \sin 30^{\circ}+\frac{1}{2} \cos 30^{\circ}+\frac{v^{2}}{g r} \frac{1}{2} \sin 30^{\circ}-\frac{h}{B} \cos 30^{\circ}\right]
\end{aligned}
$$

$$
\begin{gathered}
=20\left[\frac{0.8}{1.6} \sin 30^{\circ}+\frac{1}{2} \cos 30^{\circ}+\frac{26.667^{2}}{9.81 \times 60}\left(\frac{1}{2} \sin 30^{\circ}-\frac{0.8}{1.6} \cos 30^{\circ}\right)\right] \\
R_{1}=9.238 \mathrm{kN}
\end{gathered}
$$

Ans.
Taking summation of forces normal to road surface, we get

$$
\begin{array}{ll}
R_{1}+R_{2} & =W \cos 30^{\circ}+\frac{W}{g} \frac{v^{2}}{r} \sin 30^{\circ} \\
\therefore \quad & R_{2}=20\left[\cos 30^{\circ}+\frac{26.667^{2}}{7.81 \times 60} \sin 30^{\circ}\right]-9.238 \\
\text { i.e., } \quad R_{2}=20.164 \mathrm{kN}
\end{array}
$$

Ans.
Example 8.18 Calculate the super elevation of the rail on a curved track for a locomotive running at 60 kmph , gauge and radius of curvature being 1.68 m and 800 m respectively. Find the lateral thrust exerted on the outer rail if the speed of the locomotive is changed to 80 kmph . Weight of locomotive is 1000 kN .

## Solution.

$$
G=1.68 \mathrm{~m} \text { and } r=800 \mathrm{~m}
$$

(i) $v=60 \mathrm{kmph}=(60 \times 1000) /(60 \times 60)=16.667 \mathrm{~m} / \mathrm{s}$

If $\alpha$ is the angle of banking required, for comfortable journey, (i.e., without experiencing any lateral thrust), $\alpha$ is given by

$$
\tan \alpha=\frac{v^{2}}{g r}=\frac{16.667^{2}}{9.81 \times 800}=0.03539
$$

$\therefore \quad$ Superelevation is given by
or

$$
\begin{aligned}
\frac{e}{G} & =\tan \alpha \\
e & =G \tan \alpha=1.68 \times 0.03539 \\
& =0.0595 \mathrm{~m} \\
& =59.5 \mathrm{~mm}
\end{aligned}
$$

Ans.
(ii) When speed is 80 kmph :

$$
\begin{aligned}
v & =80 \mathrm{kmph}=\frac{80 \times 1000}{60 \times 60}=22.222 \mathrm{~m} / \mathrm{s} \\
\tan \alpha & =\frac{e}{G}=0.03539 \\
\alpha & =2.02685^{\circ} \\
\sin \alpha & =0.03537 \\
\cos \alpha & =0.99937 .
\end{aligned}
$$

Summing up the forces parallel to the line joining top of rails, we get

$$
\begin{aligned}
F+W \sin \alpha & =\frac{W}{g} \frac{v^{2}}{r} \cos \alpha \\
F & =W\left[\frac{v^{2}}{g r} \cos \alpha-\sin \alpha\right] \\
& =1000\left[\frac{22.222^{2}}{9.81 \times 800} \times 0.99937-0.03537\right]
\end{aligned}
$$

i.e.,

$$
F=27.514 \mathrm{kN}
$$

Ans.
Example 8.19 What angle of bank the pilot of an aeroplane should maintain, if he wants to fly a horizontal circular path with a radius of 1300 m at a speed of 400 kmph ? Calculate the normal force on the aeroplane under the flight condition if the plane is weighing 80 kN .
Solution. Here

$$
r=1300 \mathrm{~m} \text { and } W=8 \mathrm{kN}
$$

$$
\begin{aligned}
v & =400 \mathrm{kmph}=\frac{400 \times 1000}{60 \times 60} \\
& =111.111 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Angle of bank required is given by

$$
\begin{aligned}
\tan \alpha & =\frac{v^{2}}{g r}=\frac{111.111^{2}}{9.81 \times 1300} \\
& =0.968
\end{aligned}
$$

i.e.,

$$
\alpha=44.07^{\circ}
$$

Ans.
Referring to Fig. 8.23, the magnitude of lift under the flight condition is given by

$$
\begin{aligned}
N & =W \cos \alpha+\frac{W}{g} \cdot \frac{v^{2}}{r} \sin \alpha \\
& =80\left[\cos 44.07^{\circ}+\frac{111.111^{2}}{9.81 \times 1300} \sin 44.07^{\circ}\right]
\end{aligned}
$$

i.e.,

$$
N=111.34 \mathrm{kN}
$$

Ans.

### 8.4 KINETICS OF RIGID BODY ROTATION

Consider the wheel as shown in Fig. 8.24, rotating about its axis in clockwise direction with acceleration $\alpha$. Let $\delta m$ be mass of an element at a distance $r$ from the axis of rotation. If $\delta p$ be the resulting force on this element,
$\delta p=\delta m \times a$, where $a$ is tangential acceleration

But $a=r \alpha$, where $\alpha$ is angular acceleration.

$$
\therefore \quad \delta p=\delta m r \alpha
$$

Rotational moment $\delta M_{t}$ due to this force $\delta p$ is given by

$$
\begin{aligned}
\delta M_{t} & =\delta p \times \mathrm{r} \\
& =\delta m r^{2} \alpha
\end{aligned}
$$

$$
\therefore \quad \quad M_{t}=\Sigma \delta M_{t}=\Sigma \delta m r^{2} \alpha
$$

$$
=\alpha \Sigma \delta m r^{2}
$$

$$
=\alpha I
$$



Fig. 8.24
where $I$ is mass moment of inertia of the rotating body ( $I_{z z}$ through centroid: [Eqn. (5.20)]. Thus

$$
\begin{equation*}
M_{t}=I \alpha \tag{8.22}
\end{equation*}
$$

Equation (8.22) is called Euler's equation.
Note the similarity between the expressions $M_{t}=I \alpha$ and $F=m a$ used in linear motion. Force causes linear motion while rotational moment causes angular motion. The force is equal to the product of mass and the linear acceleration, whereas rotational moment is the product of mass moment of inertia and the angular acceleration.

## Angular Momentum

The product of the mass moment of inertia and the angular velocity of a rotating body is called Angular Momentum. Thus,

$$
\begin{equation*}
\text { Angular momentum }=I \omega \tag{8.23}
\end{equation*}
$$

## Kinetic Energy of Rotating Bodies

Consider the rotating body shown in Fig 8.25 with angular velocity $\omega$. Let $\delta m$ be mass of an element which is at a distance $r$ from the axis of rotation. Hence, if $v$ is the linear velocity of the element

$$
v=r \omega
$$

Now, kinetic energy of the elemental mass

$$
=\frac{1}{2} \delta m v^{2}
$$

$\therefore$ K.E. of the rotating body

$$
\begin{aligned}
& =\Sigma \frac{1}{2} \delta m v^{2} \\
& =\Sigma \frac{1}{2} \delta m r^{2} \omega^{2} \\
& =\frac{1}{2} \omega^{2} \Sigma \delta m r^{2}
\end{aligned}
$$

But from the definition of mass moment of inertia, [Eqn. (8.23)]

$$
\begin{align*}
I & =\Sigma \delta m r^{2} \\
\therefore \quad \text { K.E. } & =\frac{1}{2} I \omega^{2} \tag{8.24}
\end{align*}
$$

Table 8.1 gives comparison between various terms used in linear motion and rotation.

Table 8.1

| Particulars | Linear Motion | Angular Motion |
| :--- | :--- | :--- |
| Displacement | $s$ | $\theta$ |
| Initial velocity | $u$ | $\omega_{0}$ |
| Final velocity | $v$ | $\omega$ |
| Acceleration | $a$ | $\alpha$ |
| Formula for final velocity | $v=u+a t$ | $\omega=\omega_{0}+\alpha t$ |
| Formula for displacement | $s=u t+\frac{1}{2} a t^{2}$ | $\theta=\omega_{0} t+\frac{1}{2} \alpha t^{2}$ |
| Formula in terms of displacement |  |  |
| velocity and acceleration | $v^{2}-u^{2}=2 a s$ | $\omega^{2}-\omega_{0}^{2}=2 \alpha \theta$ |
| Force causing motion | $F=m a$ | $M_{t}=I \alpha$ |
| Momentum | $m v$ | $I \omega$ |
| Kinetic energy | $\frac{1}{2} m v^{2}$ | $\frac{1}{2} I \omega^{2}$ |

Example 8.20 A flywheel weighing 50 kN and having radius of gyration 1 m loses its speed from 400 rpm to 280 rpm in 2 minutes. Calculate
(i) the retarding torque acting on it
(ii) change in its kinetic energy during the above period
(iii) change in its angular momentum during the same period

## Solution.

$$
\begin{array}{ll}
\qquad \omega_{0}=400 \mathrm{rpm} & =\frac{400 \times 2 \pi}{60}=41.888 \mathrm{rad} / \mathrm{s} \\
\omega=280 \mathrm{rpm} & =\frac{280 \times 2 \pi}{60}=29.322 \mathrm{rad} / \mathrm{s} \\
\text { but, } \quad \begin{aligned}
\omega= & 2 \mathrm{~min}=120 \mathrm{sec} \\
\omega & =\omega_{0}+\alpha t
\end{aligned} \\
\therefore \quad \alpha & =\frac{\omega-\omega_{0}}{t}=\frac{29.3224-41.888}{120} \\
& =-0.1047 \mathrm{rad} / \mathrm{s}^{2}
\end{array}
$$

but,
i.e., retardation is $0.1047 \mathrm{rad} / \mathrm{s}^{2}$

Weight of flywheel $=50 \mathrm{kN}=50,000 \mathrm{~N}$
Radius of gyration $k=1 \mathrm{~m}$

$$
\begin{aligned}
\therefore \quad I & =m k^{2}=\frac{50,000}{9.81} \times 1^{2} \\
& =5096.84
\end{aligned}
$$

(i) Retarding Torque Acting on the Flywheel

$$
\begin{aligned}
& =I \alpha \\
& =5096.84 \times 0.1047 \\
& =533.74 \mathrm{~N}-\mathrm{m}
\end{aligned}
$$

Ans.
(ii) Change in Kinetic Energy

$$
\begin{aligned}
& =\text { Initial K.E. }- \text { Final K.E. } \\
& =\frac{1}{2} I \omega_{0}^{2}-\frac{1}{2} I \cdot \omega^{2} \\
& =\frac{1}{2} \times 5096.84\left(41.888^{2}-27.322^{2}\right) \\
& =22,80442.9 \mathrm{~N}-\mathrm{m}
\end{aligned}
$$

Ans.
(iii) Change in its Angular Momentum

$$
\begin{aligned}
& =I \omega_{0}-I \omega \\
& =5096.84(41.888-29.322) \\
& =64048.93
\end{aligned}
$$

Ans.
Example 8.21 A pulley weighing 400 N has a radius of 0.6 m . A block of 600 N is suspended by a tight rope wound round the pulley, the other end being attached to the pulley as shown in Fig. 8.26. Determine the resulting acceleration of the weight and the tension in the rope.
Solution. Let $a$ be the resulting acceleration and $T$ be the tension in the rope. Hence angular acceleration of pulley,

$$
\begin{equation*}
\alpha=a / r=a / 0.6=1.667 \mathrm{a} \mathrm{rad} / \mathrm{s}^{2} \tag{1}
\end{equation*}
$$

An inertia force of ( $600 / \mathrm{g}$ ) $a$ may be considered and the dynamic equilibrium condition can be written for the block as:

$$
\begin{equation*}
T=\left(600-\frac{600}{9.81} a\right) \tag{2}
\end{equation*}
$$

From kinetic equation for pulley, we get
i.e.,

$$
\begin{align*}
M_{t} & =I \alpha \\
T(0.6) & =I \times 1.667 a  \tag{3}\\
I & =\frac{M r^{2}}{2}=\frac{400}{9.81} \times \frac{0.6^{2}}{2}
\end{align*}
$$

but

From (3)

$$
\begin{aligned}
T & =\frac{400}{9.81} \times \frac{0.6^{2}}{2} \times \frac{1.667 a}{0.6} \\
& =\frac{200}{9.81} a \mathrm{~N}-\mathrm{m}
\end{aligned}
$$


(a)

Fig. 8.26

Substituting it in (2), we get

$$
\begin{aligned}
\frac{200}{9.81} a & =600-\frac{600}{9.81} a \\
a & =\frac{600 \times 9.81}{800}=7.358 \mathrm{~m} / \mathrm{s}^{2} \\
\therefore \quad T & =\frac{200}{9.81} \times 7.358=150 \mathrm{~N}
\end{aligned}
$$

Ans.

Ans.

Example 8.22 The composite pulley as shown in Fig. 8.27 weighs 800 N and has a radius of gyration of 0.6 m . The 2000 N and 4000 N blocks are attached to the pulley by inextensible strings as shown in the figure. Neglecting weight of the strings, determine the tension in the strings and angular acceleration of the pulley. Solution. Since the moment of 4000 N block is more than that of 2000 N block about the axis of rotation, the pulley rotates in anticlockwise direction as shown in Fig. 8.27(b). Let $a_{A}$ be acceleration of 4000 N block and $a_{B}$ that of 2000 N block. Then

$$
a_{A}=0.5 \alpha \text { and } a_{B}=0.75 \alpha
$$

where, $\alpha$-angular acceleration of pulley. Writing dynamic equilibrium equation for the two blocks, we get

$$
\begin{aligned}
T_{A} & =4000\left(1-\frac{a_{A}}{9.81}\right) \\
& =4000\left(1-\frac{0.5}{9.81} \alpha\right) \\
T_{B} & =2000\left(1+\frac{a_{B}}{9.81}\right) \\
& =2000\left(1+\frac{0.75 \alpha}{9.81}\right)
\end{aligned}
$$


(b)

Fig. 8.27

$$
\text { i.e., } \begin{aligned}
& T_{A} \times 0.5-T_{B} \times 0.75=\frac{800}{9.81} 0.6^{2} \alpha \\
& 4000\left(1-\frac{0.5}{9.81} \alpha\right) 0.5-2000\left(1+\frac{0.75}{9.81} \alpha\right) 0.75=\frac{800}{9.81}(0.6)^{2} \alpha \\
& 245.97 \alpha=500 \\
& \alpha=2.033 \mathrm{rad} / \mathrm{s}^{2}
\end{aligned}
$$

Ans.

$$
\begin{aligned}
T_{A} & =4000\left(1-\frac{0.5}{9.81} \times 2.03\right) \\
& =3585.58 \mathrm{~N} \\
T_{B} & =2000\left(1+\frac{0.75}{9.81} \times 2.033\right) \\
& =2310.82 \mathrm{~N}
\end{aligned}
$$

Ans.

Ans.
Example 8.23 A cylinder weighing 500 N is welded to a 1 m long uniform bar of 200 N as shown in Fig. 8.28. Determine the acceleration with which the assembly will rotate about point $A$, if released from rest in horizontal position. Determine the reaction at $A$ at this instant.

(a)

(b)

Fig. 8.28
Solution. Let $\alpha$ be the angular acceleration with which the assembly will rotate. Let $I$ be the mass moment of inertia of the assembly about the axis of rotation $A$. Using the transfer formula $I=I g+M d^{2}$, we can assemble mass moment of inertia about the axis of rotation through $A$. Mass moment of inertia of the bar about $A$

$$
\begin{aligned}
& =\frac{1}{12} \times \frac{200}{9.81} \times 1^{2}+\frac{200}{9.81} \times(0.5)^{2} \\
& =6.7968
\end{aligned}
$$

Now moment of inertia of the cylinder about $A$

$$
\begin{aligned}
& =\frac{1}{12} \times \frac{500}{9.81} \times 0.2^{2}+\frac{500}{9.81} \times 1.2^{2} \\
& =74.414
\end{aligned}
$$

$\therefore$ Mass moment of inertia of the system about $A$

$$
I=6.7958+74.41=81.2097
$$

Rotational moment about $A$ (Ref. Fig. 8.28)

$$
M_{t}=200 \times 0.5+500 \times 1.2=700 \mathrm{~N}-\mathrm{m}
$$

Equating it to $I \alpha$, we get
or

$$
81.2097 \alpha=700
$$

$$
\alpha=8.6197 \mathrm{rad} / \mathrm{s}
$$

Ans.

Instantaneous acceleration of $\operatorname{rod} A B$ is vertical and its magnitude is given by

$$
\begin{aligned}
& =r_{1} \alpha=0.5 \times 8.6197 \\
& =4.310 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Similarly the instantaneous acceleration of the cylinder is also vertical and is equal to

$$
r_{2} \alpha=1.2 \times 8.6197=10.344 \mathrm{~m} / \mathrm{s}
$$

Applying D'Alembert's dynamic equilibrium equation to the system of forces shown in Fig. 8.18(b), we get

$$
\begin{array}{ll} 
& R_{A}=200+500-\frac{200}{9.81} \times 4.3100-\frac{500}{9.81} \times 10.344 \\
\text { i.e., } \quad & R_{A}=84.934 \mathrm{~N}
\end{array}
$$

Example 8.24 Rod $A B$, weighing 200 N is welded to the $\operatorname{rod} C D$ weighing 100 N as shown in Fig. 8.29(a). The assembly is hinged at $A$ and is freely held. Determine the instantaneous vertical and horizontal reactions of $A$ when a horizontal force of 300 N acts at a distance of 0.75 m from $A$.


Fig. 8.29

## Solution.

Mass moment of inertia of $A B$ about axis of rotation

$$
\begin{aligned}
A & =\frac{1}{12} \times \frac{200}{9.81} \times 1.2^{2}+\frac{200}{9.81} \times 0.6^{2} \\
& =9.786
\end{aligned}
$$

Mass moment of inertia of $\operatorname{rod} C D$ about $A$

$$
\begin{aligned}
& =\frac{1}{12} \times \frac{100}{9.81} \times 0.6^{2}+\frac{100}{9.81} \times 1.2^{2} \\
& =147.0
\end{aligned}
$$

$\therefore$ Total mass moment of the system about $A$

$$
9.786+147.0=156.78
$$

Let $\alpha$ be the instantaneous angular acceleration. Writing the kinetic equation for the rotation about $A$, we get

$$
\begin{aligned}
I \alpha & =M_{t} \\
156.786 \alpha & =300 \times 0.75 \\
\alpha & =1.4351 \mathrm{rad} / \mathrm{s}
\end{aligned}
$$

At the instant, 300 N force is applied, the linear accelerations of $A B$ and $C D$ are horizontal and are equal to $0.6 \alpha$ and $1.2 \alpha$, respectively. Hence the inertia forces are as shown in Fig. 8.29(b) by dotted arrows. Let $V_{A}$ and $H_{A}$ be the vertical and horizontal reactions at $A$. Writing the dynamic equilibrium conditions, we get
and

$$
V_{A}=200+100=300 \mathrm{~N} \quad \text { Ans. }
$$

$$
H_{A}=300-\frac{200}{9.81} 0.6 \alpha-\frac{100}{9.81} 1.2 \alpha
$$

Substituting the value of $\alpha$, we get

$$
H_{A}=264.891 \mathrm{~N}
$$

Ans.

### 8.5 KINETICS OF RIGID BODY PLANE MOTION— D'ALEMBERT'S PRINCIPLE

The plane motion of a body can be split into a translation motion and rotational motion. Hence the equations of motion are
and

$$
\begin{aligned}
R & =m a \\
M_{t} & =I \alpha
\end{aligned}
$$

It should be remembered that for rigid body motions, the force $m a$ and rotational moment $I \alpha$ are through the geometric centres of the body. By applying ' $m a$ 'and ' $I \alpha$ ' in the reverse directions of the motion, D'Alembert looked at the above equations as the equations of dynamic equilibrium. Hence the system of forces on the body along with reverse effective forces ' $m a$ ' and ' $I \alpha$ ' through geometric centres the equilibrium equations can be used as usual.

Kinetics of rolling bodies is a special case of plane motion. If the rolling is without slipping, the frictional force is to be found as coefficient of static friction times the normal reaction $\left(F=u_{s} \mathrm{~N}\right)$. If the motion is with slipping, the frictional force is equal to coefficient of dynamic friction times the normal reaction. Examples 8.25 to 8.28 illustrate the analysis of such problems.

Example 8.25 A solid cylinder weighing 1200 N is acted upon by a force $P$ horizontally as shown in Fig. 8.30(a). Determine the maximum value of $P$ for which there will be rolling without slipping. If $P=1000 \mathrm{~N}$, determine the acceleration of the mass centre and the angular acceleration, given that the coefficient of static friction $\mu_{s}=0.2$ and the coefficient of kinetic (dynamic) friction $\mu_{k}=0.15$.


Fig. 8.30
Solution. Let the linear acceleration of mass centre be $a_{A}$ and the angular acceleration be $\alpha$. The free body diagram of the cylinder along with reverse effective forces $\frac{W}{g} a_{A}$ and $I \alpha$ is shown in Fig. 8.30(b).
(i) Maximum Value of P for Rolling Without Slipping

Since there is no slipping,

$$
\begin{aligned}
a_{A} & =r \alpha=0.8 \alpha \\
I & =\frac{1}{2} \frac{W}{g} r^{2}=\frac{1}{2} \times \frac{1200}{9.81} \times 0.8^{2} \\
& =39.144 \\
\Sigma V & =0 \rightarrow N=W=1200 \mathrm{~N}
\end{aligned}
$$

From the law of friction,

$$
\begin{align*}
F & =\mu_{s} N=0.2 \times 1200 \\
& =240 \mathrm{~N} \\
\Sigma H & =0 \rightarrow \\
P-\frac{W}{g} a_{A}+F & =0 \\
\text { i.e., } \quad P-\frac{1200}{9.81} \times 0.8 \alpha+240 & =0 \\
97.859 \alpha-P & =240 \tag{1}
\end{align*}
$$

Writing moment equilibrium equation about $C$, we get

$$
P \times 1.6-\frac{W}{g} a_{A} \times 0.8-I \alpha=0
$$

$$
P \times 1.6-\frac{1200}{9.81} \times 0.8 \alpha \times 0.8-39.144 \alpha=0
$$

i.e.,

$$
\begin{align*}
1.6 \times P & =117.431 \alpha \\
P & =73.394 \alpha \tag{2}
\end{align*}
$$

From (1) and (2), we get

$$
\begin{aligned}
97.859 \alpha-73.394 \alpha & =240 \\
\alpha & =9.81 \mathrm{rad} / \mathrm{s} \\
\therefore \quad P & =73.394 \times 9.81 \\
P & =720 \mathrm{~N}
\end{aligned}
$$

Ans.
(ii) When $P=1000 \mathrm{~N}$;

In this case as slippage occurs, the relationship $a_{A}=r \alpha$ does not hold good. In this case, we have the relationship

$$
\begin{aligned}
F & =\mu_{k} N \\
& =0.15 \times 1200=180 \mathrm{~N}
\end{aligned}
$$

Taking moment about geometric centre $A$, we get

$$
\begin{aligned}
& P \times 0.8-F \times r-I \alpha=0 \\
& 1000 \times 0.8-180 \times 0.8-39.144 \alpha=0 \\
& \alpha=16.759 \mathrm{rad} / \mathrm{s}^{2} \\
& \Sigma H=0, \text { gives } \\
& \frac{1200}{9.81} a_{A}=1000+180 \\
& a_{A}=9.647 \mathrm{rad} / \mathrm{s}^{2}
\end{aligned}
$$

Ans.

Ans.
Example 8.26 A solid cylinder of weight $W$ and radius $r$ rolls down an inclined plane which makes $\theta^{\circ}$ with the horizontal axis. Determine the minimum coefficient of friction and the acceleration of the mass centre for rolling, without slipping. Solution. The free body diagram of the cylinder along with reverse effective forces $\frac{W}{g} a_{A}$ and $I \alpha$ is shown in Fig. 8.31, where $a_{A}$ is the acceleration of the mass centre and $\alpha$ is the angular acceleration.

Now,

$$
\begin{aligned}
I & =\frac{1}{2} \frac{W}{g} r^{2} \\
a_{A} & =r \alpha
\end{aligned}
$$

Taking moment about the point of contact of cylinder with the plane, we get


Fig. 8.31

$$
\begin{aligned}
W \sin \theta \times r-\frac{W}{g} a_{A} r-I \alpha & =0 \\
W \sin \theta \times r & =\frac{W}{g} r \alpha r+\frac{1}{2} \frac{W}{g} r^{2} \alpha \\
& =\frac{3}{2} \frac{W}{g} r^{2} \alpha
\end{aligned}
$$

$$
\begin{array}{ll}
\therefore & \alpha=\frac{2}{3} \frac{g}{r} \sin \theta \\
\therefore & a_{A}=r \alpha=\frac{2}{3} g \sin \theta
\end{array}
$$

Ans.
Now, $\Sigma$ forces normal to the plane $=0$, gives

$$
\begin{array}{ll} 
& N=W \cos \theta \\
\therefore & F=\mu N=\mu W \cos \theta
\end{array}
$$

$\Sigma$ Forces parallel to the plane $=0$, gives

$$
\begin{aligned}
F+\frac{1}{2} \frac{W}{g} \sin \theta-\frac{W}{g} a_{A} & =0 \\
F & =-\frac{1}{2} \frac{W}{g} \sin \theta+\frac{W}{g} \times \frac{2}{3} g \sin \theta \\
& =\frac{1}{6} W \sin \theta
\end{aligned}
$$

Hence direction of $F$ is to be reversed.

$$
\begin{aligned}
F & =\frac{1}{6} W \sin \theta \\
\text { i.e., } \quad \mu \mathrm{W} \cos \theta & =\frac{1}{6} W \sin \theta \\
\mu & =-\frac{1}{6} \tan \theta
\end{aligned}
$$

Ans.
Example 8.27 A uniform bar 1.6 m long and weighing 300 N is held in position with its ends in contact with smooth surfaces as shown in Fig. 8.32(a). If the bar is released with no velocity, determine the angular acceleration of the rod and the reactions at ends, just after release.
Solution. Let $\alpha$ and $a$ be the angular acceleration and linear acceleration of centre of gravity of the rod, respectively. The direction of accelerations $a_{A}$ and $a_{B}$ of points $A$ and $B$ are known (Fig. 8.32). The relative velocity of $B$ with respect to $A$ is $1.6 \alpha$ and is directed at right angles to the position of rod. Hence, from vector diagram of the acceleration shown in Fig. 8.32(c), we get


Fig. 8.32
and

$$
\begin{aligned}
\frac{a_{A}}{\sin 75^{\circ}} & =\frac{a_{B}}{\sin 60^{\circ}}=\frac{1.6 \alpha}{\sin 45^{\circ}} \\
a_{A} & =2.1856 \alpha \\
a_{B} & =1.9596 \alpha
\end{aligned}
$$



Fig. 8.32(b)


Fig. 8.32(c)

Now, the relative acceleration of $G$ with respect to $A$ is directed at right angles to the rod and is equal to

$$
a_{G / A}=0.8 \alpha
$$

From the vector diagram, as shown in Fig. 8.31(d) for acceleration of $G$, we get

$$
\begin{aligned}
a_{G x} & =a_{A}-a_{G / A} \cos 60^{\circ} \\
& =2.1856 \alpha-0.8 \alpha \cos 60^{\circ}
\end{aligned}
$$

$$
=1.7856 \alpha
$$

$$
a_{G Y}=a_{G / A} \sin 60^{\circ}
$$

$$
=0.8 \alpha \sin 60^{\circ}
$$

$$
=0.6928 \alpha
$$

$\therefore \quad \tan \beta=\frac{0.6928}{1.7856}$
i.e.,

$$
\beta=21.21^{\circ}
$$



Fig. 8.32(d)
and

$$
\begin{aligned}
a_{G} & =\sqrt{1.7856^{2}+0.6928^{2}} \\
& =1.915 \alpha
\end{aligned}
$$

Now consider the kinetics of the rod.
Let $E$ be the point of interaction of the reaction $N_{A}$ and $N_{B}$. Then the required distances for taking moment about $E$ can be easily worked out. Given system of forces [Fig. 8.32(e)] are the same as the effective force system shown in Fig. 8.32(f).


Fig. 8.32(e)


Fig. 8.32(f)

$$
\text { Now, } \quad \begin{aligned}
I & =\frac{1}{12} \frac{W}{g} l^{2} \\
& =\frac{1}{12} \times \frac{300}{9.81} \times 1.6^{2} \\
& =6.5240 .
\end{aligned}
$$

Equating the moment of the two systems about $E$, we get
or

$$
\begin{aligned}
300 \times 0.6928 & =\frac{300}{9.81} \times 1.7856 \alpha+\frac{300}{9.81} \times 0.6928+6.5240 \alpha \\
& =82.3160 \alpha \\
\alpha & =2.5249 \mathrm{rad} / \mathrm{s}^{2} \quad \text { Ans }
\end{aligned}
$$

Taking horizontal components, we get

$$
\begin{aligned}
N_{B} \cos 45^{\circ} & =\frac{W}{g} a_{G x}=\frac{300}{9.81} \times 1.7856 \times 2.5249 \\
N_{B} & =194.98 \mathrm{~N}
\end{aligned}
$$

Ans.
Taking vertical components of the forces, we get

$$
\begin{aligned}
N_{A}+N_{B} \sin 45^{\circ}-300 & =\frac{W}{g} a_{G y} \\
N_{A}+194.98 \sin 45^{\circ}-300 & =\frac{300}{9.81} \times 0.6928 \times 2.5249 \\
\therefore \quad N_{B} & =215.62 \mathrm{~N}
\end{aligned}
$$

Ans.
Example 8.28 A 5 N piston is connected to the 80 mm rotating crank by a 200 mm connecting rod. The connecting rod weighs 12 N . In the position shown in Fig. 8.33(a), if a force of 4000 N acts on the piston, and crank rotates at 1800 rpm , determine the reactions at the pins $A$ and $B$.


Fig. 8.33(a)

## Solution.

Let $\angle B A C=\theta^{\circ}$. Then applying sine rule, we get

$$
\begin{aligned}
\frac{80}{\sin \theta} & =\frac{200}{\sin 60^{\circ}} \\
\theta & =20.2679^{\circ}
\end{aligned}
$$

Let $\omega$ be the angular velocity of the crank.

$$
\omega=\frac{2 \pi \times 1800}{60}=188.4956 \mathrm{rad} / \mathrm{s}
$$

$\therefore \quad$ Velocity of $B$ is $v_{B}=0.08 \omega$

$$
=15.0796 \mathrm{~m} / \mathrm{s}
$$

and is at right angles to $B C$.
Since $B C$ is having pure rotation about $C$ with uniform tangential velocity $v_{B}$. $B$ has acceleration along $B C$ equal to

$$
\begin{aligned}
a_{B} & =\frac{v_{B}^{2}}{r}=\frac{15.0796^{2}}{0.8} \\
& =2842.4292 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Now consider connecting rod $B A$ as shown in Fig. 8.33(b).
Let $\omega^{\prime}$ be the angular velocity of $B A$ about $B$. Then $v_{A / B}=0.2 \omega^{\prime}$

$$
v_{A}=v_{B}+v_{A / B}
$$

From vector diagram shown in Fig. 8.33(c)

$$
v_{A / B} \cos \theta=v_{B} \cos 60^{\circ}
$$

$$
0.2 \omega^{\prime} \cos 20.2679^{\circ}=15.0796 \cos 60^{\circ}
$$

$$
\omega^{\prime}=40.1873 \mathrm{rad} / \mathrm{s}
$$

$$
v_{A / b}=0.2 \omega=8.0375 \mathrm{~m} / \mathrm{s}
$$

and

$$
\begin{aligned}
v_{A} & =v_{B} \sin 60^{\circ}+v_{A / B} \sin \theta \\
& =15.8436 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$



Fig. 8.33(b)


Fig. 8.33(c)

Consider the acceleration of $A B$. Let $\alpha$ be angular acceleration [Fig. 8.33(d)].
Now,

$$
\begin{aligned}
a_{B} & =2842.4292 \text { making } 60^{\circ} \text { with horizontal } \\
a_{A / B} & =0.2 \alpha
\end{aligned}
$$

Now,

$$
a_{\mathrm{A}}=\vec{a}_{B}+\vec{a}_{A / B}
$$

Looking at vector diagram shown in Fig. 8.33(e),

$$
a_{B} \sin 60^{\circ}=a_{A / B} \cos \theta
$$

$$
2842.4292 \sin 60^{\circ}=0.2 \cos 20.2679^{\circ}
$$

$$
\alpha=13120.457 \mathrm{rad} / \mathrm{s}^{2}
$$



Fig. 8.33(d)


Fig. 8.33(e)

$$
\begin{aligned}
\therefore & a_{A / B}=0.2 \alpha \\
\therefore & =2624.0913 \\
\therefore & a_{A}
\end{aligned}=a_{B} \cos 60^{\circ}-a_{A / B} \sin 20.2679^{\circ} ~ 子 ~ 512.2027 \mathrm{~m} / \mathrm{s}^{2} .
$$

Now the acceleration of the centre of gravity of bar $A B$ can be obtained.
Various components of acceleration at $G$ are shown in Figs. 8.32(f) and $8.32(\mathrm{~g})$. Let $a_{x}$ be horizontal component and $a_{y}$ be vertical components of acceleration of C.G. of connecting $\operatorname{rod} A B$.


Fig. 8.33(f)


Fig. 8.33(g)

Then,

$$
\begin{aligned}
a_{x} & =a_{B} \cos 60^{\circ}+r \omega^{\prime 2} \cos \theta-r \alpha \sin \theta \\
& =1118.2109 \mathrm{~m} / \mathrm{s}^{2} \\
a_{y} & =r \alpha \cos \theta+r \omega^{\prime 2} \sin \theta-a_{B} \sin 60^{\circ} \\
& =-1174.862 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

i.e., $a_{y}$ is downward $1174.862 \mathrm{~m} / \mathrm{s}^{2}$.

Now consider the dynamic equilibrium of piston $A$ [Fig. 8.33(h)]

$$
\begin{array}{ll}
\Sigma H & =0, \text { gives } \\
H_{A} & =4000-\frac{50}{9.81} \times 512.2027 \\
\text { i.e., } \quad H_{A} & =3738.38 \mathrm{kN} \quad \text { Ans. }
\end{array}
$$



Fig. 8.33(h)

Inertia forces are: $\frac{W}{g} a_{x}=\frac{10}{9.81} \times 1118.2109$

$$
=1139.87 \mathrm{~N}
$$

$$
\begin{aligned}
\frac{W}{g} a_{y} & =\frac{10}{9.81} \times 1174.862 \\
& =1197.62 \mathrm{~N}
\end{aligned}
$$



Fig. 8.33(i)

$$
\begin{aligned}
I \alpha & =\frac{W}{g} \frac{l^{2}}{12} \alpha=\frac{10}{9.81} \times \frac{0.2^{2}}{12} \times 13120.457 \\
& =44.58 \mathrm{~N}-\mathrm{m}
\end{aligned}
$$

Taking moment about $B$, we get

$$
\begin{aligned}
& v_{A} \times 0.1876-H_{A} \times 0.0693-10 \times 0.0938 \\
& +\frac{W}{g} a_{y} \times 0.0938+\frac{W}{g} a_{x} \times 0.03478-I \alpha=0
\end{aligned}
$$

or

$$
v_{A}=813.95 \mathrm{~N}
$$

Ans.
$\Sigma V=0$, gives
$V_{B}=V_{A}+\frac{W}{g} a_{y}-w$
i.e.,

$$
V_{B}=2001.57 \mathrm{~N} \downarrow
$$

$$
H_{B}=H_{A}-\frac{W}{g} a_{x}
$$

i.e.,

$$
H_{B}=2598.51 \mathrm{~N} .
$$

Ans.

Ans.

### 8.6 IMPULSE MOMENTUM METHOD

Referring to geometric centre of the rigid body, linear impulse-momentum equation derived in Chapter 7 for kinetics of particle may be applied to rigid body motion, i.e.,

$$
\begin{align*}
\int_{0}^{t} R d t & =m v-m u \\
& =m v-m v^{\prime} \tag{8.25}
\end{align*}
$$

where $u=v^{\prime}$ is the initial velocity
holds good for rigid body motion as well.
From Euler's equation [Eqn. (8.22)], we have

$$
\begin{aligned}
M_{t} & =I \alpha \\
& =I \frac{d \omega}{d t} \\
\therefore \quad \int_{0}^{t} M_{t} d t & =\int_{w_{1}}^{w_{2}} I \frac{d \omega}{d t}
\end{aligned}
$$

where $\omega_{1}=v^{\prime}$ is the angular velocity (when $t=0$ ) and $\omega_{2}$ is the final velocity (i.e., after time interval $t$ ), then

$$
\begin{equation*}
\int_{0}^{t} M_{t} d t=I\left(\omega_{2}-\omega_{1}\right) \tag{8.26}
\end{equation*}
$$

Equation (8.26) is called angular impulse-momentum principle. It may be noted that for plane motion of rigid bodies, the angular impulse as well as angular momentum is to be applied about mass centres.

When two bodies have impact, there is neither force nor moment acting on them. Hence impulse is zero which means initial momentum is equal to final momentum. This principle is called conservation of momentum. It holds good for linear as well as angular momentum. It may be stated as follows:
and

$$
\begin{align*}
m_{1} v_{1}+m_{2} v_{2} & =m_{1} v_{1}^{\prime}+m_{2} v_{2}^{\prime}  \tag{a}\\
I_{1} \omega_{1}+I_{2} \omega_{2} & =I_{1} \omega_{1}^{\prime}+I_{2} \omega_{2}^{\prime} \tag{b}
\end{align*}
$$

It is restated that $v_{1}, v_{2}, v_{1}^{\prime}, v_{2}^{\prime}, I_{1}, I_{2}, \omega_{1}, \omega_{2}, \omega_{1}^{\prime}$ and $\omega_{2}^{\prime}$ refer to the values at geometric centre of the rigid body in motion.

All problems solved using D'Alembert's principle in Arts 8.3 to 8.5 can be solved by this method also. It may be noted that this method is more useful in the analysis of motion where time and velocities are involved.
Example 8.29 A 10 kN roller of radius 0.25 m is pulled with a force of 1.2 kN on a cricket pitch. If the roller starts from rest, determine the distance travelled by it when the velocity acquired is $3 \mathrm{~m} / \mathrm{s}$. Assume there is no slippage.
Solution. Figure 8.34(a) shows the roller and Fig. 8.34(b) shows its free body diagram. Applying impulse momentum equation, we get

$$
\begin{align*}
& \frac{10,000}{9.81}\left(v_{1}-v^{\prime}\right) & =\int_{0}^{t} R d t \\
\text { i.e., } & \frac{10,000}{9.81}(3-0) & =(1200-F) t \\
\text { i.e., } & 3058.10 & =1200 t-F t \tag{1}
\end{align*}
$$


(a)

(b)

Fig. 8.34
Now mass moment of inertia for cylindrical roller

$$
\begin{aligned}
& =\frac{M r^{2}}{2}=\frac{1}{2} \times \frac{10,000}{9.81}(0.25)^{2}=31.855 \\
\omega_{1} & =0, \omega_{2}
\end{aligned}=\frac{v}{r}=\frac{3}{0.25}=12
$$

From angular impulse-momentum principle, we have

$$
\begin{array}{rlrl}
\int M_{t} d t & =I\left(\omega_{2}-\omega_{1}\right) \\
& & \int F r d t & =31.855(12-0) \\
\text { i.e., } & F \times 0.25 t & =31.855 \times 12 \\
\therefore & F t & =1529.052 \tag{2}
\end{array}
$$

Substituting it in Eqn. (1), we get
$\therefore \quad t=3.823 \mathrm{~s}$.

Average acceleration

$$
a=\frac{3}{3.823}=0.7848 \mathrm{~m} / \mathrm{s}^{2}
$$

$\therefore \quad$ Distance travelled

$$
\begin{aligned}
s & =v_{1} t+\frac{1}{2} a t^{2} \\
& =0+\frac{1}{2} \times 0.7848(3.823)^{2} \\
s & =5.734 \mathrm{~m}
\end{aligned}
$$

Ans.
Example 8.30 A cylinder of mass $M$ and radius $r$ starts rolling down an inclined plane having angle $\theta$ with horizontal from position of rest at $A$. What is its velocity after ' $t$ ' seconds and how much distance it has moved by that time? Instead of cylinder, if it were sphere of mass $M$, what is the velocity and distance? Assume no slippage and coefficient of friction $\mu<\tan \theta$.
Solution. Figure 8.35 shows the roller on the inclined plane.

$$
\begin{aligned}
& N=W \cos \theta=M g \cos \theta \\
& F=\mu N=\mu M g \cos \theta
\end{aligned}
$$



Fig. 8.35
Component of self weight down the plane

$$
=W \sin \theta=M g \sin \theta
$$

Applying angular impulse-momentum equation in the direction down the plane, we get

$$
\int_{0}^{I} M_{t} d t=I\left(\omega_{2}-\omega_{1}\right)
$$

For cylinder

$$
I=\frac{M r^{2}}{2}
$$

$$
\omega_{1}=0, \text { given }
$$

$$
\therefore \quad \int_{0}^{t} F r d t=\frac{M r^{2}}{2} \omega_{2}
$$

i.e.,

$$
\begin{aligned}
\mu M g \cos \theta r t & =\frac{M r^{2}}{2} \omega_{2} \\
\therefore \quad \omega_{2} & =2 \frac{\mu g}{r} \cos \theta t \\
v & =r \omega_{2}=2 \mu g \cos \theta t
\end{aligned}
$$

$$
\text { Average velocity }=\frac{1}{2}(v+0)=\frac{v}{2}=\mu g \cos \theta t
$$

$\therefore$ Distance moved $\quad s=$ Average $v \times t$

$$
s=\mu g \cos \theta t^{2}
$$

Ans.

Ans.
If it is sphere instead of cylinder

$$
I=\frac{2}{5} M r^{2}
$$

$\therefore$ Impulse momentum equation yields,
or

$$
\mu M g \cos \theta r t=\frac{2}{5} M r^{2} \omega_{2}
$$

$$
\omega_{2}=2.5 \frac{\mu g}{r} \cos \theta t
$$

$\therefore \quad v=2.5 \mu \mathrm{~g} \cos \theta t$
and

$$
s=\frac{2.5}{2} \mu g \cos \theta t^{2}
$$

Ans.
Ans.

Example 8.31 Determine the velocity of the blocks shown in Fig. 8.36 after $t=5$ second, if the system starts from rest. It may be noted that the pulley is neither frictionless nor weightless. Assume the pulley as a ring of uniform-cross section.

Solution. Let the velocity of system after 5 second be $v$. Initial velocity $v_{0}=0$. Let tension in the wire connecting block of weighing 25 kN be $T_{1}$ and that on other side be $T_{2}$.

Then, linear momentum equation for block weighing 25 kN is,

$$
\begin{align*}
-\left(T_{1}-25\right) 5 & =\frac{25}{g}(v-0) \\
-T_{1}+25 & =\frac{5 v}{g} \tag{1}
\end{align*}
$$



Fig. 8.36

$$
\begin{align*}
\left(T_{2}-20\right) 5 & =\frac{20}{g}(v-0) \\
T_{2}-20 & =\frac{4 v}{g} \tag{2}
\end{align*}
$$

Adding Eqns. (1) and (2), we get

$$
\begin{equation*}
T_{2}-T_{1}+5=\frac{9 v}{g} \tag{3}
\end{equation*}
$$

Applying angular momentum equation to the pulley, we get

$$
\begin{align*}
I \omega_{2} & =\left(T_{1}-T_{2}\right) r 5 \\
& =\left(T_{1}-T_{2}\right) 0.8 \times 5=4\left(T_{1}-T_{2}\right)  \tag{4}\\
\text { But } \quad I & =\frac{M r^{2}}{2}=\frac{2}{g} \cdot \frac{(0.8)^{2}}{2}=\frac{0.64}{g}
\end{align*}
$$

Hence from Eqn. 4, we get

$$
\begin{align*}
\frac{0.64}{g} \omega_{2} & =4\left(T_{1}-T_{2}\right) \\
T_{1}-T_{2} & =\frac{0.16}{g} \omega_{2} \\
& =\frac{0.16}{g}\left(\frac{v}{0.8}\right) \quad\left[\text { Since } \quad \theta=r \omega_{2}\right] \\
& =\frac{0.2}{g} v \tag{5}
\end{align*}
$$

From Eqns. 3 and 5, we get

$$
\begin{aligned}
& -\frac{0.2}{g} v+5 & =\frac{9}{g} v \\
\therefore & v \frac{9.2}{g} & =5 \\
\therefore & v & =\frac{5 \times g}{9.2}=\frac{5 \times 9.81}{9.2} \\
\therefore & v & =5.332 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

### 8.7 WORK-ENERGY PRINCIPLE

Kinetic problems of rigid bodies may be solved by the work-energy method also. This method is ideally suited for the problems involving velocities and distances.

Since the body is rigid, no relative motion occurs between the particles of rigid body, the work done by internal forces is zero. Work done by the external forces is

$$
=\text { Force } \times \text { Distance moved in the direction of force }
$$

Work done by external moment

$$
=M \theta
$$

Kinetic energy of translation $=\frac{m v^{2}}{2}=\frac{W}{2 g} v^{2}$
Kinetic energy of rotation $=\frac{1}{2} I \omega^{2}$
However it may be noted that $v, I, \omega$ are referred to as geometric centre of the body, if the rotation is about geometric centre. If the rotation is about any other point, say $O$, as shown in Fig. 8.37, $I$ is mass moment of inertia about $O$, which is obviously


Fig. 8.37

$$
I_{0}=I+m r_{c}^{2}
$$

Hence kinetic energy of rotation about $O$ is,

$$
=\frac{1}{2} I_{0} \omega^{2}
$$

Principle of work energy states, work done in the system $=$ change of kinetic energy. In plane motion, work done in both translation and rotation should be considered. Similarly change in kinetic energy in both translation and rotation should be considered. Thus,

$$
\begin{equation*}
\text { Work done }=\mathrm{KE}_{2}-\mathrm{KE}_{1} \tag{8.28}
\end{equation*}
$$

## Conservation of Energy

Conservative system is the one in which work done by the force is independent of the path. Hence if a body is moved from position 1 to position 2, in what so ever manner,

$$
\text { Work done }=\mathrm{PE}_{1}-\mathrm{PE}_{2}
$$

where PE is the potential energy. Equating it to change in kinetic energy (Eqn. 8.28), we get

$$
\begin{array}{ll} 
& \mathrm{PE}_{1}-\mathrm{PE}_{2}=\mathrm{KE}_{2}-\mathrm{KE}_{1} \\
\text { i.e., } & \mathrm{PE}_{1}+\mathrm{KE}_{1}=\mathrm{PE}_{2}+\mathrm{KE}_{2} \tag{8.29}
\end{array}
$$

Thus the total energy is conserved. However, during the process of moving the body from position 1 to position 2, there should not be dissipation of energy in the form of heat energy, etc.

Example 8.32 Solve Example 8.29 by work-energy principle

## Solution.

$$
\begin{aligned}
w_{0} & =v_{0}=0 \\
v & =3 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

$$
\therefore \quad \omega=\frac{v}{r}=\frac{3}{0.25}=12 \mathrm{rad} / \mathrm{s}
$$

$$
I=\frac{1}{2} M r^{2}=\frac{1}{2} \times \frac{10,000}{9.81}(0.25)^{2}=31.855
$$

Final kinetic energy

$$
\begin{aligned}
& =\frac{1}{2} \frac{W}{g} v^{2}+I \omega^{2} \\
& =\frac{1}{2} \times \frac{10,000}{9.81} 3^{2}+\frac{1}{2} \times 31.855 \times 12^{2} \\
& =6880.716 \\
& \\
\text { Initial kinetic energy } & =0 \\
\therefore \quad & \quad \text { Work done }
\end{aligned}=1200 s
$$

Ans.
Example 8.33 A 1500 N block is held down with an inextensible wire wound round a solid cylinder as shown in Fig. 8.38. The radius of the cylinder is 0.1 m . If the frictional moment in the bearing is estimated to be $30 \mathrm{~N}-\mathrm{m}$, determine the moment of inertia of the cylinder, if the velocity of the block, after a fall of 1 m , is $0.25 \mathrm{~m} / \mathrm{s}$.
Solution. Work done by the block in moving 1 m

$$
=1500 \times 1=1500 \mathrm{~N}-\mathrm{m}
$$

Angular rotation of cylinder when block moves by 1 m

$$
\theta=\frac{s}{r}=\frac{1.0}{0.1}=10 \text { radians }
$$

$\therefore$ Work done by frictional moment

$$
M \theta=-30 \times 10=-300 \mathrm{~N}-\mathrm{m}
$$

Total work done $=1500-300=1200 \mathrm{~N}-\mathrm{m}$
Initial $\mathrm{KE}=0$


Fig. 8.38

$$
\omega=\frac{v}{r}=\frac{0.25}{0.1}
$$

Final

$$
\begin{aligned}
\mathrm{KE} & =\mathrm{KE} \text { of block }+\mathrm{KE} \text { of cylinder } \\
& =\frac{1}{2} \frac{1500}{9.81} 0.25^{2}+\frac{1}{2} I\left(\frac{0.25}{0.1}\right)^{2} \\
& =4.778+3.125 I
\end{aligned}
$$

$\therefore$ Equating work done to change in kinetic energy, we get

$$
\begin{array}{rlrl} 
& & 1200 & =4.778+3.125 I \\
\therefore & I & =382.47 \mathrm{~kg}-\mathrm{m}^{2}
\end{array}
$$

Example 8.34 Two blocks weighing 300 N and 450 N are hung to the ends of inextensible cable over a pulley of weight 50 N and radius 0.2 m . How much distance the blocks will move in increasing the velocity of the system from $2 \mathrm{~m} / \mathrm{s}$ to $4 \mathrm{~m} / \mathrm{s}$. What is tension in string? Use work-energy principle. Assume the pulley as a circular plate of radius 0.2 m . Refer Fig. 8.39(a).
Solution. Figure 8.39 (b) shows the free body diagrams of 300 N block, 450 N and the pulley. Let the distance moved be $s$. It was be noted that 450 N block moves downward and 300 N block moves upward. Work-energy equation for 450 N block gives,

$$
\left(450-T_{1}\right) s=\frac{1}{2} \frac{450}{9.81}\left(4^{2}-2^{2}\right)
$$


(a)

(b)

Fig. 8.39
i.e., $\quad\left(450-T_{1}\right) s=\frac{2700}{9.81}$

Work energy-equation for block 300 N gives,

$$
\text { i.e., } \left.\begin{array}{rl}
\left(T_{2}-300\right) s & =\frac{1}{2} \frac{300}{9.81}\left(4^{2}-2^{2}\right) \\
\text { For pulley, } & \left(T_{2}-300\right) s
\end{array}\right)=\frac{1800}{9.81}, ~ \begin{aligned}
r & =\frac{s}{0.2}=5 \mathrm{~s} \\
\omega & =\frac{v_{0}}{r} \\
\therefore \quad \omega_{0} & =\frac{2}{0.2}=10 \mathrm{rad} / \mathrm{s} \\
\omega_{1} & =\frac{v_{1}}{0.2}=\frac{4}{0.2}=20 \mathrm{rad} / \mathrm{s}
\end{aligned}
$$

$\therefore$ Work-energy equation for pulley is,

$$
\begin{aligned}
M_{t} \theta & =I\left(\omega_{1}^{2}-\omega_{0}^{2}\right) \\
M_{t} & =\left(T_{1}-T_{2}\right) r=\left(T_{1}-T_{2}\right) 0.2 \\
I & =\frac{m r^{2}}{2}=\frac{1}{2} \frac{50}{9.81} \times 0.2^{2}
\end{aligned}
$$

$\therefore$ Work-energy equation is
i.e.,

$$
\left(T_{1}-T_{2}\right) 0.25 s=\frac{1}{2} \frac{50}{9.81} \times 0.2^{2}\left(20^{2}-10^{2}\right)
$$

Dividing Eqn. (1) by Eqn. (2), we get

$$
\begin{align*}
\frac{450-T_{1}}{T_{2}-300} & =\frac{2700}{1800}=1.5 \\
450-T_{1} & =1.5 T_{2}-450 \\
T_{1}+1.5 T_{2} & =900 \tag{4}
\end{align*}
$$

From Eqns. (2) and (3), we have

$$
\begin{align*}
\frac{T_{2}-300}{T_{1}-T_{2}} & =\frac{1800}{9.81 \times 30.581}=6 \\
T_{2}-300 & =6 T_{1}-6 T_{2}  \tag{5}\\
7 T_{2}-6 T_{1} & =300
\end{align*}
$$

$$
\therefore \quad T_{2}-300=6 T_{1}-6 T_{2}
$$

Adding 6 times Eqn. (4) to Eqn. (5), we get

$$
\begin{array}{rlrl}
\text { i.e., } & 16 T_{2} & =5700 \\
& \therefore & T_{2} & =356.25 \mathrm{~N} \\
& \text { From Eqn. (4), } & T_{1} & =900-1.5 \times 356.25 \\
\text { i.e., } & T_{1} & =365.625 \mathrm{~N}
\end{array}
$$

Ans.

Ans.
From Eqn. (3),

$$
\begin{aligned}
(365.625-356.25) s & =30.581 \\
s & =3.262 \mathrm{~m}
\end{aligned}
$$

## Ans.

Example 8.35 A carpet rolled in the form of cylinder of radius $R$ is at rest at $A$ as shown in Fig. 8.40. When a small push is given it starts unrolling on the floor. Assuming material of carpet is inextensible and no sliding, determine the horizontal velocity of the carpet when its radius reduces to $0.6 R$.


Fig. 8.40
Solution. Rolling of the carpet is due to loss of potential energy of the cylindrical mass. As potential energy is lost, kinetic energy is gained. Hence this problem can be solved by conservation of total energy principle.

$$
\mathrm{PE}_{1}=M g R
$$

At the point of interest, the mass of the cylindrical part of carpet

$$
\begin{align*}
& =M(0.6 R)^{2} g=0.36 M g \\
\mathrm{PE}_{2} & =0.36 M g(0.6 R)=0.216 \mathrm{MgR} \\
\text { Initial KE } & =0 \\
\text { Final KE } & =\frac{1}{2}(0.36 M) v^{2}+\frac{1}{2} I \omega^{2} \tag{1}
\end{align*}
$$

where $v$ is the velocity and $\omega$ is the angular velocity.

$$
\begin{aligned}
I & =\text { Mass moment of inertia through centre of gravity } \\
& =\frac{1}{2} \times \text { Mass } \times r^{2}=\frac{1}{2}(0.36 M) \times(0.6 R)^{2} \\
& =0.0648 M R^{2} \\
\omega & =\frac{v}{r}=\frac{v}{0.6 R}
\end{aligned}
$$

$\therefore$ Equation 1 reduces to

$$
\begin{aligned}
\mathrm{KE} & =\frac{1}{2} 0.36 M v^{2}+\frac{1}{2} \times 0.0648 M R^{2}\left(\frac{v}{0.6 R}\right)^{2} \\
& =(0.18+0.09) M v^{2}=0.27 M v^{2}
\end{aligned}
$$

Equating initial total energy to final total energy, we get

$$
\begin{aligned}
M g R+0 & =0.216 M g R+0.27 M v^{2} \\
v^{2} & =\frac{0.784}{0.27} g R=2.9037 g R \\
v & =\sqrt{2.90379 g R}
\end{aligned}
$$

Ans.

## IMPORTANT DEFINITIONS AND CONCEPTS

1. Plane motion can be split into translation of the centre of gravity and rotation about an axis.
2. Translation of rigid body may be rectilinear or curvilinear. In these cases, body may be treated as translation of the mass centre and the problems can be handled as in case of kinetics of particles.
3. In case of curvilinear translation, it may be noted that always there is acceleration towards centre of curved path equal to $v^{2} / r$ where $v$ is tangential velocity. There may be tangential acceleration also.
4. At any instant it is possible to locate a point in the plane which has zero velocity and hence the plane motion of other points may be looked as pure rotation about this point. Such point is called instantaneous centre and the axis passing through this point and directed at right angles to the plane of motion is called Instantaneous Axis of Rotation.

## IMPORTANT EQUATIONS

1. In case of rectilinear motion of rigid body,

$$
\begin{aligned}
r_{B} & =r_{A}+r_{B / A} \\
v_{B} & =v_{A} \\
a_{B} & =a_{A}
\end{aligned}
$$

2. In case of curvilinear translation, radial inward acceleration $=\frac{v^{2}}{r}$.
3. In case of rotation:

$$
\begin{aligned}
\omega & =\frac{d \theta}{d t} \\
\alpha & =\frac{d^{2} \theta}{d t^{2}}=\omega \cdot \frac{d \omega}{d \theta} \\
v & =r \omega \\
a_{t} & =r \alpha \\
a_{n} & =\frac{v^{2}}{r}=r \omega^{2}
\end{aligned}
$$

4. If angular velocity is uniform,

$$
\theta=\omega t
$$

5. In case of uniformly accelerated rotation,

$$
\begin{aligned}
\omega & =\omega_{0}+\alpha t \\
\theta & =\omega_{0} t+\frac{1}{2} \alpha t^{2} \\
\omega^{2}-\omega_{0}^{2} & =2 \alpha s
\end{aligned}
$$

6. In case of curvilinear motion, inertia force (centrifugal force) is

$$
=\frac{m v^{2}}{r}=\frac{W}{g} \frac{v^{2}}{r}
$$

7. Limiting velocity of vehicle on level track
(i) From the consideration of skidding

$$
v=\sqrt{\mu g r}
$$

(ii) From the consideration of overturning is

$$
v=\sqrt{\frac{g r}{2} \frac{B}{h}}
$$

$B=$ Distance between outer and inner wheels gauge $=$ House in case of railway wagons.
8. Angle of banking for designed speed $v$ is $\tan \alpha=\frac{v^{2}}{g r}$.
9. On banked track, limiting velocity of vehicles
(i) From the consideration of skidding is

$$
v=\sqrt{g r \frac{\mu+\tan \alpha}{1-\mu \tan \alpha}}
$$

(ii) From the consideration of overturning is

$$
v=\sqrt{g r \frac{B+2 h \tan \alpha}{2 h-B \tan \alpha}}
$$

10. Rotational moment

$$
M_{t}=I \alpha
$$

11. Angular momentum $=I \omega$
12. Kinetic energy of rotation

$$
=\frac{1}{2} I \omega^{2}
$$

13. In case of plane motion,

Inertia of translation $=m a$
Inertia of rotation $=I \alpha$
14. Impulse momentum for translation is

$$
\int_{0}^{t} R d t=m v-m u=m v-m v^{\prime}
$$

Impulse momentum equation for rotation is

$$
\int_{0}^{t} M_{t} d t=I\left(\omega_{2}-\omega_{1}\right)
$$

15. Conservations of momentum equations for translation and for rotation are

$$
\begin{aligned}
& m_{1} v_{1}+m_{1} v_{2}=m_{1} v_{1}^{\prime}+m_{2} v_{2}^{\prime} \\
& I_{1} \omega_{1}+I_{2} \omega_{2}=I_{1} \omega_{1}^{\prime}+I_{2} \omega_{2}^{\prime}
\end{aligned}
$$

16. Work done by forces in case of translation $=F \times s$, where $s$ is the distance moved in the direction of force.

Angular momentum $=M \theta$
Kinetic energy of translation $=\frac{m v^{2}}{2}=\frac{W}{2 g} v^{2}$
Kinetic energy of rotation $=\frac{1}{2} I \omega^{2}$
17. Work-energy principle is

$$
W=\mathrm{KE}_{2}-\mathrm{KE}_{1}
$$

18. Conservation of energy principle is

$$
\mathrm{PE}_{1}+\mathrm{KE}_{1}=\mathrm{PE}_{2}+\mathrm{KE}_{2}
$$

## PROBLEMS FOR EXERCISE

8.1 The motion of a cam is defined by the relation $\theta=t^{3}-8 t+15$, where $\theta$ is expressed in radians and $t$ in seconds. Determine the angular displacement, angular velocity and angular acceleration after:
(a) 2 s
(b) 4 s
[Ans. (a) $\theta=7 \mathrm{rad}$ and $47 \mathrm{rad} ; \omega=4 \mathrm{rad} / \mathrm{s}$ and
(b) $40 \mathrm{rad} / \mathrm{s}, \alpha=12 \mathrm{rad} / \mathrm{s}^{2}$ and $24 \mathrm{rad} / \mathrm{s}^{2}$ ]
8.2 The angular acceleration of a flywheel is given by the expression $\alpha=6 t-t^{2}$ where $\alpha$ is in $\mathrm{rad} / \mathrm{s}^{2}$ and $t$ is in second. When will the flywheel stop momentarily prior to reversing the direction, given that the flywheel starts from rest. How many revolutions are made during this time?
[Ans. $t=9 \mathrm{~s}$, revolutions $=29.006]$
8.3 A wheel is rotating about its axis with a constant acceleration of $1 \mathrm{rad} / \mathrm{s}^{2}$. If the initial and final velocities are 50 rpm and 100 rpm , determine the time taken and number of revolutions made during this period.
[Ans. $t=5.236 \mathrm{~s}$ and revolutions $=6.545$ ]
8.4 A rotor of an electric motor is uniformly accelerated to a speed of 1800 rpm from rest for 5 seconds, and then immediately power is switched off and the rotor decelerated uniformly. If the total time elapsed from start to stop is 12.5 s , determine the number of revolutions made while (i) acceleration, (ii) deceleration. Also determine the value of deceleration.
[Ans. (i) 75 revolutions, (ii) 175 revolutions and $25.1327 \mathrm{rad} / \mathrm{s}^{2}$ ]
8.5 A wheel rotates with uniform acceleration. During 3rd and 6th seconds, wheel rotates by 8 radians and 12 radians, respectively. Determine the initial velocity and acceleration of the wheel.

$$
\left[\text { Ans. } \omega_{0}=4.667 \mathrm{rad} / \mathrm{s}, \alpha=1.333 \mathrm{rad} / \mathrm{s}^{2}\right]
$$

8.6 The pulley $B$, as shown in Fig. 8.41, is rotating at the rate of $5 \mathrm{rad} / \mathrm{s}$, clockwise. If the deceleration is observed to be 1.2 $\mathrm{rad} / \mathrm{s}^{2}$, how much distance will blocks $C$ and $D$ move before coming to rest? How many revolutions are made by pulleys $A$ and $B$ during this period? Assume that there is no slip between the wire and the pulley.
[Ans $s_{c}=4.166 \mathrm{~m}, s_{D}=8.333 \mathrm{~m}, 0.829$ revolution by $A, 1.658$ revolutions by $B$ ]
8.7 A roller weighing 1 kN starts rolling down from position $A$ on a smooth surface as shown in Fig. 8.42. What will be its velocity at $B$ and the vertical reaction on it from the surface?


Fig. 8.41


Fig. 8.42
[Ans. $17.155 \mathrm{~m} / \mathrm{s}, 2.5 \mathrm{kN}$ ]
8.8 Find at what maximum speed a vehicle can move round a flat curve of 60 m without side slip. Find also the limiting value of height of C.G. of vehicle, if overturning consideration should not limit speed of the vehicle on this curve. Given, weight of vehicle $=22 \mathrm{kN}$, base width 1.6 m and coefficient of friction between tyres and road surface $=0.5$.
[Ans. $v=17.155 \mathrm{~m} / \mathrm{s}, h=1.6 \mathrm{~m}$ ]
8.9 Find the limiting speed of vehicle on a curve of 100 m radius from the consideration of skidding and overturning
(i) if no banking is provided and
(ii) if $20^{\circ}$ banking is provided

With what speed the vehicle should negotiate the above banked curve, so that passengers do not feel any discomfort. Take weight of vehicle $=15 \mathrm{kN}$, distance between centres of inner and outer wheels $=1.5 \mathrm{~m}$, height of C.G. above road surface $=0.65 \mathrm{~m}$ and coefficient of friction $=0.5$.
[Ans. (i) With no banking $v=22.147 \mathrm{~m} / \mathrm{s}$ and
(ii) with banking $v=33.081 \mathrm{~m} / \mathrm{s}$, Design velocity $v=18.896 \mathrm{~m} / \mathrm{s}$ ]
8.10 A cyclist is riding in a horizontal circle of radius 20 m at a speed of 18 kmph . What should be the angle to the vertical of the centre line of the bicycle to ensure stability? What is the maximum velocity with which he can negotiate the curve if weight of bicycle and the man is 800 N and the C.G. is 450 mm above the road surface? Take coefficient of friction $\mu=0.6$.
[Ans. $\left.\alpha=7.26^{\circ}, v=39.06 \mathrm{kmph}\right]$
8.11 A 20 kN vehicle is going round a circular curve of radius 80 m with a velocity of 72 kmph . If the road has a banking of $25^{\circ}$, find the total frictional force developed and normal reactions at inner and outer wheels. Take base width of vehicle $B=1.5 \mathrm{~m}$, height of C.G. above road surface $=0.6 \mathrm{~m}$. Coefficient of friction $\mu=0.6$.

$$
\text { [Ans. } R_{\text {inner }}=10.903 \mathrm{kN} ; R_{\text {outer }}=11.532 \mathrm{kN} ; F=0.787 \mathrm{kN} \text { ] }
$$

8.12 A pulley weighs 500 N and has a radius of 0.75 m . A block weighing 400 N is supported by inextensible wire wound round the pulley. Determine the velocity of the block 2 second after it is released from rest. Assume that the motion is under constant acceleration.
[Ans. $12.074 \mathrm{~m} / \mathrm{s}$ ]
8.13 The pulley, as shown in Fig. 8.43, weighs 600 N and has a radius of 0.8 m . A rope passing over this pulley supports 800 N load at one end and 400 N at another end. Determine the tension in the string and the angular acceleration of the pulley if the blocks are allowed to move.
[Ans. $\alpha=3.27 \mathrm{rad} / \mathrm{s}^{2}, T_{1}=506.67 \mathrm{~N}$ and


Fig. 8.43

$$
\left.T_{2}=586.67 \mathrm{~N}\right]
$$

8.14 A solid sphere of radius 0.2 m and weighing 80 N is joined to a bar, 1.5 m long and weighing 100 N , as shown in Fig. 8.44. At what distance a horizontal force of 250 N should be applied, so that the system will get an instantaneous acceleration of $8 \mathrm{rad} / \mathrm{s}$ ? What are the reactions produced at the support.
[Ans. $V_{A}=110 \mathrm{~N} ; H_{A}=17.807 \mathrm{~N}$ ]


Fig. 8.44
8.15 The wheel of a tractor is 1 m in diameter and the tractor is travelling at 9 kmph . What are the velocities at the top and bottom of the wheel relative to (i) a person seated in tractor; (ii) a person standing on the ground.

$$
\text { [Ans. (i) } v_{\text {Top }}=2.5 \mathrm{~m} / \mathrm{s} ; v_{\text {bottom }}=2.5 \mathrm{~m} / \mathrm{s} \text {; }
$$

(ii) $\left.v_{\text {Top }}=5.0 \mathrm{~m} / \mathrm{s} ; v_{\text {bottom }}=0\right]$
8.16 $\operatorname{Rod} A B$, as shown in Fig. 8.45, is 2 m long and slides with its ends in contact with the floor and the inclined plane. At the instant when $A B$ makes $40^{\circ}$ with the horizontal, point $A$ has left ward velocity of $5 \mathrm{~m} / \mathrm{s}$. Determine the angular velocity of the rod and the velocity of end $B$.
[Ans. $\omega=2.3040 \mathrm{rad} / \mathrm{s} ; v_{B}=4.0760 \mathrm{~m} / \mathrm{s}$ ]
8.17 In Problem 8.16, if $A$ has acceleration of $3 \mathrm{~m} / \mathrm{s}^{2}$ to the left at the instant as shown in Fig. 8.45, determine angular acceleration of the rod and resultant acceleration of end $B$.
[Ans. $\alpha=3.3145 \mathrm{rad} / \mathrm{s}^{2} ; a_{B}=13.7438 \mathrm{~m} / \mathrm{s}^{2}$ ]


Fig. 8.45
8.18 Figure 8.46 shows a reciprocating pump driven by a driving wheel. If crank is 80 mm long and connecting rod 200 mm , determine the velocity of the piston in the position shown. The driving wheel rotates at 2000 rpm in anticlockwise direction.
[Ans. $15.34 \mathrm{~m} / \mathrm{s}$ ]


Fig. 8.46
8.19 Solve Problem 8.16 using instantaneous centre method.
8.20 A solid sphere of weight $W$ and radius $r$ rolls down an inclined plane, which makes angle $\theta$ to horizontal. Determine the minimum frictional resistance required to prevent slippage and also the corresponding linear acceleration of mass centre.
[Ans. $\mu=1 / 3 \tan \theta$ ]
8.21 A solid cylinder of weight 1000 N and radius 1.2 m is to be rolled up an inclined plane which makes $30^{\circ}$ with the horizontal. Determine force $P$ to be applied on the periphery in the direction parallel to the plane (see Fig. 8.47), such that it has maximum acceleration but no slippage. Take coefficient of friction is 0.2 .
[Ans. 519.61 N ]


Fig. 8.47


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